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PRELIMINARY: NOT FOR QUOTATION

”[An] important and difficult question...[is] not answered by the approach taken here: the integration of money in the theory of value...”

—— Gerard Debreu, *Theory of Value* (1959)

Abstract

General equilibrium is investigated with N commodities deliverable at T dates traded spot and futures at $\frac{1}{2}N^2T^3$ dated commodity-pairwise trading posts. Trade is a resource-using activity recovering transaction costs through the spread between bid (wholesale) and ask (retail) prices (pairwise rates of exchange). Budget constraints are enforced at each trading post separately implying demand for a carrier of value between trading posts and over time, commodity money (spot or futures). Trade in media of exchange and stores of value is the difference between gross and net inter-post trades. ”Demand for ’money’” is stocks held for retrade.

JEL Classification: C62, D50, E40

*rstarr@ucsd.edu, Economics Dept. 0508, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92093-0508, USA. Duncan Foley and I have discussed these issues repeatedly; he may not be able to escape co-authorship. The hospitality of the Federal Reserve Bank of San Francisco is gratefully acknowledged. The views in this essay do not reflect the Federal Reserve Bank of San Francisco; this essay and any errors are solely the responsibility of the author.

1 Introduction

It is well-known that the Arrow-Debreu model of Walrasian general equilibrium cannot account for money. Professor Hahn (1982) writes

”The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu.”

This paper pursues development of foundations for a theory of money based on elaborating the detail structure of an Arrow-Debreu model. As an exercise in mathematical economic theory, the object is to develop a parsimonious model in keeping with the spirit of Walrasian general equilibrium theory that results in commodity money as a medium of exchange and a store of value; money is to be a result of the model not an assumption. This is fulfillment of the long-standing theorists’ strategy that powerful theorems come from weak assumptions — the results of mild assumptions are conclusions that can be widely applied.

The underlying economics of the model is that transactions are a resource-using activity, a production process, and thus they are priced. The cost of transacting is the bid/ask spread, explicitly priced in financial markets, more obscure in most goods markets. A general equilibrium model that makes these costs explicit allows the model endogenously to distinguish liquid (low transaction cost) instruments from illiquid instruments. We ordinarily expect that the most liquid instruments will become media of exchange, Menger (1892), Starr (2008A). If they are both liquid and durable, they will become stores of value.

The elementary first step is to create a general equilibrium where there is a well defined demand for a medium of exchange and a store of value — a carrier of value between transactions and over time. This is arranged by replacing the single budget constraint of the Arrow-Debreu model with the requirement that the typical household or firm pays for its purchases directly at each of many separate transactions, and that markets reopen over time. Transactions take place at commodity-pairwise trading posts, available at a sequence of dates. This represents a formal distinction from the Arrow-Debreu model — there is not merely one budget constraint on an agent, but many at each of a sequence of dates — that makes this treatment require a separate proof of existence of equilibrium.

A well-defined demand for media of exchange and stores of value (commodity monies, not necessarily unique) arises endogenously as an outcome of the market equilibrium. Money is not a social contrivance or an assumption. It is an outcome of the equilibrium. The price system in equilibrium creates and defines money. The use of commodity money is particularly evident when the structure of demands is characterized by an absence of double coincidence of wants, Jevons (1875). Media of

exchange and stores of value are characterized as the carrier of value between transactions and over time (not fulfilling final demands or input requirements themselves), the difference between gross and net trades ¹. Related general equilibrium models with transaction cost include Foley (1970), Hahn (1971, 1973), Kurz(1974), Starrett (1973), Starr (2008). These references are discussed at greater length in section 11, below.

1.1 Structure of the Trading Post Model

In the trading post model, transactions take place at commodity pairwise trading posts (Shapley and Shubik (1977), Walras (1874)) with budget constraints (you pay for what you get in commodity terms) enforced at each post. Prices — bid (wholesale) and ask (retail) — are quoted as commodity rates of exchange. Trade is arranged by firms, typically buying at bid prices and selling at ask prices, incurring costs (resources used up in the transaction process) and recouping them through the bid/ask spread. Market equilibrium occurs when bid and ask prices at each trading post have adjusted so that all trading posts clear.

The model is a sequence economy: markets reopen over time. There is a full set of futures markets. Thus each of N goods is characterized by its delivery date, $1 \leq \tau, \theta \leq T$. In addition, there is a transaction date $t \leq \tau, \theta$. Thus the trading post $\{i(\tau), j(\theta), t\}$ is the setting at date t for trade of i deliverable at τ in exchange for j deliverable at θ . In the case where $t = \tau = \theta$, the trade is spot, not futures. At each trading post in each transaction, there is a budget constraint. Buyers pay for acquiring $i(\tau)$ by delivering $j(\theta)$ of equal value. It is likely that in a general equilibrium many of the commodity-pairwise futures markets will be inactive; traders will prefer to transact spot. That is why they carry stores of value from one period to the next. Practical availability and actual use of a futures market is endogenously determined; market activity is an outcome, not an assumption.

1.2 Structure of the Proof

The structure of the proof of existence of general equilibrium follows the approach of Arrow and Debreu (1954), Debreu (1959), and Starr (1997). The usual assumptions

¹ The present model is an alternative to the fiat money models of overlapping generations, Wallace (1980) et al., and of search, Kiyotaki and Wright (1989) et al. There a unique unbacked fiat money of positive value is typically assumed (not derived as an outcome) and presented as a bubble.

It is possible to accommodate in an Arrow-Debreu setting an intrinsically worthless paper money trading at a positive value and used as a common medium of exchange. The rationale is that taxes payable in paper money provide backing for a positive value, and low transaction cost ensures use as medium of exchange, Goldberg (2005), Smith (1776), Starr (2003A, 2003B).

of continuity, convexity (traditional but by no means innocuous in this context), and no free lunch/irreversibility are used. The price space at a trading post for exchange of one good at bid price for another at ask price is the unit 1-simplex, allowing any possible nonnegative relative price ratio. The price space for the economy as a whole then is a Cartesian product of unit 1-simplices. The attainable set of trading post transactions is compact. As in Arrow and Debreu (1954), the model considers transaction plans of firms and households artificially bounded in a compact set including the attainable set as a proper subset. Price adjustment to a fixed point with market clearing leads to equilibrium of the artificially bounded economy. But the artificial bounds are not a binding constraint in equilibrium. The equilibrium of the artificially bounded economy is as well an equilibrium of the original economy.

1.3 Conclusion: Medium(a) of Exchange and Store(s) of Value

The general equilibrium specifies each household and firm's trading plan. At the conclusion of trade, each has achieved a net trade. Gross trades include trading activity that goes to paying for acquisitions and accepting payment for sales rather than directly implementing desired net trades. A household with endowment concentrated at date τ sells its endowment at trading posts meeting at $t \leq \tau$. The household there acquires goods for consumption in the present, at $\theta = t$ or in the future, at $\theta > t$, or acquires other goods — perhaps more durable or of lower transaction cost — subsequently to retrade for spot goods in the future. Goods acquired for retrade in the same period are media of exchange; goods acquired for future trade are stores of value. It's easy to calculate gross trades and net trades at equilibrium. For households, the difference — gross trades minus net trades — represents trading activity in carriers of value between trades, media of exchange and stores of value (perhaps including some arbitrage). Since firms perform a market-making function within trading posts, identification of media of exchange and stores of value used by firms is not so straightforward. After netting out intra-post trades, the remaining difference between inter-post gross and net trades represents the firms' trade flows of media of exchange. In some examples (see Starr (2003A, 2003B)) the medium of exchange may be a single specialized commodity (the common medium of exchange). The approach of the present model is intended to provide a Walrasian general equilibrium theory of (commodity) money as a medium of exchange and store of value. It is sufficiently general to include both a unique common medium of exchange and store of value and many goods simultaneously acting as media of exchange or stores of value. In this setting, the use of (commodity) money is an outcome of the price system in equilibrium; money is a conclusion, not an assumption.

When will media of exchange actually be used in the trading post economy? Two conditions seem to be sufficient: desirability of trade, net of transaction costs; absence

of double coincidence of wants. The logic is simple. If trade is desirable at prevailing equilibrium prices (net of transaction costs including the transaction cost of media of exchange) and there is no double coincidence of wants, then in order for trade to proceed fulfilling the budget constraint at each trading post separately, media of exchange will be used as carriers of value between trading posts. However, the absence of double coincidence of wants depends on prevailing prices as well as endowments and technology. A variety of examples is presented in Starr (2008A, 2008B).

When will stores of value be used? Essentially the same statement, applied intertemporally. When buyers of goods i at one date do not have a reciprocal supply j demanded by the providers of i , the two sides of the trade will exchange for a convenient intertemporal store of value, subsequently to be retraded for the goods desired by the suppliers of i who delivered it earlier. Conversely, there are two cases where trading post equilibria will have no use of media of exchange: full double coincidence of wants (subject to direct trade experiencing no higher transaction costs than indirect trade); and a no-trade equilibrium. Again, necessary and sufficient conditions, a priori, to fulfill these characteristics are not immediately evident.

2 Commodities, Futures, and Trading Posts

There are N tradeable goods denoted $1, 2, \dots, N$. There are T time periods. Transactions take place at trading posts where goods trade for one another pairwise. $\{i(\tau), j(\theta), t\}$ (or equivalently $\{j(\theta), i(\tau), t\}$) denotes the trading post meeting at t , where i (deliverable at τ) and j (deliverable at θ) are traded for one another. Of course, for active trade to take place, we should have $t \leq \tau, \theta$. The case $t = \tau = \theta$ represents a spot market, and the cases $t < \tau$, or $t < \theta$ include futures transactions. As a matter of accounting, it is simpler to list all conceivable markets at each date, recognizing that those with $t > \tau$ or $t > \theta$ must be inactive in equilibrium. At date t therefore, all $\frac{1}{2}N^2T^2$ conceivable trading posts will be priced but we expect only $\frac{1}{2}N^2(T - t + 1)^2$ trading posts to be actively available for transactions dated t , actual activity to be determined in equilibrium. That is, futures markets may or may not be active. That's a decision to be determined in equilibrium by the market; those with low transaction costs, lower than spot markets, are likely to be active. Those with transaction costs higher than their spot counterparts are likely to be inactive.

Over the T -period horizon, there are $\frac{1}{2}N^2T^3$ current and future trading posts. For ease of accounting this includes the impossible posts, $t > \tau$ or $t > \theta$, and the useless posts $\{i(\tau), i(\tau), t\}$.

3 Prices

Goods are traded directly for one another without distinguishing any single good as 'money'. Prices are then quoted as rates of exchange between goods. We distinguish between bid (selling or wholesale) prices and ask (buying or retail) prices. Thus the ask price of a hamburger might be 5.0 chocolate bars and the bid price 3.0 chocolate bars. Note that the ask price of a chocolate bar then is the inverse of bid price of a hamburger. That is, the ask price of a chocolate bar is 0.333 hamburger and the bid price of a chocolate bar is 0.2 hamburger.

Let Δ represent the unit 1-simplex. At trading post $\{i(\tau), j(\theta), t\}$, the (relative) ask price of good i and (relative) bid price of good j are represented as $p^{\{i(\tau), j(\theta), t\}} \equiv (a_i^{\{i(\tau), j(\theta), t\}}, b_j^{\{i(\tau), j(\theta), t\}}) \in \Delta$. In a (minor) abuse of notation, the ordering of i and j in the superscript on p will matter. $p^{\{j(\theta), i(\tau), t\}} \equiv (a_j^{\{i(\tau), j(\theta), t\}}, b_i^{\{i(\tau), j(\theta), t\}}) \in \Delta$. Thus there are two operative price 1-simplices at each trading post. The price space at date t then is $\Delta^{N^2 T^2}$, the $N^2 T^2$ -fold Cartesian product of Δ with itself; its typical element is $p^t \in \Delta^{N^2 T^2}$. The T-period price space is $\Delta^{N^2 T^3}$.

4 Budget Constraints and Trading Opportunities

The budget constraint is simply that at each pairwise trading post, at prevailing prices, in each transaction, payment is given for goods received. That is, at trading post $\{i(\tau), j(\theta), t\}$, an ask/bid price pair is quoted $p^{\{i(\tau), j(\theta), t\}} \equiv (a_i^{\{i(\tau), j(\theta), t\}}, b_j^{\{i(\tau), j(\theta), t\}}) \in \Delta$ expressing the ask price of i in terms of j and a bid price of j in terms of i . A firm or household's trading plan over the T-period horizon with reopening of markets at each date is $(y, x) \in R^{2N^2 T^3}$ specifies the following transactions at trading post $\{i(\tau), j(\theta), t\}$: $y_i^{\{i(\tau), j(\theta), t\}}$ (at ask prices — retail) in i , $y_j^{\{i(\tau), j(\theta), t\}}$ (at ask prices — retail) in j , $x_i^{\{i(\tau), j(\theta), t\}}$ (at bid prices — wholesale) in i , $x_j^{\{i(\tau), j(\theta), t\}}$ (at bid prices — wholesale) in j . Positive values of these transactions are purchases. Negative values are sales. At each trading post (of two goods) there are four quantities to specify in a trading plan. Then the budget constraint facing firms and households at each trading post is that value delivered must at least equal value received. That is

$$0 \geq (a_i^{\{i(\tau), j(\theta), t\}}, b_j^{\{i(\tau), j(\theta), t\}}) \cdot (y_i^{\{i(\tau), j(\theta), t\}}, x_j^{\{i(\tau), j(\theta), t\}}),$$

$$0 \geq (a_j^{\{i(\tau), j(\theta), t\}}, b_i^{\{i(\tau), j(\theta), t\}}) \cdot (y_j^{\{i(\tau), j(\theta), t\}}, x_i^{\{i(\tau), j(\theta), t\}}) \quad (\mathbf{B})$$

(B) says that purchases of i at the bid price are repaid by sales of j at the ask price, purchases of i at the ask price are repaid by sales of j at the bid price.

The use of **(B)** as the budget constraint is the single most important distinguishing feature of the trading post model. It contrasts with the single lifetime budget constraint of the Arrow-Debreu model. Thus each trading decision plan faces N^2T^3 budget constraints of the form **(B)**. Given a price vector $p \in \Delta^{N^2T^3}$ the array of trades fulfilling **(B)** is the set of trades fulfilling the N^2T^3 local budget constraints at the trading posts. Denote this set

$$\mathbf{M}(p) \equiv \{(y, x) \in R^{2N^2T^3} | (y, x) \text{ fulfills } \mathbf{(B)} \text{ at } p \text{ for all } i, j = 1, \dots, N; 1 \leq t, \tau, \theta \leq T\}$$

5 Firms

The heavy lifting in this model is done by firms. They perform the market-making function, incurring transaction costs. The population of firms is a finite set denoted F , with typical element $f \in F$. Thus, firm f 's technology set may specify that f 's purchase of labor for delivery at date θ in exchange for $i(\tau)$ on the $\{i(\tau), \text{labor}(\theta), t\}$ market and purchase of i and j wholesale on the $\{i(\tau), j(\theta), t\}$ market allows f to sell i and j (retail) on the $\{i(\tau), j(\theta), t\}$ market. That's how f can become a market maker. If there is a sufficient difference between bid and ask prices so that f can cover the cost of its inputs with a surplus left over, that surplus becomes f 's profits, to be rebated to f 's shareholders. In the special case where $t = \tau = \theta$ the transactions are on the spot market at t .

5.1 Transaction and Production Technology

Firm f 's technology set is Y^f . We assume

$$\mathbf{P.0} \quad Y^f \subset R^{2N^2T^3}$$

The typical element of Y^f is (y^f, x^f) , a pair of N^2T^3 -dimensional vectors. The N^2T^3 -dimensional vector y^f represents f 's transactions at ask (retail) prices at all dates for all possible delivery dates; the N^2T^3 -dimensional vector x^f represents f 's transactions at bid (wholesale) prices at all market dates for delivery at all possible delivery dates. The 2-dimensional vector $y^{f\{i(\tau), j(\theta), t\}}$ represents f 's transactions at ask (retail) prices at trading post $\{i(\tau), j(\theta), t\}$; the 2-dimensional vector $x^{f\{i(\tau), j(\theta), t\}}$ represents f 's transactions at bid (wholesale) prices at trading post $\{i(\tau), j(\theta), t\}$. The typical co-ordinates $y_i^{f\{i(\tau), j(\theta), t\}}, x_i^{f\{i(\tau), j(\theta), t\}}$ are f 's action with respect to good i at the $\{i(\tau), j(\theta), t\}$ trading post. Since f may act as a wholesaler/retailer/market maker, entries anywhere in $(y^{f\{i(\tau), j(\theta), t\}}, x^{f\{i(\tau), j(\theta), t\}})$ may be positive or negative — subject of course to constraints of technology Y^f and prices $\mathbf{M}(p)$. This distinguishes

the firm from the typical household. The typical household can only sell at bid prices and buy at ask prices.

The entry $y_i^{f\{i(\tau),j(\theta),t\}}$, represents f's actions at ask prices with regard to good i at trading post $\{i(\tau), j(\theta), t\}$. $y_i^{f\{i(\tau),j(\theta),t\}} > 0$ represents a purchase of i deliverable at τ at the $\{i(\tau), j(\theta), t\}$ trading post (at the ask price). $y_i^{f\{i(\tau),j(\theta),t\}} < 0$ represents a sale of i at the ask price.

The entry $x_i^{f\{i(\tau),j(\theta),t\}}$, represents f's actions at bid prices with regard to good i at trading post $\{i(\tau), j(\theta), t\}$. $x_i^{f\{i(\tau),j(\theta),t\}} > 0$ represents a purchase of i deliverable at τ at the trading post (at the bid price). $x_i^{f\{i(\tau),j(\theta),t\}} < 0$ represents a sale of i deliverable at τ at the bid price.

A firm that is an active market-maker at $\{i(\tau), j(\theta), t\}$ will typically buy at the bid price and sell at the ask price. A firm that is not a market-maker may have to pay retail — like the rest of us — selling at the bid price and buying at the ask price.

In addition to indicating the transaction possibilities, Y^f includes the usual production possibilities. The usual assumptions on production technology apply. For each $f \in F$, assume

P.I Y^f is convex.

P.II $0 \in Y^f$, where 0 indicates the zero vector in $R^{2N^2T^3}$.

P.III Y^f is closed.

The aggregate technology set is the sum of individual firm technology sets. $Y \equiv \sum_{f \in F} Y^f$. It fulfills the familiar *no free lunch* and *irreversibility* conditions.

P.IV [(a)] if $(y, x) \in Y$ and $(y, x) \neq 0$, then $y_i^{\{i(\tau),j(\theta),t\}} + x_i^{\{i(\tau),j(\theta),t\}} > 0$ for some $i(\tau), j(\theta), t \leq \tau, \theta$.

[(b)] if $(y, x) \in Y$ and $(y, x) \neq 0$, then $-(y, x) \notin Y$.

Denote the initial resource endowment of the economy as $r \in R_+^{NT}$. Then we define the attainable production plans of the economy as

$$\hat{Y} \equiv \{(y, x) \in Y \mid r_{i(\tau)} \geq \sum_{j=1}^N \sum_{\theta=t}^T \sum_{t=1}^{\tau} (y_i^{\{i(\tau),j(\theta),t\}} + x_i^{\{i(\tau),j(\theta),t\}}) \text{ all } i = 1, 2, \dots, N\}$$

Attainable production plans for firm f can then be described as

$$\hat{Y}^f \equiv \{(y^f, x^f) \in Y^f \mid \text{there is } (y^k, x^k) \in Y^k \text{ for each } k \in F, k \neq f, \text{ so that}$$

$$[\sum_{k \in F, k \neq f} (y^k, x^k) + (y^f, x^f)] \in \hat{Y}\}.$$

Lemma 5.1: Assume P.0 - P.IV. Then $\hat{Y}$ and $\hat{Y}^f$ are closed, convex, and bounded. Proof: Starr (1997), Theorem 8.1, 8.2.

5.2 Firm Maximand and Transactions Function

The firm formulates a production plan and a trading plan. The firm cannot deliver negative goods to its shareholders, though during the course of trade it may have net sales positions in goods that it has yet to acquire. The firm's opportunity set for transactions fulfilling budget is

$$E^f(p) \equiv [\mathbf{M}(p) - Y^f].$$

Now consider $(y, x) \in E^f(p)$. In each good i , the net surplus available in good $i(\tau)$ is $w_{i(\tau)}^f \equiv \sum_{j=1}^N \sum_{\theta=t}^T \sum_{t=1}^T (y_i^{\{i(\tau), j(\theta), t\}} + x_i^{\{i(\tau), j(\theta), t\}})$ and firm f 's surplus is the vector w^f of these co-ordinates. To give this notion a functional notation, let $W(y, x) \equiv w^f \in R^{NT}$ described here.

That is, consider the firm's production, purchase, and sale possibilities, net after paying for them, and what's left is the net yield. Using the sign conventions we've adopted — purchases are positive co-ordinates, sales are negative co-ordinates — the net yield is then the negative co-ordinates (supplies) in a trading plan that are not absorbed by payments due and the net purchases not required as inputs to the firm.

At each date t , there are $\frac{1}{2}NT$ trading posts where each good $i(\tau)$ is priced (active trade depending on date t and market equilibrium), at NT rates of exchange. Hence, the notion of 'profit' is not well defined. In the absence of a single family of well-defined prices, it is difficult to characterize optimizing behavior for the firm. *Fautes de mieux* we'll give the firm a scalar maximand with argument p, y', x' . Firm f is assumed to have a real-valued, continuous maximand $v^f(p; y', x')$. We take v^f to be strictly monotone and concave in (y', x') .

The firm's optimizing choice then is

$$G^f(p) \equiv \{\operatorname{argmax} v^f(p; y', x') \in E^f(p)\}.$$

This results in the firm's market behavior (without any constraint requiring actions to stay in a bounded range) described by

$H^f(p) \equiv \{(y, x) \in \mathbf{M}(p) | [(y, x) + (y', x')] \in Y^f, (y', x') \in G^f(p)\}$. This marketed plan then results in the market and dividend plan

$$S^f(p) \equiv \{(y, x; w) | (y, x) \in H^f(p), [(y, x) + (y', x')] \in Y^f, (y', x') \in G^f(p); w = W(y', x')\}$$

The logic of this definition is that $(y', x') \geq 0$ is the surplus left over after the firm f has performed according to its technology and subject to prevailing prices.

It is possible that $S^f(p)$ is not well defined, since the opportunity set may be unbounded. In the light of Lemma 5.1, there is a constant $c > 0$ sufficiently large so that for all $f \in F$, $\hat{Y}^f$ is strictly contained in a closed ball, denoted B_c of radius c centered at the origin of $R^{2N^2T^3}$. Following the technique of Arrow and Debreu (1954), constrained market behavior for the firm will consist of limiting its production choices to $Y^f \cap B_c$. This leads to the constrained surplus

$$\tilde{E}^f(p) \equiv [[\mathbf{M}(p) \cap B_c] - [Y^f \cap B_c]]$$

$$\tilde{G}^f(p) \equiv \{\text{argmax } v^f(p; y', x') \in \tilde{E}^f(p)\}.$$

$$\tilde{H}^f(p) \equiv \{(y, x) \in \mathbf{M}(p) | [(y, x) + (y', x')] \in Y^f \cap B_c, (y', x') \in \tilde{G}^f(p)\}.$$

The firm's constrained (to B_c) market behavior then is defined as

$$\tilde{S}^f(p) \equiv \{(y, x; w) | (y, x) \in \tilde{H}^f(p), [(y, x) + (y', x')] \in Y^f \cap B_c, (y', x') \in \tilde{G}^f(p); w = W(y', x')\}.$$

Lemma 5.2: Assume P.0 - P.IV. Let $p \in \Delta^{N^2T^3}$. Then $\tilde{E}^f(p)$ is convex-valued, nonempty, upper and lower hemicontinuous.

Proof: Upper hemicontinuity and convexity follow from closedness and convexity of the underlying sets. $0 \in \tilde{E}^f(p)$ always, so nonemptiness is fulfilled. Lower hemicontinuity follows from lower hemicontinuity of $[\mathbf{M}(p) \cap B_c]$.

Lemma 5.3: Assume P.0 - P.IV. Then $\tilde{G}^f(p)$, $\tilde{H}^f(p)$, $\tilde{S}^f(p)$ are well defined, nonempty, upper hemicontinuous, and convex-valued for all $p \in \Delta^{N(N-1)}$.

Proof: Note compactness of B_c . Apply Theorem of the Maximum, continuity and concavity of v^f .

Lemma 5.4: Assume P.0 - P.IV. Let $[\tilde{G}^f(p) + \tilde{H}^f(p)] \cap \hat{Y}^f \neq \emptyset$. Then $[\tilde{G}^f(p) + \tilde{H}^f(p)] \subseteq [G^f(p) + H^f(p)]$.

Proof: Recall that B_c strictly includes $\hat{Y}^f$. Then the result follows from convexity of Y^f and $\hat{Y}^f$ and concavity of $v^f(p; y', x')$. The proof follows the model of Starr (1997) Theorem 8.3. Let $(y^*, x^*) \in \tilde{G}^f(p)$, $(y^*, x^*) \in \tilde{H}^f(p)$, $[(y^*, x^*) + (y^*, x^*)] \in \hat{Y}^f \subset B_c$. Use a proof by contradiction. Suppose not. Then there is $(y, x) \in Y^f$ so that $(y, x) - (y^o, x^o) = (y', x')$, where $v^f(p; y', x') > v^f(p; y^*, x^*)$, $(y', x') \in E^f(p)$, and $(y^o, x^o) \in \mathbf{M}(p)$. But convexity of Y^f and concavity of v^f imply that on the chord between (y^*, x^*) and (y, x) there is $[\alpha(y^*, x^*) + (1 - \alpha)(y, x)] \in B_c$ for $1 \geq \alpha > 0$ where $v^f(p; [\alpha(y^*, x^*) + (1 - \alpha)(y, x)]) > v^f(p; y^*, x^*)$. This is a contradiction.

5.3 Inclusion of constrained supply in unconstrained supply

$(y, x; w) \in \tilde{S}^f(p)$ implies $(y, x) \in B_c$, a bounded set. $w \in R_+^{NT}$ is f 's profits. By construction there is $K > 0$ so that w is contained in the nonnegative quadrant of a ball of radius K centered at the origin, denoted $B_K \subset R_+^{NT}$.

Lemma 5.5: Let $p \in \Delta^{N^2T^3}$ such that $\tilde{S}^f(p) \cap [\hat{Y}^f \times B_K] \neq \emptyset$. Then $S^f(p)$ is well defined and nonempty. Further $\tilde{S}^f(p) \subseteq S^f(p)$.

Proof: Lemma 5.4.

5.4 Nonnegativity of profits

In order to avoid boundary problems on the household consumption sets we assume that profits distributed to households are always nonnegative co-ordinatewise. There are undoubtedly elementary conditions on v^f that will assure this result, but the issues are already sufficiently complex.

P.V For each $f \in F, p \in \Delta^{N^2 T^3}$, so that $(y, x; w) \in \tilde{S}^f(p)$, it follows that $w \geq 0$.

6 Households

There is a finite set of households, H , with typical element h .

6.1 Endowment and Consumption Set

$h \in H$ has a possible consumption set, taken for simplicity to be the nonnegative quadrant of R^{NT} , R_+^{NT} . $h \in H$ is endowed with $r^h >> 0, r^h \in R_+^{NT}$ assumed to be strictly positive to avoid boundary problems. $h \in H$ has a share $\alpha^{hf} \geq 0$ of firm f , so that $\sum_{h \in H} \alpha^{hf} = 1$.

6.2 Trades and Payment Constraint

$h \in H$ chooses $(y^h, x^h) \in R^{2N^2 T^3}$ subject to the following restrictions. A household always balances its budget, sells wholesale and buys retail:

- (i) $0 \geq x_i^{h\{i(\tau), j(\theta), t\}}$ for all i, j .
- (ii) $y_i^{h\{i(\tau), j(\theta), t\}} \geq 0$ for all i, j .
- (iii) $(y^h, x^h) \in \mathbf{M}(p)$

6.3 Maximand and Demand

Household h 's share of profits from firm f is part of h 's endowment and enters directly into consumption. h decides on consumption of $c_{i(\tau)}^h$ of good $i(\tau)$ at date τ and to store $s_{i(\tau)}^h$ of his available good $i(\tau)$. From one period to the next, good $i(\tau)$ survives at the rate $\kappa_i \geq 0$. Equivalently, good i depreciates at the rate $1 - \kappa_i$ per period.

When the profits of all firms $f \in F$, w^f in $(y^f, x^f; w^f)$, are well defined, f distributes to shareholders w^f , and h 's disposable good i , allocated between storage and consumption, is

$$(iv) \ c_{i(\tau)}^{h\tau} + s_{i(\tau)}^{h,\tau} \equiv r_{i(\tau)}^h + \kappa_i s_{i(\tau-1)}^{h,\tau-1} + [\sum_{f \in F} \alpha^{hf} w^f]_{i(\tau)} + \sum_{j=1}^N \sum_{\theta=1}^T \sum_{t=1}^T [x_i^{h\{i(\tau),j(\theta),t\}} + y_i^{h\{i(\tau),j(\theta),t\}}]$$

However, prices p may be such that $S^f(p)$ is not well defined for some f . Then we may wish to discuss the constrained version of (iv), where $\tilde{w}^f$ comes from $(y^f, x^f; \tilde{w}^f) \in \tilde{S}^f(p)$.

$$(iv') \ c_{i(\tau)}^h + s_{i(\tau)}^{h,\tau} \equiv r_{i(\tau)}^h + \kappa_i s_{i(\tau-1)}^{h,\tau-1} + [\sum_{f \in F} \alpha^{hf} \tilde{w}^f]_{i(\tau)} + \sum_{j=1}^N \sum_{\theta=1}^T \sum_{t=1}^T [x_i^{h\{i(\tau),j(\theta),t\}} + y_i^{h\{i(\tau),j(\theta),t\}}]$$

In addition, h 's consumption and storage must be nonnegative.

$$(v) \ c_{i(\tau)}^h, s_{i(\tau)}^{h,\tau} \geq 0 \text{ all } i = 1, 2, \dots, N, \text{ all } \tau = 1, 2, \dots, T.$$

C.I For all $h \in H$, h 's maximand is the continuous, quasi-concave, real-valued, strictly monotone, utility function $u^h(c^h). u^h : R^{NT} \rightarrow R_+$.

h 's planned transactions function is defined as $D^h : \Delta^{N^2 T^3} \times R^{NT \# F} \rightarrow R^{2N^2 T^3}$. Let w denote $(w^1, w^2, w^3, \dots, w^f, \dots, w^{\#F})$.

$D^h(p, w) \equiv \{(y^h, x^h) \in R^{2N^2 T^3} | (y^h, x^h) \text{ maximizes } u^h(c^h), \text{ subject to (i), (ii), (iii), (iv) and (v)}\}$. h 's planned storage at τ is $s^{h,\tau}$.

However, $D^h(p, w)$ may not be well defined when opportunity sets are unbounded (when ask prices of some goods are zero) and w may not be well defined for p such that $S^f(p)$ is not well defined for some f . To treat this issue, let $B_K^{\#F}$ be the $\#F$ -fold Cartesian product of B_K , and define $\tilde{D}^h : \Delta^{N^2 T^3} \times B_K^{\#F} \rightarrow B_c$.

$\tilde{D}^h(p, w) \equiv \{(y^h, x^h) | (y^h, x^h) \text{ maximizes } u^h(c^h), \text{ subject to (i), (ii), (iii), (iv'), (v), and } (y^h, x^h) \in B_c\}$. The restriction to B_c in this definition assures that $\tilde{D}^h(p)$ represents the result of optimization on a bounded set, and is well-defined.

Lemma 6.1: Assume P.0 - P.IV, C.I. Then $\tilde{D}^h(p, w)$ is nonempty, upper hemi-continuous and convex-valued, for all $p \in \Delta^{N(N-1)}$, $w \in B_K^{\#F}$. The range of $\tilde{D}^h(p, w)$ is compact. For (p, w) such that $|(y^h, x^h)| < c$ for (some) $(y^h, x^h) \in \tilde{D}^h(p, w)$, it follows that $\tilde{D}^h(p, w) \subseteq D^h(p, w)$.

Proof: (Note to the reader: This proof includes an unfortunate confusion of notation. c without superscript denotes a large real number indicating the radius of B_c , a ball strictly containing all attainable transactions of the typical firm. c^h and c^* (with superscript) denote consumption vectors.) Apply Theorem of the Maximum, noting $r^h \gg 0$ by assumption and $w^f \geq 0$ by P.V, continuity and quasi-concavity of u^h , convexity of constraint sets defined by (i)-(v) or by (i),(ii),(iii), (iv'), (v). Inclusion of $\tilde{D}^h(p, w)$ in $D^h(p, w)$ follows the pattern of Starr (1997) Theorem 9.1(b). Proof by contradiction. Suppose not. Then there is $(y^*, x^*) \in D^h(p, w)$ with associated c^* so that $u^h(c^*) > u^h(c^h)$. But recall $|(y^h, x^h)| < c$. On the chord between (y^h, x^h) and (y^*, x^*) there is $[\alpha(y^*, x^*) + (1 - \alpha)(y^h, x^h)]$, $1 > \alpha > 0$, fulfilling (i), (ii), (iii), (iv'), (v), and $|\alpha(y^*, x^*) + (1 - \alpha)(y^h, x^h)| = c$ so that $u(\alpha c^* + (1 - \alpha)c^h) > u(c^h)$. This is a contradiction.

7 Excess Demand

Let $(p, w') \in \Delta^{N^2 T^3} \times B_K^{\#F}$. Constrained excess demand and dividends at (p, w') is defined as

$$\tilde{Z} : \Delta^{N^2 T^3} \times B_K^{\#F} \rightarrow R^{2N^2 T^3} \times B_K^{\#F}.$$

$$\tilde{Z}(p, w') \equiv \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} \tilde{D}^h(p, w'), w^1, w^2, \dots, w^f, \dots, w^{\#F}) | (y^f, x^f, w^f) \in \tilde{S}^f(p)\}.$$

Lemma 7.1: Assume P.0 - P.IV, and C.I. The range of $\tilde{Z}$ is bounded. $\tilde{Z}$ is upper hemi-continuous and convex-valued for all $(p, w') \in \Delta^{N^2 T^3} \times B_K^{\#F}$.

Lemma 7.2 (Walras' Law): Let $(p, w') \in \Delta^{N^2 T^3} \times B_K^{\#F}$. Let $(y, x, w) \in \tilde{Z}(p, w')$. Then for each $i, j = 1, \dots, N, i \neq j$, we have

$$0 = (a_i^{\{i(\tau), j(\theta), t\}}, b_j^{\{i(\tau), j(\theta), t\}}) \cdot (y_i^{\{i(\tau), j(\theta), t\}}, x_j^{\{i(\tau), j(\theta), t\}}),$$

$$0 = (a_j^{\{i(\tau), j(\theta), t\}}, b_i^{\{i(\tau), j(\theta), t\}}) \cdot (y_j^{\{i(\tau), j(\theta), t\}}, x_i^{\{i(\tau), j(\theta), t\}}) \quad (\mathbf{W}).$$

Proof : The element (y, x) of $(y, x, w) \in \tilde{Z}(p, w')$ is the sum of elements (y^f, x^f) of $\tilde{S}^f(p)$ and (y^h, x^h) of $\tilde{D}^h(p, w')$ each of which is subject to **(B)**. Note that v^f and u^h are both strictly monotone.

8 Equilibrium

Let Ξ denote a compact convex subset of $R^{2N^2 T^3}$ so that $\Xi \times B_K^{\#F}$ includes the range of $\tilde{Z}$. Let

$$z \in \Xi, z \equiv ((y_1^{\{1(1), 1(1), 1\}}, x_1^{\{1(1), 1(1), 1\}}), \dots, (y_i^{\{i(\tau), j(\theta), t\}}, x_j^{\{i(\tau), j(\theta), t\}}), \dots, (y_N^{\{N(T), N(T), T\}}, x_N^{\{N(T), N(T), T\}})).$$

Define $\rho : \Xi \rightarrow \Delta^{N^2 T^3}$

$\rho(z) \equiv \{p^o \in \Delta^{N^2 T^3} | \text{For each } i, j = 1, 2, \dots, N, \text{ each } t = 1, 2, \dots, T, \text{ each } \tau = 1, 2, \dots, t, \text{ each } \theta = \tau, \tau + 1, \dots, T, p^{o\{i(\tau), j(\theta), t\}} \in \Delta \text{ maximizes } p^{\{i(\tau), j(\theta), t\}} \cdot (y_i^{\{i(\tau), j(\theta), t\}}, x_j^{\{i(\tau), j(\theta), t\}}) \text{ subject to } p^{\{i(\tau), j(\theta), t\}} \in \Delta\}.$

Lemma 8.1: ρ is upper hemi-continuous and convex-valued for all $z \in \Xi$.

Define $\Gamma : \Delta^{N^2 T^3} \times \Xi \times B_K^{\#F} \rightarrow \Delta^{N^2 T^3} \times \Xi \times B_K^{\#F}$.

$\Gamma(p, z, w') \equiv \rho(z) \times \tilde{Z}(p, w')$.

Lemma 8.2: Assume P.0 - P.IV, and C.I. Then Γ is upper hemi-continuous and convex-valued on $\Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$. Γ has a fixed point (p^*, z^*, w^*) and $0 = z^*$.

Proof: Upper hemicontinuity and convexity are established in lemmas 7.1 and 8.1. Existence of the fixed point (p^*, z^*) then follows from the Kakutani fixed point theorem. To demonstrate that $z^* = 0$, note lemma 7.2 and strict monotonicity of u^h .

Definition: $(p^*, w^*) \in \Delta^{N^2 T^3} \times B_K^{\#F}$ is said to be an equilibrium if

$$(0, w^*) \in \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} D^h(p^*, w^*), w^1, w^2, \dots, w^f, \dots, w^{\#F}) | (y^f, x^f, w^f) \in S^f(p^*)\}$$

where 0 is the origin in $R^{2N^2 T^3}$.

Theorem 8.1: Assume P.0 - P.IV, C.I. . Then there is an equilibrium $(p^*, w^*) \in \Delta^{N^2 T^3} \times B_K^{\#F}$.

Proof: Apply Lemmas 5.5, 6.1, 8.2. Lemma 8.2 provides $(p^*, z^*, w^*) \in \Delta^{N^2 T^3} \times \Xi \times B_K^{\#F}$ so that $0 = z^*$, where

$(z^*, w^*) \in \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} \tilde{D}^h(p^*, w^*), w^1, w^2, \dots, w^f, \dots, w^{\#F}) | (y^f, x^f, w^f) \in \tilde{S}^f(p^*)\}$. Then $\tilde{S}^f(p^*) \cap [\hat{Y}^f \times B_K] \neq \emptyset$, so by Lemma 5.5, $\tilde{S}^f(p^*) \subseteq S^f(p^*)$. $0 = z^*$, implies that $|(y^{*h}, x^{*h})| < c$, so by lemma 6.1, $\tilde{D}^h(p^*, w^*) \subseteq D^h(p^*, w^*)$. But then $(0, w^*) \in \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} D^h(p^*, w^*), w^1, w^2, \dots, w^f, \dots, w^{\#F}) | (y^f, x^f, w^f) \in S^f(p^*)\}$. Then (p^*, w^*) is an equilibrium.

9 Media of Exchange

Let $(y^h, x^h) \in D^h(p, w')$ be household h 's $2N^2 T^3$ -dimensional transaction vector. The x co-ordinates are typically sales (negative sign) at bid prices; the y co-ordinates are typically purchases (positive sign) at ask prices. Then we can characterize h 's gross transactions in good $i(\tau)$ at t as

$$\sum_{j=1}^N \sum_{\theta=1}^T [y_i^{h\{i(\tau), j(\theta), t\}} - x_i^{h\{i(\tau), j(\theta), t\}}] \equiv \gamma_{i(\tau)}^{h, t}.$$

Further, the absolute value of h 's net transactions in good $i(\tau)$ at t , is

$$|\sum_{j=1}^N \sum_{\theta=1}^T [y_i^{h\{i(\tau), j(\theta), t\}} + x_i^{h\{i(\tau), j(\theta), t\}}]| \equiv \nu_{i(\tau)}^{h, t}.$$

The NT -dimensional vector $\gamma^{h, t}$ with typical element $\gamma_{i(\tau)}^{h, t}$ is h 's gross trade at t . The NT -dimensional vector $\nu^{h, t}$ with typical element $\nu_{i(\tau)}^{h, t}$ is h 's net trade vector at t (in absolute value). $\mu^{h, t} \equiv \gamma^{h, t} - \nu^{h, t}$ is h 's flow of goods as media of exchange at t , gross trades minus net trades.

Since firms perform a market-making function, buying and selling the same good at a single trading post, a more complex view of their transactions is needed to sort out trading flows used as media of exchange. In particular, for firms, we should net out offsetting transactions within a single trading post. Thus for $f \in F$, f 's gross transactions in $i(\tau)$, netting out intra-post transactions is

$$\sum_{j=1}^N \sum_{\theta=1}^T [y_i^{f\{i(\tau), j(\theta), t\}} + x_i^{f\{i(\tau), j(\theta), t\}}] \equiv \gamma_{i(\tau)}^{f, t}.$$

The corresponding net transaction is

$$|\sum_{j=1}^N \sum_{\theta=1}^T [y_i^{f\{i(\tau), j(\theta), t\}} + x_i^{f\{i(\tau), j(\theta), t\}}]| \equiv \nu_{i(\tau)}^{f, t}.$$

The NT -dimensional vector $\gamma^{f, t}$ with typical element $\gamma_{i(\tau)}^{f, t}$ is f 's gross inter-post trade. The NT -dimensional vector $\nu^{f, t}$ with typical element $\nu_{i(\tau)}^{f, t}$ is h 's net inter-post

trade vector (in absolute value). $\mu^{f,t} \equiv \gamma^{f,t} - \nu^{f,t}$ is f's flow of goods as media of exchange at t, gross (inter-post) trades minus net trades.

The total (NT-dimensional vector) flow of media of exchange among households and firms at t is then $\sum_{h \in H} \mu^{h,t} + \sum_{f \in F} \mu^{f,t}$.

Thus the trading post equilibrium establishes a well-defined demand for media of exchange as an outcome of the market equilibrium. Media of exchange (commodity monies) are characterized as goods flows acting as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades at a point in time. They may also be stores of value, but these functions are distinct.

10 Stores of Value

At date η , $1 \leq \eta \leq T$, household h's holding of spot good $i(\eta)$ is $s_{i(\eta)}^{h,\eta}$ described above. In addition, h holds a portfolio of positive and negative futures market positions in $i(\tau)$, $\tau > \eta$.

$$s_{i(\tau)}^{h,\eta} \equiv \sum_{t=1}^{\eta} \sum_{j=1}^N \sum_{\theta=1}^T [y_i^{h\{i(\tau),j(\theta),t\}} + x_i^{h\{i(\tau),j(\theta),t\}}]$$

Household h's storage vector at η is the NT-dimensional vector $s^{h,\eta}$ whose $i(\tau)$ element is $s_{i(\tau)}^{h,\eta}$. Of course, for $\eta > \tau$, $s_{i(\tau)}^{h,\eta} \equiv 0$.

$\Upsilon^h(p, w) \equiv \{(s^{h,1}, s^{h,2}, \dots, s^{h,\eta}, \dots, s^{h,T}) \in R^{NT} | s^{h,\eta} \text{ is defined above for } (y^h, x^h) \in D^h(p, w)\}$. Υ^h is h's asset and liability plan. Focusing on h's net assets (ignoring his debts), let $\Upsilon^{h+}(p, w) \equiv \{[(s^{h,1}, s^{h,2}, \dots, s^{h,\eta}, \dots, s^{h,T})]_+ | s^h \in \Upsilon^h(p, w)\}$ where the notation $[\]_+$ indicates the positive elements of the vector, negative elements replaced by zero. Then $\Upsilon^{h+}(p, w)$ is the demand for 'money' as a stock, the stock demand for tradeable assets. It is tempting to define a firm's demand for money as well — but the definition in this setting is not obvious.

Household h's (NT - dimensional) change in asset position from date $\eta - 1$ to η is $s^{h,\eta} - s^{h,\eta-1}$. The change in spot goods (dated $\eta - 1$ to those dated η) reflects two elements, transactions in spot goods and the passage of time, zeroing out the $\eta - 1$ entry and retention of spot i to η at rate κ_i . For goods dated $\tau > \eta$ (subscript τ) however, the change in asset position reflects transactions undertaken at η . Thus reductions in h's $i(\tau)$ entry represent asset sales at η . Sales of net positive positions in $s_{i(\tau)}^{h,\eta-1}$ can be said to reflect holding of $i(\tau)$ as a store of value. Sales of negative positions (becoming increasingly negative) are increases in indebtedness. Purchases in negative positions (becoming less negative) are repayments of debt. Thus changes from $\eta - 1$ to η of $s_{i(\tau)}^{h,\eta}$ are asset purchases and sales of $i(\tau)$ acting as a store of value.

11 Walrasian Equilibrium, Trading Post Equilibrium, and Demand for Media of Exchange

11.1 Transaction Costs, Essential and Inessential Sequence Economies

The issues of general equilibrium with transaction cost, efficiency of allocation and the implications for the role of money appear in Foley (1970), Hahn (1971, 1973), and Starrett (1973). Foley (1970) considers a static equilibrium with (consistent with the Arrow-Debreu treatment) a single market meeting. In Foley (1970) all of the formal structure of the Arrow-Debreu economy is maintained while the transaction process is treated as a production activity. Each of N goods has a bid and ask (wholesale and retail) price with the resulting dimensionality of the price space at $2N$. As in Debreu (1959) the count N includes futures markets for all of the relevant goods. As in Debreu (1959) there are futures markets for all of the relevant goods. Foley (1970)'s distinctive powerful insight is that this structure is mathematically equivalent to the Arrow-Debreu model. Assuming the usual continuity and convexity assumptions, a competitive equilibrium exists in the convex transaction cost economy, and the resulting allocation is Pareto efficient. The notion of Pareto efficiency here needs to take account of transaction costs: moving ownership from one firm or household to another is a resource using activity. Efficiency consists of efficient allocation net of the necessary resource cost of reassigning ownership.

Hahn (1973) treats the reopening of markets over time in a sequence economy, distinguishing between *essential* and *inessential* sequence economies. The issue treated is whether two otherwise identical economies have significantly different equilibrium prices and resource allocation depending on the character of the budget constraint: a single Arrow-Debreu budget for each household versus a time-dated sequence of budget constraints in a sequence economy. In this comparison it is necessary to take account of transaction costs, so the reference point is not the conventional Arrow-Debreu equilibrium without transaction costs, Debreu (1959). Rather, it is the allocation in an Arrow-Debreu economy with transaction costs, Foley (1970).

This paper adopts the same usage. A sequence economy trading post equilibrium is 'inessential' if the resulting allocation is Walrasian, the same as in an Arrow-Debreu-Foley economy with transaction costs. The equilibrium is inessential if the multifaceted structure of the sequential trading post budget constraint has no effect in itself on the resulting allocation of resources. Conversely, the trading post equilibrium will be described as 'essential' if the equilibrium resource allocation is non-Walrasian,

differing from a Walrasian allocation because of the structure of budget constraints.

The applicable Pareto efficiency concept is efficiency subject to technically necessary transaction costs. A trading post economy will be said to be inessential when the multi-faceted structure of budget constraints has no effect on the equilibrium allocation of resources, compared to the single budget constraint of an Arrow-Debreu model. Then the resulting allocation is a Walrasian equilibrium allocation and it is Pareto efficient by the First Fundamental Theorem of Welfare Economics. Conversely, a trading post economy is essential when the multi - faceted structure of budget constraints renders the equilibrium allocation of resources different from an Arrow-Debreu-Foley equilibrium (taking full account of the effect of transaction costs, with a complete array of futures markets). Then the equilibrium allocation will not be a Walrasian equilibrium and may be Pareto inefficient. The inefficiency arises in either of two ways: additional resources may be expended in fulfillment the multiplicity of budget constraints, or the allocation may be shifted (relative to Walrasian equilibrium) to fulfill the additional constraints. Since these circumstances represent real resource allocations to fulfill a purely administrative constraint, the reallocation is regarded as Pareto inefficient. This treatment is similar to Hahn (1973)'s treatment of sequence economies. A full development of efficiency conditions and detailed characterization of (in)essentiality is a significant topic, beyond the scope of this paper.

The array of economies subject to general equilibrium modeling includes essential and inessential trading post economies with resultant Walrasian and non-Walrasian allocations. Since the designation 'essential' or 'inessential' is based on the character of endogenous equilibrium pricing, it seems problematic to distinguish essential from inessential trading post economies a priori.

11.2 Pareto Efficiency of Trading Post Equilibrium with Transaction Costless Media of Exchange

When there is a generally available zero-transaction cost medium of exchange, and a zero-transaction cost non-depreciating store of value, the trading post equilibrium will be inessential and the resulting allocation of resources Pareto efficient (taking into account transaction costs). The allocation will be a Walrasian equilibrium. Supposing that the transaction costs of media of exchange and stores of value in advanced monetary economies are low (if not nil), the zero-cost case should be a significant limiting case.

However important, the result is not new. The presence of a costless medium of exchange and store of value means that price ratios in a trading post economy will be the same as those of the corresponding Arrow-Debreu-Foley economy. The

point of comparison is an economy with transaction costs, complete markets, efficient allocation in general equilibrium, a single budget constraint for each household and well-defined profit maximand for each firm, as in Foley (1970). Then apply the First Fundamental Theorem of Welfare Economics.

12 Conclusion

The goal in this study is to create a parsimonious model where a medium of exchange and store of value (commodity monies) can be an outcome of the (augmented) formal Arrow-Debreu general equilibrium, not an additional assumption. The trades of firms and households in a trading post economy may be characterized by many separate transactions, each fulfilling a separate budget constraint. In an economy of N commodities at T periods there are $\frac{1}{2}N^2T^3$ trading posts, one for each transaction date and for each pair of dated goods. The trading post model reformulates the budget so that each of many separate transactions fulfills its own budget constraint. This treatment generates a demand for carriers of value (media of exchange) moving among trades, Starr (2003A, 2003B), and over time (stores of value). Virtually the same axiomatic structure, Arrow and Debreu (1954), that ensures the existence of general equilibrium in the model of a unified market without transaction costs yields existence of equilibrium and a well-defined demand for media of exchange and stores of value in this disaggregated setting.

Trading post equilibria are Pareto efficient when they are simply the elaboration of an underlying Walrasian equilibrium, an inessential trading post economy; see also Hahn (1973). However, the multiplicity of separate budget constraints and the additional transaction costs incurred or avoided may skew the allocation and pricing (an essential trading post equilibrium). Then the equilibrium cannot be supported by a Walrasian price structure and the allocation will be Pareto inefficient; see also Starrett (1973).

The price system is informative not only about scarcity, desirability, and productivity. It also prices liquidity. Transaction costs generate a spread between bid and ask prices at each trading post. The bid-ask spread tells firms and households which goods are liquid, easily traded without significant loss of value, and which are illiquid, unsuitable as carriers of value between trades. The multiplicity of budget constraints creates the demand for liquidity; the bid-ask spreads signal its supply. The trading post sequence economy model endogenously generates a designation and flow of commodity money(ies) at a point and over time and a stock demand for stores of value.

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