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Author

Chu, Xuehao

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Xuehao Chu

Working Paper
UCTC No. 199

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**Endogenous Trip Scheduling:
The Henderson Approach Reformulated and Compared
with the Vickrey Approach**

Xuehao Chu

Department of Economics
University of California at Irvine
Irvine, CA 92717

*Working Paper
April 1993*

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Abstract Two approaches to modeling peak-period congestion that account for travelers' scheduling behavior have made their way into the economics literature. On the demand side of both approaches, travelers trade off a cost of travel delay against a cost of being early or late at destination in scheduling their trip. On the supply side, the Vickrey approach uses a queuing-congestion technology; the Henderson approach uses a flow-congestion technology, assuming that the travel time for any traveler is determined by the departure flow he departs with at origin. But the Henderson approach is found to have problems. This paper illustrates these problems; shows that they can be eliminated by assuming that the travel time for any traveler is determined by the arrival flow he arrives with at destination; and compares the behavior of the Vickrey and reformulated Henderson approaches both analytically and using simulations. The paper finds that the behavior of the reformulated Henderson approach varies with its elasticity of travel delay with respect to traffic flow, while the Vickrey approach lacks such a flexibility; and that the behavior of the Vickrey approach is the limit of that of the reformulated Henderson approach as the elasticity of travel delay goes to infinity.

Endogenous Trip Scheduling:
The Henderson Approach Reformulated and Compared with the Vickrey Approach¹

Xuehao Chu
Department of Economics
University of California
Irvine, CA 92717

Two approaches to modeling peak-period congestion that account for travelers' scheduling behavior have made their way into the economics literature. One approach, developed by Vickrey [17], models congestion as queuing behind a single bottleneck. It has subsequently been elaborated in Arnott, de Palma, and Lindsey (ADL) [1, 2], Braid [3], and Small [14]. I will refer to it as the Vickrey approach. The other approach, developed by Henderson [6, 7], models congestion in a flow form by applying a speed-flow function at each instant. I will refer to it as the Henderson approach.

Both approaches focus on the journey to work with a fixed number of commuters traveling on the same highway between home and work. All commuters wish to arrive at work at the same time. This is physically impossible. As a result, each commuter schedules the trip to minimize his trip cost, including a cost of travel delay and a cost of schedule delay (time early or late for work). Both approaches use simplified congestion technologies. In the Vickrey approach, a bottleneck of fixed capacity is assumed along the highway. If the departure flow from home is below capacity, travel is at the free-flow speed; otherwise, a queue develops. In the Henderson approach, speed for any commuter is constant throughout the journey and his travel time is determined solely by the departure flow he departs together. It is implicitly assumed that traffic flows departing at different times are independent. In both approaches, equilibrium obtains

when no commuter can reduce his trip cost by altering his schedule unilaterally. Both approaches have been used to investigate the effects of time-of-day pricing policies and to determine the level of optimal capacity.

Unfortunately, the solutions of the Henderson approach, as formulated in Henderson [6, 7], are not equilibria: commuters can reduce their trip cost by altering their schedule unilaterally. The subtle mistake can be easily missed because trip cost is constant at the solutions, which is necessary for equilibrium, but is often treated as sufficient. In fact, the mistake is made elsewhere in the literature on trip scheduling.²

In this paper I reformulate and compare the Henderson approach with the Vickrey approach. I make four contributions to the literature on trip scheduling. First, I illustrate the problems in the original Henderson approach. I examine the original Henderson model which prohibits lateness (Henderson [7]), and show its lack of equilibria. I then extend the original Henderson model for lateness, and find that equilibria do exist in the extended Henderson model, but have two peculiar features: 1) commuters can arrive earlier by starting later at the priced equilibria; 2) their limits are no longer equilibria as the unit cost of being late goes to infinity.

Second, I reformulate the original Henderson approach by assuming that travel time for any commuter is determined by the arrival flow he arrives with at work rather than by the departure flow he departs with from home. This reformulation no longer requires that traffic flows departing at different times from home be independent. It is easier to solve for equilibrium, and more importantly eliminates the problems in the original Henderson approach just noted.³

Third, I solve the models with closed-form solutions, while Henderson [6, 7] solves the original models numerically. These closed-form solutions make analytical studies possible for assessing the economic implications of these models and allow the reformulated Henderson approach to be readily applied to cost-benefit analyses.

Fourth, I investigate the behavior of the reformulated Henderson approach and compare it with that of the Vickrey approach, both analytically and using simulations. I find that the behavior of the reformulated Henderson approach varies with its elasticity of travel delay with respect to traffic flow, while the Vickrey approach lacks such a flexibility; and that the behavior of the Vickrey approach is the limit of that of the reformulated Henderson approach as the elasticity of travel delay goes to infinity.

There is reason to believe that the two approaches could yield vastly different answers to the same question. For example, it is interesting to economists how the standard method of benefit analysis, which ignores scheduling behavior, would bias the benefits from an incremental expansion in road capacity. The answer seems dependent on the approach used. Using the new formulation for a different scheduling problem for commuters (see footnote 3), Henderson [10] shows that the standard method will always overstate the benefits. Under the Vickrey approach, however, Small [14, pp. 134-135] shows that the standard method may either over- or underestimate the benefits.

I examine the original Henderson approach and illustrate its problems in Section I. I reformulate the original Henderson approach in Section II. I investigate the behavior of the reformulated Henderson approach, and compare it with that of the Vickrey approach in Section III. The first part of Section III reviews the Vickrey approach. I conclude in Section VI.

I. THE HENDERSON APPROACH EXAMINED

This section examines the original Henderson approach and illustrates its problems. Part A reviews the original Henderson model that prohibits lateness (Henderson [7]). An example is used to show that it lacks equilibria. Part B extends the original Henderson model for lateness. The example is used again to show that the extended Henderson model does have equilibria, but contain two peculiar features: 1) commuters can arrive earlier by starting later at the priced equilibrium; 2) their limits are no longer equilibria as the unit cost of being late goes to infinity.

A. ORIGINAL HENDERSON MODEL: LATENESS PROHIBITED

Consider the journey to work, where a fixed number of identical commuters, N , one per vehicle, travel on the same road m miles to work.⁴ All must arrive at work no later than a common work-start time t^* . Each chooses a home-departure time, t , to minimize the private trip cost, $C(t)$, which includes three parts. The first part is the travel time cost, $\alpha[m/S(t)]$, where α is the unit cost of travel time, and $m/S(t)$ is the travel time. The second part is the schedule delay cost, $\beta[t^* - t - m/S(t)]$, where β is the unit cost of schedule delay early--time wasted waiting for work to start, and $t + m/S(t)$ is the arrival time at work. The third part is the toll, $\tau(t)$, if any is imposed. So

$$C(t) = \alpha \frac{m}{S(t)} + \beta \left[t^* - t - \frac{m}{S(t)} \right] + \tau(t) . \quad (1)$$

Let R be the road capacity, $S_{\max}$ the free-flow speed, and $F(t)$ the departure rate at time t . Henderson [7] assumes a power speed-flow function given by

$$\frac{1}{S(t)} = \frac{1}{S_{\max}} + \left[\frac{F(t)}{R} \right]^{\gamma}, \quad (2)$$

following Vickrey [16]. That is, the travel time for any commuter who departs at t is determined solely by the departure flow at the same time t . This congestion technology requires that departure flows at different times be independent. The second term of (2) measures the travel delay associated with departure flow $F(t)$. The parameter γ measures the elasticity of this travel delay with respect to $F(t)$.

1. Unpriced Solution

Equilibrium obtains when no commuter can reduce his trip cost by altering his departure time unilaterally. With identical commuters, it is necessary that the private trip cost be constant across departure times, or

$$\frac{dC(t)}{dt} = (\alpha - \beta) \frac{d}{dt} \left[\frac{m}{S(t)} \right] - \beta = 0, \quad (3)$$

which, given $\alpha > \beta$, implies

$$\frac{d}{dt} \left[\frac{m}{S(t)} \right] = \frac{\beta}{\alpha - \beta} > 0. \quad (4)$$

Let C be the constant private trip cost. Those departing first at i travel at the free-flow speed: $S(i) = S_{\max}$; using (1),

$$C = \alpha T_f + \beta (t^* - i - T_f). \quad (5)$$

where $T_f \equiv m/S_{\max}$, the free-flow travel time. Solving (4) with condition (5) yields the equilibrium travel time function given by

$$\frac{m}{S(t)} = T_f + \frac{\beta}{\alpha - \beta} (t - i) . \quad (6)$$

Let n be the on-time departure time: $n + m/S(n) = t^*$, or

$$t^* - n = T_f + \frac{\beta}{\alpha - \beta} (n - i) . \quad (7)$$

All N commuters depart in $[i, n]$:

$$\int_i^n F(t) dt = N . \quad (8)$$

To solve for i, n , and C analytically, I first use (2) and (6) to write $F(t)$ in terms of t and i , and substitute the resulting $F(t)$ into (8). I then use (7) and (8) to solve for i and n , and use (5) for C . The solution is given by

$$\begin{aligned} \Phi &= \left(\frac{N}{R} \frac{1+\gamma}{\gamma} \frac{\beta}{\alpha - \beta} m^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{1+\gamma}} , \\ i &= t^* - T_f - \frac{\alpha}{\beta} \Phi , \\ n &= t^* - T_f - \Phi , \\ C &= \alpha T_f + \alpha \Phi , \end{aligned} \quad (9)$$

where Φ is the maximum travel delay, which occurs at n .⁵

Total variable cost of travel TVC , total cost of travel delay TCC , and total cost of schedule delay TSC can be calculated as:

$$TCC = \int_i^n \alpha F(t) \left[\frac{m}{S(t)} - T_f \right] dt = \alpha N \Phi \frac{1+\gamma}{1+2\gamma}, \quad (10)$$

$$TSC = \int_i^n \beta F(t) \left[t^* - t - \frac{m}{S(t)} \right] dt = \alpha N \Phi \frac{\gamma}{1+2\gamma}, \quad (11)$$

$$TVC = TCC + TSC = \alpha N \Phi. \quad (12)$$

2. Optimally Priced Solution

To minimize the total cost of transporting N commuters to work, the traffic planner chooses $F(t)$, $t \in [i, n]$, to minimize

$$\int_i^n F(t) \left[\alpha \frac{m}{S(t)} + \beta \left(t^* - t - \frac{m}{S(t)} \right) \right] dt \quad (13)$$

subject to (8). The Lagrangian of this optimal control problem, given i and n , is

$$\mathcal{L} = \int_i^n F(t) \left[\alpha \frac{m}{S(t)} + \beta \left(t^* - t - \frac{m}{S(t)} \right) \right] dt + \lambda \left[N - \int_i^n F(t) dt \right], \quad (14)$$

where λ is the Lagrangian Multiplier of (8). The first order condition with respect to $F(t)$ is given by ⁶

$$\lambda = \alpha \frac{m}{S(t)} + \beta \left[t^* - t - \frac{m}{S(t)} \right] + (\alpha - \beta) F(t) \frac{d}{dF(t)} \left[\frac{m}{S(t)} \right]. \quad (15)$$

Henderson [7] interprets λ as the social cost of transporting the marginal traveller on the road at any departure time t ; so (15) requires that this marginal social cost be equal across all departure times. Commuters privately incur the first two terms in (15); the optimal toll is equal to the third term in (15), or

$$\tau(t) = (\alpha - \beta) F(t) \frac{d}{dF(t)} \left[\frac{m}{S(t)} \right]. \quad (16)$$

Using (1), this can be written as

$$\tau(t) = (\alpha - \beta) \gamma \left[\frac{m}{S(t)} - T_f \right]. \quad (17)$$

With this toll imposed, the private trip cost becomes

$$C(t) = \alpha \frac{m}{S(t)} + \beta \left[t^* - t - \frac{m}{S(t)} \right] + (\alpha - \beta) \gamma \left[\frac{m}{S(t)} - T_f \right]. \quad (18)$$

Equation (5) also holds in the priced equilibrium. Differentiating (18) with respect to t , and solving the resulting equation with condition (5) yields the equilibrium travel time function under optimal pricing

$$\frac{m}{S(t)} = T_f + \frac{1}{1+\gamma} \frac{\beta}{\alpha - \beta} (t - i). \quad (19)$$

The equation that defines n changes from (7) to

$$t^* - n = T_f + \frac{1}{1+\gamma} \frac{\beta}{\alpha - \beta} (n - i). \quad (20)$$

I first use (2) and (18) to solve for $F(t)$, and substitute the resulting $F(t)$ into (8). I then use (8) and (20) for i and n , and use (5) for C . The solution is given by

$$\begin{aligned} i &= t^* - T_f - \Phi \frac{\alpha}{\beta} \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \\ n &= t^* - T_f - \Phi \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \\ C &= \alpha T_f + \alpha \Phi \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \end{aligned} \quad (21)$$

where Φ is given by (9). The aggregate costs can be calculated as

$$TCC = \int_i^n \alpha F(t) \left[\frac{m}{S(t)} - T_f \right] dt = \alpha N \Phi \frac{1+\gamma}{1+2\gamma} \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \quad (22)$$

$$TSC = \int_i^n \beta F(t) \left[t^* - t - \frac{m}{S(t)} \right] dt = \alpha N \Phi \frac{\gamma}{1+2\gamma} \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \quad (23)$$

$$TVC = TCC + TSC = \alpha N \Phi \left[1 + \frac{\gamma^2}{1+2\gamma} \left(1 - \frac{\beta}{\alpha} \right) \right] \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}. \quad (24)$$

3. Lack of Equilibrium

Using the example given in Table 2, this section illustrates the lack of equilibria of the original Henderson approach when lateness is prohibited.

Congestion technology (2) assumes that the travel time for any commuter is determined solely by the departure flow he departs together. This can lead to overtaking, i.e., arriving earlier by starting later. For example, a group departing a bit later than a larger group would arrive earlier. As Henderson [7] notes, by assuming $\alpha > \beta$, no overtaking can happen during the period of departures from i to n . A problem comes after the period of departures, however: commuters can reduce their private trip cost by unilaterally shifting departure to after n . I use the example given in Table 2 to illustrate the problem.

Panels a and b of Figure 1 show the cumulative departures and arrivals for the unpriced and priced solutions respectively.⁷ The slopes of the cumulative curves measure the rates of departure and arrival respectively; the horizontal distance between

the two cumulative curves measures travel time if no overtaking happens; and the horizontal distance between the curve of cumulative arrivals and a vertical line at t^* measures schedule delay. The first group travels with the free-flow speed (i.e., the slopes of the cumulative departure curves at i are zero), but incurs maximum schedule delay; the last group incurs no schedule delay, but travels with maximum travel delay; everyone departs during the period of departures.

At both solutions, no one has an incentive to shift unilaterally either across i because those departing at i travel at the free-flow speed, or among times between i and n because private trip cost is constant during this period. But if one shifts unilaterally across n , one can travel almost at the free-flow speed, and overtake the group departing at n . Instead of spending the time between n and t^* in congestion as the group departing at n does, one spends the time waiting for work to start. Since travel delay is more costly than schedule delay, this unilateral shift lowers one's private trip cost. In fact, one could unilaterally depart shortly before $t^* - m/S_{\max}$ and suffer neither travel nor schedule delay. Thus, the solutions of the original Henderson model are not equilibria.

What contributes to this lack of equilibria? The assumptions of both no lateness and travel time being determined by departure flow play a role. While assuming travel time being determined by departure flow makes overtaking possible, prohibiting lateness leads to a speed at n that is below the free-flow level. To improve the original Henderson model, I relax the assumption of no lateness next, and reconsider the assumption of travel time being determined by departure flow in Section II.

B. EXTENDED HENDERSON MODEL: LATENESS ALLOWED WITH PENALTY

To extend the original Henderson model for lateness, define t as an early departure time and $t^* - t - m/S(t)$ as schedule delay early if $t + m/S(t) - t^*$ is negative. Define t as a late departure time and $t + m/S(t) - t^*$ as schedule delay late if $t + m/S(t) - t^*$ is positive. Define n such that

$$n + \frac{m}{S(n)} - t^* = 0 . \quad (25)$$

Let v be the unit cost of schedule delay late and u the last departure time. The private trip cost is (2) for early departure times; for late departure times it is

$$C(t) = \alpha \frac{m}{S(t)} + v \left[t + \frac{m}{S(t)} - t^* \right] + \tau(t) . \quad (26)$$

1. Unpriced Solution

Those who depart at u do not incur travel delay: $S(u) = S_{\max}$, or

$$C(u) = \alpha T_f + v (u + T_f - t^*) . \quad (27)$$

The equilibrium travel time function is (6) for early departure times. For late departure times, differentiating (26) with respect to t , and solving the resulting equation with condition (27) yields

$$\frac{m}{S(t)} = T_f + \frac{v}{\alpha + v} (u - t) . \quad (28)$$

The private trip costs at i and u must be equal, and all N commuters depart between i and u . That is,

$$\begin{aligned} C(i) &= C(u) , \\ \int_i^u F(t) dt &= N . \end{aligned} \quad (29)$$

I first use (2), (26), (6), and (28) to solve for $F(t)$, and substitute the resulting $F(t)$ into (29). I then use (7) and (29) for i , n , and u , and use (5) for C . The solution is

$$\begin{aligned} \Psi &= \left(\frac{N}{R} \frac{1+\gamma}{\gamma} \frac{\delta}{\alpha} m^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{1+\gamma}} , \\ i &= t^* - T_f - \Psi \frac{\alpha}{\beta} , \\ n &= t^* - T_f - \Psi , \\ u &= t^* - T_f + \Psi \frac{\alpha}{\nu} , \\ C &= \alpha T_f + \alpha \Psi , \end{aligned} \quad (30)$$

where $\delta = \beta\nu/(\beta+\nu)$, and Ψ is the maximum travel delay, which occurs at n . The aggregate costs can be calculated as:

$$TCC = \int_i^u \alpha F(t) \left[\frac{m}{S(t)} - T_f \right] dt = \alpha N \Psi \frac{1+\gamma}{1+2\gamma} , \quad (31)$$

$$TSC = \int_i^n \beta F(t) \left[t^* - t - \frac{m}{S(t)} \right] dt + \int_n^u F(t) \left[t + \frac{m}{S(t)} - t^* \right] dt = \alpha N \Psi \frac{\gamma}{1+2\gamma} , \quad (32)$$

2. Optimally Priced Solution

If there is a marginal-cost toll to support the social optimum where the total variable cost of transporting N commuters to work is minimized, the traffic planner needs to choose $F(t)$ to minimize

$$\int_i^n F(t) \left[\alpha \frac{m}{S(t)} + \beta \left(t^* - t - \frac{m}{S(t)} \right) \right] dt + \int_n^u F(t) \left[\alpha \frac{m}{S(t)} + v \left(t + \frac{m}{S(t)} - t^* \right) \right] dt \quad (34)$$

subject to the second constraint of (29). Let λ be the Lagrangian Multiplier of this constraint; then the Lagrangian for this optimal control problem, given i and u , is

$$\begin{aligned} \mathcal{L} = & \int_i^n F(t) \left[\alpha \frac{m}{S(t)} + \beta \left(t^* - t - \frac{m}{S(t)} \right) \right] dt + \int_n^u F(t) \left[\alpha \frac{m}{S(t)} + v \left(t + \frac{m}{S(t)} - t^* \right) \right] dt \\ & + \lambda \left[N - \int_i^u F(t) dt \right]. \end{aligned} \quad (35)$$

The first order condition with respect to $F(t)$ is (15) for early departure time; for late departure time, it is

$$\lambda = \alpha \frac{m}{S(t)} + v \left(t + \frac{m}{S(t)} - t^* \right) + (\alpha + v) F(t) \frac{d}{dF(t)} \left[\frac{m}{S(t)} \right]. \quad (36)$$

The constant λ can be interpreted as the marginal social cost of departing at any time t ; so (15) and (36) requires that this cost be equal across all departure times. Commuters privately incur the first two terms in (15) for early departure times, and in (36) for late departure times. The optimal toll is (17) for early departure times; for late departure times, it is

$$\tau(t) = (\alpha + v) \gamma \left[\frac{m}{S(t)} - T_f \right]. \quad (37)$$

The private trip cost is (18) for early departure times; it is

$$C(t) = \alpha \frac{m}{S(t)} + v \left[t + \frac{m}{S(t)} - t^* \right] + (\alpha + v) \gamma \left[\frac{m}{S(t)} - T_f \right] \quad (38)$$

for late departure times.

The equilibrium travel time function for early departure times is (19); for late

departure times, differentiating (38) with respect to t , and solving the resulting equation with condition (27) yields

$$\frac{m}{S(t)} = T_f + \frac{1}{1+\gamma} \frac{v}{\alpha+v} (u-t) . \quad (39)$$

Equations in (29) still hold; but the equation that defines n changes to

$$t^*-n = T_f + \frac{1}{1+\gamma} \frac{\beta}{\alpha-\beta} (n-i) . \quad (40)$$

I first use (18), (38), (19), and (39) to get $F(t)$, and substitute the resulting $F(t)$ into (29). I then use (29) and (40) for i , n , and u , and use (5) for C . The solution is

$$\begin{aligned} \theta &= \frac{1+\gamma \frac{\alpha-\beta}{\alpha+v}}{1+\gamma} , \quad \Gamma = \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}} \left[\frac{\alpha/\delta}{\frac{\alpha-\beta}{\beta} + \frac{\alpha+v}{v} \theta^{\frac{1+\gamma}{\gamma}}} \right]^{\frac{\gamma}{1+\gamma}} , \\ i &= t^* - T_f - \frac{\alpha}{\beta} \Gamma \Psi \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] , \\ n &= t^* - T_f - \Gamma \Psi , \\ u &= t^* - T_f + \frac{\alpha}{v} \Gamma \Psi \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] , \\ C &= \alpha T_f + \alpha \Gamma \Psi \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \right] , \end{aligned} \quad (41)$$

where Ψ is given by (30). The aggregate costs can be calculated as

$$\begin{aligned} TCC &= \alpha N \Psi \Gamma \Gamma_1 \frac{1+\gamma}{1+2\gamma} , \\ TSC &= \alpha N \Psi \Gamma \left[\left(1 - \frac{\gamma}{1+2\gamma} \Gamma_1 \right) + \gamma \left(1 - \frac{\beta}{\alpha} \right) \left(1 - \frac{1+\gamma}{1+2\gamma} \Gamma_2 \right) \right] , \\ TVC &= \alpha N \Psi \Gamma \left[1 + \gamma \left(1 - \frac{\beta}{\alpha} \right) \left(1 - \frac{1+\gamma}{1+2\gamma} \Gamma_2 \right) \right] , \end{aligned} \quad (42)$$

where Γ_1 and Γ_2 are given by

$$\Gamma_1 = \frac{\frac{\alpha - \beta}{\beta} + \frac{\alpha + v}{v} \theta^{\frac{1+2\gamma}{\gamma}}}{\frac{\alpha - \beta}{\beta} + \frac{\alpha + v}{v} \theta^{\frac{1+\gamma}{\gamma}}}, \quad \Gamma_2 = \frac{\frac{(\alpha - \beta)^2}{\beta} + \frac{(\alpha + v)^2}{v} \theta^{\frac{1+2\gamma}{\gamma}}}{\frac{(\alpha - \beta)^2}{\beta} + \frac{(\alpha + v)(\alpha - \beta)}{v} \theta^{\frac{1+\gamma}{\gamma}}}. \quad (43)$$

3. Overtaking and Limiting Solution

This section first examines whether the above solutions are equilibria, and if they are, whether overtaking can happen. Congestion technology (2) requires that traffic flows that depart at different times from home be independent. But overtaking violates this independence. The section then examines the limit of each equilibrium as the unit cost of being late goes infinity. One would expect that such limits would still be equilibria if there is no problem within the model.

Panels a and b of Figure 2 show the cumulative departures and arrivals for the unpriced and priced solutions respectively, using parameter values in Table 2. The dashed lines in panel b are due to overtaking and explained below. Private trip costs are the same across departure times; speeds are at the free-flow level at both ends of the period of departures. Therefore no one has an incentive to shift unilaterally either within or outside the period; both solutions are equilibria.

To examine the possibility of overtaking in equilibrium, I examine arrival times $t + m/S(t)$. Using the equilibrium travel time functions in (6) and (28) for the unpriced equilibrium, and in (19) and (39) for the priced equilibrium, I get $d[t + m/S(t)]/dt > 0$ within the periods of early and late departures respectively. That is, in each of the periods before and after the on-time departure, the change in travel delay for those

traveling later is not enough for them to catch up given the difference in departure times. So no overtaking happens within each of the two periods.

No overtaking within each period, however, does not rule out overtaking across the two periods. Whether overtaking happens depends on whether travel delay drops suddenly across the two periods. At the unpriced equilibrium, the only cost for those departing at and immediately after n is travel time. Given identical unit costs of travel time, the condition of equal private trip costs requires travel times to be equal at and immediately after n . This rules out overtaking at the unpriced equilibrium.

At the priced equilibrium, however, the condition of equal private trip costs does not require equal travel times across n because the toll imposed can make up the difference. In fact, the toll given by (17) and (37) jumps discontinuously across n . The result is a discontinuous drop in both toll and departure rate across n . Figure 3 shows the toll schedule and departure rates. So overtaking happens across n at the priced equilibrium.

I now explain the two dashed lines in panel b of Figure 2. For ease of reference, let n' be the arrival time for those who depart just after n , n_1 the departure time for those who depart before n and arrive at n' , and n_2 the departure time for those who depart after n and arrive at t^* . Those who depart between n and n_2 overtake those who depart between n_1 and n . The two dashed lines show the cumulative arrivals for the two segments of the curve of cumulative departures from n_1 to n and from n to n_2 respectively. The real curve of cumulative arrivals between n' and t^* is obtained by adding the number of overtaking vehicles, i.e., those departing between n and n_2 and arriving between n' and t^* , to the lower dashed line.

I now examine the limits of the equilibria and show that they are no longer equilibria. When the unit cost of schedule delay late v approaches infinity, δ approaches β ; Ψ and the last departure time at both solutions become:

$$\Psi = \left(\frac{N}{R} \frac{1+\gamma}{\gamma} \frac{\beta}{\alpha} m^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{1+\gamma}}, \quad (44)$$

$$u = t^* - T_f.$$

Figure 4 shows the cumulative departures and arrivals of the limits. In the limit of the unpriced equilibrium, departures occur after the on-time departure. Those who depart after the on-time departure also arrive on time, but incur different travel delays. This cannot be true in equilibrium. This is why the unpriced solution shown in Figure 1a looks entirely different from this limit; in fact neither is a true equilibrium.

The limit of the priced equilibrium in Figure 4b is not an equilibrium either. Compare the private trip costs at i and u , at which there are no travel delay or toll. But departure at i incurs schedule delay as well as the free-flow travel time, while departure at u incurs only the free-flow travel time.

To explain the two dashed lines, let n' be the arrival time for those who depart just after n and n_1 the departure time for those who depart before n and arrive at n' . Those who depart between n and u overtake those who depart between n_1 and n . The two dashed lines show the cumulative arrivals for the two segments of the curve of cumulative departures from n_1 to n and from n to u respectively. The real curve of cumulative arrivals between n' and t^* is obtained by adding the number of overtaking vehicles, i.e., those departing between n and u and arriving between n' and t^* , to the lower dashed line.

II. THE HENDERSON APPROACH REFORMULATED

Congestion technology (2) assumes that travel time for any commuter is determined solely by the departure flow he departs with from home; and that traffic flows that depart at different times are independent. The previous section shows that with these two assumptions the original Henderson approach has problems: a) the original Henderson model that prohibits lateness lacks equilibria; b) equilibria do exist in the extended Henderson model that allows lateness, but have two peculiar features: 1) commuters can arrive earlier by starting later at the priced equilibrium; 2) their limits are no longer equilibria as the unit cost of being late goes to infinity.

The assumption of travel time being determined by departure flow plays a key role in the existence of these problems: it makes overtaking possible. When lateness is prohibited, the speed at the end of the departure period is below the free-flow level. This low speed at the end of the departure period and the possibility of overtaking after the departure period create an incentive to shift schedule across the end of the departure period. When lateness is allowed, on the other hand, overtaking that occurs in the priced equilibrium violates the assumption of independent traffic flows.

This section reformulates the original Henderson approach by assuming that the travel time for any commuter is determined solely by the arrival flow he arrives with at work. This reformulation no longer requires that traffic flows that depart at different times from home are independent.

This new formulation is just as plausible as that in Henderson [7]: both are approximations to reality. In fact, it was used, without comments, for a different

scheduling problem of commuters by Henderson [8, 9] to model production effects of staggered work hours, and by Henderson [10] to investigate the biases inherent in cost-benefit analyses of capacity expansion that ignores scheduling behavior (see footnote 3 for the nature of the scheduling problem).

So the reformulated Henderson approach modifies congestion technology (2) so that travel speed, $s(t')$, is determined by arrival flow, $f(t')$, through

$$\frac{1}{s(t')} = \frac{1}{S_{\max}} + \left[\frac{f(t')}{R} \right]^{\gamma}, \quad (2')$$

where t' is any arrival time at work. The elasticity γ may not be the same as in (2). Since the solution method for the reformulated Henderson approach is the same as for the original Henderson approach, I will use the same numbers with a prime added for similar equations. For example, the prime in (2') indicates that it is similar to (2).

A. REFORMULATED HENDERSON MODEL: LATENESS PROHIBITED

Each commuter chooses an arrival time t' , no later than t^* , to minimize

$$c(t') = \alpha \frac{m}{s(t')} + B(t^* - t') + p(t'), \quad (1')$$

where $p(t')$ is the toll schedule if any is imposed.

1. Unpriced Equilibrium

Equilibrium requires travel time to change at the following rate:

$$\frac{d}{dt'} \left[\frac{m}{s(t')} \right] = \frac{B}{\alpha} > 0. \quad (4')$$

The private trip cost of arriving first at i' is given by

$$c = \alpha T_f + \beta (t^* - i') . \quad (5')$$

The equilibrium travel time function is given by

$$\frac{m}{s(t')} = T_f + \frac{\beta}{\alpha} (t' - i') . \quad (6')$$

The last group arrives at t^* , and all N commuters arrive in $[i', t^*]$:

$$\int_{i'}^{t^*} f(t') dt' = N . \quad (8')$$

I first use (2') and (6') to get $f(t')$, and substitute the resulting $f(t')$ into (8'). I then use (8') for i' , and use (5') for c . The solution is given by

$$\begin{aligned} \Phi' &= \left(\frac{N}{R} \frac{1+\gamma}{\gamma} \frac{\beta}{\alpha} m^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{1+\gamma}} , \\ i' &= t^* - \frac{\alpha}{\beta} \Phi' , \\ c &= \alpha T_f + \alpha \Phi' , \end{aligned} \quad (9'a)$$

where Φ' is the maximum travel delay, which occurs at t^* . For later comparison, the first and last departure times, i and n , are given by

$$\begin{aligned} i &= i' - T_f = t^* - T_f - \frac{\alpha}{\beta} \Phi' , \\ n &= t^* - T_f - \Phi' . \end{aligned} \quad (9'b)$$

Once i' and c are determined, the aggregate costs can be calculated as

$$TCC = \int_{i'}^{t^*} \alpha f(t') \left[\frac{m}{s(t')} - T_f \right] dt' = \alpha N \Phi' \frac{1+\gamma}{1+2\gamma} , \quad (10')$$

$$TSC = \int_{t'}^{t^*} \beta f(t')(t^* - t') dt' = \alpha N \Phi' \frac{\gamma}{1+2\gamma}, \quad (11')$$

$$TVC = TCC + TSC = \alpha N \Phi'. \quad (12')$$

I can compare the unpriced solution of the reformulated Henderson approach, given by (9')-(12'), with that of the original Henderson approach, given by (9)-(12), assuming that the elasticities of travel delay of the two approaches in (2) and (2') are the same. The maximum travel delay Φ' , given by (9'), is less than Φ , given by (9). Given the parameter values in Table 2, $\Phi' = 0.466$ and $\Phi = 0.991$. Comparing (9) with (9'), the period of departures both starts and ends later in the reformulated Henderson approach. Comparing (10)-(12) with (10')-(12'), aggregate costs are smaller in the reformulated Henderson approach.

One possible explanation for this difference between the original and reformulated Henderson approaches is that at the unpriced solution of the original Henderson approach, commuters can lower their private trip cost by shifting departures later. Without a full adjustment of departures to reach an equilibrium, the period of departures both starts and ends too early; aggregate costs are too high.

2. Optimally Priced Equilibrium

The traffic planner chooses $f(t')$ to minimize

$$\int_{t'}^{t^*} f(t') \left[\alpha \frac{m}{s(t')} + \beta (t^* - t') \right] dt' \quad (13')$$

subject to (8'). Let λ be the Lagrangian Multiplier of (8'); then the Lagrangian for

this optimal control problem, given i' , is

$$\mathcal{L} = \int_{i'}^{t^*} f(t') \left[\alpha \frac{m}{s(t')} + \beta(t^* - t') \right] dt' + \lambda \left[N - \int_{i'}^{t^*} f(t') dt' \right]. \quad (14')$$

The first order condition with respect to $f(t')$ is

$$\lambda = \alpha \frac{m}{s(t')} + \beta(t^* - t') + \alpha f(t') \frac{d}{df(t')} \left[\frac{m}{s(t')} \right]. \quad (15')$$

The constant λ can be interpreted as the marginal social cost of arriving at any time t' ; (15') requires that this cost be equal across all arrival times. Each commuter privately incurs the first two terms in (15'); the optimal toll is the third term, or

$$p(t') = \alpha f(t') \frac{d}{df(t')} \left[\frac{m}{s(t')} \right]. \quad (16')$$

Using (2'), this can be written as

$$p(t') = \alpha \gamma \left[\frac{m}{s(t')} - T_f \right]. \quad (17')$$

With this toll imposed, the private trip cost of arriving at t' becomes

$$c(t') = \alpha \frac{m}{s(t')} + \beta(t^* - t') + \alpha \gamma \left[\frac{m}{s(t')} - T_f \right]. \quad (18')$$

The equilibrium travel time function becomes

$$\frac{m}{s(t')} = T_f + \frac{1}{1+\gamma} \frac{\beta}{\alpha} (t' - i'). \quad (19')$$

I first use (2') and (19') to solve for $f(t')$, and substitute the resulting $f(t')$ into (8'). I then use (8') for i' , and use (5') for c . The solution is given by

$$\begin{aligned}
i' &= t^* - \Phi' \frac{\alpha}{\beta} (1+\gamma)^{\frac{1}{1+\gamma}}, \\
c &= \alpha T_f + \alpha \Phi' (1+\gamma)^{\frac{1}{1+\gamma}},
\end{aligned} \tag{21'a}$$

where Φ' is given by (9'). For later comparison, i and n are given by

$$\begin{aligned}
i &= i' - T_f = t^* - T_f - \Phi' \frac{\alpha}{\beta} (1+\gamma)^{\frac{1}{1+\gamma}}, \\
n &= t^* - T_f - \Phi' \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}.
\end{aligned} \tag{21'b}$$

The aggregate costs can be calculated as

$$TCC = \int_{i'}^{t^*} \alpha f(t') \left[\frac{m}{s(t')} - T_f \right] dt' = \alpha N \Phi' \frac{1}{1+2\gamma} (1+\gamma)^{\frac{\gamma}{1+\gamma}}, \tag{22'}$$

$$TSC = \int_{i'}^{t^*} \beta f(t') (t^* - t') dt' = \alpha N \Phi' \frac{\gamma}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}}, \tag{23'}$$

$$TVC = TCC + TSC = \alpha N \Phi' \frac{1+\gamma}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}}. \tag{24'}$$

I can again compare the priced solutions of the original and reformulated Henderson approaches, given by (21)-(24) and (21')-(24') respectively, assuming the same elasticities of travel delay in the two approaches. Since Φ' is less than Φ , the reformulated Henderson approach gives a period of departures that ends later (using (21) and (21')); it also gives smaller total cost of travel delay (using (22) and (22')).

The relative values in the first departure time, total cost of schedule delay, and total variable cost of travel all depend on the relative values of $(1+\gamma)\Phi'$ and $[1+\gamma(1-\beta/\alpha)]\Phi$. Given the parameter values in Table 2, $(1+\gamma)\Phi' = 2.365$ and

$[1 + \gamma(1 - \beta/\alpha)]\Phi = 2.569$. This leads to the same result as for the unpriced solutions: the period of departures both starts and ends later, and aggregate costs are smaller in the reformulated Henderson approach than the original Henderson approach.

B. REFORMULATED HENDERSON MODEL: LATENESS ALLOWED

If one arrives after t^* , one is late by an amount $t' - t^*$. I will refer to $t' > t^*$ as late arrival times. The private cost is (1') for $t' \leq t^*$; for $t' > t^*$, it is

$$c(t') = \alpha \frac{m}{s(t')} + v(t' - t^*) + p(t') . \quad (26')$$

1. Unpriced Equilibrium

Those who arrive at the end, u' , incur no travel delay: $s(u') = S_{\max}$, or

$$c(u') = \alpha T_f + v(u' - t^*) . \quad (27')$$

The equilibrium travel time function is (6') for $t' \leq t^*$. For $t' > t^*$, differentiating (26') with respect to t' , and solving the resulting equation with condition (27') yields

$$\frac{m}{s(t')} = T_f + \frac{v}{\alpha}(u' - t') . \quad (28')$$

The private trip costs at i' and u' must be equal, and all N commuters arrive between i' and u' . That is,

$$\begin{aligned} c(i') &= c(u') , \\ \int_{i'}^{u'} f(t') dt' &= N . \end{aligned} \quad (29')$$

I first use (1') and (28') to solve for $f(t')$, and substitute the resulting $f(t')$ into

(29'); I then use (29') for i' and u' , and use (5') for c . The solution is given by

$$\begin{aligned} i' &= t^* - \Psi \frac{\alpha}{\beta}, \\ u' &= t^* + \Psi \frac{\alpha}{\nu}, \\ c &= \alpha T_f + \alpha \Psi, \end{aligned} \quad (30' a)$$

where Ψ , given by (30), is the maximum travel delay, which occurs at t^* . For later comparison, the first, on-time, and last departure times, i , n , and u , are given by

$$\begin{aligned} i &= i' - T_f = t^* - T_f - \Psi \frac{\alpha}{\beta}, \\ n &= t^* - T_f - \Psi, \\ u &= u' - T_f = t^* - T_f + \Psi \frac{\alpha}{\nu}. \end{aligned} \quad (30' b)$$

The aggregate costs can be calculated as

$$TCC = \int_{i'}^{u'} \alpha f(t') \left[\frac{m}{s(t')} - T_f \right] dt' = \alpha N \Psi \frac{1+\gamma}{1+2\gamma}, \quad (31')$$

$$TSC = \int_{i'}^{t^*} \beta f(t') (t^* - t') dt' + \int_{t^*}^{u'} \nu f(t') (t' - t^*) dt' = \alpha N \Psi \frac{\gamma}{1+2\gamma}, \quad (32')$$

$$TVC = TCC + TSC = \alpha N \Psi. \quad (33')$$

Again, I can compare the unpriced solutions of the original and reformulated Henderson approaches given by (30)-(33) and (30')-(33') respectively. The unpriced solutions are identical if the elasticities of travel delay are the same. This is no surprise because unlike the unpriced solution without lateness, the unpriced solution with lateness in the original Henderson approach is a real equilibrium without overtaking.

I can also compare the aggregate costs of the reformulated Henderson approach

at the unpriced equilibria with and without lateness, given by (31')-(33') and (10')-(12') respectively. Allowing lateness saves aggregate costs by a fraction of $(\Phi' - \Psi)/\Phi' = 1 - (v/(\beta + v))^{\gamma/(1+\gamma)}$, where Ψ and Φ' are given by (30) and (9') respectively. The larger the elasticity of travel delay γ , or the larger the relative unit costs of being early and late, the larger is this saving. Given the parameter values in Table 2, this is about a 17 percent saving.

2. Optimally Priced Solution

The traffic planner chooses $f(t')$ to minimize

$$\int_{i'}^{t^*} f(t') \left[\alpha \frac{m}{s(t')} + \beta (t^* - t') \right] dt' + \int_{t^*}^{u'} f(t') \left[\alpha \frac{m}{s(t')} + v (t' - t^*) \right] dt' \quad (34')$$

subject to the second constraint in (29'). Let λ be the Lagrangian Multiplier of this constraint; then the Lagrangian for this optimal control problem, given i' and u' , is

$$\begin{aligned} \mathcal{L} = & \int_{i'}^{t^*} f(t') \left[\alpha \frac{m}{s(t')} + \beta (t^* - t') \right] dt' + \int_{t^*}^{u'} f(t') \left[\alpha \frac{m}{s(t')} + v (t' - t^*) \right] dt' \\ & + \lambda \left[N - \int_{i'}^{u'} f(t') dt' \right]. \end{aligned} \quad (35')$$

The first order condition with respect to $f(t')$ for $t' \leq t^*$ is (15'); for $t' > t^*$, it is

$$\lambda = \alpha \frac{m}{s(t')} + v (t' - t^*) + \alpha f(t') \frac{d}{df(t')} \left[\frac{m}{s(t')} \right]. \quad (36')$$

The constant λ can be interpreted as the marginal social cost of arriving at any time t' . Commuters privately incur the first two terms in (15') for $t' \leq t^*$, and in (36') for $t' > t^*$; the optimal toll is given by (16'), the third term in (15') or (36').

The private trip cost then is (18') for $t' \leq t^*$; when $t' > t^*$, it is

$$c(t') = \alpha \frac{m}{s(t')} + v(t' - t^*) + \alpha \gamma \left[\frac{m}{s(t')} - T_f \right]. \quad (38')$$

The equilibrium travel time function for $t' \leq t^*$ is (19'); for $t' > t^*$, it is

$$\frac{m}{s(t')} = T_f + \frac{1}{1+\gamma} \frac{v}{\alpha} (u' - t'). \quad (39')$$

Solving (29') for i' and u' , and using (5') for c yields

$$\begin{aligned} i' &= t^* - \frac{\alpha}{\beta} \Psi (1+\gamma)^{\frac{1}{1+\gamma}}, \\ u' &= t^* + \frac{\alpha}{v} \Psi (1+\gamma)^{\frac{1}{1+\gamma}}, \\ c &= \alpha T_f + \alpha \Psi (1+\gamma)^{\frac{1}{1+\gamma}}, \end{aligned} \quad (41' a)$$

where Ψ is given in (30), with γ being the elasticity of travel delay with respect to arrival flow of the reformulated Henderson approach. For later comparison, the first, on-time, and last departure times, i , n , and u , are given by

$$\begin{aligned} i &= i' - T_f = t^* - T_f - \frac{\alpha}{\beta} \Psi (1+\gamma)^{\frac{1}{1+\gamma}}, \\ n &= t^* - T_f - \Psi \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}}, \\ u &= u' - T_f = t^* - T_f + \frac{\alpha}{v} \Psi (1+\gamma)^{\frac{1}{1+\gamma}}. \end{aligned} \quad (41' b)$$

The aggregate costs can be calculated as

$$TCC = \int_{i'}^{u'} \alpha f(t') \left[\frac{m}{s(t')} - T_f \right] dt' = \alpha N \Psi \frac{1}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}}, \quad (42' a)$$

$$TSC = \int_{i'}^{t^*} B f(t') (t^* - t') dt' + \int_{t^*}^{u'} v f(t') (t' - t^*) dt' = \alpha N \Psi \frac{\gamma}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}}, \quad (42' b)$$

$$TVC = TCC + TSC = \alpha N \Psi \frac{1+\gamma}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}}. \quad (42' c)$$

3. Cumulatives and Limiting Equilibria

This section uses the example in Table 2 to illustrate that the reformulated Henderson approach is free of the problems noted earlier in the original Henderson approach.

Panels a and b of Figure 5 show the cumulative arrivals and departures for the unpriced and priced solutions, respectively, without lateness. Arrivals start at i' and end at t^* . Private costs are the same across arrival times; there is no incentive for any commuter to change arrival time within $[i', t^*]$. Commuters arriving at i' travel at the free-flow speed; arriving earlier than i' is worse off. So is arriving later than t^* . So both solutions in Figure 5 are equilibria.

Panels a and b of Figure 6 show the unpriced and priced solutions with lateness. Private cost associated with any arrival time between i' and u' is the same; nobody can do better by changing arrival time within this period of arrivals. Neither can anybody outside the period because travel associated with the start and end of the period is at the free-flow speed.

The equilibria with lateness converge to those without lateness as the unit cost of being late v goes to infinity; therefore the curves of cumulative departures and arrivals for the limiting equilibria are not shown separately.

III. A COMPARISON WITH THE VICKREY APPROACH

This section compares the behavior of the reformulated Henderson approach with the Vickrey approach, focusing on: a) pattern of travel; b) five known results of the Vickrey approach. Part A reviews the Vickrey approach, based on ADL [1]; Part B presents the comparison. ADL [1] formalize the model in Vickrey [17], and focus on the characteristics of equilibrium with various pricing schemes. Since I follow ADL [1] closely in the review of the Vickrey model, I will use the term "Vickrey-ADL" model in the detailed discussions. Models with lateness are used.

A. THE VICKREY MODEL: LATENESS ALLOWED

ADL [1] set up the journey to work as follows. Travel is not congested except at a single segment of the road (the bottleneck) through which at most k vehicles can pass per unit of time; if the departure rate exceeds k , a queue develops at the bottleneck. The length of queue for those leaving home at t is given by

$$Q(t) = \int_t^{\infty} [F(u) - k] du , \quad (45)$$

where the low limit is the last departure time before t when there was no queue.

$Q(t)/k$ is their queuing delay. Let $T(t)$ be the travel time ; then

$$T(t) = T_f + \frac{Q(t)}{k} . \quad (46)$$

Each commuter chooses a departure time t to minimize the private trip cost

$$C(t) = \begin{cases} \alpha T(t) + \beta (t^* - t - T(t)) + \tau(t) & \text{for } t \leq n, \\ \alpha T(t) + v (t + T(t) - t^*) + \tau(t) & \text{for } t > n. \end{cases} \quad (47)$$

where $\tau(t)$ is the toll schedule if any is imposed.

1. Unpriced Equilibrium

As ADL [1] show, all commuters except the first and last experience congestion, and they depart home at a piecewise constant rate given by

$$F(t) = \begin{cases} \frac{\alpha k}{\alpha - \beta} & \text{for } t \leq n, \\ \frac{\alpha k}{\alpha + v} & \text{for } t > n. \end{cases} \quad (48)$$

The equilibrium travel-time function is given by

$$T(t) = \begin{cases} T_f + \frac{\beta}{\alpha - \beta} (t - i) & \text{for } t \leq n, \\ T_f + \frac{v}{\alpha + v} (u - t) & \text{for } t > n. \end{cases} \quad (49)$$

The first, one-time, and last departure times, i , n , u , can be solved with

$$\begin{aligned} N &= (n - i) \frac{\alpha k}{\alpha - \beta} + (u - n) \frac{\alpha k}{\alpha + v}, \\ \alpha T_f + \beta (t^* - i - T_f) &= \alpha T_f + v (u + T_f - t^*), \\ t^* - n &= T_f + \frac{\beta}{\alpha - \beta} (n - i). \end{aligned} \quad (50)$$

The first states that all N commuters leave home between i and u ; the second specifies that the private costs are the same at i and u ; and the last defines n .

Solving (50) yields

$$\begin{aligned} i &= t^* - T_f - \frac{\delta}{\beta} \frac{N}{k} , \\ n &= t^* - T_f - \frac{\delta}{\alpha} \frac{N}{k} , \\ u &= t^* - T_f + \frac{\delta}{v} \frac{N}{k} , \end{aligned} \quad (51)$$

where $(\delta/\alpha)(N/k)$ is the maximum queuing delay, which occurs at n . Substituting i into (50) yields the constant private trip cost

$$C = \delta \frac{N}{k} . \quad (52)$$

The aggregate costs are calculated as

$$TCC = \int_i^u \alpha F(t) [T(t) - T_f] dt = \frac{\delta}{2} \frac{N^2}{k} , \quad (53)$$

$$TSC = \int_i^n \beta F(t) [t^* - t - T(t)] dt + \int_n^u v F(t) [t + T(t) - t^*] dt = TCC , \quad (54)$$

$$TVC = TCC + TSC = \delta \frac{N^2}{k} . \quad (55)$$

At the unpriced equilibrium, total cost of schedule delay is half of total variable cost of travel; total variable cost of travel is independent of the unit cost of travel time α .

2. Optimally Priced Equilibrium

ADL [1] show that at the social optimum at which total variable cost of travel is minimized, there should be no queuing. It follows that at the social optimum,

$F(t) = k$ for $t \in [i, u]$, and speed of travel is constant at m/T_f .

This social optimum can be decentralized by a time-varying toll over $[i, u]$:

$$\tau(t) = \begin{cases} \delta \frac{N}{k} - \beta [t^* - t - T_f] & \text{for } t \leq n , \\ \delta \frac{N}{k} - v [t + T_f - t^*] & \text{for } t > n . \end{cases} \quad (56)$$

With (51), this toll can be written alternatively as

$$\tau(t) = \begin{cases} \beta (t - i) & \text{for } t \leq n , \\ v (u - t) & \text{for } t > n . \end{cases} \quad (57)$$

As ADL [1] show, this optimal toll does not change the period of arrivals, the private trip cost, or total cost of schedule delay; but it eliminates queuing and thereby cuts total variable cost of travel in half. That is, i and u are the same as in (51), C is the same as in (52), but n changes to

$$n = t^* - T_f . \quad (58)$$

The aggregate costs are $TCC = 0$, $TSC = \delta N^2/2k$, and $TVC = \delta N^2/2k$.

B. COMPARISON

The comparison focuses on: a) pattern of travel; b) five known results of the Vickrey-ADL model. Travel patterns are compared numerically. The five results are examined analytically as well as numerically.

The five results of the Vickrey-ADL model are: 1) total cost of schedule delay is half of total variable cost of travel at the unpriced equilibrium; 2) the optimal toll saves 100 percent of total cost of travel delay, 0 percent of total cost of schedule delay, and 50 percent of total variable cost of travel; 3) the optimal toll does not change the period of

arrivals; 4) the optimal toll does not change the equilibrium private trip cost; and 5) total variable cost of travel is independent of the unit cost of travel time at both the priced and unpriced equilibria.

1. *Analytical*

In the reformulated Henderson model, the five results of the Vickrey-ADL model do not hold for any finite value of γ :

1) With (32')-(33'), the ratio between total cost of schedule delay and total variable cost of travel (*SDR*) at the unpriced equilibrium is given by

$$SDR = \frac{\gamma}{1+2\gamma} < \frac{1}{2} . \quad (59)$$

2) With (31')-(33') and (42'), the fractional savings in total cost of travel delay (*STCC*), total cost of schedule delay (*STSC*), and total variable cost of travel (*STVC*) due to optimal pricing are given respectively by

$$STCC = 1 - \left(\frac{1}{1+\gamma} \right)^{\frac{\gamma}{1+\gamma}} < 1 , \quad (60)$$

$$STSC = 1 - (1+\gamma)^{\frac{1}{1+\gamma}} < 0 , \quad (61)$$

$$STVC = 1 - \frac{1+\gamma}{1+2\gamma} (1+\gamma)^{\frac{1}{1+\gamma}} < \frac{1}{2} . \quad (62)$$

STSC is negative, but both *STCC* and *STVC* are positive. The increase in total cost of schedule delay due to optimal pricing is the result of travel being spread over a wider interval; but it is more than offset by the saving in total cost of travel delay.

3) With (30') and (41'), the fractional lengthening of the period of arrivals due to optimal pricing is given by

$$\Delta(u' - i') = (1 + \gamma)^{\frac{1}{1+\gamma}} - 1 > 0 . \quad (63)$$

The optimal toll lengthens the period of arrivals by forcing the first arrival earlier and the last one later. The changes in the first and last arrival times due to optimal pricing are given respectively by

$$\begin{aligned} \Delta i' &= \Psi \frac{\alpha}{\beta} \left[1 - (1 + \gamma)^{\frac{1}{1+\gamma}} \right] < 0 , \\ \Delta u' &= \Psi \frac{\alpha}{v} \left[(1 + \gamma)^{\frac{1}{1+\gamma}} - 1 \right] > 0 . \end{aligned} \quad (64)$$

4) With (30') and (41') again, the fractional change in the private trip cost due to optimal pricing is given by

$$\Delta c = (1 + \gamma)^{\frac{1}{1+\gamma}} - 1 > 0 . \quad (65)$$

The optimal toll increases the equilibrium private trip cost.

5) With (30), (33'), and (42'), total variable cost of travel depends on the unit cost of travel time α , with a factor of $\alpha^{1/(1+\gamma)}$, at both the priced and unpriced equilibria. It also depends on the schedule-delay parameters β and v through δ , just as in the Vickrey-ADL model.

The five results hold, however, in the limit as γ goes to infinity. This is because as γ goes to infinity, Ψ goes to $(\delta/\alpha)(N/R)$ and $(1 + \gamma)^{1/(1+\gamma)}$ goes to unity. In the limit, the equilibria of the reformulated Henderson model become exactly the same as

those of the Vickrey-ADL model, with k replaced by R . The intuition behind this limiting behavior of the reformulated Henderson model is that the speed-flow function (2') approaches a shape as 1, which is exactly the relationship implicitly assumed in the Vickrey-ADL model.

2. Numerical

The models are specified to compare the patterns of travel between the two models, and to examine to what extent the five results of the Vickrey-ADL model hold in the reformulated Henderson model.

Model Specification

One difficulty of this specification is to find appropriate values of supply-side parameters of the two models so that they are comparable; only then is comparison between the two approaches helpful. The parameter values in Table 2 are for the reformulated Henderson model. The value for the bottleneck capacity is determined for the Vickrey-ADL model so that the two models yield the same total variable cost at the unpriced equilibrium. The same free-flow travel time is used in the two models.

With (30), (33'), and (55), the condition of equal total variable cost at the unpriced equilibrium yields the following level of capacity for the Vickrey-ADL model:

$$k = \frac{\delta}{\alpha \Psi} N, \quad (66)$$

where Ψ is given by (30). k defined above approaches R as γ approaches infinity.

Model Comparison

Travel patterns Results using the base set of parameter values just described are reported in Table 3 and Figures 6 and 7 . I discuss the travel patterns first.

At the unpriced equilibrium the two models yield the same travel time function. With the total variable costs of travel being equal, the first and last arrival times, as well as private trip cost, are all the same (see Table 3). This is verified by comparing equations (6') and (49). The difference between the equilibria of the two models is in departure and arrival rates (see Figure 7).

Panels a and c of Figure 6 show the cumulative departures and arrivals in the unpriced equilibrium for both models. Panels a and c of Figure 7 show the corresponding departure and arrival rates. In the Vickrey-ADL model, traffic flows enter the road at a constant rate of 3203 vehicles per hour, which is 2.6 times larger than the bottleneck capacity (1251 vehicles per hour). A queue grows steadily and speeds decline continuously until the on-time departure time n . Thereafter, traffic flows enter at a constant rate of 373 vehicles per hour, far below the bottleneck capacity. The queue shrinks and speed increases continuously until the end u . Traffic flows exit at the constant rate of the bottleneck capacity.

In the reformulated Henderson model, however, commuters enter the road at an increasing rate from 0 at the beginning i to 3988 vehicles per hour at the on-time departure time n , and at a decreasing rate thereafter from 461 at n to 0 vehicles per hour at the end u . Unlike in the Vickrey-ADL model, speed is determined by arrival flows, which exit at an increasing rate from 0 at i' to 1558 vehicles per hour at t' , and at a decreasing rate from 1558 at t' to 0 vehicles per hour at the end u' .

Panels b and d of Figure 6 depict the cumulative departures and arrivals, and those of Figure 7 depict the rates of departure and arrival at the priced equilibrium for both models. With the optimal toll, the first and last arrival times are no longer the same between the two models. Neither are their travel time functions.

Specifically, the departure rate in the Vickrey-ADL model reduces to the level of the bottleneck capacity, resulting in free-flow travel for everyone. The period of arrivals is not changed since each commuter now pays the optimal toll instead of queuing at the bottleneck.

In the presence of the optimal toll in the reformulated Henderson model, commuters leave home still at an increasing rate before the on-time departure, and at a decreasing rate thereafter. The arrive rate is smaller than at the unpriced equilibrium, increasing continuously from 0 at the beginning to 1131 vehicles per hour at t^* , and decreasing continuously to 0 at the end. The result is less severe congestion throughout. Unlike in the Vickrey-ADL model, the period of arrivals lengthens with the first arrival earlier and the last one later.

Five results With an understanding of the equilibria of both models, it is ready to numerically examine the five results listed earlier. I consider the first four only.

The first result is that total cost of schedule delay is 50 percent of total variable cost of travel at the unpriced equilibrium in the Vickrey-ADL model. Using the base set of parameter values in Table 2, total cost of schedule delay is about 44 percent of total variable cost of travel in the reformulated Henderson model.

The second result is that the optimal toll saves 100 percent in total cost of travel

delay, 0 percent in total cost of schedule delay, and 50 percent in total variable cost of travel in the Vickrey-ADL model. Again using the base set of parameter values in Table 2, the optimal toll saves about 73 percent in total cost of travel delay, minus 23 percent in total cost of schedule delay, and about 24 percent in total variable cost of travel in the reformulated Henderson model.

The third result is that the optimal toll leaves the period of arrivals unchanged in the Vickrey-ADL model. In the reformulated Henderson model with the parameter values of Table 2, however, the period of arrivals lengthens by 38 percent.

The fourth result is that the optimal toll leaves the private trip cost unchanged in the Vickrey-ADL model. With the parameter values of Table 2, the private trip cost increases by about 38 percent in the reformulated Henderson model.

As emphasized above, these percentages in the reformulated Henderson model are based on the base set of parameter values in Table 2, while those in the Vickrey-ADL model are independent of any of its parameters. One natural question is: do these percentages in the reformulated Henderson model vary with its parameters; and, if so, how? The answer is that these percentages in the reformulated Henderson model depend only on the elasticity of travel delay with respect to arrival flows, γ , as shown in equations (59)-(63) and (65).

Simulations

How do these percentages in the reformulated Henderson approach vary with γ ? The variation of these percentages with γ is shown in Figures 8-10 with γ ranging from 1 to 30. These percentages converge to those of the Vickrey-ADL

model as γ approaches ∞ (i.e., infinity). The base value for γ is 4.08 , which lies between 2.5 and 5 , a range suggested for γ (Small [14]).

Figure 8 shows the percentage ratio between total cost of schedule delay and total variable cost of travel at the unpriced equilibrium. This ratio reaches 40 and 45 percent as $\gamma = 2$ and 5 respectively. After 5, it levels off, and converges to 100 percent (the Vickrey-ADL level) as $\gamma = \infty$.

Figure 9 shows the percentage savings in aggregate costs. With $\gamma = 4.08$, percentage savings in total cost of travel delay, total cost of schedule delay, and total variable cost of travel are 73 , -23 , and 24 , respectively; as $\gamma = 5$, they are 88 , -35 , and 39 ; they converge to 100 , 0 , and 50 (the Vickrey-ADL levels) as $\gamma = \infty$.

Figure 10 shows the percentage lengthening of the period of arrivals. It is 38 percent as $\gamma = 4.08$, 35 percent as $\gamma = 5$, and 0 (the Vickrey-ADL level) as $\gamma = \infty$. The percentage increase in the private trip cost with respect to γ is not separately shown because it follows the same pattern as the percentage lengthening of the period of arrivals (see (63) and (65)).

Some comments are in order on the cusps in Figures 9 and 10 in the lengthening of the period of arrivals and in the savings of total cost of schedule delay. These cusps actually occur at $\gamma = e - 1 \approx 1.72$, where e is the base of natural logarithm. It is intuitive to have the largest increases in total cost of schedule delay and in the period of arrivals occur together because it is the spreading of arrivals that causes total cost of schedule delay to increase. But it is not intuitive that these largest increases occur at $\gamma \approx 1.72$.

VI. CONCLUSION

This paper has demonstrated that the original Henderson approach has problems: for example, the original Henderson model that prohibits lateness lacks equilibria. It showed that these problems are eliminated by assuming the number of travelers arriving together at their destination, instead of departing together at their origin, determines their travel time. It also investigated and compared the behavior of the reformulated Henderson approach with that of the Vickrey approach both analytically and using simulations.

This paper finds that the behavior of the reformulated Henderson approach varies with its elasticity of travel delay with respect to arrival flow at destination, while the Vickrey approach lacks such a flexibility; that the behavior of the Vickrey approach is the limit of that of the reformulated Henderson approach as the elasticity of travel delay goes to infinity; and that the behaviors of the two approaches are not close when the elasticity of travel delay varies between 2.5 and 5, the range suggested in the literature.

The paper also finds that giving travelers the flexibility of being late for activities at the destination can result in substantial savings. The larger the elasticity of travel delay with respect to arrival flow, or the larger the relative unit costs of being early and being late, the larger are these savings. For typical values of the elasticity and unit costs of schedule delays, these savings are about twenty percent.

The Henderson and Vickrey approaches have mainly been used to gain analytical insights on the effects of accounting for trip scheduling. Both approaches, with or without lateness, have been useful for this purpose. But the insights one gets from the

Vickrey approach can be obtained from the reformulated Henderson approach.

The two approaches will ultimately be applied as building blocks to analyze congestion situations in reality. For such applications, neither approach should be used without lateness because travelers rarely incur infinitely large penalty for being late for activities at destination. With lateness, both approaches can be useful: the reformulated Henderson approach is useful for a wide range of severity of flow congestion, while the Vickrey approach is more useful for queuing situations. Given the limiting relationship between the two approaches, however, even queuing situations may be well approximated by the reformulated Henderson approach.

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FOOTNOTES

1. This work is funded by a fellowship from the U.S. Department of Transportation and the California Department of Transportation through the University of California Transportation Center. I am grateful to Ken Small for valuable discussions and suggestions, to the referee for a helpful report, and to J. Vernon Henderson and Herbert Mohring for comments. I also thank seminar participants at University of California-Irvine, and participants of the Location and Travel Modeling sessions at the 32nd Annual Meeting of the Western Regional Science Association, February 21-25, 1993, Maui, Hawaii. All responsibility lies with me.
2. Mahmassani and Herman [11] is an example. They treat a road section as a "black box" (Henderson [7], p. 145) by assuming an immediate influence of any input flow on the average density over the whole road considered. They work with average density, speed, and flow over the whole road. Their modeling framework allows hypercongestion (i.e., a situation where the more vehicles forced into the road, the lower the average flow), and can be useful for hypercongestion-prone areas such as CBD. In fact, Vickrey [18] does just that for midtown Manhattan. But their formulation can also lead to overtaking (i.e., arriving earlier by starting later) and lack of equilibrium (Chu [4]).

Is the simultaneous occurrence of hypercongestion, overtaking and lack of equilibrium a general result or just a property of their particular formulation? Newell [12] asserts that it is a general result of the assumed immediate influence of any input flow on the average density over the whole road. This immediate influence implies an infinite group velocity--the speed at which shock waves of constant density propagate along the relevant road section. Instead, Newell works with location-specific density, speed, and flow so that a finite group velocity is implied. But doing so loses tractability.

For comparison, the Henderson approach implies a zero group velocity because vehicles entering the road at different times do not affect one another. On the other hand, group velocity is undefined in the Vickrey approach because it requires no knowledge of the flow or speed in the queue itself.

3. Henderson [8, 9] and [10] used this new formulation for a setup of the journey to work that is different from Henderson [6, 7] and models of the Vickrey approach. Each commuter chooses a work-start time either by choosing a firm with that particular work-start time or by choosing a work-start time within the current firm. Commuters start work upon arrival. The schedule delay for a chosen work-start time is measured relative to a most desired work-start time. This setup is appropriate for commuters under a flexible work hours program, but not for others. Other commuters need to choose a work-start time, and a daily commuting schedule, given the chosen work-start time. The above setup mixes these two choice problems.
4. See Table 1 for a list of symbols and their brief definition.
5. Henderson [7, p. 175] solves for i , n , and C numerically.
6. Henderson [7] ignores the derivative of the schedule delay term with respect to $F(t)$ and results in an error in his first-order condition, which replaces $(\alpha - \beta)$ by α in the second term of (15). This amounts to calculating the extra travel delay caused by the marginal commuter but failing to offset its cost by the change in schedule delay.
7. If we let t' be the arrival time associated with departure time t , $CF(t)$ the cumulative departures, and $CA(t')$ the cumulative arrivals, then the following relationships hold:

$$t' = t + \frac{m}{S(t)}, \quad CA(t') = CF(t) = \int_i^t F(x)dx.$$

TSC	total cost of schedule delay
TVC	total variable cost, the sum of TCC and TSC
$T(t)$	travel time function
u	last departure time with late arrivals
α	unit cost of travel time
β	unit cost of schedule delay early
γ	elasticity of travel delay with respect to travel flow
δ	$\beta v / (\beta + v)$
λ	Lagrangian Multiplier
v	unit cost of schedule delay late
$\tau(t)$	toll schedule
Φ	maximum travel delay without lateness
Ψ	maximum travel delay with lateness

Table 2. Parameter Values for an Example^a

Demand side	Supply side
$N = 1000$	$R = 3817$ vehicles/hour
$\alpha = \$ 6.40/\text{hour}$	$\gamma = 4.08$
$\beta = \$ 3.90/\text{hour}$	$S_{\max} = (60/2.48)$ miles/hour
$v = \$15.21/\text{hour}$	$m = 15$ miles
$t^* = 8:00$ A.M.	$T_f \equiv m/S_{\max} = 0.62$ hours = 37.2 minutes

^a Arnott et al. [1] use the same unit costs (i.e., α , β , v), which are based on Small [13]. Parameters for the supply side are based on Small [14, p. 70]. Using data from Dewees' [5] simulation experiments on city arterials, Small estimates a speed-flow curve of $T = 2.48 + 0.254(V/1000)^{4.08}$, where T (travel time) and V (traffic flow) are measured in minutes per mile and vehicles per hour, respectively. I convert the coefficient 0.254 into the denominator in the parentheses to get $R = 3817$ vehicles per hour. $S_{\max}$ is 60/2.48 miles per hour, and γ is 4.08. I set the trip distance at 15 miles, the number of commuters at 1000, and the common work start time at 8:00 A.M.

Table 3. Equilibria for Alternative Approaches and Pricing Schemes^a

	Vickrey			Henderson		
	Unpriced	Priced	%	Unpriced	Priced	%
First arrival time	7:22	7:22	NA ^b	7:22	7:07	NA ^b
Last arrival time	8:10	8:10	NA ^b	8:10	8:13	NA ^b
Peak length: minutes	48	48	0%	48	66	+38%
Private trip cost: \$/trip	2.48	2.48	0%	2.48	3.42	+38%
Average Toll: \$/trip	0	1.24	NA ^b	0	1.52	NA ^b
Total cost of travel delay: \$	1240	0000	-100%	1375	373	-73%
Total cost of schedule delay: \$	1240	1240	0%	1105	1522	+23%
Total variable cost of Travel: \$	2480	1240	-50%	2480	1895	-24%

^a Demand-side parameters for both approaches, and supply-side parameters for the reformulated Henderson approach are given in Table 2. For the Vickrey approach, supply-side parameter T_f is the same as $m/S_{\max}$ in the reformulated Henderson approach; parameter k is determined by setting its total cost of travel at the unpriced equilibrium equal to that of the reformulated Henderson approach:

$$k = 1251 \text{ vehicles/hour}$$

^b Percentage changes do not apply here.

a. Unpriced Solution

b. Optimally Priced Solution

Figure 1. Cumulative arrivals and departures: Original Henderson model without lateness.

a. Unpriced Solution

b. Optimally Priced Solution

Figure 2. Cumulative arrivals and departures: Original Henderson model with lateness.

a. Toll Schedule

b. Departure Rate

Figure 3. Toll schedule and departure rate at optimally priced solution: Henderson model with lateness.

a. Unpriced Solution

b. Optimally Priced Solution

Figure 4. Cumulative arrivals and departures: Limit of Original Henderson model with lateness as the unit cost of being late approaches infinity.

a. Unpriced Solution

b. Optimally Priced Solution

Figure 5. Cumulative arrivals and departures: Reformulated Henderson model without lateness.

Figure 6. Cumulative arrivals and departures: Reformulated Henderson (left) and Vickrey-ADL (right) models with lateness.

Figure 7. Arrival and departure rates: Reformulated Henderson (left) and Vickrey-ADL (right) models with lateness.

Figure 8. Percentage ratio of total cost of schedule delay and total variable cost of travel at unpriiced solution: Reformulated Henderson model with lateness.

Figure 9. Percentage savings in aggregate costs due to optimal pricing: Reformulated Henderson model with lateness.

Figure 10. Percentage change in length of period of arrivals due to optimal pricing: Reformulated Henderson model.