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Carbon taxes and climate commitment with non-constant time preference[*]

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Abstract

We study the Markov Perfect equilibrium to a dynamic game in which private agents choose savings and a planner chooses climate policy. All agents have the same hyperbolic discount function. Using new algorithms, we solve the game with general functional forms (apart from isoelastic utility). We also obtain an analytic solution for a log-linear specialization. With hyperbolic discounting, convex damages lead to significant strategic interactions across generations of planners. The ability to commit to policy for over a century significantly increases welfare for the first generation, but the ability to commit for only a few decades has negligible benefit.

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1 Introduction

A portion of current carbon emissions will remain in the atmosphere for thousands of years. With such a long legacy, the present discounted value of climate-related damages is highly sensitive to the discount rate. The pure rate of time preference is an important determinant of the discount rate and thus important for climate policy. The assumption of constant time preference was challenged, from its inception, on normative grounds (Samuelson 1937), and it has recently been challenged on empirical grounds. Constant time preference leads to time-consistent preferences (Koopmans 1960) which simplifies the calculation of equilibrium behavior. The accompanying advantage is especially helpful when solving Integrated Assessment Models (IAMs) that combine climate policy and private savings. Incorporating non-constant time preference (NCTP) into IAMs presents long-recognized technical challenges: one must compute a subgame perfect equilibrium among generations (Strotz 1955) and impose a fixed point condition to ensure consistency when modeling the determination of decentralized savings (Krusell et al. 2002). The paper does both, providing the first IAM with general discounting and general functional forms.

The choice between constant and non-constant time preference can have a large effect on the equilibrium carbon tax. A model with constant time preference can match observed savings rates or it can incorporate concern for distant generations, but it cannot do both. With NCTP, in contrast, agents may invest in rapidly depreciating capital at observed rates and also incur significant costs to avoid climate damages in the distant future (Gerlagh and Liski 2017). In addition, by creating a conflict between climate policy-makers at different points in time, NCTP affects how we think about the policy problem. The conflict among generations gives policy-makers in earlier periods an incentive to develop a commitment device to constrain future choices, just as in the case of private agents (Strotz 1955, Laibson 1997). We examine the effect of NCTP on equilibrium climate policy and savings, and we assess the value of commitment.

Arguments for NCTP suggest a path of rates that declines over many periods. Nevertheless, most macroeconomic applications of NCTP use quasi-hyperbolic (β, δ) discounting—a two-tier step function that approximates a general discount function (Phelps and Pollak 1968, Laibson 1997, Harris and Laibson 2001, Krusell and Smith 2003). Apart from special cases, constant discounting, quasi-hyperbolic discounting, and general discounting are not observationally equivalent, and the different types of discounting can have both a qualitative and a large quantitative impact on policy.[1] The approximation error due to the restriction to quasi-hyperbolic discounting grows in the decision horizon and is likely large for climate policy, where the relevant horizon spans centuries. In addition, the quasi-hyperbolic approximation misrepresents the nature of intergenerational decision conflict that stems from the difference between the discount rate a future agent uses and the rate the current agent would like them to use. With quasi-hyperbolic discounting, this conflict is constant after one period, while with continuously decreasing NCTP, it grows with the distance between generations.

The paper makes four contributions. First, we develop and implement novel methods to solve a model with isoelastic utility but otherwise general functional forms. We

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[1]Harstad (2016, pg. 4) illustrates the qualitative impact in an investment game. He notes: "...the result that upstream technologies should be subsidized more does *not* hold under quasi-hyperbolic discounting—which is therefore a poor approximation for hyperbolic discounting." Iverson (2012), the original version of the current paper, documents the quantitative impact of the quasi-hyperbolic restriction.

translate the dynamic game that arises with NCTP into a modified finite horizon dynamic programming problem. This programming problem is hard because the savings decision must be solved as an equilibrium fixed point problem at every stage, not as a command problem. The fixed-point problem exacerbates the instability associated with approximation methods in non-stationary dynamic programs (Judd and Cai 2014). We obtain an easily approximated, semi-analytic expression for the value function, resulting in a stable and accurate solution to a difficult numerical problem. Krusell, Kurusco and Smith (KKS; 2002) and Hiraguchi (2014) solve the one-aggregate-state savings problem under logarithmic utility and Cobb-Douglass production. Our methods permit a solution with isoelastic utility, general production and discounting functions, and endogenous climate policy.

Second, we provide the first assessment of climate policy under arbitrary NCTP. We explore the role of non-constant discounting in influencing equilibrium savings and climate policy, while varying the depreciation rate, the elasticity of intertemporal substitution, the damage function, and TFP growth. Strategic interactions across generations are important if the damage function is strongly convex in the stock of carbon. In this case, climate policies are dynamic strategic substitutes, and strong climate policy today reduces the incentive to invest in climate policy in the future. If damages are roughly linear in the stock of carbon—as in Nordhaus (2013) and Golosov et al. (GHKT; 2014)—then strategic incentives across generations are insignificant.

Third, we study the value of intertemporal policy commitment. A decision-maker's ability to commit policy in future periods resolves its conflict with those generations. With continuously decreasing discount rates, the conflict between two generations increases with the span of time separating them. This feature causes the marginal value of a longer commitment period to grow with the commitment interval. However, discounting reduces the present value of the resolution of a more distant conflict. The combination of these two forces makes the value of commitment initially convex, and later concave in the commitment period. A third force, arising from strategic interactions, is quantitatively unimportant. We find that the ability to commit for hundreds of years would indeed be valuable to today's generation, but the ability to commit for only a few decades is worth little. In view of the cost of devising a long period of intertemporal commitment, and the low value of a short period of commitment, we conclude that searching for such a device is likely futile.

Finally, we analytically solve a version of the model that adopts the log-linear assumptions in GHKT. Gerlagh and Liski (GL; 2017) extend this model to accommodate quasi-hyperbolic discounting. We obtain formulae for the equilibrium aggregate savings rate and the carbon tax under general NCTP. The formulae encompass the constant-time-preference and quasi-hyperbolic special cases considered in earlier papers. The tax formula renders transparent the effect of discounting and climate-related parameters on climate policy. We show explicitly how the log-linear structure decouples decisions across generations and between savings and climate policy. The decoupling properties give rise to the model's analytic tractability, but they limit its ability to shed light on intergenerational conflict and on the links between savings and climate policy. The log-linear results are robust to the depreciation rate of capital, but not to the elasticity of intertemporal substitution or the damage function.

The next subsection completes the literature review. Section 2 presents the model and

defines the equilibrium. Section 3 studies the analytic log-linear model. Section 4 presents and studies the more general model. Section 5 quantifies the value of commitment. Section 6 presents the new algorithms, and section 7 concludes.

1.1 Literature review

Frederick et al. (2002) conclude that empirical evidence overwhelmingly supports hyperbolic discounting over exponential discounting. Table 1 summarizes recent empirical estimates of present bias (in the quasi-hyperbolic setting, $\beta < 1$). In a field experiment, Augenblick et al. (2015) find evidence of present bias in the domain of consumption. They also show that subjects value commitment. Survey evidence suggests that people discount the distant future less heavily than the near future (Cropper et al. 1994, Layton and Levine 2003, Drupp et al. 2014). Montiel Olea and Strzalecki (2014) provide an axiomatic foundation for quasi hyperbolic discounting.[2]

Table 1: Recent estimates of the β, δ model. $\beta < 1$ implies present bias.

Article	Decision involving	β	δ
Shui and Ausubel (2005)	credit cards	0.8	0.99
Paserman (2008)	job search	$(0.4, 0.9)$	≈ 1
Fang and Silverman (2009)	labor supply and welfare	0.34	0.88
Mahajan and Tarozzi (2011)	technology adoption	0.56	0.79
Fang and Wang (2015)	mamography	$(0.51, 0.79)$	$(0.68, 0.94)$
Laibson et al. (2017)	consumption over lifecycle	0.504	0.987

Brocas et al (2003) and Bryan et al. (2010) survey the literature on commitment devices. Examples of these include mandatory pension plans with limited accessibility, mandatory minimum years worked before retirement, and externally enforced deadlines. Barro (1999) shows that short run savings falls, and long run savings rises, with the length of a commitment device.

Karp (2005) and Karp and Tsur (2007) use NCTP in analytic climate models, and Fujji and Karp (2008) use numerical methods, but these abstract from the savings decision and include only a rough approximation of climate dynamics. Iverson et al. (2015) use the formulae in Section 3 to show that, with constant discounting, damages during the next two centuries are responsible for virtually the entire carbon tax, while with NCTP, later damages account for a substantial portion of the tax. Rezai and van der Ploeg (2017) study a partial equilibrium model under the assumption that the initial planner can commit to the sequence of future policies.

Most of the empirical and macro-theory studies of NCTP take the decision-maker as a consumer or a firm. However, one of the earliest applications involves a dynasty consisting of many generations, each of which lives for a single period (Phelps and Pollack, 1968). Climate policy involves both intra- and inter-generational transfers, because current abatement expenditures can benefit a currently living agent late in their life, and also benefit people who have not yet been born. Therefore, an overlapping generations (OLG)

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[2]A growing literature studies reasons for non-constant consumption discount rates arising from the correlation between the returns to climate investments and a market portfolio (Traeger 2014, Gollier 2014, Giglio et al. 2015) Our deterministic model cannot include this feature.

structure is particularly appropriate for studying climate policy. We adapt Ekeland and Lazrak's (2010) model to the IAM setting.

2 Model

Time is discrete and runs from 0 to infinity. There is no uncertainty.

A unit continuum of identical households discount future utility with arbitrary time weights $\{\lambda_m\}_{m=0}^{\infty}$, where $\lambda_0 = 1$. The weights are unrestricted, but we emphasize decreasing time preference rates. Household welfare (assumed finite) in t is the present value of discounted utility,

$$\sum_{\tau=0}^{\infty} \lambda_\tau u(c_{t+\tau}),$$

where utility is isoelastic in consumption:

$$u(c) = \begin{cases} \frac{c^{1-\eta}-1}{1-\eta}, & \text{if } \eta > 0,\, \eta \neq 1 \\ \ln(c), & \text{if } \eta = 1. \end{cases} \tag{1}$$

Agent i at t owns capital $k_{i,t}$, and aggregate capital is

$$K_t = \int_0^1 k_{i,t}\, di.$$

We use a representative agent model and drop the agent index when the meaning is clear. Lower case symbols denote individual levels, while upper case symbols denote aggregate levels. Symbols in bold denote vectors.

Aggregate carbon emissions, E_t, affect the evolution of a vector of climate states, $\mathbf{S}_t$, which could include carbon stocks and temperature in different reservoirs. The climate states evolve according to

$$\mathbf{S}_{t+1} = \mathbf{f}\left(\mathbf{S}_t, E_t\right). \tag{2}$$

Output of the final good is

$$Y_t = F_t(K_t, E_t, \mathbf{S}_t). \tag{3}$$

$F_t\left(\cdot\right)$ is concave increasing in capital and emissions, decreasing in the climate state, with constant returns to scale in K, E, and labor (normalized to 1); the subscript t accommodates exogenous technological change. Climate damages arise from output losses, affecting consumption; we abstract from damage channels directly impacting utility, e.g. a loss in amenity value.[3] Current output equals aggregate consumption, C_t, plus investment,

$$Y_t = C_t + K_{t+1} - (1 - \delta)K_t, \tag{4}$$

where δ is the depreciation rate for capital.

Most anthropogenic carbon emissions result from burning fossil fuel, whose finite stocks are, in principle, exhaustible. We ignore stock constraints, making the model appropriate for coal, but probably not for oil. Section 3.3 discusses this assumption.

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[3]Referees' Appendix B.2 describes an extension to include population growth and damages directly affecting welfare.

2.1 Equilibrium savings and climate policy

Firms maximize profits, hiring capital and labor to equate the factors' value of marginal product to the rental rate and wage. With free entry and constant returns to scale, equilibrium profits are zero. We denote the value of output minus payments to capital as "transitory income". Under a carbon tax (or cap and trade), transitory income equals tax revenue (or quota rents) which are returned to agents in a lump sum, plus payments to labor. This income is "transitory", because from the agent's perspective it is unrelated to their savings decision. To emphasize the dynamic problem, we leave the static decisions (firms' choice of inputs) in the background until Section 4.1. There we follow GHKT in specifying clean and dirty energy sectors with mobile labor.

We begin with the households' problem and then consider the planner's problem. In period t, agent i earns income from capital, r_t, and transitory income, w_t. We study the Markov Perfect Equilibria (MPE) to the savings game: agent i chooses current savings, taking as given her future savings policies. There are typically many MPE, but we focus on the unique limit of a sequence of finite-horizon equilibria.[4] Agents have zero mass and rational expectations, so they take as given the trajectories of aggregate capital, aggregate emissions, and climate states—which depend on equilibrium climate policy.

Suppressing the agent index i, the representative agent in t takes as given its successors' savings rules, $g_s(k_s, K_s, \mathbf{S}_s; E_s)$, $s > t$, and the current and future aggregate savings rules, $G_s(K_s, \mathbf{S}_s; E_s)$, $s \geq t$. Current emissions, E_s, affects the level of savings via its effect on current income. The agent chooses k_{t+1} to maximize

$$u(r_t k_t + w_t + (1-\delta)k_t - k_{t+1}) + \tilde{V}(k_{t+1}, G_t(K_t, \mathbf{S}_t; E_s), \mathbf{S}_{t+1}, t+1), \qquad (5)$$

where

$$\tilde{V}(k_{t+1}, K_{t+1}, \mathbf{S}_{t+1}, t+1) \equiv$$
$$\sum_{j=1}^{T} \lambda_j u(r_{t+j} k_{t+j} + w_{t+j} + (1-\delta)k_{t+j} - g_{t+j}(k_{t+j}, K_{t+j}, \mathbf{S}_{t+j})). \qquad (6)$$

The optimum defines $g_t(k_t, K_t, \mathbf{S}_t; E_s)$, leading to agent welfare

$$V(k_t, K_t, \mathbf{S}_t, t) \equiv u(r_t k_t + w_t + (1-\delta)k_t - g_t(k_t, K_t, \mathbf{S}_t))$$
$$+ \tilde{V}(g_t(k_t, K_t, \mathbf{S}_t), G_t(K_t, \mathbf{S}_t; E_s), \mathbf{S}_{t+1}, t+1). \qquad (7)$$

In equilibrium, $k = K$. Agents take as given current and future emissions levels. In equilibrium, these are functions of the state variable, $K_t, \mathbf{S}_t, t$. Because the dependence of equilibrium welfare on future as well as current climate policy is important for describing the climate problem, we make it explicit:

$$V^e(K_t, \mathbf{S}_t, t; \{E_s\}_{s=t}^{\infty}) \equiv V(K_t, K_t, \mathbf{S}_t, t).$$

We turn next to the determination of climate policy. In each period a planner chooses policy to maximize welfare for the contemporaneous representative household. Because household preferences are time inconsistent, we model climate policy as a Markov perfect equilibrium of a dynamic game. Planners take the equilibrium aggregate savings rule,

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[4]For both the savings and climate policy decisions, we prove uniqueness in the log-linear model, and we find no numerical evidence to indicate non-uniqueness in the general model.

$G_t(K_t, \mathbf{S}_t; E_t)$, as given. As above, we study the infinite horizon limit of a sequence of finite horizon games.

To study the role of commitment, we introduce the following technology.

Definition 1 *(Commitment device) A planner in t with a $j-$period commitment device chooses climate policy, E_s, for periods $s = t, \ldots, t+j-1$, provided the emissions level during these periods was not already fixed by a commitment device in an earlier period.*

We denote $E_s^*(K_s, \mathbf{S}_s)$ as the equilibrium emissions policy in period s, as a function of the state variable.[5] The planner in t with $j-$period commitment anticipates that planners outside the commitment window choose $\{E_s\}_{s=t+j}^{\infty} = \{E_s^*(K_s, \mathbf{S}_s)\}_{s=t+j}^{\infty}$; $j = 1$ implies that the planner chooses policy only in the current period. The planner chooses $\{E_s\}_{s=t}^{t+j-1}$ to maximize equilibrium welfare for the period-t representative agent. This planner solves

$$\max_{\{E_s\}_{s=t}^{t+j-1}} V^e \left(K_t, \mathbf{S}_t, t; \{E_s\}_{s=t}^{t+j-1}, \{E_s^*(K_s, \mathbf{S}_s)\}_{s=t+j}^{\infty} \right). \tag{8}$$

Combining the capital market equilibrium and the climate policy equilibrium, we define a Markov Perfect equilibrium as follows.

Definition 2 *A Markov Perfect equilibrium satisfies the following for all t:*

1. *Prices, determined competitively, satisfy*

$$(\text{rental rate}) \; r_t = r_t(K_t, E_t, \mathbf{S}_t) = \frac{\partial F_t(K_t, E_t, \mathbf{S}_t)}{\partial K_t},$$

$$(\text{transitory income}) \; w_t = w_t(K_t, E_t, \mathbf{S}_t) = F_t(K_t, E_t, \mathbf{S}_t) - \frac{\partial F_t(K_t, E_t, \mathbf{S}_t)}{\partial K_t} K_t.$$

2. *The physical constraints in (2), (3), and (4) hold.*

3. *Individual savings maximizes equilibrium welfare, defined in (5) and (6).*

4. *Individual and aggregate savings are consistent: $g_t(K_t, K_t, \mathbf{S}_t; E_t) = G_t(K_t, \mathbf{S}_t; E_t)$.*

5. *The climate policy function solves (8).*

The solution to the model consists of time-dated equilibrium aggregate savings rules and emissions policies. We describe the emissions policy using the carbon tax, $\tau \equiv \frac{\partial F_t(K_t, E_t, \mathbf{S}_t)}{\partial E}\big|_{E_t = E^*}$, that supports the equilibrium emissions level E^*.

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[5]This policy also depends on the commitment window. We suppress that dependence to avoid complicating the notation.

3 Analytical solution for the log-linear case

This section presents the log-linear model and explains two properties responsible for its tractability. GHKT extend Brock and Mirman's (1972) growth model (log utility, Cobb-Douglass production in capital, and full depreciation) to include a tractable climate module. GL use a variation of the model to study climate policy when agents have quasi-hyperbolic time preferences. We extend their analysis to include general discounting. We also extend the analysis of decentralized equilibrium savings with arbitrary NCTP in Hiraguchi (2014). He studies the equilibrium problem in a one-sector, Brock-Mirman growth model, while we study the augmented climate-economy. The climate component does not change the equilibrium savings rate, but we require a different proof because Hiraguchi considers a stationary setting, while ours is non-stationary due to exogenous technological change and endogenous (time-varying) climate policy.

Assumption 1 *The log-linear model*
  (i) The utility function is logarithmic: $u(C) = \log(C)$.
  (ii) Capital depreciates 100 percent in every period: $\delta = 1$.
  (iii) Output in the final-goods sector is Cobb Douglass in capital with multiplicative climate damages:
$$Y_t = (1 - D(S_t))K_t^\alpha A_t(E_t),$$

where $A_t(E_t)$ is a time-dependent function that incorporates changes in technology and labor supply; the damage function is exponential:

$$1 - D(S_t) = \exp\left(-\gamma(S_t - \bar{S})\right).$$

  (iv) The climate vector, $\mathbf{S}$, affects output only via atmospheric carbon, denoted S, and S is linear in prior emissions:

$$S_\tau - \bar{S} = \sum_{j=0}^{\tau+t+H} (1 - d_j)E_{\tau-j},$$

where $\bar{S}$ is the preindustrial stock of atmospheric carbon, $1 - d_j$ is the portion of emissions remaining in the atmosphere after j periods, and $0 \leq d_j \leq 1$.

3.1 Equilibrium without commitment

We first consider the case in which planners choose policy for a single period. Proposition 1 shows that the limit equilibrium is unique, and it provides formulae for the equilibrium savings rule and the equilibrium carbon tax. All proofs are presented in appendix A. We define

$$\rho \equiv \sum_{t=1}^\infty \lambda_t \text{ and } \lambda_{l,m} \equiv \frac{\lambda_m}{\lambda_l} \text{ for } m \geq l.$$

We assume ρ is finite. $\lambda_{l,m}$ is the amount an agent in t would be willing to reduce utility in $t + l$ to obtain a one-unit increase in utility in $t + m$.

Proposition 1 *Suppose Assumption 1 is satisfied and there is no commitment ($j = 1$ for all periods). Then the limit equilibrium is unique. The equilibrium carbon tax and*

equilibrium savings are constant fractions of income. Equilibrium savings for an individual in t with capital k is

$$k' = \frac{\rho}{1+\rho} r_t k;$$

aggregate savings is

$$K' = sY_t, \text{ where } \quad s \equiv \frac{\alpha\rho}{\rho+1}; \tag{9}$$

and carbon taxes are

$$\tau_t = \left[\sum_{k=0}^{\infty} \lambda_k (1-d_k)\gamma \cdot \Gamma_k \right] Y_t, \tag{10}$$

where

$$\Gamma_k \equiv \frac{\sum_{m=0}^{\infty} \alpha^m \lambda_{k,k+m}}{\sum_{n=0}^{\infty} \alpha^n \lambda_n}. \tag{11}$$

Corollary 1 Intra-temporal decoupling*: The equilibrium aggregate savings rate does not depend on emissions levels; equilibrium emission levels do not depend on the aggregate savings rate.*

Corollary 2 Inter-temporal decoupling: *The equilibrium emissions policy in any period is independent of climate policy in all other periods.*

The equilibrium carbon tax is a constant fraction of output, where the constant depends on the path of time preference rates, the damage elasticity parameter, γ, the carbon cycle parameters, and the Cobb Douglass coefficient on capital in final-goods production. Notably, the tax does not depend on beliefs about future technology or emissions. The same features hold for the optimal tax in GHKT with the exception that the formula there does not depend on the Cobb Douglass coefficient on capital. With constant time preference, $\Gamma_k = 1$, and the expression in (10) reduces to the tax formula in GHKT. The formula also reduces to the tax formula in GL when discounting is quasi-hyperbolic. When time preference rates decline, $\Gamma_k > 1$, so the equilibrium tax is greater than the tax one would obtain if the constant-discounting time weights in the GHKT formula were simply replaced with nonconstant-discounting time weights.

The corollaries show how Assumption 1 shuts down interactions between different parts of the problem. Corollary 1 shows that the savings problem and the climate problem are in a significant sense decoupled. The stock invariant savings rate does not depend on the level of emissions in current and future periods, though the *level* of savings does depend on current emissions through the effect on climate damages and abatement costs. The equilibrium emission level and the equilibrium tax as a fraction of output are both independent of the aggregate savings rate. The corollaries are statements about the equilibrium policy functions, not about arbitrary policy functions. For example, the climate policy response to a non-equilibrium savings rule that depends on the climate state might depend on capital and on future savings rules.

3.1.1 Intuition

Taking the corollaries as given, we provide a heuristic derivation of the tax formula that illustrates the origin of the model's tractability and provides intuition.

Corollary 1 implies that the aggregate savings rate is stock invariant. Therefore, capital accumulates according to

$$K_{t+1} = s_t A_t (E_t) K_t^\alpha e^{-\gamma(S_t - \bar{S})}. \tag{12}$$

Taking logs gives a linear difference equation in log capital. Solving for $v \geq 1$ gives

$$\ln K_{t+v} = \alpha^v \ln K_t + \alpha^{v-1} \ln A_t (E_t) - \gamma \sum_{j=0}^{v-1} \alpha^{v-j-1} (1 - d_j) E_t + \text{``terms''}. \tag{13}$$

Here, "terms" collects all expressions that do not depend on time-t emissions or capital. We combine (4), (12), and (13) to get an expression that shows the dependence of future flow payoffs on current output and current emissions,

$$\ln C_{t+v} = \alpha^v \ln Y_t - \gamma \sum_{j=1}^{v} \alpha^{v-j} (1 - d_j) E_t + \text{``terms''}. \tag{14}$$

Next, we use (14) to obtain an expression for the marginal rate of substitution between a unit of income in t and a future period, viewed by the agent in t. We use this to define two welfare measures that maintain the time perspective of the agent in t:

$$V_t (Y_t) \equiv \sum_{v=0}^{\infty} \lambda_v \ln C_{t+v}$$

and

$$V_{t+k} (Y_{t+k}) \equiv \lambda_k \sum_{v=0}^{\infty} \lambda_{k,k+v} \ln C_{t+k+v}.$$

The functions equal agent $t's$ present discounted value of consumption from t onwards and $t + k$ onwards, respectively. Substituting from (14) gives

$$V_t (Y_t) = \left(\sum_{v=0}^{\infty} \lambda_v \alpha^v \right) \ln Y_t + \text{``terms''}$$

and

$$V_{t+k} (Y_{t+k}) = \lambda_k \sum_{v=0}^{\infty} \lambda_{k,k+v} \alpha^v \ln Y_{t+k} + \text{``terms''}.$$

Taking the derivative with respect to output gives the shadow value of income in t and $t + k$, viewed by the agent in t. The ratio of shadow values gives the marginal rate of substitution between a unit of income in t and $t + k$:

$$MRS_{t,t+k} \equiv \frac{V'_{t+k} (Y_{t+k})}{V'_t (Y_t)} = \frac{\lambda_k \sum_{v=0}^{\infty} \lambda_{k,k+v} \alpha^v}{\sum_{v=0}^{\infty} \lambda_v \alpha^v} \frac{Y_t}{Y_{t+k}} = \lambda_k \Gamma_k \frac{Y_t}{Y_{t+k}}.$$

The standard definition of MRS involves the ratio of discounted marginal utility of consumption in two periods, whereas our definition involves the ratio of discounted shadow values of income. These shadow values, and also their ratio, depend on α because an extra unit of income increases savings; the resulting increase in future consumption, and thus

the change in welfare, depends on the productivity of capital, measured by α. Consumption is proportional to income, and marginal utility equals the inverse of consumption. Therefore, under constant discounting, where $\lambda_k = \lambda$, a constant, and $\Gamma_k = 1$, $MRS_{t,t+k}$ simplifies to the standard definition.

Climate damages from an extra unit of emissions in t reduce Y_{t+k} by

$$\frac{dY_{t+k}}{dS_{t+k}} \frac{dS_{t+k}}{dE_t} = \gamma Y_{t+k} \left(1 - d_k\right).$$

Multiplying this loss by the marginal rate of substitution between periods t and $t + k$ converts the change in units of $t + k$ output to a change expressed in units of time t output:

$$\gamma Y_{t+k} \left(1 - d_k\right) MRS_{t,t+k} = \gamma Y_{t+k} \left(1 - d_k\right) \lambda_k \Gamma_k \frac{Y_t}{Y_{t+k}} = \gamma \left(1 - d_k\right) \lambda_k \Gamma_k Y_t.$$

Summing over k from 0 to ∞ gives the marginal social cost of a unit of emissions, viewed from the perspective of an agent in t. This infinite sum equals the equilibrium carbon tax in (10).

Thus, under both constant and non-constant time preference, the carbon tax equals the stream of future damages (measured in units of the consumption good) due to an extra unit of emissions, weighted by the marginal rates of substitution. Only the definitions of the marginal rates of substitution differ, as described above. This simplicity arises from the fact that in choosing current emissions, the current policy-maker does not have to think about how the resulting change in the climate state alters future savings and future climate policies; as the Corollaries state, current emissions do not alter those policies.

3.2 Equilibrium with commitment

Here we consider the problem when the planner in $t = 0$ chooses policy for j periods. The equilibrium is equivalent to one in which an appropriately defined sequence of planners choose policy without commitment, so we can solve for the commitment equilibrium by applying Proposition 1.

For each period $k < j$, periods for which policy would be chosen by the initial planner with commitment, let policy instead be chosen by a hypothetical planner without commitment, endowed with the sequence of time preference rates that the initial planner would want them to use. The hypothetical planner at k employs time weights $\{\lambda_{k,k+m}\}_{m=0}^{\infty}$. By Corollary 1, changing from the setting without commitment to the setting with commitment—thus, changing the path of equilibrium emissions—does not affect the equilibrium aggregate savings rate. The hypothetical planner's optimized continuation welfare cannot be higher than the corresponding continuation value if policy were chosen by the initial generation, because that would contradict optimality of the initial generation's policy choice with commitment. Similarly, the continuation value when policy is chosen by the initial generation cannot be higher, because that would contradict optimality of the hypothetical planner's problem. Thus, because optimal policy is unique, the policy rules are the same.

We obtain the equilibrium with commitment by substituting the discount factors $\{\lambda_{k,k+m}\}_{m=0}^{\infty}$ into the definitions in (11) for the first j periods. For simplicity, we present

the case where planners beyond the initial commitment interval choose policy for one period only; however, due to Corollary 2, this assumption does not affect the optimal policy within the commitment interval.

Proposition 2 *Suppose the planner at period $t = 0$ has a commitment device for $j > 1$, and thus can choose policy for periods $v = 0, \ldots, j - 1$. In equilibrium—as for the case without commitment—savings in every period are determined by the aggregate savings rule in (9); carbon taxes in periods $v > j - 1$ are given by (10) (with $v = t$). Within the commitment interval ($v = 0, \ldots, j - 1$) equilibrium carbon taxes are given by*

$$\tau_v = \left[\sum_{k=0}^{\infty} \lambda_{v+k} (1 - d_k) \gamma \cdot \tilde{\Gamma}_k^v \right] Y_v \tag{15}$$

where

$$\tilde{\Gamma}_k^v = \frac{\sum_{m=0}^{\infty} \alpha^m \lambda_{v+k,v+k+m}}{\sum_{n=0}^{\infty} \alpha^n \lambda_{v+n}}.$$

The two forms of decoupling in the log-linear model simplify the equilibrium construction. Commitment has no effect on the savings rule, and it alters the tax rule only during the commitment interval. These changes in the tax rules alter emissions during these periods, so they change the climate state and the capital stock inherited at period $t + j$. Subsequent tax levels, but neither the tax rules nor the emissions levels, change after $t + j$.

Formula 15 reveals another interesting property of the model. Let t be the time index at the beginning of a commitment interval of length $j \geq 1$. Denote the equilibrium tax in t given commitment period j as $\tau_t^{(j)}$.

Corollary 3 *The equilibrium tax in the first period of the commitment interval (time t) is independent of j: $\tau_t^{(1)} = \tau_t^{(j)}$ for $j \geq 1$.*

The result follows because $\tilde{\Gamma}_k^0 = \Gamma_k$. The corollary shows that time consistency is not an issue for near-term policy in the log-linear setting. Time inconsistency arises with NCTP when the agent incorrectly assumes a commitment device (the "naive" scenario). Without commitment, the actual decisions of future decision-makers differ from the choices earlier generations would have made, which typically changes the marginal payoffs of initial-period actions. But not in the log-linear model.

The result arises because equilibrium flow payoffs are linear in prior emissions. Consequently, changes in future emissions due to commitment do not change the optimal emission decision at the beginning of the commitment interval. Comparison of (10) and (15) shows that the full path of carbon tax rules in the two equilibria can differ substantially when we move beyond the initial period, while staying within the commitment interval. We return to this observation when studying the value of commitment.

Corollary 3 reflects a more general feature of the log-linear model: today's policy decisions are independent of future generations' time preference paths (Iverson (2012). This feature has a useful policy implication. If agents know less about future generations' preferences than about their own, they might entertain a range of assumptions about the former (Beltratti, Chichilnisky and Heal 1998). In most settings, the initial carbon price is sensitive to these assumptions, but in the log-linear model it is invariant to them.

3.3 Modeling Considerations and Robustness

Three aspects of climate change are particularly salient: a portion of carbon emissions persist in the atmosphere for thousands of years; temperature responds to carbon stocks with a delay; and some resource stocks are exhaustible. The analytic results in this section are robust to inclusion of the first two features, but not the third. GHKT show that Assumption 1.iv (or some other linear structure) accommodates persistent atmospheric stocks. Traeger (2016) extends GHKT, including a delayed response between temperature and carbon stocks in a model that is linear in transformed variables.

With constant discounting, Traeger (2016) also shows that the introduction of a binding constraint on fossil fuel does not affect the form of the carbon tax: it is still a constant times output, and the constant does not depend on the stock of fuel. However, the equilibrium emissions level does depend on the resource stock, and other state variables. Determining the emissions level, and thus the levels of income and of the tax, requires solving the full dynamic model.

With constant discounting, the planner who uses a carbon tax is in a first best setting, so the tax is the solution to an optimization problem. In the second best setting arising from NCTP, we need to solve an equilibrium problem, complicating the introduction of a binding resource constraint. Just as with constant discounting, the resource constraint causes the equilibrium emissions levels to depend on the state variables. The analytic tractability of the log-linear model under NCTP depends on the state-independence of equilibrium emissions. Because an exhaustible resource (with binding constraint) causes state-dependence, the proof of Proposition 1, required for all results in this section, no longer holds.

We develop the methods in Section 6.2 to solve IAMs with NCTP and state-dependent equilibria. These methods can be modified by adding an additional endogenous function, the Hotelling rent on the exhaustible resource. We leave this topic to ongoing research (Iverson and Karp, 2017), because we consider other sources of state-dependence, including more general utility and damage functions, more important. Also, the need to approximate the Hotelling rent complicates both the explanation and the numerical implementation of the extention to an exhaustible resource.

4 The general model

The decoupling properties of the log-linear model (Corollaries 1 and 2) eliminate most of the linkages between savings and the climate. However, the possible importance of such linkages motivates the use of a Ramsey general equilibrium model to study climate policy in the first place. To explore the linkages and to more convincingly assess the effect of commitment, we relax Assumption 1. A growing literature examines macro economic and climate linkages in fairly general settings under constant discounting (Nordhaus 2013, Lemoine and Traeger 2014, Kelly and Tan, 2015, Lontzek, Cai, Judd, 2015). NCTP leads to a qualitatively different problem.

KKS, for quasi-hyperbolic discounting, and Hiraguchi (2014), for general discounting, solve for equilibrium savings in the one-state log-linear growth model with one aggregate state. To our knowledge, no one has solved the full non-log-linear equilibrium problem even for the one-state growth model with quasi-hyperbolic discounting. Our model accommodates arbitrary NCTP, non-stationarity, three aggregate state variables, and a coupled

equilibrium problem that jointly determines private savings and climate policy.

To proceed, we overcome two technical challenges. First, to solve for the MPE in a non-stationary setting with arbitrary NCTP, we develop an algorithm that translates the equilibrium problem into a modified finite horizon dynamic programming problem. Second, we surmount the well-known instability of polynomial approximations applied to non-stationary dynamic programs (Judd and Cai 2014). The need to solve a fixed point problem at each stage, to obtain the equilibrium savings rule, compounds those instability problems. We exploit structure from the savings problem to obtain a semi-analytic expression for the equilibrium value function that can be accurately approximated. These algorithms constitute an important contribution of the paper, but we describe them only after first presenting the policy-relevant results.

4.1   Assumptions and calibration

To assess the robustness of results from the log-linear setting, we relax the most restrictive parts of Assumption 1: log-linear damages, log utility, and 100% depreciation of physical capital.[6] We also relax Assumption 1.v, replacing GHKT's damage function, $D\left(S\right)$, with the convex transformation $F\left(D\left(S\right)\right) \equiv \theta_1 D^{\theta_2} + \theta_3$. This change responds to recent criticisms that mainstream models understate the likely damages arising from higher temperatures (Weitzman, 2012). It also creates strategic interactions across generations of climate policy-makers, and links between the savings and climate policies. Damages are convex in temperature, which is logarithmic in carbon.[7] In GHKT's formulation, the composition of these two functions is approximately linear, shown by the solid curve in Figure 1. This damage function implies that a six degree Centigrade rise in temperature lowers output by 4%. The solid curve in Figure 1 shows the graph of $\theta_1 D^{\theta_2} + \theta_3$. We calibrate the parameters of the transformation function by requiring the level and slope of the transformed and the original damage functions to be equal at the current carbon stock and by setting damages associated with a 6 degree C temperature rise at three times the level under $D(S)$. We refer to the two damage functions as "linear" and "convex".

We abstract from resource scarcity, so it is enough to model a single composite fossil energy sector ("coal"; $i = 1$) and a single composite clean energy sector ("wind"; $i = 2$). By choice of units, emissions equal fossil fuel production: $E \equiv E_1$. Following GHKT, output in each energy sector i is linear in labor,

$$E_{i,t} = A_{i,t} N_{i,t},$$

and the CES energy composite equals

$$\tilde{E}_t = (\kappa_1 E_{1,t}^\rho + \kappa_2 E_{2,t}^\rho)^{1/\rho}.$$

Net output is Cobb Douglass in capital, labor, and energy,

$$Y_t = [1 - F\left(D\left(S\right)\right)]A_{0,t} K_t^\alpha N_{0,tt}^{1-\alpha-v} \tilde{E}^\nu.$$

---

[6]Barrage (2013) uses numerical methods to evaluate robustness of the log-linear model to alternative functional form assumptions under constant time preference.

[7]We set the climate sensitivity (the change in steady state temperature due to doubling the atmospheric carbon stock) to 3 degrees. With this assumption, the steady state increase in temperature relative to preindustrial levels, T, is $T = 3.0 \ln(S/\bar{S})/2$. A temperature increase of 6 degrees C corresponds to an increase in atmospheric carbon, $S - \bar{S}$, of about 1700 GtC.
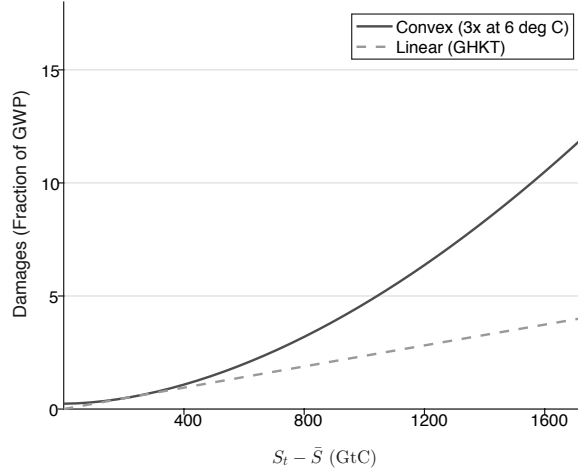
Figure 1: Linear climate damages follow calibration in GHKT; convex damages apply the calibrated convex transformation.

Labor is mobile across sectors with $N_{0,t} + N_{1,t} + N_{2,t} = 1$.

We obtain a recursive version of the carbon cycle by assuming that atmospheric carbon in t is the sum of carbon in permanent $(S_{1,t})$ and transient $(S_{2,t})$ reservoirs:

$$S_t = S_{1,t} + S_{2,t}.$$

The fraction ϕ_L of global carbon emissions enters the permanent reservoir

$$S_{1,t} = S_{1,t-1} + \phi_L E_t.$$

Remaining emissions enter the transient reservoir and decay at the rate ϕ,

$$S_{2,t} - \bar{S} = \phi(S_{2,t-1} - \bar{S}) + \phi_0(1 - \phi_L)E_t.$$

We replace the assumption of log utility with the isoelastic utility function in (1), and we allow for incomplete depreciation of physical capital: $0 < \delta \leq 1$. The general model reduces to the log-linear case when $(\theta_1, \theta_2, \theta_3, \delta, \eta) = (1, 1, 0, 1, 1)$. We use this fact to validate our numerical code, by comparing the numerical and analytic solutions in the log-linear case. The solutions coincide almost exactly.

We use a convex combination of two exponentials to represent time preference. Two interpretations are consistent with this discount function and our savings model: an infinitely lived representative agent model or a particular OLG model.[8] The OLG interpretation is compelling in the climate context, where policy creates both transfers across points in time for the same agent, and transfers across generations. A person considering these transfers might discount them at different rates.[9] For expositional convenience, we refer to the three discounting parameters as the "Nordhaus time preference rate", the

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[8]Aggregation of agents with different beliefs or time preferences also produces a convex combination of exponentials, but it requires a different model of the savings decision.

[9]Ekland and Lazrak (2010) show that an OLG model with paternalistic altruism is isomorphic to an infinitely lived agent model where the representative agent's discount function is a weighted sum of exponentials; this model is isomorphic to one in which agents have pure altruism (Karp 2017). These two
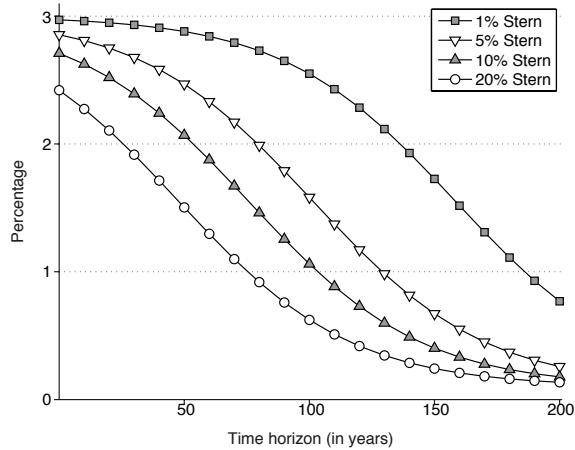
Figure 2: Comparison of alternative hyperbolic discounting paths.

"Stern time preference rate", and the "Stern weight". We set the Nordhaus rate to 3.0%, generating a real return on capital of 5.5% in DICE under log utility. We set the Stern rate to 0.1% (Stern, 2007). Holding these rates fixed and varying the Stern weight generates time preference paths that vary monotonically in the weight; $p_S = 0$ produces the constant Nordhaus time preference rate and $p_S = 1$ produces the constant Stern weight (Figure 2).

Following Nordhaus (2013), we assume that TFP grows at a declining rate:

$$A_{0,t} = A_{0,t-1}(1 + g_{A,t}), \quad g_{A,t} = \frac{g_{A,t-1}}{1 + \delta_A},$$

where δ_A and $g_{A,t}$ are decadal rates. To simplify the interpretation of results, we hold constant the technology terms for energy production, $A_{i,t}$, $i = 1, 2$. We assume that capital depreciates at 10% per year, implying a decadal rate of $\delta = 0.65$. Table 2 collects parameter values.

Table 2: Calibration summary

ϕ	ϕ_L	ϕ_0	α	ν	κ_1	κ_2	ρ	δ	K_0
0.0228	0.2	0.393	0.3	0.04	0.2	0.8	-0.058	0.65	$164,030$
$S_0(S_{1,0})$	$A_{0,0}$	$g_{A,0}$	δ_A	$A_{1,0}$	$A_{2,0}$	θ_1	θ_2	θ_3	
802(684)	17887	0.1617	0.0616	$7,693$	$1,311$	40.6	1.82	0.00236	

We vary η and the Stern weight across simulations. We also consider both zero and the calibrated TFP growth, and both the GHKT damage function and its convex transformation. The planning horizon is set to 2000 years, though we follow Barrage (2013) in

papers do not have a savings decision, but Appendix B.4 explains the modification to include privately owned capital. Schneider et al. (2012) study an OLG model with endogenous capital and finitely lived agents, but they assume that a planner discounts from the time of an agent's birth, not from the current time, thus avoiding the problem of time-inconsistent preferences. Barro's (1974) OLG model in which agents attach intrinsic importance to their immediate successor's welfare, but care about subsequent generations' welfare only insofar as it affects the immediate successor's welfare, also leads to time consistent preferences.

Table 3: Optimal tax in current dollars per ton carbon. Assumes Stern weight 0.2, TFP growth, and $\delta = 0.65$.

	$\eta = 1.0$	$\eta = 1.5$	$\eta = 2.0$
Linear damages	126	48	30
Convex damages	151	62	39

assuming that coal becomes fully decarbonized after 300 years. Because carbon persists in the atmosphere, emissions during the first 300 years affect welfare over the full planning horizon.

4.2 Equilibrium without commitment

We compute equilibria in the model without commitment, showing the response of equilibrium trajectories and the initial tax to alternative assumptions. Figure 3 shows the sensitivity of the first-period tax to η, with (solid curve) and without (dashed curve) TFP growth, under linear damages. Other simulations (not included) show that relaxing the assumption of 100-percent depreciation has negligible effect when $\eta = 1$, and a slightly larger but still small impact when $\eta > 1$. In contrast, the choice of η and TFP growth is important.

The income and the substitution effects explain the relation between η and the initial tax, and also the effect of TFP growth. Where the substitution effect dominates the income effect ($\eta < 1$), agents are more willing to incur sacrifices today to increase future consumption, so they use a higher carbon tax. Where the income effect dominates the substitution effect ($\eta > 1$), agents are less willing to make sacrifices to avoid future climate-related consumption losses.

TFP growth lowers the tax when $\eta > 1$ and raises the tax when $\eta < 1$. For $\eta = 1$, where the income and substitution effects are equal, TFP growth does not affect the tax. The intuition is the same as under constant discounting (GHKT). Higher TFP growth increases future consumption, lowering marginal utility, and increases future output, raising damages. Damages are directly proportional to output, while marginal utility is inversely proportional to output when the savings rate is stock invariant, so the two effects cancel. Introducing incomplete depreciation does not change the intuition. TFP growth increases future income. If the income effect dominates the substitution effect ($\eta > 1$), growth leads to a large reduction in the initial tax, because the agent today is unwilling to reduce current consumption to protect the richer successor from climate damages. Where the substitution effect dominates ($\eta < 1$), the agent is willing to incur even greater consumption loss to protect the climate, and magnify successors' consumption gain.

Table 3 shows how changing the damage function affects the initial tax, for different values of η, with TFP growth. Switching from linear to convex damages increases the initial tax by 20% when $\eta = 1$ and by 30% when $\eta = 2$. We consider this effect modest, in view of the large change in the damage function shown in Figure 1. The next subsection discusses this result.

Figures 4 (for linear damages) and 5 (for convex damages) graph equilibrium trajectories for the savings rate, the carbon tax, and the stocks of capital and atmospheric carbon.
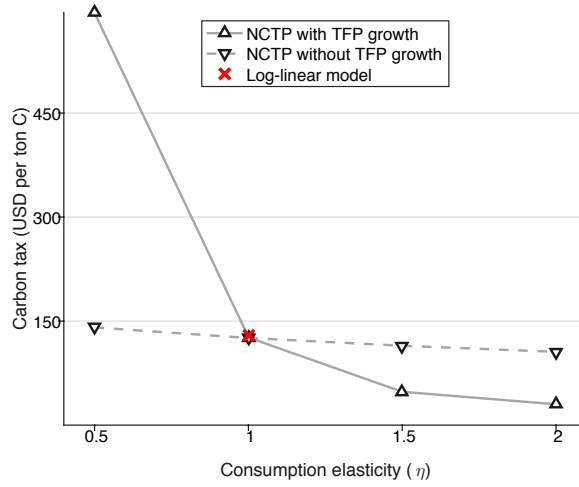
17

Figure 3: First-period tax for log-linear case and alternatives. Both grey lines have $\delta = 0.65$ and linear damages. Difference between downward- and upward-pointed triangles shows effect of TFP growth.
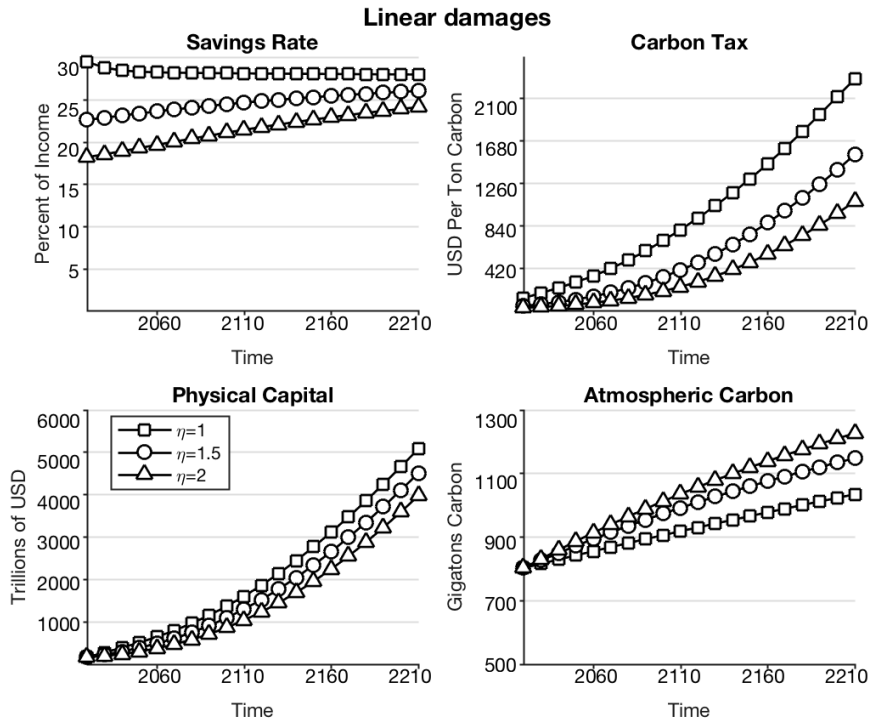

Figure 4: Panels show trajectories of policy variables and states for different values of η when the damage function matches GHKT. In addition, $\delta = .65$, the Stern weight equals 0.2, and TFP growth matches Nordhaus (2013).
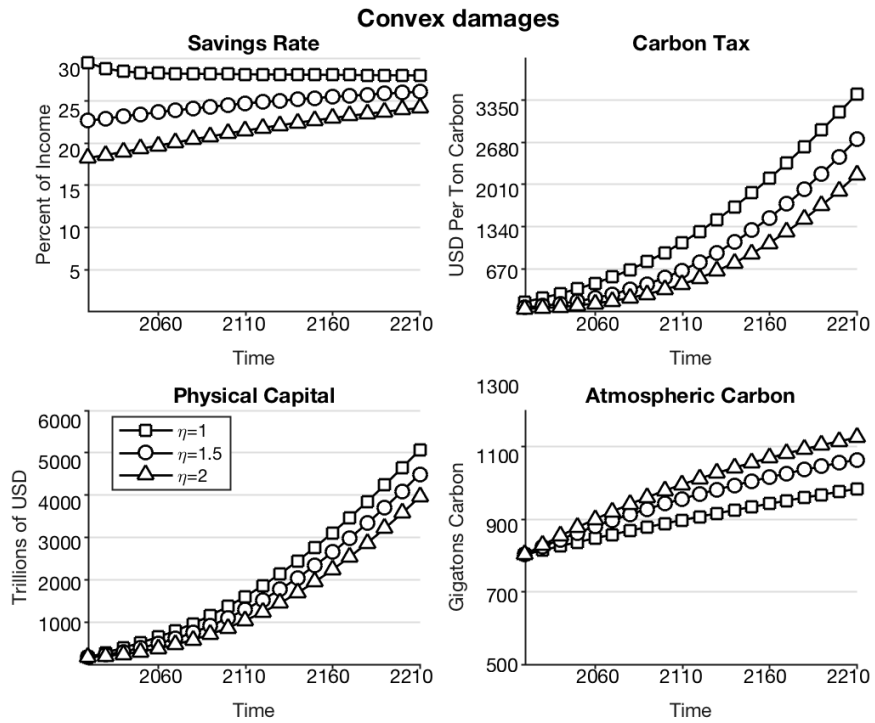
Figure 5: Panels show trajectories of policy variables and states for different values of η when the damage function is convex. In addition, $\delta = .65$, the Stern weight equals 0.2, and TFP growth matches Nordhaus (2013).
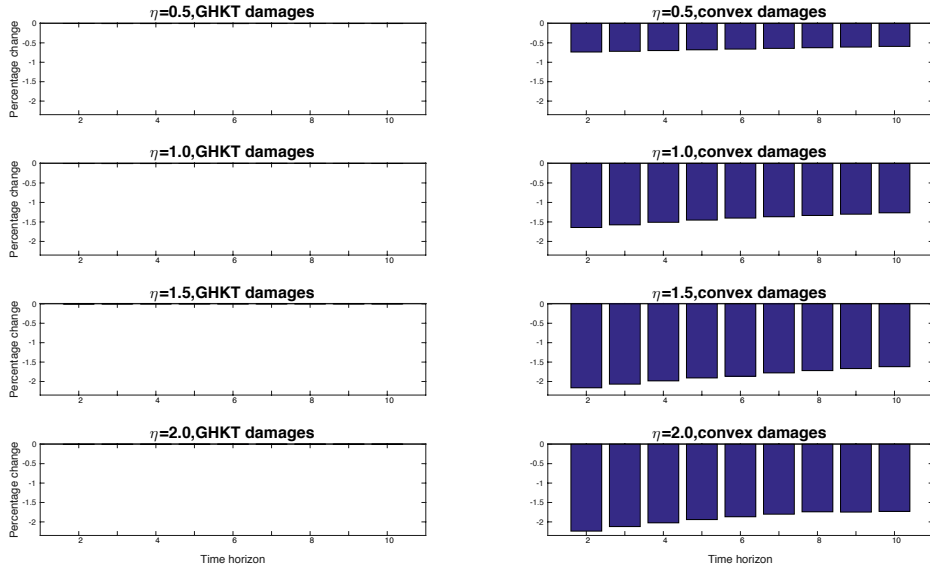
Figure 6: Impulse response when first-period emissions are doubled. Vertical axis shows percentage change in emissions relative to the original equilibrium. Horizontal axis shows time horizon in decades. Panel columns vary damages. Panel rows vary the consumption elasticity.

The path of carbon taxes increases only modestly—by about 30%—when switching from linear to convex damages. The savings rate and accumulation of physical capital are almost the same under linear and convex damages. One might expect the higher (convex) damage function to induce higher capital accumulation. However, the higher carbon tax under convex damages reduces the accumulation of carbon by about 25%. On balance, equilibrium damages are only modestly higher with convex damages. The savings rate is sensitive to the consumption elasticity, but not to damages.

4.3 Strategic interactions without commitment

Corollary 2 notes that the log-linear model shuts down strategic interactions among climate planners in different periods. Two experiments help quantify the importance of strategic interactions in the general setting. First, impulse response functions trace the equilibrium response to an exogenous doubling in first-period emissions. Second, we compare the first-period equilibrium tax with the tax in a restricted problem that shuts down strategic interactions.

Figure 6 presents impulse response functions in periods 2 through 10 associated with an exogenous doubling of first-period emissions relative to the original equilibrium. We calculate the responses using the equilibrium policy rules of future planners. All simulations assume NCTP (Stern weight of 0.2), TFP growth, and incomplete depreciation ($\delta = 0.65$). The panels differ in the assumed damage function (columns) and the consumption elasticity (rows).
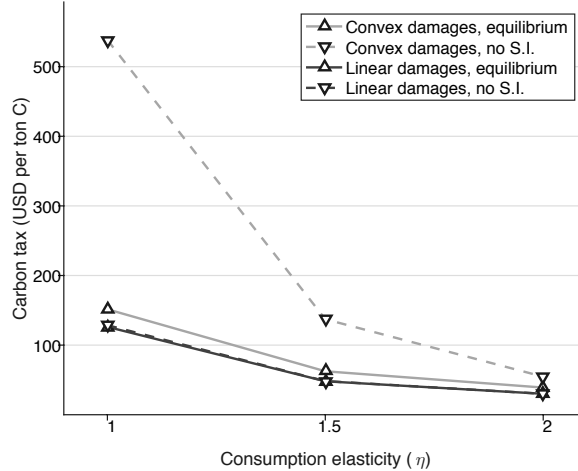
20

Figure 7: Black lines, solid and dashed, compare the equilibrium and restricted taxes when damages are linear. Grey lines compare the equilibrium and restricted taxes when damages are convex.

The results are starkly different under linear versus convex damages. With linear damages, strategic interactions remain quantitatively irrelevant, even as we relax the assumptions of 100% depreciation and log utility. With convex damages, in contrast, strategic interactions are quantitatively significant for all values of η. When $\eta = 2$, a 100% increase in first-period emissions reduces emissions in each of the next 9 periods by roughly 2%. With convex damages, emissions are dynamic strategic substitutes: higher current emissions cause future generations to face higher marginal damages, inducing them to emit less.

To quantify the impact of strategic interactions, we consider a restricted problem that fixes future emissions at their level in the original equilibrium, thereby closing the strategic channel. We also fix all savings rates at their equilibrium level, thereby limiting strategic interactions to emission decisions. Comparing the initial tax in the original and the restricted problems gives a measure of the quantitative importance of strategic interactions. To solve the restricted problem we fix labor allocations, and thus emissions, at their equilibrium level. Conditional on the period 1 state, $(K_1, \mathbf{S}_1)$, we use the equilibrium saving rate and labor allocations and the equations of motion for capital and the carbon stocks to construct the consumption trajectory, $\{\hat{C}_t\}_{t=1}^T$, and the restricted continuation value, $W^R(K_1, \mathbf{S}_1) = \sum_{t=1}^T \lambda_t u(\hat{C}_t)$. The initial decision in the restricted problem solves

$$\max_{N_0, N_1, N_2} u(C_1) + \beta_1 W^R(K_1, \mathbf{S}_1)$$

subject to the equations of motion for capital and the carbon stocks.

Figure 7 presents the results. Black curves plot the carbon taxes with linear damages, and grey curves plot carbon taxes with convex damages. Solid curves (and upward-pointed triangles) identify equilibrium carbon taxes. Dashed curves (and downward-pointed triangles) identify taxes that solve the restricted problem. Comparing the solid black and the solid grey curves (equilibrium taxes with convex versus linear damages) reinforces the message of table 3: moving from linear to convex damages has a modest effect on the equilibrium tax.
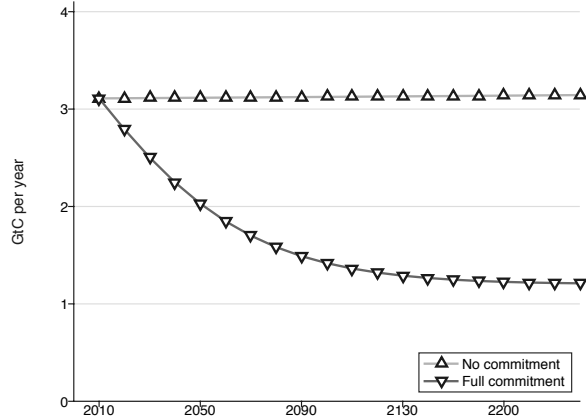
21

Figure 8: Equilibrium emissions with and without commitment. Log-linear model with TFP growth and Stern weight of 0.2.

Strong strategic incentives partly explain this modest response: the initial generation understands that higher abatement on its part would induce future generations to abate significantly less. This anticipated response lowers the initial tax. Figure 7 shows that strategic incentives are negligible under linear damages, but are strong under convex damages. These strategic incentives dampen the direct incentive to increase taxes in response to more convex damages. Under log utility and convex damages, shutting down the strategic incentive increases the initial tax by a factor of almost three. Higher values of η increase the consumption discount rate, diminishing the present value of long run differences caused by strategic interactions. The strategic effect is smaller but still significant for higher η.

5 The value of commitment

Figure 8 uses the log-linear model to plot the path of emissions in the equilibrium without commitment (upward triangles) and in the full-commitment scenario where the initial generation chooses climate policy in all future periods (downward triangles). With declining time preference, future generations do less to combat climate change than the initial generation would like. Without commitment, emissions are constant at a third the level of current global emissions.[10] With permanent commitment, emissions begin at the same level, but eventually decline to a third the no-commitment level. The large difference in emission paths suggests that a commitment device could be valuable.

We find that the value of a commitment technology is almost certainly less than the (likely) high costs of designing and implementing it. We begin with the log-linear model, where Corollaries 1 and 2 simplify the computation of commitment value and make it possible to explain this conclusion. The next subsection shows that the conclusion also holds in the general setting.

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[10]Absent commitment, emissions are constant because we assume that the TFP paths for coal, $\{A_{1,t}\}$, and wind, $\{A_{2,t}\}$, are constant. If $\{A_{2,t}\}$ grew faster than $\{A_{1,t}\}$, emissions in the equilibrium without commitment would decline, as wind becomes relatively cheaper.
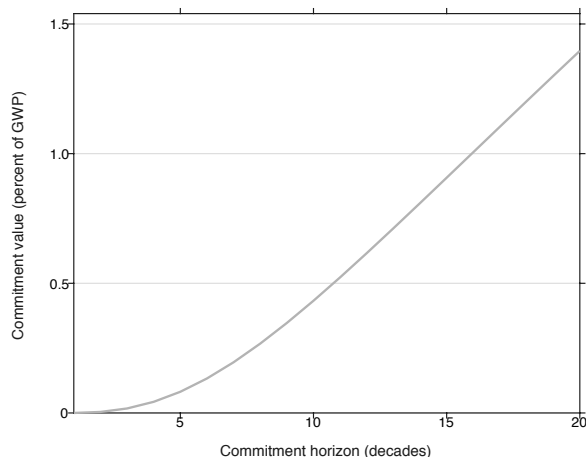
Figure 9: Commitment value as a function of commitment horizon. Log-linear model with TFP growth and Stern weight of 0.2.

5.1 The log-linear case

We define commitment value as the addition to first period consumption, absent commitment, that leads to equal welfare in the two scenarios. We calculate this equivalent variation by dividing the uncompensated difference in welfare by the initial period marginal utility, absent commitment. A permanent commitment device, illustrated in Figure 8, is worth about 16 trillion USD (2.3% of GWP) for the initial (decade-lived) generation. For comparison, the value of climate policy in the model—the increase in first generation welfare in the no-commitment equilibrium compared to welfare without climate policy— is 13.3 percent of GWP. The value of climate policy is higher than in typical economic models like DICE (Nordhaus 2013), because under decreasing time preference rates the initial generation puts relatively higher weight on long run outcomes.

Permanent commitment would be valuable, but likely infeasible. Recent history shows how difficult it is to achieve cooperation amongst nations even in the short run. Commitment to future climate policy, over long spans of time, requires current policy-makers to induce their successors to maintain "more cooperation" than successors desire. Figure 8 shows that the difference between future generations' emissions choice, and the level the current generation would choose, increases in the commitment horizon. More distant generations have more incentive to break the commitment device.

A technology with commitment horizon j enables the initial generation to commit policy for j periods. Figure 9 plots commitment value as a function of the commitment horizon, showing a convex relation for the first 150 years. The value of commitment remains near zero for a few decades and reaches one percent of GWP only after 160 years. Because the value of a short commitment period is small, and the cost of supporting a long period is probably large, efforts to devise and implement a commitment technology are likely futile.

5.1.1 Explanation for convexity

Two forces affect the curvature of the value-of-commitment graph. The first, denoted "decision conflict" arises because the discount rate falls over time. As a consequence, the difference between the discount rate that the agent at t applies to evaluate a utility transfer between $t + N - 1$ and $t + N$, and the discount rate that the agent at $t + N - 1$ uses to evaluate this transfer, grows with N. The disagreement, between the agents at t and at $t + N - 1$, about the optimal plan at $t + N - 1$ therefore also tends to grown with N.[11] From the perspective of the agent at t, a commitment period of N solves this conflict. Other things equal, the solution to the conflict is more valuable to the agent at t, the greater is the conflict. This force tends to make the value of commitment convex in the commitment period. However the serious conflicts also tend to occur farther in the future, and are thus discounted more heavily, reducing their present value for the agent at t. This "present value effect" tends to make the value of commitment concave in the commitment period.

An increase in the commitment horizon from N to $N + 1$ periods potentially alters equilibrium climate and savings policy in the first $N - 1$ periods, creating a third reason for the slope of commitment value to depend on N. However, Corollaries 1 and 2 show that this channel does not operate in the log-linear setting. Therefore, the curvature of the value of commitment depends on the relative strength of the decision conflict and the present value effect.

Proposition 3 provides a necessary and sufficient condition for convexity, using the current value continuation payoff. If generation t has an N-period commitment technology, it can choose policy in periods t through $t + N - 1$. Let $\{c_{t+s}^N\}_{s \geq 0}$ denote the sequence of equilibrium consumption when the commitment period is N. Generation t's present value of the continuation payoff beginning in $t + N$ is

$$W_{t+N}^N \equiv \sum_{s=0}^{\infty} \lambda_{N+s} \ln(c_{t+N+s}^N),$$

and the current value, viewed from the perspective of the generation in t, is

$$V_{t+N}^N \equiv \left(W_{t+N}^N\right) / \lambda_N. \tag{16}$$

The increase in the current value continuation payoff due to an additional commitment period is

$$\Delta V_{t+N}^N \equiv V_{t+N}^{N+1} - V_{t+N}^N.$$

Convexity of the commitment value requires the increment in W_{t+N}^N when moving from N-period commitment to $N + 1$-period commitment to increase in N. This condition holds if and only if the growth rate of ΔV_{t+N}^N exceeds the rate at which the time weight decays.

Proposition 3 *For the log-linear model, a necessary and sufficient condition for the commitment value curve to be convex in the commitment horizon is*

$$\frac{\Delta V_{t+N+1}^{N+1} - \Delta V_{t+N}^N}{\Delta V_{t+N}^N} \geq r_N, \tag{17}$$

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[11]With quasi-hyperbolic discounting, the disagreement conflict becomes constant after a single period. Therefore, quasi-hyperbolic discounting tends to result in a lower and less convex value of commitment.

Figure 10: Commitment value curve with TFP growth, $\delta = 0.65$ and Stern weight 0.2. Black lines show case with linear damages; grey lines show case with convex damages.

where r_N is the discount rate applied by the initial generation between period $N - 1$ and period N.

The left-side of inequality (17) captures the impact of rising decision conflict, measuring the growth of the increment to the current value continuation payoff, viewed by the generation in t. The right-side captures the fact that as the added commitment period occurs further in the future, it matters less because the initial generation discounts it more heavily. For small N the left side is greater than the right side, and the graph of commitment value is convex in N. As $N \to \infty$, the growth rate of ΔV_{N+1} approaches zero, but the discount rates approaches 0.01 (the Stern decadal discount rate). Therefore, for large N inequality (17) is reversed, and the value of commitment becomes concave. Commitment value converges to a constant as the commitment horizon goes to infinity. In the application, the value of commitment is strictly convex in the commitment period for 150 years.

5.2   The general case

Figure 10 graphs commitment value in the general setting.[12] The solid-black lines vary the consumption elasticity with linear damages. The dashed-grey lines vary the consumption elasticity with convex damages. Depreciation is incomplete ($\delta = 0.65$) and we include TFP growth. A higher consumption elasticity decreases the value of commitment, without changing the graph's convex shape. With TFP growth, higher η increases the consumption discount rate, lowering the present value of the higher future utility achieved by commitment.

Comparing the solid black and dashed grey lines shows that moving from linear to convex damages does not substantially change the shape or magnitude of the curves. This

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[12]To compute the equilibrium path with commitment, we numerically implement the backward induction sweep separately for each possible commitment horizon. When choosing policies within the commitment horizon, we impose time preferences on future decision makers consistent with those the initial generation would want them to use.

Figure 11: Carbon tax trajectory for equilibria with one-, five- and ten-period commitment. TFP growth, $\eta = 1.5$, $\delta = 0.65$. Panels differ in terms of damages.
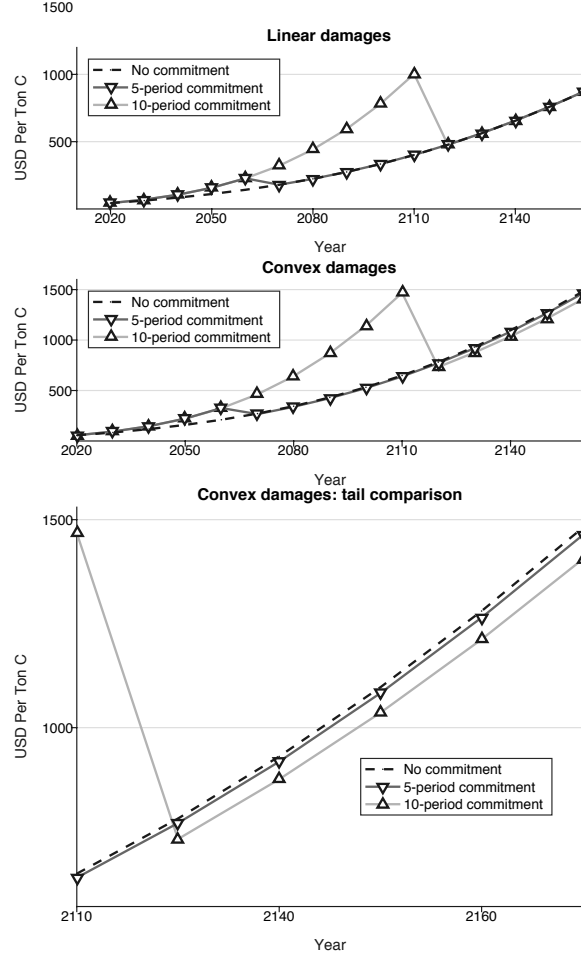

Figure 12: Tail portion of second panel from figure 11.

insensitivity is somewhat surprising. When damages are convex, absent commitment, strategic interactions induce the initial generation to emit more than it otherwise would. Because a commitment device shuts down strategic interactions within the commitment horizon, we might expect it to confer additional value when the strategic interactions are significant (i.e., with convex damages). However, those strategic interactions do not substantially increase the value of commitment, because countervailing forces operate beyond the commitment horizon. Figures 11 and 12 make this point.[13]

With linear damages, at the end of a commitment period of either 5 or 10 decades, the tax drops to almost exactly the no-commitment level. In contrast, with convex damages, the tax after a commitment interval falls below the no-commitment baseline. Figure 12, for a 10-period commitment interval, makes this reduction easier to see. Post-commitment generations abate less, the longer is the commitment period, because they inherit a lower

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[13]The figures presents results for the case with $\eta = 1.5$, though the qualitative findings remain the same for $\eta = 1$ and $\eta = 2$.

stock of carbon. The generation operating within the commitment interval anticipates this response, reducing the value of commitment.

Comparing the three pairs of curves in Figure 10 shows the interaction between η and the convexity of costs, in determining the value of commitment. The commitment value is higher under linear compared to convex damages for $\eta = 1$, but $\eta > 1$ reverses the ordering. The reversal results from the post-commitment strategic interactions noted above. Those strategic incentives operate under convex but not linear damages. The downside of commitment materializes in the future. The future matters less with higher η, so the post-commitment strategic reaction is less damaging with high η.

In the log-linear setting, we noted that there are no strategic effects. These effects are present in the general model, and potentially influence the value of commitment. However, we find that those effects are negligible (Appendix B.6). Therefore, the intuition for the convex value of commitment, obtained for the log-linear model, remains approximately correct in the general model.

6 Numerical Methods

Standard dynamic programming methods cannot be used under NCTP because agents in different periods disagree about the evaluation of subsequent continuation values. Section 6.1 modifies the standard finite horizon dynamic programming algorithm to account for agents' distinct time perspectives.

Polynomial approximation methods used to implement continuous state dynamic programming may be unstable, especially in non-stationary settings. Repeated iteration of the dynamic programming equation can amplify small wobbles in the approximated value function, causing it to "lose shape" (Cai and Judd 2014). For example, the approximation of a convex function may cease to be convex.[14] The instability is heightened in our problem, where we need to solve a fixed point at each time step to obtain the aggregate savings decision. Small non-convexities in the value function can cause the search for a fixed point to fail. In section 6.2, we build on an insight in KKS to devise an alternate approach. KKS note that the value function for agents with isoelastic utility and a linear constraint can be expressed in closed form in individual (but not aggregate) capital. They use this fact to calculate the steady state for a class of models. We adapt it to approximate the dynamic equilibrium for a broader class. We verify the accuracy of the approach by comparing the numerical solution with the analytic solution for the log-linear case.

6.1 The generalized dynamic programming equation

To simplify exposition, we present the algorithm for the current model; with additional notation, it can be used for an arbitrary dynamic model, as in Iverson (2012). To focus on the savings decision, we take the climate policy as given. In implementing the algorithm we need to consider the two types of decisions simultaneously. The model has $T + 1$

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[14]One option is to use linear splines, though this requires many more nodal points than with polynomial approximation and becomes very time consuming for problems, like ours, with several state variables. Another approach is to apply shape-preserving spline approximation methods (Cai and Judd 2014), but so far these methods have only been developed for problems with a single state variable. In any case, for the problem with NCTP, we cannot guarantee concavity of the value function on *a priori* grounds, so imposing it would amount to an *ad hoc* restriction.

periods. Working backwards, the iteration index i identifies the number of periods to go. Fixing the initial period at $t = 0$, the iteration index also accounts for calendar time, thus incorporating non-stationarity. The discount factor the agent applies to the flow j periods in the future is $\lambda_j \equiv \Pi_{i=0}^{j}\beta_i$, where β_j is the single period discount factor, and $\beta_0 = 1$.

The function $V^{(i)}(k, K, \mathbf{S})$ denotes the agent's equilibrium welfare when there are i periods to go. We construct auxiliary value functions to represent the continuation value for agents in earlier periods. The value that the agent at $T - j$ attaches to the stream of payoffs from $T - i$ through T is $W_{T-j}^{(i)}(k, K, \mathbf{S})$, for $j = i + 1, i + 2...T - 1$.[15] We need to track as many auxiliary value functions as there are periods over which the rate of time preference declines. With quasi-hyperbolic discounting, we need one auxiliary value function (Harris and Laibson, 2001). In the application, we assume the rate of time preference declines for 300 years—30 periods—so we need to save 29 auxiliary value functions at each time step. Because computing the auxiliary value functions requires no additional optimization, the computational intensity is only modestly greater than for a comparable-sized dynamic program.

Proposition 4 *At iteration i the agent solves*

$$V^{(i)}(k, K, \mathbf{S}) = \max_{k'}\left(U\left(R_i k + w_i + (1 - \delta)k - k'\right) + \beta_1 W_{T-i}^{(i-1)}(k', K', \mathbf{S}')\right) \Rightarrow$$

$$k'^* \equiv \arg\max_s\left(U\left(R_i k + w_i + (1 - \delta)k - k'\right) + \beta_1 W_{T-i}^{(i-1)}(k', K', \mathbf{S}')\right). \tag{18}$$

Using k'^* to denote the equilibrium savings rule (as distinct from the level of savings), the updating equation for the auxiliary value function is

$$W_{T-j}^{(i)}(k, K, \mathbf{S}) = U\left(R_i k + w_i + (1 - \delta)k - k'^*\right) + \beta_{j-i+1}W_{T-j}^{(i-1)}\left(k'^*, K', \mathbf{S}'\right) \tag{19}$$

for $j = i + 1, i + 2...T - 1$ with boundary condition

$$W_{T-j}^{(-1)}(k, K, \mathbf{S}) \equiv 0, \ for \ j = 0, 1, ...T - 1. \tag{20}$$

For $i = 0$ we impose the constraint $k' \geq 0$.

To implement the procedure, we first solve the static problem in the last period and use the solution to construct numerical approximations for the auxiliary value functions. We then iterate backwards, using the appropriate auxiliary value function when evaluating the "next-period outcome" from the perspective of each earlier generations. We have successfully used this procedure to compute MPE for the command economy, where a planner chooses savings. But, as noted, the procedure becomes unstable when we try to use it to solve the competitive equilibrium, which requires finding a fixed point at each iteration. The next subsection describes an alternate approach.

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[15]With non-constant discounting, $W_{T-j}^{(i)}(k, K, \mathbf{S})$ does not in general equal the discounted value of $V^{(i)}(k, K, \mathbf{S})$, because the agents at $T - i$ and at $T - j$ put different weight on the stream of payoffs from $T - i$ through T.

6.2 Semi-analytic numerical procedure

Proposition 5 provides the algorithm for the private agents' problem, taking as given the climate policy. Section 6.2.2 explains the modification to determine the climate policy. Referees' Appendix B.5 discusses some numerical details needed to impose the identity between individual and aggregate savings rules.

6.2.1 The private savings decision

Here we include individual capital as an additional state variable, apart from the aggregate states. Agents have isoelastic utility and face the linear constraint $k' = R(K, E, \mathbf{S}, i) k + w(K, E, \mathbf{S}, i) - c + (1 - \delta) k$. The iteration index i captures time. Prices depend on variables the agent takes as given, including climate policy (Section 6.2.2). We present the case $\eta \neq 1$, relegating the log case, $\eta = 1$, to Appendix A.5.

The solution has two important properties: a semi-analytic expression for the auxiliary value function and a linear savings rule. The auxiliary value functions in (18) have the following structure:

$$W_{T-j}^{(i)}(k, K, E, \mathbf{S}) = b_{i;j}(K, E, \mathbf{S}) \frac{(k + f_i(K, E, \mathbf{S}))^{1-\eta}}{1 - \eta}. \tag{21}$$

The endogenous functions $b_{i;j}(K, E, \mathbf{S})$ and $f_i(K, E, \mathbf{S})$ depend on the aggregate states and emissions, but not on individual capital k. Equilibrium net saving, $s_i(K, E, \mathbf{S})[k + \xi_i(K, E, \mathbf{S})]$, is linear in the agent's capital, so gross savings equal $k' = s_i(K, E, \mathbf{S})[k + \xi_i(K, E, \mathbf{S})] + (1 - \delta) k$. We obtain closed form expressions for $s_i(K, E, \mathbf{S})$ and $\xi_i(K, E, \mathbf{S})$ in terms of $b_{i-1}(K, E, \mathbf{S})$ and $f_{i-1}(K, E, \mathbf{S})$, which we know from the previous iteration. We also derive an updating rule for $b_{i;j}(K, E, \mathbf{S})$ and $f_i(K, E, \mathbf{S})$, and we get a boundary condition that uses the fact that savings are zero in the last period.

The procedure has three advantages. First, we need to approximate functions that depend only on the aggregate states, not individual capital, thus reducing by one the number of state variables. Second, conditional on the climate policy, obtaining the equilibrium savings rule requires, at each stage, the solution to relatively simple fixed point problems from $\Re^2 \to \Re^2$ instead of a complicated mapping in the space of functions. Third, the analytical expression for continuation values captures the curvature that these functions inherit from the curved utility function. The functions $b_i(K, \mathbf{S})$ and $f_i(K, \mathbf{S})$ are quite flat, and thus well approximated by low-dimensional polynomials, leading to a stable numerical procedure.

Proposition 5 *Taking as given the decision rules for climate policy, $E_i \equiv E_i(K, \mathbf{S})$, the equilibrium savings rule for the agent is linear in own-capital, $k' = s_i(k + \xi_i) + 1 - \delta$, and the auxiliary value function has the form:*

$$W_{T-j}^{(i)}(k, K, E, \mathbf{S}) = b_{i;j}(K, E, \mathbf{S}) \frac{(k + f_i(K, E, \mathbf{S}))^{1-\eta}}{1 - \eta}, \tag{22}$$

for $i = 0, 1, 2..T$; for each i, the index j runs over $i + 1...T - 1$. At $i = 0$ the agent the agent saves nothing (so net savings equal $-(1 - \delta) k$):

$$s_0 = -(1 - \delta) \quad and \quad \xi_0 = 0. \tag{23}$$

For $i \geq 1$ the coefficients of the savings rule are

$$s_i = \frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \tag{24}$$

$$\xi_i = \frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}}{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)} \tag{25}$$

The endogenous functions $b_{i;j}(K, E, \mathbf{S})$ (for $j = i+1, ...T-1$) and $f_i(K, E, \mathbf{S})$ obey the recursion

$$b_{i;j} = \left(\frac{R_i + (1-\delta)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}\right)^{1-\eta} \left[(\beta_1 b_{i-1;i})^{-\frac{1-\eta}{\eta}} + \beta_{j-i+1} b_{i-1;j}\right] \tag{26}$$

$$f_i = \frac{w_i + f_{i-1}}{R_i}$$

with boundary conditions

$$f_0 = \frac{w_0}{R_0 + 1 - \delta} \text{ and } b_{0;j} = (R_0 + 1 - \delta)^{1-\eta}. \tag{27}$$

6.2.2 The climate policy problem

The planner takes the aggregate states, $(K, \mathbf{S})$, all savings rules, and future climate policy rules as given. The equilibrium current savings rule, $s_i(K, E, \mathbf{S})[K + \xi_i(K, E, \mathbf{S})]$, depends on current emissions, via its effect on output. The planner chooses current emissions, $E = E_i(K, \mathbf{S})$, to maximize welfare for the representative agent, so the value functions and savings rules are evaluated where $k = K$. The planner's problem is

$$\max_E \frac{(R_i(K, \mathbf{S}, E)K + w_i(K, \mathbf{S}, E) - K')^{1-\eta}}{1-\eta} + \beta_1 b_{i-1;i}(K', \mathbf{S}') \frac{(K' + f_{i-1}(K', \mathbf{S}'))^{1-\eta}}{1-\eta}$$

subject to

$$K' = s_i(K, E, \mathbf{S})(K + \xi_i(K, E, \mathbf{S})) + (1-\delta)K,$$

$$\mathbf{S}' = H(\mathbf{S}, E).$$

Current emissions affect the next period states: $\mathbf{S}'$ through the impact on the stock accumulation equation—here denoted $H(\cdot)$—and K' through the impact of emissions on current output, and thus on current savings.

6.2.3 Demonstration of numerical stability

Figure 13 plots level sets of $b_i(K, \mathbf{S})$ and $f_i(K, \mathbf{S})$ in the initial period after iterating on the numerical procedure for 200 periods, corresponding to the 2000 year time horizon in the application. Figure 14 plots the corresponding value function that satisfies the analytical relationship between b_i and f_i and the value function. The plots show that the numerical procedure is highly stable, owing in part to the low degree of curvature in the b_i and f_i functions, along with the fact that equilibrium welfare inherits concavity directly from the analytical portion of the semi-analytic expression described in proposition 5.
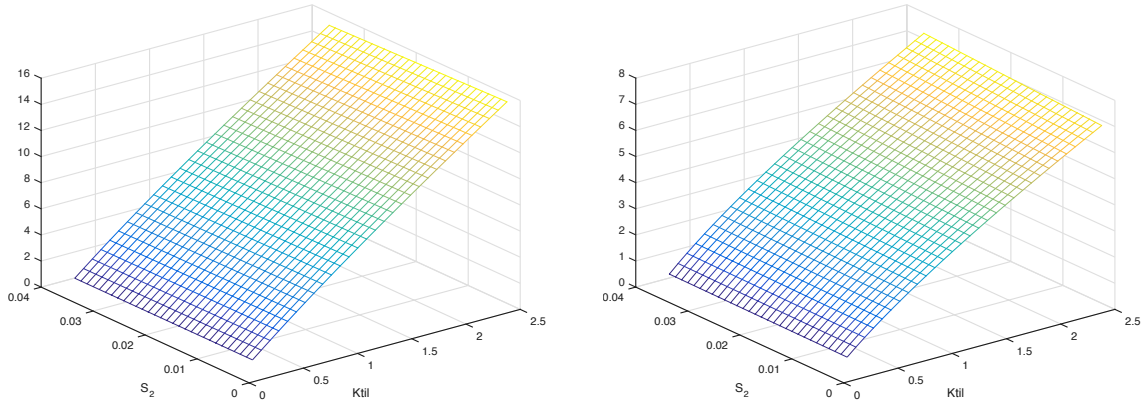
Figure 13: The function on the left is $f_i(K, \mathbf{S})$, while the function on the right is $b_i(K, \mathbf{S})$. Shown in the initial period when the model is solved with $\eta = 2$, $\delta = 0.65$, and a time horizon of 2000 years.
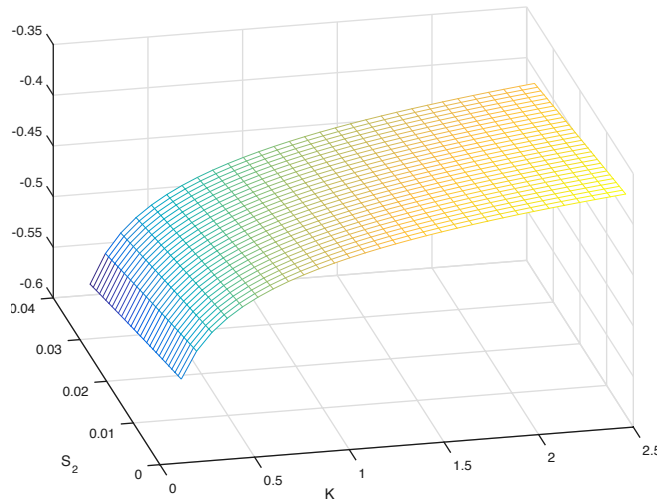

Figure 14: Value function W corresponding to the f_i and $b_{i;j}$ functions in figure 13.

7   Conclusion

Climate policy creates costs today in exchange for benefits that accrue to currently living people later in life and to subsequent generations. If people discount their own future benefits and the benefits of those born later at different rates, they have non-constant time preference. We solve the first integrated assessment model with iso-elastic utility, but otherwise general functional forms, under NCTP. The need to solve the savings decision as an equilibrium problem, instead of a command problem, creates technical difficulties that we overcome using two new algorithms. NCTP changes the nature of the policy decision. Future policy-makers do less to combat climate change than the current generation would like, so there is an incentive to devise a commitment device to force the hand of future generations. Although a commitment device lasting hundreds of years would be highly

valuable, one lasting only a few decades is worth little. Given the difficulty involved, we conclude that efforts to create and implement such a device are likely futile.

The paper also provides an analytic solution to the log-linear model. Given its convenience, it is important to consider the robustness of this model's policy prescriptions. We find that the magnitude of the carbon tax is robust to changes in the depreciation rate, but not to changes in the elasticity of intertemporal substitution or the damage function. The strong property of the log-linear model, which shuts down strategic interactions among generations, approximately holds for other values of the elasticity of intertemporal substitution and of the depreciation rate, provided damages are roughly linear in the stock of carbon. But if damages are strongly convex, policies are dynamic strategic substitutes, creating strategic incentives that reduce the equilibrium carbon tax.

Due to its analytic solution, the log-linear model can be extended at little cost to include temperature delay and uncertainty about damages. The additional state variables needed to extend the general model give rise to the curse of dimensionality. Nevertheless, it is worth considering temperature delay and a finite resource stock. Incorporating uncertainty about future technology would require a more fundamental change, but would make it possible to quantify the relative importance of two distinct reasons for a declining consumption discount rate: hyperbolic discounting and changes to the term structure of interest rates due to uncertainty.

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A Appendix

A.1 Proof of Proposition 1

We solve for the unique MPE that arises in the finite horizon version of the problem, then take the limit as the time horizon goes to infinity. Lemma 1 presents the equilibrium savings rule that obtains in the non-stationary, finite-horizon setting. Hiraguchi (2014) studies the analogous equilibrium problem in a stationary, infinite-horizon (one-state) economy with general NCTP. To link the savings equilibrium with the climate policy equilibrium in our setting and to solve for the limit equilibrium, we need to study the finite horizon equilibrium problem explicitly. In this way, the proof we present differs significantly from that in Hiraguchi (2014). Nevertheless, because the result coincides with Hiraguchi (2014) in the infinite horizon limit, we relegate our proof of the lemma to Referees' Appendix B.1.

We employ the following notation. The horizon at time 0 is H, so the remaining horizon at time t is $T = H - t$.

Lemma 1 *Suppose Assumption 1 holds and that climate policy in each period is independent of the inherited states. Then the unique equilibrium savings rule is*

$$K' = G(K,T)Y_t = \frac{\alpha \rho\left(T\right)}{1 + \rho\left(T\right)} Y_t = \frac{\rho\left(T\right)}{1 + \rho\left(T\right)} r_t K \tag{28}$$

with the definition

$$\rho\left(T\right) \equiv \sum_{t=1}^{T-1} \lambda_\tau. \tag{29}$$

Building on (28), define

$$s_t \equiv \frac{\alpha \rho\left(H - t\right)}{1 + \rho\left(H - t\right)}. \tag{30}$$

The climate planner equilibrium is constructed using an inductive proof. In period t, the inductive hypothesis states that for all subsequent periods $\tau > t$ optimal emissions, E_τ, are independent of the inherited state variables, K_τ and $S_{\tau-1}$. In addition, from Lemma 1 the sequence of savings rules $\{s_{t+j}\}_{j=0}^{T-t}$ are stock invariant so the climate planner in t takes them as given also.

The hypothesis is easy to verify for the last period. Suppose that in a given period t, it holds for all subsequent periods. Then the planner in t anticipates capital will accumulate according to

$$K_{\tau+1} = s^\tau K_\tau^\alpha A_\tau(E_\tau) \exp\left(-\gamma S_\tau\right), \ \tau = t, \ldots, T-1. \tag{31}$$

Taking logs gives a first-order linear difference equation in the log of capital. Iterating this equation, the log of capital in $\tau > t + 1$ can be written

$$\ln(K_\tau) = \alpha^{\tau-(t+1)} \ln(K_{t+1}) + \sum_{j=0}^{\tau-t-2} \alpha^{\tau-t-2-j} [\ln(s_{t+1+j}) + \ln(A_{t+1+j}(E_{t+1+j})) - \gamma S_{t+1+j}]. \tag{32}$$

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Ignoring variables that are exogenous to the decision-maker in t, this can be written

$$\ln(K_\tau) = \alpha^{\tau-(t+1)}\ln(K_{t+1}) - \sum_{j=0}^{\tau-t-2}\alpha^{\tau-t-2-j}\gamma(1-d_{1+j})E_t + \text{``terms''}. \qquad (33)$$

Even though the decision-maker in t does not control the savings rate in t, they still influence K_{t+1} via their influence on Y_t. Using (33), the flow payoff in τ can be written

$$\begin{aligned}
\ln(C_\tau) &= \ln[(1-s_\tau)K_\tau^\alpha A_\tau(E_\tau)\exp(-\gamma S_\tau)]\\
&= \alpha\ln(K_\tau) - \gamma S_\tau + \text{``terms''}\\
&= \alpha[\alpha^{\tau-(t+1)}\ln(K_{t+1}) - \sum_{j=0}^{\tau-t-2}\alpha^{\tau-t-2-j}\gamma(1-d_{1+j})E_t + \text{``terms''}] - \gamma S_\tau + \text{``terms''}
\end{aligned}$$

Combining gives

$$\ln(C_\tau) = \alpha^{\tau-t}\ln(K_{t+1}) - \sum_{j=0}^{\tau-t-1}\alpha^{\tau-t-1-j}\gamma(1-d_{1+j})E_t + \text{``terms''} \qquad (34)$$

Using this and substituting for Y_t in the planner's objective function, the planner's problem in t can be written

$$\max_{E_t}\ln\left((1-s_t)K_t^\alpha A_t(E_t)\exp\left(-\gamma\sum_{j=0}^{t+T}(1-d_j)E_{t-j}\right)\right) +$$

$$\sum_{\tau=t+1}^{T}\lambda_{\tau-t}\left[\text{``terms''} + \alpha^{\tau-t}\ln(s_t K_t^\alpha A_t(E_t)\exp\left(-\gamma\sum_{j=0}^{t+T}(1-d_j)E_{t-j}\right)) - \sum_{j=0}^{\tau-t-1}\alpha^{\tau-t-1-j}\gamma(1-d_{1+j})E_t\right]$$

Letting F_t denote the final-good production function in t, the first-order condition with respect to E_t becomes

$$\frac{\partial F_t/\partial E_t}{F_t}\left(1 + \sum_{\tau=t+1}^{T}\lambda_{\tau-t}\alpha^{\tau-t}\right) = \sum_{\tau=t+1}^{T}\lambda_{\tau-t}\sum_{j=0}^{\tau-t-1}\alpha^{\tau-t-1-j}\gamma(1-d_{1+j})$$

$$+\left(1 + \sum_{\tau=t+1}^{T}\lambda_{\tau-t}\alpha^{\tau-t}\right)\gamma(1-d_0).$$

Combining the RHS terms gives

$$\sum_{\tau=t}^{T}\lambda_{\tau-t}\sum_{j=0}^{\tau-t}\alpha^{\tau-t-j}\gamma(1-d_j).$$

Taking the limit as $T\to\infty$, the first-order condition can be written

$$\frac{\partial F_t/\partial E_t}{F_t}(1+\theta) = \sum_{\tau=t}^{T}\lambda_{\tau-t}\sum_{j=0}^{\tau-t}\alpha^{\tau-t-j}\gamma(1-d_j). \qquad (35)$$

Thus,

$$\partial F_t/\partial E_t = \frac{\sum_{\tau=t}^{T} \lambda_{\tau-t} \sum_{j=0}^{\tau-t} \alpha^{\tau-t-j}\gamma(1-d_j)}{1+\theta} Y_t \equiv \Lambda_t^s. \qquad (36)$$

Standard arguments show that competitive firms will respond to a tax on emissions by setting $\partial F_t/\partial E_t$ equal to the tax so the equilibrium tax in t is Λ_t^s. Uniqueness of the equilibrium tax follows by construction.

To verify the inductive hypothesis, rewrite (37) as

$$A_t'(E_t)/A_t(E_t) = \frac{\sum_{\tau=t}^{T} \lambda_{\tau-t} \sum_{j=0}^{\tau-t} \alpha^{\tau-t-j}\gamma(1-d_j)}{1+\theta}. \qquad (37)$$

This gives a deterministic function that defines E_t as a stock-invariant quantity that depends only on model parameters.

Next, we rewrite the expression for Λ_t^s to get the formula in the proposition. Specifically, for the infinite horizon case , we employ the following double sum identity:

$$\sum_{p=0}^{\infty} \sum_{q=0}^{p} a_{q,p-q} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{n,m}.$$

The identify can be proved by listing the terms in a two by two grid, where the index of the first sum comprise the rows and the index of the second sum comprise the columns. The left-hand side is obtained by summing the rows of the grid, while the right-hand side is obtained by summing the same set of terms diagonally.

When $t = 0$ and $T = \infty$,

$$\sum_{\tau=t+1}^{T} \lambda_{\tau-1} \sum_{j=1}^{\tau-t} \alpha^{\tau-t-j}\gamma(1-d_j) = \sum_{p=0}^{\infty} \lambda_p \sum_{j=1}^{p+1} \alpha^{p+1-j}\gamma(1-d_j) \quad = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \lambda_{p-q+q} \alpha^{p-q}\gamma(1-d_{1+q}).$$

Letting $n = q$ and $m = p - q$,

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m+n} \alpha^m \gamma(1-d_{1+n})$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_n \lambda_{n+1,n+m} \alpha^m \gamma(1-d_{1+n})$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda_n \lambda_{n+1,n+m} \alpha^m \gamma(1-d_{1+n})$$

$$= \sum_{n=0}^{\infty} \lambda_n \gamma(1-d_{1+n}) \sum_{m=0}^{\infty} \alpha^m \lambda_{n+1,n+m}$$

$$= \sum_{n=0}^{\infty} \lambda_n \gamma(1-d_{1+n}) \Gamma(n),$$

where

$$\Gamma(n) = \sum_{m=0}^{\infty} \alpha^m \lambda_{n+1,n+m}.$$

3

Combining this with (36) gives the expression for the equilibrium tax stated in the expression.

The corollaries follow from the proof above. Corollary 1 follows from the Lemma 1, together with the finding that the path of stock-invariant aggregate savings rates drops out from the determination of the equilibrium tax. Corollary 2 follows from the inductive hypothesis, which was verified above.

A.2 Proof of Proposition 3

A necessary and sufficient condition for commitment value to be convex in commitment horizon is for the increment in commitment value when moving from N-period commitment to $N + 1$-period commitment to increase in N. Suppose commitment is imposed in period 0. With N-period commitment, the taxes in $t = 0$ through $N - 1$ are τ_t^N, where the "N" superscript indicates that the tax is chosen by the initial generation with N-period commitment. After period $N - 1$, taxes are $\tau_t^{1,N}$, where the "1,N" superscript indicates that taxes are chosen by the contemporaneous generation without commitment in the equilibrium in which the initial generation commits policy N periods.

With $N + 1$-period commitment, the taxes are $\left(\{\tau_t^{N+1}\}_{t=0}^{N-1}, \tau_N^{N+1}, \{\tau_t^{1,N+1}\}_{t=N+1}^{\infty}\right)$. It follows from Corollary 1 that $\tau_t^N = \tau_t^{N+1}$ for $t = 0, \ldots, N - 1$ and $\tau_t^{1,N} = \tau_t^{1,N+1}$ for $t > N$. In addition, due to Corollary 2, savings rates are the same in all periods regardless of commitment horizon. It follows that consumption levels are the same for the first N periods with N and $N + 1$ period commitment. Letting c_t^N denote equilibrium consumption in t with N-period commitment, we have

$$c_t^{N+1} = c_t^N \text{ for } t = 0, \ldots, N - 1 \tag{38}$$

Define welfare for the generation in period 0 with j period commitment as $\tilde{W}_0^j$. Then the value of commitment is[16]

$$\tilde{W}_0^N - \tilde{W}_0^1.$$

Then the increment in commitment value when going from N-period commitment to $N + 1$-period commitment is

$$\text{INC}_{N+1} \equiv (\tilde{W}_0^{N+1} - \tilde{W}_0^1) - (\tilde{W}_0^N - \tilde{W}_0^1) = \tilde{W}_0^{N+1} - \tilde{W}_0^N.$$

---

[16]For simplicity, we denominate commitment value in utility units here, though the argument is unchanged when welfare changes are denominated in consumption units.

Using (38),

$$\text{INC}_{N+1} = \tilde{W}_0^{N+1} - \tilde{W}_0^N$$

$$= \sum_{t=0}^{\infty} \lambda_t \ln(c_t^{N+1}) - \sum_{t=0}^{\infty} \lambda_t \ln(c_t^N)$$

$$= \left(\sum_{t=0}^{N-1} \lambda_t \ln(c_t^{N+1}) + \sum_{t=N}^{\infty} \lambda_t \ln(c_t^{N+1}) \right) - \left(\sum_{t=0}^{N-1} \lambda_t \ln(c_t^N) + \sum_{t=N}^{\infty} \lambda_t \ln(c_t^N) \right)$$

$$= \left(\sum_{t=0}^{N-1} \lambda_t \ln(c_t^N) + \sum_{t=N}^{\infty} \lambda_t \ln(c_t^{N+1}) \right) - \left(\sum_{t=0}^{N-1} \lambda_t \ln(c_t^N) + \sum_{t=N}^{\infty} \lambda_t \ln(c_t^N) \right)$$

$$= \sum_{t=N}^{\infty} \lambda_t \ln(c_t^{N+1}) - \sum_{t=N}^{\infty} \lambda_t \ln(c_t^N)$$

$$= \lambda_N \left[\sum_{s=0}^{\infty} \frac{\lambda_{N+s}}{\lambda_N} \ln(c_{N+s}^{N+1}) \right] - \lambda_N \left[\sum_{s=0}^{\infty} \frac{\lambda_{N+s}}{\lambda_N} \ln(c_{N+s}^N) \right]$$

$$= \lambda_N \left[V_N^{N+1} - V_N^N \right]$$

where V_t^N is defined in (16). Thus,

It follows that $\text{INC}_{N+1} \geq \text{INC}_N$ if and only if

$$\frac{\Delta V_{N+1}^{N+1}}{\Delta V_N^N} \geq \frac{\lambda_{N-1}}{\lambda_N} \equiv 1 + r_N, \tag{39}$$

where r_N is the discount rate applied by the initial generation between period $N-1$ and period N. This is equivalent to the stated result since we assumed $t = 0$.

A.3   Proof of Proposition 4

Proposition 4 generalizes the standard dynamic programming equation. Harris and Laibson (2001) use this approach for quasi-hyperbolic discounting. Fujii and Karp (2007) use a variation as a basis for numerical work for a one-state variable stationary problem with general NCTP.

Proof. (Proposition 4) We use a proof by induction. At $i = 0$, (20) and the fact that there are no subsequent payoffs imply that the agent solves the optimization problem in the first line of (18) (under the constraint $k' \geq 0$). Therefore, the decision rule is given by the second line of (18) for $i = 0$.

The inductive hypothesis is that the first line of (18) holds at iterations $i - 1$. We need to establish that this hypothesis and the updating equation 19 imply that the first line of (18) gives the correct optimization problem at iteration i. To establish this claim, it is necessary and sufficient to confirm that $\beta_1 W_{T-i}^{(i-1)}$ is the correct continuation value at iteration i. The rest of the proof establishes this claim.

The agent's evaluation, at iteration i, of an arbitrary sequence of flow payoffs $\{U^{(m)}\}_{m=0}^{i}$ is

$$U^{(i)} + \beta_1 U^{(i-1)} + \ldots + (\beta_1 \beta_2 \ldots \beta_i) U^{(0)} = \sum_{t=0}^{i} \left(\Pi_{n=0}^t \beta_n \right) U^{(i-t)}. \tag{40}$$

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where the equality uses $\beta_0 = 1$. Denote the *equilibrium* flow payoff at iteration i as $\tilde{U}^{(i)}$. The equilibrium payoff at iteration i depends on the agent's level of capital, and current and future variables that the agent takes as exogenous (captured by the iteration index). Thus, the equilibrium values $\left\{\tilde{U}^{(m)}\right\}_{m=0}^{i}$ depend on k_i and on variables that the agent takes as exogenous.

Replacing the arbitrary functions $U^{(i)}$ with equilibrium levels of utility, $\tilde{U}^{(i)}$ and using (19) and (20) and repeated substitution, we write

$$W_{T-j}^{(i)} = \tilde{U}^{(i)} + \beta_{j-i+1} W_{T-j}^{(i-1)}$$

$$= \tilde{U}^{(i)} + \beta_{j-i+1} \left[\tilde{U}^{(i-1)} + \beta_{j-i+2} W_{T-j}^{(i-2)} \right]$$

$$= \tilde{U}^{(i)} + \beta_{j-i+1} \left[\tilde{U}^{(i-1)} + \beta_{j-i+2} \left(\tilde{U}^{(i-2)} + \beta_{j-i+3} W_{T-j}^{(i-3)} \right) \right]$$

$$\vdots$$

$$= \tilde{U}^{(i)} + \sum_{t=1}^{i} \left(\Pi_{n=0}^{t-1} \beta_{j-i+1+n} \right) \tilde{U}^{(i-t)}.$$

Thus,

$$W_{T-j}^{(i-1)} = \tilde{U}^{(i-1)} + \sum_{t=1}^{i-1} \left(\Pi_{n=0}^{t-1} \beta_{j-i+2+n} \right) \tilde{U}^{(i-1-t)}$$

Setting $j = i$ (the largest value of j when the superscript is $i-1$) gives

$$W_{T-i}^{(i-1)}(k_{i-1}) = \tilde{U}^{(i-1)} + \sum_{t=1}^{i-1} \left(\Pi_{n=0}^{t-1} \beta_{2+n} \right) \tilde{U}^{(i-1-t)} \Rightarrow$$

$$\beta_1 W_{T-i}^{(i-1)}(k_{i-1}) = \beta_1 \left[\tilde{U}^{(i-1)} + \sum_{t=1}^{i-1} \left(\Pi_{n=0}^{t-1} \beta_{2+n} \right) \tilde{U}^{(i-1-t)} \right] \tag{41}$$

$$= \sum_{t=1}^{i-1} \left(\Pi_{n=1}^{t} \beta_n \right) \tilde{U}^{(i-t)}.$$

Here we include the argument k_{i-1}, the agent's stock of capital at iteration $i-1$, in the function $W_{T-i}^{(i-1)}$ in order to emphasize that the equilibrium sequence of current and future flow payoffs depend on this stock.

At iteration i, using (40), the agent chooses k' to solve

$$\max_{k'} \left(U^{(i)}(k, k') + \beta_1 \sum_{t=1}^{i} \left(\Pi_{n=0}^{t} \beta_n \right) \tilde{U}^{(i-t)} \right) =$$

$$\max_{k'} \left(U^{(i)}(k, k') + \beta_1 W_{T-i}^{(i-1)}(k_{i-1}) \right),$$

where the equality uses (41). This equation establishes that the first line of (18) gives the correct maximand at iteration i. The agent in the current period can deviate from equilibrium, so the current flow payoff is $U^{(i)}(k, k')$. However, the sequence of future flow payoffs, $\tilde{U}^{(i-t)}$, are evaluated in equilibrium; these payoffs are functions of k'.) $\blacksquare$

A.4   Proof of Proposition 5

We use Proposition 4 to develop formulae for the endogenous savings rule in the case of isoelastic utility. The equilibrium is a linear savings rule whose coefficients depend on

variables that the agent takes as exogenous. Those variable might be either exogenous or endogenous to the model (e.g. technological change versus climate change). Single period utility is

$$U\left(c\right) = \frac{c^{1-\eta}}{1-\eta} \text{ for } \eta \neq 1; \ U\left(c\right) = \ln\left(c\right) \text{ o/w} \Rightarrow U'\left(c\right) = c^{-\eta}.$$

A linear net savings rule, $s_i\left(k + \xi_i\right)$ implies next period capital for the agent is

$$k' = s_i\left(k + \xi_i\right) + \left(1 - \delta\right)k \tag{42}$$

Consumption is

$$c_i = \left(R_i - s_i\right)k + w_i - s_i\xi_i. \tag{43}$$

Utility under this linear savings rule equals

$$\frac{\left(\left(R_i - s_i\right)k + w_i - s_i\xi_i\right)^{1-\eta}}{1-\eta} = \frac{\left(R_i - s_i\right)^{1-\eta}\left(k + \frac{w_i - s_i\xi_i}{R_i - s_i}\right)^{1-\eta}}{1-\eta} \tag{44}$$

The subscript i recognizes that the net savings rate changes with the iteration number. That change potentially arises for several reasons: changes in variables that are exogenous to the model (e.g. technology), changes in state variables that are exogenous to the agent but endogenous to the model (e.g., climate and climate policy) and also because (for finite terminal time, T) the distance from the current to the final period changes with the iteration index.

For the climate problem it is important to include the climate state and climate policies as arguments of the endogenous functions, but in the interest of clarity we begin with a simpler problem in which the only exogenous states (from the agent's perspective) are aggregate capital, K, and the index i. In this case, $R_i = R\left(K, i\right)$ and $w_i = w\left(K, i\right)$ are known functions, depending on aggregate capital stock and i (to capture exogenous-to-the-model changes).

Proof. (Sketch of Proposition 5) The proof is algebra-intensive, so we relegate the details to Referees' Appendix B.3. Here we describe the straightforward logic. We use an inductive proof. In the last period ($i = 0$) savings equal zero. With this fact, it is easy to confirm (27) and, for $i = 0$, equation 22. We then use the inductive hypothesis (equation 22 holds for $i - 1$) to write the agent's saving problem at iteration i. The first order condition to this problem implies the linear savings rule, with s_i given by (24) and ξ_i given by (25). The agent's problem is concave iff $b_{i-1;i} \geq 0$; we confirm this inequality numerically. We substitute this savings rule into the agent's dynamic programming equation to establish the recursions in the two lines of (26). ∎

A.5 Logarithmic utility

Using the linear savings and consumption rules, equations 42 and 43, utility under logarithmic preferences equals

$$\ln\left(\left(R_i - s_i\right)k + w_i - s_i\xi_i\right) = \ln\left(\left(R_i - s_i\right)\left[k + \frac{w_i - s_i\xi_i}{\left(R_i - s_i\right)}\right]\right)$$

$$= \ln\left(R_i - s_i\right) + \ln\left(k + \frac{w_i - s_i\xi_i}{\left(R_i - s_i\right)}\right). \tag{45}$$

Proposition 6 *The auxiliary value functions have the form*

$$W_{T-j}^{(i)}(k, K) = a_{i;j} + b_{i;j} \ln (k + f_i). \tag{46}$$

with coefficients

$$a_{i;j} = \ln (R_i - s_i) + \beta_{j-i+1} [a_{i-1;j} + b_{i-1;j} \ln (s_i + 1 - \delta)]$$

$$b_{i;j} = (1 + \beta_{j-i+1} b_{i-1;j}) \ \text{and} \ f_i = \frac{w_i + f_{i-1}}{R_i + (1 - \delta)} \tag{47}$$

and boundary conditions

$$a_{0;j} = \ln (R_0 + 1 - \delta)$$
$$b_{0;j} = 1 \ \text{and} \ f_0 = \frac{w_0}{R_0 + 1 - \delta}. \tag{48}$$

At $i = 0$ gross savings equal zero; for $i \geq 1$ the equilibrium savings rule is

$$k' = s_i (k + \xi_i) + (1 - \delta) k \tag{49}$$

with (for $i \geq 1$)

$$s_i = \frac{(\beta_1 b_{i-1;i} R_i) - (1 - \delta)}{(1 + \beta_1 b_{i-1;i})} \tag{50}$$

$$\xi_i = \frac{w_i \beta_1 b_{i-1;i} - f_{i-1}}{(\beta_1 b_{i-1;i} R_i) - (1 - \delta)} \tag{51}$$

Appendix B.3 provides the proof, which parallels the proof for the case $\eta \neq 1$.

B   Referees' appendix: Not intended for publication

This appendix collects technical information, not intended for publication. Appendix B.1 contains the proof of Lemma 1. This lemma provides only a slight generalization to a result in Hiraguchi (2014). However, we need this result in order to establish uniqueness of the limit equilibrium in the log-linear model. Appendix B.2 discusses a generalization of the utility function to include population growth and a climate-related amenity value. Appendix B.3 provides the detailed proofs of Propositions 5 and 6. Appendix B.4 summarizes Ekeland and Lazrak's (2010) OLG model, and explains how we modify it to include privately owned capital. Appendix B.5 explains how we impose the equilibrium condition that individual and aggregate savings are equal.

B.1   Proof of Lemma 1

The horizon at time $t = 0$ (the initial period) is H, so the remaining horizon at time t is $T = H - t$. The proof uses the formula for the agent's continuation value function

$$V^e(k, K, t, T) = (\theta(T) - \rho(T)) \ln K + \rho(T) \ln(k + \psi(T) K) + P_t(H - t), \qquad (52)$$

which uses the definitions

$$\psi(T) = (\rho(T) + 1)\left(\frac{1}{\alpha} - 1\right) \text{ and } \theta(T) \equiv \sum_{t=1}^{T-1} \alpha^\tau \lambda_\tau. \qquad (53)$$

We also show that the agent's decision rule is

$$k' = g(k, K, T) = \frac{\rho(T)}{1 + \rho(T)} r(K) k. \qquad (54)$$

We verify (52)–(54) in the course of the proof. Comparing (28) and (54) shows that the individual and society at large save the same fraction of income from capital, $r(K) K$ for society and $r(K) k$ for the representative agent. We define $P_t(H - t)$ in the course of the proof, merely noting here that this function depends on time-to-go and, via the argument t, on all state and policy variables apart from own- and aggregate capital. Thus, $P_t(H - t)$ depends on the trajectory of technology, the climate state, and climate policy.

We define $\tilde{A}_t \equiv (1 - D(S_t)) A_t(E_t)$, so that with Assumption 1 we can write output as $Y_t = \tilde{A}_t K_t^\alpha$; $\tilde{A}_t$ incorporates all variables, apart from aggregate capital, that the private agent takes as exogenous.

We use an inductive argument. We start the inductive chain by noting that at $t = H$ (where time-to-go is $T = H - t = 0$) individual and aggregate equilibrium savings are zero. We use the convention $\sum_{t=1}^{0} y_t = 0$ for any sequence $\{y_\tau\}$. Using this convention, $\rho(0) = 0$, so the individual and aggregate savings rules in (28) and (54) are indeed equilibrium savings rules. Any other identically zero function is an equilibrium decision rule at $t = H$, so there is a trivial multiplicity at $t = H$, but not at earlier times. Using $P_H(0) = 0$, (52) gives the equilibrium continuation payoff (zero) at $T = 0$.

The inductive hypothesis states that: (i) the future aggregate saving satisfies (28); future own-savings satisfy (54); and the current continuation value function is given by (52). We need to show that this hypothesis implies that (28) and (54) comprise the unique equilibrium aggregate and own savings in the current period, and that the continuation

value, from the perspective of the previous period, $t-1$, satisfies (52). The assumption in the lemma (that current climate policy does not depend on aggregate capital) means that $\left\{\tilde{A}_\tau\right\}_{\tau=t}^{H}$ does not depend on values of aggregate capital. Elements of $\left\{\tilde{A}_\tau\right\}_{\tau=t+1}^{H}$ typically do depend on future climate policy.

Given the agent's continuation value at t, (52), the agent's problem at t is

$$
V_0^e\left(k, K, t, T\right) = \max_{k'}[\ln\left(w_t + r_t k - k'\right) +
$$
$$
\left(\theta\left(T\right) - \rho\left(T\right)\right)\ln K' + \rho\left(T\right)\ln\left(k' + \psi\left(T\right)K'\right) + P_{t+1}\left(T\right)]. \tag{55}
$$

The first order condition is

$$
\frac{-1}{\left(w_t + r_t k - k'\right)} + \frac{\rho\left(T\right)}{\left(k' + \psi\left(T\right)K'\right)} = 0 \tag{56}
$$

and the second order condition is satisfied:

$$
-\left[\frac{1}{\left(w_t + r_t k - k'\right)}\right]^2 - \left[\frac{\rho\left(T\right)}{\left(k' + \psi\left(T\right)K'\right)}\right]^2 < 0.
$$

The first order condition implies

$$
k' + \psi\left(T\right)K' = \rho\left(T\right)\left(w_t + r_t k - k'\right). \tag{57}
$$

Using the expressions for the rental rate and transitory income in Definition 2.1, and Assumption 1,

$$
w_t = \left(1 - \alpha\right)\tilde{A}_t K^\alpha \frac{r_t}{\alpha \tilde{A}_t K^{\alpha-1}} = \frac{\left(1 - \alpha\right)}{\alpha}K r_t. \tag{58}
$$

Using (58) and (53) we can write (57) as

$$
k'\left(1 + \rho\left(T\right)\right) = \rho\left(T\right)r_t k + \rho\left(T\right)\frac{1-\alpha}{\alpha}r_t K - \left(\rho\left(T\right) + 1\right)\left(\frac{1}{\alpha} - 1\right)K' \Rightarrow
$$

$$
k' = \frac{\rho\left(T\right)}{\left(1 + \rho\left(T\right)\right)}r_t k + \frac{\rho\left(T\right)}{\left(1 + \rho\left(T\right)\right)}\frac{1-\alpha}{\alpha}r_t K - \left(\frac{1}{\alpha} - 1\right)K' \tag{59}
$$

We need to confirm that under the hypothesis about future behavior, there is a unique aggregate equilibrium savings rule at t. We do this in two steps. First, we use the fact that in equilibrium, where $k = K$, the individual and aggregate savings levels must be equal. Using this fact, we show that there is a unique candidate aggregate savings rule. Next we show that when the agent takes this candidate as the aggregate savings rule, his optimal decision is to save the same fraction of own-wealth as society saves of social wealth. Thus, the unique candidate equilibrium is in fact an equilibrium savings rule at t.

At t, T, society saves some fraction of income, which we denote as $G\left(t, T\right)$:[17]

$$
K' = G\left(t, T\right)Y_t = \frac{G\left(t, T\right)}{\alpha}r_t K_t. \tag{60}
$$

---

[17]We could condition the aggregate savings fraction on other state variables, but we see below that S depends on T, but not on t. Adding additional arguments to S would not change the result that the equilibrium S depends only on T.

Using this savings rule in (59) gives

$$k' = \frac{\rho(T)}{(1 + \rho(T))} r_t k_t + \left[\frac{\rho(T)}{(1 + \rho(T))} \frac{1 - \alpha}{\alpha} - \left(\frac{1}{\alpha} - 1 \right) \frac{G(t, T)}{\alpha} \right] r_t K_t. \qquad (61)$$

This relation must hold for all k, K, including at equilibrium, where $k = K$. Moreover, if $k = K$, equilibrium requires $k' = K'$. These facts, and using (60), gives

$$k' = \left[\frac{\rho(T)}{(1 + \rho(T))} + \frac{\rho(T)}{(1 + \rho(T))} \frac{1 - \alpha}{\alpha} - \left(\frac{1}{\alpha} - 1 \right) \frac{G(t, T)}{\alpha} \right] r_t K_t = \frac{G(t, T)}{\alpha} r_t K_t = K'.$$

This relation must hold for all $r_t K_t$, which implies

$$\left[\frac{\rho(T)}{(1 + \rho(T))} \frac{1}{\alpha} - \left(\frac{1}{\alpha} - 1 \right) \frac{G(t, T)}{\alpha} \right] = \frac{G(t, T)}{\alpha} \Rightarrow$$

$$\left[\frac{\rho(T)}{(1 + \rho(T))} \frac{1}{\alpha} - \frac{1}{\alpha} \frac{G(t, T)}{\alpha} \right] = 0 \Rightarrow G(t, T) = \frac{\alpha \rho(T)}{(1 + \rho(T))}.$$

Thus, we know that (28) is the unique candidate aggregate savings rule. To establish that this candidate savings rule is in fact an equilibrium, we need to confirm that it is consistent with the individual savings rule (61) for $k \gtrless K$. Using the candidate in (61) gives

$$k' = \frac{\rho(T)}{(1 + \rho(T))} r_t k + \left[\frac{\rho(T)}{(1 + \rho(T))} \frac{1 - \alpha}{\alpha} - \left(\frac{1}{\alpha} - 1 \right) \frac{\rho(T)}{(1 + \rho(T))} \right] r_t K. \qquad (62)$$

Using

$$\frac{\rho(T)}{(1 + \rho(T))} \frac{1 - \alpha}{\alpha} - \left(\frac{1}{\alpha} - 1 \right) \frac{\rho(T)}{(1 + \rho(T))} = 0,$$

in (62) produces the individual savings rule in (54), thus confirming that the unique candidate savings rule is indeed an equilibrium.

Now we show that the individual and aggregate savings rules produce the continuation value function shown in (52), for any period. This result implies that this function is indeed the continuation value function in the previous period, as the inductive argument requires.

First, we solve the difference equation for aggregate capital to write future capital as a function of current capital.

$$K_{t+1} = \frac{\rho(T)}{1 + \rho(T)} \alpha \tilde{A}_t K_t^\alpha \Rightarrow$$

$$\ln K_{t+1} = \underbrace{\ln \frac{\rho(H - t)}{1 + \rho(H - t)} + \ln \left(\alpha \tilde{A}_t \right)}_{\equiv y_t} + \alpha K_t.$$

Here we use $T = H - t$. Note that y_t depends on the parameter H, but we suppress that dependence. The definition

$$y_t \equiv \ln \frac{\rho(H - t)}{1 + \rho(H - t)} + \ln \left(\alpha \tilde{A}_t \right)$$

recognizes that t affects $\tilde{A}_t$ and also affects the remaining horizon. Use the fact that the solution to the difference equation

$$x_{t+1} = \alpha x_t + y_t$$

3

is

$$x_{t+\tau} = \alpha^\tau x_t + \sum_{j=0}^{\tau-1} \alpha^{\tau-1-j} y_{t+j}$$

or

$$x_{t+\tau} = \alpha^{\tau-1} x_{t+1} + \sum_{j=0}^{\tau-2} \alpha^{\tau-2-j} y_{t+1+j}$$

to write, for integers $\tau \geq 1$,

$$\ln K_{t+\tau} = \alpha^{\tau-1} \ln K_{t+1} + \sum_{j=0}^{\tau-2} \alpha^{\tau-2-j} y_{t+1+j}. \tag{63}$$

In period t the agent chooses c_t which produces next period capital k_{t+1}. The agent at t might begin off the equilibrium trajectory (where $k_t \neq K_t$) or she might deviation (causing $k_{t+1} \neq K_{t+1}$), but he believes that his successors will follow the equilibrium decision rule, so he believes that $\frac{k_{t+\tau}}{K_{t+\tau}} = \frac{k_{t+1}}{K_{t+1}}$ for all integers $\tau \geq 2$. Suppressing the parameter H, we write consumption at $t + \tau$ as $c_{t+\tau}$, recognizing that the time index has two types of effects, via the time-dependent function $\tilde{A}_t$ and via the time-dependent time-to-go, $H - t$. The agent believes that for integers $\tau \geq 1$

$$
\begin{aligned}
c_{t+\tau} &= w_{t+\tau} + r_{t+\tau} k_{t+\tau} - \frac{\rho\left(H - (t+\tau)\right)}{\left(1 + \rho\left(H - (t+\tau)\right)\right)} r_{t+\tau} k_{t+\tau} \\
&= \frac{(1-\alpha)}{\alpha} r_{t+\tau} K_{t+\tau} + \frac{1}{\left(1 + \rho\left(H - (t+\tau)\right)\right)} r_{t+\tau} k_{t+\tau} \\
&= \left[\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(H - (t+\tau)\right)\right)} \frac{k_{t+\tau}}{K_{t+\tau}}\right] r_{t+\tau} K_{t+\tau}.
\end{aligned}
$$

We now use the fact that equilibrium beliefs, at time t, imply $\frac{k_{t+\tau}}{K_{t+\tau}} = \frac{k_{t+1}}{K_{t+1}}$. Regardless of whether the agent enters period t with $k_t = K_t$, he could choose a level of savings that results in $k_{t+1} \neq K_{t+1}$. However, given that future agents use their equilibrium savings rule, $\frac{k_{t+\tau}}{K_{t+\tau}} = \frac{k_{t+1}}{K_{t+1}}$. This fact allows us to write

$$
\begin{aligned}
c_{t+\tau} &= \left(\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(T - 1\right)\right)} \frac{k_{t+1}}{K_{t+1}}\right) K_{t+\tau} r_{t+\tau} \Rightarrow \\
\ln c_{t+\tau} &= \ln\left(\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(T - 1\right)\right)} \frac{k_{t+1}}{K_{t+1}}\right) + \ln\left(K_{t+\tau} r_{t+\tau}\right) \\
&= \ln\left(\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(T - 1\right)\right)} \frac{k_{t+1}}{K_{t+1}}\right) + \ln\left(K_{t+\tau} \alpha \tilde{A}_{t+\tau} K_{t+\tau}^{\alpha-1}\right) \\
&= \ln\left(\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(T - 1\right)\right)} \frac{k_{t+1}}{K_{t+1}}\right) + \ln\left(\alpha \tilde{A}_{t+\tau} K_{t+\tau}^{\alpha}\right) \\
&= \ln\left(\frac{(1-\alpha)}{\alpha} + \frac{1}{\left(1 + \rho\left(T - 1\right)\right)} \frac{k_{t+1}}{K_{t+1}}\right) + \alpha \ln K_{t+\tau} + \ln\left(\alpha \tilde{A}_{t+\tau}\right)
\end{aligned}
$$

4

Using (63),

$$
\begin{aligned}
\ln c_{t+\tau} \;=\;& \ln\left(\frac{(1-\alpha)}{\alpha} + \frac{1}{(1+\rho\,(T-1))}\frac{k_{t+1}}{K_{t+1}}\right) \\
&+ \alpha\left(\alpha^{\tau-1}\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-2-j}y_{t+1+j}\right) + \ln\left(\alpha\tilde{A}_{t+\tau}\right) \\
\;=\;& \ln\left[\left(\frac{(1-\alpha)}{\alpha}K_{t+1} + \frac{1}{(1+\rho\,(T-1))}k_{t+1}\right)\frac{1}{K_{t+1}}\right] \\
&+ \left(\alpha^{\tau}\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-1-j}y_{t+1+j}\right) + \ln\left(\alpha\tilde{A}_{t+\tau}\right) \\
\;=\;& \ln\left[\left(\frac{(1-\alpha)}{\alpha}K_{t+1} + \frac{1}{(1+\rho\,(T-1))}k_{t+1}\right)\right] \\
&- \ln K_{t+1} + \left(\alpha^{\tau}\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-1-j}y_{t+1+j}\right) + \ln\left(\alpha\tilde{A}_{t+\tau}\right) \\
\;=\;& \ln\left(\frac{(1-\alpha)}{\alpha}K_{t+1} + \frac{1}{(1+\rho\,(T-1))}k_{t+1}\right) \\
&+ (\alpha^{\tau}-1)\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-1-j}y_{t+1+j} + \ln\left(\alpha\tilde{A}_{t+\tau}\right) \\
\;=\;& \ln\left[\left(\frac{(1-\alpha)}{\alpha}(1+\rho\,(T-1))K_{t+1} + k_{t+1}\right)\frac{1}{(1+\rho\,(T-1))}\right] \\
&+ (\alpha^{\tau}-1)\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-1-j}y_{t+1+j} + \ln\left(\alpha\tilde{A}_{t+\tau}\right) \\
\;=\;& \ln\left(\frac{(1-\alpha)}{\alpha}(1+\rho\,(H-(t+1)))K_{t+1} + k_{t+1}\right) + \\
&\underbrace{(\alpha^{\tau}-1)\ln K_{t+1} + \sum_{j=0}^{\tau-2}\alpha^{\tau-1-j}y_{t+1+j} + \ln\left(\alpha A_{t+\tau}\right) - \ln\left(1+\rho\,(H-(t+1))\right)}_{\equiv z_{t+\tau}(t)}
\end{aligned}
$$

Using the expression for consumption in the definition of the continuation welfare

$$V^e\left(k_{t+1}, K_{t+1}, t+1, T-1\right)$$

$$= \sum_{\tau=1}^{T-1} \lambda_\tau \ln c\left(k_{t+\tau}, K_{t+\tau}, t+\tau,\right)$$

$$= \sum_{\tau=1}^{T-1} \lambda_\tau \ln\left(\frac{(1-\alpha)}{\alpha}\left(1+\rho\left(H-(t+1)\right)\right)K_{t+1}+k_{t+1}\right) +$$

$$\sum_{\tau=1}^{T-1} \lambda_\tau\left(\alpha^\tau-1\right)\ln K_{t+1} + \sum_{\tau=1}^{T-1}\lambda_\tau z_{t+\tau}(t)$$

$$= \ln\left(\frac{(1-\alpha)}{\alpha}\left(1+\rho\left(H-(t+1)\right)\right)K_{t+1}+k_{t+1}\right)\sum_{\tau=1}^{T-1}\lambda_\tau +$$

$$\ln K_{t+1}\sum_{\tau=1}^{T-1}\lambda_\tau\left(\alpha^\tau-1\right) + \sum_{\tau=1}^{T-1}\lambda_\tau z_{t+\tau}(t).$$

Using the definitions in (29) and (53) and introducing the definition

$$P\left(T,t\right) = \sum_{\tau=1}^{T-1}\lambda_\tau\left(\alpha^\tau-1\right)\ln K_{t+1} + \sum_{\tau=1}^{T-1}\lambda_\tau z_{t+\tau}(t),$$

we confirm that (52) provides the formula for the continuation value of consumption from $T-1$ $(t=H-(T-1))$ to 0 $(t=H)$.

B.2 Generalizing the utility function

The agent's utility at iteration i depends on consumption, $c_i = R_i k + w_i + (1-\delta)k - k'$, where k is current capital and k' equals savings (next period capital). If utility at iteration i depends only on consumption, we write the utility function as $U\left(R_i k + w_i + (1-\delta)k - k'\right)$. There are at least two reasons that we might want to write the utility function more generally, as $U_i\left(R_i k + w_i + (1-\delta)k - k'\right)$, i.e. to allow a direct dependence of utility on the iteration index:

1. If utility depends explicitly on climate-related variables (or some other state variable that the agent takes as exogenous), then the subscript i in $U_i\left(\cdot\right)$ enables us to take this dependence into account.

2. We can accommodate exogenous population growth by replacing the representative agent with a dynasty. Let $p_i = p\left(T-i\right)$ be the size of the population at iteration i (calendar time $T-i$) relative to the size at iteration T (calendar time 0). Now denoting k as aggregate capital for the dynasty, $R_i k$ as the dynasty's income from capital, and w_i as their aggregate transient income, the individual member of the dynasty has consumption $\frac{R_i k + w_i + (1-\delta)k - k'}{p_i}$. The dynasty's aggregate utility at iteration i is $p_i U\left(\frac{R_i k + w_i + (1-\delta)k - k'}{p_i}\right) \equiv U_i\left(R_i k + w_i + (1-\delta)k - k'; p_i\right)$. Thus, by including an iteration index as an argument of the utility function, we can accommodate exogenous population growth.

In the interest of simplicity, we write U instead of U_i.

B.3 Detailed proofs

Here we provide the algebraic proofs of Propositions 5 and 6.

Proof. (Proposition 5) We use an inductive proof. Using the first line of (18) and (20), the agent's problem for $i = 0$ is

$$
\begin{aligned}
V^{(0)}(k, K) &= \max_{k' \geq 0} \left(U\left((R_i + 1 - \delta) k + w_i - k' \right) \right) \Rightarrow k'^{*} = 0 \Rightarrow \\
s_0(K) &\equiv -(1 - \delta) \text{ and } \xi_0(K) \equiv 0 \Rightarrow \\
W^{(0)}_{T-j}(k, K) &= \frac{\left((R_0 + 1 - \delta) k + w_0 \right)^{1-\eta}}{1 - \eta} = \frac{\left((R_0 + 1 - \delta) \left(k + \frac{w_0}{R_0 + 1 - \delta} \right) \right)^{1-\eta}}{1 - \eta} \\
&= (R_0 + 1 - \delta)^{1-\eta} \frac{\left(k + \frac{w_0}{R_0 + 1 - \delta} \right)^{1-\eta}}{1 - \eta} \Rightarrow \\
f_0(K) &= \frac{w_0}{R_0 + 1 - \delta} \text{ and } b_{0;j} = (R_0 + 1 - \delta)^{1-\eta} \quad (64)
\end{aligned}
$$

for $j = 1, 2...T - 1$. These equalities confirm (27) and, for $i = 0$, (22).

Suppose that (22), holds for $i - 1$ (which, from the previous paragraph, we know is true for $i = 1$). Using the first line of (18), the agent's problem at iteration i is

$$
\max_{k'} \left[\frac{(R_i k + w_i + (1 - \delta) k - k')^{1-\eta}}{1 - \eta} + \beta_1 \left[b_{i-1;i} \frac{(k' + f_{i-1})^{1-\eta}}{1 - \eta} \right] \right]. \quad (65)
$$

(Note that the second index in the subscript of the endogenous functions is $j = i$.) Given $\eta > 0$, a sufficient condition for concavity (in k') of the agent's objective is $b_{i-1;i} \geq 0$; we can confirm this inequality numerically.

$$
\left(R_i k + (1 - \delta) k + w_i - k' \right)^{-\eta} = \beta_1 b_{i-1;i} \left(k' + f_{i-1} \right)^{-\eta} \Rightarrow
$$

The first order condition is

$$
\left(R_i k + (1 - \delta) k + w_i - k' \right)^{-\eta} = \beta_1 b_{i-1;i} \left(k' + f_{i-1} \right)^{-\eta} \Rightarrow
$$

$$
k' - (1 - \delta) k =
$$

$$
\frac{R_i - (1 - \delta) \left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right)}{\left(1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right)} \left[k + \frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}}{R_i - (1 - \delta) \left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right)} \right] =
$$

$$
= s_i R_i (k + \xi_i)
$$

with s_i given by (24) and ξ_i given by (25).

We use this savings rule in the updating equation for $W^{(i)}_{T-j}(k, K)$, (19), together with the hypothesis (for $i - 1$), that (22) holds; we also use the equation for equilibrium

utility—(44):

$$W_{T-j}^{(i)}(k, K) = U\left(R_i k + w_i + (1-\delta)k - k'^*\right) + \beta_{j-i+1} W_{T-j}^{(i-1)}(k'^*, K')$$

$$= \frac{(R_i - s_i)^{1-\eta}\left(k + \frac{w_i - s_i \xi_i}{R_i - s_i}\right)^{1-\eta}}{1-\eta} + \beta_{j-i+1}\left[b_{i-1;j}\frac{(k' + f_{i-1})^{1-\eta}}{1-\eta}\right]$$

$$= \frac{(R_i - s_i)^{1-\eta}\left(k + \frac{w_i - s_i \xi_i}{R_i - s_i}\right)^{1-\eta}}{1-\eta} + \beta_{j-i+1}\left[b_{i-1;j}\frac{(s_i(k+\xi_i) + (1-\delta)k + f_{i-1})^{1-\eta}}{1-\eta}\right]$$

$$= \frac{(R_i - s_i)^{1-\eta}\left(k + \frac{w_i - s_i \xi_i}{R_i - s_i}\right)^{1-\eta}}{1-\eta} + \beta_{j-i+1}\left[b_{i-1;j}\frac{[s_i + 1 - \delta]^{1-\eta}\left(k + \frac{s\xi_i + f_{i-1}}{s_i + 1 - \delta}\right)^{1-\eta}}{1-\eta}\right]$$

(66)

To confirm (22) for iteration i we first need to establish that the arguments of the power functions are the same, i.e.

$$\frac{w_i - s_i \xi_i}{R_i - s_i} = \frac{s\xi_i + f_{i-1}}{s_i + 1 - \delta}$$

(67)

Using the formulae for s_i and ξ_i, equations 24 and 25, in the left side of (67), we have

$$\frac{w_i - s_i \xi_i}{R_i - s_i} = \frac{w_i - \frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}\frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}}{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}}{R_i - \frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}}$$

$$= \frac{w_i - \frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}}{R_i - \frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}}$$

$$= \frac{w_i\left[1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right] - \left(w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}\right)}{R_i\left[1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right] - \left(R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)\right)}$$

$$= \frac{w_i\left[(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right] + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} f_{i-1}}{R_i\left[(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right] + \left((1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)\right)}$$

$$= \frac{w_i + f_{i-1}}{R_i + (1-\delta)}.$$

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Making the same substitutions for the right side of (67) gives

$$\frac{s\xi_i + f_{i-1}}{s_i + 1 - \delta} = \frac{\frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1+(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}\frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}f_{i-1}}{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)} + f_{i-1}}{\frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1+(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} + 1 - \delta}$$

$$= \frac{\frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}f_{i-1}}{1+(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} + f_{i-1}}{\frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{1+(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} + 1 - \delta}$$

$$= \frac{w_i - (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}f_{i-1} + f_{i-1}\left(1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right) + (1-\delta)\left(1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}\right)}$$

$$= \frac{w_i + f_{i-1}}{R_i + (1-\delta)}.$$

We have thus confirmed that

$$\frac{w_i - s_i\xi_i}{R_i - s_i} = \frac{s\xi_i + f_{i-1}}{s_i + 1 - \delta} = \frac{w_i + f_{i-1}}{R_i + (1 - \delta)}. \tag{68}$$

Using (68), we write (66) as

$$W_{T-j}^{(i)}(k,K) = \frac{(R_i - s_i)^{1-\eta}\left(k+\frac{w_i+f_{i-1}}{R_i+(1-\delta)}\right)^{1-\eta}}{1-\eta} + \beta_{j-i+1}\left[b_{i-1;j}\frac{[s_i+1-\delta]^{1-\eta}\left(k+\frac{w_i+f_{i-1}}{R_i+(1-\delta)}\right)^{1-\eta}}{1-\eta}\right]$$

$$= \frac{1}{1-\eta}\left[(R_i - s_i)^{1-\eta} + \beta_{j-i+1}b_{i-1;j}(s_i + 1 - \delta)^{1-\eta}\right]\left(k + \frac{w_i+f_{i-1}}{R_i+(1-\delta)}\right)^{1-\eta},$$

or

$$W_{T-j}^{(i)}(k,K) = b_{i;j}\frac{(k+f_i)^{1-\eta}}{1-\eta} \quad \text{with}$$

$$f_i = \frac{w_i+f_{i-1}}{R_i+(1-\delta)} \tag{69}$$

$$b_{i;j} = (R_i - s_i)^{1-\eta} + \beta_{j-i+1}b_{i-1;j}(s_i + 1 - \delta)^{1-\eta}$$

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Using (24) we can eliminate s_i to write the recursion for $b_{i;j}$:

$$b_{i;j} =$$

$$\left(R_i - \frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \right)^{1-\eta} + \beta_{j-i+1} b_{i-1;j} \left(\frac{R_i - (1-\delta)\left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} + 1 - \delta \right)^{1-\eta}$$

$$= \left(\frac{1}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \left[R_i \left(1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right) - \left(R_i - (1-\delta) \left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right) \right) \right] \right)^{1-\eta}$$

$$+ \beta_{j-i+1} b_{i-1;j} \left(\frac{1}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \left[R_i - (1-\delta) \left((\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right) + (1-\delta) \left[1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}} \right] \right] \right)^{1-\eta}$$

$$= \left(\frac{(\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \left[R_i + (1-\delta) \right] \right)^{1-\eta} + \beta_{j-i+1} b_{i-1;j} \left(\frac{1}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \left[R_i + (1-\delta) \right] \right)^{1-\eta}$$

$$= \left(\frac{R_i + (1-\delta)}{1 + (\beta_1 b_{i-1;i})^{-\frac{1}{\eta}}} \right)^{1-\eta} \left[(\beta_1 b_{i-1;i})^{-\frac{1-\eta}{\eta}} + \beta_{j-i+1} b_{i-1;j} \right]$$

$$(70)$$

for $i = 1, 2 ... T$; for each i, the index $j = i + 1, .. T - 1$. We thus have the recursion, the first line of (26). The second line of (26) merely repeats the second line of (69). ∎

Proof. (Proposition 6) We use an inductive proof. At iteration 0 optimal gross savings are zero, so optimal net savings equals undepreciated capital: $s_0 = -(1-\delta)$ and $\xi_0 = 0$. Therefore,

$$W_{T-j}^{(0)} (k, K) = \ln (R_0 + 1 - \delta) + \ln \left(k + \frac{w_0}{R_0 + 1 - \delta} \right), \qquad (71)$$

thus establishing (46) for $i = 0$, and confirming the boundary conditions in (48). We now have the basis for the inductive chain.

Suppose that (46) holds for iteration $i - 1$. This hypothesis and the first line of (18) imply that the agent's problem at iteration i is

$$\max_{k'} \left[\ln \left(R_i k + w_i + (1-\delta) k - k' \right) + \beta_1 \left[a_{i-1;i} + b_{i-1;i} \ln \left(k' + f_{i-1} \right) \right] \right] \qquad (72)$$

The first order condition is

$$-\frac{1}{R_i k + w_i + (1-\delta) k - k'} + \frac{\beta_1 b_{i-1;i}}{k' + f_{i-1}} = 0 \qquad (73)$$

which simplifies to

$$k' - (1-\delta) k = \frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1 + \beta_1 b_{i-1;i})} \left(k + \frac{(\beta_1 b_{i-1;i}) w_i - f_{i-1}}{(\beta_1 b_{i-1;i} R_i) - (1-\delta)} \right)$$

$$= s_i (k + \xi_i)$$

with s_i and ξ_i given by equations 50 and 51.

We use this savings rule in the updating equation for $W^{(i)}_{T-j}(k,K)$, (19), together with the hypothesis that (46) holds for iteration $i-1$; we also use the equation for equilibrium utility, (45):

$$W^{(i)}_{T-j}(k,K) = U(R_i k + w_i - k'^*) + \beta_{j-i+1} W^{(i-1)}_{T-j}(k'^*, K')$$

$$= \ln(R_i - s_i) + \ln\left(k + \tfrac{w_i - s_i \xi_i}{(R_i - s_i)}\right)$$
$$+ \beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j}\ln\left(s_i(k+\xi_i) + (1-\delta)k + f_{i-1}\right)\right]$$

$$= \ln(R_i - s_i) + \ln\left(k + \tfrac{w_i - s_i \xi_i}{(R_i - s_i)}\right)$$
$$+ \beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j}\ln\left((s_i + 1 - \delta)k + s_i \xi_i + f_{i-1}\right)\right] \tag{74}$$

$$= \ln(R_i - s_i) + \ln\left(k + \tfrac{w_i - s_i \xi_i}{(R_i - s_i)}\right)$$
$$+ \beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j}\ln\left[(s_i + 1 - \delta)\left(k + \tfrac{s_i \xi_i + f_{i-1}}{s_i + 1 - \delta}\right)\right]\right]$$

$$= \ln(R_i - s_i) + \ln\left(k + \tfrac{w_i - s_i \xi_i}{(R_i - s_i)}\right)$$
$$+ \beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j}\ln(s_i + 1 - \delta)\right] + \beta_{j-i+1}b_{i-1;j}\ln\left(k + \tfrac{s_i \xi_i + f_{i-1}}{s_i + 1 - \delta}\right)$$

The right side of (74) has the form as the right side of (46) if and only if (67) holds (so that the logarithms involving k have the same arguments). This requirement is identical to the condition obtained for $\eta \neq 1$. We establish this equality using equations 50 and 51. Using these equations to eliminate s_i and ξ_i, we have

$$\frac{w_i - s_i \xi_i}{(R_i - s_i)} = \frac{w_i - \frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1+\beta_1 b_{i-1;i})}\frac{w_i \beta_1 b_{i-1;i} - f_{i-1}}{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}}{\left(R_i - \frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1+\beta_1 b_{i-1;i})}\right)}$$

$$= \frac{\frac{w_i(1+\beta_1 b_{i-1;i})}{(1+\beta_1 b_{i-1;i})} - \frac{w_i \beta_1 b_{i-1;i} - f_{i-1}}{(1+\beta_1 b_{i-1;i})}}{\left(\frac{R_i(1+\beta_1 b_{i-1;i}) - (\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1+\beta_1 b_{i-1;i})}\right)}$$

$$= \frac{w_i(1+\beta_1 b_{i-1;i}) - (w_i \beta_1 b_{i-1;i} - f_{i-1})}{R_i(1+\beta_1 b_{i-1;i}) - (\beta_1 b_{i-1;i} R_i) - (1-\delta)}$$

$$= \frac{w_i + f_{i-1}}{R_i - (1-\delta)}$$

Similarly, we have

$$\frac{s_i \xi_i + f_{i-1}}{s_i + 1 - \delta} = \frac{\frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1+\beta_1 b_{i-1;i})}\frac{w_i \beta_1 b_{i-1;i} - f_{i-1}}{(\beta_1 b_{i-1;i} R_i) - (1-\delta)} + f_{i-1}}{\frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta)}{(1+\beta_1 b_{i-1;i})} + 1 - \delta}$$

$$= \frac{\frac{w_i \beta_1 b_{i-1;i} - f_{i-1} + (1+\beta_1 b_{i-1;i})f_{i-1}}{(1+\beta_1 b_{i-1;i})}}{\frac{(\beta_1 b_{i-1;i} R_i) - (1-\delta) + (1-\delta)(1+\beta_1 b_{i-1;i})}{(1+\beta_1 b_{i-1;i})}}$$

$$= \frac{w_i \beta_1 b_{i-1;i} - f_{i-1} + (1+\beta_1 b_{i-1;i})f_{i-1}}{(\beta_1 b_{i-1;i} R_i) - (1-\delta) + (1-\delta)(1+\beta_1 b_{i-1;i})}$$

$$= \frac{(\beta_1 b_{i-1;i})(w_i + f_{i-1})}{\beta_1 b_{i-1;i}(R_i + (1-\delta))} = \frac{w_i + f_{i-1}}{R_i + (1-\delta)}$$

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These manipulations establish

$$\frac{w_i - s_i \xi_i}{(R_i - s_i)} = \frac{s_i \xi_i + f_{i-1}}{s_i + 1 - \delta} = \frac{w_i + f_{i-1}}{R_i + (1 - \delta)}$$

Substituting these expressions into the last line of (74) gives

$$W_{T-j}^{(i)}(k, K) = \ln(R_i - s_i) + \ln\left(k + \frac{w_i + f_{i-1}}{R_i + (1-\delta)}\right) +$$

$$\beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j} \ln(s_i + 1 - \delta)\right] + \beta_{j-i+1}b_{i-1;j} \ln\left(k + \frac{w_i + f_{i-1}}{R_i + (1-\delta)}\right)$$

$$= \ln(R_i - s_i) + \beta_{j-i+1}\left[a_{i-1;j} + b_{i-1;j} \ln(s_i + 1 - \delta)\right] + (1 + \beta_{j-i+1}b_{i-1;j}) \ln\left(k + \frac{w_i + f_{i-1}}{R_i + (1-\delta)}\right)$$

This equation establishes that the value function has the form in (46) with coefficients given by (47). ∎

B.4   The discount function

This appendix discusses an OLG model that gives rise to a discount function equal to a convex combination of exponentials. Because the discounting model has three parameters, we can calibrate it to represent a combination of "Nordhaus" and "Stern" discount rates.

In Ekeland and Lazrak's (2010) formulation (without privately owned capital) agents' lifetime is an exponentially distributed random variable with mortality rate θ, giving expected lifetime $\frac{1}{\theta}$; with constant population, the birth rate is also θ. Agents have the pure rate of time preference r for their own future utility, so their risk-adjusted pure rate of time preference is $r + \theta$. When an agent dies, a new agent is born. Agents with paternalistic altruism discount the utility of unborn agents, but possibly at a different rate. This model is isomorphic to one in which agents have pure, rather than paternalistic altruism, i.e. they care about their successors' welfare, not just their utility (Karp 2017). If agents with pure altruism discount their successors' welfare at rate λ, the discount factor for the representative agent is

$$D(t) = \left(\frac{r + \theta - \lambda}{r + 2\theta - \lambda}\right) e^{-(r+\theta)t} + \frac{\theta}{r + 2\theta - \lambda} e^{-(\lambda - \theta)t}. \tag{75}$$

We assume that $\lambda > \theta$ and that the utility flow is bounded for each t; these assumptions imply that welfare is bounded.

The discount rate is

$$-\frac{dD}{dt}\frac{1}{D} = \frac{(r+\theta)(r+\theta-\lambda) + \theta(\lambda - \theta)e^{-(\lambda - 2\theta - r)t}}{r + \theta - \lambda + \theta e^{-(\lambda - 2\theta - r)t}}. \tag{76}$$

The discount rate declines over time if and only if $\theta + r - \lambda > 0$, as we assume. This inequality is necessary and sufficient for the discount factor to be a convex combination (not merely a sum) of exponentials.

To incorporate privately owned capital, we reinterpret Ekeland and Lazrak's model. The representative dynasty has capital k_t, shared equally among currently living dynastic members. Members die off deterministically at rate θ and are replaced by new members,

maintaining a constant population in the dynasty. The number of currently living members still alive after t units of time is $e^{-\theta t}$. Members of a dynasty discount the future utility of currently living members at rate r and they discount the welfare of unborn members at rate λ. Agents do not care about other dynasties; they are altruistic with respect to their own progeny, but not with respect to society at large. When an agent dies, a new agent is born and inherits the deceased member's capital.

B.5 Imposing the equilibrium conditions

The factor prices, R and w, and the next period climate state, $\mathbf{S}'$, depend on current emissions, E. With a view to studying the climate problem, Proposition 5 includes the argument E in all functions, but we drop it in this section to simplify notation. We also suppress E from the functions s, ξ, b, and f. We do this while understanding that in equilibrium E is a function of $(K, \mathbf{S})$. Section 6.2.2 explains how we modify the algorithm to solve for climate policy.

Equating the individual and aggregate savings rules, while imposing the equilibrium condition $k = K$, gives

$$K' = s_i(K, \mathbf{S}) R_i(K, \mathbf{S}) (K + \xi_i(K, \mathbf{S})). \tag{77}$$

At iteration $i > 1$, we have approximations of the functions $b_{i-1;j}(K', \mathbf{S}')$ and $f_{i-1}(K', \mathbf{S}')$ from the previous iteration. To update our approximation, we use an $N-$element grid over the three dimensional state space, $(K, \mathbf{S})$, with node n given by $(K_n, \mathbf{S}_n)$, for $n = 1, 2...N$. We have to solve N fixed point problems at every iteration; these problems are mappings from $\Re^2 \to \Re^2$ (conditional on $K, \mathbf{S}$), and are straightforward.

We denote the factor prices at iteration i, evaluated at $(K_n, \mathbf{S}_n)$, as w_i^n and R_i^n. Using (24), (25) and (77), the numbers s_i^n and ξ_i^n are solutions to the implicit equations:[18]

$$s_i^n = \frac{1}{\left(\beta_1 b_{i-1;i}\left(s_i^n R_i^n \times (K_n + \xi_i^n), \mathbf{S}'\right)\right)^{-\frac{1}{\eta}} + 1} \tag{78}$$

and

$$\xi_i^n = \frac{w_i^n - \left(\beta_1 b_{i-1;i}\left(s_i^n R_i^n \times (K_n + \xi_i^n), \mathbf{S}'\right)\right)^{-\frac{1}{\eta}} f_{i-1}\left(s_i^n R_i^n \times (K_n + \xi_i^n), \mathbf{S}'\right)}{R_i^n}. \tag{79}$$

The right side of these equations involve the approximations of the functions $b_{i-1;i}$ and f_{i-1} obtained from the previous iteration.

Given s_i^n and ξ_i^n, for $n = 1, 2...N$, we compute the approximations of $b_{i;j}(K, \mathbf{S})$ and $f_i(K, \mathbf{S})$. We introduce addition notation to express the updating equations concisely. Consistent with the notation above, we use superscript n on the functions $b_{i,j}$ and f_i to denote the value of these functions at node $(K_n, \mathbf{S}_n)$. The functions $b_{i-1;j}$ and f_{i-1}, in contrast, are evaluate at the next-period state (iteration $i - 1$), which depends on the current state. We write

$$b_{i-1;j}^n = b_{i-1;j}\left(s_i^n R_i^n (k_n + \xi_i^n,), S'\right) \text{ and } f_{i-1}^n = f_{i-1}\left(s_i^n R_i^n (k_n + \xi_i^n,), S'\right),$$

---

[18] As noted above, the next-period climate state, $\mathbf{S}'$, is a function of the current state, identified by the node n; to avoid further complicating the notation, we do not put an index n on the next period climate state.
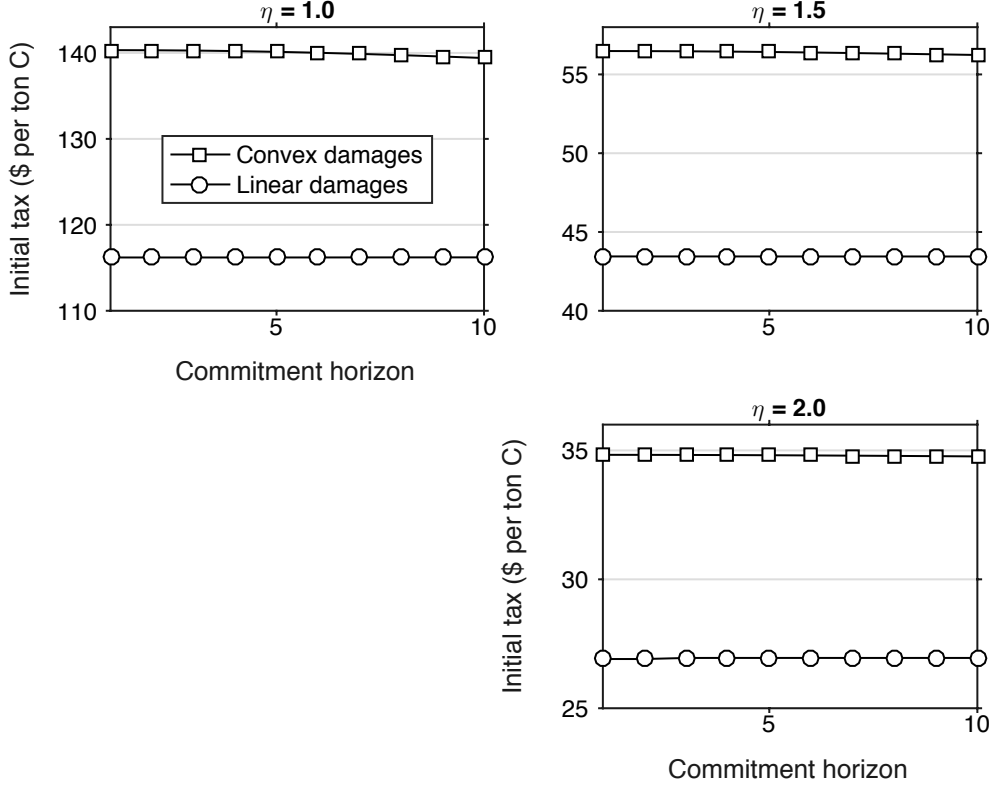
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Figure 15: First period tax for equilibria with different commitment horizons. Panels denote different η. All assume TFP growth, $\delta = 0.65$, and Stern weight 0.2.

showing the value of $b_{i-1;j}$ and f_{i-1} as functions of the current state, $(K_n, \mathbf{S}_n)$. The right sides of these two equations use the approximations obtained at the previous iteration. Again, we recognize that $\mathbf{S}'$ depends on $\mathbf{S}_n$ and current climate policy, a function of $(K_n, \mathbf{S}_n)$. Using this notation, we write the recursion (26) as

$$b_{i;j}^n = (R_i^n (1 - s_i^n))^{1-\eta} + \beta_{j-i+1} b_{i-1;j}^n \times (s_i^n R_i^n)^{1-\eta}, \tag{80}$$

and

$$f_i^n = \frac{1}{R_i^n} \left(w_i^n + f_{i-1}^n \right) \tag{81}$$

(We use (24) to simplify the first line of (26) to obtain (80).) We solve these $2N$ independent equations and then use polynomials to obtain the approximations of b_{ij} and f_i.

B.6   Impact of commitment on first-period tax

Figure 15 shows that strategic interactions that arise in the general model with commitment are quantitatively negligible. The figures plot the first-period carbon tax as a function of the horizon of the commitment technology available to the initial generation. This is show for different values of η.

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