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Total positivity for Grassmannians and amplituhedra



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# Total positivity for Grassmannians and amplituhedra

by

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## Abstract


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*Total positivity* is the mathematical study of spaces and their *positive parts*, which can have interesting combinatorial properties as well as applications in areas such as analysis, representation theory, and theoretical physics. In this dissertation, I study total positivity in the *Grassmannian* $\mathrm{Gr}_{k,n}$, which is the space of $k$-dimensional subspaces of $\mathbb{R}^n$. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of $\mathrm{Gr}_{k,n}$ where all Plücker coordinates are nonnegative. In Chapter 2, I generalize a result of Gantmakher and Krein, who showed that $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative if and only if every vector in $V$, when viewed as a sequence of $n$ numbers and ignoring any zeros, changes sign at most $k-1$ times. I characterize when the vectors in $V$ change sign at most $k-1+m$ times for any $m \geq 0$, in terms of the Plücker coordinates of $V$. I then apply this result to solve the problem of determining when *Grassmann polytopes*, generalizations of polytopes into the Grassmannian studied by Lam, are well defined. In Chapter 3, which is joint work with Lauren Williams, we study the *(tree) amplituhedron* $\mathcal{A}_{n,k,m}$, the image in $\mathrm{Gr}_{k,k+m}$ of $\mathrm{Gr}_{k,n}^{\geq 0}$ under a (map induced by a) linear map which is totally positive. It was introduced by Arkani-Hamed and Trnka in 2013 in order to give a geometric basis for computing scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We take an orthogonal point of view and define a related "B-amplituhedron" $\mathcal{B}_{n,k,m}$, which we show is isomorphic to $\mathcal{A}_{n,k,m}$, and use the results of Chapter 2 to describe the amplituhedron in terms of sign variation. Then we use this reformulation to give a cell decomposition of the amplituhedron in the case $m = 1$, using the images of a collection of distinguished cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. We also identify $\mathcal{A}_{n,k,1}$ with the complex of bounded faces of a *cyclic hyperplane arrangement*, and deduce that $\mathcal{A}_{n,k,1}$ is homeomorphic to a ball. In Chapter 4, I study the action of the cyclic group of order $n$ on $\mathrm{Gr}_{k,n}^{\geq 0}$. I show that the cyclic action has a unique fixed point, given by taking $n$ equally spaced points on the *trigonometric moment curve* (if $k$ is odd) or the *symmetric moment curve* (if $k$ is even). More generally, I show that the cyclic action on the entire *complex* Grassmannian has exactly $\binom{n}{k}$ fixed points, corresponding to $k$-subsets of $n$th roots of $(-1)^{k-1}$. I explain how these fixed points also appear in the study of the quantum cohomology ring of the Grassmannian.

# Contents

## Acknowledgments

I thank Lauren Williams, my advisor, who taught me how to be a mathematician. I also thank my family, friends, and colleagues for their guidance and support. I acknowledge financial support from an NSERC postgraduate fellowship, a Chateaubriand fellowship, the University of California, Berkeley, and the NSF grants DMS-1049513 and DMS-1600447.

# Chapter 1

# Introduction

*Total positivity* is the study of spaces and their 'positive' parts. A classical example is the case of matrices: a real matrix is *totally positive* if each of its square submatrices has positive determinant.[1] For example, the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ is totally positive, but $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ is not. There are two seminal results about totally positive matrices, proved in the 1930s. In 1930, Schoenberg [Sch30] showed that totally positive matrices *diminish variation*. That is, given $v \in \mathbb{R}^n$, let $\operatorname{var}(v)$ be the number of sign changes of $v$, viewed as a sequence of $n$ numbers and ignoring any zero components. (For example, if $v = (3, 0, 2, -1) \in \mathbb{R}^4$, we have $\operatorname{var}(v) = 1$.) Then $\operatorname{var}(Mx) \le \operatorname{var}(x)$ for any $n \times k$ totally positive matrix $M$ and $x \in \mathbb{R}^k$. In 1937, Gantmakher and Krein [GK37] showed that the eigenvalues of a square totally positive matrix are all real, positive, and distinct, and also showed that its eigenvectors satisfy certain positivity properties. This is a remarkable result, since it gives a large class of *non-symmetric* matrices closed under perturbation which are guaranteed to have real eigenvalues. While we will focus on the discrete aspects of total positivity in this dissertation, we mention that these results are discrete analogues of continuous results which had been proved by Kellogg in the 1910s [Kel16, Kel18] and Gantmakher in 1936 [Gan36] for *totally positive kernels*, i.e. continuous functions $[0, 1]^2 \to \mathbb{R}$ on the unit square satisfying an analogous positivity condition.[2]

Building on the aforementioned work, a general theory of total positivity developed over the subsequent decades. Two important works in the subject are the textbook of Gantmakher and Krein [GK50], who were especially interested in applications of totally positive matrices and kernels to the study of oscillations of mechanical systems,[3] and the textbook of Karlin

[1] The modifier 'totally' is used to distinguish totally positive matrices from *positive* matrices, i.e. matrices whose entries are all positive.

[2] Kellogg studied *symmetric* kernels, and Gantmakher generalized his results to the non-symmetric case. Kellogg arrived at his work starting from the study of *orthogonal functions*. These functions were known to possess positivity properties in many well-known examples, and he decided to make a systematic study of such positivity properties.

[3] Their book garnered sufficient interest that the U.S. Atomic Energy Commission released an English translation of the Russian original in 1961 [GK61].

[Kar68], who was interested in applications in analysis, including integral equations and interpolation of functions. We refer to the survey paper of Pinkus [Pin10] for an exposition of the early history of total positivity.

The theory of total positivity was rejuvenated in the 1990s by Lusztig. As he says in his seminal paper [Lus94], "I was introduced to the classical totally positive theory for $\mathrm{GL}_n$ by Bert Kostant, who pointed out to me its beauty and asked me about the possible connection with the positivity properties of the canonical bases." He generalized the theory of total positivity to any split, reductive, connected, real algebraic group $G$, by defining a totally positive part $G^{>0}$ and generalizing some classical results for the case $G = \mathrm{SL}_n(\mathbb{R})$ (totally positive matrices) to arbitrary $G$. For example, the fact that every totally positive matrix has distinct and positive eigenvalues may be phrased more generally as follows: every $g \in G^{>0}$ is contained in a unique orbit of $T \cap G^{>0}$ under the action of $G$ by conjugation, where $T \subseteq G$ is a maximal torus.[4] He also defined the totally positive part of any partial flag variety $G/P$ [Lus98]. Fomin and Zelevinsky, interested in finding a more explicit description of $G^{>0}$ (which Lusztig had defined in terms of certain distinguished generators), introduced *generalized minors*, and showed that these could be used to define $G^{>0}$ [FZ99, FZ00]. They observed that these minors obeyed certain three-term relations, which they abstracted to define *cluster algebras* [FZ02]. Cluster algebras have since been widely studied, and been applied in diverse areas such as representation theory, Teichmüller theory, and Poisson geometry.[5]

An important family of examples both for Lusztig's total positivity for flag varieties $G/P$, and also for cluster algebras, is provided by *Grassmannians*. The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$. Its positive part is defined in terms of certain projective coordinates $\Delta_I$ on $\mathrm{Gr}_{k,n}$, called *Plücker coordinates*. Here $I$ ranges over $\binom{[n]}{k}$, all $k$-element subsets of $[n] := \{1, 2, \dots, n\}$. Explicitly, given $V \in \mathrm{Gr}_{k,n}$, take a $k \times n$ matrix $A$ whose rows span $V$; then the $\Delta_I(V)$ are the $k \times k$ minors of the matrix $A$, where $I$ denotes the column set used. (The $\Delta_I(V)$ depend on our choice of $A$ only up to a global constant, and so indeed give well defined projective coordinates.) If all nonzero $\Delta_I(V)$ have the same sign, then $V$ is called *totally nonnegative*, and if in addition no $\Delta_I(V)$ equals zero, then $V$ is called *totally positive*. For example, if $V \in \mathrm{Gr}_{2,4}$ is the row span of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & 1 & 3, \end{bmatrix},$$

then its Plücker coordinates are

$$\Delta_{\{1,2\}}(V) = 2,\ \Delta_{\{1,3\}}(V) = 1,\ \Delta_{\{1,4\}}(V) = 2,\ \Delta_{\{2,3\}} = 0,\ \Delta_{\{2,4\}}(V) = 2,\ \Delta_{\{3,4\}}(V) = 1,$$

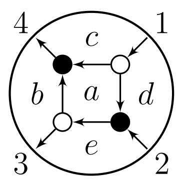
so $V$ is totally nonnegative but not totally positive. We denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$ by $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$, respectively.

Total positivity in the Grassmannian is a special case of Lusztig's theory of total positivity for partial flag varieties $G/P$, where we take $G = \mathrm{SL}_n$ and $P$ a maximal parabolic

[4]We refer to Lusztig's paper [Lus08] for a survey of his results.

[5]We refer to Williams's survey paper [Wil14] for an introduction to cluster algebras.

subgroup.[6] Lusztig defined a decomposition of the totally nonnegative part of such a $G/P$, which Rietsch proved is a cell decomposition [Rie98]. In the case of $\mathrm{Gr}_{k,n}^{\geq 0}$, each cell is given by requiring some subset of the Plücker coordinates to be strictly positive, and the rest to equal zero. There is a unique totally positive cell, namely $\mathrm{Gr}_{k,n}^{>0}$, where we require all Plücker coordinates to be positive. Later, Postnikov independently studied the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ from a combinatorial perspective, giving parametrizations of each cell in its cell decomposition using certain planar graphs drawn inside a disk, called *plabic graphs*[7]. Besides these plabic graphs, he gave many other sets of objects (all in bijection with each




$$\longmapsto \quad \begin{bmatrix} 1 & 0 & -abc & -(c+ac) \\ 0 & 1 & 1/e & acd \end{bmatrix}$$

$a, b, c, d, e > 0, abcde = 1$

Figure 1.1: A planar directed graph corresponding to the cell $\mathrm{Gr}_{2,4}^{>0}$ of the totally nonnegative Grassmannian $\mathrm{Gr}_{2,4}^{\geq 0}$.

other) which label these cells, including *decorated permutations*, *Ꞁ-diagrams*, and *Grassmann necklaces*.

In addition to being studied in the context of cluster algebras, the totally nonnegative Grassmannian and its cell decomposition has been applied to study mirror symmetry [MR], the KP equation [KW14], and particle physics [AHBC$^+$16]. However, in these exciting modern developments, the historical connection between total positivity and sign variation had been largely overlooked. While we believe this connection is interesting in its own right, the results in this dissertation indicate it also provides a useful tool in studying total positivity for Grassmannians and especially *amplituhedra*, objects which were recently introduced in particle physics. (We will describe these results in more detail below.)

The starting point for much of the work in this dissertation is the following result of Gantmakher and Krein from 1950. Recall that for $v \in \mathbb{R}^n$, $\mathrm{var}(v)$ be the number of sign changes of $v$ (viewed as a sequence of $n$ numbers, ignoring any zeros). We also define

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } 1 \leq i \leq n \text{ with } v_i \neq 0\},$$

i.e. $\overline{\mathrm{var}}(v)$ is the maximum number of sign changes of $v$ over all possible choices of sign for its zero components.

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[6]In fact it is not obvious that Lusztig's definition is the same as the one we gave; this was proved by Rietsch [Rie].

[7]*Plabic* stands for *planar bicolored* (in a plabic graph, the vertices in the interior of the disk are each colored either black or white).

**Theorem 1.0.1** (Theorems V.3 and V.1 of [GK50]).
*(i)* $V \in \mathrm{Gr}_{k,n}$ *is totally nonnegative if and only if* $\mathrm{var}(v) \le k-1$ *for all* $v \in V$.
*(ii)* $V \in \mathrm{Gr}_{k,n}$ *is totally positive if and only if* $\overline{\mathrm{var}}(v) \le k-1$ *for all* $v \in V \setminus \{0\}$.

For example, the two vectors $(1,0,0,-1)$ and $(-1,2,1,3)$ each change sign exactly once, and we can check that any vector in their span $V$ changes sign at most once, which is equivalent to $V$ being totally nonnegative. On the other hand, $\overline{\mathrm{var}}(1,0,0,-1) = 3$, so $V$ is not totally positive. Every element of $\mathrm{Gr}_{k,n}$ has a vector which changes sign at least $k-1$ times (put a $k \times n$ matrix whose rows span $V$ into reduced row echelon form, and take the alternating sum of the rows), so the totally nonnegative elements are those whose vectors change sign as few times as possible.

Theorem 1.0.1(i) also has the following geometric interpretation, due to Schoenberg and Whitney [SW51] (who also independently proved part (i)). Let $x_1, \dots, x_n \in \mathbb{R}^k$ be vectors which span $\mathbb{R}^k$, and $A$ be the $k \times n$ matrix with columns $x_1, \dots, x_n$. Then the polygonal path with vertices $x_1, \dots, x_n$ crosses any hyperplane passing through the origin at most $k-1$ times if and only if all nonzero $k \times k$ minors of $A$ have the same sign. Indeed, the number of times the path crosses the hyperplane through the origin normal to $c \in \mathbb{R}^k \setminus \{0\}$ equals $\mathrm{var}(v)$, where $v_i = \langle x_i, c \rangle$ for $i = 1, \dots, n$.

In Chapter 2, we generalize Theorem 1.0.1 from the totally nonnegative part of the Grassmannian to the entire Grassmannian, by characterizing $\max_{v \in V} \mathrm{var}(v)$ and $\max_{v \in V \setminus \{0\}} \overline{\mathrm{var}}(v)$ in terms of the Plücker coordinates of $V \in \mathrm{Gr}_{k,n}$, beyond the case that the maximum is $k-1$. Note that we may interpret $\max_{v \in V} \mathrm{var}(v)$ as the maximum number of hyperplane crossings of an associated polygonal path. First we state the result for $\max_{v \in V \setminus \{0\}} \overline{\mathrm{var}}(v)$.

**Theorem 1.0.2.** *Suppose that* $V \in \mathrm{Gr}_{k,n}$, *and* $m \ge 0$. *We have* $\overline{\mathrm{var}}(v) \le k-1+m$ *for all* $v \in V \setminus \{0\}$ *if and only if* $\overline{\mathrm{var}}((\Delta_{I \cup \{i\}}(V))_{i \in [n] \setminus I}) \le m$ *for all* $I \in \binom{[n]}{k-1}$ *such that* $\Delta_{I \cup \{i\}}(V) \ne 0$ *for some* $i \in [n]$.

(See Theorem 2.3.1.) If we take $m = 0$, then we recover Theorem 3.3.4(ii). For example, let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 2 & 1 & 4 \end{bmatrix}$, so $k = 2$. Then the fact that $\overline{\mathrm{var}}(v) \le 2$ for all $v \in V \setminus \{0\}$ is equivalent to the fact that

$$\begin{aligned} &(\Delta_{\{1,2\}}(V), \Delta_{\{1,3\}}(V), \Delta_{\{1,4\}}(V)) = (2,1,4), \\ &(\Delta_{\{1,2\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{2,4\}}(V)) = (2,4,-6), \\ &(\Delta_{\{1,3\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{3,4\}}(V)) = (1,4,-11), \\ &(\Delta_{\{1,4\}}(V), \Delta_{\{2,4\}}(V), \Delta_{\{3,4\}}(V)) = (4,-6,-11) \end{aligned}$$

each change sign at most once.

The case of $\max_{v \in V} \mathrm{var}(v)$ is more interesting, since the analogue of the necessary and sufficient condition of Theorem 1.0.2 does not hold for $\mathrm{var}(\cdot)$. We say that $V \in \mathrm{Gr}_{k,n}$ is *generic* if all Plücker coordinates of $V$ are nonzero.

**Theorem 1.0.3.** *Let $V \in \mathrm{Gr}_{k,n}$ and $m \ge 0$.*
*(i) If $\mathrm{var}(v) \le k-1+m$ for all $v \in V$, then $\mathrm{var}((\Delta_{I\cup\{i\}}(V))_{i\in[n]\setminus I}) \le m$ for all $I \in \binom{[n]}{k-1}$.*
*(ii) We can perturb $V$ into a generic $V' \in \mathrm{Gr}_{k,n}$ such that $\max_{v\in V}\mathrm{var}(v) = \max_{v\in V'}\mathrm{var}(v)$. In particular, $\mathrm{var}(v) \le m$ for all $v \in V$ if and only if $\mathrm{var}((\Delta_{I\cup\{i\}}(V'))_{i\in[n]\setminus I}) \le m-k+1$ for all $I \in \binom{[n]}{k-1}$.*

(See Section 2.3.) Another way of stating part (ii) is that for any $m \ge 0$, the generic elements in $\{V \in \mathrm{Gr}_{k,n} : \mathrm{var}(v) \le k-1+m \text{ for all } v \in V\}$ are dense. If $m = 0$, this recovers the result of Postnikov (Section 17 of [Pos]) that $\mathrm{Gr}_{k,n}^{>0}$ is dense in $\mathrm{Gr}_{k,n}^{\ge 0}$.

In Section 2.5, we use sign variation to study the cell decomposition of $\mathrm{Gr}_{k,n}^{\ge 0}$. Recall that the cell of $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ is determined by which Plücker coordinates of $V$ are nonzero. How can we determine which Plücker coordinates of $V$ are nonzero from the sign patterns of its vectors? Given $I \subseteq [n]$ and a sign vector $\omega \in \{+,-\}^I$, we say that $V$ *realizes* $\omega$ if there exists a vector in $V$ whose restriction to $I$ has signs given by $\omega$. It is not difficult to show that for any $V \in \mathrm{Gr}_{k,n}$ and $I \in \binom{[n]}{k}$, we have $\Delta_I(V) \ne 0$ if and only if $V$ realizes all $2^k$ sign vectors in $\{+,-\}^I$. Conversely, for any $\omega \in \{+,-\}^I$, if $n > k$ there exists $V \in \mathrm{Gr}_{k,n}$ such that $\Delta_I(V) = 0$ but $V$ realizes all $2^k$ sign vectors in $\{+,-\}^I$ except for $\pm\omega$. However, in the case that $V$ is totally nonnegative, we show that we only need to check $k$ particular sign vectors in $\{+,-\}^I$ to verify that $\Delta_I(V) \ne 0$.

**Theorem 1.0.4.** *For $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ and $I \in \binom{[n]}{k}$, we have $\Delta_I(V) \ne 0$ if and only if $V$ realizes the $2k$ (or $k$ up to sign) sign vectors in $\{+,-\}^I$ which have at least $k-2$ sign changes.*

As a corollary, we describe the *Grassmann necklace* of $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ in terms of sign patterns of its vectors.

In the next part of the dissertation, we study *amplituhedra*, objects recently appearing in particle physics which we now define. Let $m \ge 0$ satisfy $k+m \le n$, and $Z$ be a $(k+m)\times n$ matrix whose $(k+m)\times(k+m)$ minors are all positive. We can regard $Z$ as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^{k+m}$. This induces a map $\tilde{Z}$ on $\mathrm{Gr}_{k,n}$, which takes the subspace $V$ of $\mathbb{R}^n$ to the subspace $\{Zv : v \in V\}$ of $\mathbb{R}^{k+m}$. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\ge 0})$ in $\mathrm{Gr}_{k,k+m}$. For special values of the parameters, we recover familiar objects. If $k+m = n$, then the amplituhedron is isomorphic to the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\ge 0}$. If $k = 1$, then $\mathrm{Gr}_{1,n}^{\ge 0}$ is an $(n-1)$-simplex, and so the amplituhedron $\mathcal{A}_{n,1,m}(Z)$, being the linear projection of a simplex, is a polytope in $\mathbb{P}^m$. The positivity condition on $Z$ implies that $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* [Stu88], i.e. it is combinatorially equivalent to a polytope whose vertices lie on the *moment curve* $(1 : t : t^2 : \cdots : t^m)$. Cyclic polytopes were defined Carathéodory [Car11], and have remarkable properties. For example, they have the maximum number of faces of each dimension [McM70, Sta75].

Amplituhedra were introduced in 2013 by the physicists Arkani-Hamed and Trnka in their study of *scattering amplitudes* in particle physics [AHT14]. A scattering amplitude is a complex number whose norm squared equals the probability of observing a certain scattering process. In 2012, a collaboration of physicists and mathematicians made an astonishing

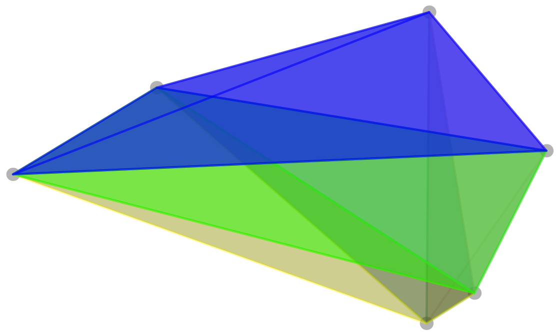

Figure 1.2: The amplituhedron $\mathcal{A}_{6,1,3}(Z)$ is a cyclic polytope with 6 vertices in $\mathbb{R}^3$.

connection between the study of scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and the totally nonnegative Grassmannian [AHBC$^+$16]. Namely, they showed that one could use the *BCFW recursion* [BCF05, BCFW05] to express the leading order term of the scattering amplitude as the sum of integrals over certain $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. (Here $n$ is the number of particles, and $k$ records the helicity of the particles.) Building on this work, Arkani-Hamed and Trnka [AHT14] asserted that one can assemble these $4k$-dimensional cells into a single object, which they call an *amplituhedron*. (In this application $m$ equals 4, though amplituhedra are interesting mathematical objects for any $m$.) They verified this statement by computer using extensive sampling, but it has not been mathematically proven. It is an important open problem in this area, and techniques employed to solve it will likely yield more insight into amplituhedra and scattering amplitudes, and hopefully total positivity more generally.

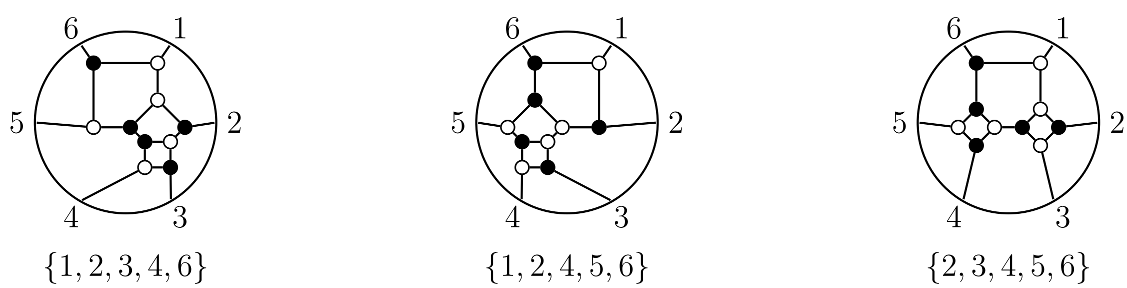


Figure 1.3: In the case $k = 1$, $n = 6$, the BCFW recursion outputs 3 *plabic graphs*, which correspond to cells of $\mathrm{Gr}_{1,6}^{\geq 0}$. Their images under $\tilde{Z}$ triangulate the cyclic polytope $\mathcal{A}_{6,1,4}(Z)$ by 3 simplices. The vertex labels of the simplices are given below each graph.

**Conjecture 1.0.5** (Arkani-Hamed, Trnka [AHT14])**.** *The amplituhedron $\mathcal{A}_{n,k,4}(Z)$ is "triangulated" by the images under $\tilde{Z}$ of the $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ coming from the BCFW recursion.*

We will have more to say about this problem, but first we return to the definition of the amplituhedron: why is it that it is well defined? That is, given $V \in \mathrm{Gr}_{k,n}$, why does the subspace $\tilde{Z}(V)$ of $\mathbb{R}^{k+m}$ have dimension $k$? This does not hold for all $(k+m) \times n$ matrices $Z$, but it holds if $Z$ has positive maximal minors (which is required in constructing the amplituhedron $\mathcal{A}_{n,k,1}(Z)$). Lam [Lam16b] proposed studying the images $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ beyond the case that $Z$ has positive maximal minors. In the case that $\tilde{Z}$ is well defined, he calls such an image a *(full) Grassmann polytope*. When $k = 1$ Grassmann polytopes are precisely polytopes in $\mathbb{P}^m$, so Grassmann polytopes generalize polytopes into the Grassmannian (while amplituhedra generalize cyclic polytopes). Lam showed that $\tilde{Z}$ is well defined if the row span of $Z$ in $\mathbb{R}^n$ contains a totally positive $k$-dimensional subspace. We use sign variation to give a necessary and sufficient condition for the map $\tilde{Z}$ to be well defined, and use Theorem 1.0.2 to translate it into a condition on the maximal minors of $Z$ (see Section 2.4).

**Theorem 1.0.6.** *Let $Z$ be a $(k+m)\times n$ matrix ($k+m \leq n$), which we also regard as a linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$, and let $W \in \mathrm{Gr}_{d,n}$ be the row span of $Z$. The following are equivalent:*
*(i) the map $\tilde{Z}$ is well defined, i.e. $\dim(\tilde{Z}(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$;*
*(ii) $\mathrm{var}(v) \geq k$ for all nonzero $v \in \ker(Z)$; and*
*(iii) $\overline{\mathrm{var}}((\Delta_{I\setminus\{i\}}(W))_{i\in I}) \leq d-k$ for all $I \in \binom{[n]}{d+1}$ such that $W|_I$ has dimension $d$.*

When I published Theorem 1.0.6, I left it as an open problem to determine whether Lam's condition is not only sufficient for $\tilde{Z}$ to be well defined, but also necessary. In March 2017, Pavel Galashin informed me that he found an example of a matrix $Z$ which does not satisfy Lam's condition, but still gives a well-defined Grassmann polytope.[8]

In Chapter 3, which is joint work with Lauren Williams, we study the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ in the case $m = 1$. Recall that when $k = 1$, the amplituhedron is a cyclic polytope in $\mathbb{P}^m$. We show that when $m = 1$, the amplituhedron can be identified with the complex of bounded faces of a certain hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$, called a *cyclic hyperplane arrangement*. As a first step in our construction, we take orthogonal complements and define (for any $m$) a related "B-amplituhedron"

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

which we show is homeomorphic to $\mathcal{A}_{n,k,m}(Z)$, where $W \in \mathrm{Gr}_{k+m,n}^{>0}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $Z$ (see Section 3.3). In the context of scattering amplitudes ($m = 4$), $W$ is the span of 4 bosonic variables and $k$ fermionic variables. Using the result of Gantmakher and Krein (Theorem 1.0.1), we can show that

$$\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \mathrm{Gr}_m(W) : k \leq \overline{\mathrm{var}}(v) \leq k+m-1 \text{ for all } v \in X \setminus \{0\}\}. \tag{1.0.7}$$

It is an important problem to determine if equality holds, since it would give an intrinsic description of the amplituhedron which does not mention $\mathrm{Gr}_{k,n}^{\geq 0}$. If equality does hold, then

[8] Lam has proposed calling the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ a *tame* Grassmann polytope if $Z$ satisfies his condition, and a *wild* Grassmann polytope otherwise.

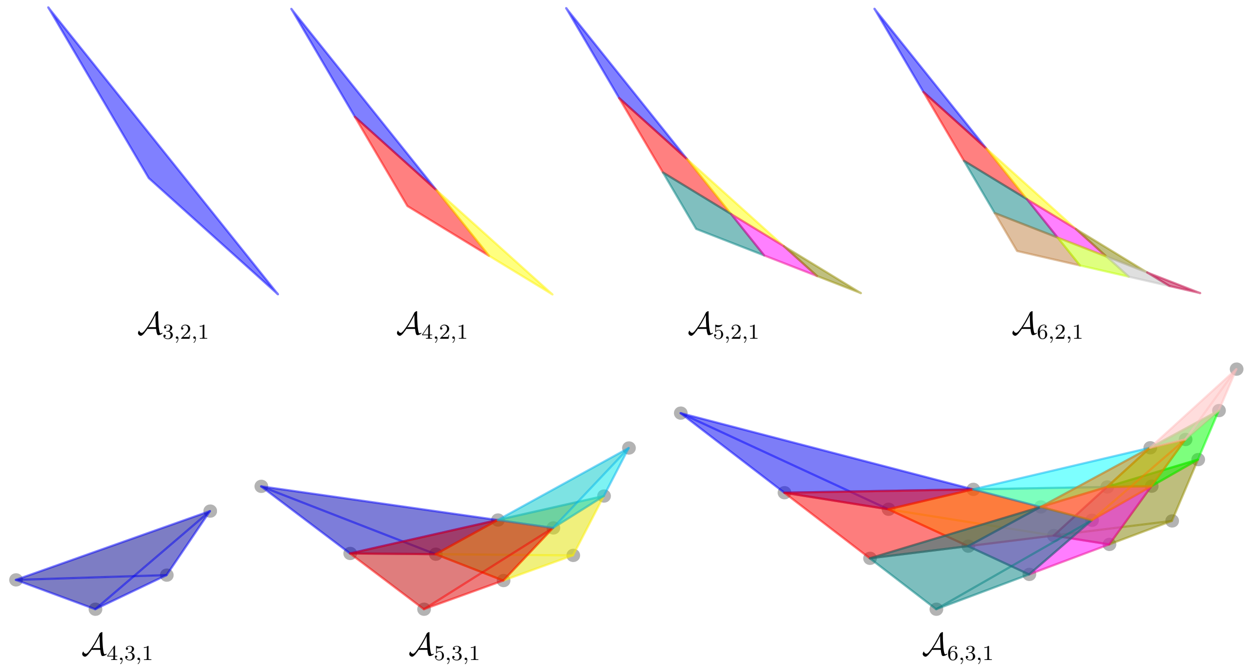


Figure 1.4: The amplituhedron $\mathcal{A}_{n,k,1}(Z)$ as the complex of bounded faces of a cyclic hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$, for $k = 2, 3$ and $n \leq 6$.

we can apply the results of Chapter 2 to give an alternative description in terms of sign changes of Plücker coordinates (see Corollary 3.3.22). This description is similar to one independently conjectured by Arkani-Hamed, Thomas, and Trnka [AHTT].

In the case $m = 1$, it follows from Lemma 2.4.1 that equality holds in (1.0.7), i.e.

$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

Modeling the $m = 4$ case, we define a BCFW-like recursion in the case $m = 1$, which we use to produce a subset of $k$-dimensional "BCFW cells" of $\mathrm{Gr}_{k,n}^{\geq 0}$, whose images we show triangulate the $m = 1$ amplituhedron.

**Theorem 1.0.8.** *Let $Z$ be a $(k+1) \times n$ matrix with positive $(k+1) \times (k+1)$ minors, and $W \in \mathrm{Gr}_{k+1,n}^{>0}$ the row span of $Z$.*
*(i) The amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to the subcomplex of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ induced by the $k$-dimensional $m = 1$ BCFW cells.*
*(ii) The amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to the bounded complex of a cyclic hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$.*

We describe when two cells of $\mathcal{A}_{n,k,1}(Z)$ are adjacent, and which cells lie on the boundary. We also determine when an arbitrary cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ is mapped injectively by $\tilde{Z}$ to the amplituhedron $\mathcal{A}_{n,k,1}(Z)$, and if so, describe its image. We then show that our construction in part (ii) extends to tame Grassmann polytopes in the case $m = 1$: from $Z$, we can construct a hyperplane arrangement whose bounded complex is homeomorphic to $\tilde{Z}(\mathrm{Gr}_{k,n}^{\ge 0})$.

It is known that the totally nonnegative Grassmannian has a remarkably simple topology: it is contractible with boundary a sphere [RW10], and its poset of cells is Eulerian [Wil07]. While there are not yet any general results in this direction beyond the case $k = 1$, calculations of Euler characteristics [FGMT15] indicate that the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is likely also topologically very nice. Theorem 1.0.8(ii), together with a result of Dong [Don08], implies that the $m = 1$ amplituhedron is homeomorphic to a closed ball.

In Chapter 4, the final part of this dissertation, we study the *cyclic symmetry* of the Grassmannian and its totally nonnegative part. For this part, it makes more sense for us to work with the *complex Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{C})$ of $k$-dimensional subspaces of $\mathbb{C}^n$. We regard the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\ge 0}$ as the subset of $\mathrm{Gr}_{k,n}(\mathbb{C})$ where all Plücker coordinates are real and nonnegative.

For each $k$ and $n$, we define an action of the cyclic group of order $n$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$, as follows. We let $\sigma \in \mathrm{GL}_n(\mathbb{C})$ be given by

$$\sigma(v) := (v_2, v_3, \dots, v_n, (-1)^{k-1}v_1) \quad \text{for } v = (v_1, \dots, v_n) \in \mathbb{C}^n,$$

and for $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, we denote by $\sigma(V)$ the subspace $\{\sigma(v) : v \in V\} \in \mathrm{Gr}_{k,n}(\mathbb{C})$. In terms of Plücker coordinates, $\sigma$ acts on $\mathrm{Gr}_{k,n}(\mathbb{C})$ by rotating the index set $[n]$. Hence $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ is an automorphism of order $n$, which restricts to an automorphism of $\mathrm{Gr}_{k,n}^{\ge 0}$. This provides a "cyclic symmetry" which is manifest in much of the combinatorics unique to total positivity in the Grassmannian (as opposed to e.g. total positivity for matrices). For example, the cyclic action acts on plabic graphs (see Figure 1.3), which index cells of $\mathrm{Gr}_{k,n}^{\ge 0}$, by shifting the labels of the boundary vertices. Also, the poset of cells of $\mathrm{Gr}_{k,n}^{\ge 0}$ is described in terms of decorated permutations by an analogue of Bruhat order called *circular Bruhat order* (see Section 17 of [Pos]). Our main result in Chapter 4 is the following.

**Theorem 1.0.9.** *The cyclic shift map $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ has exactly $\binom{n}{k}$ fixed points, each of the form* $\mathrm{span}\{(1, z_j, \dots, z_j^{n-1}) : 1 \le j \le k\}$ *for some $k$ distinct $n$th roots $z_1, \dots, z_k \in \mathbb{C}$ of $(-1)^{k-1}$. Precisely one of these fixed points is totally nonnegative, corresponding to the $k$ roots $z_1, \dots, z_k$ closest to* 1 *on the unit circle.*

We remark that the proof of the uniqueness of the totally nonnegative fixed point uses Gantmakher and Krein's result (Theorem 3.3.4(i)).

There is another particularly nice description of this unique totally nonnegative fixed point $V_{k,n} \in \mathrm{Gr}_{k,n}^{\ge 0}$. Define $f_k : \mathbb{R} \to \mathbb{R}^k$ by

$$f_k(\theta) := \begin{cases} \left(1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \dots, \cos\left(\frac{k-1}{2}\theta\right), \sin\left(\frac{k-1}{2}\theta\right)\right), & \text{if } k \text{ is odd} \\ \left(\cos\left(\frac{1}{2}\theta\right), \sin\left(\frac{1}{2}\theta\right), \cos\left(\frac{3}{2}\theta\right), \sin\left(\frac{3}{2}\theta\right), \dots, \cos\left(\frac{k-1}{2}\theta\right), \sin\left(\frac{k-1}{2}\theta\right)\right), & \text{if } k \text{ is even} \end{cases}.$$

Note that $f_k(\theta + 2\pi) = (-1)^{k-1} f_k(\theta)$. For odd $k$, the curve in $\mathbb{R}^{k-1}$ formed from $f_k$ by deleting the first component is the *trigonometric moment curve*, and for even $k$, the curve

$f_k$ is the *symmetric moment curve*. For example, $f_2$ is the unit circle in $\mathbb{R}^2$. These curves have a rich history, which we discuss in Remark 4.1.4. The fixed point $V_{k,n}$ is represented by any $k \times n$ matrix whose columns are $f_k(\theta_1), \dots, f_k(\theta_n)$, such that the points $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ are equally spaced on the real line, i.e. $\theta_{j+1} - \theta_j = \frac{2\pi}{n}$ for $1 \le j \le n-1$.

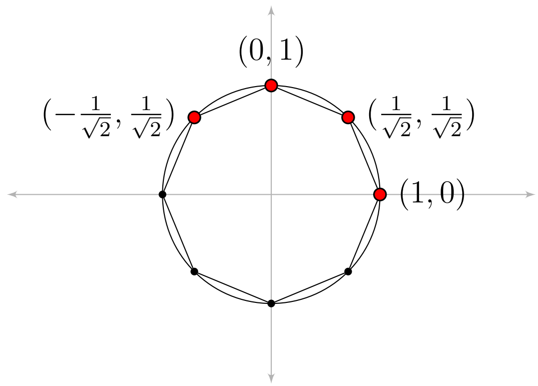


Figure 1.5: The unique totally nonnegative cyclically symmetric element of $\mathrm{Gr}_{2,4}^{\ge 0}$ is represented by the matrix $\begin{bmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix}$, which comes from taking four consecutive points on the regular unit octagon.

We also have the following explicit formula for the Plücker coordinates of the totally nonnegative fixed point $V_{k,n}$:

$$\Delta_I(V_{k,n}) = \prod_{1 \le r < s \le k} \sin\left(\tfrac{i_s - i_r}{n}\pi\right) \qquad \text{for all } k\text{-subsets } I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}.$$

Since $\sin(\theta) > 0$ for $0 < \theta < \pi$, this directly implies that $V_{k,n}$ is totally positive.

The other fixed points of $\sigma$ are also interesting. Remarkably, they arise in quantum cohomology. The *quantum cohomology ring* of $\mathrm{Gr}_{k,n}(\mathbb{C})$ is a deformation of the cohomology ring by an indeterminate $q$. In unpublished work, Peterson discovered that this ring is isomorphic to the coordinate ring of a certain subvariety $\mathcal{Y}_{k,n}$ of $\mathrm{GL}_n(\mathbb{C})$. This was proved by Rietsch [Rie01]. Under her isomorphism, the indeterminate $q$ corresponds to a map $\mathcal{Y}_{k,n} \to \mathbb{C}$, and the specialization at $q = 1$ of the quantum cohomology ring corresponds to the ring of $\mathbb{C}$-valued functions on the fiber in $\mathcal{Y}_{k,n}$ over $q = 1$. This fiber has size $\binom{n}{k}$, and it turns out that there is a natural embedding of $\mathcal{Y}_{k,n}$ into the (affine cone over) $\mathrm{Gr}_{k,n}(\mathbb{C})$ which identifies the fiber with the fixed points of $\sigma$. Moreover, we can rewrite a formula of Bertram [Ber97] for *Gromov-Witten invariants* (generalized intersection numbers) of Schubert varieties in terms of the Plücker coordinates of the fixed points of $\sigma$. This makes manifest the so-called 'hidden symmetry' of these Gromov-Witten invariants, i.e. they are invariant (up to an appropriate change of degree) under the cyclic action of the ground set $[n]$. This hidden symmetry first appeared in the work of Seidel [Sei97], and was further studied by Agnihotri and Woodward (see Section 7 of [AW98]) and Postnikov (see Section 6.2 of [Pos05])[9].

Finally, we use the ideas behind proving Theorem 1.0.9 to construct many fixed points of the *twist map* on the Grassmannian. This is an automorphism of $\mathrm{Gr}_{k,n}(\mathbb{C})$ which appears in the study of the cluster-algebraic structure of the Grassmannian [MS16, MS], by relating the $\mathcal{A}$-*cluster structure* and the $\mathcal{X}$-*cluster structure* of $\mathrm{Gr}_{k,n}(\mathbb{C})$. Marsh and Scott [MS16] also showed that the twist map can be implemented by a *maximal green sequence*, a special sequence of mutations in a cluster algebra which has been studied because of its importance in the study of quiver representations. The element $V_{k,n}$ is one of the fixed points of the twist map which we identify, and the unique totally nonnegative one. It is an interesting open problem to classify all fixed points of the twist map, and to determine whether $V_{k,n}$ is the only totally positive fixed point.

[9] Postnikov says the hidden symmetry "comes from symmetries of the extended Dynkin diagram of type $A_{n-1}$, which is an $n$-circle." [Pos05]

# Chapter 2

# Sign variation in the Grassmannian

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## 2.1 Introduction and main results

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$. Given $V \in \mathrm{Gr}_{k,n}$, take a $k \times n$ matrix $X$ whose rows span $V$; then for $k$-subsets $I \subseteq \{1, \dots, n\}$, we let $\Delta_I(V)$ be the $k \times k$ minor of $X$ restricted to the columns in $I$, called a *Plücker coordinate*. (The $\Delta_I(V)$ depend on our choice of $X$ only up to a global constant.) If all nonzero $\Delta_I(V)$ have the same sign, then $V$ is called *totally nonnegative*, and if in addition no $\Delta_I(V)$ equals zero, then $V$ is called *totally positive*. For example, the span $V$ of $(1, 0, 0, -1)$ and $(-1, 2, 1, 3)$ is a totally nonnegative element of $\mathrm{Gr}_{2,4}$, but $V$ is not totally positive since $\Delta_{\{2,3\}}(V) = 0$.

The set $\mathrm{Gr}_{k,n}^{\geq 0}$ of totally nonnegative $V \in \mathrm{Gr}_{k,n}$, called the *totally nonnegative Grassmannian*, has become a hot topic in algebraic combinatorics in the past two decades. The general algebraic study of total positivity for split reductive connected algebraic groups $G$ over $\mathbb{R}$, and partial flag varieties $G/P$, was initiated by Lusztig [Lus94], of which $\mathrm{Gr}_{k,n}^{\geq 0}$ corresponds to the special case $G/P = \mathrm{Gr}_{k,n}$. Of particular interest is the stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$ according to whether each $\Delta_I$ is zero or nonzero. This stratification is a cell decomposition, which was conjectured by Lusztig [Lus94] and proved by Rietsch [Rie98] (for the general case $G/P$), and later understood combinatorially by Postnikov [Pos].

This general theory traces its origin to the study of *totally positive matrices* in the 1930s, in the context of oscillation theory in analysis. Here positivity conditions on matrices can imply special oscillation and spectral properties. A well-known result of this kind is due to Gantmakher and Krein [GK37], which states that if an $n \times n$ matrix $X$ is *totally positive* (i.e. all $\binom{2n}{n}$ minors of $X$ are positive), then the $n$ eigenvalues of $X$ are distinct positive reals.

Gantmakher and Krein [GK50] also gave a characterization of (what would later be called) the totally nonnegative and totally positive Grassmannians in terms of sign variation. To state their result, we introduce some notation. For $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times $v$ (viewed as a sequence of $n$ numbers, ignoring any zeros) changes sign, and let

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } 1 \le i \le n \text{ with } v_i \neq 0\}.$$

(We use the convention $\mathrm{var}(0) := -1$.) For example, if $v := (1, -1, 0, -2) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

**Theorem 2.1.1** (Chapter V, Theorems 3 and 1 of [GK50]).
*(i) $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative if and only if $\mathrm{var}(v) \le k-1$ for all $v \in V$.*
*(ii) $V \in \mathrm{Gr}_{k,n}$ is totally positive if and only if $\overline{\mathrm{var}}(v) \le k-1$ for all $v \in V \setminus \{0\}$.*

(Part (i) above was proved independently by Schoenberg and Whitney [SW51].) For example, the two vectors $(1, 0, 0, -1)$ and $(-1, 2, 1, 3)$ each change sign exactly once, and we can check that any vector in their span $V$ changes sign at most once, which is equivalent to $V$ being totally nonnegative. On the other hand, $\overline{\mathrm{var}}((1, 0, 0, -1)) = 3$, so $V$ is not totally positive. Every element of $\mathrm{Gr}_{k,n}$ has a vector which changes sign at least $k-1$ times (put a $k \times n$ matrix whose rows span $V$ into reduced row echelon form, and take the alternating sum of the rows), so the totally nonnegative elements are those whose vectors change sign as few times as possible.

The results of the chapter are organized as follows. In Section 2.3, we generalize Theorem 2.1.1 from the totally nonnegative Grassmannian to the entire Grassmannian, by giving a criterion for when $\mathrm{var}(v) \le m$ for all $v \in V$, or when $\overline{\mathrm{var}}(v) \le m$ for all $v \in V \setminus \{0\}$, in terms of the Plücker coordinates of $V$. (Theorem 2.1.1 is the case $m = k-1$.) As an application of our results, in Section 2.4 we examine the construction of *amplituhedra* introduced by Arkani-Hamed and Trnka [AHT14]. In Section 2.5, we show how to use the sign patterns of vectors in a totally nonnegative $V$ to determine the cell of $V$ in the cell decomposition of $\mathrm{Gr}_{k,n}^{\ge 0}$.

We briefly mention here that all of our results hold more generally for *oriented matroids*, and we prove them in this context. In this section we state our results in terms of the Grassmannian, so as to make them as accessible as possible. We introduce oriented matroids and the basic results about them we will need in Section 2.2. See Remark 2.1.13 at the end of this section for further comments about oriented matroids.

We now describe our main results. We let $[n] := \{1, 2, \ldots, n\}$, and denote by $\binom{[n]}{r}$ the set of $r$-subsets of $[n]$.

**Theorem 2.1.2.** *Suppose that $V \in \mathrm{Gr}_{k,n}$, and $m \ge k-1$.*
*(i) If $\mathrm{var}(v) \le m$ for all $v \in V$, then $\mathrm{var}((\Delta_{I \cup \{i\}}(V))_{i \in [n] \setminus I}) \le m-k+1$ for all $I \in \binom{[n]}{k-1}$.*
*(ii) We have $\overline{\mathrm{var}}(v) \le m$ for all $v \in V \setminus \{0\}$ if and only if $\overline{\mathrm{var}}((\Delta_{I \cup \{i\}}(V))_{i \in [n] \setminus I}) \le m-k+1$ for all $I \in \binom{[n]}{k-1}$ such that $\Delta_{I \cup \{i\}}(V) \neq 0$ for some $i \in [n]$.*

(See Theorem 2.3.1.) If we take $m := k-1$, then we recover Theorem 2.1.1; see Corollary 2.3.4 for the details.

**Example 2.1.3.** Let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 2 & 1 & 4 \end{bmatrix}$, so $k := 2$. Then by Theorem 2.1.2(ii), the fact that $\overline{\mathrm{var}}(v) \le 2 =: m$ for all $v \in V \setminus \{0\}$ is equivalent to the fact that the 4 sequences

$$\begin{aligned}
(\Delta_{\{1,2\}}(V), \Delta_{\{1,3\}}(V), \Delta_{\{1,4\}}(V)) &= (2, 1, 4), \\
(\Delta_{\{1,2\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{2,4\}}(V)) &= (2, 4, -6), \\
(\Delta_{\{1,3\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{3,4\}}(V)) &= (1, 4, -11), \\
(\Delta_{\{1,4\}}(V), \Delta_{\{2,4\}}(V), \Delta_{\{3,4\}}(V)) &= (4, -6, -11)
\end{aligned}$$

each change sign at most $m - k + 1 = 1$ time. ◊

We say that $V \in \mathrm{Gr}_{k,n}$ is *generic* if all Plücker coordinates of $V$ are nonzero. If $V$ is generic, then (ii) above implies that the converse of (i) holds. The converse of (i) does not hold in general (see Example 2.3.2); however, if $V \in \mathrm{Gr}_{k,n}$ is not generic and $\mathrm{var}(v) \le m$ for all $v \in V$, then we show how to perturb $V$ into a generic $V' \in \mathrm{Gr}_{k,n}$ while maintaining the property $\mathrm{var}(v) \le m$ for all $v \in V'$. Working backwards, we can then apply Theorem 2.1.2(i) to $V'$ in order to test whether $\mathrm{var}(v) \le m$ for all $v \in V$. The precise statement is as follows.

**Theorem 2.1.4.** *Given $V \in \mathrm{Gr}_{k,n}$, we can perturb $V$ into a generic $V' \in \mathrm{Gr}_{k,n}$ such that $\max_{v\in V} \mathrm{var}(v) = \max_{v\in V'} \mathrm{var}(v)$. In particular, for $m \ge k-1$ we have $\mathrm{var}(v) \le m$ for all $v \in V$ if and only if $\mathrm{var}((\Delta_{I\cup\{i\}}(V'))_{i\in[n]\setminus I}) \le m - k + 1$ for all $I \in \binom{[n]}{k-1}$.*

*Thus in $\{V \in \mathrm{Gr}_{k,n} : \mathrm{var}(v) \le m \text{ for all } v \in V\}$, the generic elements are dense.*

(See Theorem 2.3.14 and Theorem 2.3.15.) In the special case $m = k-1$, we recover the result of Postnikov (Section 17 of [Pos]) that the totally positive Grassmannian is dense in the totally nonnegative Grassmannian.

Theorem 2.3.14 in fact gives an algorithm for perturbing $V$ into a generic $V'$. It involves taking a $k \times n$ matrix $X$ whose rows span $V$, and repeatedly adding a very small multiple of a column of $X$ to an adjacent column (and taking the row span of the resulting matrix). We show that repeating the sequence $1 \to_+ 2, 2 \to_+ 3, \dots, (n-1) \to_+ n, n \to_+ (n-1), (n-1) \to_+ (n-2), \dots, 2 \to_+ 1$ of adjacent-column perturbations $k$ times in order from left to right is sufficient to obtain a generic $V'$, where $i \to_+ j$ denotes adding a very small positive multiple of column $i$ to column $j$. We give several other sequences of adjacent-column perturbations which work; see Theorem 2.3.14.

We use these results to study *amplituhedra*, introduced by Arkani-Hamed and Trnka [AHT14] to help calculate scattering amplitudes in theoretical physics. They consider the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ on the totally nonnegative Grassmannian induced by a given linear map $Z : \mathbb{R}^n \to \mathbb{R}^r$. Note that this map is not necessarily well defined, since $Z$ may send a $k$-dimensional subspace to a subspace of lesser dimension. In part to preclude this possibility, Arkani-Hamed and Trnka require that $k \le r$ and $Z$ has positive $r \times r$ minors (when viewed as an $r \times n$ matrix), and call the image of the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ a *(tree) amplituhedron*. Lam [Lam16b] showed more generally that the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ is well defined if the row span of $Z$ (regarded as an $r \times n$ matrix) has a totally positive $k$-dimensional subspace, in which case he calls the image a *(full) Grassmann polytope*. (When $k = 1$, Grassmann polytopes are precisely polytopes in projective space.) We use sign variation to give a necessary and sufficient condition for the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ to be well defined, and in particular recover the sufficient conditions of Arkani-Hamed and Trnka, and Lam.

**Theorem 2.1.5.** *Suppose that $k, n, r \in \mathbb{N}$ with $n \ge k, r$, and that $Z : \mathbb{R}^n \to \mathbb{R}^r$ is a linear map, which we also regard as an $r \times n$ matrix. Let $d$ be the rank of $Z$ and $W \in \mathrm{Gr}_{d,n}$ the row span of $Z$, so that $W^\perp = \ker(Z) \in \mathrm{Gr}_{n-d,n}$. The following are equivalent:*
*(i) the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined, i.e. $\dim(Z(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\ge 0}$;*
*(ii) $\mathrm{var}(v) \ge k$ for all nonzero $v \in \ker(Z)$; and*
*(iii) $\overline{\mathrm{var}}((\Delta_{I \setminus \{i\}}(W))_{i \in I}) \le d - k$ for all $I \in \binom{[n]}{d+1}$ such that $W|_I$ has dimension $d$.*

(See Theorem 2.4.2.) We remark that the equivalence of (ii) and (iii) above is equivalent to Theorem 2.1.2(ii).

We now describe our results about the cell decomposition of $\mathrm{Gr}_{k,n}^{\ge 0}$. Given $V \in \mathrm{Gr}_{k,n}$, we define the *matroid* $M(V)$ of $V$ as the set of $I \in \binom{[n]}{k}$ such that $\Delta_I(V)$ is nonzero. If $V$ is totally nonnegative, we also call $M(V)$ a *positroid*. The stratification of $\mathrm{Gr}_{k,n}^{\ge 0}$ by positroids (i.e. its partition into equivalence classes, where $V \sim W$ if and only if $M(V) = M(W)$) is a cell decomposition [Rie98, Pos].

How can we determine the matroid of $V \in \mathrm{Gr}_{k,n}$ from the sign patterns of vectors in $V$? Given $I \subseteq [n]$ and a sign vector $\omega \in \{+, -\}^I$, we say that $V$ *realizes* $\omega$ if there exists a vector in $V$ whose restriction to $I$ has signs given by $\omega$. For example, if $(2, 3, -2, -1) \in V$, then $V$ realizes $(+, -, -)$ on $\{1, 3, 4\}$. Note that $V$ realizes $\omega$ if and only if $V$ realizes $-\omega$. It is not difficult to show that for all $I \in \binom{[n]}{k}$, we have $I \in M(V)$ if and only if $V$ realizes all $2^k$ sign vectors in $\{+, -\}^I$. Furthermore, in order to determine whether $I$ is in $M(V)$ from which sign vectors $V$ realizes in $\{+, -\}^I$, we potentially have to check all $2^{k-1}$ pairs of sign vectors (each sign vector and its negation), since given any $\omega \in \{+, -\}^I$ (and assuming $n > k$), there exists $V \in \mathrm{Gr}_{k,n}$ which realizes all $2^k$ sign vectors in $\{+, -\}^I$ except for $\pm\omega$. (See Remark 2.5.7.) However, in the case that $V$ is totally nonnegative, we show that we need only check $k$ particular sign vectors in $\{+, -\}^I$ to verify that $\Delta_I(V) \ne 0$.

**Theorem 2.1.6.** *For $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ and $I \in \binom{[n]}{k}$, we have $I \in M(V)$ if and only if $V$ realizes the $2k$ (or $k$ up to sign) sign vectors in $\{+, -\}^I$ which alternate in sign between every pair of consecutive components, with at most one exceptional pair.*

(See Corollary 2.5.6.) For example, if $k = 5$, these $2k$ sign vectors are $(+, -, +, -, +)$, $(+, +, -, +, -)$, $(+, -, -, +, -)$, $(+, -, +, +, -)$, $(+, -, +, -, -)$, and their negations.

**Example 2.1.7.** Let $V \in \mathrm{Gr}_{3,5}^{\geq 0}$ be the row span of the matrix $\begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Theorem 2.1.6 implies that for all $I \in \binom{[5]}{3}$, we have $\Delta_I(V) \neq 0$ if and only if $V$ realizes the 3 sign vectors $(+,-,+)$, $(+,+,-)$, $(+,-,-)$ on $I$. For $I = \{1,3,5\}$, the vectors $(2,1,-1,0,3)$, $(2,1,1,-4,-1)$, $(2,1,-1,-4,-1) \in V$ realize the sign vectors $(+,-,+)$, $(+,+,-)$, $(+,-,-)$ on $I$, so $\Delta_{\{1,3,5\}}(V) \neq 0$. (We do not need to check that $(+,+,+)$, the remaining sign vector in $\{+,-\}^I$ up to sign, is realized.) For $I = \{1,4,5\}$, the vectors $(2,1,0,-1,2), (2,1,0,-4,-1) \in V$ realize the sign vectors $(+,-,+), (+,-,-)$ on $I$, but no vector in $V$ realizes the sign vector $(+,+,-)$ on $I$, so $\Delta_{\{1,4,5\}}(V) = 0$. $\Diamond$

We now describe another way to recover the positroid of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ from the sign patterns of vectors in $V$. We begin by showing how to obtain the *Schubert cell* of $V$, which is labeled by the lexicographic minimum of $M(V)$. To state this result, we introduce some notation. For $v \in \mathbb{R}^n$ and $I \subseteq [n]$, we say that *$v$ strictly alternates in sign on $I$* if $v|_I$ has no zero components, and alternates in sign between consecutive components. Let $A(V)$ denote the set of $I \in \binom{[n]}{k}$ such that some vector in $V$ strictly alternates in sign on $I$. Note that if $I \in M(V)$ then $V|_I = \mathbb{R}^I$, so $M(V) \subseteq A(V)$. We also define the *Gale partial order* $\leq_{\mathrm{Gale}}$ on $\binom{[n]}{k}$ by $\{i_1 < \cdots < i_k\} \leq_{\mathrm{Gale}} \{j_1 < \cdots < j_k\}$ if and only if $i_1 \leq j_1, i_2 \leq j_2, \dots, i_k \leq j_k$.

**Theorem 2.1.8.** *For $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, the lexicographic minimum of $M(V)$ equals the Gale minimum of $A(V)$.*

(See Theorem 2.5.1.) We remark that the lexicographic minimum of $M(V)$ is also the Gale minimum of $M(V)$, but $A(V)$ does not necessarily equal $M(V)$ (see Example 2.1.9). We also note that if $V \in \mathrm{Gr}_{k,n}$ is not totally nonnegative, then $A(V)$ does not necessarily have a Gale minimum (see Example 2.5.3).

**Example 2.1.9.** Let $V \in \mathrm{Gr}_{3,5}^{\geq 0}$ be the row span of the matrix $\begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, as in Example 2.1.7. Theorem 2.1.8 implies that the lexicographic minimum $\{1,3,4\}$ of $M(V)$ equals the Gale minimum of

$$A(V) = \{\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}.$$

Note that $\{2,4,5\} \in A(V) \setminus M(V)$. $\Diamond$

By the cyclic symmetry of the totally nonnegative Grassmannian, we can then use Theorem 2.1.8 to recover the *Grassmann necklace* of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ (see Corollary 2.5.5), which in turn determines the positroid of $V$ by a result of Postnikov (Theorem 17.1 of [Pos]).

**Remark 2.1.10.** We can easily reinterpret results about upper bounds on var in terms of lower bounds on $\overline{\mathrm{var}}$, and upper bounds on $\overline{\mathrm{var}}$ in terms of lower bounds on var, by the following two facts.

**Lemma 2.1.11.** *(i) [GK50] For* $v \in \mathbb{R}^n \setminus \{0\}$*, we have*

$$\mathrm{var}(v) + \overline{\mathrm{var}}(\mathrm{alt}(v)) = n - 1,$$

*where* $\mathrm{alt}(v) := (v_1, -v_2, v_3, -v_4, \dots, (-1)^{n-1}v_n) \in \mathbb{R}^n$.
*(ii) [Hil90][Hoc75] Given* $V \in \mathrm{Gr}_{k,n}$*, let* $V^\perp \in \mathrm{Gr}_{n-k,n}$ *be the orthogonal complement of* $V$*. Then* $V$ *and* $\mathrm{alt}(V^\perp)$ *have the same Plücker coordinates:*

$$\Delta_I(V) = \Delta_{[n]\setminus I}(\mathrm{alt}(V^\perp)) \qquad \text{for all } I \in \binom{[n]}{k}.$$

(Part (i) is stated without proof as equation (67) in Chapter II of [GK50]; see equation (5.1) of [And87] for a proof. The earliest statement of part (ii) we found in the literature is at the beginning of Section 7 of [Hoc75]. Hochster does not give a proof, and says that this result "was basically known to Hilbert." The idea is that if $[I_k|A]$ is a $k \times n$ matrix whose rows span $V \in \mathrm{Gr}_{k,n}$, where $A$ is a $k \times (n-k)$ matrix, then $V^\perp$ is the row span of the matrix $[A^T| - I_{n-k}]$. This idea appears implicitly in equation (14) of [Hil90], and more explicitly in Theorem 2.2.8 of [Oxl11] and Proposition 3.1(i) of [MR14]. I thank a referee for pointing out the reference [MR14].) For example, we get the following dual formulation of Gantmakher and Krein's result (Theorem 2.1.1).

**Corollary 2.1.12** (Chapter V, Theorems 7 and 6 of [GK50])**.**
*(i)* $V \in \mathrm{Gr}_{k,n}$ *is totally nonnegative if and only if* $\overline{\mathrm{var}}(v) \ge k$ *for all* $v \in V^\perp \setminus \{0\}$.
*(ii)* $V \in \mathrm{Gr}_{k,n}$ *is totally positive if and only if* $\mathrm{var}(v) \ge k$ *for all* $v \in V^\perp \setminus \{0\}$.

**Remark 2.1.13.** The natural framework in which to consider sign patterns of vectors in $V$, and signs of the Plücker coordinates of $V$, is that of *oriented matroids*. Our results hold, and are proven, in this context, and are more general because while every subspace gives rise to an oriented matroid, not every oriented matroid comes from a subspace. (The totally nonnegative Grassmannian is a special case; the analogue of a totally nonnegative subspace is a *positively oriented matroid*, and Ardila, Rincón, and Williams [ARW16] recently showed that each positively oriented matroid comes from a totally nonnegative subspace. Hence there is no added generality gained here in passing from the Grassmannian to oriented matroids.)

Those already familiar with oriented matroids can use the following dictionary to reinterpret the results stated in this section:

| Subspaces | Oriented matroids |
|---|---|
| sign vectors of vectors in $V$ | *covectors* of $\mathcal{M}(V)$ |
| $\Delta(V)$, up to sign | the *chirotope* $\chi_{\mathcal{M}(V)}$ |
| $V$ is generic | $\mathcal{M}(V)$ is *uniform* |
| $V'$ is a perturbation of $V$ | there is a weak map from $\mathcal{M}(V')$ to $\mathcal{M}(V)$ |
| the closure of $S \subseteq \mathrm{Gr}_{k,n}$ | images of weak maps from $\mathcal{M}(V')$, over $V' \in S$ |
| the orthogonal complement $V^\perp$ of $V$ | the dual of $\mathcal{M}(V)$ |

We also generalize to oriented matroids the operation of adding a very small multiple of a column (of a $k \times n$ matrix whose rows span $V \in \mathrm{Gr}_{k,n}$) to an adjacent column; see Definition 2.3.6.

For those unfamiliar with oriented matroids, we give an introduction in Section 2.2, biased toward the tools we need. For a thorough introduction to oriented matroids, see the book [BLVS+99].

## 2.2 Introduction to oriented matroids

In this section, we introduce oriented matroids, and much of the notation and tools that we use in our proofs. A comprehensive account of the theory of oriented matroids, and our reference throughout, is the book by Björner, Las Vergnas, Sturmfels, White, and Ziegler [BLVS+99]. We begin by describing oriented matroids coming from subspaces of $\mathbb{R}^n$ (i.e. *realizable* oriented matroids), which will serve as motivation for the exposition to follow.

For $\alpha \in \mathbb{R}$ we define

$$\mathrm{sign}(\alpha) := \begin{cases} 0, & \text{if } \alpha = 0 \\ +, & \text{if } \alpha > 0 \\ -, & \text{if } \alpha < 0 \end{cases},$$

and for $x \in \mathbb{R}^E$ we define the sign vector $\mathrm{sign}(x) \in \{0, +, -\}^E$ by $\mathrm{sign}(x)_e := \mathrm{sign}(x_e)$ for $e \in E$. We will sometimes use 1 and $-1$ in place of $+$ and $-$. For example, $\mathrm{sign}(5, 0, -1, 2) = (+, 0, -, +) = (1, 0, -1, 1)$. Given a sign vector $X \in \{0, +, -\}^E$, the *support* of $X$ is the subset $\underline{X} := \{e \in E : X_e \neq 0\}$ of $E$. We can think of $X$ as giving a sign to each element of $\underline{X}$ (some authors call $X$ a *signed subset*). We also define $-X \in \{0, +, -\}^E$ by $(-X)_e := -X_e$ for $e \in E$. For example, $X = (+, 0, -, +) \in \{0, +, -\}^4$ has support $\{1, 3, 4\}$, and $-X = (-, 0, +, -)$.

**Definition 2.2.1** (realizable oriented matroids; 1.2 of [BLVS+99])**.** Let $E$ be a finite set and $V$ a $k$-dimensional subspace of $\mathbb{R}^E$. The *(realizable) oriented matroid* $\mathcal{M}(V)$ associated to $V$ is uniquely determined by $E$ (the *ground set* of $\mathcal{M}(V)$) and any one of the following three objects:
• the set $\mathcal{V}^* := \{\mathrm{sign}(v) : v \in V\}$, called the *covectors* of $\mathcal{M}(V)$; or
• the set $\mathcal{C}^* := \{X \in \mathcal{V}^* : X \text{ has minimal nonempty support}\}$, called the *cocircuits* of $\mathcal{M}(V)$; or
• the function $\chi : E^k \to \{0, +, -\}$ (up to multiplication by $\pm 1$), called the *chirotope* of $\mathcal{M}(V)$, where $\chi(i_1, \dots, i_k) := \mathrm{sign}(\det([x^{(i_1)}|\cdots|x^{(i_k)}]))$ $(i_1, \dots, i_k \in E)$ for some fixed $k \times E$ matrix $[x^{(i)} : i \in E]$ whose rows span $V$.
The *rank* of $\mathcal{M}(V)$ is $k$.

**Example 2.2.2.** Let $V \in \mathrm{Gr}_{2,3}$ be the row span of the matrix $\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$. Then $\mathcal{M}(V)$ is an oriented matroid of rank $k := 2$ with ground set $E := \{1, 2, 3\}$. Note that $(+, +, -)$ is a covector of $\mathcal{M}(V)$, because it is the sign vector of e.g. $(3, 3, -1) \in V$. The covectors of $\mathcal{M}(V)$ are

$$(0,0,0),(0,+,-),(0,-,+),(+,0,+),(-,0,-),(+,+,0),(-,-,0),$$
$$(+,+,-),(-,+,-),(+,-,+),(-,-,+),(+,+,+),(-,-,-).$$

The cocircuits of $\mathcal{M}(V)$ are the covectors with minimal nonempty support, i.e.

$$(0,+,-),(0,-,+),(+,0,+),(-,0,-),(+,+,0),(-,-,0).$$

The chirotope $\chi$ of $\mathcal{M}(V)$ is given (up to sign) by

$$\begin{aligned}
\chi(1,2) &= \operatorname{sign}(\Delta_{\{1,2\}}(V)) = \operatorname{sign}(3) = +,\\
\chi(1,3) &= \operatorname{sign}(\Delta_{\{1,3\}}(V)) = \operatorname{sign}(-3) = -,\\
\chi(2,3) &= \operatorname{sign}(\Delta_{\{2,3\}}(V)) = \operatorname{sign}(-2) = -,
\end{aligned}$$

and by the fact that $\chi$ is *alternating*, i.e. swapping two arguments multiplies the result by $-1$. The fact that the Plücker coordinates $\Delta_I(V)$ are defined only up to multiplication by a global nonzero constant explains why the chirotope is defined only up to sign. $\diamondsuit$

**Definition 2.2.3** (oriented matroid, cocircuit axioms; 3.2.1 of [BLVS$^+$99]). An *oriented matroid* $\mathcal{M}$ is an ordered pair $\mathcal{M} = (E, \mathcal{C}^*)$, where $E$ is a finite set and $\mathcal{C}^* \subseteq 2^{\{0,+,-\}^E}$ satisfies the following four axioms:
(C0) every sign vector in $\mathcal{C}^*$ has nonempty support;
(C1) $\mathcal{C}^* = -\mathcal{C}^*$;
(C2) if $X, Y \in \mathcal{C}^*$ with $\underline{X} \subseteq \underline{Y}$, then $X = \pm Y$;
(C3) if $X, Y \in \mathcal{C}^*$ and $a \in E$ such that $X \neq -Y$ and $X_a = -Y_a \neq 0$, then there exists $Z \in \mathcal{C}^*$ such that $Z_a = 0$, and $Z_b = X_b$ or $Z_b = Y_b$ for all $b \in \underline{Z}$.

The set $E$ is called the *ground set* of $\mathcal{M}$, and the sign vectors in $\mathcal{C}^*$ are called the *cocircuits* of $\mathcal{M}$.

We denote the cocircuits of $\mathcal{M}$ by $\mathcal{C}^*(\mathcal{M})$. (The superscript $*$ is present to indicate that cocircuits are *circuits* of the *dual* of $\mathcal{M}$.) We remark that not all oriented matroids are realizable; see 1.5.1 of [BLVS$^+$99] for an example of a non-realizable oriented matroid.

**Example 2.2.4.** The sign vectors

$$(0,+,-),(0,-,+),(+,0,+),(-,0,-),(+,-,0),(-,+,0)$$

are *not* the cocircuits of an oriented matroid, because e.g. (C3) above fails when we take $X = (0,+,-)$, $Y = (+,0,+)$, $a = 3$. $\diamondsuit$

For sign vectors $X, Y \in \{0,+,-\}^E$, define the *composition* $X \circ Y$ as the sign vector in $\{0,+,-\}^E$ given by

$$(X \circ Y)_e := \begin{cases} X_e, & \text{if } X_e \neq 0 \\ Y_e, & \text{if } X_e = 0 \end{cases} \qquad \text{for } e \in E.$$

We can think of $X \circ Y$ as being formed by starting with $X$ and recording $Y$ in the empty slots of $X$, or by starting with $Y$ and overwriting $X$ on top. In general, $X \circ Y \neq Y \circ X$; if the composition of sign vectors $X^{(1)}, \dots, X^{(r)}$ of $E$ does not depend on the order of composition, we say that $X^{(1)}, \dots, X^{(r)}$ are *conformal*. For example, $(+,0,-)$ and $(0,+,-)$ are conformal.

A *covector* of an oriented matroid $\mathcal{M}$ is a composition of some (finite number of) cocircuits of $\mathcal{M}$. (We include the empty composition, which is the zero sign vector.) We let $\mathcal{V}^*(\mathcal{M})$ denote the set of covectors of $\mathcal{M}$. Note that by (C2) of Definition 2.2.3, we can recover the cocircuits of $\mathcal{M}$ as the covectors with minimal nonempty support. A key property of covectors is the following conformality property.

**Proposition 2.2.5** (conformality for covectors; 3.7.2 of [BLVS$^+$99])**.** *Suppose that $X$ is a covector of the oriented matroid $\mathcal{M}$. Then $X = C^{(1)} \circ \cdots \circ C^{(r)}$ for some conformal cocircuits $C^{(1)}, \dots, C^{(r)}$ of $\mathcal{M}$.*

There are axioms which characterize when a set of sign vectors in $\{0,+,-\}^E$ is the set of covectors of an oriented matroid; see 3.7.5 of [BLVS$^+$99].

**Definition 2.2.6** (basis, rank; pp. 124, 115 of [BLVS$^+$99])**.** Let $\mathcal{M}$ be an oriented matroid with ground set $E$. A *basis of* $\mathcal{M}$ is a minimal $B \subseteq E$ such that $B \cap \underline{C} \neq \emptyset$ for every cocircuit $C$ of $\mathcal{M}$. All bases of $\mathcal{M}$ have the same size $k \geq 0$, called the *rank* of $\mathcal{M}$.

$\mathcal{M}$ determines a unique orientation on its bases (up to a global sign).

**Definition 2.2.7** (chirotope; 3.5.1, 3.5.2 of [BLVS$^+$99])**.** Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $E$. Then there exists a function $\chi_{\mathcal{M}} : E^k \to \{0,+,-\}$ (called the *chirotope* of $\mathcal{M}$), unique up to sign, satisfying the following properties:
(i) $\chi_{\mathcal{M}}$ is *alternating*, i.e. $\chi_{\mathcal{M}}(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \operatorname{sgn}(\sigma)\chi_{\mathcal{M}}(i_1, \dots, i_k)$ for $i_1, \dots, i_k \in E$ and $\sigma \in \mathfrak{S}_k$;
(ii) $\chi_{\mathcal{M}}(i_1, \dots, i_k) = 0$ if $\{i_1, \dots, i_k\} \subseteq E$ is not a basis of $\mathcal{M}$; and
(iii) if $\{a, i_1, \dots, i_{k-1}\}, \{b, i_1, \dots, i_{k-1}\} \subseteq E$ are bases of $\mathcal{M}$ and $C$ is a cocircuit of $\mathcal{M}$ with $i_1, \dots, i_{k-1} \notin \underline{C}$, then $\chi_{\mathcal{M}}(a, i_1, \dots, i_{k-1}) = C_a C_b \chi_{\mathcal{M}}(b, i_1, \dots, i_{k-1})$.

$\mathcal{M}$ is uniquely determined by $\chi_{\mathcal{M}}$ up to sign (3.5.2 of [BLVS$^+$99]). For the axioms characterizing chirotopes of oriented matroids, see 3.5.3, 3.5.4 of [BLVS$^+$99]. If $E$ is totally ordered (for example, $E = [n]$ ordered by $1 < 2 < \cdots < n$), we let $\chi_{\mathcal{M}}(I)$ denote $\chi_{\mathcal{M}}(i_1, \dots, i_k)$ for $I \in \binom{E}{k}$ ($I = \{i_1, \dots, i_k\}$, $i_1 < \cdots < i_k$), and set $\chi_{\mathcal{M}}(J) := 0$ for $J \subseteq E$ with $|J| < k$. In this case $\chi_{\mathcal{M}}$ gives an *orientation* (either $+$ or $-$) to each basis of $\mathcal{M}$.

The relation (iii) above between $\chi_{\mathcal{M}}$ and the cocircuits of $\mathcal{M}$ is called the *pivoting property*; we state it in the following useful form.

**Proposition 2.2.8** (pivoting property; 3.5.1, 3.5.2 of [BLVS+99])**.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with a totally ordered ground set $E$, $I \in \binom{E}{k-1}$, and $a, b \in E$. If $I \cup \{a\}$ and $I \cup \{b\}$ are bases of $\mathcal{M}$, then there exists a cocircuit $C$ of $\mathcal{M}$ with $I \cap \underline{C} = \emptyset$ (unique up to sign), whence $a, b \in \underline{C}$, and*

$$\chi_{\mathcal{M}}(I \cup \{a\}) = (-1)^{|\{i \in I : i \text{ is strictly between } a \text{ and } b\}|} C_a C_b \chi_{\mathcal{M}}(I \cup \{b\}). \tag{2.2.9}$$

*Conversely, if there exists a cocircuit $C$ of $\mathcal{M}$ with $I \cap \underline{C} = \emptyset$ and $b \in \underline{C}$, then (2.2.9) holds.*

Only the first part of Proposition 2.2.8 is proved in [BLVS+99], so we prove the converse.

**Proof** (of converse)**.** Let $C$ be a cocircuit of $\mathcal{M}$ with $I \cap \underline{C} = \emptyset$ and $b \in \underline{C}$. First suppose that $I \cup \{b\}$ is not a basis of $\mathcal{M}$; we must show that $I \cup \{a\}$ is also not a basis. By Definition 2.2.6 there exists a cocircuit $D$ of $\mathcal{M}$ with $(I \cup \{b\}) \cap \underline{D} = \emptyset$. If $a \notin \underline{C}$ or $a \notin \underline{D}$, then we immediately get that $I \cup \{a\}$ is not a basis. Otherwise we have $a \in \underline{C} \cup \underline{D}$, and $C \neq \pm D$ since $b \in \underline{C} \setminus \underline{D}$. Hence we may apply (C3) of Definition 2.2.3 to obtain a cocircuit of $\mathcal{M}$ whose support is contained in $(\underline{C} \cup \underline{D}) \setminus \{a\} \subseteq E \setminus (I \cup \{a\})$, whence $I \cup \{a\}$ is not a basis of $\mathcal{M}$. Similarly, if $a \in \underline{C}$ and $I \cup \{a\}$ is not a basis of $\mathcal{M}$, then $I \cup \{b\}$ is not a basis, giving (2.2.9). Also, if $I \cup \{a\}$ and $I \cup \{b\}$ are both bases of $\mathcal{M}$, then (2.2.9) follows from the first part of this result. The remaining case is when $I \cup \{b\}$ is a basis of $\mathcal{M}$, $I \cup \{a\}$ is not a basis, and $a \notin \underline{C}$, whence both sides of (2.2.9) are zero. ■

Now we introduce restriction of oriented matroids; for a realizable oriented matroid $\mathcal{M}(V)$, this corresponds to restricting $V$ to a subset of the canonical coordinates.

**Definition 2.2.10** (restriction; 3.7.11, 3.4.9, pp. 133-134 of [BLVS+99])**.** Let $\mathcal{M}$ be an oriented matroid with ground set $E$, and $F \subseteq E$. The *restriction of $\mathcal{M}$ to $F$*, denoted by $\mathcal{M}|_F$ or $\mathcal{M} \setminus G$ (where $G := E \setminus F$), is the oriented matroid with ground set $F$ and covectors $\{X|_F : X \in \mathcal{V}^*(\mathcal{M})\}$. The bases of $\mathcal{M}|_F$ are the maximal elements of $\{B \cap F : B \text{ is a basis of } \mathcal{M}\}$. The chirotope $\chi_{\mathcal{M}|_F}$ is given as follows. Let $k, l$ be the ranks of $\mathcal{M}, \mathcal{M}|_F$, respectively, and take $i_1, \dots, i_{k-l} \in E \setminus F$ such that $F \cup \{i_1, \dots, i_{k-l}\}$ contains a basis of $\mathcal{M}$. Then

$$\chi_{\mathcal{M}|_F}(j_1, \dots, j_l) = \chi_{\mathcal{M}}(j_1, \dots, j_l, i_1, \dots, i_{k-l}) \qquad \text{for } j_1, \dots, j_l \in F.$$

If $V$ is a subspace of $\mathbb{R}^E$, then $\mathcal{M}(V)|_F = \mathcal{M}(V|_F)$.

We conclude by describing a partial order on oriented matroids with a fixed ground set. Geometrically, for point configurations, moving up in the partial order corresponds to moving the points of the configuration into more general position. (A configuration of $n$ points in $\mathbb{R}^k$ gives rise to a subspace of $\mathbb{R}^n$, and hence an oriented matroid with ground set $[n]$, by writing the points as the columns of a $k \times n$ matrix and taking the row span of this matrix.) We use the partial order on sign vectors given by $X \le Y$ if and only if $Y = X \circ Y$ ($X, Y \in \{0, +, -\}^E$), i.e. $X_e = Y_e$ for all $e \in E$ such that $X_e \neq 0$. This also defines a partial order on chirotopes, regarded as sign vectors in $\{0, +, -\}^{E^k}$.

**Definition 2.2.11** (partial order on oriented matroids; 7.7.5 of [BLVS+99])**.** Let $\mathcal{M}, \mathcal{N}$ be oriented matroids with ground set $E$. We say that $\mathcal{M} \le \mathcal{N}$ if for every covector $X$ of $\mathcal{M}$, there exists a covector $Y$ of $\mathcal{N}$ with $X \le Y$. Then $\le$ is a partial order on oriented matroids with ground set $E$. If $\mathcal{M}$ and $\mathcal{N}$ have the same rank, then $\mathcal{M} \le \mathcal{N}$ if and only if $\chi_{\mathcal{M}} \le \pm\chi_{\mathcal{N}}$.

The standard terminology for $\mathcal{M} \le \mathcal{N}$ is that there is a *weak map* from $\mathcal{N}$ to $\mathcal{M}$.

## 2.3 Relating sign changes of covectors and the chirotope

Recall that given a sign vector $X \in \{0,+,-\}^E$ over a totally ordered set $E$, the number of sign changes of $X$ (ignoring any zeros) is denoted by $\mathrm{var}(X)$, and $\overline{\mathrm{var}}(X) := \max_{Y \ge X} \mathrm{var}(Y)$. The goal of this section is to give, for any oriented matroid $\mathcal{M}$ with a totally ordered ground set, a criterion for when $\mathrm{var}(X) \le m$ for all covectors $X$ of $\mathcal{M}$, or when $\overline{\mathrm{var}}(X) \le m$ for all nonzero covectors $X$ of $\mathcal{M}$, in terms of the chirotope of $\mathcal{M}$. Theorem 2.3.1 provides such a criterion in the latter case, as well as in the former case if $\mathcal{M}$ is *uniform*, i.e. every $k$-subset of its ground set is a basis (where $k$ is the rank of $\mathcal{M}$). (Hence $V \in \mathrm{Gr}_{k,n}$ is generic if and only if $\mathcal{M}(V)$ is uniform.) For non-uniform $\mathcal{M}$, we then show (Theorem 2.3.14) how to perturb $\mathcal{M}$ into a generic uniform matroid $\mathcal{N}$ so that we may apply the criterion in Theorem 2.3.1 to determine whether $\mathrm{var}(X) \le m$ for all covectors $X$ of $\mathcal{M}$.

We remark that while var is weakly increasing (i.e. $\mathrm{var}(X) \le \mathrm{var}(Y)$ if $X \le Y$), $\overline{\mathrm{var}}$ is weakly decreasing, which helps explain why var and $\overline{\mathrm{var}}$ require such different treatments.

**Theorem 2.3.1.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $[n]$, and $m \ge k-1$.*
*(i) If $\mathrm{var}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M})$, then $\mathrm{var}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}) \le m-k+1$ for all $I \in \binom{[n]}{k-1}$.*
*(ii) We have $\overline{\mathrm{var}}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M}) \setminus \{0\}$ if and only if $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}) \le m-k+1$ for all $I \in \binom{[n]}{k-1}$ such that $I \cup \{i\}$ is a basis of $\mathcal{M}$ for some $i \in [n]$.*

For an example using this theorem, see Example 2.1.3. Note that (ii) above implies that if $\mathcal{M}$ is uniform, then the converse of (i) holds. However, the converse of (i) does not hold in general, as shown in Example 2.3.2. Example 2.3.3 shows that the condition "$I \cup \{i\}$ is a basis of $\mathcal{M}$ for some $i \in [n]$" (equivalently, that the sequence $(\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}$ is nonzero) in (ii) is necessary. Also note that there is no loss of generality in the assumption $m \ge k-1$, because there exists a covector of $\mathcal{M}$ which changes sign at least $k-1$ times; in fact, if $B \in \binom{[n]}{k}$ is any basis of $\mathcal{M}$, then there exists a covector of $\mathcal{M}$ which strictly alternates in sign on $B$. (This follows from Definition 2.2.10: $\mathcal{M}|_B$ is the uniform oriented matroid of rank $k$ with ground set $B$, and so $\mathcal{V}^*(\mathcal{M})|_B = \mathcal{V}^*(\mathcal{M}|_B) = \{0,+,-\}^B$.)

**Example 2.3.2.** Let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, so $k := 2$. Note that the 4 sequences of Plücker coordinates

$$\begin{aligned}
&(\Delta_{\{1,2\}}(V), \Delta_{\{1,3\}}(V), \Delta_{\{1,4\}}(V)) = (1, 0, 1), \\
&(\Delta_{\{1,2\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{2,4\}}(V)) = (1, -1, 0), \\
&(\Delta_{\{1,3\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{3,4\}}(V)) = (0, -1, 1), \\
&(\Delta_{\{1,4\}}(V), \Delta_{\{2,4\}}(V), \Delta_{\{3,4\}}(V)) = (1, 0, 1)
\end{aligned}$$

each change sign at most $m - k + 1 = 1$ time (where we take $m := 2$), but the vector $(1, -1, 1, -1) \in V$ changes sign 3 times. Hence the converse to Theorem 2.3.1(i) does not hold. However, if we were forced to pick a sign for, say, $\Delta_{\{1,3\}}(V)$, then either the first or third sequence above would change sign twice. This motivates the introduction of perturbations below. ♢

**Example 2.3.3.** Let $V \in \mathrm{Gr}_{3,5}$ be the row span of the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\mathcal{M}(V)$ satisfies the equivalent conditions of Theorem 2.3.1(ii) with $m := 3$, i.e. $\overline{\mathrm{var}}(v) \le 3$ for all $v \in V \setminus \{0\}$, and $\overline{\mathrm{var}}((\Delta_{I \cup \{i\}}(V))_{i \in [5] \setminus I}) \le 1$ for all $I \in \binom{[5]}{2}$ such that $\Delta_{I\cup\{i\}}(V) \neq 0$ for some $i \in [n]$. We cannot remove the condition "$\Delta_{I\cup\{i\}}(V) \neq 0$ for some $i \in [n]$," because taking $J := \{1, 2\}$ we have $\Delta_{J\cup\{i\}}(V) = 0$ for all $i \in [n]$, and so $\overline{\mathrm{var}}((\Delta_{J\cup\{i\}}(V))_{i\in[5]\setminus J}) = 2$. ♢

**Proof** (of Theorem 2.3.1)**.** The idea is to use the fact that if $X \in \{0, +, -\}^n$ with $\mathrm{var}(X) = r$, then there exists $A \in \binom{[n]}{r+1}$ such that $X$ strictly alternates in sign on $A$. We restrict our attention to an appropriate choice of $A$, using Definition 2.2.6 and the pivoting property (Proposition 2.2.8) to relate cocircuits, bases, and the chirotope.

(i) Suppose that $I \in \binom{[n]}{k-1}$ such that $\mathrm{var}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}) \ge m - k + 2$. Take $A \in \binom{[n] \setminus I}{m-k+3}$ such that $(\chi_{\mathcal{M}}(I \cup \{i\}))_{i\in[n]\setminus I}$ strictly alternates in sign on $A$. Fix $a \in A$, and for the remainder of this proof, for $i, j \in [n]$ let $[i, j)$ denote the interval of integers from $i$ (inclusively) to $j$ (exclusively), i.e. $\{i, i+1, \ldots, j-1\}$ if $j \ge i$ and $\{j+1, j+2, \ldots, i\}$ if $j \le i$. By Definition 2.2.6, for $i \in I$ there exists a cocircuit $C^{(i)}$ of $\mathcal{M}$ with $((I \cup \{a\}) \setminus \{i\}) \cap \underline{C^{(i)}} = \emptyset$; since $I \cup \{a\}$ is a basis of $\mathcal{M}$ we have $i \in \underline{C^{(i)}}$, so we may assume that $C^{(i)}_i = (-1)^{|(I \cup A) \cap [a,i)|}$. Also let $D$ be a cocircuit of $\mathcal{M}$ with $I \cap \underline{D} = \emptyset$; since $a \in \underline{D}$ we may assume that $D_a = 1$. Then for $b \in [n]$, the pivoting property (Proposition 2.2.8) gives

$$\chi_{\mathcal{M}}(I \cup \{b\}) = (-1)^{|I \cap [a,b)|} D_a D_b \chi_{\mathcal{M}}(I \cup \{a\}).$$

Because $(\chi_{\mathcal{M}}(I \cup \{i\}))_{i\in A}$ strictly alternates in sign, we have

$$\chi_{\mathcal{M}}(I \cup \{b\}) = (-1)^{|A \cap [a,b)|} \chi_{\mathcal{M}}(I \cup \{a\}) \qquad \text{for } b \in A,$$

so $D_b = (-1)^{|(I \cup A) \cap [a,b)|}$. Now let $X$ be the covector $D \circ C^{(i_1)} \circ \cdots \circ C^{(i_{k-1})}$ of $\mathcal{M}$, where $I = \{i_1, \dots, i_{k-1}\}$. Then $X_i = (-1)^{|(I \cup A) \cap [a,i)|}$ for $i \in I \cup A$, so $X$ strictly alternates in sign on $I \cup A$, giving $\mathrm{var}(X) \geq m+1$.

(ii) ($\Rightarrow$): Suppose that $I \in \binom{[n]}{k-1}$ such that $I \cup \{i\}$ is a basis of $\mathcal{M}$ for some $i \in [n]$, and $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}) \geq m-k+2$. We proceed as in the proof of (i). Take $A \in \binom{[n] \setminus I}{m-k+3}$ such that $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in A}) = m-k+2$ and $I \cup \{a\}$ is a basis of $\mathcal{M}$ for some $a \in A$; fix such an $a \in A$. By Definition 2.2.6 there exists a cocircuit $D$ of $\mathcal{M}$ with $I \cap \underline{D} = \emptyset$; since $a \in \underline{D}$ we may assume that $D_a = 1$. Then for $b \in [n]$, the pivoting property (Proposition 2.2.8) gives
$$\chi_{\mathcal{M}}(I \cup \{b\}) = (-1)^{|I \cap [a,b)|} D_a D_b \chi_{\mathcal{M}}(I \cup \{a\}).$$
Because $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in A}) = m-k+2$, for $b \in A$ either $\chi_{\mathcal{M}}(I \cup \{b\}) = 0$ or
$$\chi_{\mathcal{M}}(I \cup \{b\}) = (-1)^{|A \cap [a,b)|} \chi_{\mathcal{M}}(I \cup \{a\}),$$
whence either $D_b = 0$ or $D_b = (-1)^{|(I \cup A) \cap [a,b)|}$. Hence $D|_{I \cup A} \leq X$, where $X$ is the sign vector in $\{0,+,-\}^{I \cup A}$ with $X_a = 1$ which strictly alternates in sign. This gives $\overline{\mathrm{var}}(D) \geq \mathrm{var}(X) = m+1$.

($\Leftarrow$): Suppose that $\overline{\mathrm{var}}(X) \geq m+1$ for some nonzero covector $X$ of $\mathcal{M}$. By Proposition 2.2.5 there exists a cocircuit $C$ of $\mathcal{M}$ with $C \leq X$, whence $\overline{\mathrm{var}}(C) \geq m+1$. We consider two cases. First suppose that $|\underline{C}| \leq n-m-1$. Take $a \in \underline{C}$, and note that by (C2) of Definition 2.2.3, $([n] \setminus \underline{C}) \cup \{a\}$ has nonempty intersection with the support of every cocircuit of $\mathcal{M}$. Hence by Definition 2.2.6, some subset of $([n] \setminus \underline{C}) \cup \{a\}$ is a basis of $\mathcal{M}$, which we may write as $I \cup \{a\}$ for some $I \in \binom{[n] \setminus C}{k-1}$. Then $I \cup \{i\}$ is not a basis of $\mathcal{M}$ for $i \in [n] \setminus \underline{C}$, whence $(\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}$ has at least $m-k+2$ zero components. This gives $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in [n] \setminus I}) \geq m-k+2$, completing the proof.

Now suppose instead that $|\underline{C}| \geq n-m-1$. There exists $J \in \binom{[n]}{m+2}$ with $\overline{\mathrm{var}}(C|_J) = m+1$; take such a $J$ which minimizes $|J \cap \underline{C}|$. It follows that $[n] \setminus \underline{C} \subseteq J$. (Otherwise there exists $e \in [n] \setminus (J \cup \underline{C})$, whence letting $e'$ equal either $\min_{f \in J \cap \underline{C}, f > e} f$ or $\max_{f \in J \cap \underline{C}, f < e} f$, at least one of which exists because $|\underline{C}| \geq n-m-1$, we have $\overline{\mathrm{var}}(C|_{(J \setminus \{e'\}) \cup \{e\}}) = m+1$, contradicting our choice of $J$.) Since $|\underline{C}| \geq n-m-1$, we may take $j \in J \cap \underline{C}$. Note that by (C2) of Definition 2.2.3, $([n] \setminus \underline{C}) \cup \{j\}$ has nonempty intersection with the support of every cocircuit of $\mathcal{M}$. Hence by Definition 2.2.6, some subset of $([n] \setminus \underline{C}) \cup \{j\}$ is a basis of $\mathcal{M}$, which we may write as $I \cup \{j\}$ for some $I \in \binom{[n] \setminus C}{k-1}$. In particular, we have $I \subseteq J$.

By the pivoting property (Proposition 2.2.8), we have
$$\chi_{\mathcal{M}}(I \cup \{i\}) = (-1)^{|I \cap [j,i)|} C_i C_j \chi_{\mathcal{M}}(I \cup \{j\}) \qquad \text{for } i \in [n].$$
Also, since $C$ weakly alternates in sign on $J$, for $i \in J$ we have either $C_i = 0$ or $C_i = (-1)^{J \cap [j,i)} C_j$. Hence for $i \in J \setminus I$ we have either $\chi_{\mathcal{M}}(I \cup \{i\}) = 0$ or $\chi_{\mathcal{M}}(I \cup \{i\}) = (-1)^{(J \setminus I) \cap [j,i)} \chi_{\mathcal{M}}(I \cup \{j\})$, whence $(\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in J \setminus I}$ weakly alternates in sign on $J \setminus I$, i.e. $\overline{\mathrm{var}}((\chi_{\mathcal{M}}(I \cup \{i\}))_{i \in J \setminus I}) = m-k+2$. ■

We call an oriented matroid $\mathcal{M}$ with a totally ordered ground set *positively oriented* if every basis of $\mathcal{M}$ has the same orientation, and *alternating* if $\mathcal{M}$ is positively oriented and uniform. Hence $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative if and only if $\mathcal{M}(V)$ is positively oriented, and $V$ is totally positive if and only if $\mathcal{M}(V)$ is alternating. We now obtain the generalization of Gantmakher and Krein's characterization (Theorem 2.1.1) to oriented matroids, as a consequence of Theorem 2.3.1 in the special case $m := k - 1$.

**Corollary 2.3.4.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $[n]$.*
*(i) $\mathcal{M}$ is positively oriented if and only if $\mathrm{var}(X) \le k-1$ for all $X \in \mathcal{V}^*(\mathcal{M})$.*
*(ii) $\mathcal{M}$ is alternating if and only if $\overline{\mathrm{var}}(X) \le k-1$ for all $X \in \mathcal{V}^*(\mathcal{M}) \setminus \{0\}$.*

We remark that the forward directions of (i) and (ii) above follow from Theorem 2.1.1 and Ardila, Rincón, and Williams' result [ARW16] that every positively oriented matroid is realizable. (The converses do not so follow, because we do not know *a priori* that an oriented matroid $\mathcal{M}$ satisfying $\mathrm{var}(X) \le k-1$ for all $X \in \mathcal{V}^*(\mathcal{M})$ is realizable.) Part (ii) above is implicit in the literature (cf. [BLV78], 9.4 of [BLVS+99], and [CD00]), though we have not seen it explicitly stated and proven in this form.

**Proof.** (i) ($\Rightarrow$): Suppose that $\mathcal{M}$ is positively oriented, and let $\mathcal{N}$ be the uniform positively oriented matroid of rank $k$ with ground set $[n]$. Then by Theorem 2.3.1(ii) with $m := k-1$, we have $\overline{\mathrm{var}}(Y) \le k-1$ for all $Y \in \mathcal{V}^*(\mathcal{N}) \setminus \{0\}$. Now given any $X \in \mathcal{V}^*(\mathcal{M})$, since $\mathcal{M} \le \mathcal{N}$ there exists $Y \in \mathcal{V}^*(\mathcal{N})$ with $X \le Y$ (Definition 2.2.11), whence $\mathrm{var}(X) \le \mathrm{var}(Y) \le k-1$.

($\Leftarrow$): Suppose that $\mathrm{var}(X) \le k-1$ for all $X \in \mathcal{V}^*(\mathcal{M})$. Then by Theorem 2.3.1(i) with $m := k-1$, any two bases of $\mathcal{M}$ which have $k-1$ elements in common have the same orientation. Hence it will suffice to show that given any two bases $I$, $J$ of $\mathcal{M}$, there exist bases $I_0 := I, I_1, \dots, I_{r-1}, I_r := J$ of $\mathcal{M}$ such that $|I_{s-1} \cap I_s| \ge k-1$ for all $s \in [r]$. This follows from the *basis exchange* axiom for (oriented) matroids (p. 81 of [BLVS+99]): if $A$ and $B$ are bases of an (oriented) matroid and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\}$ is a basis.

(ii) The forward direction follows from Theorem 2.3.1(ii) with $m := k-1$. For the converse, suppose that $\overline{\mathrm{var}}(X) \le k-1$ for all $X \in \mathcal{V}^*(\mathcal{M}) \setminus \{0\}$. Then $\mathcal{M}$ is positively oriented by part (i) of this result. Also, if there exists $I \in \binom{[n]}{k}$ which is not a basis of $\mathcal{M}$, then by Definition 2.2.6 there exists a cocircuit $C$ of $\mathcal{M}$ with $I \cap \underline{C} = \emptyset$, whence $\overline{\mathrm{var}}(C) \ge k$, a contradiction. Hence $\mathcal{M}$ is uniform. ■

We have already observed that the converse to Theorem 2.3.1(i) holds when $\mathcal{M}$ is a uniform oriented matroid, but not in general. Our goal in the remainder of the section is to prove a necessary and sufficient condition for having $\mathrm{var}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M})$. Namely, we give an algorithm for perturbing any oriented matroid $\mathcal{M}$ with a totally ordered ground set into a uniform $\mathcal{N} \ge \mathcal{M}$ of the same rank, such that $\max_{X \in \mathcal{V}^*(\mathcal{M})} \mathrm{var}(X) = \max_{Y \in \mathcal{V}^*(\mathcal{N})} \mathrm{var}(Y)$; we then apply Theorem 2.3.1 to $\mathcal{N}$ to determine $\max_{X \in \mathcal{V}^*(\mathcal{M})} \mathrm{var}(X)$ (Theorem 2.3.14). In the case of realizable oriented matroids $\mathcal{M}(V)$ ($V \in \mathrm{Gr}_{k,n}$), this perturbation involves repeatedly adding a very small multiple of one column of a $k \times n$

matrix whose rows span $V$ to an adjacent column (and taking the row span of the resulting matrix). These perturbations generalize to all oriented matroids, as we explain below.

Let $\mathcal{M}$ be an oriented matroid with ground set $E$. A *single element extension of* $\mathcal{M}$ *at* $a$ is an oriented matroid $\widetilde{\mathcal{M}}$ with ground set $E \sqcup \{a\}$ (where $\sqcup$ denotes disjoint union) and the same rank as $\mathcal{M}$, such that $\widetilde{\mathcal{M}}|_E = \mathcal{M}$. (Some authors allow $\widetilde{\mathcal{M}}$ to have rank greater than $\mathcal{M}$.) Las Vergnas [LV78] studied single element extensions; we use his results as stated in [BLVS$^+$99]. For a sign vector $X \in \{0,+,-\}^E$ and $y \in \{0,+,-\}$, let $(X,y)_a \in \{0,+,-\}^{E\sqcup\{a\}}$ denote the sign vector whose restriction to $E$ is $X$ and whose $a$th component is $y$.

**Lemma 2.3.5** (cocircuits of single element extensions; 7.1.4 of [BLVS$^+$99])**.** *Suppose that the oriented matroid* $\widetilde{\mathcal{M}}$ *is the single element extension of* $\mathcal{M}$ *at* $a$*, where* $\mathcal{M}$ *has ground set* $E$ *and rank* $k$*. Then there exists a unique function* $\sigma : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}$ *such that* $(C,\sigma(C))_a$ *is a cocircuit of* $\widetilde{\mathcal{M}}$ *for all cocircuits* $C$ *of* $\mathcal{M}$*. We have* $\mathcal{C}^*(\widetilde{\mathcal{M}}) = \mathcal{C}_1 \sqcup \mathcal{C}_2$*, where*

$$\mathcal{C}_1 := \{(C,\sigma(C))_a : C \in \mathcal{C}^*(\mathcal{M})\},$$
$$\mathcal{C}_2 := \left\{(C \circ D, 0)_a : \begin{array}{l} C, D \in \mathcal{C}^*(\mathcal{M}) \text{ are conformal}, \sigma(C) = -\sigma(D) \neq 0, \text{ and} \\ |B \setminus (\underline{C} \cup \underline{D})| \geq k-2 \text{ for some basis } B \text{ of } \mathcal{M} \end{array}\right\}.$$

In this case we say that $\widetilde{\mathcal{M}}$ is the single element extension of $\mathcal{M}$ at $a$ *by* $\sigma$. In general, not all functions $\sigma : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}$ give rise to single element extensions. However, for $e \in E$ the evaluation function $\phi_e : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}, C \mapsto C_e$ and its negation $-\phi_e$ are guaranteed to give single element extensions (7.1.8 of [BLVS$^+$99]). (Geometrically, the single element extension by $\phi_e$ duplicates coordinate $e$ at the new coordinate $a$.) Also, if the two functions $\sigma, \tau : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}$ each give rise to a single element extension of $\mathcal{M}$, then so does the composition $\sigma \circ \tau$ (7.2.2 of [BLVS$^+$99]), and the extension of $\mathcal{M}$ by $\sigma$ is less than or equal to (under Definition 2.2.11) the extension of $\mathcal{M}$ by $\sigma \circ \tau$ (7.7.8 of [BLVS$^+$99]). (The composition $\sigma \circ \tau : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}$ is defined just as for covectors, by

$$(\sigma \circ \tau)(C) := \begin{cases} \sigma(C), & \text{if } \sigma(C) \neq 0 \\ \tau(C), & \text{if } \sigma(C) = 0 \end{cases} \qquad \text{for } C \in \mathcal{C}^*(\mathcal{M}).)$$

**Definition 2.3.6** ($i \to_\epsilon j$-perturbation)**.** Let $\mathcal{M}$ be an oriented matroid with ground set $E$, $i, j \in E$, and $\epsilon \in \{+,-\}$. If $i = j$ or $j$ is a coloop of $\mathcal{M}$, set $\mathcal{N} := \mathcal{M}$. (A *coloop* $c$ of an (oriented) matroid is an element of its ground set which is in every basis.) Otherwise, the restriction $\mathcal{M} \setminus \{j\}$ has the same rank as $\mathcal{M}$, so $\mathcal{M}$ is the single element extension of $\mathcal{M} \setminus \{j\}$ at $j$ by some $\sigma : \mathcal{C}^*(\mathcal{M} \setminus \{j\}) \to \{0,+,-\}$. Let $\mathcal{N}$ be the single element extension of $\mathcal{M} \setminus \{j\}$ at $j$ by $\sigma \circ \epsilon\phi_i$, which is well defined and satisfies $\mathcal{N} \geq \mathcal{M}$ by the preceding discussion. We call $\mathcal{N}$ the $i \to_\epsilon j$*-perturbation of* $\mathcal{M}$.

We now prove several properties of $i \to_\epsilon j$-perturbation.

**Lemma 2.3.7** (chirotope of the $i \to_\epsilon j$-perturbation)**.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with a totally ordered ground set $E$, and $\mathcal{N}$ is the $i \to_\epsilon j$-perturbation of $\mathcal{M}$ (where $i, j \in E$ and $\epsilon \in \{+,-\}$). Then the chirotope of $\mathcal{N}$ is given by*

$$\chi_{\mathcal{N}}(I) = \begin{cases} (-1)^{|I \cap (i,j)|}\epsilon\chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\}), & \text{if } i \notin I,\ j \in I, \text{ and } \chi_{\mathcal{M}}(I) = 0 \\ \chi_{\mathcal{M}}(I), & \text{otherwise} \end{cases}$$

*for $I \in \binom{E}{k}$, where $(i,j)$ denotes the set of elements of $E$ strictly between $i$ and $j$.*

**Proof.** If $i = j$ or $j$ is a coloop of $\mathcal{M}$, then $\mathcal{N} = \mathcal{M}$ and the result is clear, so we may assume that $i \neq j$ and $j$ is not a coloop of $\mathcal{M}$. Let $I \in \binom{E}{k}$. If $j \notin I$, we have $\chi_{\mathcal{N}}(I) = \chi_{\mathcal{M}}(I)$ because $\mathcal{M} \setminus \{j\} = \mathcal{N} \setminus \{j\}$. Also, if $\chi_{\mathcal{M}}(I) \neq 0$, we have $\chi_{\mathcal{N}}(I) = \chi_{\mathcal{M}}(I)$ because $\mathcal{M} \leq \mathcal{N}$. Hence we may assume that $j \in I$ and $\chi_{\mathcal{M}}(I) = 0$. Then by Definition 2.2.6 there exists a cocircuit $C$ of $\mathcal{M}$ with $I \cap \underline{C} = \emptyset$. In particular, $C_j = 0$. If $C_i = 0$, then by Lemma 2.3.5 $C$ is also a cocircuit of $\mathcal{N}$, whence by Definition 2.2.6 both $I$ and $(I \setminus \{j\}) \cup \{i\}$ are not bases of $\mathcal{M}$ or $\mathcal{N}$, giving $\chi_{\mathcal{N}}(I) = \chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\}) = \chi_{\mathcal{M}}(I) = 0$.

Suppose instead that $C_i \neq 0$. In particular $i \notin I$, so we must show that $\chi_{\mathcal{N}}(I) = (-1)^{|I \cap (i,j)|}\epsilon\chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\})$. By Lemma 2.3.5, we get a cocircuit $D$ of $\mathcal{N}$ such that $D_e = C_e$ for $e \in E \setminus \{j\}$, and either $D_j = \epsilon C_i$ (if $C \in \mathcal{C}_1$) or $D_j = 0$ (if $C \in \mathcal{C}_2$). Hence by the pivoting property (Proposition 2.2.8), we have

$$\chi_{\mathcal{N}}(I) = (-1)^{|I \cap (i,j)|}D_iD_j\chi_{\mathcal{N}}((I \setminus \{j\}) \cup \{i\}),$$

and $\chi_{\mathcal{N}}((I \setminus \{j\}) \cup \{i\}) = \chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\})$ since $j \notin (I \setminus \{j\}) \cup \{i\}$. If $C \in \mathcal{C}_1$, then $D_iD_j = \epsilon$, giving $\chi_{\mathcal{N}}(I) = (-1)^{|I \cap (i,j)|}\epsilon\chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\})$. Now suppose that $C \in \mathcal{C}_2$. Then $D_j = 0$, giving $\chi_{\mathcal{N}}(I) = 0$; we must show that $\chi_{\mathcal{M}}((I \setminus \{j\}) \cup \{i\}) = 0$. Since $C \in \mathcal{C}_2$, we can write $C = (X \circ Y, 0)_j$ for some conformal cocircuits $X, Y$ of $\mathcal{M} \setminus \{j\}$ with $\sigma(X) = -\sigma(Y) \neq 0$, where $\mathcal{M}$ is the single element extension of $\mathcal{M} \setminus \{j\}$ by $\sigma$. From $I \cap \underline{C} = \emptyset$ we get $I \cap \underline{X} = I \cap \underline{Y} = \emptyset$. Also, by Lemma 2.3.5, $(X, \sigma(X))_j$ and $(Y, \sigma(Y))_j$ are cocircuits of $\mathcal{M}$. Hence if $i \notin \underline{X}$ or $i \notin \underline{Y}$, then $(I \setminus \{j\}) \cup \{i\}$ is not a basis of $\mathcal{M}$ by Definition 2.2.6. Otherwise we have $X_i = Y_i \neq 0$ (since $X$ and $Y$ are conformal), and $X \neq Y$ (since $\sigma(X) \neq \sigma(Y)$). Then by (C3) of Definition 2.2.3, there exists a cocircuit of $\mathcal{M}$ whose support is contained in $(\underline{X} \cup \underline{Y} \cup \{j\}) \setminus \{i\} \subseteq E \setminus ((I \setminus \{j\}) \cup \{i\})$, whence $(I \setminus \{j\}) \cup \{i\}$ is not a basis of $\mathcal{M}$. ∎

**Corollary 2.3.8** (geometric interpretation of $i \to_\epsilon j$-perturbation)**.** *Suppose that $V \in \mathrm{Gr}_{k,n}$, $i, j \in [n]$, and $\epsilon \in \{+,-\}$. For $\alpha \in \mathbb{R}$, let $W(\alpha) \in \mathrm{Gr}_{k,n}$ be the row span of the $k \times n$ matrix $[x^{(1)}|\cdots|x^{(j-1)}|(x^{(j)} + \alpha x^{(i)})|x^{(j+1)}|\cdots|x^{(n)}]$, where $[x^{(1)}|\cdots|x^{(n)}]$ is a $k \times n$ matrix whose rows span $V$. (Note that $W(\alpha)$ does not depend on the choice of matrix.) Then for all $\alpha \in \mathbb{R}$ with sign $\epsilon$ such that $\Delta_I(W(\alpha))$ has the same sign as $\Delta_I(V)$ for all $I \in \binom{[n]}{k}$ with $\Delta_I(V) \neq 0$, $\mathcal{M}(W(\alpha))$ is the $i \to_\epsilon j$-perturbation of $\mathcal{M}(V)$.*

Note that the possible values of $\alpha$ form an open interval between 0 and some number, or $\pm\infty$, with sign $\epsilon$.

**Example 2.3.9.** Let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -1 & 4 \end{bmatrix}$, and for $\alpha < 0$ let $W(\alpha) \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & 2 & \alpha \\ 0 & 3 & -1 & 4 \end{bmatrix}$. Note that the $\{3,4\}$-minor of the first matrix equals 8, and the $\{3,4\}$-minor of the second matrix equals $8+\alpha$, so we should pick $\alpha > -8$ so that these minors agree in sign. In fact, for all $\alpha \in (-8,0)$ the corresponding minors of the two matrices agree in sign whenever the first minor is nonzero, whence $\mathcal{M}(W(\alpha))$ equals the $1 \to_{-} 4$-perturbation of $\mathcal{M}(V)$. $\diamondsuit$

**Proof** (of Corollary 2.3.8)**.** Note that for $I \in \binom{[n]}{k}$ and $\alpha \in \mathbb{R}$, we have

$$\Delta_I(W(\alpha)) = \begin{cases} \Delta_I(V) + (-1)^{|I \cap (i,j)|}\alpha\Delta_{(I \setminus \{j\}) \cup \{i\}}(V), & \text{if } i \notin I \text{ and } j \in I \\ \Delta_I(V), & \text{otherwise} \end{cases}, \tag{2.3.10}$$

where $(i,j)$ denotes the set of elements of $[n]$ strictly between $i$ and $j$. Hence the result follows from Lemma 2.3.7. ∎

We observe that certain $i \to_{\epsilon} j$-perturbations do not increase sign variation.

**Lemma 2.3.11** (sign variation and $i \to_{\epsilon} j$-perturbation)**.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $[n]$, and $m \ge k-1$.*
*(i) Let $\mathcal{N}$ be either the $(i+1) \to_{+} i$-perturbation of $\mathcal{M}$ ($i \in [n-1]$), the $i \to_{+} (i+1)$-perturbation of $\mathcal{M}$ ($i \in [n-1]$), the $1 \to_{(-1)^m} n$-perturbation of $\mathcal{M}$, or the $n \to_{(-1)^m} 1$-perturbation of $\mathcal{M}$. If $\mathrm{var}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M})$, then $\mathrm{var}(Y) \le m$ for all $Y \in \mathcal{V}^*(\mathcal{N})$.*
*(ii) Suppose that $\mathcal{P} \ge \mathcal{M}$ has rank $k$. If $\overline{\mathrm{var}}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M}) \setminus \{0\}$, then $\overline{\mathrm{var}}(Y) \le m$ for all $Y \in \mathcal{V}^*(\mathcal{P}) \setminus \{0\}$.*

Note that (i) above does not hold for any other $i \to_{\epsilon} j$-perturbations of $\mathcal{M}$ (assuming $i \neq j$); for counterexamples, we can take $k := 1$ and $m \in \{0,1\}$.

**Proof.** (i) Note that for $X \in \{0,+,-\}^n$, we have $\mathrm{var}(X) = \mathrm{var}((X_n, X_{n-1}, \ldots, X_1))$, and if $\mathrm{var}(X) \le m$ then $\mathrm{var}((X_2, X_3, \ldots, X_n, (-1)^m X_1)) \le m$. By this cyclic symmetry, it will suffice to prove the result assuming that $\mathcal{N}$ is the $2 \to_{+} 1$-perturbation of $\mathcal{M}$. Suppose that $\mathrm{var}(X) \le m$ for all $X \in \mathcal{V}^*(\mathcal{M})$, but there exists a covector $Y$ of $\mathcal{N}$ with $\mathrm{var}(Y) \ge m+1$. We will derive a contradiction by showing that $Y$ is a covector of $\mathcal{M}$.

Since $\mathcal{M} \setminus \{1\} = \mathcal{N} \setminus \{1\}$, by Definition 2.2.10 we have $X|_{[n]\setminus\{1\}} = Y|_{[n]\setminus\{1\}}$ for some covector $X$ of $\mathcal{M}$. From $\mathrm{var}(X) \le m$, we get that $Y_1 \neq 0, Y_2$. Write $\mathcal{M}$ as the single element extension of $\mathcal{M} \setminus \{1\}$ by $\sigma : \mathcal{C}^*(\mathcal{M} \setminus \{1\}) \to \{0,+,-\}$. Since $Y$ is a composition of conformal cocircuits of $\mathcal{N}$ (Proposition 2.2.5), by Lemma 2.3.5 we have a composition of conformal cocircuits

$$Y = (C^{(1)}, (\sigma \circ \phi_2)(C^{(1)}))_1 \circ \cdots \circ (C^{(r)}, (\sigma \circ \phi_2)(C^{(r)}))_1 \circ (D^{(1)} \circ E^{(1)}, 0)_1 \circ \cdots \circ (D^{(s)} \circ E^{(s)}, 0)_1$$

for some cocircuits $C^{(1)}, \dots, C^{(r)}, D^{(1)}, E^{(1)}, \dots, D^{(s)}, E^{(s)}$ of $\mathcal{M} \setminus \{1\}$. If $Y_2 = 0$, then $(\sigma \circ \phi_2)(C^{(t)}) = \sigma(C^{(t)})$ for $t \in [r]$, whence by Lemma 2.3.5

$$\begin{aligned} Y = (C^{(1)}, \sigma(C^{(1)}))_1 \circ \cdots \circ (C^{(r)}, \sigma(C^{(r)}))_1 \circ \\ (D^{(1)}, \sigma(D^{(1)}))_1 \circ (E^{(1)}, \sigma(E^{(1)}))_1 \circ \cdots \circ (D^{(s)}, \sigma(D^{(s)}))_1 \circ (E^{(s)}, \sigma(E^{(s)}))_1 \end{aligned}$$

is a covector of $\mathcal{M}$, a contradiction. Hence $Y_1 = -Y_2$. In particular, because the composition above is of conformal cocircuits, we have $C_2^{(t)} \neq Y_1$ and $(\sigma \circ \phi_2)(C^{(t)}) \neq -Y_1$ for $t \in [r]$. We also have $\sigma(C^{(u)}) = Y_1$ for some $u \in [r]$, since otherwise $(\sigma \circ \phi_2)(C^{(t)}) = \phi_2(C^{(t)}) \neq Y_1$ for all $t \in [r]$. This gives

$$\begin{aligned} Y = (C^{(u)}, \sigma(C^{(u)}))_1 \circ (C^{(1)}, \sigma(C^{(1)}))_1 \circ \cdots \circ (C^{(r)}, \sigma(C^{(r)}))_1 \circ \\ (D^{(1)}, \sigma(D^{(1)}))_1 \circ (E^{(1)}, \sigma(E^{(1)}))_1 \circ \cdots \circ (D^{(s)}, \sigma(D^{(s)}))_1 \circ (E^{(s)}, \sigma(E^{(s)}))_1, \end{aligned}$$

so $Y$ is a covector of $\mathcal{M}$ by Lemma 2.3.5, a contradiction.

(ii) This follows from a general fact about oriented matroids $\mathcal{A}$ and $\mathcal{B}$ with the same rank and ground set (7.7.5 of [BLVS+99]): $\mathcal{A} \leq \mathcal{B}$ if and only if for all nonzero covectors $Y$ of $\mathcal{B}$, there exists a nonzero covector $X$ of $\mathcal{A}$ with $X \leq Y$. ■

We now explain how to perturb an oriented matroid into a uniform oriented matroid by repeatedly applying $i \to_\epsilon j$-perturbations.

**Proposition 2.3.12** (uniform perturbation). *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $[n]$.*
*(i) The oriented matroid obtained from $\mathcal{M}$ by applying any $k(2n-k-1)$ consecutive perturbations of the sequence*

$$\dots, (n-1) \to n, n \to 1, 1 \to 2, 2 \to 3, \dots, (n-1) \to n, n \to 1, 1 \to 2, \dots$$

*in order from left to right is uniform. (Here an $i \to j$-perturbation denotes either of the $i \to_\epsilon j$-perturbations, for $\epsilon \in \{+,-\}$.)*
*(ii) The oriented matroid obtained from $\mathcal{M}$ by applying any $(n-k)(n+k-1)$ consecutive perturbations of the sequence*

$$\dots, n \to (n-1), 1 \to n, 2 \to 1, 3 \to 2, \dots, n \to (n-1), 1 \to n, 2 \to 1, \dots$$

*in order from left to right is uniform.*
*(iii) The oriented matroid obtained from $\mathcal{M}$ by applying the sequence of perturbations*

$$1 \to 2, 2 \to 3, \dots, (n-1) \to n, n \to (n-1), (n-1) \to (n-2), \dots, 2 \to 1$$

*in order from left to right $k$ times is uniform.*
*(iv) The oriented matroid obtained from $\mathcal{M}$ by applying the sequence of perturbations*

$$2 \to 1, 3 \to 2, \dots, n \to (n-1), (n-1) \to n, (n-2) \to (n-1), \dots, 1 \to 2$$

*in order from left to right $n-k$ times is uniform.*

Thus we have four specific algorithms for perturbing $\mathcal{M}$ into a uniform oriented matroid, each using at most $2n^2$ perturbations. For example, if $k := 1$ and $n := 3$, then applying any of the following sequences of perturbations to $\mathcal{M}$, in order from left to right, produces a uniform oriented matroid:

- $1 \to 2, 2 \to 3, 3 \to 1, 1 \to 2$ (by (i)); or
- $2 \to 3, 3 \to 1, 1 \to 2, 2 \to 3$ (by (i)); or
- $3 \to 1, 1 \to 2, 2 \to 3, 3 \to 1$ (by (i)); or
- $2 \to 1, 3 \to 2, 1 \to 3, 2 \to 1, 3 \to 2, 1 \to 3$ (by (ii)); or
- $3 \to 2, 1 \to 3, 2 \to 1, 3 \to 2, 1 \to 3, 2 \to 1$ (by (ii)); or
- $1 \to 3, 2 \to 1, 3 \to 2, 1 \to 3, 2 \to 1, 3 \to 2$ (by (ii)); or
- $1 \to 2, 2 \to 3, 3 \to 2, 2 \to 1$ (by (iii)); or
- $2 \to 1, 3 \to 2, 2 \to 3, 1 \to 2, 2 \to 1, 3 \to 2, 2 \to 3, 1 \to 2$ (by (iv)).

**Example 2.3.13.** Let $V \in \mathrm{Gr}_{2,3}$ be the row span of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that the vectors in $V$ change sign at most $m := 1$ time. Now $V$ is not generic, because $\Delta_{\{2,3\}}(V) = 0$. We can perturb $V$ into a generic subspace by applying a $3 \to_- 1$-perturbation, giving the row span of

$$\begin{bmatrix} 1 & 3 & 0 \\ \alpha & 0 & 1 \end{bmatrix} \qquad (\alpha < 0),$$

or by applying a $3 \to_+ 2$-perturbation, giving the row span of

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & \beta & 1 \end{bmatrix} \qquad (\beta > 0).$$

The vectors in either of these generic subspaces change sign at most once, as guaranteed by Lemma 2.3.11. Note that we cannot make $V$ generic by applying only $1 \to 2$- and $2 \to 3$-perturbations. $\diamondsuit$

**Proof** (of Proposition 2.3.12)**.** Let $\mathcal{N}$ be an oriented matroid of rank $k$ with ground set $[n]$. The *dual* $\mathcal{N}^*$ of $\mathcal{N}$ is the oriented matroid of rank $n-k$ with ground set $[n]$ whose chirotope is given by $\chi_{\mathcal{N}^*}(J) = (-1)^{\sum_{j \in J} j} \chi_{\mathcal{N}}([n] \setminus J)$ for $J \in \binom{[n]}{n-k}$. Note that $\mathcal{N}$ is uniform if and only if $\mathcal{N}^*$ is uniform. Also, Lemma 2.3.7 implies that the dual of the $i \to_\epsilon j$-perturbation of $\mathcal{N}$ is the $j \to_{-\epsilon} i$-perturbation of $\mathcal{N}^*$. Hence statements (i) and (ii) are dual, and statements (iii) and (iv) are dual. We will prove (ii) and (iv).

A *hyperplane* of an (oriented) matroid is a maximal subset of its ground set which contains no basis. Note that by Definition 2.2.6 and (C2) of Definition 2.2.3, hyperplanes are precisely the complements of supports of cocircuits. Now suppose that we have a collection of functions, each of which, given an oriented matroid $\mathcal{P}$ of rank $k$ with ground set $[n]$, produces an oriented matroid $\mathcal{P}' \geq \mathcal{P}$ of rank $k$, such that no hyperplane of $\mathcal{P}$ of size at least $k$ is a hyperplane of $\mathcal{P}'$. Note that every basis of $\mathcal{P}$ is a basis of $\mathcal{P}'$ (by Definition 2.2.11), so every hyperplane of $\mathcal{P}'$ is contained in a hyperplane of $\mathcal{P}$. Hence the maximum size of a hyperplane of $\mathcal{P}'$ is less than the maximum size of a hyperplane of $\mathcal{P}$, unless every hyperplane of $\mathcal{P}$ has size less than $k$ (i.e. $\mathcal{P}$ is uniform). By applying such a function $n-k$ times (possibly a different function in our collection each time), we obtain a uniform oriented matroid. Thus to prove (ii), it suffices to show that for all $i \in \mathbb{Z}$, applying the sequence of perturbations

$$(i+1) \to i, (i+2) \to (i+1), \dots, (i+n+k-1) \to (i+n+k-2)$$

(where we read the indices modulo $n$) in order from left to right is such a function (we then apply this function for $i = j, j+(n+k-1), j+2(n+k-1), \dots, j+(n-k-1)(n+k-1)$ for any $j \in \mathbb{Z}$). Similarly, to prove (iv), it suffices to show that applying the sequence of perturbations

$$2 \to 1, 3 \to 2, \dots, n \to (n-1), (n-1) \to n, (n-2) \to (n-1), \dots, 1 \to 2$$

in order from left to right is such a function. To this end, we prove the following claim.

*Claim.* Suppose that $\mathcal{P}$ is an oriented matroid of rank $k$ with ground set $[n]$, and $I \subseteq [n]$ is a hyperplane of $\mathcal{P}$ with $|I| \ge k$. Take $a \in I$ and $b \in [k]$ such that
- $a$ is not a coloop of $\mathcal{P}|_I$;
- $a+1, a+2, \dots, a+b-1 \in I$ are coloops of $\mathcal{P}|_I$; and
- $a+b \notin I$,

where we read the indices modulo $n$. Then for all $\mathcal{Q} \ge \mathcal{P}$ of rank $k$, $I$ is not a hyperplane the oriented matroid obtained from $\mathcal{Q}$ by applying the sequence of perturbations $(a+1) \to a, (a+2) \to (a+1), \dots, (a+b) \to (a+b-1)$ in order from left to right, where we read the indices modulo $n$.

*Proof of Claim.* First note that $I \neq [n]$, and $\mathcal{P}|_I$ has at most $k-1$ coloops (otherwise the rank of $\mathcal{P}|_I$ would be at least $k$), so such $a$ and $b$ exist. Also note that for any oriented matroids $\mathcal{A} \le \mathcal{B}$ of equal rank with ground set $[n]$, by Lemma 2.3.7 the $i \to_\epsilon j$-perturbation of $\mathcal{A}$ is less than or equal to the $i \to_\epsilon j$-perturbation of $\mathcal{B}$, for all $i, j \in [n]$ and $\epsilon \in \{+,-\}$. Hence it will suffice to prove the claim assuming that $\mathcal{Q} = \mathcal{P}$.

Let $\mathcal{P}^{(0)} := \mathcal{P}$, and define $\mathcal{P}^{(c)}$ recursively for $c = 1, \dots, b$ as either of the $(a+c) \to_\epsilon (a+c-1)$-perturbations of $\mathcal{P}^{(c-1)}$ for $\epsilon \in \{+,-\}$. Also let $J \in \binom{I}{k-1}$ be a basis of $\mathcal{P}|_I$ which does not contain $a$. Since $a+1, a+2, \dots, a+b-1$ are coloops of $\mathcal{P}|_I$, they are in $J$.

We claim that $(J \cup \{a, a+b\}) \setminus \{a+c\}$ is a basis of $\mathcal{P}^{(c)}$ for $0 \le c \le b$. Let us prove this by induction on $c$. For the base case $c = 0$, we must show that $J \cup \{a+b\}$ is a basis of $\mathcal{P}$. If not, then by Definition 2.2.6 there exists a cocircuit $C$ of $\mathcal{P}$ with $(J \cup \{a+b\}) \cap \underline{C} = \emptyset$. We then have $C_I = 0$. (Otherwise there exists a cocircuit $D$ of $\mathcal{P}_I$ with $D \le C_I$ by Proposition 2.2.5, whence $J \cap \underline{D} = \emptyset$, contradicting Definition 2.2.6 since $J$ is a basis of $\mathcal{P}_I$.) This gives $\underline{C} \subset [n] \setminus I$, which contradicts (C2) of Definition 2.2.3 because $[n] \setminus I$ is the support of a cocircuit of $\mathcal{P}$. For the induction step, suppose that $c \in [b]$ and $(J \cup \{a, a+b\}) \setminus \{a+c-1\}$ is a basis of $\mathcal{P}^{(c-1)}$. By Lemma 2.3.7, we have

$$\chi_{\mathcal{P}^{(c)}}((J \cup \{a, a+b\}) \setminus \{a+c\}) =$$

$$\begin{cases} \pm\chi_{\mathcal{P}^{(c-1)}}((J \cup \{a, a+b\}) \setminus \{a+c-1\}), & \text{if } \chi_{\mathcal{P}^{(c-1)}}((J \cup \{a, a+b\}) \setminus \{a+c\}) = 0 \\ \chi_{\mathcal{P}^{(c-1)}}((J \cup \{a, a+b\}) \setminus \{a+c\}), & \text{otherwise} \end{cases}.$$

In the first case we have $\chi_{\mathcal{P}^{(c)}}((J \cup \{a, a+b\}) \setminus \{a+c\}) \neq 0$ by the induction hypothesis, while in the second case $(J \cup \{a, a+b\}) \setminus \{a+c\}$ is a basis of $\mathcal{P}^{(c-1)}$, and hence also of $\mathcal{P}^{(c)} \geq \mathcal{P}^{(c-1)}$. This completes the induction. Taking $c := b$ we get that $J \cup \{a\}$ is a basis of $\mathcal{P}^{(c)}$, and so $I$ is not a hyperplane of $\mathcal{P}^{(c)}$. □

Note that for any $a \in \mathbb{Z}$ and $b \in [k]$, the sequence $(a+1) \to a, (a+2) \to (a+1), \ldots, (a+b) \to (a+b-1)$ is a consecutive subsequence of

$$(i+1) \to i, (i+2) \to (i+1), \ldots, (i+n+k-1) \to (i+n+k-2)$$

for all $i \in \mathbb{Z}$ (where we read the indices modulo $n$). This proves (ii).

For (iv), let $\mathcal{P}$ be an oriented matroid of rank $k$ with ground set $[n]$, and $I \subseteq [n]$ a hyperplane of $\mathcal{P}$ with $|I| \geq k$. It will suffice to show that $I$ is not a hyperplane of the oriented matroid $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying the sequence of perturbations

$$2 \to 1, 3 \to 2, \ldots, n \to (n-1), (n-1) \to n, (n-2) \to (n-1), \ldots, 1 \to 2$$

in order from left to right. To this end, take $i \in [n] \setminus I$. If there exists an element of $[1, i] \cap I$ which is not a coloop of $\mathcal{P}|_I$, then we may take $a$ and $b$ as in the statement of the claim such that we also have $1 \leq a < a+b \leq i$; then $I$ is not a hyperplane of the oriented matroid obtained from any $\mathcal{Q} \geq \mathcal{P}$ by applying the sequence of perturbations $(a+1) \to a, (a+2) \to (a+1), \ldots, (a+b) \to (a+b-1)$ in order from left to right, whence $I$ is not a hyperplane of $\mathcal{P}'$. Otherwise, there exists an element of $[i, n] \cap I$ which is not a coloop of $\mathcal{P}|_I$, whence we take $a' \in [i, n] \cap I$ and $b' \in [k]$ such that
- $a'$ is not a coloop of $\mathcal{P}|_I$;
- $a'-1, a'-2, \ldots, a'-b'+1 \in I$ are coloops of $\mathcal{P}|_I$; and
- $a'-b' \notin I$.

We have $a'-b' \geq i$, and by the claim $I$ is not a hyperplane of the oriented matroid obtained from any $\mathcal{Q} \geq \mathcal{P}$ by applying the sequence of perturbations $(a'-1) \to a', (a'-2) \to (a'-1), \ldots, (a'-b') \to (a'-b'+1)$ in order from left to right, whence $I$ is not a hyperplane of $\mathcal{P}'$. ■

We are now ready to give a necessary and sufficient condition that $\mathrm{var}(X) \leq m$ for all $X \in \mathcal{V}^*(\mathcal{M})$.

**Theorem 2.3.14.** *Suppose that $\mathcal{M}$ is an oriented matroid of rank $k$ with ground set $[n]$, and $m \geq k-1$. Let $\mathcal{N}$ be any oriented matroid obtained from $\mathcal{M}$ by applying one of the following sequences of perturbations:*
- *any $k(2n-k-1)$ consecutive perturbations of the sequence*

$$\ldots, (n-1) \to_{+} n, n \to_{(-1)^m} 1, 1 \to_{+} 2, 2 \to_{+} 3, \ldots, (n-1) \to_{+} n, n \to_{(-1)^m} 1, 1 \to_{+} 2, \ldots$$

*in order from left to right; or*
• *any* $(n-k)(n+k-1)$ *consecutive perturbations of the sequence*

$$\dots, n \to_+ (n-1), 1 \to_{(-1)^m} n, 2 \to_+ 1, 3 \to_+ 2, \dots, n \to_+ (n-1), 1 \to_{(-1)^m} n, 2 \to_+ 1, \dots$$

*in order from left to right; or*
• *the sequence of perturbations*

$$1 \to_+ 2, 2 \to_+ 3, \dots, (n-1) \to_+ n, n \to_+ (n-1), (n-1) \to_+ (n-2), \dots, 2 \to_+ 1$$

*in order from left to right* $k$ *times; or*
• *the sequence of perturbations*

$$2 \to_+ 1, 3 \to_+ 2, \dots, n \to_+ (n-1), (n-1) \to_+ n, (n-2) \to_+ (n-1), \dots, 1 \to_+ 2$$

*in order from left to right* $n-k$ *times.*

*Then* $\mathcal{N}$ *is uniform, and the following are equivalent:*
*(i)* $\mathrm{var}(X) \le m$ *for all* $X \in \mathcal{V}^*(\mathcal{M})$*;*
*(ii)* $\mathrm{var}(Y) \le m$ *for all* $Y \in \mathcal{V}^*(\mathcal{N})$*; and*
*(iii)* $\mathrm{var}((\chi_{\mathcal{N}}(I \cup \{i\}))_{i \in [n] \setminus I}) \le m-k+1$ *for all* $I \in \binom{[n]}{k-1}$.

Note that the first two sequences of perturbations take advantage of the cyclic symmetry of sign variation, but they depend on (the parity of) $m$, whereas the last two sequences do not. Note that none of the sequences depend on $\mathcal{M}$ (only on $n$ and $k$, and perhaps $m$).

**Proof.** Proposition 2.3.12 implies that $\mathcal{N}$ is uniform. We have (i) ⇒ (ii) by Lemma 2.3.11, (ii) ⇒ (i) by Definition 2.2.11, (ii) ⇒ (iii) by Theorem 2.3.1(i), and (iii) ⇒ (ii) by Theorem 2.3.1(ii) (since $\mathcal{N}$ is uniform). ■

We can interpret this statement as a closure result in the space of oriented matroids (or the Grassmannian $\mathrm{Gr}_{k,n}$), where the *closure* of a set $S$ of oriented matroids is $\{\mathcal{M} : \mathcal{M} \le \mathcal{N} \text{ for some } \mathcal{N} \in S\}$. ($\mathrm{Gr}_{k,n}$ has the classical topology.)

**Theorem 2.3.15.** *Let* $n \ge k \ge 0$ *and* $m \ge k-1$.
*(i) Let* $S$ *be the set of oriented matroids* $\mathcal{M}$ *of rank* $k$ *with ground set* $[n]$ *satisfying* $\mathrm{var}(X) \le m$ *for all* $X \in \mathcal{V}^*(\mathcal{M})$*. Then the closure of the set of uniform elements of* $S$ *(in the space of oriented matroids of rank* $k$ *with ground set* $[n]$*) equals* $S$*.*
*(ii) Let* $T := \{V \in \mathrm{Gr}_{k,n} : \mathrm{var}(v) \le m \text{ for all } v \in V\}$*. Then the closure in* $\mathrm{Gr}_{k,n}$ *of the set of generic elements of* $T$ *equals* $T$*.*

**Proof.** Theorem 2.3.14 implies (i). For (ii), note that the closure in $\mathrm{Gr}_{k,n}$ of the generic elements of $T$ is contained in $T$. Conversely, given $V \in T$ we can construct (by Theorem 2.3.14) a sequence $\mathcal{M}_0 := \mathcal{M}(V), \mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_r$ of elements of $S$ such that $\mathcal{M}_s$ is the $i_s \to_{\epsilon_s} j_s$-perturbation of $\mathcal{M}_{s-1}$ (for some $i_s, j_s \in [n]$ and $\epsilon_s \in \{+,-\}$) for all $s \in [r]$, and $\mathcal{M}_r$ is uniform. For $\alpha > 0$, let $V_0(\alpha) := V$, and define $V_s(\alpha) \in \mathrm{Gr}_{k,n}$ recursively for $s = 1, \ldots, r$ as the row span of the $k \times n$ matrix $[x^{(1)}|\cdots|x^{(j_s-1)}|(x^{(j_s)} + \epsilon_s\alpha^{2^{s-1}}x^{(i_s)})|x^{(j_s+1)}|\cdots|x^{(n)}]$, where $[x^{(1)}|\cdots|x^{(n)}]$ is a $k \times n$ matrix whose rows span $V_{s-1}(\alpha)$. Note that for $0 \le s \le r$, every Plücker coordinate of $V_s(\alpha)$ is a polynomial in $\alpha$ of degree at most $2^s - 1$; we can prove this by induction on $s$, using (2.3.10).

*Claim.* Let $s \in [r]$ and $I \in \binom{[n]}{k}$. Then for $\alpha > 0$ sufficiently small, either $\Delta_I(V_{s-1}(\alpha)) = 0$, or $\Delta_I(V_s(\alpha))$ and $\Delta_I(V_{s-1}(\alpha))$ are nonzero with the same sign.

*Proof of Claim.* Regard $\Delta_I(V_{s-1}(\alpha))$ as a polynomial in $\alpha$. If this polynomial is zero then the claim is proven, so suppose that this polynomial is nonzero, and write $\Delta_I(V_{s-1}(\alpha)) = c\alpha^d + O(\alpha^{d+1})$ (as $\alpha \to 0$) for some $d \le 2^{s-1} - 1$ and $c \ne 0$. Then by (2.3.10) we have $\Delta_I(V_s(\alpha)) = \Delta_I(V_{s-1}(\alpha)) + O(\alpha^{2^{s-1}}) = c\alpha^d + O(\alpha^{d+1})$. Hence for $\alpha > 0$ sufficiently small, we have $\mathrm{sign}(\Delta_I(V_s(\alpha))) = \mathrm{sign}(\Delta_I(V_{s-1}(\alpha))) = \mathrm{sign}(c)$. □

Thus by Corollary 2.3.8, for $\alpha > 0$ sufficiently small we have $\mathcal{M}(V_s(\alpha)) = \mathcal{M}_s$ for all $s \in [r]$, whence $V_r(\alpha)$ is generic and $V_r(\alpha) \in T$. Taking $\alpha \to 0$ shows explicitly that $V$ is in the closure of $T$. ■

## 2.4 Defining amplituhedra and Grassmann polytopes

Let $k, n, r \in \mathbb{N}$ with $n \ge k, r$, and let $Z : \mathbb{R}^n \to \mathbb{R}^r$ be a linear map, which we also regard as an $r \times n$ matrix. Arkani-Hamed and Trnka [AHT14] consider the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ on the totally nonnegative Grassmannian. Explicitly, if $X$ is a $k \times n$ matrix whose row span is $V \in \mathrm{Gr}_{k,n}^{\ge 0}$, then $Z(V)$ is the row span of the $k \times r$ matrix $XZ^T$. In the case that $k \le r$ and all $r \times r$ minors of $Z$ are positive, Arkani-Hamed and Trnka call the image of this map a *(tree) amplituhedron*, and use it to calculate scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (taking $r := k + 4$). One motivation they provide for requiring that $k \le r$ and $Z$ have positive $r \times r$ minors is to guarantee that the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined, i.e. that $Z(V)$ has dimension $k$ for all $V \in \mathrm{Gr}_{k,n}^{\ge 0}$. As a more general sufficient condition for this map to be well defined, Lam [Lam16b] requires that the row span of $Z$ has a $k$-dimensional subspace which is totally positive. (It is not obvious that Arkani-Hamed and Trnka's condition is indeed a special case of Lam's; see Section 15.1 of [Lam16b].) In the case that the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined, Lam calls the image a *(full) Grassmann polytope*, since in the case $k = 1$ Grassmann polytopes are precisely polytopes in the projective space $\mathrm{Gr}_{1,r} = \mathbb{P}^{r-1}$ (and the amplituhedra are projective *cyclic polytopes*). In this section we give (Theorem 2.4.2) a necessary and sufficient condition for the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ to be well defined, in terms of sign variation; we are able to translate this into a condition on the maximal minors of $Z$ using the results of Section 2.3. As a consequence, we recover Arkani-Hamed and Trnka's and Lam's sufficient conditions. To be thorough, we similarly determine when the map $\mathrm{Gr}_{k,n}^{>0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ on the totally positive Grassmannian is well defined (Theorem 2.4.4).

**Lemma 2.4.1.** *Let $v \in \mathbb{R}^n \setminus \{0\}$ and $k \le n$.*
*(i) There exists an element of $\mathrm{Gr}_{k,n}^{\ge 0}$ containing $v$ if and only if $\mathrm{var}(v) \le k-1$.*
*(ii) There exists an element of $\mathrm{Gr}_{k,n}^{>0}$ containing $v$ if and only if $\overline{\mathrm{var}}(v) \le k-1$.*

**Proof.** The forward directions of (i) and (ii) follow from Gantmakher and Krein's result (Theorem 2.1.1). For the reverse direction of (i), suppose that $\mathrm{var}(v) \le k-1$. Then we may partition $[n]$ into pairwise disjoint nonempty intervals of integers $I_1, \dots, I_k$, such that for all $j \in [k]$ the components of $v|_{I_j}$ are all nonnegative or all nonpositive. For $j \in [k]$, let $w^{(j)} \in \mathbb{R}^n$ have support $I_j$ such that $w^{(j)}|_{I_j}$ equals $v|_{I_j}$ if $v|_{I_j} \neq 0$, and $e_{I_j}$ otherwise. Then $\mathrm{span}(\{w^{(j)} : j \in [k]\}) \in \mathrm{Gr}_{k,n}^{\ge 0}$ contains $v$. (For example, if $v = (2,5,0,-1,-4,-1,0,0,3)$ and $k = 4$, then we may take $I_1 := \{1,2,3\}, I_2 := \{4,5,6\}, I_3 := \{7,8\}, I_4 := \{9\}$, whence our subspace is the row span of the matrix

$$\begin{bmatrix} 2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix};$$

note that $v$ is the sum of rows 1, 2, and 4.)

Now we prove the reverse direction of (ii). The point is that by rescaling the basis vectors of $\mathbb{R}^n$ (the *torus action* on the Grassmannian), we need only determine the sign vectors appearing in totally positive subspaces.

*Claim* ([GK50], [BLVS+99]). Let $V \in \mathrm{Gr}_{k,n}^{>0}$.
(i) $\{\mathrm{sign}(v) : v \in V\} = \{X \in \{0,+,-\}^n : \overline{\mathrm{var}}(X) \le k-1\} \cup \{0\}$.
(ii) $\{\mathrm{sign}(w) : w \in V^\perp\} = \{X \in \{0,+,-\}^n : \mathrm{var}(X) \ge k\} \cup \{0\}$.

In the terminology of oriented matroids, the sets in (i) and (ii) are the covectors and vectors, respectively, of $\mathcal{M}(V)$.

*Proof of Claim.* This essentially follows from known results, as follows. First recall that by Lemma 2.1.11(ii), $V$ is totally positive if and only if $\mathrm{alt}(V^\perp)$ is totally positive. Hence parts (i) and (ii) of the claim are equivalent by Lemma 2.1.11(i). Let us prove (ii). The containment $\subseteq$ follows from Gantmakher and Krein's result (Corollary 2.1.12(ii)). For the containment $\supseteq$, given $X \in \{0,+,-\}^n$ with $\mathrm{var}(X) \ge k$, take $I \in \binom{[n]}{k+1}$ such that $X$ alternates in sign on $I$. By Proposition 9.4.1 of [BLVS+99], there exists $w \in V^\perp$ such that $\mathrm{sign}(w|_I) = X|_I$ and $\mathrm{sign}(w|_{[n]\setminus I}) = 0$. Now for each $j \in [n] \setminus I$, take $v^{(j)} \in V^\perp$ such that $v^{(j)}_j = 1$ and $v^{(j)}|_{[n]\setminus(I\cup\{j\})} = 0$. (For example, fix any $h \in I$, whence $\Delta_{([n]\setminus I)\cup\{h\}}(V^\perp) \neq 0$ since $V^\perp$ is generic. Then take any $(n-k) \times n$ matrix whose rows span $V^\perp$, and row reduce it so that we get an identity matrix in the columns $([n] \setminus I) \cup \{h\}$. Then we let $v^{(j)}$ for $j \in [n] \setminus I$ be the row of this matrix whose pivot column is $j$.) By perturbing $w$ by $v^{(j)}$ for $j \in [n] \setminus I$ so that $w_j$ has sign $X_j$, we obtain a vector in $V^\perp$ with sign vector $X$. $\square$

Suppose that $\overline{\mathrm{var}}(v) \le k-1$. Take any $V \in \mathrm{Gr}_{k,n}^{>0}$ (e.g. let $V$ be the row span of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{k-1} & t_2^{k-1} & \cdots & t_n^{k-1} \end{bmatrix},$$

where $t_1 < \cdots < t_n$). Then the oriented matroid $\mathcal{M}(V)$ defined by $V$ is the alternating oriented matroid of rank $k$ with ground set $[n]$, whence $\mathrm{sign}(v)$ is a covector of $\mathcal{M}$ by the claim. That is (cf. Definition 2.2.1), there exist $\alpha_1, \ldots, \alpha_n > 0$ such that $(\alpha_1 v_1, \ldots, \alpha_n v_n) \in V$. Then $\{(\frac{w_1}{\alpha_1}, \ldots, \frac{w_n}{\alpha_n}) : w \in V\} \in \mathrm{Gr}_{k,n}^{>0}$ contains $v$. ■

**Theorem 2.4.2.** *Suppose that $k, n, r \in \mathbb{N}$ with $n \ge k, r$, and that $Z : \mathbb{R}^n \to \mathbb{R}^r$ is a linear map, which we also regard as an $r \times n$ matrix. Let $d$ be the rank of $Z$ and $W \in \mathrm{Gr}_{d,n}$ the row span of $Z$, so that $W^\perp = \ker(Z) \in \mathrm{Gr}_{n-d,n}$. The following are equivalent:*
*(i) the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined, i.e. $\dim(Z(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\ge 0}$;*
*(ii) $\mathrm{var}(v) \ge k$ for all nonzero $v \in \ker(Z)$; and*
*(iii) $\overline{\mathrm{var}}((\Delta_{I\setminus\{i\}}(W))_{i\in I}) \le d-k$ for all $I \in \binom{[n]}{d+1}$ such that $W|_I$ has dimension $d$.*

We explain how to use Theorem 2.4.2 to deduce the sufficient conditions of Arkani-Hamed and Trnka, and of Lam, for the map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ to be well defined. Note that if the $r \times r$ minors of $Z$ are all positive, then $d = r$ and $W$ is totally positive, so the condition (iii) holds for any $k \le r$. Alternatively, by Corollary 2.1.12(ii), we have $\mathrm{var}(v) \ge r$ for all nonzero $v \in \ker(Z)$, so the condition (ii) holds for any $k \le r$. This recovers the sufficient condition of Arkani-Hamed and Trnka [AHT14]. On the other hand, if $W$ has a subspace $V \in \mathrm{Gr}_{k,n}^{>0}$, then by Corollary 2.1.12(ii) we have $\mathrm{var}(v) \ge k$ for all $v \in V^\perp \setminus \{0\}$, which implies condition (ii) above since $\ker(Z) = W^\perp \subseteq V^\perp$. This recovers the sufficient condition of Lam [Lam16b]. However, our result does not show why Arkani-Hamed and Trnka's condition is a special case of Lam's. Indeed, it is an interesting open problem to determine whether or not Lam's sufficient condition is also necessary, i.e. whether the condition $\mathrm{var}(v) \ge k$ for all nonzero $v \in W^\perp$ implies that $W$ has a totally positive $k$-dimensional subspace. [1]

**Example 2.4.3.** Let $Z : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map given by the matrix $\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 3 \end{bmatrix}$ (so $n = 4$, $d = r = 2$), and let $W \in \mathrm{Gr}_{2,4}$ be the row span of this matrix. Let us use Theorem 2.4.2(iii) to determine for which $k$ ($0 \le k \le 4$) the map $\mathrm{Gr}_{k,4}^{\ge 0} \to \mathrm{Gr}_{k,2}$ induced by $Z$ is well defined. The 4 relevant sequences of Plücker coordinates (as $I$ ranges over $\binom{[4]}{3}$) are

$$(\Delta_{\{2,3\}}(W), \Delta_{\{1,3\}}(W), \Delta_{\{1,2\}}(W)) = (-1, -3, 5),$$

[1] After this work was published, Pavel Galashin informed me that he found an example showing that Lam's condition is strictly stronger than mine. Thomas Lam has proposed using my condition for the definition of a 'Grassmann polytope', calling those defined by $Z$ satisfying his condition 'tame' Grassmann polytopes.

$$(\Delta_{\{2,4\}}(W), \Delta_{\{1,4\}}(W), \Delta_{\{1,2\}}(W)) = (-5, 5, 5),$$
$$(\Delta_{\{3,4\}}(W), \Delta_{\{1,4\}}(W), \Delta_{\{1,3\}}(W)) = (4, 5, -3),$$
$$(\Delta_{\{3,4\}}(W), \Delta_{\{2,4\}}(W), \Delta_{\{2,3\}}(W)) = (4, -5, -1).$$

The maximum number of sign changes among these 4 sequences is 1, which is at most $2-k$ if and only if $k \le 1$. Hence the map is well defined if and only if $k \le 1$.

Note that for $k \ge 2$, the proof of Lemma 2.4.1(i) shows how to explicitly construct $V \in \mathrm{Gr}_{k,4}^{\ge 0}$ with $\dim(Z(V)) < k$: take a nonzero $v \in \ker(Z)$ with $\mathrm{var}(v) \le 1$, and extend $v$ to $V \in \mathrm{Gr}_{k,4}^{\ge 0}$. For example, if $k = 2$ we can take $v = (1, -3, -5, 0) \in \ker(Z)$ and extend it to the row span $V \in \mathrm{Gr}_{2,4}^{\ge 0}$ of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -5 & 0 \end{bmatrix}$. Note that $Z(V)$ is the span of $(2, 1)$, so $\dim(Z(V)) = 1 < \dim(V)$. $\diamondsuit$

**Proof** (of Theorem 2.4.2)**.** (i) $\Leftrightarrow$ (ii): The map $\mathrm{Gr}_{k,n}^{\ge 0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined if and only if for all $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ and $v \in V \setminus \{0\}$, we have $Z(v) \ne 0$. This condition is equivalent to (ii) above by Lemma 2.4.1(i).

(ii) $\Leftrightarrow$ (iii): This is exactly the dual statement of (the realizable case of) Theorem 2.3.1(ii). Explicitly, recall that $\mathrm{alt} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\mathrm{alt}(v) := (v_1, -v_2, v_3, \dots, (-1)^{n-1}v_n)$ for $v \in \mathbb{R}^n$. By Lemma 2.1.11(i), the condition (ii) is equivalent to $\overline{\mathrm{var}}(v) \le n-k-1$ for all $v \in \mathrm{alt}(\ker(Z)) \setminus \{0\}$, which is in turn equivalent to $\overline{\mathrm{var}}((\Delta_{J\cup\{i\}}(\mathrm{alt}(\ker(Z))))_{i\in[n]\setminus J}) \le d-k$ for all $J \in \binom{[n]}{n-d-1}$ such that $\Delta_{J\cup\{i\}}(\mathrm{alt}(\ker(Z))) \ne 0$ for some $i \in [n]$ by Theorem 2.3.1(ii). This condition is precisely (iii) above, since $\Delta_K(W) = \Delta_{[n]\setminus K}(\mathrm{alt}(\ker(Z)))$ for all $K \in \binom{[n]}{d}$ by Lemma 2.1.11(ii). ■

We give the analogue of Theorem 2.4.2 for the map induced by $Z$ not on $\mathrm{Gr}_{k,n}^{\ge 0}$, but on $\mathrm{Gr}_{k,n}^{>0}$.

**Theorem 2.4.4.** *Suppose that $k, n, r \in \mathbb{N}$ with $n \ge k, r$, and that $Z : \mathbb{R}^n \to \mathbb{R}^r$ is a linear map, which we also regard as an $r \times n$ matrix. Let $d$ be the rank of $Z$ and $W \in \mathrm{Gr}_{d,n}$ the row span of $Z$, so that $W^\perp = \ker(Z) \in \mathrm{Gr}_{n-d,n}$. The following are equivalent:*
*(i) the map $\mathrm{Gr}_{k,n}^{>0} \to \mathrm{Gr}_{k,r}$ induced by $Z$ is well defined, i.e. $\dim(Z(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{>0}$;*
*(ii) $\overline{\mathrm{var}}(v) \ge k$ for all nonzero $v \in \ker(Z)$; and*
*(iii) there exists a generic perturbation $W' \in \mathrm{Gr}_{d,n}$ of $W$ such that $\mathrm{var}((\Delta_{I\setminus\{i\}}(W'))_{i\in I}) \le d-k$ for all $I \in \binom{[n]}{d+1}$.*

We omit the proof, since it is similar to that of Theorem 2.4.2; we only mention that instead of Lemma 2.4.1(i) we use Lemma 2.4.1(ii), and along with Theorem 2.3.1(ii) we also use Theorem 2.3.15.

## 2.5 Positroids from sign vectors

Recall that the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ has a cell decomposition, where the *positroid cell* of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ is determined by $M(V) := \{I \in \binom{[n]}{k} : \Delta_I(V) \neq 0\}$. The goal of this section is show how to obtain the positroid cell of a given $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ from the sign vectors of $V$ (i.e. $\mathcal{V}^*(\mathcal{M}(V))$). Note that $M(V)$ is the set of bases of $\mathcal{M}(V)$, so $\mathcal{V}^*(\mathcal{M}(V))$ determines $M(V)$ by the theory of oriented matroids. However, this does not exploit the fact that $V$ is totally nonnegative. We now describe two other ways to recover $M(V)$ from the sign vectors of $V$, both of which require $V$ to be totally nonnegative.

We begin by examining the *Schubert cell* of $V$, which is labeled by the lexicographic minimum of $M(V)$. Recall that the *Gale partial order* $\leq_{\mathrm{Gale}}$ on $\binom{[n]}{k}$ is defined by

$$I \leq_{\mathrm{Gale}} J \quad\iff\quad i_1 \leq j_1,\ i_2 \leq j_2,\ \dots,\ i_k \leq j_k$$

for subsets $I = \{i_1, \dots, i_k\}$ $(i_1 < \cdots < i_k)$, $J = \{j_1, \dots, j_k\}$ $(j_1 < \cdots < j_k)$ of $[n]$. Note that $I \leq_{\mathrm{Gale}} J$ if and only if $|I \cap [m]| \geq |J \cap [m]|$ for all $m \in [n]$. Also recall that for $V \in \mathrm{Gr}_{k,n}$, $A(V)$ is the set of $I \in \binom{[n]}{k}$ such that some vector in $V$ strictly alternates in sign on $I$. Note that if $I \in M(V)$ then $V|_I = \mathbb{R}^I$, so $M(V) \subseteq A(V)$. We can obtain the Schubert cell of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ from $A(V)$ as follows.

**Theorem 2.5.1** (Schubert cell from sign vectors)**.** *For $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, the lexicographic minimum of $M(V)$ equals the Gale minimum of $A(V)$.*

We remark that the lexicographic minimum of $M(V)$ is also the Gale minimum of $M(V)$, for all $V \in \mathrm{Gr}_{k,n}$. (In general, the lexicographically minimal basis of any matroid with a totally ordered ground set is also a Gale minimum [Gal63].) However, $A(V)$ does not necessarily equal $M(V)$ (see Example 2.1.9 or Example 2.5.2), nor does $A(V)$ necessarily uniquely determine $M(V)$ (see Example 2.5.2). Also, if $V$ is not totally nonnegative, then $A(V)$ does not necessarily have a Gale minimum (see Example 2.5.3).

**Example 2.5.2.** Let $V, W \in \mathrm{Gr}_{2,3}^{\geq 0}$ be the row spans of the matrices $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, respectively. Then $A(V) = A(W) = \binom{[3]}{2}$, but $M(V) \neq M(W)$ since $\{1,3\} \in M(W) \setminus M(V)$. $\lozenge$

**Example 2.5.3.** Let $V \in \mathrm{Gr}_{3,6}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, which is not totally nonnegative. Then $(1,-1,-2,-3,1,0) \in V$ strictly alternates in sign on $\{1,2,5\}$, and $(3,2,-1,1,3,0) \in V$ strictly alternates in sign on $\{1,3,4\}$, but no vector in $V$ strictly alternates in sign on $\{1,2,3\}$ or $\{1,2,4\}$. Hence $A(V)$ has no Gale minimum. $\lozenge$

**Proof.** Given $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, let $I$ be the lexicographic minimum of $M(V)$.

*Claim.* Let $m \in [n]$ and $l := |I \cap [m]|$. Then $V|_{[m]} \in \mathrm{Gr}_{l,m}^{\geq 0}$.

*Proof of Claim.* Express $V$ as the row span of a $k \times n$ matrix $X = [x^{(1)}|\cdots|x^{(n)}]$ whose restriction to the columns in $I$ is an identity matrix. Note that $V|_{[m]}$ is the row span of the first $m$ columns of $X$. Since $\{x^{(i)} : i \in I \cap [m]\}$ is linearly independent, we may extend $I \cap [m]$ to $B \in M(V|_{[m]})$, and then extend $B$ to $B' \in M(V)$. Since $I$ is the Gale minimum of $M(V)$ [Gal63], we have $I \leq_{\mathrm{Gale}} B'$. In particular $|I \cap [m]| \geq |B' \cap [m]|$, so $B = I \cap [m]$. Hence $\dim(V|_{[m]}) = l$, and the entries in the first $m$ columns of $X$ past the $l$th row are all zero. It follows that $V|_{[m]}$ is the row span of the submatrix of $X$ formed by the first $l$ rows and the first $m$ columns. Since the restriction of $X$ to the columns in $I$ is an identity matrix, we see that $\Delta_K(V|_{[m]}) = \Delta_{K \cup (I \setminus [m])}(V) \geq 0$ for $K \in \binom{[m]}{l}$. □

Hence if $v \in V$ strictly alternates in sign on $J \in \binom{[n]}{k}$, by Theorem 2.1.1 we get $|I \cap [m]| - 1 \geq \mathrm{var}(v|_{[m]}) \geq |J \cap [m]| - 1$ for all $m \in [n]$, whence $I \leq_{\mathrm{Gale}} J$. ■

Given $n \geq 0$, for $j \in [n]$ let $\leq_j$ be the total order on $[n]$ defined by $j <_j j+1 <_j \cdots <_j n <_j 1 <_j \cdots <_j j-1$. Then for $V \in \mathrm{Gr}_{k,n}$, we let $I_j$ ($j \in [n]$) denote the lexicographic minimum of $M(V)$ with respect to $\leq_j$. The tuple $(I_1, \ldots, I_n)$ is called the *Grassmann necklace* of $V$. For example, if $V \in \mathrm{Gr}_{2,4}$ is generic, then the Grassmann necklace of $V$ is $(\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\})$. The Grassmann necklace is of special interest to us because of a result of Postnikov (Theorem 17.1 of [Pos]), which implies that if $V$ is totally nonnegative, the positroid cell of $V$ is determined by its Grassmann necklace. Oh [Oh11] explicitly described $M(V)$ in terms of the Grassmann necklace of $V$, for $V \in \mathrm{Gr}_{k,n}^{\geq 0}$.

**Theorem 2.5.4** ([Oh11])**.** *Suppose that $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ has Grassmann necklace $(I_1, \ldots, I_n) \in \binom{[n]}{k}^n$. Then*

$$M(V) = \left\{ J \in \binom{[n]}{k} : I_j \leq_{j\text{-Gale}} J \text{ for all } j \in [n] \right\}.$$

(Here $\leq_{j\text{-Gale}}$ denotes the Gale order on $\binom{[n]}{k}$ induced by $\leq_j$.)

We can generalize Theorem 2.5.1 to the Grassmann necklace $(I_1, \ldots, I_n)$ of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ as follows. For $j \in [n]$, we define $V_j$ as the row span of the cyclically shifted $k \times n$ matrix $[x^{(j)}|x^{(j+1)}|\cdots|x^{(n)}|(-1)^{k-1}x^{(1)}|\cdots|(-1)^{k-1}x^{(j-1)}]$, where $[x^{(1)}|\cdots|x^{(n)}]$ is a $k \times n$ matrix whose rows span $V$. Note that $V_j$ does not depend on our choice of matrix, and since $V$ is totally nonnegative so is $V_j$. Then $\{i - j + 1 \pmod{n} : i \in I_j\}$ is the lexicographic minimum of $M(V_j)$, and so applying Theorem 2.5.1 to $V_j$ gives the following result.

**Corollary 2.5.5** (Grassmann necklace from sign vectors)**.** *Suppose that $V \in \mathrm{Gr}_{k,n}^{\geq 0}$. For $j \in [n]$, let $A_j$ be the set of $J \in \binom{[n]}{k}$ such that some vector in $V$ strictly alternates in sign on $J$ except precisely from component* $\max(J \cap [1, j))$ *to component* $\min(J \cap [j, n])$ *(if both components exist). Then $A_j$ has a $j$-Gale minimum $I_j$ for all $j \in [n]$, and $(I_1, \ldots, I_n)$ is the Grassmann necklace of $V$.*

For example, if $n := 5$, $J := \{1,3,4,5\}$, and $j := 3$, then $(1,1,1,-1,1)$ strictly alternates in sign on $J$ except precisely from component $\max(J \cap [1,j))$ to component $\min(J \cap [j,n])$, but $(1,1,-1,1,-1)$ does not. (If $j \le \min(J)$ or $j > \max(J)$, then the condition reduces to "strictly alternates in sign on $J$.")

With Oh's result (Theorem 2.5.4), we get the following corollary.

**Corollary 2.5.6.** *Suppose that $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ has Grassmann necklace $(I_1, \dots, I_n) \in \binom{[n]}{k}^n$, and $J \in \binom{[n]}{k}$. Then the following are equivalent:*
*(i) $J \in M(V)$;*
*(ii) $I_j \le_{j\text{-Gale}} J$ for all $j \in [n]$; and*
*(iii) $V$ realizes all $2k$ sign vectors in $\{+,-\}^J$ which alternate in sign between every pair of consecutive components, with at most one exceptional pair.*

For example, if $k = 5$ the $2k$ sign vectors in (iii) above are $(+,-,+,-,+)$, $(+,+,-,+,-)$, $(+,-,-,+,-)$, $(+,-,+,+,-)$, $(+,-,+,-,-)$, and their negations. Since $V$ realizes a sign vector if and only if $V$ realizes its negation, we need only check $k$ sign vectors in (iii) up to sign.

**Proof.** We have (i) $\Rightarrow$ (iii) since $J \in M(V)$ implies $V|_J = \mathbb{R}^J$, (iii) $\Rightarrow$ (ii) by Corollary 2.5.5(ii), and (ii) $\Rightarrow$ (i) by Oh's result (Theorem 2.5.4). ■

We can prove (iii) $\Rightarrow$ (i) directly from Theorem 2.1.1, as follows. Suppose that (iii) holds, but $J \notin M(V)$. Then there exists $v \in V \setminus \{0\}$ with $v|_J = 0$; take $j \in [n]$ such that $v_j \neq 0$. Then (iii) guarantees the existence of a vector $w \in V$ which strictly alternates in sign on $J$ except precisely from component $\max(J \cap [1,j))$ to component $\min(J \cap [j,n])$ (if both components exist). Adding a sufficiently large multiple of $\pm v$ to $w$ gives a vector in $V$ which strictly alternates in sign on $J \cup \{j\}$, contradicting Theorem 2.1.1. This establishes the equivalence of (i) and (iii) without appealing to Oh's result (Theorem 2.5.4). The implication (i) $\Rightarrow$ (ii) is a general fact about matroids [Gal63]. We would be interested to see a direct proof of (ii) $\Rightarrow$ (iii) (and hence of Corollary 2.5.6) which is substantially different from Oh's proof, using the tools of sign variation.

**Remark 2.5.7.** We remark that (iii) $\Rightarrow$ (i) does not necessarily hold when $V$ is not totally nonnegative; in fact, it is possible that $V$ realizes all $2^k$ sign vectors in $\{+,-\}^J$ except two, but $J \notin M(V)$. To see this, given $J \in \binom{[n]}{k}$, let $v \in \mathbb{R}^J$ have no zero components, and take $V \in \mathrm{Gr}_{k,n}$ such that $V|_J = \{v\}^\perp$ (which is always possible, assuming $n > k$). That is, $J \notin M(V)$ and $V|_J = \{w \in \mathbb{R}^J : \sum_{j \in J} v_j w_j = 0\}$. We see that if $w \in \mathbb{R}^J$ satisfies $\mathrm{sign}(w) = \mathrm{sign}(v)$, then $\sum_{j \in J} v_j w_j > 0$, and so $w \notin V|_J$. Similarly, if $\mathrm{sign}(w) = -\mathrm{sign}(v)$ then $w \notin V|_J$. Conversely, given $\omega \in \{+,-\}^J$ with $\omega \neq \pm\mathrm{sign}(v)$, let us construct $w \in V|_J$ with $\mathrm{sign}(w) = \omega$. Take $a, b \in J$ such that $\mathrm{sign}(v_a)\omega_a \neq \mathrm{sign}(v_b)\omega_b$. For $j \in J \setminus \{a,b\}$ let $w_j$ be any real number with sign $\omega_j$, then take $w_b$ with sign $\omega_b$ and sufficiently large magnitude that $\mathrm{sign}(\sum_{j \in J \setminus \{a\}} v_j w_j) = \mathrm{sign}(v_b)\omega_b$, and set $w_a := -\frac{\sum_{j \in J \setminus \{a\}} v_j w_j}{v_a}$. Thus $V$ realizes all sign vectors in $\{+,-\}^J$ except for precisely $\pm\mathrm{sign}(v)$.

On the other hand, if $V$ realizes all $2^k$ sign vectors in $\{+,-\}^J$, then $J \in M(V)$. Indeed, if $J \notin M(V)$ then we may take $v \in (V|_J)^\perp \setminus \{0\}$, whence $V$ does not realize any $\omega \in \{+,-\}^J$ satisfying $\mathrm{sign}(v) \le \omega$.

# Chapter 3

# The $m = 1$ amplituhedron and cyclic hyperplane arrangements

The work in this chapter is joint with Lauren Williams, and has been posted on the arXiv [KW]. I am grateful to her for allowing me to include this work in my dissertation. This work is an offshoot of a larger ongoing project which is joint with Yan Zhang, and we thank him for many helpful conversations. We also thank Richard Stanley for providing a reference on hyperplane arrangements, Nima Arkani-Hamed, Hugh Thomas, and Jaroslav Trnka for sharing their results, Thomas Lam for giving useful comments about Proposition 3.8.4, Pavel Galashin for resolving Problem 3.3.14, and anonymous referees for their feedback.

## 3.1 Introduction

The totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ consisting of points with all Plücker coordinates nonnegative. Following seminal work of Lusztig [Lus94], as well as by Fomin and Zelevinsky [FZ99], Postnikov initiated the combinatorial study of $\mathrm{Gr}_{k,n}^{\geq 0}$ and its cell decomposition [Pos]. Since then the totally nonnegative Grassmannian has found applications in diverse contexts such as mirror symmetry [MR], soliton solutions to the KP equation [KW14], and scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [AHBC$^+$16].

Building on [AHBC$^+$16], Arkani-Hamed and Trnka [AHT14] recently introduced a beautiful new mathematical object called the *(tree) amplituhedron*, which is the image of the totally nonnegative Grassmannian under a particular map.

**Definition 3.1.1.** Let $Z$ be a $(k+m)\times n$ real matrix whose maximal minors are all positive, where $m \geq 0$ is fixed with $k+m \leq n$. Then it induces a map

$$\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$$

defined by

$$\tilde{Z}(\langle v_1, \dots, v_k\rangle) := \langle Z(v_1), \dots, Z(v_k)\rangle,$$

where $\langle v_1, \ldots, v_k \rangle$ is an element of $\mathrm{Gr}_{k,n}^{\geq 0}$ written as the span of $k$ basis vectors.[1] The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is defined to be the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ inside $\mathrm{Gr}_{k,k+m}$.

In special cases the amplituhedron recovers familiar objects. If $Z$ is a square matrix, i.e. $k+m = n$, then $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to the totally nonnegative Grassmannian. If $k = 1$, then $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in projective space [Stu88].

While the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is an interesting mathematical object for any $m$, the case of immediate relevance to physics is $m = 4$. In this case, it provides a geometric basis for the computation of *scattering amplitudes* in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. These amplitudes are complex numbers related to the probability of observing a certain scattering process of $n$ particles. It is expected that such amplitudes can be expressed (modulo higher-order terms) as an integral over the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. This statement would follow from the conjecture of Arkani-Hamed and Trnka [AHT14] that the images of a certain collection of $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ provide a "triangulation" of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. More specifically, the BCFW recurrence [BCF05, BCFW05] provides one way to compute scattering amplitudes. Translated into the Grassmannian formulation of [AHBC+16], the terms in the BCFW recurrence can be identified with a collection of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$. If the images of these *BCFW cells* in $\mathcal{A}_{n,k,4}(Z)$ fit together in a nice way, then we can combine the contributions from each term into a single integral over $\mathcal{A}_{n,k,4}(Z)$.

In this chapter, we study the amplituhedron $\mathcal{A}_{n,k,1}(Z)$ for $m = 1$. We find that this object is already interesting and non-trivial. Since $\mathcal{A}_{n,k,1}(Z) \subseteq \mathrm{Gr}_{k,k+1}$, it is convenient to take orthogonal complements and work with lines rather than $k$-planes in $\mathbb{R}^{k+1}$. This leads us to define a related "B-amplituhedron"

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

which is homeomorphic to $\mathcal{A}_{n,k,m}(Z)$, where $W$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $Z$ (Section 3.3). In the context of scattering amplitudes ($m = 4$), $W$ is the span of 4 bosonic variables and $k$ fermionic variables. Building on the results of Section 2.3, we use this reformulation to give a description of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ in terms of sign variation (Section 3.3).

Modeling the $m = 4$ case, we define a BCFW-like recursion in the case $m = 1$, which we use to produce a subset of $k$-dimensional "BCFW cells" of $\mathrm{Gr}_{k,n}^{\geq 0}$ (Section 3.4). The set of all cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ are in bijection with various combinatorial objects, including Ꮣ-diagrams and decorated permutations, so we describe our $m = 1$ BCFW cells in terms of these objects. We then show that their images triangulate the $m = 1$ amplituhedron; more specifically, we show that $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a $k$-dimensional subcomplex of the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ (Section 3.5). See Figure 3.6 for $\mathcal{A}_{4,2,1}(Z)$ as a subcomplex of $\mathrm{Gr}_{2,4}^{\geq 0}$.

[1] The fact that $Z$ has positive maximal minors ensures that $\tilde{Z}$ is well defined [AHT14]. See Theorem 2.4.2 for a characterization of when a matrix $Z$ gives rise to a well-defined map $\tilde{Z}$.

We also show that $\mathcal{A}_{n,k,1}(Z)$ can be identified with the complex of bounded faces of a certain hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^k$, called a *cyclic hyperplane arrangement* (Section 3.6; see also Figure 1.4). We use this description of the $m = 1$ amplituhedron to describe how its cells fit together (Section 3.7).

It is known that the totally nonnegative Grassmannian has a remarkably simple topology: it is contractible with boundary a sphere [RW10], and its poset of cells is Eulerian [Wil07]. While there are not yet any general results in this direction, calculations of Euler characteristics [FGMT15] indicate that the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is likely also topologically very nice. Our description of $\mathcal{A}_{n,k,1}(Z)$ as the complex of bounded faces of a hyperplane arrangement, together with a result of Dong [Don08], implies that the $m = 1$ amplituhedron is homeomorphic to a closed ball (Corollary 3.6.18).

Since $k + m \leq n$, the map

$$\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$$

is far from injective in general. We determine when an arbitrary cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ is mapped injectively by $\tilde{Z}$ into $\mathcal{A}_{n,k,1}(Z)$, and in this case we describe its image in $\mathcal{A}_{n,k,1}(Z)$ (Section 3.8).

Finally, we discuss to what extent our results hold in the setting of *Grassmann polytopes* (Section 3.9). Grassmann polytopes are generalizations of amplituhedra obtained by relaxing the positivity condition on the matrix $Z$ [Lam16b].

## 3.2 Background on the totally nonnegative Grassmannian

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the space of all $k$-dimensional subspaces of $\mathbb{R}^n$, for $0 \leq k \leq n$. An element of $\mathrm{Gr}_{k,n}$ can be viewed as a $k \times n$ matrix of rank $k$, modulo left multiplication by invertible $k \times k$ matrices. That is, two $k \times n$ matrices of rank $k$ represent the same point in $\mathrm{Gr}_{k,n}$ if and only if they can be obtained from each other by invertible row operations.

Let $[n]$ denote $\{1, \dots, n\}$, and $\binom{[n]}{k}$ the set of all $k$-element subsets of $[n]$. Given $V \in \mathrm{Gr}_{k,n}$ represented by a $k \times n$ matrix $A$, for $I \in \binom{[n]}{k}$ we let $\Delta_I(V)$ be the maximal minor of $A$ located in the column set $I$. The $\Delta_I(V)$ do not depend on our choice of matrix $A$ (up to simultaneous rescaling by a nonzero constant), and are called the *Plücker coordinates* of $V$.

**Definition 3.2.1** (Section 3 of [Pos])**.** We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and *totally positive* if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. The set of all totally nonnegative $V \in \mathrm{Gr}_{k,n}$ is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$, and the set of all totally positive $V$ is the *totally positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$. For $M \subseteq \binom{[n]}{k}$, the *positroid cell* $S_M$ is the set of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ with the prescribed collection of Plücker coordinates strictly positive (i.e. $\Delta_I(V) > 0$ for all $I \in M$), and the remaining Plücker coordinates equal to zero (i.e. $\Delta_J(V) = 0$ for all $J \in \binom{[n]}{k} \setminus M$). We call $M$ a *positroid* if $S_M$ is nonempty. We let $Q_{k,n}$ denote the poset on the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ defined by $S_M \leq S_{M'}$ if and only if $S_M \subseteq \overline{S_{M'}}$.

**Remark 3.2.2.** There is an action of the "positive torus" $T_{>0} = \mathbb{R}^n_{>0}$ on $\mathrm{Gr}^{\geq 0}_{k,n}$. Concretely, if $A$ is a $k \times n$ matrix representing an element of $\mathrm{Gr}^{\geq 0}_{k,n}$, then the positive torus acts on $A$ by rescaling its columns. If $\mathbf{t} = (t_1, \ldots, t_n) \in T_{>0}$ and $A$ represents an element of $S_M$, then $\mathbf{t} \cdot A$ also represents an element of $S_M$.

The fact that each nonempty $S_M$ is a topological cell is due to Postnikov [Pos]. Moreover, it was shown in [PSW09] that the cells glue together to form a CW decomposition of $\mathrm{Gr}^{\geq 0}_{k,n}$.

## Combinatorial objects parameterizing cells

In [Pos], Postnikov gave several families of combinatorial objects in bijection with cells of the totally nonnegative Grassmannian. In this section we will start by defining ⅃*-diagrams*, *decorated permutations*, and equivalence classes of *reduced plabic graphs*, and give (compatible) bijections among all these objects. This will give us a canonical way to label each positroid by a ⅃-diagram, a decorated permutation, and an equivalence class of plabic graphs.

**Definition 3.2.3.** A *decorated permutation* of the set $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are colored either black or white. We denote a black fixed point by $\pi(i) = \underline{i}$ and a white fixed point by $\pi(i) = \overline{i}$. An *anti-excedance* of the decorated permutation $\pi$ is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \overline{i}$ (i.e. $i$ is a white fixed point).

**Definition 3.2.4.** Fix $k$ and $n$. Given a partition $\lambda$, we let $Y_\lambda$ denote the Young diagram associated to $\lambda$. A ⅃*-diagram* (or Le-diagram) $D$ of shape $\lambda$ and type $(k, n)$ is a Young diagram of shape $Y_\lambda$ contained in a $k \times (n-k)$ rectangle, whose boxes are filled with 0's and $+$'s in such a way that the ⅃*-property* is satisfied: there is no 0 which has a $+$ above it in the same column and a $+$ to its left in the same row. See Figure 3.1 for an example of a ⅃-diagram.

**Lemma 3.2.5** (Section 20 of [Pos])**.** *The following algorithm is a bijection between ⅃-diagrams $D$ of type $(k, n)$ and decorated permutations $\pi$ on $[n]$ with exactly $k$ anti-excedances.*

1. *Replace each $+$ in the ⅃-diagram $D$ with an elbow joint , and each $0$ in $D$ with a cross .*
2. *Note that the southeast border of $Y_\lambda$ gives rise to a length-$n$ path from the northeast corner to the southwest corner of the $k \times (n-k)$ rectangle. Label the edges of this path with the numbers $1$ through $n$.*
3. *Now label the edges of the north and west border of $Y_\lambda$ so that opposite horizontal edges and opposite vertical edges have the same label.*
4. *View the resulting 'pipe dream' as a permutation $\pi = \pi(D)$ on $[n]$, by following the 'pipes' from the southeastern border to the northwest border of the Young diagram. If the pipe originating at label $i$ ends at the label $j$, we define $\pi(i) := j$.*

5. *If $\pi(i) = i$ and $i$ labels two horizontal (respectively, vertical) edges of $Y_\lambda$, then $\pi(i) := \underline{i}$ (respectively, $\pi(i) := \overline{i}$).*

Figure 3.1 illustrates this procedure.

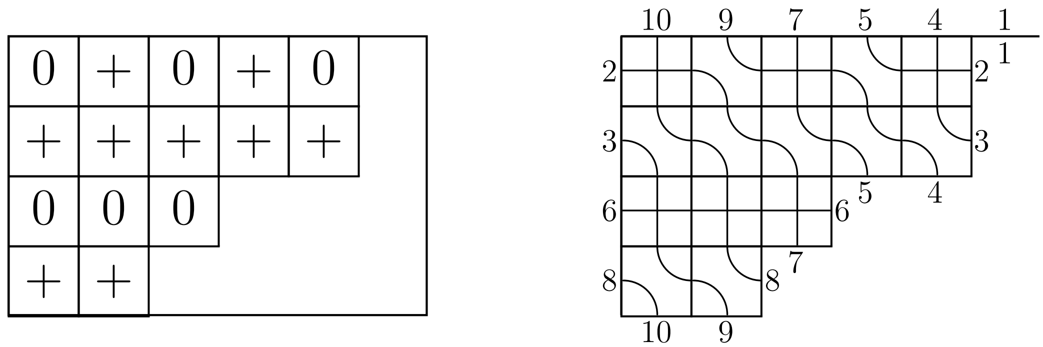



Figure 3.1: A ⅃-diagram with $\lambda = (5,5,3,2)$, $n = 10$, and $k = 4$, and its corresponding pipe dream with $\pi = (\underline{1}, 5, 4, 9, 7, \overline{6}, 2, 10, 3, 8)$.

**Definition 3.2.6.** A *plabic graph*[2] is an undirected planar graph $G$ drawn inside a disk (considered modulo homotopy) with $n$ *boundary vertices* on the boundary of the disk, labeled $1, \ldots, n$ in clockwise order, as well as some colored *internal vertices*. These internal vertices are strictly inside the disk and are each colored either black or white. Moreover, each boundary vertex $i$ in $G$ is incident to a single edge. If a boundary vertex is adjacent to a leaf (a vertex of degree 1), we refer to that leaf as a *lollipop*.

A *perfect orientation* $\mathcal{O}$ of a plabic graph $G$ is a choice of orientation of each of its edges such that each black internal vertex $u$ is incident to exactly one edge directed away from $u$, and each white internal vertex $v$ is incident to exactly one edge directed towards $v$. A plabic graph is called *perfectly orientable* if it admits a perfect orientation. Let $G_\mathcal{O}$ denote the directed graph associated with a perfect orientation $\mathcal{O}$ of $G$. Since each boundary vertex is incident to a single edge, it is either a *source* (if it is incident to an outgoing edge) or a *sink* (if it is incident to an incoming edge) in $G_\mathcal{O}$. The *source set* $I_\mathcal{O} \subset [n]$ is the set of boundary vertices which are sources in $G_\mathcal{O}$.

Figure 3.3 shows a plabic graph with a perfect orientation. In that example, $I_\mathcal{O} = \{2, 3, 6, 8\}$.

All perfect orientations of a fixed plabic graph $G$ have source sets of the same size $k$, where $k - (n-k) = \sum \mathrm{color}(v) \cdot (\deg(v) - 2)$. Here the sum is over all internal vertices $v$,

[2] "Plabic" stands for *planar bi-colored*.

where $\mathrm{color}(v) = 1$ if $v$ is black, and $\mathrm{color}(v) = -1$ if $v$ is white; see Lemma 9.4 of [Pos]. In this case we say that $G$ is of *type* $(k, n)$.

The following construction of Postnikov (Sections 6 and 20 of [Pos]) associates a perfectly orientable plabic graph to any ⅃-diagram.

**Definition 3.2.7.** Let $D$ be a ⅃-diagram and $\pi$ its decorated permutation. Delete the 0's of $D$, and replace each $+$ with a vertex. From each vertex we construct a hook which goes east and south, to the border of the Young diagram. The resulting diagram is called the *hook diagram* $H(D)$. After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by $1, \dots, n$, we obtain a planar graph in a disk, with $n$ boundary vertices and one internal vertex for each $+$ of $D$. Then we replace the local region around each internal vertex as in Figure 3.2, and add a black (respectively, white) lollipop for each black (respectively, white) fixed point of $\pi$. This gives rise to a plabic graph which we call $G(D)$. By orienting the edges of $G(D)$ down and to the left, we obtain a perfect orientation.

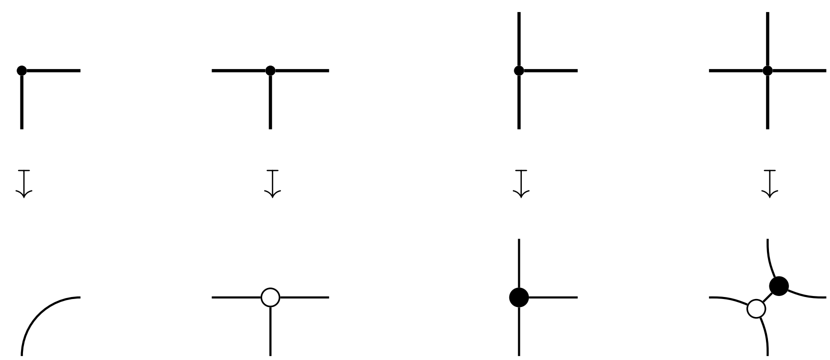


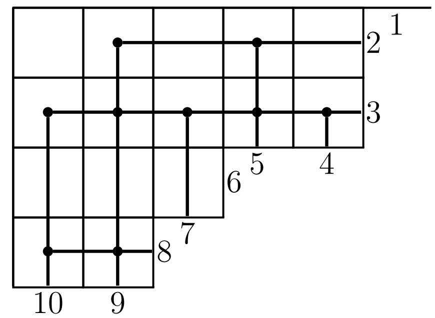
Figure 3.2: Local substitutions for getting the plabic graph $G(D)$ from the hook diagram $H(D)$.

Figure 3.3a depicts the hook diagram $H(D)$ corresponding to the ⅃-diagram $D$ from Figure 3.1, and Figure 3.3b shows the corresponding plabic graph $G(D)$.

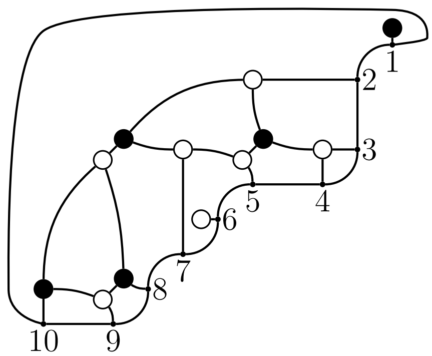
More generally, each ⅃-diagram $D$ is associated with a family of *reduced plabic graphs* consisting of $G(D)$ together with other plabic graphs which can be obtained from $G(D)$ by certain *moves*; see Section 12 of [Pos].

From the plabic graph constructed in Definition 3.2.7 (and more generally from a reduced plabic graph $G$), one may read off the corresponding decorated permutation $\pi_G$ as follows.

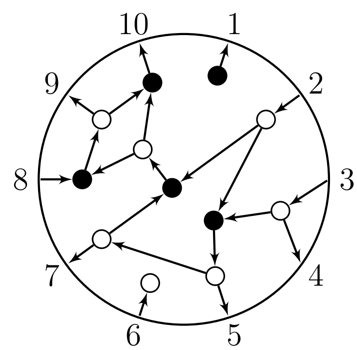
**Definition 3.2.8.** Let $G$ be a reduced plabic graph as above with boundary vertices $1, \dots, n$. The *trip from $i$* is the path obtained by starting from $i$ and traveling along edges of $G$ according to the rule that each time we reach an internal black vertex we turn (maximally) right, and each time we reach an internal white vertex we turn (maximally) left. This trip




(a) The hook diagram $H(D)$.




(b) The plabic graph $G(D)$.




(c) The plabic graph $G(D)$ redrawn and perfectly oriented.

Figure 3.3: The hook diagram and plabic graph associated to the Ꞁ-diagram $D$ from Figure 3.1.

ends at some boundary vertex $\pi(i)$. By Section 13 of [Pos], the fact that $G$ is reduced implies that each fixed point of $\pi$ is attached to a lollipop; we color each fixed point by the color of its lollipop. In this way we obtain the *decorated permutation* $\pi_G = (\pi(1), \dots, \pi(n))$ of $G$.

We invite the reader to verify that when we apply these rules to plabic graph $G$ of Figure 3.3b, we obtain the decorated permutation $\pi_G = (\underline{1}, 5, 4, 9, 7, \overline{6}, 2, 10, 3, 8)$.

## Matroids and positroids

A matroid is a combinatorial object which unifies several notions of independence. Among the many equivalent ways of defining a matroid we will adopt the point of view of bases, which is one of the most convenient for the study of positroids. We refer the reader to [Oxl11] for an in-depth introduction to matroid theory.

**Definition 3.2.9.** Let $E$ be a finite set. A *matroid with ground set $E$* is a subset $M \subseteq 2^E$ satisfying the *basis exchange axiom*:

if $B, B' \in M$ and $b \in B \setminus B'$, then there exists $b' \in B' \setminus B$ such that $(B \setminus \{b\}) \cup \{b'\} \in M$.

The elements of $M$ are called *bases*. All bases of $M$ have the same size, called the *rank* of $M$. We say that $i \in E$ is a *loop* if $i$ is contained in no basis of $M$, and a *coloop* if $i$ is contained in every basis of $M$.

**Example 3.2.10.** Let $A$ be a $k \times n$ matrix of rank $k$ with entries in a field $\mathbb{F}$. Then the subsets $B \in \binom{[n]}{k}$ such that the columns $B$ of $A$ are linearly independent form the bases of a matroid $M(A)$ with rank $k$ and ground set $[n]$. In terms of the Grassmannian, the rows of $A$ span an element of $\mathrm{Gr}_{k,n}(\mathbb{F})$, whose nonzero Plücker coordinates are indexed by $M(A)$. For example, the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$$

over $\mathbb{F} := \mathbb{Q}$ gives rise to the matroid $M(A) = \{\{1,2\},\{1,3\},\{1,5\},\{2,3\},\{3,5\}\}$ with ground set $[5]$. In this example, 4 is a loop of $M(A)$, and $M(A)$ has no coloops.

Matroids arising in this way are called *representable* (over $\mathbb{F}$). $\diamondsuit$

**Example 3.2.11.** Given $k \in \mathbb{N}$ and a finite set $E$, all $k$-subsets of $E$ form a matroid $M$, called the *uniform matroid* (of rank $k$ with ground set $E$). Note that $M$ can be represented over any infinite field, by a generic $k \times n$ matrix. $\diamondsuit$

Recall the definition of a positroid from Definition 3.2.1. In the language of matroid theory, a positroid is a matroid representable by an element of the totally nonnegative Grassmannian. Every perfectly orientable plabic graph gives rise to a positroid as follows.

**Definition 3.2.12** (Proposition 11.7 of [Pos])**.** Let $G$ be a plabic graph of type $(k, n)$. Then we have a positroid $M_G$ on $[n]$ whose bases are precisely

$$\{I_{\mathcal{O}} : \mathcal{O} \text{ is a perfect orientation of } G\},$$

where $I_{\mathcal{O}}$ is the set of sources of $\mathcal{O}$.

If $D$ is a Ⅎ-diagram contained in a $k \times (n-k)$ rectangle, we let $M(D)$ denote the positroid $M_{G(D)}$ of the plabic graph $G(D)$ from Definition 3.2.7.

Postnikov (Theorem 17.1 of [Pos]) showed that every positroid can be realized as $M(D)$ for some Ⅎ-diagram $D$. We observe that we can describe the loops and coloops of $M(D)$ in terms of $D$ as follows: $i$ is a loop if and only if $i$ labels a horizontal step whose column contains only 0's, and $i$ is a coloop if and only if $i$ labels a vertical step in the southeast border whose row contains only 0's.

We introduce some further notions from matroid theory which we will use later: duality, direct sum, connectedness, restriction, and a partial order.

**Definition 3.2.13.** Let $M$ be a matroid with ground set $E$. Then $\{E \setminus B : B \in M\}$ is the set of bases of a matroid $M^*$ with ground set $E$, called the *dual* of $M$.

See Section 2 of [Oxl11] for a proof that $M^*$ is indeed a matroid. We make the following observations about matroid duality:

- $(M^*)^* = M$;
- the ranks of $M$ and $M^*$ sum to $|E|$;
- $i \in E$ is a loop of $M$ if and only if $i$ is a coloop of $M^*$;
- if $E = [n]$, then $M$ is a positroid if and only if $M^*$ is a positroid (see Lemma 3.3.3(ii)).

**Example 3.2.14.** Let $A := \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$, as in Example 3.2.10. Then

$$M(A)^* = \{\{3,4,5\},\{2,4,5\},\{2,3,4\},\{1,4,5\},\{1,2,4\}\},$$

and is represented by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -2 & 0 & 1 & 0 & -3 \end{bmatrix},$$

whose rows are orthogonal to the rows of $A$. ◊

**Definition 3.2.15.** Let $M$ and $N$ be matroids with ground sets $E$ and $F$, respectively. The *direct sum* $M \oplus N$ is the matroid with ground set $E \sqcup F$, and bases $\{B \sqcup C : B \in M, C \in N\}$. The rank of $M \oplus N$ is the sum of the ranks of $M$ and $N$.

A matroid is *connected* if we cannot write it as the direct sum of two matroids whose ground sets are nonempty. Any matroid $M$ can be written uniquely (up to permuting the summands) as the direct sum of connected matroids, whose ground sets are the *connected components* of $M$; see Corollary 4.2.9 of [Oxl11].

**Example 3.2.16.** Consider the matrix $A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and its associated matroid $M(A) = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$. We have $M(A) = M_1 \oplus M_2$, where $M_1$ is the uniform matroid of rank 1 with ground set $\{1,4\}$, and $M_2$ is the uniform matroid of rank 1 with ground set $\{2,3\}$. Since $M_1$ and $M_2$ are connected, the connected components of $M(A)$ are $\{1,4\}$ and $\{2,3\}$. In particular, $M(A)$ is disconnected. ◊

**Definition 3.2.17.** Let $M$ be a matroid with ground set $E$. For a subset $F \subseteq E$, the *restriction* $M|_F$ of $M$ to $F$ is the matroid with ground set $F$ whose bases are the inclusion-maximal sets of $\{B \cap F : B \in M\}$; see p. 20 of [Oxl11].

For example, if $M$ and $N$ are matroids with ground sets $E$ and $F$, respectively, then the restriction of $M \oplus N$ to $E$ is $M$, and the restriction of $M \oplus N$ to $F$ is $N$.

**Definition 3.2.18.** We define a partial order on matroids of rank $k$ with ground set $E$ as follows: $M' \leq M$ if and only if every basis of $M'$ is a basis of $M$. (In the matroid theory literature, one says that the identity map on $E$ is a *weak map* from $M$ to $M'$.)

Note that $M' \leq M$ if and only if $M'^* \leq M^*$. This partial order, restricted to positroids of rank $k$ with ground set $[n]$, recovers the poset $Q_{k,n}$ of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ coming from containment of closures (see Section 17 of [Pos]). The poset $Q_{2,4}$ for $\mathrm{Gr}_{2,4}^{\geq 0}$ is shown in Figure 3.6.

**Remark 3.2.19.** All bijections that we have defined in this section are compatible. This gives us a canonical way to label each positroid of rank $k$ with ground set $[n]$ by a set of bases, a decorated permutation, a Ⅎ-diagram, and an equivalence class of reduced plabic graphs. The partial order on positroids (Definition 3.2.18) gives a partial order on these other objects (of type $(k, n)$).

## 3.3 A complementary view of the amplituhedron $\mathcal{A}_{n,k,m}$

### Background on sign variation

**Definition 3.3.1.** Given $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times $v$ changes sign, when viewed as a sequence of $n$ numbers and ignoring any zeros. We use the convention $\mathrm{var}(0) := -1$. We also define

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\},$$

i.e. $\overline{\mathrm{var}}(v)$ is the maximum number of times $v$ changes sign after we choose a sign for each zero component.

For example, if $v := (4, -1, 0, -2) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

We now explain how $\mathrm{var}(\cdot)$ and $\overline{\mathrm{var}}(\cdot)$ are dual to each other.

**Definition 3.3.2.** We define $\mathrm{alt} : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathrm{alt}(v) := (v_1, -v_2, v_3, -v_4, \ldots, (-1)^{n-1}v_n)$ for $v \in \mathbb{R}^n$. If $S \subseteq \mathbb{R}^n$, we let $\mathrm{alt}(S)$ denote $\{\mathrm{alt}(v) : v \in S\}$.

**Lemma 3.3.3** (Duality via alt)**.**
*(i)*[3] *[GK50] We have* $\mathrm{var}(v) + \overline{\mathrm{var}}(\mathrm{alt}(v)) = n - 1$ *for all* $v \in \mathbb{R}^n \setminus \{0\}$.
*(ii)*[4] *[Hil90, Hoc75] Given* $V \in \mathrm{Gr}_{k,n}$*, let* $V^\perp \in \mathrm{Gr}_{n-k,n}$ *be the orthogonal complement of* $V$.

[3] This result is stated without proof as Equation II.(67) of [GK50]. See Equation (5.1) of [And87] for a proof.

[4] The earliest reference we found for this result is Section 7 of [Hoc75], where it appears without proof. Hochster says this result "was basically known to Hilbert." The idea is that if $[I_k|A]$ is a $k \times n$ matrix whose rows span $V \in \mathrm{Gr}_{k,n}$, where $A$ is a $k \times (n-k)$ matrix, then $V^\perp$ is the row span of the matrix $[A^T| - I_{n-k}]$. This appears implicitly in Equation (14) of [Hil90], and more explicitly in Theorem 2.2.8 of [Oxl11] and Proposition 3.1(i) of [MR14].

*Then $V$ and $\mathrm{alt}(V^\perp)$ have the same Plücker coordinates, i.e. $\Delta_I(V) = \Delta_{[n]\setminus I}(\mathrm{alt}(V^\perp))$ for all $I \in \binom{[n]}{k}$.*

Note that part (ii) above implies that a subspace $V$ is totally nonnegative if and only if $\mathrm{alt}(V^\perp)$ is totally nonnegative, and totally positive if and only if $\mathrm{alt}(V^\perp)$ is totally positive.

The following result of Gantmakher and Krein, which characterizes totally nonnegative and totally positive subspaces in terms of sign variation, will be essential for us.

**Theorem 3.3.4** (Theorems V.3, V.7, V.1, V.6 of [GK50])**.** *Let $V \in \mathrm{Gr}_{k,n}$.*
*(i) $V \in \mathrm{Gr}_{k,n}^{\geq 0} \iff \mathrm{var}(v) \leq k-1$ for all $v \in V \iff \overline{\mathrm{var}}(w) \geq k$ for all $w \in V^\perp \setminus \{0\}$.*
*(ii) $V \in \mathrm{Gr}_{k,n}^{>0} \iff \overline{\mathrm{var}}(v) \leq k-1$ for all $v \in V \setminus \{0\} \iff \mathrm{var}(w) \geq k$ for all $w \in V^\perp \setminus \{0\}$.*

**Corollary 3.3.5.** *If $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ and $W \in \mathrm{Gr}_{r,n}^{>0}$, where $r \geq k$, then $V \cap W^\perp = \{0\}$.*

**Proof.** By Theorem 3.3.4, $v \in V$ implies that $\mathrm{var}(v) \leq k-1$. And $w \in W^\perp \setminus \{0\}$ implies that $\mathrm{var}(w) \geq r \geq k$. Therefore $V \cap W^\perp = \{0\}$. ■

We will also need to know which sign vectors appear in elements of $\mathrm{Gr}_{k,n}^{>0}$.

**Definition 3.3.6.** For $t \in \mathbb{R}$ we define

$$\mathrm{sign}(t) := \begin{cases} 0, & \text{if } t = 0 \\ +, & \text{if } t > 0 \\ -, & \text{if } t < 0 \end{cases}.$$

(We will sometimes use 1 and $-1$ in place of $+$ and $-$.) Given $v \in \mathbb{R}^n$, define the *sign vector* $\mathrm{sign}(v) \in \{0,+,-\}^n$ of $v$ by $\mathrm{sign}(v)_i := \mathrm{sign}(v_i)$ for $i \in [n]$. For example, $\mathrm{sign}(5,0,-1,2) = (+,0,-,+) = (1,0,-1,1)$. If $S \subseteq \mathbb{R}^n$, we let $\mathrm{sign}(S)$ denote $\{\mathrm{sign}(v) : v \in S\}$.

**Lemma 3.3.7.** *Suppose that $V \in \mathrm{Gr}_{k,n}^{>0}$ with orthogonal complement $V^\perp$.*
*(i)* $\mathrm{sign}(V) = \{\sigma \in \{0,+,-\}^n : \overline{\mathrm{var}}(\sigma) \leq k-1\} \cup \{0\}$.
*(ii)* $\mathrm{sign}(V^\perp) = \{\sigma \in \{0,+,-\}^n : \mathrm{var}(\sigma) \geq k\} \cup \{0\}$.

This result essentially follows from Theorem 3.3.4, Proposition 9.4.1 of [BLVS+99], and Lemma 3.3.3. For a more thorough explanation, see the claim in the proof of Lemma 2.4.1.

## An orthogonally complementary view of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is a subset of $\mathrm{Gr}_{k,k+m}$. Since we are considering the case $m = 1$, it will be convenient for us to take orthogonal complements and work with subspaces of dimension $m$, rather than codimension $m$. To this end, we define an object $\mathcal{B}_{n,k,m}(W) \subseteq \mathrm{Gr}_m(W)$ for $W \in \mathrm{Gr}_{k+m,n}^{>0}$, which we show is homeomorphic to $\mathcal{A}_{n,k,m}(Z)$ (Proposition 3.3.12), where $W = \mathrm{rowspan}(Z)$. We remark that in the context of scattering amplitudes when $m = 4$, $W$ is the subspace of $\mathbb{R}^n$ spanned by 4 bosonic variables and $k$ fermionic variables.

**Definition 3.3.8.** Given $W \in \mathrm{Gr}_{k+m,n}^{>0}$, let

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

where $\mathrm{Gr}_m(W)$ denotes the subset of $\mathrm{Gr}_{m,n}$ of elements $X \in \mathrm{Gr}_{m,n}$ with $X \subseteq W$. Let us show that $\mathcal{B}_{n,k,m}(W)$ is well defined, i.e. $\dim(V^\perp \cap W) = m$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$. By Corollary 3.3.5 we have $V \cap W^\perp = \{0\}$, so the sum $V + W^\perp$ is direct. Hence $\dim(V + W^\perp) = \dim(V) + \dim(W^\perp) = n - m$. Since $(V^\perp \cap W)^\perp = V + W^\perp$, we get $\dim(V^\perp \cap W) = m$. (We remark that this is the same idea used in Section 2.4 to determine when, given an arbitrary linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$, the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ is well defined in $\mathrm{Gr}_{k,k+m}$.)

**Remark 3.3.9.** While we were preparing this manuscript, we noticed that a similar construction appeared in Lam's definition of *universal amplituhedron varieties* Section 18 of [Lam16b]. There are two main differences between his construction and ours. First, Lam allows $Z$ to vary (hence the term "universal"). Second, he works with complex varieties, and does not impose any positivity conditions on $V$ or $Z$ (rather, he restricts $V$ to lie in a closed complex positroid cell in $\mathrm{Gr}_{k,n}(\mathbb{C})$). Correspondingly he works with rational maps, while we will need our maps to be well defined everywhere.

We now show that $\mathcal{B}_{n,k,m}(W)$ is homeomorphic to $\mathcal{A}_{n,k,m}(Z)$, where $Z$ is any $(k+m) \times n$ matrix ($n \geq k + m$) with positive maximal minors and row span $W$. The idea is that we obtain $\mathcal{B}_{n,k,m}(W)$ from $\mathcal{A}_{n,k,m}(Z) \subseteq \mathrm{Gr}_{k,k+m}$ by taking orthogonal complements in $\mathbb{R}^{k+m}$, and then applying an isomorphism from $\mathbb{R}^{k+m}$ to $W$, so that our subspaces lie in $W$, not $\mathbb{R}^{k+m}$.

**Lemma 3.3.10.** *Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ be a surjective linear map, which we also regard as a $(k + m) \times n$ matrix, and let $W \in \mathrm{Gr}_{k+m,n}$ be the row span of $Z$. Then the map $f_Z : \mathrm{Gr}_m(W) \to \mathrm{Gr}_{k,k+m}$ given by*

$$f_Z(X) := Z(X^\perp) = \{Z(x) : x \in X^\perp\} \quad \textit{for all } X \in \mathrm{Gr}_m(W)$$

*is well defined and an isomorphism. (Here $X^\perp \in \mathrm{Gr}_{n-m,n}$ denotes the orthogonal complement of $X$ in $\mathbb{R}^n$; we use the notation $Z(X^\perp)$, and not $\tilde{Z}(X^\perp)$, because $\dim(X^\perp) \neq k$.)*

*Moreover, for $X \in \mathrm{Gr}_m(W)$ with corresponding point $Y := f_Z(X) \in \mathrm{Gr}_{k,k+m}$, we can write the Plücker coordinates of $X$ (as an element of $\mathrm{Gr}_{m,n}$) in terms of $Y$ and $Z$, as follows. Let $z_1, \ldots, z_n \in \mathbb{R}^{k+m}$ be the columns of the $(k+m) \times n$ matrix $Z$, and $y_1, \ldots, y_k \in \mathbb{R}^{k+m}$ be a basis of $Y$. Then for $1 \le i_1 < \cdots < i_m \le n$, we have*

$$\Delta_{\{i_1,\ldots,i_m\}}(X) = \det([y_1 \,|\, \cdots \,|\, y_k \,|\, z_{i_1} \,|\, \cdots \,|\, z_{i_m}]). \tag{3.3.11}$$

The formula (3.3.11) is stated in Section 18 of [Lam16b], though the proof is deferred to a forthcoming paper.

**Proof.** First let us show that $f_Z$ is well defined, i.e. for $X \in \mathrm{Gr}_m(W)$, we have $\dim(f_Z(X)) = k$. Since $X \subseteq W$, we have $W^\perp \subseteq X^\perp$, so we can write $X^\perp = V \oplus W^\perp$ for some $V \in \mathrm{Gr}_{k,n}$. Since $\ker(Z) = W^\perp$, we have $Z(X^\perp) = Z(V + W^\perp) = Z(V)$, and then $V \cap \ker(Z) = V \cap W^\perp = \{0\}$ implies $\dim(Z(V)) = k$.

To see that $f_Z$ is injective, suppose that we have $X, X' \in \mathrm{Gr}_m(W)$ with $Z(X^\perp) = Z(X'^\perp)$. Then as above we have $X^\perp = V \oplus W^\perp$ and $(X')^\perp = V' \oplus W^\perp$ where $V \cap \ker(Z) = \{0\} = V' \cap \ker(Z)$. But then $Z(X^\perp) = Z(X'^\perp)$ implies that $Z(V) = Z(V')$, which implies that $V = V'$ and hence $X = X'$. Now we describe the inverse of $f_Z$. Given $Y \in \mathrm{Gr}_{k,k+m}$, consider the subspace $Z^{-1}(Y) = \{v \in \mathbb{R}^n : Z(v) \in Y\}$. Since $Z^{-1}(Y)$ contains $\ker(Z) = W^\perp$, which has dimension $n - k - m$, and $\dim(Y) = k$, we have $\dim(Z^{-1}(Y)) = n - m$. Therefore we can write $Z^{-1}(Y) = X^\perp$ for some $X \in \mathrm{Gr}_m(W)$, and then $Y = f_Z(X)$. It follows that $f_Z$ is invertible, and hence an isomorphism.

Now given $X \in \mathrm{Gr}_m(W)$ and $Y := f_Z(X) \in \mathrm{Gr}_{k,k+m}$, we prove (3.3.11). Fix column vectors $y_1, \ldots, y_k \in \mathbb{R}^{k+m}$ which form a basis of $Y$ and $x_1, \ldots, x_{n-m} \in \mathbb{R}^n$ which form a basis of $X^\perp$. For $i \in [n-m]$, we can write $Z(x_i) = \sum_{j=1}^k C_{i,j} y_j$ for some $C_{i,j} \in \mathbb{R}$. Let $D$ be the $(n-m) \times n$ matrix whose rows are $x_1^T, \ldots, x_{n-m}^T$. Then the elements of $\mathrm{Gr}_{n-m,n+k}$ and $\mathrm{Gr}_{k+m,n+k}$ which are given by the row spans of

$$[-C \,|\, D] \qquad \text{and} \qquad [y_1 \,|\, \cdots \,|\, y_k \,|\, Z],$$

respectively, are orthogonally complementary. Given $I = \{i_1 < \cdots < i_m\} \subseteq [n]$ with complement $[n] \setminus I = \{j_1 < \cdots < j_{n-m}\}$, applying Lemma 3.3.3(ii) twice gives

$$\begin{aligned} \Delta_I(X) = \Delta_{[n]\setminus I}(\mathrm{alt}(X^\perp)) = \Delta_{\{k+j_1,\ldots,k+j_{n-m}\}}(\mathrm{alt}([-C \,|\, D])) = \\ \Delta_{\{1,\ldots,k,k+i_1,\ldots,k+i_m\}}([y_1 \,|\, \cdots \,|\, y_k \,|\, Z]) = \det([y_1 \,|\, \cdots \,|\, y_k \,|\, z_{i_1} \,|\, \cdots \,|\, z_{i_m}]). \quad \blacksquare \end{aligned}$$

**Proposition 3.3.12.** *Suppose that $Z$ is a $(k+m) \times n$ matrix ($n \ge k+m$) with positive maximal minors, and $W \in \mathrm{Gr}_{k+m,n}^{>0}$ is the row span of $Z$. Then the map $f_Z$ from Lemma 3.3.10 restricts to a homeomorphism from $\mathcal{B}_{n,k,m}(W)$ onto $\mathcal{A}_{n,k,m}(Z)$, which sends $V^\perp \cap W$ to $\tilde{Z}(V)$ for all $V \in \mathrm{Gr}_{k,n}^{\ge 0}$. The Plücker coordinates of $V^\perp \cap W$ can be written in terms of $\tilde{Z}(V)$ and $Z$ by* (3.3.11).

**Example 3.3.13.** Let $(n, k, m) := (4, 2, 1)$, and $Z : \mathbb{R}^4 \to \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [z_1 \,|\, z_2 \,|\, z_3 \,|\, z_4],$$

whose $3 \times 3$ minors are all positive. Also let $V \in \mathrm{Gr}_{2,4}^{\ge 0}$ be the row span of the matrix $\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}$, where $a, b \ge 0$, and define $Y := \tilde{Z}(V)$. We can explicitly find a basis $y_1, y_2 \in \mathbb{R}^3$ of $Y$ as follows:

$$[y_1 \,|\, y_2] := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 0 \\ 0 & 1 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & b \\ a & -b \\ 0 & 1+b \end{bmatrix}.$$

Now let $X := V^\perp \cap \mathrm{rowspan}(Z)$, so that $X \in \mathcal{B}_{4,2,1}(\mathrm{rowspan}(Z))$ is mapped to $Y \in \mathcal{A}_{4,2,1}(Z)$ under the homeomorphism of Proposition 3.3.12. We can write $X$ as the line spanned by $(a(b+1), -(b+1), -b(a+1), a+1)$. We can check that we have

$$\Delta_{\{1\}}(X) = \begin{vmatrix} 1 & b & 1 \\ a & -b & 0 \\ 0 & 1+b & 0 \end{vmatrix}, \qquad \Delta_{\{3\}}(X) = \begin{vmatrix} 1 & b & 0 \\ a & -b & 0 \\ 0 & 1+b & 1 \end{vmatrix},$$

$$\Delta_{\{2\}}(X) = \begin{vmatrix} 1 & b & 0 \\ a & -b & 1 \\ 0 & 1+b & 0 \end{vmatrix}, \qquad \Delta_{\{4\}}(X) = \begin{vmatrix} 1 & b & 1 \\ a & -b & -1 \\ 0 & 1+b & 1 \end{vmatrix},$$

as asserted by (3.3.11). (Here $|M|$ denotes the determinant of $M$.) $\diamondsuit$

**Proof** (of Proposition 3.3.12)**.** Given $V \in \mathrm{Gr}_{k,n}^{\ge 0}$, since $\ker(Z) = W^\perp$ we have

$$f_Z(V^\perp \cap W) = Z((V^\perp \cap W)^\perp) = Z(V + W^\perp) = Z(V) = \tilde{Z}(V).$$

Thus the image $f_Z(\mathcal{B}_{n,k,m}(W))$ equals $\mathcal{A}_{n,k,m}(Z)$, and the result follows from Lemma 3.3.10. ■

## A hypothetical intrinsic description of the amplituhedron

We now give a description of the amplituhedron $\mathcal{B}_{n,k,1}(W)$ which does not mention $\mathrm{Gr}_{k,n}^{\ge 0}$. This description will extend to $\mathcal{B}_{n,k,m}(W)$ for $m > 1$ if part (i) of the following problem has a positive answer.

**Problem 3.3.14.** *Let $V \in \mathrm{Gr}_{m,n}$, and $l \ge m$.*
*(i) If $\mathrm{var}(v) \le l - 1$ for all $v \in V$, can we extend $V$ to an element of $\mathrm{Gr}_{l,n}^{\ge 0}$?*
*(ii) If $\overline{\mathrm{var}}(v) \le l - 1$ for all $v \in V \setminus \{0\}$, can we extend $V$ to an element of $\mathrm{Gr}_{l,n}^{> 0}$?*

**Lemma 3.3.15.** *For $W \in \mathrm{Gr}_{k+m,n}^{>0}$, we have*

$$\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \mathrm{Gr}_m(W) : k \le \overline{\mathrm{var}}(v) \le k+m-1 \text{ for all } v \in X \setminus \{0\}\}.$$

*If Problem 3.3.14(i) has a positive answer for $l = n-k$, then equality holds.*

In calculating $\overline{\mathrm{var}}(v)$ for $v \in X$ (where $X \in \mathrm{Gr}_m(W)$), we regard $v$ as a vector in $\mathbb{R}^n$. We remark that showing equality holds in Lemma 3.3.15 does not resolve Problem 3.3.14(i), because $W \in \mathrm{Gr}_{k+m,n}$ is not arbitrary, but is required to be totally positive.

**Proof.** Given $X \in \mathcal{B}_{n,k,m}(W)$, we can write $X = V^\perp \cap W$ for some $V \in \mathrm{Gr}_{k,n}^{\ge 0}$. Then for any $v \in X \setminus \{0\}$, we have $\overline{\mathrm{var}}(v) \ge k$ by Theorem 3.3.4(i) applied to $V$, and $\overline{\mathrm{var}}(v) \le k+m-1$ by Theorem 3.3.4(ii) applied to $W$. This proves the containment. Now suppose that Problem 3.3.14(i) has a positive answer for $l = n-k$. Let $X \in \mathrm{Gr}_m(W)$ with $\overline{\mathrm{var}}(v) \ge k$ for all $v \in X \setminus \{0\}$. Then $\mathrm{var}(w) \le n-k-1$ for all $w \in \mathrm{alt}(X) \setminus \{0\}$ by Lemma 3.3.3(i). Hence we can extend $\mathrm{alt}(X)$ to an element of $\mathrm{Gr}_{n-k,n}^{\ge 0}$, which by Lemma 3.3.3(ii) we can write as $\mathrm{alt}(V^\perp)$ for some $V \in \mathrm{Gr}_{k,n}^{\ge 0}$. Since $\mathrm{alt}(X) \subseteq \mathrm{alt}(V^\perp)$, we have $X \subseteq V^\perp$, whence $X \subseteq V^\perp \cap W$. Since $\dim(X) = m = \dim(V^\perp \cap W)$, we have $X = V^\perp \cap W \in \mathcal{B}_{n,k,m}(W)$. ■

**Problem 3.3.16.** *Do we have equality in Lemma 3.3.15? In other words, is it true that for $W \in \mathrm{Gr}_{k+m,n}^{>0}$, we have*

$$\mathcal{B}_{n,k,m}(W) = \{X \in \mathrm{Gr}_m(W) : k \le \overline{\mathrm{var}}(v) \le k+m-1 \text{ for all } v \in X \setminus \{0\}\}?$$

**Remark 3.3.17.** We observe that Problem 3.3.16 has a positive answer in the extreme cases $k = 0$ (whence $\mathcal{B}_{n,k,m}(W) = \{W\}$), $m = 0$ (whence $\mathcal{B}_{n,k,m}(W) = \{\{0\}\}$), and $k+m = n$ (whence $\mathcal{B}_{n,k,m}(W) = \{V^\perp : V \in \mathrm{Gr}_{k,n}^{\ge 0}\}$). Also, by Lemma 2.4.1, both parts (i) and (ii) of Problem 3.3.14 have a positive answer for $m = 1$ (and all $l$ and $n$), and so Problem 3.3.16 has a positive answer for $m = 1$.

This gives the following explicit description of $\mathcal{B}_{n,k,1}(W)$, which will be important for us in our study of the structure of $\mathcal{B}_{n,k,1}(W)$.

**Corollary 3.3.18.** *For $W \in \mathrm{Gr}_{k+1,n}^{>0}$, we have*

$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

**Remark 3.3.19.** We give some background on Problem 3.3.14. Lam (see Section 15 of [Lam16b]) considered images $\tilde{Z}(\mathrm{Gr}_{k,n}^{\ge 0}) \subseteq \mathrm{Gr}_{k,k+m}$ of the totally nonnegative Grassmannian induced by linear maps $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ more general than those appearing in the construction of amplituhedra (where $Z$ is represented by a matrix with positive maximal minors).

We still need a positivity condition on $Z$ so that the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ is contained in $\mathrm{Gr}_{k,k+m}$, i.e. that

$$\dim(Z(V)) = k \quad \text{for all } V \in \mathrm{Gr}_{k,n}^{\geq 0}. \tag{3.3.20}$$

Lam (Proposition 15.2 of [Lam16b]) showed that (3.3.20) holds if the row span of $Z$ has a subspace in $\mathrm{Gr}_{k,n}^{>0}$, in which case he calls $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ a *(full) Grassmann polytope*, since in the case $k = 1$ Grassmann polytopes are precisely polytopes in $\mathbb{P}^m$. It is an open problem to determine whether Lam's condition is equivalent to (3.3.20). It turns out that this problem is equivalent to Problem 3.3.14(ii); see Theorem 2.4.2 and the discussion which follows it. Thus Problem 3.3.14(ii) is fundamental to the study of Grassmann polytopes. In the current context, we are more concerned with part (i) of Problem 3.3.14, though part (ii) will reappear in Section 3.3.

We now translate the hypothetical description of Problem 3.3.16 into one in terms of Plücker coordinates. This is reminiscent of a description of the amplituhedron which was conjectured by Arkani-Hamed, Thomas, and Trnka [AHTT]. We thank Jara Trnka for telling us about this conjecture.

**Proposition 3.3.21.** *Let $W \in \mathrm{Gr}_{k+m,n}^{>0}$, and define the open subset of $\mathrm{Gr}_m(W)$*

$$\mathcal{G} := \{X \in \mathrm{Gr}_m(W) : \mathrm{var}(v) \geq k \text{ for all } v \in X \setminus \{0\}\}.$$

*We have $\mathrm{int}(\mathcal{B}_{n,k,m}(W)) \subseteq \mathcal{G}$ and $\mathcal{B}_{n,k,m}(W) \subseteq \overline{\mathcal{G}}$, where $\mathrm{int}(\cdot)$ and $\overline{\cdot}$ denote interior and closure. If equality holds in Lemma 3.3.15, then both containments above are equalities. Independently, we can describe $\mathcal{G}$ in terms of Plücker coordinates:*

$$\mathcal{G} = \left\{ X \in \mathrm{Gr}_m(W) : \begin{array}{l} \mathrm{var}(((-1)^{|I \cap [j]|}\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}) \geq k \text{ for all } I \in \binom{[n]}{m-1} \text{ such} \\ \text{that the sequence } (\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I} \text{ is not identically zero} \end{array} \right\}.$$

We prove Proposition 3.3.21 below. First we translate this description to $\mathcal{A}_{n,k,m}(Z)$ using Lemma 3.3.10, make some remarks, and give an example.

**Corollary 3.3.22.** *Let $Z$ be a $(k+m) \times n$ matrix whose maximal minors are all positive, with row span $W \in \mathrm{Gr}_{k+m,n}^{>0}$. Let $\mathcal{F} := f_Z(\mathcal{G})$ be the image of the set $\mathcal{G}$ from Proposition 3.3.21 under the map $f_Z$ from Lemma 3.3.10, and $z_1, \dots, z_n \in \mathbb{R}^{k+m}$ be the columns of $Z$. Then*

$$\mathcal{F} = \left\{ \langle y_1, \dots, y_k \rangle : \begin{array}{l} \mathrm{var}((\det([y_1 \mid \cdots \mid y_k \mid z_{i_1} \mid \cdots \mid z_{i_{m-1}} \mid z_j]))_{j \in [n] \setminus I}) \geq k \\ \text{for all } I = \{i_1 < \cdots < i_{m-1}\} \in \binom{[n]}{m-1} \text{ such that} \\ \text{this sequence of minors is not identically zero} \end{array} \right\}.$$

*We have $\mathrm{int}(\mathcal{A}_{n,k,m}(Z)) \subseteq \mathcal{F}$ and $\mathcal{A}_{n,k,m}(Z) \subseteq \overline{\mathcal{F}}$. If equality holds in Lemma 3.3.15, then both containments above are equalities.*

As stated, the descriptions of $\mathcal{F}$ and $\mathcal{G}$ require checking $\binom{n}{m-1}$ sequences of minors. Does it suffice to check fewer sequences?

**Remark 3.3.23.** For all $X \in \mathrm{Gr}_m(W)$, we have $\overline{\mathrm{var}}(v) \le k+m-1$ for all $v \in X \setminus \{0\}$ by Theorem 3.3.4(ii), whence $\overline{\mathrm{var}}((\Delta_{I\cup\{j\}}(X))_{j\in[n]\setminus I}) \le k$ for all $I \in \binom{[n]}{m-1}$ such that the sequence $(\Delta_{I\cup\{i\}}(X))_{j\in[n]\setminus I}$ is not identically zero (see Theorem 2.3.1). Note that the sequence $(\Delta_{I\cup\{j\}}(X))_{j\in[n]\setminus I}$ is the one in the description of $\mathcal{G}$, without the sign $(-1)^{|I\cap[j]|}$. For the sequence in the description of $\mathcal{F}$, this sign change corresponds to moving $z_j$ from the right end of the matrix into its proper relative position in the submatrix of $Z$. If $I \in \binom{[n]}{m-1}$ satisfies the condition that $(-1)^{|I\cap[j]|}$ is the same for all $j \in [n] \setminus I$ (called *Gale's evenness condition* [Gal63]), then these two sequences are the same up to multiplication by $\pm 1$.

**Example 3.3.24.** Let $(n,k,m) := (5,1,2)$, and $Z : \mathbb{R}^5 \to \mathbb{R}^3$ be given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} =: [z_1 \,|\, z_2 \,|\, z_3 \,|\, z_4 \,|\, z_5],$$

whose $3 \times 3$ minors are all positive. In this case $\mathcal{A}_{5,1,2}(Z)$ is a convex pentagon in $\mathbb{P}^2 = \mathrm{Gr}_1(\mathbb{R}^3)$. Given $y \in \mathbb{R}^3$, Corollary 3.3.22 says that we have $\langle y\rangle \in \mathcal{F}$ if and only if each sequence $(\det([y \,|\, z_i \,|\, z_j]))_{j\in[5]\setminus\{i\}}$, for $i \in [5]$, is either identically zero or changes sign at least once. For example, if $i = 3$, this sequence is

$$\left( \begin{vmatrix} y_1 & 0 & 1 \\ y_2 & 0 & 0 \\ y_3 & 1 & 0 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 0 \\ y_2 & 0 & 1 \\ y_3 & 1 & 0 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 1 \\ y_2 & 0 & -1 \\ y_3 & 1 & 1 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 3 \\ y_2 & 0 & -2 \\ y_3 & 1 & 1 \end{vmatrix} \right) = (y_2, -y_1, y_1+y_2, 2y_1+3y_2).$$

Geometrically, the sequence corresponding to $i$ records where the point $\langle y\rangle \in \mathbb{P}^2$ lies in relation to each of the 4 lines joining vertex $i$ to another vertex of the pentagon. If this sequence does not change sign, then $\langle y\rangle$ lies on the same side of all 4 of these lines, i.e. the line segment between $\langle y\rangle$ and vertex $i$ does not intersect the interior of the pentagon. We see that $\mathcal{F}$ is the interior of the pentagon. In general, $\mathcal{F}$ is the interior of $\mathcal{A}_{n,k,m}(Z)$ if $k = 1$, independently of Problem 3.3.14(i). $\diamondsuit$

**Proof** (of Proposition 3.3.21)**.** First we prove $\mathrm{int}(\mathcal{B}_{n,k,m}(W)) \subseteq \mathcal{G}$, i.e. given $X \in \mathcal{B}_{n,k,m}(W) \setminus \mathcal{G}$, we have $X \notin \mathrm{int}(\mathcal{B}_{n,k,m}(W))$. Since $X \notin \mathcal{G}$, there exists $v \in X \setminus \{0\}$ with $\mathrm{var}(v) < k$. Let $\sigma := \mathrm{sign}(v)$, and take $\tau \in \{+,-\}^n$ such that $\tau \ge \sigma$ and $\mathrm{var}(\tau) = \mathrm{var}(\sigma)$. Then by Lemma 3.3.7, there exists $w \in W$ with $\mathrm{sign}(w) = \tau$. Now extend $v$ to a basis $v, v_2, \ldots, v_m$ of $X$, and for $t > 0$ let $X_t := \mathrm{span}(v + tw, v_2, \ldots, v_m)$, so that $X_t \in \mathrm{Gr}_m(W)$ except for at most one value of $t$. Since $\overline{\mathrm{var}}(v+tw) = \overline{\mathrm{var}}(\tau) = \mathrm{var}(v) < k$, we have $X_t \notin \mathcal{B}_{n,k,m}(W)$ by Lemma 3.3.15. But $\{X_t \in \mathrm{Gr}_m(W) : t > 0\}$ intersects every neighborhood of $X$, so $X \notin \mathrm{int}(\mathcal{B}_{n,k,m}(W))$. The fact that $\mathcal{B}_{n,k,m}(W) \subseteq \overline{\mathcal{G}}$ follows from these two facts:

- $\mathrm{Gr}_{k,n}^{\geq 0} = \overline{\mathrm{Gr}_{k,n}^{>0}}$ (see Section 17 of [Pos]);
- if $V \in \mathrm{Gr}_{k,n}^{>0}$, then $V^\perp \cap W \in \mathcal{G}$ (by Theorem 3.3.4(ii)).

Conversely, if equality holds in Lemma 3.3.15, then $\mathcal{G} \subseteq \mathcal{B}_{n,k,m}(W)$. Since $\mathcal{G}$ is open and $\mathcal{B}_{n,k,m}(W)$ is closed, we get the reverse containments $\mathcal{G} \subseteq \mathrm{int}(\mathcal{B}_{n,k,m}(W))$ and $\overline{\mathcal{G}} \subseteq \mathcal{B}_{n,k,m}(W)$.

Now we describe $\mathcal{G}$ in terms of Plücker coordinates. By Lemma 3.3.3(i), we have

$$\mathcal{G} = \{X \in \mathrm{Gr}_m(W) : \overline{\mathrm{var}}(v) \leq n-k-1 \text{ for all } v \in \mathrm{alt}(X) \setminus \{0\}\}.$$

Theorem 2.3.1(ii) states that for $X' \in \mathrm{Gr}_{m,n}$, we have $\overline{\mathrm{var}}(v) \leq n-k-1$ for all $v \in X' \setminus \{0\}$ if and only if $\overline{\mathrm{var}}((\Delta_{I\cup\{j\}}(X'))_{j\in[n]\setminus I}) \leq n-k-m$ for all $I \in \binom{[n]}{m-1}$ such that the sequence $(\Delta_{I\cup\{j\}}(X'))_{j\in[n]\setminus I}$ is not identically zero. Also by Lemma 3.3.3(i), we have $\overline{\mathrm{var}}(p) + \mathrm{var}(\mathrm{alt}(p)) = n-m$ for all nonzero $p \in \mathbb{R}^{[n]\setminus I}$ ($I \in \binom{[n]}{m-1}$), where alt acts on $\mathbb{R}^{[n]\setminus I}$ by changing the sign of every second component. We get

$$\mathcal{G} = \left\{ X \in \mathrm{Gr}_m(W) : \begin{array}{l} \mathrm{var}(\mathrm{alt}(\Delta_{I\cup\{j\}}(\mathrm{alt}(X)))_{j\in[n]\setminus I}) \geq k \text{ for all } I \in \binom{[n]}{m-1} \text{ such} \\ \text{that the sequence } (\Delta_{I\cup\{j\}}(X))_{j\in[n]\setminus I} \text{ is not identically zero} \end{array} \right\}.$$

In order to obtain the desired description of $\mathcal{G}$, it remains to show that given $I \in \binom{[n]}{m-1}$ such that the sequence $(\Delta_{I\cup\{j\}}(X))_{j\in[n]\setminus I}$ is not identically zero, we have

$$\mathrm{var}(\mathrm{alt}(\Delta_{I\cup\{j\}}(\mathrm{alt}(X)))_{j\in[n]\setminus I}) = \mathrm{var}(((-1)^{|I\cap[j]|}\Delta_{I\cup\{j\}}(X))_{j\in[n]\setminus I}). \tag{3.3.25}$$

To this end, write $I = \{i_1, \dots, i_{m-1}\} \in \binom{[n]}{m-1}$. Then for $j \in [n] \setminus I$, component $j$ of $\mathrm{alt}(\Delta_{I\cup\{j'\}}(\mathrm{alt}(X)))_{j'\in[n]\setminus I} \in \mathbb{R}^{[n]\setminus I}$ equals

$$(-1)^{|([n]\setminus I)\cap[j]|-1}(-1)^{(i_1-1)+\cdots+(i_{m-1}-1)+(j-1)}\Delta_{I\cup\{j\}}(X) = \epsilon_I(-1)^{|I\cap[j]|}\Delta_{I\cup\{j\}}(X),$$

where $\epsilon_I := (-1)^{(i_1-1)+\cdots+(i_{m-1}-1)} = \pm 1$ does not depend on $j$. This gives (3.3.25). ■

**Proof** (of Corollary 3.3.22)**.** Applying (3.3.11) to the description of $\mathcal{G}$ in Proposition 3.3.21 gives

$$\mathcal{F} = \left\{ \langle y_1, \dots, y_k \rangle : \begin{array}{l} \mathrm{var}(((-1)^{|I\cap[j]|}\det([y_1 \,|\, \cdots \,|\, y_k \,|\, Z_{I\cup\{j\}}]))_{j\in[n]\setminus I}) \geq k \\ \text{for all } I = \{i_1 < \cdots < i_{m-1}\} \in \binom{[n]}{m-1} \text{ such that} \\ \text{this sequence of minors is not identically zero} \end{array} \right\},$$

where $Z_J$ denotes the submatrix of $Z$ with columns $J$, for $J \subseteq [n]$. Moving the column of $Z_{I\cup\{j\}}$ labeled by $j$ to the right end of the matrix introduces a sign $(-1)^{|I\cap(j,n]|}$ in the determinant. Since $(-1)^{|I\cap[j]|}(-1)^{|I\cap(j,n]|} = (-1)^{|I|}$, which does not depend on $j$, we obtain the stated description of $\mathcal{F}$. The rest follows from Proposition 3.3.21, using Lemma 3.3.10. ■

## Removing $k$ from the definition of the amplituhedron

In the definition of $\mathcal{B}_{n,k,1}(W)$ (Definition 3.3.8), if we let $W$ and $k$ vary, then we obtain the following object:

$$\widehat{\mathcal{B}}_{m,n} := \bigcup_{0 \le k \le n-m} \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\ge 0}, W \in \mathrm{Gr}_{k+m,n}^{>0}\} \subseteq \mathrm{Gr}_{m,n}\,. \tag{3.3.26}$$

By Theorem 3.3.4, we have

$$\widehat{\mathcal{B}}_{m,n} \subseteq \bigcup_{0 \le k \le n-m} \{X \in \mathrm{Gr}_{m,n} : k \le \overline{\mathrm{var}}(v) \le k+m-1 \text{ for all } v \in X \setminus \{0\}\}, \tag{3.3.27}$$

and equality holds if both parts of Problem 3.3.14 have a positive answer. (The proof is similar to that of Lemma 3.3.15.) If $m = 1$, then Problem 3.3.14 has a positive answer, and so $\widehat{\mathcal{B}}_{1,n} = \mathrm{Gr}_{1,n}$. However $\widehat{\mathcal{B}}_{m,n} \neq \mathrm{Gr}_{m,n}$ for $m \ge 2$.

Our motivation for considering $\widehat{\mathcal{B}}_{m,n}$ is that in the BCFW recursion [BCFW05], which conjecturally provides a "triangulation" of $\mathcal{A}_{n,k,4}(Z)$, the parameter $k$ is allowed to vary. This suggests that the diagrams appearing in the BCFW recursion corresponding to a given $n$ might label pieces of an object which somehow encompasses $\mathcal{A}_{n,k,4}(Z)$ for all $k$. Could $\widehat{\mathcal{B}}_{4,n}$ be such an object?

As a first result about $\widehat{\mathcal{B}}_{m,n}$, we prove that the union in (3.3.26) is disjoint if $m \ge 1$. It suffices to show that the union in (3.3.27) is disjoint, which follows from the lemma below. It also follows from the lemma that the union in (3.3.27) consists of the elements of $\mathrm{Gr}_{m,n}$ whose range of $\overline{\mathrm{var}}(\cdot)$ (over nonzero vectors) is contained in as small an interval as possible. That is, for $X \in \mathrm{Gr}_{m,n}$ with $m \ge 1$, we have $\sup_{v,w \in X\setminus\{0\}} \overline{\mathrm{var}}(v) - \overline{\mathrm{var}}(w) \ge m-1$, and equality holds if and only if $X$ is contained in the union in (3.3.27).

**Lemma 3.3.28.** *Suppose that $X \in \mathrm{Gr}_{m,n}$ with $m \ge 1$, and $k \ge 0$ such that $k \le \overline{\mathrm{var}}(v) \le k+m-1$ for all $v \in X \setminus \{0\}$. Then $\overline{\mathrm{var}}(v) = k+m-1$ for some $v \in X \setminus \{0\}$ and $\overline{\mathrm{var}}(w) = k$ for some $w \in X \setminus \{0\}$.*

**Proof.** Let $v^{(1)}, \dots, v^{(m)}$ be the rows of an $m \times n$ matrix whose rows span $X$, after it has been put into reduced row echelon form. That is, if $i_1 < \cdots < i_m$ index the pivot columns of this matrix, then we have $v_{i_s}^{(r)} = \delta_{r,s}$ for all $r, s \in [m]$, and $v_j^{(r)} = 0$ for all $r \in [m]$ and $j < i_r$. Let $v := v^{(m)}$, and note that $\overline{\mathrm{var}}(v) = \overline{\mathrm{var}}(v|_{[i_m,n]}) + i_m - 1$. Now let

$$w := v + t\sum_{r=1}^{m-1} \epsilon_r v^{(r)},$$

where $t > 0$ is sufficiently small that in positions $j \in [i_m, n]$ where $v$ is nonzero, $w_j$ has the same sign as $v_j$, and $\epsilon_r \in \{1, -1\}$ is chosen to be 1 precisely if $|\{i \in \mathbb{Z} : i_r < i < i_{r+1}\}|$ is even. Then

$$k \le \overline{\mathrm{var}}(w) \le \overline{\mathrm{var}}(v|_{[i_m,n]}) + i_m - m = \overline{\mathrm{var}}(v) - m + 1 \le (k+m-1) - m + 1 = k,$$

and hence equality holds everywhere above. ∎

## 3.4 A BCFW-like recursion for $m = 1$

In the case that $m = 4$, the *BCFW recursion* (named after Britto, Cachazo, Feng, and Witten [BCF05, BCFW05]) can be viewed as a procedure which outputs a subset of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ whose images conjecturally "triangulate" the amplituhedron $\mathcal{A}_{n,k,4}(Z)$ [AHT14]. The procedure is described in Section 2 of [AHBC+16] as an operation on plabic graphs.

In this section we give an $m = 1$ analogue of the BCFW recursion, which naturally leads us to a subset of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ that we call the $m = 1$ *BCFW cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$. We remark that unlike the recursion for $m = 4$ as described in [AHBC+16], there is no 'shift' to the decorated permutation involved. These cells can be easily described in terms of their Ʇ-diagrams, positroids, or decorated permutations. As we will show in Section 3.5, the images of these cells "triangulate" the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}(Z)$. More specifically, they each map injectively into $\mathcal{A}_{n,k,1}(Z)$, and their images are disjoint and together form a dense subset of $\mathcal{A}_{n,k,1}(Z)$. In fact, we will show that the BCFW cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ plus the cells in their boundaries give rise to a cell decomposition of $\mathcal{A}_{n,k,1}(Z)$.

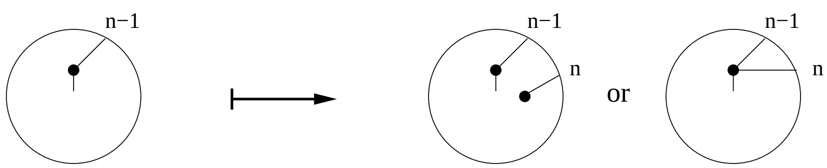



Figure 3.4: The $m = 1$ BCFW recursion: add a new vertex $n$, which is incident either to a black lollipop or to the edge adjacent to vertex $n - 1$.

**Definition 3.4.1.** The *BCFW recursion for* $m = 1$ is defined as follows.

- We start from the plabic graph with one boundary vertex, incident to a black lollipop. This is our graph for $n = 1$ (with $k = 0$).
- Given a plabic graph with $n - 1$ boundary vertices produced by our recursion (where $n \geq 2$), we perform one of the following two operations: we add a new boundary vertex $n$ which is incident either to a black lollipop, or to the edge adjacent to boundary vertex $n - 1$. See Figure 3.4. The first operation preserves the $k$ statistic, while the second operation increases it by 1.

We refer to the set of all plabic graphs with fixed $n$ and $k$ statistics produced in this way as the $m = 1$ *BCFW cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$. See Figure 3.5 for all $m = 1$ BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ with $n \leq 4$.

The following lemma is easy to verify by inspection, using the bijections between plabic graphs, Ʇ-diagrams, and decorated permutations which we gave in Section 3.2.

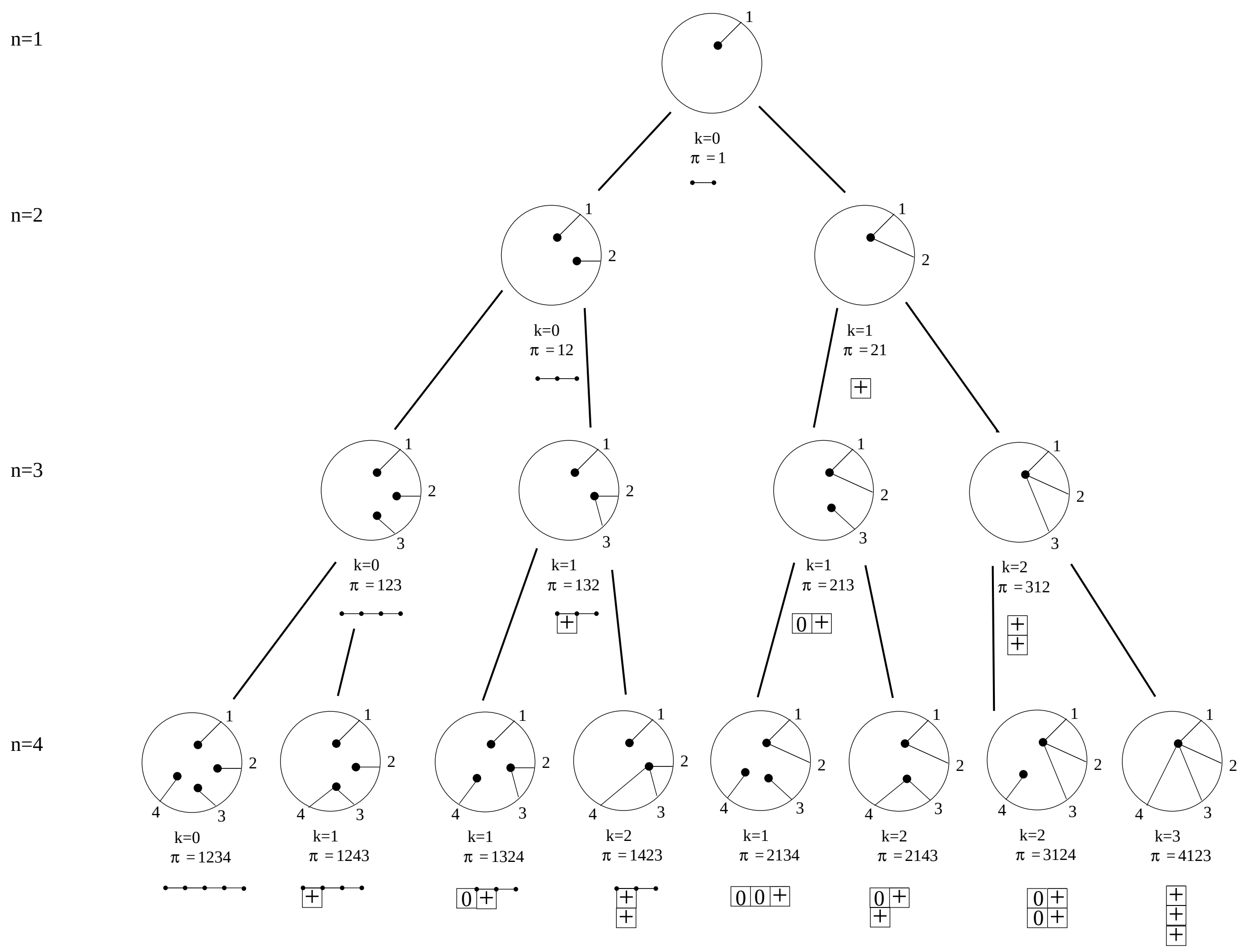



Figure 3.5: The $m = 1$ BCFW-style recursion for $n = 1, 2, 3, 4$.

**Lemma 3.4.2.** *The $m = 1$ BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ are indexed by the ⅃-diagrams of type $(k, n)$ such that each of the $k$ rows contains a unique $+$, which is at the far right of the row. The decorated permutation of such a ⅃-diagram $D$ can be written in cycle notation as*

$$\pi(D) = (i_1, i_1 - 1, i_1 - 2, \ldots, 1)(i_2, i_2 - 1, \ldots, i_1 + 1) \cdots (n, n - 1, \ldots, i_{n-k-1} + 1),$$

*where $i_1 < i_2 < \cdots < i_{n-k-1} < i_{n-k} = n$ label the horizontal steps of the southeast border of $D$ (read northeast to southwest), and all fixed points are colored black. In particular, the number of $m = 1$ BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ equals $\binom{n-1}{k}$.*

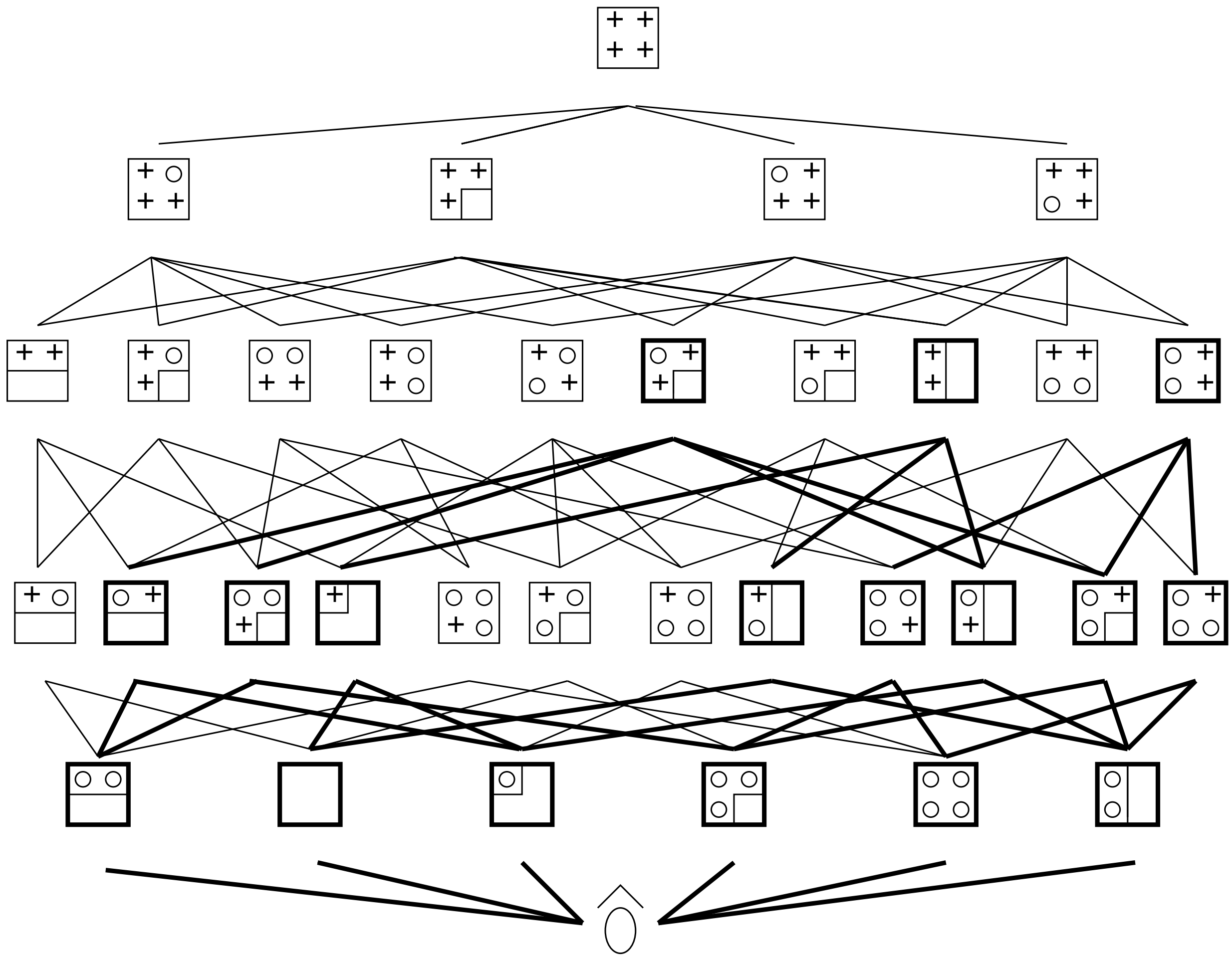


Figure 3.6: The poset $Q_{2,4}$ of cells of $\mathrm{Gr}_{2,4}^{\geq 0}$, where each cell is identified with the corresponding Ꞁ-diagram. The bold edges indicate the subcomplex (an induced subposet) which gets identified with the amplituhedron $\mathcal{A}_{4,2,1}(Z)$.

## 3.5 $\mathcal{A}_{n,k,1}$ as a subcomplex of the totally nonnegative Grassmannian

In this section we show that the amplituhedron $\mathcal{B}_{n,k,1}(W)$ (for $W \in \mathrm{Gr}_{k+1,n}^{>0}$) is isomorphic to a subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$. We begin by defining a stratification of $\mathcal{B}_{n,k,1}(W)$, whose strata are indexed by sign vectors $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ and have a natural poset structure. We will also define a poset of certain Ꞁ-diagrams $\overline{\mathcal{D}_{n,k,1}}$ and a poset of positroids $\overline{\mathcal{M}_{n,k,1}}$ and show that all three posets are isomorphic. Finally we will give an isomorphism between the amplituhedron $\mathcal{B}_{n,k,1}(W)$ and the subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$ indexed by the cells associated to $\overline{\mathcal{D}_{n,k,1}}$, which induces an isomorphism on the posets of closures of strata, ordered by containment.

Recall that by Corollary 3.3.18, we have

$$\mathcal{B}_{n,k,1}(W) = \{w \in W \setminus \{0\} : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W),$$

where we identify a nonzero vector in $W$ with the line it spans in $\mathbb{P}(W)$. We define a stratification of $\mathcal{B}_{n,k,1}(W)$ using sign vectors.

**Definition 3.5.1.** Let $\overline{\mathrm{Sign}_{n,k,1}}$ denote the set of nonzero sign vectors $\sigma \in \{0,+,-\}^n$ with $\overline{\mathrm{var}}(\sigma) = k$, such that if $i \in [n]$ indexes the first nonzero component of $\sigma$, then $\sigma_i = (-1)^{i-1}$. (Equivalently, the first nonzero component of $\mathrm{alt}(\sigma)$ equals $+$.) Also let $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ denote the set of nonzero sign vectors $\sigma \in \{0,+,-\}^n$ with $\overline{\mathrm{var}}(\sigma) = k$, modulo multiplication by $\pm 1$. We let $\mathrm{Sign}_{n,k,1}$ and $\mathbb{P}\mathrm{Sign}_{n,k,1}$ be the subsets of $\overline{\mathrm{Sign}_{n,k,1}}$ and $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ consisting of vectors with no zero components.

**Definition 3.5.2.** We stratify the amplituhedron $\mathcal{B}_{n,k,1}(W)$ by $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$, i.e. its strata are $\mathcal{B}_\sigma(W) := \{w \in W \setminus \{0\} : \mathrm{sign}(w) = \pm\sigma\}$ for $\sigma \in \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$. All strata are nonempty by Lemma 3.3.7(i). The strata are partially ordered by containment of closures, i.e. $\mathcal{B}_\sigma(W) \le \mathcal{B}_\tau(W)$ if and only if $\mathcal{B}_\sigma(W) \subseteq \overline{\mathcal{B}_\tau(W)}$.

This stratification for $\mathcal{B}_{5,2,1}(W)$ is shown in Figure 3.8. We will show in Theorem 3.5.17 and Theorem 3.6.16 that the partial order on strata corresponds to a very natural partial order on $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$, which we now describe.

**Definition 3.5.3.** We define a partial order on the set of sign vectors $\{0,+,-\}^n$ as follows: $\sigma \le \tau$ if and only if $\sigma_i = \tau_i$ for all $i \in [n]$ such that $\sigma_i \neq 0$. Equivalently, $\sigma \le \tau$ if and only if we can obtain $\sigma$ by setting some components of $\tau$ to 0. This gives a partial order on $\overline{\mathrm{Sign}_{n,k,1}}$ by restriction. And for nonzero sign vectors $\sigma, \tau$ representing elements in $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$, we say that $\sigma \le \tau$ if and only if $\sigma \le \tau$ or $\sigma \le -\tau$ in $\overline{\mathrm{Sign}_{n,k,1}}$.

For example, $(+,0,+,0,+) \le (+,-,+,+,+)$, but $(+,0,+,0,+) \not\le (+,-,0,+,+)$. Figure 3.8 shows $\overline{\mathrm{Sign}_{5,2,1}}$ as labels of the bounded faces of a hyperplane arrangement.

We will now show that $\overline{\mathrm{Sign}_{n,k,1}}$ and $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ are isomorphic as posets. Our reason for using both posets is as follows. Since $\mathcal{B}_{n,k,1}(W)$ is a subset of the projective space $\mathbb{P}(W)$, the sign vectors used to index strata should be considered modulo multiplication by $\pm 1$, which leads us naturally to $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$. However, in Theorem 3.6.16 we will show that $\mathcal{B}_{n,k,1}(W)$ is isomorphic to the bounded complex of a hyperplane arrangement, and the bounded faces of this arrangement are labeled by sign vectors *not* considered modulo multiplication by $\pm 1$. To prove this result, we will need to work with $\overline{\mathrm{Sign}_{n,k,1}}$, which requires a more careful analysis.

**Lemma 3.5.4.**
*(i) The map* $\overline{\mathrm{Sign}_{n,k,1}} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ *is an isomorphism of posets.*
*(ii) Conversely, suppose that* $P$ *is an induced subposet of* $\{\sigma \in \{0,+,-\}^n \setminus \{0\} : \overline{\mathrm{var}}(\sigma) = k\}$ *such that the map* $P \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ *is an isomorphism of posets. Then* $P$ *equals* $\overline{\mathrm{Sign}_{n,k,1}}$ *or* $-\overline{\mathrm{Sign}_{n,k,1}}$.

This says that $\pm\overline{\mathrm{Sign}_{n,k,1}}$ are the unique liftings of $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ to $\{0,+,-\}^n$ which preserve its poset structure. For example, if $n = 2, k = 1$, then the lifting $P := \{(+,-),(+,0),(0,+)\}$ does not have the same poset structure as $\overline{\mathrm{Sign}_{2,1,1}} = \{(+,-),(+,0),(0,-)\}$, since $P$ has two maximal elements, but $\overline{\mathrm{Sign}_{2,1,1}}$ has the unique maximum $(+,-)$.

**Proof.** (i) The map $\overline{\mathrm{Sign}_{n,k,1}} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ is a bijection and a poset homomorphism. To show that it is a poset isomorphism, we must show that there do not exist $\sigma, \tau \in \overline{\mathrm{Sign}_{n,k,1}}$ with $\sigma \le -\tau$. Suppose otherwise that such $\sigma$ and $\tau$ exist. Let $\sigma' := \mathrm{alt}(\sigma)$ and $\tau' := \mathrm{alt}(\tau)$. By Lemma 3.3.3(i), we have $\mathrm{var}(\sigma') = \mathrm{var}(\tau') = n-k-1$. Let $i, j \in [n]$ index the first nonzero components of $\sigma', \tau'$, respectively, so that $\sigma'_i = \tau'_j = +$. But also by our assumption, $\sigma'_i = -\tau'_i$. Since $j < i$, $-\tau'$ changes sign at least once from $j$ to $i$. This implies $\mathrm{var}(-\tau') > \mathrm{var}(\sigma')$, a contradiction.

(ii) Let $G$ be the graph with vertex set $\mathbb{P}\mathrm{Sign}_{n,k,1}$, where distinct $\sigma, \tau \in \mathbb{P}\mathrm{Sign}_{n,k,1}$ are adjacent if and only if $\sigma$ differs in a single component from either $\tau$ or $-\tau$.

*Claim.* $G$ is connected.

*Proof of Claim.* Let us show that every vertex $\sigma$ of $G$ is connected to

$$\widehat{\sigma} := (+,-,+,-,\dots,(-1)^{k-1},(-1)^k,(-1)^k,\dots),$$

i.e. $\widehat{\sigma}$ is the sign vector which alternates in sign on $[k+1]$, and is constant thereafter. Take $i \in [n]$ maximum such that $\sigma$ alternates in sign on $[i]$. If $i = k+1$, then $\sigma = \widehat{\sigma}$. Otherwise, take $j > i$ minimum with $\sigma_j \ne \sigma_i$. That is, on the interval $[i,j]$, $\sigma$ equals $(+,+,\dots,+,-)$ up to sign. By performing sign flips at components $j-1, j-2, \dots, i+1$, we obtain a sign vector which equals $(+,-,-,\dots,-)$ on $[i,j]$, and hence alternates in sign on $[i+1]$. Repeating this procedure for $i+1, i+2, \dots, k$, we obtain $\widehat{\sigma}$. For example, if $\sigma = (+,+,+,-,-,+,-)$ (where $n = 7$, $k = 3$), then we obtain $\widehat{\sigma}$ as follows:

$$\sigma \overset{i=1}{\longmapsto} (+,-,-,-,-,+,-) \overset{i=2}{\longmapsto} (+,-,+,+,+,+,-) \overset{i=3}{\longmapsto} (+,-,+,-,-,-,-) = \widehat{\sigma}.$$

This proves the claim. □

Let $H$ be the corresponding graph for $P$, i.e. $H$ has vertex set $Q := P \cap \{+,-\}^n$, and two distinct sign vectors are adjacent in $H$ if and only if they differ in a single component. Note that $G$ (respectively $H$) depends only on the poset $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ (respectively $P$): two distinct sign vectors are adjacent if and only if they cover a common sign vector in the poset. Since $P \cong \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ and $G$ is connected by sign flips, we get that $H$ is connected by sign flips. Note that we can never flip the first component, which would change sign variation. Hence all elements of $Q$ have the same first component. After replacing $P$ with $-P$ if necessary, we may assume that $\sigma_1 = +$ for all $\sigma \in Q$.

Let $\sigma \in P$. We will show that $\sigma_i = (-1)^{i-1}$, where $i$ indexes the first nonzero component of $\sigma$. Since $\overline{\mathrm{var}}(\sigma) = k$, we can extend $\sigma$ to $\tau \in \{+,-\}^n$ (i.e. $\tau \ge \sigma$) with $\mathrm{var}(\tau) = k$. Then $\tau$ alternates in sign on $[i]$, i.e. it equals $((-1)^{i-1}\sigma_i, (-1)^{i-2}\sigma_i, \dots, \sigma_i)$ on $[i]$. Now since

$P \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ is a poset isomorphism, exactly one of $\tau, -\tau$ is in $P$. If $-\tau \in P$, then $\sigma \nleq -\tau$ in $P$ but $\sigma \leq -\tau$ in $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$, a contradiction. Hence $\tau$ is in $P$ (and hence in $Q$), whence $\tau_1 = +$, i.e. $\sigma_i = (-1)^{i-1}$. ■

We now define a subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$ which will turn out to be isomorphic to $\mathcal{B}_{n,k,1}(W)$.

**Definition 3.5.5.** Let $\mathcal{D}_{n,k,1}$ (respectively, $\overline{\mathcal{D}_{n,k,1}}$) be the set of Ⅎ-diagrams contained in a $k \times (n-k)$ rectangle whose rows each have precisely one $+$ (respectively, at most one $+$), and each $+$ appears at the right end of its row. For $D \in \overline{\mathcal{D}_{n,k,1}}$, we let $\dim(D) := \dim(S_D)$ be the number of $+$'s in $D$.

Note that $\mathcal{D}_{n,k,1}$ indexes the $m = 1$ BCFW cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ by Lemma 3.4.2.

Since Ⅎ-diagrams index the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, $\overline{\mathcal{D}_{n,k,1}}$ has a poset structure as a subposet of $Q_{k,n}$. However, it is more convenient for us to define our own partial order on $\overline{\mathcal{D}_{n,k,1}}$, as follows; we then show in Lemma 3.5.13 that our partial order agrees with the one coming from $Q_{k,n}$, and that our poset on $\overline{\mathcal{D}_{n,k,1}}$ is in fact an order ideal (a downset) of $Q_{k,n}$.

**Definition 3.5.6.** We define a partial order on $\overline{\mathcal{D}_{n,k,1}}$, with the following cover relations $\lessdot$:

- (Type 1) Let $D \in \overline{\mathcal{D}_{n,k,1}}$, where $\dim(D) \geq 1$, and choose some $+$ in $D$ which has no $+$'s below it in the same column. We obtain $D'$ from $D$ by deleting the box containing that $+$ and every box below it and in the same column. Then $D' \lessdot D$.
- (Type 2) Let $D \in \overline{\mathcal{D}_{n,k,1}}$, where $\dim(D) \geq 1$, and choose some $+$ in $D$. We obtain $D'$ from $D$ by replacing the $+$ with a 0. Then $D' \lessdot D$.

We now give a poset isomorphism $\Omega_{DS} : \overline{\mathcal{D}_{n,k,1}} \to \overline{\mathrm{Sign}_{n,k,1}}$.

**Definition 3.5.7.** Let $D \in \mathcal{D}_{n,k,1}$ with Young diagram $Y_\lambda$, where the steps of the southeast border of $Y_\lambda$ are labeled from 1 to $n$. Then we define $\sigma(D) \in \mathrm{Sign}_{n,k,1}$ recursively by setting:

- $\sigma_1 := +$;
- $\sigma_{i+1} = \sigma_i$ if and only if $i$ is the label of a horizontal step of $Y_\lambda$, $i = 1, \dots, n-1$.

Now let $D \in \overline{\mathcal{D}_{n,k,1}}$ with Young diagram $Y_\lambda$. We obtain $\widehat{D}$ from $D$ by putting a $+$ at the far right of every row which has no $+$. To define $\sigma(D)$, we first compute $\sigma(\widehat{D})$ as above, but then for every vertical step $i$ corresponding to an all-zero row of $D$, we set $\sigma(D)_i = 0$. This gives a map $\Omega_{DS} : \overline{\mathcal{D}_{n,k,1}} \to \overline{\mathrm{Sign}_{n,k,1}}$ defined by $\Omega_{DS}(D) = \sigma(D)$.

For examples of this bijection, compare Figure 3.9 with Figure 3.8.

**Lemma 3.5.8.** *The map $\Omega_{DS}$ from Definition 3.5.7 is an isomorphism of posets.*

**Proof.** First we show that $\Omega_{DS}$ is well defined, i.e. given $D \in \overline{\mathcal{D}_{n,k,1}}$ we have $\sigma(D)_i = (-1)^{i-1}$, where $i \in [n]$ indexes the first nonzero component of $\sigma(D)$. Indeed, we have that $1, \dots, i-1$ label vertical steps of the southeast border of $D$ whose rows contain only 0's, and $i$ labels either a vertical step whose row contains a $+$, or a horizontal step. Hence when we lift $D$ to $\widehat{D}$ in Definition 3.5.7, $\sigma(\widehat{D})$ alternates in sign on $[1, i]$, whence $\sigma(D)_i = \sigma(\widehat{D})_i = (-1)^{i-1}$.

To show that $\Omega_{DS}$ is a bijection, we describe its inverse. Given $\sigma \in \overline{\mathrm{Sign}_{n,k,1}}$, we construct $D \in \overline{\mathcal{D}_{n,k,1}}$ with $\sigma(D) = \sigma$. We first lift $\sigma$ to $\widehat{\sigma} \in \mathrm{Sign}_{n,k,1}$ by reading $\sigma$ from right to left:

- Set $\widehat{\sigma}_n := (-1)^k$.
- For $i = n-1, \dots 1$, if $\sigma_i = 0$, we set $\widehat{\sigma}_i := -\widehat{\sigma}_{i+1}$, and otherwise we set $\widehat{\sigma}_i := \sigma_i$.

The reason we set $\widehat{\sigma}_n := (-1)^k$ is that if $\tau \in \{+,-\}^n$ with $\tau \geq \sigma$ and $\mathrm{var}(\tau) = k$, then $\tau_1 = +$ since $\tau \in \mathrm{Sign}_{n,k,1}$, which implies that $\tau_n = (-1)^k$. The same reasoning implies that $\widehat{\sigma} \in \mathrm{Sign}_{n,k,1}$. Now we construct $D$ as follows. The vertical steps of the southeast border of $D$ are in bijection with positions $i$ of $\widehat{\sigma}$ such that $\widehat{\sigma}_i = -\widehat{\sigma}_{i+1}$. In each row with vertical step labeled $i$, where $\sigma_i \neq 0$, we place a $+$ in the far right. We then fill the remaining boxes with 0's. Note that $\sigma(D) = \sigma$. Moreover, we know that 0's of $\sigma$ must correspond to vertical steps in the southeast border of $D$, which uniquely determines $D$. Therefore $\Omega_{DS}$ is a bijection.

We can check that $\Omega_{DS}$ is a poset homomorphism. It remains to check that its inverse is a poset homomorphism. To this end, let $\sigma' \lessdot \sigma$ be a cover relation in $\overline{\mathrm{Sign}_{n,k,1}}$, so that $\sigma'$ is obtained from $\sigma$ by setting some $\sigma_i$ to 0. Construct $D \in \overline{\mathcal{D}_{n,k,1}}$ as above with $\sigma(D) = \sigma$. Since $\sigma_i \neq 0$, either $i$ labels a horizontal step in the southeast border of $D$ and $i-1$ labels a vertical step (otherwise we would have $\overline{\mathrm{var}}(\sigma') > \overline{\mathrm{var}}(\sigma)$), or $i$ labels a vertical step and there is a $+$ in that row. In the first case, let $D' \lessdot D$ be the Type 1 cover relation obtained by deleting the box containing the lowest $+$ in column of $D$ labeled by $i$ (and all boxes below it). In the second case, let $D' \lessdot D$ be the Type 2 cover relation obtained by replacing the unique $+$ in the row labeled by $i$ with a 0. In either case we have $\sigma(D') = \sigma'$. ■

We now introduce the positroids corresponding to $\overline{\mathcal{D}_{n,k,1}}$.

**Definition 3.5.9.** Let $\mathcal{M}_{n,k,1}$ be the set of matroids of rank $k$ with ground set $[n]$ which are direct sums $M_1 \oplus \cdots \oplus M_{n-k}$, where $E_1 \sqcup \cdots \sqcup E_{n-k}$ is a partition of $[n]$ into nonempty intervals, and $M_j$ is the uniform matroid of rank $|E_j| - 1$ with ground set $E_j$, for $j \in [n-k]$. Let $\overline{\mathcal{M}_{n,k,1}}$ be the order ideal of matroids of $\mathcal{M}_{n,k,1}$, i.e. the set of matroids $M'$ of rank $k$ with ground set $[n]$ such that $M' \leq M$ for some $M \in \mathcal{M}_{n,k,1}$. (The partial order was defined in Definition 3.2.18.)

We also define $\mathcal{M}^*_{n,k,1} := \{M^* : M \in \mathcal{M}_{n,k,1}\}$ and $\overline{\mathcal{M}^*_{n,k,1}} := \{M^* : M \in \overline{\mathcal{M}_{n,k,1}}\}$, so that $\overline{\mathcal{M}^*_{n,k,1}}$ is the set of matroids $M'$ of rank $n-k$ with ground set $[n]$ such that $M' \leq M$ for some $M \in \mathcal{M}^*_{n,k,1}$. Since taking duals commutes with direct sum (see Proposition 4.2.21 of [Oxl11]), $\mathcal{M}^*_{n,k,1}$ is the set of matroids of rank $k$ with ground set $[n]$ which are direct sums $M_1 \oplus \cdots \oplus M_{n-k}$, where $E_1 \sqcup \cdots \sqcup E_{n-k}$ is a partition of $[n]$ into nonempty intervals, and $M_j$ is the uniform matroid of rank 1 with ground set $E_j$, for $j \in [n-k]$.

The matroids in $\overline{\mathcal{M}_{n,k,1}}$ and $\overline{\mathcal{M}^*_{n,k,1}}$ are in fact positroids; see Definition 3.5.12. (Alternatively, using Lemma 3.5.10, we can easily construct a matrix representing any matroid in $\overline{\mathcal{M}^*_{n,k,1}}$.) This implies that $\overline{\mathcal{M}_{n,k,1}}$ indexes an order ideal in $Q_{k,n}$, the poset of cells of $\mathrm{Gr}^{\geq 0}_{k,n}$.

**Lemma 3.5.10.** *The matroids in* $\overline{\mathcal{M}_{n,k,1}}$ *and* $\overline{\mathcal{M}^*_{n,k,1}}$ *correspond to pairs* $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$*, where*

- $E_1 \sqcup \cdots \sqcup E_{n-k}$ *is a partition of* $[n]$ *into nonempty intervals and* $C \subseteq [n]$*;*
- *for all* $j \in [n-k]$*,* $E_j \setminus C$ *is nonempty; and*
- $\max(E_j) \notin C$ *whenever* $\max(E_j) \neq n$*.*

*The matroid* $M \in \overline{\mathcal{M}_{n,k,1}}$ *associated to* $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ *is the direct sum* $M_1 \oplus \cdots \oplus M_{n-k}$*, where* $M_j$ *(*$j \in [n-k]$*) is the matroid with ground set* $E_j$ *such that* $E_j \cap C$ *are coloops and the restriction of* $M_j$ *to* $E_j \setminus C$ *is a uniform matroid of rank* $|E_j \setminus C| - 1$*.*

*And the matroid* $M^* \in \overline{\mathcal{M}^*_{n,k,1}}$ *associated to* $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ *is the direct sum* $M^*_1 \oplus \cdots \oplus M^*_{n-k}$*, where* $M^*_j$ *(*$j \in [n-k]$*) is the matroid with ground set* $E_j$ *such that* $E_j \cap C$ *are loops and the restriction of* $M_j$ *to* $E_j \setminus C$ *is a uniform matroid of rank* $1$*.*

**Proof.** The description of $\overline{\mathcal{M}_{n,k,1}}$ follows from the description of $\overline{\mathcal{M}^*_{n,k,1}}$, so we prove the latter. In considering $\overline{\mathcal{M}^*_{n,k,1}}$, we will use the following claim.

*Claim.* Let $M$ be a matroid with ground set $F$ and connected components $F_1, \dots, F_l$, and let $M' \leq M$. Then each $F_j$ ($j \in [l]$) is a union of connected components of $M'$, and $M'|_{F_j} \leq M|_{F_j}$.

*Proof of Claim.* By Proposition 4.1.2 of [Oxl11] (see also Proposition 7.2 of [ARW16]), elements $i, j$ of the ground set are in the same connected component of a matroid $N$ if and only if there exist bases $B, B'$ of $N$ with $B' = (B \setminus \{i\}) \cup \{j\}$. It follows that if $i, j \in F$ are in the same connected component of $M'$, then they are in the same connected component of $M$. The fact that $M'|_{F_j} \leq M|_{F_j}$ for $j \in [l]$ follows from Definition 3.2.17, as long as $M'|_{F_j}$ and $M|_{F_j}$ have the same rank. But this is true since $\sum_{j=1}^{l} \mathrm{rank}(M'|_{F_j}) = \mathrm{rank}(M') = \mathrm{rank}(M) = \sum_{j=1}^{l} \mathrm{rank}(M|_{F_j})$. □

Now note that if $M$ is a uniform matroid of rank 1, then the matroids $M'$ satisfying $M' \leq M$ are precisely all matroids of rank 1 with the same ground set as $M$. Hence by the claim, the elements of $\overline{\mathcal{M}^*_{n,k,1}}$ are obtained precisely by taking a matroid $M_1 \oplus \cdots \oplus M_{n-k} \in \mathcal{M}_{n,k,1}$ with ground set $E_1 \sqcup \cdots \sqcup E_{n-k}$, and choosing some subset $C \subseteq [n]$ of the ground set, satisfying $E_j \setminus C \neq \emptyset$ for all $j \in [n-k]$, to turn into loops (i.e. we delete all bases with a nonempty intersection with $C$). The condition that $\max(E_j) \notin C$ if $\max(E_j) \neq n$ comes from our convention that a loop which appears 'between' two intervals is associated to the interval on its right. ■

**Remark 3.5.11.** Using Lemma 3.5.10, one can write down the generating function for the stratification of $\mathcal{B}_{n,k,1}(W)$ with respect to dimension; see Corollary 3.6.21. We will give a different proof of Corollary 3.6.21, using the fact that $\mathcal{B}_{n,k,1}(W)$ is isomorphic to the bounded complex of a generic hyperplane arrangement, whose rank generating function is known.

We now give a poset isomorphism $\Omega_{DM} : \overline{\mathcal{D}_{n,k,1}} \to \overline{\mathcal{M}_{n,k,1}}$.

**Definition 3.5.12.** Given $D \in \overline{\mathcal{D}_{n,k,1}}$, we label the southeast border of $D$ by the numbers 1 through $n$. Let the labels of the horizontal steps be denoted by $h_1, \dots, h_{n-k}$. Then set $E_1 := \{1, 2, \dots, h_1\}$, $E_j := \{h_{j-1} + 1, h_{j-1} + 2, \dots, h_j\}$ for $1 < j < n - k$, and $E_{n-k} := \{h_{n-k-1}+1, h_{n-k-1}+2, \dots, n\}$. Let $C$ be the set of labels of all vertical steps indexing rows of $D$ with no $+$'s. Then $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ determines a positroid in $\overline{\mathcal{M}_{n,k,1}}$ as in Lemma 3.5.10, which we denote by $M(D)$. By inspection, we see that the map $\Omega_{DM} : D \mapsto M(D)$ is a bijection, and we will denote the inverse by $\Omega_{DM}^{-1} : M \mapsto D(M)$.

We observe that $M(D)$ is precisely the positroid of $D$ defined in Definition 3.2.12. In particular, the elements of $\overline{\mathcal{M}_{n,k,1}}$ and $\overline{\mathcal{M}^*_{n,k,1}}$ are all positroids. See Figure 3.7. The white lollipops correspond to coloops and the black lollipops correspond to loops. By considering perfect orientations, it is easy to see that every component of the graph which is not a white lollipop gives rise to a uniform matroid of corank 1.

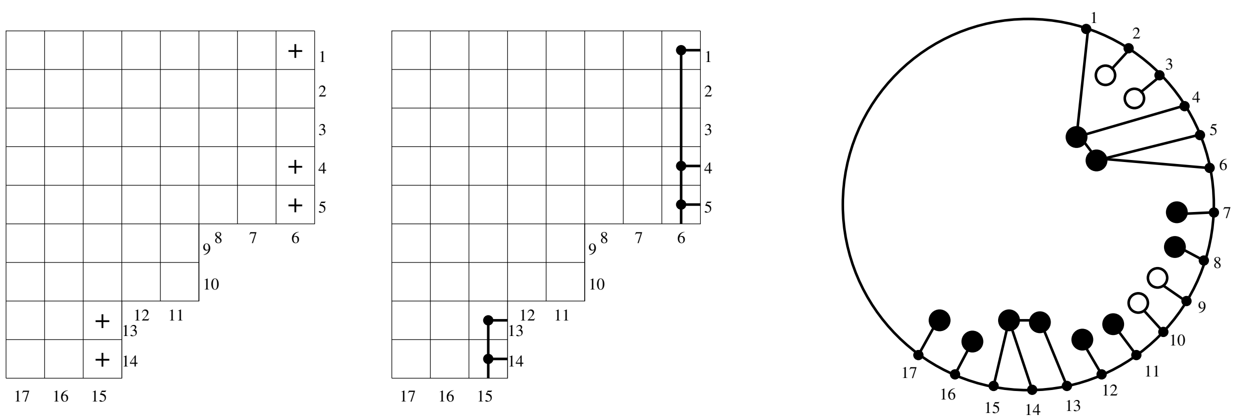


Figure 3.7: Going from a ꓲ-diagram in $\overline{\mathcal{D}_{n,k,1}}$ to the corresponding plabic graph, which in turn determines the positroid in $\overline{\mathcal{M}_{n,k,1}}$.

**Lemma 3.5.13.** *The map $\Omega_{DM}$ from Definition 3.5.12 is an isomorphism of posets. In particular, $\overline{\mathcal{D}_{n,k,1}}$ can be identified with an order ideal of $Q_{k,n}$, the poset of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.*

**Proof.** By inspection, $\Omega_{DM}$ is a bijection and a poset homomorphism. To see that its inverse is a poset homomorphism, let $M' \lessdot M$ be a cover relation in $\overline{\mathcal{M}_{n,k,1}}$, and take $D \in \overline{\mathcal{D}_{n,k,1}}$ with $M = M(D)$. Let $E_1 \sqcup \cdots \sqcup E_{n-k}$ correspond to $M$ as in Lemma 3.5.10. Then we obtain $M'$ from $M$ by taking some interval $E_j$ which contains at least 2 elements which are not coloops, and turning some $i \in E_j$ into a coloop. If $i$ is the greatest element of $E_j$ not already a coloop, then $i$ labels a horizontal step in the southeast border of $D$. In this case, let $D' \lessdot D$ be the Type 1 cover relation given by deleting the box containing the lowest $+$ in column of $D$ labeled by $i$ (and all boxes below it). Otherwise $i$ labels a vertical step in the southeast border of $D$ whose row contains a $+$; let $D' \lessdot D$ be the Type 2 cover relation given by replacing this $+$ with a 0. Then $M(D') = M'$. ■

By composing the bijections from Definitions (3.5.7) and (3.5.12), we obtain the following result.

**Corollary 3.5.14.** *The map $\Omega_{DS} \circ {\Omega_{DM}}^{-1} : \overline{\mathcal{M}_{n,k,1}} \to \overline{\mathrm{Sign}_{n,k,1}}, M \mapsto \sigma(M)$ is an isomorphism of posets, where we define $\sigma(M) := \sigma(D)$ for $D \in \overline{\mathcal{D}_{n,k,1}}$ with $M = M(D)$. We can compute $\sigma(M)$ from $M$ as follows. If $M \in \mathcal{M}_{n,k,1}$ is the direct sum $M_1 \oplus \cdots \oplus M_{n-k}$ with ground set $E_1 \sqcup \cdots \sqcup E_{n-k}$, then $\sigma(M)$ is uniquely determined by the following two properties:*

- *$\sigma(M)_1 = +$;*
- *$\sigma(M)_i = \sigma(M)_{i+1}$ if and only if $i$ and $i+1$ are in different blocks of $E_1 \sqcup \cdots \sqcup E_{n-k}$.*

*And if $M \in \overline{\mathcal{M}_{n,k,1}}$ is the direct sum $M_1 \oplus \cdots \oplus M_{n-k}$ with ground set $E_1 \sqcup \cdots \sqcup E_{n-k}$ and coloops at $C$, then $\sigma(M)$ is given exactly as above, except that $\sigma(M)_i = 0$ for each $i \in C$.*

For example, if $M \in \mathcal{M}_{8,5,1}$ is associated to $E_1 \sqcup E_2 \sqcup E_3$ with $E_1 = \{1,2,3\}, E_2 = \{4,5,6\}, E_3 = \{7,8\}$, then $\sigma(M)$ equals $(+,-,+,+,-,+,+,-)$.

**Definition 3.5.15.** Let $\mathcal{S} := \bigsqcup_{M \in \overline{\mathcal{M}_{n,k,1}}} S_M = \overline{\bigsqcup_{M \in \mathcal{M}_{n,k,1}} S_M}$ be the subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$ corresponding to $\mathcal{M}_{n,k,1}$. We define a map (*cf. Definition 3.3.8*)

$$\phi_W : \mathcal{S} \to \mathcal{B}_{n,k,1}(W), \quad V \mapsto V^\perp \cap W.$$

**Proposition 3.5.16.** *If $V \in S_M$ for $M \in \overline{\mathcal{M}_{n,k,1}}$, then $V^\perp \cap W \in \mathcal{B}_{\sigma(M)}(W)$. In other words, the map $\phi_W$ from Definition 3.5.15 induces the map $M \mapsto \sigma(M)$ on strata.*

Technically $\sigma(M)$ is an element of $\overline{\mathrm{Sign}_{n,k,1}}$, while $\mathcal{B}_{n,k,1}(W)$ is stratified by $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$. By Lemma 3.5.4(i) the map $\overline{\mathrm{Sign}_{n,k,1}} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ is a poset isomorphism, so we need not concern ourselves with this distinction.

**Proof.** We first consider the case that $M \in \mathcal{M}_{n,k,1}$. Let us describe the sign vectors of $V^\perp$ for $V \in S_M$. Write $M = M_1 \oplus \cdots \oplus M_{n-k}$, where $E_1 \sqcup \cdots \sqcup E_{n-k}$ is a partition of $[n]$ into nonempty intervals, and $M_j$ is the uniform matroid of rank $|E_j| - 1$ with ground set $E_j$, for $j \in [n-k]$. By Lemma 3.3.7(ii), if $V_j \in S_{M_j}$ then $\mathrm{sign}(V_j^\perp) = \{\sigma \in \{0,+,-\}^{E_j} : \mathrm{var}(\sigma) \geq |E_j| - 1\} \cup \{0\}$, i.e.

$$\mathrm{sign}(V_j^\perp) = \{0, (+,-,+,-,\dots), (-,+,-,+,\dots)\}.$$

Hence for $V \in S_M$, we have $\sigma \in \operatorname{sign}(V^\perp)$ if and only if $\sigma|_{E_j}$ equals 0 or strictly alternates in sign, for all $j \in [n-k]$.

Recall from Corollary 3.3.18 that

$$\mathcal{B}_{n,k,1}(W) = \{w \in W \setminus \{0\} : \overline{\operatorname{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

Note that there is a unique nonzero $\sigma \in \operatorname{sign}(V^\perp)$ (modulo multiplication by $\pm 1$) with $\overline{\operatorname{var}}(\sigma) = k$: $\sigma$ has no zero components, and $\sigma_i = \sigma_{i+1}$ if and only if $i$ and $i+1$ are in different blocks of $E_1 \sqcup \cdots \sqcup E_{n-k}$, for all $i \in [n-1]$. This is precisely the definition of $\sigma(M)$. Therefore we must have $V^\perp \cap W \in \mathcal{B}_{\sigma(M)}(W)$.

Now we consider the general case of sign vectors of $V'^\perp$, where $M' \in \overline{\mathcal{M}_{n,k,1}}$ and $V' \in S_{M'}$. We have $M' \le M$ for some $M \in \mathcal{M}_{n,k,1}$, and $M'$ is obtained from $M$ by making some subset $C \subseteq [n]$ of the ground set coloops. Thus the sign vectors of $V'^\perp$ are precisely obtained from those of $V^\perp$ by setting the components indexed by $C$ to 0. In particular, we again have a unique nonzero $\sigma' \in \operatorname{sign}(V'^\perp)$ (modulo multiplication by $\pm 1$) with $\overline{\operatorname{var}}(\sigma') = k$, which we obtain from $\sigma(M)$ by setting the components $C$ to 0. Then $\sigma' = \sigma(M')$, and therefore $V'^\perp \cap W \in \mathcal{B}_{\sigma(M')}(W)$. ■

**Theorem 3.5.17.** *The map $\phi_W$ from Definition 3.5.15 is a homeomorphism which induces a poset isomorphism on the stratifications of $\mathcal{S}$ and $\mathcal{B}_{n,k,1}(W)$.*

**Proof.** We know from Proposition 3.5.16 that $\phi_W$ induces a poset isomorphism on the strata. To show that $\phi_W$ is a bijection, we construct the inverse map $\phi_W^{-1} : \mathcal{B}_{n,k,1} \to \mathcal{S}$ as follows. Given an element of $\mathcal{B}_{n,k,1}(W)$ spanned by $w \in W \setminus \{0\}$ (so $\overline{\operatorname{var}}(w) = k$), let $\sigma$ be either $\operatorname{sign}(w)$ or $-\operatorname{sign}(w)$, whichever is in $\overline{\operatorname{Sign}_{n,k,1}}$. Also let $M \in \overline{\mathcal{M}_{n,k,1}}$ be the positroid such that $\sigma(M) = \sigma$, corresponding to $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ in Lemma 3.5.10. If $V \in \mathcal{S}$ with $\phi_W(V) = w$, then $V \in S_M$ because $\phi_W$ induces a poset isomorphism on the strata. Since $E_j \setminus C \neq \emptyset$ for all $j \in [n-k]$, $\sigma$ is nonzero when restricted to any interval $E_j$. Hence the unique $V \in \mathcal{S}$ with $\phi_W(V) = w$ is determined by the conditions

$$V^\perp|_{E_j} = \operatorname{span}(w|_{E_j}) \qquad \text{for all } j \in [n-k].$$

Explicitly, $V^\perp$ has the basis $w^{(1)}, \ldots, w^{(n-k)}$, where $w^{(i)}|_{E_j} = \delta_{i,j} w|_{E_j}$ for $i, j \in [n-k]$. Thus $\phi_W$ is invertible, with an inverse which is piecewise polynomial (each stratum is a domain of polynomiality). The map $\phi_W$ is continuous, and therefore a homeomorphism. ■

## 3.6 $\mathcal{A}_{n,k,1}$ as the bounded complex of a cyclic hyperplane arrangement

We show that the $m = 1$ amplituhedron $\mathcal{B}_{n,k,1}(W)$ (or $\mathcal{A}_{n,k,1}(Z)$) is homeomorphic to the complex of bounded faces of a *cyclic hyperplane arrangement* of $n$ hyperplanes in $\mathbb{R}^k$. It then follows from a result of Dong [Don08] that it is homeomorphic to a ball. This story is somewhat analogous to that of $k = 1$ amplituhedra $\mathcal{A}_{n,1,m}$, which are *cyclic polytopes* with $n$ vertices in $\mathbb{P}^m$.[5] (We do not know whether this is a coincidence, or a specific instance of some form of duality for amplituhedra.) Cyclic hyperplane arrangements have been studied by Shannon [Sha79], Ziegler [Zie93], Ramírez Alfonsín [RA99], and Forge and Ramírez Alfonsín [FRA01]. For an introduction to hyperplane arrangements, see [Sta07].

**Remark 3.6.1.** In the literature, a *cyclic hyperplane arrangement* of $n$ hyperplanes in $\mathbb{R}^k$ is usually defined to be an arrangement with hyperplanes

$$\{x \in \mathbb{R}^k : t_i x_1 + t_i^2 x_2 + \cdots + t_i^k x_k + 1 = 0\} \qquad (i \in [n]), \tag{3.6.2}$$

where $0 < t_1 < \cdots < t_n$. We will need to consider more general hyperplane arrangements, whose hyperplanes are of the form

$$\{x \in \mathbb{R}^k : A_{i,1} x_1 + \cdots + A_{i,k} x_k + A_{i,0} = 0\} \qquad (i \in [n]), \tag{3.6.3}$$

such that

$$\mathrm{colspan}((A_{i,j})_{1 \le i \le n, 1 \le j \le k}) \in \mathrm{Gr}_{k,n}^{>0} \qquad \text{and} \qquad \mathrm{colspan}((A_{i,j})_{1 \le i \le n, 0 \le j \le k}) \in \mathrm{Gr}_{k+1,n}^{>0}\,.$$

(We will require that this latter subspace is $W$.) The hyperplane arrangement (3.6.2) is of this form by Vandermonde's identity. Theorem 3.6.16 implies that all hyperplane arrangements of the form (3.6.3) are isomorphic, so we will not be concerned with the distinction between (3.6.2) and (3.6.3). This is analogous to the situation for cyclic polytopes (see [Stu88]).

**Definition 3.6.4.** An arrangement $\{H_1, \dots, H_n\}$ of hyperplanes in $\mathbb{R}^k$ is called *generic* if for all $I \subseteq [n]$, we have $\dim(\bigcap_{i \in I} H_i) = n - |I|$ if $|I| \le k$, and $\bigcap_{i \in I} H_i = \emptyset$ if $|I| > k$.

**Remark 3.6.5.** Cyclic polytopes have many faces of each dimension, in the sense of the upper bound theorem of McMullen [McM70] and Stanley [Sta75]. An analogous property of cyclic hyperplane arrangements is that they have few simplicial faces of each dimension, in the sense of Shannon [Sha79]. Note that it does not make sense to look at the total number of faces, because the number of faces of a given dimension of a generic hyperplane arrangement depends only on its dimension and the number of hyperplanes [Buc43].

In order to define our hyperplane arrangement, we will use the following convention.

[5] In fact, the $k = 1$ amplituhedra $\mathcal{A}_{n,1,m}$ are precisely the *alternating polytopes* of dimension $m$ with $n$ vertices in $\mathbb{P}^m$, as follows from work of Sturmfels [Stu88]. Alternating polytopes are cyclic polytopes which have the additional property that every induced subpolytope is also cyclic. See pp. 396-397 of [BLVS+99] for an example of a cyclic polytope which is not alternating.

**Definition 3.6.6.** Suppose that $V \in \mathrm{Gr}_{k,n}^{>0}$ and $w \in \mathbb{R}^n \setminus V$ such that $V + w \in \mathrm{Gr}_{k+1,n}^{>0}$. Let $u \in V^\perp$ be the orthogonal projection of $w$ to $V^\perp$, i.e. $w - u \in V$. Since $u \neq 0$, Theorem 3.3.4(ii) implies that $\mathrm{var}(u) \geq k$. But since $u$ also lies in $V + w \in \mathrm{Gr}_{k+1,n}^{>0}$, Theorem 3.3.4(ii) implies that $\overline{\mathrm{var}}(u) \leq k$. Therefore $\mathrm{var}(u) = \overline{\mathrm{var}}(u) = k$. But $\mathrm{var}(u) = \overline{\mathrm{var}}(u)$ implies that the first component $u_1$ of $u$ is nonzero. We call $w$ *positively oriented with respect to* $V$ if $u_1 > 0$, and *negatively oriented* if $u_1 < 0$.

**Example 3.6.7.** Let $V \in \mathrm{Gr}_{1,3}^{>0}$ be the span of $(1,1,1)$ (so $n = 3$, $k = 1$), and $w := (1,2,4)$, so that $V + w \in \mathrm{Gr}_{2,3}^{>0}$. Then the projection of $w$ to $V^\perp$ is $u := (-\frac{4}{3}, -\frac{1}{3}, \frac{5}{3})$, since $w - u = \frac{7}{3}(1,1,1) \in V$. Since $u_1 = -\frac{4}{3} < 0$, $w$ is negatively oriented with respect to $V$. $\diamondsuit$

Let us show that the cyclic hyperplane arrangements defined in (3.6.2) satisfy this positive orientation property. Although we will not need Proposition 3.6.8 in what follows, it will mean that our future characterization of face labels of $\mathcal{H}^W$ in Proposition 3.6.14 applies to "classical" cyclic hyperplane arrangements (3.6.2). (This is due to the assumption in Definition 3.6.10 that $w^{(0)}$ is positively oriented with respect to $V$.)

**Proposition 3.6.8.** *Let $0 < t_1 < \cdots < t_n$, and $V \in \mathrm{Gr}_{k,n}^{>0}$ be the span of the vectors $(t_1^j, \ldots, t_n^j)$ for $1 \leq j \leq k$. Then $w = (1, \ldots, 1)$ is positively oriented with respect to $V$.*

**Proof.** We define the $n \times (k+1)$ matrix $A$ with entries $A_{i,j} := t_i^j$ $(1 \leq i \leq n, 0 \leq j \leq k)$. Then $A$ is a *totally positive matrix*, i.e. all its minors are positive. Indeed, for $I = \{i_1 < \cdots < i_l\} \subseteq [n]$ and $J = \{j_1 < \cdots < j_l\} \subseteq \{0, 1, \ldots, k\}$, the classical definition of Schur functions implies that

$$\det(A_{I,J}) = s_{(j_l - l + 1, j_{l-1} - l + 2, \ldots, j_1)}(t_{i_1}, \ldots, t_{i_l}) \prod_{r,s \in [l],\, r<s} (t_{i_s} - t_{i_r}).$$

Since Schur functions are monomial-positive, $\det(A_{I,J}) > 0$.

Because $V + w$ is the column span of $A$, and $V$ is the span of the last $k$ columns of $A$, we get that $V \in \mathrm{Gr}_{k,n}^{>0}$ and $V + w \in \mathrm{Gr}_{k+1,n}^{>0}$. As in Definition 3.6.6, we let $u \in V^\perp$ be the projection of $w$ to $V^\perp$. That is, $w - u \in V$ and $\mathrm{var}(u) = \overline{\mathrm{var}}(u) = k$. We must show that $u_1 > 0$. We will use the following properties of totally positive $n \times (k+1)$ matrices $A$:

- $\mathrm{var}(Ax) \leq \mathrm{var}(x)$ for all $x \in \mathbb{R}^{k+1}$ [Sch30];
- if $\mathrm{var}(Ax) = \mathrm{var}(x)$, then the first nonzero components of $Ax$ and $x$ have the same sign (Theorem V.5 of [GK50]).

Since $w - u \in V$, we can write $u = Ax$ for some $x \in \mathbb{R}^{k+1}$ with $x_1 = 1$. And because $\mathrm{var}(u) = k$, we have $\mathrm{var}(x) = k$, whence the first nonzero component of $u$ is positive. $\blacksquare$

We will also need the following result of Rietsch.[6]

**Lemma 3.6.9** ([Rie]). *If $W \in \mathrm{Gr}_{k+m,n}^{>0}$, where $m \geq 0$, then $W$ contains a subspace in $\mathrm{Gr}_{k,n}^{>0}$.*

We now define our hyperplane arrangement. A hyperplane arrangement $\mathcal{H}$ partitions its ambient space into *faces*; maximal faces (equivalently, connected components of the complement of $\mathcal{H}$) are called *regions*.

**Definition 3.6.10.** Given $W \in \mathrm{Gr}_{k+1,n}^{>0}$, by Lemma 3.6.9 we can find $w^{(1)}, \dots, w^{(k)} \in W$ such that $V := \mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$. We extend $\{w^{(1)}, \dots, w^{(k)}\}$ to a basis $\{w^{(0)}, w^{(1)}, \dots, w^{(k)}\}$ of $W$; after replacing $w^{(0)}$ with $-w^{(0)}$ if necessary, we assume that $w^{(0)}$ is positively oriented with respect to $V$ (see Definition 3.6.6).

We let $\mathcal{H}^W$ be the hyperplane arrangement in $\mathbb{R}^k$ with hyperplanes

$$H_i := \{x \in \mathbb{R}^k : w_i^{(1)} x_1 + \cdots + w_i^{(k)} x_k + w_i^{(0)} = 0\} \text{ for } i \in [n].$$

Note that $\mathcal{H}^W$ is generic by the first three sentences of the proof of Proposition 5.13 of [Sta07].[7] Also note that $\mathcal{H}^W$ depends not only on $W$ but also on our choice of basis of $W$.

Given $w \in W$, we let $\langle w \rangle \in \mathbb{P}(W)$ denote the line spanned by $w$. We define the maps

$$\begin{aligned} \Psi_{\mathcal{H}^W} &: \mathbb{R}^k \to \mathbb{P}(W), & x &\mapsto \langle x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)} \rangle, \\ \psi_{\mathcal{H}^W} &: \mathbb{R}^k \to \{0, +, -\}^n, & x &\mapsto \mathrm{sign}(x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)}). \end{aligned}$$

Note that the faces of $\mathcal{H}^W$ are precisely the nonempty fibers of $\psi_{\mathcal{H}^W}$. If $\sigma \in \{0, +, -\}^n$ has a nonempty preimage under $\psi_{\mathcal{H}^W}$, we call this fiber the face of $\mathcal{H}^W$ *labeled by* $\sigma$. When we identify faces with labels in this way, the face poset of $\mathcal{H}^W$ is an induced subposet of the sign vectors $\{0, +, -\}^n$ (see Definition 3.5.3).

Finally, we let $B(\mathcal{H}^W)$ be the subcomplex of bounded faces of $\mathcal{H}^W$. We denote the set of sign vectors which label the bounded faces of $\mathcal{H}^W$ by $\mathcal{V}_{B(\mathcal{H}^W)}$.

**Remark 3.6.11.** By Proposition 3.6.8, the classical cyclic hyperplane arrangement (3.6.2) is an example of such an $\mathcal{H}^W$ from Definition 3.6.10.

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[6] Rietsch [Rie] in fact proved that the totally nonnegative part $\mathrm{Fl}_n^{\geq 0}$ of the complete flag variety (as defined by Lusztig in Section 8 of [Lus94]) projects surjectively onto $\mathrm{Gr}_{k,n}^{\geq 0}$, and that the Lusztig-Rietsch stratification of $\mathrm{Fl}_n^{\geq 0}$ projects onto Postnikov's stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$. In particular, given $V \in \mathrm{Gr}_{k,n}^{>0}$, there exists a complete flag $V_0 \subset V_1 \subset \cdots \subset V_n$ in the totally positive part $\mathrm{Fl}_n^{>0}$ of $\mathrm{Fl}_n^{\geq 0}$ with $V_k = V$. This immediately implies Lemma 3.6.9, because if $V_0 \subset V_1 \subset \cdots \subset V_n \in \mathrm{Fl}_n^{>0}$ then $V_j \in \mathrm{Gr}_{j,n}^{>0}$ for $0 \leq j \leq n$. (See Corollary 7.2 of [TW15] for a related result.) An alternative proof of Rietsch's result was given by Talaska and Williams (see Theorem 6.6 of [TW13]), by relating Postnikov's parameterizations of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ (see Theorem 6.5 of [Pos]) to Marsh and Rietsch's parameterizations of the Lusztig-Rietsch cells [MR04]. A direct proof of Lemma 3.6.9 using similar tools was given in Lemma 15.6 of [Lam16b].

[7] We thank Richard Stanley for pointing out this argument to us.

We will show that $\Psi_{\mathcal{H}^W}$ gives a homeomorphism from $B(\mathcal{H}^W)$ to $\mathcal{B}_{n,k,1}(W)$ (Theorem 3.6.16). The key to the proof is establishing that $\mathcal{V}_{B(\mathcal{H}^W)} = \overline{\mathrm{Sign}_{n,k,1}}$, which we do in Proposition 3.6.14. (Recall that $\overline{\mathrm{Sign}_{n,k,1}}$ is the set of nonzero sign vectors $\sigma \in \{0,+,-\}^n$ such that $\overline{\mathrm{var}}(\sigma) = k$, and if $i \in [n]$ indexes the first nonzero component of $\sigma$, then $\sigma_i = (-1)^{i-1}$. See Definition 3.5.1.)

In what follows, we fix $W \in \mathrm{Gr}_{k+1,n}^{>0}$, as well as a basis $\{w^{(0)}, w^{(1)}, \dots, w^{(k)}\}$ of $W$ and a corresponding hyperplane arrangement $\mathcal{H}^W$, as in Definition 3.6.10.

**Lemma 3.6.12.**
*(i) If $\sigma \in \{+,-\}^n$ satisfies $\mathrm{var}(\sigma) \le k-1$, then $\sigma$ labels an unbounded region of $\mathcal{H}^W$.*
*(ii) If $\sigma \in \{0,+,-\}^n$ labels an unbounded face of $\mathcal{H}^W$, then $\overline{\mathrm{var}}(\sigma) \le k-1$.*

**Proof.** (i) Suppose that $\sigma \in \{+,-\}^n$ with $\mathrm{var}(\sigma) \le k-1$. Since $\mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$, Lemma 3.3.7(i) implies that we can write $\sigma = \mathrm{sign}(w)$ for some $w \in \mathrm{span}(w^{(1)}, \dots, w^{(k)})$. Then for all $t > 0$ sufficiently large we have $\mathrm{sign}(tw + w^{(0)}) = \sigma$, whence $\psi_{\mathcal{H}^W}^{-1}(\sigma)$ is unbounded.

(ii) Given $\sigma \in \{0,+,-\}^n$ such that $\psi_{\mathcal{H}^W}^{-1}(\sigma)$ is an unbounded face of $\mathcal{H}^W$, take a sequence $(x^{(j)})_{j\in\mathbb{N}}$ in $\psi_{\mathcal{H}^W}^{-1}(\sigma)$ with $\lim_{j\to\infty} \|x^{(j)}\| = \infty$.

*Claim.* $\lim_{j\to\infty} \|x_1^{(j)}w^{(1)} + \cdots + x_k^{(j)}w^{(k)} + w^{(0)}\| = \infty$.

*Proof of Claim.* Let $c \in \mathbb{R}$ be the minimum value of $\|x_1w^{(1)} + \cdots + x_kw^{(k)}\|$ on the compact set $\{x \in \mathbb{R}^k : \|x\| = 1\}$. Since $w^{(1)}, \dots, w^{(k)}$ are linearly independent, we have $c > 0$. Hence by the triangle inequality,

$$\|x_1^{(j)}w^{(1)} + \cdots + x_k^{(j)}w^{(k)} + w^{(0)}\| \ge \|x_1^{(j)}w^{(1)} + \cdots + x_k^{(j)}w^{(k)}\| - \|w^{(0)}\| \ge c\|x^{(j)}\| - \|w^{(0)}\|$$

for $j \in \mathbb{N}$. The claim follows when we send $j$ to $\infty$. □

By the claim, the following sequence is well defined for $j$ sufficiently large:

$$\left(\frac{x_1^{(j)}w^{(1)} + \cdots + x_k^{(j)}w^{(k)} + w^{(0)}}{\|x_1^{(j)}w^{(1)} + \cdots + x_k^{(j)}w^{(k)} + w^{(0)}\|}\right)_{j\in\mathbb{N}}.$$

Since $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is compact, this sequence has a convergent subsequence; we let $v \in \mathbb{R}^n$ denote one of its limit points. Then since $\lim_{j\to\infty} \|x^{(j)}\| = \infty$, we have $v \in \mathrm{span}(w^{(1)}, \dots, w^{(k)}) \setminus \{0\}$. We get $\overline{\mathrm{var}}(\sigma) \le \overline{\mathrm{var}}(v) \le k-1$, where the first inequality holds since $\mathrm{sign}(v)$ is obtained from $\mathrm{sign}(\sigma)$ by possibly setting some nonzero components to zero, and the second inequality follows from Theorem 3.3.4(ii). ■

**Lemma 3.6.13.** *The map $\mathcal{V}_{B(\mathcal{H}^W)} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}, \sigma \mapsto \sigma$ is a poset isomorphism. In other words, the face poset of the bounded faces of $\mathcal{H}^W$ is isomorphic to the face poset of $\mathcal{B}_{n,k,1}(W)$.*

**Proof.** First let us show that the map $\mathcal{V}_{B(\mathcal{H}^W)} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ is well defined, i.e. given $\sigma \in \mathcal{V}_{B(\mathcal{H}^W)}$, we have $\overline{\mathrm{var}}(\sigma) = k$. Since $B(\mathcal{H}^W)$ equals the closure of the union of the bounded regions of $\mathcal{H}^W$ [Don08] (see also Chapter 1, Exercise 7(d) of [Sta07]), the face labeled by $\sigma$ is contained in the closure of some bounded region of $\mathcal{H}^W$, labeled by, say, $\tau \in \{+,-\}^n$. Therefore $\tau \geq \sigma$, where the partial order is the one on $\{0,+,-\}^n$ from Definition 3.5.3. By Lemma 3.6.12(i) we have $\mathrm{var}(\tau) \geq k$, and so $\overline{\mathrm{var}}(\sigma) \geq \mathrm{var}(\tau) \geq k$. But by Theorem 3.3.4(ii) we also have $\overline{\mathrm{var}}(\sigma) \leq k$, so $\overline{\mathrm{var}}(\sigma) = k$, as desired.

Therefore the map $\mathcal{V}_{B(\mathcal{H}^W)} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ is a poset homomorphism. To see that it is surjective, note that if $\sigma \in \{0,+,-\}^n \setminus \{0\}$ satisfies $\overline{\mathrm{var}}(\sigma) = k$, then $\sigma = \mathrm{sign}(w)$ for some $w \in W$ by Lemma 3.3.7(i). When we write $w$ in terms of the basis $w^{(0)}, w^{(1)}, \dots, w^{(k)}$, the coefficient of $w^{(0)}$ is nonzero. (Otherwise $w \in \mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$, implying $\overline{\mathrm{var}}(w) \leq k-1$ by Theorem 3.3.4(ii).) Rescaling $w$ by a positive real number so that this coefficient is $\pm 1$, we see that we can write

$$w = x_1 w^{(1)} + \cdots + x_k w^{(k)} \pm w^{(0)}$$

for some $x \in \mathbb{R}^k$. Therefore either $\sigma$ or $-\sigma$ labels a face of $\mathcal{H}^W$, and such a face is bounded by Lemma 3.6.12(ii).

It remains to show that the map $\mathcal{V}_{B(\mathcal{H}^W)} \to \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}}$ is injective and that its inverse is a poset homomorphism. It suffices to prove that there do not exist $\sigma, \tau \in \{0,+,-\}^n$ labeling bounded faces of $\mathcal{H}^W$ such that $\sigma \leq -\tau$. Suppose otherwise that there exist such $\sigma, \tau$. Take $x, y \in \mathbb{R}^k$ with $\psi_{\mathcal{H}^W}(x) = \sigma$ and $\psi_{\mathcal{H}^W}(y) = \tau$. Subtracting, we get $\mathrm{sign}((x_1 - y_1)w^{(1)} + \cdots + (x_k - y_k)w^{(k)}) = -\tau$. Since $\mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$, Theorem 3.3.4(ii) implies that $\overline{\mathrm{var}}(\tau) \leq k-1$. But we showed in the first paragraph that $\overline{\mathrm{var}}(\tau) = k$. ∎

**Proposition 3.6.14** (The face labels of $\mathcal{H}^W$)**.**
*(i) The labels of the bounded faces of $\mathcal{H}^W$ are precisely $\overline{\mathrm{Sign}_{n,k,1}}$, i.e. $\mathcal{V}_{B(\mathcal{H}^W)} = \overline{\mathrm{Sign}_{n,k,1}}$.*
*(ii) The labels of the unbounded faces of $\mathcal{H}^W$ are precisely $\sigma \in \{0,+,-\}^n$ with $\overline{\mathrm{var}}(\sigma) \leq k-1$.*

**Example 3.6.15.** Let $n := 5, k := 2, m := 1$, and $W \in \mathrm{Gr}_{3,5}^{>0}$ have the basis

$$w^{(0)} := (-1,-1,-1,-1,-1), \quad w^{(1)} := (0,1,2,3,4), \quad w^{(2)} := (10,6,3,1,0).$$

Note that $V := \mathrm{span}(w^{(1)}, w^{(2)}) \in \mathrm{Gr}_{2,5}^{>0}$. The projection $u$ of $w^{(0)}$ to $V^\perp$ is

$$u := w^{(0)} + \tfrac{1}{831}(232 w^{(1)} + 90 w^{(2)}) = \tfrac{1}{831}(69, -59, -97, -45, 97).$$

Since $u_1 > 0$, by Definition 3.6.6 $w^{(0)}$ is positively oriented with respect to $V$.

The hyperplane arrangement $\mathcal{H}^W$ from Definition 3.6.10 consists of 5 lines in $\mathbb{R}^2$:

$$\ell_1\colon\ 10y = 1, \qquad \ell_2\colon\ x + 6y = 1, \qquad \ell_3\colon\ 2x + 3y = 1, \qquad \ell_4\colon\ 3x + y = 1, \qquad \ell_5\colon\ 4x = 1.$$

See Figure 3.8 and Figure 3.9 for $B(\mathcal{H}^W)$ labeled by sign vectors and Ꞁ-diagrams, respectively. (The positive side of each line is above and to the right of it.) By Proposition 3.6.14, the

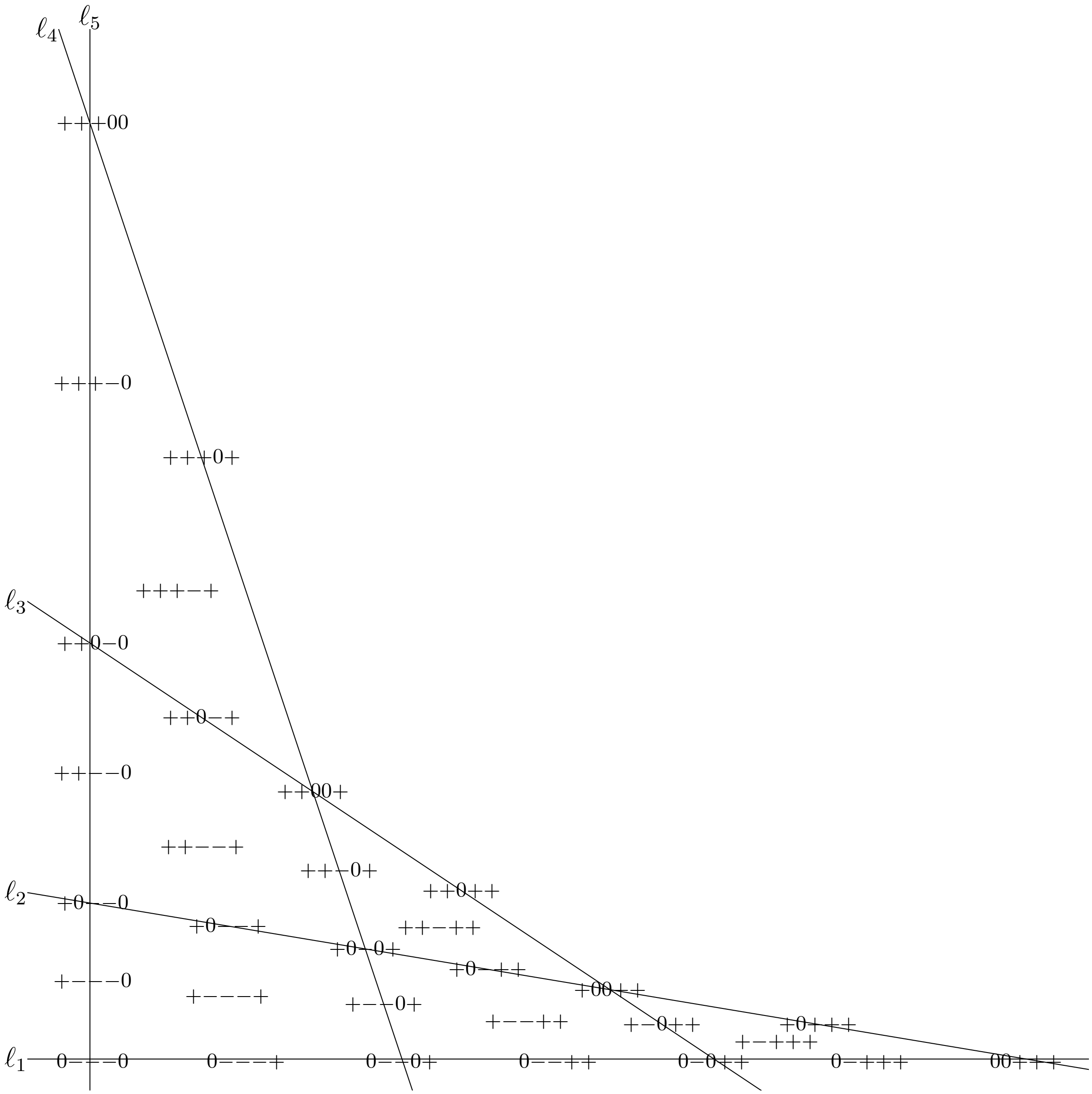


Figure 3.8: The hyperplane arrangement $\mathcal{H}^W$ from Example 3.6.15, with $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$. Its bounded faces are labeled by sign vectors.

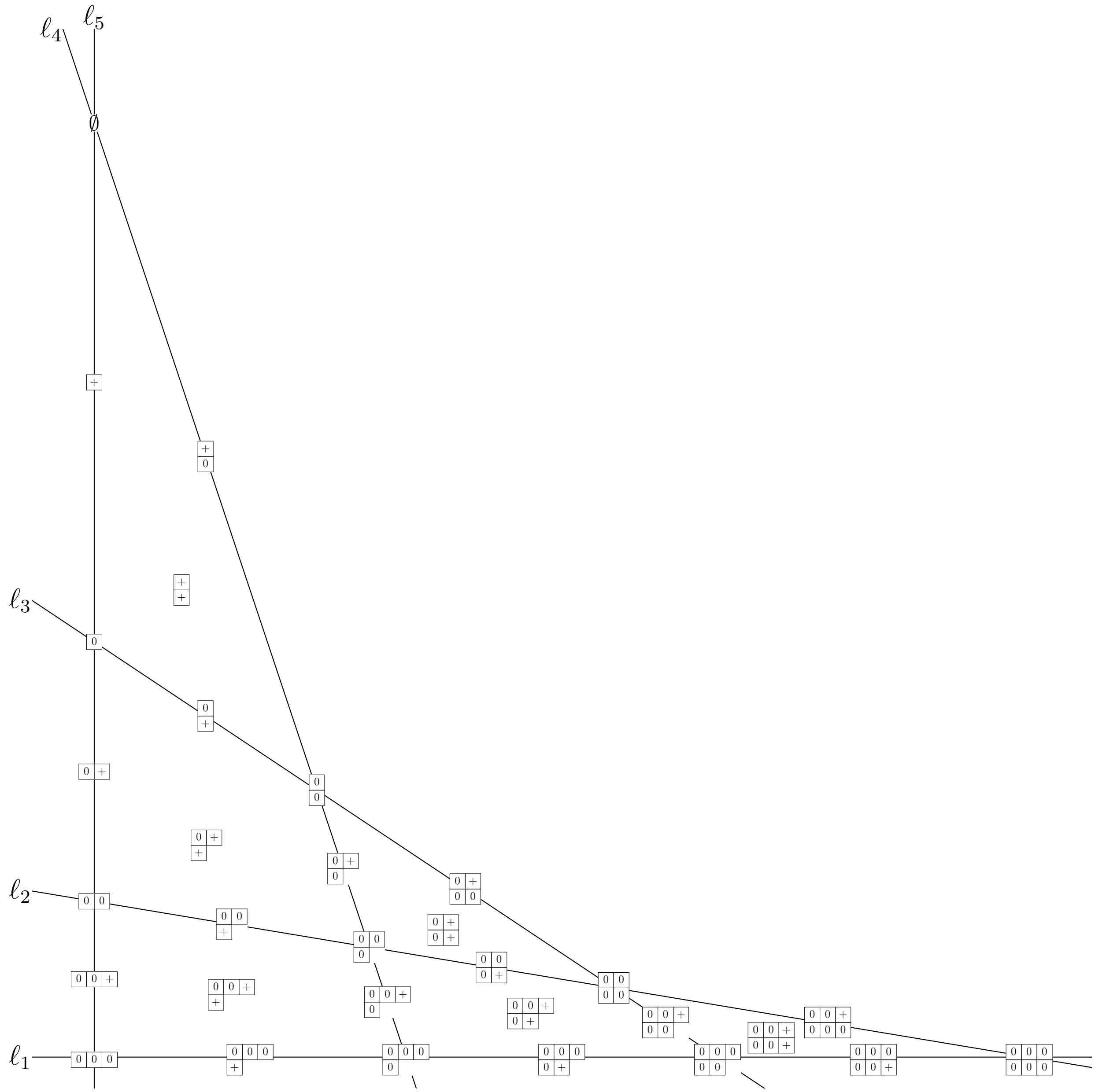


Figure 3.9: The hyperplane arrangement $\mathcal{H}^W$ from Example 3.6.15, with $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$. Its bounded faces are labeled by ⅃-diagrams.

bounded faces of $\mathcal{H}^W$ are labeled by sign vectors in $\overline{\mathrm{Sign}_{5,2,1}}$, and the bounded regions by sign vectors in $\mathrm{Sign}_{5,2,1}$. The unbounded regions are labeled by the sign vectors $\sigma \in \{0, +, -\}^5$ satisfying $\overline{\mathrm{var}}(\sigma) \le 1$. By Theorem 3.6.16, we have $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$. $\Diamond$

**Proof** (of Proposition 3.6.14)**.** (i) By Lemma 3.6.13 and Lemma 3.5.4(ii), $\mathcal{V}_{B(\mathcal{H}^W)}$ equals either $\overline{\mathrm{Sign}_{n,k,1}}$ or $-\overline{\mathrm{Sign}_{n,k,1}}$. We must rule out the latter possibility. Recall that by construction, $w^{(0)}$ is positively oriented with respect to $V := \mathrm{span}(w^{(1)}, \dots, w^{(k)})$. According to Definition 3.6.6, there exist $x_1, \dots, x_k \in \mathbb{R}$ such that $u := x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)}$ satisfies $\mathrm{var}(u) = \overline{\mathrm{var}}(u) = k$ and $u_1 > 0$. Let $\sigma := \mathrm{sign}(u)$. Then $\sigma$ labels a face of $\mathcal{H}^W$, which is a bounded face by Lemma 3.6.12(ii). We get $\sigma \in \mathcal{V}_{B(\mathcal{H}^W)}$, and $\sigma_1 = +$ implies $\mathcal{V}_{B(\mathcal{H}^W)} \neq -\overline{\mathrm{Sign}_{n,k,1}}$.

(ii) One direction follows from Lemma 3.6.12(ii). For the other direction, we will use the following fact about generic hyperplane arrangements $\mathcal{H}$ in $\mathbb{R}^k$: if $\tau$ labels a face of $\mathcal{H}$, then $\sigma$ also labels a face of $\mathcal{H}$ for all $\sigma \ge \tau$. (This follows from the fact that the normal vectors of any $k$ or fewer hyperplanes are linearly independent.)

Given $\sigma \in \{0, +, -\}^n$ with $\overline{\mathrm{var}}(\sigma) \le k - 1$, we must show that $\sigma$ labels a face of $\mathcal{H}^W$ (whence this face is unbounded by part (i)). Since $\mathcal{H}^W$ is generic, it suffices to construct $\tau \le \sigma$ which labels a face of $\mathcal{H}^W$. Our strategy will be to modify the sign vector $\sigma$ until we get a sign vector $\tau \le \sigma$ with $\tau \in \overline{\mathrm{Sign}_{n,k,1}}$, which then implies that $\tau$ labels a bounded face of $\mathcal{H}^W$ by part (i).

Set $\sigma' := \mathrm{alt}(\sigma)$. By Lemma 3.3.3(i), we have $\mathrm{var}(\sigma') \ge n - k$. In particular, the fact that $k < n$ implies that $\sigma'$ has a positive component. Take $i \in [n]$ minimum with $\sigma'_i = +$, and let $\sigma''$ be obtained from $\sigma'$ by setting to zero all components $j$ with $j < i$. Note that $\mathrm{var}(\sigma'') \ge \mathrm{var}(\sigma') - 1 \ge n - k - 1$. Now we repeatedly set the last nonzero component of $\sigma''$ to zero, until we obtain a sign vector $\tau'$ with $\mathrm{var}(\tau') = n - k - 1$. Letting $\tau := \mathrm{alt}(\tau')$, we have $\tau \le \sigma$, and $\overline{\mathrm{var}}(\tau) = k$ by Lemma 3.3.3(i). Since the first nonzero component of $\tau'$ equals $+$, we have $\tau \in \overline{\mathrm{Sign}_{n,k,1}}$, as desired. ■

We are now ready to show that $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$.

**Theorem 3.6.16.** *The restriction of $\Psi_{\mathcal{H}^W}$ to $B(\mathcal{H}^W)$ is a homeomorphism from $B(\mathcal{H}^W)$ to $\mathcal{B}_{n,k,1}(W)$, and induces an isomorphism of posets on the strata of $B(\mathcal{H}^W)$ and $\mathcal{B}_{n,k,1}(W)$. Explicitly, $\Psi_{\mathcal{H}^W}$ sends the stratum $\psi^{-1}_{\mathcal{H}^W}(\sigma)$ of $B(\mathcal{H}^W)$ to the stratum $\mathcal{B}_\sigma(W)$ of $\mathcal{B}_{n,k,1}(W)$, for all $\sigma \in \overline{\mathrm{Sign}_{n,k,1}}$.*

**Proof.** To see that $\Psi_{\mathcal{H}^W}$ is injective, note that if $\Psi_{\mathcal{H}^W}(x) = \Psi_{\mathcal{H}^W}(y)$ for some $x, y \in \mathbb{R}^k$, then there exists $t \in \mathbb{R} \setminus \{0\}$ such that $x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)} = t(y_1 w^{(1)} + \cdots + y_k w^{(k)} + w^{(0)})$. The linear independence of the vectors $w^{(0)}, w^{(1)}, \dots, w^{(k)}$ implies that $t = 1$ and $x_i = y_i$ for all $i \in [k]$, so $x = y$.

Recall that by Corollary 3.3.18, $\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\}$. Let us show that $\Psi_{\mathcal{H}^W}(\mathbb{R}^k)$ contains $\mathcal{B}_{n,k,1}(W)$. It suffices to observe that when we express an element of $\mathcal{B}_{n,k,1}(W)$ as a linear combination of $w^{(0)}, w^{(1)}, \dots, w^{(k)}$, the coefficient of $w^{(0)}$ is nonzero. This follows from Theorem 3.3.4(ii), and the fact that $\mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$.

Now we let $\mathcal{Q} := \Psi_{\mathcal{H}^W}^{-1}(\mathcal{B}_{n,k,1}(W))$. We have shown that $\Psi_{\mathcal{H}^W}$ is a homeomorphism from $\mathcal{Q}$ to $\mathcal{B}_{n,k,1}$. Recall that $\mathcal{B}_{n,k,1}(W)$ is stratified by the sign vectors in $\overline{\mathbb{P}\mathrm{Sign}_{n,k,1}(W)}$. Therefore $\mathcal{Q}$ is the union of the faces of $\mathcal{H}^W$ labeled by $\sigma \in \{0, +, -\}^n$ satisfying $\overline{\mathrm{var}}(\sigma) = k$. By Proposition 3.6.14, this is precisely $B(\mathcal{H}^W)$. The fact that $\Psi_{\mathcal{H}^W}$ induces a poset isomorphism on strata follows from Lemma 3.6.13. ■

**Remark 3.6.17.** It follows from Theorem 3.6.16 that the amplituhedron $\mathcal{B}_{n,k,1}$ is a regular cell complex, and in particular its strata $\mathcal{B}_\sigma(W)$ are homeomorphic to open balls. Using the results of Section 3.5, we can also index the cells of the amplituhedron by Ⅎ-diagrams and matroids, in which case we will use the notation $\mathcal{B}_D(W)$ and $\mathcal{B}_M(W)$, respectively.

**Corollary 3.6.18.** *The amplituhedron $\mathcal{B}_{n,k,1}(W)$ (and also $\mathcal{A}_{n,k,1}(Z)$) is homeomorphic to a ball of dimension $k$.*

**Proof.** This follows from Theorem 3.6.16 together with Dong's result (Theorem 3.1 of [Don08]) that the bounded complex of a uniform affine oriented matroid (of which the bounded complex of a generic hyperplane arrangement is a special case) is a piecewise linear ball. ■

Recall from Theorem 3.5.17 that the amplituhedron $\mathcal{B}_{n,k,1}(W)$ is homeomorphic to a subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$; indeed, $\mathcal{B}_{n,k,1}(W)$ inherits a cell decomposition whose face poset is an induced subposet of the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$. Another way to prove Corollary 3.6.18 would be to show that the face poset of $\mathcal{B}_{n,k,1}(W)$, with a new top element $\hat{1}$ adjoined, is *shellable*. Then since the cell decomposition of $\mathcal{B}_{n,k,1}(W)$ is regular, and its face poset is *pure* and *subthin* (this follows from Theorem 3.6.16), a result of Björner (Proposition 4.3(c) of [Bjö84]) would imply that $\mathcal{B}_{n,k,1}(W)$ is homeomorphic to a ball.

**Problem 3.6.19.** *Show that the face poset of $\mathcal{B}_{n,k,1}(W)$ with a top element $\hat{1}$ adjoined is shellable, e.g. by finding an EL-labeling.*

**Remark 3.6.20.** Note that in earlier work the second author proved that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is thin and shellable [Wil07], which shows that the same is true for any induced subposet. However, this does not solve Problem 3.6.19, because after adjoining $\hat{1}$, the face poset of $\mathcal{B}_{n,k,1}(W)$ is no longer an induced subposet of the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$.

As a further corollary of Theorem 3.6.16, we obtain the generating function for the stratification of $\mathcal{B}_{n,k,1}(W)$ with respect to dimension, since Buck found the corresponding generating function of $B(\mathcal{H})$ for a generic hyperplane arrangement $\mathcal{H}$ (which only depends on its dimension and the number of hyperplanes).

**Corollary 3.6.21** ([Buc43]). *Let $f_{n,k,1}(q) := \sum_{\text{strata } S \text{ of } \mathcal{B}_{n,k,1}(W)} q^{\dim(S)} \in \mathbb{N}[q]$ be the generating function for the stratification of $\mathcal{B}_{n,k,1}(W)$, with respect to dimension. Then*

$$f_{n,k,1}(q) = \sum_{i=0}^{k} \binom{n-k-1+i}{i}\binom{n}{k-i} q^i = \sum_{j=0}^{k} \binom{n-k-1+j}{j}(1+q)^j.$$

For example, we have $f_{5,3,1}(q) = 4q^3 + 15q^2 + 20q + 10$, which we invite the reader to verify from Figure 1.4.

**Proof.** By the corollary to Theorem 3 of [Buc43], for $i \in \mathbb{N}$ the coefficient of $q^i$ in $f_{n,k,1}(q)$ equals $\frac{k+1}{n-k+i}\binom{k}{i}\binom{n}{k+1}$, which we can rewrite as $\binom{n-k-1+i}{i}\binom{n}{k-i}$. This gives the first sum above. For the last equality above, note that the coefficient of $q^i$ in $\sum_{j=0}^{k}\binom{n-k-1+j}{j}(1+q)^j$ equals

$$\sum_{j=i}^{k}\binom{n-k-1+j}{j}\binom{j}{i} = \binom{n-k-1+i}{i}\sum_{j=i}^{k}\binom{n-k-1+j}{n-k-1+i}$$
$$= \binom{n-k-1+i}{i}\binom{n}{n-k+i}$$

by the hockey-stick identity. ■

**Remark 3.6.22.** By substituting $q = -1$ into the last expression in Corollary 3.6.21, it is easy to check that the Euler characteristic of $\mathcal{B}_{n,k,1}(W)$ equals 1.

## 3.7 How cells of $\mathcal{A}_{n,k,1}$ fit together

In this section we will address how cells of the $m = 1$ amplituhedron fit together. In particular, we will explicitly work out when two maximal cells are adjacent, and which cells lie in the boundary of $\mathcal{B}_{n,k,1}(W)$, in terms of Ⅎ-diagrams, sign vectors, positroids, and decorated permutations. See Figure 3.8 and Figure 3.9 for examples when $n = 5$ and $k = 2$.

**Proposition 3.7.1** (Adjacency of maximal cells in the $m = 1$ amplituhedron).
*Given $D_1, D_2 \in \mathcal{D}_{n,k,1}$, the following are equivalent:*

*(i) the cells $\mathcal{B}_{D_1}(W)$ and $\mathcal{B}_{D_2}(W)$ in $\mathcal{B}_{n,k,1}(W)$ are adjacent, i.e. their closures intersect in a cell $\mathcal{B}_{D'}(W)$ of codimension 1, where $D' \in \overline{\mathcal{D}_{n,k,1}}$ is necessarily unique;*

*(ii) the Young diagrams of $D_1$ and $D_2$ differ by a single box, in which case we obtain $D'$ from either $D_1$ or $D_2$ by including this box with a 0 inside it;*

*(iii) there exists $2 \le i \le n-1$ such that the sign vectors $\sigma(D_1)$ and $\sigma(D_2)$ differ precisely in component $i$, in which case we obtain $\sigma(D')$ from either $\sigma(D_1)$ or $\sigma(D_2)$ by setting component $i$ to 0;*

*(iv) there exists $2 \leq i \leq n-1$ such that we can obtain the partition of $[n]$ for $M(D_1)$ (in the sense of Definition 3.5.9) from the partition of $[n]$ for $M(D_2)$ by moving $i$ from one interval to another, in which case we obtain $M(D')$ from either $M(D_1)$ or $M(D_2)$ by turning $i$ into a coloop;*

*(v) there exists $2 \leq i \leq n-1$ such that $\pi(D_2) = s_{i-1}\pi(D_1)s_i$, in which case $\pi(D')$ equals either $\pi(D_1)s_i = s_{i-1}\pi(D_2)$ or $s_{i-1}\pi(D_1) = \pi(D_2)s_i$, whichever has exactly $k-1$ inversions. (Here $s_j$ denotes the simple transposition exchanging $j$ and $j+1$, and all fixed points are colored black.)*

**Proof.** The uniqueness of $D'$ and the equivalence (i) $\Leftrightarrow$ (iii) follows from the fact that $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$ (Theorem 3.6.16); note that since $\sigma(D_1), \sigma(D_2) \in \mathrm{Sign}_{n,k,1}$, the sign vectors cannot differ in their first or last component. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 3.5.8, and (ii) $\Leftrightarrow$ (iv) follows from the bijection in Definition 3.5.12. The equivalence (ii) $\Leftrightarrow$ (v) follows from the bijection in Lemma 3.4.2, using the following explicit description of $\pi' := s_{i-1}\pi s_i$ for any $m = 1$ BCFW permutation $\pi$:

- if $i$ is the minimum value in its cycle and not a fixed point, then we obtain $\pi'$ from $\pi$ in cycle notation by moving $i$ to the cycle with $i-1$, to the left of $i-1$;
- if $i$ is the maximum value in its cycle and not a fixed point, then we obtain $\pi'$ from $\pi$ in cycle notation by moving $i$ to the cycle with $i+1$, to the right of $i+1$;
- if $\pi(i) = i$, then $\pi'$ has $k+1$ anti-excedances (and hence does not index a cell of $\mathrm{Gr}_{k,n}^{\geq 0}$, by Lemma 3.2.5);
- if $i-1$, $i$, and $i+1$ are all in the same cycle, then $\pi' = \pi$. ■

Since our stratification of the amplituhedron $\mathcal{B}_{n,k,1}(W)$ is a regular cell decomposition of a ball (see Remark 3.6.17 and Corollary 3.6.18), it is interesting to characterize which cells (necessarily of codimension at least 1) comprise its boundary. Note that by our identification of cell complexes $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$, every cell of $\mathcal{B}_{n,k,1}(W)$ lies in either the interior or the boundary of $\mathcal{B}_{n,k,1}(W)$.

**Proposition 3.7.2** (Boundary of the $m = 1$ amplituhedron)**.**
*Given $D \in \overline{\mathcal{D}_{n,k,1}}$, the following are equivalent:*

*(i) the cell $\mathcal{B}_D(W)$ of $\mathcal{B}_{n,k,1}(W)$ is contained in the interior;*

*(ii) $D$ has $k$ (nonempty) rows, and for all $r \in [k]$ such that row $r$ of $D$ has no $+$'s, row $r-1$ of $D$ is longer than row $r$ (where row $0$ has length $n-k$);*

*(iii) $\mathrm{var}(\sigma(D)) = k$;*

*(iv) $C \subseteq \{\min(E_1), \ldots, \min(E_{n-k})\} \setminus \{1\}$, where $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ corresponds to $M(D)$ as in Lemma 3.5.10;*

*(v) if $i \in [n]$ is a white fixed point of $\pi$, then $2 \le i \le n-1$ and $i-1$ is not an anti-excedance of $\pi$.*

We remark that the result applies even when $D$ is in $\mathcal{D}_{n,k,1}$ (i.e. the corresponding cell has full dimension), in which case all of the above properties hold.

**Proof.** We fix a hyperplane arrangement $\mathcal{H}^W$ as in Definition 3.6.10, so that $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$ by Theorem 3.6.16, and we let $\sigma$ denote $\sigma(D)$.

(i) $\Rightarrow$ (iii): Suppose that $\mathrm{var}(\sigma) \neq k$. Then $\mathrm{var}(\sigma) < k$, and we can construct $\tau \in \{+,-\}^n$ with $\tau \ge \sigma$ and $\mathrm{var}(\tau) = \mathrm{var}(\sigma)$. For example, do the following repeatedly: take $i \in [n]$ such that component $i$ is zero but either component $i-1$ or $i+1$ is nonzero, and make component $i$ nonzero and equal to either component $i-1$ or $i+1$. Then $\tau$ labels an unbounded face of $\mathcal{H}^W$ by Proposition 3.6.14(ii), whose closure contains the face labeled by $\sigma$.

(iii) $\Rightarrow$ (i): The faces of $\mathcal{H}^W$ whose closure contains the face labeled by $\sigma$ are labeled by $\tau \in \{+,-\}^n$ with $\tau \ge \sigma$. If $\mathrm{var}(\sigma) = k$ then $\overline{\mathrm{var}}(\tau) \ge \mathrm{var}(\tau) \ge \mathrm{var}(\sigma) = k$, whence the face labeled by $\tau$ is bounded by Proposition 3.6.14(ii).

(ii) $\Leftrightarrow$ (iii): Observe that $\mathrm{var}(\sigma) = \overline{\mathrm{var}}(\sigma)$ if and only if for all $i \in [n]$ such that $\sigma_i = 0$, we have that $i \neq 1, n$, and that $\sigma_{i-1}, \sigma_{i+1}$ are nonzero and of opposite sign. This condition is equivalent to (ii) by Lemma 3.5.8.

(ii) $\Leftrightarrow$ (iv): This follows from the bijection in Definition 3.5.12.

(ii) $\Leftrightarrow$ (v): This follows from the bijection in Lemma 3.2.5. ■

## 3.8 The image in $\mathcal{A}_{n,k,1}$ of an arbitrary cell of $\mathrm{Gr}_{k,n}^{\ge 0}$

In this section we study the image in the $m = 1$ amplituhedron of an arbitrary cell of the totally nonnegative Grassmannian. In particular, we describe the image of an arbitrary cell in terms of strata of $\mathcal{B}_{n,k,1}(W)$ (Lemma 3.8.2), we compute the dimension of the image of an arbitrary cell (Proposition 3.8.4), and we characterize the cells which map injectively to the $m = 1$ amplituhedron (Theorem 3.8.10). Since we have a regular cell decomposition of the amplituhedron $\mathcal{B}_{n,k,1}(W)$ using the $m = 1$ BCFW cells and their closures (which can be indexed by the Ⅎ-diagrams in $\overline{\mathcal{D}_{n,k,1}}$), it is also natural to ask how to describe the image of an arbitrary cell of $\mathrm{Gr}_{k,n}^{\ge 0}$ in terms of $\overline{\mathcal{D}_{n,k,1}}$. We answer this question (Theorem 3.8.10) for cells which map injectively into the amplituhedron.

Let us fix a subspace $W \in \mathrm{Gr}_{k+1,n}^{>0}$ for the remainder of the section. Given a Ⅎ-diagram $D$ inside a $k \times (n-k)$ rectangle, let $S_D$ denote its corresponding positroid cell, i.e. the cell $S_{M(D)}$ from Definition 3.2.1, where $M(D)$ is the positroid corresponding to $D$ from Section 3.2. Recall from Remark 3.6.17 that

$$\mathcal{B}_D(W) := \{V^\perp \cap W : V \in S_D\}$$

is the image of $S_D$ in $\mathcal{B}_{n,k,1}(W)$. (It is equivalent to study the image $\tilde{Z}(S_D)$ in $\mathcal{A}_{n,k,1}(Z)$ by Proposition 3.3.12, where $Z$ is any $(k+1) \times n$ matrix whose rows span $W$, but we will find it more convenient to work in $\mathcal{B}_{n,k,1}(W)$.)

**Definition 3.8.1.** Let $D$ be a ⅃-diagram of type $(k, n)$. Fix $V \in S_D$ and define

$$\mathcal{V}(D) := \{\mathrm{sign}(v) : v \in V^\perp\} \subseteq \{0, +, -\}^n.$$

In terms of oriented matroids [BLVS$^+$99], $\mathcal{V}(D)$ is the set of *vectors* of the *positive orientation* of $M(D)$, and so does not depend on our choice of $V \in S_D$.

A basic observation is that $\mathcal{B}_D(W)$ depends precisely on the sign vectors in $\mathcal{V}(D)$ which minimize $\overline{\mathrm{var}}(\cdot)$.

**Lemma 3.8.2.** *For $D$ a ⅃-diagram of type $(k, n)$, we have*

$$\mathcal{B}_D(W) = \bigcup \{\mathcal{B}_\sigma(W) : \sigma \in \mathcal{V}(D) \textit{ with } \overline{\mathrm{var}}(\sigma) = k\}.$$

Recall that $\mathcal{B}_\sigma(W)$ is the $\sigma$-stratum of $\mathcal{B}_{n,k,1}(W)$ from Definition 3.5.2. While Lemma 3.8.2 is not very explicit (it requires being able to compute the sign vectors in $\mathcal{V}(D)$), we will give a more concrete description of the images of certain cells in Theorem 3.8.10.

**Proof.** By Corollary 3.3.18, the left-hand side is contained in the right-hand side. Conversely, given $\sigma \in \mathcal{V}(D)$ with $\overline{\mathrm{var}}(\sigma) = k$ and an element $\mathrm{span}(w) \in \mathcal{B}_\sigma(W)$ (where $w \in W \setminus \{0\}$), let us show that $\mathrm{span}(w) \in \mathcal{B}_D(W)$. Take any $V \in S_D$. Since $\sigma \in \mathcal{V}(D)$, there exists $v \in V^\perp$ with $\mathrm{sign}(v) = \sigma$. Since $\mathrm{sign}(v) = \mathrm{sign}(w)$, there exist $c_1, \dots, c_n > 0$ such that $(c_1 v_1, \dots, c_n v_n) = w$. We use the positive torus action (see Remark 3.2.2) to define $V' := \{(\frac{x_1}{c_1}, \dots, \frac{x_n}{c_n}) : x \in V\} \in S_D$, so that $(c_1 v_1, \dots, c_n v_n) \in V'^\perp \cap W$. Since $\dim(V'^\perp \cap W) = 1$, we get $\mathrm{span}(w) = V'^\perp \cap W \in \mathcal{B}_D(W)$. ■

**Remark 3.8.3.** If $M$ is the positroid corresponding to a ⅃-diagram $D$, with dual positroid $M^*$, then by Lemma 3.3.3(ii) we have $\mathcal{V}(D) = \{\mathrm{alt}(\mathrm{sign}(u)) : u \in U\}$ for any $U \in S_{M^*}$. Therefore by Lemma 3.3.3(i), determining which sign vectors in $\mathcal{V}(D)$ minimize $\overline{\mathrm{var}}(\cdot)$ is equivalent to determining which sign vectors in $\mathrm{sign}(U)$ (for any $U \in S_{M^*}$) maximize $\mathrm{var}(\cdot)$.

Recall that $\overline{\mathcal{D}_{n,k,1}}$ is the set of ⅃-diagrams with at most one + per row, and each + appears at the right end of its row. We showed in Proposition 3.5.16 that for $D \in \overline{\mathcal{D}_{n,k,1}}$, $\mathcal{V}(D)$ contains a unique sign vector $\sigma$ (up to multiplication by $\pm 1$) with $\overline{\mathrm{var}}(\sigma) = k$, which we denoted by $\sigma(D)$ (Definition 3.5.7). In this case, we have $\mathcal{B}_D(W) = \mathcal{B}_{\sigma(D)}(W)$, as verified by Lemma 3.8.2. Also note that by Theorem 3.5.17, the dimension of $\mathcal{B}_D(W)$ is the number of +'s in $D$ (see Figure 3.9). We now give a formula for $\dim(\mathcal{B}_D(W))$ for any ⅃-diagram $D$.

**Proposition 3.8.4** (Dimension of the image of an arbitrary cell)**.**
*Let $D$ be a ⅃-diagram of type $(k, n)$. Then the dimension of $\mathcal{B}_D(W)$ is the number of rows of $D$ which contain a +.*

This implies that a cell of $\mathrm{Gr}_{k,n}^{\ge 0}$ has its dimension preserved when mapped by $\tilde{Z}$ to the $m = 1$ amplituhedron if and only if its ⅃-diagram has at most one $+$ in each row. Lam (Theorem 4.2 of [Lam16a]) gave an alternative criterion for general $m$ in terms of the affine Stanley symmetric function associated to the decorated permutation of the cell. This is related to the notion of *kinematical support* (see Chapter 10 of [AHBC$^+$16] and Definition 4.3 of [Lam16a]).

**Proof.** Label the steps of the southeast border of $D$ by $1, \dots, n$ from northeast to southwest, and denote by $I \subseteq [n]$ the set of $i \in [n]$ such that $i$ labels a vertical step whose row contains no $+$'s. We will show that the codimension of $\mathcal{B}_D(W)$ equals $|I|$.

It follows from Definition 3.2.12 that $I$ is the set of coloops of $M(D)$, i.e. $I = \{i \in [n] : e^{(i)} \in V\}$ for any $V \in S_D$, where $e^{(i)}$ denotes the $i$th unit vector. Hence $I = \{i \in [n] : \sigma_i = 0 \text{ for all } \sigma \in \mathcal{V}(D)\}$. Now let $\mathcal{H}^W$ be a hyperplane arrangement from Definition 3.6.10, so that $B(\mathcal{H}^W)$ is homeomorphic to $\mathcal{B}_{n,k,1}(W)$ by Theorem 3.6.16. By Lemma 3.8.2 we have

$$\mathcal{B}_D(W) \subseteq \bigcup \{\mathcal{B}_\sigma(W) : \sigma \in \overline{\mathbb{P}\mathrm{Sign}_{n,k,1}} \text{ with } \sigma_i = 0 \text{ for all } i \in I\},$$

so the image of $\mathcal{B}_D(W)$ in $B(\mathcal{H}^W)$ under the homeomorphism $\mathcal{B}_{n,k,1}(W) \to B(\mathcal{H}^W)$ is contained in $\bigcap_{i \in I} H_i$ by Theorem 3.6.16. Since $\mathcal{H}^W$ is generic (Definition 3.6.4), the codimension of $\bigcap_{i \in I} H_i$ equals $|I|$, so the codimension of $\mathcal{B}_D(W)$ is at least $|I|$.

Conversely, note that any $X \in \mathrm{Gr}_{l,n}$ contains a vector which changes sign at least $l - 1$ times: put an $l \times n$ matrix whose rows span $X$ into reduced row echelon form, and take the alternating sum of the rows. Now fix any $V \in S_D$. The element $\mathrm{alt}(V^\perp) \in \mathrm{Gr}_{n-k,n}$ contains a vector $\mathrm{alt}(v)$ (for some $v \in V^\perp$) which changes sign at least $n-k-1$ times. Since for all $i \in [n] \setminus I$ there exists $w \in V^\perp$ with $w_i \neq 0$, we may perturb $v \in V^\perp$ to make the components $[n] \setminus I$ all nonzero, without changing the sign of any nonzero components of $v$. We obtain a vector $v' \in V^\perp$ satisfying $\mathrm{var}(\mathrm{alt}(v')) \ge n - k - 1$ and $v'_i \neq 0$ for all $i \in [n] \setminus I$. By Lemma 3.3.3(i), we have $\overline{\mathrm{var}}(v') \le k$, and since $\overline{\mathrm{var}}(v') \ge k$ by Theorem 3.3.4(i), we get $\overline{\mathrm{var}}(v') = k$. Letting $\sigma := \mathrm{sign}(v')$, we have $\mathcal{B}_\sigma(W) \subseteq \mathcal{B}_D(W)$ by Lemma 3.8.2. The codimension of $\mathcal{B}_\sigma(W)$ equals the number of zero components of $\sigma$, which is at most $|I|$. Hence the codimension of $\mathcal{B}_D(W)$ is at most $|I|$. ■

Next we determine which cells $S_D$ are mapped injectively to the $m = 1$ amplituhedron, and explicitly describe the images of such cells.

**Definition 3.8.5.** Let $\overline{\mathcal{L}_{n,k,1}}$ denote the set of ⅃-diagrams of type $(k, n)$ which have at most one $+$ in each row, and which satisfy the L-*condition*: there is no 0 which has a $+$ above it in the same column and a $+$ to the right in the same row. (However, we do not require each $+$ to appear at the right end of its row.) We let $\mathcal{L}_{n,k,1}$ denote the subset of $\overline{\mathcal{L}_{n,k,1}}$ of ⅃-diagrams with exactly $k$ $+$'s.

Note that $\mathcal{D}_{n,k,1} \subseteq \mathcal{L}_{n,k,1}$ and $\overline{\mathcal{D}_{n,k,1}} \subseteq \overline{\mathcal{L}_{n,k,1}}$. For example, we have

$$\mathcal{D}_{2,2,1} = \left\{ \begin{array}{|c|c|}\hline 0 & + \\ \hline 0 & + \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline 0 & + \\ \hline + \\ \cline{1-1} \end{array}\,, \begin{array}{|c|}\hline + \\ \hline + \\ \hline \end{array} \right\} \quad \text{and} \quad \mathcal{L}_{2,2,1} = \mathcal{D}_{2,2,1} \sqcup \left\{ \begin{array}{|c|c|}\hline + & 0 \\ \hline + & 0 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline + & 0 \\ \hline + \\ \cline{1-1} \end{array} \right\}.$$

**Remark 3.8.6.** It is not hard to see that $\mathcal{L}_{n,k,1}$ (respectively, $\overline{\mathcal{L}_{n,k,1}}$) consists of the ⅃-diagrams we obtain from diagrams in $\mathcal{D}_{n',k,1}$ (respectively, $\overline{\mathcal{D}_{n',k,1}}$) by inserting $n-n'$ columns of all 0's, as $n'$ ranges over all $n' \leq n$.

**Definition 3.8.7.** Given a ⅃-diagram $D \in \overline{\mathcal{L}_{n,k,1}}$, we define a set of ⅃-diagrams Slide$(D)$ as follows. If $D \in \mathcal{L}_{n,k,1}$ (i.e. $D$ has no zero rows), then we let Slide$(D)$ be the set of ⅃-diagrams which can be obtained from $D$ by doing the following for each $+$ of $D$.

(1) Slide the $+$ weakly to the right somewhere in the same row, say from box $b$ to box $b'$, such that the southeast corner of box $b'$ lies on the southeast border of $D$.

(2) Remove all boxes to the right of box $b'$ in the same row.

(3) If $b' \neq b$ and the entire lower edge of $b'$ lies on the southeast border of $D$, we can choose to remove box $b'$ (or not).

More generally, if $D \in \overline{\mathcal{L}_{n,k,1}}$, we label the steps of the southeast border of $D$ by $1, \dots, n$, and let $I \subseteq [n]$ be the set of $i$ which label a vertical step whose row contains no $+$'s. Let $D'$ be the ⅃-diagram obtained by deleting the rows corresponding to $I$ from $D$, so $D'$ has no zero rows. Then we define Slide$(D)$ as a set of ⅃-diagrams in bijection with Slide$(D')$, where given a ⅃-diagram in Slide$(D')$ we obtain the corresponding ⅃-diagram in Slide$(D)$ by adding a row of all 0's for each $i \in I$, such that its vertical step on the southeast border gets labeled by $i$ when we label the southeast border by $1, \dots, n$. Note that Slide$(D) \subseteq \overline{\mathcal{D}_{n,k,1}}$.

**Example 3.8.8.** Let

$$D := \begin{array}{|c|c|c|c|} \hline 0 & + & 0 & 0 \\ \hline 0 & 0 & 0 & \\ \cline{1-3} 0 & + & 0 & \\ \cline{1-3} 0 & + & 0 & \\ \cline{1-3} + & & & \\ \cline{1-1} \end{array} \in \overline{\mathcal{L}_{5,9,1}}.$$

Then Slide$(D)$ equals the set of ⅃-diagrams appearing below.

$$\begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & + & \\ \cline{1-2} + & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & + \\ \hline + & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & \\ \cline{1-2} + & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & \\ \cline{1-3} 0 & 0 & + & \\ \cline{1-3} 0 & + & & \\ \cline{1-2} + & & & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & + & \\ \cline{1-2} + & & \\ \cline{1-1} \end{array}$$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | + |
| 0 | 0 | 0 | |
| 0 | 0 | + | |
| 0 | 0 | + | |
| + | | | |

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | + |
| 0 | 0 | 0 | |
| 0 | 0 | + | |
| 0 | 0 | | |
| + | | | |

| | | |
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| 0 | 0 | 0 |
| 0 | 0 | 0 |
| 0 | 0 | + |
| 0 | 0 | + |
| + | | |

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 0 | 0 |
| 0 | 0 | + |
| 0 | 0 | |
| + | | |

$\lozenge$

**Example 3.8.9.** If $D \in \overline{\mathcal{D}_{n,k,1}}$, then each $+$ of $D$ is in the rightmost box of its row, so we cannot slide it further to the right. Hence $\mathrm{Slide}(D) = \{D\}$. $\lozenge$

**Theorem 3.8.10.** *Let $D$ be a ⅃-diagram of type $(k, n)$. Then the map $S_D \to \mathcal{B}_{n,k,1}(W), V \mapsto V^\perp \cap W$ is injective on the cell $S_D$ if and only if $D \in \overline{\mathcal{L}_{n,k,1}}$. In this case, we have*

$$\mathcal{B}_D(W) = \bigsqcup_{D' \in \mathrm{Slide}(D)} \mathcal{B}_{D'}(W). \tag{3.8.11}$$

We will prove Theorem 3.8.10 over the remainder of the section, divided into several steps. First, we consider two examples.

**Example 3.8.12.** Let $D$ be the ⅃-diagram from Example 3.8.8. Then Theorem 3.8.10 asserts that $S_D$ maps injectively to the $m = 1$ amplituhedron, and that its image is the disjoint union of the images of the 9 cells corresponding to the ⅃-diagrams in $\mathrm{Slide}(D)$. $\lozenge$

**Example 3.8.13.** Theorem 3.8.10 implies that if two cells of $\mathrm{Gr}_{k,n}^{\ge 0}$ map injectively to the $m = 1$ amplituhedron, then their images are distinct. This can fail to hold if we do not assume that the cells map injectively. For example, if $n = 3$ and $k = 1$, then the cells $\{(1 : 0 : a) : a > 0\}$ and $\{(1 : b : c) : b, c > 0\}$ of $\mathrm{Gr}_{1,3}^{\ge 0}$ have the same image by Lemma 3.8.2, namely $\mathcal{B}_{(+,-,-)}(W) \sqcup \mathcal{B}_{(+,0,-)}(W) \sqcup \mathcal{B}_{(+,+,-)}(W)$. $\lozenge$

Now we begin proving Theorem 3.8.10. Let us characterize the ⅃-diagrams in $\overline{\mathcal{L}_{n,k,1}}$ by a matroid-theoretic condition.

**Definition 3.8.14.** Let $D$ be a ⅃-diagram of type $(k, n)$. The set of sign vectors $\mathcal{V}(D)$ from Definition 3.8.1 is an induced subposet of $\{0, +, -\}^n$ from Definition 3.5.3. We call the minimal elements of $\mathcal{V}(D) \setminus \{0\}$ the *circuits*[8] of $D$, and denote the set of circuits by $\mathcal{C}(D)$. A more concrete way to think about circuits is the following. Fix $V \in S_D$ and an $(n-k) \times n$ matrix $A$ whose rows span $V^\perp$. For a subset $I \in \binom{[n]}{n-k}$ such that columns $I$ of $A$ form a basis for $\mathbb{R}^{n-k}$, let $A(I)$ be the $(n-k) \times n$ matrix whose rows span $V^\perp$ which we obtain by

[8]In the language of oriented matroids, $\mathcal{C}(D)$ is the set of (signed) circuits of the oriented matroid represented by any $V \in S_D$. The supports of circuits are precisely the (unsigned) circuits of the (unoriented) matroid $M(V)$. What we call a circuit will always be a sign vector.

row reducing $A$ so as to get an identity matrix in columns $I$. Then the sign vectors of rows appearing in some matrix $A(I)$ are precisely the circuits of $D$ (up to sign).

We call $i \in [n]$ a *loop* of $D$ if the unit vector $e^{(i)}$ is contained in $\mathcal{V}(D)$, and a *coloop* if $\sigma_i = 0$ for all $\sigma \in \mathcal{V}(D)$. These are precisely the loops and coloops (from Definition 3.2.9) of the positroid $M(D)$ from Definition 3.2.12. Alternatively, loops label the columns of $D$ which contain only 0's, and coloops label the rows which contain only 0's.

**Lemma 3.8.15.** *Let $D$ be a ⅃-diagram of type $(k, n)$. Then $D \in \overline{\mathcal{L}_{n,k,1}}$ if and only if there do not exist $\sigma \in \mathcal{C}(D)$ and $a < b < c$ in $[n]$ such that*

*(i) $\sigma_a, \sigma_c \neq 0$;*

*(ii) $\sigma_b = 0$; and*

*(iii) $b$ is neither a loop nor a coloop of $D$.*

**Proof.** Let $D'$ be the ⅃-diagram contained in a $k' \times (n' - k')$ rectangle obtained from $D$ by deleting all rows and columns which contain only 0's. Equivalently, the positroid $M(D')$ is obtained from $M(D)$ by restricting the ground set $[n]$ (as in Definition 3.2.17) to elements which are neither loops nor coloops. Then we obtain $\mathcal{C}(D')$ from $\mathcal{C}(D)$ by deleting the circuits $\pm e^{(i)}$ corresponding to loops $i$, restricting all circuits from $[n]$ to $\{i \in [n] : i \text{ is neither a loop nor a coloop of } D\}$, and multiplying certain components by $-1$. Note that $D'$ has no loops or coloops, and $D \in \overline{\mathcal{L}_{n,k,1}}$ if and only if $D' \in \mathcal{D}_{n',k',1}$. Hence it suffices to prove that $D' \in \mathcal{D}_{n',k',1}$ if and only if there do not exist $\sigma \in \mathcal{C}(D')$ and $a < b < c$ in $[n]$ such that $\sigma_a, \sigma_c \neq 0$ and $\sigma_b = 0$.

($\Rightarrow$): Suppose that $D' \in \mathcal{D}_{n',k',1'}$. Let us fix $V' \in S_{D'}$, so that the representable matroid $M(V'^{\perp})$ (defined in Example 3.2.10) equals $M(D')^*$. By Lemma 3.5.10, $M(D')^*$ is the direct sum of uniform matroids of rank 1 whose ground sets are all intervals. That is, we can write $V'^{\perp}$ as the row span of an $(n' - k') \times n'$ matrix of the form

$$\begin{bmatrix} * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * & \cdots \\ & \vdots & & & \vdots & & & \vdots & & \ddots \end{bmatrix}, \tag{3.8.16}$$

where all the $*$'s are nonzero. The sign vectors of these rows, and their negations, are precisely the circuits of $D'$. We see that the circuits of $D'$ satisfy the required condition.

($\Leftarrow$): Suppose that there do not exist $\sigma \in \mathcal{C}(D')$ and $a < b < c$ in $[n]$ such that $\sigma_a, \sigma_c \neq 0$ and $\sigma_b = 0$. Let us again take some $V' \in S_{D'}$, and put an $(n' - k') \times n'$ matrix whose rows span $V'^{\perp}$ into reduced row echelon form. Then the sign vector of each row of this matrix is a circuit of $D'$, and so the nonzero entries in each row form a consecutive block. It follows that this matrix is of the form (3.8.16), where all the $*$'s are nonzero. (The matrix has no zero columns since $D'$ has no coloops.) Hence the matroid $M(D')^* = M(V'^{\perp})$ is the direct sum of uniform matroids of rank 1 whose ground sets are all intervals, whence $D' \in \mathcal{D}_{n',k',1}$ by Lemma 3.5.10. ■

**Lemma 3.8.17.** *Let $D$ be a* Ⅎ*-diagram of type $(k, n)$. Suppose that there exist a circuit $\sigma \in \mathcal{C}(D)$ and $a < b < c$ in $[n]$ such that*

*(i) $\sigma_a, \sigma_c \neq 0$;*

*(ii) $\sigma_b = 0$; and*

*(iii) $b$ is not a coloop of $D$.*

*Then there exists $\tau \in \mathcal{V}(D)$ with $\overline{\mathrm{var}}(\tau) = k$ and $\tau_b = 0$.*

We will use such a sign vector $\tau$ in Corollary 3.8.18 to provide a certificate of non-injectivity, if $D \notin \overline{\mathcal{L}_{n,k,1}}$. In proving Corollary 3.8.18, we will want to require that $b$ (satisfying $\tau_b = 0$) is not a loop of $D$, as in Lemma 3.8.15. But in order to prove just Lemma 3.8.17, we do not need to assume that $b$ is not a loop.

**Proof.** Fix $V \in S_D$ and an $(n-k) \times n$ matrix $A$ whose rows span $U := \mathrm{alt}(V^{\perp})$. By Lemma 3.3.3(i), it suffices to show that there exists $w \in U$ with $\mathrm{var}(w) = n - k - 1$ and $w_b = 0$. We will use the fact that $U \in \mathrm{Gr}_{n-k,n}^{\geq 0}$, by Lemma 3.3.3(ii). The idea is to take $u \in U$ with sign vector $\sigma' := \mathrm{alt}(\sigma)$, and perturb it so that it has $n - k - 1$ sign changes. The presence of both $a$ and $c$ will allow us to get a 'free' sign change without 'using' $b$.

Let $J := \{j \in [n] : \sigma_j = 0\}$. The fact that $\sigma$ is a circuit implies that columns $J$ of $A$ do not span $\mathbb{R}^{n-k}$, but columns $J \cup \{a\}$ do span $\mathbb{R}^{n-k}$. Also, since $b$ is not a coloop of $D$, column $b$ of $A$ is nonzero. Hence we can extend $\{a, b\}$ to $I \in \binom{J \cup \{a\}}{n-k}$ such that columns $I$ of $A$ form a basis of $\mathbb{R}^{n-k}$. Let $A(I)$ denote the $(n-k) \times n$ matrix whose rows span $U$ which we obtain by row reducing $A$ so as to get an identity matrix in columns $I$. We denote by $v^{(i)}$ the row of $A(I)$ whose pivot column is $i$, i.e. $v_i^{(i)} = 1$ and $v_j^{(i)} = 0$ for all $j \in I \setminus \{i\}$.

*Claim.* $\sigma'_c = (-1)^{|I \cap (a,c)|}\sigma'_a$, where $(a, c)$ denotes the open interval between $a$ and $c$.

*Proof of Claim.* Write $I = \{i_1 < \cdots < i_{n-k}\}$, so that $a = i_r$ for some $r \in [n-k]$. Since $u_{i_s} = 0$ for all $s \in [n-k] \setminus \{r\}$, we can use $v^{(i_s)}$ to perturb $u$, without changing the sign of $u_c$, so that component $i_s$ gets the sign

$$\begin{cases} (-1)^{s-r}\sigma'_a, & \text{if } i_s < c \\ (-1)^{s-r+1}\sigma'_a, & \text{if } i_s > c \end{cases}.$$

Since the resulting vector $u'$ lies in $U \in \mathrm{Gr}_{n-k,n}^{\geq 0}$, we get $\mathrm{var}(u') \leq n - k - 1$ by Theorem 3.3.4(i). Hence $u'$ does not alternate in sign on $I \cup \{c\}$, and so $\sigma'_c \neq -(-1)^{|I \cap (a,c)|}\sigma'_a$. □

We obtain our desired vector $w \in U$ as follows. Write $I \setminus \{b\} = \{i'_1 < \cdots < i'_{n-k-1}\}$, so that $a = i'_r$ for some $r \in [n-k]$. Then for $s \in [n-k-1] \setminus \{r\}$, we use $v^{(i'_s)}$ to perturb $u$, without changing the sign of $u_c$, so that component $i'_s$ gets sign

$$\begin{cases} (-1)^{s-r}\sigma'_a, & \text{if } i'_s < c \\ (-1)^{s-r+1}\sigma'_a, & \text{if } i'_s > c \end{cases}.$$

The resulting vector $w$ alternates in sign on $(I \setminus \{b\}) \cup \{c\}$ by the claim, so $\mathrm{var}(w) \geq n-k-1$. We have $\mathrm{var}(w) \leq n-k-1$ by Theorem 3.3.4(i), and $w_b = 0$ by construction. ■

**Corollary 3.8.18.** *Let $D$ be a Ⅎ-diagram of type $(k, n)$ which is not in $\overline{\mathcal{L}_{n,k,1}}$. Then the map $S_D \to \mathcal{B}_{n,k,1}(W), V \mapsto V^\perp \cap W$ is not injective.*

**Proof.** Fix $V \in S_D$. By Lemma 3.8.15 and Lemma 3.8.17, there exist $w \in V^\perp$ and $b \in [n]$ such that $\overline{\mathrm{var}}(w) = k$, $w_b = 0$, and $b$ is neither a loop nor a coloop of $D$. The strategy of the proof is to use the positive torus action (see Remark 3.2.2) so as to get from $w$ an element of $\mathcal{B}_{n,k,1}(W)$, and then use the positive torus action again in component $b$ to get many elements of $S_D$ which map to our chosen element of $\mathcal{B}_{n,k,1}(W)$.

Let $\sigma := \mathrm{sign}(w)$, so that by Lemma 3.3.7(i) there exists $w' \in W$ with $\mathrm{sign}(w') = \sigma$. Letting $w = (w_1, \ldots, w_n)$, we can therefore write $w' = (c_1 w_1, \ldots, c_n w_n)$ for some $c_1, \ldots, c_n > 0$. Letting $V' := \{(\frac{v_1}{c_1}, \ldots, \frac{v_n}{c_n}) : v \in V\} \in S_D$, we have $w' \in V'^\perp$. Then for $t > 0$, we define

$$V'_t := \{(v_1, \ldots, v_{b-1}, tv_b, v_{b+1}, \ldots, v_n) : v \in V'\} \in S_D.$$

*Claim.* The $V'_t$ $(t > 0)$ are all distinct.

*Proof of Claim.* Let $M(D)$ denote the matroid from Definition 3.2.12. Since $b$ is not a loop or a coloop of $D$, there exist $I, J \in M(D)$ with $b \in I$ and $b \notin J$. Then $\frac{\Delta_I(V'_t)}{\Delta_J(V'_t)} \cdot \frac{\Delta_J(V')}{\Delta_I(V')} = t$ for $t > 0$. □

Since $w' \in V_t'^\perp \cap W$ and $\dim(V_t'^\perp \cap W) = 1$, the elements $V'_t$ $(t > 0)$ all map to the line spanned by $w'$ in $\mathcal{B}_{n,k,1}(W)$. ■

We now show that if $D \in \overline{\mathcal{L}_{n,k,1}}$, then $S_D$ maps injectively to the $m = 1$ amplituhedron, and we also determine its image. We need to understand the sign vectors $\mathcal{V}(D)$ for such $D$. This can be done, thanks to the fact that when we remove enough loops (columns containing only 0's) from $D$, we obtain an element of $\overline{\mathcal{D}_{n,k,1}}$ (see Remark 3.8.6). We first consider the case that $D$ is in $\mathcal{L}_{n,k,1}$ (i.e. $D$ has no coloops) and its $+$'s lie in a single column.

**Lemma 3.8.19.** *Suppose that $D \in \mathcal{L}_{n,k,1}$ with $k \geq 1$ such that the $+$'s of $D$ lie in a single column. Then the map $S_D \to \mathcal{B}_{n,k,1}(W), V \mapsto V^\perp \cap W$ is injective on $S_D$, and $\mathcal{B}_D(W) = \bigsqcup_{D' \in \mathrm{Slide}(D)} \mathcal{B}_{D'}(W)$.*

**Proof.** Let $L \in \binom{[n]}{n-k-1}$ denote the set of loops of $D$ (from Definition 3.8.14). That is, if we label the steps of the southeast border of $D$ from 1 to $n$, then $L$ is the set of labels of the horizontal steps, excluding the label corresponding to the column of $+$'s. Then $M(D)$ is a matroid of rank $k$, which is uniform when restricted to $[n] \setminus L$ and has set of loops $L$. It follows from Lemma 3.3.7(ii) that $\sigma \in \mathcal{V}(D)$ if and only if

$$\sigma|_{[n]\setminus L} \in \{0, (+,-,+,-,\cdots), (-,+,-,+,\cdots)\}.$$

(The sign of $\sigma_i$ for $i \in L$ is arbitrary.)

To compute $\mathcal{B}_D(W)$, we use Lemma 3.8.2. We need to identify those $\sigma \in \mathcal{V}(D)$ satisfying $\overline{\mathrm{var}}(\sigma) = k$. Let us first consider the example $D = \boxed{+}\boxed{0}\boxed{0}$, so $n = 4$, $k = 1$, $L = \{2, 3\}$. We have

$$\mathcal{V}(D) = \{(+,*,*,-)\} \cup \{(-,*,*,+)\} \cup \{(0,*,*,0)\},$$

where each $*$ can be 0, $+$, or $-$. Since $\overline{\mathrm{var}}(0,*,*,0) \geq 2$, the sign vectors $\sigma \in \mathcal{V}(D)$ with $\overline{\mathrm{var}}(\sigma) = 1$ (modulo multiplication by $\pm 1$) are

$$(+,+,+,-), (+,+,-,-), (+,-,-,-), (+,+,0,-), (+,0,-,-),$$

that is, a sequence of $+$'s followed by $-$'s, possibly with one 0 in between. The corresponding Ⅎ-diagrams (from Definition 3.5.7) are

$$\boxed{+}\,, \quad \boxed{0}\boxed{+}\,, \quad \boxed{0}\boxed{0}\boxed{+}\,, \quad \boxed{0}\,, \quad \boxed{0}\boxed{0}\,,$$

respectively.

Now we describe what happens in general. Write $[n] \setminus L = \{r_1 < \cdots < r_{k+1}\}$. Given $\sigma \in \{0, +, -\}^n$, we have $\sigma \in \mathcal{V}(D)$ and $\overline{\mathrm{var}}(\sigma) = k$ if and only if

- $\sigma$ alternates in sign on $[n] \setminus L$;
- $\sigma_i = \sigma_{r_1}$ for all $i < r_1$;
- $\sigma_i = \sigma_{r_{k+1}}$ for all $i > r_{k+1}$; and
- for all $j \in [k]$, there exists an integer $s_j \in [r_j, r_{j+1} - 1]$ such that $\sigma_i = \sigma_{r_j}$ for all $i \in [r_j, s_j]$, $\sigma_i = \sigma_{r_{j+1}}$ for all $i \in [s_j + 2, r_{j+1}]$, and if $s_j + 1 \neq r_{j+1}$ then $\sigma_{s_j+1}$ equals either 0 or $\sigma_{r_{j+1}}$.

We obtain the Ⅎ-diagram $\Omega_{DS}^{-1}(\sigma)$ (see Definition 3.5.7) by a slide as in Definition 3.8.7, as follows. In step (1), we slide the $+$ in the $j$th row of $D$ so that it is $s_j - r_j$ boxes to the left of the rightmost box in the row. In step (2), we remove the boxes to the right of this $+$. In step (3), we remove the box $b'$ containing this $+$ if and only if $\sigma_{s_j+1} = 0$. (Note that the case when $b' \neq b$ and the bottom edge of $b'$ lies on the southeast border of $D$ corresponds to $s_j + 1 \neq r_{j+1}$.) Thus $\Omega_{DS}^{-1}$ restricted to $\{\sigma \in \mathcal{V}(D) : \overline{\mathrm{var}}(\sigma) = k\}$ is a bijection $\{\sigma \in \mathcal{V}(D) : \overline{\mathrm{var}}(\sigma) = k\} \to \mathrm{Slide}(D)$, with inverse $\Omega_{DS}$. Then Lemma 3.8.2 implies (3.8.11).

To see that the map $S_D \to \mathcal{B}_{n,k,1}(W)$ is injective, suppose that $w \in W$ with $\mathrm{sign}(w) \in \mathcal{V}(D)$ and $\overline{\mathrm{var}}(w) = k$. We have just observed that $w$ is nonzero when restricted to $[n] \setminus L$, so the vectors $w$ and $e^{(i)}$ for $i \in L$ are linearly independent. Their span is an element of $\mathrm{Gr}_{n-k,n}$, whose orthogonal complement is the unique $V \in S_D$ with $V^\perp \cap W = \mathrm{span}(w)$. ■

**Proof** (of Theorem 3.8.10)**.** If $D \notin \overline{\mathcal{L}_{n,k,1}}$, then the map $S_D \to \mathcal{B}_{n,k,1}(W)$ is not injective by Corollary 3.8.18. Now suppose that $D \in \overline{\mathcal{L}_{n,k,1}}$. We must show that the map $S_D \to \mathcal{B}_{n,k,1}(W)$ is injective, and that its image equals $\bigsqcup_{D' \in \mathrm{Slide}(D)} \mathcal{B}_{D'}(W)$. We first show how to reduce to the case that $D \in \mathcal{L}_{n,k,1}$, i.e. $D$ has no coloops (see Definition 3.8.14). Let us suppose that $i \in [n]$ is a coloop of $D$, and explain what happens when we delete $i$. In terms of ⅃-diagrams, $i$ labels a row of $D$ which contains only 0's, and deleting this row gives a ⅃-diagram $D'$ indexing a cell of $\mathrm{Gr}_{k-1,n-1}^{\ge 0}$. In terms of cells, we have the map

$$\begin{gathered} \{V^\perp : V \in S_D\} \to \{V'^\perp : V' \in S_{D'}\}, \\ \{(v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_n)\} \mapsto \{(v_1, \dots, v_{i-1}, -v_{i+1}, \dots, -v_n)\}. \end{gathered}$$

In terms of the subspace $W \in \mathrm{Gr}_{k+1,n}^{>0}$, we map it to

$$W' := \{(w_1, \dots, w_{i-1}, -w_{i+1}, \dots, -w_n) : w \in W \text{ with } w_i = 0\},$$

where $W' \in \mathrm{Gr}_{k,n-1}^{>0}$ by Theorem 3.3.4(ii). Note that the map $V \mapsto V^\perp \cap W$ is injective on $S_D$ if and only if the map $V' \mapsto V'^\perp \cap W'$ is injective on $S_{D'}$. We also have a bijection $\mathrm{Slide}(D) \to \mathrm{Slide}(D')$ given by deleting the row labeled by $i$. By Lemma 3.8.2, this shows that the result for $D'$ implies the result for $D$. Hence we may assume that $D$ has no coloops, i.e. $D \in \mathcal{L}_{n,k,1}$.

In this case, we can decompose $D$ as follows: there exist ⅃-diagrams $D_1, \dots, D_l$, where each $D_j$ has no coloops and all its $+$'s lie in a single column, such that we get $D$ by gluing together $D_1, \dots, D_l$ from northeast to southwest (so that the southwest corner of $D_j$ and the northeast corner of $D_{j+1}$ coincide, for all $j \in [l-1]$), and filling the space above and to the left of the resulting diagram with 0's. For example, if $D = \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & + \\ \hline + & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$, then $D_1 = \begin{array}{|c|c|} \hline 0 & + \\ \hline 0 & + \\ \hline \end{array}$ and $D_2 = \boxed{+}$. We have $M(D) = M(D_1) \oplus \cdots \oplus M(D_l)$, so any $V \in S_D$ can be written uniquely as $V_1 \oplus \cdots \oplus V_l$, where $V_j \in S_{D_j}$. Now for $j \in [l]$, let $E_j$ be the ground set of $M(D_j)$, and let $U_j$ denote the orthogonal complement of $V_j$ in $\mathbb{R}^{E_j}$, so that $V^\perp = U_1 \oplus \cdots \oplus U_l$. It follows that the vectors $w \in V^\perp$ satisfying $\overline{\mathrm{var}}(w) = k$ can be written precisely as $w = u_1 + \cdots + u_l$, where $u_j \in U_j$ with $\overline{\mathrm{var}}(u_j) = \dim(V_j)$ for all $j \in [l]$, and the signs of the $u_j$'s are such that we never get an extra sign change from $u_j$ to $u_{j+1}$ for $j \in [l-1]$. Thus it suffices to verify injectivity and (3.8.11) locally for each $D_j$, which we can do thanks to Lemma 3.8.19. ■

## 3.9 Relaxing to Grassmann polytopes

In this section we discuss what happens when we relax the condition that $W \in \mathrm{Gr}_{k+m,n}$ is totally positive, in the sense of Lam's *Grassmann polytopes* [Lam16b].

**Definition 3.9.1** (Definition 15.1 of [Lam16b])**.** Let $Z$ be a real $r \times n$ matrix with row span $W \in \mathrm{Gr}_{k+m,n}$, where $k + m \leq r$ (we allow $Z$ to not have full row rank). Suppose that $W$ contains a totally positive $k$-dimensional subspace. Then by Proposition 15.2 of [Lam16b], the map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ is well defined, i.e. $\dim(Z(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ (see Remark 3.3.19). We call the image $\tilde{Z}(\overline{S_M}) \subseteq \mathrm{Gr}_{k,r}$ of the closure of a cell $S_M$ of $\mathrm{Gr}_{k,n}^{\geq 0}$ a *Grassmann polytope*. When $\overline{S_M} = \mathrm{Gr}_{k,n}^{\geq 0}$, i.e. $M$ is the uniform matroid of rank $k$, we call the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ a *full Grassmann polytope*.

Analogously, given a cell $S_M$ of $\mathrm{Gr}_{k,n}^{\geq 0}$, let us define a *Grassmann arrangement* as

$$\{V^{\perp} \cap W : V \in \overline{S_M}\} \subseteq \mathrm{Gr}_m(W).$$

We call this a *full Grassmann arrangement* in the case that $\overline{S_M} = \mathrm{Gr}_{k,n}^{\geq 0}$. A Grassmann arrangement is well defined, and homeomorphic to the corresponding Grassmann polytope, by the same arguments which appear in Section 3.3.

The amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is a full Grassmann polytope (see Lemma 3.6.9). In the case $k = 1$, Grassmann polytopes are precisely polytopes in the projective space $\mathrm{Gr}_{1,r} = \mathbb{P}^{r-1}$. Therefore, Grassmann polytopes are a generalization of polytopes into Grassmannians.

Lemma 3.3.15 generalizes to any full Grassmann arrangement; the proof is the same. In particular, in the case that $m = 1$, the analogue of Corollary 3.3.18 holds.

**Proposition 3.9.2.** *For $m = 1$, the full Grassmann arrangement can be described as follows:*

$$\{V^{\perp} \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) \geq k\} \subseteq \mathbb{P}(W).$$

As in Section 3.6, if $m = 1$ we can define a hyperplane arrangement $\mathcal{H}^W$ depending on $W$, and show that the full Grassmann arrangement above is homeomorphic to $B(\mathcal{H}^W)$. In slightly more detail, we take a $k$-dimensional totally positive subspace $W'$ of $W$ as provided by Definition 3.9.1, and let $w^{(1)}, \dots, w^{(k)}$ be its basis. We extend this to a basis $w^{(0)}, w^{(1)}, \dots, w^{(k)}$ of $W$, and define $\mathcal{H}^W$ as in Definition 3.6.10 (we ignore the requirement that $w^{(0)}$ is positively oriented). Then Lemma 3.6.12 holds for $\mathcal{H}^W$, with the same proof. This implies (as in the first paragraph of the proof of Lemma 3.6.13) that the face labels of $B(\mathcal{H}^W)$ are precisely the sign vectors $\sigma \in \{0, +, -\}^n$ such that $\overline{\mathrm{var}}(\sigma) \geq k$ and $\psi_{\mathcal{H}^W}^{-1}(\sigma) \neq \emptyset$. We deduce that $\Psi_{\mathcal{H}^W}$ restricted to $B(\mathcal{H}^W)$ is a homeomorphism onto the full Grassmann arrangement of $W$, as claimed. If all Plücker coordinates of $W$ are nonzero, this implies that the full Grassmann arrangement of $W$ (for $m = 1$) is homeomorphic to a ball, as in Corollary 3.6.18. We remark that Lam has conjectured that every Grassmann polytope is contractible (p. 112 of [Lam16b]).

We observe that as $W$ varies, we do not recover all bounded complexes of hyperplane arrangements in this way, since the condition that $W \in \mathrm{Gr}_{k+1,n}$ contains a subspace in $\mathrm{Gr}_{k,n}^{>0}$ is very restrictive. Indeed, if $\sigma$ labels a face of $B(\mathcal{H}^W)$ for such $W$, then $\overline{\mathrm{var}}(\sigma) \geq k$.

**Remark 3.9.3.** As in Remark 3.3.19, the map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ is well defined if and only if $\mathrm{var}(v) \geq k$ for all nonzero $v \in \ker(Z)$ (see Theorem 2.4.2). Hence we could have defined Grassmann arrangements using this (possibly) more general class of matrices $Z$. However, in the construction of $\mathcal{H}^W$ above for $m = 1$, it was essential that $W$ (the row span of $Z$) should have a totally positive $k$-dimensional subspace. Indeed, we can see from the proof of Lemma 3.6.12 that classifying the labels of bounded and unbounded faces of $\mathcal{H}^W$ uses the fact that $\mathrm{span}(w^{(1)}, \dots, w^{(k)})$ is totally positive.

Recall from Remark 3.3.19 that it is unknown whether there exists a matrix $Z$ such that $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ is well defined, but the row span of $Z$ does not contain a totally positive $k$-dimensional subspace. The construction of $\mathcal{H}^W$ above gives further motivation for resolving this problem, which is equivalent to Problem 3.3.14(ii).

# Chapter 4

# Cyclic symmetry, moment curves, and quantum cohomology

I thank Gabriel Frieden, Michael Gekhtman, Rachel Karpman, Lauren Williams, Alexander Postnikov, Konstanze Rietsch, and Franco Vargas Pallete for enlightening conversations which influenced the work in this chapter.

## 4.1 Introduction

In this chapter we will work with the *complex Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{C})$, which is the set of $k$-dimensional subspaces of $\mathbb{C}^n$. We may view the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\ge 0}$ as the subset of $\mathrm{Gr}_{k,n}(\mathbb{C})$ where all Plücker coordinates are real and nonnegative.

One of the remarkable properties of the totally nonnegative Grassmannian is its *cyclic symmetry.* For fixed $k$ and $n$, we define the *(left) cyclic shift map* $\sigma \in \mathrm{GL}_n(\mathbb{C})$ by

$$\sigma(v) := (v_2, v_3, \dots, v_n, (-1)^{k-1}v_1) \quad \text{for } v = (v_1, \dots, v_n) \in \mathbb{C}^n.$$

Given $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, we denote by $\sigma(V)$ the subspace $\{\sigma(v) : v \in V\} \in \mathrm{Gr}_{k,n}(\mathbb{C})$. In terms of Plücker coordinates, $\sigma$ acts on $\mathrm{Gr}_{k,n}(\mathbb{C})$ by rotating the index set $\{1, \dots, n\}$. Hence $\sigma$ is an automorphism of $\mathrm{Gr}_{k,n}(\mathbb{C})$ of order $n$, which restricts to an automorphism of $\mathrm{Gr}_{k,n}^{\ge 0}$. The main result of this chapter is the following.

**Theorem 4.1.1.** *The cyclic shift map $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ has exactly $\binom{n}{k}$ fixed points, each of the form $\mathrm{span}\{(1, z_j, \dots, z_j^{n-1}) : 1 \le j \le k\}$, where $z_1, \dots, z_k \in \mathbb{C}$ are some $k$ distinct $n$th roots of $(-1)^{k-1}$. Precisely one of these fixed points is totally nonnegative, corresponding to the $k$ roots $z_1, \dots, z_k$ closest to $1$ on the unit circle.*

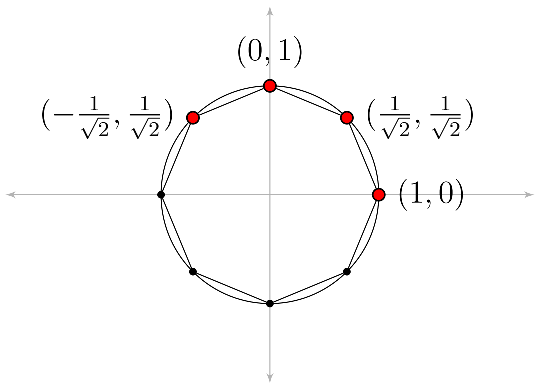
Let $V_{k,n} \in \mathrm{Gr}_{k,n}^{\ge 0}$ denote the unique totally nonnegative fixed point of $\sigma$. There is an elegant way to describe $V_{k,n}$ in terms of certain real curves. Define $f_k : \mathbb{R} \to \mathbb{R}^k$ by

$$f_k(\theta) := \begin{cases} \left(1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \dots, \cos\left(\frac{k-1}{2}\theta\right), \sin\left(\frac{k-1}{2}\theta\right)\right), & \text{if } k \text{ is odd} \\ \left(\cos\left(\frac{1}{2}\theta\right), \sin\left(\frac{1}{2}\theta\right), \cos\left(\frac{3}{2}\theta\right), \sin\left(\frac{3}{2}\theta\right), \dots, \cos\left(\frac{k-1}{2}\theta\right), \sin\left(\frac{k-1}{2}\theta\right)\right), & \text{if } k \text{ is even} \end{cases}.$$

Note that $f_k(\theta + 2\pi) = (-1)^{k-1} f_k(\theta)$. For odd $k$, the curve in $\mathbb{R}^{k-1}$ formed from $f_k$ by deleting the first component is the *trigonometric moment curve*, and for even $k$, the curve $f_k$ is the *symmetric moment curve*. These curves have a rich history, which we discuss in Remark 4.1.4. The fixed point $V_{k,n}$ is represented by any $k \times n$ matrix whose columns are $f_k(\theta_1), \ldots, f_k(\theta_n)$, such that the points $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi$ are equally spaced on the real line, i.e. $\theta_{j+1} - \theta_j = \frac{2\pi}{n}$ for $1 \le j \le n-1$. We also have the following explicit formula for its Plücker coordinates:

$$\Delta_I(V_{k,n}) = \prod_{1 \le r < s \le k} \sin\left(\tfrac{i_s - i_r}{n}\pi\right) \qquad \text{for all } k\text{-subsets } I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}. \tag{4.1.2}$$

**Example 4.1.3.** Let $k := 2$, $n := 4$. Then $f_k$ is the unit circle in the plane, and the fixed point $V_{2,4}$ is represented by the matrix $\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \end{bmatrix}$, whose columns correspond to four consecutive points on the regular unit octagon:

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Alternatively, we can represent $V_{2,4}$ by the matrix $\begin{bmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^{-1} & \zeta^{-2} & \zeta^{-3} \end{bmatrix}$, where $\zeta := e^{\frac{i\pi}{4}}$ and $\zeta^{-1}$ are the two fourth roots of $-1$ closest to 1 on the unit circle. $\diamondsuit$

We observe that by Theorem 4.1.1, we can recover the curve $f_k$ (up to an automorphism of $\mathbb{R}^k$) from the cyclic symmetry of $\mathrm{Gr}_{k,n}^{\ge 0}$, by taking a sort of limit of $V_{k,n}$ as $n \to \infty$. We prove Theorem 4.1.1, and the properties of $V_{k,n}$ stated above, in Section 4.2. We also discuss a connection to arrangements of equal minors in the totally nonnegative Grassmannian (see Section 4.2).

Remarkably, the $\binom{n}{k}$ fixed points of $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ also arise in quantum cohomology. The *quantum cohomology ring* of $\mathrm{Gr}_{k,n}(\mathbb{C})$ is a deformation of the cohomology ring by an indeterminate $q$. In unpublished work, Peterson discovered that this ring is isomorphic to the coordinate ring of a certain subvariety $\mathcal{Y}_{k,n}$ of $\mathrm{GL}_n(\mathbb{C})$. This was proved by Rietsch [Rie01]. Under her isomorphism, the indeterminate $q$ corresponds to a map $\mathcal{Y}_{k,n} \to \mathbb{C}$, and the specialization at $q = 1$ of the quantum cohomology ring corresponds to the ring of $\mathbb{C}$-valued functions on the fiber in $\mathcal{Y}_{k,n}$ over $q = 1$. This fiber has size $\binom{n}{k}$, and it turns out that there is a natural embedding of $\mathcal{Y}_{k,n}$ into the (affine cone over) $\mathrm{Gr}_{k,n}(\mathbb{C})$ which identifies the fiber with the fixed points of $\sigma$. Moreover, we can rewrite a formula of Bertram [Ber97] for Gromov-Witten invariants of Schubert varieties in terms of the Plücker coordinates of the fixed points of $\sigma$. We do this in Section 4.3, and also explain how the results of Section 4.2 give an alternative proof of an explicit description of Rietsch of the totally nonnegative part of $\mathcal{Y}_{k,n}$, which can also be stated in terms of Schur polynomials evaluated at roots of unity.

In Section 4.4, we construct many fixed points of the *twist map* on $\mathrm{Gr}_{k,n}(\mathbb{C})$, which appears in the study of the cluster-algebraic structure of the Grassmannian [MS16, MS]. The element $V_{k,n}$ is one of the fixed points we identify, and the unique totally nonnegative one. It is an open problem to classify all fixed points of the twist map, and to determine whether $V_{k,n}$ is the only totally positive fixed point.

**Remark 4.1.4.** The curves $f_k$ have an interesting history. For odd $k$, the curve in $\mathbb{R}^{k-1}$ formed from $f_k$ by deleting the first component is the *trigonometric moment curve.* (Perhaps $f_k$ should be regarded as a curve in $\mathbb{P}^{k-1}$, however, for our purposes we need $f_k$ to give vectors in $\mathbb{R}^k$.) Carathéodory [Car11] used such curves, along with the *moment curves* $t \mapsto (t, t^2, \dots, t^{k-1})$, to define *cyclic polytopes* of even dimension. These polytopes have many faces, in the sense of the upper bound theorem of McMullen [McM70] and Stanley [Sta75]. For even $k$, the curve $f_k$ is the *symmetric moment curve*, first studied by Nudel'man [Nud75] in order to resolve an isoperimetric problem. Before we discuss his result, we mention that Barvinok and Novik [BN08] used symmetric moment curves to define *bicyclic polytopes* of even dimension, which are centrally symmetric analogues of cyclic polytopes. Together with Lee [BLN13], they showed that bicyclic polytopes have a record number of faces among centrally symmetric polytopes.

Now we describe the isoperimetric problem to which we alluded earlier. A curve $g : S \to \mathbb{R}^k$ defined on an interval $S \subseteq \mathbb{R}$ is called *convex on* $\mathbb{R}^k$ if the determinants

$$\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ g(t_0) & g(t_1) & \cdots & g(t_k) \end{bmatrix}\right) \qquad (t_0 < t_1 < \cdots < t_k \text{ in } S)$$

are either all nonnegative or all nonpositive, and not all zero.[1] Let $L$ denote the length of such a curve, and $V$ the volume of its convex hull. The *isoperimetric problem for convex curves*, first considered by Schoenberg [Sch54], is to find an upper bound for $V$ in terms of $L$ (given a fixed $k$). There are three cases, depending on the parity of $k$ and whether the curve is closed or not. We present them in chronological order of their resolution, and give an extremal curve (i.e. one which achieves equality in the isoperimetric inequality) in each case.

[1] If we also require all the determinants to be nonzero, then we obtain the definition of a *strictly convex* curve. Such curves have been widely studied under the various names *curves of order k* [Jue15], *monotone curves* [Hje14], *strictly comonotone curves* [Mot60], *alternating curves* [BLVS+99], and *hyperconvex curves* [Lab06].

| case | reference | isoperimetric inequality | extremal curve, for $\theta \in [0, 2\pi]$ |
|---|---|---|---|
| $k$ even; closed curves | [Sch54] | $V \leq \frac{L^k}{(\pi k)^{\frac{k}{2}} k! (\frac{k}{2})!}$ | $\left(\cos(\theta), \sin(\theta), \frac{\cos(2\theta)}{2}, \frac{\sin(2\theta)}{2}, \ldots, \frac{\cos(\frac{k}{2}\theta)}{\frac{k}{2}}, \frac{\sin(\frac{k}{2}\theta)}{\frac{k}{2}}\right)$ |
| $k$ odd | Theorem III.8.6 of [KN73] | $V \leq \frac{L^k}{\pi^{\frac{k-1}{2}} k^{\frac{k}{2}} k! (\frac{k-1}{2})!}$ | $\left(\frac{\theta}{\sqrt{2}}, \cos(\theta), \sin(\theta), \frac{\cos(2\theta)}{2}, \frac{\sin(2\theta)}{2}, \ldots, \frac{\cos(\frac{k-1}{2}\theta)}{\frac{k-1}{2}}, \frac{\sin(\frac{k-1}{2}\theta)}{\frac{k-1}{2}}\right)$ |
| $k$ even | [Nud75] | $V \leq \frac{L^k}{(\frac{\pi k}{2})^{\frac{k}{2}} k! (k-1)!!}$ | $\left(\cos\left(\frac{1}{2}\theta\right), \sin\left(\frac{1}{2}\theta\right), \frac{\cos(\frac{3}{2}\theta)}{3}, \frac{\sin(\frac{3}{2}\theta)}{3}, \ldots, \frac{\cos(\frac{k-1}{2}\theta)}{k-1}, \frac{\sin(\frac{k-1}{2}\theta)}{k-1}\right)$ |

Note the similarity of these curves to $f_{k-1}$ or $f_k$. There is no fourth case, because there are no closed curves convex on $\mathbb{R}^k$ if $k$ is odd. We mention that in the first two cases above, the extremal curve is unique (modulo affine isometries of $\mathbb{R}^k$). It is not known if uniqueness holds in the third case as well.

There is a discrete version of this isoperimetric problem, where we fix an integer $n > k$ and consider only those curves convex on $\mathbb{R}^k$ which are piecewise linear with $n$ segments, i.e. a polygonal path with $n + 1$ vertices. The convex hull of such a curve is precisely an *alternating polytope* [Stu88], a special kind of cyclic polytope.[2] Hence we may interpret this formulation as an isoperimetric problem for alternating polytopes. This problem was proposed by Krein and Nudel'man (see the discussion after Theorem III.8.6 of [KN73]) and solved by Nudel'man (Theorems III, II, and IV of [Nud75]). We present his results for the three cases in the same order as above.

| case | isoperimetric inequality | vertices of extremal piecewise linear curve, for $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, 2\pi$ |
|---|---|---|
| $k$ even, $n$ segments; closed curves | $V \leq \frac{L^k}{k^{\frac{k}{2}} k! \prod_{j=1}^{\frac{k}{2}} n \tan(\frac{j\pi}{n})}$ | $\left(\frac{\cos(\theta)}{\sin(\frac{\pi}{n})}, \frac{\sin(\theta)}{\sin(\frac{\pi}{n})}, \frac{\cos(2\theta)}{\sin(\frac{2\pi}{n})}, \frac{\sin(2\theta)}{\sin(\frac{2\pi}{n})}, \ldots, \frac{\cos(\frac{k}{2}\theta)}{\sin(\frac{k\pi}{2n})}, \frac{\sin(\frac{k}{2}\theta)}{\sin(\frac{k\pi}{2n})}\right)$ |
| $k$ odd, $n$ segments | $V \leq \frac{L^k}{k^{\frac{k}{2}} k! \prod_{j=1}^{\frac{k-1}{2}} n \tan(\frac{j\pi}{n})}$ | $\left(\frac{n\theta}{\sqrt{2}\pi}, \frac{\cos(\theta)}{\sin(\frac{\pi}{n})}, \frac{\sin(\theta)}{\sin(\frac{\pi}{n})}, \frac{\cos(2\theta)}{\sin(\frac{2\pi}{n})}, \frac{\sin(2\theta)}{\sin(\frac{2\pi}{n})}, \ldots, \frac{\cos(\frac{k}{2}\theta)}{\sin(\frac{k\pi}{2n})}, \frac{\sin(\frac{k}{2}\theta)}{\sin(\frac{k\pi}{2n})}\right)$ |
| $k$ even, $n$ segments | $V \leq \frac{L^k}{k^{\frac{k}{2}} k! \prod_{j=1}^{\frac{k}{2}} n \tan(\frac{(2j-1)\pi}{2n})}$ | $\left(\frac{\cos(\frac{1}{2}\theta)}{\sin(\frac{\pi}{2n})}, \frac{\sin(\frac{1}{2}\theta)}{\sin(\frac{\pi}{2n})}, \frac{\cos(\frac{3}{2}\theta)}{\sin(\frac{3\pi}{2n})}, \frac{\sin(\frac{3}{2}\theta)}{\sin(\frac{3\pi}{2n})}, \ldots, \frac{\cos(\frac{k-1}{2}\theta)}{\sin(\frac{(k-1)\pi}{2n})}, \frac{\sin(\frac{k-1}{2}\theta)}{\sin(\frac{(k-1)\pi}{2n})}\right)$ |

In all three cases, the extremal curve is unique. By taking $n \to \infty$, we recover the isoperimetric inequalities in the continuous case, and in fact this was how Nudel'man resolved the problem for even $k$. However, when we pass to the limit we are not able to conclude that

[2]A *cyclic polytope* of dimension $k$ is, by definition, one whose face lattice is the same as a polytope whose vertices lie on the moment curve $t \mapsto (t, t^2, \ldots, t^k)$ for $t > 0$. An *alternating polytope* is one whose every subpolytope (the convex hull of a subset of its vertices) is cyclic. The description stated above in terms of convex curves is due to Sturmfels [Stu88]. It follows from a result of Shemer (Theorem 2.12 of [She82]) that every cyclic polytope of even dimension is alternating. For an example of a cyclic polytope (necessarily of odd dimension) which is not alternating, see pp. 396-397 of [BLVS+99].

the resulting extremal curve is unique. We observe that Nudel'man's constructions in the first and third cases above are very similar to the definitions of $V_{k+1,n}$ and $V_{k,n}$, respectively. It is difficult to make a precise statement in this vein, since a real matrix representing $V_{k,n}$ is unique modulo linear automorphisms of $\mathbb{R}^k$, while the extremal curves above are unique modulo affine isometries of $\mathbb{R}^k$.

We remark that before Nudel'man's work [Nud75], it was not known what the extremal curve for even $k$ would look like. (Krein and Nudel'man [KN73] speculated that it might be half of the extremal curve in the closed case, which is true for $k = 2$ but false in general.) Similarly, before Barvinok and Novik's work [BN08], it was not known that the symmetric moment curve would give the 'correct' centrally symmetric analogue of the cyclic polytope. We view Theorem 4.1.1, and indeed Lemma 4.2.1, as further confirmation that trigonometric moment curves and symmetric moment curves belong in the same family.

## 4.2 Fixed points of the cyclic shift map

In this section we identify all fixed points of the cyclic shift map $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$. We begin by establishing the positivity properties of $V_{k,n} \in \mathrm{Gr}_{k,n}(\mathbb{C})$, which we will show is the unique totally nonnegative fixed point. We define $V_{k,n}$ as the element of $\mathrm{Gr}_{k,n}(\mathbb{C})$ represented by the $k \times n$ matrix $[f_k(\theta + \frac{2\pi}{n}) \,|\, f_k(\theta + \frac{4\pi}{n}) \,|\, \cdots \,|\, f_k(\theta + 2\pi)]$ for any $\theta \in \mathbb{R}$. The fact that $V_{k,n}$ does not depend on $\theta$ follows from the formula (4.1.2) for its Plücker coordinates, which in turn is a consequence of the following lemma.

**Lemma 4.2.1.** *[Sco79] For $k \ge 0$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, we have*

$$\det(f_k(\theta_1), \dots, f_k(\theta_k)) = 2^{\lfloor (k-1)^2/2 \rfloor} \prod_{1 \le r < s \le k} \sin\left(\tfrac{\theta_s - \theta_r}{2}\right).$$

Scott [Sco79] stated this result for odd $k$, and outlined a proof, which also handles the case of even $k$. It is short, so we give it here. We use $r$ and $s$ to index the rows and columns of a matrix, rather than $i$ and $j$, since we reserve $i$ to denote $\sqrt{-1}$.

**Proof.** Define $g_k : \mathbb{R} \to \mathbb{R}^k$ by $g_k(\theta) := (e^{-\frac{k-1}{2}i\theta}, e^{-\frac{k-3}{2}i\theta}, \dots, e^{\frac{k-3}{2}i\theta}, e^{\frac{k-1}{2}i\theta})$. We regard $f_k(\theta)$ and $g_k(\theta)$ as column vectors. Using

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} e^{-i\theta} \\ e^{i\theta} \end{bmatrix}, \tag{4.2.2}$$

we can get from the matrix $[f_k(\theta_1) \,|\, \cdots \,|\, f_k(\theta_k)]$ to $[g_k(\theta_1) \,|\, \cdots \,|\, g_k(\theta_k)]$, introducing a factor of $\pm(2i)^{-\lfloor k/2 \rfloor}$ when we take determinants. (The sign $\pm$ is $-$ if $k$ and $\frac{k-1}{2}$ are both odd, and $+$ otherwise.) When we multiply each column of $[g_k(\theta_1) \,|\, \cdots \,|\, g_k(\theta_k)]$ by a constant so that its first entry is 1, we obtain a Vandermonde matrix, whose determinant is well known:

$$\det([g_k(\theta_1) \,|\, \cdots \,|\, g_k(\theta_k)]) = \prod_{s=1}^{k} e^{-\frac{k-1}{2}i\theta_s} \cdot \det\big((e^{i\theta_s})^{r-1}\big)_{1 \le r,s \le k}$$

$$= \prod_{s=1}^{k} e^{-\frac{k-1}{2}i\theta_s} \prod_{1\le r<s\le k} e^{i\theta_s} - e^{i\theta_r} = \prod_{1\le r<s\le k} e^{-\frac{i\theta_r}{2}} e^{-\frac{i\theta_s}{2}} (e^{i\theta_s} - e^{i\theta_r}) = \prod_{1\le r<s\le k} 2i\sin\left(\tfrac{\theta_s-\theta_r}{2}\right).$$

We can check that $\pm(2i)^{-\lfloor k/2\rfloor}(2i)^{\binom{k}{2}} = 2^{\lfloor (k-1)^2/2\rfloor}$. ∎

We will use the following lemma to show that among the fixed points we identify, only $V_{k,n}$ can be totally nonnegative. First we make a useful definition.

**Definition 4.2.3.** We say that $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$ is *real* if all (ratios of) Plücker coordinates of $V$ are real, or equivalently if $V$ is closed under complex conjugation.

**Lemma 4.2.4.** *Suppose that* $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ *with* $n > k \ge 1$, *and* $(1, z, \dots, z^{n-1}) \in V$ *for some* $z \in \mathbb{C}^\times$. *Then* $|\arg(z)| \le \frac{k-1}{n-1}\pi$, *where* $\arg : \mathbb{C}^\times \to (-\pi, \pi]$ *denotes the argument function.*

We remark that the upper bound $\frac{k-1}{n-1}\pi$ is optimal: given any $z \in \mathbb{C}^\times$ with $|\arg(z)| \le \frac{k-1}{n-1}\pi$, there exists $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ with $(1, z, \dots, z^{n-1}) \in V$. To see this, first rescale $z$ by an element of $\mathbb{R}_{>0}$ so that $|z| = 1$, whence the general case follows using the *torus action* (see Remark 4.4.3). If $z = 1$, we let $V$ be represented by the $k \times n$ matrix $(s^{r-1})_{1\le r\le k, 1\le s\le n}$. Otherwise, given $\rho > 0$, let $V$ be represented by the $k \times n$ matrix with columns $f_k(\theta)$, for $\theta = 0, \rho, 2\rho, \dots, (n-1)\rho$. If $\rho \le \frac{2\pi}{n-1}$, then $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ by Lemma 4.2.1. On the other hand, if $\rho = \frac{2|\arg(z)|}{k-1}$, then $(1, z, \dots, z^{n-1}) \in V$ by (4.2.2). (Note that if $\rho = \frac{2\pi}{n}$, then $V = V_{k,n}$.)

**Proof** (of Lemma 4.2.4)**.** Since $V$ is real, defining $\theta := |\arg(z)|$ and letting $\epsilon \in \{1, -1\}$ be the sign of $\arg(z)$, we have

$$(\cos(\phi), |z|\cos(\theta+\phi), \dots, |z|^{n-1}\cos((n-1)\theta+\phi)) = \frac{e^{\epsilon i\phi}(1, z, \dots, z^{n-1}) + \overline{e^{\epsilon i\phi}(1, z, \dots, z^{n-1})}}{2} \in V$$

for all $\phi \in \mathbb{R}$. Note that for $\phi < \frac{\pi}{2}$ sufficiently close to $\frac{\pi}{2}$, the vector $(\cos(\phi), |z|\cos(\theta + \phi), \dots, |z|^{n-1}\cos((n-1)\theta+\phi))$ changes sign at least $\frac{(n-1)\theta}{\pi}$ times, whence $\frac{(n-1)\theta}{\pi} \le k-1$ by Theorem 3.3.4(i). ∎

We introduce some notation in order to describe the fixed points of $\sigma$.

**Definition 4.2.5.** Given a $k$-subset $S \subseteq \mathbb{C}$, define $V_S \in \mathrm{Gr}_{k,n}(\mathbb{C})$ as the subspace with basis $\{(1, z, \dots, z^{n-1}) : z \in S\}$, which is well defined by Vandermonde's determinantal identity. We also denote the $k$-subset $\left\{e^{-i\frac{(k-1)\pi}{n}}, e^{-i\frac{(k-3)\pi}{n}}, \dots, e^{i\frac{(k-3)\pi}{n}}, e^{i\frac{(k-1)\pi}{n}}\right\}$ of $n$th roots of $(-1)^{k-1}$ closest to 1 on the unit circle by $S_{k,n}$. Note that by (4.2.2), we have $V_{k,n} = V_{S_{k,n}}$.

**Theorem 4.2.6.** *The map $S \mapsto V_S$ is a bijection from the set of $k$-subsets of $n$th roots of $(-1)^{k-1}$ to the set of fixed points of $\sigma$. Therefore $\sigma$ has exactly $\binom{n}{k}$ fixed points. The unique totally nonnegative fixed point is $V_{k,n}$, corresponding to the $k$-subset $S_{k,n}$.*

**Proof.** First suppose that $\sigma(V) = V$, and let $A$ be a $k \times n$ matrix representing $V$. Write out $A$ in columns as $[a^{(1)} \,|\, \cdots \,|\, a^{(n)}]$. Since $\sigma(V) = V$, there exists $g \in \mathrm{GL}_k(\mathbb{C})$ such that $gA = [a^{(2)} \,|\, a^{(3)} \,|\, \cdots \,|\, a^{(n)} \,|\, (-1)^{k-1}a^{(1)}]$. Then

$$ga^{(s)} = a^{(s+1)} \quad \text{for } 1 \le s \le n-1, \quad \text{and} \quad ga^{(n)} = (-1)^{k-1}a^{(1)}.$$

In particular, $g^n a^{(s)} = (-1)^{k-1}a^{(s)}$ for $1 \le s \le n$, and since the columns of $A$ span $\mathbb{C}^k$, we get $g^n = (-1)^{k-1}I_k$. Therefore the minimal polynomial of $g$ divides $x^n - (-1)^{k-1}$, whose complex zeros are distinct. Hence $g$ is diagonalizable over $\mathbb{C}$, and its eigenvalues $z_1, \ldots, z_k \in \mathbb{C}$ are $n$th roots of $(-1)^{k-1}$. Also, the matrix $A$ uniquely determines $g$, so $V$ determines $g$ up to conjugation by an element of $\mathrm{GL}_k(\mathbb{C})$. Hence $V$ uniquely determines the eigenvalues $z_1, \ldots, z_k$.

Now write $hgh^{-1} = d$ for some $h \in \mathrm{GL}_k(\mathbb{C})$, where $d \in \mathrm{GL}_k(\mathbb{C})$ is the diagonal matrix with diagonal entries $z_1, \ldots, z_k$. Letting $b := ha^{(1)} \in \mathbb{C}^k$, we get

$$hA = [ha^{(1)} \,|\, hga^{(1)} \,|\, \cdots \,|\, hg^{n-1}a^{(1)}] = [ha^{(1)} \,|\, dha^{(1)} \,|\, \cdots \,|\, d^{n-1}ha^{(1)}] = [b \,|\, db \,|\, \cdots \,|\, d^{n-1}b].$$

We have $b_r \neq 0$ for all $r \in \{1, \ldots, k\}$, since otherwise row $r$ of $hA$ would be zero, contradicting the fact that $\mathrm{rank}(hA) = k$. Multiplying $hA$ on the left by the diagonal matrix in $\mathrm{GL}_k(\mathbb{C})$ with diagonal entries $\frac{1}{b_1}, \ldots, \frac{1}{b_k}$ (which commutes with $d$), we obtain the $k \times n$ matrix whose $r$th row is $(1, z_r, \ldots, z_r^{n-1})$. The rank of this matrix is $k$, so $z_1, \ldots, z_k$ are distinct. Hence $V = V_S$, where $S := \{z_1, \ldots, z_k\}$ is uniquely determined by $V$.

Conversely, suppose that $S \subseteq \mathbb{C}$ is a subset of $k$ distinct $n$th roots of $(-1)^{k-1}$. Since $\sigma(1, z, \ldots, z^{n-1}) = z(1, z, \ldots, z^{n-1})$ for all $z \in \mathbb{C}$ with $z^n = (-1)^{k-1}$, we have $\sigma(V_S) = V_S$. Suppose further that $V_S$ is totally nonnegative. Then by Lemma 4.2.4 we have $S = S_{k,n}$, whence $V_S = V_{k,n}$. (This also follows by a result of Rietsch; see Proposition 4.2.9 in Section 4.2.) We already showed in (4.1.2) that $V_{k,n}$ is totally positive. ■

We conclude this section with two remarks surrounding Theorem 4.2.6.

## Schur polynomials evaluated at roots of unity

The *Schur polynomial* of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (where $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$) with at most $k$ parts is defined by

$$s_\lambda(x_1, \ldots, x_k) := \frac{\det(x_r^{\lambda_{k+1-s}+s-1})_{1 \le r,s \le k}}{\det(x_r^{s-1})_{1 \le r,s \le k}}. \tag{4.2.7}$$

In particular, let $z_1, \ldots, z_k \in \mathbb{C}$ be distinct, $S := \{z_1, \ldots, z_k\}$, and $\mathcal{P}_{k,n}$ denote the set of partitions with at most $k$ parts and parts at most $n-k$ (i.e. partitions whose Young diagram fits inside the $k \times (n-k)$ rectangle). Then we have

$$s_\lambda(z_1, \ldots, z_k) = \frac{\Delta_{\{\lambda_k+1, \lambda_{k-1}+2, \ldots, \lambda_1+k\}}(V_S)}{\Delta_{\{1,\ldots,k\}}(V_S)} \quad \text{for all } \lambda \in \mathcal{P}_{k,n}. \tag{4.2.8}$$

As a consequence of Lemma 4.2.4, we obtain the following result of Rietsch.

**Proposition 4.2.9** (Theorem 8.4 of [Rie01])**.** *Let $0 \le k < n$, $t \in \mathbb{C}^\times$, and $z_1, \ldots, z_k \in \mathbb{C}$ be distinct $n$th roots of $t$. Then $s_\lambda(z_1, \ldots, z_k) \ge 0$ for all $\lambda \in \mathcal{P}_{k,n}$ if and only if $\{z_1, \ldots, z_k\} = |t|^{1/n} S_{k,n}$. In this case,*

$$s_\lambda(z_1, \ldots, z_k) = |t|^{|\lambda|/n} \prod_{1 \le r < s \le k} \frac{\sin\left(\frac{\lambda_r - \lambda_s + s - r}{n}\pi\right)}{\sin\left(\frac{s-r}{n}\pi\right)} \quad \textit{for all } \lambda \in \mathcal{P}_{k,n}. \tag{4.2.10}$$

**Proof.** Suppose that $s_\lambda(z_1, \ldots, z_k) \ge 0$ for all $\lambda \in \mathcal{P}_{k,n}$. Then letting $S := \{z_1, \ldots, z_k\}$, we have $V_S \in \mathrm{Gr}_{k,n}^{\ge 0}$. In particular, $V_S$ is real (see Definition 4.2.3), and so $(1, \overline{z_1}, \ldots, \overline{z_1}^{n-1}) \in V_S$, giving $V_{S \cup \{\overline{z_1}\}} \subseteq V_S$. Therefore $\overline{z_1} \in S$ (otherwise $V_{S \cup \{\overline{z_1}\}}$ would have dimension $k+1$), whence $t$ is real. Now Lemma 4.2.4 implies $S = |t|^{1/n} S_{k,n}$, and (4.2.10) follows from (4.1.2). ■

Rietsch proved this result in her study of the quantum cohomology of $\mathrm{Gr}_{k,n}(\mathbb{C})$. We give the context of her work in Section 4.3, which is surprisingly different from ours, yet leads to some of the same combinatorics. We remark that Rietsch's proof of Proposition 4.2.9 uses orthogonality for Schur polynomials evaluated at roots of unity, whereas our argument relies on the result of Gantmakher and Krein which we used to prove Lemma 4.2.4. Proposition 4.2.9 provides an alternative way to conclude the proof of Theorem 4.2.6, rather than using Lemma 4.2.4.

We also mention that in their study of symmetric group characters, Orellana and Zabrocki [OZ] consider evaluations of symmetric polynomials at certain other $k$-multisubsets of roots of unity (namely, those which are the eigenvalues of a $k \times k$ permutation matrix).

## Arrangements of equal minors

Farber and Postnikov [FP16] studied the possible arrangements of equal and unequal Plücker coordinates among totally positive elements of $\mathrm{Gr}_{k,n}(\mathbb{C})$. By (4.1.2), $V_{k,n}$ has many pairs of equal Plücker coordinates: if $I$ and $J$ are $k$-subsets of $\{1, \ldots, n\}$ which are cyclic shifts of each other modulo $n$, then $\Delta_I(V_{k,n}) = \Delta_J(V_{k,n})$. (This does not hold for all $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$ fixed by $\sigma$. Indeed, we have $\sigma(V) = V$ if and only if there exists $\zeta \in \mathbb{C}$ with $\zeta^n = 1$ such that $\Delta_{\{i_1+j, \ldots, i_k+j\}}(V) = \zeta^j \Delta_{\{i_1, \ldots, i_k\}}(V)$ for all $j \in \mathbb{Z}$ and distinct $i_1, \ldots, i_k$ modulo $n$.) We do not know if the converse is true, i.e. if $\Delta_I(V_{k,n}) = \Delta_J(V_{k,n})$ implies that $I$ and $J$ are cyclic shifts of each other modulo $n$.

We expect that the minimum Plücker coordinates of $V_{k,n}$ (after we have rescaled its Plücker coordinates to all be positive real numbers) are indexed by $\{1, \ldots, k\}$ and its cyclic shifts. In general, we expect that for a $k$-subset $I \subseteq \{1, \ldots, n\}$, the Plücker coordinate $\Delta_I(V_{k,n})$ measures 'how spread out' are the elements of $I$ modulo $n$.

## 4.3 Quantum cohomology of the Grassmannian

In this section we explain how the fixed points of the cyclic shift map $\sigma$ appear in the theory of quantum cohomology. The *(small) quantum cohomology ring* $QH^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$ of $\mathrm{Gr}_{k,n}(\mathbb{C})$ is a deformation of the cohomology ring $H^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$ by an indeterminate $q$. Recall that $\mathcal{P}_{k,n}$ is the set of partitions whose Young diagram fits inside the $k \times (n-k)$ rectangle. For $\lambda \in \mathcal{P}_{k,n}$, let $\lambda^\vee := (n-k-\lambda_k, n-k-\lambda_{k-1}, \ldots, n-k-\lambda_1)$ denote the partition whose Young diagram (rotated by 180°) is the complement of the Young diagram of $\lambda$ inside the $k \times (n-k)$ rectangle. Then $H^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$ has the *Schubert basis* $\{\sigma_\lambda : \lambda \in \mathcal{P}_{k,n}\}$, where $\sigma_\lambda$ is the cohomology class of the *Schubert variety* $X_\lambda$. Multiplication in $QH^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$ is given by

$$\sigma_\lambda \cdot \sigma_\mu := \sum_{d \geq 0} \sum_{\nu \in \mathcal{P}_{k,n}} \langle X_\lambda, X_\mu, X_{\nu^\vee} \rangle_d q^d \sigma_\nu,$$

where $\langle X_\lambda, X_\mu, X_{\nu^\vee} \rangle_d \in \mathbb{N}$ is a *Gromov-Witten invariant*, which is a generalized intersection number of the Schubert varieties $X_\lambda$, $X_\mu$, and $X_{\nu^\vee}$. If $d = 0$ then the Gromov-Witten invariant is the usual intersection number given by the Littlewood-Richardson rule, and so the specialization at $q = 0$ recovers the cup product in $H^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$. *Quantum Schubert calculus* involves the study of these Gromov-Witten invariants. See [Ber97] for more details.

In unpublished work, Peterson defined a subvariety $\mathcal{Y}_{k,n}$ of $\mathrm{GL}_n(\mathbb{C})$ whose coordinate ring is isomorphic to $QH^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$. This fact was proved by Rietsch [Rie01], who characterized $\mathcal{Y}_{k,n}$ as follows (see Lemma 3.7 of [Rie01]). For $z_1, \ldots, z_k \in \mathbb{C}$, define the *Toeplitz matrix*

$$u_{k,n}(z_1, \ldots, z_k) := (e_{s-r}(z_1, \ldots, z_k))_{1 \leq r,s \leq n} \in \mathrm{GL}_n(\mathbb{C}),$$

where $e_j(x_1, \ldots, x_k) := \sum_{1 \leq i_1 < \cdots < i_j \leq k} x_{i_1} \cdots x_{i_j}$ is the $j$th elementary symmetric polynomial for $j \geq 0$, and $e_j(x_1, \ldots, x_k) := 0$ for $j < 0$. Then

$$\mathcal{Y}_{k,n} = \{I_n\} \cup \{u_{k,n}(z_1, \ldots, z_k) : z_1, \ldots, z_k \in \mathbb{C} \text{ are distinct and } z_1^n = \cdots = z_k^n\} \subseteq \mathrm{GL}_n(\mathbb{C}).$$

Rietsch's isomorphism $QH^*(\mathrm{Gr}_{k,n}(\mathbb{C})) \to \mathbb{C}[\mathcal{Y}_{k,n}]$ sends the indeterminate $q$ to the map $\mathcal{Y}_{k,n} \to \mathbb{C}$ given by $q(I_n) = 0$ and $q(u(z_1, \ldots, z_k)) = (-1)^{k-1} z_1^n$ for distinct $z_1, \ldots, z_k$ with $z_1^n = \cdots = z_k^n$. We can identify the specialization of $QH^*(\mathrm{Gr}_{k,n}(\mathbb{C}))$ at $q = 1$ with the ring of $\mathbb{C}$-valued functions on the fiber of $\mathcal{Y}_{k,n}$ over $q = 1$, and this fiber equals

$$\{u(z_1, \ldots, z_k) : \{z_1, \ldots, z_k\} \text{ is a } k\text{-subset of } n\text{th roots of } (-1)^{k-1}\}$$

(see Section 11 of [Rie01]).

We also have the map $\mathcal{Y}_{k,n} \to \mathrm{Gr}_{k,n}(\mathbb{C})$ which sends $g$ to the subspace $V$ with Plücker coordinates $\Delta_I(V) = \det(g_{[n]\setminus I,\{k+1,k+2,\dots,n\}})$ for all $k$-subsets $I \subseteq \{1,\dots,n\}$. (Explicitly, $V$ is orthogonal to the last $n-k$ columns of $g$ under the bilinear form $\langle v,w\rangle := v_1w_1 - v_2w_2 + \cdots + (-1)^{n-1}v_nw_n$ on $\mathbb{C}^n$; see Lemma 1.11(ii) of [Kar17] for a discussion of this fact.) By the dual Jacobi-Trudi identity (see Corollary 7.16.2 of [Sta99]) and (4.2.8), this map sends $u(z_1,\dots,z_k)$ to $V_{\{z_1,\dots,z_k\}}$ (defined in Definition 4.2.5). Therefore we have the following alternative description of the fixed points of the cyclic shift map $\sigma$.

**Corollary 4.3.1.** *The set of fixed points of the cyclic shift map $\sigma$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ corresponds to the fiber of $\mathcal{Y}_{k,n}$ over $q = 1$.*

Another connection between the fixed points of $\sigma$ and quantum cohomology is evident in a formula of Bertram for the Gromov-Witten invariants $\langle X_{\lambda^{(1)}},\dots,X_{\lambda^{(l)}}\rangle_d$ for Schubert varieties (see Section 5 of [Ber97]). Bertram expressed his formula in terms of Schur polynomials evaluated at roots of unity, which using (4.2.8) we can rewrite as follows.

**Theorem 4.3.2** (Section 5 of [Ber97])**.** *Let $\lambda^{(1)},\dots,\lambda^{(l)} \in \mathcal{P}_{k,n}$ and $d \ge 0$. For $1 \le j \le l$, define $I_j := \{\lambda^{(j)}_k+1, \lambda^{(j)}_{k-1}+2,\dots,\lambda^{(j)}_1+k\} \subseteq \{1,\dots,n\}$. If $|\lambda^{(1)}|+\cdots+|\lambda^{(l)}| = k(n-k)+dn$, then*

$$\langle X_{\lambda^{(1)}},\dots,X_{\lambda^{(l)}}\rangle_d = \frac{1}{n^k}\sum_{S=\{z_1,\dots,z_k\}}(z_1\cdots z_k)\prod_{r\neq s} z_r - z_s \prod_{j=1}^{l}\frac{\Delta_{I_j}(V_S)}{\Delta_{\{1,\dots,k\}}(V_S)},$$

*where the sum is over all $k$-subsets $S \subseteq \mathbb{C}$ of distinct $n$th roots of $(-1)^{k-1}$, and $V_S$ is defined in Definition 4.2.5. If $|\lambda^{(1)}|+\cdots+|\lambda^{(l)}| \neq k(n-k)+dn$, then $\langle X_{\lambda^{(1)}},\dots,X_{\lambda^{(l)}}\rangle_d = 0$.*

This also makes manifest the so-called 'hidden symmetry' of these Gromov-Witten invariants, i.e. they are invariant (up to an appropriate change of $d$) under the cyclic actions on each of the partitions $\lambda^{(j)}$, which translates the set $I_j$ modulo $n$. This hidden symmetry first appeared in the work of Seidel [Sei97], and was further studied by Agnihotri and Woodward (see Section 7 of [AW98]) and Postnikov (see Section 6.2 of [Pos05]).

Finally, we note that Rietsch used her result Proposition 4.2.9 to derive an explicit description of the *totally nonnegative part* $\mathcal{Y}^{\ge 0}_{k,n}$ of $\mathcal{Y}_{k,n}$, which is defined as the elements of $\mathcal{Y}_{k,n}$ whose submatrices all have a nonnegative determinant. Therefore, our argument for Proposition 4.2.9 gives a different proof of this description.

**Theorem 4.3.3** (Theorem 8.4 of [Rie01])**.** *For distinct $z_1,\dots,z_k \in \mathbb{C}$ with $z_1^n = \cdots = z_k^n$, we have $u_{k,n}(z_1,\dots,z_k) \in \mathcal{Y}^{\ge 0}_{k,n}$ if and only if $V_{\{z_1,\dots,z_k\}} \in \mathrm{Gr}^{\ge 0}_{k,n}$. Therefore by Proposition 4.2.9, writing $S_{k,n} = \{y_1,\dots,y_k\}$, we have $\mathcal{Y}^{\ge 0}_{k,n} = \{I_n\} \cup \{u_{k,n}(ty_1,\dots,ty_k) : t > 0\}$.*

The same ideas can be used to prove the result of Aissen, Schoenberg, and Whitney [ASW52] that for any $z_1,\dots,z_k \in \mathbb{C}$, we have $z_1,\dots,z_k \ge 0$ if and only if $u_{k,n}(z_1,\dots,z_k) \in \mathcal{Y}^{\ge 0}_{k,n}$ for all $n > k$ (see also Section 8.3 of [Kar68]). In this context, one normally interprets the matrix entries $e_j(z_1,\dots,z_k)$ of $u_{k,n}(z_1,\dots,z_k)$ as the coefficients of the polynomial with zeros

$-\frac{1}{z_1}, \ldots, -\frac{1}{z_k}$. We also mention that Rietsch gives an explicit factorization of the element $u_{k,n}(ty_1, \ldots, ty_k)$ into elementary matrices (see Proposition 9.3 of [Rie01]).

**Remark 4.3.4.** Rietsch proves the following interesting inequality (Proposition 11.1 of [Rie01]). For any distinct $z_1, \ldots, z_k \in \mathbb{C}$ such that $z_1^n = \cdots = z_k^n$ and $|z_1| = 1$, we have

$$|s_\lambda(z_1, \ldots, z_k)| \le s_\lambda(S_{k,n}) = \prod_{1 \le r < s \le k} \frac{\sin\left(\frac{\lambda_r - \lambda_s + s - r}{n}\pi\right)}{\sin\left(\frac{s-r}{n}\pi\right)} \quad \text{for all } \lambda \in \mathcal{P}_{k,n}.$$

It would be interesting to find a common interpretation of such inequalities and the discrete isoperimetric inequalities of Nudel'man we discussed in Remark 4.1.4.

## 4.4 Fixed points of the twist map

In this section we construct many fixed points of the *twist map* on $\mathrm{Gr}_{k,n}(\mathbb{C})$, which appears in the study of the cluster-algebraic structure of the Grassmannian. We show that among the fixed points we identify, there is a unique totally nonnegative one, namely $V_{k,n}$. We leave open the problem of determining all fixed points of the twist map on $\mathrm{Gr}_{k,n}(\mathbb{C})$.

The twist map on $\mathrm{Gr}_{k,n}(\mathbb{C})$ was introduced by Marsh and Scott [MS16], as the Grassmannian analogue of the *twist map* on double Bruhat cells defined by Fomin and Zelevinsky [FZ99]. It was later studied by Muller and Speyer [MS] using a slightly different definition. It relates the *$\mathcal{A}$-cluster structure* and *$\mathcal{X}$-cluster structure* of $\mathrm{Gr}_{k,n}(\mathbb{C})$, and more generally, of positroid varieties. We adopt here the conventions of Muller and Speyer. See Remark 4.4.6 for a discussion of how our results apply to Marsh and Scott's twist map.

**Definition 4.4.1.** Let

$$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}_{k,n}(\mathbb{C}) : \Delta_{\{j,j+1,\ldots,j+k-1\}}(V) \neq 0 \text{ for } 1 \le j \le n\} \subseteq \mathrm{Gr}_{k,n}(\mathbb{C}),$$

where we read the indices modulo $n$.[3] Given $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, let $A = [a^{(1)} \,|\, \cdots \,|\, a^{(n)}]$ be a $k \times n$ matrix representing $V$. For $1 \le j \le n$, we define the $k \times k$ matrix $A_j := [a^{(j)} \,|\, a^{(j+1)} \,|\, \cdots \,|\, a^{(j+k-1)}]$, where we read the indices modulo $n$. The *(right) twist* $\tau : \Pi^\circ_{k,n} \to \Pi^\circ_{k,n}$ is defined such that $\tau(V)$ is represented by the $k \times n$ matrix whose $j$th column (for $1 \le j \le n$) equals the first column of $(A_j^{-1})^T$. The twist $\tau$ is an automorphism of $\Pi^\circ_{k,n}$, and its inverse is the analogously defined *left twist* (see Corollary 6.8 of [MS]). (Muller and Speyer in fact define $\tau$ on all of $\mathrm{Gr}_{k,n}(\mathbb{C})$, but we will only consider fixed points in $\Pi^\circ_{k,n}$.)

[3] $\Pi^\circ_{k,n}$ is the top-dimensional stratum of the *positroid stratification* of $\mathrm{Gr}_{k,n}(\mathbb{C})$ studied by Knutson, Lam, and Speyer [KLS13]. This is the projection to $\mathrm{Gr}_{k,n}(\mathbb{C})$ of the stratification of the complete flag variety into intersections of opposite Schubert cells, which had first been considered by Lusztig [Lus94]. He conjectured, and Rietsch proved [Rie98], that when restricted to $\mathrm{Gr}^{\ge 0}_{k,n}$ it gives a cell decomposition.

**Example 4.4.2.** Let $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ be represented by the matrix $A = \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 3 \end{bmatrix}$. Then $\tau(V)$ is represented by the matrix $\begin{bmatrix} 1 & 1 & \frac{3}{4} & 0 \\ -\frac{1}{2} & 0 & 1 & \frac{1}{3} \end{bmatrix}$. ◊

**Remark 4.4.3.** Let the torus $(\mathbb{C}^\times)^n$ act on $\mathrm{Gr}_{k,n}(\mathbb{C})$ by rescaling columns, i.e. given $t \in (\mathbb{C}^\times)^n$, we define $t(V) := \{(t_1 v_1, \ldots, t_n v_n) : v \in V\}$. Then $\tau$ induces an automorphism on the quotient $\Pi^\circ_{k,n}/(\mathbb{C}^\times)^n$ of order dividing $2n$ (see Section 4 of [MS16]). We are also interested in determining the fixed points of the twist map on $\Pi^\circ_{k,n}/(\mathbb{C}^\times)^n$, however, our methods do not identify any fixed point in $\Pi^\circ_{k,n}/(\mathbb{C}^\times)^n$ which does not come from a fixed point in $\Pi^\circ_{k,n}$.

**Remark 4.4.4.** The action of the subgroup $\{1,-1\}^n$ of $(\mathbb{C}^\times)^n$ on $\Pi^\circ_{k,n}$ commutes with $\tau$. Therefore, if $V \in \Pi^\circ_{k,n}$ is fixed by $\tau$, then so is $t(V)$ for any $t \in \{1,-1\}^n \subseteq (\mathbb{C}^\times)^n$.

**Theorem 4.4.5.** *Let $S \subseteq \mathbb{C}$ be a $k$-subset closed under inversion (i.e. $S = \{z^{-1} : z \in S\}$), and $\epsilon \in \{1,-1\}$ such that $z^n = \epsilon$ for all $z \in S$. Then $V_S$ (defined in Definition 4.2.5) is an element of $\Pi^\circ_{k,n}$ fixed by the twist map $\tau$, and moreover $V_S \in \mathrm{Gr}^{\geq 0}_{k,n}$ if and only if $V_S = V_{k,n}$.*

Note that all fixed points of $\tau$ we identify above are real (see Definition 4.2.3). Hence Theorem 4.2.6 implies that all real fixed points of $\sigma$ are also fixed by $\tau$.

**Proof.** Write $S = \{z_1, \ldots, z_k\}$, let $A$ be the $k \times n$ matrix $(z_r^{s-1})_{1 \leq r \leq k, 1 \leq s \leq n}$, and for $1 \leq j \leq n$ define $B_j := (z_r^{j+s-2})_{1 \leq r,s \leq k}$. Since $z_1^n = \cdots = z_k^n = \epsilon$, we obtain $B_j$ from $A_j := [a^{(j)} \,|\, a^{(j+1)} \,|\, \cdots \,|\, a^{(j+k-1)}]$ by multiplying some subset of columns $2, \ldots, k$ by $\epsilon$. Hence the first columns of $(B_j^{-1})^T$ and $(A_j^{-1})^T$ are equal. By the classical adjoint description of the matrix inverse, and Vandermonde's determinantal identity, the $(p,1)$ entry of $(B_j^{-1})^T$ equals

$$(-1)^{p-1} \frac{\det(z_r^{j+s-2})_{\substack{r=1,\ldots,p-1,p+1,\ldots,k \\ s=2,\ldots,k}}}{\det(z_r^{j+s-2})_{1 \leq r,s \leq k}} = (-1)^{p-1} \frac{(z_1 \cdots z_k)^j z_p^{-j} \displaystyle\prod_{\substack{1 \leq r < s \leq k, \\ r,s \neq p}} z_s - z_r}{(z_1 \cdots z_k)^{j-1} \displaystyle\prod_{1 \leq r < s \leq k} z_s - z_r} = \frac{(z_1 \cdots z_k) z_p^{-j}}{\displaystyle\prod_{\substack{1 \leq s \leq k, \\ s \neq p}} z_s - z_p}.$$

This calculation also shows that $\det(A_j)$ (which equals $\pm\det(B_j)$) is nonzero, so $V_S$ is indeed in $\Pi^\circ_{k,n}$. By factoring out $\frac{z_1 \cdots z_k}{z_p \prod_{1 \leq s \leq k, s \neq p} z_s - z_p}$ from row $p$ for $1 \leq p \leq k$, we see that $\tau(V_S)$ is represented by the $k \times n$ matrix $(z_p^{-(j-1)})_{1 \leq p \leq k, 1 \leq j \leq n}$. Since $S$ is closed under inversion, this matrix has the same rows as $A$. Hence $\tau(V_S) = V_S$. Finally, Lemma 4.2.4 implies $V_S \in \mathrm{Gr}^{\geq 0}_{k,n}$ if and only if $V_S = V_{k,n}$. ■

**Remark 4.4.6.** The twist map of Marsh and Scott restricted to $\Pi^\circ_{k,n}$ is defined as in Definition 4.4.1, except that in the definition we take the first column of $\Delta_{\{j,j+1,\dots,j+k-1\}}(V)(A_j^{-1})^T$ rather than the first column of $(A_j^{-1})^T$. An argument similar to the proof above shows that if $S \subseteq \mathbb{C}$ is a subset of size $k$ such that $z^n = (-1)^{k-1}$ for all $z \in S$ and $S = \{\prod_{z' \in S \setminus \{z\}} z' : z \in S\}$, then $V_S$ is an element of $\Pi^\circ_{k,n}$ fixed by Marsh and Scott's twist map. In particular, $V_{k,n}$ is such a fixed point.

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