

UC Berkeley

UC Berkeley Electronic Theses and Dissertations

Title

Essays on Delegated Portfolio Management and Optimal Contracting

Permalink

<https://escholarship.org/uc/item/9th5f577>

Author

Leung, Raymond Chi Wai

Publication Date

2016

Peer reviewed|Thesis/dissertation

Essays on Delegated Portfolio Management and Optimal Contracting

by

Raymond Chi Wai Leung

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Business Administration

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Associate Professor Gustavo Manso, Chair

Professor Robert M. Anderson

Professor Christine Parlour

Spring 2016

Essays on Delegated Portfolio Management and Optimal Contracting

Copyright 2016
by
Raymond Chi Wai Leung

Abstract

Essays on Delegated Portfolio Management and Optimal Contracting

by

Raymond Chi Wai Leung

Doctor of Philosophy in Business Administration

University of California, Berkeley

Associate Professor Gustavo Manso, Chair

This dissertation is a compilation of three papers that investigate the role of optimal contracting in a delegated portfolio management setting. While the study of optimal contracts in classical principal-agent setup has been extensively studied, relatively few have been studied in the context of delegated portfolio management in finance. And even delegated portfolio management papers in finance, there are still several open questions and unresolved issues that are beyond the scope of a standard principal-agent problem.

In Chapter 1, I study a continuous-time principal-agent problem with drift and stochastic volatility control. While the problem with drift-only control by an agent has been extensively studied recently, very few existing papers allow an agent to endogenously influence volatility. Endogenous volatility control is particularly important in delegated portfolio management settings as volatility is one of the defining aspects of modern financial portfolio management.

In Chapter 2, I study a model that encompasses dynamic agency, delegated portfolio management and asset pricing. Traditionally, the fields of “asset pricing” and “corporate finance” are studied independently of each other. However, as the modern portfolio management industry blooms in size and influence, the role of the portfolio manager and the contracts that are extended to them arguably has a role in the securities that they invest in, and hence in equilibrium, the asset pricing implications of the market overall. This paper is an attempt to bridge “asset pricing” and “corporate finance” (specifically interpreted to mean delegated portfolio management contracting) into one.

In Chapter 3, I study whether a principal investor is better off delegating most of his money to a single portfolio manager (centralized delegation), as opposed to multiple portfolio managers (decentralized delegation), especially when there is the possible presence of moral hazard. With the size of the hedge fund industry and growing empirical support that moral hazard is a growing risk among hedge fund managers, it becomes imperative to understand when an investor decides to delegate his money, should it be delegated in a more centralized or decentralized fashion.

Dedicated to my family and friends.

Contents

Contents	ii
1 Continuous-Time Principal-Agent Problem with Drift and Stochastic Volatility Control: With Applications to Delegated Portfolio Management	1
1.1 Introduction	3
1.2 Related Literature	7
1.3 Model Motivation in Discrete-Time	9
1.4 Model Outline	12
1.5 First Best	18
1.6 Continuation value and Incentive compatible contracts	19
1.7 Principal's Problem	23
1.8 Optimal Contract Discussion	26
1.9 Delegated portfolio management	36
1.10 Conclusion	41
Appendices	43
1.A Assorted Remarks	43
1.B Selected important special cases	44
1.C Proofs of Section 1.6	47
1.D Principal's value function	56
2 Dynamic Agency, Delegated Portfolio Management and Asset Pricing	69
2.1 Introduction	71
2.2 Related Literature	72
2.3 Model setup	75
2.4 Financial markets equilibrium	83
2.5 Free rider problem	84
2.6 Contracting Problem	85
2.7 Retirement and Quitting	87
2.8 Investor's optimization problem	88
2.9 Optimal pie sharing rule contract	90

2.10	Asset Pricing Dynamics	92
2.11	Numerical Solutions	94
2.12	Conclusion	102
Appendices		103
2.A	Probabilistic setup	103
2.B	General Proofs	104
2.C	Proofs to investor's optimization problem	109
2.D	Markov Control Construction	113
2.E	Proofs to Asset Pricing Dynamics	115
2.F	Special Case: CRRA utility with $\gamma = 2$	117
2.G	Technical details to the numerical solution	119
2.H	Additional illustrations	120
3	Centralized versus Decentralized Delegated Portfolio Management under Moral Hazard	126
3.1	Introduction	128
3.2	Literature Review	134
3.3	Static Model Setup	135
3.4	First Best	141
3.5	Second Best	146
3.6	Dynamic Model	157
3.7	First Best in Dynamics	163
3.8	Second Best in Dynamics	165
3.9	Conclusion	175
Appendices		178
3.A	Proofs for Section 3.4	178
3.B	Additional Results and Proofs for Section 3.5	179
Appendices		186
3.A	Principle of Dynamic Programming	186
3.B	Dynamic Centralized Delegation	186
3.C	Dynamic Centralized Delegation in First Best	188
3.D	Dynamic Centralized Delegation in Second Best	195
3.E	Distribution restrictions in second best	196
3.F	Dependence Modeling and Copulas	197
3.G	Additional Results and Proofs for Section 3.7	202
3.H	Proofs for Section 3.H	210
Bibliography		215

Acknowledgments

I am deeply indebted to my committee members Robert M. Anderson, Gustavo Manso and Christine Parlour for their sincere and committed support during my doctoral studies. First and foremost, with my sincerest heart, I regard both Bob and Gustavo to be the co-chairs of my committee; it is unfortunate that due to some idiosyncrasies, this formality cannot happen. I thank Bob for the countless hours he had spent on guiding me through some of the most technically challenging aspects of my research ideas. Bob has also been very gracious in inviting me to numerous selective research seminars and conferences that broadened my intellectual horizon. I thank Gustavo for the countless hours he had spent on challenging the economic intuition and motivation for my research ideas. I thank Gustavo so much for encouraging me and giving me space to pursue ideas that perhaps take a little bit long to come to fruition. I thank Christine for teaching me that good finance papers are simple models with deep insights (rather than complex models with convoluted insights). It is truly a blessing that I had the good fortune to be advised by both Bob and Gustavo during my time here at Berkeley.

I also thank the finance faculty of the Haas School of Business for their endless patience and support. In particular, I am very grateful for Nicolae Garleanu on being tough, yet putting in full effort to work through my ideas. As well, I thank Brett Green and William Fuchs for their guidance during the earlier parts of my studies.

I thank for my family and friends for their endless support. I thank my mom and dad, Shui Kau Yu and Siu Fong Leung, for always being there and providing endless love and support for all of my life decisions. I thank my one and only Zeyu Li for being patiently there through my best of times and also the worst of times.

Chapter 1

Continuous-Time Principal-Agent Problem with Drift and Stochastic Volatility Control: With Applications to Delegated Portfolio Management

SEPTEMBER 21, 2014

Chapter Abstract

We study a continuous-time principal-agent problem where the agent can privately and meaningfully choose both the drift and volatility of a cash flow, while the principal only continuously observes the managed cash flows over time. Our model contributes a result that is hitherto relatively unexplored in both the continuous-time dynamic contracting and the delegated portfolio management literatures. Firstly, even though there is no direct moral hazard conflict between the principal and the agent on their preferred volatility choices, but to avoid inefficient termination and compensation from excess diffusion, this first best choice is not reached; this is the “reverse moral hazard” effect. Secondly, the dollar incentives the principal gives to the agent critically depends on the volatility choice, endogenous quasi-risk aversion of the principal, and the elasticity to the exogenous factor level; this is the “risk adjusted sensitivity” (RAS) effect. In a delegated portfolio management context, our model suggests outside investors should prefer funds such that: (i) the investment fund has an “internal fund” available only to management; (ii) the “external fund” for the outside investors closely tracks the value of the internal fund; and (iii) has dynamic incentive fee schemes, and these fees can be interpreted via Black-Scholes “greeks”.

Chapter Acknowledgements

I am deeply indebted to and grateful for the countless hours, encouragement, patience and comments of both of my advisers, Robert M. Anderson and Gustavo Manso, on this project

and other research endeavors. I also thank Xin Guo, Dmitry Livdan, Alexei Tchisty, and Johan Walden for very helpful discussions. I particularly thank Yuliy Sannikov for his excellent comments and questions. I thank David Feldman for his excellent comments and for pointing out some errors in an earlier draft. Comments and suggestions from the seminar attendees of the UC Berkeley Center for Risk Management Research, UC Berkeley-Haas Real Estate Student Seminar, UC Berkeley-Haas Finance Student Seminar, Berkeley-Stanford Joint Finance Student Seminar Spring 2014, LBS Trans-Atlantic Doctoral Conference 2014, EconCon 2014 conference at Princeton University, and the 26th Annual Northern Finance Association PhD Program in Ottawa are gratefully appreciated. All remaining errors are mine and mine alone.

1.1 Introduction

As of 2009, \$71.3 trillion is invested into managed portfolios worldwide ¹, and the vast majority of these managed portfolios are under active management. Despite the prevalence and importance of delegated portfolio management in the modern capital markets, surprisingly little is known about its optimal contracting characteristics in a dynamic environment. The difficulty of approaching these problems is succinctly captured by a remark in Cuoco and Kaniel (2011):

“A distinctive feature of the agency problem arising from portfolio management is that the agent’s actions (the investment strategy and possibly the effort spent acquiring information about securities’ returns) affect both the drift and volatility of the relevant state variable (the value of the managed portfolio), although realistically the drift and the volatility cannot be chosen independently. This makes the problem significantly more complex than the one considered in the classic paper by Holmström and Milgrom (1987) and its extensions. With a couple of exceptions, as noted by Stracca (2006) in his recent survey of the literature on delegated portfolio management, ‘the literature has reached more negative rather than constructive results, and the search for an optimal contract has proved to be inconclusive even in the most simple settings’ ”.

To emphasize the point, the ability to influence the volatility of a managed cash flow is critical in a delegated portfolio management context. Indeed, numerous papers have recognized that risk shifting ² behavior of the portfolio manager as an important source of moral hazard that is typically not present in traditional principal-agent contexts, such as employer-employee and landlord-tenant relationships.

Here, we present a continuous-time principal-agent model that represents both a first step in the literature in dynamic contracting theory whereby the agent can explicitly and meaningfully privately choose volatility, and equally important, also as a first step into understanding dynamic contracting environment in the context of delegated portfolio management. We consider a dynamic contracting environment in continuous-time with a risk-neutral agent and a risk-neutral principal, whereby the agent can privately choose effort and volatility levels that affect both the mean and overall risk of the cash flows. The principal can continuously observe the cash flows, but not the hidden choices of effort and volatility that the agent chooses. In line with the literature, the agent enjoys a private benefit from exerting low levels of effort (“job shirking”), and we also further assume that the agent enjoys a private benefit from choosing high level volatility control (“lazy quality management”). The expected payoff of the managed cash flows (i.e. the drift) is a “reward function” of both the effort and volatility chosen by the agent. This is to roughly capture the classical “risk-reward trade off” intuition

¹ Wermers (2011)

² See Stoughton (1993) and Admati and Pfleiderer (1997); see also Stracca (2006) for a summary on how the delegated portfolio management problem presents unique challenges that are not present in standard principal-agent problems.

of financial economics, particularly in portfolio choice theory. Here, effort is a binary choice but volatility is chosen from a closed interval. In a certain sense, in the context of delegated portfolio management, we can view the agent as continuously managing a portfolio whose per-period return follows a classical single factor model, whereby the overall expected return is dependent on effort (i.e. “alpha”) and volatility choice (i.e. “beta”) of a priced systematic factor that is *unobservable* to the principal off-equilibrium. See Section 1.3 for details. The problem of drift only control, to various degrees of sophistication, has been extensively studied in recent years (see Section 1.2 for a literature review).

However, to the best knowledge of this author, continuous volatility control when the principal can continuously observe the cash flows has been given little to no attention in the models commonly used in the literature, and for good reason. Specifically, if the principal can continuously observe the cash flows and the agent directly controls the volatility term of the cash flow, the principal can compute the quadratic variation of the cash flow process, and thereby infer directly the choice of volatility that the agent has been using. Hence, if indeed the agent has deviated from the principal’s prescribed volatility level, the principal could in effect apply a “grim trigger” strategy and punish the agent indefinitely thereon. By using this argument, the agent will never have any incentive to deviate from the principal’s desired and prescribed volatility level, and thus, volatility control by the agent can essentially be abstracted away. Yet economically, the absence of meaningful volatility control by the agent is very unsatisfying. There are several important situations where allowing the agent to influence the volatility of the cash flows is economically significant; for example, the classical considerations of asset substitution in corporate finance ³ and risk shifting in delegated portfolio management. Thus to have any meaningful volatility control by the agent, a richer model of the agent managed cash flows is required.

The key ingredient of our model is to introduce an exogenous *factor level* component to the overall diffusion term of the managed cash flows. In particular, we will allow the instantaneous diffusion of the cash flows be a product of an exogenous factor level process that is completely not managed by the agent, and a component that is directly controlled by the agent. The agent can observe this exogenous factor term, and off equilibrium, this exogenous factor level is unobservable to the principal, even though at equilibrium becomes observable to the principal. Thus when the principal computes the quadratic variation of this cash flow process, the principal can at best observe the overall product of an unobservable exogenous factor level and a controlled volatility term, but not the two components separately. So economically, even if the principal observes high instantaneous cash flow volatility through the computation of the quadratic variation, the principal cannot disentangle whether this high volatility is due to a high realization of the exogenous factor level, high volatility control by the agent, or both. Clearly, the principal should only punish or reward the agent for the endogenous volatility control by the agent and not for the exogenous component. As well, in line with the models of drift-only control by the agent, the principal must put the agent at risk to induce the agent to work according to the principal’s desired and prescribed plan.

³Say Leland (1998), among many others.

However, given that the agent can choose the volatility of the cash flows, the agent can effectively undo or weaken some of the risks that the principal imposes on him. Thus, the incentives involved in a model with combined drift and volatility control are, perhaps understandably, considerably more difficult than a case with only drift control.

Let us focus the discussion of our model to the concrete important application of delegated portfolio management. The risk-neutral portfolio manager (i.e. agent) is assumed to have access a priced systematic factor that is unobservable to the risk-neutral outside investors (i.e. principal) and also exogenous to the manager's control. For instance, a "global macro" hedge fund manager can have access to this "global macro" priced systematic factor, but due to various institutional frictions, say trading costs or information asymmetry, even if the investors recognize the presence of this "global macro" factor that prices assets and has a positive risk premium, the investors cannot directly trade on it nor can they directly observe it. Furthermore, the manager needs to: (i) privately choose the factor loading (i.e. the volatility choice) onto this priced systematic factor; and (ii) privately choose the level of effort to realize his skills to deliver excess returns. The manager's factor loading choice is private because broadly speaking, the factor loading is the investment strategy of the manager and it is not in the incentive of the manager to completely reveal his proprietary investment positions. Moreover, in our model, the manager enjoys a private benefit from choosing a high factor loading level, which can be broadly thought of as "lazy quality management"; that is, the manager could save on actively managing the portfolio to achieve low systematic risk by not engaging into active hedging or use of derivative contracts. We look for an equilibrium where the investor wants to induce the manager to always exert high effort, and to endogenously choose the optimal factor loading.

Next, we discuss economically the form of the optimal contract. Our model suggests the manager should manage an "internal fund" that is only available to management, while offering an "external fund" only available to the outside investors. In particular, the external fund is viewed as a financial derivative contract, with the "underlying assets" being the internal fund and the exogenous systematic factor. It is worth emphasizing here that, off equilibrium, the investor can neither observe the value of this exogenous systematic factor nor the manager's private choice of the factor loading. But in equilibrium, only after the investors provide sufficient incentives to the manager, will the manager report this factor level to the investors, along with the value of the internal fund at all points in time. That is to say, endogenously and when in equilibrium, the manager will report to the investors the appropriate factor index to benchmark his internal fund's performance. The investors will terminate the manager when the value of the internal fund is sufficiently low, and/or when the level of the systematic factor is sufficiently low; as when either situation happens, the value of the external fund becomes too low and the investor will find it optimal to terminate the agent and obtain the fund's (inefficient) liquidation value, and the manager will obtain a retirement value. But if the value of the internal fund is sufficiently high, that is when it hits a performance benchmark that is dependent on the level of the exogenous systematic factor, a performance bonus is paid from the investors to the managers.

Let's discuss the determinants of the value of the internal fund. The investors need to

dynamically incentivize and impose sufficient amount of risk to the manager’s welfare to ensure the manager will dynamically make those latent choices that the investors desire. However, since the manager can directly alter the level of uncertainty in this managed portfolio through the manager’s choice of the factor loading on the systematic factor, the manager effectively can partially undo or weaken the amount of risk the investor imposes on him. Hence, this is why the amount of risk the investors subject the manager to must be in the form of a dynamic performance fee (i.e. *risk adjusted sensitivity (RAS)*); this dynamic fee is precisely the proportion of the managed returns that the manager is entitled to through the internal fund. So, the value of the internal fund equals to this proportional dynamic performance fee, multiplied by the excess returns of the managed portfolio, plus the risk free rate, less the manager’s private cost of implementing the principal’s desired investment strategy.

Thus, the key economic driver of the value of the internal fund is this dynamic performance fee and we now describe its composition. This dynamic performance fee depends on three components: (i) the factor loading onto the systematic factor chosen by the manager; (ii) the investors’ “endogenous” risk tolerance to the value of the internal fund; and (iii) the investors’ elasticity to the systematic factor. All three of these components effectively relate to the overall risk of the external fund. Component (i) is the direct effect, that the systematic factor loading chosen by the manager directly affects the riskiness of the returns received by the investors. The remaining two components are indirect effects. Component (ii) effectively captures how the value of the external fund can change depending on the value of the internal fund. In financial derivatives parlance, we can view component (ii) as the “Black-Scholes Gamma” of the value of the external fund with respect to the value of the internal fund. But given that the investors here are risk-neutral, this can also be interpreted as “endogenous quasi risk tolerance”. Component (iii) effectively captures the remaining other driver that affects investors’ returns in this economy, that being how sensitive is the value of the external fund to the systematic factor level. But the investors must also furthermore take into the account of the effect of the systematic factor level on the internal fund in providing dynamic incentives. Hence, we can view component (iii) as the “Black-Scholes Delta-of-Delta”; that is, we first consider the BS-Delta of the value of the external fund with respect to the value of the internal fund, and then subsequently consider the BS-Delta of that with respect to the level of the systematic factor.

Finally, it remains to consider the investment strategy that is chosen in equilibrium. Given that the principal is risk-neutral, in the first-best equilibrium, he would actually prefer the highest possible loading onto the systematic factor. And furthermore, given the manager’s “lazy quality management” private benefits, the manager also privately prefers the highest possible loading onto the systematic risk factor. Thus, it may appear that there is no moral hazard conflict on the factor loading choice, that being the highest possible one, since it is mutually preferred by both the manager and the investor in first best. However, the first best outcome is not implementable in equilibrium. Here, termination of the manager is not desirable since the liquidation value of the fund is inefficient. This roughly captures that without the presence of the skilled manager, the outside capital markets are only willing

to pay a low fair price for the assets of this portfolio. But if the investors were to recommend the manager to always take on the highest possible systematic factor loading, this raises the risk of the managed portfolio to its highest level, it increases the risk of the internal fund, and so making it more likely to hit the retirement value of the manager, and thus terminating the contract inefficiently. Thus, in equilibrium, it is not necessarily optimal for the investors to recommend and implement the first best highest factor loading choice. This is precisely the *reverse moral hazard* effect. Indeed, in equilibrium, the optimal choice of the systematic factor loading is given by a trade off of: (a) the direct private benefits to the manager of this choice; (b) direct payoffs to the investor from this choice; and (c) incentives due to the manager, adjusting for the dynamic performance fee. Component (b) is the direct effect to the investors, which are the additional returns that are delivered by choosing a higher loading on the systematic factor. Component (a) is a direct effect to the manager, but this direct effect is adjusted by the value of the internal fund and the value of the external fund. That is to say, component (a) equals to the manager’s marginal private benefit of selecting a higher systematic loading, multiplied by the “Black-Scholes Delta” of the value of the external fund with respect to the value of the internal fund. Finally, component (c) is the risk adjustment due to the manager through the dynamic performance fee that we had described earlier.

Finally, we note the resemblance between our proposed optimal contract implementation in the context of delegated portfolio management to current market practices. For instance, numerous banks (at least prior to the Volcker Rule) also run proprietary trading desks, which are effectively internal funds that are only available to employees and management. Several hedge funds also engage into this practice. For example, the hedge fund firm Renaissance Technologies runs three funds that are open to outside investors, but also run a separate fund, the Medallion Fund, that is only open to its employees (see Zuckerman (2013)). Darolles and Gourieroux (2014a) and Darolles and Gourieroux (2014b) also discuss at length the practice of this internal fund in the hedge fund industry.

1.2 Related Literature

This paper contributes to: (i) a growing literature on continuous-time principal-agent problems; and (ii) continuous-time delegated portfolio management problems.

One of the first papers that considered a continuous-time principal agent problem is Holmström and Milgrom (1987). Recent papers in the continuous-time principal-agent problem include DeMarzo and Sannikov (2006) (of which DeMarzo and Fishman (2007) is the discrete-time counterpart), Biais, Mariotti, Plantin, and Rochet (2007), Sannikov (2008), He (2009), Adrian and Westerfield (2009), Hoffmann and Pfeil (2010), Grochulski and Zhang (2011), He (2011), Williams (2011), DeMarzo, Fishman, He, and Wang (2012), He (2012), Szydlowski (2012), Miao and Zhang (2013), Miao and Rivera (2013), Zhu (2013), DeMarzo, Livdan, and Tchisty (2013), Giat and Subramanian (2013) and Hoffmann and Pfeil (2013). We note that Biais, Mariotti, and Rochet (2011), Sannikov (2012a), and Sannikov (2013) all give an excellent survey and overview of the current state in this literature. Please see

Table 1.B.1 in Section 1.B for a selected survey of the models used in the literature; note that even though the table enumerates the agent’s managed cash flow form, these papers often have very different assumptions on the preferences of the agent and the principal, and some also have different assumptions of the timing in which the principal can observe the cash flows.

All the aforementioned papers allow the agent to manage a cash flow in the form of a stochastic differential equation, of various levels of complexity, but the common setup is that the agent can only exert effort to influence the drift of the cash flow but not its volatility. But in these sorts of papers, the volatility parameter is held constant, known both to the principal and the agent. This is without loss of generality in the case when the noise term of the cash flow is driven exclusively by Brownian motion. DeMarzo, Livdan, and Tchisty (2013) is an interesting example whereby the cash flows have a jump component and the agent can influence the jump, but nonetheless, the agent still does not (and cannot meaningfully) influence the volatility. To our best knowledge, there are some notable exceptions and we will describe these below in Section 1.B but regardless, none of them allow for meaningful volatility control as in our context.

The setup of this model lends itself naturally to delegated portfolio management problems. As emphasized by Stoughton (1993) and Admati and Pfleiderer (1997), and summarized in Stracca (2006), delegated portfolio management problems present challenges that are not commonly considered in standard principal-agent problems; in particular, the portfolio manager has the ability to influence both the expected return and also volatility of the managed returns or cash flows. While managing expected return part, usually modeled as moral hazard hidden effort selection, is common in standard principal-agent problems, managing volatility is not. Ou-Yang (2003) is one of the key models in the delegated portfolio management literature but modeled in continuous time and we will further discuss this case in Section 1.B.

The problem of “risk shifting”, namely changing volatility of the managed cash flows, is well recognized as a key moral hazard component in the delegated portfolio management literature. Basak, Pavlova, and Shapiro (2007) considers a portfolio manager’s risk taking incentives induced by an increasing and convex relationship of fund flows to relative performance, and how this objective could give rise to risk-shifting incentives. However, it should be noted that the contract in Basak et al. (2007) is exogenously given and there is no explicit principal-agent modeling. Other papers that investigate into risk shifting behavior by portfolio managers include: Chevalier and Ellison (1997), Rauh (2008), Giambona and Golec (2009), Hodder and Jackwerth (2009), Foster and Young (2010) and Huang et al. (2011). Stracca (2006) and Ang (2012) offer excellent recent surveys on the literature in delegated portfolio management.

1.3 Model Motivation in Discrete-Time

To motivate the continuous-time model of the paper, and also to highlight how existing drift-only control principal-agent models in the literature cannot be used to appropriately model delegation portfolio management problems, we first draw an analogy to a simple two period discrete-time asset pricing factor model. Suppose in the current period, a group of investors hire a portfolio manager to manage a portfolio that will deliver excess returns $R - r_f$ next period that is observable to the investor, where r_f is the risk free return. The investors know a priori that the manager has skill so ex-ante the investors are willing to invest into the portfolio manager. What is not known to the investors is whether the manager is exerting sufficient effort to maximize his skills to deliver positive excess returns.⁴

Transparent investment funds

Suppose further the fund operates like a *mutual fund*, so that the investment holdings of the fund are effectively transparent, and so the investors know and can observe what is the appropriate market factor, say $R_M - r_f$, to price the portfolio, and moreover, the investors can precisely choose their desired factor loading for themselves.⁵ And suppose the managed portfolio returns are driven by a factor form,

$$R - r_f = \alpha(e) + \beta_M^0 \times (R_M - r_f) + \varepsilon, \quad (1.3.1)$$

where $\alpha(e)$ represents the excess returns of the managed portfolio that is increasing in the level of effort e that the manager exerts, β_M^0 is the factor loading onto the factor $R_M - r_f$, and ε is the idiosyncratic risk with zero mean, and independent of $R_M - r_f$ and effort e .⁶ Since β_M^0 had been a priori selected by the principal, there is no need to condition on this anymore. Thus, conditioning on the effort e the manager will exert, the investors' expected returns from this managed portfolio is,

$$\mathbb{E}[R - r_f \mid e] = \alpha(e) + \beta_M^0 \mathbb{E}[R_M - r_f], \quad (1.3.2)$$

⁴ Throughout the paper, we will interchange the terms “principal” with “investor”, and “agent” with “manager”, especially when we discuss the implications in a delegated portfolio management setting. It is hoped that no confusion should arise.

⁵ Indeed, commercial services like Morningstar regularly report the appropriate investment style or factor (i.e. “value”, “growth”, “big cap”, etc.) of the majority of mutual funds available, and they also report a CAPM beta value to the investors. Furthermore, by regulation, mutual funds are required to periodically disclose their exact holdings.

⁶ Another way to view (1.3.1) is to view the mutual funds types are indexed by their factor loading $\{\beta_{M,j}^0\}_j$. But there the investors can perfectly see the type at $t = 0$, and so according to their preferences, select their desired type, say β_M^0 . Once this type has been selected, the investors then proceed to construct contracts to motivate the managers to exert high effort to maximize their skills. But it should be noted that this type selection argument is only valid because the investors know precisely the appropriate market factor is $R_M - r_f$, and hence can compute for themselves the expected risk premium $\mathbb{E}[R_M - r_f]$. If this were unknown, then this argument does not hold.

where we assumed that effort choice e is independent of the market risk factor R_M . Thus, we see that the expected returns, conditional on effort e is increasing in effort e . Note here that since the investors know the appropriate risk factor $R_M - r_f$, he can also compute the expected risk premium $\mathbb{E}[R_M - r_f]$. That is, the entire term $\beta_M^0 \mathbb{E}[R_M - r_f]$ and its two individual multiplicative components, are common knowledge to both the manager and the investor. Referring to Table 1.B.1 of Section 1.B, the prototypical model in the existing continuous-time principal-agent literature takes the form,

$$dY_t = \mu_t dt + \sigma dB_t, \tag{1.3.3}$$

where μ_t is a choice that the agent can privately select, and the models in the literature specify μ_t to various degrees of sophistication; see also Section 1.B for further discussion. Mapping (1.3.3) to the asset pricing model in (1.3.1), existing drift-only control models can effectively be viewed as,⁷

$$\begin{aligned} dY_t &\approx R - r_f - \beta_M^0 \mathbb{E}[R_M - r_f], \\ \mu_t dt &\approx \mathbb{E}[R - r_f | e] = \alpha(e) + \beta_M^0 \mathbb{E}[R_M - r_f], \\ \sigma dB_t &\approx \beta_M^0 (R_M - r_f) - \beta_M^0 \mathbb{E}[R_M - r_f] + \varepsilon. \end{aligned} \tag{1.3.4}$$

In particular, we note that for the noise term $\sigma dB_t \approx \beta_M^0 (R_M - r_f) - \beta_M^0 \mathbb{E}[R_M - r_f] + \varepsilon$, as mentioned earlier, the risk loading β_M^0 is a priori known to the investor, the fact that there is positive risk premium $\mathbb{E}[R_M - r_f] > 0$ associated with this factor is also known, and this is a pure noise term that does not convey any other information to the investor.

“Black box” investment funds

While the viewpoint (1.3.1), and by extension the continuous-time formulation (1.3.3) with no volatility control, may be plausible for, say, mutual funds that have fairly transparent investment procedures, this is not the case for numerous other delegated portfolio management practices in the market. Most notably, *hedge funds* and *private equity funds*, unlike mutual funds, are not subject to regulation to reveal their investment positions or trading strategies. And indeed, the investment strategies and positions of these funds are precisely their “secret sauce” or “black box”, of which they are very protective of its details. As such, unlike (1.3.1), a far more appropriate model here is the form,

$$R - r_f = \alpha(e) + \beta_Z \times (R_Z - r_f) + \varepsilon, \tag{1.3.5}$$

where $R_Z - r_f$ is the excess return of an exogenously priced factor that is observable to the manager, but *unobservable* to the investor, and β_Z is the factor loading the manager

⁷ Note that we “demean” by $\beta_M^0 \mathbb{E}[R_M - r_f]$ primarily to balance the third equation of (1.3.4). That is in that third equation, since σdB_t has mean zero, if we did not demean, the right hand side has non-zero mean as we have positive risk premia. But if we were to demean as written, then both the left-hand side and right-hand side balance.

can privately and endogenously control. Concretely speaking, if a hedge fund manager claims that it is a “global macro fund” but does not disclose its actual positions and trading strategies, there is no way the investor can infer what is the appropriate risk factor to benchmark the fund at. And indeed, even looking at other peer “global macro funds” only give at best a noisy proxy to what the fund in question is actually doing. In particular, that means that unlike the case of mutual funds as per (1.3.1), the investor cannot ex-ante view and select the factor loading of the fund. Thus, conditional on the effort e and the factor loading β_Z as chosen by the manager, the expected returns of the managed portfolio are,

$$\mathbb{E}[R - r_f \mid e, \beta_Z] = \alpha(e) + \beta_Z \mathbb{E}[R_Z - r_f], \quad (1.3.6)$$

where the effort e and factor loading β_Z are independent of the risk factor $R_Z - r_f$. Note that unlike (1.3.2), the last term $\beta_Z \mathbb{E}[R_Z - r_f]$ is *not* common knowledge to both the investor and the manager. Indeed, even if the investor knew that there is a positive risk premium $\mathbb{E}[R_Z - r_f] > 0$ associated with the risk factor $R_Z - r_f$, the investor still does not know which factor loading β_Z the manager chose, nor necessarily how high that risk premium $\mathbb{E}[R_Z - r_f]$ is.

In all, this means to have a principal-agent model that represents the practices of hedge funds, private equity firms, and other “secret sauce” investment funds, we need at least two additional ingredients, on top of the skill term $\alpha(e)$: (i) exogenous factor term $R_Z - r_f$ observable to the manager but unobservable to the investor; and (ii) endogenous factor loading term β_Z that can be privately controlled by the manager. This represents the starting point of our continuous-time model, and we will also continue our discussion of this discrete-time motivation in Section 1.4.

Remark 1.3.1. *There’s a slight caveat that prevents a complete mapping from the two-period discrete time model to the fully dynamic continuous time model. In particular, in a two-period discrete time model, there is no economic need to discern between “returns” and “portfolio values”. That is, we may think of the final period returns in the two-period model as simply the value of the portfolio when its initial portfolio investment is one dollar. And if the initial portfolio investment was not one dollar, we simply just scale up or down. In the actual dynamic continuous-time model, the principal’s payoff will have the form $\int e^{-rt} dY_t$. Thus, if we use the exact mapping of (1.3.4), and interpret dY_t as per period excess returns, it would seem like the principal wants to motivate the agent to maximize per period excess returns, which is at odds with general economic intuition and the large asset pricing literature in portfolio-consumption choice. Hence, in the application of delegated portfolio management, we will interpret Y_t as the value of a managed portfolio at time t , and dY_t as the instantaneous changes of the value of the portfolio at time t . Implicit however, in this interpretation, is that the principal gave a fixed dollar amount at time $t = 0$ for the agent to manage, and there are no capital withdrawal or injection into the portfolio. And hence, this is why simply having the principal “myopically” motivate the agent to maximize expected excess returns is an appropriate interpretation. In general, as far as we are aware, there are yet any full fledged dynamic portfolio-consumption choice models with agency in the literature.*

1.4 Model Outline

Let (Ω, \mathcal{F}, P) be a complete probability space and let $\{B_t\}_{t \geq 0}$ be a standard Brownian motion on this probability space and let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by this Brownian motion, suitably augmented. We will write E as the expectation operator under probability measure P . The agent can choose an *action process* $A = \{(e_t, \sigma_t)\}_{t \geq 0}$, where for all times $t \geq 0$, $(e_t, \sigma_t) \in \{e_L, e_H\} \times [\sigma_L, \sigma_H]$, where $e_H > e_L > 0$ and $\sigma_H > \sigma_L > 0$. We will call $\{e_t\}_{t \geq 0}$ the *effort control (process)* and $\{\sigma_t\}_{t \geq 0}$ *volatility control (process)*.^{8 9} Consider a function $\kappa : \{e_L, e_H\} \times [\sigma_L, \sigma_H] \rightarrow [\mu_L, \mu_H]$ that maps both the effort and volatility chosen by the agent to the (expected) return of the (cumulative) *cash flow process* Y ; that is, consider $(e, \sigma) \mapsto \kappa(e, \sigma) = \mu$. We will call κ as the *reward function* and we will discuss further on the assumptions and properties of this function in Section 1.4 below. The cash flow process Y has dynamics that depend on the agent's action process¹⁰,

$$dY_t = \kappa(e_t, \sigma_t)dt + \sigma_t dM_t, \quad Y_0 = y_0 \quad (1.4.1)$$

$$dM_t = M_t dB_t, \quad M_0 = m_0, \quad (1.4.2)$$

where $m_0 > 0$ and note we have denoted $\mu_t := \kappa(e_t, \sigma_t)$. Given an action process A , we will call $\{\mu_t\}_{t \geq 0} = \{\kappa(e_t, \sigma_t)\}_{t \geq 0}$ the *drift (process)* of the cash flow Y ¹¹. The principal cannot observe the agent's action process but can only observe the cash flow Y . The agent can also observe the cash flow Y . Let $\{\mathcal{F}_t^Y\}_{t \geq 0}$ be the (suitably augmented) filtration generated by the cash flow process Y , which represents the principal's information set. The extra term M_t in (1.4.1) and its dynamics (1.4.2) is different from the prevailing literature (see Table 1.B.1). We shall call $\{M_t\}_{t \geq 0}$ as the *exogenous factor*.

Finally, for illustrative purposes only, Figure 1.1 plots this cash flow process against some other cash flow processes that have been used in the literature.

Both the principal and the agent are risk neutral. The principal discounts time at rate $r_1 > 0$ and the agent discounts time at rate $r_0 > 0$. As per DeMarzo and Sannikov (2006), we assume that the agent is less patient than investors; so we assume $r_0 > r_1$. The principal needs to compensate the agent and is modeled via the $\{\mathcal{F}_t^Y\}_{t \geq 0}$ -adapted stochastic process $X = \{X_t\}_{t \geq 0}$ and assuming limited liability, we restrict the compensations to be non-negative, so $dX_t \geq 0$. The principal also has the ability to terminate the agent at some

⁸ Of course, strictly speaking in the usual language of stochastic differential equations, we would call $\sigma_t m_t$ as the (stochastic) volatility of Y . However, since in this setup, m_t is an exogenous process, and only σ_t is being directly controlled by the agent, it would be more natural to think and call σ_t as *volatility* that is being managed by the agent.

⁹ Throughout this article, we will use e to denote the effort parameter / process, and use e to denote the exponential function.

¹⁰ Specifically, the action process A is progressively measurable with respect to $\{\mathcal{F}_t\}$.

¹¹ Throughout the paper, we will interchange the notation μ_t and $\kappa(e_t, \sigma_t)$ to denote the drift part of the cash flow process. This is for notational brevity. Thus, given an action process $A = \{(e_t, \sigma_t)\}_{t \geq 0}$, we will also with some abuse of notation, also call and denote $A = \{(\mu_t, \sigma_t)\}_{t \geq 0}$ as the action process, with the understanding that $\mu_t \equiv \kappa(e_t, \sigma_t)$.

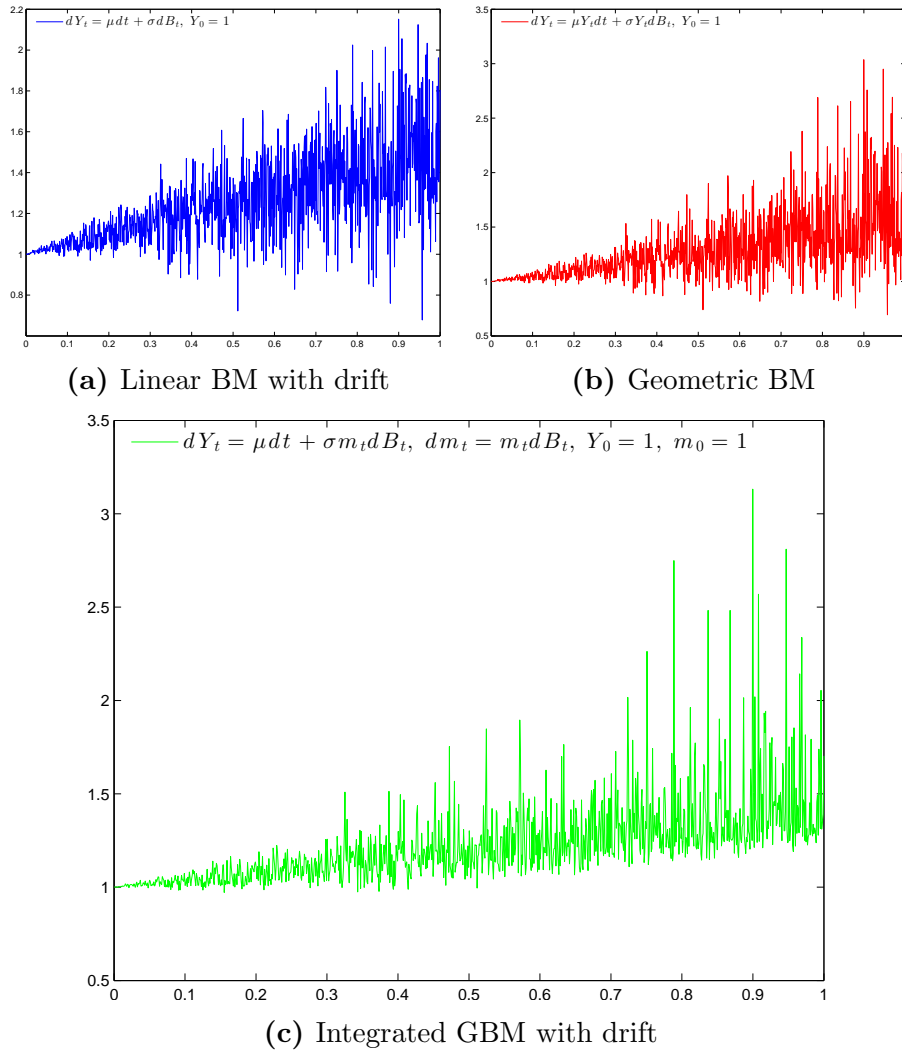


Figure 1.1: Illustrations of various types of commonly used cash flow processes. Here, the (constant) parameters are chosen to be $\mu = 0.5, \sigma = 0.3, Y_0 = 1, m_0 = 1$, and we simulate over 1000 discrete evenly spaced points over the time interval $t \in [0, 1]$. Subfigure (1.1a) describes the *linear Brownian motion with drift* cash flow process, $dY_t = \mu dt + \sigma dB_t$, that is used in the models by Holmström and Milgrom (1987), DeMarzo and Sannikov (2006), Sannikov (2008) and several others; see Table 1.B.1. Subfigure 1.1b describes the *geometric Brownian motion* process, $dY_t = \mu Y_t dt + \sigma Y_t dB_t$, that is used by He (2009) (note, He (2009) calls this the *firm value process*). Finally, subfigure (1.1c) describes an *integrated Geometric Brownian motion with drift*, $dY_t = \mu dt + \sigma dM_t, dM_t = M_t dB_t$, as it will be used in this paper. Note also the parameters used in generating this figure are for illustrative purposes only. Unless specified otherwise, these parameters are *not* enforced throughout the paper.

\mathcal{F}_t^Y -measurable random time $\tau \in [0, \infty]$. Upon termination, the firm is liquidated for value¹² $L > 0$ and the agent receives retirement value¹³ $R > 0$. A *contract* is the tuple (A, X, τ) , which specifies the recommended action process A , a compensation for the agent X and the termination time τ .

Fix a contract (A, X, τ) and suppose the agent follows the principal's recommended action A . The *agent's payoff* at time $t = 0$,

$$W_0(A) := \mathbb{E}^A \left[\int_0^\tau e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) + e^{-r_0 \tau} R \right], \quad (1.4.3)$$

where $\phi_e, \phi_\sigma > 0$ are constants known to both the principal and the agent. Here, we denote \mathbb{E}^A as the expectation under the probability measure \mathbb{P}^A induced by the agent's chosen action process A . We will further discuss the properties of the agent's payoffs and incentives in Section 1.4. Likewise, the *principal's payoff* at time $t = 0$ is,¹⁴

$$\mathbb{E}^A \left[\int_0^\tau e^{-r_1 t} (dY_t - dX_t) + e^{-r_1 \tau} L \right] = \mathbb{E}^A \left[\int_0^\tau e^{-r_1 t} \kappa(e_t, \sigma_t) dt - \int_0^\tau e^{-r_1 t} dX_t + e^{-r_1 \tau} L \right]. \quad (1.4.4)$$

Further discussion of the conflict between the agent and principal is deferred until Section 1.4. We collect some assorted remarks, largely technical in nature, about the model in Section 1.A.

Mapping back to the discrete-time model

Mapping back to the discrete-time specification in Section 1.3, and in particular to (1.3.5), we can map the terms analogously as,

$$\begin{aligned} dY_t &\approx R - r_f - \mathbb{E}[\beta_Z(R_Z - r_f)], \\ \kappa(e_t, \sigma_t) dt &\approx \mathbb{E}[R - r_f \mid e, \beta_Z] = \alpha(e) + \beta_Z \mathbb{E}[R_Z - r_f], \\ \sigma_t dM_t &\approx \beta_Z(R_Z - r_f) - \mathbb{E}[\beta_Z(R_Z - r_f)] + \varepsilon. \end{aligned} \quad (1.4.5)$$

As discussed, the specification (1.3.5) and now to the continuous-time extension of (1.4.1), (1.4.2), can be viewed as a more appropriate model of delegated portfolio management than existing drift-only control models in the literature. Specifically, observe that the diffusion term $\sigma_t dM_t \approx \beta_Z(R_Z - r_f) - \mathbb{E}[\beta_Z(R_Z - r_f)] + \varepsilon$ does indeed contain information that the investor does not know, unlike that of (1.3.4). In particular, the investor can at best only

¹² We will be more specific about this liquidation value L in Section 1.5.

¹³ We will be more specific regarding this retirement value R in Section 1.6.

¹⁴ In the second equality, we applied Doob's Optional Stopping Theorem. While τ could have been unbounded (i.e. never terminating the agent) but a standard argument using bounded sequences of stopping times and an usual application of the Dominated Convergence Theorem will also show the result. We will omit these details.

form an expectation $\mathbb{E}[\beta_Z(R_Z - r_f)]$ of the product of the systematic factor loading β_Z , interacted with the risk premium $R_Z - r_f$, but cannot completely disentangle the two in expectation. But this implies that it allows an opportunity for the manager to privately deviate from the investor preferred choice of the systematic factor loading β_Z .

Remark 1.4.1. *Although slightly beyond the original motivation scope of the paper, it should be noted that in modeling the managed cash flows dY_t , we have that the expected value $\kappa(e_t, \sigma_t)dt$ is dependent on both effort and volatility, and this modeling form has found precedence in the recent empirical and theoretical asset pricing literature. In Buraschi, Kosowski, and Srivastava (2013), the authors note:*

“...[The traditional alpha measure that is independent of beta] raises the question of how well a reduced-form alpha measures the true managerial skill of a hedge fund manager. An answer to this question depends on the determinants of the optimal allocation θ_{At}^ made by that manager. If the optimal allocation is constant and determined exclusively by the risk and return characteristics of the investment opportunity set (as in a traditional Merton model without agency distortions), then reduced-form alpha is an unbiased estimate of managerial skill. However, if the optimal allocation is influenced by nonlinear agency contracts, then reduced-form alpha is a misspecified estimate of true skill. For instance, a high reduced-form alpha could be the fortunate result of too much leverage as managers aim to maximize their incentive options. Of course, high leverage increases not only the manager’s expected return (because of the call option) but also the likelihood of large negative returns.”*

Thus, Buraschi et al. (2013) suggests that the managed portfolio alpha, under the influence of “nonlinear agency contracts”, could depend on leverage and investment opportunities.

In addition, although absent of any agency considerations, Frazzini and Pedersen (2014) also considers an overlapping generations model that implies a factor model structure for risky asset returns, and show that the alpha term could depend explicitly on the factor loading. Also in empirical research of hedge fund performance, Bollen and Whaley (2009) also notes:

“Accurate appraisal of hedge fund performance must recognize the freedom with which managers shift asset classes, strategies, and leverage in response to changing market conditions and arbitrage opportunities. The standard measure of performance is the abnormal return defined by a hedge fund’s exposure to risk factors. If exposures are assumed constant when, in fact, they vary through time, estimated abnormal returns may be incorrect.”

Reward Function κ

We need to be more specific about the way the agent can control the drift and volatility of the cash flows. It should be noted in drift-control only models, the specification of the drift

is usually quite simple (i.e. linear). But in our case, given the volatility control, we must be more careful in modeling and giving economic meaning to link the volatility and drift controls. Note that one possible characterization is to have the agent control drifts and volatilities that are completely unrelated to each other. But this case is not economically meaningful since it destroys the traditional link of risk-return trade-offs of financial economics, particularly that of portfolio choice theory. We will now more specifically define the reward function κ as follows.

Definition 1.4.1. A strictly positive real valued function $\kappa : \{e_L, e_H\} \times [\sigma_L, \sigma_H] \rightarrow [\mu_L, \mu_H]$, $(e, \sigma) \mapsto \kappa(e, \sigma) = \mu$, that is twice-continuously differentiable in the second argument, is called a *reward (drift) function* if it that has the following properties:

- (a) *Higher effort, higher reward:* $\kappa(e_H, \sigma) > \kappa(e_L, \sigma)$, for all σ ;
- (b) *Higher risk, higher reward but at decreasing rate:* $\kappa_\sigma(e, \sigma) > 0$ and $\kappa_{\sigma\sigma}(e, \sigma) < 0$, for all (e, σ) ;
- (c) *Risk cannot substitute for effort:* $\kappa(e_H, \sigma) > \kappa(e_L, \sigma')$ for all σ, σ' .

The requirement (a) is natural to interpret; that is, if the agent exerts higher effort to running the project, then the expected payoff should be higher, regardless of the choice of volatility. Requirement (b) is the traditional risk-reward type trade off. One would expect that by choosing a riskier project (higher volatility) over a safer project (lower volatility), it is so that one could enjoy higher expected returns, but we impose that the rate of return from increasing risk is decreasing. Thus, requirements (a) and (b) should have good natural interpretations. Requirement (c) means that exerting high effort always gives a higher return, regardless of the level of risk taken. Effectively, that means that effort and risk are not “substitute goods”; hence, this requirement explicitly rules out a case where the agent can exert low effort and take on a high level of risk such that this return is equal or greater to one with high effort and any level of risk. In the delegated portfolio management context, this means high managerial skills always deliver better returns any form of “financial engineering”. Note that clearly (c) implies (a) but we write them out separately as (a) is effectively the only requirement assumed in the controlled drift-only models (i.e. when κ is a function only of effort e). More generically, early studies between project selection (viewed as volatility in the current context), risk and effort can be found in Lambert (1986) and Hirshleifer and Suh (1992).

We now given an example that satisfies Definition 1.4.1 and hence directly showing that the set of reward functions is nonempty.

Example 1.4.1. Consider the reward function of the form,

$$\kappa(e, \sigma) = \frac{e - e_L}{e_H - e_L} \alpha_1 \log \sigma_H + \alpha_1 e^{\alpha_0(e - e_L)} \log \sigma,$$

where $\alpha_0, \alpha_1 > 0$ are deterministic constants. Here, we restrict σ_L, σ_H such that $\sigma_L = c$, $\sigma_H \approx 1.763$, where $1 < c < \sigma_H$, and that $\phi_e, \phi_\sigma > 0$ are such that $\frac{\phi_e}{\phi_\sigma} \frac{1}{e_H} (e_H - e_L) > \frac{\sigma_H}{\sigma_L}$. Note here that $\mu_L = \kappa(e_L, \sigma_L)$ and $\mu_H = \kappa(e_H, \sigma_H)$. See Figure 1.2 for an illustration.

Proof. It is immediate that Example 1.4.1 satisfies the conditions of Definition 1.4.1. For the proofs of all other statements in the paper, please see the Appendix. ■

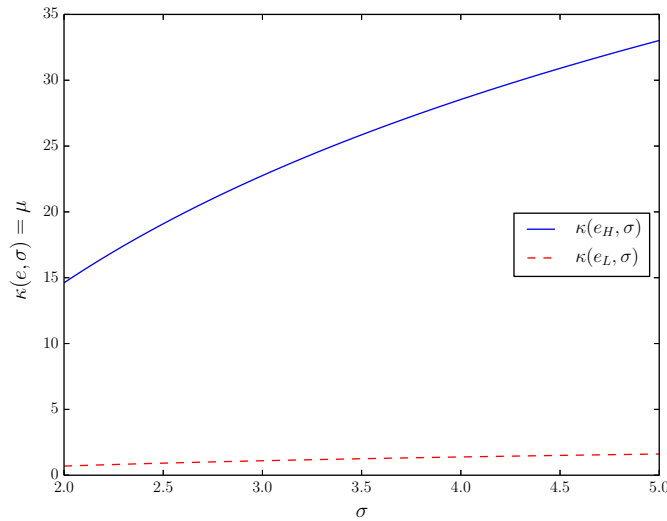


Figure 1.2: Illustration of Example 1.4.1 with the parameters: $\alpha_0 = 1, \alpha_1 = 1, e_L = 2, e_H = 5, \sigma_L = 2, \sigma_H = 5$.

Principal and Agent Conflict

With the reward function specified in Definition 1.4.1, we are now ready to discuss the sources of conflicts between the principal and the agent. From the agent’s payoff in (1.4.3), we see that the agent dislikes exerting high effort $e_t = e_H$ and likes to exert low effort $e_t = e_L$, and the agent likes to choose high volatility $\sigma_t = \sigma_H$ and dislikes to choose low volatility $\sigma_t = \sigma_L$. In contrast, from the principal’s payoff in (1.4.4), and the properties of the reward function as given in Definition 1.4.1, the principal likes high effort $e_t = e_H$ and dislikes low effort $e_t = e_L$. Moreover, by the properties of the reward function, and also effectively by the risk neutrality of the principal, the principal also seems to like high volatility σ_t . The assumption that the agent likes to job shirk while the principal does not is common in the principal-agent literature.

However, in this context, the specification of volatility warrants more discussion. It seems like since both the agent and the principal prefers higher level of volatility, then volatility is not a source of moral hazard conflict between the principal and the agent. However, this

is not entirely correct. As it is standard in the principal-agent literature, to incentivize the agent, the principal must put the agent's payoff at risk, and specifically meaning the agent's payoff must be sensitive to the agent's managed cash flows. However, such sensitivity here is also further affected both by the agent's volatility choice σ_t and also the exogenous factor M_t . Hence, even though both the principal and the agent prefers the volatility choices in the same direction, but since volatility choice also affects the overall uncertainty in this economy, this uncertainty indirectly causes the conflict between the agent and the principal. We will have more to say about this important feature of volatility choice in Section 1.8. Indeed, this is one of our core paper's core results of "reverse moral hazard".

1.5 First Best

Let's begin by characterizing the first best result. At this point, we should further impose a restriction on the recovery value L of the firm upon termination. In line with the literature, we will assume that termination is inefficient so that never terminating $\tau = +\infty$ is indeed optimal in the first best case.

Assumption 1.5.1. *We assume that termination is inefficient. That is, the salvage value of the firm L is such that,*

$$0 < L < \int_0^\infty e^{-r_1 t} \kappa(e_L, \sigma_L) dt = \frac{\kappa(e_L, \sigma_L)}{r_1}. \quad (1.5.1)$$

Recalling Definition 1.4.1, the right-hand side of (1.5.1) is precisely the "worst case" indefinite payoff scenario for the principal.

Suppose the principal can and will operate the firm without the agent. In this case, the principal does not need to pay any compensation, so $X \equiv 0$, nor is there any need for termination, so $\tau \equiv +\infty$. Recalling (1.4.4), the principal has the optimization problem,

$$b_0^{FB} := \sup_{e, \sigma} \mathbb{E} \left[\int_0^\infty e^{-r_1 t} \kappa(e_t, \sigma_t) dt \right] = \sup_{e, \sigma} \int_0^\infty e^{-r_1 t} \kappa(e_t, \sigma_t) dt. \quad (1.5.2)$$

Proposition 1.5.2. *Suppose there are no agency conflicts so the principal does not need to hire the agent to run the firm. Then the principal will pay zero compensation, $X \equiv 0$, and never terminate, $\tau \equiv +\infty$. The principal will always exert high effort at all times, so $e_t \equiv e_H$ for all t , and always choose high volatility $\sigma_t = \sigma_H$ for all times. The first best value of the firm b_0^{FB} at time $t = 0$ is,*

$$b_0^{FB} = \int_0^\infty e^{-r_1 t} \kappa(e_H, \sigma_H) dt = \frac{\kappa(e_H, \sigma_H)}{r_1}. \quad (1.5.3)$$

The first best value of the firm is deterministic and stationary. That is, the time $t = 0$ value b_0^{FB} does not depend on any state variable, and this is economically intuitive. Given that the principal only derives payoff from the reward function $\kappa(e_t, \sigma_t)$, there are no exogenous state variables (namely, say the exogenous factor M) involved.

1.6 Continuation value and Incentive compatible contracts

Now we proceed to the main focus of the paper. As it is standard in the literature, following the arguments like DeMarzo and Sannikov (2006) and Sannikov (2008), we consider the agent's continuation value as a state variable to capture the dynamic incentive compatibility constraints. However, again because of the richer volatility setup of (1.4.1) than the ones in the current literature, we must take more care in deriving the results.

Throughout this section, let's fix an arbitrary contract (A, X, τ) . In particular, note that the action process¹⁵ has the form, $A = \{(e_t, \sigma_t)\}_{t \geq 0}$. As noted in footnote 11, defining $\mu_t \equiv \kappa(e_t, \sigma_t)$, we will also call $A = \{(\mu_t, \sigma_t)\}_{t \geq 0}$ as the action process. Define, the agent's time t continuation value (or promised value),

$$W_t(A) := \mathbb{E}^A \left[\int_t^\tau e^{-r_0(s-t)} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0(\tau-t)} R \mid \mathcal{F}_t^Y \right]. \quad (1.6.1)$$

Note here on the left-hand side of (1.6.1), we have suppressed the notation for the dependence on the payment X and termination time τ , but retained the notation emphasis on the action process A .

Incentive compatible contracts

Definition 1.6.1. A contract (A, X, τ) at time 0 with expected agent payoff $W_0(A)$ is *incentive compatible* if

(1) (a) $W_t(A) \geq R$ for all times $t \leq \tau$; , where the retirement value $R > 0$ is such that,

$$R > \frac{1}{r_0} \phi_\sigma \left(\frac{\sigma_H}{\sigma_L} - 1 \right); \quad (1.6.2)$$

(b) $M_t \geq \underline{m}$, for all times $t \leq \tau$, where $\underline{m} > 0$; and

(2) $W_0(A) \geq W_0(A^\dagger)$, for all other action processes A^\dagger .

The *optimal contracting problem* is to find an incentive-compatible contract that maximizes the principal's time 0 expected payoff.

Requirement (1) of Definition 1.6.1 is also more aptly called agent's *individual participation* (IR) constraint. In particular, (1a) says that the agent's continuation value W_t must

¹⁵ We should be clear on the word "fixed" action process here. Although the agent *chooses* an action process that just needs to be $\{\mathcal{F}_t\}$ -adapted, but when the principal fixes a recommended action A to the agent, this recommended action process A is known to the principal and hence A is also $\{\mathcal{F}_t^Y\}$ -adapted. That is, the recommended action must be known to the principal but any general deviation away from the recommended action by agent is not known to the principal.

be at all times greater than or equal to the agent's reservation value R . This is a standard definition of the IR constraint in the literature. The addition of (1b) warrants slightly more discussion as this is not standard in the literature. Requirement (1b) effectively requires when the agent manages the cash flows dY_t , the agent will only manage it only when the exogenous factor level M_t at any point in time t is not too low, and in particular it must be greater than this lower bound \underline{m} . See further detailed discussions and motivations in Remark 1.A.1 in Section 1.A, where we also discuss the justification of the retirement value R in (1.6.2). Finally, requirement (2) is the usual incentive compatibility condition in the literature.

Continuation value dynamics

The dynamics of the agent's continuation value is given as follows.

Theorem 1.6.1. *Fix a contract (A, X, τ) . Then for $t \in (0, \tau)$, the agent's continuation value $W_t(A)$ of (1.6.1) has dynamics,*

$$dW_t(A) = r_0 W_t(A) dt - \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) + \beta_t (dY_t - \mu_t dt) + d\epsilon_t^{\perp, A}, \quad (1.6.3)$$

where $\epsilon_t^{\perp, A} := \int_0^t e^{r_0 s} dV_s^{\perp, A}$, $\mu_t := \kappa(e_t, \sigma_t)$, and where β_t and $V_t^{\perp, A}$ are given in Proposition 1.C.3.

Let's discuss economic meaning of the dynamics of the agent's continuation value as characterized in (1.6.3) of Theorem 1.6.1. Here, β_t represents the sensitivity of the agent's continuation value to output dY_t . When the agent takes the recommended action process A as given in the contract, the term $dY_t - \mu_t dt = dY_t - \kappa(e_t, \sigma_t) dt = \sigma_t M_t dB_t$ is a mean-zero noise term. The term $d\epsilon_t^{\perp, A}$ (explained in more detail below) also has mean zero. Economically and intuitively (though mathematically incorrect), we can view (1.6.3) in this alternative way:

$$\mathbb{E}_t[r_0 W_t(A) dt] \approx \mathbb{E}_t[dW_t(A)] + \mathbb{E}_t \left[dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right] \quad (1.6.4)$$

Hence, viewed in this way, we can think of the expected growth of the agent's continuation value $\mathbb{E}_t[r_0 W_t(A) dt]$, when the agent follows the recommended action process A , can be decomposed into the expected change from the previous continuation value $\mathbb{E}_t[dW_t(A)]$, plus the expected compensation from the principal $\mathbb{E}_t[dX_t]$, and plus the expected benefits from taking not the highest effort ($e_t \neq e_H$) and not the lowest volatility ($\sigma_t \neq \sigma_L$), which yields a strictly positive value $\mathbb{E}_t \left[\left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right]$. Note that if the agent were to really take the highest effort level ($e_t = e_H$) and the lowest volatility ($\sigma_t = \sigma_L$), then the agent's private benefits vanishes.

The economic interpretation of (1.6.3) in terms of a logic like (1.6.4) is similar across models with only drift control, say for instance, DeMarzo and Sannikov (2006) and Sannikov (2008). However, the economic interpretations of the two noise terms $\beta_t(dY_t - \mu_t dt)$ and $d\epsilon_t^{\perp, A}$ warrant more discussion. Firstly, observe that we have two noise terms here, rather than one, as in essentially all the papers with drift-only control. Secondly, while the interpretation of β_t as the sensitivity of the agent's continuation value to output here is still in line with the existing models, the noise term $dY_t - \mu_t dt$ (being multiplied by the sensitivity β_t) is different. Note also (as we will see in subsequent development) β_t still retains the interpretation as the minimal amount of risk the principal wants to subject and incentivize the agent, as in line with the literature. However, since the agent can control the volatility σ_t , and if we read the diffusion term of the agent's continuation value process dW_t as $\beta_t(dY_t - \mu_t dt) = \beta_t \sigma_t M_t dB_t$, then we see that even if the principal can dictate the sensitivity β_t for the agent, the agent still has the ability to "counteract" this dictation by choosing a volatility level σ_t to shift the overall diffusion term $\beta_t \sigma_t M_t$ (recall M is exogenous). Thus, we can already see that in a model where the agent can control volatility σ_t , the principal's tools to incentivize the agent may be weakened, as compared to a model where the agent can only control the drift. This effect is distinctly not present in models without volatility control. Finally, the additional term $d\epsilon_t^{\perp, A}$ is also related to the fact that in this model the agent can control volatility. Recalling that the cash flows are of the form $dY_t = \mu_t dt + \sigma_t M_t dB_t$. If there were no volatility control, so the cash flow takes on the form $dY_t = \mu_t dt + M_t dB_t$, then uncertainty (as seen by the diffusion term M_t) cannot be dictated by the agent. However, in this current case, the diffusion term in the cash flow is $\sigma_t M_t$, meaning that the agent can actually endogenously change the uncertainty of the cash flows, and as seen from the discussions with regards to the quadratic variation, this change of uncertainty cannot be detected by the principal if the agent does not follow an incentive compatible action process. Hence, that is why the term $d\epsilon_t^{\perp, A}$ is there to capture this source of extra (orthogonal) uncertainty (see Proposition 1.C.3). Note that, as shown in Lemma 1.7.3, when we consider incentive compatible contracts, this term $d\epsilon_t^{\perp, A}$ will become identically zero. The economic intuition is simply that when the principal offers incentive compatible contracts, as opposed to any arbitrary contracts to the agent, the principal knows that the agent will have no incentive deviate from the principal's recommendations. In particular, this also implies the principal can see the instantaneous diffusion of the cash flows $\sigma_t M_t$ and hence there is no source of extra uncertainty that we'd described earlier.

Incentive compatibility conditions

Now, the following is a necessary and sufficient condition to characterize the incentive compatible contracts in this context.

Lemma 1.6.2. *Fix a contract (A, X, τ) and consider the process β as given in Proposition 1.C.3. Then we have the following equivalence.*

(i) The action $A = \{(e_t, \sigma_t)\}_{t \geq 0}$ is such that,

$$0 \geq -\frac{\phi_e}{e_H}(e' - e_t) + \frac{\phi_\sigma}{\sigma_L}(\sigma' - \sigma_t) + \beta_t(\kappa(e', \sigma') - \kappa(e_t, \sigma_t)) \quad (1.6.5)$$

for all $(e', \sigma') \in \{e_L, e_H\} \times [\sigma_L, \sigma_H]$.

(ii) Contract (A, X, τ) is incentive compatible.

The following corollary is a simple rewriting of Lemma 1.6.2 but will be useful for the subsequent discussion.

Corollary 1.6.3. *Under the same setup of Lemma 1.6.2, if $\beta = \{\beta_t\}$ is a nonnegative process, then a given action process $A = \{(e_t, \sigma_t)\}$ is incentive compatible if and only if for all times t :*

(i) If $e_t = e_H$,

$$\beta_t \geq \frac{1}{\kappa(e_H, \sigma_t) - \kappa(e_L, \sigma_H)} \left[\frac{\phi_e}{e_H}(e_H - e_L) + \frac{\phi_\sigma}{\sigma_H}(\sigma_H - \sigma_t) \right]. \quad (1.6.6)$$

(ii) If $e_t = e_L$,

$$0 \leq \beta_t \leq \frac{1}{\kappa(e_H, \sigma_H) - \kappa(e_L, \sigma_t)} \left[\frac{\phi_e}{e_H}(e_H - e_L) + \frac{\phi_\sigma}{\sigma_L}(\sigma_t - \sigma_H) \right] \quad (1.6.7)$$

Remark 1.6.4. *We should note that in Corollary 1.6.3, the right hand side of (1.6.6) is strictly positive for all choices of σ_t . Also, recalling Definition 1.4.1, the right hand side of the second inequality of (1.6.7) is also strictly positive for all choices of σ_t .*

Remark 1.6.5. *At this point, we can make a direct comparison to the case when only the drift, but not the volatility, is under the agent's control. That drift only control case has been considered in DeMarzo and Sannikov (2006) and He (2009) but given our current linear cost form, a more direct comparison is with He (2009). It should be noted that in He (2009), the agent manages a geometric Brownian motion (which He (2009) regards as firm value, rather than cash flow). Nonetheless, consider He (2009, Proposition 1) and they derive the analogous necessary and sufficient condition to be,*

$$\beta_t \geq \phi_\mu \frac{\sigma}{\mu_H}. \quad (1.6.8)$$

Note in (1.6.8), the multiplicative factor by σ is to reflect the fact that the agent managed process in He (2009) is a geometric Brownian motion (with managed drift μ_t and unmanaged constant volatility σ), rather than our linear setup.

What is most striking about the characterization in (1.6.8) and Lemma 1.6.2(i) is that on the right-hand side of (1.6.8), there are no other agent choice variables involved; indeed, the

entire incentive-compatible contract is characterized by this single — perhaps rather “static” — inequality. In contrast, on the right-hand side of the inequality in Lemma 1.6.2(i), there still remains a choice variable by the agent; indeed, this type of characterization is very similar to the one that is provided in Sannikov (2008, Appendix A, Proposition 2), even though in that problem, there is still no volatility control.

Remark 1.6.6. *Despite the addition of volatility control, and in particular that volatility σ_t is chosen from an interval $[\sigma_L, \sigma_H]$ in our model, it might seem surprising that incentive compatibility can still be completely be characterized by two inequalities, (1.6.6) and (1.6.7) of Corollary 1.6.3, much alike binary hidden effort or drift choice models of DeMarzo and Sannikov (2006) and He (2009). Economically, it is because the volatility choice σ_t here is not a direct source of moral hazard conflict. That is, both the principal and the agent prefer the same direction of volatility, even though they may disagree on the level. Hence, the principal need not be concerned with providing direct incentives by altering the sensitivity β_t , and hence the optimal choice of sensitivity β_t should just focus on providing incentives to motivate the correct effort level, and since there are just two effort choices here, this corresponds to the two inequalities. As mentioned earlier, the volatility choice is an indirect source of moral hazard conflict. In particular, even though the principal and the agent may agree on the general direction of volatility choices¹⁶, the fact that the agent can directly alter the uncertainty of this economy implies the agent’s volatility choice complicates the principal’s task of providing incentives to the agent.*

Remark 1.6.7. *At this point, one might step back and ponder about this question: The principal here can only observe a one-dimensional managed cash flow Y , but why is it that it can provide incentives to induce the agent to make the appropriate choices for a two-dimensional moral hazard term (e, σ) , that being effort and volatility choices? The essential explanation lies in the monotonicity of the reward function $\kappa(e, \sigma)$ in both arguments and also the way that the volatility term σ enters linearly into the diffusion term $\sigma_t dM_t$ of the managed cash flows dY_t . For instance, if we had considered an alternative reward function form, say like $\kappa(e, \sigma m)$, for $M_t = m$, and that the diffusion term is more complicated, like in the form $\sigma_t Y_t dM_t$, then we can see that the above argument will not hold.*

1.7 Principal’s Problem

Once the incentive compatible contracts have been characterized as in Lemma 1.6.2, we are now ready to consider the principal’s problem.

¹⁶ A far more difficult characterization happens when the principal and the agent *disagree* on their preferences of the volatility level. This is left for future research.

Strengthening the IC condition

If we take the necessary and sufficient IC condition as characterized by Corollary 1.6.3, it will be difficult to ensure that the resulting principal's value function will be concave in the agent's continuation value w . Hence, we will strengthen the IC condition and consider a sufficient IC condition for Corollary 1.6.3, and also we will restrict the set of sensitivities to be bounded above.

Assumption 1.7.1. *Suppose we restrict the set of sensitivities to be,*

$$\mathcal{B} := \{\beta : K \geq \beta \geq \underline{\beta}\}, \quad (1.7.1)$$

for some sufficiently large $K > 0$, and where we define,

$$\underline{\beta} := \frac{1}{\kappa(e_H, \sigma_L) - \kappa(e_L, \sigma_H)} \left[\frac{\phi_e}{e_H} (e_H - e_L) + \frac{\phi_\sigma}{\sigma_L} (\sigma_H - \sigma_L) \right]. \quad (1.7.2)$$

Remark 1.7.2. *Note that since $\sigma \mapsto \frac{1}{\kappa(e_H, \sigma) - \kappa(e_L, \sigma_H)} \left[\frac{\phi_e}{e_H} (e_H - e_L) + \frac{\phi_\sigma}{\sigma_L} (\sigma_H - \sigma) \right]$ is monotonically decreasing, it is clear that (1.7.1) is a sufficient condition that satisfies the IC condition as characterized in Corollary 1.6.3. Furthermore, also note that $\underline{\beta} > 0$. As well, we impose an upper bound K on \mathcal{B} to ensure that the set \mathcal{B} is compact. If the set \mathcal{B} is not upper bounded, and in particular not compact, then it is conjectured that most of the arguments henceforth will still go through but one might need more sophisticated proof techniques.*

Principal's optimization problem

Henceforth, we will restrict our attention to incentive compatible contracts. And when we write the probability measure \mathbb{P} and expectation \mathbb{E} and other processes where there is dependence on the action process A , we will denote them without the superscript A notation. The following result significantly simplifies the principal's optimization problem.

Lemma 1.7.3. *Fix an incentive compatible contract (A, X, τ) . Then under the recommended action A , we have that,*

- (i) $\{\mathcal{F}_t^Y\}_{t \geq 0} = \{\mathcal{F}_t\}_{t \geq 0}$, where $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by Brownian motion B .
- (ii) $d\varepsilon_t^\perp \equiv 0$, \mathbb{P} -a.s.

Thus, with Lemma 1.7.3(ii) in hand, we are now ready to consider the principal's optimization problem. For the remainder of the discussion, we will only consider the case when the principal wants to induce the agent to choose high effort $e_t = e_H$ at all times t , but the principal still needs to induce the agent to optimally choose the volatility level σ_t . In binary

effort models like DeMarzo and Sannikov (2006) and He (2009), the authors also look for an always high effort implementation of the optimal contract. Zhu (2013) considers the model of DeMarzo and Sannikov (2006) and shows that it is possible to induce the agent to choose high effort and switch to low effort shirking at times. We acknowledge this possibility that inducing the agent to shirk could yield a potentially higher payoff for the principal, but for this paper we will only look for an always high effort equilibrium. It should be noted that we do not place such a restriction on the volatility choice; for instance, we do not insist on looking for an equilibrium where the principal induces the agent to always choose high volatility. That is because in our model, as discussed earlier, there is no *direct* moral hazard conflict between the principal and agent's desired direction of volatility. However, there remains an *indirect* moral hazard conflict arising due to volatility choice as the agent can effectively alter the level of uncertainty directly in this economy and thereby making it more difficult for the principal to provide incentives.

Recalling the principal's time $t = 0$ payoff form in (1.4.4), the *principal's optimization problem*, when the principal desires to induce always high effort $e_t = e_H$, is thus,

$$\tilde{v}(w, m) := \sup_{\sigma, X, \beta, \tau} \mathbb{E} \left[\int_0^\tau e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^\tau e^{-r_1 t} dX_t + e^{-r_1 \tau} L \right], \quad (\text{P}')$$

subject to state value dynamics,

$$\begin{cases} dW_t = \left[r_0 W_t dt - \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt - dX_t + \beta_t \sigma_t M_t dB_t, & W_0 = w, \\ dM_t = M_t dB_t, & M_0 = m, \end{cases} \quad (\text{S})$$

Optimal termination time

In (P') the principal maximizes over the set of effort control processes $e = \{e_t\}$ where $e_t \in \{e_L, e_H\}$, volatility control processes $\sigma = \{\sigma_t\}$ where $\sigma_t \in [\sigma_L, \sigma_H]$, compensation processes $X = \{X_t\}$ which is cadlag and nondecreasing, the sensitivity process $\beta = \{\beta_t\}$, where $\beta_t \in \mathcal{B}$, and the termination stopping time τ ; of course, all of the above must be $\{\mathcal{F}_t\}$ -adapted.

At this point, we will note the following. Let's considered the *relaxed principal's optimization problem*, of which we simply remove maximizing over τ in (P'). That is to say, consider,

$$\begin{cases} v(W_0, M_0) := \sup_{e, \sigma, X, \beta} \mathbb{E} \left[\int_0^\tau e^{-r_1 t} \kappa(e_t, \sigma_t) dt - \int_0^\tau e^{-r_1 t} dX_t + e^{-r_1 \tau} L \right], \\ \tau := \inf\{t \geq 0 : W_t \leq R \text{ or } M_t \leq \underline{m}\}, \end{cases} \quad (\text{P})$$

subject to the state dynamics (S).

Using an argument similar to Cvitanić and Zhang (2012, Chapter 7, Lemma 7.3.2), and also in accordance to the intuition that the principal would want to hire the agent as long

as the agent is getting paid at least his outside option of R (i.e. individual participation constraint), we can show that the problem of (P) subject to (S), and the problem of (P') subject to (S), are equivalent.

Heuristic HJB

Considering problem (P) subject to (S), this is a stochastic optimal control problem with continuous controls (i.e. the volatility recommendation σ , and the sensitivity β) and singular controls (i.e. compensation process X). Hence, standard results in the optimal control literature suggests the value function v is a solution to the Hamilton-Bellman-Jacobi (HJB) equation,

$$\begin{aligned} \max \left\{ -r_1 \psi(w, m) + \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} \psi)(w, m; \sigma, \beta) + \kappa(e_H, \sigma)], \right. \\ \left. - \psi_w(w, m) - 1 \right\} = 0. \end{aligned} \quad (1.7.3)$$

And here, \mathcal{L}_{e_H} is the second order differential operator,

$$\begin{aligned} (\mathcal{L}_{e_H} \xi)(w, m, ; \sigma, \beta) := \left[r_0 w - \phi_{\sigma} \left(\frac{\sigma}{\sigma_L} - 1 \right) \right] \xi_w(w, m) + \frac{1}{2} m^2 \xi_{mm}(w, m) \\ + \beta \sigma m^2 \xi_{wm}(w, m) + \frac{1}{2} \beta^2 \sigma^2 m^2 \xi_{ww}(w, m), \end{aligned} \quad (1.7.4)$$

and where we maximize over $\sigma \in [\sigma_L, \sigma_H]$ and $\beta \in \mathcal{B}$. Also, where not specified, when we write \max_{σ} and \sup_{β} , for notational brevity, it is understood that we are maximizing over $\sigma \in [\sigma_L, \sigma_H]$ and $\beta \in \mathcal{B}$. For convenience, we will also denote the set of admissible controls at initial state (w, m) as $\mathcal{A}_{w,m}$, with a typical control element denoted as $\alpha = (\sigma, X, \beta)$.

Let us denote the state space for the agent's continuation value as $\Gamma_W := (R, \infty)$, and the state space for the exogenous factor as $\Gamma_M := (\underline{m}, \infty)$, and the overall state space be $\Gamma := \Gamma_W \times \Gamma_M$. The appropriate boundary conditions of this problem are:

$$v(w, m) = L, \quad \text{for } (w, m) \in \partial\Gamma. \quad (1.7.5)$$

Key illustrations of the value function

Detailed properties of the value function are showed in Section 1.D. The most critical qualitative behaviors of the value function are shown in Figures 1.3 and 1.4. See also the illustration in Figure 1.5 the numerical solution.

1.8 Optimal Contract Discussion

In this section we will heuristically discuss the properties of the optimal contract and the implemented actions. The emphasis is on the economic intuition and hence we will suppress

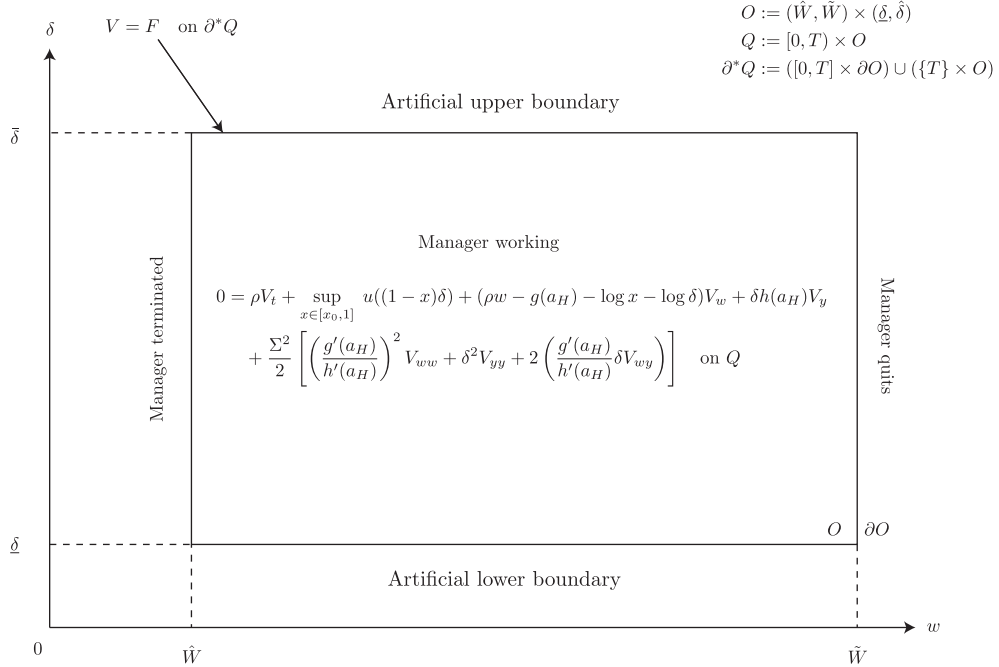


Figure 1.3: Illustration of the state space Γ . The *continuation region* is the set \mathcal{C} , and the *payment condition* is the set \mathcal{D} . Here, the *free (moving) boundary* that separates between the continuation region and the payment condition is $m \mapsto \bar{W}(m)$. It should be noted that the shape of \bar{W} as drawn is only meant to be illustrative.

the mathematical details in this section. In particular, for the sake of discussion in this section, we will assume outright that the value function is sufficiently smooth such that all the partial derivatives make sense.

Optimal sensitivity

Let's first begin by discussing the optimal choice of sensitivity β . Fix any $\sigma \in [\sigma_L, \sigma_H]$. From the HJB equation (1.7.3), when (w, m) is in the no payment region, the optimal choice of sensitivity must thus be,

$$\sup_{\beta \in \mathcal{B}} \beta \sigma v_{wm}(w, m) + \frac{1}{2} \beta^2 \sigma^2 v_{ww}(w, m), \tag{1.8.1}$$

and recall the definition of \mathcal{B} in (1.7.1).

Before we proceed to discuss the form of the optimal sensitivity β choice in the optimization problem (1.8.1), let's first discuss the diffusion term of the agent's continuation value

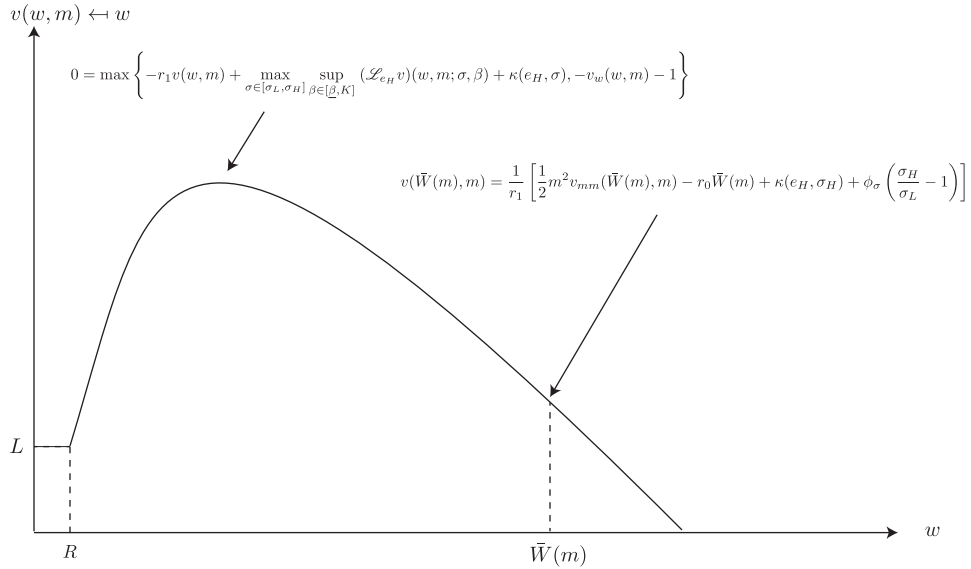
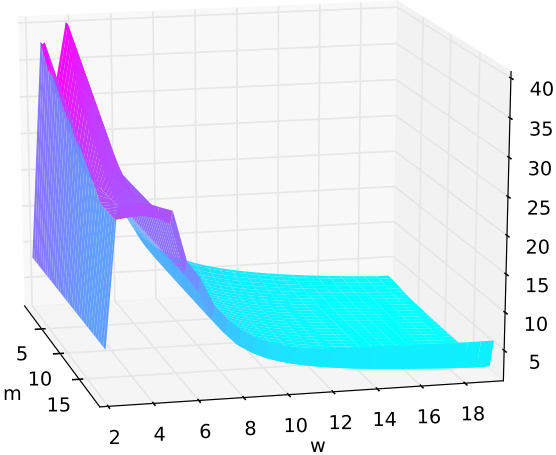


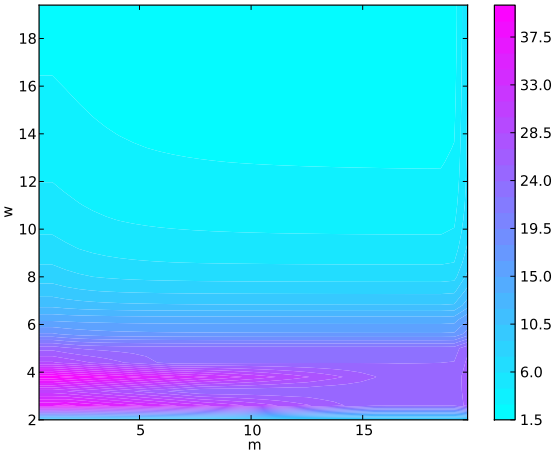
Figure 1.4: Illustration of the value function in the w -slice. That is, for each $m \in \Gamma_M$, we plot the value function $w \mapsto v(w, m)$.

dynamics dW_t in (S), and in particular, highlight how this makes our model significantly different from the drift-only control models. If we recall back to agent’s continuation value dynamics dW_t (S), the overall diffusion term is $\beta_t \sigma_t M_t dB_t$. Hence, even focusing on the choices of (β, σ) on the agent’s continuation value diffusion term alone, we see several effects at play. On the one hand, the principal wants to provide the cheapest or lowest amount of sensitivity β to induce the agent to adhere to his recommended actions. But on the other hand, the amount of risk (i.e. the diffusion term of the agent’s continuation value) is not solely just based on the principal’s imposed sensitivity β . It is indeed determined by the product of the sensitivity β , volatility choice σ , and the exogenous factor level $M_t = m$. That is to say, in contrast to drift-only control models, where the total amount of risk (i.e. again, meaning the diffusion term of the agent’s continuation value) is of the form $\beta_t dB_t$, so the principal can directly dictate the amount of risk he wants to subject the agent to through the choice of sensitivity β . In contrast, in our case, the principal’s choice of sensitivity β is *not* the only source of risk the agent is facing — the agent faces the *product* $\beta_t \sigma_t M_t$, of which σ_t remains to be a term that the principal wants to recommend and dictate for direct payoff reasons, and M_t is an exogenous factor level not controlled by the agent nor the principal. In all, that is to say when the principal wants to provide incentives through the sensitivity β , the principal must thus take into account providing incentives for an optimal volatility choice σ , and also the exogenous factor level m . It is precisely in this sense, the ability for the principal to provide incentives to the agent to induce the agent to take on the recommended action is weakened, relative to a drift-only control model.

Once we understand the incentive concerns in the diffusion term of the agent’s continu-



(a) Value function $v(w, m)$.



(b) Contour plot of $v(w, m)$.

Figure 1.5: Numerical solutions. We use the methods of a controlled Markov Chain approximation (the key reference here being Kushner and Dupuis (2001)) to directly compute the value function, and this is not dependent on the HJB-PDE formulation. We use the same reward function $\kappa(e, \sigma)$ as in Example 1.4.1 with the same parameter values as in Figure 1.2. The additional parameters that were used: $r_0 = 2, r_1 = 1.5, \phi_e = 0.3, \phi_\sigma = 0.5$

ation value dynamics dW_t , we can now be more specific about what the principal needs to consider in choosing the optimal sensitivity β , as in the optimization problem (1.8.1). Again, we immediately note several effects that are distinctly not present in drift-only control models. The optimal choice of sensitivity β now clearly depends on the volatility choice σ , the cross marginal effect $v_{wm}(w, m)$ of the agent's continuation value $W_t = w$ and the exogenous factor level $M_t = m$, and the second order effect $v_{ww}(w, m)$ of the agent's continuation value. Let's assume that $v_{ww}(w, m) < 0$ in the no payment region, implying that the principal, although is risk neutral, becomes "endogenously quasi risk averse" with respect to the agent's continuation value. Then the objective function (1.8.1) is a concave quadratic continuous function in β over a compact convex set \mathcal{B} . Thus, a unique maximizer $\beta^*(\sigma; w, m)$ exists. Let us also define the sets on $[\sigma_L, \sigma_H]$,

$$\mathcal{G}_L(w, m) := \left\{ \sigma \in [\sigma_L, \sigma_H] : -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} < \underline{\beta} \right\} \quad (1.8.2a)$$

$$\mathcal{G}_M(w, m) := \left\{ \sigma \in [\sigma_L, \sigma_H] : K \geq -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} \geq \underline{\beta} \right\} \quad (1.8.2b)$$

$$\mathcal{G}_H(w, m) := \left\{ \sigma \in [\sigma_L, \sigma_H] : -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} > K \right\}. \quad (1.8.2c)$$

Note that $\cup_{j \in \{L, M, H\}} \mathcal{G}_j(w, m) = [\sigma_L, \sigma_H]$, and $\mathcal{G}_j(w, m) \cap \mathcal{G}_k(w, m) = \emptyset$ for $j, k \in \{L, M, H\}, j \neq k$. Then by a usual constrained optimization argument, we see that the optimal sensitivity choice $\beta^*(\sigma; w, m)$ is given by the following.¹⁷

Proposition 1.8.1. *The optimal choice of sensitivity associated with the optimization problem (1.8.1) is,*

$$\beta^*(\sigma; w, m) = \begin{cases} \underline{\beta}, & \text{if } \sigma \in \mathcal{G}_L(w, m) \\ -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)}, & \text{if } \sigma \in \mathcal{G}_M(w, m) \\ K, & \text{if } \sigma \in \mathcal{G}_H(w, m). \end{cases} \quad (1.8.3)$$

Proof. The proof is immediate by the usual constrained optimization methods via the Kuhn-Tucker conditions. ■

As it is with drift-only control models, the object β is the sensitivity, or "incentives", that the principal must subject and provide to the agent in order to induce the agent to take the principal's desired action. In this case, we see that the sensitivity β that the principal wants to subject the agent to is determined by two distinct channels: (i) the level of the exogenous factor at $M_t = m$; and (ii) the volatility level $\sigma_t = \sigma$ that should be implemented. For the rest of this discussion, let's hold the recommended volatility level σ as fixed. Also, we recognize that when (w, m) is in the no payment, we only have that $v_w(w, m) \geq -1$.

¹⁷ We denote $j \in \{L, M, H\}$ for the sets $\mathcal{G}_j(w, m)$ to, respectively, mean "low", "medium" and "high". The reason is that if $j = L$ and $\sigma \in \mathcal{G}_L(w, m)$, then the optimal sensitivity $\beta^*(\sigma; w, m)$ is chosen to be the one at the lowest value; and likewise for the other cases of j .

Thus, it is distinctly possible that $v_w(w, m) = 0$. We will suppose that $v_w(w, m) > 0$ for the sake of this economic discussion, but this is not enforced anywhere else.¹⁸

In the expressions (1.8.2), we see that the object $-\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)}$ plays a significant role to determining the optimal choice of sensitivity in (1.8.3). To highlight its importance, we will label the term $-\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)}$ as *risk adjusted sensitivity* (RAS). Economically, we can define RAS as follows:

$$\begin{aligned}
 RAS &:= -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} \\
 &= \underbrace{\frac{1}{\sigma}}_{\text{(a) Precision of volatility choice}} \times \underbrace{\left(-\frac{v_w(w, m)}{v_{ww}(w, m)}\right)}_{\text{(b) "Endogenous" risk tolerance to agent's continuation value}} \times \underbrace{\left(\frac{v_{wm}(w, m)}{v_w(w, m)}\right)}_{\text{(c) "Elasticity of exogenous factor"}}. \tag{1.8.4}
 \end{aligned}$$

We see that RAS depends on three different terms: (a) precision of volatility choice; (b) “risk tolerance”; and (c) “elasticity of exogenous factor”.

Let’s first discuss the economic channel for which RAS would induce the optimal sensitivity β to be low, that is $\beta^*(\sigma; w, m) = \underline{\beta}$ in (1.8.3). Noting the form of $\mathcal{G}_L(w, m)$, in order for the set to be nonempty, we see that while $-v_{ww}(w, m) > 0$, given that $\underline{\beta} > 0$, there are no particular sign restrictions on $v_{wm}(w, m)$. Economically, this is the case when the principal does not care or want exposure to the exogenous factor level, that is roughly to say, the principal is relatively “inelastic” to the exogenous factor $M_t = m$. Once the principal does not care about the exogenous factor. then indeed, we return to the perhaps more familiar economic logic of drift-only control models. As well, the precision of volatility choice here must be relatively low, so that the choice of volatility is relatively high. Also, we can infer here that the principal must have low risk tolerance relative to the agent’s continuation value w , which implies the principal wants to achieve the lowest overall volatility of cash flow diffusion, and this is achieved when the principal subjects the agent to the lowest sensitivity $\beta^*(\sigma; w, m) = \underline{\beta}$.

Next, let’s discuss the economic channel for which RAS would induce the optimal sensitivity β choice to be high, that is $\beta^*(\sigma; w, m) = K$ in (1.8.3). Firstly, noting the form of $\mathcal{G}_H(w, m)$ in (1.8.2c), we see that since $-v_{ww}(w, m) > 0$, if we $v_{wm}(w, m) \leq 0$, then the set $\mathcal{G}_H(w, m) = \emptyset$. So let us suppose and discuss the case when $\mathcal{G}_H(w, m) \neq \emptyset$, which implies $v_{wm}(w, m) > 0$. If $\sigma \in \mathcal{G}_H(w, m)$, then it implies that the precision of volatility choice is high, or that the volatility choice is relatively low. Furthermore, the “risk tolerance” term must also be relatively high, and the “elasticity of exogenous factor” is also relatively high. This is effectively the scenario when the cash flow volatility σ is relatively low, the principal

¹⁸ Indeed, for the rest of this discussion, the sign and value of $v_w(w, m)$ is largely irrelevant. However, including the term $v_w(w, m)$ allows us to identify terms like $-v_w(w, m)/v_{ww}(w, m)$ as “risk tolerance” as it is traditionally defined (when we view the agent’s continuation value $W_t = w$ as a “consumption good”)¹⁹, and view $v_{wm}(w, m)/v_w(w, m)$ as an “elasticity” in the traditional economic sense. But even without this normalization by $v_w(w, m)$, all of the economic reasoning here goes through, except that it may not be appropriate to keep on using the traditional economic labels.

is relatively risk tolerance and so is willing to take on more risk, and hence is willing to let the exogenous factor to give the extra “risk” bump, and so justifying why $v_{wm}(w, m) > 0$. In such a case, the principal wants to put the highest sensitivity or incentives to the agent.

Finally, let’s discuss the economic channel for which RAS would induce the optimal sensitivity β choice to be medium, that is $\beta^*(\sigma; w, m) = \text{RAS} = -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)}$. And it is through this medium case, which is effectively the interior solution to the optimization problem in (1.8.1), why we think the label *risk adjusted sensitivity* (RAS) is appropriate. This case of which $\sigma \in \mathcal{G}_M(w, m)$ as in (1.8.2b) is exactly the “Goldilocks zone” and the optimal sensitivity $\beta^*(\sigma; w, m) = -\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)}$ is almost like a “Goldilocks” sensitivity. That is, the precision of volatility choice is neither too high nor too low, the risk tolerance of the principal is neither too high nor too low, and the principal’s appetite for the exogenous factor is neither too high nor too low.

In all, the above discussion not only suggests that a model with volatility control differs substantially to drift-only control models on how the optimal sensitivity β should be chosen, but equally important, our model suggests that how it is chosen is through the decomposition of the RAS term in (1.8.4).

Optimal volatility

Once the optimal sensitivity has been characterized, as discussed in Section 1.8 and in (1.8.3), we are now ready to discuss the optimal volatility σ choice. The choice of volatility here also highlights an interesting economic result — while the principal and the agent both desire higher volatility as seen from their direct payoffs, so seemingly there is no moral hazard conflict, but there still exist a distinctive presence of a *reverse moral hazard* effect.

Firstly, it should be noted that while we have emphasized and focused on the case when the principal implements the high effort $e_t \equiv e_H$ at all times, in the case of volatility control σ , it is a priori unclear whether it is possible to say which fixed volatility level that is prevalent at all time is optimal for both the agent and the principal. Recall again from the discussion in Section 1.5, the first best action is indeed to implement high effort $e_t \equiv e_H$ at all times, and also high volatility $\sigma_t \equiv \sigma_H$ at all times.

But it is perhaps difficult to motivate and justify how and why the principal would find it desirable to implement the first best volatility choice, that being the high volatility choice, at all times. It is here that we can pinpoint the source of this *reverse moral hazard* effect. Recalling the payoffs of both the agent in (1.4.3) and principal in (1.4.4), it would *appear* that there is no direct moral hazard conflict between the principal and the agent in volatility choice at all. That is, both the principal and agent strictly prefer higher levels of volatility. Thus, it might appear that in the optimal contract, the principal want to implement the first best level of volatility, namely setting $\sigma_t \equiv \sigma_H$ to the highest level, at all times. But in light of the discussion in Section 1.8 of the overall risk or diffusion term of the agent’s continuation value dW_t , we can see that fixing at the high volatility choice at all times leads to the overall diffusion term $\beta_t \sigma_H M_t dB_t$. Recall again that in this setup, termination is inefficient. In particular, that implies up to the IR conditions of the agent being met, economically the

principal would want to keep the agent employed as long as possible. Then there are now two tensions. While the principal's instantaneous direct payoff $\kappa(e_H, \sigma_t) = \kappa(e_H, \sigma_H)$ is maximized when choosing $\sigma_t = \sigma_H$ (again, the first best result), if the principal recommends a higher volatility σ_t at time t , it also boosts the probability that the agent's continuation value W_t will hit the termination boundary (i.e. when the first time when $W_t = R$), and thereby the principal will only get the inefficient liquidation value L . This is precisely the *reverse moral hazard* effect. By the IR condition, so $W_t \geq R$, it is clear that it is better for the agent to be employed than to be terminated. But to keep employment, even though the agent desires a higher volatility choice through his private benefits, the agent must at the same time also desire lower volatility to maintain employment. Similarly, while the principal obtains a higher direct payoff from recommending a higher volatility choice, it is endogenously in the interest of the principal to not recommend too high of a volatility choice for fear of terminating the agent and receiving the inefficient liquidation value.

In all, this implies that it is *not necessarily* optimal for the principal to recommend the first best volatility choice at all times. And indeed, the above discussion highly suggests that the *reverse moral hazard* effect will endogenously lead the principal to shade down the choice of volatility. And also, by choosing a higher volatility σ_t , and also observing the decomposition of RAS in (1.8.3) and the sets (1.8.2), it also implies that the choice of sensitivity β will tend to be lower. In all, and already suggested in Section 1.8, there is an interplay of effects between the optimal choice of volatility and optimal choice of sensitivity. Let us make this precise below.

With the optimal sensitivity choice $\beta^*(\sigma; w, m)$ characterized in (1.8.3), we define first the objective function,

$$\begin{aligned} G(\sigma; w, m) := & \phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m) + \beta^*(\sigma; w, m) \sigma m^2 v_{wm}(w, m) \\ & + \frac{1}{2} \beta^*(\sigma; w, m) \sigma^2 m^2 v_{ww}(w, m) + \kappa(e_H, \sigma), \end{aligned} \quad (1.8.5)$$

and the optimization problem,

$$\max_{\sigma \in [\sigma_L, \sigma_H] = \cup_j \mathcal{G}_j(w, m)} G(\sigma; w, m). \quad (1.8.6)$$

Economically, we see that when the principal recommends the volatility choice, there are several effects at play. The first three terms of (1.8.5) are for the principal to internalize the agent's concerns. The first term $\phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m)$ are the direct payoffs to the agent for choosing volatility level σ , multiplied by the weight $v_w(w, m)$. The weight $v_w(w, m) \geq -1$ represents the marginal value of the principal's value function with respect to an increase to the agent's continuation value. So if $v_w(w, m) > 0$, then it is marginally beneficial for the principal to increase the agent's continuation value, and in that case, the principal would prefer to recommend a higher volatility choice, which is also a private benefit again for the agent; vice-versa, if $-1 \leq v_w(w, m) < 0$, then the principal would want to decrease the

agent's continuation value, and this is achieved by picking a lower volatility level that incurs a private benefit cost to the agent. And if $v_w(w, m) = 0$, then the principal is indifferent. The second and third terms $\beta^*(\sigma; w, m)\sigma m^2 v_{wm}(w, m) + \frac{1}{2}\beta^*(\sigma; w, m)\sigma^2 m^2 v_{ww}(w, m)$ of (1.8.5) capture the sensitivity effects as discussed in Section 1.8, which effectively captures the cost to providing incentives to the agent, except now the exogenous factor level effect $M_t = m$ is now explicitly present. Finally, the last term $\kappa(e_H, \sigma)$ captures the principal's concerns. In all, that means in the problem of choosing and recommending the optimal volatility, the principal must trade off the agent's incentives, the cost and sensitivity to providing correct incentives to the agent, and also the principal's own desired preferences.

At this point, to consider the optimization problem (1.8.5), we effectively need to partition the volatility control σ into three different regions, according to (1.8.2) and accordingly change the value of $\beta^*(\sigma; w, m)$ as given in (1.8.3). While $\mathcal{G}_M(w, m)$ is clearly a compact subset of $[\sigma_L, \sigma_H]$, it is clear that $\mathcal{G}_L(w, m)$ and $\mathcal{G}_M(w, m)$ are just half-open interval subsets of $[\sigma_L, \sigma_H]$. So from an optimization perspective, optimizing over non-compact intervals might have serious non-existence issues. However, this is not a concern in our current case. For instance, recalling (1.8.2c), if we pick $\sigma \in \mathcal{G}_H(w, m)$ such that $\beta^*(\sigma; w, m) = K$, and if it is indeed the optimizer σ is at the boundary K of the set $\mathcal{G}_H(w, m)$, then that effectively means $-\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} = K$ and hence we are no different from optimizing over the closure $\overline{\mathcal{G}_H(w, m)}$ or evaluating the objective function at $\sigma \in \mathcal{G}_M(w, m)$ such that $-\frac{v_{wm}(w, m)}{\sigma v_{ww}(w, m)} = K$. Either case, the optimal sensitivity is $\beta^*(\sigma; w, m) = K$ and so the overall objective function $G(\sigma; w, m)$ of (1.8.5) remains the same. Similar arguments apply to the case when we consider $\mathcal{G}_L(w, m)$ of (1.8.2a). Thus, with this argument in mind, we modify our optimization problem and consider,

$$\max_{\sigma \in \cup_j \overline{\mathcal{G}_j(w, m)}} G(\sigma; w, m). \quad (1.8.7)$$

We will consider the optimization each case at a time. When we pick $\sigma \in \overline{\mathcal{G}_j(w, m)}$, for $j = L, M, H$, then the objective function respectively becomes,

$$\begin{aligned} G(\sigma; w, m)|_{\overline{\mathcal{G}_L(w, m)}} &= \phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m) + p(\sigma)\sigma m^2 v_{wm}(w, m) \\ &\quad + \frac{1}{2}p(\sigma)^2 \sigma^2 m^2 v_{ww}(w, m) + \kappa(e_H, \sigma), \end{aligned} \quad (1.8.8a)$$

$$G(\sigma; w, m)|_{\overline{\mathcal{G}_M(w, m)}} = \phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m) - \frac{1}{2} \frac{v_{wm}(w, m)^2}{v_{ww}(w, m)} m^2 + \kappa(e_H, \sigma), \quad (1.8.8b)$$

$$\begin{aligned} G(\sigma; w, m)|_{\overline{\mathcal{G}_H(w, m)}} &= \phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m) + K\sigma m^2 v_{wm}(w, m) \\ &\quad + \frac{1}{2}K^2 \sigma^2 m^2 v_{ww}(w, m) + \kappa(e_H, \sigma). \end{aligned} \quad (1.8.8c)$$

From the forms in (1.8.8), we see that in general there are no closed form and simple analytic expressions of the optimal choice of volatility $\sigma^*(w, m)$. Moreover, one needs to compute the (set of) optimizers $\sigma_j^*(w, m)$ for each case $j = H, M, L$, substitute the optimizer back into the objective function $G(\sigma_j^*(w, m); w, m)|_{\overline{\mathcal{G}_j(w, m)}}$, and once that is complete, the optimal volatility choice is the set,

$$\sigma^*(w, m) \in \arg \max_{j \in \{L, M, H\}} G(\sigma_j^*(w, m); w, m)|_{\overline{\mathcal{G}_j(w, m)}}. \quad (1.8.9)$$

Remark 1.8.2. *As a general remark, it should not be surprising that the optimal volatility choice in the form (1.8.9) is rather this complicated. Indeed, if one observes the drift-only control model of Zhu (2013), which is based off of the model of DeMarzo and Sannikov (2006), in which the agent has a binary choice of effort, it is readily seen that it is not trivial and indeed rather challenging to characterize the optimal effort choice. Here, we have already simplified matters substantially by concentrating on implementing the always high effort case, but nonetheless, even allowing for volatility to be optimally implemented, the resulting optimal volatility recommendation is nonetheless rather complicated to observe.*

Remark 1.8.3 (Difficulty of direct application of verification theorem). *As we conclude the discussion of the optimal choice of sensitivity in Section 1.8 and optimal choice in this Section 1.8, we can now remark the tremendous difficulty in applying the traditional “verification theorem” to conjecture the existence of a smooth solution of the HJB (1.7.3) that actually coincides with the value function in (P). A classical method is the verification theorem argument, or the “guess and verify” argument, where one conjectures that a PDE that solves HJB equation subject to some well thought out and economically motivated boundary conditions, and from the HJB, one takes the first order conditions to obtain the optimal controls, and substitute these controls back into the original HJB equation. There, one then proceeds to directly solve the PDE by constructing an explicit solution and thereby directly proving existence and also smoothness. Then essentially by Ito’s lemma argument, one can then verify that the HJB is a supersolution of the value function, and hence under the optimal controls, the HJB is the solution to the value function. This type of argument is fairly prevalent in the finance literature, especially in asset pricing theory, and also in continuous-time principal agent problems where there is a single state variable, so the problem is an ODE rather than a PDE.*

However, we see here once we substitute the optimal sensitivity $\beta^(\sigma; w, m)$ in (1.8.3) and optimal volatility $\sigma^*(w, m)$ in (1.8.9) back into the HJB equation (1.7.3), the resulting HJB is sufficiently complex that it is difficult to see how we can indeed obtain the existence of a smooth solution that satisfies the necessary boundary conditions. And this is especially why in the technical proofs, we have to proceed through a more roundabout way via viscosity solutions to show existence, and then “upgrade” our smoothness results.*

1.9 Delegated portfolio management

Motivation

We are now ready to consider the concrete application of the model in the context of delegated portfolio management. Suppose we regard the principal as outside investors of a managed portfolio, and regard the agent as the portfolio manager. In this context, we will explicitly assume that the portfolio manager has skill and can exert costly private effort to search for and achieve higher cash flow payoffs in the managed portfolio. That is, the portfolio manager can directly control the drift of the cash flows. Furthermore, we assume the portfolio manager has available assorted tools and financial instruments to engage into hedging and speculating behavior that can change the overall volatility of the cash flows. The portfolio itself is also subject systematically subject to an exogenous market wide or industry wide factor that the portfolio manager cannot control. Hence, portfolio volatility is comprised of a manager specific choice in volatility, reflecting risk management practices, and an exogenous market or industry factor. A delegated portfolio management problem framed in a principal-agent setting has also been considered by Ou-Yang (2003); other recent models that consider delegated portfolio management problems include van Binsbergen et al. (2008), Dybvig et al. (2010) and Cvitanić, Possamai, and Touzi (2014). In this section, we will relabel and call the agent as the *manager*, and the principal as *investor*.

The main idea here is to have the portfolio manager to have sufficient “skin in the game” through his own investments. Specifically, consider an investment firm whereby the investment manager operates two different investment funds: an *external fund* fund that is available to outside investors and an *internal fund* that is only available to management. Suppose we regard the continuation value W_t as the value of an *internal fund* that is only available to the portfolio manager but not to the outside investor. This form of internal fund that is only available to insiders of the firm, and not outside investors, is also an observed market practice. For instance, numerous banks (at least prior to the Volcker Rule) also run proprietary trading desks, which are effectively internal hedge funds. Several hedge funds also engage into this practice. For example, the hedge fund firm Renaissance Technologies runs three funds that are open to outside investors, but also run a separate fund, the Medallion Fund, that is only open to its employees (see Zuckerman (2013)). Darolles and Gourieroux (2014a) and Darolles and Gourieroux (2014b) also discuss at length the practice of this internal fund in the hedge fund industry. The external fund has value $v(W_t, M_t)$. In particular, this specifically implies that the value of the external fund is explicitly dependent on the value of the internal fund W_t and also the exogenous factor level M_t . It is in this sense that we view the external fund as a “financial derivative contract” written on the two underlying “assets”, them being the internal fund and the exogenous factor.

Remark 1.9.1. *Asserting that the portfolio volatility overall is directly influenced by the portfolio manager’s risk management practices and market volatility should be reasonable. However, asserting that the portfolio manager has skill, and moreover that such skill surely*

Variable	Contract implementation interpretation
W_t	Value of the internal fund
M_t	Factor benchmark
$v(W_t, M_t)$	Value of the external fund
σ_t	Hedging and investment strategy
β_t	Dynamic performance incentive fees
dX_t	Performance bonus
$(\bar{W}(M_t), M_t)$	Benchmarked superior performance bonus payout mark

Table 1.1: We apply our general model to the concrete application of delegated portfolio management. This table lists how we should interpret the various variables in this specific context of delegated portfolio management.

translates to higher cash flow payoffs is a strong assumption. Indeed, the question of whether persistent manager skill exists or not is still heavily debated in empirical research. The literature on this question is too vast to survey here with any justice, but some recent papers here include: Fama and French (2010) and Barras et al. (2010).

Interpreting the continuation value

For given sensitivity $\beta = \{\beta_t\}$ and volatility $\sigma = \{\sigma_t\}$, the continuation value dynamics dW_t in (S) in the no payment region so $dX_t \equiv 0$, we can rewrite the expression as,

$$\begin{aligned}
 dW_t &= \left[r_0 W_t - \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt + \beta_t \sigma_t dM_t \\
 &= \left[r_0 W_t - \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt + \beta_t (dY_t - \kappa(e_H, \sigma_t) dt) \\
 &= \beta_t dY_t + r_0 W_t dt - \left[\phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) + \beta_t \kappa(e_H, \sigma_t) \right] dt. \tag{1.9.1}
 \end{aligned}$$

With the context of delegated portfolio management, we will interpret the continuation value dynamics dW_t via the expression form of (1.9.1). See Table 1.1 for a summary and quick reference.

Since W is the value of the internal fund, the expression (1.9.1) suggests that the value of the internal fund is driven by the amount of ownership $\beta_t dY_t$ the agent has of the underlying investment technology, and plus an investment $r_0 W_t dt$ into a riskfree asset that pays off at a rate r_0 . Note here that in this context, we can interpret β_t as the *dynamic incentive fees* of ownership the manager owns of the investment opportunity dY_t , so $\beta_t dY_t$ is the total dollar exposure the manager has to the managed cash flows. The latter two terms of (1.9.1) represent the cost of implementing an investment strategy σ_t . The term $\phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) dt$ can be thought of as the direct cost of implementing the investment strategy σ_t ; for instance,

this could represent the direct trading costs or managerial monitoring costs. The term $\beta_t \kappa(e_H, \sigma_t) dt$ represents the proportional expected return of implementing the strategy σ_t . So the manager's internal fund value only gets a positive bump if $\beta_t (dY_t - \kappa(e_H, \sigma_t) dt) > 0$, and since the ownership amount $\beta_t > 0$, then we have that $\beta_t (dY_t - \kappa(e_H, \sigma_t) dt) > 0$ if and only if $dY_t - \kappa(e_H, \sigma_t) > 0$. That is to say, the manager captures the positive excess returns over the expected return of the investment strategy, only if the investment strategy performs extraordinarily well.

Since we focus on implementing an always high effort action $e_t \equiv e_H$, other than the dollar incentives β_t , the remaining control policy here is the volatility σ_t . Again, we interpret β_t as the dollar incentives of ownership, or *dynamic incentive fees*, the manager owns of the managed investment opportunity dY_t , so $\beta_t dY_t$ is the total dollar exposure the manager has to the managed cash flows. In addition, here we may broadly interpret σ_t as *investment strategies*. To be more specific, once in equilibrium we implement an always high effort $e_t \equiv e_H$ action, then it effectively implies that the manager is already exerting costly skill to find the set investment opportunities with good returns. However, even after exerting skill to find this investment opportunity set, the manager still needs to choose the specific investments from the set, and it is here we interpret σ_t as the opportunities available to the manager. Specifically, we will let σ_t be effectively a parameter captures both the set of investment opportunities and hedging strategies.

Now using the results in Section 1.8, under the optimally chosen sensitivity $\beta^*(\sigma; w, m)$ in (1.8.3) and optimally chosen volatility $\sigma^*(w, m)$ in (1.8.9), and the set forms $\mathcal{G}_j(w, m)$ in (1.8.2), we can thus write (1.9.1) as,

$$\begin{aligned}
 dW_t = & \sum_{j \in \{L, M, H\}} \left\{ \beta^*(\sigma^*(W_t, M_t); W_t, M_t) dY_t + r_0 W_t dt - \left[\phi_\sigma \left(\frac{\sigma^*(W_t, M_t)}{\sigma_L} - 1 \right) \right. \right. \\
 & \left. \left. + \beta^*(\sigma^*(W_t, M_t); W_t, M_t) \kappa(e_H, \sigma^*(W_t, M_t)) \right] dt \right\} \mathbf{1}_{\sigma^*(W_t, M_t) \in \mathcal{G}_j(W_t, M_t)}. \tag{1.9.2}
 \end{aligned}$$

Economically, the form of (1.9.2) implies the following contractual implementation. The outside investors offer the manager an initial start up fund value of W_0 at $t = 0$ and in return, the manager commits to the following *dynamic incentive fee* compensation scheme as represented via the optimal dollar incentives $\beta^*(\sigma^*(w, m); w, m)$, viewed as a map from (w, m) . That is to say dependent on the value of the internal fund $W_t = w$ and also the exogenous factor level $M_t = m$, the manager will choose a different investment strategy $\sigma^*(w, m)$. And dependent on this strategy, the internal fund will only get different dollar exposures of the managed cash flows dY_t . For instance, for those investment strategies such that $\sigma^*(w, m) \in \mathcal{G}_L(w, m)$, the manager gets a low dollar incentive $\beta^*(\sigma^*(w, m); w, m) = \underline{\beta}$; for those investment strategies $\sigma^*(w, m) \in \mathcal{G}_M(w, m)$, the manager gets a medium dollar incentive $\beta^*(\sigma^*(w, m); w, m) = -\frac{v_{wm}(w, m)}{\sigma^*(w, m)v_{ww}(w, m)}$; and finally, for those investment strategies $\sigma^*(w, m) \in \mathcal{G}_H(w, m)$, the manager gets a high dollar incentive $\beta^*(\sigma^*(w, m); w, m) = K$.

The critical real-world implication of the above is as follows — investors should contract on the value of the internal fund and the exogenous factor level. Again, take hedge funds as a

prototypical example. Hedge fund investment strategies are essentially completely black box, and effectively that means even investors into the fund most often have no idea what types of investment strategies the manager is employing. Hence, that makes directly contracting on investment strategies to be highly unrealistic and impossible. In our framework however, the investor only needs to contract on two things: the value of the internal fund $W_t = w$ and the stochastic factor level $M_t = m$. That is, the investor writes a contract not on the investment strategy that maps to the dollar incentives $\sigma \mapsto \beta^*(\sigma; w, m)$, but rather directly from the value of the internal fund and the stochastic factor level $(w, m) \mapsto \beta^*(\sigma^*(w, m); w, m)$, and we emphasize that $\beta^*(\sigma^*(w, m); w, m)$ only depends on (w, m) , and indeed only has three relatively “small” sets of values, as given by (1.8.3).

However, while the investors can certainly contract on the value of the internal fund $W_t = w$, it is unclear how the investors can contract on the factor level $M_t = m$. In particular, recall that off equilibrium, the investors cannot observe the exogenous factor level. Thus, to complete the optimal contract implementation, we further require the manager to directly and truthfully report the exogenous factor level to the investors. In practice, that translates to the manager reporting periodically some factor benchmark index to the investors.

Thus, with the internal fund value $W_t = w$ and the exogenous factor level $M_t = m$ known to the investors, the investors can just adjust the level of the dynamic incentive fee $\beta^*(\sigma^*(w, m); w, m)$ accordingly, without knowledge of the actual employed investment strategy $\sigma^*(w, m)$. The advantage of this implementation is that the manager does *not* need to report to the investors their actual employed investment strategy, which is usually what is observed in practice in the case of hedge funds. Moreover, if the internal fund does well, so when $W_t = w$ hits the payment boundary $\bar{W}(M_t) = \bar{W}(m)$, the external investors will directly compensate manager²⁰

As a result of the above discussion, the value of the external fund to the investors is $v(w, m)$, when the value of the internal fund is $W_t = w$ and the level of the exogenous factor is $M_t = m$. But economically and conceptually, what does it mean by the value of the external fund is a function of the value of the internal fund and the level of the exogenous factor? Borrowing the language of financial derivatives, we effectively can view the external fund as a *derivative*, where the underlying asset here is written on the value of the internal fund and level of the exogenous factor, with two associated barriers. The lower barrier is the first time $(W_t, M_t) = (w, m)$ hits the level $(w, m) = (R, m)$ or $(w, m) = (w, \underline{m})$; that is, either when the value of the internal fund goes bust (i.e. $W_t = R$), or when the exogenous factor level is sufficiently low (i.e. $M_t = \underline{m}$) that the manager effectively walks away from the firm. The upper barrier is the moving barrier $(\bar{W}(M_t), M_t)$ that determines the optimal capital injection or compensation scheme. However, despite this discussion, a direct implication here is that the investment strategy of the manager still remains a black-box to the investor. Another direct implication here is that the investment strategy of the external fund will closely track that of the internal fund’s investment strategy.

²⁰ In this setup, injecting more capital dX_t into the internal fund is equivalent to compensating the manager, as we assume the manager derives all utility from maximizing the value of the internal fund.

Finally, we should observe what is *not* optimal or feasible in this context. Most notably, the perhaps “easier” contractual setup would be that there is a single investment fund, managed by the manager, for which managers and investors commonly invest to. In this context, this is not possible. As a thought experiment, suppose this were true, meaning the internal fund and the external fund are exactly identical. But because the manager can privately select effort and volatility (again, broadly interpreted as investment opportunity), and by limited liability, the manager would effectively have incentives to gamble (i.e. choose the highest volatility) and exert the lowest effort. Focusing on the volatility choice, although by the form of the reward function $\kappa(e, \sigma)$ it may appear that it too is desirable for the investor to choose the highest volatility, the discussion in Section 1.8 argued that this is not the case. As well, and this is perhaps a more cynical view of managers, suppose there does exist only a single common fund, and recalling the black box nature of the investment strategy, how can investors ensure that the managers will not privately squirrel away the best available investment opportunities and leave the subpar investment opportunities to the common fund? However, the establishment of an internal only fund for the manager with an external fund that closely tracks the investment strategy of the internal fund does mitigate this concerns. That is, although in equilibrium, the manager will collect some “information rent”, namely in the form of keeping the best investment strategies still for the internal fund, but if the investment strategy of the external fund commits to following that of the internal fund, the external fund still benefits from the exposure of those good investment strategies.

Black-Scholes-Merton “greeks” interpretation

Once we view the external fund as a financial derivative contract written on the internal fund and the exogenous factor, we can obtain interesting interpretations and understandings of the RAS and also the optimal investment strategy in the paper. This is essentially made possible by the “greeks”²¹ of the classical papers of Black and Scholes (1973) and Merton (1973).

Recalling the RAS, and suppose we drop the normalization $v_w(w, m)$ as in (1.8.4), we see that,

$$RAS := \frac{-v_{wm}(w, m)}{\sigma v_{ww}(w, m)} = \underbrace{\frac{1}{\sigma}}_{\text{(a) Precision of investment strategy choice}} \times \underbrace{\left(-\frac{1}{v_{ww}(w, m)}\right)}_{\text{(b) Negative reciprocal of "gamma"}} \times \underbrace{v_{wm}(w, m)}_{\text{(c) "Cross-gamma"}}$$

This effectively implies that our dynamic performance fees depend on: (a) the investment choice taken by the manager; (b) the “gamma” of the value of the external fund relative to the value of the internal fund; and (c) the “cross-gamma” of the value of the external fund

²¹ In the Black-Scholes-Merton framework, if $V = V(t, S)$ is the value of an option at time t and with underlying asset value $S_t = S$, then $\Delta = \partial V / \partial S$ is called the *Delta* of the option, and $\Gamma = \partial^2 V / \partial S^2$ is called the *Gamma* of the option. And if there are two underlying assets, so the value of the option is $V = V(t, S_1, S_2)$, then $\partial^2 V / \partial S_1 \partial S_2$ is the *Cross-Gamma* of the option.

relative to the value of the internal fund and the external factor. That is, component (a) links to the direct choice of the manager, component (b) links to incentive provisions for the manager, and component (c) links to the external factor that is outside of the manager’s controls.

Much more interestingly, however, is how we can think about the optimal investment strategy. Recalling (1.8.5) which described the objective function to choosing the investment strategy, we can now re-interpret as,

$$\begin{aligned}
 G(\sigma; w, m) := & \underbrace{\phi_\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right) v_w(w, m)}_{\text{(A) Direct private payoffs to manager, adjusted by "delta"}} + \underbrace{\kappa(e_H, \sigma)}_{\text{(B) Direct payoffs to the investor}} \\
 & \underbrace{\beta^*(\sigma, w, m)\sigma m^2 v_{wm}(w, m) + \frac{1}{2}\beta^*(\sigma, w, m)\sigma^2 m^2 v_{ww}(w, m)}_{\text{(C) Incentives due to the manager, adjusting for dynamic performance fees}}
 \end{aligned}$$

The expression above suggests the investment strategy must trade-off the direct benefits to the manager, direct benefits to the investor, and also the incentives for the manager. Component (B) here is straightforward, as it is simply the direct benefit to the investor, and this is strictly increasing in σ . However, as mentioned regarding the “reverse moral hazard” effect, we see that taking the highest value $\sigma = \sigma_H$ may not necessarily be optimal, as one must furthermore consider components (A) and (C). Component (A) can be seen as the marginal benefit $\sigma \left(\frac{\sigma}{\sigma_L} - 1 \right)$ of choosing investment strategy σ , multiplied by the Black-Scholes-Merton “Delta” of the value of the external fund relative to the value of the internal fund. That is to say, there is an interaction effect between taking on higher σ , which the manager enjoys, but that could either be amplified or suppressed by how such a change alters the value of the internal fund, which in turn impacts the value of the external fund. Component (C) is the incentives term for the manager, and this effectively relates back to our earlier discussion of RAS.

1.10 Conclusion

We studied continuous-time principal-agent problem where the agent can continuously choose the drift and volatility parameters, while the principal continuously observes and receives the resulting controlled cash flows. The key ingredient yielding a meaningful private volatility control by the agent, in that the agent’s deviation cannot be easily detected by the principal’s computation of the quadratic variation of the cash flows, is via the introduction of an exogenous factor level. Hence effectively, even though the principal can infer the overall instantaneous diffusions of the cash flows, the principal cannot disentangle the component that is due to the agent’s endogenous volatility control and the exogenous factor level. As a result, beyond merely hidden drift or effort control, the principal must provide incentives

now for both inducing the desired effort and volatility. Most importantly, as a concrete application, our current model provides a first step to considering the dynamic contracting environment in the context of delegated portfolio management.

By introducing this meaningful sense of volatility control, we now open a new economic channel for researchers to study continuous-time principal-agent problems. In particular, there are further questions one can consider between the interplay of effort and volatility. In particular, there are several questions that this framework researchers could consider:

- The current model assumes the principal is risk neutral. However, once we give the agent the meaningful ability to privately select volatility, an immediate and relevant extension is to consider a case when the principal is risk averse.
- In the context of delegated portfolio management, since prices of risky assets jump (i.e. “disaster” states), it will be interesting to pair, say, our current model to that of DeMarzo et al. (2013), where the agent can also influence the likelihood of a disaster state occurring.

In all, we feel that this model is an important step to the growing literature of continuous-time dynamic contracting, and also to our better understanding of delegated portfolio management contracting practices.

Appendix

1.A Assorted Remarks

Single Brownian motion

In (1.4.1) and (1.4.2), we use a single Brownian motion B , rather than say two different Brownian motions. In particular, this might be surprising coming from an asset pricing perspective; in asset pricing applications, say like the classical Heston (1993) model, if S is the price of an asset, then it has say the dynamics,

$$\begin{aligned} dS_t &= \mu_S(t, S_t)dt + \sigma_S(t, \sigma_t, \nu_t)dB_{1t} \\ d\nu_t &= \mu_\nu(t, \nu_t)dt + \sigma_\nu(t, \nu_t)dB_{2t}, \end{aligned}$$

where B_1 and B_2 are correlated Brownian motions (with possibly zero correlation). In this type of specification, we see that the stochastic volatility dynamics of price S are also further driven by the process ν . In contrast, the specification of (1.4.1) and (1.4.2) uses the same Brownian motion. Indeed, in our specification, if we were to use two different Brownian motions, then economically, then there is little hope for an equilibrium. Economically, recall here the principal can only observe a single source of information (i.e. the cash flow Y_t over time), but if there are two sources of risk (i.e. two Brownian motions), then the agent has far too much room to deviate from the principal's recommended actions. Indeed, we will see the importance of using a single Brownian motion in Lemma 1.7.3.

Stochastic Time Change

We can actually view (1.4.1) and (1.4.2) as a time-changed process. We will not use this fact elsewhere in the paper. Since we can write (1.4.1) and (1.4.2) as $dY_t = \kappa(e_t, \sigma_t)dt + \sigma_t dM_t$, but since M is a geometric Brownian motion, by the Dambis-Dubins-Schwartz theorem²² and by expanding the filtration and changing the probability space if needed, there exists a stochastic time change $\{T(t)\}_{t \geq 0}$ given by $T(t) := \inf\{u : \int_0^u \exp\{-v + 2B_v\}dv > t\}$, and another Brownian motion Z , such that $M_t = Z_{T(t)}$. Hence, we can rewrite the cash flows as,

$$dY_t = \kappa(e_t, \sigma_t)dt + \sigma_t dZ_{T(t)}.$$

Indeed, more is true. Since M is a geometric Brownian motion, by Lamperti's relation²³, we can further write M as a time-changed squared Bessel process.

Further discussions of Definition 1.6.1

We collect some additional discussions of Definition 1.6.1 here so as to not block the overall reading flow of the main text.

Remark 1.A.1. *Let's discuss the economic justification of (1b). If the exogenous factor level M , which again is a geometric Brownian motion so $M > 0$, is too low, say when $M_t \approx 0$ (even though $M_t = 0$ happens on a set of measure zero), then all sources of uncertainty in this economy vanishes. Indeed, suppose in the extreme that we indeed have $M \equiv 0$. And when that happens, the managed cash flows thus become $dY_t = \kappa(e_t, \sigma_t)dt$, without any additional noise term. But this implies the principal, upon observing cash flows dY_t continuously over time, can precisely detect the choice of effort e_t and choice of*

²² See Revuz and Yor (2005, Chapter V, §1, Theorem 1.6) for the precise statement.

²³ See Revuz and Yor (2005, Chapter XI, §1, Exercise 1.28).

volatility σ_t that the agent has chosen²⁴, and clearly then, the principal would instruct the agent to choose the first best effort and volatility choices. However, first best effort and volatility choices are clearly not beneficial for the agent.

Without the presence of uncertainty (so when $M \equiv 0$), the principal no longer needs to compensate the agent, $X \equiv 0$. Mapping back to the context of delegated portfolio management and recalling the discussion in Section 1.3, when $M \equiv 0$, it is equivalent to saying the outside investor is getting precisely zero premia for the factor exposure of this particular managed fund. In that case, the investor has no particular reason to compensate the manager for management anymore. Indeed, in this case, if the manager does not choose the first best case of highest effort ($e_t \equiv e_H$) and choose the appropriate investment opportunity ($\sigma_t \equiv \sigma_H$), the investor will simply walk away. Anticipating this, the agent is conceivably better off to “walk away” from managing the project before the exogenous factor level M is too low, namely at \underline{m} , and still manage to extract some information rent from the principal. More precisely, the agent’s retirement value R is such that,

$$\begin{aligned}
 & \text{Payoff to agent with positive exogenous factor } M \geq \underline{m} > 0, \text{ and arbitrary actions} \\
 & \underbrace{\int_0^\tau e^{-r_0 t} dX_t + \int_0^\tau e^{-r_1 \tau} \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] dt}_{\geq R} \\
 & > \underbrace{\int_0^\infty e^{-r_0 t} \phi_\sigma \left(\frac{\sigma_H}{\sigma_L} - 1 \right) dt}_{\text{Payoff to agent with always zero exogenous factor } M \equiv 0, \text{ and first best actions}} = \frac{1}{r_0} \phi_\sigma \left(\frac{\sigma_H}{\sigma_L} - 1 \right).
 \end{aligned}$$

This discussion hence also justifies the retirement value R as specified in (1.6.2).

Remark 1.A.2. With Remark 1.A.1 in mind, we should also consider the counter case. What if the retirement value R is too low? Specifically, what if, unlike (1.6.2), R is such that,

$$\frac{1}{r_0} \phi_\sigma \left(\frac{\sigma_H}{\sigma_L} - 1 \right) \geq R > 0. \tag{1.A.1}$$

If the retirement value R is such that (1.A.1) holds, when the exogenous factor level is identically zero $M \equiv 0$, we will see that it may be not optimal for the agent to walk away from the contract. That is, if $M \equiv 0$, again as per the argument in Remark 1.A.1, the principal can credibly instruct the agent to take on the first best action $e_t \equiv e_H$ and $\sigma_t \equiv \sigma_H$, and pay zero compensation $X \equiv 0$. That is, the agent simply then receives the instantaneous private benefit of $\phi_\sigma \left(\frac{\sigma_H}{\sigma_L} - 1 \right) dt$. Even though the agent knows that there are some positive information rent to extract from the principal if the exogenous factor level is strictly positive, but even at the identically zero exogenous factor level case, his instantaneous private benefit still exceeds the outside retirement value R . So if (1.A.1) holds, it implies there is a possibility that the principal can give the agent zero compensation and yet the agent will still happily remain employed.

We rule this case out. Specifically, we assume that at time $t = 0$, the agent can anticipate such effects, and negotiate, ex-ante, with the outside labor market to secure a sufficiently high retirement value R that satisfies (1.6.2), rather than a low retirement value of (1.A.1). In the context of delegated portfolio management, we may think of a high retirement value R to represent an outside fund management opportunity that’s available to the portfolio manager.

1.B Selected important special cases

Several papers in the literature have some level of volatility control in the dynamic principal-agent problem in continuous-time. However, all of them place various levels of restrictive assumptions on the way the agent can control volatility, which is not imposed in our setup.

Sung (1995, 2004)

The papers by Sung (1995) and Sung (2004) are the closest in terms of volatility control but still does not resolve the problem that we have in mind for this paper.

Let’s first review Sung (1995). The author considers a finite time horizon, $[0, 1]$. The cash flows under management is of the form $dY_t = \mu_t dt + \sigma_t dB_t$, where the drift μ_t and volatility σ_t are under the agent’s control. A contract is signed between the principal and the agent at time $t = 0$ and the agent is compensated at time $t = 1$. The agent incurs an integrated

²⁴ This is possible since the reward function κ of Definition 1.4.1 is bijective and non-crossing in effort e and volatility σ .

	Cash flow dynamics	References
Brownian motion with controlled drift	$dY_t = \boldsymbol{\mu}_t dt + \sigma dB_t$	Holmström and Milgrom (1987), Biais, Mariotti, Plantin, and Rochet (2007), Sannikov (2008), DeMarzo and Sannikov (2006), Adrian and Westerfield (2009), Zhu (2013)
Geometric Brownian motion with controlled drift	$dY_t = \boldsymbol{\mu}_t Y_t dt + \sigma Y_t dB_t$	He (2009)
General Ito diffusion with controlled drift	$dY_t = f(t, Y, \boldsymbol{\mu}) dt + \sigma(t, Y) dB_t$	Schaettler and Sung (1993)
Brownian motion with controlled drift and controlled jump	$dY_t = (\alpha + \rho \boldsymbol{\mu}_t) dt + \sigma dB_t - D \boldsymbol{\mu}_t dN_t$, α, ρ, D constants	DeMarzo, Livdan, and Tchistiyi (2013)
Brownian motion with controlled drift via long run incentives	$dY_t = \delta_t dt + \sigma dB_t$, $\delta_t = \int_0^t f(t-s) \boldsymbol{\mu}_s ds$	Sannikov (2012c)
Controlled Poisson intensity	$dY_t = C dN_t$, N has intensity process $\{\boldsymbol{\mu}_t\}_{t \geq 0}$, $C > 0$ constant	He (2012)
Linear Ito diffusion with controlled drift and volatility	$dY_t = f(\boldsymbol{\mu}_t, \boldsymbol{\sigma}_t) dt + \boldsymbol{\sigma}_t dB_t$	(*) Sung (1995, 2004)
Geometric Brownian motion with same control on drift and volatility	$dY_t = [rY_t + \boldsymbol{\mu}_t(\alpha - r)] dt + \boldsymbol{\mu}_t \kappa dB_t$, r, α, κ constants	(*) Ou-Yang (2003)
Geometric Brownian motion with controlled drift and volatility	$dY_t = \kappa Y_t dt + \delta \boldsymbol{\mu}_t dt + \alpha \boldsymbol{\sigma}_t V_t dt + \boldsymbol{\sigma}_t V_t dB_t$, κ, δ, α constants	(*) Cadenillas, Cvitanic, and Zapatero (2004)
Multidimensional Brownian motion with drift and volatility control, but scalar observations	$dY_t = \boldsymbol{\mu}_t^\top \sigma(bdt + dB_t)$, $\boldsymbol{\mu}, \sigma, b, B$ are multidimensional but Y is a scalar	(*) Cvitanic, Possamai, and Touzi (2014)

Table 1.B.1: A selected survey of agent's managed cash flows in the existing literature. Here, B denotes a standard Brownian motion and N denotes a Poisson process. For consistency, the notations here differ from that of the original papers. The agent's control (in **bold** for emphasis) is $\boldsymbol{\mu} = \{\boldsymbol{\mu}_t\}_{t \geq 0}$ (and where relevant, $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_t\}_{t \geq 0}$). For the (*) starred cases where volatility is (seemed to be) under control, please see section 1.B for discussions of their key caveats.

cost $\int_0^1 c(\mu_t, \sigma_t) dt$ for choosing between the drift μ_t and σ_t over the investment horizon $[0, 1]$. The principal is restricted to compensate the agent according to an exogenously fixed “salary function” S . The author considers two cases. In the first case, the principal observes the entire path of $\{Y_t\}_{t \in [0, 1]}$ and hence can also observe σ_t across this path (i.e. via the quadratic variation of Y). Hence, there is no need for the principal to provide incentives to control the volatility σ_t . In the second case, the principal can only observe the ending cash flow value Y_1 . In such a case, the principal cannot know what volatility control σ_t the agent had chosen over the investment horizon $[0, 1]$.

Sung (2004) is similar to Sung (1995), except that the author allows for a more general specification of the cash flow process, and restrict to the second case setup of Sung (1995), whereby the principal can only observe the initial Y_0 cash flow and the terminal Y_1 cash flow. Specifically, Sung (2004)’s specification is of the form $dY_t = f(\mu_t, \sigma_t) + \sigma_t dB_t$, where the agent controls both the drift μ_t and volatility σ_t . The details in the preferences of the agent and principal differ slightly between Sung (1995) and Sung (2004) and we defer the reader to the actual papers for details.

Unlike Sung (1995) and Sung (2004), in this paper we will explicitly allow the principal to observe the agent’s managed cash flow process at all times.

Ou-Yang (2003)

In Ou-Yang (2003), the principal-agent problem is the in form of an investment manager (i.e. agent) has to manage a portfolio for an investor (i.e. principal). Asset returns follow the familiar geometric Brownian motion and together with the risk free asset, it induces a wealth process for the portfolio. The agent can choose the portfolio process and the conflict arises when the investor cannot observe the manager’s chosen portfolio policy ²⁵. In this setup, the portfolio choice variable is attached to the diffusion term of the wealth process. But as duly noted in Ou-Yang (2003, Page 178): “If the investor observes both the stock price vector $P(t)$ and the wealth process $W(t)$ of the portfolio continuously, then she can infer precisely the manager’s portfolio policy vector $A(t)$ from the fact that the instantaneous covariance between $W(t)$ and $P(t)$ equals $\text{diag}(P)\sigma\sigma^T A(t)$. Since $\sigma\sigma^T$ is invertible by assumption, the manager’s policy vector $A(t)$ is completely determined. Hence we must assume that the investor does not observe the wealth and stock processes simultaneously.” Hence, through this, rather strong, assumption in restricting what the investor can observe over time, volatility can be controlled without detection by the principal. In this paper, we will not impose such a strong assumption that restricts the principal’s information set.

Cadenillas, Cvitanić, and Zapatero (2004, 2007)

Cadenillas et al. (2004) does allow the agent to have explicit drift and volatility control but the compensation type is exogenously given. Using the original notation of Cadenillas et al. (2004, Equation (2)), the agent (manager) manages the value of assets V under the agent’s management evolves according to,

$$dV_t = \mu V_t dt + \delta \mu_t dt + \alpha v_t V_t dt + v_t V_t dW_t,$$

where u and v are the agent’s controls. Moreover, the paper assumes that while effort (drift) u control for the agent is costly, project selection (volatility) v control incurs no cost on the agent, but only implicitly matters to the agent through the principal’s compensation. The principal (outside investors) is exogenously allowed to only compensate the agent with stock that becomes vested at a terminal time T . The principal simply needs to choose the number of shares of stock to give to the agent and the level of debt of the firm ²⁶. In all, taken in this light, Cadenillas et al. (2004)’s interesting approach of the problem does not have the key ingredients that are present in this paper. Firstly, we do not exogenously fix what the compensation contract the principal must give to the agent, and indeed the compensation structure is dynamic and endogenous. And secondly, the agent

²⁵ The wealth process, referring to Ou-Yang (2003, Equation (1)), has the form,

$$dW(t) = [rW(t) + A(t)(\mu - r)]dt + A(t)\sigma dB_t,$$

where r is the risk free rate, and μ is the expected return of risky assets, and $A(t)$ is the portfolio choice policy. Note in particular the choice variable $A(t)$ enters both into the drift and volatility of the wealth process.

²⁶ It should be noted that in a similar spirit, Carpenter (2000) considers a delegated portfolio choice problem of which through the portfolio choice policy, the agent can choose the volatility of the value of the asset portfolio. There are no private benefits or costs to the agent in choosing a particular portfolio choice policy. Exogenously, the principal compensates the agent (only) with a call option with the strike price being the terminal value of the managed portfolio.

does incur private benefits (or equivalently, negative private costs) of controlling the unobservable volatility level of cash flows and so the principal must provide incentives on both effort and volatility.

Cadenillas et al. (2007) allows for the agent to control both the drift and volatility but they explicitly consider a first-best risk-sharing setup whereby agency problems are absent.

Cvitanic, Possamai, and Touzi (2014)

The closest work in the literature to our paper is the contemporary work by Cvitanic, Possamai, and Touzi (2014).²⁷ The goal of that paper is also to investigate under what conditions would there be meaningful volatility control by the agent. The authors propose that if the principal can only observe managed cash flows Y continuously overtime, and if the agent controls a *vector* of volatilities $\nu \in \mathbb{R}_+^d$, so that the managed cash flows have the form,

$$Y_t = \int_0^t \nu_s \cdot (bd_t + dB_t),$$

where B is a d -dimensional Brownian motion, and $b \in \mathbb{R}^d$ is a common knowledge constant vector. Again, since the principal can continuously observe the managed cash flows Y , then the principal can compute the quadratic variation of Y and obtain an (integrated) matrix,

$$d[Y]_t = \nu_t \cdot \nu_t dt,$$

where, of course, $\nu \cdot \nu'$ is a scalar. But given that the principal can only observe Y , there is no way the principal can decipher the individual managed elements of $\nu \cdot \nu$. Thus, this yields to a setup where there is meaningful volatility control. It should be immediately noted that in the setup of Cvitanic et al. (2014), the dimensionality d plays a critical role. That is, if $d = 1$, then we collapse back to the case where there is no meaningful volatility control. Hence, in their setup, the dimensionality must be $d \geq 2$.

Cvitanic et al. (2014) considers a setup where the agent is only paid at a final deterministic time T and both the principal and the agent have identical CARA utility, and the agent has a quadratic cost in volatility choices. In contrast, our paper considers only risk neutral principal and agent, but allow for intertemporal compensation from the principal to the agent, endogenous termination of the agent, and also private effort and volatility choices by the agent. Moreover, fundamentally, our methods for “hiding” volatility control are fundamentally different in that Cvitanic et al. (2014) relies on a dimensionality argument, whereas this paper relies on reconsidering economically the modeling method of the noise term. It could be an interesting extension for future research to combine both of these approaches.

1.C Proofs of Section 1.6

First let's define,

$$V_t(A) := \mathbb{E}^A \left[\int_0^\tau e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0(\tau-t)} R \mid \mathcal{F}_t^Y \right] \quad (1.C.1)$$

It is important to note that both (1.6.1) and (1.C.1), which represents the continuation value of the agent at time t if the action process A is being taken, are conditioned on the filtration $\{\mathcal{F}_t^Y\}$. That is, the information observable to the principal. In particular, the filtration that is being conditioned on is not a Brownian one. This is a key and important departure from the usual papers in continuous-time principal agent problems. In providing incentives to the agent, since the principal can only observe the cash flows Y , this implies the continuation value of the agent, from the perspective of the principal, can only condition on the information $\{\mathcal{F}_t^Y\}$ generated by the cash flows Y , and hence (1.6.1) and (1.C.1) have the correct conditioning.

Remark 1.C.1. Consider the typical setup of DeMarzo and Sannikov (2006) in the form (1.C.2),

$$dY_t = \mu_t dt + \sigma dB_t. \quad (1.C.2)$$

For a fixed recommended action $\{\mu_t\}_{t \geq 0}$, and since the principal observes the cash flows $\{Y_t\}_{t \geq 0}$, by simply rearranging terms, we see that,

$$\frac{dY_t - \mu_t dt}{\sigma} = dB_t.$$

²⁷ This author was not aware of the presence of this paper (with a working date of March 7, 2014) until mid April 2014, but by then, a well working draft of this current paper had already been written and indeed circulated in small private circles.

Indeed, this is the key step to the analysis of both DeMarzo and Sannikov (2006) and Sannikov (2008). And so, in this case, once the action process $\{\mu_t\}_{t \geq 0}$ is fixed²⁸, then the left-hand side is completely observable by the principal. And thus, in these types of setup, the information set available to the principal $\{\mathcal{F}_t^Y\}_{t \geq 0}$ is exactly identical to the Brownian information set $\{\mathcal{F}_t\}_{t \geq 0}$, and so we do achieve $\mathbb{E}^A[\cdot | \mathcal{F}_t^Y] = \mathbb{E}[\cdot | \mathcal{F}_t]$. That is, in words, if the principal knows the action process and can observe the cash flows, that means he must also know the Brownian motion noise.

However, in this current setup of (1.4.1) and (1.4.2), this is clearly not the case. In particular, even if the agent knows the action process and the cash flows, he does not know the Brownian motion noise, and so we have a clear inequality, $\mathbb{E}^A[\cdot | \mathcal{F}_t^Y] \neq \mathbb{E}^A[\cdot | \mathcal{F}_t]$. To see this, observe that even if we repeat the above rearranging computation,

$$dY_t - \mu_t dt = \sigma_t M_t dB_t,$$

where specifically, we cannot “divide” over the generically not constant over time volatility choice σ_t (i.e. simply write out the above SDE in its integrated form). Recall also that m is not observable to the principal. And from this expression, we see that even if an action process $A = \{(\mu_t, \sigma_t)\}_{t \geq 0}$ is known to the principal, his information set \mathcal{F}_t^Y cannot equal to the information set generated by Brownian motion \mathcal{F}_t .

A trivial rewriting

In light of Remark 1.C.1, it motivates for the following rewriting. The idea is to *not* think of noise driven by Brownian motion but rather driven by a more general continuous martingale process. In particular, observe, trivially, from (1.4.1) and (1.4.2), we can write,

$$dY_t = \mu_t dt + \sigma_t dM_t. \quad (1.C.3)$$

In particular, note that from (1.4.2), it is a geometric Brownian motion with zero drift and unit volatility. Thus, we have an explicit solution,

$$M_t = M_0 \exp\left\{-\frac{1}{2}t + B_t\right\}, \quad t \geq 0. \quad (1.C.4)$$

For all the proofs that follow, we take, without loss of generality that $M_0 \equiv 1$; the proof goes through with a generic $M_0 = m_0$ but we just have to carry more algebra.

Proposition 1.C.2. *For a fixed contract (A, X, τ) , the stochastic process $t \mapsto \int_0^t \sigma_s dm_s$ is an $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale.*

Proof. First, from (1.C.3) and again since the action process A is held fixed, we write,

$$Y_t - \mu_t dt = \sigma_t dM_t. \quad (1.C.5)$$

From (1.C.4), we see immediately that M is a true $(P, \{\mathcal{F}_t\})$ -martingale (i.e. the Dolean exponential with respect to Brownian motion). Since σ_t is $\{\mathcal{F}_t\}$ -adapted and since m is a P -square integrable continuous martingale, then this implies that $t \mapsto \int_0^t \sigma_s dM_s$ is also a $(P, \{\mathcal{F}_t\})$ -martingale.

Now, let's show that $t \mapsto \int_0^t \sigma_s dM_s$ is also a $(P, \{\mathcal{F}_t^Y\})$ -martingale. Observe that $t \mapsto \int_0^t \sigma_s dM_s$ is \mathcal{F}_t^Y -adapted. But this is immediate by viewing the left-hand side of (1.C.5) and recalling footnote 15. It remains to verify the martingale property. Pick any time $t_1 \geq t_0$. Since $t \mapsto \int_0^t \sigma_s dM_s$ is a martingale, then we immediately have that,

$$E\left[\int_{t_0}^{t_1} \sigma_s dM_s \mid \mathcal{F}_{t_0}\right] = 0.$$

But by applying the Law of Iterated Expectations,

$$E\left[\int_{t_0}^{t_1} \sigma_s dM_s \mid \mathcal{F}_{t_0}^Y\right] = E\left[E\left[\int_{t_0}^{t_1} \sigma_s dM_s \mid \mathcal{F}_{t_0}\right] \mid \mathcal{F}_{t_0}^Y\right] = 0,$$

so the martingale property holds for $\{\mathcal{F}_t^Y\}$. Finally, the fact that $t \mapsto \int_0^t \sigma_s dM_s$ is a $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale follows immediately. \blacksquare

This trivial rewriting of (1.C.3) and observation of Proposition 1.C.2 that $\sigma_t dM_t$ is an $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale are the key steps to the beginning of our analysis. Indeed, to emphasize the importance of the observation in Proposition 1.C.2, we will in the subsequent few sections, denote explicitly the $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale as,

$$\sigma_t dM_t^A = Y_t - \mu_t dt. \quad (1.C.6)$$

²⁸ More precisely, this means we work under the induced probability measure \mathbb{P}^A , where $A = \{\mu_t\}_{t \geq 0}$.

Martingale Representation Theorem

With Proposition 1.C.2 in mind and also the cash flow process in the form (1.C.3), we can now state the following key proposition.

Proposition 1.C.3. *Fix a contract (A, X, τ) . Then there exist processes $\{\beta_t\}_{t \geq 0}$ and $\{V_t^{\perp, A}\}_{t \geq 0}$ such that the dynamics of $V_t(A)$ in (1.C.1) can be written as,*

$$dV_t(A) = e^{-r_0 t} \beta_t \sigma_t dM_t^A + dV_t^{\perp, A}, \quad (1.C.7)$$

where β_t is some predictable process such that $\int_0^t (e^{-r_0 s} \beta_s)^2 d[\int_0^\cdot \sigma_u dM_u^A]_s < \infty$, \mathbb{P}^A -a.s., and $V_t^{\perp, A}$ is continuous and orthogonal to $t \mapsto \int_0^t \sigma_s dM_s^A$ (i.e. meaning, $[V^{\perp, A}, \int_0^\cdot \sigma_s dM_s^A]_t = 0$, \mathbb{P}^A -a.s.), and $V_0^{\perp, A} = 0$.

Proof. It is easy to verify that (1.C.1) is a $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale (i.e. Doob's martingale). Then the result is an immediate consequence of Proposition 1.C.2 and the (general) martingale representation theorem. For the precise statement for this general martingale representation theorem result, please see Hunt and Kennedy (2004, Chapter 5, Theorem 5.37), Protter (2005, Chapter IV, Section 3, Corollary 1) and Revuz and Yor (2005, Chapter V, Section 4, Lemma 4.2). ■

Remark 1.C.4. *In the term $e^{-r_0 t} \beta_t \sigma_t dM_t^A$, the time discount factor term $e^{-r_0 t}$ is merely a convenient normalization; this is also done in Sannikov (2008). Also, strictly speaking, β is clearly dependent on the choice of the action process A but will suppress it for notational convenience when the context is clear. We keep the notation A on the orthogonal process $V_t^{\perp, A}$ as the choice of A will make a meaningful difference in the subsequent discussions.*

Compared to the papers like Holmström and Milgrom (1987), DeMarzo and Sannikov (2006), Sannikov (2008), and others, they all invoke a martingale representation theorem for the case when the filtration is generated by Brownian motion (i.e. see Karatzas and Shreve (1991, Chapter 3, Theorem 4.2), among others); recall again the discussion in Remark 1.C.1 on why a Brownian filtration setup here is inappropriate. Specifically in the Brownian case, the orthogonal term, denoted above as $V_t^{\perp, A}$, would be identically zero. It is also worth noting that in this line of continuous-time principal-agent literature, there have been some notable cases where the filtration is not Brownian. For instance, in Sannikov (2007, Proposition 1), the “extra” orthogonal term is interpreted as a public randomization device.

Dynamics of the agent's continuation value

With Proposition 1.C.3 in mind, the following is an easy application of Ito's lemma.

Proof of Theorem 1.6.1. From (1.C.1), we can write,

$$V_t(A) = \int_0^t e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] \right) ds + e^{-r_0 t} W_t(A). \quad (1.C.8)$$

Applying Ito's lemma, we obtain,

$$dV_t(A) = e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) + \underbrace{d(e^{-r_0 t} W_t(A))}_{-r_0 e^{-r_0 t} W_t dt + e^{-r_0 t} dW_t}. \quad (1.C.9)$$

Equating (1.C.7) with (1.C.9), we obtain,

$$\begin{aligned} & e^{-r_0 t} \beta_t \sigma_t dM_t^A + dV_t^{\perp, A} \\ &= e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) - r_0 e^{-r_0 t} W_t(A) dt + e^{-r_0 t} dW_t(A) \end{aligned} \quad (1.C.10)$$

Defining $\epsilon_t^{\perp, A} := \int_0^t e^{r_0 s} dV_s^{\perp, A}$, rearranging, and recalling that $\sigma_t dM_t = Y_t - \mu_t dt$, we obtain,

$$dW_t(A) = r_0 W_t(A) dt - \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) + \beta_t (dY_t - \mu_t dt) + d\epsilon_t^{\perp, A}, \quad (1.C.11)$$

and we are done. ■

Checking for Deviations

With Theorem 1.6.1 on hand, we are now ready to give a simple condition (hopefully that is both sufficient and necessary) to pin down the incentive compatibility constraints of the agent. Fix two action processes²⁹ $A = \{(\mu_t, \sigma_t)\}_{t \geq 0}$ and $A^\dagger = \{(\mu_t^\dagger, \sigma_t^\dagger)\}_{t \geq 0}$.

If the agent plays A , then the agent's time zero continuation value is $W_0(A)$ as in (1.6.1). But suppose the agent deviates to A^\dagger . Specifically, the cash flow processes under the two different action processes evolve as³⁰,

$$\text{Under } \mathbb{P}^A: \quad dY_t = \overline{\mu}_t dt + \sigma_t dM_t^A. \quad (1.C.12)$$

$$\text{Under } \mathbb{P}^{A^\dagger}: \quad dY_t = \mu_t^\dagger dt + \sigma_t^\dagger dM_t^{A^\dagger}. \quad (1.C.13)$$

Phrased in this light, this strongly calls for a change-of-measure type analysis. To do so, we need to invoke a stronger version of Girsanov's theorem, which is usually applied in a Brownian setting. Here, we will use the slightly more general Girsanov-Meyer theorem. We will reiterate it here for reference:

Girsanov-Meyer Theorem. *Let P and Q be equivalent measures. Let X be a continuous (classical) semimartingale under P with decomposition $X = M + A$. Then X is also a continuous (classical) semimartingale under Q and has decomposition $X = L + C$, where*

$$L_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s \quad (1.C.14)$$

is a Q local martingale, and $C = X - L$ is a Q finite variation process.

Proof. For details of the theorem and its proof, please see Protter (2005, Chapter III, Section 8, Theorem 39). ■

Using the Girsanov-Meyer theorem as a guide, and we can set $P = \mathbb{P}^A$ and $Q = \mathbb{P}^{A^\dagger}$, then it is natural to set,

$$A_t = \int_0^t \mu_s ds, \quad M_s = \int_0^t \sigma_s dM_s^A, \quad (1.C.15)$$

and,

$$C_t = \int_0^t \mu_s^\dagger ds, \quad L_s = \int_0^t \sigma_s^\dagger dM_s^{A^\dagger}. \quad (1.C.16)$$

Then, to ensure that the correct change of measure is possible, it remains to identify the process Z .

In particular, we define the Radon-Nikodym derivative as,

$$Z_t = \mathbb{E}^A \left[\frac{d\mathbb{P}^{A^\dagger}}{d\mathbb{P}^A} \mid \mathcal{F}_t^Y \right]. \quad (1.C.17)$$

To continue the discussion, we will need the following mild technical assumption that we have an appropriate kernel.

Assumption 1.C.5. *Suppose associated with the Radon-Nikodym derivative in (1.C.17), there exists a square integrable process $\{\varphi_t\}_{t \geq 0}$ such that,*

$$dZ_t = \varphi_t Z_t \sigma_t dM_t^A. \quad (1.C.18)$$

With Assumption 1.C.5 on hand, and letting $N_t := \varphi_t \sigma_t dM_t^A$, we see that,

$$Z_t = \mathcal{E}(N)_t, \quad (1.C.19)$$

²⁹ Again, recalling the notation convention in footnote 11, what we mean here is to fix two action processes $A = \{(e_t, \sigma_t)\}_{t \geq 0}$ and $A^\dagger = \{(e_t^\dagger, \sigma_t^\dagger)\}_{t \geq 0}$, then define $\mu_t \equiv \kappa(e_t, \sigma_t)$ and $\mu_t^\dagger \equiv \kappa(e_t^\dagger, \sigma_t^\dagger)$.

³⁰ To be precise, we mean the following and please see Proposition 1.C.2 again. Under \mathbb{P}^A , we have the $(\mathbb{P}^A, \{\mathcal{F}_t^Y\})$ -martingale $\sigma_t dM_t^A = dY_t - \mu_t dt$; and under \mathbb{P}^{A^\dagger} , we have the $(\mathbb{P}^{A^\dagger}, \{\mathcal{F}_t^Y\})$ -martingale $\sigma_t^\dagger dM_t^{A^\dagger} = dY_t - \mu_t^\dagger dt$. Rearrange to obtain the two displayed equations.

the Dolean's exponential³¹ for the process N . For now, let's suppose that Z is a true $\{\mathcal{F}_t^Y\}$ -martingale but we will verify this in the subsequent discussion³². Thus, it remains to find φ_t . Note in the above, we have two expressions for L , (1.C.14) as given in the Girsanov-Meyer theorem statement, and also in (1.C.16). Equating these two expressions for L , we find,

$$L_t = \int_0^t \sigma_s dM_s^A - \int_0^t \varphi_s d \left[\int_0^\cdot \sigma_u dM_u^A \right]_s = \int_0^t \sigma_s^\dagger dM_s^{A^\dagger}, \quad (1.C.20)$$

and since $dM_t^A = M_t^A dB_t^A$ then we immediately have by the quadratic variation of Brownian motion,

$$d \left[\int_0^\cdot \sigma_u dM_u^A \right]_t = \sigma_t^2 (M_t^A)^2 dt.$$

Rewriting everything in differential form and substituting, we thus have that,

$$\sigma_t dM_t^A - \varphi_t \sigma_t^2 (M_t^A)^2 dt = \sigma_t^\dagger dM_t^{A^\dagger}.$$

But recall again that $\sigma_t dM_t^A = dY_t - \mu_t dt$ and $\sigma_t^\dagger dM_t^{A^\dagger} = dY_t - \mu_t^\dagger dt$, so we substitute,

$$dY_t - \mu_t dt - \varphi_t \sigma_t^2 (M_t^A)^2 dt = dY_t - \mu_t^\dagger dt.$$

Canceling terms and equating, we have thus,

$$\int_0^t \varphi_s \sigma_s^2 M_s^2 ds = \int_0^t (\mu_s^\dagger - \mu_s) ds.$$

But since the integrands on the left-hand side and the right-hand side are well bounded, this immediately implies the integrands must equal. And thus, we have that the Girsanov kernel is,

$$\varphi_t = \frac{\mu_t^\dagger - \mu_t}{\sigma_t^2 (M_t^A)^2}. \quad (1.C.21)$$

Now in particular, we have the following important relationship.

Lemma 1.C.6. *Fix a contract (A, X, τ) . Suppose the agent considers the recommended action process A and fixes another action process A^\dagger . The agent considers a deviation from A to A^\dagger . Then the noise terms are related in the following manner:*

$$\sigma_t dM_t^A = (\mu_t^\dagger - \mu_t) dt + \sigma_t^\dagger dM_t^{A^\dagger}. \quad (1.C.22)$$

Proof. Using the Girsanov kernel in (1.C.21), simply substitute,

$$\begin{aligned} \sigma_t dM_t^A - \varphi_t \sigma_t^2 (M_t^A)^2 dt &= \sigma_t dM_t^A - \frac{(\mu_t^\dagger - \mu_t)}{\sigma_t^2 (M_t^A)^2} \sigma_t^2 (M_t^A)^2 dt \\ &= \sigma_t^\dagger dM_t^{A^\dagger}. \end{aligned}$$

Rearrange, and we get (1.C.22). ■

³¹ The Dolean's exponential $\mathcal{E}(N)_t$ for a continuous local martingale S is given by,

$$\mathcal{E}(S)_t := \exp \left\{ S_t - S_0 - \frac{1}{2} [S]_t \right\}.$$

³² Of course, with the equivalence of \mathbb{P}^A and \mathbb{P}^{A^\dagger} , there is no need to specify for which probability measure is Z a martingale.

Change of Measure and Novikov's Criterion

However, an important task remains — we need to verify that the Dolean's exponential $\mathcal{E}(N)$ for N is a true martingale to allow for a valid change of measure. With the Girsanov kernel given in (1.C.21), and recalling again that $dM_t^A = M_t^A dB_t^A$, we substitute back to see that,

$$dN_t = \frac{\mu_t^\dagger - \mu_t}{\sigma_t(M_t^A)^2} M_t^A dB_t^A = \frac{\mu_t^\dagger - \mu_t}{\sigma_t M_t^A} dB_t^A$$

But this implies N has quadratic variation,

$$[N]_t = \int_0^t \frac{(\mu_s^\dagger - \mu_s)^2}{\sigma_s^2 (M_s^A)^2} ds.$$

In all, that means to ensure that $\mathcal{E}(N)$ is a true martingale, a sufficient condition is to ensure that the Novikov's criterion³³ holds. For what follows, consider a fixed deterministic time horizon $T < \infty$. The *Novikov's criterion* requires,

$$\mathbb{E}^A \left[\exp \left\{ \frac{1}{2} \int_0^T \frac{(\mu_t^\dagger - \mu_t)^2}{\sigma_t^2 (M_t^A)^2} dt \right\} \right] < \infty. \quad (1.C.23)$$

Lemma 1.C.7. *The Novikov criterion (1.C.23) holds for finite time horizon $T < \infty$ and infinite time horizon $T = \infty$. And thus, $\mathcal{E}(N)_t$ is a true martingale for all $t \in [0, \infty]$.*

Proof. Recall again that M^A itself is a geometric Brownian motion with zero drift and unit volatility. Thus, we can immediately write $M_t^A = \exp \left\{ -\frac{1}{2}t + B_t^A \right\}$; we have taken, without loss of generality $M_0 = 1$. Furthermore, since we know that $\mu_t, \mu_t^\dagger \in [\mu_L, \mu_H]$ and likewise $\sigma_t, \sigma_t^\dagger \in [\sigma_L, \sigma_H]$, then we clearly have,

$$|\mu_t^2 - \mu_t^\dagger|^2 \leq 2\mu_H^2, \quad \frac{1}{\sigma_L^2} \geq \frac{1}{\sigma_t^2}, \quad \frac{1}{\sigma_L^2} \geq \frac{1}{(\sigma_t^\dagger)^2}.$$

Substituting,

$$\begin{aligned} \mathbb{E}^A \left[\exp \left\{ \frac{1}{2} \int_0^T \frac{(\mu_t^\dagger - \mu_t)^2}{\sigma_t^2 (M_t^A)^2} dt \right\} \right] &\leq \mathbb{E}^A \left[\exp \left\{ \frac{1}{2} \frac{2\mu_H^2}{\sigma_L^2} \int_0^T \frac{1}{(M_t^A)^2} dt \right\} \right] \\ &= e^{\mu_H^2/\sigma_L^2} \mathbb{E}^A \left[\exp \left\{ \int_0^T \frac{1}{(M_t^A)^2} dt \right\} \right] \\ &= e^{\mu_H^2/\sigma_L^2} \mathbb{E}^A \left[\exp \left\{ \int_0^T e^{t-2B_t^A} dt \right\} \right] \end{aligned}$$

Thus, the problem reduces now to proving that, $\mathbb{E}^A \left[\exp \left\{ \int_0^T e^{t-2B_t^A} dt \right\} \right] < \infty$. It should be noted that this is a non-trivial problem since we essentially have to show that the expectation of the exponential of an integrated geometric Brownian motion is finite. As noted from Yor (1992), the aforementioned problem is essentially the same as investigating the properties of,

$$\int_0^t e^{aB_s + bs} ds, \quad a, b \in \mathbb{R}.$$

But by scaling properties of Brownian motion³⁴ B , it suffices to consider the process,

$$A_t^{(\nu)} := \int_0^t e^{2(B_s + \nu s)} ds, \quad \nu \in \mathbb{R}. \quad (1.C.24)$$

Hence, to solve our problem of showing (1.C.23), it is equivalent to showing that,

$$\mathbb{E} \left[\exp \left\{ A_t^{(\nu)} \right\} \right] < \infty. \quad (1.C.25)$$

³³ See Protter (2005, Chapter III, Section 8, Theorem 45), Revuz and Yor (2005, Proposition 1.15, Corollary 1.16) and also Duffie (2001, Appendix D)

³⁴ For this discussion, it suffices to suppress the dependence on action process A .

But to show that (1.C.25) is finite is equivalent to showing that the Laplace transform (moment generating function) of $A_t^{(\nu)}$ is well defined and finite. But using Yor (1992, Equation (7.e)) as pointed out by Kim (2004) (see also Albanese and Lawi (2005)), we are ensured that the aforementioned Laplace transform is well defined and finite.

Thus, this implies that (1.C.23) does indeed hold for each finite $T < \infty$. However, given in this model, we allow for a termination time τ that could be finite (i.e. terminating the agent at some time) or infinite (i.e. never terminating the agent), considering the deterministic finite time case is insufficient. But invoking Revuz and Yor (2005, Chapter VIII, §1, Corollary 1.16), we can extend the discussion from finite time interval $[0, T]$ for $T < \infty$ to $[0, \infty]$. Thus, this shows that $\mathcal{E}(N)_t$ is a martingale for all times $t \in [0, \infty]$. ■

The following lemma will be useful when we further characterize the effects of deviation in the subsequent discussion.

Lemma 1.C.8. *Fix a contract (A, X, τ) . Consider the setup and the process $\epsilon^{\perp, A}$ as defined in Theorem 1.6.1. Fix another action process A^\dagger . The stochastic process $\epsilon^{\perp, A}$ is a $\{\mathcal{F}_t^Y\}$ -martingale under both probability measures \mathbb{P}^A and \mathbb{P}^{A^\dagger} .*

Proof. The fact that $\epsilon^{\perp, A}$ is a $(\mathbb{P}^A, \mathcal{F}_t^Y)$ -martingale is immediate from the fact that by Proposition 1.C.3, $V^{\perp, A}$ is a $(\mathbb{P}^A, \mathcal{F}_t^Y)$ martingale.

Thus, it remains to prove that $\epsilon_t^{\perp, A}$ is a $(\mathbb{P}^{A^\dagger}, \mathcal{F}_t^Y)$ -martingale. Clearly, this is equivalent to showing that $V_t^{\perp, A}$ is an (\mathbb{P}^{A^\dagger}) -martingale. But thanks to Lemma 1.C.7, this is equivalent to showing that for any $s \geq t$, we have,

$$V_t^{\perp, A} = \mathbb{E}^{A^\dagger} [V_s^{\perp, A} | \mathcal{F}_t^Y] = \mathbb{E}^A \left[V_s^{\perp, A} \frac{d\mathbb{P}^{A^\dagger}}{d\mathbb{P}^A} \Big|_{\mathcal{F}_s^Y} \Big| \mathcal{F}_t^Y \right]$$

Hence, we see that it suffices to prove that the stochastic process as a product, $t \mapsto V_t^{\perp, A} \frac{d\mathbb{P}^{A^\dagger}}{d\mathbb{P}^A} \Big|_{\mathcal{F}_t^Y}$ is an \mathcal{F}_t^Y -martingale ³⁵

But we observe that $V_t^{\perp, A}$ is a square-integrable martingale and likewise for $d\mathbb{P}^{A^\dagger}/d\mathbb{P}^A \Big|_{\mathcal{F}_t^Y} = \mathcal{E}(N)_t$. Then using the notion of strongly orthogonal martingales in, say, Protter (2005, Chapter IV, §3), to show that the product $V_t^{\perp, A} \mathcal{E}(N)_t$ is a martingale, it is equivalent to showing,

$$[V^{\perp, A}, \mathcal{E}(N)]_t \text{ is a uniformly integrable martingale.}$$

But observe that since $dV_t^{\perp, A}$ and $\sigma_t dM_t^A$ are orthogonal (i.e. recall Proposition 1.C.3), then computing the quadratic covariation (here, for convenience, we use the differential notation),

$$\begin{aligned} dV_t^{\perp, A} d\mathcal{E}(N)_t &= dV_t^{\perp, A} \mathcal{E}(N)_t \varphi_t \sigma_t dM_t^A \\ &= \mathcal{E}(N)_t \varphi_t \underbrace{(dV_t^{\perp, A})(\sigma_t dM_t^A)}_{=0} \\ &= 0. \end{aligned}$$

Hence, we have that $[V^{\perp, A}, \mathcal{E}(N)]_t = 0$ (i.e. a constant stochastic process, which is a trivial martingale). Thus, we have that $V^{\perp, A}$ is also a martingale under \mathbb{P}^{A^\dagger} , in addition to being a martingale under \mathbb{P}^A . ■

Characterizing Deviations

Proof of Lemma 1.6.2. As before, fix a contract (A, X, τ) . And fix another action process A^\dagger . Consider a deviation from A to A^\dagger . Let's do some preliminary computations before showing the equivalence of (i) and (ii). Define,

$$\hat{V}_t := \int_0^t e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s^\dagger}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0 t} W_t(A), \quad (1.C.26)$$

which is the time t expectation of the agent's total payoff if he experienced the cost of effort from the action process A^\dagger before time t , and plans to follow the recommended action process A after time t . Let's write the dynamics of \hat{V}_t under the measure

³⁵ Note by the equivalence in measures, we no longer need to be explicit about the probability measure for which this is a martingale.

\mathbb{P}^{A^\dagger} . Observe that in differential form, and noting the expression in (1.C.10), and equating,

$$\begin{aligned} d\hat{V}_t &= e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t^\dagger}{\sigma_L} - 1 \right) \right] dt \right) + d(e^{-r_0 t} W_t(A)) \\ &= e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t^\dagger}{\sigma_L} - 1 \right) \right] dt \right) \\ &\quad + e^{-r_0 t} \beta_t^A \sigma_t dm_t^A + dV_t^{\perp, A} - e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right). \end{aligned} \quad (1.C.27)$$

But using (1.C.22) of Lemma (1.C.6), and collecting terms, we can further rewrite (1.C.27) as,

$$\begin{aligned} d\hat{V}_t &= e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t^\dagger}{\sigma_L} - 1 \right) \right] dt \right) \\ &\quad + e^{-r_0 t} \beta_t \left\{ [\kappa(e_t^\dagger, \sigma_t^\dagger) - \kappa(e_t, \sigma_t)] dt + \sigma_t^\dagger dm_t^{A^\dagger} \right\} + dV_t^{\perp, A} \\ &\quad - e^{-r_0 t} \left(dX_t + \left[\phi_e \left(1 - \frac{e_t}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt \right) \\ &= e^{-r_0 t} \left[-\phi_e \frac{e_t^\dagger - e_t}{e_H} + \phi_\sigma \frac{\sigma_t^\dagger - \sigma_t}{\sigma_L} + \beta_t [\kappa(e_t^\dagger, \sigma_t^\dagger) - \kappa(e_t, \sigma_t)] \right] dt \\ &\quad + dV_t^{\perp, A} + e^{-r_0 t} \beta_t \sigma_t^\dagger dM_t^{A^\dagger} \end{aligned} \quad (1.C.28)$$

Thus, writing in integrated form for both (1.C.26) and (1.C.28), and equating,

$$\begin{aligned} \hat{V}_t &= \hat{V}_0 + \int_0^t e^{-r_0 s} \left[-\phi_e \frac{e_s^\dagger - e_s}{e_H} + \phi_\sigma \frac{\sigma_s^\dagger - \sigma_s}{\sigma_L} + \beta_s [\kappa(e_s^\dagger, \sigma_s^\dagger) - \kappa(e_s, \sigma_s)] \right] ds \\ &\quad + V_t^{\perp, A} + \int_0^t e^{-r_0 s} \beta_s \sigma_s^\dagger dM_s^{A^\dagger} \end{aligned} \quad (1.C.29)$$

$$= \hat{V}_0 + \int_0^t e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s^\dagger}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0 t} W_t(A). \quad (1.C.30)$$

Let's consider taking the time 0 expectation of \hat{V}_t in (1.C.29) above under \mathbb{P}^{A^\dagger} . From Lemma 1.C.8, we have that $\mathbb{E}^{A^\dagger}[V_t^{\perp, A}] = 0$. Furthermore, note that the stochastic process $t \mapsto e^{-r_0 t} \beta_t \sigma_t^\dagger$ is an \mathcal{F}_t^Y square integrable martingale, since the integrator $m_t^{A^\dagger}$ is a square integrable martingale and the terms in the integrand are well bounded; see Protter (2005, Chapter IV, §2, Theorem 11). Thus, in expectation, the last two terms in the sum of (1.C.29) vanish. Thus, picking any two times $t \geq t_0 \geq 0$ in mind, observe that, under condition (i),

$$\begin{aligned} &\mathbb{E}^{A^\dagger} \left[\hat{V}_t - \hat{V}_{t_0} \mid \mathcal{F}_{t_0}^Y \right] \\ &= \mathbb{E}^{A^\dagger} \left[\int_{t_0}^t e^{-r_0 s} \left[-\phi_e \frac{e_s^\dagger - e_s}{e_H} + \phi_\sigma \frac{\sigma_s^\dagger - \sigma_s}{\sigma_L} + \beta_s [\kappa(e_s^\dagger, \sigma_s^\dagger) - \kappa(e_s, \sigma_s)] \right] ds \mid \mathcal{F}_{t_0}^Y \right] \\ &\leq 0. \end{aligned}$$

Specifically, this implies that \hat{V}_t is an \mathcal{F}_t^Y -supermartingale (under both probability measures \mathbb{P}^A and \mathbb{P}^{A^\dagger} , by Lemma 1.C.7).

Now, let's show that (i) \implies (ii). The above supermartingale property implies, $\mathbb{E}^{A^\dagger}[\hat{V}_t] \leq \hat{V}_0 = W_0(A)$. Rewriting the left-hand side of the this inequality, and using (1.C.30), we have that,

$$\begin{aligned} &\mathbb{E}^{A^\dagger} \left[\int_0^t e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s^\dagger}{\sigma_L} - 1 \right) \right] ds \right) \right] + \mathbb{E}^{A^\dagger} [e^{-r_0 t} W_t(A)] \\ &\leq \hat{V}_0 \\ &= W_0(A) \\ &= \mathbb{E}^A \left[\int_0^\tau e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0 \tau} R \right]. \end{aligned}$$

Since $t \geq 0$ was arbitrary, set it to $t = \tau$ and note that $W_\tau(A) = R$ a.s. (both in \mathbb{P}^A and in \mathbb{P}^{A^\dagger}), then we have that,

$$\begin{aligned} & \mathbb{E}^{A^\dagger} \left[\int_0^\tau e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s^\dagger}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s^\dagger}{\sigma_L} - 1 \right) \right] ds \right) + e^{-r_0 \tau} R \right] \\ & \leq \mathbb{E}^A \left[\int_0^\tau e^{-r_0 s} \left(dX_s + \left[\phi_e \left(1 - \frac{e_s}{e_H} \right) + \phi_\sigma \left(\frac{\sigma_s}{\sigma_L} - 1 \right) \right] \right) ds + e^{-r_0 \tau} R \right] \end{aligned}$$

Thus, if the recommended action process is A , then it is not optimal for the agent to deviate to A^\dagger . Thus, (i) \implies (ii) holds.

Let's show that (ii) \implies (i) holds. We will prove by contrapositive. Suppose that (i) does not hold on a set of non-zero measure (again, both under \mathbb{P}^A or \mathbb{P}^{A^\dagger}). Let's show that deviating away from A is optimal (i.e. A is suboptimal). On the set of times with non-zero measure such that (1.6.5) does not hold for some time s and some $(e, \sigma) \in \{e_L, e_H\} \times [\sigma_L, \sigma_H]$. But this implies we can construct an action process \tilde{A} such that there would exist some time t' such that,

$$\mathbb{E}^{\tilde{A}}[\hat{V}_{t'}] > \hat{V}_0 = W_0(A).$$

But since the agent gets utility $\mathbb{E}^{\tilde{A}}[\hat{V}_{t'}]$ if he follows \tilde{A} until time t' and switches to A , the action process A is suboptimal. This shows that (ii) \implies (i). \blacksquare

Proof of Corollary 1.6.3. Consider (1.6.6). To consider a deviation from the recommended action (e_t, σ_t) to the deviated action (e', σ') , that is $(e_t, \sigma_t) \neq (e', \sigma')$, we have three cases to consider:

- (i) $e_t \neq e', \sigma_t \neq \sigma'$;
- (ii) $e_t = e', \sigma_t \neq \sigma'$; and
- (iii) $e_t \neq e', \sigma_t = \sigma'$.

Case (i): Suppose $e_t \neq e'$ and $\sigma_t \neq \sigma'$. Let's prove the case when $e_t = e_H$. Then we must have that $e' = e_L$. So, we have,

$$0 \geq -\frac{\phi_e}{e_H}(e_L - e_H) + \frac{\phi_\sigma}{\sigma_L}(\sigma' - \sigma_t) + \beta_t [\kappa(e_L, \sigma') - \kappa(e_H, \sigma_t)].$$

But from Definition 1.4.1(c), we have $\kappa(e_L, \sigma') - \kappa(e_H, \sigma_t) < 0$. Rearranging the above, we have that,

$$\begin{aligned} \beta_t & \geq \frac{1}{\kappa(e_L, \sigma') - \kappa(e_H, \sigma_t)} \left[\frac{\phi_e}{e_H}(e_L - e_H) - \frac{\phi_\sigma}{\sigma_L}(\sigma' - \sigma_t) \right] \\ & = \frac{1}{\kappa(e_H, \sigma_t) - \kappa(e_L, \sigma')} \left[\frac{\phi_e}{e_H}(e_H - e_L) + \frac{\phi_\sigma}{\sigma_L}(\sigma' - \sigma_t) \right], \end{aligned} \tag{1.C.31}$$

for all $\sigma' \in [\sigma_L, \sigma_H]$. But we observe immediately that the inequality (1.C.31) holds if and only if (1.6.6) holds. This shows the equivalence of (1.6.6) with (1.6.5) when $e_t = e_H$. The case of when $e_t = e_L$ is proved similarly.

Case (ii): Suppose that $e_t = e' = e$ and $\sigma_t \neq \sigma'$. Suppose first if $\sigma' > \sigma_t$, which implies $\kappa(e, \sigma') - \kappa(e, \sigma_t) > 0$. Then we have,

$$\frac{\phi_\sigma}{\sigma_L}(\sigma_t - \sigma') \geq \beta_t (\kappa(e, \sigma') - \kappa(e, \sigma_t)),$$

which implies that $0 > \beta_t$ — contradiction. That is to say, if the principal's recommended volatility is σ_t , the agent will not deviate to a higher volatility $\sigma' > \sigma_t$. Next, if $\sigma' < \sigma_t$, so $\kappa(e, \sigma') - \kappa(e, \sigma_t) < 0$, then we have,

$$\frac{\phi_\sigma}{\sigma_L}(\sigma_t - \sigma') \geq \beta_t (\kappa(e, \sigma') - \kappa(e, \sigma_t)),$$

which implies β_t is greater than or equal to some strictly negative term. But nonnegativity of β_t , this imposes no restriction on β_t .

Case (iii): Suppose $e_t \neq e'$, and $\sigma_t = \sigma' = \sigma$. Consider first the case when $e_t = e_H$, so $e' = e_L$, which implies $\kappa(e_L, \sigma) - \kappa(e_H, \sigma) < 0$. Then we have,

$$0 \geq -\frac{\phi_e}{e_H}(e_L - e_H) + \beta_t (\kappa(e_L, \sigma) - \kappa(e_H, \sigma)),$$

implying,

$$\beta_t \geq \frac{1}{\kappa(e_H, \sigma) - \kappa(e_L, \sigma)} \frac{\phi_e}{e_H} (e_H - e_L). \quad (1.C.32)$$

But we also have that $\kappa(e_H, \sigma) - \kappa(e_L, \sigma) \geq \kappa(e_H, \sigma) - \kappa(e_L, \sigma_H)$, which then is equivalent to the following chain of inequalities,

$$\begin{aligned} \frac{1}{\kappa(e_H, \sigma) - \kappa(e_L, \sigma)} \frac{\phi_e}{e_H} (e_H - e_L) &\leq \frac{1}{\kappa(e_H, \sigma) - \kappa(e_L, \sigma)} \left[\frac{\phi_e}{e_H} (e_H - e_L) + \frac{\phi_\sigma}{\sigma_L} (\sigma_h - \sigma) \right] \\ &\leq \frac{1}{\kappa(e_H, \sigma) - \kappa(e_L, \sigma_H)} \left[\frac{\phi_e}{e_H} (e_H - e_L) + \frac{\phi_\sigma}{\sigma_L} (\sigma_H - \sigma) \right], \end{aligned}$$

and hence the condition (1.6.6) covers Case (iii) when $e_t = e_H$. The case of when $e_t = e_L$ is similar. ■

1.D Principal's value function

Properties of the value function

First we obtain some basic properties of the value function \tilde{v} .

Proposition 1.D.1. *The value function \tilde{v} is concave in w . That is, for all $m > \underline{m}$, and $w^1, w^2 > R$ and $\lambda \in [0, 1]$,*

$$\lambda \tilde{v}(w^1, m) + (1 - \lambda) \tilde{v}(w^2, m) \leq \tilde{v}(\lambda w^1 + (1 - \lambda) w^2, m).$$

Proof of Proposition 1.D.1. Pick any $W^j \geq R = 0$ and controls $\sigma^j, X^j, \beta^j, \tau^j$, for $j = 1, 2$, and in particular pick $\tau^1 = \tau^2 \equiv \tau^0$, from the admissible control set. Then from (S), we have the dynamics,

$$dW_t^j = \left[r_0 W_t^j - \phi_\sigma \left(\frac{\sigma_t^j}{\sigma_L} - 1 \right) \right] dt - dX_t^j + \beta_t^j \sigma_t^j dM_t,$$

for any $j = 1, 2$. Fix any $\lambda \in [0, 1]$. Multiplying and summing, we obtain,

$$\begin{aligned} d(\lambda W_t^1 + (1 - \lambda) W_t^2) &= \left[r_0 (\lambda W_t^1 + (1 - \lambda) W_t^2) - \phi_\sigma \left(\frac{\lambda \sigma_t^1 + (1 - \lambda) \sigma_t^2}{\sigma_L} - 1 \right) \right] dt \\ &\quad - d(\lambda X_t^1 + (1 - \lambda) X_t^2) + (\lambda \beta_t^1 \sigma_t^1 + (1 - \lambda) \beta_t^2 \sigma_t^2) dM_t. \end{aligned}$$

Now, let us define,

$$\beta_t := \frac{\lambda \beta_t^1 \sigma_t^1 + (1 - \lambda) \beta_t^2 \sigma_t^2}{\lambda \sigma_t^1 + (1 - \lambda) \sigma_t^2}. \quad (1.D.1)$$

Let's show that $\beta_t \in \mathcal{B}$, as given in (1.7.1). That is, let's show that,

$$K \geq \beta_t \geq \underline{\beta}. \quad (1.D.2)$$

The upper bound is clear since $\beta_t^1, \beta_t^2 \leq K$. But the lower bound is also clear since $\beta_t^1, \beta_t^2 \geq \underline{\beta}$. Thus, $\beta_t \in \mathcal{B}$.

Hence, we have that if $(\sigma^j, X^j, \beta^j, \tau^j) \in \mathcal{A}_{w^j, m}$, $j = 1, 2$, then

$$(\lambda \sigma^1 + (1 - \lambda) \sigma^2, \lambda X^1 + (1 - \lambda) X^2, \beta, \tau) \in \mathcal{A}_{\lambda(w^1, m) + (1 - \lambda)(w^2, m)},$$

where β is as constructed in (1.D.1).

Thus, this implies by optimality, and concavity of $\kappa(e_H, \sigma)$ in σ ,

$$\begin{aligned} &\lambda \mathbb{E} \left[\int_0^{\tau^0} e^{-r_1 t} \kappa(e_H, \sigma_t^1) dt - \int_0^{\tau^0} e^{-r_1 t} dX_t^1 + e^{-r_1 \tau^0} L \right] \\ &+ (1 - \lambda) \mathbb{E} \left[\int_0^{\tau^0} e^{-r_1 t} \kappa(e_H, \sigma_t^2) dt - \int_0^{\tau^0} e^{-r_1 t} dX_t^2 + e^{-r_1 \tau^0} L \right] \\ &\leq \mathbb{E} \left[\int_0^{\tau^0} e^{-r_1 t} \kappa(e_H, \lambda \sigma_t^1 + (1 - \lambda) \sigma_t^2) dt - \int_0^{\tau^0} e^{-r_1 t} d(\lambda X_t^1 + (1 - \lambda) X_t^2) + e^{-r_1 \tau^0} L \right] \\ &\leq \tilde{v}(\lambda w^1 + (1 - \lambda) w^2, m). \end{aligned}$$

Take the supremum to the above over the admissible set of controls, and we obtain, $\lambda\bar{v}(w^1, m) + (1 - \lambda)\bar{v}(w^2, m) \leq \bar{v}(\lambda w^1 + (1 - \lambda)w^2, m)$, as desired. \blacksquare

Proposition 1.D.1 is not only mathematically important, but also economically critical. In the model of DeMarzo and Sannikov (2006), it was explicitly shown that the principal's value function as a function of the agent's continuation value is concave, and thus, public randomization does not improve the payoff for the principal. Note that public randomization is effectively concavification of the principal's value function. However, in DeMarzo, Livdan, and Tchisty (2013), the authors show that in the case the agent manages a cash flows with a jump component (interpreted as "disasters"), then public randomization does indeed improve the value for the principal. Economically, public randomization implies the following.³⁶ To induce the current agent to work, the manager could effectively flip a coin every morning, and if the coin lands in heads, the manager keeps the current agent employed, but if the coin lands in tails, the manager fires the current agent and finds an identical agent in the labor market to replacement the outgoing agent. This coin flipping act implies that the principal is indifferent to the identity of the agent, as long as there does exist a competitive labor market of identical agents, and the principal can frictionlessly hire and fire agents from this labor market pool. Also equally important, the production technology of the firm is completely independent of the identity of the agent. Effectively, that means the firm is effectively a factory with a fixed production technology, and the agent is simply hired to spend effort (or not) to press a button in the factory; it does not matter who presses that button.

However, in the context of delegated portfolio management, this public randomization argument does not hold. In particular, for investment firms, the technology *is* the agent. Effectively, investment firms, and hedge funds in particular, live and die by the investment manager. Threatening the investment manager via the aforementioned coin flipping exercise is not credible, as the manager knows if he is fired, the firm also collapses with him. Thus, the importance of Proposition 1.D.1 is that the principal does not need to resort to a public randomization device to achieve a better outcome, as if otherwise, this public randomization device is not even feasible.

Further properties of the value function

Lemma 1.D.2. *For any $(w^i, m^i) \in \Gamma$, $i = 1, 2$, and $\lambda \in [0, 1]$, if $(\sigma^i, \beta^i, X^i) \in \mathcal{A}_{w^i, m^i}$, then there does not exist some β such that $(\lambda\sigma^1 + (1 - \lambda)\sigma^2, \beta, \lambda X^1 + (1 - \lambda)X^2) \in \mathcal{A}_{\lambda(w^1, m^1) + (1 - \lambda)(w^2, m^2)}$.*

Proof of Lemma 1.D.2. We proceed by contradiction. Fix any (w^i, m^i) , $i = 1, 2$ and $\lambda \in [0, 1]$. Without loss of generality, let us pick $m^2 > m^1$. Then there exists some admissible controls (σ^i, β^i, X^i) such that,

$$dW_t^i = \left(r_0 W_t^i - \phi_\sigma \left(\frac{\sigma_t^i}{\sigma_L} - 1 \right) \right) dt - dX_t^i + \beta_t^i \sigma_t^i dM_t^i.$$

In particular, we must have that there exist some point (w, m) such that $\lambda(w^1, m^1) + (1 - \lambda)(w^2, m^2) = (w, m)$ and some admissible controls (σ, β, X) associated with the point (w, m) . In particular, multiplying by λ and summing, we must have that,

$$\begin{aligned} d(\lambda W_t^1 + (1 - \lambda)W_t^2) &= \left(r_0(\lambda W_t^1 + (1 - \lambda)W_t^2) - \phi_\sigma \left(\frac{\lambda\sigma_t^1 + (1 - \lambda)\sigma_t^2}{\sigma_L} - 1 \right) \right) dt \\ &\quad - d(\lambda X_t^1 + (1 - \lambda)X_t^2) + \lambda\beta_t^1 \sigma_t^1 dM_t^1 + (1 - \lambda)\beta_t^2 \sigma_t^2 dM_t^2. \end{aligned} \tag{1.D.3}$$

But since M^1, M^2 are both geometric Brownian motions on the same underlying Brownian motion term except for different initial conditions, so $M_t^i = m^i e^{-1/2t + B_t}$, the diffusion term above in (1.D.3) can be rewritten as,

$$\begin{aligned} \lambda\beta_t^1 \sigma_t^1 dM_t^1 + (1 - \lambda)\beta_t^2 \sigma_t^2 dM_t^2 &= \lambda\beta_t^1 \sigma_t^1 m^1 e^{-1/2t + B_t} dB_t + (1 - \lambda)\beta_t^2 \sigma_t^2 m^2 e^{-1/2t + B_t} dB_t \\ &= (\lambda\beta_t^1 \sigma_t^1 m^1 + (1 - \lambda)\beta_t^2 \sigma_t^2 m^2) e^{-1/2t + B_t} dB_t. \end{aligned}$$

But if there exist some admissible control σ associated with (w, m) , then from the $\phi_\sigma \left(\frac{\lambda\sigma_t^1 + (1 - \lambda)\sigma_t^2}{\sigma_L} - 1 \right) dt$ term, it means this admissible volatility control σ must be $\sigma_t = \lambda\sigma_t^1 + (1 - \lambda)\sigma_t^2$. As well, the admissible compensation must be $X = \lambda X^1 + (1 - \lambda)X^2$. Then from this form, it implies the admissible sensitivity β must thus be the form, for $m = \lambda m^1 + (1 - \lambda)m^2$,

$$\beta_t [\lambda\sigma_t^1 + (1 - \lambda)\sigma_t^2] [\lambda m^1 + (1 - \lambda)m^2] = \lambda\beta_t^1 \sigma_t^1 m^1 + (1 - \lambda)\beta_t^2 \sigma_t^2 m^2,$$

³⁶ I thank Dmitry Livdan for pointing this out.

or that,

$$\beta_t = \frac{\lambda\beta_t^1\sigma_t^1m^1 + (1-\lambda)\beta_t^2\sigma_t^2m^2}{[\lambda\sigma_t^1 + (1-\lambda)\sigma_t^2][\lambda m^1 + (1-\lambda)m^2]}. \quad (1.D.4)$$

If this β_t is admissible, it must thus be in \mathcal{B} . But the lower bound in \mathcal{B} cannot hold for β_t of (1.D.4). To see this, since $\beta_t^i \in \mathcal{B}$, $i = 1, 2$, we have that,

$$\beta_t \geq \beta \frac{\lambda\sigma_t^1m^1 + (1-\lambda)\sigma_t^2m^2}{[\lambda\sigma_t^1 + (1-\lambda)\sigma_t^2][\lambda m^1 + (1-\lambda)m^2]}.$$

Thus, in order for β_t to be admissible, we must thus have that,

$$\frac{\lambda\sigma_t^1m^1 + (1-\lambda)\sigma_t^2m^2}{[\lambda\sigma_t^1 + (1-\lambda)\sigma_t^2][\lambda m^1 + (1-\lambda)m^2]} \geq 1. \quad (1.D.5)$$

Rearranging and after some algebra, (1.D.5) implies,

$$m^2(\sigma_t^2 - \sigma_t^1) \geq m^1(\sigma_t^2 - \sigma_t^1). \quad (1.D.6)$$

Recall we had assumed, without loss of generality, $m^2 > m^1$ — contradiction, this is impossible to hold for all choices of $\sigma_t^1, \sigma_t^2 \in [\sigma_L, \sigma_H]$ for all times t . In particular, it suffices to pick those times t and controls such that $\sigma_t^2 < \sigma_t^1$ and the above inequality will imply $m^2 \leq m^1$. Thus, there does not exist an admissible control β associated with the point $(w, m) = \lambda(w^1, m^1) + (1-\lambda)(w^2, m^2)$. ■

Remark 1.D.3. *The significance of Lemma 1.D.2 is that it is not possible that the value function is concave in the coordinate pair of (w, m) . Since Γ is clearly a convex set, that means it must be that for any $(w^i, m^i) \in \Gamma$, $i = 1, 2$ and $\lambda \in [0, 1]$ we can for sure find a point (w, m) such that $(w, m) = \lambda(w^1, m^1) + (1-\lambda)(w^2, m^2)$. The difficult in making the concavity argument of the value function is that from those controls (σ^i, β^i, X^i) associated with point (w^i, m^i) , can we find or construct a control (σ, β, X) associated with the point (w, m) , which again is a convex combination of (w^i, m^i) . Lemma 1.D.2 shows that we cannot. However, to be clear, that is not to say there does not exist an admissible associated with the point (w, m) . Lemma 1.D.2 merely states that if (w, m) is a convex combination of (w^i, m^i) , $i = 1, 2$, that admissible control associated with the point (w, m) cannot be constructed out of the controls associated with (σ^i, β^i, X^i) .*

Finally, we note that Lemma 1.D.2 is not contradicting Proposition 1.D.1. In particular, Proposition 1.D.1 is not claiming concavity in the coordinate pair (w, m) , but rather it is claiming that if we hold the exogenous factor level m fixed and look at the w -slice of the state space, then the value function is concave in the w -direction, with respect to the agent's continuation value. We summarize and formalize this below in Corollary 1.D.4.

Corollary 1.D.4. *The value function v is not concave on Γ .*

The next result shows that the value function is decreasing in the exogenous factor level.

Proposition 1.D.5. *The value function is decreasing in the exogenous factor level. That is, for any $w \in \Gamma_W$, $m_1, m_2 \in \Gamma_M$ with $m_2 \geq m_1$, we have,*

$$v(w, m_1) \geq v(w, m_2). \quad (1.D.7)$$

Proof of Proposition 1.D.5. Fix any $w \in \Gamma_W$ and fix any $m_1, m_2 \in \Gamma_M$ and let's suppose $m_2 \geq m_1$. Pick the admissible control as follows. Pick an arbitrary volatility choice $\sigma = \{\sigma_t\}$ and let $\sigma^1 = \sigma^2 = \sigma$ and also pick an arbitrary sensitivity choice $\beta = \{\beta_t\}$ and let $\beta^1 = \beta^2 = \beta$. For the compensation process, pick an arbitrary compensation process $X = \{X_t\}$, and set X^1, X^2 such that,

$$\begin{aligned} X_t^1 &= X_t, \\ X_t^2 &= X_t \mathbf{1}t \in (0, \tau_1). \end{aligned}$$

Then we have the associated state variable dynamics associated with those controls as,

$$\begin{aligned} dW_t^i &= \left[r_0 W_t^i - \phi_\sigma \left(\frac{\sigma_t}{\sigma_L} - 1 \right) \right] dt - dX_t^i + \beta_t \sigma_t dM_t^i \\ dM_t^i &= M_t^i dB_t, \end{aligned}$$

where $(W_0^i, M_0^i) = (w, m_i)$, $i = 1, 2$. Let τ^i be the associated hitting time of the form,

$$\tau^i := \inf \{t \geq 0 : W_t^i \leq R \text{ or } M_t^i \leq \underline{m}\}, \quad (1.D.8)$$

corresponding to the stopping time form in (P).

But recalling that M^i is a geometric Brownian motion, that implies the diffusion terms of dW_t^i is simply $\beta_t \sigma_t dM_t^i = \beta_t \sigma_t M_t^i dB_t = \beta_t \sigma_t m_i e^{-1/2t + B_t} dB_t = \beta_t \sigma_t m_i dM_t$, where M is a geometric Brownian motion on B with zero drift and unit variance and with initial value $M_0 = 1$. But given that $m_2 \geq m_1$, it implies that the diffusion term of dW_t^2 is weakly greater than that of dW_t^1 . But recalling (1.D.8), and since $W_0^1 = W_0^2 = w$, it implies that we have $\tau^1 \geq \tau^2$; that is, with a higher diffusion (from dW_t^2), and on the same Brownian path B_t , it is likely the first time to get bumped out of the region in (1.D.8) comes before that of one with a lower diffusion (from dW_t^1).

Then consider that,

$$\begin{aligned} & \mathbb{E} \left[\int_0^{\tau^1} e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^{\tau^1} e^{-r_1 t} dX_t^1 + e^{-r_1 \tau^1} L \right] - \mathbb{E} \left[\int_0^{\tau^2} e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^{\tau^2} e^{-r_1 t} dX_t^2 + e^{-r_1 \tau^2} L \right] \\ &= \mathbb{E} \left[\int_{\tau^2}^{\tau^1} \kappa(e_H, \sigma_t) dt + (e^{-r_1 \tau^1} - e^{-r_1 \tau^2}) L \right] \\ &\geq 0. \end{aligned}$$

But rearranging the above, and recalling the chosen admissible controls were arbitrary, and by optimality, we have that,

$$v(w, m_1) \geq \mathbb{E} \left[\int_0^{\tau^1} e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^{\tau^1} e^{-r_1 t} dX_t^1 + e^{-r_1 \tau^1} L \right] \geq \mathbb{E} \left[\int_0^{\tau^2} e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^{\tau^2} e^{-r_1 t} dX_t^2 + e^{-r_1 \tau^2} L \right].$$

And again by arbitrariness of the admissible controls and optimality, we have that,

$$v(w, m_1) \geq v(w, m_2) \geq \mathbb{E} \left[\int_0^{\tau^2} e^{-r_1 t} \kappa(e_H, \sigma_t) dt - \int_0^{\tau^2} e^{-r_1 t} dX_t^2 + e^{-r_1 \tau^2} L \right].$$

This concludes the proof. ■

The next result provides a lower bound on the value function v and directly shows that the value function is positive.

Proposition 1.D.6. For any $(w, m) \in \Gamma$, define the processes \tilde{W}, \tilde{M} , given by

$$\begin{aligned} d\tilde{W}_t &= r_0 \tilde{W}_t dt + \underline{\beta} \sigma_L d\tilde{M}_t, & \tilde{W}_0 &= w \\ d\tilde{M}_t &= \tilde{M}_t dB_t, & \tilde{M}_0 &= m, \end{aligned}$$

Define the hitting time θ as,

$$\theta := \inf \{t \geq 0 : (\tilde{W}_t, \tilde{M}_t) \notin \bar{\Gamma}\}.$$

Then,

$$\frac{\kappa(e_H, \sigma_L)}{r_1} - \mathbb{E}[e^{-r_1 \theta}] \left(\frac{\kappa(e_H, \sigma_L)}{r_1} - L \right) \leq v(w, m), \quad (1.D.9)$$

where $\kappa(e_H, \sigma_L)/r_1 - L > 0$, and holds with equality if and only if $(w, m) \in \partial\Gamma$, in which case $\theta = 0$, and $v(w, m) = L$. Thus, the value function is bounded below by a finite, positive constant.

Proof of Proposition 1.D.6. Fix any $(w, m) \in \Gamma$. Pick the controls (σ, X, β) as $\sigma_t \equiv \sigma_L$, $X \equiv 0$ and $\beta_t \equiv \underline{\beta}$ for all times t . Then the state variables thus becomes,

$$\begin{aligned} d\tilde{W}_t &= r_0 \tilde{W}_t dt + \underline{\beta} \sigma_L d\tilde{M}_t, & \tilde{W}_0 &= w \\ d\tilde{M}_t &= \tilde{M}_t dB_t, & \tilde{M}_0 &= m. \end{aligned}$$

With these choices of controls, the principal's payoff is thus,

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau} e^{-r_1 t} \kappa(e_H, \sigma_L) dt + e^{-r_1 \tau} L \right] &= \kappa(e_H, \sigma_L) \mathbb{E} \left[\frac{1 - e^{-r_1 \tau}}{r} \right] + L \mathbb{E}[e^{-r_1 \tau}] \\ &= \frac{\kappa(e_H, \sigma_L)}{r_1} - \mathbb{E}[e^{-r_1 \tau}] \left(\frac{\kappa(e_H, \sigma_L)}{r_1} - L \right), \end{aligned}$$

where note that $\kappa(e_H, \sigma_L)/r_1 - L > 0$ by Assumption 1.5.1 that $\kappa(e_L, \sigma_L)/r_1 > L$ and we have $\kappa(e_H, \sigma_L) > \kappa(e_L, \sigma_L)$ by Definition 1.4.1. Now, since the stopping time $\tau =: \theta$ is now viewed as,

$$\theta := \inf \{t \geq 0 : W_t \leq 0 \text{ or } M_t \leq 0\} = \inf \{t \geq 0 : (W_t, M_t) \notin \bar{\Gamma}\}.$$

Since $(w, m) \in \Gamma$ was arbitrary, then we clearly have (1.D.9) as desired. \blacksquare

Remark 1.D.7. Economically, the value on the left hand side of the inequality (1.D.9) represents the following. The term $\kappa(e_H, \sigma_L)/r_1$ is the “second worst” value of the firm, in which the agent effectively chooses the lowest possible volatility $\sigma_t \equiv \sigma_L$ for all times t . We note that it is “second worst” value because the absolute “worst” value of the firm is $\kappa(e_L, \sigma_L)/r_1$, that is when the lowest effort $e_t = e_L$ is exerted at all times, but note in this discussion we are concentrating on implementing the high effort e_H contract. However, the agent is still running the firm and recall from Assumption 1.5.1 that terminating the firm remains to be inefficient. Hence, the term $\kappa(e_H, \sigma_L)/r_1 - L$ effectively represents the premium the principal has to give up to the agent to operate the firm, even at its “second worst” value. However, to maintain IR constraints of the agent, the principal will only allow the agent to run the firm up until the stopping time θ .

Comparison Principle

We first establish a comparison principle for the value function v .

Proposition 1.D.8. Suppose ψ is a smooth solution on Γ that satisfies $\psi(0, m) \geq L$ for all $m > 0$, and also satisfies,

$$\max \left\{ -r_1 \psi(w, m) + \max_{\sigma} \sup_{\beta} (\mathcal{L}_{e_H} \psi)(w, m; \sigma, \beta) + \kappa(e_H, \sigma), -\psi_w(w, m) - 1 \right\} \leq 0, \quad (1.D.10)$$

Then we have that,

$$\psi \geq v.$$

Proof of Proposition 1.D.8. Fix an initial state $(w_0, m_0) \in \Gamma$ and select an arbitrary admissible control $\alpha = (\sigma, \beta, X) \in \mathcal{A}_{w_0, m_0}$. Furthermore, for $k, n \in \mathbb{N}$, set $\theta_k := \inf\{t \geq 0 : W_t \geq k \text{ or } W_t \leq 1/k\}$, and $\rho_n := \inf\{t \geq 0 : M_t \geq n \text{ or } M_t \leq 1/n\}$. Then we have that $\theta_k, \rho_n \uparrow \infty$ as $k, n \rightarrow \infty$. Now by Ito’s formula, we have that,

$$\begin{aligned} & e^{-r_1 \tau \wedge \theta_k \wedge \rho_n} \psi(W_{\tau \wedge \theta_k \wedge \rho_n}, m_{\tau \wedge \theta_k \wedge \rho_n}) \\ &= \psi(w_0, m_0) + \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} [-r_1 \psi(W_{s-}, M_s) + (\mathcal{L}_{e_H} \psi)(W_{s-}, M_s; \sigma_s, \beta_s) + \kappa(e_H, \sigma_s)] ds \\ & \quad - \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} \kappa(e_H, \sigma_s) ds + \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} \psi_w(W_{s-}, M_s) \beta_s \sigma_s dM_s + \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} \psi_m(W_{s-}, M_s) dM_s \\ & \quad + \sum_{0 \leq s \leq \tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} (\psi(W_s, M_s) - \psi(W_{s-}, M_s)) - \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} \psi_w(W_{s-}, M_s) dX_s^c, \end{aligned}$$

where X^c is the continuous part of X . Using the mean value theorem and since ψ satisfies the variational inequality (1.D.10), we have that $\psi_w \geq -1$, and moreover since for times $s \in [0, \tau \wedge \theta_k \wedge \rho_n]$, all the integrands in the diffusion terms are bounded, and noting also that the controls are also in a compact set, and so taking expectations and rearrange, we obtain,

$$\psi(w_0, m_0) \geq \mathbb{E} e^{-r_1 \tau \wedge \theta_k \wedge \rho_n} \psi(W_{\tau \wedge \theta_k \wedge \rho_n}, m_{\tau \wedge \theta_k \wedge \rho_n}) + \mathbb{E} \left[\int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} \kappa(e_H, \sigma_s) ds - \int_0^{\tau \wedge \theta_k \wedge \rho_n} e^{-r_1 s} dX_s \right].$$

Now, take $k, n \rightarrow \infty$ and applying Fatou’s lemma, we obtain, and recalling $\psi(W_\tau, m_\tau) = \psi(0, m_\tau) \geq L$,

$$\psi(w_0, m_0) \geq \mathbb{E} \left[\int_0^\tau e^{-r_1 s} \kappa(e_H, \sigma_s) ds - \int_0^\tau e^{-r_1 s} dX_s + e^{-r_1 \tau} L \right].$$

Since the set of admissible controls \mathcal{A}_{w_0, m_0} were arbitrary, taking the supremum on the right hand side of the inequality above, then we are done. \blacksquare

With Proposition 1.D.8 on hand, we can now derive some easy growth conditions on the value function v .

Corollary 1.D.9. For all $(w, m) \in \Gamma$, the value function v satisfies,

$$v(w, m) \leq (1 + r_0)w + m + \frac{\kappa(e_H, \sigma_H)}{r_1}.$$

Proof of Corollary 1.D.9. Take $\psi(w, m) := (1 + r_0)w + m + \frac{\kappa(e_H, \sigma_H)}{r_1}$ on Γ and $\psi(w, m) = L$ for $w \leq 0$ and $m > 0$. Then clearly ψ is smooth on Γ and moreover, $\psi_{ww} = \psi_{wm} = \psi_{mm} = 0$, and $\psi_w = 1 + r_0$. Observing (1.D.10), we have that,

$$-\psi_w(w, m) - 1 = -(1 + r_0) - 1 < 0,$$

and

$$\begin{aligned} & -r_1\psi(w, m) + \max_{\sigma} \sup_{\beta} (\mathcal{A}_{e_H} \psi)(w, m; \sigma, \beta) + \kappa(e_H, \sigma_H) \\ & \leq -r_1 \left((1 + r_0)w + m + \frac{\kappa(e_H, \sigma_H)}{r_1} \right) + r_0w + \kappa(e_H, \sigma_H) \\ & = (r_0 - r_1(1 + r_0))w - r_1m \\ & \leq 0, \end{aligned}$$

since we have $r_0/r_1 < 1 + r_0$. Then this choice of ψ satisfies the hypothesis of Proposition 1.D.8 and we are done. \blacksquare

Viscosity solution

Overview

Unlike the approach by the existing continuous-time principal-agent problem literature where either the value function of the principal is only dependent on one single state variable³⁷, namely the agent's continuation value, or there are multiple state variables but can be shown that the value function can be written in such a way that dynamic programming only applies to a single state variable³⁸. In particular, because there is only one relevant state variable in considering dynamic programming, the literature can rely on the extensive literature on existence and uniqueness results of ODE theory, and in some cases even compute explicitly the form of the principal's value function from the ODE form.

However in our case, it is not evident or perhaps even possible, to consider a rewriting to reduce the two state variables of the agent's continuation value W and the exogenous factor level M to a single state variable case. As a result the conventional and classical approach of the "verification theorem" does not apply. In particular, it means unlike the extensive results from ODE theory that can ensure existence and uniqueness of smooth solutions, we cannot a priori assume that there will exist a smooth solution (namely $C^2(\Gamma)$) such that we can take the first order conditions in (1.7.3), substitute the maximizer back into (1.7.3) and hope that there will exist a C^2 solution that still satisfies the highly nonlinear HJB PDE (1.7.3). Without existence of such a C^2 solution to the HJB PDE (1.7.3), a verification theorem to show that the solution to the HJB PDE (1.7.3) is indeed the value function (P) may likely fail. Thus, we must use more general techniques to understand the value function (P) and the HJB PDE (1.7.3) and hence we will consider *viscosity solution* methods.

To this end, we will first define the PDE operator F . Let us define,

$$\begin{aligned} F(w, m, u, p, A) := \max \left\{ & -r_1u + \max_{\sigma} \sup_{\beta} \left(\left[r_0w - \phi_{\sigma} \left(\frac{\sigma}{\sigma_L} - 1 \right) \right] p_w \right. \right. \\ & \left. \left. + \frac{1}{2}m^2 A_{mm} + \beta \sigma m^2 A_{wm} + \frac{1}{2}\beta^2 \sigma^2 m^2 A_{ww} + \kappa(e_H, \sigma) \right), \right. \\ & \left. -p_w - 1 \right\}. \end{aligned} \quad (1.D.11)$$

Hence, the HJB PDE in (1.7.3) is the rewriting,

$$F(w, m, v, Dv, D^2v) = 0, \quad (1.D.12)$$

where we denote Dv as the gradient vector and D^2v as the Hessian matrix of v , respectively. We will now show that the value function v of (P) can be understood as the viscosity solution to (1.D.12).

We now give some definitions.

³⁷ There are many examples here. Most notably, DeMarzo and Sannikov (2006), Sannikov (2008), He (2009), among many others.

³⁸ He, Wei, and Yu (2014) is an interesting recent example.

Definition 1.D.1. We say that u is a *viscosity supersolution* of (1.D.12) in Γ if, for every $(w, m) \in \Gamma$ and $\varphi \in C^2(\bar{\Gamma})$ such that (w, m) is a local minimum of $u - \varphi$ in $\bar{\Gamma}$, then

$$F(w, m, u, D\varphi, D^2\varphi) \geq 0. \quad (1.D.13)$$

We say that u is a *viscosity subsolution* of (1.D.12) in Γ if, for every $(w, m) \in \Gamma$ and $\varphi \in C^2(\bar{\Gamma})$ such that (w, m) is a local maximum of $u - \varphi$ in $\bar{\Gamma}$, then

$$F(w, m, u, D\varphi, D^2\varphi) \leq 0. \quad (1.D.14)$$

We say that u is a *viscosity solution* of (1.D.12) in Γ if it is both a viscosity supersolution and viscosity subsolution.

It is widely known that it is without loss of generality at the point (w, m) in the definition above to take $v(w, m) = \varphi(w, m)$ and also to replace local optimality with global optimality in the above.

Dynamic Programming Principle (DPP)

We will also assume and state without proof the *Dynamic Programming Principle* (DPP).

Theorem 1.D.10 (Dynamic Programming Principle). *For every initial state $(w, m) \in \Gamma$ and every stopping time θ ,*

$$v(w, m) = \sup_{\alpha \in \mathcal{A}_{w, m}} \mathbb{E}_0 \left[\int_0^{\tau \wedge \theta} e^{-r_1 s} \kappa(e_H, \sigma_s) ds - \int_0^{\tau \wedge \theta} e^{-r_1 s} dX_s + e^{-r_1 \tau \wedge \theta} v(W_{\tau \wedge \theta}, M_{\tau \wedge \theta}) \right].$$

Value function as a viscosity solution

Proposition 1.D.11. *The value function v of (P) is the unique viscosity solution of (1.D.12) in Γ .*

Remark 1.D.12. *In the proof of Proposition 1.D.11, we directly show that v is both a viscosity subsolution and a viscosity super solution of (1.D.12), and thus by definition, v is a viscosity solution of (1.D.12).³⁹ The proof for uniqueness is lengthy and technical. Hence, on a first pass, we will omit the proof for uniqueness.*

Proof to Proposition 1.D.11, Viscosity Subsolution. Fix any $(w, m) \in \Gamma$ and let $\varphi \in C^2(\bar{\Gamma})$ with $v - \varphi$ is a local max in $\bar{\Gamma}$ and $v(w, m) = \varphi(w, m)$. By Theorem 1.D.10, if we pick any $x \in (0, w]$ with $X \equiv x$, then we have that,

$$\varphi(w, m) = v(w, m) \geq v(w - x, m) - x \geq \varphi(w - x, m) - x.$$

Rearrange and take $x \downarrow 0$, then we have

$$\varphi_w(w, m) \geq -1. \quad (1.D.15)$$

Next, fix any constant $\bar{\beta}, \bar{\sigma}$ in the control set, and set $\beta_t \equiv \bar{\beta}$ and $\sigma_t \equiv \bar{\sigma}$, and let $X_t \equiv 0$ for all times t . Let (W, M) be the state variables with those associated control policies. Fix any $h > 0$. Define $\tau_\rho := \inf \{t \geq 0 : (W_t, M_t) \notin B_\rho(w, m) \cap \Gamma\}$, where for $\rho > 0$ sufficiently small, $B_\rho(w, m)$ is the ball centered at (w, m) with radius ρ . Then from Theorem 1.D.10, and noting that

³⁹ The proof ideas are largely inspired by and inherited from Yong and Zhou (1999), Fleming and Soner (2006), Budhiraja and Ross (2008) and Ly Vath, Pham, and Villeneuve (2008).

$\tau \wedge \tau_\rho = \tau_\rho$, and applying Ito's lemma, we have that,

$$\begin{aligned}
 0 &\geq \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} \kappa(e_H, \bar{\sigma}) ds + e^{-r_1 \tau_\rho \wedge h} v(W_{\tau_\rho \wedge h}, M_{\tau_\rho \wedge h}) \right] - v(w, m) \\
 &\geq \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} \kappa(e_H, \bar{\sigma}) ds + e^{-r_1 \tau_\rho \wedge h} [v(W_{\tau_\rho \wedge h}, M_{\tau_\rho \wedge h}) - \varphi(W_{\tau_\rho \wedge h}, M_{\tau_\rho \wedge h}) + \varphi(W_{\tau_\rho \wedge h}, M_{\tau_\rho \wedge h})] \right] - v(w, m) \\
 &\geq \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} \kappa(e_H, \bar{\sigma}) ds + e^{-r_1 \tau_\rho \wedge h} \underbrace{[v(w, m) - \varphi(w, m)]}_{=0} + \varphi(W_{\tau_\rho \wedge h}, M_{\tau_\rho \wedge h}) \right] - \underbrace{v(w, m)}_{=\varphi(w, m)} \\
 &= \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} \kappa(e_H, \bar{\sigma}) ds + \varphi(w, m) + \int_0^{\tau_\rho \wedge h} e^{-r_1 s} [-r_1 \varphi(W_s, M_s) + (\mathcal{L}_{e_H} \varphi)(W_s, M_s; \bar{\sigma}, \bar{\beta})] ds \right. \\
 &\quad \left. + \int_0^{\tau_\rho \wedge h} e^{-r_1 s} \varphi_w(W_s, M_s) \bar{\beta} \bar{\sigma} dM_s + \int_0^{\tau_\rho \wedge h} e^{-r_1 s} \varphi_m(W_s, M_s) dM_s \right] - \varphi(w, m) \\
 &= \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} [-r_1 \varphi(W_s, M_s) + (\mathcal{L}_{e_H} \varphi)(W_s, M_s; \bar{\sigma}, \bar{\beta}) + \kappa(e_H, \bar{\sigma})] ds \right] \\
 &\geq \mathbb{E} \left[\int_0^{\tau_\rho \wedge h} e^{-r_1 s} \inf_{B_\rho(w, m)} [-r_1 \varphi(\tilde{w}, \tilde{m}) + (\mathcal{L}_{e_H} \varphi)(\tilde{w}, \tilde{m}; \bar{\beta}, \bar{\beta}) + \kappa(e_H, \bar{\sigma})] ds \right] \\
 &= \mathbb{E} \left[\frac{1 - e^{-r_1 \tau_\rho \wedge h}}{r_1} \right] \inf_{B_\rho(w, m)} [-r_1 \varphi(\tilde{w}, \tilde{m}) + (\mathcal{L}_{e_H} \varphi)(\tilde{w}, \tilde{m}; \bar{\beta}, \bar{\beta}) + \kappa(e_H, \bar{\sigma})].
 \end{aligned}$$

Since with $X \equiv 0$, then the state variable process (W, M) are continuous and hence $\tau_\rho > 0$. By dominated convergence theorem, let $h \downarrow 0$ and we have,

$$\mathbb{E} \left[\frac{1 - e^{-r_1 \tau_\rho \wedge h}}{h} \right] \rightarrow r_1.$$

As well, dividing the above inequality by h , and letting $\rho \downarrow 0$ so $\tau_\rho \rightarrow \infty$ and $B_\rho(w, m) \rightarrow \{(w, m)\}$ and $h \downarrow 0$, and recall $v(w, m) = \varphi(w, m)$, we obtain,

$$0 \geq -r_1 v(w, m) + (\mathcal{L}_{e_H} \varphi)(w, m; \bar{\sigma}, \bar{\beta}) + \kappa(e_H, \bar{\sigma}).$$

But since the choice of $\bar{\beta}, \bar{\sigma}$ were arbitrary, the above also implies,

$$0 \geq -r_1 v(w, m) + \max_\sigma \sup_\beta (\mathcal{L}_{e_H} \varphi)(w, m; \sigma, \beta) + \kappa(e_H, \sigma). \quad (1.D.16)$$

Putting (1.D.16) and (1.D.15) together and we are done. ■

Proof to Proposition 1.D.11, Viscosity Supersolution. Let $\varphi \in C^2(\bar{\Gamma})$ and (\hat{w}, \hat{m}) be a local minimizer of $v - \varphi$ on Γ with $v(\hat{w}, \hat{m}) = \varphi(\hat{w}, \hat{m})$. We need to show that,

$$F(\hat{w}, \hat{m}, \varphi, D\varphi, D^2\varphi) \geq 0 \quad (1.D.17)$$

For contradiction, suppose not. Then the left hand side of (1.D.17) is strictly negative and by smoothness of φ , there exists $\delta, \gamma > 0$ satisfying,

$$F(w, m, \varphi, D\varphi, D^2\varphi) \leq -\gamma, \quad (w, m) \in B_\delta(\hat{w}, \hat{m}), \quad (1.D.18)$$

where $B_\delta(\hat{w}, \hat{m}) := \{(w, m) : \|(w, m) - (\hat{w}, \hat{m})\|_2 < \delta\}$. Since Γ is an open set, by changing δ , if necessary, we may assume that $B_\delta(\hat{w}, \hat{m}) \subset \Gamma$.

Fix an arbitrary control $\alpha = (\sigma, X, \beta) \in \mathcal{A}_{\hat{w}, \hat{m}}$, and let θ be the first exist time of (W, M) from $B_\delta(\hat{w}, \hat{m})$. Since $B_\delta(\hat{w}, \hat{m}) \subset \Gamma$, we have that $\theta < \tau$.

Let W^c, X^c denote the continuous parts of W, X , respectively, and noting that $\Delta W_t := W_t - W_{t-} = -\Delta X_t := -(X_t - X_{t-})$. By the continuity of sample paths, $M_s = M_{s-}$. Now by Ito's lemma and taking expectations, we have that,

$$\begin{aligned}
 &\mathbb{E} e^{-r_1 \theta^-} \varphi(W_{\theta^-}, M_{\theta^-}) - \varphi(\hat{w}, \hat{m}) \\
 &= \mathbb{E} \int_0^{\theta^-} e^{-r_1 s} [-r_1 \varphi(W_s, M_s) + (\mathcal{L}_{e_H} \varphi)(W_s, M_s; \sigma_s, \beta_s) + \kappa(e_H, \sigma_s)] ds \\
 &\quad - \mathbb{E} \int_0^{\theta^-} e^{-r_1 s} \varphi_w(W_s, M_s) dX_s^c + \mathbb{E} \sum_{0 \leq s < \theta^-} e^{-r_1 s} [\varphi(W_s, M_s) - \varphi(W_{s-}, M_s)]
 \end{aligned} \quad (1.D.19)$$

But for $0 \leq s < \theta^-$, (1.D.18) implies,

$$-r_1 \varphi(W_s, M_s) + (\mathcal{L}_{e_H} \varphi)(W_s, M_s; \sigma_s, \beta_s) \leq -\gamma, \quad (1.D.20)$$

$$-\varphi_w(W_s, M_s) - 1 \leq -\gamma. \quad (1.D.21)$$

And using the mean value theorem and (1.D.21), we obtain,

$$\varphi(W_s, M_s) - \varphi(W_{s-}, M_s) \leq (1 - \gamma) \Delta X_s. \quad (1.D.22)$$

Substituting (1.D.20), (1.D.21) and (1.D.22) and noting that $X_t = X_t^c + \Delta X_t$, we obtain,

$$\begin{aligned} & \mathbb{E} e^{-r_1 \theta^-} \varphi(W_{\theta^-}, M_\theta) - \varphi(\hat{w}, \hat{m}) \\ & \leq -\mathbb{E} \int_0^{\theta^-} e^{-r_1 s} \kappa(e_H, \sigma_s) ds + \mathbb{E} \int_0^{\theta^-} e^{-r_1 s} (1 - \gamma) dX_s - \mathbb{E} \int_0^{\theta^-} e^{-r_1 s} \gamma ds. \end{aligned} \quad (1.D.23)$$

Note that while $(W_{\theta^-}, M_{\theta^-}) \in \bar{B}_\delta(\hat{w}, \hat{m})$, we have that (W_θ, M_θ) is either on the boundary $\partial \bar{B}_\delta(\hat{w}, \hat{m})$ or out of $\bar{B}_\delta(\hat{w}, \hat{m})$. However, there exists some random variable $\lambda \in [0, 1]$ such that,

$$(W^\lambda, M^\lambda) := (W_{\theta^-} + \lambda \Delta W_\theta, M_\theta) = (W_{\theta^-} - \lambda \Delta X_\theta, M_\theta) \in \partial \bar{B}_\delta(\hat{w}, \hat{m}). \quad (1.D.24)$$

And again by the mean value theorem and (1.D.21), we have that,

$$\varphi(W^\lambda, M^\lambda) - \varphi(W_{\theta^-}, M_{\theta^-}) \leq (1 - \gamma) \lambda \Delta X_\theta. \quad (1.D.25)$$

Note also that,

$$\begin{aligned} W^\lambda &= W_{\theta^-} - \lambda \Delta X_\theta \\ &= (W_\theta - \Delta W_\theta) - \lambda \Delta X_\theta \\ &= W_\theta + \Delta X_\theta - \lambda \Delta X_\theta \\ &= W_\theta + (1 - \lambda) \Delta X_\theta. \end{aligned} \quad (1.D.26)$$

From (1.D.26) and properties of the value function, we also have that,

$$v(W^\lambda, M^\lambda) \geq v(W_\theta, M_\theta) - (1 - \lambda) \Delta X_\theta. \quad (1.D.27)$$

And since $v - \varphi$ is a local min at (\hat{w}, \hat{m}) , with $v(\hat{w}, \hat{m}) = \varphi(\hat{w}, \hat{m})$, so we have that,

$$v(W^\lambda, M^\lambda) \leq \varphi(W^\lambda, M^\lambda). \quad (1.D.28)$$

So from (1.D.25), (1.D.27) and (1.D.28), we obtain,

$$\begin{aligned} \varphi(W_{\theta^-}, M_{\theta^-}) &\geq \varphi(W^\lambda, M^\lambda) - (1 - \gamma) \lambda \Delta X_\theta \\ &\geq v(W^\lambda, M^\lambda) - (1 - \gamma) \lambda \Delta X_\theta \\ &\geq v(W_\theta, M_\theta) - (1 - \lambda) \Delta X_\theta - (1 - \gamma) \lambda \Delta X_\theta \\ &= v(W_\theta, M_\theta) - (1 - \lambda \gamma) \Delta X_\theta \end{aligned} \quad (1.D.29)$$

Substituting (1.D.29) into (1.D.23), and rearranging, we thus have,

$$\begin{aligned} \varphi(\hat{w}, \hat{m}) &\geq \mathbb{E} \left[\int_0^{\theta^-} e^{-r_1 s} \kappa(e_H, \sigma_s) ds - \int_0^{\theta^-} e^{-r_1 s} dX_s - e^{-r_1 \theta^-} \Delta X_\theta + e^{-r_1 \theta^-} v(W_\theta, M_\theta) \right] \\ &\quad + \mathbb{E} \left[\int_0^{\theta^-} e^{-r_1 s} dX_s - \int_0^{\theta^-} e^{-r_1 s} ds + e^{-r_1 \theta^-} \lambda \Delta X_\theta \right]. \end{aligned} \quad (1.D.30)$$

Now suppose we can show that there exists a constant $c_0 > 0$ such that,

$$\mathbb{E} \left[\int_0^{\theta^-} e^{-r_1 s} dX_s - \int_0^{\theta^-} e^{-r_1 s} ds + e^{-r_1 \theta^-} \lambda \Delta X_\theta \right] \geq c_0. \quad (1.D.31)$$

Suppose for now that (1.D.31) is true. Then from (1.D.30), using (1.D.31), recalling that the chosen controls were arbitrary so we may take the supremum over all admissible controls, and using Theorem 1.D.10 we have that,

$$\varphi(\hat{w}, \hat{m}) \geq \gamma c_0 + v(\hat{w}, \hat{m}), \quad (1.D.32)$$

implying that $\varphi(\hat{w}, \hat{m}) - v(\hat{w}, \hat{m}) \geq \gamma c_0 > 0$ — contradiction, since we had assumed that $\varphi(\hat{w}, \hat{m}) = v(\hat{w}, \hat{m})$.

So, the proof is complete once we can prove the existence of the constant $c_0 > 0$ that satisfies (1.D.31). To this end, let us define the C^2 function,

$$\psi(w, m) := c_0 \left(1 - \frac{\|(w, m) - (\hat{w}, \hat{m})\|_2^2}{\delta^2} \right), \quad (1.D.33)$$

where,

$$c_0 := C_0 \wedge (\delta/2), \quad (1.D.34)$$

and C_0 is given by,

$$C_0 := \delta^2 (\kappa(e_H, \sigma_L) - r_1) \times \inf_{\bar{\sigma} \in [\sigma_L, \sigma_H]} \inf_{(\tilde{w}, \tilde{m}) \in \mathcal{B}_\delta(\hat{w}, \hat{m})} \left[2r_0 \tilde{w}(\tilde{w} - \hat{w}) - 2\phi_\sigma \left(\frac{\bar{\sigma}}{\sigma_L} - 1 \right) (\tilde{w} - \hat{w}) + \tilde{m}^2(\tilde{m} - \hat{m}) + K^2 \sigma_H^2 \right]^{-1}. \quad (1.D.35)$$

Then a direct (but somewhat messy) computation will show that for any admissible choice $(\bar{\beta}, \bar{\sigma})$, (1.D.33) satisfies,

$$\begin{cases} \max \{ 1 - \psi_w, \psi - 1, -[-r_1 \psi + (\mathcal{L}_{e_H} \psi)(\cdot, \cdot; \bar{\sigma}, \bar{\beta})] - 1 \} \leq 0, & \text{on } \bar{B}_\delta(\hat{w}, \hat{m}), \\ \psi = 0, & \text{on } \partial \bar{B}_\delta(\hat{w}, \hat{m}). \end{cases} \quad (1.D.36)$$

By Ito's lemma applied to $e^{-r_1 \theta^-} \psi(W_{\theta^-}, M_{\theta^-})$, taking expectations and rearranging, we will arrive at (1.D.31).

This completes the proof. \blacksquare

Regularity upgrade

Once we have obtained Proposition 1.D.11 and thus we can understand the value function v as the viscosity solution to the HJB PDE (1.7.3), we are now ready to “upgrade” our results. First we give a “partial” C^1 result.

Proposition 1.D.13. *The value function v is C^1 in the w -direction; that is, for each $m \in \Gamma_M$, the partial derivative $v_w(w, m)$ exists for all $w \in \Gamma_W$, and is continuous in w .*

Proof of Proposition 1.D.13. Fix any $m_0 \in \Gamma_m$. Define the limits,

$$\nabla_w^+ v(w, m_0) := \lim_{\delta \downarrow 0} \frac{v(w + \delta, m_0) - v(w, m_0)}{\delta}, \quad (1.D.37a)$$

$$\nabla_w^- v(w, m_0) := \lim_{\delta \downarrow 0} \frac{v(w, m_0) - v(w - \delta, m_0)}{\delta}. \quad (1.D.37b)$$

By Proposition 1.D.1, the map $w \mapsto v(w, m_0)$ is concave in the w -direction. Thus from standard results in convex analysis (see, for instance, Rockafellar (1970)), the limits (1.D.37) exist. We want to show that $\nabla_w^+ v(w, m_0) = \nabla_w^- v(w, m_0)$, and hence equals $v_w(w, m_0)$ for $w \in \Gamma_w$. We proceed in three steps.

Step 1. Let's show that $\nabla_w^+ v(w, m_0) \geq \nabla_w^- v(w, m_0)$. For contradiction, suppose there exist some $w_0 \in \Gamma_w$ such that $\nabla_w^+ v(w_0, m_0) < \nabla_w^- v(w_0, m_0)$. Fix any nonzero $q \in (\nabla_w^+ v(w_0, m_0), \nabla_w^- v(w_0, m_0))$, and any $\varepsilon > 0$. Consider the function,

$$\varphi(w, m) := v(w_0, m_0) + q \left(1 + \frac{4m_0^2}{\varepsilon q w_0} + \frac{4K^2 \sigma_L^2 m_0^2}{\varepsilon q w_0} \right) (w - w_0) - \frac{1}{2\varepsilon} (w - w_0)^2 - \frac{1}{2\varepsilon} (m - m_0)^2.$$

Then φ is quadratic and concave in (w, m) , and then clearly (w_0, m_0) is a local maximum of $v - \varphi$, with $v(w_0, m_0) = \varphi(w_0, m_0)$, $\varphi_w(w_0, m_0) = q \left(1 + \frac{4m_0^2}{\varepsilon qw_0} + \frac{4K^2\sigma_L^2 m_0^2}{\varepsilon qw_0} \right)$, $\varphi_{wm} = 0$, and $\varphi_{ww}(w_0, m_0) = \varphi_{mm}(w_0, m_0) = -\frac{1}{\varepsilon}$. By the viscosity subsolution property of v , and suboptimality,

$$\begin{aligned} 0 &\geq F(w_0, m_0, \varphi, D\varphi, D^2\varphi) \\ &= \max \left\{ -r_1 v(w_0, m_0) + \max_{\sigma} \sup_{\beta} \left\{ \left[r_0 w_0 - \phi_{\sigma} \left(\frac{\sigma}{\sigma_L} - 1 \right) \right] \cdot q \left(1 + \frac{4m_0^2}{\varepsilon qw_0} + \frac{4K^2\sigma_L^2 m_0^2}{\varepsilon qw_0} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} m_0^2 \cdot \left(-\frac{1}{\varepsilon} \right) + \beta \sigma m_0^2 \cdot 0 + \frac{1}{2} \beta^2 \sigma^2 m_0^2 \cdot \left(-\frac{1}{\varepsilon} \right) + \kappa(e_H, \sigma) \right\}, \right. \\ &\quad \left. - q \left(1 + \frac{4m_0^2}{\varepsilon qw_0} + \frac{4K^2\sigma_L^2 m_0^2}{\varepsilon qw_0} \right) - 1 \right\} \\ &\geq -r_1 v(w_0, m_0) + r_0 w_0 q \left(1 + \frac{4m_0^2}{\varepsilon qw_0} + \frac{4K^2\sigma_L^2 m_0^2}{\varepsilon qw_0} \right) - \frac{m_0^2}{2\varepsilon} - \frac{K^2\sigma_L^2 m_0^2}{2\varepsilon} + \kappa(e_H, \sigma_L). \end{aligned}$$

Rearranging the above, we will have,

$$0 \geq -\varepsilon r_1 v(w_0, m_0) + \varepsilon r_0 w_0 q + \varepsilon \kappa(e_H, \sigma_L) + m_0^2 \left(4r_0 - \frac{1}{2} \right) + K^2 \sigma_L^2 m_0^2 \left(4r_0 - \frac{1}{2} \right).$$

Take $\varepsilon \downarrow 0$, then the above implies,

$$0 \geq m_0^2 \left(4r_0 - \frac{1}{2} \right) + K^2 \sigma_L^2 m_0^2 \left(4r_0 - \frac{1}{2} \right),$$

— contradiction, as we recall $r_0 \in (0, 1)$. Thus, we have $\nabla_w^+ v(w, m_0) \geq \nabla_w^- v(w, m_0)$ for $w \in \Gamma_w$.

Step 2. Now, it remains to show $\nabla_w^+ v(w, m_0) \leq \nabla_w^- v(w, m_0)$. But since for each $m_0 \in \Gamma_m$, $v(w, m_0)$ is concave in the w -direction, by again standard results from convex analysis (see Rockafellar (1970)), the desired inequality $\nabla_w^+ v(w, m_0) \leq \nabla_w^- v(w, m_0)$ immediately holds.

Step 3. By Step 1 and 2, we have that $\nabla_w^+ v(w, m_0) = \nabla_w^- v(w, m_0) \equiv v_w(w, m_0)$. But furthermore, since we have concavity in the w -direction and also continuity, this implies $v_w(w, m_0)$ is also continuous. Thus, we have C^1 in the w -direction. ■

Now, we give a C^2 regularity upgrade.

Proposition 1.D.14. *Define the sets,*

$$\mathcal{D} := \{(w, m) \in \Gamma : v_w(w, m) = -1\}, \quad (1.D.38)$$

$$\mathcal{C} := \Gamma \setminus \mathcal{D}. \quad (1.D.39)$$

Then,

1. v is C^2 in the w -direction on $\mathcal{C} \cup \mathcal{D}^o$.

2. In the classical C^2 solution sense, we have,

$$-r_1 v + \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} v)(\cdot, \cdot; \sigma, \beta) + \kappa(e_H, \sigma)] = 0, \quad \text{on } \mathcal{C}. \quad (1.D.40)$$

Proof of Proposition 1.D.14. Part 1. It is clear that v is C^2 in the w -direction on \mathcal{D}^o . That is, for $(w, m) \in \mathcal{D}^o$, we have $v_w(w, m) = -1$. Note that by Proposition 1.D.13, the expression $v_w(w, m)$ makes sense. But moreover, since the right hand side of $v_w(w, m) = -1$ is a constant, which is trivially differentiable in the w -direction, and so it also implies $v_w(w, m)$ is trivially C^1 , and so v is C^2 in the w -direction.

It remains to prove that v is C^2 in the w -direction on \mathcal{C} .

Part 2. Let's first show that v is a viscosity solution to,

$$-r_1 v(w, m) + \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} v)(w, m; \sigma, \beta) + \kappa(e_H, \sigma)] = 0, \quad (w, m) \in \mathcal{C}. \quad (1.D.41)$$

Viscosity supersolution. Indeed, let $(\hat{w}, \hat{m}) \in \mathcal{C}$ and φ be a C^2 function on \mathcal{C} such that (\hat{w}, \hat{m}) is a local minimum of $v - \varphi$ with $v(\hat{w}, \hat{m}) = \varphi(\hat{w}, \hat{m})$. But that means by first order conditions, and again recalling Proposition 1.D.13, it implies we have

$0 = \frac{\partial}{\partial w}(v - \varphi)(\hat{w}, \hat{m})$; so in particular, we have that for $(\hat{w}, \hat{m}) \in \mathcal{C}$, $\varphi_w(\hat{w}, \hat{m}) = v_w(\hat{w}, \hat{m}) < -1$. And thus from the viscosity supersolution property of v from Proposition 1.D.11, we have,

$$-r_1 \varphi(\hat{w}, \hat{m}) + \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} v)(\hat{w}, \hat{m}; \sigma, \beta) + \kappa(e_H, \sigma)] \geq 0.$$

This shows the desired viscosity supersolution property.

Viscosity subsolution. The subsolution property is immediate by the fact v is (at least) a viscosity subsolution to (1.7.3), as given by Proposition 1.D.11. Thus, v is also a viscosity solution to (1.D.41).

Now, fix any arbitrarily bounded set $O \subset \mathcal{C}$. Consider the nonlinear Dirichlet boundary value problem,

$$-r_1 \xi + \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} \xi)(\cdot, \cdot; \sigma, \beta) + \kappa(e_H, \sigma)] = 0, \quad \text{on } O, \quad (1.D.42a)$$

$$\xi = v, \quad \text{on } \partial O. \quad (1.D.42b)$$

In particular, we see that for any $a = (a_1, a_2) \in \mathbb{R}^2$, we have that by extracting out the coefficients to the second order derivative and cross derivative terms of ξ in (1.D.42), and for any β, σ in the admissible choice set,

$$\begin{aligned} \beta^2 \sigma^2 m^2 a_1^2 + 2\beta \sigma m^2 a_1 a_2 + m^2 a_2^2 &\geq \underline{\beta}^2 \underline{\sigma}_L^2 \underline{m}^2 a_1^2 + 2\underline{\beta} \underline{\sigma}_L \underline{m} a_1 a_2 + \underline{m}^2 a_2^2 \\ &\geq C(a_1^2 + 2a_1 a_2 + a_2^2) \\ &= C \|a\|_2^2, \end{aligned}$$

where the constant $C := \min\{\underline{\beta}^2 \underline{\sigma}_L^2 \underline{m}^2, 2\underline{\beta} \underline{\sigma}_L \underline{m}, \underline{m}^2\} > 0$ and $\|\cdot\|_2$ is the standard \mathbb{R}^2 Euclidean norm. In particular, this shows PDE (1.D.42) is *uniformly elliptic* (see Remark 1.D.15) with Dirichlet boundary data. Thus, standard classical existence and uniqueness results are available (see Evans (1983), Fleming and Soner (2006), and Evans (2010)). Hence, a unique C^2 solution ξ on O to (1.D.42) exists. But from the standard uniqueness results of viscosity solution to (1.D.42), this implies we have $v = \xi$ on O . From the arbitrariness of O , this proves that v is C^2 smooth on Γ . ■

Remark 1.D.15. *We can now see the significance of the IR constraint in Definition 1.6.1(1b). If in contrast, we do not have the requirement that $M_t \geq \underline{m}$, so that the state space in question is $(w, m) \in (0, \infty) \times (0, \infty)$ rather than $(0, \infty) \times (\underline{m}, \infty)$, then there does not exist a strictly positive constant C for which $\beta^1 \sigma^2 m^1 a_1^2 + 2\beta \sigma m^2 a_1 a_2 + m^2 a_2^2 \geq C \|a\|_2^2$ can hold, in which case the PDE is known as being degenerate. The essential problem is that when $m = 0$, the nature of the state variable dynamics significantly changes (i.e. from being fully stochastic to fully deterministic).*

See also Figure 1.3 for an illustration of the state space Γ . And also see Figure 1.4 for an illustration, for each fixed $m \in \Gamma_M$, the value function $w \mapsto v(w, m)$.

Remark 1.D.16. *The set \mathcal{D} of (1.D.38) is the payment condition and \mathcal{C} is the continuation region (i.e. no payment condition).*

Remark 1.D.17. *Proposition 1.D.14 shows that in the continuation region \mathcal{C} , the value function v is C^2 smooth in both (w, m) . Together with the concavity of the value function in the w -direction from Proposition 1.D.1, it implies that in (1.D.40), when we optimize over the volatility σ choice and the sensitivity β choice, we can use the usual first order conditions to uniquely characterize them. Thus, Proposition 1.D.1 and Proposition 1.D.14 show that the discussions in Section 1.8 are on meaningful grounds.*

Free boundary

We first introduce the *free (moving) boundary*,

$$\partial^* := \{(\bar{W}(m), m) : m \in \Gamma_M\}, \quad (1.D.43)$$

where \bar{W} is the map from Γ_M to Γ_W , defined by,

$$\bar{W}(m) := \sup \{w \in \Gamma_W : v_w(w, m) = -1\}, \quad m \in \Gamma_M. \quad (1.D.44)$$

Through a rather technical and elaborate argument similar to Soner and Shreve (1989), one can show that \bar{W} is finite and indeed twice continuously differentiable. We omit the proof here, but the argument should follow, in spirit and actuality, from Soner and Shreve (1989).

Then we can have a further regularity upgrade of our earlier results. Note that for each $m \in \Gamma_M$, we can partition $\Gamma_W = (R, \infty) = (R, \bar{W}(m)) \cup [\bar{W}(m), \infty)$. Moreover, note that the set $(R, \bar{W}(m)) \times \Gamma_M = \mathcal{C}$, by construction. Define the function V on Γ as follows. For each $m \in \Gamma_M$, define,

$$V(w, m) \tag{1.D.45}$$

$$:= \begin{cases} \frac{1}{r_1} \max_{\sigma} \sup_{\beta} [(\mathcal{L}_{e_H} v)(w, m; \sigma, \beta) + \kappa(e_H, \sigma)], & w \in (R, \bar{W}(m)), \\ \bar{W}(m) - w + \frac{1}{r_1} \left[-r_0 \bar{W}(m) + \phi_{\sigma} \left(\frac{\sigma_H}{\sigma_L} - 1 \right) + \frac{1}{2} m^2 v_{mm}(\bar{W}(m), m) + \kappa(e_H, \sigma_H) \right], & w \in [\bar{W}(m), \infty). \end{cases} \tag{1.D.46}$$

In particular, we have simply taken the value function v , and extracted out the dynamics in the continuation region \mathcal{C} as in Proposition 1.D.14(1.D.40), and then on the payment condition region, linearly extrapolated the value at the slope -1 . By the smoothness of \bar{W} , we have that V is also a viscosity solution to the HJB PDE (1.7.3). By uniqueness of viscosity solutions, this implies that $V \equiv v$ on Γ , but we note that V is C^2 in the w -direction on Γ_W .

In particular, when we evaluate $w = \bar{W}(m)$ for any $m \in \Gamma_M$ from (1.D.46), we obtain,

$$V(\bar{W}(m), m) = \frac{1}{r_1} \left[-r_0 \bar{W}(m) + \phi_{\sigma} \left(\frac{\sigma_H}{\sigma_L} - 1 \right) + \frac{1}{2} m^2 v_{mm}(\bar{W}(m), m) + \kappa(e_H, \sigma_H) \right], \quad m \in \Gamma_M. \tag{1.D.47}$$

But recalling the terminal condition $V = v = L$ on $\partial\Gamma$, and in particular if we take $m \rightarrow \underline{m}$ in (1.D.47), we have,

$$L = \frac{1}{r_1} \left[-r_0 \bar{W}(\underline{m}) + \phi_{\sigma} \left(\frac{\sigma_H}{\sigma_L} - 1 \right) + \frac{1}{2} \underline{m}^2 v_{mm}(\bar{W}(\underline{m}), \underline{m}) + \kappa(e_H, \sigma_H) \right]. \tag{1.D.48}$$

But if we view the (moving) free boundary $m \mapsto \bar{W}(m)$ as the object of interest, then (1.D.47) identifies the nonlinear ODE.

Chapter 2

Dynamic Agency, Delegated Portfolio Management and Asset Pricing

MAY 17, 2015

Chapter Abstract

We study a dynamic contracting problem in continuous-time dynamically complete market general equilibrium, whereby an investor must delegate all his portfolio choice problems to a manager. This framework is one of the first attempts to attack a combined dynamic contracting and dynamic asset pricing problem. The portfolio manager can exert costly private monitoring effort costs to increase the expected dividend growth rate of a representative firm. The investor can only observe the dividends of the firm over time, and will consider a pie sharing rule contract over the dividends of consumption goods to dynamically incentivize the manager. The key result is that dynamic moral hazard and dynamic optimal contracting endogenously generates stochastic volatility in the asset returns, and substantial state varying stochasticity in the market price of risk and the risk free rate; this is in sharp contrast to an economy without the presence of agency and dynamic contracts where the market price of risk, risk free rate and asset volatility are all constant. Our results raise the question whether a traditionally viewed “idiosyncratic” risk, namely incentives and compensation contracts of fund managers, are priced in that they do affect asset pricing in equilibrium.

Chapter Acknowledgements

I am deeply indebted to and grateful for the countless hours, encouragement, patience and comments of both of my advisers, Robert M. Anderson and Gustavo Manso, on this project and other research endeavors. I am indebted to Nicolae Gârleanu’s detailed thoughts and critical suggestions that have substantially improved this paper. I also thank Dae-Il Kang, Martin Lettau, Dmitry Livdan and Philipp Strack for the helpful comments and constructive suggestions. Comments and suggestions from the seminar attendees of the Fall 2014

UC Berkeley-Haas Finance Student Seminar, and the 2014 Korean National Pension Service International Forum for Specialists on Public Pension Fund Management are gratefully appreciated. All errors are mine and mine alone.

2.1 Introduction

A significant part of wealth in the modern economy are held in delegated portfolios that are managed by institutional fund managers on behalf of their client investors.¹ But also as a rise of this concentrated institutional ownership of assets, institutional fund managers play a dual role, both as portfolio managers who make portfolio weighting decisions, *and* also as monitors or shareholder activists who can exert pressure to improve the corporate governance or even operational performance on their invested firms. Yet, traditional asset pricing theory stipulates that asset prices are directly determined by individual households, and as a result, there is no need to consider agency friction in portfolio allocation decisions. Moreover, given that each household is small relative to the economy, it can be safely assumed that the household has insufficient voting power to influence the actions of the invested firm's manager. But with the ever growing size of the delegated portfolio management industry and also the monitoring or shareholder activism roles these fund managers play, the agency conflict between the fund manager and the investor becomes a significant concern. In recent years, a rich theory of continuous-time dynamic contracting arose in the literature, and has been successfully applied in the context of corporate finance. Yet surprisingly from the literature, we know little about the intersection between asset pricing and optimal contracting. This paper represents one of the first step to understanding the interaction between dynamic asset pricing, delegated portfolio management and dynamic optimal contracting. In particular, we use methodologies available almost exclusively to continuous-time asset pricing and also continuous-time dynamic contracting to derive the results, which would be extremely difficult to obtain in a discrete-time context.

Suppose there are two individuals in this economy, a risk averse portfolio manager (the agent) and a risk averse investor (the principal). The investor does not have any access to the financial markets and must delegate all portfolio decisions to the manager. The investor contracts with the manager for a fixed finite time horizon and there is intermediate consumption for both the manager and the investor. There is a risk free asset, and a single risky asset in this economy that is a claimant to a stream of dividends of a firm (i.e. like Lucas (1978)). The manager has two roles, both as a monitor and also as a portfolio decision maker. As a monitor, the manager can privately exert costly effort to increase the expected dividend growth rate of the firm. But as a portfolio manager, the manager needs to also decide the amount of wealth to allocate between the risky asset and the risk free asset. The investor offers a *pie sharing contract* to the manager over the share of dividends they can share at each time period. The investor only observes the value of the risky asset dividends and has no other pieces of information; specifically, the investor cannot observe the private effort the manager exerts in monitoring, nor can the investor observe the specific portfolio choices made by the manager.

The results suggest that institutional investors do matter for asset pricing. In particular, we find that all the key asset pricing primitives have an adjustment for dynamic moral

¹ See Wermers (2011).

hazard and dynamic contracts, above and beyond the base case agency free levels. The key channel is that the manager's per period consumption is precisely proportional to the current dividend level (which is influenced by the manager's actions), multiplied by the pie sharing contract (which is influenced by the investor). This cross effect is what endogenously drives the asset pricing dynamics, as there is a need to endogenously punish or reward the manager. When the manager's incentives are aligned with the investor's, the asset pricing dynamics (i.e. price-dividend ratio, Sharpe ratio, risk premium, risk free rate, and risky asset return volatility) are all equivalent to their agency-free constant value counterparts. When the manager's incentives are misaligned with the manager's, and if it becomes relatively costly to marginally compensate the manager, then the investor will choose to compensate the agent at the lowest possible rate possible to continue the relationship, and this will also lead to asset pricing dynamics that are equal to their agency-free constant value counterparts. However, when the manager's incentives are again misaligned with the manager's, but the cost of compensating the manager is not too high, then the investor will dynamically compensate the agent, and this will generate time varying and state dependent asset pricing dynamics. In terms of contracting, our results suggest that the performance and compensation fees that the manager receives should be back loaded and past performance dependent (i.e. pay the manager more towards the end of the contracting period). This model also enjoys some empirical confirmations from the existing literature, and also generates a few testable empirical implications. On the relationship between institutional investors and asset volatility, Gabaix, Gopikrishnan, Plerou, and Stanley (2006) show that institutional investors matter for generating excess stock market volatility. In a similar line, Christoffersen, Musto, and Wermers (2014) review the literature on investor fund flows, institutional managers and asset pricing.

2.2 Related Literature

This paper contributes to the literature in (i) the interaction between contracting and asset pricing; (ii) institutional shareholder activism and corporate governance; and (iii) delegated portfolio management. At the end, we will also briefly discuss how our paper fits broadly to asset pricing and also the continuous-time dynamic moral hazard literature.

Interaction Between Contracting and Asset Pricing

There are only a few studies that examine the interaction between asset pricing and optimal contracting. The companion papers of He and Krishnamurthy (2012, 2013) are interesting steps towards this direction, but the contract forms that they consider are *not* long-term dynamic contracts. In particular, Section 5.3 of He and Krishnamurthy (2012) has that (emphasis ours):

“For tractability reasons, in this paper we focus on short-term contracts. . . . [I]t will be interesting to develop models that marry the dynamic financial contracting

models with the dynamic asset pricing models. We are unaware of papers in the literature that accomplish this.”

Also, Gromb and Vayanos (2010) has that (emphasis ours):

“The constraints stem from moral hazard, and contracts [between fund managers and investors] are restricted to be static... *Extending this line of research to dynamic contracts, while retaining the tractability that is necessary to compute asset prices in general equilibrium, would be an important step forward...* Work along these lines could take constraints as given and so proceed in parallel with work on optimal contracting — although an important objective should remain that the two lines of research eventually merge.”

Thus, as evident from the above quotes by He and Krishnamurthy (2012) and Gromb and Vayanos (2010), this “marriage” between dynamic asset pricing in general equilibrium and dynamic optimal contracting remains an important open question in the finance literature.

Cuoco and Kaniel (2011) is one of the first papers to explicitly consider how contracts of managers can affect equilibrium asset prices in a delegated portfolio management setting. In particular, the paper takes exogenously several commonly used contract form and derives the return properties of both the benchmark and non-benchmark securities. Although the paper does discuss briefly the implications of optimal benchmarking, there was no formal attempt to consider derive such optimal contracts.

Basak and Pavlova (2013) motivates the preference form of an institutional investor to include the effects of benchmarking via a private diversion of managed wealth, using the Edmans and Gabaix (2011) assumption that the manager “takes action after noise”. Vayanos and Woolley (2013) argues that fund flows between investment funds can explain momentum and reversal, but they are relatively silent on the question of optimal contracting. Buffa, Vayanos, and Woolley (2013) explicitly considers an optimal contracting and asset pricing framework, but the contracts that they consider are essentially static, and not dynamic; moreover, they restrict the equilibrium asset prices to be affine in the shocks to the dividends process, and a priori, it is unclear why this necessarily needs to hold in equilibrium.

Sung and Wan (2013) is an interesting contribution to this line of thinking by considering a discrete-time two-period general equilibrium model with many firms, each firm produces an identical numeraire good, and each firm is owned by a principal that hires an agent to manage the firm; furthermore, each individual in the economy can trade bonds and stocks in the financial markets. Sung and Wan (2013)’s contribution is interesting in that it directly embeds a general equilibrium framework with a contracting framework, but is not the delegated portfolio management and shareholder activism problem that this paper considers.

Ou-Yang (2005) is also another attempt to integrate asset pricing with moral hazard, and indeed is most similar in spirit to this paper. However, the crucial difference between Ou-Yang (2005) and this paper is that, in Ou-Yang (2005), there is a distinct separation between the agent (i.e. the manager who can exert effort to influence the growth rate of

dividends) and the principal (i.e. who makes portfolio allocation choices and designs compensation schemes for the agent); in contrast, in this paper, *both* the portfolio choice and effort choice are made by the agent (manager), and the principal (investor) has no access to the financial markets and the only action the principal has is to design compensation contracts for the agent. Thus, whereas Ou-Yang (2005) is in some sense closer towards the classical principal-agent environment of Holmström and Milgrom (1987), this paper is closer to the motivation of delegated portfolio management and shareholder activism. Furthermore, from an asset pricing perspective, Ou-Yang (2005) explicitly fixes the risk free rate to the constant, and hence the author is in effect silent on the moral hazard effects on the risk free rate. As well, Ou-Yang (2005) assumes a priori that the equilibrium asset prices are affine in the dividends, but whereas in this paper, all the asset pricing quantities are derived and proved by first principles, and hence we can say meaningfully the endogenous effects of moral hazard on the risk premia and the market price of risk. And from the optimal contracting perspective, the principal can only compensate the agent (based on a history of observed managed dividends) at the end of the contracting period, but whereas we allow for full dynamic optimal contracting over all periods of time.

Gorton, He, and Huang (2014) is a recent study to model “managed Lucas trees”, in which a manager is hired to tend to the trees and the manager trades shares with the investors. The model is particularly interesting in that the manager here is explicitly acknowledged to be “big” in that when the manager trades shares with the investors, his trading has price impact and so the manager must also take this into account. Thus, this is a significant departure from the aforementioned relevant studies. Kaniel and Kondor (2013) considers a “delegated Lucas tree” (indeed, similar to our setup here) and consider the portfolio choice dynamics, but they consider an exogenous contract form for the manager and so does not discuss dynamic optimal contracts. Dybvig et al. (2010) also has a similar spirit to our paper in that they consider the effects of agency in portfolio management, assuming complete financial markets in a single period framework. However, the authors do not consider the full general equilibrium asset pricing implications and also, they do not have full dynamic contracts as we do given the time horizon setup they work with.

Delegated Portfolio Management

The setup of this model lends itself naturally to delegated portfolio management problems. As emphasized by Stoughton (1993) and Admati and Pfleiderer (1997), and summarized in Stracca (2006), delegated portfolio management problems present challenges that are not commonly considered in standard principal-agent problems; in particular, the portfolio manager has the ability to influence both the expected return and also volatility of the managed returns or cash flows. While managing expected return part, usually modeled as moral hazard hidden effort selection, is common in standard principal-agent problems, managing volatility is not. Ou-Yang (2003) is one of the key models in the delegated portfolio management literature, but does not consider the full asset pricing implications. Other recent models that consider delegated portfolio management problems include van Binsbergen, Brandt, and

Koijen (2008), Dybvig, Farnsworth, and Carpenter (2010), Pástor and Stambaugh (2012), Stambaugh (2014), Cvitanić, Possamaï, and Touzi (2014) and Leung (2014).

Although for the majority of the paper, we will keep in mind a “hedge fund” as a prototypical example of the fund that the manager manages, our model motivation here can easily be reinterpreted for a “private equity fund”. Thus here, we will briefly mention how our model fits in with that literature. Specifically, rather than through a shareholder activism channel to motivate why the manager must exert privately costly effort to improve the operations of the portfolio firms, we can think of the private equity manager as the direct operational manager of the portfolio firm, but must also simultaneously make portfolio allocation decisions. Kaplan and Schoar (2005) and Metrick and Yasuda (2010) are two recent empirical studies that examine the performance of private equity funds.

Asset Pricing and Dynamic Contracting

Martingale Methods and Dynamically Complete Markets

Clearly the literature of asset pricing is vast, and an excellent treatment is given in Duffie (2001). What is crucial to our paper is the application of martingale methods in solving the portfolio-consumption choice problem, as opposed to dynamic programming methods (say as per Merton (1969)). Under dynamically complete markets, the seminal contributions are in Cox and Huang (1989, 1991), and Huang and Pagès (1992). As well, we also used results in the existence of dynamically complete markets in continuous-time, of which recent contributions are in Anderson and Raimondo (2008) and Hugonnier, Malamud, and Trubowitz (2012).

Continuous-Time Dynamic Moral Hazard

Our paper here is also a contribution to the growing literature of continuous-time principal-agent problems, or dynamic moral hazard problems. The seminal contribution here is Holmström and Milgrom (1987), but recently, more sophisticated martingale representation methods have been put forth by Sannikov (2008) and has since been widely adopted in this strand of the literature. Recent interesting applications of these methods are found in corporate finance (say of DeMarzo and Sannikov (2006), He (2009), and Edmans, Gabaix, Sadzik, and Sannikov (2012)) and in macrofinance (say of Brunnermeier and Sannikov (2013)). The surveys of Sannikov (2012b, 2013) and Biais, Mariotti, and Rochet (2011) summarize the current state of this research.

2.3 Model setup

Let’s fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There are two individuals in this economy, a principal (whom we will call as the *investor*) and an agent (whom we will call as the *manager*). There exists a single firm that produces a stream of dividends $D = \{D_t\}$ of the consumption good.

² The financial markets consist of two securities: a risk free asset, and a risky asset that is a claimant on the stream of dividends produced by this firm. The investor has no access to the financial markets nor has any labor income, but is initially endowed with the entire supply (which we will subsequently normalize to unity) of this risky asset. The risk free asset is in zero net supply. The investor has no managerial expertise in monitoring the firm, and given that the investor has no access to the financial markets, the investor delegates the management of his wealth portfolio to the manager. ³ ⁴ The delegation time is for a finite T , and so the time span in consideration is $[0, T]$. Since the investor bestowed the entire supply share of equity of this firm to the manager, the manager now is a majority (or even the sole) shareholder of this firm.

Dividends

The single firm outputs a dividend stream $D = \{D_t\}$ with dynamics,

$$\frac{dD_t}{D_t} = h(A_t)dt + \Sigma dZ_t^A, \quad (2.3.1)$$

where $A = \{A_t\}$ will be the costly private shareholder activism effort exerted by the manager. We interpret the \mathbb{P}^A -Brownian motion Z^A in the sense as discussed in Section 2.A. In particular, we will make take the following conditions on the dividend drift:

Assumption 2.3.1 (Dividend drift and effort). *The set of feasible effort choices is the closed interval $[a_L, a_H]$, where $0 < a_L < a_H < \infty$. The reward function $h : [a_L, a_H] \rightarrow \mathbb{R}_{++}$ is $C^1([a_L, a_H])$, strictly positive and strictly increasing, and $h(a_L) - \Sigma^2/2 > 0$.*

Next, we make the following assumption on the form of the equilibrium gains process.

Assumption 2.3.2 (Gains process). *The price of the risky asset is S and the gains process are such that,*

$$\frac{dS_t + D_t dt}{S_t} = \mu(t)dt + \sigma(t)dZ_t^A, \quad (2.3.2)$$

where the drift μ and volatility σ are $\{\mathcal{F}_t^A\}$ -adapted processes that are to be determined in equilibrium.

² Essentially this is a Lucas (1978) tree, but it will be under management, as discussed below.

³ Alternatively, we can think of the investor as giving all his voting rights via proxy to the manager, and furthermore, allow the manager to make portfolio allocation decisions on the investor's wealth portfolio.

⁴ We should note that in contrast, He and Krishnamurthy (2012, 2013) does indeed allow the investor to directly invest into the risk free asset, but not the risky asset. But due to differences in how they model the moral hazard problem and also the contracting environment, allowing the investor here to have (partial) access to the financial markets substantially complicates our problem. It is conjectured that one could attempt to modify the approach considered by Basak and Cuoco (1998) to this paper, but nonetheless, the presence of the moral hazard and contracting problem substantially complicates the analysis. We leave this problem for future research.

Interpretations of effort $A = \{A_t\}$

There are two interpretations of “effort” $A = \{A_t\}$ in this context, and both are in accordance to the delegated portfolio management context and literature.

The first interpretation is *monitoring*. That is, there is a single firm and the portfolio manager needs to continuously over time monitor the management of this firm. Monitoring is private and costly, and higher monitoring A_t increases the expected operational performance of the invested firm, and thus yields higher expected dividend growth. For each monitoring plan $A = \{A_t\}$, the manager then in turn chooses a dynamic portfolio strategy. This idea of the financial intermediary, which in this case is our portfolio manager, acts as a monitor for investment projects goes back to the seminal paper of Diamond (1984). Moreover, this is the same interpretation that of *shareholder activism*, in that portfolio managers that have amassed a substantial equity stake into their portfolio firms, can “use their voting power as a catalyst for corporate change” (Yermack (2010)). Indeed, in particular for hedge funds, Yermack (2010) observes that:

“... [H]edge funds have a number of intrinsic advantages as activists when compared to pension funds, mutual funds, or other institutional investors. Hedge funds have no diversification requirement, enabling them to concentrate assets in a few target companies. Hedge funds can invest in illiquid securities, because their own investors cannot withdraw their capital on short notice, and hedge funds face less comprehensive ownership disclosure requirements than other institutions, enabling them to operate with greater secrecy and flexibility.”

Gillan and Starks (2007) review the evolution of shareholder activism in the United States.

The second interpretation is *investment opportunity search*. The manager can privately search and invest into one single risky investment, chosen from an infinite pool of choices $A^\omega = \{A_t^\omega\}$, where each investment opportunity is indexed by $\omega \in \mathbb{R}$. Private searching and selecting this risky investment opportunity ω incurs private cost $\mathbb{E}^{A^\omega} [\int_0^T e^{-\rho t} g(A_s^\omega) ds]$ at $t = 0$. In return, the investment opportunity with dividend dynamics (2.3.1), with $A = A^\omega$, becomes available, and the manager then in turn needs to further make portfolio investment decisions. The idea of searching and investing into a particular asset was explored in Vayanos and Wang (2007).

With these two types of interpretations in mind, but for the sake of brevity, we will generically call the manager’s choice of $A = \{A_t\}$ simply as “effort”.

Manager (Agent)

The manager has time separable logarithmic utility over consumption with subjective time discount factor ρ . From the role as a portfolio choice allocator, the manager restricts his choices to *self-financing portfolios* and from the returns of the managed portfolio P , he draws his consumption C_M . The manager will allocate α dollars into the risk free asset and will

allocate θ dollars into the risky asset. Furthermore, manager needs to choose the amount of costly private effort A he will exert.

Thus, the manager has objective function,

$$\sup_{C_M, (\alpha, \theta), A} \mathbb{E}_0 \left[\int_0^T e^{-\rho t} \log(C_M(t)) dt - \int_0^T e^{-\rho t} g(A_t) dt \right], \quad (2.3.3)$$

subject to the *self-financing condition*,

$$dP_t = \alpha(t)r_t dt + \theta(t) \left(\frac{dS_t + D_t dt}{S_t} \right) - C_M(t) dt \quad (2.3.4)$$

$$= \alpha(t)r_t dt + \theta(t) (\mu(t) dt + \sigma(t) dZ_t^A) - C_M(t) dt, \quad (2.3.5)$$

where the last equality follows from (2.3.2).

Assumption 2.3.3 (Manager's private cost function). *The manager's private cost function $g : [a_L, a_H] \rightarrow \mathbb{R}_{++}$ is strictly positive, $C^2([a_L, a_H])$, and strictly convex.*

The manager has an *individual rationality* (IR) participation constraint. That is, the manager's objective (2.3.3) must be greater than or equal to his constant outside option of $\hat{W} \geq 0$. That is,

$$\sup_{C_M, (\alpha, \theta), A} \mathbb{E}_0 \left[\int_0^T e^{-\rho t} \log(C_M(t)) dt - \int_0^T e^{-\rho t} g(A_t) dt \right] \geq \hat{W}. \quad (2.3.6)$$

To ensure that this is a feasible contracting relationship between the manager and the investor, we need some parameter restrictions to ensure that the IR constraint can be satisfied. That is to say, if the contracting parameters are too small, but the outside option is too large, then it will never be in the interest of the manager to engage into this contracting relationship. The following assumption ensures that it is weakly beneficial for the manager to consider this contract.

Assumption 2.3.4 (IR parameters). *The parameters $x_0, D_0, a_L, a_H, \Sigma, \rho, T, \hat{W}$ and functions g, h are such that*

$$\frac{\log x_0 + \log D_0 - g(a_L)}{\rho} (1 - e^{-\rho T}) + \frac{h(a_H) - \Sigma^2/2}{\rho^2} (1 - e^{-\rho T} (1 + \rho T)) \geq \hat{W}. \quad (2.3.7)$$

Investor (Principal)

As discussed, the investor has no access to the financial markets and nor does the investor have any labor income. Thus, consumption C_I for the investor must come from the dividends distribution of the consumption good. However, to incentivize and reward the manager, the investor will offer a *pie sharing rule contract* X , of which $X(t) \in [x_0, 1]$ for all t . That is to

say, for each unit D_t of dividends output by the firm, the manager will be entitled to $X(t)D_t$ units, while the investor will be entitled to $(1 - X(t))D_t$. We will think of $0 < x_0 \ll 1$ as the *minimal* piece of pie that the investor must share with the manager to maintain the contractual relationship. See Section 2.3 for more subsequent discussion of this pie sharing contract. To represent the fact that the investor is the ultimate owner of the managed portfolio and that the manager is an agent, the investor derives utility over the terminal value of the portfolio P_T ; this utility over the terminal value of the managed portfolio is explicitly absent in the manager's optimization problem in (2.3.3). Finally, the investor has a time separable utility function u satisfying Assumption 2.3.5 below, and also has the same time subjective discount factor $\rho > 0$ as the manager.⁵ See also Remark 2.3.6 for a discussion for the utility form u of the investor.

Thus, the investor has objective function,

$$\begin{aligned} & \sup_X \mathbb{E}_0 \left[\int_0^T e^{-\rho t} u(C_I(t)) dt + e^{-rT} u(P_T) \right] \\ & = \sup_X \mathbb{E}_0 \left[\int_0^T e^{-\rho t} u((1 - X(t))D_t) dt + e^{-rT} u(P_T) \right], \end{aligned} \quad (2.3.8)$$

where we have applied the market clearing condition in the equality; see Definition 2.3.1 below. Also, the utility function u satisfies the following conditions.

Assumption 2.3.5 (Investor's utility function). *The investor's utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the following conditions:*

- (i) u is $C^2(\mathbb{R}_+)$;
- (ii) $u(0) = \underline{u}$ for some constant $\underline{u} \in \mathbb{R}$;
- (iii) $\lim_{c \rightarrow \infty} u(c) = \bar{u}$ for \bar{u} with $\underline{u} < \bar{u} \leq \infty$;
- (iv) u is strictly increasing and strictly concave; and
- (v) $\lim_{c \rightarrow 0^+} u'(c) = +\infty$.

Remark 2.3.6. *One might naturally wonder why is it that in the manager's intertemporal utility specification (2.3.3) we deliberately constrain ourselves to using log utility, but for the investor's intertemporal utility specification (2.3.8) we allow for much greater flexibility. From a general conceptual perspective, we naturally can freely change the manager's utility from the logarithmic specification to something more general. However, as we shall see in subsequent development, the logarithmic form yields vastly more traceable solutions than some other general utilities. Thus, the choice of logarithmic utility in (2.3.3) is largely out of tractability and convenience.*

⁵ DeMarzo and Sannikov (2006) and He (2009) considers the difference in patience level between the manager and investor.

Example utility functions

The standard examples of utility functions that satisfy Assumption 2.3.5 include:

Example 2.3.1 (Exponential utility). Consider,

$$u(c) = \underline{u} - \frac{e^{-\gamma c}}{\gamma}, \quad (2.3.9)$$

for some constant $\underline{u} > 0$, and risk aversion coefficient $\gamma > 0$.

Example 2.3.2 (Power / CRRA utility). Consider,

$$u(c) = \underline{u} + \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad (2.3.10)$$

for some constant $\underline{u} \in \mathbb{R}$, and risk aversion coefficient $\gamma > 0$, $\gamma \neq 1$. In particular, we do not allow for the standard logarithmic utility (i.e. when $\gamma = 1$).⁶ For the subsequent numerical implementation, we will in particular focus on this CRRA utility with $\gamma = 2$.

Discussion of the pie sharing rule contract

We will now discuss the limitations and implications of focusing on a pie sharing rule contract X . Unlike the dynamic moral hazard literature of say DeMarzo and Sannikov (2006) and Sannikov (2008), where the compensation form from the principal to the agent is rather general, here it seems like we are far more specific and restrictive on the compensation form. But given the explicit risk averse preferences of the manager and also his managed portfolio wealth dynamics (2.3.5), it is economically senseless to compensate the manager into his objective function, as in the cases considered by DeMarzo and Sannikov (2006) and He (2009), among others, which assume the agent is not only risk neutral, but does not have a managed portfolio.⁷

As it will be seen below, we will be assuming the existence of a dynamically complete market (see Assumption 2.3.7). Anticipating this assumption, it also restricts the set of

⁶ This is just for the subsequent dynamic programming implementation. That is, more broadly, it is undesirable for our purposes to consider investor's utility forms that are not defined at $c = 0$, clearly of which logarithmic utility is an example. And clearly, the exponential utility (2.3.9) and power utility (2.3.10) do not have this undesirable property.

⁷ That is, if the investor offers an income stream \tilde{Y} , it is senseless to consider a form like,

$$E_0 \left[\int_0^T e^{-\rho t} [(\log(C_M(t)) - g(A_t)) + d\tilde{Y}_t] \right],$$

where the compensation $d\tilde{Y}$ directly enters into the manager's objective function. This is very at odds with the self-financing condition of the portfolio dynamics, and hence is economically senseless. But this form is indeed a common compensation form in the dynamic moral hazard literature, which is valid when there is no portfolio management considerations and the agent is risk neutral.

meaningful compensation form that we can consider. For instance, suppose the manager can be compensated directly through a labor income stream that enters into the self-financing wealth dynamics (2.3.5). Note that, naturally, the presence of a labor income stream will increase the managed wealth portfolio and thereby allow the manager to draw higher consumption, and hence derive utility.

But for this labor income stream, we need to split into two different cases for discussion. The first case is that this labor income stream consists of *marketable payoffs*, that is, payoffs for which it *can* be replicated by a portfolio of marketable securities, and in this case being some combination of the risk free asset and the risky asset. But here since the manager is also a portfolio choice allocator, and in the presence of dynamically complete markets, this means the manager can freely undo the marketed payoff compensation that's offered to him through different portfolio allocation choices. Hence, for the purpose of offering incentives to the agent, offering marketed payoffs in a dynamically complete market is not useful to provide incentives. The second case is that this labor income stream consist of *non-marketable payoffs*, that is, these payoffs *cannot* be replicated through a portfolio of the risk free and risky asset, because the manager is prevented from selling his labor income in the securities market.⁸ But from the results of He and Pagès (1993), Cuoco (1997) and others on consumption-portfolio choice problems, the presence of this stochastic non-marketed labor income generates incomplete markets, which will then be at odds with our Assumption 2.3.7. And economically from the standpoint of delegated portfolio management, it is not clear how one can understand the this non-marketed stochastic labor income as a compensation from the investor to the manager. In particular since in the portfolio management industry, the majority of the fees the manager generates come from the performance of the managed portfolio and a percentage of assets under management,⁹ so meaning that the majority of the manager's compensation is indeed derived from marketed assets, and hence are marketed themselves. Indeed, the main motivation for non-marketed stochastic labor income comes from a household portfolio-consumption problem whereby, as quoting from He and Pagès (1993), "the individual has limited opportunities to borrow against future labor income and cannot totally insure the risk of income fluctuations". This description does not seem to be the most fitting for the case of portfolio managers.

The advantage of using the described *pie sharing rule contract* is that it is consistent with the notion of market clearing of the consumption good in the classical general equilibrium literature.¹⁰ Moreover, it is also economically meaningful in the case of delegated portfolio management. That is to say, we can think of the pie sharing rule as payouts from the managed portfolio, of which the investor gets to get some of the payouts and the remainder goes to the portfolio manager. Thus, in summary, given the aforementioned discussion, the pie sharing rule contract in this current economy seems to be an economically robust and

⁸ Clearly, if the manager is able to sell his labor income in the securities market, we are back to the first case. Thus effectively, this second case is a case where we must have borrowing constraints to make this statement meaningful.

⁹ See Wermers (2011) for a discussion.

¹⁰ Of, say, the standard reference like Mas-Colell, Whinston, and Green (1995).

meaningful contract form. Finally, we will also discuss additional contractual tools that are available at the disposal to both the manager and investor in Section 2.7, where we will allow for actions like quitting, termination, takeovers and retirement.

Financial Markets

First we give the standard definition of market clearing.

Definition 2.3.1 (Market clearing). We say the *consumption good market clears* if, for any pie sharing contract X ,

$$D = C_M + C_I = C_M + (1 - X)D \quad (2.3.11)$$

Suppose the risk free assets are in zero net supply, and we normalize the supply of risky assets to be one. Then the *securities market clears* if,

$$\theta(t) = S_t, \quad (2.3.12a)$$

$$\alpha(t) = 0. \quad (2.3.12b)$$

Next, we state an assumption that is the most critical to this entire paper.

Assumption 2.3.7 (Dynamically complete markets). *For any effort process $A = \{A_t\}$ chosen by the manager, the financial markets are dynamically complete with no arbitrage. Thus, for each effort process $A = \{A_t\}$, a state price density process $\xi^A =: \xi = \{\xi_t\}$ exists and is unique, and has dynamics,*

$$d\xi_t = -\xi_t(r_t dt + \kappa_t dZ_t^A), \quad (2.3.13)$$

where the market price of risk κ is defined as,

$$\kappa_t := \frac{\mu(t) - r_t}{\sigma(t)}. \quad (2.3.14)$$

It should be remarked that Assumption 2.3.7 is the most critical assumption of the paper. That is, this assumption is actually making a stronger claim than what is usually considered in the asset pricing theory literature. For instance, in the asset pricing literature and relative to our context, for a *fixed single* dividend process D , Anderson and Raimondo (2008) *proves* the conditions needed for the existence of dynamically complete markets. Here, specifically, we require that for *any* arbitrarily chosen effort process A , which then affects the Brownian motion Z^A , we still have a dynamically complete financial markets. Finally, we note that in Dybvig et al. (2010), they also consider the effects of agency in portfolio management (but not in full general equilibrium) in a single period framework and they also further assume complete financial markets.

Equilibrium concept

Finally, in the discussion above, we are now ready to summarize and define our concept of equilibrium in this economy.

Definition 2.3.2 (Complete market competitive subgame perfect Nash equilibrium). A *complete market subgame perfect Nash equilibrium* (or *equilibrium* for short) $(\mu, \sigma, S, r, \alpha, \theta, A, X, C_M, C_I)$ is one such that:

1. The drift and volatility pair (μ, θ) admit a solution such that the risky asset gains process (2.3.2) holds and admits the existence and uniqueness of the state price density ξ^A of Assumption 2.3.7;
2. The consumptions (C_M, C_I) , portfolios (α, θ) , risky asset price S and risk free rate r satisfy the conditions for market clearing of Definition 2.3.1 holds;
3. The manager's choice variables $(C_M, (\alpha, \theta), A)$ satisfy the manager's individual rationality (IR) condition (2.3.6);
4. The manager's choice variables $(C_M, (\alpha, \theta), A)$ is *incentive compatible (IC)*, meaning it is a solution to the manager's problem (2.3.3) and (2.3.5); and
5. The investor's pie sharing rule contract X is a solution to the investor's problem (2.3.8).

Remark 2.3.8. *The problem solving strategy is to separate the problem into two distinct steps — we first solve the portfolio-consumption choice problem, and then second we solve the dynamic contracting problem.*

2.4 Financial markets equilibrium

We begin by solving the portfolio-consumption choice problem and Proposition 2.4.1 is the main result of this section.

Proposition 2.4.1. *Fix any given effort A and contract X . Then,*

- (i) *The manager's dollar amount portfolio choice into the risk free asset and risky asset are respectively,*

$$\alpha(t) = 0 \quad \text{and} \quad \theta(t) = P_t = S_t. \quad (2.4.1)$$

- (ii) *The parameters satisfy,*

$$\kappa_t = \sigma(t), \quad (2.4.2)$$

$$\mu(t) = r_t + \sigma(t)^2. \quad (2.4.3)$$

(iii) The risky asset price satisfies,

$$S_t = \frac{1}{\rho} X(t) D_t. \quad (2.4.4)$$

Thus, using the results of Proposition 2.4.1, we can rewrite the manager's objective function from (2.3.3) as,

$$\sup_A \mathbb{E}_0 \left[\int_0^T e^{-\rho t} [\log(X(t) D_t) - g(A_t)] dt \right]. \quad (2.4.5)$$

The key observation of Proposition 2.4.1 is (2.4.4), where $X(t) D_t$ is exactly proportional to (up to the subjective time discount rate ρ) the equilibrium consumption level $C_M(t)$ of the manager. Furthermore, this implies the equilibrium consumption level of the manager is affected by two channels — his effort to influence the expected growth rate of dividends D_t , and the pie sharing rule $X(t)$ that is influenced by the investor. This cross interaction between the actions of the manager and the investor is what drives the key results in the implications of the asset pricing dynamics.

2.5 Free rider problem

Before we proceed to consider the contracting problem, with Proposition 2.4.1 and recalling the manager's objective (2.3.3) subject to (2.3.5), we first study the source of the free riding problem in this economy. Suppose the manager is given a deterministic and constant contract $X(t) \equiv \bar{x} \in [x_0, 1]$. In particular, the case when $\bar{x} = 1$ corresponds exactly to the case when the manager is the sole owner of the asset in this economy.

Proposition 2.5.1. *Suppose the pie sharing rule contract is $X(t) \equiv \bar{x} \in [x_0, 1]$.*

(i) *Then the manager's payoff (2.3.3) equals to,*

$$\frac{\log \bar{x}}{\rho} (1 - e^{-\rho T}) + U(0, m), \quad (2.5.1)$$

where $U : [0, T] \times (-\infty, \infty)$ satisfies the Hamilton-Bellman-Jacobi (HJB) partial differential equation (PDE) on $[0, T] \times (-\infty, \infty)$,

$$\rho U = U_t + \max_{a \in [a_L, a_H]} (m - g(a)) + (h(a) - \Sigma^2/2) U_m. \quad (2.5.2)$$

(ii) *If g, h, a_L, a_H, ρ, T are such that $g'(a_H)/h'(a_H) > T e^{-\rho T}$, then the optimal effort is $A_t^* \in [a_L, a_H]$; that is, if the marginal cost to marginal benefit ratio is sufficiently high at the highest effort, the manager will never choose the highest effort a_H .*

In many ways, the result of Proposition 2.5.1(ii) is intuitive. If there were neither time nor stochastic variability in the pie sharing contract X , meaning the compensation to the manager is “not risky”, so that the manager is effectively the sole owner of the asset, the manager will clearly fully internalize both the costs and benefits of this private effort. That is to say, if the marginal private costs for exerting the highest effort greatly exceeds the marginal benefit of exerting the highest effort, it is optimal for the manager to not choose the highest effort. But this is precisely the source of the *free rider problem*.¹¹ The investor does not bear any of the private costs and reaps all the benefits, and it is clear that the shareholder would want the highest effort to be implemented at all times. But with non-risky compensations (i.e. when $X(t) \equiv \bar{x} \in [x_0, 1]$), there is no incentive for the manager to choose the highest effort whatsoever. This already clearly suggests that to implement an always high effort contract, we must consider stochastic contracts to put the manager at risk. This is in accordance to standard principal-agent problem intuition and results as per, say, Laffont and Martimort (2001) and Bolton and Dewatripont (2004).

To emphasize the free rider inefficiency, as per Proposition 2.5.1(ii), we impose the following assumption henceforth.

Assumption 2.5.2 (Free rider problem). *The parameters g, h, a_L, a_H, ρ, T are such that,*

$$\frac{g'(a_H)}{h'(a_H)} > Te^{-\rho T}. \quad (2.5.3)$$

The condition (2.5.3) of Assumption 2.5.2 is indeed relatively weak. For T moderately large, we have that $Te^{-\rho T} \approx 0$. Hence, the condition (2.5.3) essentially requires $g'(a_H)/h'(a_H) \gtrsim 0$. That is, as mentioned above, we just need that the marginal cost to marginal benefit ratio at the highest effort to be sufficiently high.

2.6 Contracting Problem

Now let’s consider the core section and main results of the paper. The following is a convenient and easy result.

Lemma 2.6.1 (Observational equivalence between stock price and dividends). *For any effort A and contract X , to the investor there is an unique observational equivalence between the risky asset price S and the dividends D .*

Thus, in light of Lemma 2.6.1, we will assume henceforth that the investor can continuously observe dividends D and its dynamics (2.3.1).

¹¹ A recent paper by Edmans and Manso (2010) considers free rider problems in corporate governance in the presence of multiple small block shareholders. We are not taking this direction here as in our framework, there is really only a “single” block holder (of voting rights), that being the manager.

Incentive Compatibility

With Lemma 2.6.1 on hand, we are now ready to consider the set of incentive compatible contracts. The development of this section resembles the discussion of Sannikov (2008).

Proposition 2.6.2. *For any given pie sharing rule contract X and any effort A , define the manager's continuation value,*

$$W_t(X, A) =: W_t = \mathbb{E}_t^A \left[\int_t^T e^{-\rho s} [\log(X(s)D_s) - g(A_s)] ds \right], \quad (2.6.1a)$$

$$W_0(X, A) =: W_0 = \hat{W}. \quad (2.6.1b)$$

where $\mathbb{E}_t^A[\cdot] =: \mathbb{E}^A[\cdot | \mathcal{F}_t^{Z^A}]$ is the conditional expectation generated by the probability measure induced by action A , conditioning on the filtration generated by dividends D .

(i) Then there exists η such that the manager's continuation value has the dynamics,

$$dW_t = [\rho W_t + g(A_t) - \log(X(t)D_t)] dt + \eta_t dZ_t^A \quad (2.6.2a)$$

$$= [\rho W_t + g(A_t) - \log(X(t)D_t)] dt + \frac{\eta_t D_t}{\Sigma} \left(\frac{dD_t}{D_t} - h(A_t) dt \right). \quad (2.6.2b)$$

(ii) An effort process A is incentive compatible for the manager if and only if η is such that,

$$A_t = \arg \max_{a \in [a_L, a_H]} \frac{\eta_t}{\Sigma} h(a) - g(a). \quad (2.6.3)$$

(iii) Suppose we seek an equilibrium to implement high effort $A_t \equiv a_H$ at all times. Then η must be such that,

$$\eta_t \geq \Sigma \frac{g'(a_H)}{h'(a_H)}. \quad (2.6.4)$$

Once we have characterized the incentive compatibility conditions as per Proposition 2.6.1, we are now ready to consider the investor's optimization problem. We will focus the formulation where the investor must contract with the manager for the duration of length T , and during that time, the investor can change the pie sharing contract rule. Here, we will make a broad simplification and subsequently discuss its implications and restrictions.

Assumption 2.6.3 (Constantly High Effort Equilibrium). *There exists an equilibrium (in the sense of Definition 2.3.2) for which it is optimal for the manager to implement an incentive compatible equilibrium for which:*

(i) The manager always exerts high effort $A_t \equiv a_H$; and

(ii) The sensitivity η in (2.6.4) of Proposition 2.6.1 to implement the said equilibrium can be held at equality, so

$$\eta_t \equiv \Sigma \frac{g'(a_H)}{h'(a_H)}.$$

Condition (i) of Assumption 2.6.3 is the equilibrium effort choice if the investor could operate the asset without any private costs g . Condition (ii) essentially says the performance sensitivity imposed on the manager can be implemented at the cheapest and minimal one. What is in particular ruled out by Assumption 2.6.3 is job shirking by the manager; that is to say, allowing the manager to choose $A_t < a_H$. We will assume and enforce Assumption 2.6.3 henceforth, unless specified otherwise.

2.7 Retirement and Quitting

The setup as of now will not let us pin down the contract form. In particular, we will need boundary conditions to our dynamic programming problem, else there are infinitely many number of solutions to our problem. Thus, we will impose the following conditions that are similar in spirit to Sannikov (2008). In particular, we will allow the investor to *retire* the manager, and also allow the manager to quit. That is to say, we consider the following *retirement function* F .

Proposition 2.7.1 (Retirement Function). *Suppose at time $\tau \in (0, T)$, when dividends $D_\tau = \delta$, the investor wants to indefinitely offer the manager a continuation value $w \in (-\infty, \infty)$ such that the manager will always exert the lowest effort $A_t \equiv a_L$, for $t \in [\tau, T]$. Then the investor's value function will thus equal to F , called the retirement function, given by,*

$$F(\tau, w, \delta) := \mathbb{E}_\tau^{a_L} \left[\int_\tau^T e^{-\rho t} u((1 - \bar{x}(\tau, w, \delta)) D_t^L) dt + e^{-\rho T} u(\rho^{-1} D_T^L) \mid D_\tau^L = \delta \right], \quad (2.7.1)$$

where,

$$\bar{x}(\tau, w, \delta) := (\tilde{x}(\tau, w, \delta) \wedge 1) \vee x_0 \quad (2.7.2a)$$

$$\tilde{x}(\tau, w, \delta) := \exp \left\{ \frac{\rho w e^{\rho \tau}}{1 - e^{-\rho(T-\tau)}} - \log \delta + g(a_L) + \frac{h(a_L) - \Sigma^2/2}{\rho} \left(1 - \frac{\rho(T-\tau)e^{\rho(T-\tau)}}{1 - e^{-\rho(T-\tau)}} \right) \right\}. \quad (2.7.2b)$$

and

$$D_t^L = \delta \exp \{ (h(a_L) - \Sigma^2/2)(t - \tau) + \Sigma(Z_t^{a_L} - Z_\tau^{a_L}) \}. \quad (2.7.3)$$

Effectively, the retirement function F is the worst possible outcome for the investor. That is, F is the payoff to the investor when the manager indefinitely chooses the lowest effort $A_t \equiv a_L$, and the investor in turn gives the manager a time invariant pie \bar{x} .

Example 2.7.1 (Power / CRRA Utility). Suppose the investor has power utility of the form in Example 2.3.2 with parameters such that,

$$\rho - (1 - \gamma) (h(a_L) - \Sigma^2(1 - 2\gamma)/2) > 0.$$

Then the retirement function R is given by,

$$\begin{aligned} F(\tau, w, \delta) = & \left(\frac{u}{1 - \gamma} - \frac{1}{1 - \gamma} \right) \frac{1 - e^{-\rho(T-\tau)}}{\rho} \\ & + \frac{(1 - \bar{x}(\tau, w, \log \delta))^{1-\gamma} \delta^{1-\gamma}}{1 - \gamma} e^{-\rho\tau} \frac{1 - \exp\{\rho - (1 - \gamma)(h(a_L) - \Sigma^2(1 - 2\gamma)/2)\}}{\rho - (1 - \gamma)(h(a_L) - \Sigma^2(1 - 2\gamma)/2)} \\ & + e^{-\rho T} \left(\frac{u}{1 - \gamma} + \frac{(\rho^{-1})^{1-\gamma} \delta^{1-\gamma} \exp\{(1 - \gamma)(h(a_L) - \Sigma^2(1 - 2\gamma)/2)(T - \tau)\}}{1 - \gamma} \right) \end{aligned} \quad (2.7.4)$$

and \bar{x} is as given in Proposition 2.7.1.

2.8 Investor's optimization problem

Thus in summary and recalling (2.3.8), and Proposition 2.4.1 that at $t = T$ we have $P_T = X(T)D_T/\rho$, the investor's optimization problem is now of the form,

$$\sup_X \mathbb{E}_0 \left[\int_0^T e^{-\rho t} u((1 - X(t))D_t) dt + e^{-\rho T} u(\rho^{-1} X(T)D_T) \right].$$

Here we make the obvious observation that at the terminal time $t = T$, since the manager has no claims to the final dividend nor the final portfolio wealth, the investor will clearly set $X(T) = 1$ optimally. Thus, we can simplify the investor's optimization problem above to,

$$\sup_X \mathbb{E}_0 \left[\int_0^T e^{-\rho t} u((1 - X(t))D_t) dt + e^{-\rho T} u(\rho^{-1} D_T) \right], \quad (2.8.1)$$

subject to state variables,

$$dW_t = [\rho W_t - g(a_H) - \log X(t) - \log D_t] dt + \Sigma \frac{g'(a_H)}{h'(a_H)} dZ_t^{a_H}, \quad W_0 = w, \quad (2.8.2a)$$

$$dD_t = D_t h(a_H) dt + D_t \Sigma dZ_t^{a_H}, \quad D_0 = \delta \quad (2.8.2b)$$

Dynamic programming

Then we have the following standard principle of dynamic programming result. Denote the sets,

$$O := (\hat{W}, \tilde{W}) \times (\underline{\delta}, \bar{\delta}), \quad (2.8.3a)$$

$$Q := [0, T] \times O, \quad (2.8.3b)$$

$$\partial^* Q := ([0, T] \times \partial O) \cup (\{T\} \times O), \quad (2.8.3c)$$

where ∂ denotes the boundary of a set.

Proposition 2.8.1. *Let the value function¹² associated with the optimization problem (2.8.1) and (2.8.2) be denoted as $V(t, w, y) =: V(t, w, y)$, and we will use subscripts to denote partial derivatives.¹³ The value function V satisfies the Hamilton-Jacobi-Bellman (HJB) equation on Q ,*

$$\begin{aligned} V_t = & \sup_{x \in [x_0, 1]} \rho V + u((1-x)\delta) + (\rho w - g(a_H) - \log x - \log \delta) V_w + h(a_H) \delta V_y \\ & + \frac{\Sigma^2}{2} \left[\left(\frac{g'(a_H)}{h'(a_H)} \right)^2 V_{ww} + \delta^2 V_{yy} + 2 \left(\frac{g'(a_H)}{h'(a_H)} \right) \delta V_{wy} \right]. \end{aligned} \quad (2.8.4)$$

subject to boundary conditions,

$$V = F \quad \text{on } \partial^* Q. \quad (2.8.5)$$

See also Figure 2.1 for an illustration.

Remark on the economic richness of this setup

Actually, more is true than what is shown in Proposition 2.8.1. Similar to Sannikov (2008), we can actually consider alternative contractual environments by changing the boundary conditions. That is to say, if we maintain the same boundaries of the form (2.8.3), but change the boundary function F (i.e. the retirement function of Section 2.7) with another smooth function \tilde{F} , the general results thereafter on existence and smoothness still holds. For our purposes, what is most critical is imposing the lower and upper bounds so that the state space Q is indeed bounded. Unfortunately, that comes at the cost in terms of the economic richness that we can consider in the model. For instance, in Sannikov (2008) and

¹² In the actual proofs, we will actually consider a “smoothed” version F_k of the retirement function F . Essentially, the issue is that F itself is not differentiable in all the arguments. However, for the purpose of obtaining the existence and uniqueness of smooth solutions to the dynamic programming problem, we simply replace F with a smoothed version F_k so that the function becomes differentiable. Qualitatively and economically, this does not change the results. See the appendix for details.

¹³ To be clear, we will denote $V_t := \partial V / \partial t$, $V_w := \partial V / \partial w$ and $V_y := \partial V / \partial y$. Note that for the derivative with respect to the dividend state variable, we denote the partial derivative by “ V_y ” instead of, say, “ V_δ ”. Similar comments apply to the other higher order partial and cross derivatives.

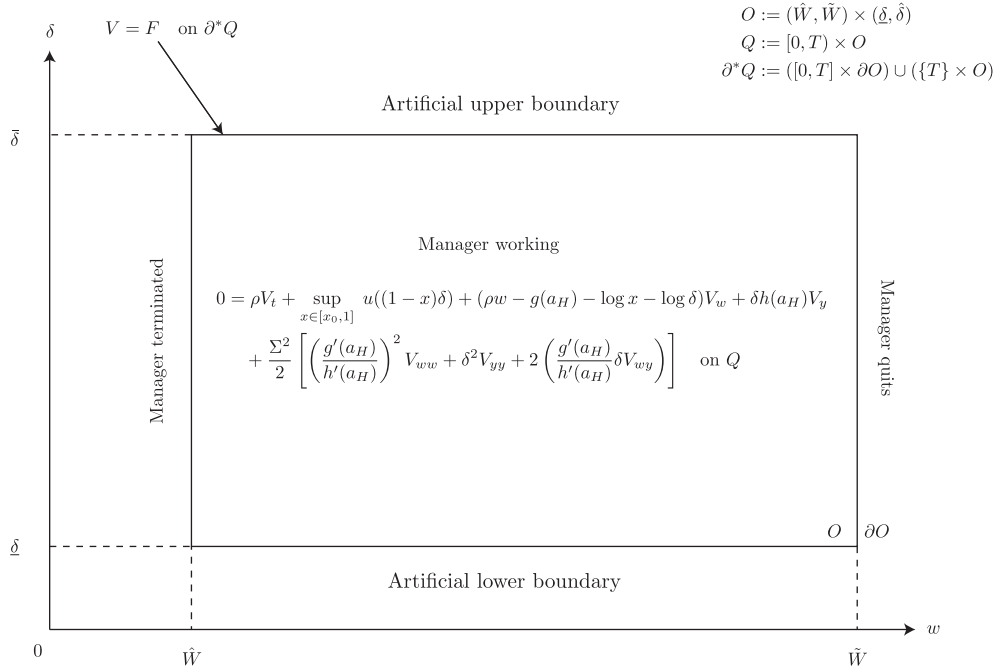


Figure 2.1: Illustration of the state space to the optimization problem in Proposition 2.8.1.

He (2009), the authors use the *smooth-pasting* and *value-matching* conditions to characterize the optimal termination point of the agent's continuation value. That type of analysis is only held in solid rigorous grounds when we deal with *ordinary* differential equations, as in those papers. However, in this setting, given that we have an explicit *partial* differential equations setting with one time variable and two state variables, it is a priori unclear whether the *smooth-pasting* and *value-matching* conditions as per Dixit (1993); Dixit and Pindyck (1994) are still valid. Finally, if we were to consider an unbounded state space, we would require far stronger conditions on the growth behavior of all the relevant functions to ensure that we do have a smooth solution to the HJB in Proposition 2.8.1, which we do not have, but the trade-off here is that we have far more explicit characterizations on the asset pricing side of the model.

2.9 Optimal pie sharing rule contract

We will define some sets, although they seem very unmotivated, are essentially related to the Lagrange multipliers, and the sufficient and necessary conditions of optimality. Recall we denote $V_w := V_w(t, w, \delta)$, and using Lemma 2.D.3, let's also denote $\tilde{X} := \tilde{X}(t, w, \delta) =$

$I(-V_w(t, w, \delta)/\delta, \delta) \in (x_0, 1)$.

Definition 2.9.1. We will define the following sets on Q ,

$$E := \{(t, w, \delta) : V_w(t, w, \delta) \geq 0\} \quad (2.9.1a)$$

$$EI := \{(t, w, \delta) : V_w(t, w, \delta) < 0 \text{ and } -V_w(t, w, \delta) \geq x_0\delta u'((1-x_0)\delta)\} \quad (2.9.1b)$$

$$EI^c = \{(t, w, \delta) : V_w(t, w, \delta) < 0 \text{ and } -V_w(t, w, \delta) < x_0\delta u'((1-x_0)\delta)\} \quad (2.9.1c)$$

Note that the above sets completely partition Q , and each of them are mutually exclusive to each other.

Now we come to the explicit construction and computation of the optimal Markovian pie sharing rule X .

Proposition 2.9.1. *The optimal Markovian pie sharing contract rule is given by,*

$$X(t, w, \delta) = \begin{cases} x_0, & \text{on } E \cup EI^c, \\ \tilde{X}(t, w, \delta), & \text{on } EI. \end{cases} \quad (2.9.2)$$

Economically and intuitively speaking, the optimal pie sharing contract takes on the following form. Firstly, recall that $V_w := V_w(t, w, \delta)$ is the change of the investor's value function with respect to the change of the manager's continuation value $W_t = w$. So on E , when $V_w \geq 0$, given that the investor's value function V is increasing with the continuation's valuation value w , so this is the case when the manager's and the investor's interests are perfectly aligned, there is no need to provide additional incentives to motivate the manager, and hence, it is optimal to choose the lowest pie sharing x_0 .

Secondly, suppose now $V_w < 0$, that is, when the agent's continuation value w increases, the principal's value function V decreases, or conversely, when the agent's continuation value w decreases, the principal's value function V increases. This is precisely the case when the interests of the manager and the investor are misaligned, and hence the investor may now need to provide incentives to the manager. Observe that $u'((1-x_0)\delta)$ is the investor's instantaneous marginal utility when he enjoys the highest possible share $(1-x_0)\delta$ of the dividends, but $x_0\delta$ is the lowest possible portion of the dividends that are due to the manager. Thus, we can think of $u'((1-x_0)\delta)$ as the "price" of providing lowest incentives, and $x_0\delta$ is the "quantity" of lowest possible incentives to be provided.

So when $V_w < 0$, the investor needs to compare the case in EI when

$$-V_w \geq x_0\delta u'((1-x_0)\delta),$$

and the case in EI^c when

$$-V_w < x_0\delta u'((1-x_0)\delta).$$

In EI , the condition says that the benefit $-V_w > 0$ of correcting the alignment of interests between the investor and manager is weakly greater than the total instantaneous benefit

$x_0\delta u'((1-x_0)\delta)$ of the investor taking the highest possible pie share, and so meaning in EI , it is beneficial for the investor to compensate the agent $\tilde{X}(t, w, \delta) > x_0$. Analogously in EI^c , the benefit $-V_w > 0$ of correcting the alignment of interests between the investor and the manager is strictly smaller than the total instantaneous benefit of the investor taking the highest possible pie share, and hence on EI^c , the investor is better off by taking the highest possible pie share $1-x_0$, and give the lowest possible pie share x_0 to the manager.

2.10 Asset Pricing Dynamics

With the results of Section 2.9 on hand, we are now ready to consider the asset pricing dynamics and solve for the equilibrium μ, σ as considered in (2.3.2). In particular, in light of Proposition 2.9.2, there are clearly three types of values the contract X can take on, and they all are associated with three different state space values (t, w, δ) .

Proposition 2.10.1. *Denote,*

$$\tilde{X}(t, w, \delta) := I\left(-\frac{V_w(t, w, \delta)}{\delta}, \delta\right), \quad (2.10.1)$$

where the definition of I is given in Lemma 2.D.3. Note that \tilde{X} is effectively the interior solution of the pie sharing contract rule on $[x_0, 1]$. For any $(t, W_t, D_t) \in Q$, for notational convenience, $\tilde{X} := \tilde{X}(t, W_t, D_t)$, $\tilde{X}_t := \tilde{X}_t(t, W_t, D_t)$, $\tilde{X}_w := \tilde{X}_w(t, W_t, D_t)$ and likewise for all the other cross and partial derivatives. For any set K in Q , we will also denote the indicator $\mathbf{1}_K := \mathbf{1}_K(t, W_t, D_t)$.

(i) The equilibrium gains process (2.3.2) drift $\mu(t) = \mu(t, W_t, D_t)$ is given by,

$$\begin{aligned} & \mu(t, W_t, D_t) \\ &= \left(\frac{\rho}{x_0} + h(a_H)\right) \mathbf{1}_{E \cup EI^c} \\ &+ \left(\frac{\rho}{\tilde{X}} + h(a_H) + \frac{1}{\tilde{X}} \left[\tilde{X}_t + \tilde{X}_w(\rho W_t - g(a_H) - \log \tilde{X} - \log D_t) + D_t \tilde{X}_y h(a_H) \right] \right. \\ &\quad \left. + \frac{1}{\tilde{X}} \frac{\Sigma^2}{2} \left[\tilde{X}_{ww} \left(\frac{g'(a_H)}{h'(a_H)}\right)^2 + (2\tilde{X}_y + D_t \tilde{X}_{yy}) + 2(\tilde{X}_w + D_t \tilde{X}_{wy}) \frac{g'(a_H)}{h'(a_H)} \right] \right) \mathbf{1}_{EI}. \end{aligned}$$

(ii) The equilibrium gains process (2.3.2) volatility $\sigma(t) = \sigma(t, W_t, D_t)$ is given by,

$$\sigma(t, W_t, D_t) = \Sigma \mathbf{1}_{E \cup EI^c} + \left(\Sigma + \frac{\Sigma}{\tilde{X}} \left[\tilde{X}_w \frac{g'(a_H)}{h'(a_H)} + D_t \tilde{X}_y \right] \right) \mathbf{1}_{EI}.$$

(iii) The equilibrium market price of risk is given by,

$$\kappa_t = \kappa(t, W_t, D_t) = \sigma(t, W_t, D_t), \quad (2.10.2)$$

where the expression for σ is as given in (ii).

(iv) The equilibrium risk free rate is given by,

$$r_t = r(t, W_t, D_t) = \mu(t, W_t, D_t) - \sigma(t, W_t, D_t)^2. \quad (2.10.3)$$

Remark 2.10.2. Finally with Proposition 2.10.1 we can now fully explain why it is crucial in this paper that we must have smooth solutions to the investor's value function V , while such condition, while desirable, is strictly not necessary for general optimal contracting problems, where we can use weaker notions of solutions like viscosity solutions. Effectively, we need two conditions to make everything fall in place: (i) the existence of Markovian optimal controls; and (ii) of these Markovian controls, they are smooth in the state variables. The existence of the Markovian optimal controls is afforded by the smoothness of the value function. The smoothness of the state variables is afforded essentially by the primitive parameters of the model. In particular, it is precisely the smoothness of optimal control in the state variables that allow us to apply validly Ito's lemma¹⁴ to derive the equilibrium risky asset gains process drift μ and volatility σ , and thereby "complete the loop" of the model.

Discussion of the asset pricing dynamics

Again, the key result of this section and the paper is Proposition 2.10.1 where we show and summarize the implications of dynamic moral hazard and contracting on the dynamic asset pricing primitive parameters.

Drift μ

If we consider the drift μ of the risky asset gains process, we clearly see that the value of the drift takes on three different values depending on the time and state values $(t, W_t, D_t) = (t, w, \delta)$. The cases E or EI^c where the pie sharing rule takes on the lowest constant ($X = x_0$), we have the standard result that the drift is identically constant, resulting from the log preferences of the manager. Roughly speaking, this drift value is essentially equal expected growth rate of the dividends (i.e. $h(a_H)$), plus the patience of the manager ρ , modified by the proportion X of the dividend consumption good is entitled to. This is the result if there were no agency conflicts in the market. The most interesting case is the intermediate case when $X \in (x_0, 1)$, which happens when the time and state values are in EI . Rewriting the

¹⁴ Recall that for a function $f(t, y)$, Ito's lemma requires that f is $C^{1,2}$.

result of Proposition 2.10.1 when we are on EI ,

$$\begin{aligned}
 & \overbrace{\frac{\rho}{\tilde{X}} + h(a_H)}^{\text{Agency-free expected return}} \\
 & + \overbrace{\frac{1}{\tilde{X}} \left[\tilde{X}_t + \tilde{X}_w(\rho W_t - g(a_H) - \log \tilde{X} - \log D_t) + D_t \tilde{X}_y h(a_H) \right]}^{\text{Risk premium due to dynamic agency and contracting}} \\
 & + \overbrace{\frac{1}{\tilde{X}} \frac{\Sigma^2}{2} \left[\tilde{X}_{ww} \left(\frac{g'(a_H)}{h'(a_H)} \right)^2 + (2\tilde{X}_y + D_t \tilde{X}_{yy}) + 2(\tilde{X}_w + D_t \tilde{X}_{wy}) \frac{g'(a_H)}{h'(a_H)} \right]}^{\text{Risk adjustment, modified for marginal cost to marginal benefit ratio}}.
 \end{aligned} \tag{2.10.4}$$

Volatility σ / Market price of risk κ

Similarly, if we consider the volatility σ of the risky asset gains process, we see that again it depends on the time and state values. The cases E or EI^c when the pie sharing rule takes on the lowest constant, we have that the asset volatility is identically equal to the constant dividend volatility, which again is the case when we have no agency conflicts in the market. Things are more interesting when we have the intermediate case when the time and state values are in EI . Rewriting the result of Proposition 2.10.1 when we are on EI ,

$$\underbrace{\Sigma}_{\text{Agency-free volatility}} + \overbrace{\frac{\Sigma}{\tilde{X}} \left[\tilde{X}_w \frac{g'(a_H)}{h'(a_H)} + D_t \tilde{X}_y \right]}^{\text{Adjustment for dynamic moral hazard and optimal contracting}}. \tag{2.10.5}$$

It should be noted that by Proposition 2.4.1, the market price of risk κ and the risky asset return volatility σ are identical.

2.11 Numerical Solutions

It is clear that the value function V in Proposition 2.8.1 would have no closed form solutions, which thus also leads to no closed form solutions for the optimizer X which characterizes the optimal pie sharing contract, and also the important asset pricing primitive parameters as per Proposition 2.4.1 and Proposition 2.10.1. Hence, we must resort to numerical solutions to continue our investigation. We in particular focus on the investor's utility u being of CRRA with risk aversion parameter $\gamma = 2$. See also Section 2.F for more details of this case. We plot the value function (Figures 2.1, 2.2, 2.3, 2.4 and 2.5) at various points of time t .

Remark 2.11.1. *It should be explicitly be noted that no specific attempts are made to “calibrate” these parameters to match any real world empirical moments. But rather, the core parameters are chosen merely as an illustration of the model. Specifically, one might wonder*

about why we choose a very high drift value for the dividend growth rate under the highest effort case. That is only for numerical stability purposes. The main takeaway from these numerical illustrations should be more of its qualitative than quantitative effects.

Investor's value function $V(t, w, \delta)$

The value function confirms several economic intuitions regarding the setup of this problem. At the end of the contracting period, when $t = T$, the principal has essentially a zero value function. This is to be expected since for moderate values of the subjective time discount rate ρ and moderate values of contract time T , $e^{-\rho T} u(\rho^{-1} D_T)$ is close to (but not equal to) zero. And as time rolls back to the initial time point $t = 0$, we see that the value function increases in value. Again, this is anticipated since the investor has more time to enjoy consumption of his share of the managed Lucas fruit. We note that at $t = 0$, the value function $V(t, w, \delta)$ exhibits also several behavior that is anticipated from this model setup. The value function is increasing in the dividends δ realization. Clearly, with a higher dividend realization, all else being equal, the investor can derive higher utility over its consumption. But with respect to the continuation value, we see that the value function is concave in the w -direction; that is, holding time t and the dividend level δ fixed, $V(t, \cdot, \delta)$ is concave in w . This is visually most apparent when $t = 0$, but it is indeed true at all times t .

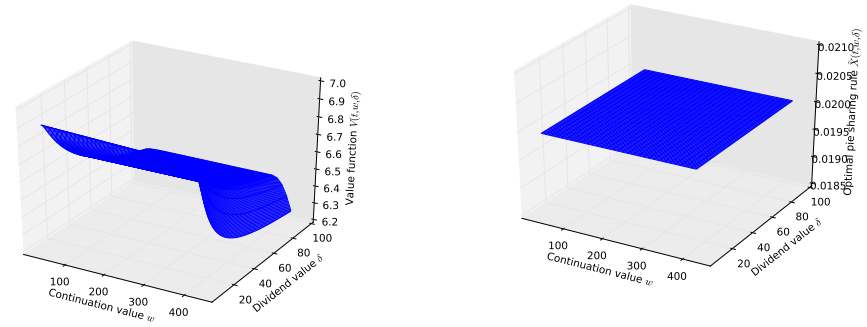
Optimal pie sharing rule contract $X^*(t, w, \delta)$

The most interesting results come from the optimal pie sharing rule $X^*(t, w, \delta)$, particularly since both the solution to the contracting problem and the implications of asset pricing all strongly depend on the pie sharing rule. The most obvious characteristic of the pie sharing contract is its *back loaded* nature. That is, near the ending time periods, there are vastly more states than near the initial periods where the investor will pay the manager an amount strictly greater than the minimum pie sharing portion x_0 . Furthermore, we see even for fixed time t , there is substantial variation in the optimal pie sharing contract across (w, δ) , suggesting thus that the payment the investor offers to the manager is past performance dependent (i.e. depending on the continuation value w of the manager), and also market dependent (i.e. depending on the dividends δ).

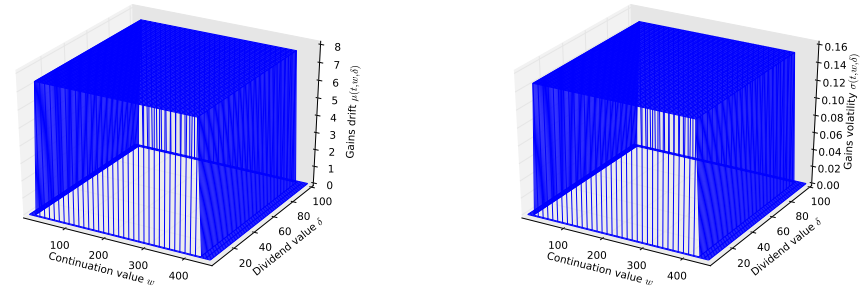
Asset pricing parameters $\mu(t, w, \delta)$ and $\sigma(t, w, \delta)$

Let's now turn to the key asset pricing implications of this model. In addition to the figures below, we also show several other figures that emphasize upon the asset pricing implications. Recall from Proposition 2.4.1 and Proposition 2.10.1, we note that the market price of risk κ_t , in equilibrium, is identical to the gains volatility σ_t . And also, we have that the price-dividend ratio $S_t/D_t = X(t)/\rho$, and hence this means the price-dividend is equal to the optimal pie sharing rule, inverse scaled by the subjective time discount factor ρ . Hence, it suffices to concentrate our focus to the gains parameters μ_t and σ_t . From the figures, we

see the qualitative result that in the regions where the optimal pie sharing rule $X^*(t, w, \delta)$ exceeds the minimum pie sharing level x_0 , dynamic changes occur in both the gains drift μ_t and the gains volatility σ_t . When the optimal pie sharing rule exceeds the minimum pie sharing level, this is exactly when the investor needs to provide incentives to the manager. Qualitatively, we see the result that when the investor needs to provide incentives to the manager, the expected gains drift μ_t decreases, and the gains volatility σ_t increases. This is precisely the channel in which institutional managers' incentives can affect asset prices in general equilibrium.

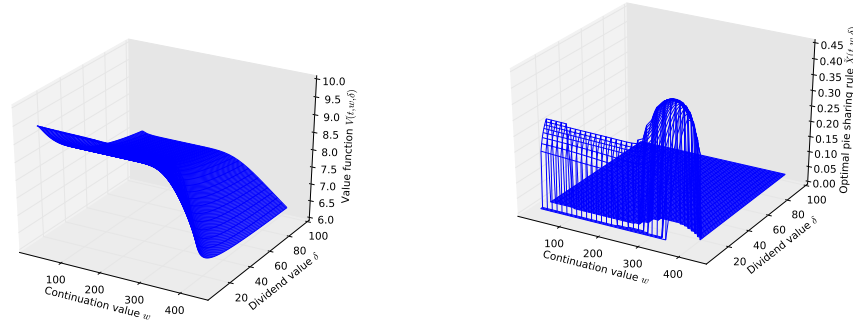


(a) Investor's value function $V(t, w, \delta)$. (b) Optimal pie sharing rule contract $X^*(t, w, \delta)$.

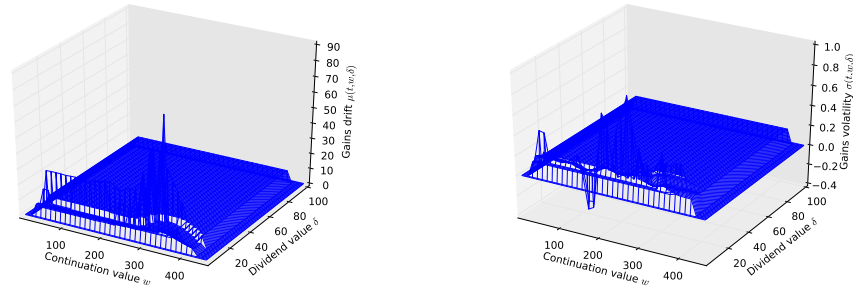


(c) Gains drift $\mu(t, w, \delta)$. (d) Gains volatility $\sigma(t, w, \delta)$.

Figure 2.1: The shown time value is at $t = 0.01$. The x -axis is the manager's continuation value $W_t = w$, the y -axis is the dividend value $D_t = \delta$. The investor's utility is CRRA, $u(c) = \frac{c}{1-\gamma}$. The key parameters are: $\rho = 0.10\underline{u} = 10, \gamma = 2, x_0 = 0.02, [a_L, a_H] = [0.36, 0.95], h(a) = g(a) = e^a, \Sigma = 0.15, T = 6$.



(a) Investor's value function $V(t, w, \delta)$. (b) Optimal pie sharing rule contract $X^*(t, w, \delta)$.



(c) Gains drift $\mu(t, w, \delta)$. (d) Gains volatility $\sigma(t, w, \delta)$.

Figure 2.2: The shown time value is at $t = 0.4$. The x -axis is the manager's continuation value $W_t = w$, the y -axis is the dividend value $D_t = \delta$. The investor's utility is CRRA, $u(c) = \frac{c}{1-\gamma}$. The key parameters are: $\rho = 0.10$, $\underline{u} = 10$, $\gamma = 2$, $x_0 = 0.02$, $[a_L, a_H] = [0.36, 0.95]$, $h(a) = g(a) = e^a$, $\Sigma = 0.15$, $T = 6$.

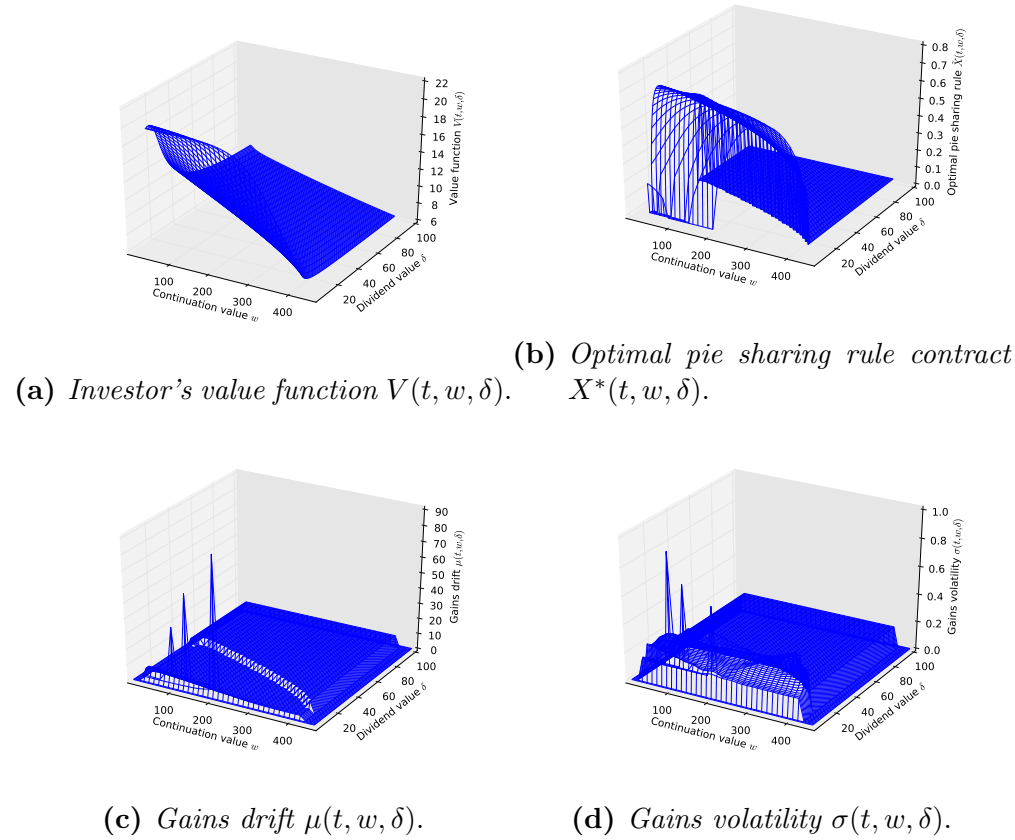


Figure 2.3: The shown time value is at $t = 1.6$. The x -axis is the manager's continuation value $W_t = w$, the y -axis is the dividend value $D_t = \delta$. The investor's utility is CRRA, $u(c) = \frac{c}{1-\gamma}$. The key parameters are: $\rho = 0.10$, $\underline{u} = 10$, $\gamma = 2$, $x_0 = 0.02$, $[a_L, a_H] = [0.36, 0.95]$, $h(a) = g(a) = e^a$, $\Sigma = 0.15$, $T = 6$.

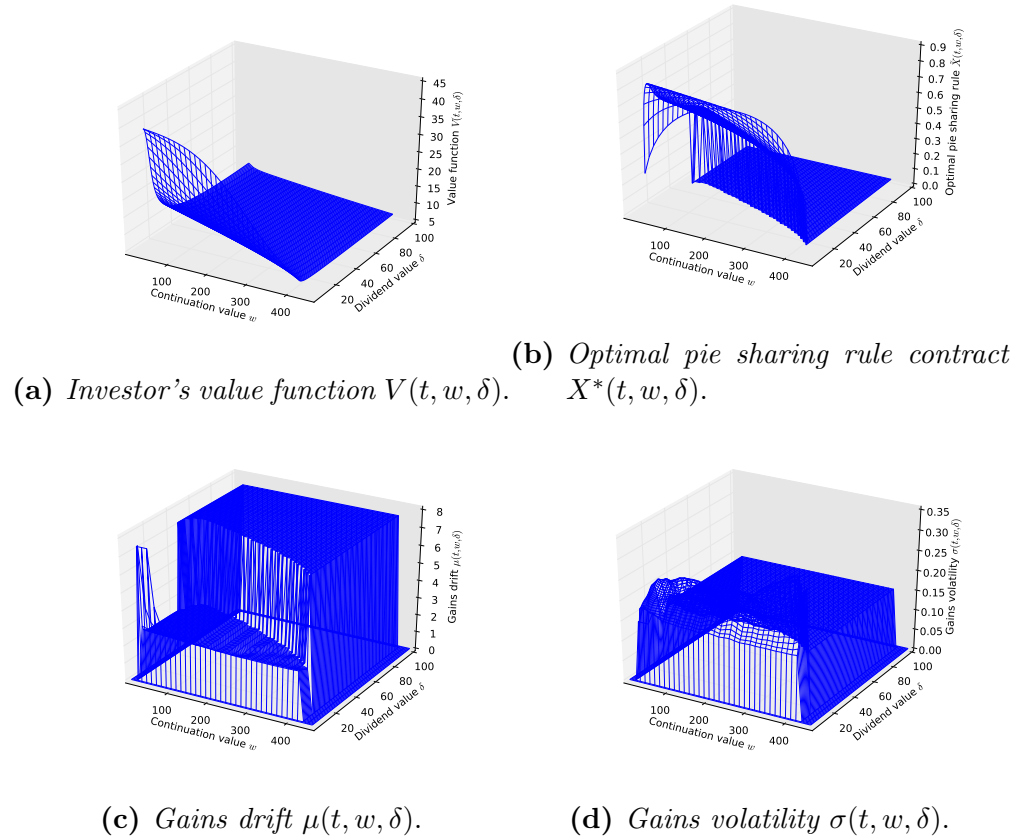
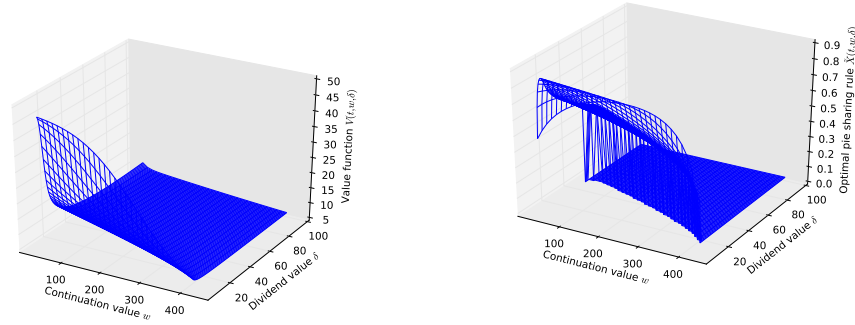
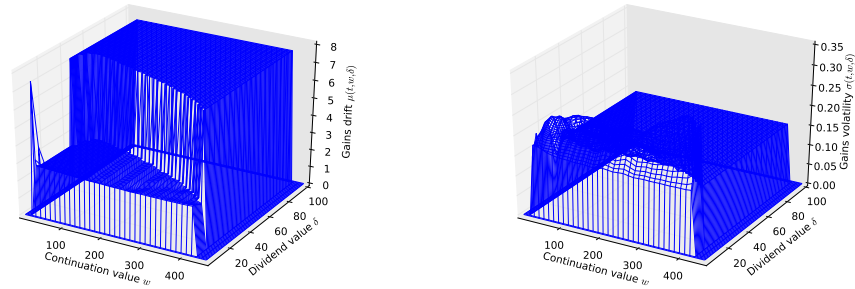


Figure 2.4: The shown time value is at $t = 4.0$. The x -axis is the manager's continuation value $W_t = w$, the y -axis is the dividend value $D_t = \delta$. The investor's utility is CRRA, $u(c) = \frac{c}{1-\gamma}$. The key parameters are: $\rho = 0.10$, $\underline{u} = 10$, $\gamma = 2$, $x_0 = 0.02$, $[a_L, a_H] = [0.36, 0.95]$, $h(a) = g(a) = e^a$, $\Sigma = 0.15$, $T = 6$.



(a) Investor's value function $V(t, w, \delta)$. (b) Optimal pie sharing rule contract $X^*(t, w, \delta)$.



(c) Gains drift $\mu(t, w, \delta)$. (d) Gains volatility $\sigma(t, w, \delta)$.

Figure 2.5: The shown time value is at $t = 5.0$. The x -axis is the manager's continuation value $W_t = w$, the y -axis is the dividend value $D_t = \delta$. The investor's utility is CRRA, $u(c) = \frac{c}{1-\gamma}$. The key parameters are: $\rho = 0.10$, $\underline{u} = 10$, $\gamma = 2$, $x_0 = 0.02$, $[a_L, a_H] = [0.36, 0.95]$, $h(a) = g(a) = e^a$, $\Sigma = 0.15$, $T = 6$.

2.12 Conclusion

We have presented a model where an investor must delegate all of his investment decisions to an institutional investment manager. This model incorporates elements of both dynamic agency, dynamic optimal contracting and dynamic asset pricing in general equilibrium. The results strongly suggest that the incentives and the dynamic incentive contracts of institutional managers do indeed affect asset prices in general equilibrium. More importantly, the results suggest that dynamic incentives and dynamic contracts imply that in equilibrium, the key asset pricing parameters such as the equilibrium asset drift, volatility, market price of risk and risk free rate become endogenously stochastic over and above the agency-free benchmark case.

Although not directly investigated in our model, but our qualitative results strongly suggest that for those financial assets that are managed or heavily invested upon by institutional managers, its “idiosyncratic risks” would be priced. That is to say, classical asset pricing intuition would suggest that the compensation contracts of the firm or that of the institutional fund managers are considered idiosyncratic and can be diversified away by the representative household investor. However, this view is limiting. In particular, if there are frictions that prevent the household investor from directly accessing the capital markets, and specifically, the investor must somehow depend on portfolio delegation to an institutional manager, as it is shown in this paper, the dynamic incentives and contracts of this manager could indeed affect asset prices in equilibrium. Indeed, following up on the empirical literature say by Campbell et al. (2001), Goyal and Santa-Clara (2003), Bali et al. (2005) and others where they suggest that idiosyncratic risk of assets could be priced, our results suggest that a potential avenue for further investigation could be the linkage between institutional investor participation of the asset and that asset’s pricing (or not) of its idiosyncratic risk.

Appendix

2.A Probabilistic setup

Remark 2.A.1. *These technical details will become crucial in the subsequent development and hence we need to be clear on the exact probabilistic setup of the problem. We take a similar discussion here to Cvitanić and Zhang (2012), Chapter 5.*

Let's recall we had fixed a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Z be a standard Brownian motion on this probability space, and let $\mathcal{F}^Z = \{\mathcal{F}_t^Z\}_{0 \leq t \leq T}$ be the filtration on $[0, T]$ generated by Z . Define the process Y by,

$$Y_t := \int_0^t \Sigma dZ_s = \Sigma Z_t. \quad (2.A.1)$$

For any given \mathcal{F}^Z -adapted effort process $A = \{A_t\}$ (this will be further elaborated below), define,

$$Z_t^A := Z_t - \int_0^t \left(\frac{h(A_s) - \Sigma^2/2}{\Sigma} \right) ds, \quad (2.A.2a)$$

$$M_t^A := \exp \left\{ \int_0^t \left(\frac{h(A_s) - \Sigma^2/2}{\Sigma} \right) dZ_s - \frac{1}{2} \int_0^t \left(\frac{h(A_s) - \Sigma^2/2}{\Sigma} \right)^2 ds \right\}, \quad (2.A.2b)$$

$$\mathbb{P}^A(G) := \mathbb{E}[M_T^A \mathbf{1}_G], \quad (2.A.2c)$$

where \mathbb{E} is the expectation under the probability measure \mathbb{P} , G is any \mathcal{F} -measurable event, and $\mathbf{1}$ is the indicator function. And since for all t , $A_t = a \in [a_L, a_H]$, we are ensured that M^A is a martingale, and hence by Girsanov's theorem, we have that \mathbb{P}^A is a probability measure and Z^A is a \mathbb{P}^A -Brownian motion. Then we have that,

$$dY_t = \Sigma dZ_t = \Sigma \left(dZ_t^A + \frac{h(A_t) - \Sigma^2/2}{\Sigma} \right) dt = \left(h(A_t) - \frac{\Sigma^2}{2} \right) dt + \Sigma dZ_t^A. \quad (2.A.3)$$

Note that we view the triple (Y, Z^A, \mathbb{P}^A) as a weak solution to the stochastic differential equation,

$$dY_t = \left(h(A_t) - \frac{\Sigma^2}{2} \right) dt + \Sigma dZ_t^A. \quad (2.A.4)$$

Now, for this given effort process A , we define $D_t := e^{Y_t}$ and by Ito's lemma, we have that

$$\begin{aligned} dD_t &= D_t \left[\left(h(A_t) - \frac{\Sigma^2}{2} \right) dt + \Sigma dZ_t^A \right] + \frac{\Sigma^2}{2} D_t dt \\ &= h(A_t) D_t dt + \Sigma D_t dZ_t^A \end{aligned} \quad (2.A.5)$$

Thus, when we write the dividends process in (2.3.1), it is understood in the sense of (2.A.5) and the preceding development leading to that expression.

“Soft” Retirement Function

We note that the retirement function F (2.7.1) is not differentiable in the arguments due to the form of \bar{x} in (2.7.2). Again, the non-differentiability comes because we are constrained such that the pie sharing rule must lie in $[x_0, 1]$. However, for subsequent development (which are completely technical), smoothness is a highly desirable technical property. Thus, we will make the following technical compromise by replacing the “hard” maximum and minimum functions by their “soft” counterparts.

Definition 2.A.1 (Soft max / Soft min). Fix any $a, b \in \mathbb{R}$. For any softening parameter $k > 0$, define the *soft max* $M_k : \mathbb{R} \rightarrow \mathbb{R}$ as,

$$M_k(a, b) := \frac{1}{k} \log(e^{ka} + e^{kb}), \quad (2.A.6)$$

and define the *soft min* $m_k : \mathbb{R} \rightarrow \mathbb{R}$ as,

$$m_k(a, b) := -M_k(-a, -b). \quad (2.A.7)$$

Definition 2.A.2 (“Soft” Retirement Function). Fix any softening parameter $k > 0$. Then from Proposition 2.7.1, we define the *soft retirement function* $(\tau, w, \delta) \mapsto R_k(\tau, w, \delta)$ in the same form as (2.7.1), except we replace \bar{x} in (2.7.2) with the softened version \bar{x}_k as,

$$\bar{x}_k(\tau, w, \delta) := m_k(M_k(\bar{x}(\tau, w, \delta), 1), x_0),$$

where \bar{x} is as given in (2.7.2).

It should be emphasized that the difference between the “hard” retirement function of Proposition 2.7.1 and that of the “soft” retirement function in Definition 2.A.2 is purely technical and has no economic content. That is to say, when we generically refer to “retirement function” in economic intuition discussions, we could refer to either, even though in the actual implementation, we will use the “soft” version for its smoothness properties.

2.B General Proofs

Proof of Proposition 2.4.1. Choose any effort $A = \{A_t\}$ as fixed and consider the manager’s portfolio choice problem in (2.3.3). By dynamically complete markets, there exists a unique state price density $\xi^A =: \xi$ with dynamics (2.3.13). By Cox and Huang (1989, 1991), we can consider the static optimization problem pointwise as,

$$\begin{aligned} & \max_{C_M(t)} e^{-\rho t} \log(C_M(t)) \\ & \text{subject to } \xi_t C_M(t) = \xi_0 w_0. \end{aligned} \quad (2.B.1)$$

So we can use the standard Lagrangian method to solve, and using first order conditions, we have that the manager’s optimal consumption satisfies,

$$C_M(t) = \frac{e^{-\rho t}}{\lambda_0 \xi_t}, \quad (2.B.2)$$

where $\lambda_0 > 0$ is the Lagrange multiplier. But by no arbitrage, the value P of the portfolio satisfies,

$$\begin{aligned} P_t &= \mathbb{E}_t \left[\int_t^T \frac{\xi_s}{\xi_t} C_M(s) ds \right] \\ &= \mathbb{E}_t \left[\int_t^T \frac{\xi_s e^{-\rho s}}{\xi_t \lambda_0 \xi_s} ds \right] \\ &= \frac{1}{\xi_t \lambda_0} \mathbb{E}_t \left[\int_t^T e^{-\rho s} ds \right] \\ &= \frac{1}{\xi_t \lambda_0} \frac{e^{-\rho t} - e^{-\rho T}}{\rho} \end{aligned} \quad (2.B.3)$$

$$= C_M(t) \frac{1 - e^{-\rho(T-t)}}{\rho}, \quad (2.B.4)$$

where the last equality follows from (2.B.2).

Using (2.B.4) and Ito's lemma, we have that,

$$\begin{aligned}
 dP_t &= d\left(C_M(t)\frac{1-e^{-\rho(T-t)}}{\rho}\right) \\
 &= \frac{1}{\rho}\left[dC_M(t) - d(C_M(t)e^{-\rho(T-t)})\right] \\
 &= \frac{1}{\rho}\left[-\rho C_M(t)e^{-\rho(T-t)}dt + (1-e^{-\rho(T-t)})dC_M(t)\right] \\
 &= -C_M(t)dt + P_t(r_t + \kappa_t^2)dt + P_t\kappa_t dZ_t.
 \end{aligned}$$

But since,

$$\begin{aligned}
 dC_M(t) &= d\left(\frac{e^{-\rho t}}{\lambda_0\xi_t}\right) \\
 &= \frac{1}{\lambda_0}\left[-\rho e^{-\rho t}\frac{1}{\xi_t}dt + e^{-\rho t}d\left(\frac{1}{\xi_t}\right)\right] \\
 &= \frac{1}{\lambda_0}\left[-\rho e^{-\rho t}dt + e^{-\rho t}\left(\frac{1}{\xi_t}(r_t + \kappa_t^2)dt + \frac{1}{\xi_t}\kappa_t dZ_t\right)\right]
 \end{aligned}$$

Hence, substituting back, we have,

$$dP_t = -C_M e^{-\rho(T-t)}dt - \rho P_t dt + P_t(r_t + \kappa_t^2)dt + P_t\kappa_t dZ_t. \quad (2.B.5)$$

Taking (2.B.5) and recalling the self-financing condition (2.3.5), and matching drift and volatility, and simplifying, we have that,

$$\alpha(t)r_t + \theta(t)\mu(t) = P_t(r_t + \kappa_t^2), \quad (2.B.6a)$$

$$\theta(t)\sigma(t) = P_t\kappa_t. \quad (2.B.6b)$$

Apply the securities market clearing condition (2.3.12a) and (2.3.12b), then (2.B.6a), (2.B.6b) imply that

$$\kappa_t = \sigma(t), \quad (2.B.7a)$$

$$\theta(t) = P_t = S_t, \quad (2.B.7b)$$

$$\mu(t) = r_t + \kappa_t^2 = r_t + \sigma(t)^2. \quad (2.B.7c)$$

Now, using the consumption goods market clearing condition (2.3.11), we have that $C_M(t) = D_t - (1 - X(t))D_t = X(t)D_t$. Using (2.B.7b) and (2.B.4), we obtain thus,

$$S_t = P_t = \frac{1}{\rho}X(t)D_t. \quad (2.B.8)$$

■

Proof to Proposition 2.5.1. Suppose the pie sharing rule contract is $X(t) \equiv \bar{x} \in [x_0, 1]$. Then the manager's optimization problem can be rewritten as,

$$\begin{aligned}
 &\sup_A \mathbb{E}_0 \left[\int_0^T e^{-\rho t} [\log(\bar{x}D_t) - g(A_t)] dt \right] \\
 &= \sup_A \mathbb{E}_0 \left[\int_0^T e^{-\rho t} \left[\log x + \log D_0 + \int_0^t (h(A_s) - \Sigma^2/2) ds + \int_0^t \Sigma dZ_s^A - g(A_t) \right] dt \right] \\
 &= \sup_A \int_0^T e^{-\rho t} \left[\log x + \log D_0 + \int_0^t (h(A_s) - \Sigma^2/2) ds - g(A_t) \right] dt.
 \end{aligned} \quad (2.B.9)$$

Define the state transition as,

$$\frac{dM_s^A}{ds} = h(A_s) - \Sigma^2/2, \quad (2.B.10a)$$

$$M_t = \log D_t = m \in (-\infty, \infty). \quad (2.B.10b)$$

Define

$$U(t, m) := \sup_A \int_t^T e^{-\rho(T-t)} [M_s^A - g(A_s)] dt \quad (2.B.11)$$

Thus, (2.B.9) can be written as (2.5.1).

Hence, it remains to study the value function U . It should be noted that we now have on hand a *deterministic* optimal control problem.¹⁵ Thus, U satisfies the HJB-PDE (2.5.2) on $[0, T] \times (-\infty, \infty)$.¹⁶ Thus (i) holds.

Next we consider (ii). The optimizing control thus satisfies the optimization problem,

$$\max_{a \in [a_L, a_H]} -g(a) + h(a)U_m,$$

and this is well posed by the smoothness conditions on h and g as per Assumptions 2.3.1 and 2.3.3.¹⁷ This is a standard constrained optimization problem. Using the Kuhn-Tucker theorem and proceeding, we can verify that,

$$a^* \in \begin{cases} \{a_H\} & \text{when } U_m > g'(a_H)/h'(a_H), \\ \{a_L\} & \text{when } U_m < g'(a_L)/h'(a_L), \\ \{(a_L, a_H) : g'(a)/h'(a) = U_m\} & \text{otherwise.} \end{cases} \quad (2.B.12)$$

But under the condition $g'(a_H)/h'(a_H) > Te^{-\rho T}$, that means the condition for the highest effort case $a^* = a_H$ will never be feasible. Thus, (ii) holds. ■

Lemma 2.B.1. *The value function U of (2.B.11) in the proof to Proposition 2.5.1 is such that for $(t, m) \in [0, T] \times (-\infty, \infty)$*

(i) $U_m(t, m) = e^{-\rho(T-t)(T-t)}$;

(ii) $U_m(0, m) \geq U_m(t, m)$

Proof to Lemma 2.B.1. Following the proof of Proposition 2.5.1, observe that for any $\varepsilon > 0$,

$$\begin{aligned} \frac{U(t, m + \varepsilon) - U(t, m)}{\varepsilon} &= \frac{1}{\varepsilon} \int_t^T e^{\rho(T-t)} [(m + \varepsilon) - m] ds \\ &= e^{-\rho(T-t)} (T - t). \end{aligned}$$

Taking $\varepsilon \downarrow 0$, we clearly see that, $U_m(t, m) = e^{-\rho(T-t)(T-t)}$, and in particular, $U_m > 0$ and we have that $U_m(0, m) \geq U_m(t, m)$ for all $t \in [0, T]$. ■

Proof to Lemma 2.6.1. Suppose the investor can only observe the stock prices S . But by dynamically complete markets, the investor can anticipate the investment strategy of the manager, and hence deduce the results of Proposition 2.4.1. But given that the investor obviously knows his own contract offer X , and using thus Proposition 2.4.1(iii), that is equivalent to observing the dividends D . ■

¹⁵ Given that the focus of the paper is *not* the trivial case when the manager is the sole owner of the asset, we will omit checking the technical regularity conditions to ensure the existence and uniqueness of this deterministic optimal control problem. However, we will be far more careful in the second best case in the subsequent development, which is the main focus of this paper. Details of deterministic optimal control problems can be found in Fleming and Soner (2006).

¹⁶ Also note that the value function here must clearly satisfy the terminal condition $U(T, m) = 0$. We also should further impose the initial condition $U(0, m) = \hat{W}$ so that it satisfies the manager's IR constraints (2.3.6). However, we place no further boundary conditions on m . We explicitly acknowledge that this implies there will be, in general, infinite number of solutions that satisfy the first order Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). That is, here, we are only providing a *generalized solution*, and not the *specific solution*, which will require us to further provide boundary conditions. But given the purpose of this proposition is to illustrate the form of the action A_t , we will not further push for the full discussion here.

¹⁷ Technically, we had also implicitly used a condition on the sign of U_m which ensures that we indeed have global concavity of this objective function; see the simple technical Lemma 2.B.1 below.

Proof of Proposition 2.6.2. First let's consider (i). Fix any action process A and contract X . Define,

$$V_t = \mathbb{E}_t^A \left[\int_0^T e^{-\rho s} [\log(X(s)D_s) - g(A_s)] \right] \quad (2.B.13)$$

$$= \int_0^t e^{-\rho s} [\log(X(s)D_s) - g(A_s)] ds + e^{-\rho t} W_t(X, A). \quad (2.B.14)$$

Now, since V is a \mathbb{P}^A -martingale, then by the Martingale Representation Theorem, say from Sannikov (2008), there exists η such that,

$$dV_t = e^{-\rho t} \eta_t dZ_t^A \quad (2.B.15)$$

First, applying Ito's lemma to (2.B.14), we have,

$$\begin{aligned} dV_t &= e^{-\rho t} [\log(X(t)D_t) - g(A_t)] dt + d(e^{-\rho t} W_t) \\ &= e^{-\rho t} [\log(X(t)D_t) - g(A_t)] dt - \rho e^{-\rho t} W_t dt + e^{-\rho t} dW_t. \end{aligned} \quad (2.B.16)$$

Equating (2.B.15) with (2.B.16) and solving for dW_t , we have (2.6.2a), (2.6.2b).

Next let's consider (ii). Suppose the investor wants to implement action process A . But suppose the manager considers an arbitrary different effort process A^* , such that $A_s \neq A_s^*$ for all s . For any given point time $t > 0$, suppose instead the manager considers the action \tilde{A} defined by,

$$\tilde{A}_s = \begin{cases} A_s^*, & \text{for } s \in [0, t] \\ A_s, & \text{for } s \in (t, T]. \end{cases}$$

Consider the manager's payoff \hat{V} when he considers the action process \tilde{A} . Then, using (2.B.14), we have that,

$$\hat{V}_t = \hat{V}_0 + \int_0^t e^{-\rho s} [\log(X(s)D_s) - g(A_s^*)] ds + e^{-\rho t} W_t(X, A). \quad (2.B.17)$$

Using (2.A.2), we see that the \mathbb{P}^A -Brownian motion Z^A and the \mathbb{P}^{A^*} -Brownian motion Z^{A^*} are given as,

$$\begin{aligned} Z_t^A &= Z_t - \int_0^t \left(\frac{h(A_s) - \Sigma^2/2}{\Sigma} \right) ds \\ Z_t^{A^*} &= Z_t - \int_0^t \left(\frac{h(A_s^*) - \Sigma^2/2}{\Sigma} \right) ds, \end{aligned}$$

and equating the above to Z_t and solving, we have that,

$$Z_t^A = Z_t^{A^*} + \frac{1}{\Sigma} \int_0^t (h(A_s^*) - h(A_s)) ds. \quad (2.B.18)$$

Applying Ito's lemma to (2.B.17), and using (2.B.18),

$$\begin{aligned} d\hat{V}_t &= e^{-\rho t} [\log(X(t)D_t) - g(A_t^*)] dt + d(e^{-\rho t} W_t(X, A)) \\ &= e^{-\rho t} [\log(X(t)D_t) - g(A_t^*)] dt - \rho e^{-\rho t} W_t(X, A) dt \\ &\quad + e^{-\rho t} \left\{ [\rho W_t(A, X) + g(A_t) - \log(X(t)D_t)] dt + \eta_t dZ_t^A \right\} \\ &= e^{-\rho t} [g(A_t) - g(A_t^*)] dt + e^{-\rho t} \eta_t dZ_t^A \\ &= e^{-\rho t} [g(A_t) - g(A_t^*)] dt + e^{-\rho t} \eta_t \frac{1}{\Sigma} [(h(A_t^*) - h(A_t)) dt + dZ_t^{A^*}] \\ &= e^{-\rho t} \left[g(A_t) - g(A_t^*) + \frac{\eta_t}{\Sigma} (h(A_t^*) - h(A_t)) \right] dt + e^{-\rho t} \eta_t dZ_t^{A^*}. \end{aligned} \quad (2.B.19)$$

To ensure that the manager has no incentive to deviate, we need the drift of (2.B.19) be such that at all time points t ,

$$A_t \in \arg \max_{a \in [a_L, a_H]} \frac{\eta_t}{\Sigma} h(a) - g(a),$$

and given the conditions on g and h , we are ensured that the unique optimizer is an interior solution, and thus we achieve equality in (2.6.3). The condition (2.6.4) is exactly the condition one needs by considering the Lagrangian to achieve the corner solution $a = a_H$. ■

Proof to Proposition 2.7.1. If the manager is retired at time $\tau \in (0, T)$, and at retirement holds a fixed $X_t \equiv \tilde{x} \in (x_0, 1)$ for all times $t \in [\tau, T]$ and the manager chooses the lowest effort $A_t \equiv a_L$, then the dividends process will follow, for $D_\tau^L = \delta$,

$$D_t^L = \delta + \int_\tau^t D_u^L h(a_L) du + \int_\tau^t D_u^L \Sigma dZ_u^{a_L}, \quad t \in [\tau, T].$$

Then from (2.4.5), we have the manager's payoff thus satisfies,

$$\begin{aligned} & \mathbb{E}_\tau^{a_L} \left[\int_\tau^T e^{-\rho t} \left(\log(\tilde{x} D_t^L) - g(a_L) \right) dt \mid D_\tau = \delta \right] \\ &= \mathbb{E}_\tau^{a_L} \left[\int_\tau^T e^{-\rho t} \left(\log \tilde{x} + \log D_\tau + (h(a_L) - \Sigma^2/2)(t - \tau) + \Sigma(Z_t^{a_L} - Z_\tau^{a_L}) - g(a_L) \right) dt \mid D_\tau = \delta \right] \\ &= \frac{\log \tilde{x} + \log \delta - g(a_L)}{\rho} e^{-\rho \tau} (1 - e^{-\rho(T-\tau)}) + \frac{h(a_L) - \Sigma^2/2}{\rho^2} e^{-\rho \tau} \left(1 - e^{-\rho(T-\tau)} (1 + \rho(T-\tau)) \right). \end{aligned} \quad (2.B.20)$$

Now, for any promised continuation value $w \in (-\infty, \infty)$ for the manager, and in the above since $\tilde{x} \in (x_0, 1)$ was arbitrary, let us find the function $\tilde{x} = \tilde{x}(t, w, \delta)$ such that w equates to the manager's payoff (2.B.20). Solving, we have that,

$$\begin{aligned} \tilde{x} &= \tilde{x}(\tau, w, \delta) \\ &:= \exp \left\{ \frac{\rho w e^{\rho \tau}}{1 - e^{-\rho(T-\tau)}} - \log \delta + g(a_L) - \frac{h(a_L) - \Sigma^2/2}{\rho} \left(1 - \frac{\rho(T-\tau) e^{\rho(T-\tau)}}{1 - e^{-\rho(T-\tau)}} \right) \right\}. \end{aligned} \quad (2.B.21)$$

But since the pie sharing rule is constrained in $[x_0, 1]$, to include the endpoints, we modify \tilde{x} to consider the bounded version \bar{x} , defined as,

$$\bar{x}(\tau, w, \delta) := (\tilde{x}(\tau, w, \delta) \wedge 1) \vee x_0. \quad (2.B.22)$$

And likewise, from (2.3.8), we have the investor's payoff,

$$R(\tau, w, \delta) := \mathbb{E}_\tau^{a_L} \left[\int_\tau^T e^{-\rho t} u((1 - \bar{x}(\tau, w, \delta)) D_t^L) dt + e^{-\rho T} u(\rho^{-1} D_T^L) \mid D_\tau = \delta \right] \quad (2.B.23)$$

■

Proof to Example 2.7.1. Observe that if utility u has the power utility form as in Example 2.3.2, then we compute that, for any $\bar{x} \in [x_0, 1]$, and $t \in [\tau, T]$ and $D_\tau = \delta$,

$$\begin{aligned} u((1 - \bar{x}) D_t^L) &= \underline{u} + \frac{[(1 - \bar{x}) D_\tau \exp \{ h(a_L)(t - \tau) - (\Sigma^2/2)(t - \tau) + \Sigma(Z_t - Z_\tau) \}]^{1-\gamma} - 1}{1 - \gamma} \\ &= \underline{u} + \frac{(1 - \bar{x})^{1-\gamma} \delta^{1-\gamma} \exp \{ (1 - \gamma) (h(a_L)(t - \tau) - (\Sigma^2/2)(t - \tau) + \Sigma(Z_t - Z_\tau)) \} - 1}{1 - \gamma}. \end{aligned}$$

Take expectation, and for convenience, denote $\mathbb{E}_\tau^{a_L}[\cdot] := \mathbb{E}_\tau^{a_L}[\cdot \mid D_\tau = \delta]$,

$$\begin{aligned} & \mathbb{E}_\tau^{a_L} \left[e^{(1-\gamma)(h(a_L)(t-\tau) - \Sigma^2/2(t-\tau))} e^{(1-\gamma)\Sigma(Z_t - Z_\tau)} \right] \\ &= e^{(1-\gamma)(h(a_L)(t-\tau) - \Sigma^2/2(t-\tau))} \mathbb{E}_\tau^{a_L} \left[e^{(1-\gamma)\Sigma(Z_t - Z_\tau)} \right] \\ &= e^{(1-\gamma)(h(a_L)(t-\tau) - \Sigma^2/2(t-\tau))} e^{(1-\gamma)^2 \Sigma^2 (t-\tau)} \\ &= \exp \{ (1 - \gamma) (h(a_L) - \Sigma^2(1 - 2\gamma)/2) (t - \tau) \}. \end{aligned}$$

which implies,

$$\mathbb{E}_\tau^{a_L} [u((1 - \bar{x}) D_t^L)] = \underline{u} + \frac{(1 - \bar{x})^{1-\gamma} \delta^{1-\gamma} \exp \{ (1 - \gamma) (h(a_L) - \Sigma^2(1 - 2\gamma)/2) (t - \tau) \} - 1}{1 - \gamma}.$$

Using Fubini's theorem and applying the expressions above, we can compute the retirement function of (2.7.1) as,

$$\begin{aligned}
R(\tau, w, \delta) &= \int_{\tau}^T e^{-\rho t} \left(\underline{u} + \frac{(1 - \bar{x}(\tau, w, \log \delta))^{1-\gamma} \delta^{1-\gamma} \exp\{(1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)(t-\tau)\} - 1}{1-\gamma} \right) dt \\
&\quad + e^{-\rho T} \left(\underline{u} + \frac{(\rho^{-1})^{1-\gamma} \delta^{1-\gamma} \exp\{(1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)(T-\tau)\}}{1-\gamma} \right) \\
&= \left(\underline{u} - \frac{1}{1-\gamma} \right) \frac{1 - e^{-\rho(T-\tau)}}{\rho} \\
&\quad + \frac{(1 - \bar{x}(\tau, w, \log \delta))^{1-\gamma} \delta^{1-\gamma}}{1-\gamma} \int_{\tau}^T e^{-\rho t} \exp\{(1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)(t-\tau)\} dt \\
&\quad + e^{-\rho T} \left(\underline{u} + \frac{(\rho^{-1})^{1-\gamma} \delta^{1-\gamma} \exp\{(1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)(T-\tau)\}}{1-\gamma} \right) \\
&= \left(\underline{u} - \frac{1}{1-\gamma} \right) \frac{1 - e^{-\rho(T-\tau)}}{\rho} \\
&\quad + \frac{(1 - \bar{x}(\tau, w, \log \delta))^{1-\gamma} \delta^{1-\gamma}}{1-\gamma} e^{-\rho \tau} \frac{1 - \exp\{\rho - (1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)\}}{\rho - (1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)} \\
&\quad + e^{-\rho T} \left(\underline{u} + \frac{(\rho^{-1})^{1-\gamma} \delta^{1-\gamma} \exp\{(1-\gamma)(h(a_L) - \Sigma^2(1-2\gamma)/2)(T-\tau)\}}{1-\gamma} \right).
\end{aligned}$$

■

2.C Proofs to investor's optimization problem

Regularity of the HJB and the investor's value function

General remark on the motivations for regularity

Beyond economic motivations as given in Section 2.7, there is a distinct technical and economic reason why we need to constrain the state space to be bounded, although the cost of this might be some loss in economic richness. For instance, while papers like Sannikov (2008) and DeMarzo and Sannikov (2006) use more complicated contractual possibilities, namely that of *optimal* termination¹⁸ of the manager, we explicitly do not pursue this route here. Firstly, if we were to consider such free-boundary problems, it is easy to conjecture or see that the optimal termination of the manager will be a surface of the form $(t, \delta) \mapsto \bar{W}(t, \delta)$, and a priori, it is difficult to prove the existence of such an object, and even given existence, it is difficult to say much about its uniqueness and smoothness properties. The key difference between our paper and the earlier work in contracting is that we are explicitly dealing with *two* state variables, rather than one, and this adds substantial difficulty to the problem. Thus, at the risk of sacrificing some economic richness of the problem, we consider simpler contracts here.

Secondly, and also extremely important both economically and technically, we need to ensure that our value function is sufficiently nice. That is to say, we are explicitly using strong conditions such that we can ensure the investor's value function V has a unique smooth classical solution. This is critical as, unlike the pure dynamic moral hazard problems in the literature, our problem does not end at the characterization of the value function. Indeed, we need a far more specific understanding of the resulting optimal controls. While a general characterization of the contracting possibilities and utility forms imply that we can use more advanced weak solution PDE concepts, such as *viscosity solutions*¹⁹, to understand the value function, but in general the resulting optimal controls may not be sufficiently smooth nor Markov. But since from Proposition 2.4.1, (2.3.2) and Assumption 2.3.7, we need to essentially apply Ito's formula to match drift and diffusion terms to solve for the equilibrium asset price dynamics and risk free asset price dynamics. Thus, to have any hope for such a goal to completing the circle and finishing the asset pricing problem, we must have strong (and perhaps overtly strong) conditions so that the value function yields a classical smooth solution, and that the resulting optimal controls are smooth and Markovian. For instance, Strulovici

¹⁸ Essentially through the value matching and smooth pasting conditions in the free-boundary problem to find the optimal termination point.

¹⁹ See, for instance, Fleming and Soner (2006) and Crandall, Ishii, and Lions (1992).

and Szydlowski (2014) also emphasizes the point that for economic applications, smoothness of the value function is a critical property.²⁰ See also Rincón-Zapatero and Santos (2012) for a discussion.

Standard conditions

It will be helpful to write our state variables in vector form. Define $R_t := (W_t, D_t)^\top$, and it has dynamics,²¹

$$dR_t = \mathbf{a}(t, R_t, x(t))dt + \mathbf{b}(t, R_t, x(t))dB_t, \quad (2.C.1)$$

where for $r = (r_1, r_2) = (w, \delta)$, we have the drift and volatility of the stacked state variables as,

$$\mathbf{a}(t, r, x) = \begin{pmatrix} \rho r_1 - g(a_H) - \log x - \log r_2 \\ g(a_H)r_2 \end{pmatrix}, \quad (2.C.2a)$$

$$\mathbf{b}(t, r, x) = \Sigma \begin{pmatrix} g'(a_H)/h'(a_H) \\ r_2 \end{pmatrix}. \quad (2.C.2b)$$

We will also denote the control value space as,²²

$$U := [x_0, 1]. \quad (2.C.3)$$

It is clear that \mathbf{a} and \mathbf{b} are both²³ $C^1(\bar{Q} \times U)$. For what follows, we will denote C as an universal constant that may vary line by line. For Propositions 2.C.1 and 2.C.2, we will merely note the qualitative description here, and leave the exact specification in the proof, especially since the qualitative behavior is what is most important for us and not the exact specification.

Lemma 2.C.1 (State regularity conditions). *The state dynamics drift and volatility terms satisfy on Q ,*

(ii) *A Lipschitz condition in time and state;*

(iii) *A growth condition in state and control.*

Proof to Lemma 2.C.1. We show that \mathbf{a} and \mathbf{b} satisfy a Lipschitz condition on Q . In particular, denoting $\|\cdot\|_2$ as the standard Euclidean 2-norm²⁴, for any $(t', r', x'), (t'', r'', x'') \in Q \times U$,

$$\begin{aligned} \|\mathbf{a}(t', r', x') - \mathbf{a}(t'', r'', x'')\|_2 &= \left\| \begin{pmatrix} \rho(r'_1 - r''_1) + (\log x'' - \log x') + (\log r'_2 - \log r''_2) \\ g(a_H)(r'_2 - r''_2) \end{pmatrix} \right\|_2 \\ &\leq C(|t' - t''| + \|r' - r''\|_2), \end{aligned}$$

where the last inequality follows since U is compact, and that $y \mapsto \log y$ is Lipschitz on a domain of the form $[\varepsilon, \infty)$ for any $\varepsilon > 0$. Likewise for the diffusion,

$$\|\mathbf{b}(t', r', x') - \mathbf{b}(t'', r'', x'')\|_2 = \left\| \begin{pmatrix} 0 \\ \Sigma(r'_2 - r''_2) \end{pmatrix} \right\|_2 \leq C(|t' - t''| + \|r' - r''\|_2),$$

where the last inequality follows trivially. Thus (i) holds.

²⁰ It should be noted that Strulovici and Szydlowski (2014) only considers *one* dimensional state dynamics. We are working with *two* dimensional state dynamics. Hence, the results from Strulovici and Szydlowski (2014) cannot be directly applied to here.

²¹ This notation here is not to be confused with the equilibrium interest rate $r = \{r_t\}$. There should be no notation confusion in this section as we will make no reference to the interest rate.

²² This U as the control value space should *not* be confused with the manager's value function in Section 2.5 when we considered a constant deterministic contract for the manager. There should be no confusion in the notational here since we do not use any results from Section 2.5 in this section. This notation U is used here as it is conventional in the control theory literature to denote the control value space.

²³ As is standard, we denote the *closure* of a set A by \bar{A} .

²⁴ That is, if $y = (y_1, y_2) \in \mathbb{R}^2$, then $\|y\|_2 := \sqrt{y_1^2 + y_2^2}$.

Next, let's show the desired growth condition. Note that, for $(t, r, x) \in Q$ we have that,

$$\|\mathbf{a}(t, r, x)\|_2 = \left\| \begin{pmatrix} \rho r_1 - g(a_H) - \log x - \log r_2 \\ g(a_H)r_2 \end{pmatrix} \right\|_2 \leq C(1 + \|r\|_2 + |x|),$$

where the last inequality follows since U is compact and we have that on Q , $\log \underline{\delta} < \log r_2$. Likewise,

$$\|\mathbf{b}(t, r, x)\|_2 = \left\| \Sigma \begin{pmatrix} g'(a_H)/h'(a_H) \\ r_2 \end{pmatrix} \right\|_2 \leq C(1 + \|r\|_2 + |x|),$$

where the last inequality is obvious. Thus (ii) holds. This completes the proof. \blacksquare

Lemma 2.C.2 (Objective function regularity conditions). *With the given state variable dynamics, both the intertemporal utility function and terminal bequest function form satisfy a polynomial growth condition in state and control.*

Proof to Lemma 2.C.2. Define $\tilde{u}(x, \delta) =: \tilde{u}(x, r_2) = u((1-x)r_2)$. Then by the multivariable mean-value theorem, for any $(t, r, x) \in Q$, there exists some $c \in (0, 1)$ such that,

$$\begin{aligned} |\tilde{u}(x, r_2) - \tilde{u}(0, r_2)| &\leq \|\nabla \tilde{u}((1-c)x + c \cdot 0, (1-c)r_2 + cr_2)\|_2 \|(x-0, r_2-r_2)\|_2 \\ &= \|\nabla \tilde{u}((1-c)x, r_2)\|_2 \|(x, 0)\|_2, \end{aligned}$$

where ∇ is the gradient. And by the chain rule,

$$\nabla \tilde{u}((1-c)x, r_2) = u'((1-c)xr_2) \begin{pmatrix} -cr_2 \\ (1-c)x \end{pmatrix}.$$

Using the above, and the triangle inequality, we thus have,

$$\begin{aligned} |e^{-\rho t} u((1-x)r_2)| &= |e^{-\rho t}| |\tilde{u}(x, r_2)| \\ &\leq 1 \cdot |\tilde{u}(x, r_2) - \tilde{u}(0, r_2) + \tilde{u}(0, r_2)| \\ &\leq |\tilde{u}(x, r_2) - \tilde{u}(0, r_2)| + |\tilde{u}(0, r_2)| \\ &\leq \left\| u'((1-c)xr_2) \begin{pmatrix} cr_2 \\ (1-c)x \end{pmatrix} \right\|_2 \|(x, 0)\|_2 + |\tilde{u}(0, r_2)| \\ &\leq C(1 + \|r\|_2^m + |x|^m), \end{aligned}$$

for some power $m > 0$. Note the last inequality follows from the smoothness of u in Assumption 2.3.5 and that on Q and U , the set of values r_2 and x can take on are bounded.

Since we use the same functional form for the intertemporal utility and the terminal bequest function, essentially the same computation shows that

$$\left| e^{-\rho T} u(\rho^{-1}r_2) \right| \leq C(1 + \|r\|_2^m).$$

It should be noted here that the precise power coefficient $m > 0$ is not particularly important here; what is important is the qualitative polynomial growth behavior. This completes the proof. \blacksquare

For the subsequent discussion, let's also define the variance-covariance matrix Ω . Define,²⁵

$$\Omega(t, r, x) := \mathbf{b}(t, r, x)\mathbf{b}(t, r, x)^\top = \Sigma^2 \begin{pmatrix} (g'(a_H)/h'(a_H))^2 & g'(a_H)/h'(a_H)r_2 \\ g'(a_H)/h'(a_H)r_2 & r_2^2 \end{pmatrix},$$

and also denote $\Omega = [\omega_{ij}]_{i,j=1,2}$.

Lemma 2.C.3 (Uniform ellipticity). *The HJB equation (2.8.4) is uniformly elliptic on Q .*

²⁵ This Ω for the variance-covariance matrix is not to be confused with the state space Ω of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. No confusion should arise in this section as we do not refer to the state space Ω here.

Proof to Lemma 2.C.3. The *uniform ellipticity* condition requires that for any $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and any $(t, r, x) \in Q \times U$ that,

$$\sum_{i,j=1,2} \omega_{ij}(t, r, x) \psi_i \psi_j \geq C \|\psi\|_2^2, \quad (2.C.4)$$

for some constant C .

In our case, fix any $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and without loss of generality, assume $\psi_1 \geq \psi_2$. And observe that for $(t, r, x) \in Q$, and in particular that $r_2 > \underline{\delta}$,

$$\begin{aligned} \sum_{i,j=1,2} \omega_{ij}(t, r, x) \psi_i \psi_j &= \Sigma^2 \left(\frac{g'(a_H)}{h'(a_H)} \right)^2 \psi_1^2 + 2\Sigma^2 \frac{g'(a_H)}{h'(a_H)} r_2 \psi_1 \psi_2 + \Sigma^2 r_2 \psi_2^2 \\ &> \Sigma^2 \left(\frac{g'(a_H)}{h'(a_H)} \right)^2 \psi_1^2 + 2\Sigma^2 \frac{g'(a_H)}{h'(a_H)} \underline{\delta} \psi_1 \psi_2 + \Sigma^2 \underline{\delta} \psi_2^2 \\ &\geq C (\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2) \\ &\geq C (\psi_1^2 + 2\psi_2^2 + \psi_2^2) \\ &\geq C (\psi_1^2 + \psi_2^2) \\ &= C \|\psi\|_2^2, \end{aligned}$$

for some universal constant C . Thus, we indeed have the uniform ellipticity condition (2.C.4). \blacksquare

Finally, we document some miscellaneous properties and smoothness properties of the relevant functions in our setup. The proposition is stated without proof, as the proof for each item is obvious by inspection.

Lemma 2.C.4 (Misc. properties and smoothness of input functions). *We have,*

- (i) U is compact;
- (ii) O is bounded with ∂O a manifold of class C^3 ;
- (iii) For $\varphi = \Omega, \mathbf{a}, u$, the function φ and its partial derivatives $\varphi_t, \varphi_{r_i}, \varphi_{r_i r_j}$ are continuous on $\bar{Q} \times U$, for $i, j = 1, 2$;
- (iv) For each smooth parameter $k > 0$, the boundary condition function ²⁶ $F_k \in C^3([0, T] \times \mathbb{R}^2)$.

Remark 2.C.5 (Smoothness at the boundary). *Lemma 2.C.4(iv) deserves a remark. Recall again the motivation for defining the “soft” retirement function F_k , in contrast to the “hard” retirement function F . We see that it is precisely due to the boundary condition $V^k =: V = F_k$ that we want high level of smoothness at the boundary to ensure that we do indeed have a resulting smooth solution V^k . Specifically, if we were to use the original hard retirement function F , and given its lack of smoothness (again, coming from the hard max and hard min), we cannot possibly expect a smooth solution V .*

Finally, we now come to the main result of this section.

Proposition 2.C.6 (Existence and uniqueness of smooth HJB equation). *For any smoothing parameter $k > 0$, there exists a unique solution $V := V^k \in C^{1,2}(Q) \cap C(\bar{Q})$ to (2.8.4) and (2.8.5).*

Proof to Proposition 2.C.6. This follows from Fleming and Soner (2006), Section IV, and specifically IV.4. That is, the results of Lemmas 2.C.1, 2.C.2, 2.C.3, and 2.C.4 satisfy the regularity conditions for the existence of a classical solution to the HJB-PDE (2.8.4) with the assigned boundary conditions (2.8.5). \blacksquare

Remark 2.C.7. *Note that thanks to these strong regularity conditions, we can thus apply results and methods from the theory of second order nonlinear parabolic partial differential equations, instead of weak solution concepts such as viscosity solutions.*

²⁶ Strictly speaking, the soft retirement function F_k is a map with domain $[0, T] \times [\hat{W}, \tilde{W}] \times [\underline{\delta}, \bar{\delta}]$. However, by inspection of the function form of the hard retirement function R of Proposition 2.7.1 and its soft extension in Definition 2.A.2, we can clearly see that the domains can be extended as $[0, T] \times \mathbb{R}^2$. This extension is simply a technical condition to ensure smoothness properties, even though the actual state space is the much smaller $[0, T] \times [\hat{W}, \tilde{W}] \times [\underline{\delta}, \bar{\delta}] = \bar{Q}$. We will use some slight abuse of notations here for brevity.

2.D Markov Control Construction

Once we have verified that the conditions as outlined in Section 2.C are satisfied, then we can use the results as outlined in Fleming and Soner (2006), Section IV. In particular, details here will be brief, in particular since the details are very well documented in Fleming and Soner (2006), Section IV. We just summarize the key results as follows:

Existence of Markovian Feedback Optimal Control

Proposition 2.D.1. *Consider the setup as in Proposition 2.8.1. Then,*

- (i) *The solution V as in Proposition 2.C.6 is the solution to the value function (2.8.1) subject to the state variables (2.8.2).*
- (ii) *The optimal control X^* exists and is Markov, and hence $X^* = X^*(t, w, \delta)$.*

Proof to Proposition 2.D.1. We observe that,

- (i) Taking the result from Proposition 2.C.6 and given the smoothness result that we obtain, we simply apply the standard Verification Theorem argument as per Theorem 3.1 of Fleming and Soner (2006).
- (ii) The conditions we have laid forth in Section 2.C satisfy the hypothesis of Theorem 4.4 of Fleming and Soner (2006), which the optimal control must be Markovian. ■

Remark 2.D.2. *In Section 2.D below, we will explicitly construct that Markovian control and thereby directly show uniqueness, instead of mere existence as per Proposition 2.D.1. Essentially, our uniqueness result comes from the smoothness and concavity of the relevant functions, and that we are maximizing over a compact control value space.*

Construction of Controls

Given the existence of the Markovian feedback controls as per Proposition 2.D.1, we are now ready to explicitly characterize them. Referring back to the HJB equation on Q in (2.8.4), for each $(t, w, \delta) \in Q$, and denoting $V_w := V_w(t, w, \delta)$ for notational convenience, we see that the control optimization problem is of the form, and recall we had denoted $U = [x_0, 1]$,

$$\max_{x \in U} \ell^{t, w, \delta}(x) := u((1-x)\delta) - (\log x)V_w, \quad (2.D.1)$$

We see that (2.D.1) is a single variate constrained optimization problem over the compact set U and for each fixed (w, δ) , the map $x \mapsto u((1-x)\delta) - (\log x)V_w$ is continuous and smooth, so the extreme value theorem guarantees the existence of a solution, and moreover, the Karush-Kuhn-Tucker theory²⁷ applies.

Given the amount of notation ahead, we should note that (2.D.1) is conceptually standard, but the difficulty and the computational burden essentially lies in the sign and value of V_w . Unfortunately, a priori, it is difficult to analytically show the qualitative behavior of $V_w(t, w, \delta)$ as (t, w, δ) varies.

We will need a simple technical lemma relating to the inverse function of the utility.

Lemma 2.D.3. *For any $\delta > 0$, the map $(x_0, 1) \ni x \mapsto xu'((1-x)\delta) \in \mathbb{R}$ has an unique smooth inverse $I(\cdot, \delta)$.*

Proof. Proof to Lemma 2.D.3 Fix any $\delta > 0$, and simply define,

$$f(x, \delta) := xu'((1-x)\delta).$$

An easy application of the chain rule has that,

$$f_x(x, \delta) = -x\delta u''((1-x)\delta) + u'((1-x)\delta) > 0,$$

where the inequality follows from Assumption 2.3.5, and in particular we have the standard utility forms $u'' < 0$ and $u' > 0$. But that means for any $\delta > 0$, the map $x \mapsto f(x, \delta)$ is strictly monotonically increasing and hence there exists a unique inverse $I(\cdot, \delta)$. But given that f is smooth in both x, δ , the inverse function theorem shows that the inverse I is also smooth in both of its arguments. ■

²⁷ See standard references like de la Fuente (2000).

Proof to Proposition 2.9.1. Considering the optimization problem (2.D.1). We will need to consider three cases. Fix any $(t, w, \delta) \in Q$ and denote $V_w := V_w(t, w, \delta)$.

Case (a): $V_w = 0$. In this case, we have that $\ell^{t,w,\delta}(x) = u((1-x)\delta)$. And computing the first derivative,

$$\frac{d}{dx}\ell^{t,w,\delta}(x) = -\delta u'((1-x)\delta) < 0,$$

which shows that on U , $\ell^{t,w,\delta}$ is monotonically decreasing. Hence, its maximizer must be $x^* = x_0$ when $V_w = 0$. This completes Case (a).

Case (b): $V_w > 0$. In this case, again computing the first derivative, we have that,

$$\frac{d}{dx}\ell^{t,w,\delta}(x) = -\delta u'((1-x)\delta) - \frac{1}{x}V_w < 0,$$

which again shows that on U , $\ell^{t,w,\delta}$ is monotonically decreasing. Hence, its maximizer must be $x^* = x_0$ when $V_w > 0$. Thus, this also implies when $V_w \geq 0$, the optimizer must be $x^* = x^*(t, w, \delta) = x_0$, and this is the condition for the set E of Definition 2.9.1. This completes Case (b).

Case (c): $V_w < 0$. In this case, we note that when $V_w < 0$, then $x \mapsto -(\log x)V_w$ is globally concave. And also, $x \mapsto u((1-x)\delta)$ is also globally concave. The sum of concave functions is concave, and hence, $\ell^{t,w,\delta}$ is globally concave. Hence, if there is a critical point to $\ell^{t,w,\delta}$ on U , this point is the only candidate point for an interior global maximizer (i.e. not minimizer). Thus, the Karush-Kuhn-Tucker conditions apply here and are both sufficient and necessary for maximization optimality.

Consider the Lagrange multipliers λ_H and λ_L associated with the constraints $x - 1 \leq 0$ and $x_0 - x \leq 0$, respectively. We define the Lagrangian,

$$\mathcal{L}^{t,w,\delta} := \ell^{t,w,\delta}(x) - \lambda_H(x - 1) - \lambda_L(x_0 - x).$$

By first order conditions,

$$\begin{aligned} 0 &= \frac{d}{dx}\mathcal{L}^{t,w,\delta} \\ &= \frac{d}{dx}\ell^{t,w,\delta}(x) - \frac{1}{x}V_w - \lambda_H + \lambda_L \\ &= -\delta u'((1-x)\delta) - \frac{1}{x}V_w - \lambda_H + \lambda_L. \end{aligned}$$

For the complementary slackness conditions, consider when $\lambda_H > 0$ and so $x = 1$, and we have,

$$0 < \lambda_H = -\delta'((1-1)\delta) - V_w = -\delta u'(0) - V_w$$

— contradiction, since $\lim_{c \rightarrow 0^+} u'(c) = +\infty$. Thus, the point $x = 1$ is never optimal.

When $\lambda_L > 0$ and $x = x_0$, we have,

$$0 > -\lambda_L = -\delta u'((1-x_0)\delta) - \frac{1}{x_0}V_w(t, w, \delta),$$

which, after rearranging, is the condition of the set EI^c of Definition 2.9.1.

Thus for the interior solution, which is the condition of the set EI of Definition 2.9.1, which solves,

$$0 = -\delta u'((1-x)\delta) - \frac{1}{x}V_w,$$

rearranging and using Lemma 2.D.3, we have that the interior solution must be,

$$x^* = \tilde{X}(t, w, \delta) = I\left(-\frac{V_w}{\delta}, \delta\right).$$

This completes Case (c).

Thus, putting Cases (a), (b) and (c) together, we have the result as claimed. ■

2.E Proofs to Asset Pricing Dynamics

Regularity upgrade assumption

Before we proceed, there remains an important technicality that we need to consider. We will first state this as an assumption and then subsequently make some remarks.

Assumption 2.E.1. *We assume that the value function V in Proposition 2.C.6 is such that $V_{tw}, V_{www}, V_{yyy}, V_{wyy} \in C(\bar{Q})$.*

As we shall subsequently see when we consider the asset pricing dynamics, we will need higher orders of differentiability for the value function V to apply Ito's lemma on it than just $C^{1,2}(Q) \cap C(\bar{Q})$ as given in Proposition 2.C.6. In particular, we will need $V_{tw}, V_{www}, V_{yyy}, V_{wyy}$ to exist and smooth. Note that this regularity upgrade was not needed in the discussion in Section 2.D. Rather than going through the very technical details of obtaining this regularity upgrade, as per Krylov (1987), Chapters 4 and 5, we will simply make Assumption 2.E.1.

Remark 2.E.2. *At least from an intuitive (but not rigorous) this level, Assumption 2.E.1 is not as far fetched as it initially appears. Most notably, we consider the follow “fauz derivation” as an illustration, which will not be used anywhere else in the paper (and hence can be omitted from reading, if desired).*

The first five partial derivatives $V_t, V_w, V_{ww}, V_{yy}, V_{wy}$ are well defined immediately by Proposition 2.C.6. Now, consider the four partial derivatives $V_{tw}, V_{www}, V_{yyy}, V_{wyy}$. From the HJB equation (2.8.4), rearrange as,

$$\begin{aligned} & \sup_{x \in [x_0, 1]} [u((1-x)\delta) - (\log x)V_w] - \rho V + (\rho w - g(a_H) - \log \delta)V_w + h(a_H)\delta V_y \\ &= - \left\{ V_t + \frac{\Sigma^2}{2} \left[\left(\frac{g'(a_H)}{h'(a_H)} \right)^2 V_{ww} + \delta^2 V_{yy} + 2 \left(\frac{g'(a_H)}{h'(a_H)} \right) \delta V_{wy} \right] \right\}. \end{aligned} \quad (2.E.1)$$

Now, if we denote $f(x; t, w, \delta) := u((1-x)\delta) - (\log x)V_w(t, w, \delta)$, and consider the value function associated with the optimization problem to the left-hand side of (2.E.1),

$$W(t, w, \delta) := \sup_{x \in [x_0, 1]} f(x; t, w, \delta). \quad (2.E.2)$$

The result of Proposition 2.C.6 implies that f satisfies the Envelope Theorem conditions of Milgrom and Segal (2002), Theorem 2. Thus, we conclude that W is Lebesgue-a.e. differentiable. Now, the other two terms in the sum of the left-hand side of (2.E.1) are clearly differentiable in $(w, \delta) \in O$, again by Proposition 2.C.6.

Thus, it implies the entire left-hand side of (2.E.2) is differentiable in $(w, \delta) \in O$ Lebesgue-a.e. But this implies the entire term of the right-hand side of (2.E.1) is differentiable in $(w, \delta) \in O$ Lebesgue-a.e.

However, this “fauz derivation” does not lead to our desired result. Most notably, we see that if a function h is differentiable, and if $h = f + g$, it does not necessarily imply that f, g are both differentiable, as f, g could be in some sense filling each other's roughness as a sum.²⁸ Hence, we have “almost” the result that we need, but not quite.

Remark 2.E.3. *Another route that we could take is via the method of viscosity solutions and a stochastic verification theorem. It is well known that the HJB PDE can be viewed as a viscosity solution as per Crandall, Ishii, and Lions (1992). Indeed, it is also known that if a classical solution exists, this is also identical to the viscosity solution to the HJB PDE, as per Fleming and Soner (2006). Hence, if we pause the analysis at the viscosity sense of solutions, and using the stochastic verification theorems of Zhou et al. (1997) in a viscosity framework, we could also construct feedback controls. Then in this sense, the constructed feedback controls will be functions of the highly smooth test functions that approximate the value function, and in particular, we will have no issues of understanding the higher orders of smoothness of the value functions. But this approach is slightly undesirable from an economic perspective, especially when it comes to the asset pricing dynamics. As it will be made clear when we derive the asset pricing dynamics, if we understand the value function from a viscosity sense, we in principle have an infinite number (indexed by the sequence of test functions) of drift and volatility terms of the underlying risky asset gains process, even though each of them converge to the viscosity sense of our solution. Economically, this sounds somewhat at odds with our intuition of the law of one price (which is a consequence of no-arbitrage conditions, which we have here). Indeed, this somewhat awkward economic interpretation of viscosity solutions is highlighted also in Strulovici and Szydlowski (2014).*

²⁸ Clearly, the elementary converse is true; that if f, g are differentiable, then $h = f + g$ is differentiable. But we are not looking for this direction of implications here.

Preliminary computations revolving $\tilde{X}(t, w, \delta)$

We collect some computations revolving $\tilde{X}(t, w, \delta)$, as defined in (2.10.1). Note that it is precisely in these computations where we need the implications of Assumption 2.E.1. Since $(\ell, \delta) \mapsto I(\ell, \delta)$, we denote I_i as the partial derivative to the i th function argument, and I_{ij} as the ij th partial derivative, for $i, j = 1, 2$.

The following is a direct application of the chain rule, valid thanks to Assumption 2.E.1, which we will state without the obvious proof.

Lemma 2.E.4. *Consider the optimal pie sharing rule $\tilde{X}(t, w, \delta)$ as in (2.10.1). Suppose further that I is differentiable in both arguments. Then we have that,*

$$\begin{aligned}\tilde{X}_w &= I_1(-V_w/\delta, \delta) [-V_{ww}/\delta] \\ \tilde{X}_t &= I_1(-V_w/\delta, \delta) [-V_{tw}/\delta] \\ \tilde{X}_y &= I_1(-V_w/\delta, \delta) [V_w/\delta^2 - V_{wy}/\delta] + I_2(-V_w/\delta, \delta) \\ \tilde{X}_{ww} &= I_1(-V_w/\delta, \delta) [-V_{www}/\delta] + I_{11}(-V_w/\delta, \delta) [V_{ww}/\delta]^2 \\ \tilde{X}_{yy} &= [I_{11}(-V_w/\delta, \delta) (V_w/\delta^2 - V_{wy}/\delta) + 2I_{12}(-V_w/\delta, \delta)] [V_w/\delta^2 - V_{wy}/\delta] \\ &\quad + I_1(-V_w/\delta, \delta) [-2V_w/\delta^3 + V_{wyy}/\delta + 2V_{wy}/\delta^2] + I_{22}(-V_w/\delta, \delta) \\ \tilde{X}_{wy} &= I_1(-V_w/\delta, \delta) [V_{ww}/\delta^2 - 2V_{wwy}/\delta] - I_{11}(-V_w/\delta, \delta) (V_{ww}/\delta) [V_w/\delta^2 - 2V_{wy}/\delta] \\ &\quad - I_{12}(-V_w/\delta, \delta) [V_{ww}/\delta].\end{aligned}$$

Application of Ito's lemma

Proof to Proposition 2.10.1. We have three cases to consider, each corresponding to the related indicator functions. Recall Proposition 2.4.1 and (2.3.2). Consider a small time interval $[t_0, t_1]$ for $0 \leq t_0 < t_1 < T$

Case (1): Suppose the set $\{(t, W_t, D_t) \in \mathcal{E}_H : t \in [t_0, t_1]\} \neq \emptyset$. Then by Proposition 2.9.2, $X \equiv 1$ on this set. Then from Proposition 2.4.1, we have that $S_t = \frac{1}{\rho} D_t$. Then we have that on one hand, for $t \in [t_0, t_1]$,

$$\begin{aligned}dS_t + D_t dt &= \frac{1}{\rho} dD_t + D_t dt \\ &= \frac{1}{\rho} (D_t h(a_H) dt + D_t \Sigma dZ_t^{a_H}) + D_t dt \\ &= D_t (\rho^{-1} h(a_H) + 1) dt + D_t \Sigma dZ_t^{a_H}.\end{aligned}$$

Equating with (2.3.2), it implies that we have,

$$\begin{aligned}\mu(t) &= \rho + h(a_H), \\ \sigma(t) &= \Sigma.\end{aligned}$$

Case (2): Suppose the set $\{(t, W_t, D_t) \in \mathcal{E}_L : t \in [t_0, t_1]\} \neq \emptyset$. An analogous computation to Case (1) shows that,

$$\begin{aligned}\mu(t) &= \frac{\rho}{x_0} + h(a_H) \\ \sigma(t) &= \Sigma.\end{aligned}$$

Case (3): Suppose the set $\{(t, W_t, D_t) \in \mathcal{E}_M : t \in [t_0, t_1]\} \neq \emptyset$. Then we have $S_t = \frac{1}{\rho} \tilde{X}(t, W_t, D_t) D_t$. Now, applying Ito's lemma,

$$\begin{aligned}\rho dS_t &= \tilde{X}_t D_t dt + \tilde{X}_w D_t \left[(\rho W_t - g(a_H) - \log \tilde{X} - \log D_t) dt + \Sigma \frac{g'(a_H)}{h'(a_H)} dZ_t^{a_H} \right] \\ &\quad + (\tilde{X} + D_t \tilde{X}_y) (h(a_H) D_t dt + \Sigma D_t dZ_t^{a_H}) \\ &\quad + \frac{1}{2} \left[\tilde{X}_{ww} D_t \left(\Sigma \frac{g'(a_H)}{h'(a_H)} \right)^2 + (2\tilde{X}_y + D_t \tilde{X}_{yy}) (\Sigma D_t)^2 + 2(\tilde{X}_w + D_t \tilde{X}_{wy}) \Sigma^2 \frac{g'(a_H)}{h'(a_H)} D_t \right] dt.\end{aligned}$$

Matching drift to (2.3.2), we have that,

$$\begin{aligned} S_t \mu(t) &= \frac{1}{\rho} \left[\tilde{X}_t D_t + \tilde{X}_w D_t (\rho W_t - g(a_H) - \log \tilde{X} - \log D_t) + (\tilde{X} + D_t \tilde{X}_y) h(a_H) D_t \right. \\ &\quad + (\tilde{X} + D_t \tilde{X}_y) h(a_H) D_t \\ &\quad + \frac{1}{2} \left(\tilde{X}_{ww} D_t \left(\Sigma \frac{g'(a_H)}{h'(a_H)} \right)^2 + (2\tilde{X}_y + D_t \tilde{X}_{yy}) (\Sigma D_t)^2 \right. \\ &\quad \left. \left. + 2(\tilde{X}_w + D_t \tilde{X}_{wy}) \Sigma^2 \frac{g'(a_H)}{h'(a_H)} D_t \right) \right] + D_t. \end{aligned}$$

But recalling again that $S_t = \frac{1}{\rho} \tilde{X} D_t$, we arrange and simplify to obtain,

$$\begin{aligned} \mu(t) &= \frac{1}{\tilde{X}} \left[\rho + \tilde{X}_t + \tilde{X}_w (\rho W_t - g(a_H) - \log \tilde{X} - \log D_t) + (\tilde{X} + D_t \tilde{X}_y) h(a_H) \right. \\ &\quad \left. + \frac{\Sigma^2}{2} \left(\tilde{X}_{ww} \left(\frac{g'(a_H)}{h'(a_H)} \right)^2 + (2\tilde{X}_y + D_t \tilde{X}_{yy}) D_t + 2(\tilde{X}_w + D_t \tilde{X}_{wy}) \frac{g'(a_H)}{h'(a_H)} \right) \right]. \end{aligned}$$

And matching volatility to (2.3.2), and rearranging, we have that,

$$\sigma(t) = \Sigma + \frac{\Sigma}{\tilde{X}} \left[\tilde{X}_w \frac{g'(a_H)}{h'(a_H)} + D_t \tilde{X}_y \right]$$

Thus, the above shows that (i) and (ii) hold.

Now for (iii) and (iv), those are simply restatements of Proposition 2.4.1(ii). ■

2.F Special Case: CRRA utility with $\gamma = 2$

Here, we'll investigate a special case of the principal's utility u with CRRA form with coefficient of relative risk aversion $\gamma = 2$; see Example 2.3.2. In particular, with the CRRA preferences with $\gamma = 2$, we can be more specific of the results optimal pie sharing rule contract as discussed in Section 2.9.

We next revisit Lemma 2.D.3, Proposition 2.9.1, Definition 2.9.1 and Lemma 2.E.4.

Corollary 2.F.1. *Suppose the principal's utility function u is of a CRRA form with $\gamma = 2$ as in Example 2.3.2. Then,*

(a) *For $(t, w, \delta) \in EI$, and denoting $V_w := V_w(t, w, \delta)$, the interior optimal pie sharing rule of Proposition 2.9.1 can be written as,*

$$\tilde{X}(t, w, \delta) = - \frac{(\delta^{-1} - 2V_w) - \sqrt{\delta^{-2} - 4\delta^{-1}V_w}}{2V_w}. \quad (2.F.1)$$

(b) *For $(t, w, \delta) \in EI$, and denoting $V_w := V_w(t, w, \delta)$, the function I of Lemma 2.D.3 can be written as,*

$$I(y, \delta) = \frac{\delta^{-2} + 2y - \delta^{-1} \sqrt{\delta^{-2} + 4y}}{2y}. \quad (2.F.2)$$

(c) For $(t, w, \delta) \in EI$, and denoting $V_w := V_w(t, w, \delta)$, the derivatives of I in Lemma 2.E.4 can be computed as,

$$I_1(y, \delta) = -(1/2)y^{-2} \left(\delta^{-2} + 2y - \delta^{-1}(\delta^{-2} + 4y)^{1/2} \right) + y^{-1} \left(1 - \delta^{-1}(\delta^{-2} + 4y)^{-1/2} \right) \quad (2.F.3)$$

$$I_2(y, \delta) = (1/2)y^{-1} \left(-2\delta^{-3} + \delta^{-4}(\delta^{-2} + 4y)^{-1/2} + \delta^{-2}(\delta^{-2} + 4y)^{1/2} \right) \quad (2.F.4)$$

$$I_{11}(y, \delta) = - \left[y^{-2} [1 - \delta^{-1}(\delta^{-2} + 4y)^{-1/2}] - y^{-3} [\delta^{-2} + 2y - \delta^{-1}(\delta^{-2} + 4y)^{1/2}] \right] + \left[2y^{-1} \delta^{-1}(\delta^{-2} + 4y)^{-3/2} - y^{-2} [1 - \delta^{-1}(\delta^{-2} + 4y)^{-1/2}] \right] \quad (2.F.5)$$

$$I_{22}(y, \delta) = (1/2)y^{-1} \delta^{-3} \left(6\delta^{-1} + \delta^{-4}(\delta^{-2} + 4y)^{-3/2} - \delta^{-2}(\delta^{-2} + 4y)^{-1/2} - 2(\delta^{-2} + 4y)^{1/2} \right) \quad (2.F.6)$$

$$I_{12}(y, \delta) = -(1/2)y^{-2} \left[-2\delta^{-3} + \delta^{-4}(\delta^{-2} + 4y)^{-1/2} + \delta^{-2}(\delta^{-2} + 4y)^{1/2} \right] + y^{-1} \left[-\delta^{-4}(\delta^{-2} + 4y)^{-3/2} + \delta^{-2}(\delta^{-2} + 4y)^{-1/2} \right]. \quad (2.F.7)$$

Proof. Here, we have that,

$$u(c) = \underline{u} + \frac{c^{1-2} - 1}{1-2} = \underline{u} + 1 - c^{-1}.$$

And clearly, $u'(c) = c^{-2}$ and $u''(c) = -2c^{-3}$.

Let's first consider part (a). Recalling Definition 2.9.1 and consider $(t, w, \delta) \in EI$ for which we have an interior solution $\tilde{X}(t, w, \delta)$ to our problem, and denote $V_w := V_w(t, w, \delta)$. Hence, repeating the proof and some notations of Proposition 2.9.1, we have the first order conditions,

$$0 = \frac{d}{dx} \ell^{t, w, x}(x) = -\frac{1}{(1-x)^2 \delta} - \frac{1}{x} V_w. \quad (2.F.8)$$

But by rearranging, we note that (2.F.8) can be rewritten as,

$$\begin{aligned} 0 &= \delta^{-1}x + V_w(1-x)^2 \\ &= \delta^{-1}x + V_w(1-2x+x^2) \\ &= V_w x^2 + (\delta^{-1} - 2V_w)x + V_w. \end{aligned} \quad (2.F.9)$$

which we recognize is a quadratic equation in x . Hence, the quadratic equation applies to (2.F.9) and we have,

$$x = \frac{-(\delta^{-1} - 2V_w) \pm \sqrt{(\delta^{-1} - 2V_w)^2 - 4V_w^2}}{2V_w} = \frac{-(\delta^{-1} - 2V_w) \pm \sqrt{\delta^{-2} - 4\delta^{-1}V_w}}{2V_w}. \quad (2.F.10)$$

Let x_+, x_- be respectively the positive root and negative root of (2.F.10). That is,

$$\begin{aligned} x_+ &:= \frac{(\delta^{-1} - 2V_w) - \sqrt{\delta^{-2} - 4\delta^{-1}V_w}}{-2V_w}, \\ x_- &:= \frac{(\delta^{-1} - 2V_w) + \sqrt{\delta^{-2} - 4\delta^{-1}V_w}}{-2V_w}. \end{aligned}$$

Given that the optimization problem at hand is globally concave, if an interior solution exists, there can only be one. Thus, we need to decide which root to take as our interior solution. But recall that on EI , for which in particular we have $V_w < 0$, this implies the discriminant $\delta^{-2} - 4\delta^{-1}V_w > 0$, since clearly $\delta > 0$, and hence we have real and not complex solutions. Thus, $x_+, x_- > 0$. Moreover, note here that $x_- > x_+$.

To be a candidate solution, we must have (at least) that $x < 1$. However, let's show that x_- fails this criterion and hence cannot be the solution. To see this, note that,

$$\begin{aligned} x_- < 1 &\iff \frac{\delta^{-1} - 2V_w + \sqrt{\delta^{-2} - 4\delta^{-1}V_w}}{-2V_w} < 1 \\ &\iff \delta^{-1} - 2V_w + \sqrt{\delta^{-2} - 4\delta^{-1}V_w} < -2V_w \\ &\iff \delta^{-1} + \sqrt{\delta^{-2} - 4\delta^{-1}V_w} < 0 \quad \text{--- contradiction.} \end{aligned}$$

Hence, x_- cannot be the candidate solution. This leaves x_+ . But we must verify that $x_+ \in (x_0, 1)$ so that it is the only possible and admissible interior solution. Let's first verify that $x_+ < 1$. To see this,

$$\begin{aligned} x_+ < 1 &\iff \frac{\delta^{-1} - 2V_w - \sqrt{\delta^{-1} - 4\delta^{-1}V_w}}{-2V_w} < 1 \\ &\iff \delta^{-1} - 2V_w - \sqrt{\delta^{-2} - 4\delta^{-1}V_w} < -2V_w \\ &\iff \delta^{-1} < \sqrt{\delta^{-1} - 4\delta^{-1}V_w} \\ &\iff 0 < -4\delta^{-1}V_w, \end{aligned}$$

which holds, recalling that $V_w < 0$. Hence, $x_+ < 1$ holds. Next, we verify that $x_+ > x_0$. To see this,

$$\begin{aligned} x_0 < x_+ &\iff x_0 < \frac{\delta^{-1} - 2V_w + \sqrt{\delta^{-1} - 4\delta^{-1}V_w}}{-2V_w} \\ &\iff \sqrt{\delta^{-2} - 4\delta^{-1}V_w} < \delta^{-1} - 2V_w(1 - x_0) \\ &\iff \delta^{-2} - 4\delta^{-1}V_w < (\delta^{-1} - 2V_w(1 - x_0))^2 \\ &\iff 4\delta^{-1}V_w(1 - x_0) - 4\delta^{-1}V_w < 4V_w^2(1 - x_0)^2 \\ &\iff V_w < -x_0\delta^{-1}(1 - x_0)^2, \end{aligned}$$

where the last statement is exactly the condition of the set EI . Hence, $x_+ > x_0$, and hence $x_+ = \tilde{X}(t, w, \delta) \in (x_0, 1)$ of (2.F.1) is the only interior solution, if it exists, to our optimization problem.

Let's consider part (b). The expression (2.F.1) also allows us to compute the exact form of the function I as characterized by Lemma (2.D.3). Evidently, we have that $x = \tilde{X}(t, w, \delta)$ is equal to $I(y, \delta)$ in (2.F.2) when we evaluate at $y = -V_w/\delta$. Thus part (b) holds.

For part (c), we revisit the computations in Lemma 2.E.4, but this is just a direct computation based off of the result of part (b). \blacksquare

2.G Technical details to the numerical solution

Here, we present the technical details to the numerical solution as discussed in the main text. We proceed to numerically compute the problem in the following steps:

1. We begin the numerical solution from the results of Proposition 2.8.1, and in particular the HJB equation (2.8.4) and the boundary conditions (2.8.5).
2. We discretize the state space O in step sizes dx (for the w -direction), dy (for the δ -direction), and dt (for the t -direction).
3. We replace each of the derivatives that appear in the HJB equation in (2.8.4) with their finite difference counterparts. For the optimization procedure, we discretize the control space over some number of points, and grid search for the optimizer. Most notably, we do not take first order conditions and substitute back the optimizer, as this procedure is known to be highly numerically unstable. Hence, we prefer to discretize and then optimize, rather than optimize and then discretize.
4. Once the discretized value function is found, we use first order conditions to find the (numerical) optimizer $X^*(t, w, \delta)$, which is our optimal pie sharing rule.
5. The subsequent step is to find the various derivatives $X^*(t, w, \delta)$ necessary to compute the equilibrium gains parameters $\mu(t, w, \delta)$ and $\sigma(t, w, \delta)$ as in Proposition 2.10.1. Given that we are using an CRRA specification with relative risk aversion parameter of $\gamma = 2$ as discussed in Section 2.F, we can indeed write the various derivatives of $X(t, w, \delta)$ in terms of the various derivatives of the (numerical) value function $V(t, w, \delta)$. However, after some numerical investigation, we discovered that the results are highly numerically unstable and full of numerical errors due to discretization.
6. Hence, we will take a different approach. We will take the (numerical) optimizer $X^*(t, w, \delta)$ and use the procedure Nakamura, Wang, and Wang (2008). That is, we will take the (numerical) optimizer $X^*(t, w, \delta)$, compute its regularized version, use that to compute the desired derivatives, and finally obtain our desired gains parameters $\mu(t, w, \delta)$ and $\sigma(t, w, \delta)$. This procedure will smooth out the numerical instabilities.

2.H Additional illustrations

For the purpose of example and robustness, here we give further illustrations of the comparative statics associated with the value function and the optimal pie sharing rule. The parameters specifications here are identical to that of Section 2.11.

Derivatives of the value function

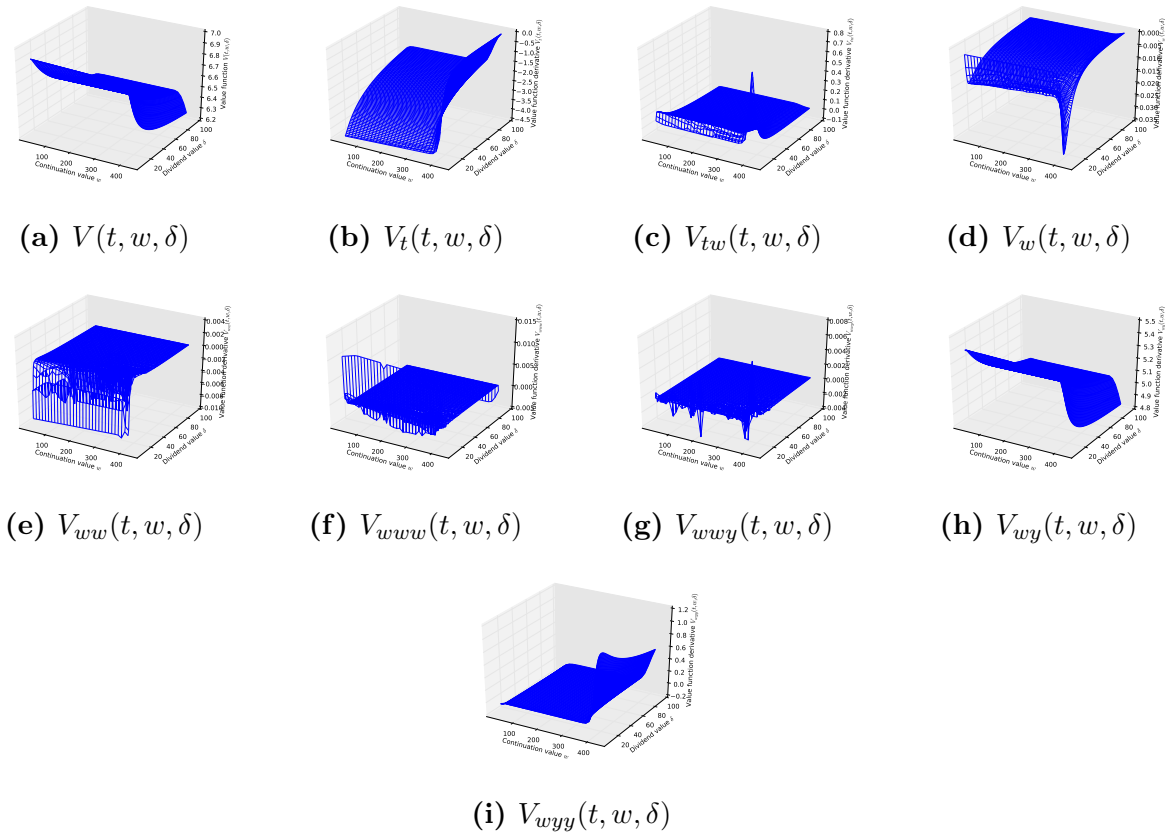


Figure 2.H.1: At $t = 0.01$, for various derivatives of the value function $V(t, w, \delta)$

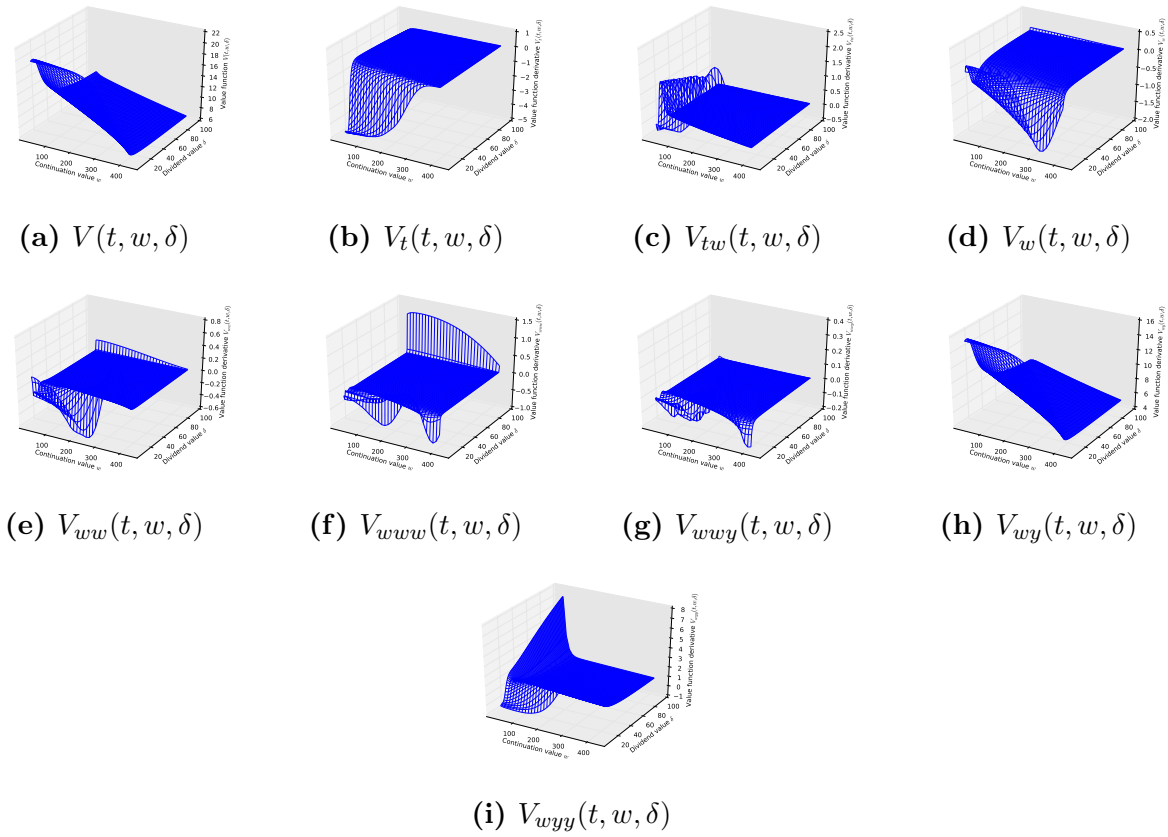


Figure 2.H.2: At $t = 1.6$, for various derivatives of the value function $V(t, w, \delta)$

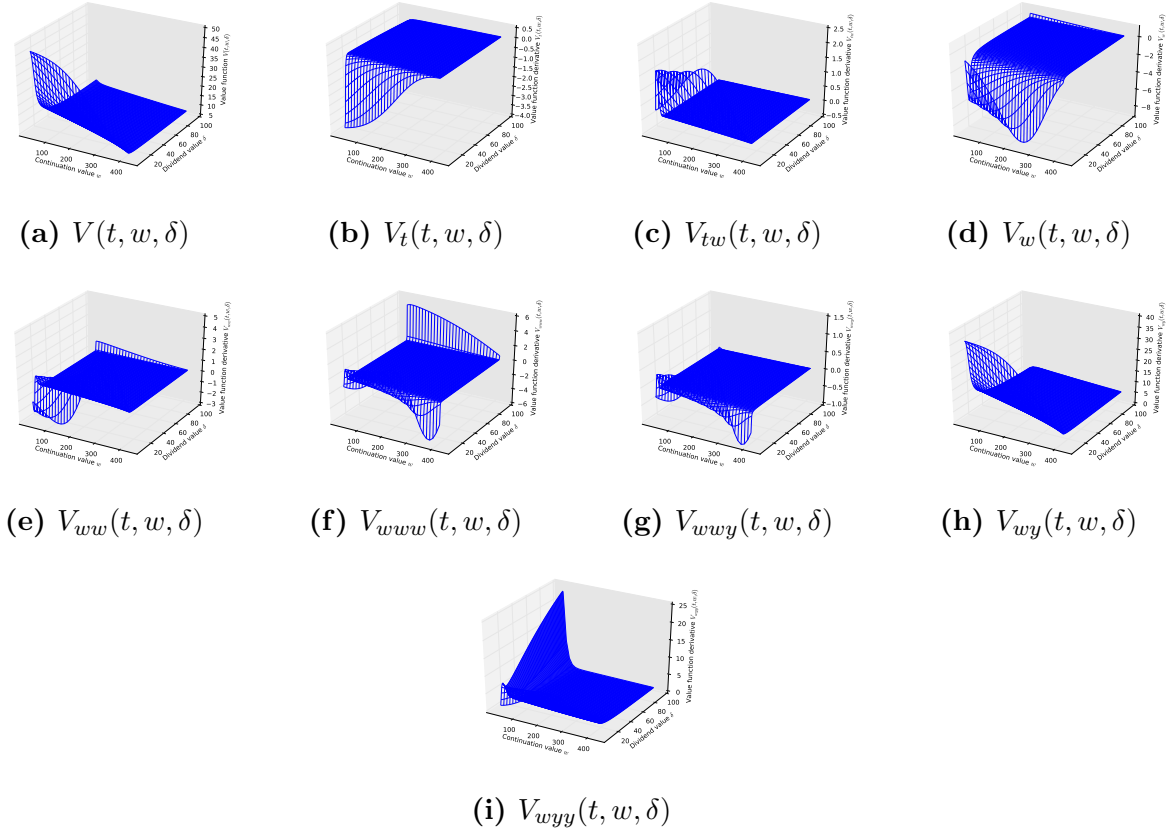
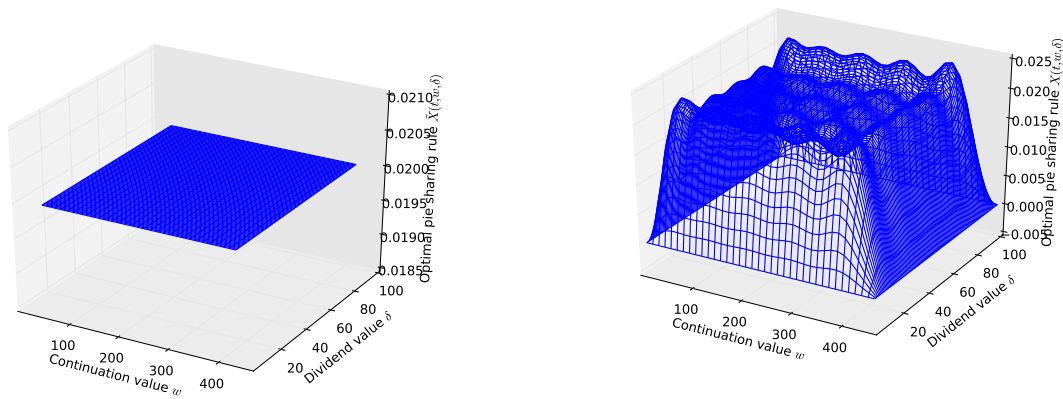


Figure 2.H.3: At $t = 5.0$, for various derivatives of the value function $V(t, w, \delta)$

Regularization of the Nakamura et al. (2008) procedure

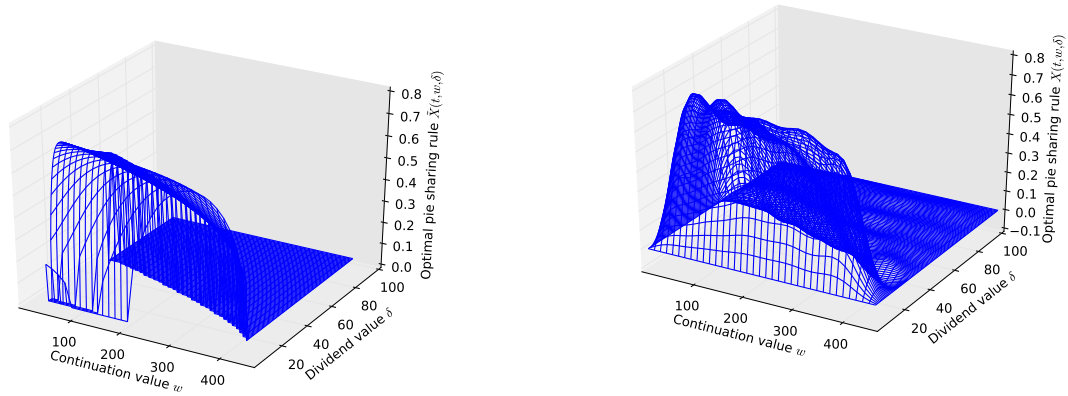
As mentioned, the sole reason for taking a regularized version of the optimal pie sharing rule $X^*(t, w, \delta)$ is that we need its associated derivatives in computing the equilibrium gains drift $\mu(t, w, \delta)$ and volatility $\sigma(t, w, \delta)$. However, as it is known in the numerical computation literature (see Nakamura et al. (2008) for a discussion), even small numerical errors of the underlying function could magnify to nontrivial amount of errors when we further compute numerical derivatives of this function. In our case, the source of the numerical errors largely come from the discretization procedure that we take in numerically solving the HJB PDE. We had initially taken this more “direct” approach, hoping to compute the numerical derivatives of the value function and with the closed form solutions available for the case of the optimal pie sharing rule as expressed as functions of the derivatives of the value function, we could just plug in the numerical counterparts. However, indeed, we find that the numerical errors from the results are far too large to be acceptable. Hence, we took a far more computationally intensive regularization approach like Nakamura et al. (2008).

Here, we will first show the optimal pie sharing rule $X^*(t, w, \delta)$ computed as Proposition 2.9.1 using the numerical derivative to $V_w(t, w, \delta)$ directly substituted. Using this as the input, we feed this to the Nakamura et al. (2008) procedure, and show the resulting regularized version. In the main text, the gains parameters $\mu(t, w, \delta)$ and $\sigma(t, w, \delta)$ are computed exclusively on the regularized version. Note that in the Nakamura et al. (2008) version, it seems that the picture “rises” from the edges whereas the original version does not; this is a mere graphical artifact but note that the levels of both versions are indeed in line with each other.



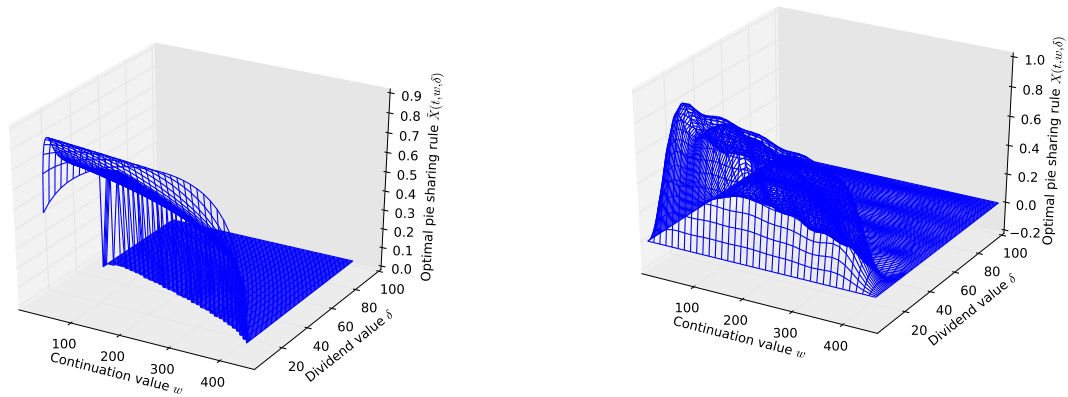
(a) $X^*(t, w, \delta)$ based on analytical solution. (b) $X^*(t, w, \delta)$ based on Nakamura et al. (2008) procedure.

Figure 2.H.4: At $t = 0.01$. We show the optimal pie sharing rule $X^*(t, w, \delta)$ based on the analytical solution in Proposition 2.9.1 and the numerical derivative $V_w(t, w, \delta)$ to the numerical value function $V(t, w, \delta)$. We also show the regularized version of the optimal pie sharing rule based on the Nakamura et al. (2008) procedure.



(a) $X^*(t, w, \delta)$ based on analytical solution. (b) $X^*(t, w, \delta)$ based on Nakamura et al. (2008) procedure.

Figure 2.H.5: At $t = 1.6$. We show the optimal pie sharing rule $X^*(t, w, \delta)$ based on the analytical solution in Proposition 2.9.1 and the numerical derivative $V_w(t, w, \delta)$ to the numerical value function $V(t, w, \delta)$. We also show the regularized version of the optimal pie sharing rule based on the Nakamura et al. (2008) procedure.



(a) $X^*(t, w, \delta)$ based on analytical solution. (b) $X^*(t, w, \delta)$ based on Nakamura et al. (2008) procedure.

Figure 2.H.6: At $t = 5.0$. We show the optimal pie sharing rule $X^*(t, w, \delta)$ based on the analytical solution in Proposition 2.9.1 and the numerical derivative $V_w(t, w, \delta)$ to the numerical value function $V(t, w, \delta)$. We also show the regularized version of the optimal pie sharing rule based on the Nakamura et al. (2008) procedure.

Chapter 3

Centralized versus Decentralized Delegated Portfolio Management under Moral Hazard

DECEMBER 5, 2015

Chapter Abstract

If an investor wants to invest into two asset classes, should he delegate to a single portfolio manager to manage both asset classes (centralized delegation)? Or should he delegate to two managers, each of whom exclusively manages one asset class (decentralized delegation)? Optimal risk sharing and portfolio choice discretion delineate the difference between centralization versus decentralization. Asset classes whose returns are negatively correlated and have high volatilities will favor centralization. But if the two asset classes have very different mean returns, this disfavors centralization: the single manager may disregard portfolios implementing the investor's desired investments and prefer portfolios in alternative investments. Thus, the investor must pay the single manager high performance fees to disincentivize deviation. Decentralization eliminates this necessity because one manager cannot trade another manager's asset class, and the investor contracts with each manager individually. But in decentralization, it may be impossible to implement the investor's desired investments because managers deviate without considering the correlation between the managers' returns. This last problem can be resolved in a dynamic setting, in which the investor's wealth "intertemporally glues" together the managers' wealths to provide the correct incentives.

Chapter Acknowledgements

Special thanks to my committee members Robert M. Anderson, Gustavo Manso and Christine A. Parlour for their endless support. I am deeply indebted to and grateful for the countless hours, encouragement, patience and feedbacks of both of my co-advisers, Robert

M. Anderson and Gustavo Manso, on this project and other research endeavors. I also thank Nicolae Garleanu, Adair Morse, David Sraer, David Thesmar, and Johan Walden for very helpful comments and suggestions. All errors are mine and mine alone.

3.1 Introduction

Delegated portfolio management is a core activity in the modern financial markets — but what is the optimal form of delegation? If an investor wants to access, say, an Asian macro strategy and an European macro strategy, should the investor delegate the execution of these two strategies to a single global strategy manager (*centralized delegation*)? Or should he delegate separately to an Asian strategy manager and also an European strategy manager (*decentralized delegation*)? Moreover, suppose there is moral hazard risk that instead of delivering the advertised Asian macro strategy, managers could privately deviate to a passive Asian equity index that the investor could have accessed without delegation. And suppose analogous moral hazard risks could occur in the European asset class. In the presence of such moral hazard problem, which form of delegation is better: centralized delegation or decentralized delegation?

William F. Sharpe was the first to coin the term “decentralized investment management” in his Presidential Address to the *American Finance Association* 1981 Annual Meeting. Furthermore, Sharpe concludes with:

There is, of course, much more to this problem [of decentralized investment management]. We have assumed away many important aspects of the principal-agent relationships(s). . . . In short, we have clearly provided necessary and sufficient conditions for the traditional final sentence in such a paper: More research on this subject is needed. (Sharpe (1981, page 233))

Clearly the general literature in principal-agent theory, and also to its specific applications to delegated portfolio management, has significantly advanced in the years since 1981. Yet to the best my knowledge, the problem of understanding the similarities and differences between centralized versus decentralized delegation with the presence of moral hazard remains unexplored, and its solution properties remain elusive. In particular, substantial recent empirical evidence (see literature review in Section 3.2 below) suggests that moral hazard risks are strongly present in hedge funds, via the forms of fraud, operational risk, misrepresentation of investment strategies and conflicting evidence of managerial effort in generating alpha. Thus, the key contribution of this paper is an attempt to explore a question opened by Sharpe from decades ago, and this question is made ever more imperative in the modern financial markets.

In this economy, there are two classes of individuals: a single *Principal* and multiple *Managers*. The Principal is initially endowed with a single unit of wealth, while Managers have zero initial wealth. All individuals are risk averse with mean-variance preferences over terminal wealth. The Principal has a strict desire for Managers to be *compliant* and implement a specific pair of investment strategies (say Asian active macro and European active macro strategies, from the previous example), and the Principal must delegate to Managers to access these strategies. However, implementing these strategies will incur private costs for the Managers. Moreover, Managers could deviate to alternative *deviant* strategies (say

Asian passive market index and European passive market index) that are privately costless but have lower mean returns and different correlation structure than the Principal's desired pair of strategies. For simplicity, we take an extreme assumption that the Principal would abandon delegation if his desired strategy pair cannot be implemented. Largely for tractability in the model, we emphasize again that the Principal will only want to implement his desired pair of strategies and he will not entertain other strategy pairs.

In the presence of such moral hazard over investment strategies within each of the two asset classes, the Principal needs to decide which form of delegation is best. In the first option, the Principal can choose *centralized delegation*: the Principal will delegate all initial wealth to a single Manager C (say a global strategy manager). Manager C will have two actions: investment strategy choice and portfolio allocation choice. Manager C will first need to select a strategy pair, one strategy from each asset class. Then secondly, taking any offered contract into account, Manager C will construct portfolio weights between this strategy pair. In return, the Principal will compensate Manager C with a linear contract over the net returns of the resulting portfolio.

Alternatively, in the second option, the Principal can choose *decentralized delegation*: the Principal will make a portfolio choice and decide how much of his initial wealth to delegate to Manager A (say, an Asian asset manager) who will exclusively manage one asset class, and delegate the rest to Manager B (say, an European asset manager) who will exclusively manage the other asset class. Both Managers can only pick one strategy from their respective asset classes. The Principal will compensate these two Managers also with linear contracts over the net returns from their respective asset class.

We make clear on the action differences between centralization and decentralization. In centralization, Manager C has both strategy choice and portfolio choice, while the Principal only has contract choice. In decentralization, Manager A and Manager B have strategy choice within their own asset class, while the Principal has both asset allocation choice and contract choice.

Static delegation

We begin with a static delegation model, where the contract begins today and terminates one period later.

In first best with no moral hazard risk, where the Principal can observe and directly contract on the Managers' strategy choices in each asset class, the comparison of centralized versus decentralized delegation is a simple question of optimal risk sharing. High return correlation of returns between the Principal's preferred strategy pair will favor decentralization, and low correlation will favor centralization. Given any contract, the single Manager C will pick portfolios between this strategy pair, again with one strategy from each asset class. And since the contract is linear over the net returns of the portfolios, then low correlation between the strategies will lead to lower overall portfolio volatility, which then implies lower contract volatility for Manager C . As Manager C is also risk averse, it becomes cheaper for the Principal to compensate him because of reduced risk compensations. In contrast, if the

correlation is high, delegating to Manager C will increase his contract volatility, which in turn means the Principal must increase the risk compensation.

In contrast, for decentralization, the Principal picks portfolio weights over his initial wealth to delegate to Manager A and Manager B , in addition to offering separate linear contracts for each Manager over their strategy's net returns. When the correlation of the Principal's desired strategy pair is high, the Principal can pick portfolio weights to spread out the risk between himself, Manager A and Manager B to optimally risk share. In contrast, when the correlation is low, since Manager A 's and Manager B 's contracts only depend on their own strategy returns, only the Principal can capture the low correlation diversification benefit and thus the risk sharing benefit is reduced for decentralization. This first best result illustrates the idea that risk management "defines the boundaries" of a firm. In centralization, risk management is handled exclusively by Manager C , since only Manager C picks portfolios between strategies. Whereas in decentralization, the Principal handles risk management himself since only he picks portfolios.

Next, we consider the second best case where moral hazard is distinctly present, in that Managers can privately choose their investment strategies. We highlight three specific components that affect the Principal's decision for centralized delegation versus decentralized delegation under moral hazard: (i) investment opportunity set; (ii) Managers' risk aversions; and (iii) Managers' private costs.

With respect to (i), we claim that a wide investment opportunity set strongly *disfavors* centralized delegation. For any given contract, Manager C makes portfolio weight choices and also strategy pair choices. When Manager C deviates away from the Principal's desired strategy pair, the deviant pair of strategies will generically have different mean returns with some correlation level. Given that the Manager C has mean-variance preferences, he will naturally put greater portfolio weights to the deviant strategy of one asset class with a higher mean return and a lower portfolio weight to the deviant strategy of another asset class with a lower mean return. This generates a "long-short" trading profit benefit for Manager C out of the deviant strategy pair that is not enjoyed by the Principal; again, the Principal has a strict desire for Manager C to be compliant and to implement the Principal's desired strategy pair, and will not entertain any other deviant strategy pairs. Thus to ensure compliance, the Principal must compensate Manager C with higher performance fees as an opportunity cost for Manager C 's foregone long-short trading profits, along with Manager C 's private costs for implementing the Principal's desired strategy pair. That is to say, if the investment opportunity set is so "wide" that the mean return differences between the two asset classes are large, it will strongly disfavor centralized delegation due to increased performance fees the Principal must compensate. In contrast, under decentralized delegation, even if Manager A or Manager B deviates, they can only deviate in strategies within their own asset class. So the aforementioned long-short opportunity cost in centralization simply does not exist for them due to restriction in their respective investment opportunity set. Hence, to ensure compliance from the decentralized Managers, the Principal simply needs to compensate for their private costs, and the mean and volatility differences between the compliant and deviant strategies in their respective asset classes.

With respect to (ii) Managers' risk aversions, lower managerial risk aversion will *disfavor* centralized delegation. The effect of risk aversion on Manager *C* is intimately linked to (i) the investment opportunity set restrictions. As Manager *C* becomes less risk averse, the less he cares about volatility of the overall portfolio and likewise on the correlation between investment strategies across asset classes; the only thing that becomes relevant are simply the mean return differences between the deviant strategy pairs. Thus, in the extreme limit when Manager *C* becomes risk neutral, for any given contract, Manager *C* will simply take an infinite long position into the deviant strategy from one asset class with highest possible mean, and take an infinite short position into the deviant investment strategy from another asset class with lowest possible mean. Without any portfolio constraints on Manager *C*, this potentially infinitely large long-short trading profit would be too high of an opportunity cost for the Principal to compensate, and thereby resulting in the nonexistence of a contract to implement the Principal's desired strategy pair. We should note that this result is starkly different from standard principal-agent theories¹ where it is generically cheaper for a Principal to contract with a less risk averse agent since the Principal then saves on the required risk premium compensation. In our case, the result is completely reversed: a less risk averse Manager *C* is actually more expensive to compensate, and in the limit when Manager *C* is risk neutral, it may become infinitely costly to compensate him. These effects are completely driven by Manager *C*'s relaxed investment opportunity set; in particular, when the Principal is risk averse while Manager *C* is risk neutral. When Manager *C* can have less restrictive access to financial markets — namely that he can freely construct portfolio weights between asset classes — he can modify the risk and reward effects of the contract to his desire, and in particular, can offload the contract risks onto the financial markets, and thereby distort the incentive effects of the contract. In contrast, in decentralized delegation and again extending the discussion from (i), Manager *A* and Manager *B* cannot trade each other's asset class, and thus are severely restricted in their respective investment opportunity set. When the two Managers consider a deviation, they are concerned with the differences in the strategies' means and volatilities within their own asset class. Thus decentralized delegation is much closer to a standard principal-(multi)agent problem whereby compensation to less risk averse agents could be reduced.²

With respect to (iii) Managers' private costs, high private costs *disfavors* decentralized delegation. In centralization, by being compliant and implementing the Principal's desired strategy pair, Manager *C* must incur a high private cost. But for any given any contract, Manager *C* is still a risk averse individual; Manager *C*'s portfolio choice behavior is similar to the Principal if the Principal were to have direct access to his desired strategy pair. Hence, Manager *C* also prefers portfolio choices that generates a high portfolio mean return and low portfolio volatility to generate an optimal mean-variance trade-off for his performance

¹For instance, the standard references of Laffont and Martimort (2001) and Bolton and Dewatripont (2004).

²In the model, we assume that volatilities of all investment strategies are equivalent and hence the risk aversion term will not even appear in the incentive compatibility constraints for decentralized delegation. But from the model, it is evident this will be true when the strategies have different volatilities.

fees. Thus, Manager *C* acts like a quasi-Principal for any given contract, and while private costs certainly affect the contracting environment, they only play a second order effect in centralization. In contrast, in decentralization, Manager *A* and Manager *B* are restricted in their investment opportunity sets. So when Manager *A* or Manager *B* consider a deviation, the private costs play a first order effect as in standard principal-agent theories. Indeed, when private costs are sufficiently high for Manager *A* and Manager *B* to implement the Principal's desired pair of strategies, a contract may fail to exist; in contrast, for those same high private cost levels, a contract may still exist for centralized delegation.

In all, under static delegation, the relaxed investment opportunity set in centralized delegation and the restricted investment opportunity set in decentralized delegation is the critical source of difference between the contracting environment of centralization and decentralization. And indeed, this trickles down to why Managers' risk aversions and Managers' private costs have different implications to the contracting environments.

The above discussion illustrates various trade-offs between centralization versus decentralization. However, there is one important special case, which has critical implications for risk management practices for the Principal, whereby decentralization is surely worse than centralization. In decentralization, again, linear contracts are offered over Manager *A*'s and Manager *B*'s respective strategy's net returns. In determining a deviation, Manager *A* and Manager *B* are only concerned with the mean return and volatility differences between the Principal's desired strategy and the deviant strategy within their own asset class. Indeed, these differences are the benefits to Managers for compliance. Nonetheless, the Principal still wants a particular strategy pair from each of the asset classes to be implemented because it correlates favorably with some in-situ background investments that the Principal already holds.³ However, suppose if these differences are small, then the benefits to the Managers for compliance are small, but the private costs for implementing the Principal's desired strategy remain high. In this case, the Managers will surely deviate and thus, no contract will exist to implement the Principal's pair of desired strategies in decentralization. Fundamentally, this is because only the Principal can capture the diversification benefits of the two asset classes, while Managers completely ignore these benefits when considering a deviation because one Manager's compensation does not depend on another Manager's strategy. Effectively, we need a contracting mechanism that only depends on the Managers' own strategy returns, and nothing else, to link the Managers' wealths despite the presence of moral hazard.

³ Continuing from the previous example with the Asian asset class, one could argue that an Asian macro strategy could have a mean return that is only marginally higher than that of the Asian passive equities index. And likewise for the European asset class. However, the broad passive Asian and European passive indices will tend to have large index weights into constituent members that are conglomerates with global business operations. In contrast, suppose Managers construct the Asian and European macro strategies to have investments in firms whose business scopes are largely confined to their geographies. Furthermore, suppose the Principal already has some background in-situ investments in some developing economies. Then investing an extra marginal unit of wealth into a broad Asian and European index may correlate less favorably with the developing economies due to the presence of those conglomerate firms in those passive indices. In contrast, investing into those Asian and European macro funds, even if they have low excess mean returns over their respective indices, would correlate more favorably with the investments in the developing economies.

Dynamic delegation

A dynamic delegation model with committed reinvestment is a possible solution. The key idea here is through committed reinvestments, the Principal's intermediate wealth becomes an "intertemporal glue" that links the terminal wealths between Manager *A* and Manager *B*.

Suppose Manager *A*'s and Manager *B*'s strategy choice from their respective asset classes are chosen and committed to at the initial contracting date $t = 0$. Once the strategies have been committed, subsequent per-period returns will be generated from this strategy only.⁴ Furthermore, the Principal also commit to future portfolio policies and contracts. All individuals have mean-variance preferences over terminal wealths at $t = 2$, and there is no intermediate consumption. The Principal allocates portions of his initial wealth to Manager *A* and Manager *B* at $t = 0$, then subsequently, one-period returns are generated and performance fees are collected at $t = 1$. Then also at $t = 1$, the Principal collects all the returns from the Managers and aggregates them into a single pot of intermediate wealth. From this pot of intermediate wealth, the Principal reinvests (quite possibly different proportions than that of $t = 0$) into Manager *A* and Manager *B* at $t = 1$. Finally, at $t = 2$, one-period returns are generated and performance fees are collected by the Managers. But then $t = 2$ terminal wealths of Manager *A* and Manager *B* will depend on the portfolio weights and performance fees the Principal had allocated to them at $t = 1$, which then depends on the level of intermediate wealth that the Principal had at $t = 1$ available for reinvestment. Furthermore, the intermediate aggregated wealth at $t = 1$ depends on the investment strategies that Manager *A* and Manager *B* had committed to at $t = 0$. Thus, using the Principal's intermediate aggregated wealth as an "intertemporal glue", Manager *A*'s and Manager *B*'s $t = 2$ terminal wealth will depend on each other's committed strategies at $t = 0$. Moreover, through this "intertemporal glue", if one Manager deviates from the Principal's desired investment strategy choices at $t = 0$, it could potentially hurt both of them simultaneously at $t = 2$. Note that in this mechanism, the fees the Principal pays to Manager *A* and Manager *B* are still dependent only on the respective Manager's strategy returns. To have a comparison against dynamic decentralized delegation, we also consider the dynamic model for centralized delegation whereby the Principal reinvests back into Manager *C* in an analogous fashion.

Overview

A literature review is in Section 3.2. We introduce the model setup in Section 3.3. The first best results are in Section 3.4. The second best results, which are the core contributions of the paper, are in Section 3.5. From there, we motivate and discuss the dynamic decentralized delegation model in Section 3.6. The first best results for the dynamic decentralized delegation are in Section 3.7, while the second best results are in Section 3.8. All proofs are

⁴ For instance, if an Asian macro strategy was chosen at $t = 0$, the subsequent period returns (from $t = 0$ to $t = 1$, and $t = 1$ to $t = 2$) will be from this Asian macro strategy.

in the Appendix. Moreover, we also have an Online Appendix for additional results for the dynamic centralized model. All the proofs and additional results on the dynamic models are also in the Online Appendix. We conclude in Section 3.9.

3.2 Literature Review

Sharpe (1981) is the seminal paper that coined the term “decentralized investment management”. In particular, he argues an investor would prefer decentralization over centralization for “diversification of style” and “diversification of judgment”.⁵ But beyond these two reasons, Barry and Starks (1984) also add that risk sharing is another motive for preferring decentralization over centralization. Elton and Gruber (2004) recognize that decentralized delegation is a very real issue faced by practitioners and offer conditions under which “a central decision maker can make optimal decisions without requiring decentralized decision makers to reveal estimates of security returns”. More recently, van Binsbergen, Brandt, and Koijen (2008) study the decentralization problem in continuous time and derive the optimal wealth that the investor should allocate between decentralized managers. None of the references above have studied a moral hazard problem of any form. The goal of our paper is to study the similarities and differences of centralization versus decentralization under an explicit moral hazard problem, whereby both the contract and the portfolio policies are endogenously determined.

Our problem clearly belongs to the vast literature of delegated portfolio management. Stracca (2006) offers a survey on the theory findings of delegated portfolio management. The problem of moral hazard in delegated portfolio management, but only to delegation of a *single* agent, has been studied at least since Bhattacharya and Pfleiderer (1985) and ?. These papers usually information based, whereby the principal delegates to an agent because the agent can exert private costly effort to acquire a signal of the future value of a security. Instead, in our paper, we do not take the private costly information acquisition route and rather assume that the principal delegates because the principal has access restrictions to the financial markets. A recent paper by He and Xiong (2013) also assumes the principal has restricted access to the financial markets, and delegates to an agent who will both acquire a signal about an asset’s future return, and also make an investment decision. Our model is not about costly private information acquisition. Nonetheless, the authors reach a similar conclusion that if there is too much flexibility in what an agent can do — like our single Manager C in centralized delegation — it will be more costly to the principal to induce the agent for correct decisions. Indeed, like our paper, the inability of the principal to contract on the agent’s portfolio choice in He and Xiong (2013) is a critical source of moral hazard. But in our paper, the portfolio choice dimension will form a critical difference between

⁵ While Sharpe (1981) notes that some of these concepts were already discussed in Rosenberg (1977) and Rudd and Rosenberg (1980), we view that Sharpe (1981) makes the ideas more transparent and offers a clearer call for research directions.

centralization versus decentralization, and moreover, we show that there are important cases where centralization is indeed preferred over decentralization.

The empirical question of whether moral hazard is present in investment managers is a subject of substantial research. Although our paper is not specific to the type of funds being delegated to, the prototypical example we have in mind is hedge funds. Getmansky, Lee, and Lo (2015) and Agarwal, Mullally, and Naik (2015) are recent survey papers of the hedge fund industry. In particular, strong empirical evidence suggests that moral hazard is a substantial concern in hedge fund. Patton (2009a) argue that a quarter of funds that advertise themselves as “market neutral” have significant exposures to the market factor. Brown, Goetzmann, Liang, and Schwarz (2008, 2012) and Brown, Goetzmann, Liang, and Schwarz (2009) argue that proper due diligence to the extent of reducing operational risks of hedge funds is a source of alpha. Bollen and Pool (2012) constructs several performance flags based on hedge fund return patterns as indicators of increased fraud risks.

There is a small but growing empirical literature on comparing the effectiveness of centralized versus decentralized delegation. Blake, Rossi, Timmermann, Tonks, and Wermers (2013) document that pension fund managers have gravitated from a centralized delegation model to a decentralized delegation model. In the context of mutual funds, Dass, Nanda, and Wang (2013) compare the performance of sole- and team-managed balanced funds. Similarly, Kacperczyk and Seru (2012) ask whether centrally managed or decentrally managed mutual funds perform better.

Our model also fits into the broad literature of optimal delegation forms. The recent work by Gromb and Martimort (2007) discuss the optimal design of contracts for experts who can privately collect a signal. The paper there focuses on risk neutral individuals with limited liability and economies of scale of private costs. Whereas in this paper, we explicitly focus on how risk aversion can play a critical role in portfolio choice, and there is no economies of scale in private costs. Some key earlier work on delegation to multiple agents are Demski and Sappington (1984), Demski, Sappington, and Spiller (1988) and Holmström and Milgrom (1991), but these papers do not explicitly consider the issue of portfolio choice and access to financial markets in the moral hazard formulation.

3.3 Static Model Setup

Individuals, Assets and Moral Hazard

There are two time periods $t = 0$ and $t = 1$. There are two classes of individuals: a single *Principal* and three *Managers* A , B and C . The Principal is initially endowed with \$1 unit of wealth, and Managers have \$0 initial wealth. Both the Principal and the Managers have mean-variance preferences over their own terminal wealth. The Principal has a risk aversion parameter of $\eta_P > 0$, while the Managers have a risk aversion parameter of $\eta_M > 0$.

There are two risky asset classes, indexed by θ and τ . Within each asset class, there are two specific investment strategies $\{H, L\}$. Thus, for asset class θ , the specific investment

strategies are $\{\theta_H, \theta_L\}$, and for asset class τ , they are $\{\tau_H, \tau_L\}$. We denote the net return of any particular investment strategy to be R_i for $i \in \{\theta_H, \theta_L, \tau_H, \tau_L\}$.

The Principal has no access to the financial markets and must delegate to the Managers for access. Here, we will make an assumption on the investment strategy the Principal strictly prefers from each asset class. Please also see Remark 3.3.3 for a discussion of the importance and restrictions of this assumption.

Assumption 3.3.1. *The Principal has a strict preference to implement the strategy pair (θ_H, τ_H) over any other strategy pairs.*

Motivated by Assumption 3.3.1, we will call the “H” investment strategies to be *compliant*⁶, and the “L” strategies to be *deviant*⁷. Likewise, we will call the strategy pair (θ_H, τ_H) to be the *compliant* strategy pair, and call any strategy pair $(\theta, \tau) \in \mathcal{S}_{-(\theta_H, \tau_H)}$ to be *deviant* strategy pairs. As a concrete example, we may think of θ as the Asian equities asset class and τ as European equities. Then θ_H can represent an active Asian macro equities strategy, while θ_L is a passive Asian market index. Analogously, the τ_H strategy can represent an active European macro equities, while τ_L can represent a passive European market index.

With some abuse of notations, we will also use θ and τ to index the investment strategies under their respective asset classes θ and τ . Thus, we will write $\theta \in \{\theta_H, \theta_L\}$ to denote θ is an investment strategy from $\{\theta_H, \theta_L\}$ of the asset class θ . Analogous comments for the expression $\tau \in \{\tau_H, \tau_L\}$. And we will write R_θ to denote the net return of a strategy $\theta \in \{\theta_H, \theta_L\}$ in the asset class θ . Again, analogous comments for the notation R_τ for $\tau \in \{\tau_H, \tau_L\}$. Thus, the set of all possible strategy pair combinations from these two asset classes is $\mathcal{S} := \{(\theta_H, \tau_H), (\theta_H, \tau_L), (\theta_L, \tau_H), (\theta_L, \tau_L)\}$. We will denote the set of strategy pairs that exclude the compliant pair as $\mathcal{S}_{-(\theta_H, \tau_H)} := \mathcal{S} \setminus \{(\theta_H, \tau_H)\}$.

For each asset class, the Managers can privately choose the investment strategy and they incur a private cost for implementing the Principal’s desired strategies. The private cost structure for choosing (θ, τ) is,

$$c(\theta) = \begin{cases} c > 0, & \theta = \theta_H, \\ 0, & \theta = \theta_L \end{cases} \quad \text{and} \quad c(\tau) = \begin{cases} c > 0, & \tau = \tau_H, \\ 0, & \tau = \tau_L. \end{cases} \quad (3.3.1)$$

We may think of the source of this private as “effort”, in that the Managers need to exert private costs to actively manage a more complex investment strategy for any given asset class.⁸

We will, respectively, denote the means and variances of $\theta \in \{\theta_H, \theta_L\}$ as, $\mu_\theta := \mathbb{E}[R_\theta]$, $\sigma_\theta^2 := \text{Var}(R_\theta)$, with analogous notations for $\tau \in \{\tau_H, \tau_L\}$. And we will denote the correlations of the pairs (θ, τ) as $\rho_{\theta\tau} := \text{Corr}(R_\theta, R_\tau)$, for $(\theta, \tau) \in \mathcal{S}$.

⁶ The term *compliant* refers to the strategies that the Principal strictly prefers the Managers to implement.

⁷ The term *deviant* refers the strategies that the Principal strictly prefers the Managers to *not* implement.

⁸ Here, we have assumed that both asset classes θ and τ have identical private costs but this can be readily relaxed without affecting the qualitative results.

We make the following assumptions on the moments of the investment strategies.

Assumption 3.3.2. *Assume that,*

1. *The compliant strategies have identical means⁹, $\mu \equiv \mu_{\theta_H} = \mu_{\tau_H}$. Also, compliant strategies have higher means than the deviant ones,*

$$\begin{aligned}\Delta\mu_\theta &:= \mu_{\theta_H} - \mu_{\theta_L} = \mu - \mu_{\theta_L} > 0, \\ \Delta\mu_\tau &:= \mu_{\tau_H} - \mu_{\tau_L} = \mu - \mu_{\tau_L} > 0.\end{aligned}$$

2. *The volatilities of all investment strategies are identical,¹⁰*

$$\sigma^2 \equiv \sigma_\theta^2 = \sigma_\tau^2, \quad \text{for all } \theta, \tau.$$

3. *No perfect correlations between the investment strategies,*

$$|\rho_{\theta\tau}| < 1, \quad (\theta, \tau) \in \mathcal{S}.$$

Remark 3.3.3. *Assumption 3.3.1 is a critical assumption of the paper, and it is clearly done with some loss of generality. Furthermore, the assumption has both technical and economic content. From a technical perspective, and as we shall see in the next section, this assumption simplifies the objective function of the Principal. Without this assumption, the Principal will need to cycle through all four possible strategy pairs $(\theta, \tau) \in \mathcal{S}$ to compute which pair yields the highest value function for himself. This exercise will then exclusively depend on the parameter values. While this is not difficult to do from a technical perspective, it is not particularly economically insightful. Furthermore, the parameter conditions to ensure that (θ_H, τ_H) is the optimal pair is also not economically interesting.*

Economically, however, this assumption can be motivated in one of the two following ways. Firstly, this assumption may be justified if the Principal has some in place background investments that correlate favorably with the compliant pair (θ_H, τ_H) . Thus, he wants to delegate to Managers that will implement (θ_H, τ_H) for him.

Secondly, this motivation can be economically justified if the Principal actually has partial access to the financial markets. Suppose the Principal can actually directly and costlessly access both of the deviant strategies of each asset class, so θ_L and τ_L . Continuing from the opening example with Asian equities and European equities, θ_L would represent a passive Asian index and τ_L would represent a passive European index that the Principal could access directly without delegation. Thus, the Principal would only want to delegate to implement his preferred strategy pair (θ_H, τ_H) , which are, respectively, the Asian macro and European macro strategies from the opening example. Thus, for instance, if the parameters are such that the

⁹ The equivalent means assumption can be easily relaxed at the expense of more complicated expressions of the results.

¹⁰ The equivalent volatility assumption can be easily relaxed at the expense of more complicated expressions of the results.

strategy pair (θ_H, τ_L) yields higher value for the Principal than (θ_H, τ_H) , then the Principal only needs one outside Manager to manage the asset class θ ; analogous comments also apply for the other deviant strategy pairs. If this were the case, we would have no meaningful discussion of centralized versus decentralized delegation as in our context.

Thus, for the remainder of the paper, Assumption 3.3.1 is strictly enforced.

Delegation forms

In the presence of moral hazard over investment strategy choices for each asset class, how should the Principal delegate? For the rest of the paper, we will focus on two forms of delegation — *centralized* delegation and *decentralized* delegation. In all forms of delegation, the Principal will offer a *linear contract* over the portfolio’s net returns.

Centralized delegation

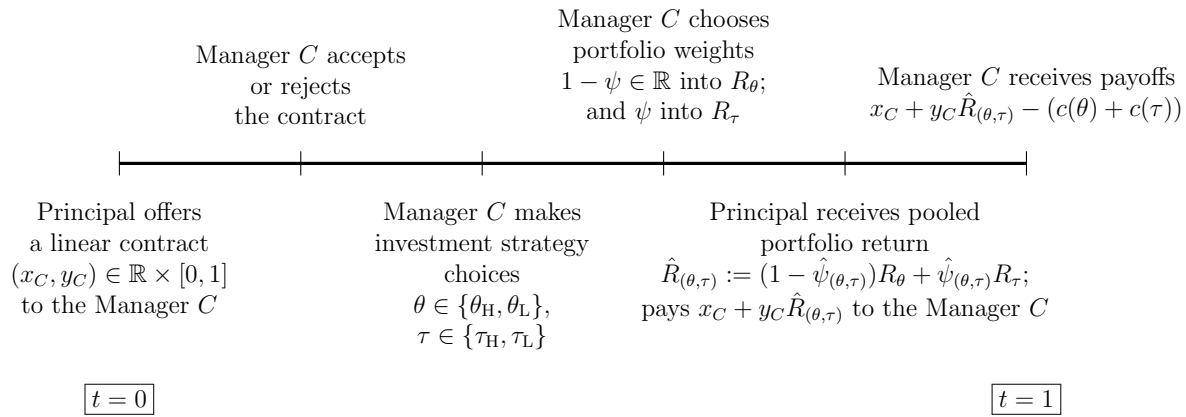


Figure 3.1: Centralized delegation timeline. Manager C has a strategy choice from each asset class, and also has a portfolio choice between those selected pair of strategies. The Principal only has a contract design choice.

In *centralized delegation*, the Principal delegates all initial wealth to a single Manager C . In the previous example, Manager C can be a global strategy manager who manages both the Asian and European asset class. Manager C will be responsible for managing both asset class θ and τ . Given any contract, Manager C will have both a strategy choice and a portfolio choice. Firstly, from each of the two asset classes, Manager C will pick a strategy $\theta \in \{\theta_H, \tau_H\}$ and a strategy $\tau \in \{\tau_H, \tau_L\}$. Secondly, for each chosen strategy pair (θ, τ) , Manager C will pick portfolio weight $1 - \psi$ into strategy θ , and portfolio weight ψ into strategy τ . The resulting portfolio $(1 - \hat{\psi}_{(\theta, \tau)}, \hat{\psi}_{(\theta, \tau)})$ will have a net return $\hat{R}_{(\theta, \tau)}$. In return for Manager C 's services, the Principal offers a linear contract $x_C + y_C \hat{R}_{(\theta, \tau)}$ over the portfolio net return, where $(x_C, y_C) \in \mathbb{R} \times [0, 1]$. Thus, x_C is a *fixed (percentage) fee*, while y_C is a *performance (percentage) fee*. See Figure 3.1 for a timeline.

Thus, the optimization problem under centralized delegation is as follows.

$$\sup_{(x_C, y_C) \in \mathbb{R} \times [0, 1]} \mathbb{E}[W_{cP}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_{cP}^{(\theta_H, \tau_H)}), \quad (\text{Cen})$$

subject to,

$$W_{cP}^{(\theta, \tau)} := 1 + \hat{R}_{(\theta, \tau)} - (x_C + y_C \hat{R}_{(\theta, \tau)}), \quad (3.3.2a)$$

$$W_C^{(\theta, \tau)} := -(c(\theta) + c(\tau)) + x_C + y_C \hat{R}_{(\theta, \tau)}, \quad (3.3.2b)$$

$$\tilde{W}_C^{(\theta, \tau)} := -(c(\theta) + c(\tau)) + x_C + y_C ((1 - \psi)R_\theta + \psi R_\tau), \quad (3.3.2c)$$

$$\hat{\psi}_{(\theta, \tau)} := \arg \sup_{\psi \in \mathbb{R}} \mathbb{E}[\tilde{W}_C^{(\theta, \tau)}] - \frac{\eta_M}{2} \text{Var}(\tilde{W}_C^{(\theta, \tau)}), \quad (3.3.2d)$$

$$\hat{R}_{(\theta, \tau)} := (1 - \hat{\psi}_{(\theta, \tau)})R_\theta + \hat{\psi}_{(\theta, \tau)}R_\tau, \quad (3.3.2e)$$

$$0 \leq \mathbb{E}[W_C^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}(W_C^{(\theta_H, \tau_H)}), \quad (3.3.2f)$$

$$(\theta_H, \tau_H) = \arg \max_{(\theta', \tau') \in \mathcal{S}} \mathbb{E}[W_C^{(\theta', \tau')}] - \frac{\eta_M}{2} \text{Var}(W_C^{(\theta', \tau')}). \quad (3.3.2g)$$

In (Cen), the Principal maximizes his mean-variance utility over terminal wealth (3.3.2a), which is equal to the return from the Manager C managed portfolio, less the fees that the Principal pays. In the maximization, the Principal needs to pick the optimal fixed fees x_C and optimal performance fees y_C as part of the linear contract. Given the linear contract, Manager C will construct the optimal portfolio, as in (3.3.2d), out of the two strategies, one each from the two asset classes, to obtain the portfolio returns in (3.3.2e). In return for Manager C 's service, his terminal wealth is (3.3.2b). The contract must be such that Manager C is willing to participate and so (3.3.2f) is Manager C 's individual rationality constraint. In the second best case, the Principal's desired strategy pair (θ_H, τ_H) must also be incentive compatible for Manager C , which is (3.3.2g).

Decentralized delegation

In *decentralized delegation*, the Principal delegates wealth to two different individuals, Manager A and Manager B . Manager A is responsible for only managing asset class θ , and Manager B is responsible for only managing asset class τ . Thus, following the earlier example, Manager A is an Asian asset class manager, while Manager B is an European asset class manager. The Principal will allocate $1 - \pi$ portion of his initial wealth to Manager A and π proportion to Manager B . In return for the two individuals services, the Principal will offer a linear contract $(1 - \pi)(x_A + y_A R_\theta)$ to Manager A , and $\pi(x_B + y_B R_\tau)$ to Manager B , where $(x_A, y_A), (x_B, y_B) \in \mathbb{R} \times [0, 1]$.¹¹ Thus, x_A, x_B represent the *fixed (percentage)*

¹¹ To actually have a feasible contract, we actually require that $\pi \geq 0$ and $1 - \pi \geq 0$. Else, without this requirement, the Principal could demand infinitely large claw back payments from the two Managers. We will see in the subsequent that these conditions do not bind in the presence of the individual rationality constraints.

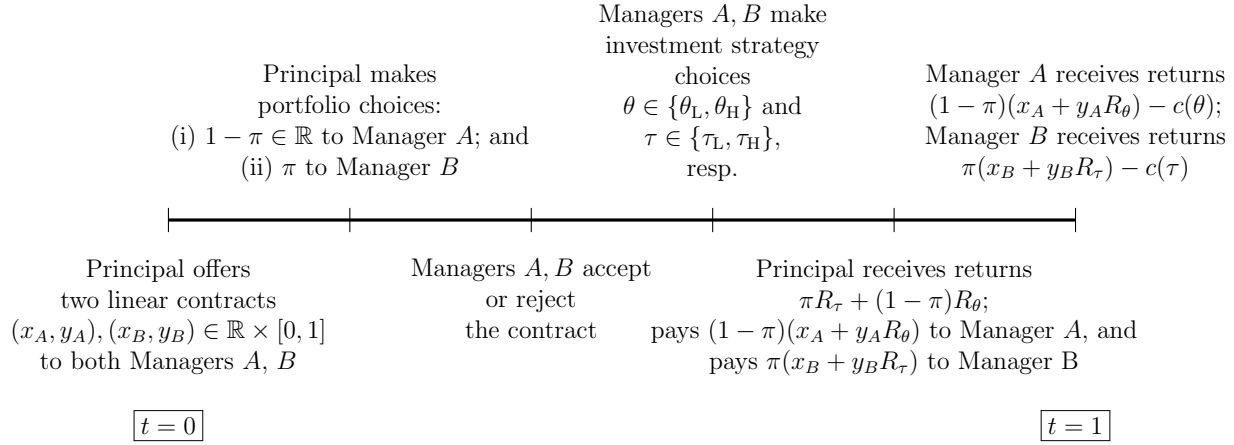


Figure 3.2: Decentralized delegation time line. In contrast to centralized delegation of Figure 3.1, the Principal now has both contract design choice and portfolio choice. Manager A and Manager B only have strategy choices within their own asset class.

fees for, respectively, Manager A and Manager B , whereas y_A, y_B represent the *performance (percentage) fees*. Please see Figure 3.2 for the time line.

Thus, the optimization problem under decentralized delegation is as follows.

$$\sup_{(x_A, y_A), (x_B, y_B) \in \mathbb{R} \times [0, 1]} \sup_{\pi \in [0, 1]} \mathbb{E}[W_P^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_P^{(\theta_H, \tau_H)}) \quad (\text{Dec})$$

subject to,

$$W_P^{(\theta, \tau)} := 1 + \pi R_\tau + (1 - \pi)R_\theta - \pi(x_B + y_B R_\tau) - (1 - \pi)(x_A + y_A R_\theta) \quad (3.3.3a)$$

$$W_A^\theta := (1 - \pi)(x_A + y_A R_\theta) - c(\theta) \quad (3.3.3b)$$

$$W_B^\tau := \pi(x_B + y_B R_\tau) - c(\tau) \quad (3.3.3c)$$

$$0 \leq \mathbb{E}[W_A^{\theta_H}] - \frac{\eta_M}{2} \text{Var}(W_A^{\theta_H}), \quad \text{and} \quad 0 \leq \mathbb{E}[W_B^{\tau_H}] - \frac{\eta_M}{2} \text{Var}(W_B^{\tau_H}) \quad (3.3.3d)$$

$$\theta_H = \arg \max_{\theta' \in \{\theta_H, \theta_L\}} \mathbb{E}[W_A^{\theta'}] - \frac{\eta_M}{2} \text{Var}(W_A^{\theta'}) \quad (3.3.3e)$$

$$\tau_H = \arg \max_{\tau' \in \{\tau_H, \tau_L\}} \mathbb{E}[W_B^{\tau'}] - \frac{\eta_M}{2} \text{Var}(W_B^{\tau'}) \quad (3.3.3f)$$

The Principal's objective (Dec) is to pick the optimal linear contracts to compensate the two Managers, and also to pick the optimal portfolio policy to decide how much wealth to allocate to the Managers' strategies. The Principal's terminal return (3.3.3a) is equal to the portfolio $(1 - \pi, \pi)$ that the Principal decides to allocate to Manager A and B 's strategy returns (R_θ, R_τ) , less the fees owed. Both (3.3.3b) and (3.3.3c) represent, respectively, Manager A and Manager B 's terminal wealth. The two Managers' participation constraints are in (3.3.3d). To induce Manager A and Manager B to pick the Principal's desired strategy pair (θ_H, τ_H) , the Managers' incentive compatibility constraints are in (3.3.3e) and (3.3.3f).

3.4 First Best

Let us begin by considering the first best setup, whereby the Principal can directly observe and contract on the private investment strategy choices of the Managers.

Centralized Delegation in First Best

For the first best centralized delegation case, consider problem (Cen) without the incentive compatibility constraint (3.3.2g).

Proposition 3.4.1. *Consider the first best centralized delegation problem; that is, problem (Cen) without the incentive compatibility constraint (3.3.2g). Fix any strategy pair $(\theta, \tau) \in \mathcal{S}$.*

(a) *Given any linear contract $(x_C, y_C) \in \mathbb{R} \times [0, 1]$, the optimal portfolio weight to strategy τ of Manager C is,¹²*

$$\hat{\psi}_{(\theta, \tau)} = \frac{1}{2} \left(1 + \frac{1}{y_C} \frac{\mu_\tau - \mu_\theta}{\eta_M \sigma^2 (1 - \rho_{\theta\tau})} \right). \quad (3.4.1)$$

(b) *For any given contract (x_C, y_C) , the mean and variance of the portfolio return are given by,*

$$\begin{aligned} \mathbb{E}[\hat{R}_{(\theta, \tau)}] &= \frac{1}{y_C} \frac{(\mu_\theta - \mu_\tau)^2}{2\eta_M \sigma^2 (1 - \rho_{\theta\tau})} + \frac{\mu_\theta + \mu_\tau}{2}, \\ \text{Var}(\hat{R}_{(\theta, \tau)}) &= \frac{1}{y_C^2} \frac{(\mu_\theta - \mu_\tau)^2}{2\eta_M^2 \sigma^2 (1 - \rho_{\theta\tau})} + \sigma^2 (1 + \rho_{\theta\tau}). \end{aligned}$$

(c) *The optimal fixed and performance fees are, respectively,*

$$\hat{x}_C((\theta, \tau), y_C) = (c(\theta) + c(\tau)) - y_C \mathbb{E}[\hat{R}_{(\theta, \tau)}] + \frac{\eta_M}{2} y_C^2 \text{Var}(\hat{R}_{(\theta, \tau)}), \quad \text{for any } y_C \in [0, 1] \quad (3.4.2)$$

$$\hat{y}_C^{FB} = \frac{\eta_P}{\eta_P + \eta_M},$$

and so under the optimal contract, the optimal portfolio is,

$$\hat{\psi}_{(\theta, \tau)} = \hat{\psi}_{(\theta, \tau)}(\hat{y}_C^{FB}) = \frac{1}{2} \left(1 + \frac{\eta_P + \eta_M}{\eta_P \eta_M} \frac{\mu_\tau - \mu_\theta}{1 - \rho_{\theta\tau}} \right).$$

¹²If one needs the value of $\hat{\psi}_{\theta\tau}(y_C)$ at $y_C = 0$, we will define it via the limit; that is, $\hat{\psi}_{(\theta, \tau)}(0) := \lim_{y_C \downarrow 0} \hat{\psi}_{(\theta, \tau)}(y)$. However, as we shall see, the optimal performance fee y generically will not be reached at 0 (i.e. due to individual rationality of Manager C), and hence the point 0 is not really of concern. For subsequent expressions in this proposition that involves y_C in the denominator, define it through the same limiting argument.

(d) The Principal's value function for implementing (θ_H, τ_H) ,

$$\mathbb{E}[W_{cP}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_{cP}^{(\theta_H, \tau_H)}) \Big|_{FB} = -2c + 1 + \mu - \frac{1}{4} \frac{\eta_P \eta_M}{\eta_P + \eta_M} \sigma^2 (1 + \rho_{\theta_H, \tau_H}).$$

(e) For any contract (x_C, y_C) , the Manager C's utility for implementing investment strategy pair (θ, τ) is,

$$\begin{aligned} & \mathbb{E}[W_C^{(\theta, \tau)}] - \frac{\eta_M}{2} \text{Var}(W_C^{(\theta, \tau)}) \\ &= -(c(\theta) + c(\tau)) + x_C + \frac{(\mu_\theta - \mu_\tau)^2}{4\eta_M \sigma^2 (1 - \rho_{\theta\tau})} + \frac{1}{2}(\mu_\theta + \mu_\tau)y_C - \frac{1}{4}\eta_M \sigma^2 (1 + \rho_{\theta\tau})y_C^2, \end{aligned}$$

and in particular for (θ_H, τ_H) , it is,

$$\mathbb{E}[W_C^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}(W_C^{(\theta_H, \tau_H)}) = -2c + x_C + \mu y_C - \frac{1}{4}\eta_M \sigma^2 (1 + \rho_{\theta_H, \tau_H})y_C^2.$$

For any given contract, the portfolio weight $\hat{\psi}_{(\theta, \tau)}$ into strategy τ made by Manager C will clearly be independent of the fixed fees x_C and only be dependent on the performance fee y_C . For instance, suppose strategy τ has a higher mean return than strategy θ , so $\mu_\tau > \mu_\theta$ (the converse case is analogous). Then in this case, naturally Manager C will allocate higher portfolio weights $\hat{\psi}_{(\theta, \tau)}$ to strategy τ and less to strategy θ ; and if the strategies have high correlations $\rho_{\theta\tau}$, it induces the Manager C to almost take a “long-short” strategy whereby even more weights are allocated to τ and less are to θ .

Indeed, for any contract (x_C, y_C) and any strategy pair $(\theta, \tau) \in \mathcal{S}$, we observe the expected wealth and risk aversion adjusted wealth volatility for Manager C, after he has chosen the optimal portfolio, are respectively,

$$\begin{aligned} \mathbb{E}[W_C^{(\theta, \tau)}] &= -(c(\theta) + c(\tau)) + x_C + y_C \cdot \underbrace{\left[\frac{1}{y_C} \frac{(\mu_\theta - \mu_\tau)^2}{2\eta_M \sigma^2 (1 - \rho_{\theta\tau})} + \frac{\mu_\theta + \mu_\tau}{2} \right]}_{\mathbb{E}\hat{R}_{(\theta, \tau)}} \\ &= -(c(\theta) + c(\tau)) + x_C + \frac{(\mu_\theta - \mu_\tau)^2}{2\eta_M \sigma^2 (1 - \rho_{\theta\tau})} + \frac{\mu_\theta + \mu_\tau}{2} y_C, \\ \frac{\eta_M}{2} \text{Var}(W_C^{(\theta, \tau)}) &= \frac{\eta_M}{2} \text{Var}(y_C \hat{R}_{(\theta, \tau)}) \\ &= \frac{\eta_M}{2} y_C^2 \cdot \underbrace{\left[\frac{1}{y_C^2} \frac{(\mu_\theta - \mu_\tau)^2}{2\eta_M^2 \sigma^2 (1 - \rho_{\theta\tau})} + \sigma^2 (1 + \rho_{\theta\tau}) \right]}_{\text{Var}(\hat{R}_{(\theta, \tau)})} \\ &= \frac{(\mu_\theta - \mu_\tau)^2}{4\eta_M \sigma^2 (1 - \rho_{\theta\tau})} + \frac{\eta_M}{2} \sigma^2 (1 + \rho_{\theta\tau}) y_C^2. \end{aligned}$$

Firstly, the risk-adjusted long-short trading profits for Manager C will be the term

$$\left(\frac{1}{2} - \frac{1}{4}\right) \frac{(\mu_\theta - \mu_\tau)^2}{\eta_M \sigma^2 (1 - \rho_{\theta\tau})} = \frac{(\mu_\theta - \mu_\tau)^2}{4\eta_M \sigma^2 (1 - \rho_{\theta\tau})},$$

and these trading profits are completely independent of any contract (x_C, y_C) the Principal offers. Secondly, there are contract mean and volatility effects for implementing the strategy pair (θ, τ) . The contract mean effect is $\frac{\mu_\theta + \mu_\tau}{2} y_C$, which is the expected performance fee payoff to Manager C . The contract volatility effect, adjusted for Manager C 's risk aversion, is $\frac{\eta_M}{2} \sigma^2 (1 + \rho_{\theta\tau}) y_C^2$. In particular, for any contract and any strategy pair, the expected wealth of Manager C is $\mathbb{E}[W_C^{(\theta, \tau)}]$ and its risk adjusted wealth volatility is $\frac{\eta_M}{2} \text{Var}(W_C^{(\theta, \tau)})$. However, the performance fees y_C only enter through the contract mean and contract volatility effect as discussed above, but is independent of the long-short trading profits for Manager C . These effects are related to Manager C 's "relaxed investment opportunity set" and we will have more to say about incentivization in Section 3.5 when moral hazard is present.

The optimal fixed fees \hat{x}_C are to simply compensate for Manager C 's private costs for taking on investment strategy pairs (θ, τ) , less Manager C 's share of the returns, and plus a volatility adjustment. The performance fees \hat{y}_C is equal to the ratio of Principal's risk aversion η_P over the sum of both the Principal and Manager C 's risk aversion $\eta_P + \eta_M$. This performance fee form is directly from optimal risk sharing of the linear contract form.

Decentralized Delegation in First Best

Next, we consider the first best decentralized delegation case. That is, consider the problem (Dec) without the incentive compatibility constraints (3.3.3e) and (3.3.3f).

Proposition 3.4.2. *Consider the first best centralized delegation problem; that is, problem (Dec) without the incentive compatibility constraints (3.3.3e) and (3.3.3f). For any investment strategy θ, τ , define the quantities:*

$$\begin{aligned} \pi_{(\theta, \tau)}^o &:= \frac{1}{2} \left[1 + \frac{(\mu_\tau - \mu_\theta)(\eta_M + \eta_P(1 - \rho_{\theta\tau}))}{\eta_P \eta_M \sigma^2 (1 - \rho_{\theta\tau})} \right], \\ y_{A, (\theta, \tau)}^o &:= \frac{\eta_P [(\mu_\theta - \mu_\tau)(1 - \rho_{\theta\tau})(\eta_M + \eta_P(1 + \rho_{\theta\tau})) + \eta_P \eta_M \sigma^2 (1 - \rho_{\theta\tau}^2)]}{(\mu_\theta - \mu_\tau) [(\eta_M + \eta_P)^2 - \eta_P^2 \rho_{\theta\tau}^2] + \eta_P \eta_M \sigma^2 (1 - \rho_{\theta\tau})(\eta_M + \eta_P(1 + \rho_{\theta\tau}))}, \\ y_{B, (\theta, \tau)}^o &:= \frac{\eta_P [(\mu_\theta - \mu_\tau)(1 - \rho_{\theta\tau})(\eta_M + \eta_P(1 + \rho_{\theta\tau})) - \eta_P \eta_M \sigma^2 (1 - \rho_{\theta\tau}^2)]}{(\eta_M + \eta_P(1 + \rho_{\theta\tau})) [(\mu_\theta - \mu_\tau)(\eta_M + \eta_P(1 - \rho_{\theta\tau})) - \eta_P \eta_M \sigma^2 (1 - \rho_{\theta\tau})]}. \end{aligned}$$

Then,

(a) For any portfolio π allocated to Manager B and performance fees (y_A, y_B) , the optimal fixed fees are,

$$\hat{x}_A(\theta, \pi, y_A) = \frac{1}{1 - \pi} \left[c(\theta) - (1 - \pi) y_A \mu_\theta + \frac{\eta_M}{2} y_A^2 (1 - \pi)^2 \sigma^2 \right] \quad (3.4.3a)$$

$$\hat{x}_B(\tau, \pi, y_B) = \frac{1}{\pi} \left[c(\tau) - \pi y_B \mu_\tau + \frac{\eta_M}{2} y_B^2 \pi^2 \sigma^2 \right] \quad (3.4.3b)$$

(b) If $(\pi_{(\theta,\tau)}^o, y_{A,(\theta,\tau)}^o, y_{B,(\theta,\tau)}^o) \in (0, 1)^3$, then the optimal portfolio policy and optimal performance fee policy of the Principal for implementing strategy (θ, τ) are

$$(\pi_{(\theta,\tau)}^o, y_{A,(\theta,\tau)}^o, y_{B,(\theta,\tau)}^o).$$

(c) In particular, for implementing (θ_H, τ_H) , the optimal portfolio and performance fee policies are,

$$(\hat{\pi}^{FB}, \hat{y}_A^{FB}, \hat{y}_B^{FB}) = \left(\frac{1}{2}, \frac{\eta_P(1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H})}, \frac{\eta_P(1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H})} \right),$$

and the Principal's value function is,

$$\mathbb{E}[W_P^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_P^{(\theta_H, \tau_H)}) \Big|_{FB} = -2c + \mu - \frac{1}{4} \frac{\eta_P \eta_M \sigma^2 (1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H})}.$$

In decentralization, the Principal will allocate equal amount of wealth into both Manager A and Manager B , and moreover, the performance fees the Principal will pay to them will be equal as well. This result is immediate since from Assumption 3.3.2, we had assumed that the compliant strategy pair (θ_H, τ_H) have identical means and identical volatilities. Unlike the performance fees of centralization in Proposition 3.4.1, where the performance fees are simply $\eta_P/(\eta_P + \eta_M)$, the performance fees in decentralization must take into account the correlations ρ_{θ_H, τ_H} of the strategies. Thus, in centralization, risk management is internalized by the single Manager C , and hence the resulting performance fees only need to depend on the risk aversions of the individuals. However, with decentralization, the Principal must handle risk management himself and thus the performance fees must reflect the correlations of strategies, in addition to the individuals' respective risk aversions.

Risk management defines “the boundaries of a firm”

The first best results of centralized delegation in Proposition 3.4.1 and that of decentralized delegation in Proposition 3.4.2 illustrate the idea that risk management defines the “boundaries of a firm”. Indeed, the boundaries of a firm are central ideas in economics since Coase (1937), and the contemporary resurgence of these ideas can be traced to Williamson (1975, 1985). See Holmström and Roberts (1998) and Williamson (2002) for excellent surveys.

In centralization, the boundary of a “firm” is *fixed and wide*. Here, we interpret a “firm” to be Manager C , and its boundaries to be the activities that Manager C could take on. The Principal delegates all wealth to Manager C , and in this sense, the boundary of a “firm” is fixed. But the boundary of a firm is wide in the sense that given any contract, Manager C handles portfolio choice between the investment strategy pair desired by the Principal. In first best centralization, while the Principal can observe and contract on the investment strategy choices made by Manager C , the Principal cannot contract on the specific portfolio

choices made by Manager C . Thus, risk management is, in effect, non-contractible. In our setup, largely because it will help in analytical tractability for the second best case of Section 3.5, we assume that the strategy pair (θ_H, τ_H) have equivalent mean returns μ and also equivalent volatility σ ; see again Assumption 3.3.2. Again, both the Principal and Manager C have mean-variance preferences. And suppose, hypothetically, the Principal could have direct access to the capital markets. Then both the hypothetical Principal and Manager C will load equal portfolio weights into the two strategies, regardless of their risk aversions η_M and η_P ; in this special case, the Principal and Manager C will identically agree on portfolio choice. However, it is not difficult to see that if the mean returns are different, or that the volatilities are different, Manager C 's portfolio choice would differ to that of the hypothetical Principal. Moreover, even in first best, while the Principal can contract on the strategy pair (θ, τ) Manager C implements, the Principal cannot contract on Manager C 's portfolio choice. Moreover, this difference in the portfolio weight choices between Manager C and this hypothetical Principal stems from the difference of their risk aversion and also the performance fees offered to Manager C . In all, without ability to contract on portfolio choice (or more broadly speaking, risk management), the boundary of Manager C is effectively fixed and wide, and Principal will use contracts to simply optimally risk share.

The discussion for decentralization is far simpler. In decentralization, the boundary of a “firm” is *flexible and narrow*. Manager A and Manager B can only operate within their own asset class, and in this sense, their boundaries are narrow. In contrast to centralization, wealth allocations and hence also risk management, is exclusively handled by the Principal. The amount of wealth the Principal allocates to each Manager thus dictates the influence each Manager has on the Principal's welfare, and it is in this sense that the boundary of a “firm” is flexible.

As we shall see in Section 3.5 where moral hazard is present, these two core differences between the boundary of a “firm” in centralization versus decentralization will have key implications for their optimal contracts and its existence.

Comparison between Centralized Delegation versus Decentralized Delegation in First Best

Now we can compare centralized delegation versus decentralized delegation under first best.

Proposition 3.4.3. *The difference between the Principal's value function in first best decentralized delegation and first best centralized delegation is,*

$$\begin{aligned} & \left(\mathbb{E}[W_P^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_P^{(\theta_H, \tau_H)}) \right) \Big|_{FB} - \left(\mathbb{E}[W_{cP}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}(W_{cP}^{(\theta_H, \tau_H)}) \right) \Big|_{FB} \\ &= \frac{\eta_M \eta_P^2 \rho_{\theta_H, \tau_H} (1 + \rho_{\theta_H, \tau_H}) \sigma^2}{4(\eta_P + \eta_M)(\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H}))}. \end{aligned}$$

Thus, decentralized delegation is better than centralized delegation if and only if the returns of the Principal's desired strategy pair (θ_H, τ_H) are strictly positively correlated $\rho_{\theta_H, \tau_H} > 0$;

conversely, centralized delegation is better than decentralized delegation if and only if the strategies are strictly negatively correlated $\rho_{\theta_H, \tau_H} < 0$; and the two forms of delegation are equivalent when the strategies are uncorrelated $\rho_{\theta_H, \tau_H} = 0$.

When the correlation between the Principal's desired strategy pair (θ_H, τ_H) is strictly negative, $\rho_{\theta_H, \tau_H} < 0$, delegating to a single Manager C is beneficial. Given that Manager C will be putting long positions into both investment strategies θ_H and τ_H , and since Manager C is also risk averse, a strictly negative correlation ρ_{θ_H, τ_H} lowers the contract volatility for Manager C , and thereby it is cheaper for the Principal to risk share with Manager C . But when the correlations become strictly positive, $\rho_{\theta_H, \tau_H} > 0$, the reverse happens, and decentralized delegation is more appealing to the Principal. When the correlations become positive, delegating to a single Manager C actually increases Manager C 's contract volatility, and thereby making it more expensive for the Principal to risk share. However, with decentralized delegation, neither Manager A nor Manager B directly absorb the positive correlation effects. Thus, the Principal, via use his own portfolio choice, can make it cheaper to risk share with the decentralized Managers. And finally, in the case when the strategies are uncorrelated, $\rho_{\theta_H, \tau_H} = 0$, both centralized and decentralized delegation are identical, since neither the centralized or decentralized Managers(s) are affected by the correlation structure directly for the purpose of risk sharing.

3.5 Second Best

We come to one of the core results of the paper. Here, we assume the Principal cannot directly observe nor contract on the specific investment strategies that the Managers choose within each asset class. In both the second best centralized delegation (Proposition 3.5.1) and second best decentralized delegation (Proposition 3.5.2), the key emphasis will be, respectively, the performance fees and the optimal portfolio policies. In contrast, the fixed fees (i.e. x_C in centralization; and x_A, x_B in decentralization) are actually relatively straightforward. In both cases, the optimal fixed fees will ensure the Managers will participate and accept the contract. Furthermore, the fixed fees will compensate the Managers for their private costs, less the expected performance fee amount payoff, plus a volatility adjustment. This fixed fee form is standard in all linear contracting setups, and hence we will focus our paper on the performance fees and the portfolios.

Centralized delegation

Let's first state the second best results for centralized delegation.

Proposition 3.5.1. *Consider the second best centralized delegation problem (Cen). Then:*

- (a) *For any performance fee $y_C \in [0, 1]$, the optimal fixed fees has the same form as that of first best in (3.4.2) of Proposition 3.4.1.*

- (b) For any performance fees $y_C \in [0, 1]$ and investment strategy pair $(\theta, \tau) \in \mathcal{S}$, the optimal portfolio $\hat{\psi}_{(\theta, \tau)}$ chosen by Manager C is the same as the first best form (3.4.1) of Proposition 3.4.1. Indeed, the portfolio weight Manager C will allocate to strategy τ in the strategy pair (θ, τ) is,

$$\begin{aligned}\hat{\psi}_{(\theta_H, \tau_H)} &\equiv \frac{1}{2}, \\ \hat{\psi}_{(\theta', \tau')} &= \frac{1}{2} \left(1 + \frac{1}{y_C} \frac{\mu_{\tau'} - \mu_{\theta'}}{\eta_M \sigma^2 (1 - \rho_{\theta', \tau'})} \right), \quad (\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}\end{aligned}$$

- (c) The (three) incentive compatibility constraints on the performance fees y_C for inducing Manager C to implement the strategy pair (θ_H, τ_H) are,

$$\begin{aligned}-2c + \frac{1}{2}(\mu_{\theta_H} + \mu_{\tau_H})y_C - \frac{1}{4}\eta_M \sigma^2(1 + \rho_{\theta_H, \tau_H})y_C^2 \\ \geq -(c(\theta') + c(\tau')) + \frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2(1 - \rho_{\theta', \tau'})} + \frac{1}{2}(\mu_{\theta'} + \mu_{\tau'})y_C - \frac{1}{4}\eta_M \sigma^2(1 + \rho_{\theta', \tau'})y_C^2,\end{aligned}\tag{3.5.1}$$

for $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. The three incentive compatibility constraints (3.5.1) can be equivalently written as a single constraint,

$$\begin{aligned}0 \geq \max_{(\theta', \tau')} \left\{ -(c(\theta') + c(\tau') - 2c) + \frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2(1 - \rho_{\theta', \tau'})} \right. \\ \left. + \frac{1}{2}(\mu_{\theta'} - \mu_{\theta_H} + \mu_{\tau'} - \mu_{\tau_H})y_C - \frac{1}{4}\eta_M^2 \sigma^2(\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H})y_C^2 \right\},\end{aligned}\tag{3.5.2}$$

where the maximum is taken over the possible deviant strategy pairs $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. Recall from Assumption 3.3.2, $\mu \equiv \mu_{\theta_H} = \mu_{\tau_H}$.

- (d) If the net cost for Manager C to comply and implement the Principal's desired strategy pair (τ_H, τ_H) is sufficient low, the Principal will achieve first best. That is, if

$$\begin{aligned}0 > \max_{(\theta', \tau')} \left\{ -(c(\theta') + c(\tau') - 2c) + \frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2(1 - \rho_{\theta', \tau'})} \right. \\ \left. + \frac{1}{2} \frac{\eta}{\eta + \eta_M}(\mu_{\theta'} - \mu_{\theta_H} + \mu_{\tau'} - \mu_{\tau_H}) - \frac{1}{4} \left(\frac{\eta}{\eta + \eta_M} \right)^2 \eta_M^2 \sigma^2(\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H}) \right\}\end{aligned}\tag{3.5.3}$$

then then the optimal performance fee is $\hat{y}_C = \hat{y}_C^{FB}$.

- (e) Suppose the net costs for compliance to Manager C is sufficiently high; that is, replace $>$ in (3.5.3) with \leq . If an optimal performance fee $\hat{y}_C \in [0, 1]$ exists, there necessarily exists some (unique) pair of deviant strategy pair $(\theta^b, \tau^b) \in \mathcal{S}_{-(\theta_H, \tau_H)}$ that yields the highest net deviation benefit for Manager C. Furthermore, consider the following two conditions on (θ^b, τ^b) .

(i) For any strategy pair $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$, define (i.e. the quadratic discriminant),

$$D_{(\theta', \tau')} := \frac{1}{4} (\mu_{\theta'} - \mu_{\theta_H} + \mu_{\tau'} - \mu_{\tau_H})^2 + \left[- (c(\theta') + c(\tau') - 2c) + \frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2 (1 - \rho_{\theta', \tau'})} \right] \eta_M^2 \sigma^2 (\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H}). \quad (3.5.4)$$

Suppose the strategy pair (θ^b, τ^b) is such that,

$$D_{(\theta^b, \tau^b)} \geq 0.$$

(ii) For the pair (θ^b, τ^b) , define (i.e. the positive quadratic root),

$$\tilde{y}_{+, (\theta^b, \tau^b)} := \frac{1}{2} \left(- \frac{\mu_{\theta^b} - \mu_{\theta_H} + \mu_{\tau^b} - \mu_{\tau_H}}{2} + \sqrt{D_{(\theta^b, \tau^b)}} \right) \times \left[- (c(\theta^b) + c(\tau^b) - 2c) + \frac{1}{4} \frac{(\mu_{\theta^b} - \mu_{\tau^b})^2}{\eta_M \sigma^2 (1 - \rho_{\theta^b, \tau^b})} \right]^{-1}. \quad (3.5.5)$$

Suppose the pair is (θ^b, τ^b) is such that,

$$\tilde{y}_{+, (\theta^b, \tau^b)} \in [0, 1].$$

If both conditions (i) and (ii) hold, then the second best performance fee is $\hat{y}_C = \tilde{y}_{+, (\theta^b, \tau^b)}$. If neither condition (i) nor (ii) hold, then no second best contract will exist for centralized delegation.

The right hand side of the incentive compatibility condition (3.5.2) is the “net cost” for Manager C for being compliant instead of being deviant. Firstly, we have the standard private costs effect: by being compliant and picking (θ_H, τ_H) , Manager C needs to incur private costs of $c(\theta_H) + c(\tau_H) = 2c$, but by deviating to $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$, the private costs are strictly lowered to $c(\theta') + c(\tau')$; hence, $2c - (c(\theta') + c(\tau'))$ represents the net private costs for complying instead of deviating. These effects would be standard in practically all standard principal-agent models. However, there are three additional effects that arise solely because of Manager C 's ability to take an arbitrary contract offered by the Principal, and then subsequently trade upon it.

Secondly, incentive compatibility for Manager C also comes in the form of the net return differences from implementing the compliant investment strategy pair (θ_H, τ_H) versus the deviant pair $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. From implementing the compliant pair and recalling the optimal portfolio choices, Manager C gains an expected performance fee payoff of $(\mu_{\theta_H} + \mu_{\tau_H})y/2 = \mu y$, whereas by implementing a deviant pair, Manager C has the expected performance fee payoff of $(\mu_{\theta'} + \mu_{\tau'})y/2$. But recall for the compliant investment strategies, $\mu = \mu_{\theta_H} > \mu_{\theta'}$ and $\mu = \mu_{\tau_H} > \mu_{\tau'}$. Thus, by being compliant, Manager C enjoys a net gain of $(2\mu - \mu_{\theta'} - \mu_{\tau'})y/2$ in higher performance fee payoffs.

Finally, the Principal wants to incentivize Manager C as cheap as possible, which is equivalent to binding the incentive compatibility constraints with the minimal net costs to Manager C across all possible strategy deviations.

For the remaining terms in the incentive compatibility constraint (3.5.2) we will discuss them in Section 3.5, when we compare centralization versus decentralization. See also Corollary 3.B.1 for the explicit conditions on the parameters under which which strategy pair (θ^b, τ^b) is the most profitable deviation for Manager C .

Decentralized delegation

Next, let's state the second best result for decentralized delegation.

Proposition 3.5.2. *Consider the second best decentralized delegation case; that is consider, problem (Dec) in its entirety.*

- (a) *For any portfolio π and performance fee policies (y_A, y_B) , the optimal fixed fees have the form (3.4.3) of first best decentralization in Proposition 3.4.2.*
- (b) *The incentive compatibility conditions to induce the Principal's desired strategy pair (θ_H, τ_H) are,*

$$0 \geq c - (1 - \pi)y_A\Delta\mu_\theta, \quad (3.5.6a)$$

$$0 \geq c - \pi y_B\Delta\mu_\tau. \quad (3.5.6b)$$

- (c) *Suppose the private costs c are sufficiently high¹³, then the second best decentralized optimal policies are,*

$$(\hat{\pi}, \hat{y}_A, \hat{y}_B) = \left(\frac{1}{2} \left[1 + \frac{\Delta\mu_\tau - \Delta\mu_\theta}{\Delta\mu_\theta\Delta\mu_\tau} c \right], \frac{2\Delta\mu_\tau c}{c(\Delta\mu_\tau - \Delta\mu_\theta) + \Delta\mu_\theta\Delta\mu_\tau}, \frac{2\Delta\mu_\theta c}{c(\Delta\mu_\theta - \Delta\mu_\tau) + \Delta\mu_\theta\Delta\mu_\tau} \right).$$

We will defer discussing the decentralized contracting environment and the incentive compatibility constraints (3.5.6) in the next section, Section 3.5, when we compare centralization versus decentralization.

Comparison between Centralization versus Decentralization

Now, we can compare the similarities and differences in contracting between centralization and decentralization.

¹³ The precise conditions for this are in Proposition 3.B.2(aiv). See also Proposition 3.B.2 for further details of the optimal policies of second best decentralized delegation.

Investment opportunity set

The relaxation or restriction in the investment opportunity set is the key incentive difference between centralization and decentralization.

Let's first consider centralized delegation. Firstly, Manager C enjoys a relaxed investment opportunity set. The term $\frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2 (1 - \rho_{\theta', \tau'})}$ in Manager C 's incentive compatibility constraint (3.5.2) represents the long-short trading benefit for Manager C by deviating to (θ', τ') . Under the compliant strategy pair (θ_H, τ_H) , we had assumed that they have equivalent means μ and also equivalent volatility σ , and thus Manager C would put equal weights into both investment strategies, and hence the optimal portfolio weights would be independent of Manager C 's risk aversion, strategies' volatility σ , and the correlation ρ_{θ_H, τ_H} . In contrast, under the deviant investment strategy pairs $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$, they have potentially different means, and hence the correlation structure $\rho_{\theta', \tau'}$ and the volatility σ contribute to a long-short strategy motive for Manager C ; see again the portfolio form $\hat{\psi}_{(\theta', \tau')}$ of Proposition 3.4.1. This constitutes a benefit for Manager C that is foregone by being compliant, and hence is an opportunity cost for Manager C that the Principal needs to compensate for in the form of higher performance fees.

Secondly, incentive compatibility for Manager C also comes in the form of differences in the contract volatility under the compliant strategy pair and that of deviant strategy pairs. For any investment strategy pair (θ, τ) , the contract volatility for Manager C is $\sigma^2(1 + \rho_{\theta, \tau})y_C^2$. And thus, adjusting for Manager C 's risk aversion, the term $-\frac{1}{4}\eta_M\sigma^2(\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H})y_C^2$ is the net change in contract volatility for Manager C from taking the compliant pair (θ_H, τ_H) versus a deviant pair $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. The signs of the correlations matter. If $\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H} > 0$, that is the deviant strategy $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$ has a strictly higher correlation than the compliant strategy pair, then this represents a net benefit for Manager C ; that is, since Manager C is risk averse, picking the compliant strategy pair with a lower correlation is beneficial, so being compliant reduces the contract volatility. For the reverse case, when $\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H} < 0$, being compliant increases the contract volatility.

Finally, there is an interaction between the contract volatility and the long-short trading benefit for Manager C . On the one hand, a higher correlation $\rho_{\theta', \tau'}$ increases the contract volatility for Manager C when consideration a deviation to (θ', τ') , and is thus detrimental to Manager C . But on the other hand, a higher $\rho_{\theta', \tau'}$ also increases the long-short trading benefit for Manager C , and is thus beneficial for Manager C . These interaction effects, again, are only afforded to Manager C due to his ability to modify the contract via his relaxed investment opportunity set.

In contrast, Manager A and Manager B under decentralization face a far more restricted investment opportunity set as compared to Manager C under centralization. And accordingly, the incentive compatibility constraints of decentralization take on a far more simpler form than centralization. Given any contract, Manager A and Manager B will only care about the mean and volatility differences¹⁴ between the strategies' returns in their respec-

¹⁴ From Assumption 3.3.2, we assumed that all strategies have identical volatility σ . We further discuss

tive asset classes. And these differences in the moments between the Principal's desired strategy versus that of the deviant strategy are the benefits for Manager *A* and Manager *B* for deviation. We will see the restricted investment opportunity set in decentralization will translate to different implications than centralization for the managers' risk aversions and their private costs.

Managers' risk aversions

Related to the investment opportunity set, the Managers' risk aversion also play an opposite role in contracting under centralization versus decentralization.

In centralization, suppose Manager *C* becomes less risk averse, so $\eta_M \downarrow 0$. Firstly, when this happens, Manager *C* becomes less concerned with the volatility difference of the contract $\frac{1}{4}\eta_M^2\sigma^2(\rho_{\theta',\tau'} - \rho_{\theta_H,\tau_H})y_C^2 \rightarrow 0$, for all deviant strategy pairs $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. Secondly, when Manager *C* considers a deviation, Manager *C* cares less about the volatilities of the deviant strategy pairs (θ', τ') and also less of the correlation of their returns $\rho_{\theta',\tau'}$. And indeed, as Manager *C* becomes less risk averse, he only cares about the absolute difference $|\mu_{\theta'} - \mu_{\tau'}|$ between the deviant strategies. In the limit when Manager *C* becomes risk neutral, he will take an infinitely large long position into strategy with highest mean, and take an infinitely large short position into the strategy with the lowest mean. Thus as $\eta_M \downarrow 0$, the long-short trading profit would become infinitely large, $\frac{1}{4}\frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M\sigma^2(1 - \rho_{\theta',\tau'})} \uparrow +\infty$. When this happens, the cost for the Principal to compensate Manager *C* to ensure his compliance will be excessively high, and thus a contract to implement the Principal's desired investment strategy pair (θ_H, τ_H) will fail to exist.

The above result is completely the opposite of standard principal-agent theories.¹⁵ The literature suggests that it should be cheaper for a principal to compensate a less risk averse agent, because of the lower risk premium the principal needs to pay the agent for bearing risk. Here it is the reverse — the less risk averse Manager *C* becomes, the more expensive it is to compensate him. This is again due to the relaxed investment opportunity set in centralized delegation. For any given contract, Manager *C* can simply use the financial markets to modify the intended incentives of the contract.¹⁶

In contrast, in decentralized delegation, as Manager *A* and Manager *B* become less risk averse, an optimal contract may still exist. Indeed in this regard, unlike centralization, decentralization is much closer to a standard principal-(multi)agent problem. In the model, we have assumed that the volatility σ of all investment strategies are equivalent. Thus, the incentive compatibility constraints (3.5.6) under decentralization do not involve the Managers' risk aversion. And even if we were to assume the volatilities of investment strategies are different, it is straightforward to see that the right-hand side of (3.5.6) would simply have additional terms $+\frac{\eta_M}{2}(1 - \pi)^2 y_A^2 (\sigma_{\theta_H}^2 - \sigma_{\theta_L}^2)$ and $+\frac{\eta_M}{2}\pi^2 y_B^2 (\sigma_{\tau_H}^2 - \sigma_{\tau_L}^2)$ for Manager *A*

this implication in Section 3.5 below.

¹⁵ Say Laffont and Martimort (2001) and Bolton and Dewatripont (2004).

¹⁶ Admati and Pfleiderer (1997, Section V) makes a related point that benchmarked compensations are not relevant to soliciting effort.

and Manager B , respectively. Depending on the sign of $\sigma_{\theta_H}^2 - \sigma_{\theta_L}^2$ and $\sigma_{\tau_H}^2 - \sigma_{\tau_L}^2$, the Principal either pays additional fees for increased volatility risk imposed on the Managers, or get savings in fees for decreased volatility risk. Regardless, as $\eta_M \downarrow 0$, we collapse back to our current case of (3.5.6). Thus, given Managers have restricted investment opportunity sets, Managers' risk aversion η_M play the standard role in the usual principal-agent literature under decentralization.

Managers' private costs

Managers' private costs play a differentiating effect on centralization and decentralization.

Decentralized delegation cannot tolerate high levels of private costs c by both Manager A and Manager B before no contract to implement the Principal's desired strategy pair (θ_H, τ_H) can exist. Unlike centralized delegation where the Manager C can take an arbitrary contract and trade it to maximize the risk-return trade-offs for himself first, this is distinctly not the case for decentralized delegation. In decentralized delegation, both Manager A and Manager B have completely dedicated themselves to one particular strategy from their respective asset classes, and cannot further form portfolios to maximize risk-return trade-offs. Thus, although Manager A and Manager B are truly risk averse, from the perspective of incentive compatibility, they behave like risk neutral individuals. That is to say, both Manager A and Manager B only care about the private costs c and also the mean return differences $\Delta\mu_\theta$ and $\Delta\mu_\tau$ between the compliant strategy and the deviant strategy in their own asset class, and do not care about second moment effects of volatility nor correlation and even their own risk aversions.¹⁷ And due to this "risk neutrality" in determining incentive compatibility, contracting with decentralized individuals with high private costs could become prohibitively costly, and so much so that a contract to implement the Principal's desired strategy pair (θ_H, τ_H) could fail to exist.

In contrast, centralized delegation can tolerate a higher level of private costs c before no contract can exist. Given any contract, since Manager C is risk averse, he will pick portfolios that generate a high portfolio mean return and a low portfolio volatility. Indeed, save for the differences in risk aversion levels between the Principal and Manager C , the portfolio choice behavior of Manager C is analogous to that of the Principal, were the Principal to have direct access to the financial markets. Thus, Manager C behaves like a "quasi-Principal" and hence, private costs c only play a second order effect. This is why for moderately high levels of private costs c , the compliant Manager C must pay $2c$ and yet a centralized contract will still exist for Principal to implement his desired strategy pair (θ_H, τ_H) . In sharp contrast, for these same moderately high levels of private costs c , decentralized contracts may fail for Manager A and Manager B .

¹⁷ As discussed earlier, we had assumed all strategies have equivalent volatilities. But it is not difficult to see that even if strategies in each of the asset classes have different volatilities, the fact that private costs c will still play a first order effect in Manager's consideration for deviation in decentralization.

Numerical illustrations

To gain a fuller understanding of the differences and similarities between second best centralized delegation and second best decentralized delegation, we now turn to some numerical illustrations of our results. As one can surmise from Proposition 3.5.1 and Proposition 3.5.2, it is easiest to display these results in a numerical and graphical fashion. Thankfully, despite the perhaps complex structures of the optimal portfolios and fees, and thus extending to their respective Principal's value functions, in that they are often highly nonlinear in the economic parameters of interest, the results are nonetheless rather straightforward to compute numerically; especially since we actually do have closed form analytical answers for all of the results. A more explicit analytical solution to the difference in value functions between centralization and decentralization is available under the extreme case when there is only moral hazard over mean returns; see Section 3.B.

The base parameters that we will use in the numerical illustrations are given in Table 3.1, unless plotted otherwise. In the figures below, we need to distinguish between two different

Principal's risk aversion parameter	η_P	5
Managers' risk aversion parameter	η_M	3
Compliant investment strategies' mean returns, $\mu \equiv \mu_{\theta_H} = \mu_{\tau_H}$	μ	0.25
Mean return on deviant strategy θ_L	μ_{θ_L}	0.10
Mean return on deviant strategy τ_L	μ_{τ_L}	0.08
Volatility of all strategies	σ	0.40
Correlation coefficient of compliant pair (θ_H, τ_H)	ρ_{θ_H, τ_H}	0.20
Correlation coefficient of deviant pair (θ_H, τ_L)	ρ_{θ_H, τ_L}	0.30
Correlation coefficient of deviant pair (θ_L, τ_H)	ρ_{θ_L, τ_H}	0.30
Correlation coefficient of deviant pair (θ_L, τ_L)	ρ_{θ_L, τ_L}	0.30
Managers' private costs	c	0.06

Table 3.1: The base parameter assumptions used in Section 3.5.

types of “better”. The first type is when contracts for implementing (θ_H, τ_H) exist for both centralization and decentralization; the darker colors indicate which form of delegation is better under this circumstance. The second type is when contracts for implementing (θ_H, τ_H) does not exist under one form of delegation, while it does exist for another form of delegation. In this second type, the form of delegation that has contract existence is better, by default; this is indicated by the lighter colors.

In Figure 3.1, we see that high correlation ρ_{θ_H, τ_H} for the compliant strategy pair (θ_H, τ_H) will favor decentralization, while low correlation will favor centralization. This is inherited from the optimal risk sharing result of first best in Proposition 3.4.3. However, in the presence of moral hazard, when ρ_{θ_H, τ_H} is sufficiently high, a centralized contract to implement (θ_H, τ_H) for the Principal will not exist. Recalling the discussion on the investment opportunity set in Section 3.5, for any performance fee $y_C \in [0, 1]$, the term $-\frac{1}{4}\eta_M^2\sigma^2(\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H})y_C^2$ is the difference between the contract volatility for Manager C implementing a deviant strategy

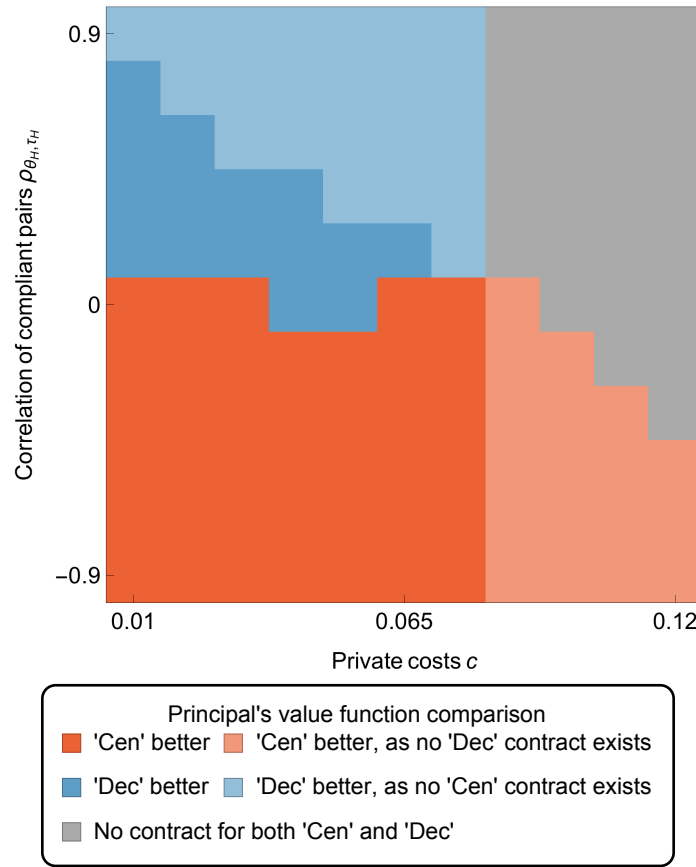


Figure 3.1: Comparing the Principal's value function under centralization versus decentralization: compliant strategy pair correlation ρ_{θ_H, τ_H} versus private costs c .

pair $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$, versus that of the compliant strategy pair (θ_H, τ_H) . As the correlation ρ_{θ_H, τ_H} of the compliant strategy pair increases, Manager C will incur a high contract volatility for being compliant, whereas a low contract volatility for being deviant. Thus, when the correlation ρ_{θ_H, τ_H} is sufficiently high, Manager C will surely deviate for any performance fee y_C to lower the contract volatility for himself, which then leads to nonexistence of a contract in centralization to implement the Principal's desired strategy pair (θ_H, τ_H) . For decentralization, high private costs c will also lead to contract nonexistence for implementing the Principal's desired strategy pair (θ_H, τ_H) ; this effect is as discussed in Section 3.5.

In Figure 3.2, we again see the effects of the relaxed investment opportunity set of Section 3.5 under centralization. In this example, consider the deviant strategy τ_L of the asset class τ (the case for the strategy θ_L of the asset class θ is similar). Recall the long-short opportunity cost under centralization is $\frac{1}{4} \frac{(\mu'_\theta - \mu'_\tau)^2}{\eta_M \sigma^2 (1 - \rho_{\theta', \tau'})}$ for the deviant strategy pairs $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. When the deviant strategy is $\tau' = \tau_L$, if its mean return μ_{τ_L} is low, Manager C can take small long or even short positions in τ_L to finance large positions in strategies in the asset class θ . So, as μ_{τ_L} decreases, the long-short opportunity cost for Man-

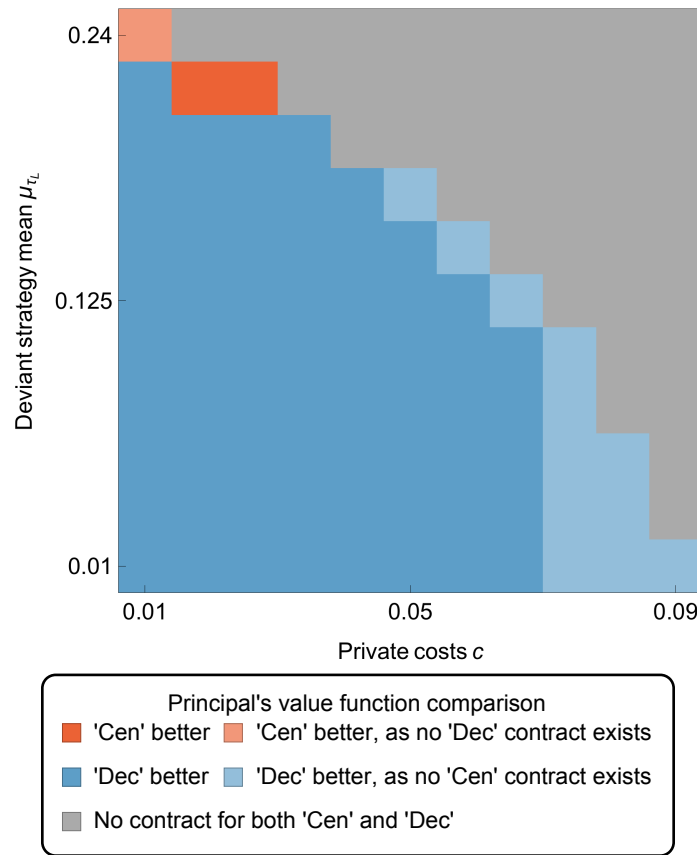


Figure 3.2: Comparing the Principal’s value function under centralization versus decentralization: deviant strategy mean return μ_{τ_L} versus private costs c .

ager C increases, making centralized delegation unfavorable. In contrast, this opportunity cost does not exist in decentralization. Furthermore, recall that Manager B is responsible for managing asset class τ . As μ_{τ_L} decreases, the expected performance fee payoff $y_B \mu_{\tau_L}$ for Manager B when he deviates from the compliant strategy τ_H to the deviant strategy τ_L also decreases, and thereby making deviation less profitable for him. Thus when this happens, the performance fees for Manager B could reach that of the first best result, and thereby making decentralization favorable. However, we note that as μ_{τ_L} increases and as it approaches the mean return $\mu_{\tau_H} = \mu$ of the compliant strategy τ_H , the payoff in performance fees for Manager B to be compliant and deviant become similar. However, Manager B still needs to incur a private cost c to implement the Principal’s desired strategy; in such a case when the net benefit for being compliant rather than deviant is small, while Manager B still needs to incur private costs c , Manager B will for sure deviate. As a result, a decentralized contract for implementing the Principal’s desired strategy pair (θ_H, τ_H) could fail to exist, as Manager B will for sure deviate. This observation here will motivate our discussion of the dynamic model in Section 3.6.

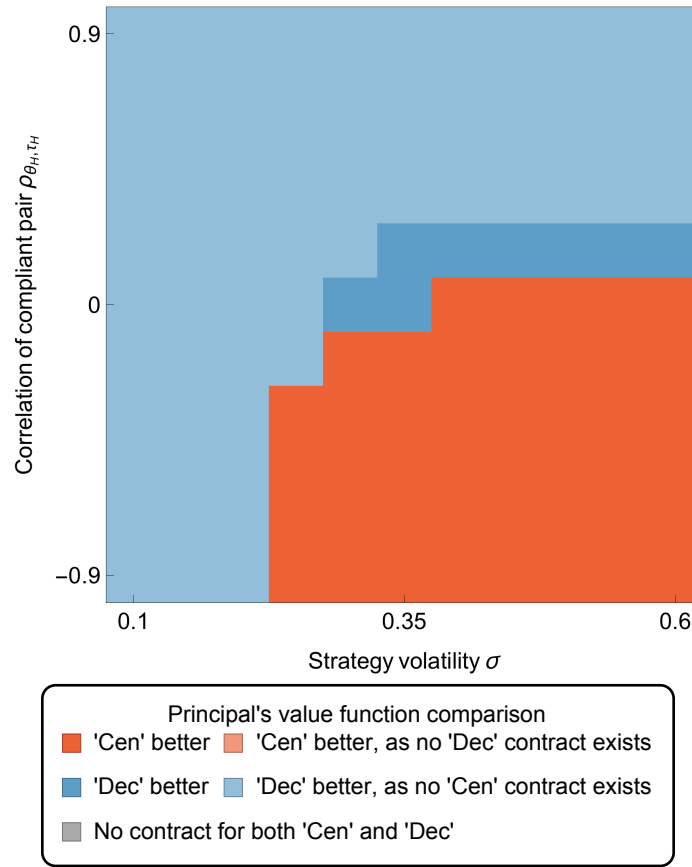


Figure 3.3: Comparing the Principal’s value function under centralization versus decentralization: compliant strategy pair correlation ρ_{θ_H, τ_H} versus strategy volatility σ .

In Figure 3.3, we see the effects of strategy volatility σ and the correlation ρ_{θ_H, τ_H} of the compliant pair on the contracting environment. For low correlations, as volatility σ increases, it will favor centralization because of the optimal risk sharing effect as discussed even in the first best setup of Proposition 3.4.3. As already discussed in Figure 3.1, high correlation ρ_{θ_H, τ_H} of the compliant strategy pair will increase the contract volatility for Manager C . Here, volatility also brings about another perspective on this long-short opportunity cost. As volatility σ decreases across all strategies, Manager C will care even more about the mean difference between the deviant strategy pairs, and thus place more extreme long and short positions. This increases the opportunity cost for Manager C to be compliant, and thereby making centralization unfavorable.

In Figure 3.4, we study the effects of the Principal risk aversion η_P and the Managers’ risk aversion η_M on the contracting environment. As discussed in Section 3.5, in centralization when Manager C becomes less risk averse, he will more extreme long-short profits in the deviant strategy pairs, and it will become ever more costly for Principal to induce Manager C to be compliant. In decentralization, thanks to Assumption 3.3.2 that volatilities are

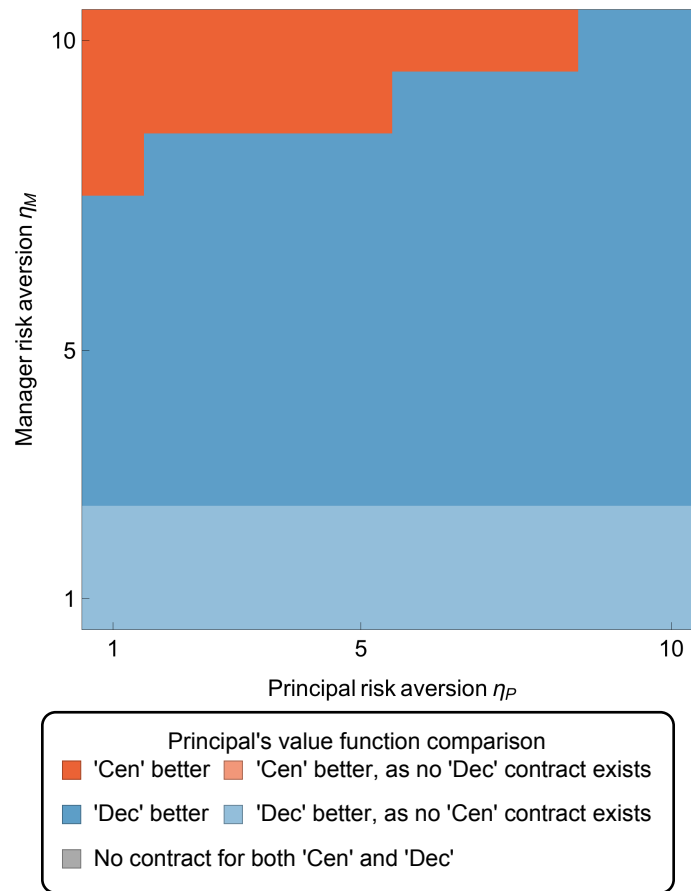


Figure 3.4: Comparing the Principal’s value function under centralization versus decentralization: Principal’s risk aversion η_P versus Managers’ risk aversion η_M .

identical across all strategies, Manager *A* and Manager *B* will not factor in their risk aversion in a deviation. Note that in one extreme when Manager *C* is highly risk averse while the Principal is relatively less risk averse, centralization will be favored.

3.6 Dynamic Model

Motivation

In Section 3.5, we discussed the various effects of moral hazard on *static* centralized delegation and decentralized delegation under second best. However, in spite of our discussions, there is however one distinct important case where decentralized delegation is particularly fragile compared to centralization.

Failure of static decentralized delegation

Suppose the strategy returns within the asset class θ and τ have similar mean returns. That is, suppose $\Delta\mu_\theta \approx 0$ and $\Delta\mu_\tau \approx 0$. To fix ideas on when this is possible, recall again the motivating example discussed in Footnote 3 of the Introduction. Here, we can be more concrete with that example. Recalling the centralized delegation incentive compatibility constraint (3.5.1), in this case when the mean strategy returns within each asset class are similar, a contract to induce the centralized Manager C to take the Principal's desired strategy pair (θ_H, τ_H) over other deviant strategy pairs will exist. Indeed in this case when $\Delta\mu_\theta \approx 0$ and $\Delta\mu_\tau \approx 0$, the opportunity cost for foregone long-short trading profits are actually reduced, and thereby making centralized delegation even more attractive. However, in this case the incentive compatibility constraints (3.5.6) for Manager A and Manager B , respectively, in decentralized delegation are,

$$0 \geq c - (1 - \pi)y_A\Delta\mu_\theta \approx c \quad \text{and} \quad 0 \gtrsim c - \pi y_B\Delta\mu_\tau \approx c.$$

The condition $0 \gtrsim c$ is clearly impossible to satisfy unless the private costs are trivially small, $c \approx 0$. For Manager A , the difference in mean returns, $\Delta\mu_\theta$, between the compliant strategy θ_H and the deviant strategy θ_L represents the benefit to Manager A for compliance in terms of higher expected performance fees. And when $\Delta\mu_\theta \approx 0$, the benefit for being compliant is small while Manager A still needs to incur a private cost c . In this situation, it is impossible for the Principal to incentivize Manager A to implement the Principal's desired strategy θ_H , as Manager A will surely deviate. Similar observations hold for Manager B with respect to his asset class τ . Economically, this is because neither of the decentralized Managers are affected by the strategies' correlations when considering a deviation. Indeed, in a static decentralization, only the Principal reaps the diversification benefits via the strategies' return correlation.

This issue hints at a severe loss of efficiency for the decentralized delegation form. Here, we have a setup whereby the decentralized Managers cannot be motivated and coordinated to take on the compliant strategy pair for the Principal. An obvious and correct response to this is to simply declare that for these asset classes with such mean return properties, centralized delegation is by default better than decentralized delegation. However, this is a rather unsatisfying response, and it is of interest to study mechanisms whereby we can still correctly incentivize Managers in decentralized delegation.

Remark 3.6.1. *When dynamics are available, a conceivable natural solution to this problem is through the use of benchmarks to ensure that Managers had indeed implemented the desired strategy. Indeed, van Binsbergen et al. (2008) have argued that a carefully designed benchmark can be used to align the incentives of multiple decentralized delegated portfolio managers.¹⁸ However, the use of benchmarks is problematic in light of the discussion of Admati and Pfleiderer (1997), who argue that benchmarks are not useful in providing incentives to Managers,*

¹⁸ One should note, however, van Binsbergen et al. (2008) is not a model of moral hazard. And thus, what they call as "aligning incentives" is really simply better risk sharing in a first best case.

and also echoed in the discussion earlier in Section 3.5, allowing the Managers to access the financial markets could allow them to modify the effects of the contract. And most importantly, from an empirical perspective, direct benchmarked compensation contracts are simply not prevalent in hedge funds and private equity type investments whereby the contracts are usually based on the raw returns of the managed assets; see Getmansky, Lee, and Lo (2015) for a recent review. Heinkel and Stoughton (1994) also consider a dynamic two-period delegated portfolio management problem with linear contracts to incentivize managers to acquire information; however, the authors critically assume the managers are risk neutral, and hence is silent to our intertemporal income hedging motive of the Managers that we emphasize as a key economic channel, in both centralized and decentralized delegation.

Furthermore, from a contracting theory perspective, this problem has two obvious candidate solutions even in a static setup. The first one is via a team based contracting scheme¹⁹ where the two decentralized Managers take on disjoint actions but their compensation is from a common source; for instance, consider when Manager A's and Manager B's compensations are dependent not on their own fund returns but actually on the Principal's terminal wealth. The second one is via a tournament contracting scheme²⁰ where Manager A's compensation depends on not only his own fund performance but also that of Manager B, and vice-versa. However, neither the first type nor the second type of compensation are observed in practice. The current practice remains that a fund's compensation is only dependent its own return performance.²¹

In all, a direct expansion of the static contracting space will not solve the problem of coordinating decentralized Managers return strategy dependence in the presence of moral hazard.

Joint incentivization via reinvestments

The use of reinvestments in a dynamic model is a potential mechanism to induce the correct incentivization in decentralized delegation. Suppose the Principal commits to contracting with the Managers for one additional period. For each asset class, strategies chosen by Managers have a committed long term effect: once strategies (θ, τ) have been chosen and committed to at the beginning of the contract, the same set of strategies will be executed in both the first period and in the subsequent period.

It should be strongly emphasized that by contracting for another period, we are *not* relying on the arguments of repeated interactions as in the repeated dynamic principal-agent

¹⁹See Marschak (1955) and Marschak and Radner (1972).

²⁰See Nalebuff and Stiglitz (1983) and more recently for an explicit example in portfolio delegation and competition for fund flows, Basak and Makarov (2014).

²¹It is beyond the scope of the paper to study why is it that the current practice does not incorporate these alternative compensation schemes as suggested by contracting theory. However, this author speculates that these team based and tournament based compensation schemes strongly depend on all agents in the game to know of other agents' existence and characteristics. This condition could be difficult to execute in practice.

literature.²² The strategy choice is long term so the Principal need not perform statistical inference over time to learn the true action taking by the Managers and thereby punish or reward accordingly in subsequent periods. Moreover, the key mechanism that we require is the path dependence of Principal's wealth on the Managers' long term compensation. In decentralization, once we allow for a subsequent period of contracting, the more wealth the Principal has at the end of the initial period, the more wealth the Principal can allocate to both Managers for investing in the subsequent period. Thus, the next period compensation for the multiple Managers would heavily depend on the Principal's wealth level from the previous period. Thus, this path dependence in next period compensation would induce a hedging motive for the multiple Managers in the initial period. In particular, this hedging motive comes from the intertemporal covariance between a particular Manager's future wealth and the Principal's wealth, of which this compounds in all Managers' strategy pair correlations. Through this mechanism, the decentralized Managers will become concerned about the correlation of strategies amongst each other when considering a deviation.

Additional assumptions in dynamics

The economic setup of the dynamic model is essentially identical to that of the static model in Section 3.3, but we will need a few adjustments to account for the temporal nature of the dynamic problem. The individuals in this economy are still equivalent to before. Instead of contracting for just periods $t = 0, 1$, the contracting period is now extended to $t = 0, 1, 2$.

The asset class θ with investment strategies $\{\theta_L, \theta_H\}$ now has per-period-returns $R_{\theta,t}$, for $t = 1, 2$; and likewise, the asset class τ with strategies $\{\tau_L, \tau_H\}$ now has per-period-returns $R_{\tau,t}$ for $t = 1, 2$. We should note that this is *not* a dynamic nor repeated moral hazard model in the usual principal-agent literature. In particular, the Managers could *not* make private choices on the strategies θ, τ in both periods $t = 0$ and $t = 1$. Rather, this is a model where the Managers commit to a particular strategy at $t = 0$ but simply contracts with the Principal for two periods.

In what follows, we will denote \mathbb{E}_t as the time t conditional expectation, Var_t as the time t conditional variance, Cov_t as the time t conditional covariance, and Corr_t as the time t conditional correlation, for $t = 0, 1$.

We will make two assumptions.

Assumption 3.6.2. *Let $R_{\theta,t}$ and $R_{\tau,t}$ be the period $t = 1, 2$ returns of investment strategies θ, τ . We assume the time $t = 1$ returns of all investment strategies are independent of their $t = 2$ counterparts.*

Assumption 3.6.3. *Assume Assumption 3.6.2. Furthermore assume the following.*

- (i) *The conditional means of all strategies are equivalent across time; that is, $\mu_\theta \equiv \mathbb{E}_t[R_{\theta,t+1}]$ and $\mu_\tau \equiv \mathbb{E}_t[R_{\tau,t+1}]$ for $t = 0, 1$.*

²² Say, for instance, Mailath and Samuelson (2006).

- (ii) The conditional volatility of all strategies are equivalent across time; that is, $0 < \sigma^2 \equiv \text{Var}_t(R_{\theta,t+1}) = \text{Var}_t(R_{\tau,t+1})$ for all strategies θ, τ and all time $t = 0, 1$.
- (iii) The conditional correlations of strategy pairs are equivalent across time; that is, $\rho_{\theta,\tau} \equiv \text{Corr}_t(R_{\theta,t+1}, R_{\tau,t+1})$ for all strategies θ, τ and all time $t = 0, 1$.

As we begin to discuss dynamic portfolio choice and dynamic contracting, it becomes quite clear the fashion in which state variables statistically relate to each other across time are critically important. Assumption 3.6.2 immediately rules out stochastic and time varying means, volatility and correlation between investment strategies, which is admittedly the strongest assumption above. While we do impose the assumption of independent distributions across time, we do not need to impose identical distributions. In particular, it is not difficult to actually extend our current model to allow non-identical (but independent) distributions across time. But allowing some moments (i.e. Assumption 3.6.3) to be equivalent across time, makes some of the discussion easier, and can be extended to be more general case at the cost of some loss of tractability.

Dynamic Decentralized Delegation

In all, the optimization problem for decentralized delegation is as follows. Please see Figure 3.1 for the time line.

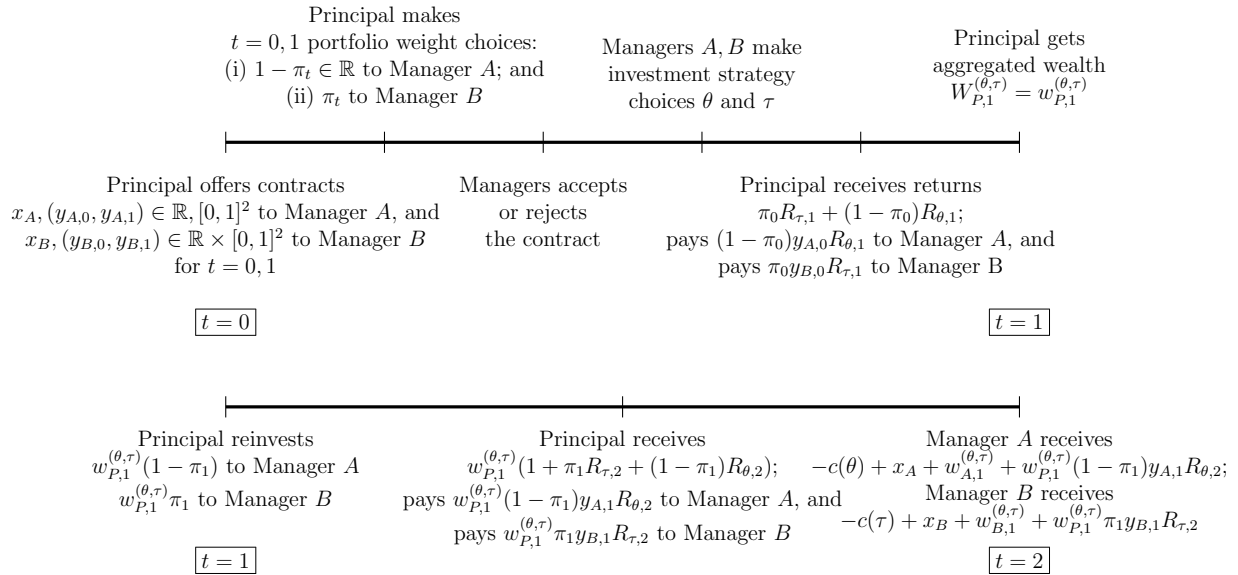


Figure 3.1: Dynamic decentralized delegation time line.

$$\sup_{\substack{x_A, x_B \in \mathbb{R}, \\ y_{A,0}, y_{A,1} \in [0,1], \\ y_{B,0}, y_{B,1} \in [0,1]}} \sup_{\pi_0, \pi_1} -x_A - x_B + \mathbb{E}_0[W_{P,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{P,2}^{(\theta_H, \tau_H)}), \quad (\text{DynDec})$$

subject to,

$$W_{P,1}^{(\theta, \tau)} := 1 + (1 - \pi_0)R_{\theta,1} + \pi_0 R_{\tau,1} - (\pi_0 y_{B,0} R_{\tau,1} + (1 - \pi_0) y_{A,0} R_{\theta,1}), \quad (3.6.1a)$$

$$W_{P,2}^{(\theta, \tau)} := W_{P,1}^{(\theta, \tau)} [1 + (1 - \pi_1)R_{\theta,2} + \pi_1 R_{\tau,2} - ((1 - \pi_1) y_{A,1} R_{\theta,2} + \pi_1 y_{B,1} R_{\tau,2})], \quad (3.6.1b)$$

$$W_{A,1}^{(\theta, \tau)} := (1 - \pi_0) y_{A,0} R_{\theta,1}, \quad (3.6.1c)$$

$$W_{A,2}^{(\theta, \tau)} := W_{A,1}^{(\theta, \tau)} + W_{P,1}^{(\theta, \tau)} (1 - \pi_1) y_{A,1} R_{\theta,2}, \quad (3.6.1d)$$

$$W_{B,1}^{(\theta, \tau)} := \pi_0 y_{B,0} R_{\tau,1}, \quad (3.6.1e)$$

$$W_{B,2}^{(\theta, \tau)} := W_{B,1}^{(\theta, \tau)} + W_{P,1}^{(\theta, \tau)} \pi_1 y_{B,1} R_{\tau,2}, \quad (3.6.1f)$$

$$0 \leq x_A - c + \mathbb{E}_0[W_{A,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(W_{A,2}^{(\theta_H, \tau_H)}), \quad (3.6.1g)$$

$$0 \leq x_B - c + \mathbb{E}_0[W_{B,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(W_{B,2}^{(\theta_H, \tau_H)}), \quad (3.6.1h)$$

$$\theta_H = \arg \max_{\theta'} x_A - c(\theta') + \mathbb{E}_0[W_{A,2}^{(\theta', \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(W_{A,2}^{(\theta', \tau_H)}), \quad (3.6.1i)$$

$$\tau_H = \arg \max_{\tau'} x_B - c(\tau') + \mathbb{E}_0[W_{B,2}^{(\theta_H, \tau')}] - \frac{\eta_M}{2} \text{Var}_0(W_{B,2}^{(\theta_H, \tau')}). \quad (3.6.1j)$$

See also Remark 3.6.5 for notes on the slight differences between the economic setup here of the dynamic model and that of the static model in Section 3.3.

In the decentralized objective function of the Principal in (DynDec), the Principal needs to choose the optimal fixed fees x_A, x_B and the $t = 0$ and $t = 1$ performance fees $y_{A,0}, y_{A,1}$ and $y_{B,0}, y_{B,1}$, respectively for Manager A and B , and the portfolio weights π_0, π_1 for Manager B . After Manager A and Manager B are paid their performance fees $y_{A,0}, y_{B,0}$ at $t = 1$, we assume for simplicity that neither Manager will reinvest their collected fees. The budget constraints for the Principal are (3.6.1a), (3.6.1b), and the budget constraints for Manager A and B are respectively (3.6.1c), (3.6.1d) and (3.6.1e), (3.6.1f). Given that both Manager A and Manager B are initially endowed with zero amount of wealth, (3.6.1g) and (3.6.1h) are their individual rationality constraints. Likewise, (3.6.1i) and (3.6.1j) are their respective incentive compatibility constraints for the Principal to induce the two Managers to take on the Principal's strict preferred (θ_H, τ_H) investment strategy pair.

Remark 3.6.4. *Let's clarify some of the notations and the interpretations of the timing.*

- *This is a model of commitment and thus the Principal commits to both Manager A and Manager B on his current and future portfolio and performance fee policies.*
- *To be clear on the returns notation, we write $R_{\theta,1}$ to be the net return from $t = 0$ to $t = 1$ for strategy θ , and $R_{\theta,2}$ to be the net return from $t = 1$ to $t = 2$. Analogous comments apply for strategy τ with net returns notation $R_{\tau,t}$.*

- We should discuss the notation of the portfolio and performance fee policies. At $t = 0$, the Principal will allocate $1 - \pi_0$ of his initial wealth to Manager A, and π_0 to Manager B. The Principal will also at $t = 0$ promise performance fees $y_{A,0}$ and $y_{B,0}$ to Manager A and Manager B, respectively, at $t = 1$. Thus for instance, the total performance fee payoff for Manager A at $t = 1$ for implementing strategy θ is $(1 - \pi_0)y_{A,0}R_{\theta,1}$. We will call π_0 to be the $t = 0$ portfolio policy, and $y_{A,0}$ be the $t = 0$ performance fee policy to Manager A, even though the performance fees are actually paid at $t = 1$. We keep this timing convention notation because the randomness in the returns are indeed realized at $t = 1$. Thus, the portfolio and performance policy must be known in advanced, at $t = 0$, in order for the model to be nontrivial. Hence, we refer to policies at the time of decision, and not at the time of payment. Likewise, we will call $1 - \pi_1$ be the $t = 1$ portfolio policy to Manager A and $y_{A,1}$ be the $t = 1$ performance fee policy, even though the actual performance fee payoff to Manager A is at $t = 2$ for the quantity $w_{P,1}^{(\theta,\tau)}(1 - \pi_1)y_{A,1}R_{\theta,2}$. Analogous comments apply for Manager B.

Remark 3.6.5. While the economic setup in the dynamic model is largely identical to that of the static model setup in Section 3.3, there are two minor differences that are worth noting. Firstly, we have assumed that the fixed fees x_A, x_B to Manager A and Manager B, respectively, are a lump sum figure and only paid at the contract terminal date $t = 2$. That is, only performance fees are paid at $t = 1$ and $t = 2$. This is largely to simplify the analysis. Secondly, the private costs $c(\theta)$ and $c(\tau)$ are — akin to the fixed fees — are now paid at $t = 2$. Thus, even though the Managers commit to their investment strategy choice at $t = 0$, the costs are only incurred at $t = 2$. This, again, is largely to simplify the problem. Alternatively, we may think of the private cost not as a wealth cost but as a utility cost incurred at $t = 0$.

Finally, in light of the modifications of the timing of the fixed fees and private costs as discussed above, as opposed to the static model of Section 3.3, we slightly reinterpret the budget constraints $W_{k,t}^{(\theta,\tau)}$ the individual k at time t . Whereas the wealths in the static model are inclusive of the private costs and the fixed fees, in the dynamic model we interpret the wealths are exclusive of the private costs and the fixed fees. This largely is done to simplify the notations.

Remark 3.6.6. We discuss the analogous dynamic centralized delegation model in Section 3.B. As the core motivation for introducing a dynamic model is to fix an issue specific to decentralized delegation in the static model, we will thus keep the main discussion of a dynamic model to decentralized delegation.

3.7 First Best in Dynamics

Dynamic Decentralized Delegation in First Best

Let us begin by considering the first best setup, whereby the Principal can directly observe and contract on the private investment strategy choices of the Managers.

Proposition 3.7.1. *Consider the first best dynamic decentralized delegation problem; so consider (DynDec) without the incentive compatibility constraints (3.6.1i) and (3.6.1j). Recall that the Principal wants to implement strategy pair (θ_H, τ_H) .*

(a) *For any portfolio and performance fee policies $(\pi_0, \pi_1, y_{A,0}, y_{A,1}, y_{B,0}, y_{B,1})$, the optimal fixed fees for Manager A and Manager B, are, respectively,*

$$\begin{aligned}\hat{x}_{A,(\theta_H, \tau_H)} &= c - \mathbb{E}_0[W_{A,2}^{(\theta_H, \tau_H)}] + \frac{\eta_M}{2} \text{Var}_0(W_{A,2}^{(\theta_H, \tau_H)}), \\ \hat{x}_{B,(\theta_H, \tau_H)} &= c - \mathbb{E}_0[W_{B,2}^{(\theta_H, \tau_H)}] + \frac{\eta_M}{2} \text{Var}_0(W_{B,2}^{(\theta_H, \tau_H)}).\end{aligned}$$

(b) *The $t = 1$ optimal policies are given as follows.*

(i) *The optimal $t = 1$ portfolio to allocate to Manager B is independent of fees and wealth effects,*

$$\hat{\pi}_1^{FB} = \frac{1}{2}.$$

(ii) *The optimal $t = 1$ performance fee policies chosen by the Principal to compensate Managers A and B are equivalent, and they are,*

$$\hat{y}_{A,1}^{FB} = \hat{y}_{B,1}^{FB} = \frac{\eta_P(1 + \rho_{\theta_H, \tau_H})}{\eta_P(1 + \rho_{\theta_H, \tau_H}) + \eta_M}.$$

(c) *The $t = 0$ optimal policies are given as follows.*

(i) *The optimal $t = 0$ portfolio to allocate to Manager B is again independent of fees and wealth effects,*

$$\hat{\pi}_0^{FB} = \frac{1}{2}.$$

(ii) *The optimal $t = 0$ (interior solution) performance fee policies chosen by the Principal to compensate Manager A are,²³*

$$\hat{y}_{A,0}^{FB} = \frac{\hat{y}_{A,0}^N}{\hat{y}_{A,0}^D},$$

²³ We use “N” for numerator and “D” for denominator.

provided that $\hat{y}_{A,0}^{FB} \in (0, 1)$, and where,

$$\begin{aligned} \hat{y}_{A,0}^N &:= 2\eta_P^2(1 + \rho_{\theta_H, \tau_H})^2[\eta_P\sigma^2(1 + \rho_{\theta_H, \tau_H}) - 2\mu^2] \\ &\quad + \eta_P\eta_M(1 + \rho_{\theta_H, \tau_H}) \left[-8\mu^2 + \eta_P\sigma^2(1 + \rho_{\theta_H, \tau_H}) \times \right. \\ &\quad \left. [4 + \mu(5 - \rho_{\theta_H, \tau_H} + \mu(3 + \rho_{\theta_H, \tau_H}))] + \eta_P\sigma^4(1 + \rho_{\theta_H, \tau_H})^2 \right] \\ &\quad + \eta_M^2 \left[-4\mu^2 + \eta_P(2 + \mu(5 + 4\mu - \rho_{\theta_H, \tau_H}))(1 + \rho_{\theta_H, \tau_H})\sigma^2 \right. \\ &\quad \left. + \eta_P\sigma^4(1 + \rho_{\theta_H, \tau_H})^2 \right], \\ \hat{y}_{A,0}^D &:= \sigma^2 \left[2\eta_P^3(1 + \rho_{\theta_H, \tau_H})^3 + 2\eta_M^3 \right. \\ &\quad \left. + \eta_M\eta_P^2(1 + \rho_{\theta_H, \tau_H})^2[6 + \mu(2 - 2\rho_{\theta_H, \tau_H} + \mu(3 + \rho_{\theta_H, \tau_H})) + \sigma^2(1 + \rho_{\theta_H, \tau_H})] \right. \\ &\quad \left. + \eta_P\eta_M^2(1 + \rho_{\theta_H, \tau_H})[6 + 4\mu^2 + 2\mu(1 - \rho_{\theta_H, \tau_H}) + (1 + \rho_{\theta_H, \tau_H})\sigma^2] \right]. \end{aligned}$$

The performance fees to compensate Manager B are identical, $\hat{y}_{B,0}^{FB} = \hat{y}_{A,0}^{FB}$.

Observing Proposition 3.7.1, under the Principal's desired pair (θ_H, τ_H) , and recalling Assumption 3.6.2 and Assumption 3.6.3, given that both strategies have equivalent means, it is no surprise that the $t = 1$ portfolio policy $\hat{\pi}_1^{FB}$ would place equal weights between Manager A and Manager B . And indeed, the $t = 1$ optimal performance fees $\hat{y}_{A,1}^{FB}$ and $\hat{y}_{B,1}^{FB}$ for Manager A and Manager B , respectively, are identical in form to that of the static first best decentralized delegation problem of Proposition 3.4.2. For the $t = 0$ policies, and again given the return assumptions of Assumption 3.6.2 and Assumption 3.6.3, the Principal will still place equal portfolio weights $\hat{\pi}_0^{FB}$ into Manager A and Manager B . Given that Manager A and Manager B have identical risk aversion η_M and identical outside options, it is no surprise that their $t = 0$ performance fees $\hat{y}_{A,0}^{FB}$ and $\hat{y}_{B,0}^{FB}$ would be identical. However, even though the portfolio policies of $\hat{\pi}_0^{FB}$ and $\hat{\pi}_1^{FB}$ are identical across time, the performance fees are not. The $t = 0$ performance fees will affect the amount of wealth $W_{P,1}^{(\theta_H, \tau_H)}$ available for reinvestment at $t = 1$, and hence affecting the terminal $t = 2$ wealths of all individuals involved; thus the $t = 0$ performance fees $\hat{y}_{A,0}^{FB}$ and $\hat{y}_{B,0}^{FB}$ must take into account this intertemporal wealth hedging channel, and hence why these $t = 0$ performance fee policies will differ from that of the $t = 1$ policy (since one period later at $t = 2$ is the terminal contracting date). See Section 3.C for numerical illustrations of Proposition 3.7.1.

3.8 Second Best in Dynamics

Now we consider the second best dynamic delegation problem in its entirety.

Dynamic Decentralized Delegation in Second Best

Proposition 3.8.1. *Consider the second best decentralized delegation problem (DynDec) in its entirety.*

(a) *For any portfolio and performance fee $(\pi_0, \pi_1, y_{A,0}, y_{A,1}, y_{B,0}, y_{B,1})$, the optimal fixed fees for Manager A and Manager B have the same form as that of the first best result of Proposition 3.7.1.*

(b) *Consider the $t = 1$ optimal policies. Suppose the $t = 1$ realized wealth under strategy pair $(\theta, \tau) \in \mathcal{S}$ is $W_{P,1}^{(\theta,\tau)} = w_{P,1}^{(\theta,\tau)}$. Then for some constants $\lambda_A, \lambda_B \in \mathbb{R}$:*

(i) *The optimal $t = 1$ portfolio is,*

$$\hat{\pi}_1^{\lambda_A, \lambda_B} = \frac{\hat{\pi}_1^{N, \lambda_A, \lambda_B}}{\hat{\pi}_1^{D, \lambda_A, \lambda_B}}, \quad (3.8.1)$$

where the analytical forms of the numerator and denominator can be found in (3.H.8) of the Appendix.

(ii) *The optimal $t = 1$ performance fee to compensate Manager A is,*

$$\hat{y}_{A,1}^{\lambda_A, \lambda_B} = \frac{\hat{y}_{A,1}^{N, \lambda_A, \lambda_B}}{\hat{y}_{A,1}^{D, \lambda_A, \lambda_B}}, \quad (3.8.2)$$

where the analytical forms of the numerator and denominator can be found in (3.H.9) of the Appendix.

(iii) *The optimal $t = 1$ performance fee to compensate Manager B is,*

$$\hat{y}_{B,1}^{\lambda_A, \lambda_B} = \frac{\hat{y}_{B,1}^{N, \lambda_A, \lambda_B}}{\hat{y}_{B,1}^{D, \lambda_A, \lambda_B}}, \quad (3.8.3)$$

where the analytical forms of the numerator and denominator can be found in (3.H.10) of the Appendix.

(c) *The $t = 1$ continuation utilities for the Principal, Manager A and Manager B under the strategy pair $(\theta, \tau) \in \mathcal{S}$ are as follows. For notational simplicity, we will denote*

$\hat{\pi}_1 := \hat{\pi}_1^{\lambda_A, \lambda_B}$, $\hat{y}_{A,1} := \hat{y}_{A,1}^{\lambda_A, \lambda_B}$, $\hat{y}_{B,1} := \hat{y}_{B,1}^{\lambda_A, \lambda_B}$. Then,

$$\begin{aligned} \hat{U}_{P,1}^{(\theta_H, \tau_H)} &:= w_{P,1}^{(\theta_H, \tau_H)} (1 + [1 - (1 - \hat{\pi}_1)\hat{y}_{A,1} - \hat{\pi}_1\hat{y}_{B,1}]\mu) \\ &\quad - \frac{\eta_P}{2} \sigma^2 (w_{P,1}^{(\theta_H, \tau_H)})^2 \left[(1 - \hat{\pi}_1)^2 (1 - \hat{y}_{A,1})^2 + \hat{\pi}_1^2 (1 - \hat{y}_{B,1})^2 \right. \\ &\quad \left. - 2\hat{\pi}_1(1 - \hat{\pi}_1)(1 - \hat{y}_{A,1})(1 - \hat{y}_{B,1})\rho_{\theta_H, \tau_H} \right], \\ \hat{U}_{A,1}^{(\theta_H, \tau_H)} &:= w_{A,1}^{(\theta_H, \tau_H)} + (1 - \hat{\pi}_1)\hat{y}_{A,1}w_{P,1}^{(\theta_H, \tau_H)}\mu - \frac{\eta_M}{2} \sigma^2 (1 - \hat{\pi}_1)^2 \hat{y}_{A,1}^2 (w_{P,1}^{(\theta_H, \tau_H)})^2, \\ \hat{U}_{A,1}^{(\theta_L, \tau_H)} &:= w_{A,1}^{(\theta_L, \tau_H)} + (1 - \hat{\pi}_1)\hat{y}_{A,1}w_{P,1}^{(\theta_L, \tau_H)}\mu_{\tau_L} - \frac{\eta_M}{2} \sigma^2 (1 - \hat{\pi}_1)^2 \hat{\pi}_1^2 (w_{P,1}^{(\theta_L, \tau_H)})^2, \\ \hat{U}_{B,1}^{(\theta_H, \tau_H)} &:= w_{B,1}^{(\theta_H, \tau_H)} + \hat{\pi}_1\hat{y}_{B,1}w_{P,1}^{(\theta_H, \tau_H)}\mu - \frac{\eta_M}{2} \sigma^2 \hat{\pi}_1^2 \hat{y}_{B,1}^2 (w_{P,1}^{(\theta_H, \tau_H)})^2, \\ \hat{U}_{B,1}^{(\theta_H, \tau_L)} &:= w_{B,1}^{(\theta_H, \tau_L)} + \hat{\pi}_1\hat{y}_{B,1}w_{P,1}^{(\theta_H, \tau_L)}\mu_{\tau_L} - \frac{\eta_M}{2} \sigma^2 \hat{\pi}_1^2 \hat{y}_{B,1}^2 (w_{P,1}^{(\theta_H, \tau_L)})^2, \end{aligned}$$

Note that the first two sum terms in all the continuation utility expressions above (excluding the term involving $-\eta_M \sigma^2 / 2$) are equal to $\mathbb{E}_1 W_{k,2}^{(\theta, \tau)}$ for individuals $k = P, A, B$ and for various respective strategy pairs (θ, τ) . Define also the $t = 1$ aggregated continuation utility,

$$\begin{aligned} \hat{U}_{P,1}^{\lambda_A, \lambda_B} &:= \hat{U}_{P,1}^{(\theta_H, \tau_H)} + \hat{U}_{A,1}^{(\theta_H, \tau_H)} + \hat{U}_{B,1}^{(\theta_H, \tau_H)} \\ &\quad - \lambda_A \left(\hat{U}_{A,1}^{(\theta_L, \tau_H)} - (-c + \hat{U}_{A,1}^{(\theta_H, \tau_H)}) \right) \\ &\quad - \lambda_B \left(\hat{U}_{B,1}^{(\theta_H, \tau_L)} - (-c + \hat{U}_{B,1}^{(\theta_H, \tau_H)}) \right). \end{aligned} \quad (3.8.4)$$

(d) The set of optimal $t = 0$ policies $(\hat{\pi}_0^{\lambda_A, \lambda_B}, \hat{y}_{A,0}^{\lambda_A, \lambda_B}, \hat{y}_{B,0}^{\lambda_A, \lambda_B})$ is the solution to the following optimization problem,

$$\begin{aligned} \hat{U}_{P,0}^{\lambda_A, \lambda_B} &= \sup_{y_{A,0}, y_{B,0} \in [0,1]} \sup_{\pi_0 \in \mathbb{R}} \mathbb{E}_0 [\hat{U}_{P,1}^{\lambda_A, \lambda_B}] - \frac{\eta_P}{2} \text{Var}_0 (\mathbb{E}_1 W_{P,2}^{(\theta_H, \tau_H)}) \\ &\quad - \frac{\eta_M}{2} \text{Var}_0 (\mathbb{E}_1 W_{A,2}^{(\theta_H, \tau_H)}) - \frac{\eta_M}{2} \text{Var}_0 (\mathbb{E}_1 W_{B,2}^{(\theta_H, \tau_H)}) \\ &\quad - \lambda_A \left[-\frac{\eta_M}{2} \text{Var}_0 (W_{A,2}^{(\theta_L, \tau_H)}) + \frac{\eta_M}{2} \text{Var}_0 (W_{A,2}^{(\theta_H, \tau_H)}) \right] \\ &\quad - \lambda_B \left[-\frac{\eta_M}{2} \text{Var}_0 (W_{B,2}^{(\theta_H, \tau_L)}) + \frac{\eta_M}{2} \text{Var}_0 (W_{B,2}^{(\theta_H, \tau_H)}) \right]. \end{aligned} \quad (3.8.5)$$

(e) The optimal constants $\hat{\lambda}_A, \hat{\lambda}_B \in \mathbb{R}$ are the solution to the problem,

$$\inf_{\lambda'_A, \lambda'_B \in \mathbb{R},} \hat{U}_{P,0}^{\lambda'_A, \lambda'_B}.$$

and at least one
of them are
nonzero

The analytical forms (see Section 3.H) of these $t = 1$ optimal policies all have a fractional form, where both the numerator and denominator are separately nonlinear in the $t = 1$ realized wealths of both compliant and deviant strategy pairs. As we explain below, this fractional form of the optimal policies at $t = 1$ will drive the value-at-risk (VaR) constraint²⁴ interpretation of the $t = 0$ portfolio and performance policies.

The $t = 1$ portfolio and performance fee policies

Let's begin with a discussion of the $t = 1$ portfolio and performance fee policies $(\pi_1, y_{A,1}, y_{B,1})$. To make the argument concrete, for any given contract, let $\tilde{U}_{P,t}^{(\theta_H, \tau_H)}$ be the Principal's time $t = 0, 1$ continuation utility under the compliant investment strategy pair (θ_H, τ_H) , and let $U_{A,t}^{(\theta, \tau)}$ and $U_{B,t}^{(\theta, \tau)}$ be the time $t = 0, 1$ continuation utilities of both Manager A and Manager B under a general strategy pair $(\theta, \tau) \in \mathcal{S}$. In particular, the $t = 1$ continuation utilities for the Principal, Manager A and Manager B , respectively, under the compliant investment strategy pair (θ_H, τ_H) are,

$$\begin{aligned}\tilde{U}_{P,1}^{(\theta_H, \tau_H)} &= \mathbb{E}_1[W_{P,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_1(W_{P,2}^{(\theta_H, \tau_H)}), \\ U_{A,1}^{(\theta_H, \tau_H)} &= \mathbb{E}_1[W_{A,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_1(W_{A,2}^{(\theta_H, \tau_H)}), \\ U_{B,1}^{(\theta_H, \tau_H)} &= \mathbb{E}_1[W_{B,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_1(W_{B,2}^{(\theta_H, \tau_H)}).\end{aligned}$$

As it is the case with first best, the Principal wants to design portfolio and performance fee policies to optimally risk share at the lowest possible cost possible with Manager A and Manager B . This implies that, on the one hand, Principal wants to pick policies such that the sum of all individuals' continuation utilities $\tilde{U}_{P,1}^{(\theta_H, \tau_H)} + U_{A,1}^{(\theta_H, \tau_H)} + U_{B,1}^{(\theta_H, \tau_H)}$ is maximized.

However, Manager A and Manager B could have deviated at $t = 0$. If Manager A was compliant and Manager B was deviant, the resulting strategy pair would be (θ_H, τ_L) ; if Manager A was deviant and Manager B was compliant, the resulting strategy pair would be (θ_L, τ_H) . Since the Principal only wants to implement (θ_H, τ_H) , there is no circumstance where Manager A and Manager B would deviate to the pair (θ_L, τ_L) . Thus, the incentive compatibility constraints at $t = 0$ to implement (θ_H, τ_H) are,

$$\begin{aligned}-c + U_{A,0}^{(\theta_H, \tau_H)} &\geq U_{A,0}^{(\theta_L, \tau_H)}, \\ -c + U_{B,0}^{(\theta_H, \tau_H)} &\geq U_{B,0}^{(\theta_H, \tau_L)},\end{aligned}$$

where, for reasonable parameters in equilibrium, both constraints will bind. Now, let $W_{P,1}^{(\theta_L, \tau_H)} = w_{P,1}^{(\theta_L, \tau_H)}$ and $W_{P,1}^{(\theta_H, \tau_L)} = w_{P,1}^{(\theta_H, \tau_L)}$ be the Principal's realized wealths at $t = 1$ under the two deviant strategy pairs (θ_L, τ_H) and (θ_H, τ_L) , respectively.

Here we argue why the Principal's wealth can be used as an "intertemporal glue" to bridge the payoffs of the two Managers even in the presence of moral hazard. For instance from the

²⁴ See Jorion (2006) for a survey of the value-at-risk (VaR) literature.

budget constraint (3.6.1d), for any strategy pair (θ, τ) , Manager A 's expected performance fee payoff at $t = 2$ is $w_{P,1}^{(\theta,\tau)}(1 - \pi_1)y_{A,1}R_{\theta,2}$. This expected payoff to Manager A depends on Manager A 's strategy return $R_{\theta,2}$ from $t = 1$ to $t = 2$, the Principal's portfolio allocation $1 - \pi_1$ at $t = 1$ to Manager A , performance fees $y_{A,1}$, and also on the Principal's $t = 1$ realized wealth $W_{P,1}^{(\theta,\tau)} = w_{P,1}^{(\theta,\tau)}$ available for reinvestment when strategy pair (θ, τ) was implemented at $t = 0$. However, the strategy pair choice (θ, τ) at $t = 0$ by the Managers is private, and the Principal only wants to implement the strategy pair (θ_H, τ_H) . Thus, the $t = 1$ portfolio policy π_1 and the performance fee $(y_{A,1}, y_{B,1})$ policy chosen by the Principal — which are committed to at $t = 0$ — must induce Manager A to prefer implementing strategy θ_H over θ_L , regardless of what strategy $\tau \in \{\tau_H, \tau_L\}$ Manager B takes. Analogous remarks apply for incentivizing Manager B . Assigning Lagrange multipliers $\lambda_A, \lambda_B \in \mathbb{R} \setminus \{0\}$ to the *equality* binding constraints, the Principal's $t = 1$ optimization problem will thus be maximizing the $t = 1$ policies $(\pi_1, y_{A,1}, y_{B,1})$ over,²⁵

$$\begin{aligned} U_{P,1}^{\lambda_A, \lambda_B} = & \tilde{U}_{P,1}^{(\theta_H, \tau_H)} + U_{A,1}^{(\theta_H, \tau_H)} + U_{B,1}^{(\theta_H, \tau_H)} \\ & - \lambda_A \left(U_{A,1}^{(\theta_L, \tau_L)} - (-c + U_{A,1}^{(\theta_H, \tau_H)}) \right) - \lambda_B \left(U_{B,1}^{(\theta_H, \tau_L)} - (-c + U_{B,1}^{(\theta_H, \tau_H)}) \right). \end{aligned}$$

We optimize the $t = 1$ policies to get $(\hat{\pi}_1^{\lambda_A, \lambda_B}, \hat{y}_{A,1}^{\lambda_A, \lambda_B}, \hat{y}_{B,1}^{\lambda_A, \lambda_B})$ as per (3.8.1), (3.8.2) and (3.8.3). The $t = 1$ portfolio and performance fee policies $(\hat{\pi}_1^{\lambda_A, \lambda_B}, \hat{y}_{A,1}^{\lambda_A, \lambda_B}, \hat{y}_{B,1}^{\lambda_A, \lambda_B})$ will be contingent on the realized wealths $W_{P,1}^{(\theta_H, \tau_H)} = w_{P,1}^{(\theta_H, \tau_H)}$ of the compliant strategy pair (θ_H, τ_H) , and also the realized wealths $W_{P,1}^{(\theta_H, \tau_L)} = w_{P,1}^{(\theta_L, \tau_H)}$ (when Manager B deviates) and $W_{P,1}^{(\theta_L, \tau_H)} = w_{P,1}^{(\theta_L, \tau_H)}$ (when Manager A deviates). The resulting optimized $t = 1$ aggregated continuation utility is $\hat{U}_{P,1}^{\lambda_A, \lambda_B}$ as per (3.8.4). To ensure that both Manager A and Manager B would be compliant in equilibrium, the Principal must compensate both Manager A and Manager B for the difference in the mean and volatility in their performance fee payoffs under the compliant strategy pair (θ_H, τ_H) versus that of the deviant strategy pairs (θ', τ') . This effect is intuitively quite similar to the opportunity cost effect of foregone alternative wealth realizations as per second best static centralization of Proposition 3.5.1. The Lagrange multipliers λ_A, λ_B give the appropriate scaling of said compensation difference.

Moreover, the optimal policies $(\hat{\pi}_1^{\lambda_A, \lambda_B}, \hat{y}_{A,1}^{\lambda_A, \lambda_B}, \hat{y}_{B,1}^{\lambda_A, \lambda_B})$ will take on a fractional form, where the numerator and denominator are nonlinear in the Principal's $t = 1$ realized wealth $w_{P,1}^{(\theta_H, \tau_H)}$ under the compliant strategy pair, and are also nonlinear in the wealths $w_{P,1}^{(\theta_H, \tau_L)}$ and $w_{P,1}^{(\theta_L, \tau_H)}$ that are realized under the deviant strategy pairs (θ_H, τ_L) and (θ_L, τ_H) , respectively. The fractional forms are inherent from the mean-variance preferences of all individuals involved. Even from the classical Markowitz (1952) mean-variance formulation, it is immediate that the optimal portfolio policy there would follow a fractional form, where the numerator involves the mean returns of securities, and the denominator involves the volatilities of the securities

²⁵ See Section 3.A for the *dynamic programming principle* specific to mean-variance preferences so that the optimal policies are *time consistent*.

and the initial wealth of an investor. A similar effect is at play here. To Manager A , for instance, $w_{P,1}^{(\theta_H, \tau_H)}(1 - \hat{\pi}_1^{\lambda_A, \lambda_B})\hat{y}_{A,1}^{\lambda_A, \lambda_B}$ is precisely the amount of after performance fees wealth that the Principal allocates to Manager A , and multiplying this term out, we can see that the numerator of this expression involves the risk-adjusted mean payoffs to the Principal, Manager A and Manager B under the wealths of the compliant strategy pair and wealths of the deviant strategy pairs, and the denominator involves the volatility of such payoffs and also the correlations with respect to the return strategies.

One should note that in the first best case of Proposition 3.7.1, when no moral hazard is present and so $\lambda_A, \lambda_B = 0$, the optimal $t = 1$ portfolio and performance fee policies have the simple form,

$$(\hat{\pi}_1^{FB}, \hat{y}_{1,A}^{FB}, \hat{y}_{1,B}^{FB}) = \left(\frac{1}{2}, \frac{\eta(1 + \rho_{\theta_H, \tau_H})}{\eta(1 + \rho_{\theta_H, \tau_H}) + \eta_M}, \frac{\eta(1 + \rho_{\theta_H, \tau_H})}{\eta(1 + \rho_{\theta_H, \tau_H}) + \eta_M} \right).$$

Under first best, the only motive for the Principal is to optimally risk share and indeed, the Principal would invest equally into both Manager A and Manager B , and the resulting performance fees are simply a re-weighting of their respective risk aversions by the correlation of the compliant strategy pair. In particular, in first best, no wealth effects are involved in the $t = 1$ optimal policies, whereas in second best, the wealth effects driven by strategy pair deviations are distinctly present.

The $t = 0$ portfolio and performance fee policies

We now need to determine the $t = 0$ portfolio and performance fee policies $(\pi_0, y_{A,0}, y_{B,0})$. At this point, at least from a mechanical perspective, it is apparent why the full joint distribution of compliant and deviant return strategies $(R_{\theta_H,1}, R_{\tau_H,1}, R_{\theta_L,1}, R_{\tau_L,1})$ will be needed. The $t = 1$ optimal policies are a fractional form of the $t = 1$ wealths under compliant and deviant strategies of the Principal, which in turn depends on the $t = 0$ policies $(\pi_0, y_{A,0}, y_{B,0})$. Given that we are using a mean-variance framework, it implies that we need to compute the $t = 0$ expectation and variance of $t = 1$ fractional form random variables to arrive at the $t = 0$ optimal policies. Given that ratios of expectations are not the same as expectations of ratios,²⁶ we will need the full joint distribution of the strategies' returns to compute such moments. The above summarizes the technical reason for why full joint distributions of the returns are needed, beyond the first and second moments. But the economic reason is most interesting.

Economically, moral hazard implies a value-at-risk (VaR) type constraint on the $t = 0$ portfolio and performance fee policies $(\pi_0, y_{A,0}, y_{B,0})$. By the principle of dynamic programming for mean-variance preferences (see Section 3.A), the Principal's $t = 0$ policies $(\pi_0, y_{A,0}, y_{B,0})$ is the solution to the problem (3.8.5). Firstly, the $t = 0$ policies will have an intertemporal hedging effect on the individuals' $t = 2$ terminal wealth volatility under the compliant strategy pair (θ_H, τ_H) and these are to be minimized, as seen in $\frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{P,2}^{(\theta_H, \tau_H)}) + \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta_H, \tau_H)}) + \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta_H, \tau_H)})$. In these $t = 0$ variance

²⁶ Simply put, $\mathbb{E}[X/Y] \neq \mathbb{E}[X]/\mathbb{E}[Y]$, and $\text{Var}(X/Y) \neq \text{Var}(X)/\text{Var}(Y)$.

computations, the “intertemporal glue” effect of the Principal’s wealth become apparent; these $t = 0$ variance computations will involve the $t = 1$ optimal policies, which then affect the wealths ²⁷ $W_{P,1}^{(\theta_H, \tau_H)}$ under the compliant strategy pair, and the wealths $W_{P,1}^{(\theta_H, \tau_L)}$ and $W_{P,1}^{(\theta_L, \tau_H)}$ under the deviant strategy pairs. That is, the Principal’s wealths (both under compliant and deviant strategy pairs) enters into the covariance term of Manager A ’s and Manager B ’s wealths, and thereby through this “intertemporal glue”, Manager A ’s and Manager B ’s terminal wealth volatility are connected to each other. This effect is starkly absent in static decentralization. Secondly, there is a direct intertemporal hedging effect from $t = 0$ policies that Manager A and Manager B ’s $t = 2$ terminal wealth volatilities under the compliant strategy pair (θ_H, τ_H) is weakly lower than that of the deviant strategy pairs; this is reflected in the condition $\lambda_A(-\frac{\eta_M}{2}\text{Var}_0(W_{A,2}^{(\theta_L, \tau_H)}) + \frac{\eta_M}{2}\text{Var}_0(W_{A,2}^{(\theta_H, \tau_H)}))$ for Manager A , and $\lambda_B(-\frac{\eta_M}{2}\text{Var}_0(W_{B,2}^{(\theta_H, \tau_L)}) + \frac{\eta_M}{2}\text{Var}_0(W_{A,2}^{(\theta_H, \tau_H)}))$ for Manager B . Thirdly and finally, the $t = 0$ policies must ensure that the continuation utility $\mathbb{E}_0[\hat{U}_{P,0}^{\lambda_A, \lambda_B}]$ is maximized. These three points places not only restrictions on the types of $t = 0$ portfolios and performance fees the Principal can offer, but also these are also intimately linked with the return distribution — not just moments — of the compliant and deviant strategies.

Illustration of VaR constraints effects due to moral hazard

As we can see from Figures 3.1, 3.2, 3.3 and 3.4 (and see also its caption descriptions), whether a high or low realization in the wealths $(w_{P,1}^{(\theta_H, \tau_H)}, w_{P,1}^{(\theta_L, \tau_H)})$ of deviant strategy pairs at $t = 1$ makes a substantial difference in the $t = 1$ optimal policies. Effectively, if an extremely low realization of wealths under deviant strategy pairs are realized, to maintain incentive compatibility for those affected Managers, the Principal must redirect further extreme wealth allocations and performance fees to ensure that the terminal wealths between the continuation utilities under compliant strategy pair and the continuation utilities under deviant strategy pairs are equal. In contrast, when positive shocks are realized, the Principal need not consider extreme wealth allocation and performance fees for incentive compatibility. Clearly, the $t = 0$ optimal policies $(\pi_0, y_{A,0}, y_{B,0})$ of the Principal will endogenously alter the distribution of $t = 1$ wealths under the compliant and deviant strategy pairs; but as seen in those figures, the actual distribution of the strategy returns also play a critical input.

Thus, due to extreme differences between the $t = 1$ optimal policies response to the positive and negative shocks, which then propagate back to the $t = 0$ optimal policy choices, we can thus see that the incentive compatibility constraints are precisely akin to a VaR constraint on the $t = 0$ portfolio policies, where downside return shocks play a far more prominent role than upside return shocks. Furthermore, this is why extreme downside tail risks and joint probability tail dependence play a critical role in the contracting environment, and this motivates the discussion in Section 3.8.

²⁷ Note these $t = 1$ wealths are random from the perspective at $t = 0$.

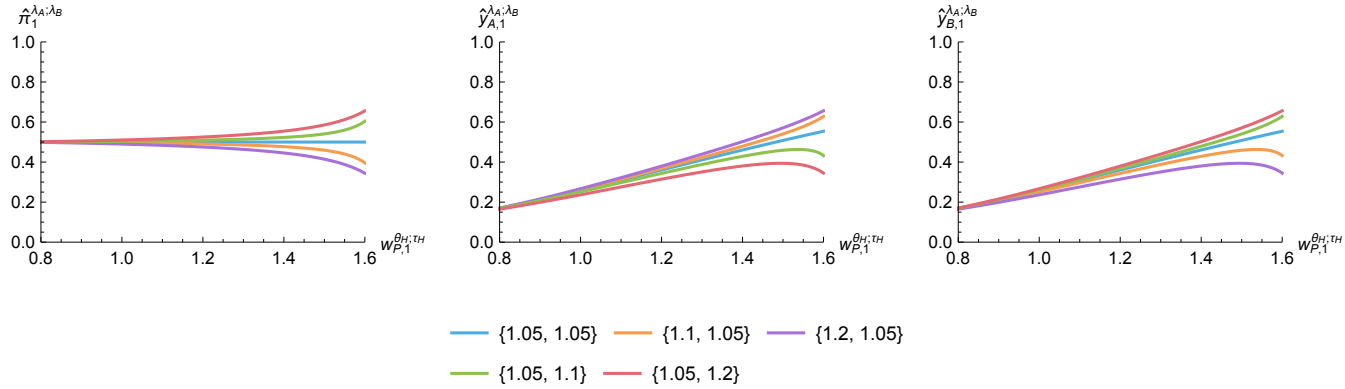


Figure 3.1: The $t = 1$ optimal policies of second best decentralized delegation when the wealths under deviant strategy pairs (θ_H, τ_L) and (θ_L, τ_H) have a positive shock. The base parameters are equivalent to that of Table 3.1 and with the amendment Table 3.1, which simply resets the deviant strategies to have equivalent means. We set the Lagrange multipliers $(\lambda_A, \lambda_B) = (-1.7, -1.7)$ as these are the numerical results associated with this set of base parameters when we actually numerically compute for the value function of the Principal. The vertical axis plots the respective $t = 1$ policies $(\hat{\pi}_1^{\lambda_A, \lambda_B}, \hat{y}_{A,1}^{\lambda_A, \lambda_B}, \hat{y}_{B,1}^{\lambda_A, \lambda_B})$, while the horizontal axis plots the $t = 1$ realized values $W_{P,1}^{(\theta_H, \tau_H)} = w_{P,1}^{(\theta_H, \tau_H)}$ of the Principal under the compliant strategy pair (θ_H, τ_H) . The legended pairs of values correspond to various scenarios of $t = 1$ realized wealths $(W_{P,1}^{(\theta_H, \tau_L)}, W_{P,1}^{(\theta_H, \tau_L)}) = (w_{P,1}^{(\theta_H, \tau_L)}, w_{P,1}^{(\theta_L, \tau_H)})$ under deviant strategy pairs; these wealths represent a positive realization to the returns $(R_{\theta_H}, R_{\tau_L})$ and $(R_{\theta_L}, R_{\tau_H})$, such that the realized wealths, respectively, $w_{P,1}^{(\theta_H, \tau_L)}$ and $w_{P,1}^{(\theta_L, \tau_H)}$ are greater than the Principal's $t = 0$ initial wealth of \$1. As an illustrative example, when the wealth $w_{P,1}^{(\theta_H, \tau_L)}$ (i.e. Manager A compliant, Manager B deviant) is higher than $w_{P,1}^{(\theta_L, \tau_H)}$ (i.e. Manager A deviant, Manager B compliant) and as the wealth $w_{P,1}^{(\theta_H, \tau_H)}$ under the compliant strategy pair increases, the Principal places less wealth $\hat{\pi}_1^{\lambda_A, \lambda_B}$ into Manager B and more wealth into Manager A . Furthermore, higher wealths $w_{P,1}^{(\theta_H, \tau_H)}$ will increase the $t = 1$ performance fees $(\hat{y}_{A,1}^{\lambda_A, \lambda_B}, \hat{y}_{B,1}^{\lambda_A, \lambda_B})$ for both Manager A and Manager B . However when the deviant wealth $w_{P,1}^{(\theta_H, \tau_L)}$ is high, it uniformly shifts up the performance fee $\hat{y}_{A,1}^{\lambda_A, \lambda_B}$ to Manager A , and uniformly shifts down the performance fee $\hat{y}_{B,1}^{\lambda_A, \lambda_B}$ to Manager B . Thus, in this case, there is fund in-flow and higher performance fees to the compliant Manager A , and fund out-flow and lower performance fees to the deviant Manager B . This is the case where the Principal is rewarding for compliance and punishing for deviance. But it is not necessarily true that being complaint necessarily implies higher fund flows. Suppose, however, the situation is reversed in that $w_{P,1}^{(\theta_H, \tau_L)}$ is lower than $w_{P,1}^{(\theta_L, \tau_H)}$. Then in this case, the Principal will allocate higher portfolio weights $\hat{\pi}_1^{\lambda_A, \lambda_B}$ to Manager B , and lower portfolio weights to Manager A . Moreover, the performance fees $\hat{y}_{A,1}^{\lambda_A, \lambda_B}$ to Manager A uniformly decreases, while the performance fees $\hat{y}_{B,1}^{\lambda_A, \lambda_B}$ to Manager B uniformly increases. Thus in this case, even though Manager A was complaint while Manager B was deviant, there is now fund out-flow and lower performance fees to the compliant Manager A , and fund in-flow and higher fee performance fees to the deviant Manager B . This is the case where the Principal is essentially rewarding for luck.

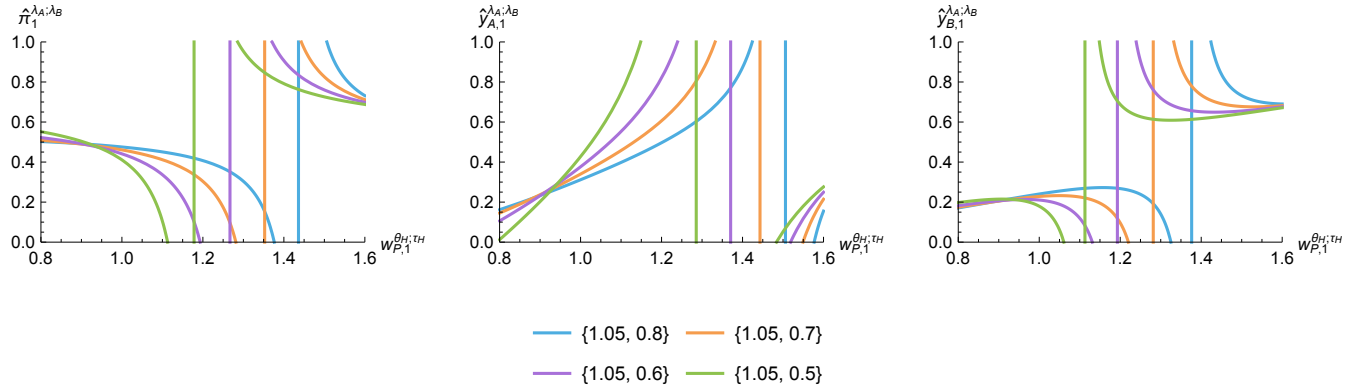


Figure 3.2: The $t = 1$ optimal policies of second best decentralized delegation when the wealths under deviant strategy pairs (θ_H, τ_L) and (θ_L, τ_H) have a negative shock. The setup and layout of this figure is identical to that of Figure 3.1. However, we assume in this case the wealth $w_{P,1}^{(\theta_H, \tau_L)}$ under the deviant strategy pair (θ_H, τ_L) had a positive shock due to positive realizations of $(R_{\theta_H}, R_{\tau_L})$, whereas the wealth $w_{P,1}^{(\theta_L, \tau_H)}$ under the deviant strategy pair (θ_L, τ_H) had a negative shock due to negative realizations of $(R_{\theta_L}, R_{\tau_H})$. It is evident that the negative shock case here is significantly different than that of the positive shock case of Figure 3.1. The discontinuity in the optimal performance fee process marks a cutoff in the wealth realization $w_{P,1}^{(\theta_H, \tau_H)}$ under the compliant strategy pair (θ_H, τ_H) . To the left of this cutoff the geometry of the portfolio policies are reversed to that of the right of this cutoff. Namely, for the portfolio policy $\hat{\pi}_1^{\lambda_A, \lambda_B}$, to the left of the cutoff, it is concave-like in the on-equilibrium wealth $w_{P,1}^{(\theta_H, \tau_H)}$, whereas to the right of the cutoff, it is convex-like; the similar can be said for the two performance fee policies. In this case, $w_{P,1}^{(\theta_L, \tau_H)}$ suffered a negative shock, and the deviant strategy pair (θ_L, τ_H) is the case when Manager A had deviated while Manager B was compliant. Given this, to ensure incentive compatibility for Manager A, the Principal must thus boost the wealth allocations to Manager A. Hence, as the compliant wealth $w_{P,1}^{(\theta_H, \tau_H)}$ increases, the Principal will reduce portfolio allocations $\hat{\pi}_1^{\lambda_A, \lambda_B}$ to Manager B and redirect them to Manager A. Likewise, the Principal will increase the performance fees $\hat{y}_{A,1}^{\lambda_A, \lambda_B}$ to Manager A, while decreasing the performance fees $\hat{y}_{B,1}^{\lambda_A, \lambda_B}$ to Manager B. Note also that there are indeed ranges of compliant wealths $w_{P,1}^{(\theta_H, \tau_H)}$ where no performance fees in $[0, 1]$ will exist, and these are the regions where a contract will not exist for decentralized delegation, and this is especially true if the realized wealths $w_{P,1}^{(\theta_H, \tau_H)}$ are not sufficiently high.

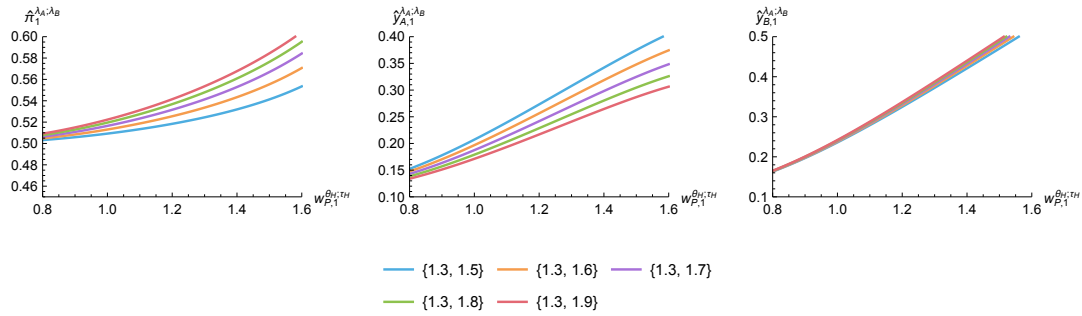


Figure 3.3: This is similar to Figure 3.1 except that both off-equilibrium wealths $(w_{P,1}^{(\theta_H, \tau_H)}, w_{P,1}^{(\theta_L, \tau_H)})$ enjoy a large positive shock.

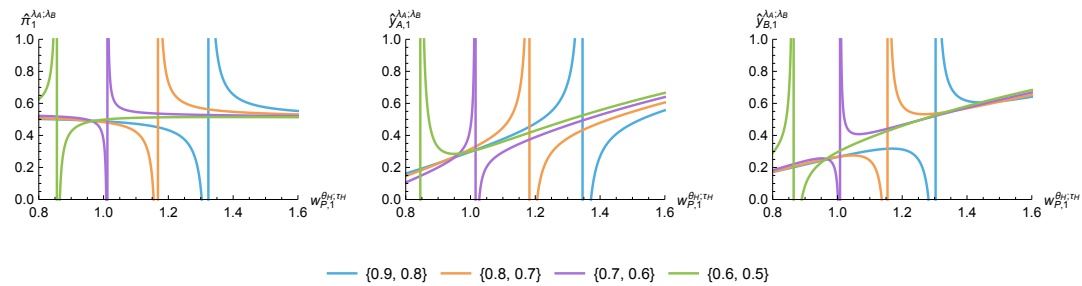


Figure 3.4: This is similar to Figure 3.2 except that both off-equilibrium wealths $(w_{P,1}^{(\theta_H, \tau_H)}, w_{P,1}^{(\theta_L, \tau_H)})$ suffer a negative shock.

Numerical illustrations of joint tail probabilities via copulas

As discussed in Section 3.8, the incentive compatibility constraints in dynamic decentralized delegation are akin to VaR constraints on the $t = 0$ portfolio and performance fee policies. Furthermore, as discussed and even further illustrated in Figures 3.1, 3.2, 3.3 and 3.4, we see that extreme downside risks are a particular concern. This motivates us to investigate how exactly do joint tail probabilities affect the contracting environment. For this purpose, we will need a method to model the joint tail probabilities, of which a natural tool is via *copulas*. Two numerical illustrations are given in Figure 3.5 and Figure 3.6. Please see details in Section 3.F.

To focus on the risk channel, we will assume that the deviant strategies θ_L and τ_L now have identical means. That is, for the numerical parameters in the following illustrations, we use the parameters of Table 3.1 but further amended with that of Table 3.1.

Mean return on deviant strategy θ_L	μ_{θ_L}	0.08
Mean return on deviant strategy τ_L	μ_{τ_L}	0.08

Table 3.1: The base parameters are amended to that of Table 3.1.

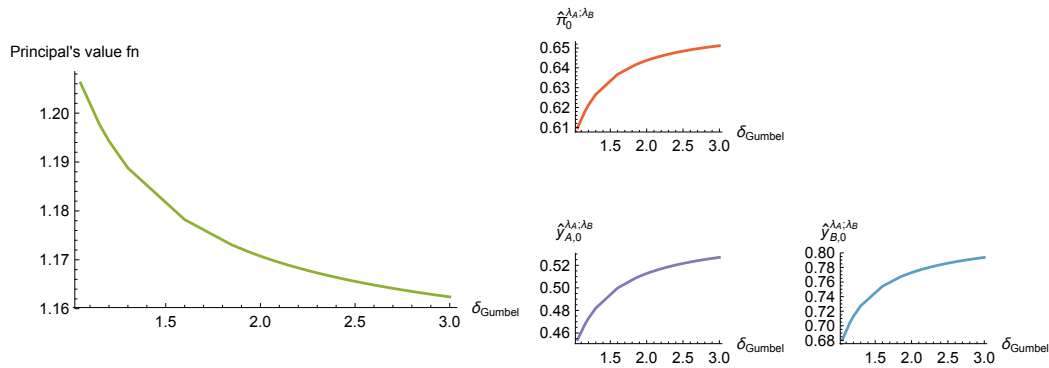


Figure 3.5: Numerical illustration of the Principal’s $t = 0$ value function along with the $t = 0$ optimal policies against various parameters of the Gumbel-Hougaard copula. The base parameters are the same as Table 3.1 along with the amendment Table 3.1. We assume the marginal distributions of the strategy returns $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$ follow by a discrete approximation to the normal distribution. The joint distribution is modeled by the Gumbel-Hougaard copula, in which it has a parameter $\delta_{\text{Gumbel}} \geq 1$. See Example 3.F.3 of Section 3.F for details. A low δ_{Gumbel} parameter implies the joint distribution is nearly independent, whereas a higher value implies increasing upper tail dependence of the joint distribution. As the joint returns have increasing upper tail dependence, the Principal’s value function decreases.

3.9 Conclusion

We study a problem of centralized delegation versus decentralized delegation, where there is moral hazard risk over the investment strategy choice within each asset class. In the static model under first best, it is simply a matter of which form of delegation that offers better risk sharing with respect to the investment strategies.

With the presence of moral hazard in centralized delegation, the Principal needs to compensate the single Manager C for the private costs of taking the Principal’s desired strategy pair but also the opportunity cost for any foregone long-short trading profits from deviant strategy pairs the Manager could have enjoyed. This implies if the Manager’s investment opportunity set is too wide, in that the mean return differences of the asset classes under management by Manager C are large, or that the Manager C is nearly risk neutral, no centralized contract will exist to implement the Principal’s desired strategy pair. In decentralized delegation, the restricted investment opportunity sets of the respective Manager A and Manager B confine their deviations to their own asset classes. Thus, the aforementioned

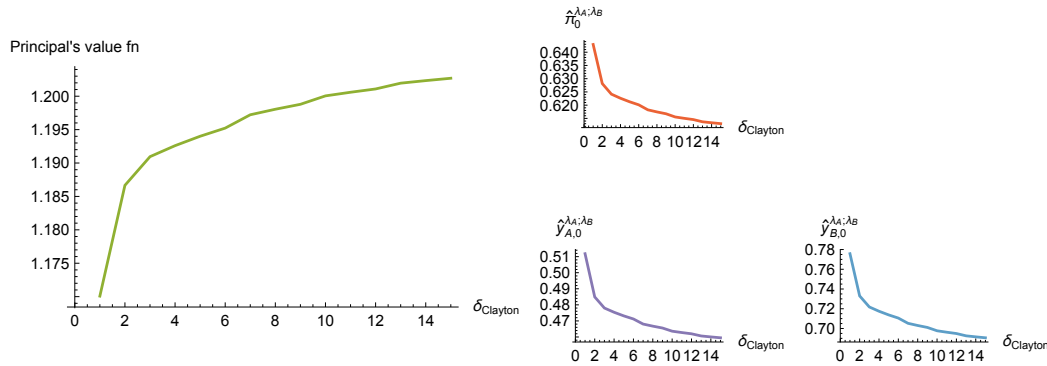


Figure 3.6: Numerical illustration of the Principal’s $t = 0$ value function along with the $t = 0$ optimal policies against various parameters of the Clayton copula. The base parameters are the same as Table 3.1 along with the amendment Table 3.1. We assume the marginal distributions of the strategy returns $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$ follow by a discrete approximation to the normal distribution. The joint distribution is modeled by the Clayton copula, in which it has a parameter $\delta_{\text{Clayton}} > 0$. See Example 3.F.1 of Section 3.F for details. A low δ_{Clayton} parameter implies the joint distribution has high lower tail dependence, whereas a higher value implies the joint distribution is almost independent. As the joint returns have increasing lower tail dependence, the Principal’s value function decreases.

opportunity cost in centralization simply does not exist in decentralization. But when the decentralized Manager A and Manager B consider a strategy deviation within their own respective asset classes, they only care for the mean and volatility differences between the compliant and deviant strategies, and do not take into account the correlation of their joint strategies. Hence, if the strategies within the asset classes have similar mean returns, it may be impossible for the Principal to induce his desired strategy pair with the decentralized Managers because only the Principal can capture the diversification benefits of the strategies’ correlations, and not Manager A and Manager B themselves.

In a dynamic decentralization model with committed reinvestments, Manager A and Manager B will have a motive to intertemporally hedge their future wealths with that of the Principal. Using the Principal’s intertemporal wealth as a bridge between multiple Managers’ payoffs, the Principal can thus incentivize the Managers to implement his desired investment strategy pair. The dynamic model shows that the incentive compatibility constraints can be viewed as value-at-risk constraints on the Principal’s portfolio and performance fee policies. Thus, even though individuals have mean-variance preferences and only linear contracts are considered, the Principal endogenously requires the knowledge of the full joint probability distribution of the compliant and deviant strategy pair returns, beyond just the first and second moments. The analogous dynamic centralized delegation model is studied in Section 3.B. Via copulas, we numerically investigate how tail dependence of strategy returns

affect the dynamic contracting environment.

It would be interesting to further study this problem of centralized versus decentralized delegation with a more general contract space, even in a static model. This paper focuses the contract space to be linear in returns for both centralization and decentralization. But as emphasized in the paper, the restriction versus relaxation in the investment opportunity set in centralization versus decentralization, respectively, drives most of the differences between the two delegation forms. This highly suggests that a “one size fits all” contract space for both delegation forms is inappropriate. In particular, this is to say that the problem suggests that the optimal contract in a more general contract space should look rather different in centralization versus decentralization. However, as also noted in the paper, when Managers can actively access the financial markets to modify the incentive effects, the determination of an optimal contract is really a joint problem of optimal contract design *and* optimal financial market restriction. We leave this interesting problem to future research.

Appendix

3.A Proofs for Section 3.4

Proof of Proposition 3.4.1. (a) Using first order sufficient and necessary conditions, we can see that the value to (3.3.2d) will be given by (3.4.1).

(b) Substitute in the optimal portfolio found above into the mean and variance expressions.

(c) Since the fixed fee $x_C \in \mathbb{R}$ is linear respect to the Principal's objective function, it implies that in equilibrium, the individual rationality constraint (3.3.2f) constraint binds. This implies the Principal's objective function can be rewritten as,

$$\begin{aligned} \mathbb{E}[W_{cP}^{(\theta,\tau)}] - \frac{\eta}{2} \text{Var}(W_{cP}^{(\theta,\tau)}) &= 1 + \mathbb{E}[\hat{R}_{(\theta,\tau)}] - (c(\theta) + c(\tau)) \\ &\quad - \frac{\eta_P}{2} y_C^2 \text{Var}(\hat{R}_{(\theta,\tau)}) - \frac{\eta_M}{2} (1 - y_C)^2 \text{Var}(\hat{R}_{(\theta,\tau)}). \end{aligned}$$

Now, by first order conditions on y_C , we see that the above becomes a fourth order polynomial (i.e. quartic) equation, and has the following roots,

$$y_C \in \left\{ \frac{\eta_P}{\eta_M + \eta_P}, -\frac{(\mu_\theta - \mu_\tau)^{2/3}}{(\eta_P^2 \sigma^4 (1 - \rho_{\theta\tau}^2))^{1/3}}, \pm \frac{(-1)^{2/3} (\mu_\theta - \mu_\tau)^{2/3}}{(\eta_P^2 \sigma^2 (1 - \rho_{\theta\tau}^2))^{1/3}} \right\}.$$

The first root is clearly in $(0, 1)$; the second root is negative and hence not in $(0, 1)$; the third and fourth roots (with \pm) are not in \mathbb{R} since $(-1)^{2/3} \in \mathbb{C}$. Thus, an interior solution exists and is uniquely given by the first root.²⁸

(d) Simply substitute in the optimal fixed and optimal fees found earlier.

(e) Analogous to the above. ■

Proof of Proposition 3.4.2. (a) By binding the (IR) constraints (3.3.3d), we obtain the optimal fixed fee form, and we can rewrite the Principal's objective function as,

$$\begin{aligned} \mathbb{E}[W_P^{(\theta,\tau)}] - \frac{\eta}{2} \text{Var}(W_P^{(\theta,\tau)}) &= -(c(\theta) + c(\tau)) + 1 + \pi(\mu_\tau - \mu_\theta) + \mu_\theta - \frac{\eta_M}{2} y_B^2 \pi^2 \sigma^2 - \frac{\eta_M}{2} y_A^2 (1 - \pi)^2 \sigma^2 \\ &\quad - \frac{\eta_P}{2} \left[\pi^2 ((1 - y_B)^2 \sigma^2 + (1 - y_A)^2 \sigma^2 - 2(1 - y_B)(1 - y_A) \rho_{\theta\tau} \sigma^2) \right. \\ &\quad \left. + 2\pi(1 - y_A) ((1 - y_B) \rho_{\theta\tau} \sigma^2 - (1 - y_A) \sigma^2) \right. \\ &\quad \left. + (1 - y_A)^2 \sigma^2 \right]. \end{aligned} \tag{3.A.1}$$

²⁸ It should be noted that in general, quartic equations (and naturally arising here because of first order conditions) are notoriously difficult to obtain simple and explicit solutions for. It is conjectured that if one extends to consider more than two risky investment strategies, or that we extend to more general non-linear contracts, it would be difficult to obtain a closed form contract for even first best centralized delegation. Indeed, the most difficult step in the proof of this Proposition 3.4.1 is this step, as everything else is straightforward. It was actually somewhat surprising to this author that despite a rather complicated first order condition, an economically sensible and intuitive solution for the performance fee arises.

- (b) By first order conditions applied to (3.A.1), we arrive at three different stationary points of (π, b, q) ,

$$(\pi, y_A, y_B) \in \left\{ \begin{aligned} & \left(0, \frac{\eta_P}{\eta_M + \eta_P}, 1 + \frac{(\eta_P + \eta_M)(\mu_\theta - \mu_\tau) - \eta_P \eta_M \sigma^2}{\eta_P \eta_M \rho_{\theta\tau} \sigma^2} \right), \\ & \left(1, 1 + \frac{(\eta_M + \eta_P)(\mu_\tau - \mu_\theta) - \eta_P \eta_M \sigma^2}{\eta_P \eta_M \rho_{\theta\tau} \sigma^2}, \frac{\eta_P}{\eta_M + \eta_P} \right), \\ & (\pi^o, y_A^o, y_B^o) \end{aligned} \right\}$$

The first and second stationary points, which would imply zero wealth invested into either of the agents, will violate the individual rationality constraint (3.3.3d). Thus, only the third stationary point is a candidate for an interior solution.

- (c) This is simply applying Assumption 3.3.2. The value function computation is straightforward. ■

Proof of Proposition 3.4.3. Use Proposition 3.4.1 and Proposition 3.4.2. ■

3.B Additional Results and Proofs for Section 3.5

Proof of Proposition 3.5.1. (a) This is the same proof as that of Proposition 3.4.1.

- (b) This is evident since the arguments in Proposition 3.4.1 for deriving Manager C 's optimal portfolio choice holds true for any arbitrary contract.
- (c) By Assumption 3.3.2 and Proposition 3.4.1, if the Principal wants to implement and induce the investment strategy pair (θ_H, τ_H) , then the Principal needs to write a contract that prevents Manager C from taking on the deviant strategies $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$. These are captured by the incentive compatibility constraints in (3.5.1). One should note that these three constraints can be collapsed to a single one by equivalently writing,

$$\begin{aligned} & -2c + \frac{1}{2}(\mu_{\theta_H} + \mu_{\tau_H})y - \frac{1}{4}\eta_M \sigma^2(1 + \rho_{\theta_H, \tau_H})y^2 \\ & \geq \max_{(\theta', \tau')} \left\{ -c(\theta') + c(\tau') + \frac{1}{4} \frac{(\mu_{\theta'} - \mu_{\tau'})^2}{\eta_M \sigma^2(1 - \rho_{\theta', \tau'})} + \frac{1}{2}(\mu_{\theta'} + \mu_{\tau'})y - \frac{1}{4}\eta_M \sigma^2(1 + \rho_{\theta', \tau'})y^2 \right\}, \end{aligned} \quad (3.B.1)$$

where we take the maximum on the right hand side over $(\theta', \tau') \in \mathcal{S}_{-(\theta_H, \tau_H)}$, which is clearly then equivalent to (3.5.2).

Note that by Assumption 3.3.2, we have that $\mu_{\theta'} \leq \mu_{\theta_H}$ and $\mu_{\tau'} \leq \mu_{\tau_H}$, and where at least one of these two inequalities are strict, and hence $\mu_{\theta'} - \mu_{\theta_H} + \mu_{\tau'} - \mu_{\tau_H} < 0$. Likewise, $c(\theta') + c(\tau') - 2c < 0$. However, since we only assume that the correlations $\rho_{\theta, \tau}$ for all investment strategy pairs (θ, τ) are different, and in particular no special sign and order restrictions, so we have that if $\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H} > 0$, then the component is concave in y , and if $\rho_{\theta', \tau'} - \rho_{\theta_H, \tau_H} < 0$, it is convex in y . Thus, we have a pointwise maximum of convex and/or concave functions, and in general, one has no particular geometric form of this.

- (d) From the condition (3.5.2), we substitute in the first best solution to check the condition under which none of the incentive compatibility constraints will bind. This is condition (3.5.3).
- (e) Suppose the conditions on the private costs (3.5.3) are such that a first best solution will not be attained in second best. While we could indeed proceed to use Kuhn-Tucker conditions (with say three Kuhn-Tucker multipliers) to solve for the optimal solution, we can proceed with a much more geometric proof here. Firstly, by (3.5.1) or equivalently (3.5.2), it is clear that when a binding solution (that is in $[0, 1]$) exists, only one of the constraints will bind. Suppose that $(\theta^b, \tau^b) \in \mathcal{S}_{-(\theta_H, \tau_H)}$ is the pair of deviant investment strategies for which its associated incentive compatibility constraint binds.

Given the quadratic form of constraints, we are motivated to define the discriminant for the binding deviant pairs (θ', τ') (3.5.4). Notice that the sign of the discriminant is heavily dependent on the sign of $\rho_{\theta^b, \tau^b} - \rho_{\theta_H, \tau_H}$. Provided that

$D_{(\theta^b, \tau^b)} \geq 0$, so that roots will exist for the quadratic associated with the binding incentive compatibility constraint (θ^b, τ^b) , we compute the roots as,

$$\begin{aligned} \tilde{y}_{\pm, (\theta^b, \tau^b)} &= \frac{1}{2} \left(-\frac{\mu_{\theta^b} - \mu_{\theta_H} + \mu_{\tau^b} - \mu_{\tau_H}}{2} \pm \sqrt{D_{(\theta^b, \tau^b)}} \right) \\ &\quad \times \left[-c(\theta^b) + c(\tau^b) - 2c + \frac{1}{4} \frac{(\mu_{\theta^b} - \mu_{\tau^b})^2}{\eta_M \sigma^2 (1 - \rho_{\theta_H, \tau_H})} \right]^{-1}. \end{aligned}$$

With our current assumptions, it is not difficult to show that the negative root $\tilde{y}_{-, (\theta^b, \tau^b)} < 0$. Thus, let's focus on the positive root $\tilde{y}_{+, (\theta^b, \tau^b)}$ of (3.5.5). We must now recall that our solution must be confined in $[0, 1]$. Hence, a second best solution will exist only if $\tilde{y}_{+, (\theta^b, \tau^b)} \in [0, 1]$, and likewise, if $\tilde{y}_{+, (\theta^b, \tau^b)} \notin [0, 1]$, then no second best solution will exist. ■

Corollary 3.B.1. *Consider the second best centralized delegation setup in Proposition 3.5.1, and suppose the conditions (i.e. conditions (i) and (ii) of part (d)) for the existence of a second best contract holds. In particular, recall (3.5.5). Then the most profitable deviant investment strategy (θ^b, τ^b) for Manager C is the following and given under the following conditions, which then leads to the optimal performance fee $\hat{y}_C = \tilde{y}_{+, (\theta', \tau')}$.*

(a) *The optimal performance fee is $\tilde{y}_{+, (\theta_H, \tau_L)}$ when,*

$$\begin{aligned} 0 \geq \max \left\{ c + \frac{1}{4} \frac{(\mu_{\theta_L} - \mu)^2}{\eta_M \sigma^2 (1 - \rho_{\theta_L, \tau_H})} - \frac{1}{2} \Delta \mu_{\tau} \tilde{y}_{+, (\theta_H, \tau_L)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_L, \tau_H} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_H, \tau_L)}^2, \right. \\ \left. 2c + \frac{1}{4} \frac{(\mu_{\theta_L} - \mu_{\tau_L})^2}{\eta_M \sigma^2 (1 - \rho_{\theta_L, \tau_L})} - \frac{1}{2} (\Delta \mu_{\theta} + \Delta \mu_{\tau}) \tilde{y}_{+, (\theta_H, \tau_L)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_L, \tau_L} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_H, \tau_L)}^2 \right\}. \end{aligned}$$

(b) *The optimal performance fee is $\tilde{y}_{+, (\theta_L, \tau_H)}$ when,*

$$\begin{aligned} 0 \geq \max \left\{ c + \frac{1}{4} \frac{(\mu - \mu_{\tau_L})^2}{\eta_M \sigma^2 (1 - \rho_{\theta_H, \tau_L})} - \frac{1}{2} \Delta \mu_{\theta} \tilde{y}_{+, (\theta_L, \tau_H)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_H, \tau_L} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_L, \tau_H)}^2, \right. \\ \left. 2c + \frac{1}{4} \frac{(\mu_{\theta_L} - \mu_{\tau_L})^2}{\eta_M \sigma^2 (1 - \rho_{\theta_L, \tau_L})} - \frac{1}{2} (\Delta \mu_{\theta} + \Delta \mu_{\tau}) \tilde{y}_{+, (\theta_L, \tau_H)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_L, \tau_L} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_L, \tau_H)}^2 \right\}. \end{aligned}$$

(c) *The optimal performance fee is $\tilde{y}_{+, (\theta_L, \tau_L)}$ when,*

$$\begin{aligned} 0 \geq \max \left\{ c + \frac{1}{4} \frac{(\mu_{\theta_L} - \mu)^2}{\eta_M \sigma^2 (1 - \rho_{\theta_L, \tau_H})} - \frac{1}{2} \Delta \mu_{\tau} \tilde{y}_{+, (\theta_L, \tau_L)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_L, \tau_H} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_L, \tau_L)}^2, \right. \\ \left. c + \frac{1}{4} \frac{(\mu - \mu_{\tau_L})^2}{\eta_M \sigma^2 (1 - \rho_{\theta_H, \tau_L})} - \frac{1}{2} \Delta \mu_{\theta} \tilde{y}_{+, (\theta_L, \tau_L)} - \frac{1}{4} \eta_M^2 \sigma^2 (\rho_{\theta_H, \tau_L} - \rho_{\theta_H, \tau_H}) \tilde{y}_{+, (\theta_L, \tau_L)}^2 \right\}. \end{aligned}$$

Proof of Corollary 3.B.1. This is simply rewriting out the condition (3.5.2) more explicitly. ■

Proof of Proposition 3.5.2. (a) This is simply by binding the (IR) constraints (3.3.3d).

(b) This is simply rewriting the (IC) constraints (3.3.3e), (3.3.3f).

(c) This will be seen as a special case of Proposition 3.B.2(aiv). ■

Proposition 3.B.2. *Recall the setup of Proposition 3.5.2.*

(a) *Consider the following conditions on the private cost c imply the optimal second best decentralized delegation optimal portfolio and performance fee policies $(\hat{\pi}, \hat{y}_A, \hat{y}_B)$ have the following form:*

(i) If,

$$0 < c \leq \eta_P \Delta\mu_\theta \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) \min \left\{ \frac{1}{\eta_P \Delta\mu_\theta (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\tau (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))}, \frac{1}{\eta_P \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\theta (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \right\},$$

then,

$$(\hat{\pi}, \hat{y}_A, \hat{y}_B) = (\hat{\pi}^{FB}, \hat{y}_A^{FB}, \hat{y}_B^{FB}) = \left(\frac{1}{2}, \frac{\eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})}, \frac{\eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})} \right).$$

(ii) If,

$$\begin{aligned} & \frac{\Delta\mu_\theta \Delta\mu_\tau \eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_P \Delta\mu_\theta (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\tau (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \\ & < c \\ & \leq \frac{\Delta\mu_\theta \Delta\mu_\tau \eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_P \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\theta (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \wedge \Delta\mu_\theta, \end{aligned}$$

then,

$$\begin{aligned} & (\hat{\pi}, \hat{y}_A, \hat{y}_B) \\ & = \left(\frac{(\Delta\mu_\theta - c) (\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))}{\Delta\mu_\theta (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))}, \frac{c (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))}{\eta_M \Delta\mu_\theta + c [\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})]}, \frac{\eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})} \right). \end{aligned}$$

(iii) If,

$$\begin{aligned} & \frac{\Delta\mu_\theta \Delta\mu_\tau \eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_P \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\theta (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \\ & < c \\ & \leq \frac{\eta_P \Delta\mu_\theta \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H})}{\eta_P \Delta\mu_\theta (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\tau (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \wedge \Delta\mu_\tau \end{aligned}$$

then,

$$\begin{aligned} & (\hat{\pi}, \hat{y}_A, \hat{y}_B) \\ & = \left(\frac{\eta_M \Delta\mu_\tau + c [\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})]}{\Delta\mu_\tau [2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})]}, \frac{\eta_P (1 + \rho_{\theta_H, \tau_H})}{\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})}, \frac{c [2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})]}{\eta_M \Delta\mu_\tau + c [\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H})]} \right). \end{aligned}$$

(iv) If,

$$\begin{aligned} & \eta_P \Delta\mu_\theta \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) \max \left\{ \frac{1}{\eta_P \Delta\mu_\theta (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\tau (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))}, \frac{1}{\eta_P \Delta\mu_\tau (1 + \rho_{\theta_H, \tau_H}) + \Delta\mu_\theta (2\eta_M + \eta_P (1 + \rho_{\theta_H, \tau_H}))} \right\} \\ & < c < \frac{\Delta\mu_\theta \Delta\mu_\tau}{\Delta\mu_\theta + \Delta\mu_\tau}, \end{aligned}$$

then,

$$(\hat{\pi}, \hat{y}_A, \hat{y}_B) = \left(\frac{1}{2} \left[1 + \frac{\Delta\mu_\tau - \Delta\mu_\theta}{\Delta\mu_\theta \Delta\mu_\tau} c \right], \frac{2\Delta\mu_\tau c}{c(\Delta\mu_\tau - \Delta\mu_\theta) + \Delta\mu_\theta \Delta\mu_\tau}, \frac{2\Delta\mu_\theta c}{c(\Delta\mu_\theta - \Delta\mu_\tau) + \Delta\mu_\theta \Delta\mu_\tau} \right).$$

- (v) *Else if none of the conditions above are satisfied, then there does not exist an optimal second best decentralized delegation contract.*

Proof of Proposition 3.B.2. (a) After binding the (IR) constraints (3.3.3d) into the Principal's objective function, it remains that the portfolio and performance fee policy (π, y_A, y_B) have to respect the (IC) constraints (3.5.6), and the box constraints $(\pi, y_B, y_A) \in \mathbb{R} \times [0, 1]^2$. However, we observe that (π, y_A, y_B) being on the boundary of $[0, 1]^3$ would immediately violate either the (IR) constraints, the (IC) constraints, or both. Hence, for a solution to exist, (π, y_A, y_B) must be in the interior of $[0, 1]^3$, that being $(0, 1)^3$. Hence, given a feasible solution $(\pi, y_A, y_B) \in (0, 1)^3$, we must then cycle through the $2 \times 2 = 4$ cases where either the (IC) constraint (3.5.6a) of Manager A bind or not, and whether (IC) constraint (3.5.6b) of Manager B bind or not.

- (i) This is the case when we obtain an interior solution and neither (IC) of Manager A nor (IC) of Manager B bind. Substitute in the first best solution from Proposition 3.4.2, under Assumption 3.3.2, into (3.5.6) and replace \geq with $>$ to get the conditions on the private costs c .
- (ii) This is the case when only (IC) of Manager A binds and when that of Manager B does not bind. This happens when, after substituting the first best solution into (3.5.6), and we obtain,

$$\begin{aligned}\Delta\mu_\theta\eta_P(1 + \rho_{\theta_H, \tau_H}) &> 2c(\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H})), \\ \Delta\mu_\tau\eta_P(1 + \rho_{\theta_H, \tau_H}) &\leq 2c(\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H})).\end{aligned}$$

The binding condition also allows for us to get the portfolio policy π as a function of y_A . Via first order conditions on the objective function, substitute back and then we solve for (π, y_A, y_B) . However, we still need to satisfy the interior box constraints $(\pi, y_A, y_B) \in (0, 1)^3$. We have $y_A \in (0, 1)$ holding. Here, $y_B > 0$ and to have $y_B < 1$, we need,

$$c < \Delta\mu_\tau.$$

Under such condition, we would also have $\pi \in (0, 1)$. Putting those three conditions on the private cost c together yields the displayed condition.

- (iii) This is the case when (IC) of Manager A does not bind, but that of Manager B does bind. The argument is completely analogous to the previous one.
- (iv) This is the case when both (IC)'s of Manager A and Manager B bind. Here, we need to differentiate between two sub-cases — when $\Delta\mu_\theta = \Delta\mu_\tau$ and when $\Delta\mu_\theta \neq \Delta\mu_\tau$.

If $\Delta\mu_\theta = \Delta\mu_\tau \equiv \Delta\mu$, then we immediately have that $(\pi, y_A, y_B) = (1/2, 2c/\Delta\mu, 2c/\Delta\mu)$. So, the condition to ensure that $(\pi, y_A, y_B) \in (0, 1)^3$ is clearly when,

$$c < \frac{\Delta\mu}{2}.$$

Suppose $\Delta\mu_\theta \neq \Delta\mu_\tau$, and without loss of generality, suppose $\Delta\mu_\theta > \Delta\mu_\tau$. To have $y_A > 0$, we would need,

$$\frac{\Delta\mu_\theta\Delta\mu_\tau}{\Delta\mu_\theta - \Delta\mu_\tau} > c,$$

and to have $y_A < 1$, one would need,

$$c < \frac{\Delta\mu_\theta\Delta\mu_\tau}{\Delta\mu_\theta + \Delta\mu_\tau}.$$

Finally, to have $\pi > 0$, we would need,

$$c < \frac{\Delta\mu_\theta\Delta\mu_\tau}{\Delta\mu_\theta - \Delta\mu_\tau}.$$

Putting these conditions together implies we need,

$$\frac{\eta(1 + \rho_{\theta_H, \tau_H})}{2(\eta_M + \eta_P(1 + \rho_{\theta_H, \tau_H}))} \Delta\mu_\theta \leq c < \min \left\{ \frac{\Delta\mu_\theta\Delta\mu_\tau}{\Delta\mu_\theta - \Delta\mu_\tau}, \frac{\Delta\mu_\theta\Delta\mu_\tau}{\Delta\mu_\theta + \Delta\mu_\tau} \right\}.$$

Simplifying and generalizing to the case when $\Delta\mu_\theta < \Delta\mu_\tau$, we have the displayed condition. ■

When there is only moral hazard over mean returns

An interesting special case that neither neither potentially favors nor biases centralized delegation is when there is no moral hazard over correlations, $\rho \equiv \rho_{\theta\tau}$ for all (θ, τ) , and the potential mean return losses between the two investment strategies are identical, $\Delta\mu \equiv \Delta\mu_\theta = \Delta\mu_\tau$. In this case, the incentive compatibility constraints (3.5.6) of decentralized delegation have the form,

$$0 \geq c - (1 - \pi)y_A\Delta\mu, \quad (3.B.2a)$$

$$0 \geq c - \pi y_B\Delta\mu, \quad (3.B.2b)$$

which is effectively the same form as before, but the incentive compatibility constraint (3.5.1) for centralized delegation reduces to,

$$0 \geq \max_{(\theta', \tau')} \{2c - (c(\theta') + c(\tau')) + \Delta\mu y_C\} = 2c + \Delta\mu y_C \quad (3.B.3)$$

Thus, in this special case for centralized delegation, the centralized Manager C has incentives that are very much aligned with the Principal, as the alternative investment strategies (θ_L, τ_L) have the same mean $\mu_{\theta_L} = \mu_{\tau_L} = \mu - \Delta\mu$, same volatility and same correlations, this implies that a long-short strategy is not profitable.

Corollary 3.B.3. *Assume that there is no moral hazard over correlations $\rho \equiv \rho_{\theta\tau}$ for all strategy pairs $(\theta, \tau) \in \mathcal{S}$, and the mean return differences between the two strategies are identical, $\Delta\mu \equiv \Delta\mu_\theta = \Delta\mu_\tau > 0$.*

(a) Consider the second best centralized delegation problem.

(i) The optimal performance fee is,

$$\hat{y}_C = \begin{cases} \hat{y}_C^{FB}, & 0 < c < \frac{1}{2} \frac{\eta_P}{\eta_P + \eta_M} \Delta\mu \\ \frac{2c}{\Delta\mu}, & \frac{1}{2} \frac{\eta_P}{\eta_P + \eta_M} \Delta\mu \leq c < \frac{\Delta\mu}{2} \\ \emptyset, & \text{otherwise.} \end{cases}$$

(ii) The associated Principal's value function in second best centralized delegation is,

$$\begin{aligned} & \mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \\ &= \begin{cases} -2c + \mu - \frac{\eta_P \eta_M}{4(\eta_M + \eta_P)} \sigma^2 (1 + \rho), & 0 < c < \frac{1}{2} \frac{\eta}{\eta + \eta_M} \Delta\mu, \\ -2c + \mu - \frac{[4\eta_M c^2 + \eta_P (\Delta\mu - 2c)^2] (1 + \rho) \sigma^2}{4(\Delta\mu)^2}, & \frac{1}{2} \frac{\eta_P}{\eta_P + \eta_M} \Delta\mu \leq c < \frac{\Delta\mu}{2}, \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Consider the second best decentralized delegation problem.

(i) The optimal portfolio and performance fee policies are,

$$(\hat{\pi}, \hat{y}_A, \hat{y}_B) = \begin{cases} \left(\frac{1}{2}, \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)}, \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)} \right), & 0 < c < \frac{1}{2} \frac{\eta(1+\rho)}{\eta_M + \eta_P(1+\rho)} \Delta\mu, \\ \left(\frac{1}{2}, \frac{2c}{\Delta\mu}, \frac{2c}{\Delta\mu} \right), & \frac{1}{2} \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)} \leq c < \frac{\Delta\mu}{2}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

(ii) The associated Principal's value function in second best decentralized delegation is,

$$\begin{aligned} & \mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \\ &= \begin{cases} -2c + \mu - \frac{\eta_P \eta_M (1 + \rho) \sigma^2}{4(\eta_M + \eta_P (1 + \rho))}, & 0 < c < \frac{1}{2} \frac{\eta_P (1 + \rho)}{\eta_M + \eta_P (1 + \rho)} \Delta\mu, \\ -2c + \mu - \frac{\sigma^2 [4(\eta_M + \eta_P (1 + \rho)) c^2 - \eta_P (1 + \rho) \Delta\mu (\Delta\mu - 4c)]}{4(\Delta\mu)^2}, & \frac{1}{2} \frac{\eta_P (1 + \rho)}{\eta_M + \eta_P (1 + \rho)} \leq c < \frac{\Delta\mu}{2}, \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Let's compute the difference between the Principal's value function under second best decentralized delegation and that of second best decentralized delegation.

(i) Suppose $\rho \in (-1, 0)$. Then $\frac{\eta_P}{\eta_P + \eta_M} > \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)}$, and,

$$\begin{aligned} & \left(\mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \right) - \left(\mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \right) \\ &= \begin{cases} \frac{\eta_M \eta_P^2 \rho (1+\rho) \sigma^2}{4(\eta_M + \eta_P)(\eta_M + \eta_P(1+\rho))}, & 0 < c < \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)}, \\ \frac{\sigma^2 [4c \Delta \mu \eta_P (\eta_M + \eta_P) - 4c^2 (\eta_M + \eta_P)(\eta_M + \eta_P(1+\rho)) - (\Delta \mu)^2 \eta_P^2 (1+\rho)]}{4(\eta_P + \eta_M)(\Delta \mu)^2}, & \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)} \leq c < \frac{\eta_P}{\eta_P + \eta_M}, \\ \frac{\eta_M c^2 \sigma^2}{(\Delta \mu)^2} \rho, & \frac{\eta_P}{\eta_P + \eta_M} \leq c < \frac{\Delta \mu}{2}, \\ \text{undefined}, & \text{otherwise.} \end{cases} \end{aligned}$$

In particular, for all $c \in (0, \Delta \mu / 2)$,

$$\left(\mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \right) - \left(\mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \right) < 0.$$

(ii) Suppose $\rho \in (0, 1)$. Then $\frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)} > \frac{\eta_P}{\eta_P + \eta_M}$, and,

$$\begin{aligned} & \left(\mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \right) - \left(\mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \right) \\ &= \begin{cases} \frac{\eta_M \eta_P^2 \rho (1+\rho) \sigma^2}{4(\eta_M + \eta_P)(\eta_M + \eta_P(1+\rho))}, & 0 < c < \frac{1}{2} \frac{\eta_P}{\eta_P + \eta_M}, \\ \frac{\sigma^2 (1+\rho)}{4} \left(\eta_P + \frac{4c[c(\eta_M + \eta_P) - \eta_P \Delta \mu]}{(\Delta \mu)^2} - \frac{\eta_P \eta_M}{\eta_M + \eta_P(1+\rho)} \right), & \frac{1}{2} \frac{\eta_P}{\eta_M + \eta_P} \leq c < \frac{1}{2} \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)}, \\ \frac{\eta_M c^2 \sigma^2}{(\Delta \mu)^2} \rho, & \frac{1}{2} \frac{\eta_P(1+\rho)}{\eta_M + \eta_P(1+\rho)} \leq c < \frac{\Delta \mu}{2}, \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

In particular, for all $c \in (0, \Delta \mu / 2)$,

$$\left(\mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \right) - \left(\mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \right) > 0.$$

(iii) If $\rho = 0$, then for all $c \in (0, \Delta \mu / 2)$,

$$\left(\mathbb{E}[W_P] - \frac{\eta_P}{2} \text{Var}(W_P) \Big|_{SB, (\theta_H, \tau_H)} \right) - \left(\mathbb{E}[W_{cP}] - \frac{\eta_P}{2} \text{Var}(W_{cP}) \Big|_{SB, (\theta_H, \tau_H)} \right) = 0.$$

Proof of Corollary 3.B.3. This is a special case of Proposition 3.5.1 and Proposition 3.5.2. ■

Corollary 3.B.3 illustrates that when there is no moral hazard over correlations, $\rho \equiv \rho_{\theta_H, \tau_H} = \rho_{\theta_L, \tau_L}$, and that the mean return losses due to moral hazard are equal, $\Delta \mu \equiv \Delta \mu_\theta = \Delta \mu_\tau$, then this substantially aligns the interests of the centrally delegated single Manager C . And as a result, in this special case, our results are essentially identical to the first best case of Proposition 3.4.3 that we had studied earlier. In particular, centralized delegation is favored when the correlations are negative $\rho < 0$, decentralized delegation is favored when the correlations are positive $\rho > 0$, and both forms of delegation are equal when the investment strategies are uncorrelated $\rho = 0$.

This Online Appendix continues its section numbering from the Appendix of the main text.

Appendix

3.A Principle of Dynamic Programming

As it is well known in the literature ²⁹ of extending mean-variance analysis to multiple periods, the mean-variance utility over terminal wealth is not directly amenable to a recursive form for dynamic programming. Here, however, we will take effectively the discrete-time approach as motivated by Basak and Chabakauri (2010).

Consider time periods $t = 0, 1, \dots, T$. Consider the $t = 0$ mean-variance utility over a time $t = T$ random variable W_T ,

$$U_0 := \mathbb{E}_0[W_T] - \frac{\eta}{2} \text{Var}_0(W_T).$$

Also let us define the time t continuation utility,

$$U_t := \mathbb{E}_t[W_T] - \frac{\eta}{2} \text{Var}_t(W_T), \quad \text{for } t = 0, 1, \dots, T, \quad (3.A.1)$$

and note that $U_T = W_T$. The *law of total variance* states that,

$$\text{Var}_t(W_T) = \mathbb{E}_t[\text{Var}_{t+1}(W_T)] + \text{Var}_t(\mathbb{E}_{t+1}[W_T]), \quad \text{for } t = 0, 1, \dots, T-1. \quad (3.A.2)$$

Substituting (3.A.2) into (3.A.1), and using the law of iterated expectations, we observe that,

$$\begin{aligned} U_t &= \mathbb{E}_t[W_T] - \frac{\eta}{2} (\mathbb{E}_t[\text{Var}_{t+1}(W_T)] + \text{Var}_t(\mathbb{E}_{t+1}[W_T])) \\ &= \mathbb{E}_t[W_T] - \frac{\eta}{2} \mathbb{E}_t[\text{Var}_{t+1}(W_T)] - \frac{\eta}{2} \text{Var}_t(\mathbb{E}_{t+1}[W_T]) \\ &= \mathbb{E}_t \left[\mathbb{E}_{t+1}[W_T] - \frac{\eta}{2} \text{Var}_{t+1}(W_T) \right] - \frac{\eta}{2} \text{Var}_t(\mathbb{E}_{t+1}[W_T]) \\ &= \mathbb{E}_t[U_{t+1}] - \frac{\eta}{2} \text{Var}_t(\mathbb{E}_{t+1}[W_T]). \end{aligned}$$

Thus, the equations that gives the backward recursive relationship are,

$$\begin{aligned} U_T &= W_T, \\ U_t &= \mathbb{E}_t[U_{t+1}] - \frac{\eta}{2} \text{Var}_t(\mathbb{E}_{t+1}[W_T]), \quad t = 0, 1, \dots, T-1. \end{aligned} \quad (3.A.3)$$

In the actual application considered in this paper, we will use $T = 2$ and so we will consider time periods $t = 0, 1, 2$.

3.B Dynamic Centralized Delegation

In the dynamic centralized delegation model, the Principal will offer to Manager C a long term contract, consisting of a fixed fee ³⁰ $x_C \in \mathbb{R}$ paid at $t = 2$, and multi-period performance fees $(y_{C,0}, y_{C,1}) \in \times [0, 1]$ over, respectively, the ending period

²⁹ Please see Li and Ng (2000), Basak and Chabakauri (2010), Basak and Chabakauri (2012) and others.

³⁰ Paying the fixed fee at the end of the contracting period substantially simplifies the discussion below. If a performance fee is also paid at the beginning of the contracting period $t = 0$, then this fixed fee also factors into the portfolio choice of Manager C at $t = 1$. This is clearly very possible, but this is not the *raison d'être* of extending our discussion from a simple one period model to a dynamic two period model.

wealth $W_{cP,1}$ and $W_{cP,2}$. Manager C will choose to either accept or reject the contract. If Manager C accepts and commits to the contract, then at $t = 0$ the Principal gives his initial wealth of \$1 to the single centralized Manager C who has \$0 initial wealth. Then Manager C will commit to a long term strategy $(\theta, \tau) \in \mathcal{S}$. Subsequently, Manager C will choose portfolio weights $(1 - \psi_0, \psi_0)$ into the strategies with a period return $(R_{\theta,1}, R_{\tau,1})$. At the end of period $t = 1$, the Principal will pay $y_{C,0}\hat{R}_{(\theta,\tau),1}$ to Manager C , where $\hat{R}_{(\theta,\tau)}$ is the managed portfolio of Manager C . Thus, after fees, the $t = 1$ wealth of the Principal is $W_{cP,1}$, and the $t = 1$ wealth of Manager C is $W_{C,1}$. For simplicity, with the available wealth of $W_{C,1}$, Manager C will make no further investments. At $t = 1$, the Principal will reinvest with Manager C again, and so Manager C will choose portfolio weights $(1 - \psi_1, \psi_1)$ into the strategies with returns $(R_{\theta,2}, R_{\tau,2})$. At the end of the period, Manager C will be paid $W_{cP,1}y_{C,1}\hat{R}_{(\theta,\tau),2}$.

In all, the optimization problem for centralized delegation is as follows. Please see Figure 3.B.1 for a time line.

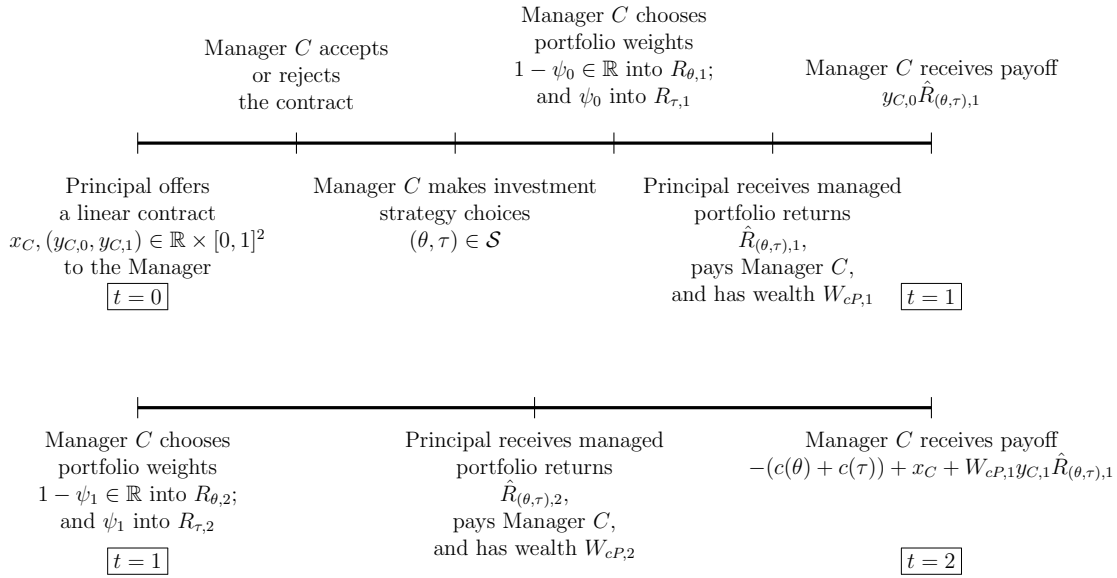


Figure 3.B.1: Dynamic centralized delegation time line.

$$\sup_{\substack{x_C \in \mathbb{R}, \\ y_{C,0}, y_{C,1} \in [0,1]}} -x_C + \mathbb{E}_0[W_{cP,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}), \quad (\text{DynCen})$$

subject to,

$$W_{cP,1}^{(\theta, \tau)} := 1 + \hat{R}_{1,(\theta, \tau)} - y_{C,0} \hat{R}_{1,(\theta, \tau)}, \quad (3.B.1a)$$

$$W_{cP,2}^{(\theta, \tau)} := W_{cP,1}^{(\theta, \tau)} \left[1 + \hat{R}_{2,(\theta, \tau)} - y_{C,1} \hat{R}_{2,(\theta, \tau)} \right], \quad (3.B.1b)$$

$$W_{C,1}^{(\theta, \tau)} := y_{C,0} \hat{R}_{1,(\theta, \tau)}, \quad (3.B.1c)$$

$$W_{C,2}^{(\theta, \tau)} := W_{C,1}^{(\theta, \tau)} + y_{C,1} W_{cP,1}^{(\theta, \tau)} \hat{R}_{2,(\theta, \tau)}, \quad (3.B.1d)$$

$$\{\hat{\psi}_{t,(\theta, \tau)}\}_{t=0,1} := \arg \max_{\substack{\{\psi_t\}_{t=0,1}, \\ \psi_t \in \mathbb{R}}} \mathbb{E}_0[\tilde{W}_{C,2}^{(\theta, \tau)}] - \frac{\eta_M}{2} \text{Var}_0(\tilde{W}_{C,2}^{(\theta, \tau)}), \quad (3.B.1e)$$

$$\tilde{W}_{C,1}^{(\theta, \tau)} := y_{C,0} (\psi_0 R_{\tau,1} + (1 - \psi_0) R_{\theta,1}) \quad (3.B.1f)$$

$$\tilde{W}_{C,2}^{(\theta, \tau)} := W_{C,1}^{(\theta, \tau)} + y_{C,1} W_{cP,1}^{(\theta, \tau)} (\psi_1 R_{\tau,2} + (1 - \psi_1) R_{\theta,2}), \quad (3.B.1g)$$

$$\hat{R}_{t,(\theta, \tau)} := (1 - \hat{\psi}_{(\theta, \tau), t-1}) R_{\theta, t} + \hat{\psi}_{(\theta, \tau), t-1} R_{\tau, t}, \quad t = 1, 2, \quad (3.B.1h)$$

$$0 \leq x_C - (c(\theta) + c(\tau)) + \mathbb{E}_0[W_{C,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(W_{C,2}^{(\theta_H, \tau_H)}), \quad (3.B.1i)$$

$$(\theta_H, \tau_H) = \arg \max_{(\theta', \tau') \in \mathcal{S}} x_C - (c(\theta') + c(\tau')) + \mathbb{E}_0[W_{C,2}^{(\theta', \tau')}] - \frac{\eta_M}{2} \text{Var}_0(W_{C,2}^{(\theta', \tau')}). \quad (3.B.1j)$$

In the objective function of (DynCen), the Principal wants to maximize the $t = 2$ terminal wealth, by choosing the optimal $t = 0$ and $t = 1$ performance fees $y_{C,0}$ and $y_{C,1}$, respectively, and also the optimal fixed fee x_C . Again, recalling from Assumption 3.3.1, we assume the Principal only wants to implement the strategy pair (θ_H, τ_H) . Here, (3.B.1a) and (3.B.1b) are the Principal's $t = 1$ and $t = 2$ budget constraints; here we assume that there is no intermediate consumption and the Principal will reinvest all the $t = 1$ payoffs back into Manager C . Given any contract, Manager C will choose the optimal $t = 0$ and $t = 1$ portfolios, in accordance to the budget constraints (3.B.1f) and (3.B.1g), and the portfolio choice problem (3.B.1e), which results in the portfolio return (3.B.1h). The budget constraints for Manager C , after substituting in the optimal portfolio choices, are (3.B.1c) and (3.B.1d); we assume that Manager C will not reinvest his $t = 1$ wealth. Since Manager C has zero initial wealth, (3.B.1i) is his individual rationality constraint. And due to moral hazard, (3.B.1j) is the incentive compatibility constraint for inducing Manager C to choose the Principal's strict preference for the strategy pairs (θ_H, τ_H) .

3.C Dynamic Centralized Delegation in First Best

For the first best centralized delegation case, consider problem (DynCen) without the incentive compatibility constraint (3.B.1j).

Proposition 3.C.1. *Consider the first best centralized delegation problem (DynCen) but without the incentive compatibility constraints (3.B.1j). Fix any investment strategy pair $(\theta, \tau) \in \mathcal{S}$. Assume Assumption 3.6.2.*

(a) *Fix any arbitrary contract $(x_C, \{y_{C,0}, y_{C,1}\})$ and let's consider the optimal portfolio policy as chosen by Manager C .*

(i) *Suppose the realized wealth of the Principal at $t = 1$ is $W_{cP,1}^{(\theta, \tau)} = w_{cP,1}^{(\theta, \tau)}$. Then the $t = 1$ optimal portfolio is,*

$$\hat{\psi}_{1,(\theta, \tau)} := \frac{\mathbb{E}_1 R_{\tau,2} - \mathbb{E}_1 R_{\theta,2} + \eta_M (\text{Var}_1(R_{\theta,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) y_{C,1} w_{cP,1}^{(\theta, \tau)}}{\eta_M \text{Var}_1(R_{\theta,2} - R_{\tau,2}) y_{C,1} w_{cP,1}^{(\theta, \tau)}} \quad (3.C.1)$$

(ii) *The $t = 0$ optimal portfolio policy chosen by Manager C is the solution to the following optimization problem. Suppose the realized $t = 1$ wealth for the Principal and Manager C are, respectively, $W_{cP,1}^{(\theta, \tau)} = w_{cP,1}^{(\theta, \tau)}$, $W_{C,1}^{(\theta, \tau)} =$*

$w_{C,1}^{(\theta,\tau)}$. Define,

$$\begin{aligned}
\hat{U}_{C,1}^{(\theta,\tau)} &:= \hat{U}_{C,1}^{(\theta,\tau)}(y_{C,0}, y_{C,1}) \\
&= \sup_{\psi_1 \in \mathbb{R}} \mathbb{E}_1[\tilde{W}_{C,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_1(\tilde{W}_{C,2}^{(\theta,\tau)}) \\
&= \tilde{w}_{C,1}^{(\theta,\tau)} + \frac{(\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2})^2}{2\eta_M \text{Var}_1(R_{\theta,2} - R_{\tau,2})} \\
&\quad + \frac{(\mathbb{E}_1 R_{\tau,2} + \mathbb{E}_1 R_{\theta,2})(\text{Var}_1(R_{\theta,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2}))}{\text{Var}_1(R_{\theta,2} - R_{\tau,2})} y_{C,1} w_{cP,1}^{(\theta,\tau)} \\
&\quad - \frac{\text{Var}_1(R_{\theta,2})\text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2}{\text{Var}_1(R_{\theta,2} - R_{\tau,2})} y_{C,1}^2 (w_{cP,1}^{(\theta,\tau)})^2
\end{aligned} \tag{3.C.2}$$

And also consider, for any arbitrary $t = 0$ portfolio $\psi_0 \in \mathbb{R}$, the $t = 0$ variance of $\mathbb{E}_1 \tilde{W}_{C,2}^{(\theta,\tau)}$ is,

$$\begin{aligned}
\text{Var}_0(\mathbb{E}_1 \tilde{W}_{C,2}^{(\theta,\tau)}) &= \frac{1}{(\eta_P + \eta_M)^2 \text{Var}_1(R_{\theta,2} - R_{\tau,2})} \left[y_{C,0}(\eta + \eta_M) \text{Var}_1(R_{\theta,2} - R_{\tau,2}) \right. \\
&\quad + \eta(1 - y_{C,0}) \mathbb{E}_1[R_{\tau,2}](\text{Var}_1(R_{\theta,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) \\
&\quad \left. + \eta(1 - y_{C,0}) \mathbb{E}_1[R_{\theta,2}](\text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) \right]^2 \\
&\quad \times (\text{Var}_1(R_{\theta,2})(1 - \psi_0)^2 + \text{Var}_1(R_{\tau,2})\psi_0^2 + 2\psi_0(1 - \psi_0)\text{Cov}_1(R_{\theta,2}, R_{\tau,2}))
\end{aligned} \tag{3.C.3}$$

The $t = 0$ optimal portfolio policy $\hat{\psi}_0 = \hat{\psi}_0(y_{C,0})$, as a function of an arbitrary $t = 0$ performance fee $y_{C,0}$, chosen by Manager C is solves the following quadratic concave objective in $\psi_0 \in \mathbb{R}$,

$$\hat{U}_{C,0} = \sup_{\psi_0 \in \mathbb{R}} \mathbb{E}_0[\hat{U}_{C,1}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 \tilde{W}_{C,2}). \tag{3.C.4}$$

(b) Let's consider the optimal contract offered by the Principal.

(i) For any performance fees $y_{C,0}, y_{C,1} \in [0, 1]$, the optimal fixed fee is,

$$\hat{x}_{C,(\theta,\tau)} := \hat{x}_{C,(\theta,\tau)}(y_{C,0}, y_{C,1}) = (c(\theta) + c(\tau)) - \mathbb{E}_0[W_{C,2}^{(\theta,\tau)}] + \frac{\eta_M}{2} \text{Var}_0(W_{C,2}^{(\theta,\tau)}). \tag{3.C.5}$$

(ii) The $t = 1$ optimal performance fee is,

$$\hat{y}_{C,1}^{FB} = \frac{\eta_P}{\eta_P + \eta_M}. \tag{3.C.6}$$

(iii) The $t = 0$ optimal performance fee is given by the following. Suppose also that the $t = 1$ realized value of the Principal's wealth is $w_{cP,1}^{(\theta,\tau)}$. Firstly, define the portfolio return variances $\hat{\Sigma}_{1,(\theta,\tau)}$, and recall $\hat{\psi}_1 = \hat{\psi}_1(\hat{y}_{C,1}^{FB})$,

$$\hat{\Sigma}_{1,(\theta,\tau)} := (1 - \hat{\psi}_1)^2 \text{Var}_1(R_{\theta,2}) + \hat{\psi}_1^2 \text{Var}_1(R_{\tau,2}) + 2\hat{\psi}_1(1 - \hat{\psi}_1) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}). \tag{3.C.7}$$

Then define also the $t = 1$ continuation utility value for the Principal,

$$\hat{U}_{cP,1}^{(\theta,\tau)} = (1 + (1 - \hat{y}_{C,1}))[(1 - \hat{\psi}_1)\mathbb{E}_1 R_{\theta,2} + \hat{\psi}_1 \mathbb{E}_1 R_{\tau,2}] - \frac{\eta_P}{2} (1 - \hat{y}_{C,1}^{FB})^2 \hat{\Sigma}_{1,(\theta,\tau)} \tag{3.C.8}$$

The $t = 0$ optimal fee $\hat{y}_{C,0}$ is the solution to the problem,

$$\hat{U}_{cP,0} = \sup_{y_{C,0} \in [0,1]} \mathbb{E}_0[\hat{U}_{cP,1}^{(\theta,\tau)}] + \mathbb{E}_0[\hat{U}_{C,1}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{cP,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta,\tau)}), \tag{3.C.9}$$

where $W_{C,t}^{(\theta,\tau)}$ is Manager C's time t wealth as per (3.B.1c), (3.B.1d) after choosing the optimal portfolios, and $\hat{U}_{C,1}^{(\theta,\tau)}$ is Manager C's $t = 1$ continuation utility value (3.C.4) after substituting in the optimal $t = 1$ performance fee $\hat{y}_{C,1}^{FB}$.

Let's begin by discussing the optimal portfolio policies (3.C.1) at $t = 1$ of the Manager C under first best dynamic centralized delegation as per Proposition 3.C.1. At $t = 1$, the optimal portfolio choice form by Manager C is nearly identical to that of a static mean-variance optimizer, and indeed, the solution form is fairly similar to the static model in Proposition 3.4.1. The beginning wealth at $t = 1$ for Manager C is precisely $y_{C,1} w_{P,1}^{(\theta,\tau)}$. Based on this level of wealth at $t = 1$ and risk aversion η_M , Manager C constructs the optimal portfolio to optimize the one period ahead mean-variance of terminal wealth. Anticipating this, the Principal simply offers a performance contract $\hat{y}_{C,1}^{FB}$ at $t = 1$ of (3.C.6) to optimally risk share with Manager C in accordance to their respective risk aversions η_P and η_M . In all, the $t = 1$ policies are essentially akin to that of the static centralized delegation model of Proposition 3.4.1.

The optimal portfolio policy (3.C.4) at $t = 0$ of Manager C is slightly more nuanced. At $t = 0$, for any given $t = 0$ performance fee $y_{C,0}$, Manager C 's hedging motive of future income will matter. Namely, Manager C wants to choose portfolios $\psi_0 = \psi_0(y_{C,0})$ at $t = 0$ such that:

- Maximizes the $t = 0$ expectation of the $t = 1$ continuation value $\tilde{U}_{C,1}$. In particular, the continuation value $\tilde{U}_{C,1}$ is equal to:
 - Manager C 's next period (final period) wealth $W_{C,1}^{(\theta,\tau)}$;
 - Constant term relating to the benefits of executing a long-short strategy;
 - The $t = 0$ portfolio choice effects on the $t = 2$ performance fees for Manager C . Such $t = 2$ performance fees depend on the level $t = 1$ wealth $W_{cP,1}^{(\theta,\tau)}$ of the Principal, which of course, depends on Manager C 's $t = 0$ portfolio choice; and
 - Given that Manager C is risk averse, there is also a contract volatility term $(W_{cP,1}^{(\theta,\tau)})^2$ in desiring lower volatility in the $t = 2$ performance fees.
- Intertemporal hedging motive $\text{Var}_0(\mathbb{E}_1 W_{C,1}^{(\theta,\tau)})$ relating to the $t = 0$ variance of the next period $t = 1$ expectation on the terminal period $t = 2$ wealth.

Once the optimal portfolio policies $\hat{\psi}_0$ of Manager C has been determined, the Principal's optimal performance fee choice procedure is as follows. At $t = 1$, the Principal simply wants to optimally risk share based on the Principal and Manager C 's risk aversion parameter and offers the performance fee $\hat{y}_{C,1}^{FB}$ of (3.C.6). However, at $t = 0$, the optimal performance fee is chosen, again, to optimally risk share with Manager C but now taking into account the intertemporal hedging motive of both the Principal himself and also that of Manager C :

- The Principal wants to choose small performance fees to maximize his next period $t = 1$ expected wealth and minimize his $t = 1$ wealth volatility; $\mathbb{E}_0[\hat{U}_{cP,1}^{(\theta,\tau)}] = \mathbb{E}_0[W_{cP,1}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,1}^{(\theta,\tau)})$.
- The Principal's $t = 0$ fees affects his $t = 1$ wealth, and that in turn affects both the amount of resulting wealth in $t = 2$, depending on the portfolio policy of Manager C and the then realized returns. Thus, the Principal's intertemporal hedging motive is to choose performance fees to minimize the terminal wealth volatility $-\frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{cP,2}^{(\theta,\tau)})$.
- The Principal has a strong risk sharing motive with Manager C to minimize Manager C 's continuation utility $\mathbb{E}_0[\hat{U}_{C,1}^{(\theta,\tau)}]$ and an intertemporal incentive motive to minimize Manager C 's terminal date $t = 2$ wealth volatility $-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta,\tau)})$; note that by the form of the performance fees, lower wealth volatility $\text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta,\tau)})$ for Manager C also implies a lower wealth volatility $\text{Var}_0(\mathbb{E}_1 W_{cP,2}^{(\theta,\tau)})$ for the Principal, and again since all individuals are risk averse, this is beneficial for the Principal.

By now explicitly imposing Assumption 3.6.3, we get greater transparency of the solution form.

Corollary 3.C.2. *Consider again the first best dynamic centralized delegation problem (DynCen) and Proposition 3.C.1. In addition to Assumption 3.6.2, assume also Assumption 3.6.3. Recall that the Principal wants to implement investment strategy pairs (θ_H, τ_H) .*

(a) *The optimal portfolio policies chosen by Manager C are given as follows.*

- (i) *The $t = 1$ optimal portfolio chosen by Manager C is independent of the performance fees y_1 , and is,*

$$\hat{\psi}_1^{FB} = \frac{1}{2}.$$

(ii) The $t = 0$ optimal portfolio chosen by Manager C is independent of the performance fees y_0 , and is,

$$\hat{\psi}_0^{FB} = \frac{1}{2}.$$

(b) The optimal contract chosen by the Principal is given as follows.

(i) The optimal fixed fee form is given in (3.C.5).

(ii) The $t = 1$ optimal performance fee chosen by the Principal is,

$$\hat{y}_{C,1}^{FB} = \frac{\eta_P}{\eta_P + \eta_M}.$$

(iii) The $t = 0$ optimal (interior solution) performance fee chosen by the Principal is,³¹

$$\hat{y}_{C,0}^{FB} = \frac{\hat{y}_{C,0}^N}{\hat{y}_{C,0}^D}, \quad (3.C.10)$$

provided that $\hat{y}_{C,0}^{FB} \in (0, 1)$, and where,

$$\begin{aligned} \hat{y}_{C,0}^N &= 2\eta_P(-2\mu^2 + \eta_P\sigma^2) + (-4\mu^2 + 2\eta_P[1 + 2\mu(1 + \mu)]\sigma^2 + \eta_P\sigma^4)\eta_M \\ &\quad + 2\eta_P\sigma^2(\eta_P + [1 + 2\mu(1 + \mu) + \sigma^2]\eta_M)\rho_{\theta_H, \tau_H} + \eta_P\eta_M\sigma^4\rho_{\theta_H, \tau_H}^2, \\ \hat{y}_{C,0}^D &= \sigma^2(1 + \rho_{\theta_H, \tau_H})[2\eta_P^2 + 2\eta_M^2 + \eta_P\eta_M(4(1 + \mu^2) + \sigma^2(1 + \rho_{\theta_H, \tau_H}))]. \end{aligned}$$

See Section 3.C for numerical illustrations of Corollary 3.C.2.

Numerical Illustrations of Dynamic Delegation in First Best

Using the analytical solutions for first best dynamic delegation from Corollary 3.C.2 and Proposition 3.7.1, we can easily numerically illustrate the Principal's value functions under centralization and decentralization, and also their associated optimal policies at $t = 0$. It should be noted that from Corollary 3.C.2 and Proposition 3.7.1, the $t = 1$ optimal policies for both centralization and decentralization take on explicit and simple forms. Hence, we will focus the numerical illustrations on the $t = 0$ optimal policies, for which its comparative statics may not be obvious at first glance. The optimal portfolios at $t = 0$ for both centralization and decentralization take on a simple explicit form, and hence are not plotted. In decentralization, the $t = 0$ optimal performance fee for both Manager A and Manager B are identical, and hence only one of them is plotted. The base parameters are all identical to that of Table 3.1.

³¹ i.e. “ N ” for numerator, and “ D ” for denominator.

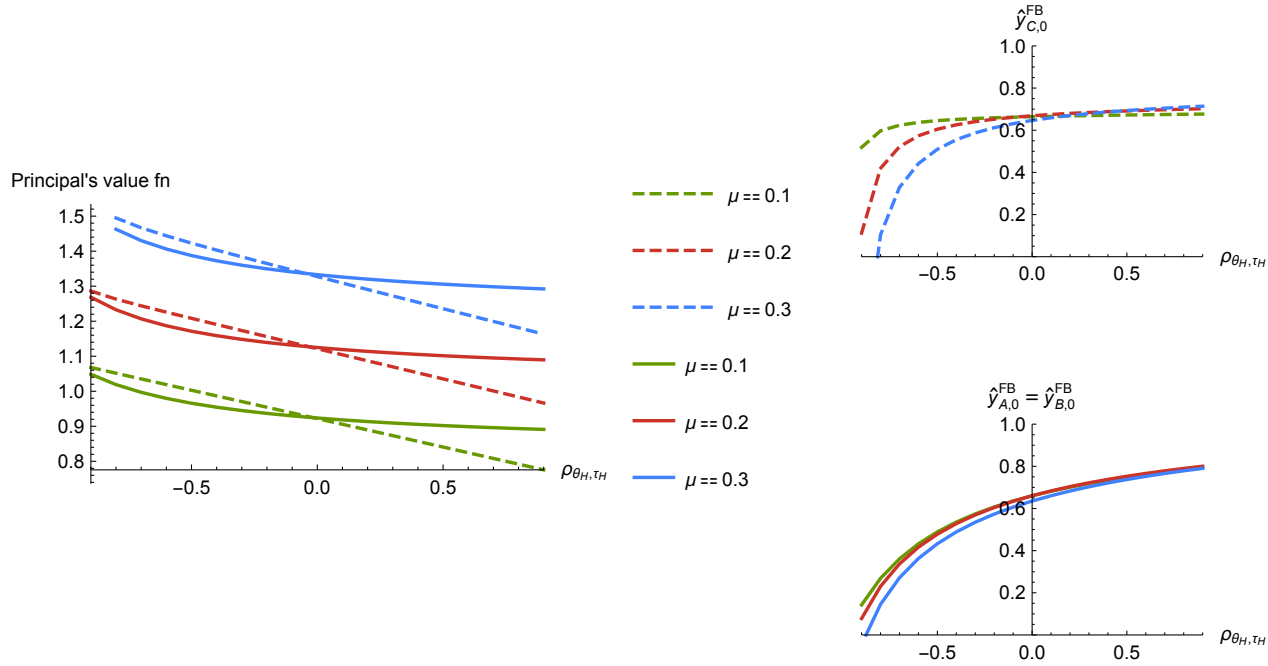


Figure 3.C.1: Plot of the first best dynamic delegation model against the compliant investment strategy pair's correlations ρ_{θ_H, τ_H} , with several scenarios on the means μ of the compliant investment strategy. Similar to the static first best case of Proposition 3.4.3, due to optimal risk sharing, higher correlations ρ_{θ_H, τ_H} favor decentralization, while lower correlations favor centralization. And also, naturally, higher mean returns μ will increase the Principal's $t = 0$ value function. While the first best performance fees in the centralized static model of Proposition 3.4.1 only consist of the Principal's and Manager C 's risk aversions, this is clearly not the case for the $t = 0$ centralization performance fees due to the intertemporal hedging incentive of all individuals involved.

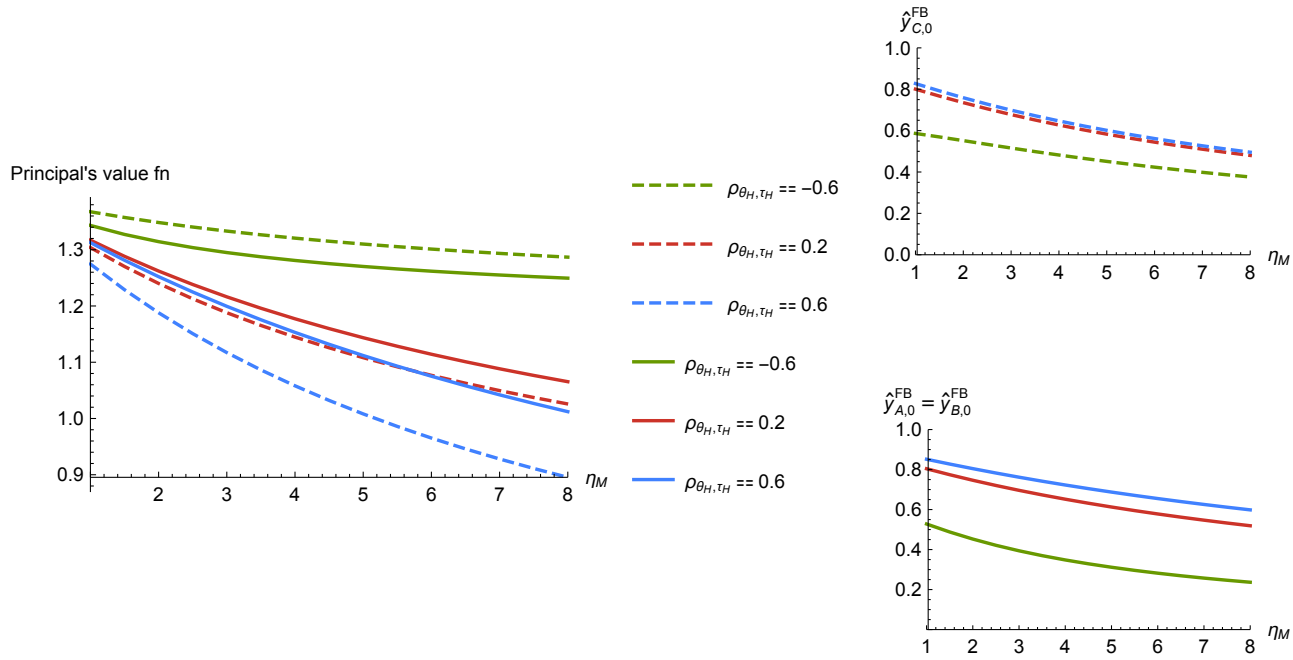


Figure 3.C.2: Plot of the first best dynamic delegation model against the Managers' risk aversion η_M , with several scenarios on the correlations of the compliant investment strategy pair ρ_{θ_H, τ_H} . As Managers' risk aversion η_M increases, it becomes more expensive to compensate the Managers for taking on the volatility of the contract. This is the same effect as per the first best static delegation model of Section 3.4.

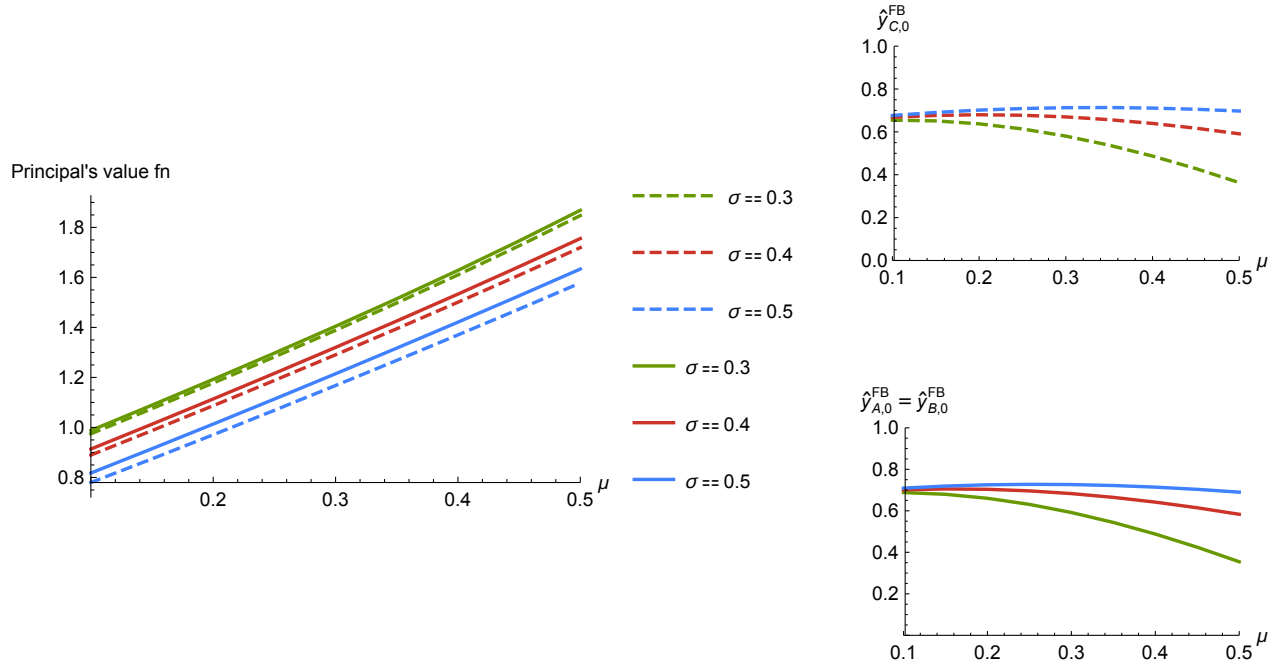


Figure 3.C.3: Plot of the first best dynamic delegation model against the compliant strategies' mean return μ , with several scenarios of the strategies' volatilities σ . Given that all individuals have mean-variance preferences, it is no surprise that the Principal's $t = 0$ value function increases with higher mean μ , and is lower with a higher volatility σ . However, as compared to the first best static delegation model of Section 3.4, especially that of Proposition 3.4.1 for centralization and Proposition 3.4.2 for decentralization, the optimal static performance fees distinctly do not depend on the mean return μ . In this first best dynamic delegation model, as the mean return μ of the compliant strategies, the $t = 0$ performance fees under both first best dynamic centralization and decentralization decrease. Likewise, in the first best static delegation model, largely thanks to Assumption 3.3.2, the volatility σ of returns also does not enter the static first best performance fees. In contrast, here in first best dynamic delegation, as volatility σ increases, the $t = 0$ performance fees decreases under both centralization and decentralization.

3.D Dynamic Centralized Delegation in Second Best

Proposition 3.D.1. *Consider the second best centralized delegation problem (DynCen) in its entirety. Assume Assumption 3.6.2 and Assumption 3.6.3. Note and recall that the Principal wants to implement the investment strategy pair (θ_H, τ_H) .*

- (a) *For any given contract $(x_C, \{y_{C,0}, y_{C,1}\})$ and any investment strategy pair (θ, τ) , the optimal portfolio policies $\hat{\psi}_{t,(\theta,\tau)}$ for $t = 0, 1$ is equivalent to the form as in Proposition 3.C.1 for first best centralized delegation.*
- (b) *The optimal fixed fee $\hat{x}_{C,(\theta_H, \tau_H)}$ is equivalent to the form (3.C.5), when evaluated at $(\theta, \tau) = (\theta_H, \tau_H)$, for any performance fees (y_0, y_1) .*
- (c) *Suppose the $t = 1$ realized wealths are $W_{cP,1}^{(\theta,\tau)} = w_{cP,1}^{(\theta,\tau)}$ and $W_{C,1}^{(\theta,\tau)} = w_{C,1}^{(\theta,\tau)}$, for the Principal and Manager C, respectively. Then there exists some vector³² $\lambda_C = (\lambda_C^{(\theta_L, \tau_L)}, \lambda_C^{(\theta_H, \tau_H)}, \lambda_C^{(\theta_b, \tau_b)}) \in \mathbb{R}^3$ of which only one of the elements is nonzero and the other two will be zero, and let $\lambda_C^{(\theta^b, \tau^b)}$ be that nonzero element. Then the optimal $t = 1$ performance fee is,*

$$\begin{aligned} \hat{y}_{C,1}^{\lambda_C} &:= \hat{y}_{C,1}^{\lambda_C}(w_{P,1}^{(\theta_H, \tau_H)}, w_{P,1}^{(\theta^b, \tau^b)}) \\ &= \frac{w_{P,1}^{(\theta_H, \tau_H)} \left[2\lambda_C^{(\theta^b, \tau^b)} \mu + w_{P,1}^{(\theta_H, \tau_H)} \eta_P \sigma^2 (1 + \rho_{\theta_H, \tau_H}) \right] - \lambda_C^{(\theta^b, \tau^b)} (\mu_{\theta^b} + \mu_{\tau^b}) w_{P,1}^{(\theta^b, \tau^b)}}{\sigma^2 \left[(\eta_P + \eta_M (1 + \lambda_C^{(\theta^b, \tau^b)})) (1 + \rho_{\theta_H, \tau_H}) (w_{P,1}^{(\theta_H, \tau_H)})^2 - \lambda_C^{(\theta^b, \tau^b)} \eta_M (w_{P,1}^{(\theta^b, \tau^b)})^2 (1 + \rho_{\theta^b, \tau^b}) \right]}. \end{aligned} \quad (3.D.1)$$

- (d) *The $t = 1$ continuation value of Manager C taking on investment strategy pair (θ, τ) and the Principal are,*

$$\begin{aligned} \hat{U}_{C,1}^{(\theta,\tau)} &= U_{C,1}^{(\theta,\tau)} \Big|_{y_{C,1} = \hat{y}_{C,1}^{\lambda_C}} \\ &= w_{C,1}^{(\theta,\tau)} + \hat{y}_{C,1}^{\lambda_C} w_{cP,1}^{(\theta,\tau)} \left[(1 - \hat{\psi}_{1,(\theta,\tau)}) \mu_{\theta} + \hat{\psi}_{1,(\theta,\tau)} \mu_{\tau} \right] - \frac{\eta_M}{2} \left(\hat{y}_{C,1}^{\lambda_C} w_{P,1}^{(\theta,\tau)} \right)^2 \hat{\Sigma}_{1,(\theta,\tau)}, \end{aligned} \quad (3.D.2a)$$

$$\begin{aligned} \hat{U}_{cP,1}^{(\theta_H, \tau_H)} &= \tilde{U}_{cP,1}^{(\theta_H, \tau_H)} \Big|_{y_{C,1} = \hat{y}_{C,1}^{\lambda_C}} \\ &= w_{cP,1}^{(\theta_H, \tau_H)} \left(1 + (1 - \hat{y}_{C,1}^{\lambda_C}) \mu \right) - \frac{\eta_P}{2} \left((1 - \hat{y}_{C,1}^{\lambda_C}) w_{P,1}^{(\theta_H, \tau_H)} \right)^2 \hat{\Sigma}_{1,(\theta_H, \tau_H)}, \end{aligned} \quad (3.D.2b)$$

where we recall that the optimal $t = 1$ performance fee $\hat{y}_{C,1}^{\lambda_C} = \hat{y}_{C,1}^{\lambda_C}(w_{cP,1}^{(\theta_H, \tau_H)}, w_{cP,1}^{(\theta^b, \tau^b)})$.

- (e) *The optimal $t = 0$ performance fee $\hat{y}_{C,0}^{\lambda_C} \in [0, 1]$ is the solution to the following optimization problem,*

$$\begin{aligned} \hat{U}_{cP,0}^{\lambda_C} &= \sup_{y_{C,0} \in [0,1]} \mathbb{E}_0[\hat{U}_{C,1}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) \\ &\quad + \mathbb{E}_0[\hat{U}_{cP,1}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}) \\ &\quad - \lambda_C^{(\theta^b, \tau^b)} \left[\mathbb{E}_0[\hat{U}_{C,1}^{(\theta^b, \tau^b)}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta^b, \tau^b)}) \right] \\ &\quad - \left((c(\theta^b) + c(\tau^b)) - 2c + \mathbb{E}_0[\hat{U}_{C,1}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) \right). \end{aligned} \quad (3.D.3)$$

³² As one can see in the proofs, this is the vector of Lagrange multipliers associated with the three possible incentive compatibility constraints. As argued in the static centralized delegation model, if a binding solution exists, only one of these will bind. And hence only the binding constraint will have a nonzero unsigned Lagrange multiplier, while the slack constraints will have zero valued multipliers. If the optimal solution is non-binding, then we return back to the first best case, and in that case, $\lambda_C = \mathbf{0}$ will be the zero vector — we refer this case back to the first best setup of Proposition 3.D.1 and is not treated here for succinctness in exposition.

(f) The optimal constant $\hat{\lambda}_C = (\hat{\lambda}_C^{(\theta_L, \tau_L)}, \lambda_C^{(\theta_H, \tau_H)}, \lambda_C^{(\theta_L, \tau_H)})$ is the solution to,

$$\inf_{\substack{\lambda'_C = (\lambda'_C{}^1, \lambda'_C{}^2, \lambda'_C{}^3) \in \mathbb{R}^3, \\ \text{only one of } \lambda'_C{}^k \text{ is nonzero}}} \hat{U}_{cP,0}^{\lambda'_C} \quad (3.D.4)$$

If a finite value is not reached for the infimum, no second best contract will exist.

At $t = 1$, for any arbitrary $t = 1$ performance fee y_1 , Manager C who is taking on the compliant investment strategy pair (θ_H, τ_H) will be entitled to the total performance fee amount of $y_{C,1} w_{cP,1}^{(\theta_H, \tau_H)}$, where $W_{cP,1}^{(\theta_H, \tau_H)} = w_{cP,1}^{(\theta_H, \tau_H)}$ is the Principal's $t = 1$ realized wealth. But if Manager C decides to deviate to the deviant investment strategy pair (θ_H, τ_H) , then Manager C 's total performance fee becomes $y_{C,1} w_{cP,1}^{(\theta^b, \tau^b)}$. Because of the long term investment strategy (θ, τ) commitment by Manager C at $t = 0$, there is a strong path dependence on the Principal's $t = 1$ wealth $w_{cP,1}^{(\theta, \tau)}$, which then affects Manager C 's $t = 1$ performance fee compensation. Thus, analogous to the idea in the static centralization model of Proposition 3.5.1, opportunity costs for Manager C associated with foregone investment strategy benefits by deviating is no longer a simple difference between means and volatilities, but rather the whole possible paths of returns form such opportunity costs. Hence, in addition to the usual risk sharing arguments that already hold in first best, the Principal needs to balance out the cost of Manager's deviant strategy. In all, the optimal $t = 1$ performance fee is precisely (3.D.1), and as usual λ_C represents the shadow price on Manager C 's incentive compatibility.

Now let's discuss the $t = 0$ optimal performance fee choice in (3.D.3). From the Principal's perspective, the optimal performance fee $\hat{y}_{C,0}$ serves two objectives: (i) optimal risk sharing; and (ii) incentive compatibility. As it was also true in the first best case of Proposition 3.C.1, the Principal wants to choose $t = 0$ performance fee to maximize his continuation utility $\mathbb{E}_0[\hat{U}_{cP,1}^{(\theta_H, \tau_H)}]$ while minimizing Manager C 's continuation utility $\mathbb{E}_0[\hat{U}_{C,1}^{(\theta_H, \tau_H)}]$. Since both the Principal and Manager C are both risk averse, again as per the first best case of Proposition 3.C.1, the Principal has an intertemporal hedging motive, in which case the $t = 0$ performance fees should minimize the terminal wealth volatility $-(\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)})) + \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)})$. But under second best, the performance fees must also induce Manager C to take on the Principal's strictly preferred strategy pair (θ_H, τ_H) , which are the terms multiplied by $\lambda_C^{(\theta^b, \tau^b)}$. In particular, in equilibrium, the performance fees are set such that Manager C 's private costs, the continuation value and Manager C 's intertemporal hedging motive $-2c + \mathbb{E}_0[\hat{U}_{C,1}^{(\theta^b, \tau^b)} - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta^b, \tau^b)})]$ under (θ_H, τ_H) , would equate to the payoff for Manager C under the most profitable investment deviation pair (θ^b, τ^b) , which consists of the private costs, continuation value and the intertemporal hedging motive, $-(c(\theta^b) + c(\tau^b)) + \mathbb{E}_0[\hat{U}_{C,1}^{(\theta^b, \tau^b)} - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta^b, \tau^b)})]$. Taking all these effects in account, it implies also that the determination of the optimal $t = 0$ performance fee $\hat{y}_{C,0}^{\lambda_C}$ must also depend on the full joint distribution of both the compliant and deviant strategy returns $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$.

The economic reason for why the full joint distribution is perhaps most interesting. The incentive compatibility constraint of Manager C act as an endogenous value-at-risk (VaR) constraint on the performance fee policies across time. The equilibrium wealth paths if Manager C is compliant is $W_{C,t}^{(\theta_H, \tau_H)}$, and the wealth paths of the most profitable deviation is $W_{C,t}^{(\theta^b, \tau^b)}$. To incentivize Manager C , it implies that the performance fees over time must be constructed such that the terminal mean-variance of $W_{C,2}^{(\theta_H, \tau_H)}$, taking into account Manager C 's private costs, must weakly exceed that of $W_{C,2}^{(\theta^b, \tau^b)}$, while due to the individuals' risk aversions, intertemporal wealth smoothing is also taken into account. As it is common with VaR type constraints, the tail probabilities of returns are of first order importance. In particular, we are now concerned with the joint tail probabilities wealths under the on-equilibrium compliant strategies and off-equilibrium deviant strategies.

The technical reason for why the full joint distribution is required is that we are now dealing with ratios of random variables. One can observe that $\hat{y}_{C,1}^{\lambda_C} W_{cP,1}^{(\theta_H, \tau_H)}$ is a ratio of function of wealths $W_{P,1}^{(\theta_H, \tau_H)}$ and $W_{P,1}^{(\theta^b, \tau^b)}$. Thus when the Principal needs to decide on the $t = 0$ optimal performance fee $\hat{y}_{C,0}$, the Principal needs to consider the $t = 0$ expectation of his $t = 1$ continuation value $\hat{U}_{cP,1}^{\lambda_C}$. And since the expectation of a ratio is generically not equal to the ratio of expectations, it means that one would indeed need the full multivariate distribution of $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$. Indeed, only when $\lambda_C = \mathbf{0}$ (i.e. the incentive compatibility constraint is non-binding, or effectively the first best setup), does $\hat{y}_{C,1}^{\lambda_C} |_{\lambda_C = \mathbf{0}}$ become linear in $W_{cP,1}^{(\theta_H, \tau_H)}$ and hence overall quadratic in the wealth for $\hat{y}_{C,1}^{\lambda_C} W_{cP,1}^{(\theta_H, \tau_H)} |_{\lambda_C = \mathbf{0}}$, and hence only in this case, it suffices to just consider the first and second moments of the returns.

3.E Distribution restrictions in second best

Throughout this paper, we have been relatively silent on the existence of the moment quantities involved in both the optimization of our static and dynamic models. This is especially since in the static models (Section 3.3), as we work with mean-variance

preferences, it is clear that having well defined and finite first and second moments of the investment strategy returns (R_θ, R_τ) will suffice for our optimization problem. And indeed, again in the first best dynamic models for both centralized and decentralized delegation, first and second moments existence will also suffice; it should be noted that even though we had made strong independence and identical distribution assumptions (i.e. Assumption 3.6.2 and Assumption 3.6.3), even if we relax these assumptions, it is clear that as long as certain conditional versions of first and second moments exist, everything will still pass through.

But as we discuss the second best dynamic delegation problem, the ratio of functions involving the investment strategy returns will naturally arise from the incentive compatibility constraints. This immediately places a strong restriction on the forms of multivariate distributions $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$ that are permissible in order to have finite first and second moments in the computation of the agents' continuation utilities and variances of wealth. As an important special case, this immediately rules out $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$ having a jointly Gaussian distribution.³³ This is worthy of mention since numerous theoretical and empirical papers in the asset pricing literature that implicitly or explicitly invoke Gaussian assumptions in the distribution of returns. The key point here is that the introduction of moral hazard in delegated portfolio management, through the incentive compatibility constraints, should give the researcher further pause on how one should think about the returns distribution of not only the equilibrium investment strategies, but also the off-equilibrium returns distributions. We have further remarks on this issue of modeling joint distribution of returns in Section 3.F, where we will use copulas to model said joint dependence.

3.F Dependence Modeling and Copulas

As noted in Section 3.8, where we'd discussed centralized and decentralized delegation under second best, simply knowing the first and second moments of the investment return strategies of R_θ, R_τ for each of θ, τ is not sufficient — one needs to have the full joint distribution $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$ of the return strategies. From Assumption 3.3.2 of Section 3.3, and Assumption 3.6.2 and Assumption 3.6.3 of Section 3.6, we have already in place several restrictions on the moments of the investment strategies, which implies that we already have some a priori restrictions on their respective marginal distributions. To further model the joint distribution of these investment strategies, when we already have some specified restrictions on their marginal distributions, the most direct method is via *copulas*.

Why Copulas?

The first order of business is to answer a seemingly obvious question — if one wants a multivariate distribution involving four random variables, isn't the multivariate Gaussian the most convenient and obvious choice?

Why not multivariate Gaussian? Why discrete distributions and copulas?

The reader might naturally wonder why would one not use a four dimensional joint Gaussian distribution on $(R_{\theta_H}, R_{\tau_H}, R_{\theta_L}, R_{\tau_L})$, where we can conveniently impose our restrictions on the means and the variances, and then subsequently model the correlations. While this statement is true in principle, but in practice we encounter several issues, both on the mathematical aspect and also on the numerical computation aspect. As seen in Proposition 3.D.1 and Proposition 3.8.1, we require to take the expectation of a ratio of random variables. And for Gaussian distributions, it is well known that something as simple as X/Y , where X and Y are independent standard normals, follows the Cauchy distribution, of which no moments exist. A similar issue arises precisely in this setting, as already mentioned in Section 3.E. As a result, it is highly inconvenient or even wrong to use the seemingly innocuous multivariate Gaussian distribution in our model.

The second reason is a numerical issue. Even if we brush aside probabilistic and integration issues on the existence of moments, from a numerical computation perspective, computing the moments of a ratio still brings substantial challenges. For one, computing a multivariate integral using a four dimensional joint Gaussian is rather computationally intensive, and when the integrands involve ratios of such random variables, we also have to handle numerical stability issues. Indeed, one could make the same remark for general multivariate continuous distributions (say, elliptical distributions) where the density function is non-trivial to compute and subsequently numerically integrate. And we also need to recall, since we are optimizing over an endogenous choice of optimal portfolio and performance fee policies, that means we need to iteratively compute such moments numerous times. In all, this brings about a highly computationally intensive and delicate task.

For these two reasons above, it is far easier to use discrete distributions so that: (i) We do not need to worry about moment existence and integral convergence issues, since we'll be working with finite sums; and (ii) Discrete distributions with small number of states are far more computationally easily to execute than that of continuous distributions with infinite number of states. And once we recognize the mathematical and numerical practical need to step away from the multivariate Gaussian distributions, where the correlation matrix being the single quantity that governs all dependence behaviors, one then needs to

³³ For instance, it is well known that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[1/X^k]$ does not exist for any integer k .

be more delicate in modeling the joint dependence of random variables. As far as this author is aware, the most direct and well-established method is via copulas.

Copulas — bare basics

The study of copulas is well established and extensive; see Joe (2014) and Nelsen (2007), and for specific applications in finance, see Embrechts et al. (2003) and Patton (2009b, 2012).

Here, we make no attempt to summarize the theories but rather just extract out the minimalistic bare elements that are necessary to achieve two goals for the purpose of this paper: (i) A way to construct joint distributions from the marginal distributions of random variables; and (ii) Parametric copula choices that can qualitatively inform us on the dependence behavior of the random variables.

For this section only, let's denote $\mathbf{I} := [0, 1]$ to be the unit interval, let $\bar{\mathbb{R}}$ be the extended real line and denote $\text{Ran } f$ to be the range of a function f .

We start with the definition of a copula.

Definition 3.F.1 (Nelsen (2007), Definition 2.10.6). An n -dimensional copula (or n -copula) is a function $C : \mathbf{I}^n \rightarrow \mathbf{I}$ such that:

1. For every $\mathbf{u} \in \mathbf{I}^n$,

$$C(\mathbf{u}) = 0 \text{ if at least one coordinate of } \mathbf{u} \text{ is } 0,$$

and,

$$\text{if all coordinates of } \mathbf{u} \text{ are } 1 \text{ except } u_k, \text{ then } C(\mathbf{u}) = u_k;$$

2. For all $\mathbf{u}_0 = (u_{0,1}, \dots, u_{0,n}) \in \mathbf{I}^n$ and $\mathbf{u}_1 = (u_{1,1}, \dots, u_{1,n}) \in \mathbf{I}^n$ such that $\mathbf{u}_0 \leq \mathbf{u}_1$ (i.e. $u_{0,j} \leq u_{1,j}$ for all $j = 1, \dots, n$),

$$\sum_{i_1=0,1} \dots \sum_{i_n=0,1} (-1)^{i_1+\dots+i_n} C(u_{i_1,1}, \dots, u_{i_n,n}) \geq 0.$$

The next key theorem connects copulas to multivariate distributions.

Theorem 3.F.1 (Sklar's theorem in n -dimensions; Nelsen (2007), Theorem 2.10.9). *Let H be an n -dimensional distribution function with margins F_1, F_2, \dots, F_n . Then there exists an n -copula C such that for all $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \bar{\mathbb{R}}^n$,*

$$H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)). \tag{3.F.1}$$

If F_1, F_2, \dots, F_n are all continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran } F_1 \times \text{Ran } F_2 \times \dots \times \text{Ran } F_n$. Conversely, if C is an n -copula and F_1, F_2, \dots, F_n are distribution functions, then the function H defined by (3.F.1) is an n -dimensional distribution function with margins F_1, F_2, \dots, F_n .

To be concrete, let's consider the probability mass function when the marginal distributions are discrete. For notational simplicity, we assume, only for the expression below, that the random vector \mathbf{Y} is discrete and moreover that $\mathbf{Y} \in \mathbb{N}^n$, where \mathbb{N} is the set of natural numbers. For the actual application in mind, the support of our random vector will be discrete and finite, and it should be clear from the below that the notation there is straightforward to modify. The probability mass function (pmf) of $\mathbf{Y} = (Y_1, \dots, Y_n)$, and where F_j is the marginal distribution of Y_j , for $j = 1, \dots, n$, is given by the 2^n finite differences,

$$\begin{aligned} \mathbb{P}(\mathbf{Y} = \mathbf{y}) &= \sum_{i_1=0,1} \dots \sum_{i_n=0,1} (-1)^{i_1+\dots+i_n} \mathbb{P}(Y_1 \leq y_1 - i_1, \dots, Y_n \leq y_n - i_n) \\ &= \sum_{i_1=0,1} \dots \sum_{i_n=0,1} (-1)^{i_1+\dots+i_n} C(F_1(y_1 - i_1), \dots, F_n(y_n - i_n)). \end{aligned}$$

For our model, we have $n = 4$ and that Y_j 's will be the investment strategy returns, which will be defined on a finite discrete support (which only has few states). Given that the marginal F_j 's will be defined on finite discrete supports, these should be relatively quick to compute numerically. However, even if the F_j 's are quick to compute, we must not forget that we will need to evaluate these $F_j(\cdot)$'s on the copula C with n arguments, before then computing the 2^n finite differences. And this is only for one possible value of the \mathbf{y} 's. And since we need to be working with moments of ratios of random variables, we need to consider summing over all possibilities of \mathbf{y} 's with respect to the pmf $\mathbb{P}(\mathbf{Y} = \mathbf{y})$. Thus, it is imperative that we pick a copula C that is very quick to compute.

Remark 3.F.2. As already motivated above, we do not consider numerically intensive procedures that do not admit closed form solutions of the copula C . An example of a copula without closed form solutions are elliptical copulas (in which Gaussian copulas are a special case), of which the pmf of a multivariate discrete random variable \mathbf{Y} has the form,

$$\mathbb{P}(\mathbf{Y} = \mathbf{y}) = \int_{\Phi^{-1}(F_1^-)}^{\Phi^{-1}(F_1^+)} \cdots \int_{\Phi^{-1}(F_n^-)}^{\Phi^{-1}(F_n^+)} \phi_n(x_1, \dots, x_n; \Gamma) dx_1 \cdots dx_n,$$

where here, $\phi_n(\cdot; \Gamma)$ denotes the probability density function (pdf) of an n -dimensional elliptical distribution at location 0 and with scale parameter (correlation matrix) Γ , and Φ^{-1} denotes the inverse cumulative distribution function (cdf) of the univariate margins of the said elliptical distribution; and $F_j^+ := \mathbb{P}(Y_j \leq y_j)$ and $F_j^- := \mathbb{P}(Y_j \leq y_j - 1)$. Clearly, there are no closed form solutions for Φ^{-1} . See Joe (2014) for details.

The main point for the purpose of our paper is that these distributions that do not admit closed form solutions for the copula entails highly numerically intensive computations (i.e. for both Φ^{-1} , and the multivariate numerical integration) for even one single computation of $\mathbb{P}(\mathbf{Y} = \mathbf{y})$. In particular, we need to consider expectations of the form $\mathbb{E}[g(\mathbf{Y}; v)] = \sum g(\mathbf{y}; v) \mathbb{P}(\mathbf{Y} = \mathbf{y})$, where for our purposes, g itself already is somewhat complicated in \mathbf{y} and may not have closed form solutions, and moreover, that we need to further numerically optimize over the endogenous variable(s) v . All such numerical computations make this copula family numerically unsuitable for our paper — even though we fully acknowledge that, with sufficient computing resources, it would be interesting to explore this copula family since the correlation scale parameter Γ allows for a richer dependence structure than the Archimedean family that we discuss below.

Archimedean copulas

For the purpose of this paper, we will only consider the family of *Archimedean* copulas.³⁴ Let's begin with a technical definition.

Definition 3.F.2 (Completely monotonic function; Nelsen (2007), Definition 4.6.1.). A function $g(t)$ is *completely monotonic* on an interval J if it is continuous there and has derivatives of all orders that alternate in sign; i.e., if it satisfies,

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0,$$

for all t in the interior of J and $k = 0, 1, 2, \dots$

Now, we can give the definition of an n -dimensional (exchangeable) Archimedean copula.³⁵

Definition 3.F.3 (Multivariate Archimedean copula; Nelsen (2007), Theorem 4.6.2.). Let φ be a continuous strictly decreasing function from \mathbf{I} to $[0, \infty]$ such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} denote the inverse of φ . If C is the function from \mathbf{I}^n to \mathbf{I} given by,

$$C(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \dots + \varphi(u_n)), \tag{3.F.2}$$

then C is an n -copula for all $n \geq 2$ if and only if φ^{-1} is completely monotonic on $[0, \infty)$. The function φ is called the *generator* of the copula.³⁶

There are numerous properties associated with the Archimedean copula (see Nelsen (2007)) but for our purposes, one of the most restrictive implications of this copula is that it implies an *exchangeable* distribution. That is, if the multivariate distribution of (Y_1, \dots, Y_n) is constructed from an Archimedean copula, then it is equivalent in distribution to $(Y_{\sigma(1)}, \dots, Y_{\sigma(n)})$, where $\sigma(1), \dots, \sigma(n)$ are any permutations of $1, \dots, n$. This is admittedly restrictive. But in return, we get a multivariate distribution that is based on a single parameter that can then generate various tail dependence behaviors; these tail dependence³⁷ behaviors are what's most important for the purpose of our application.

³⁴ In particular, we do not consider the multivariate Gaussian copula (see Remark 3.F.2), which is widely popular in mathematical finance (say Li (2000)), largely only because of computational speed problems for our model at hand. Nelsen (2007) and Joe (1997, 2014) are standard references that contain excellent overviews of various types of copulas and their properties.

³⁵ The theorem statement of Theorem 4.6.2. in Nelsen (2007) yields the definition of a multivariate Archimedean copula.

³⁶ For most purposes and also in the literature, this generator is parametrized by a single scalar δ .

³⁷ There are several different measures of *tail dependence* and details can be found in Joe (2014). For our purposes, a qualitative description suffices. We say that a pair of random variables exhibit *upper tail*

Examples

In what follows, we will list out both the bivariate and multivariate forms of the classical Archimedean copulas. We will use these examples as fundamental building blocks in the subsequent constructions of hierarchical Archimedean copulas. In the examples below, we will also record the inverse of the generator as it will become useful in the subsequent sections.

Example 3.F.1 (Clayton). For $0 \leq \delta < \infty$, the *bivariate Clayton* copula, is,

$$C(u, v; \delta) = \left(u^{-\delta} + v^{-\delta} - 1 \right)^{-1/\delta}, \quad u, v \in \mathbf{I}, \tag{3.F.3}$$

where its generator is,

$$\varphi(s; \delta) = (1 + s)^{-1/\delta}, \tag{3.F.4a}$$

$$\varphi^{-1}(t; \delta) = t^{-\delta} - 1, \quad t \in \mathbf{I}. \tag{3.F.4b}$$

The *multivariate Clayton* copula is,

$$C(\mathbf{u}; \delta) = \left[u_1^{-\delta} + \dots + u_n^{-\delta} - (n - 1) \right]^{-1/\delta}, \quad \mathbf{u} \in \mathbf{I}^n. \tag{3.F.5}$$

See Joe (2014, Section 4.6.1) for details.

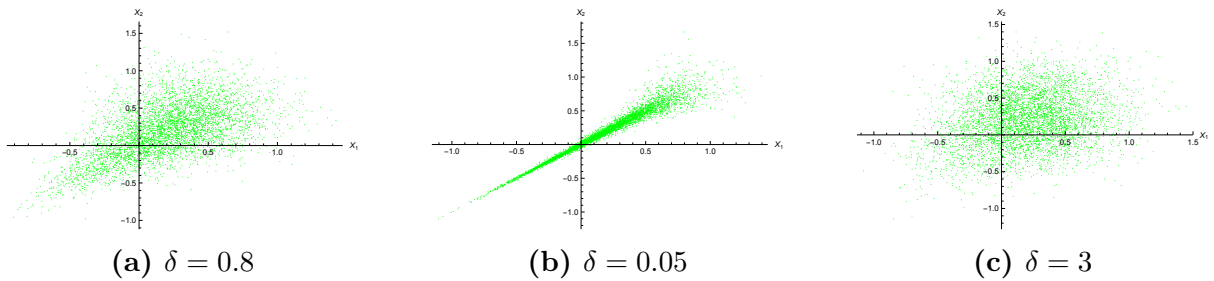


Figure 3.F.1: A scatter plot of (X_1, X_2) , whose joint distribution is generated by the Clayton copula of Example 3.F.1 with various parameters. The marginal distributions are $X_i \sim \mathcal{N}(0.20, 0.35)$ for $i = 1, 2$.

Example 3.F.2 (Frank). For $-\infty < \delta < \infty$, the *bivariate Frank* copula, is,

$$C(u, v; \delta) = -\delta^{-1} \log \left(\frac{1 - e^{-\delta} - (1 - e^{-\delta u})(1 - e^{-\delta v})}{1 - e^{-\delta}} \right), \quad u, v \in \mathbf{I}, \tag{3.F.6}$$

dependence when one realizes high extreme values, the other also realizes high extreme values. Likewise, we say a pair of random variables exhibit *lower tail dependence* when one realizes low extreme values, the other also realizes low extreme values. It should also be noted that by their constructions, it is fairly difficult for Archimedean copulas to generate distributions whereby one random variable realizes high extreme values, while the other one realizes low extreme values. Thus, when one computes (Pearson's) correlation on such random variables, they will tend to be nonnegative. But it should be noted that correlation computations obscure a critical qualitative behavior. Namely, it is possible to have high positive correlations with or without tail dependence. Moreover, if two random variables exhibit upper or lower tail dependencies, they'll both result in positive (Pearson's) correlation, and yet their qualitative behaviors are completely different and indeed opposing.

where its generator is,

$$\varphi(s; \delta) = -\delta^{-1} \log \left(1 - (1 - e^{-\delta})e^{-s} \right), \quad (3.F.7a)$$

$$\varphi^{-1}(t; \delta) = -\log \frac{1 - e^{-\delta t}}{1 - e^{-\delta}}, \quad t \in \mathbf{I}. \quad (3.F.7b)$$

The *multivariate Frank* copula is,

$$C(\mathbf{u}; \delta) = -\delta^{-1} \log \left(1 - \frac{\prod_{j=1}^n (1 - e^{-\delta u_j})}{(1 - e^{-\delta})^{n-1}} \right), \quad \mathbf{u} \in \mathbf{I}^n. \quad (3.F.8)$$

See Joe (2014, Section 4.5.1) for details.

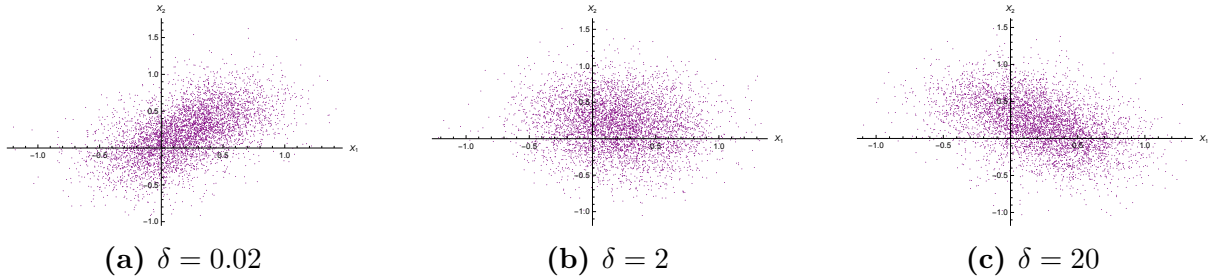


Figure 3.F.2: A scatter plot of (X_1, X_2) , whose joint distribution is generated by the Frank copula of Example 3.F.2 with various parameters. The marginal distributions are $X_i \sim \mathcal{N}(0.20, 0.35)$ for $i = 1, 2$.

Example 3.F.3 (Gumbel). For $1 \leq \delta < \infty$, the *bivariate Gumbel* copula is,

$$C(u, v; \delta) = \exp \left\{ - \left([-\log u]^\delta + [-\log v]^\delta \right)^{1/\delta} \right\}, \quad u, v \in \mathbf{I}, \quad (3.F.9)$$

where its generator is,

$$\varphi(s; \delta) = \exp(-s^{1/\delta}), \quad (3.F.10a)$$

$$\varphi^{-1}(t; \delta) = (-\log t)^\delta, \quad t \in \mathbf{I}. \quad (3.F.10b)$$

$$(3.F.11)$$

The *multivariate Gumbel* copula is,

$$C(\mathbf{u}; \delta) = \exp \left\{ - \left(\sum_{j=1}^n [-\log u_j]^\delta \right)^{1/\delta} \right\}, \quad \mathbf{u} \in \mathbf{I}^n. \quad (3.F.12)$$

See Joe (2014, Section 4.8.1) for details.

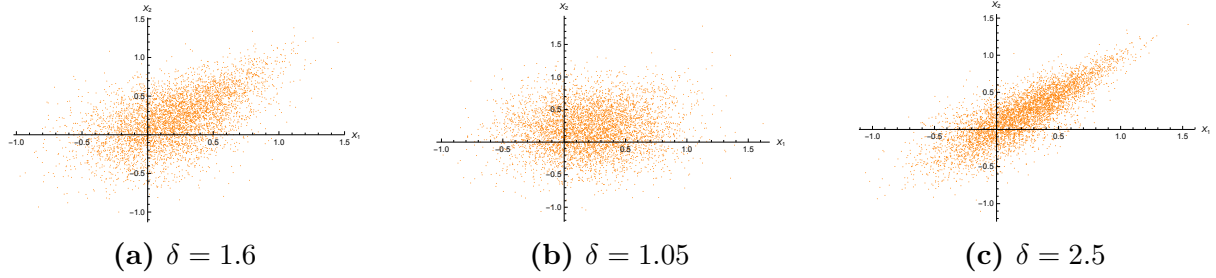


Figure 3.F.3: A scatter plot of (X_1, X_2) , whose joint distribution is generated by the Gumbel copula of Example 3.F.3 with various parameters. The marginal distributions are $X_i \sim \mathcal{N}(0.20, 0.35)$ for $i = 1, 2$.

Approximating continuous marginals by discrete marginals

As mentioned at the beginning of Section 3.F, due to both moment existence issues on the theoretical end and also on numerical computation issues in practice, in this paper we will use discrete distributions rather than continuous distributions. In particular, in light of the copula discussion above, we just need to focus on describing a method to construct marginal distributions F_i 's, and the joint distribution will then be applied via the copula.

So suppose we have finite number of states.³⁸ Rather than assigning probability weights in a potentially arbitrary fashion, we will use a more systematic method to construct the probabilities. We will construct a discrete distribution that matches the moments of a corresponding parametric continuous distribution. We follow the procedure described in Luceno (1999); although we acknowledge that the core ideas are much older and are already described in Abramowitz and Stegun (1972) (see also Stoer and Bulirsch (2002)). In particular, we approximate the Gaussian distribution. In the actual applications to this paper, we will model the returns of all strategies R_θ, R_τ to have Gaussian marginals, with moments matching Assumption 3.3.2, Assumption 3.6.2 and Assumption 3.6.3.

3.G Additional Results and Proofs for Section 3.7

Proof of Proposition 3.C.1. (a) Fix any investment strategy pair (θ, τ) .

- (i) For any given contract $(x, \{y_{C,0}, y_{C,1}\})$, the problem of solving for the portfolio policy of Manager C is,

$$\sup_{\psi_0, \psi_1 \in \mathbb{R}} -(c(\theta) + c(\tau)) + \mathbb{E}_0[x + \tilde{W}_{C,2}^{(\theta, \tau)}] - \frac{\eta_M}{2} \text{Var}_0(\tilde{W}_{C,2}^{(\theta, \tau)}) \quad (3.G.1)$$

$$=: -(c(\theta) + c(\tau)) + x_C + \hat{U}_{C,0}^{(\theta, \tau)}, \quad (3.G.2)$$

where recall that $\tilde{W}_{C,t}^{(\theta, \tau)} = \tilde{W}_{C,t}^{(\theta, \tau)}(\psi_0, \psi_1; x_C, \{y_{C,0}, y_{C,1}\})$, as per (3.B.1f), (3.B.1g), is the time t wealth of Manager C (except for the fixed fee x_C) for an arbitrary contract, and also an arbitrary portfolio policy. Using the dynamic programming principle as per Section 3.A, we are motivated to recursively define, for $t = 0, 1$,

$$\tilde{U}_{C,t}^{(\theta, \tau)} := \tilde{U}_{C,t}^{(\theta, \tau)}(y_{C,0}, y_{C,1}) = \mathbb{E}_t \tilde{U}_{C,t+1} - \frac{\eta_M}{2} \text{Var}_t(\mathbb{E}_{t+1} \tilde{W}_{C,2}^{(\theta, \tau)}). \quad (3.G.3)$$

Thus, at $t = 1$, Manager C faces the portfolio choice problem,

$$\hat{U}_{C,1}^{(\theta, \tau)} = \sup_{\psi_1 \in \mathbb{R}} \tilde{U}_{C,1}^{(\theta, \tau)}. \quad (3.G.4)$$

Note that, clearly by the fixed nature of the fixed fee x , it does not affect Manager C 's portfolio choice at any period t . Optimizing, the resulting $t = 1$ portfolio is (3.C.1). After substituting the $t = 1$ optimal portfolio back into the objective function, the resulting $t = 1$ value function for Manager C is (3.C.2).

³⁸ In the application, we'll be taking small number of states, say three.

- (ii) Next, again holding for an arbitrary contract, we consider the $t = 0$ portfolio policy as chosen by Manager C , and indeed that is the problem,

$$\hat{U}_{C,0}^{(\theta,\tau)} = \sup_{\psi_0 \in \mathbb{R}} \mathbb{E}_0[\hat{U}_{C,1}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 \tilde{W}_{C,2}^{(\theta,\tau)}), \quad (3.G.5)$$

which is (3.C.4).

Substitute the optimal portfolio choices back into Manager C 's wealth constraints, of which we then have $W_{C,t}^{(\theta,\tau)} = \tilde{W}_{C,t}^{(\theta,\tau)}(\hat{\psi}_0, \hat{\psi}_1; x_C, \{y_{C,0}, y_{C,1}\})$ as per (3.B.1c), (3.B.1d).

- (b) Thus from before, for any given arbitrary contract $(x_C, \{y_{C,0}, y_{C,1}\})$, we have solved for Manager C 's optimal portfolio policies $\hat{\psi}_{t,(\theta,\tau)}(x_C, \{y_{C,0}, y_{C,1}\})$ for all times $t = 0, 1$. We now solve the optimal contract as chosen by the Principal.

- (i) From the individual rationality constraint (3.B.1i), it will bind and implying the fixed fee x_C is (3.C.5).
 (ii) Let us now optimize over the performance fees $(y_{C,0}, y_{C,1})$. Recalling the objective function (DynCen), and substituting in the fixed fee $\hat{x}_{C,(\theta,\tau)}$ form, the optimization problem of the Principal at $t = 0$ is,

$$\begin{aligned} \hat{U}_{cP,0}^{(\theta,\tau)} &= \sup_{\substack{x_C \in \mathbb{R}, \\ y_{C,0}, y_{C,1} \in [0,1]}} -x_C + \mathbb{E}_0[W_{cP,2}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta,\tau)}) \\ &= \sup_{y_{C,0}, y_{C,1} \in [0,1]} -(c(\theta) + c(\tau)) + \mathbb{E}_0[W_{C,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_0(W_{C,2}^{(\theta,\tau)}) \\ &\quad + \mathbb{E}_0[W_{cP,2}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta,\tau)}) \\ &= \sup_{y_{C,0}, y_{C,1} \in [0,1]} -(c(\theta) + c(\tau)) + \tilde{U}_{P,0}^{(\theta,\tau)} + U_{C,0}, \end{aligned}$$

where we have defined, for any $y_{C,0}, y_{C,1} \in [0, 1]$,

$$\tilde{U}_{cP,0}^{(\theta,\tau)} := \mathbb{E}_0[W_{cP,2}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta,\tau)}), \quad (3.G.6)$$

$$U_{C,0}^{(\theta,\tau)} := \mathbb{E}_0[W_{C,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_0(W_{C,2}^{(\theta,\tau)}). \quad (3.G.7)$$

Now, applying the dynamic programming principle of Section 3.A, this motivates the definition that for $t = 0, 1$,

$$\tilde{U}_{cP,t}^{(\theta,\tau)} = \mathbb{E}_t \tilde{U}_{cP,t+1} - \frac{\eta_P}{2} \text{Var}_t(\mathbb{E}_{t+1} W_{cP,2}), \quad (3.G.8a)$$

$$U_{C,t}^{(\theta,\tau)} = \mathbb{E}_t U_{C,t+1} - \frac{\eta_M}{2} \text{Var}_t(\mathbb{E}_{t+1} W_{C,2}). \quad (3.G.8b)$$

This implies we can now consider the problem

$$\hat{U}_{cP,1}^{(\theta,\tau)} = \sup_{y_{C,1} \in [0,1]} \tilde{U}_{cP,1}^{(\theta,\tau)} + U_{C,1}^{(\theta,\tau)}, \quad (3.G.9)$$

to determine the $t = 1$ fees. Taking first order conditions for optimization, and actually analogous to the static centralized delegation problem in first best of Proposition 3.4.1, the first order conditions associated with the performance fees $y_{C,1}$ will have four roots, them being,

$$\left\{ \begin{aligned} &\frac{\eta_P}{\eta_P + \eta_M}, - \frac{(\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2})^{2/3}}{(W_{cP,1}^2 \eta_M^2 (\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2})) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2)^{1/3}}, \\ &\pm \frac{(-1)^{2/3} (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2})^{2/3}}{(W_{cP,1}^2 \eta_M^2 (\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2})) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2)^{1/3}} \end{aligned} \right\}$$

but the only real-valued solution in $[0, 1]$ is clearly the first one. Thus, we have determined Principal's $t = 1$ optimal performance fee policy (3.C.6). Let $\hat{U}_{cP,1}^{(\theta,\tau)}$ be the value of $\tilde{U}_{cP,1}^{(\theta,\tau)}$ evaluated at $\hat{y}_{C,1}$, and also let $\hat{U}_{C,1}^{(\theta,\tau)}$ be the value of $U_{C,1}^{(\theta,\tau)}$ evaluated at $\hat{y}_{C,1}$, so that $\hat{U}_{cP,1}^{(\theta,\tau)} = \hat{\tilde{U}}_{cP,1}^{(\theta,\tau)} + \hat{U}_{C,1}^{(\theta,\tau)}$.

- (iii) Now, we continue to optimize for the $t = 0$ performance fees $y_{C,0} \in [0, 1]$. From the dynamic programming principle, we can consider the problem,

$$\hat{U}_{cP,0}^{(\theta,\tau)} = \sup_{y_{C,0} \in [0,1]} \mathbb{E}_0[\hat{U}_{cP,1}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{cP,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta,\tau)}), \quad (3.G.10)$$

which is (3.C.9). ■

Proof of Corollary 3.C.2. The $t = 1$ results are a simple application of Assumption 3.6.3 to the results of Proposition 3.C.1. However, by using Assumption 3.6.3, we can get substantial simplifications and clarity in the $t = 0$ results. One can readily verify that with the simplifications provided by Assumption 3.6.3, the objective function for the $t = 0$ fees optimization becomes a concave quadratic in y_0 , and thus if an interior maximizer exists in $(0, 1)$, then it must be unique and can be characterized by first order conditions. ■

Decentralization

Proposition 3.G.1. *Consider the first best decentralized delegation problem (DynDec) but without the incentive compatibility constraints (3.6.1i) and (3.6.1j). Fix any strategy pair $(\theta, \tau) \in S$. Assume Assumption 3.6.2.*

- (a) *For any portfolio policy and performance fees $(\pi_0, \pi_1, y_{A,0}, y_{A,1}, y_{A,0}, y_{A,1})$, the optimal fixed fees for Manager A and Manager B are, respectively,*

$$\hat{x}_{A,(\theta,\tau)} = c(\theta) - \mathbb{E}_0[W_{A,2}^{(\theta,\tau)}] + \frac{\eta_M}{2} \text{Var}_0(W_{A,2}^{(\theta,\tau)}), \quad (3.G.11a)$$

$$\hat{x}_{B,(\theta,\tau)} = c(\tau) - \mathbb{E}_0[W_{B,2}^{(\theta,\tau)}] + \frac{\eta_M}{2} \text{Var}_0(W_{B,2}^{(\theta,\tau)}). \quad (3.G.11b)$$

- (b) *Suppose the $t = 1$ realized value of the Principal's wealth is $W_{P,1}^{(\theta,\tau)} = w_{P,1}^{(\theta,\tau)}$. Then the $t = 1$ optimal policies are given as follows.*

- (i) *The $t = 1$ optimal portfolio chosen by the Principal is ³⁹,*

$$\hat{\pi}_{1,(\theta,\tau)} = \frac{\hat{\pi}_1^{1N}}{\hat{\pi}_1^{1D}} + \frac{\hat{\pi}_1^{2N}}{\hat{\pi}_1^{2D}}, \quad (3.G.12)$$

provided that $\hat{\pi}_{1,(\theta,\tau)} \in (0, 1)$ and where,

$$\begin{aligned} \hat{\pi}_1^{1N} &:= \text{Var}_1(R_{\theta,2}) \left[(\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right. \\ &\quad \left. - \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (\eta_M \text{Var}_1(R_{\tau,2}) + \eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) \right], \\ \hat{\pi}_1^{1D} &:= (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})] \\ &\quad - 2\eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \\ &\quad - \eta_P (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2, \end{aligned}$$

and,

$$\begin{aligned} \hat{\pi}_1^{2N} &:= (\mathbb{E}_1 R_{\tau,2} - \mathbb{E}_1 R_{\theta,2}) \left[\eta_P^2 \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 - (\eta_P + \eta_M)^2 \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right], \\ \hat{\pi}_1^{2D} &:= \eta_P \eta_M w_{P,1}^{(\theta,\tau)} \hat{\pi}_1^{1D}. \end{aligned}$$

³⁹ i.e. “N” for numerator, and “D” for denominator.

(ii) The optimal $t = 1$ performance fee chosen by the Principal to compensate Manager A is,

$$\hat{y}_{A,1,(\theta,\tau)} = \frac{\hat{y}_{A,1}^N}{\hat{y}_{A,1}^D}, \quad (3.G.13)$$

provided that $\hat{y}_{A,1,(\theta,\tau)} \in (0, 1)$, and where,

$$\begin{aligned} \hat{y}_{A,1}^N &:= \eta_P^2 \eta_M \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] w_{P,1} \\ &\quad + \eta_P (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \left[(\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right. \\ &\quad \left. - \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (\eta_M \text{Var}_1(R_{\tau,2}) + \eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) \right], \\ \hat{y}_{A,1}^D &:= (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) [(\eta_P + \eta_M)^2 \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \eta_P^2 \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\ &\quad + \eta_P \eta_M \text{Var}_1(R_{\tau,2}) \left[-\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 \right. \\ &\quad \left. + \text{Var}_1(R_{\theta,2}) [(\eta_P + \eta_M) \text{Var}_1(R_{\tau,2}) - \eta_M \text{Cov}_1(R_{\theta,2}, R_{\tau,2})] \right] w_{P,1}^{(\theta,\tau)}. \end{aligned}$$

(iii) The optimal $t = 1$ performance fee chosen by the Principal to compensate Manager B is,

$$\hat{y}_{B,1,(\theta,\tau)} = \frac{\hat{y}_{B,1}^N}{\hat{y}_{B,1}^D}, \quad (3.G.14)$$

provided that $\hat{y}_{B,1,(\theta,\tau)} \in (0, 1)$, and where,

$$\begin{aligned} \hat{y}_{B,1}^N &:= \eta_P^2 \eta_M \text{Var}_1(R_{\theta,2}) [-\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) + \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] w_{P,1}^{(\theta,\tau)} \\ &\quad + \eta_P (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \left[-\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 \right. \\ &\quad \left. + \text{Var}_1(R_{\theta,2}) [(\eta_P + \eta_M) \text{Var}_1(R_{\tau,2}) - \eta_M \text{Cov}_1(R_{\theta,2}, R_{\tau,2})] \right], \\ \hat{y}_{B,1}^D &:= (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) [(\eta_P + \eta_M)^2 \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \eta_P^2 \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\ &\quad + \eta_P \eta_M \text{Var}_1(R_{\theta,2}) \left[-(\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right. \\ &\quad \left. + \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (\eta_M \text{Var}_1(R_{\tau,2}) + \eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})) \right] w_{P,1}^{(\theta,\tau)}. \end{aligned}$$

(c) The $t = 0$ optimal portfolio and fee policies $(\hat{\pi}_{0,(\theta,\tau)}, \hat{y}_{A,0,(\theta,\tau)}, \hat{y}_{B,0,(\theta,\tau)})$ are obtained by solving the following.

(i) Define the $t = 1$ continuation utilities for the Principal, Manager A and Manager B, respectively:

$$\tilde{U}_{P,1}^{(\theta,\tau)} := \mathbb{E}_1 W_{P,2}^{(\theta,\tau)} - \frac{\eta_P}{2} \text{Var}_1(W_{P,2}^{(\theta,\tau)}), \quad (3.G.15a)$$

$$U_{A,1}^{(\theta,\tau)} := \mathbb{E}_1 W_{A,2}^{(\theta,\tau)} - \frac{\eta_M}{2} \text{Var}_1(W_{A,2}^{(\theta,\tau)}), \quad (3.G.15b)$$

$$U_{B,1}^{(\theta,\tau)} := \mathbb{E}_1 W_{B,2}^{(\theta,\tau)} - \frac{\eta_M}{2} \text{Var}_1(W_{B,2}^{(\theta,\tau)}), \quad (3.G.15c)$$

where the $t = 2$ wealth expressions, $W_{P,2}^{(\theta,\tau)}, W_{A,2}^{(\theta,\tau)}, W_{B,2}^{(\theta,\tau)}$, have substituted in the $t = 1$ optimal policies $(\hat{\pi}_{1,(\theta,\tau)}, \hat{y}_{A,1,(\theta,\tau)}, \hat{y}_{B,1,(\theta,\tau)})$ found above. Then one can write,

$$\tilde{U}_{P,1}^{(\theta,\tau)} = -\frac{u_P^{1N}}{u_P^{1D}} (W_{P,1}^{(\theta,\tau)})^2 + \frac{u_P^{2N}}{u_P^{2D}} W_{P,1}^{(\theta,\tau)} + \frac{u_P^{3N}}{u_P^{3D}}, \quad (3.G.16a)$$

$$U_{A,1}^{(\theta,\tau)} = W_{A,1}^{(\theta,\tau)} - \frac{u_A^{1N}}{u_A^{1D}} (W_{P,1}^{(\theta,\tau)})^2 - \frac{u_A^{2N}}{u_A^{2D}} W_{P,1}^{(\theta,\tau)} + \frac{u_A^{3N}}{u_A^{3D}}, \quad (3.G.16b)$$

$$U_{B,1}^{(\theta,\tau)} = W_{B,1}^{(\theta,\tau)} - \frac{u_B^{1N}}{u_B^{1D}} (W_{P,1}^{(\theta,\tau)})^2 - \frac{u_B^{2N}}{u_B^{2D}} W_{P,1}^{(\theta,\tau)} + \frac{u_B^{3N}}{u_B^{3D}}, \quad (3.G.16c)$$

where u_k^{ij} , for $k = P, A, B$, $i = 1, 2, 3$ and $j = N, D$, are constants that relate to the risk aversion parameters (η_P, η_M) of the Principal and Managers A and B, and to the time $t = 1$ expectations, variances and covariances of the investment strategy returns pair $(R_{\theta,2}, R_{\tau,2})$ at $t = 2$. In particular, u_k^{1N}, u_k^{1D} , for $k = P, A, B$, are strictly positive terms. (The precise analytical forms of these u_k^{ij} 's are in the proof).

(ii) After substituting in the optimal $t = 1$ policies, the $t = 1$ expectation of the $t = 2$ terminal wealths are,

$$\mathbb{E}_1 W_{P,2}^{(\theta,\tau)} = \frac{w_P^{1N}}{w_P^{1D}} W_{P,1}^{(\theta,\tau)} + \frac{w_P^{2N}}{w_P^{2D}}, \quad (3.G.17a)$$

$$\mathbb{E}_1 W_{A,2}^{(\theta,\tau)} = W_{A,1}^{(\theta,\tau)} - \frac{w_A^{1N}}{w_A^{1D}} W_{P,1}^{(\theta,\tau)} + \frac{w_A^{2N}}{w_A^{2D}}, \quad (3.G.17b)$$

$$\mathbb{E}_1 W_{B,2}^{(\theta,\tau)} = W_{B,1}^{(\theta,\tau)} - \frac{w_B^{1N}}{w_B^{1D}} W_{P,1}^{(\theta,\tau)} + \frac{w_B^{2N}}{w_B^{2D}}, \quad (3.G.17c)$$

where w_k^{ij} , for $k = P, A, B$, $i = 1, 2$ and $j = N, D$, are constants that relate to the risk aversion parameters (η_P, η_M) of the Principal and Managers A and B , and to the time $t = 1$ expectations, variances and covariances of the investment strategy returns pair $(R_{\theta,2}, R_{\tau,2})$ at $t = 2$. (The precise analytical forms of these w_k^{ij} 's are in the proof).

(iii) The optimal $t = 0$ portfolio and performance fees policies $(\hat{\pi}_0, (\theta, \tau), \hat{y}_{A,0}, (\theta, \tau), \hat{y}_{B,0}, (\theta, \tau))$ are obtained by maximizing $(\pi_0, y_{A,0}, y_{B,0})$,

$$\begin{aligned} \hat{U}_{P,0}^{(\theta,\tau)} = & \sup_{y_{A,0}, y_{B,0} \in [0,1]} \sup_{\pi_0 \in \mathbb{R}} \mathbb{E}_0 \left[\tilde{U}_{P,1}^{(\theta,\tau)} + U_{A,1}^{(\theta,\tau)} + U_{B,1}^{(\theta,\tau)} \right] \\ & - \frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{P,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta,\tau)}). \end{aligned} \quad (3.G.18)$$

Let's begin by discussing the $t = 1$ optimal portfolio and fees policies, (3.G.12), (3.G.13), and (3.G.14), of the Principal in first best dynamic decentralized delegation as per Proposition 3.G.1. At $t = 1$, the realized wealths of the Principal, Manager A and Manager B become, respectively, $W_{P,1}^{(\theta,\tau)} = w_{P,1}^{(\theta,\tau)}$, $W_{A,1}^{(\theta,\tau)} = w_{A,1}^{(\theta,\tau)}$ and $W_{B,1}^{(\theta,\tau)} = w_{B,1}^{(\theta,\tau)}$. By the linearity of the contracts offered and since $t = 2$ is the terminal contracting date, it implies that from the Principal's perspective, the $t = 1$ wealths $w_{A,1}^{(\theta,\tau)}$, $w_{B,1}^{(\theta,\tau)}$ of Managers A and B do not come into his decision making. Thus at $t = 1$, the Principal simply needs to choose portfolios π_1 to maximize his $t = 2$ returns, while simultaneously using the portfolio and fee policies $(\pi_1, y_{A,1}, y_{B,1})$ to risk share with Managers A and B . We should note, however, for a generic pair of investment strategies (θ, τ) , there is a distinct Principal $t = 1$ wealth effect $w_{P,1}^{(\theta,\tau)}$ that enters into the optimal portfolio $\hat{\pi}_{1,(\theta,\tau)}$ and the optimal performance fees $(\hat{y}_{A,1}, (\theta, \tau), \hat{y}_{B,1}, (\theta, \tau))$.

Next, let's consider the $t = 0$ optimal policies (3.G.18) for the Principal. The Principal needs to take into account the motives of himself and the other two agents. In particular, by defining the continuation utilities (3.G.16) of the Principal, Manager A and Manager B , the $t = 0$ optimal portfolio and performance fee policies effectively maximize the Principal's continuation utility $\mathbb{E}_0[\tilde{U}_{P,1}^{(\theta,\tau)}]$, while minimizing Manager A 's and Manager B 's continuation utilities $\mathbb{E}_0[U_{A,1}^{(\theta,\tau)} + U_{B,1}^{(\theta,\tau)}]$. Simultaneously, given that all individuals have mean-variance preferences over terminal wealth, and hence an intertemporal hedging motive is in effect, the Principal's optimal $t = 0$ policies must minimize the volatility of all individuals' terminal wealths $\mathbb{E}_1 W_{k,2}^{(\theta,\tau)}$, for $k = P, A, B$.

Proof of Proposition 3.G.1. (a) As it is usual, the individual rationality constraints (3.6.1g), (3.6.1h) for both Manager A and B will bind. This pins down the optimal fixed fees for Manager A and B as given in (3.G.11).

(b) From the objective function (DynDec) and using the optimal form of the fixed fees (3.G.11), we consider the optimization problem,

$$\begin{aligned} & \sup_{x_A, x_B \in \mathbb{R},} \sup_{\pi_0, \pi_1} -x_A - x_B + \mathbb{E}_0[W_{P,2}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{P,2}^{(\theta,\tau)}) \\ & \sup_{y_{A,0}, y_{A,1} \in [0,1],} \sup_{y_{B,0}, y_{B,1} \in [0,1]} \\ = & \sup_{y_{A,0}, y_{A,1} \in [0,1],} \sup_{y_{B,0}, y_{B,1} \in [0,1]} \sup_{\pi_0, \pi_1 \in \mathbb{R}} -(c(\theta) + c(\tau)) + \mathbb{E}_0[W_{A,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_0(W_{A,2}^{(\theta,\tau)}) \\ & + \mathbb{E}_0[W_{B,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_0(W_{B,2}^{(\theta,\tau)}) \\ & + \mathbb{E}_0[W_{P,2}^{(\theta,\tau)}] - \frac{\eta_P}{2} \text{Var}_0(W_{P,2}^{(\theta,\tau)}). \end{aligned} \quad (3.G.19)$$

At this point, motivated by the dynamic programming principle of Section 3.A, let us further define, recursively for $t = 0, 1$,

$$\begin{aligned} \tilde{U}_{P,t}^{(\theta,\tau)} &= \mathbb{E}_t \tilde{U}_{P,t+1}^{(\theta,\tau)} - \frac{\eta_P}{2} \text{Var}_t(\mathbb{E}_{t+1} W_{P,T}^{(\theta,\tau)}), \\ U_{A,t}^{(\theta,\tau)} &= \mathbb{E}_t U_{A,t+1}^{(\theta,\tau)} - \frac{\eta_M}{2} \text{Var}_t(\mathbb{E}_{t+1} W_{A,T}^{(\theta,\tau)}), \\ U_{B,t}^{(\theta,\tau)} &= \mathbb{E}_t U_{B,t+1}^{(\theta,\tau)} - \frac{\eta_M}{2} \text{Var}_t(\mathbb{E}_{t+1} W_{B,T}^{(\theta,\tau)}). \end{aligned}$$

Define,

$$\hat{U}_{P,0}^{(\theta,\tau)} = \sup_{\substack{y_{A,0}, y_{A,1} \in [0,1], \\ y_{B,0}, y_{B,1} \in [0,1]}} \sup_{\pi_0, \pi_1 \in \mathbb{R}} \tilde{U}_{P,0}^{(\theta,\tau)} + U_{A,0}^{(\theta,\tau)} + U_{B,0}^{(\theta,\tau)},$$

so that our optimization problem (3.G.19) can be rewritten as,

$$-(c(\theta) + c(\tau)) + \hat{U}_{P,0}^{(\theta,\tau)}. \quad (3.G.20)$$

Now we can consider the Principal's problem at $t = 1$. Using the budget constraints, the Principal's objective is to maximize over portfolio policies π_1 and the fees $(y_{A,1}, y_{B,1}) \in [0, 1]^2$,

$$\begin{aligned} & \tilde{U}_{P,1}^{(\theta,\tau)} + U_{A,1}^{(\theta,\tau)} + U_{B,1}^{(\theta,\tau)} \\ &= \mathbb{E}_1 [W_{P,2}^{(\theta,\tau)} + W_{A,2}^{(\theta,\tau)} + W_{B,2}^{(\theta,\tau)}] - \frac{\eta_M}{2} \text{Var}_1(W_{A,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_1(W_{B,2}^{(\theta,\tau)}) - \frac{\eta_P}{2} \text{Var}_1(W_{P,2}^{(\theta,\tau)}) \\ &= \mathbb{E}_1 \left[w_{P,1}^{(\theta,\tau)} (1 + \pi_1(1 - y_{B,1})R_{\tau,2} + (1 - \pi_1)(1 - y_{A,1})R_{\theta,2}) \right] \\ & \quad + \mathbb{E}_1 \left[w_{A,1}^{(\theta,\tau)} + w_{P,1}^{(\theta,\tau)}(1 - \pi_1)y_{A,1}R_{\theta,2} \right] - \frac{\eta_M}{2} (w_{P,1}^{(\theta,\tau)})^2 (1 - \pi_1)^2 y_{A,1}^2 \text{Var}_1(R_{\theta,2}) \\ & \quad + \mathbb{E}_1 \left[w_{B,1}^{(\theta,\tau)} + w_{P,1}^{(\theta,\tau)} \pi_1 y_{B,1} R_{\tau,2} \right] - \frac{\eta_M}{2} (w_{P,1}^{(\theta,\tau)})^2 \pi_1^2 y_{B,1}^2 \text{Var}_1(R_{\tau,2}) \\ & \quad - \frac{\eta_P}{2} (w_{P,1}^{(\theta,\tau)})^2 \left[\pi_1^2 (1 - y_{B,1})^2 \text{Var}_1(R_{\tau,2}) + (1 - \pi_1)^2 (1 - y_{A,1})^2 \text{Var}_1(R_{\theta,2}) \right. \\ & \quad \left. + 2\pi_1(1 - \pi_1)(1 - y_{B,1})(1 - y_{A,1}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \right] \\ &= w_{A,1}^{(\theta,\tau)} + w_{B,1}^{(\theta,\tau)} + w_{P,1}^{(\theta,\tau)} [1 + \pi_1 R_{\tau,2} + (1 - \pi_1)R_{\theta,2}] \\ & \quad - \frac{\eta_M}{2} (w_{P,1}^{(\theta,\tau)})^2 (1 - \pi_1)^2 y_{A,1}^2 \text{Var}_1(R_{\theta,2}) \\ & \quad - \frac{\eta_M}{2} (w_{P,1}^{(\theta,\tau)})^2 \pi_1^2 y_{B,1}^2 \text{Var}_1(R_{\tau,2}) \\ & \quad - \frac{\eta_P}{2} (w_{P,1}^{(\theta,\tau)})^2 \left[\pi_1^2 (1 - y_{B,1})^2 \text{Var}_1(R_{\tau,2}) + (1 - \pi_1)^2 (1 - y_{A,1})^2 \text{Var}_1(R_{\theta,2}) \right. \\ & \quad \left. + 2\pi_1(1 - \pi_1)(1 - y_{B,1})(1 - y_{A,1}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \right]. \end{aligned}$$

Optimizing for an interior solution over $(\pi_1, y_{A,1}, y_{B,1})$ we get the described solution. Substitute the optimal $t = 1$ policies back into $\tilde{U}_{P,1}^{(\theta,\tau)}$, $U_{A,1}^{(\theta,\tau)}$, $U_{B,1}^{(\theta,\tau)}$ and denote them, respectively as, $\hat{U}_{P,1}^{(\theta,\tau)}$, $\hat{U}_{A,1}^{(\theta,\tau)}$, $\hat{U}_{B,1}^{(\theta,\tau)}$. After substituting and simplifying, this results in the expressions (3.G.16).

(c) Next, we consider the $t = 0$ optimal portfolio and performance fee policies. At $t = 0$, the Principal considers the problem,

$$\begin{aligned} \hat{U}_{P,0} &= \sup_{y_{A,0}, y_{B,0} \in [0,1]} \sup_{\pi_0 \in \mathbb{R}} \mathbb{E}_0 [\hat{U}_{P,1}^{(\theta,\tau)} + \hat{U}_{A,1}^{(\theta,\tau)} + \hat{U}_{B,1}^{(\theta,\tau)}] \\ & \quad - \frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{P,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta,\tau)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta,\tau)}). \end{aligned} \quad (3.G.21)$$

Firstly, we should note that after substituting in the optimal portfolio and fee policies $(\hat{\pi}_1, \hat{y}_{A,1}, \hat{y}_{B,1})$ and substituting them back and simplifying, we get (3.G.16). Where for the terms involved in $U_{A,1}$,

$$\begin{aligned} u_A^{1N} &:= \eta_P^2 \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2})^2 (\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2)^2, \\ u_A^{1D} &:= 2 \left[-(\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) \right. \\ & \quad \left. + 2\eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \right. \\ & \quad \left. + \eta_P (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 \right]^2 \end{aligned}$$

and,

$$\begin{aligned}
u_A^{2N} &:= \eta_P \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\
&\quad \times \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 [\text{Var}_1(R_{\theta,2}) \mathbb{E}_1 R_{\tau,2} + \text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2}] \right. \\
&\quad \quad + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \mathbb{E}_1 R_{\tau,2} + \text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2}] \right], \\
u_A^{2D} &:= \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) \right. \\
&\quad \quad + 2\eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) \right]^2,
\end{aligned}$$

and,

$$\begin{aligned}
u_A^{3N} &:= (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \times \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 + \eta_M \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \right. \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right] \\
&\quad \times \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 [2\text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2} + \text{Var}_1(R_{\theta,2}) (\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2})] \right. \\
&\quad \quad + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (3\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) [2\text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2} + (\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\theta,2})] \right], \\
u_A^{3D} &:= 2\eta_M \left[\eta_P (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \right. \\
&\quad \quad \left. + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Var}_1(R_{\theta,2} - R_{\tau,2}) \right]^2.
\end{aligned}$$

For the terms involved in $U_{B,1}$,

$$\begin{aligned}
u_B^{1N} &:= \eta_P^2 \eta_M \text{Var}_1(R_{\theta,2})^2 \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2], \\
u_B^{1D} &:= u_A^{1D},
\end{aligned}$$

and,

$$\begin{aligned}
u_B^{2N} &:= u_A^{2N}, \\
u_B^{2D} &:= u_A^{2D},
\end{aligned}$$

and,

$$\begin{aligned}
u_B^{3N} &:= (\mathbb{E}_1 R_{\tau,2} - \mathbb{E}_1 R_{\theta,2}) \times \left[\eta \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 + \eta_M \text{Var}_1(R_{\theta,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \right. \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right] \\
&\quad \times \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 [2\text{Var}_1(R_{\theta,2}) \mathbb{E}_1 R_{\tau,2} + \text{Var}_1(R_{\tau,2}) (\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2})] \right. \\
&\quad \quad + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (\mathbb{E}_1 R_{\theta,2} + 3\mathbb{E}_1 R_{\tau,2}) \\
&\quad \quad \left. - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) [2\text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2} + (\mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\tau,2})] \right], \\
u_B^{3D} &:= u_A^{3D}.
\end{aligned}$$

For the terms involving $\tilde{U}_{P,1}$,

$$\begin{aligned} u_P^{1N} &:= \eta_P \eta_M^2 \text{Var}_1(R_{\theta,2})^2 \text{Var}_1(R_{\tau,2})^2 \text{Var}_1(R_{\theta,2} - R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2], \\ u_P^{1D} &:= 2 \left[\eta_P (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \right. \\ &\quad \left. + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Var}_1(R_{\theta,2} - R_{\tau,2}) \right]^2, \end{aligned}$$

and,

$$\begin{aligned} u_P^{2N} &:= \eta_P^2 (\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2}))^2 [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2]^2 \\ &\quad + \eta_P \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Var}_1(R_{\theta,2} - R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\ &\quad \times [(2 + \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\theta,2}) + (2 + \mathbb{E}_1 R_{\theta,2}) \text{Var}_1(R_{\tau,2})] \\ &\quad + \eta_M^2 \text{Var}_1(R_{\theta,2})^2 \text{Var}_1(R_{\tau,2})^2 \text{Var}_1(R_{\theta,2} - R_{\tau,2}) \times \\ &\quad [(1 + \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\theta,2}) + (1 + \mathbb{E}_1 R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - (2 + \mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2})], \\ u_P^{2D} &:= u_A^{2D}. \end{aligned}$$

Furthermore, the $t = 1$ expectation of the $t = 2$ wealth, after substituting in the $t = 1$ optimal policies are given in (3.G.17). The terms involved in $W_{A,2}$ are,

$$\begin{aligned} w_A^{1N} &:= \eta_P \mathbb{E}_1 [R_{\theta,2}] \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2], \\ w_A^{1D} &:= \eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 [\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})] \\ &\quad + 2\eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) \\ &\quad - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})], \end{aligned}$$

and,

$$\begin{aligned} w_A^{2N} &:= \mathbb{E}_1 [R_{\theta,2}] (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 \right. \\ &\quad \left. + \eta_M \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right], \\ w_A^{2D} &:= \eta_M w_A^{1D}. \end{aligned}$$

The terms involving in $W_{B,2}$ are,

$$\begin{aligned} w_A^{1N} &:= \eta_P \mathbb{E}_1 [R_{\tau,2}] \text{Var}_1(R_{\theta,2}) [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2], \\ w_B^{1D} &:= w_A^{1D}, \end{aligned}$$

and,

$$\begin{aligned} w_B^{2N} &:= \mathbb{E}_1 [R_{\tau,2}] (\mathbb{E}_1 R_{\tau,2} - \mathbb{E}_1 R_{\theta,2}) \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2 \right. \\ &\quad \left. + \eta_M \text{Var}_1(R_{\tau,2}) \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) - (\eta_P + \eta_M) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right], \\ w_B^{2D} &:= \eta_M w_A^{1D}. \end{aligned}$$

The terms involving $W_{P,1}$ are,

$$\begin{aligned} w_P^{1N} &:= \eta_P [\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})] [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\ &\quad + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \left[(1 + \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\theta,2}) + (1 + \mathbb{E}_1 R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right. \\ &\quad \left. - \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) (2 + \mathbb{E}_1 R_{\theta,2} + \mathbb{E}_1 R_{\tau,2}) \right], \\ w_P^{1D} &:= \eta_P [\text{Var}_1(R_{\theta,2}) + \text{Var}_1(R_{\tau,2})] [\text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) - \text{Cov}_1(R_{\theta,2}, R_{\tau,2})^2] \\ &\quad + \eta_M \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \text{Var}_1(R_{\theta,2} - R_{\tau,2}), \end{aligned}$$

and,

$$w_P^{2N} := (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \left[\eta_P \text{Cov}_1(R_{\theta,2}, R_{\tau,2}) [\text{Var}_1(R_{\theta,2}) \mathbb{E}_1 R_{\tau,2} - \text{Var}_1(R_{\tau,2}) \mathbb{E}_1 R_{\theta,2}] \right. \\ \left. - (\eta_P + \eta_M) (\mathbb{E}_1 R_{\theta,2} - \mathbb{E}_1 R_{\tau,2}) \text{Var}_1(R_{\theta,2}) \text{Var}_1(R_{\tau,2}) \right], \\ w_P^{2D} := \eta_P w_A^{1D}.$$

■

Proof of Proposition 3.7.1. This is clearly a special case of Proposition 3.7.1. ■

3.H Proofs for Section 3.H

Proof of Proposition 3.D.1. (a) For any given contract $(x_A, \{y_{A,0}, y_{A,1}\})$ and any investment strategy pairs $(\theta, \tau) \in \mathcal{S}$, it is clear that Manager C 's optimal portfolio choice would be equivalent to the form as given in Proposition 3.C.1.

(b) Recall that the Principal wants to induce the strategy pair (θ_H, τ_H) . Thus, by binding the individual rationality constraint (3.B.1i), we obtain the optimal fixed fee form as per Proposition 3.C.1.

(c) Now we consider the Principal's second best optimization problem. Recall that the Principal wants to implement the strategy pair (θ_H, τ_H) , but Manager C could deviate to $(\theta, \tau) \in \mathcal{S}_{-(\theta_H, \tau_H)}$. As it was in the case in static centralized delegation, although we have three incentive compatibility constraints, clearly only one of them will bind, while the others will be slack. Following Bellman (1956), let us introduce the Lagrange multipliers $\lambda_C^{(\theta_H, \tau_L)}, \lambda_C^{(\theta_L, \tau_H)}, \lambda_C^{(\theta_L, \tau_L)}$ into our dynamic optimization problem. Recall also the notations from the proof of Proposition 3.C.1 where we had denoted $\hat{U}_{C,t}^{(\theta, \tau)}$ as Manager C 's time t continuation value for implementing strategy (θ, τ) along the optimal portfolio strategies $\hat{\psi}_{t,(\theta, \tau)}$; note and recall also that $\hat{U}_{C,t}^{(\theta, \tau)}$ is still a function of the performance fees (y_0, y_1) . Also note that using our notations, we can rewrite the incentive compatibility constraints (3.B.1j) as,

$$-2c + U_{C,0}^{(\theta_H, \tau_H)} \geq -c + U_{C,0}^{(\theta_H, \tau_L)}, \quad (3.H.1a)$$

$$-2c + U_{C,0}^{(\theta_H, \tau_H)} \geq -c + U_{C,0}^{(\theta_L, \tau_H)}, \quad (3.H.1b)$$

$$-2c + U_{C,0}^{(\theta_H, \tau_H)} \geq U_{C,0}^{(\theta_L, \tau_L)}. \quad (3.H.1c)$$

Thus at $t = 0$, recalling the objective function form (DynCen), substituting in the optimal fixed fee form, and incorporating the rewritten form of the incentive compatibility constraint, the Principal considers the sequence of problems indexed by $\lambda_C := (\lambda_C^{(\theta_H, \tau_L)}, \lambda_C^{(\theta_L, \tau_H)}, \lambda_C^{(\theta_L, \tau_L)}) \in \mathbb{R}^3$,

$$\begin{aligned} & \sup_{\substack{x_A \in \mathbb{R}, \\ y_{A,0}, y_{A,1} \in [0,1]}} -x_A + \mathbb{E}_0[W_{cP,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}) - \lambda_C^{(\theta_H, \tau_L)} \left[\hat{U}_{C,0}^{(\theta_H, \tau_L)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] \\ & \quad - \lambda_C^{(\theta_L, \tau_H)} \left[\hat{U}_{C,0}^{(\theta_L, \tau_H)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] - \lambda_C^{(\theta_L, \tau_L)} \left[\hat{U}_{C,0}^{(\theta_L, \tau_L)} - \left(-c + \hat{U}_{C,0}^{(\theta_L, \tau_H)} \right) \right] \\ & = \sup_{y_{A,0}, y_{A,1} \in [0,1]} -2c + \mathbb{E}_0[W_{cP,2}^{(\theta_H, \tau_H)}] - \frac{\eta_M}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}) \\ & \quad + \mathbb{E}_0[W_{cP,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}) - \lambda_C^{(\theta_L, \tau_L)} \left[\hat{U}_{C,0}^{(\theta_L, \tau_L)} - \left(-2c + U_{C,0}^{(\theta_H, \tau_H)} \right) \right] \\ & \quad - \lambda_C^{(\theta_H, \tau_L)} \left[\hat{U}_{C,0}^{(\theta_H, \tau_L)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] - \lambda_C^{(\theta_L, \tau_H)} \left[\hat{U}_{C,0}^{(\theta_L, \tau_H)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] \\ & =: -2c + \hat{U}_{cP,0}^{\lambda_C}. \end{aligned} \quad (3.H.2)$$

In particular, again as motivated by the dynamic programming principle of Section 3.A, let's recall the notations (3.G.8). In particular, this implies that if we focus on the value function term $\hat{U}_{cP,0}^{\lambda_C}$, and denoting $U_{cP,0}^{\lambda_C}$ as the associated objective

function, we can consider the recursive relationship,

$$\begin{aligned} \hat{U}_{cP,1}^{\lambda_C} = & \sup_{y_{A,1} \in [0,1]} U_{C,1}^{(\theta_H, \tau_H)} + \tilde{U}_{cP,1}^{(\theta_H, \tau_H)} - \lambda_C^{(\theta_L, \tau_L)} \left[U_{C,1}^{(\theta_L, \tau_L)} - \left(-2c + U_{C,1}^{(\theta_H, \tau_H)} \right) \right] \\ & - \lambda_C^{(\theta_H, \tau_L)} \left[\hat{U}_{C,0}^{(\theta_H, \tau_L)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] - \lambda_C^{(\theta_L, \tau_H)} \left[\hat{U}_{C,0}^{(\theta_L, \tau_H)} - \left(-c + \hat{U}_{C,0}^{(\theta_H, \tau_H)} \right) \right] \end{aligned} \quad (3.H.3)$$

$$\begin{aligned} \hat{U}_{cP,0}^{\lambda_C} = & \sup_{y_{A,0} \in [0,1]} \mathbb{E}_0[\hat{U}_{cP,1}^{\lambda_C}] - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) - \frac{\eta_P}{2} \text{Var}_0(W_{cP,2}^{(\theta_H, \tau_H)}) \\ & - \lambda_C^{(\theta_L, \tau_L)} \left[-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_L, \tau_L)}) - \left(-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) \right) \right] \\ & - \lambda_C^{(\theta_H, \tau_L)} \left[-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_L)}) - \left(-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) \right) \right] \\ & - \lambda_C^{(\theta_L, \tau_H)} \left[-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_L, \tau_H)}) - \left(-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{C,2}^{(\theta_H, \tau_H)}) \right) \right] \end{aligned} \quad (3.H.4)$$

$$\begin{aligned} = & \sup_{y_{A,0} \in [0,1]} U_{C,0}^{(\theta_H, \tau_H)} + \tilde{U}_{cP,0}^{(\theta_H, \tau_H)} - \lambda_C^{(\theta_L, \tau_L)} \left[U_{C,0}^{(\theta_L, \tau_L)} - \left(-2c + U_{C,0}^{(\theta_H, \tau_H)} \right) \right] \\ & - \lambda_C^{(\theta_H, \tau_L)} \left[U_{C,0}^{(\theta_H, \tau_L)} - \left(-c + U_{C,0}^{(\theta_H, \tau_H)} \right) \right] \\ & - \lambda_C^{(\theta_L, \tau_H)} \left[U_{C,0}^{(\theta_L, \tau_H)} - \left(-c + U_{C,0}^{(\theta_H, \tau_H)} \right) \right], \end{aligned} \quad (3.H.5)$$

where we have denoted $\hat{U}_{C,1}^{(\theta, \tau)}$ as the value of Manager C 's continuation value after substituting in the optimal $t = 1$ performance fee \hat{y}_1 into $U_{C,1}^{(\theta, \tau)}$, and likewise for $\hat{U}_{cP,1}^{(\theta_H, \tau_H)}$. But as it was argued in the static centralized delegation case, we know that at equilibrium, if a binding solution exists, only one of the three incentive compatibility constraints will bind. And thus, to further save on notations, if (θ^b, τ^b) is the pair of deviant strategies associated with the binding incentive compatibility constraints, we allow that Lagrange multiplier $\lambda_C^{(\theta^b, \tau^b)}$ to be nonzero, and set the remaining other two to zero. Then after this simplification, we get the displayed equations we see in the proposition.

And also note that, with some abuse of notations, the wealth values $W_{C,2}^{(\theta, \tau)}$ and $W_{cP,2}^{(\theta, \tau)}$ in (3.H.3) and (3.H.4) are different; in (3.H.3), after conditioning on the $t = 1$ realized wealths of both the Principal and Manager C , those $t = 2$ wealth terms are only a function of $y_{A,0}$; in (3.H.4), the resulting optimal fee $\hat{y}_{A,1}$ has been substituted in, and since at $t = 0$, the $t = 2$ wealth is a function of $t = 1$ wealth, those $t = 2$ wealth terms are only a function of $y_{A,0}$. Finally in (3.H.5), we simply reuse the notations in the proof of Proposition 3.C.1, and in particular noting that in $U_{C,0}^{(\theta, \tau)}$ and $\tilde{U}_{cP,0}^{(\theta_H, \tau_H)}$ is the $t = 0$ utility value for Manager C and the Principal, after substituting for Manager C 's optimal portfolio policy for any arbitrary contract; hence in the dynamic programming formulation, $U_{C,0}^{(\theta, \tau)}$ and $\tilde{U}_{cP,0}^{(\theta_H, \tau_H)}$ are only a function of $y_{A,0}$.

Indeed, taking first order conditions, which is both sufficient and necessary for optimization here, the optimal $t = 1$ performance fees are (3.D.1).

- (d) With the $t = 1$ optimal performance fees (3.D.1), we substitute it back to the $t = 1$ continuation value of the Principal and Manager C ; that is, and recalling the optimal portfolio policy form $\hat{\psi}_{1,(\theta, \tau)} := \hat{\psi}_{1,(\theta, \tau)}(y_{A,1})$ of (3.C.1), we get (3.D.2).
- (e) Substituting those expressions back into (3.D.3), and further substituting in the budget constraints (3.B.1a) and (3.B.1c), we can find an optimizer $\hat{y}_{A,0}^{\lambda_C} \in [0, 1]$.
- (f) Now, once $\hat{y}_{A,0}^{\lambda_C}$ has been found, then we obtain the $t = 0$ value function $\hat{U}_{cP,0}^{\lambda_C}$ for the Principal. At this point, we still need to choose the optimal λ_C . Assuming that the conditions for Strong Duality Theorem holds,⁴⁰ then the optimal λ_C is precisely the solution to (3.D.4). ■

Proof of Proposition 3.8.1. The proof is quite analogous to that of Proposition 3.D.1 but repeated here for completeness.

- (a) By the same arguments as in Proposition 3.D.1, the individual rationality constraints (3.6.1g) and (3.6.1h) will bind. This pins down the form of the fixed fees, and indeed it has the same form as the first best form of Proposition 3.G.1.

⁴⁰ It is relatively easy to check that the constraint qualifications conditions should hold here, and hence one can expect strong duality to hold. See, for instance, Bertsekas (1999).

- (b) We now consider the Principal's second best optimization problem. Again, following Bellman (1956), we introduce the Lagrange multipliers λ_A, λ_B associated with the incentive compatibility constraints (3.6.1i), (3.6.1j), respectively. We will also recycle the notations from the proof of Proposition 3.G.1 of the first best case. Note that using those notations, we can write the incentive compatibility constraints (3.6.1i) and (3.6.1j), respectively, as,

$$-c + U_{A,0}^{(\theta_H, \tau_H)} \geq U_{A,0}^{(\theta_L, \tau_H)}, \quad (3.H.6a)$$

$$-c + U_{B,0}^{(\theta_H, \tau_H)} \geq U_{B,0}^{(\theta_H, \tau_L)}. \quad (3.H.6b)$$

Thus, the Principal's optimization problem from (DynDec), binding the individual rationality constraints and thereby substituting the optimal fixed fees,

$$\begin{aligned} & \sup_{\substack{x_A, x_B \in \mathbb{R}, \\ y_{A,0}, y_{B,1} \in [0,1], \\ y_{B,0}, y_{B,1} \in [0,1]}} \sup_{\pi_0, \pi_1} -x_A - x_B + \mathbb{E}_0[W_{P,2}^{(\theta_H, \tau_H)}] - \frac{\eta_P}{2} \text{Var}_0(W_{P,2}^{(\theta_H, \tau_H)}) \\ & \quad - \lambda_A \left(U_{A,0}^{(\theta_L, \tau_H)} - (-c + U_{A,0}^{(\theta_H, \tau_H)}) \right) \\ & \quad - \lambda_B \left(U_{B,0}^{(\theta_H, \tau_L)} - (-c + U_{B,0}^{(\theta_H, \tau_H)}) \right) \\ & = \sup_{\substack{y_{A,0}, y_{A,1} \in [0,1], \\ y_{B,0}, y_{B,1} \in [0,1]}} \sup_{\pi_0, \pi_1} -2c + U_{A,0}^{(\theta_H, \tau_H)} + U_{B,0}^{(\theta_H, \tau_H)} + \tilde{U}_{P,0}^{(\theta_H, \tau_H)} \\ & \quad - \lambda_A \left(U_{A,0}^{(\theta_L, \tau_H)} - (-c + U_{A,0}^{(\theta_H, \tau_H)}) \right) - \lambda_B \left(U_{B,0}^{(\theta_H, \tau_L)} - (-c + U_{B,0}^{(\theta_H, \tau_H)}) \right) \\ & =: -2c + \hat{U}_{P,0}^{\lambda_A, \lambda_B}. \end{aligned}$$

By the dynamic programming principle of Section 3.A, and also recall the analogous argument in the proof of Proposition 3.G.1, we lead to consider the following sequence of problems:

$$\begin{aligned} \hat{U}_{P,1}^{\lambda_A, \lambda_B} & = \sup_{\substack{\pi_1 \in \mathbb{R}, \\ y_{A,1}, y_{B,1} \in [0,1]}} \tilde{U}_{P,1}^{(\theta_H, \tau_H)} + U_{A,1}^{(\theta_H, \tau_H)} + U_{B,1}^{(\theta_H, \tau_H)} \\ & \quad - \lambda_A \left(U_{A,1}^{(\theta_L, \tau_H)} - (-c + U_{A,1}^{(\theta_H, \tau_H)}) \right) - \lambda_B \left(U_{B,1}^{(\theta_H, \tau_L)} - (-c + U_{B,1}^{(\theta_H, \tau_H)}) \right), \end{aligned} \quad (3.H.7a)$$

$$\begin{aligned} \hat{U}_{P,0}^{\lambda_A, \lambda_B} & = \sup_{\substack{\pi_0 \in \mathbb{R}, \\ y_{A,0}, y_{B,0} \in [0,1]}} \mathbb{E}_0[\hat{U}_{P,1}^{\lambda_A, \lambda_B}] - \frac{\eta_P}{2} \text{Var}_0(\mathbb{E}_1 W_{P,2}^{(\theta_H, \tau_H)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta_H, \tau_H)}) - \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta_H, \tau_H)}) \\ & \quad - \lambda_A \left(-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta_L, \tau_H)}) + \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{A,2}^{(\theta_H, \tau_H)}) \right) \\ & \quad - \lambda_B \left(-\frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta_H, \tau_L)}) + \frac{\eta_M}{2} \text{Var}_0(\mathbb{E}_1 W_{B,2}^{(\theta_H, \tau_H)}) \right) \end{aligned} \quad (3.H.7b)$$

Considering the optimizing problem (3.H.7a), and optimizing for the $t = 1$ policies $(\pi_1, y_{A,1}, y_{B,1})$, we arrive at (3.8.1),

(3.8.2) and (3.8.3). In particular,

$$\begin{aligned}
& \hat{\pi}_1^{N,\lambda_A,\lambda_B} \\
&= w_{P,1}^{(\theta_H,\tau_H)} \left[-(w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M) (\lambda_A - \lambda_B) + (w_{P,1}^{(\theta_H,\tau_L)} - w_{P,1}^{(\theta_L,\tau_H)}) (w_{P,1}^{(\theta_H,\tau_L)} + w_{P,1}^{(\theta_L,\tau_H)}) \eta_M \lambda_A \lambda_B \right] \mu \\
&\quad \eta_M \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \right) \left(-(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_B)) \sigma^2 \right) \\
&\quad - w_{P,1}^{(\theta_H,\tau_L)} \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_A)) \right) \lambda_B \mu \tau_L \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \left[w_{P,1}^{(\theta_H,\tau_H)} (-\lambda_A + \lambda_B) \mu + \eta_M \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \right) \sigma^2 - w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L \right] \rho_{\theta_H,\tau_H} \\
&\quad + w_{P,1}^{(\theta_L,\tau_H)} \lambda_A \mu \theta_L \left(-(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_B)) + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \rho_{\theta_H,\tau_H} \right), \tag{3.H.8a}
\end{aligned}$$

$$\begin{aligned}
& \hat{\pi}_1^{D,\lambda_A,\lambda_B} \\
&= \eta_M \sigma^2 \left[2(w_{P,1}^{(\theta_H,\tau_H)})^4 (\eta_P + \eta_M) \right. \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)})^2 (w_{P,1}^{(\theta_H,\tau_H)} - w_{P,1}^{(\theta_L,\tau_H)}) (w_{P,1}^{(\theta_H,\tau_H)} + w_{P,1}^{(\theta_L,\tau_H)}) (\eta_P + 2\eta_P \eta_M) \lambda_A \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)} - w_{P,1}^{(\theta_H,\tau_L)}) (w_{P,1}^{(\theta_H,\tau_H)} + w_{P,1}^{(\theta_H,\tau_L)}) \left(-2(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + 2\eta_M (1 + \lambda_A)) \right) \lambda_B \\
&\quad \left. + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A - (w_{P,1}^{(\theta_H,\tau_L)})^2 \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (2 + \lambda_A + \lambda_B) \right) \rho_{\theta_H,\tau_H} \right]. \tag{3.H.8b}
\end{aligned}$$

And,

$$\begin{aligned}
& \hat{y}_{A,1}^{N,\lambda_A,\lambda_B} \\
&= -w_{P,1}^{(\theta_L,\tau_H)} \lambda_A \mu \theta_L \left(-2(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + 2\eta_M (1 + \lambda_B)) + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \rho_{\theta_H,\tau_H} \right) \\
&\quad + w_{P,1}^{(\theta_H,\tau_H)} \left[\left(-2(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_A \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 [\eta_P (\lambda_A - \lambda_B) + 2\eta_M \lambda_A (1 + \lambda_B)] \right) \mu \right. \\
&\quad + w_{P,1}^{(\theta_H,\tau_H)} \eta_P \eta_M \left(-(w_{P,1}^{(\theta_H,\tau_L)})^2 \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_B) \right) \sigma^2 \\
&\quad + w_{P,1}^{(\theta_H,\tau_H)} \eta_P (w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L + [w_{P,1}^{(\theta_H,\tau_H)} (\lambda_A - \lambda_B) \mu \\
&\quad \left. + \eta_M (-(w_{P,1}^{(\theta_H,\tau_L)})^2 \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_B)) \sigma^2 + w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L] \rho_{\theta_H,\tau_H} \right), \tag{3.H.9a}
\end{aligned}$$

$$\begin{aligned}
& \hat{y}_{A,1}^{D,\lambda_A,\lambda_B} \\
&= w_{P,1}^{(\theta_H,\tau_H)} \left[(w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M) (\lambda_A - \lambda_B) + (-(w_{P,1}^{(\theta_H,\tau_L)})^2 + (w_{P,1}^{(\theta_L,\tau_H)})^2) \eta_M \lambda_A \lambda_B \right] \mu \\
&\quad + \eta_M \left[-(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_A)) \right] \left[-(w_{P,1}^{(\theta_H,\tau_L)})^2 \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_B) \right] \sigma^2 \\
&\quad + w_{P,1}^{(\theta_H,\tau_L)} \left[-(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_A)) \right] \lambda_B \mu \tau_L \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \left[w_{P,1}^{(\theta_H,\tau_H)} (\lambda_A - \lambda_B) \mu + \eta_M \left(-(w_{P,1}^{(\theta_H,\tau_L)})^2 \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_B) \right) \sigma^2 + w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L \right] \rho_{\theta_H,\tau_H} \\
&\quad - w_{P,1}^{(\theta_L,\tau_H)} \lambda_A \mu \theta_L \left[-(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_B)) + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \rho_{\theta_H,\tau_H} \right]. \tag{3.H.9b}
\end{aligned}$$

And,

$$\begin{aligned}
& \hat{y}_{B,1}^{N,\lambda_A,\lambda_B} \\
&= w_{P,1}^{(\theta_H,\tau_H)} \left[\left(-2(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (-\eta_P \lambda_A + (\eta_P + 2\eta_M(1 + \lambda_A)) \lambda_B) \right) \mu \right. \\
&\quad + w_{P,1}^{(\theta_H,\tau_H)} \eta_P \eta_M \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \sigma^2 \right) \\
&\quad \left. - w_{P,1}^{(\theta_H,\tau_L)} \left[-2(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + 2\eta_M(1 + \lambda_A)) \right] \lambda_B \mu \tau_L \right. \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \left[w_{P,1}^{(\theta_H,\tau_H)} (-\lambda_A + \lambda_B) \mu + \eta_M \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \right) \sigma^2 - w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L \right] \rho_{\theta_H,\tau_H} \\
&\quad \left. + (w_{P,1}^{(\theta_H,\tau_H)})^2 w_{P,1}^{(\theta_L,\tau_H)} \eta_P \lambda_A \mu \theta_L (1 + \rho_{\theta_H,\tau_H}) \right], \tag{3.H.10a}
\end{aligned}$$

$$\begin{aligned}
& \hat{y}_{B,1}^{D,\lambda_A,\lambda_B} \\
&= w_{P,1}^{(\theta_H,\tau_H)} \left[-(w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M) (\lambda_A - \lambda_B) + (w_{P,1}^{(\theta_H,\tau_L)} - w_{P,1}^{(\theta_L,\tau_H)}) (w_{P,1}^{(\theta_H,\tau_L)} + w_{P,1}^{(\theta_L,\tau_H)}) \eta_M \lambda_A \lambda_B \right] \mu \\
&\quad + \eta_M \left[-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \right] \left[-(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_B)) \right] \sigma^2 \\
&\quad - w_{P,1}^{(\theta_H,\tau_L)} \left[-(w_{P,1}^{(\theta_L,\tau_H)})^2 \eta_M \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_A)) \right] \lambda_B \mu \theta_L \\
&\quad + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \left[w_{P,1}^{(\theta_H,\tau_H)} (-\lambda_A + \lambda_B) \mu + \eta_M \left(-(w_{P,1}^{(\theta_L,\tau_H)})^2 \lambda_A + (w_{P,1}^{(\theta_H,\tau_H)})^2 (1 + \lambda_A) \right) \sigma^2 - w_{P,1}^{(\theta_H,\tau_L)} \lambda_B \mu \tau_L \right] \rho_{\theta_H,\tau_H} \\
&\quad + w_{P,1}^{(\theta_L,\tau_H)} \lambda_A \mu \theta_L \left[-(w_{P,1}^{(\theta_H,\tau_L)})^2 \eta_M \lambda_B + (w_{P,1}^{(\theta_H,\tau_H)})^2 (\eta_P + \eta_M (1 + \lambda_B)) + (w_{P,1}^{(\theta_H,\tau_H)})^2 \eta_P \rho_{\theta_H,\tau_H} \right]. \tag{3.H.10b}
\end{aligned}$$

- (c) This is just to gather some results in preparation for the next subpart by substituting in the aforementioned $t = 1$ optimal policies.
- (d) This is equivalent to (3.H.7b).
- (e) This is immediate by the Strong Duality theorem.

■

Bibliography

- ABRAMOWITZ, M. AND I. A. STEGUN (1972): *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, Dover Publications, 10th ed.
- ADMATI, A. R. AND P. PFLEIDERER (1997): “Does It All Add Up? Benchmarks and the Compensation of Active Portfolio Managers,” *The Journal of Business*, 70, 323–350.
- ADRIAN, T. AND M. M. WESTERFIELD (2009): “Disagreement and Learning in a Dynamic Contracting Model,” *Review of Financial Studies*, 22, 3873–3906.
- AGARWAL, V., K. A. MULLALLY, AND N. Y. NAIK (2015): “Hedge Funds: A Survey of the Academic Literature,” *Foundations and Trends in Finance*, 1–120.
- ALBANESE, C. AND S. LAWI (2005): “Laplace Transforms for Integrals of Markov Processes,” *Markov Processes Related Fields*, 11, 677–724.
- ANDERSON, R. M. AND R. C. RAIMONDO (2008): “Equilibrium in Continuous-Time Financial Markets: Endogenously Dynamically Complete Markets,” *Econometrica*, 76, 841–907.
- ANG, A. (2012): “Asset Owners and Delegated Managers,” Tech. rep.
- BALI, T. G., N. CAKICI, X. S. YAN, AND Z. ZHANG (2005): “Does Idiosyncratic Risk Really Matter?” *Journal of Finance*, LX.
- BARRAS, L., O. SCAILLET, AND R. WERMERS (2010): “False discoveries in mutual fund performance: Measuring luck in estimated alphas,” *Journal of Finance*, LXV, 179–216.
- BARRY, C. B. AND L. T. STARKS (1984): “Investment Management and Risk Sharing with Multiple Managers,” *The Journal of Finance*, 39, 477–491.
- BASAK, S. AND G. CHABAKAURI (2010): “Dynamic Mean-Variance Asset Allocation,” *Review of Financial Studies*, 23, 2970–3016.
- (2012): “Dynamic Hedging in Incomplete Markets: A Simple Solution,” *Review of Financial Studies*, 25, 1845–1896.

- BASAK, S. AND D. CUOCO (1998): “An Equilibrium Model with Restricted Stock Market,” *Review of Financial Studies*, 11, 309–341.
- BASAK, S. AND D. MAKAROV (2014): “Strategic Asset Allocation in Money Management,” *The Journal of Finance*, 69, 179–217.
- BASAK, S. AND A. PAVLOVA (2013): “Asset Prices and Institutional Investors,” *American Economic Review*, 103, 1728–1758.
- BASAK, S., A. PAVLOVA, AND A. SHAPIRO (2007): “Optimal Asset Allocation and Risk Shifting in Money Management,” *Review of Financial Studies*, 20, 1583–1621.
- BELLMAN, R. (1956): “Dynamic Programming and Lagrange Multipliers.” *Proceedings of the National Academy of Sciences of the United States of America*, 42, 767–769.
- BERTSEKAS, D. P. (1999): *Nonlinear Programming*, Athena Scientific, 2nd ed.
- BHATTACHARYA, S. AND P. PFLEIDERER (1985): “Delegated Portfolio Management,” *Journal of Economic Theory*, 36, 1–25.
- BIAIS, B., T. MARIOTTI, G. PLANTIN, AND J.-C. ROCHET (2007): “Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications,” *The Review of Economic Studies*, 74, 345–390.
- BIAIS, B., T. MARIOTTI, AND J.-C. ROCHET (2011): “Dynamic Financial Contracting,” in *10th World Congress of the Econometric Society*.
- BLACK, F. AND M. SCHOLES (1973): “The Pricing of Options and Corporate Liabilities,” *The Journal of Political Economy*, 81, 637–654.
- BLAKE, D., A. G. ROSSI, A. TIMMERMANN, I. TONKS, AND R. WERMERS (2013): “Decentralized Investment Management: Evidence from the Pension Fund Industry,” *Journal of Finance*, 68, 1133–1178.
- BOLLEN, N. P. B. AND V. K. POOL (2012): “Suspicious Patterns in Hedge Fund Returns and the Risk of Fraud,” *Review of Financial Studies*, 25, 2673–2702.
- BOLLEN, N. P. B. AND R. E. WHALEY (2009): “Hedge Fund Risk Dynamics: Implications for Performance Appraisal,” *Journal of Finance*, LXIV, 985–1035.
- BOLTON, P. AND M. DEWATRIPONT (2004): *Contract Theory*, MIT Press.
- BROWN, S., W. N. GOETZMANN, B. LIANG, AND C. SCHWARZ (2008): “Mandatory disclosure and operational risk: Evidence from hedge fund registration,” *Journal of Finance*, 63, 2785–2815.
- (2012): “Trust and delegation,” *Journal of Financial Economics*, 103, 221–234.

- BROWN, S. J., W. N. GOETZMANN, B. LIANG, AND C. SCHWARZ (2009): "Estimating Operational Risk for Hedge Funds: The-Score," *Financial Analysts Journal*, 65, 43–53.
- BRUNNERMEIER, M. K. AND Y. SANNIKOV (2013): "A Macroeconomic Model with a Financial Sector," *American Economic Review (Forthcoming)*.
- BUDHIRAJA, A. AND K. ROSS (2008): "Optimal stopping and free boundary characterizations for some Brownian control problems," *The Annals of Applied Probability*, 18, 2367–2391.
- BUFFA, A. M., D. VAYANOS, AND P. WOOLLEY (2013): "Asset Management Contracts and Equilibrium Prices," *Working Paper*, December.
- BURASCHI, A., R. KOSOWSKI, AND W. SRITRAKUL (2013): "Incentives and Endogenous Risk Taking: A Structural View on Hedge Fund Alphas," *Journal of Finance*.
- CADENILLAS, A., J. CVITANIĆ, AND F. ZAPATERO (2004): "Leverage decision and manager compensation with choice of effort and volatility," *Journal of Financial Economics*, 73, 71–92.
- (2007): "Optimal risk-sharing with effort and project choice," *Journal of Economic Theory*, 133, 403–440.
- CAMPBELL, J. Y., M. LETTAU, B. G. MALKIEL, AND Y. XU (2001): "Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk," *Journal of Finance*, LVI.
- CARPENTER, J. N. (2000): "Does Option Compensation Increase Managerial Risk Appetite?" *Journal of Finance*, LV, 2311–2331.
- CHEVALIER, J. AND G. ELLISON (1997): "Risk Taking by Mutual Funds as a Response to Incentives Risk," *Journal of Political Economy*, 105, 1167–1200.
- CHRISTOFFERSEN, S. E. K., D. K. MUSTO, AND R. WERMERS (2014): "Investor Flows to Asset Managers: Causes and Consequences," *Annual Review of Financial Economics*, 6, 1–22.
- COASE, R. H. (1937): "The Nature of the Firm," *Economica*, 4, 386–405.
- COX, J. C. AND C.-F. HUANG (1989): "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.
- (1991): "A variational problem arising in financial economics," *Journal of Mathematical Economics*, 20, 465–487.

- CRANDALL, M. G., H. ISHII, AND P.-L. LIONS (1992): “User’s Guide to Viscosity Solutions of Second Order Partial Differential Equations,” *Bulletin of the American Mathematical Society*, 27, 1–67.
- CUOCO, D. (1997): “Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income,” *Journal of Economic Theory*, 72, 33–73.
- CUOCO, D. AND R. KANIEL (2011): “Equilibrium prices in the presence of delegated portfolio management,” *Journal of Financial Economics*, 101, 264–296.
- CVITANIĆ, J., D. POSSAMAÏ, AND N. TOUZI (2014): “Moral Hazard in Dynamic Risk Management,” *Working Paper*.
- CVITANIĆ, J. AND J. ZHANG (2012): *Contract Theory in Continuous-Time Models*, Springer.
- DAROLLES, S. AND C. GOURIEROUX (2014a): “The Effects of Management and Provision Accounts on Hedge Fund Returns — Part I: The High Water Mark Scheme,” in *Modeling Dependence in Econometrics*, ed. by V.-N. Huynh, V. Kreinovich, and S. Sriboonchitta, Cham: Springer International Publishing, vol. 251 of *Advances in Intelligent Systems and Computing*, 23–45.
- (2014b): “The Effects of Management and Provision Accounts on Hedge Fund Returns — Part II: The Loss Carry Forward Scheme,” in *Modeling Dependence in Econometrics*, ed. by V.-N. Huynh, V. Kreinovich, and S. Sriboonchitta, Cham: Springer International Publishing, vol. 251 of *Advances in Intelligent Systems and Computing*, 47–62.
- DASS, N., V. NANDA, AND Q. WANG (2013): “Allocation of decision rights and the investment strategy of mutual funds,” *Journal of Financial Economics*, 110, 254–277.
- DE LA FUENTE, A. (2000): *Mathematical Methods and Models for Economists*, Cambridge University Press.
- DEMARZO, P. M. AND M. FISHMAN (2007): “Optimal Long-Term Financial Contracting,” *Review of Financial Studies*, 20, 2079–2128.
- DEMARZO, P. M., M. J. FISHMAN, Z. HE, AND N. WANG (2012): “Dynamic Agency and the q Theory of Investment,” *Journal of Finance*, LXVII.
- DEMARZO, P. M., D. LIVDAN, AND A. TCHISTYI (2013): “Risking Other People’s Money: Gambling, Limited Liability, and Optimal Incentives,” *Working Paper*, 1–38.
- DEMARZO, P. M. AND Y. SANNIKOV (2006): “Optimal Security Design and Dynamic Capital Structure in a Continuous Time Agency Model,” *Journal of Finance*, LXI, 2681–2724.

- DEMSKI, J. S. AND D. SAPPINGTON (1984): "Optimal incentive contracts with multiple agents," *Journal of Economic Theory*, 33, 152–171.
- DEMSKI, J. S., D. E. SAPPINGTON, AND P. T. SPILLER (1988): "Incentive schemes with multiple agents and bankruptcy constraints," *Journal of Economic Theory*, 44, 156–167.
- DIAMOND, D. W. (1984): "Financial Intermediation and Delegated Monitoring," *Review of Economic Studies*, 51, 393–414.
- DIXIT, A. K. (1993): *Art of Smooth Pasting*, Routledge.
- DIXIT, A. K. AND R. S. PINDYCK (1994): *Investments under Uncertainty*, Princeton University Press, 1st ed.
- DUFFIE, D. (2001): *Dynamic Asset Pricing Theory*, Princeton University Press, 3 ed.
- DYBVIIG, P. H., H. K. FARNSWORTH, AND J. N. CARPENTER (2010): "Portfolio Performance and Agency," *Review of Financial Studies*, 23, 1–23.
- EDMANS, A. AND X. GABAIX (2011): "Tractability in Incentive Contracting," *Review of Financial Studies*, 24, 2865–2894.
- EDMANS, A., X. GABAIX, T. SADZIK, AND Y. SANNIKOV (2012): "Dynamic CEO Compensation," *Journal of Finance*, LXVII, 1603–1647.
- EDMANS, A. AND G. MANSO (2010): "Governance Through Trading and Intervention: A Theory of Multiple Blockholders," *Review of Financial Studies*, 24, 2395–2428.
- ELTON, E. J. AND M. J. GRUBER (2004): "Portfolio Construction Centralized Optimum with Decentralized Portfolio Management," *Journal of Financial and Quantitative Analysis*, 39, 481–494.
- EMBRECHTS, P., F. LINDSKOG, AND A. J. MCNEIL (2003): "Modeling Dependence with Copulas and Applications to Risk Management," in *Handbook of Heavy Tailed Distributions in Finance*, ed. by S. T. Rachev, Elsevier B.V., chap. 8, 329–384.
- EVANS, L. C. (1983): "Classical Solutions of The Hamilton-Jacobi-Bellman Equation for Uniformly Elliptic Operators," *Transactions of the American Mathematical Society*, 275, 245–255.
- (2010): *Partial Differential Equations*, American Mathematical Society, 2nd ed.
- FAMA, E. F. AND K. R. FRENCH (2010): "Luck versus Skill in the Cross-Section of Mutual Fund Returns," *Journal of Finance*, LXV, 1915–1947.
- FLEMING, W. H. AND H. M. SONER (2006): *Controlled Markov Processes and Viscosity Solutions (Stochastic Modelling and Applied Probability)*, Springer.

- FOSTER, D. P. AND H. P. YOUNG (2010): "Gaming Performance Fees by Portfolio Managers," *Quarterly Journal of Economics*, 125, 1435–1458.
- FRAZZINI, A. AND L. H. PEDERSEN (2014): "Betting against beta," *Journal of Financial Economics*, 111, 1–25.
- GABAIX, X., P. GOPIKRISHNAN, V. PLEROU, AND H. E. STANLEY (2006): "Institutional Investors and Stock Market Volatility," *Quarterly Journal of Economics*, May, 461–504.
- GETMANSKY, M., P. A. LEE, AND A. W. LO (2015): "Hedge Funds: A Dynamic Industry In Transition," *NBER Working Paper*.
- GIAMBONA, E. AND J. GOLEC (2009): "Mutual fund volatility timing and management fees," *Journal of Banking & Finance*, 33, 589–599.
- GIAT, Y. AND A. SUBRAMANIAN (2013): "Dynamic contracting under imperfect public information and asymmetric beliefs," *Journal of Economic Dynamics and Control*, 37, 2833–2861.
- GILLAN, S. L. AND L. T. STARKS (2007): "The Evolution of Shareholder Activism in the United States," *Journal of Applied Corporate Finance*, 19, 55–73.
- GORTON, G. B., P. HE, AND L. HUANG (2014): "Agency-based asset pricing," *Journal of Economic Theory*, 149, 311–349.
- GOYAL, A. AND P. SANTA-CLARA (2003): "Idiosyncratic Risk Matters!" *The Journal of Finance*, 58, 975–1008.
- GROCHULSKI, B. AND Y. ZHANG (2011): "Optimal risk sharing and borrowing constraints in a continuous-time model with limited commitment," *Journal of Economic Theory*, 146, 2356–2388.
- GROMB, D. AND D. MARTIMORT (2007): "Collusion and the organization of delegated expertise," *Journal of Economic Theory*, 137, 271–299.
- GROMB, D. AND D. VAYANOS (2010): "Limits of arbitrage," *Annual Review of Financial Economics*, 2, 251–275.
- HE, H. AND H. PAGÈS (1993): "Labor income, borrowing constraints, and equilibrium asset prices," *Economic Theory*, 3, 663–696.
- HE, Z. (2009): "Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion," *Review of Financial Studies*, 22, 859–892.
- (2011): "A model of dynamic compensation and capital structure," *Journal of Financial Economics*, 100, 351–366.

- (2012): “Dynamic Compensation Contracts with Private Savings,” *Review of Financial Studies*, 25, 1494–1549.
- HE, Z. AND A. KRISHNAMURTHY (2012): “A Model of Capital and Crises,” *The Review of Economic Studies*, 79, 735–777.
- (2013): “Intermediary asset pricing,” *American Economic Review*, 103, 732–770.
- HE, Z., B. WEI, AND J. YU (2014): “Optimal Long-term Contracting with Learning,” *Working Paper*.
- HE, Z. AND W. XIONG (2013): “Delegated asset management, investment mandates, and capital immobility,” *Journal of Financial Economics*, 107, 239–258.
- HEINKEL, R. AND N. M. STOUGHTON (1994): “The Dynamics of Portfolio Management Contracts,” *Review of Financial Studies*, 7, 351–387.
- HESTON, S. (1993): “A closed-form solution for options with stochastic volatility with applications to bond and currency options,” *Review of Financial Studies*, 6, 327–343.
- HIRSHLEIFER, D. AND Y. SUH (1992): “Risk, Managerial Effort, and Project Choice,” *Journal of Financial Intermediation*, 2, 308–345.
- HODDER, J. E. AND J. C. JACKWERTH (2009): “Incentive Contracts and Hedge Fund Management,” *Journal of Financial and Quantitative Analysis*, 42, 811.
- HOFFMANN, F. AND S. PFEIL (2010): “Reward for Luck in a Dynamic Agency Model,” *Review of Financial Studies*, 23, 3329–3345.
- (2013): “Delegated Investment in a Dynamic Agency Model,” *Working Paper*, August.
- HOLMSTRÖM, B. AND P. MILGROM (1987): “Aggregation and linearity in the provision of intertemporal incentives,” *Econometrica*, 55, 303–328.
- (1991): “Multitask principal-agent analyses: Incentive contracts, asset ownership, and job design,” *Journal of Law, Economics and Organization*, 7, 24–52.
- HOLMSTRÖM, B. AND J. ROBERTS (1998): “The Boundaries of the Firm Revisited,” *Journal of Economic Perspectives*, 12, 73–94.
- HUANG, C.-F. AND H. PAGÈS (1992): “Optimal consumption and portfolio policies with an infinite horizon: Existence and convergence,” *The Annals of Applied Probability*, 2, 36–64.
- HUANG, J., C. SIALM, AND H. ZHANG (2011): “Risk Shifting and Mutual Fund Performance,” *Review of Financial Studies*, 24, 2575–2616.

- HUGONNIER, J., S. MALAMUD, AND E. TRUBOWITZ (2012): “Endogenous Completeness of Diffusion Driven Equilibrium Markets,” *Econometrica*, 80, 1249–1270.
- HUNT, P. AND J. KENNEDY (2004): *Financial Derivatives in Theory and Practice*, John Wiley & Sons, Ltd., revised ed.
- JOE, H. (1997): *Multivariate Models and Multivariate Dependence Concepts*, Chapman and Hall/CRC.
- (2014): *Dependence Modeling with Copulas*, Chapman and Hall/CRC, 1st ed.
- JORION, P. (2006): *Value at Risk: The New Benchmark for Managing Financial Risk*, McGraw-Hill, 3rd ed.
- KACPERCZYK, M. AND A. SERU (2012): “Does Firm Organization Matter? Evidence from Centralized and Decentralized Mutual Funds,” *Working Paper*, 1–40.
- KANIEL, R. AND P. KONDOR (2013): “The Delegated Lucas Tree,” *Review of Financial Studies*, 26, 929–984.
- KAPLAN, S. N. AND A. SCHOAR (2005): “Private Equity Performance: Returns, Persistence, and Capital Flows,” *Journal of Finance*, LX, 1791–1823.
- KARATZAS, I. AND S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus*, Springer, 2nd ed.
- KIM, K. (2004): “Moment Generating Function of the Reciprocal of an Integral of Geometric Brownian Motion,” *Proceedings of the American Mathematical Society*, 132, 2753–2759.
- KRYLOV, N. V. (1987): *Nonlinear Elliptic and Parabolic Equations of the Second Order (Mathematics and its Applications, Book 7)*, Springer.
- KUSHNER, H. J. AND P. G. DUPUIS (2001): *Numerical Methods for Stochastic Control Problems in Continuous Time (Stochastic Modelling and Applied Probability)*, Springer, 2 ed.
- LAFFONT, J. J. AND D. MARTIMORT (2001): *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.
- LAMBERT, R. A. (1986): “Executive effort and Selection of Risky Projects,” *The RAND Journal of Economics*, 17, 77–88.
- LELAND, H. E. (1998): “Agency Costs, Risk Management, and Capital Structure,” *Journal of Finance*, LIII, 1213–1243.
- LEUNG, R. C. W. (2014): “Continuous-Time Principal-Agent Problem with Drift and Stochastic Volatility Control : With Applications to Delegated Portfolio Management,” *Working Paper*.

- LI, D. AND W.-L. NG (2000): “Optimal Dynamic Portfolio Selection: Multiperiod Mean-Variance Formulation,” *Mathematical Finance*, 10, 387–406.
- LI, D. X. (2000): “On Default Correlation: A Copula Function Approach,” *The Journal of Fixed Income*, 9, 43–54.
- LUCAS, R. E. (1978): “Asset prices in an exchange economy,” *Econometrica*, 46, 1429–1445.
- LUCENO, A. (1999): “Discrete Approximations to Continuous Univariate Distributions — An Alternative to Simulation,” *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, 61, 345–352.
- LY VATH, V., H. PHAM, AND S. VILLENEUVE (2008): “A mixed singular/switching control problem for a dividend policy with reversible technology investment,” *The Annals of Applied Probability*, 18, 1164–1200.
- MAILATH, G. J. AND L. SAMUELSON (2006): *Repeated Games and Reputations: Long-Run Relationships*, Oxford University Press, USA.
- MARKOWITZ, H. (1952): “Portfolio Selection,” *Journal of Finance*, 33, 77–91.
- MARSCHAK, J. (1955): “Elements for a Theory of Teams,” *Management Science*, 1, 127–137.
- MARSCHAK, J. AND R. RADNER (1972): *Economic Theory of Teams*, Yale University Press.
- MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic Theory*, Oxford University Press, USA.
- MERTON, R. C. (1969): “Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case,” *The Review of Economics and Statistics*, 51, 247.
- (1973): “Theory of rational option pricing,” *The Bell Journal of Economics and Management Science*, 4, 141–183.
- METRICK, A. AND A. YASUDA (2010): “The Economics of Private Equity Funds,” *Review of Financial Studies*, 23, 2303–2341.
- MIAO, J. AND A. RIVERA (2013): “Robust Contracts in Continuous Time,” *Working Paper*, April.
- MIAO, J. AND Y. ZHANG (2013): “A Duality Approach to Continuous-Time Contracting Problems with Limited Commitment,” *Working Paper*, September.
- MILGROM, P. R. AND I. SEGAL (2002): “Envelope Theorems for Arbitrary Choice Sets,” *Econometrica*, 70, 583–601.

- NAKAMURA, G., S. WANG, AND Y. WANG (2008): “Numerical differentiation for the second order derivatives of functions of two variables,” *Journal of Computational and Applied Mathematics*, 212, 341–358.
- NALEBUFF, B. J. AND J. E. STIGLITZ (1983): “Prizes and Incentives: Towards a General Theory of Compensation and Competition,” *The Bell Journal of Economics*, 14, 21–43.
- NELSEN, R. B. (2007): *An Introduction to Copulas*, Springer, 2nd ed.
- OU-YANG, H. (2003): “Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem,” *Review of Financial Studies*, 16, 173–208.
- (2005): “An Equilibrium Model of Asset Pricing and Moral Hazard,” *Review of Financial Studies*, 18, 1253–1303.
- PÁSTOR, L. AND R. F. STAMBAUGH (2012): “On the Size of the Active Management Industry,” *Journal of Political Economy*, 120, 740–781.
- PATTON, A. J. (2009a): “Are ”market neutral” hedge funds really market neutral?” *Review of Financial Studies*, 22, 2295–2330.
- (2009b): “Copula-Based Models for Financial Time Series,” in *Handbook of Financial Time Series*, ed. by T. G. Andersen, R. A. Davis, J.-P. Kreiss, and T. Mikosch, Springer, 767–785.
- (2012): “A review of copula models for economic time series,” *Journal of Multivariate Analysis*, 110, 4–18.
- PROTTER, P. E. (2005): *Stochastic Integration and Differential Equations*, Springer, 2nd ed.
- RAUH, J. D. (2008): “Risk Shifting versus Risk Management: Investment Policy in Corporate Pension Plans,” *Review of Financial Studies*, 22, 2687–2733.
- REVUZ, D. AND M. YOR (2005): *Continuous Martingales and Brownian Motion*, Springer.
- RINCÓN-ZAPATERO, J. P. AND M. S. SANTOS (2012): “Differentiability of the value function in continuous-time economic models,” *Journal of Mathematical Analysis and Applications*, 394, 305–323.
- ROCKAFELLAR, R. T. (1970): *Convex Analysis (Princeton Landmarks in Mathematics and Physics)*, Princeton University Press.
- ROSENBERG, B. (1977): “Institutional Investment with Multiple Portfolio Managers,” in *Proceedings of the Seminar on the Analysis of Security Prices*, University of Chicago, 55–160.

- RUDD, A. AND B. ROSENBERG (1980): "The 'Market Model' In Investment Management," *Journal of Finance*, 35, 597–607.
- SANNIKOV, Y. (2007): "Games with Imperfectly Observable Actions in Continuous Time," *Econometrica*, 75, 1285–1329.
- (2008): "A Continuous-Time Version of the Principal-Agent Problem," *Review of Economic Studies*, 75, 957–984.
- (2012a): "Contracts: The Theory of Dynamic Principal-Agent Relationships and the Continuous-Time Approach," in *10th World Congress of the Econometric Society*.
- (2012b): "Dynamic Security Design and Corporate Financing," *Handbook of Economics and Finance, Volume 2*, 1–56.
- (2012c): "Moral Hazard and Long-Run Incentives," *Working Paper*, December, 1–36.
- (2013): "Dynamic Security Design and Corporate Financing," in *Handbook of the Economics of Finance*, ed. by G. M. Constantinides, M. Harris, and R. Stulz, Elsevier B.V., vol. 2, 71–122.
- SCHAETTLER, H. AND J. SUNG (1993): "The First-Order Approach to the Continuous Time Principal Agent Problem with Exponential Utility," *Journal of Economic Theory*, 61, 331–371.
- SHARPE, W. (1981): "Decentralized Investment Management," *Journal of Finance*, 36, 217–234.
- SONER, H. M. AND S. E. SHREVE (1989): "Regularity of the value function for a two-dimensional singular stochastic control problem," *SIAM Journal on Control and Optimization*, 27, 876–907.
- STAMBAUGH, R. F. (2014): "Investment Noise and Trends," *Working Paper (AFA 2014 Presidential Address)*.
- STOER, J. AND R. BULIRSCH (2002): *Introduction to Numerical Analysis*, Springer, 3rd ed.
- STOUGHTON, N. M. (1993): "Moral Hazard and the Portfolio Management Problem," *Journal of Finance*, 48, 2009–2028.
- STRACCA, L. (2006): "Delegated Portfolio Management: A Survey of the Theoretical Literature," *Journal of Economic Surveys*, 20, 823–848.
- STRULOVICI, B. AND M. SZYDŁOWSKI (2014): "On the Smoothness of Value Functions and the Existence of Optimal Strategies," *Working Paper*, 1–49.

- SUNG, J. (1995): “Linearity with Project Selection and Controllable Diffusion Rate in Continuous-Time,” *The RAND Journal of Economics*, 26, 720–743.
- (2004): “Optimal Contracts Under Adverse Selection and Moral Hazard: A Continuous-Time Approach,” *Review of Financial Studies*, 18, 1021–1073.
- SUNG, J. AND X. WAN (2013): “A General Equilibrium Model of a Multifirm Moral-Hazard Economy With Financial Markets,” *Mathematical Finance*, 00, n/a–n/a.
- SZYDLOWSKI, M. (2012): “Incentives, Project Choice and Dynamic Multitasking,” *Working Paper*, 1–51.
- VAN BINSBERGEN, J. H., M. W. BRANDT, AND R. S. J. KOIJEN (2008): “Optimal Decentralized Investment Management,” *Journal of Finance*, 63, 1849–1895.
- VAYANOS, D. AND T. WANG (2007): “Search and endogenous concentration of liquidity in asset markets,” *Journal of Economic Theory*, 136, 66–104.
- VAYANOS, D. AND P. WOOLLEY (2013): “An Institutional Theory of Momentum and Reversal,” *Review of Financial Studies*, 26, 1087–1145.
- WERMERS, R. (2011): “Performance Measurement of Mutual Funds, Hedge Funds, and Institutional Accounts,” *Annual Review of Financial Economics*, 3, 537–574.
- WILLIAMS, N. (2011): “Persistent Private Information,” *Econometrica*, 79, 1233–1275.
- WILLIAMSON, O. E. (1975): *Markets and Hierarchies: Analysis and Antitrust Implications*, Free Press.
- (1985): *The Economic Institutions of Capitalism*, Free Press.
- (2002): “The Theory of the Firm as Governance Structure: From Choice to Contract,” *Journal of Economic Perspectives*, 16, 171–195.
- YERMACK, D. (2010): “Shareholder Voting and Corporate Governance,” *Annual Review of Financial Economics*, 2, 103–125.
- YONG, J. AND X. Y. ZHOU (1999): *Stochastic Controls: Hamiltonian Systems and HJB Equations (Stochastic Modelling and Applied Probability)*, Springer.
- YOR, M. (1992): “On Some Exponential Functionals of Brownian Motion,” *Advances in Applied Probability*, 24, 509–531.
- ZHOU, X. Y., J. YONG, AND X. LI (1997): “Stochastic Verification Theorems within the Framework of Viscosity Solutions,” *SIAM Journal on Control and Optimization*, 35, 243–253.

ZHU, J. Y. (2013): “Optimal Contracts with Shirking,” *The Review of Economic Studies*, 80, 812–839.

ZUCKERMAN, G. (2013): “Yen Bets Don’t Add Up for a Fund Giant,” .