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Asymptotically Symmetric Metrics and Ricci Flows

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ASYMPTOTICALLY SYMMETRIC METRICS AND RICCI FLOWS

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by

Yufei Shan

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Abstract

ASYMPTOTICALLY SYMMETRIC METRICS AND RICCI FLOWS

by

Yufei Shan

This thesis presents a comprehensive investigation into the properties of asymptotically hyperbolic manifolds and provides an exact definition for asymptotically symmetric manifolds.

Chapter 1 begins with a thorough classification of symmetric spaces of non-compact type, as detailed in Section 1.1. Utilizing parabolic geometry, we then explore the boundary geometry of symmetric spaces of non-compact type, aiming to precisely define asymptotically symmetric manifolds in Section 1.2.

Chapter 2 focuses on the perturbation existence of asymptotically hyperbolic Einstein manifolds. Following the methodology proposed by O. Biquard, we present the conceptual proof of perturbation existence for general asymptotically symmetric manifolds, as outlined in their work [5].

In Chapter 3, we examine the stability of asymptotically hyperbolic Einstein manifolds under normalized Ricci flow. Drawing on R. Bamler's research [1], we establish a reduction of the stability problem to estimating the heat kernel for the Lichnerowicz operator (refer to Lemma 3.2.2). Furthermore, we discuss the underlying ideas behind proving these heat kernel estimates.

Finally, in the last chapter, we introduce our improved result on long-time existence, building upon the work presented in [42]. This enhancement in long-time existence demonstrates the significant contributions made by this thesis.

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Chapter 1

Preliminary

In this chapter, we shall introduce some basic knowledge that is needed for the following chapters. In the section 1.1, we shall talk about symmetric spaces and the corresponding Lie algebra following the book [25] of S.Helgason. In the section 1.2, we will identify the boundary geometry of symmetric spaces of the non-compact type as a **model** parabolic geometry. By this identification, O.Biquard tried to define the so called asymptotically symmetric spaces in [5] at least the rank one case about which we shall talk in the section 1.1.6. The references for this section are the book [12] of the A.Čap and J.Slovák and the book [5] of O.Biquard. In the section 1.4, we shall introduce the spectrum theory and the semi-group theory. The first one will be used to discuss the spectrum of the Laplacian operator on the AH manifolds and the second one will be used to show the exponential decay of the heat kernel. The references for this section are the books [46] of M.Taylor. In the section 1.3, we shall introduce some basic concepts and methods about microlocal analysis and semi-classical analysis which will be used in the paper [48] of A.Vasy and the paper [33] of R.Mazzeo and R.Melrose to show the meromorphic continuation of the modified Laplacian (See the chapter 3 for details). And in the paper

[44] of AS.Barreto, A.Vasy and R.Melrose, they also make use of the semi-classical analysis to show the high energy resolvent estimate on AH manifold (See the chapter 3 for details). In the section 1.4.3, we shall talk about the Newton's method which is used to relate linear operators with non-linear operators. By the newton's method, we can see the existence of the solution to an non-linear equation is mainly determined by the invertibility of its linearization operator. We shall make use of this to talk about the existence of the solutions to Einstein equations in the Chapter 2.

§ 1.1 Symmetric spaces and the semi-simple Lie algebra

In this section, we first review some basic notions and facts about the the Lie group and the Lie algebra. With these knowldege and notions, we shall review some basic concepts about globally symmetric spaces of noncompact type. In particular, we will follow the ideal of R.Bamler (see [2]) to write down the Laplacian operator for tensor in the "spherical coordinate" by the root system of its corresponding Lie algebra.

Then, we will introduce the famous classification result of globally symmetric spaces which was first accomplished by E.Cartan. Here, we follow the book [25] of S.Helgason.

Finally, we will use the rank 1 symmetric spaces of noncompact type as concrete examples. We will specifically write down the Laplacian operator on the symmetric two tensor bundle under the spherical coordinate for the hyperbolic case and the complex hyperbolic case.

1.1.1 Lie groups and Lie algebras

In this subsection, we will review some basic knowledge of the Lie group and Lie algebras. Specially, we will review how the Lie algebra determine a corresponding Lie group (See the theorem 1.1.2 and the theorem 1.1.5).

A **Lie group** is a smooth manifold G (without boundary) that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \longrightarrow G$ and inversion map $i : G \longrightarrow G$ given by $m(g, h) = gh \quad i(g) = g^{-1}$ are both smooth. (See p.151Lee2013)

Moreover, any element $g \in G$ defines maps $L_g, R_g : G \longrightarrow G$, called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh, \quad R_g(h) = hg \quad \text{for any } h \in G$$

Thus, it is natural to introduce the concept of the right invariant vector field. Let X be a vector field on a Lie group G and $X(g)$ is the vector on $g \in G$. Then X is called **right invariant vector field**, if

$$dR_h[X(g)] = X(hg)$$

for arbitrary $g, h \in G$. ($d(R_h)$ is the tangent map of R_h) (p.46 [8]). It is straightforward to verify that the right (or left) invariant vector field forms a Lie algebra over $\mathbb{R}$ under the Lie bracket for vector fields (Proposition 7.1 in [28]). We denote this Lie algebra corresponding to the right (or left) invariant vector fields as $\mathfrak{g}$ (See p.1 in [28] for the definition of the Lie algebra). By the definition of the right (or left) invariant vector field, it is easy to see that $\dim(\mathfrak{g}) = \dim(T_eG)$.

By the integral curve generated by the right (or left) invariant vector field, it is natural to introduce the definition of the exponential map. For $\forall\, v \in T_eG$, let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the one-parameter group generated by the right (or left) invariant vector field $X(g) = dR_g(v)$, the exponential map is defined by the following map (Section 8 in [8])

$$\begin{aligned} \exp : T_eG &\longrightarrow G \\ v &\mapsto \varphi_e(1) \end{aligned}$$

There are some basic properties about the exponential map such as (1) $\exp(tv) = \varphi_e(t)$, $t \in \mathbb{R}$; (2) $\exp[(t_1 + t_2)v] = \exp(t_1v)\dot{\exp}(t_2v)$, $t_1, t_2 \in \mathbb{R}$; (3) $\exp(-tv) = [\exp(tv)]^{-1}$, $t \in \mathbb{R}$. One of the most important properties is

Proposition 1.1.1 ([8], Proposition 8.2)**.** *Let G and H be two Lie group and $f : G \to H$ be homomorphism. And the following diagram is commutative.*

$$\begin{array}{ccc} T_eG & \xrightarrow{(df)_e} & T_eH \\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\quad f \quad} & H \end{array}$$

where the tangent map $(df)_e$ can be thought of as a Lie algebra homomorphism.

For simplicity, we just use homomorphism and isomorphism instead of Lie group homomorphism and Lie group isomorphism without confusing.

From the previous facts, we see how to obtain a Lie algebra from a Lie group. The following theorems will show us how to recover a Lie group from a Lie algebra and in what sense this Lie group is unique.

Theorem 1.1.2 ([8], Theorem 13.3)**.** ***(One to one correspondence between simply connected Lie group and Lie algebra)*** *There is a one to one correspondence between* *isomorphism classes of Lie algebras and isomorphism classes of simply connected Lie groups.*

Proof: The above theorem follows from a theorem of Ado who prove that every Lie algebra has a faithful representation in $gl(n, \mathbb{R})$ for some n. □

Then, Given a Lie group, the Lie subalgebra of Lie algebra for the Lie group correspond an unique Lie subgroup. Before stating the theorem, we first introduce the definition of the Lie subgroups.

Definition 1.1.3 ([25], p.112)**. (Lie subgroups and subalgebras)** Let G and H be a Lie group. If there exists a inclusion (i.e. inclusion means injective) Lie group homomorphism $i : H \to G$ such that the tangent map of i is injective. Then, H is called the **Lie subgroup** of G. Moreover, if the topology of H is the induced topology of G, then H is called **topological Lie subgroup** of G.

Remark 1.1.4. The topology of H might not be the induced topology of G. For example, consider the Lie group $(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ and the Lie subgroup (at, bt), where $a, b \in \mathbb{R} - \mathbb{Q}$ are fixed and t is arbitrary real number.

Theorem 1.1.5 ([25], Theorem 2.1, Ch II)**. *(One to one corresponding between connected Lie subgroups and Lie subalgebras)*** *Let G be a Lie group. If H is a Lie subgroups of G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$, the Lie algebra of G. Each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G. Moreover, if $\mathfrak{h}$ is a ideal of $\mathfrak{g}$, then H is a normal Lie subgroup of G.*

1.1.2 The semisimple Lie algbra and root system

There is natural symmetric two form on the Lie algebra, the so-called Killing form. With this form, we will introduce the semisimple Lie algebra which is a Lie algebra with an non-degenerate Killing form. The semisimple Lie algebra can always be uniquely decomposed into a direct sum of its simple ideals. Therefore, we can classify the semisimple Lie algebras by simple Lie algebras. In order to classify the simple Lie algebras, we need to introduce the root system which is also important to the find a good frame to simplify the expression of the Laplacian operator for tensor on the global symmetric space of noncompact type. We will show that the Lie algebra over $\mathbb{C}$ is uniquely determined (up to isomorphism) by its root system and its Cartan subalgebra. By the Dynkin Diagrams, we can classify all the root systems. Therefore, so do the simple Lie algebra over $\mathbb{C}$. By the real form, we can get corresponding facts for the Lie algebra over $\mathbb{R}$. The reason we concern about the Lie algebra over $\mathbb{R}$ is that all the Lie algebras induced by the Lie groups are on $\mathbb{R}$. Let us start with the definition of the Killing form.

Definition 1.1.6 ([25], p.131)**. (Killing Form)** Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. Denoting by Tr the trace of a vector space endomorphism we consider the bilinear form $B(X,Y) = Tr(adXadY)$ on $\mathfrak{g}\times\mathfrak{g}$. The form B is called the Killing form of $\mathfrak{g}$. It is clearly symmetric.

Then, we will introduce the definition of semisimple Lie algbra

Definition 1.1.7 ([25], p.131)**. (Semisimple and simple)** A Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is called semisimple if the Killing form B of $\mathfrak{g}$ is nondegenerate. We shall call a Lie algebra $\mathfrak{g} \neq \{0\}$ if it is semisimple and has no ideals except $\{0\}$ and $\{\mathfrak{g}\}$.

A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple).

The following proposition is essential for the semisimple Lie algebra. It shows that the semsimple Lie algebra can always be decomposed into a direct sum of two ideals if it has non-trivial ideal.

Proposition 1.1.8 ([25], Propostion 6.2)**.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra,* α *an ideal in* $\mathfrak{g}$*. Let* $\alpha^{\perp}$ *denote the set of elements* $X \in \mathfrak{g}$ *which are orthogonal to* α *with respect to* B*. Then* α *is semisimple,* $\alpha^{\perp}$ *is an ideal and*

$$\mathfrak{g} = \alpha \oplus \alpha^{\perp}$$

Then, the following two corollaries are just straightforward by the previous proposition.

Corollary 1.1.9 ([25], Corollary 6.2)**.** *A semisimple Lie algebra has center* $\{0\}$.

Proof: For simsimple Lie algebra, $\mathfrak{g}$, the center is $\mathfrak{g}^{\perp}$. □

Corollary 1.1.10 ([25], Corollary 6.3)**.** *A semisimple Lie algebra* $\mathfrak{g}$ *is the direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

where $\mathfrak{g}_i$ $(1 \leq i \leq r)$ *are all the simple ideals in* $\mathfrak{g}$*. Each ideal* α *of* $\mathfrak{g}$ *is the direct sum of certain* $\mathfrak{g}_i$.

Next, we will introduce the root system, which is important to the classification of the simple Lie algebra. Let us start with the definition of the Cartan subalgebra and root system.

Definition 1.1.11 ([25], p.163)**.** Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. A **Cartan subalgebra** of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfying that (1) $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$; (2) For each $H \in \mathfrak{h}$, the endomorphism $ad(H)$ of $\mathfrak{g}$ is semisimple (diagonalizable).

The Cartan subalgebra always exists (See Theorem 4.1, Ch III in [25]). Let α be a linear function on the complex vector space $\mathbb{h}$ (Cartan subalgebra). Let $\mathfrak{g}^{\alpha}$ denote the linear subspace of $\mathfrak{g}$ given by

$$g^{\alpha} = \{X \in \mathfrak{g} \ : \ [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

The linear function α is called a **root** if $g^{\alpha} \neq 0$ and g^{α} is called a **root space** (See p.165 in [25]). The set of all the nonzero roots is denoted as Δ (See p.166 in [25]). There are some basic concept for the root.

Proposition 1.1.12 ([25], Theorem 4.2, Theorem 4.3)**.** *(1) The restriction of the Killing form B to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate.*

(2) The only roots proportional to α are $-\alpha, 0, \alpha$.

(3) Suppose $\alpha + \beta \neq 0$. Then $[g^{\alpha}, g^{\beta}] = g^{\alpha+\beta}$

By the (1) of the above proposition, for each $\alpha \in \Delta$ there exists an unique elements $H_{\alpha} \in \mathfrak{h}$ such that $B(H, H_{\alpha}) = \alpha(H)$ for all $H \in \mathfrak{h}$. We put $\langle \lambda, \mu \rangle = B(H_{\lambda}, H_{\mu})$. Let $\mathfrak{h}_{\mathbf{R}} = \sum_{\alpha \in \Delta} \mathbf{R} H_{\alpha}$. Then, we have that B is real and strictly positive definite on $\mathfrak{h}_{R} \times \mathfrak{h}_{R}$ and the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\mathbf{R}} \oplus \sqrt{-1}\mathfrak{h}_{\mathbf{R}}$ (See Theorem 4.4, CH III in [25] for the proof). Moreover, we have

$$\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{and} \quad \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Delta \quad \text{for any } \alpha, \beta \in \Delta$$

(See p.40 in [28] for more details). Now, we can the abstract definition of the root system which is actually a description of the inherent construction of the set of roots.

Definition 1.1.13 ([28], p.42)**.** A subset Φ of the Euclidean space $(E, (.,.))$ is called a **root system** in E if the following axioms are satisfied

(1) Φ is finite, spans E, and does not contain 0.

(2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(3) If $\alpha, \beta \in \Phi$, then $\beta - (2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle)\alpha \in \Phi$.

(4) $2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$.

Then, we see that the root set Δ is a root system with the inner product defined by the corresponding Killing form. As we mentioned, a semisimple Lie algebra over $\mathbb{C}$ is determined (up to isomorphism) by means of a Cartan subalgebras and the corresponding root system. (See Theorem 5.4 in [25] for details). Finally, we can use the Dynkin Diagrams to classify the root system. So do the semisimple Lie algebra. (See Section 3, Ch X in [25])

Next, we will review the real form of semisimple Lie algebra which is served as a bridge between the Lie algbra over $\mathbb{C}$ and that over $\mathbb{R}$. First, let us introduce the definition of the complexification.

Definition 1.1.14 ([25], p.179)**. (Complexification)** Let $\mathfrak{g}_0$ be a Lie algebra over $\mathbb{R}$. The complex vector space $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ consists of all symbols $X + iY$, where $X, Y \in \mathfrak{g}_0$. We define the bracket operation in $\mathfrak{g}$ by

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([Y, Z] + [X, T])$$

$\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is a Lie algebra with the above bracket over $\mathbb{C}$. $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is called the **complexification** of the Lie algebra $\mathfrak{g}_0$. Moreover, denote the Lie algebra of $\mathfrak{g}$ over $\mathbb{R}$ as $\mathfrak{g}^{\mathbb{R}}$.

The Killing form of the complexification has the following relation

Proposition 1.1.15 ([25], lemma 6.1)**.** *Let K_0, K, $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{g}_0$, $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{R}}$ respectively. Then*

$$K_0(X,Y) = K(X,Y) \ for\ X,Y \in \mathfrak{g}_0$$

$$K^{\mathbb{R}}(X,Y) = 2\mathrm{Re}(K(X,Y)) \ for\ X,Y \in \mathfrak{g}^{\mathbb{R}}$$

Then, we can define the real form.

Definition 1.1.16 ([25], p.180)**. (Real form)** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A **real form** of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ such that the complexification of $\mathfrak{g}_0$ is $\mathfrak{g}$. And the mapping σ of $\mathfrak{g}$ onto itself given by $\sigma : X + iY \to X - iY$ $(X, Y \in \mathfrak{g}_0)$ is called the **conjugation** of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

1.1.3 The symmetric space and its classification

In this section, we first identity the globally symmteric space with the so called effective symmetric Lie algebra. Then, we will classify the effective orthogonal symmetric Lie algebra by the classification of the semsimple Lie algebra over $\mathbb{C}$. First, Let us start with the definitin of the globally symmetric space.

Let (M, g) be a Riemannian manifold and $p \in M$. We call an isometry $\Phi : M \to M$ with $\Phi(p) = p$, a **reflection** at p, if $d\Phi|_p = -\mathrm{id}_{T_pM}$. A Riemmaninan manifold (M, g)

is called **Riemannian locally symmetric space** if for arbitrary point $p \in M$ there exist a neighborhood of $p \in M$, U_p, and a reflection Φ_p at p in U_p. Moreover, if Φ_p can be extended into a reflection on M, then (M, g) is called **Riemannian globally symmetric space** (See p.205 [25]). The most important property of the Riemannian symmetric space is that the Riemannian curvature of the locally symmetric space is parallel, i.e. $\nabla Rm \equiv 0$.

Next, we will introduce the definition of the effective orthogonal symmetric Lie algebra which can be thought of as a Lie algebra description of the Riemmannian globally symmetric space.

Definition 1.1.17 ([25], p.213)**. (Orthogonal symmetric Lie algebra)** A pair $(\mathfrak{g}, \mathfrak{s})$ is called an **orthogonal symmetric Lie algebra** if

(1) $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$.

(2) $\mathfrak{s}$ is an involutive automorphism of $\mathfrak{g}$

(3) $\mathfrak{l}$, the set of fixed points of $\mathfrak{s}$, is a compactly imbedded subalgebra of $\mathfrak{g}$

Moreover, it is said to be **effective** if, in addition, $\mathfrak{l} \cap \mathfrak{c} = \{0\}$ ($\mathfrak{c}$ is the center of $\mathfrak{g}$). A pair (G, K), where G is a connected Lie group with Lie algebra $\mathfrak{g}$, and K is a Lie subgroup of G with Lie algebra $\mathfrak{l}$, is said to be associated with the orthogongal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{s})$.

By the [Proposition 3.5, Proposition 3.6, Ch IV] in [25], there is one to one correspondence between the Riemannian globally symmetric spaces and the effective orthogonal symmetric Lie algebras.

Next, we will decompose the symmetric space by its effective orthogonal symmetric Lie algebra.

Definition 1.1.18 ([25], p.230)**.** Let $(\mathfrak{g}, \mathfrak{s})$ be an effective orthogonal symmetric Lie algebra. $\mathfrak{g} = \mathfrak{p} \otimes \mathfrak{l}$ be the decomposition of $\mathfrak{s}$ into the eigenspaces of $\mathfrak{s}$ for the eigenvalue $+1$ and -1 respectively.

(a) If $\mathfrak{g}$ is compact and semisimple, $(\mathfrak{g}, \mathfrak{s})$ is said to be the compact type.

(b) If $\mathfrak{g}$ is noncompact and semisimple and $\mathfrak{g} = \mathfrak{p} \otimes \mathfrak{l}$ is a Cartan decomposition of $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{s})$ is said to be of the noncompact type.

(c) If $\mathfrak{l}$ is an Abelian ideal in $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{s})$ is said to be of the Euclidean type.

Then, we have the cooresponding definition of symmetric spaces.

Definition 1.1.19 ([25], p.230)**.** Let $(\mathfrak{g}, \mathfrak{s})$ be an orthogonal symmetric Lie algebra and suppose the pair (G, H) is associated with $(\mathfrak{g}, \mathfrak{s})$. The pair (G, H) is said to be of the **compact type**, **noncompact type**, or **Euclidean type** according to the type of $(\mathfrak{g}, \mathfrak{s})$.

The basic decomposition of the symmetric space is following

Theorem 1.1.20 ([25], p.244)**.** *Let M be a simply connected Riemannian globally symmetric space. Then M is a product*

$$M = M_0 \times M_- \times M_+$$

where M_0 is a Euclidean space, M_- and M_+ are Riemannian globally symmetric of the compact and noncompact type, respectively.

We can also use the section curvature to get the compactness by the following theorem.

Theorem 1.1.21 ([25], p.241)**.** ***(Sectional curvature)*** *Let $(\mathfrak{g}, \mathfrak{s})$ be an orthogonal symmetric Lie algebra and suppose that the pair (G, K) is associated with $(\mathfrak{g}, \mathfrak{s})$. We assume* *that K is connected and closed. Let Q be an arbitrary G-invariant Riemannian structure on G/K (such a Q exists).*

(i) If (G, K) is of the compact type, then G/K has sectional curvature everywhere $\geqslant 0$

(i) If (G, K) is of the noncompact type, then G/K has sectional curvature everywhere $\leqslant 0$

(i) If (G, K) is of the Euclidean type, then G/K has sectional curvature everywhere $= 0$

Moreover, we can further decompose the compact and the noncompact symmetric space. First, we will introduce the definition of the irreducible symmetric spaces

Definition 1.1.22. Let $(\mathfrak{g}, \mathfrak{s})$ be an orthogonal symmetric Lie algebra, $\mathfrak{l}$ and $\mathfrak{p}$ the eigenspaces of s for the eigenvalues $+1$ and -1, respectively; $(\mathfrak{g}, \mathfrak{s})$ is said to be **irreducible** if the two following conditions are satisfied:

(i) $\mathfrak{g}$ is semisimple and $\mathfrak{l}$ contains no ideal $\neq \{0\}$ of $\mathfrak{g}$

(ii) The algebra $ad_{\mathfrak{g}}(\mathfrak{l})$ acts irreducibly on $\mathfrak{p}$.

Definition 1.1.23. Let (G, K) be a pair associated with $(\mathfrak{g}, \mathfrak{s})$; then (G, K) is said to be **irreducible** if $(\mathfrak{g}, \mathfrak{s})$ is irreducible.

Theorem 1.1.24. *Let M be a simply connected Riemannian globally symmetric space of the compact type or the noncompact type. Then M is a product*

$$M = M_1 \times \ldots \times M_r$$

where the factors M_i are irreducible. If M is Hermitian, then each M_i is Hermitian.

It turns out that the classification of the irreducible orthogonal symmetric Lie algebras is determined by the classification of the simple Lie algebra on $\mathbb{C}$. See the [25]. By the [p.518, Table V] [25], we can see for the rank one case are

$$SU(1,n)/U(n),\ SO(1,n)/O(n),\ Sp(1,n)/sp(1)\times sp(n),\ F_4^{-20}/Spin_9$$

1.1.4 The symmetric space of noncompact type

In this section, we will introduce the properties of the symmetric space of noncompact type and use the root system to derive the spherical coordinate.

Let M be a symmetric space with noncompact case, then by the previous discussion we have a Lie algebra and a decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$

Definition 1.1.25 ([2], Section 3)**. (Maximal abelian algebra)**$\mathfrak{a} \subseteq \mathfrak{p}$ which is not contained in a bigger abelian subalgebra in $\mathfrak{p}$.

Definition 1.1.26 ([2], Section 3)**. The rank of symmetric space** The dimension $r = \dim \mathfrak{a}$ is called the rank of the symmetric space M.

Definition 1.1.27 ([2], Section 3)**. (Root)** We say that $\alpha \in \mathfrak{a}^*$ is a root of $\mathfrak{g}$ relative to $\mathfrak{a}$ if $\alpha \neq 0$ and there exists some $X \neq 0 \in \mathfrak{g}$ such that $[v, X] = \alpha(v)X$ for any $v \in \mathfrak{a}$. Denote Δ the set of all the root. And denote $\mathfrak{g}_\alpha$ the corresponding eigenspace about α.

Definition 1.1.28 ([2], Section 3)**. (Positive root)** Let $v_0 \in \alpha$ be an arbitrary vector such that $\alpha(v_0) \neq 0$ for all nonzero $\alpha \in \Delta$ and define the set of positive roots by $\Delta_+ = \{\alpha \in \Delta : a(v_0) > 0\}$.

The existence of the involution θ_* implies $-\Delta = \Delta$and the involution θ_* maps $\mathfrak{g}_a$ to $\mathfrak{g}_{-a}$. So if we set and $\mathfrak{p}_a = (\mathfrak{g}_a \oplus \mathfrak{g}_{-a}) \cap \mathfrak{p}$ and $\mathfrak{l}_a = (\mathfrak{g}_a \oplus \mathfrak{g}_{-a}) \cap \mathfrak{l}$. Let $v_0 \in \alpha$ be an arbitrary vector such that $a(v_0) \neq 0$ for all nonzero $a \in \Delta$and define the set of positive roots by $\Delta_+ = \{a \in \Delta : a(v_0) > 0\}$.

Then we have the following root space decomposition

$$\begin{aligned} \mathfrak{g} =& \alpha \oplus_{a\in\Delta_+} (\mathfrak{g}_a \oplus \mathfrak{g}_{-a}) \oplus \mathfrak{l}_0 \\ =& \mathfrak{p} \oplus \mathfrak{l} = \alpha \oplus_{a\in\Delta_+} (\mathfrak{p}_a \oplus \mathfrak{l}_a) \oplus \mathfrak{l}_0 \end{aligned}$$

These splittings are orthogonal with respect to the Killing form. The subspace $\mathfrak{l}_0$ is a Lie algebra. Its geometric meaning will be described below. Using the Jacobi identity, we can conclude that for any two $a, b \in \Delta$, we have $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$. Hence $\mathfrak{n} = \mathfrak{n}_+ = \oplus_{a\in\Delta_+}\mathfrak{g}_a$ and $\mathfrak{n}_- = \oplus_{a\in\Delta_+}g_{-a}$ are the nilpotent algebras with $\theta_*(\mathfrak{n}_+) = \mathfrak{n}_-$. The spaces $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are isotropic with respect to the Killing form, but on $\mathfrak{n} \oplus \mathfrak{n}_-$

$$(.,.) = - < ., \theta_* >$$

is a positive definite scalar product.

Next, we will introduce the **(Orthonormal Basis)** on the symmetric space of non-compace type. Let $a_1, \cdots, a_{n-r}$ be the roots of Δ_+ be the roots of Δ_+ occurring with the appropriate multiplicities and let $x_1, \cdots, x_{n-r}$ be an orthonormal basis of $\mathfrak{n}_+$ with respect to $(.,.)$ such that $x_i \in \mathfrak{g}_{a_i}$. Then $[x_i, x_j] \in g_{a_i+a_j}$. So $< x_i, y_j >= -\delta_{ij}$ and $< x_i, x_j >= 0$ and $[x_i, y_j] \in \mathfrak{g}_{a_i-a_j}$ and $[y_i, y_j] \in \mathfrak{g}_{-a_i-a_j}$. We set

$$\mathfrak{p}_i = \frac{1}{\sqrt{2}}(x_i - y_i), \quad k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$$

Hence, $p_1, \cdots, p_{n-r}$ form an orthonormal basis of the orthogonal complement $\alpha^\perp$ of α

in $\mathfrak{p}$ and $k_1, \cdots, k_{n-r}$ are a negative orthonormal basis of the orthogonal complement of $\mathfrak{l}_0$ in $\mathfrak{l}$.

We also choose an orthonormal basis $v_1, \cdots, v_r$ of a with respect to $< ., . >$. Observe that $\theta_*[x_i, y_i] = [y_i, x_i] = -[x_i, y_i]$, hence $[x_i, y_i] \in \mathfrak{p}$. Moreover, $[x_i, y_i] \in \mathfrak{g}_0$, so $[x_i, y_i] \in \alpha$. Since for any $v \in \alpha$, we have

$$< [x_i, y_i], v >= - < [x_i, v], y_i >= -a_i(v) < x_i, y_i >= a_i(v)$$

We obtain

$$[x_i, y_i] = a_i^*$$

Finally, we apply our knowledge on the infinitesimal structure to find out more about the global geometry of M. The subgroup $A = \exp(\alpha) < G$ corresponding to a is abelian and isomorphic to $\mathbb{R}^r$. The orbit $F = Ap_0$ is a geodesic submanifold of M isometric to $\mathbb{R}^r$ and is called a maximal flat of M. The subgroup $K_0 = \exp(\mathfrak{l}_0) < K$corresponding to $\mathfrak{l}_0$ is the point stabilizer of the flat F. Observe that there are symmetric spaces with trivial K_0, such as $SL(n) \setminus SO(n)$, however many symmetric spaces, e.g. hyperbolic space $\mathbb{H}^n$ $(n \geq 3)$, have nontrivial K_0. The stabilizer (not the point stabilizer) $StabK(F)$ of the flat F however consists of several components of K_0. Forming the quotient $W = StabK(F)/K_0$ yields a discrete group, called the Weyl group. It follows that the orbit $K(p)$ of every point $p \in M$ under the isotropy group K intersects F in a nonempty set which is invariant under W. Moreover, F can be decomposed into fundamental domains for the action of W, which are called Weyl chambers, and W is generated by reflections along the walls of an arbitrary Weyl chamber. Finally, consider the subgroups N resp. N_- corresponding to $\mathfrak{n}$ resp. $\mathfrak{n}_-$. The product subgroups $P = AN$ and $P_- = AN_-$ are called Borel subgroups. They act simply transitively on M and stabilize a Weyl chamber at infinity in the geodesic compactification.

Then, in order to introduce the spherical coordinate, we will identity the symmetric space with the homogeneous space.

Definition 1.1.29 ([12], definition 5.2.1)**. (Homogeneous vector bundle)** Let G be a Lie group and let K be a closed subgroup of G. Let $M = G/K$. A vector bundle E over M is called a **homogeneous vector bundle** if G acts on E on the left and the G action satisfies

(1) $gE_x = E_{gx}$ for x in M, g in G

(2) The mapping from E_x to E_{gx} induced by g is linear for g in G and x in M.

Remark 1.1.30. We shall give a basic construction that describes all homogeneous vector bundles over M. Let (ρ, E_0) be a finite dimensional representation of K. Let K act on the right on $G \times E_0$ as follows:

$$(g, v)k = (gk, \rho(k)^{-1}(v)) \quad for\ g \in G,\ v \in E_0\ and\ k \in K$$

We set $E = G \times_\rho E_0 = (G \times E_0)/\sim$ where

$$(gk, v) \sim (g, \rho\,(k)\,v)$$

Then E is a homogeneous vector bundle on M. In fact, all the homogeneous vector bundle on M can be constructed by the above process.

Remark 1.1.31. In fact, we can regard M as the base of a right K-principal bundle

$\pi : G \longrightarrow M$. Then, homogeneous vector bundle actually is the associated vector bundle of this principal bundle and the representation ρ.

Remark 1.1.32. We have the following commutative diagram

$$\begin{array}{ccc} G \times E_0 & \xrightarrow{\pi_1} & G \times_\rho E_0 = E \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_M} \\ G & \xrightarrow{\pi} & G/K = M \end{array}$$

where p_G is the projection of $G \times E_0$ on G, P_M is the projection of the homogeneous vector bundle E on the base manifold M, and π_1 is the quotient map with the equivalent relation $\sim$.

Definition 1.1.33 ([12])**. (Section of homogeneous vector bundle)** Let G be a Lie group and let K be a closed subgroup of G. Let $M = G/K$ and $E = G \times_\rho E_0$ be a homogeneous vector bundle on M where E_0 is a finite dimensional vector space and $\rho : K \longrightarrow GL(E_0)$ is a representation of K on E_0. A smooth map $f : M \longrightarrow E$ is called **section of** E if

$$\pi \circ f = id_M$$

where $\pi : G \longrightarrow G/K = M$ is the natural quotient map. And denote all the smooth section of E as $C^\infty(M.E)$.

Definition 1.1.34 ([12])**. (The lift of section)** Let $f : M \longrightarrow E$ be a section of a homogeneous vector bundle E on the manifold $M = G/K$ where G is a Lie group and K is a Lie subgroup of G. Then $\tilde{f} : G \longrightarrow G \times E_0$ is called the lift of the section f if $\pi_1 \circ \tilde{f} = f \circ \pi$.

Remark 1.1.35. The lift of $f : M \longrightarrow E$, $\tilde{f} : G \longrightarrow G \times E_0$, can be thought of as a section on the trivial bundle $G \times E_0$. On the other hand, if one of the section $\tilde{f} : G \longrightarrow G \times E_0$

satisfies that $\tilde{f}(gk) = \rho(k^{-1})\tilde{f}(g)$, then there exists an unique section $f : M \to E$, such that $\tilde{f}$ is the lift of f.

Let $g \in G$ and $f : M \to E$ be a section on the homogeneous vector bundle. Then, g can induce another section $g_*(f)$ on the same homogeneous vector bundle by

$$g_*(f) : M = G/K \to E = G \times_\rho E_0, \quad [h] \mapsto [g^{-1}h, \tilde{f}(g^{-1}h)] \quad for\ h \in G$$

where $[g^{-1}h]$ is the equivalent class in G/K, and $[g^{-1}h, \tilde{f}(g^{-1}h)]$ is the equivalent class in $G \times_\rho E_0$. $\tilde{f} : G \to G \times E_0$ is the lift of $f : M \to E$. In particularly $g_*(f)$ is called the **push-forward** of section f. Obviously, The lift of $g_*(f)$ is

$$g_*\tilde{(}f)(h) = \tilde{f}(g^{-1}h)$$

Definition 1.1.36 ([12]). **(Lie derivative)** Let $f : M \to E$ be a section of a homogeneous vector bundle E on the manifold $M = G/K$ where G is a Lie group and K is a Lie subgroup of G. And let $\tilde{f} : G \to G \times E_0$ be the lift of f. Suppose that X is a vector field on $M = G/K$ and $\tilde{X}$ is a vector field on G such that $d\pi(\tilde{X}) = X$. Then, a section on the homogeneous vector bundle, $f' : M \to E$, is called the Lie derivative of the section $f : M \to E$ along the direction X if the lift of f' is

$$d\tilde{f}(\tilde{X})(g) = \lim_{t \to 0} \frac{\tilde{f}(\varphi(t,g)) - \tilde{f}(\varphi(0,g))}{t}$$

where $\varphi(t,g)$ is the one-parameter group generated by the vector field $\tilde{X}$. Denote this section $\mathcal{L}_X f$.

Remark 1.1.37. Since $\tilde{f}' = d\tilde{f}(\tilde{X})$ satisfies that $\tilde{f}'(gk) = \rho(k)^{-1}\tilde{f}'(g)$ for $g \in G$ and $k \in K$, by the Remark 1.20, there exists an unique section of of homogeneous vector bundle $f' : M \to E$ such that the lift of this section is $\tilde{f}'$.

Proposition 1.1.38 ([2], Section 3)**.** *Let X and Y are two different vector field on $M = G/K$. And let $f : M \to E$ is a section of the homogeneous vector bundle E. Then, we have*

$$\mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f = -\mathcal{L}_{[X,Y]} f$$

Then, we will show that the tangent bundle for symmetric space, $(M, g) = G/K$, can be thought of as a homogeneous vector bundle with the representation of K on $\mathfrak{p}$. This G is the isometric group of (M, g) and K is the isotropic group fixing a fixed point p_0. Let $\mathfrak{g}$ be the Lie algebra of G. Then, by the previous section, we have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$, where $\mathfrak{l}$ is the Lie algebra of K. The reference is from [5] of O.Biquard.

(1) We the following natural map

$$\begin{aligned} \pi : G \times M &\longrightarrow M \\ (g, p) &\mapsto g(p) \end{aligned}$$

and

$$\begin{aligned} \pi_p : G &\longrightarrow M \\ g &\mapsto g(p) \end{aligned}$$

Then, consider the tangent map of π_{p_0},

$$\begin{aligned} (d\pi_{p_0})_e : T_e G &\longrightarrow T_{p_0} M \\ v &\mapsto (d\pi_{p_0})_e(v) = \frac{d}{dt} \exp(vt)(p_0)|_{t=0} \end{aligned}$$

Moreover, the kernel of $(d\pi_{p_0})_e$ is $\mathfrak{l}$ and $(d\pi_{p_0})_e$ is surjective. Therefore, $\mathfrak{p} \cong T_{p_0} M$ Let

$\Gamma(M)$ be the all the vector field on M. Then, we have the following map

$$(d\pi)_e : T_eG \longrightarrow \Gamma(M)$$
$$v \mapsto X(p) = (d\pi_p)_e(v) = \frac{d}{dt}\exp vt(p)|_{t=0}$$

$(d\pi)_e$ is a linear map. And $Ker((d\pi)_e) = 0$. Therefore $(d\pi)_e$ is an injective. Moreover, the image of $(d\pi)_e$ is Killing field and $(d\pi)_e$ preserve the Lie bracket. This can be easily checked by the definition.

Remark 1.1.39. In general, $(d\pi_{p_0})e$ gives an isomorphism between $\mathfrak{p}$ and $T_{p_0}M$ and $(d\pi)_e$ gives an isomorphism between $\mathfrak{g}$ and the Killing field of (M, g).

(2) We have the following map

$$G \times T_{p_0}M \longrightarrow TM$$
$$(g, u) \mapsto g_*(u)$$

If $g_1(u_1) = g_2(u_2)$, then $g_1 = g_2h$, then $(g_2h)_*(u_1) = (g_2)_*(u_2)$ and $h_*(v_1) = v_2$.

(3) Define the equivalent relation $\sim$ on $G \times T_{p_0}M$ by $(gh, u) = (g, h_*(u))$. Then

$$[(G \times T_{p_0}M)/\sim] \cong TM$$

(4)(Representation) Let $u \in T_{p_0}M$. Then, there exists an unique $v \in \mathfrak{p}$ such that $(d\pi_{p_0})_e(v) = u$. Let $h \in K$. We have

$$h_*(u) = h_*((d\pi_{p_0})_e(v)) = \frac{d}{dt}h\exp(vt)(p_0)|_{t=0} = \frac{d}{dt}h\exp(vt)h^{-1}(p_0)|_{t=0}$$
$$= (d\pi_{p_0})_e[Ad_*(h)(v)]|_{p_0}$$

Define the representation $\rho_0 : K \longrightarrow GL(T_{p_0}M)$

$$\rho_0(h)(u) = (d\pi_{p_0})_e[Ad_*(h)((d\pi_{p_0})_e^{-1})(u)]$$

Then the induced representation on Lie algebra $\mathfrak{l}$ is

$$\rho_{0*}(k)(v) = (d\pi_{p_0})_e[ad(h)((d\pi_{p_0})_e^{-1})(u)]$$

Moreover, we have $h_*(u) = \rho_0(h)(u)$.

(5)(Associative vector bundle) We can think of the tangent bundle as the associative vector bundle

$$G \times_{\rho_0} T_{p_0}M \cong TM$$

which is nothing but the homogeneous vector bundle on M

(6)(The lift section) Let $f \in C^\infty(M, TM)$ be a global section. Then the lift of the section f is defined by

$$\begin{aligned} \tilde{f} : G &\longrightarrow T_{p_0}M \\ g &\mapsto g_*^{-1}(f(p)) \end{aligned}$$

where $g(p) = p_0$.

Remark 1.1.40. Every global section can uniquely determine a lift by the above. Inversely, for a function $\tilde{f} \in C^\infty(G, T_{p_0}M)$ satisfying that $\tilde{f}(gh) = \rho_0(h^{-1})\tilde{f}(g)$ can uniquely determine a global section as well. Moreover, we have $f(p) = g_*(\tilde{f}(g))$.

(7)(Push-forward) Let $f \in C^\infty(M, TM)$ and $\tilde{f} \in C^\infty(G, T_{p_0M})$ be its lift. Then for

$h, g \in G \cong ISO(M)$ and $p = g(p_0)$, we have

$$\begin{aligned}\widetilde{h_*(f)}(g) =& g_*^{-1}(h_*(f)(p)) = g_*^{-1}h_*(f)(g^{-1}(p))\\ =& g_*^{-1}h_*(f(h^{-1}gg^{-1})(p)) = g_*^{-1}h_*(f(h^{-1}g)(p_0))\\ =& \tilde{f}(h^{-1}g)\end{aligned}$$

Actually, in the above process, we just make use of a fact that $h_*(f)(h(p)) = h_*(f(p))$.

(8)(Lie derivative) Let $f \in C^\infty(M, TM)$ and $v \in \mathfrak{p}$. Then by the definition of the Lie derivative

$$\mathcal{L}_{d\pi_e(v)}f|_p = \frac{d}{dt}(\exp{(-vt)})_* f|_p$$

By the definition of the lift section,

$$f(p) = g_*(\tilde{f}(g))$$

Therefore,

$$\begin{aligned}\mathcal{L}_{d\pi_e(v)}f|_p =& \frac{d}{dt}(\exp{(-vt)})_*(\exp{(vt)}g)_*\tilde{f}(\exp(vt)g)|_{t=0}\\ =& g_*\frac{d}{dt}\tilde{f}(\exp{(vt)}g)|_{t=0} = g_*(v|_g(\tilde{f}))\end{aligned}$$

where $g \in G$ satisfying that $g(p_0) = p$. Therefore,

$$\widetilde{\mathcal{L}_{d\pi_e(v)}f}|_p = d\tilde{f}(v)|_g$$

Remark 1.1.41. We see from **(8)** that the Lie derivative of this homogeneous vector bundle is equivalent to the Lie derivative of the tangent bundle.

For cotangent bundle, from the point of the representation, we only need to take the dual representation of the representation corresponding to the tangent bundle.

Definition 1.1.42. (Dual representation) Let (ρ, V) be a representation of a group G on a finite vector space V, then the dual representation ρ^* is defined over dual vector space V^* as follows

$$\rho^*(g) = \rho(g^{-1})^* \quad for\ all\ g \in G$$

where $\rho(g^{-1})^*$ is the dual operator of $\rho(g^{-1})$ on the vector space V^*

Notice that the Riemannian metric g on a Riemannian metric actually can be thought of as a section on the tensor bundle $TM^* \otimes TM^*$. And Tensor bundle corresponds to the tensor product representation.

Definition 1.1.43. (Tensor product representation) Let (ρ_1, V_1), (ρ_2, V_2) be two linear representations of a group G. then their **tensor product representation** is a linear representation ρ_{12} of G on $V_1 \otimes V_2$ defined as follow

$$\rho_{12}(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$$

for $g \in G$, $v_1 \in V_1$ and $v_2 \in V_2$

Then, we will define a on the homogeneous vector bundle $E = G \times_\rho E_0$ of noncompact symmetric space $(M, g) = G/K$. This G is the isometric group of (M, g) and K is the isotropic group fixing a fixed point p_0. Let $\mathfrak{g}$ be the Lie algebra of G. Then, by the previous section, we have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$, where $\mathfrak{l}$ is the Lie algebra of K and $\mathfrak{g}$ is semisimple. We will see that on the tangent bundle (Since tangent bundle can also be thought of as a homogeneous vector bundle), this connection is the Levi-Civita connection.

Definition 1.1.44 ([12]). **(Maurer-Cartan form)** Let $\mathbf{g} \cong \mathrm{T}_eG$ be the tangent space of a Lie group G at the identity (its Lie algebra as right invariant vector field). G acts on itself by left translation

$$L : G \times G \longrightarrow G$$

such that for a given $g \in G$ we have

$$L_g : G \longrightarrow G \quad \text{where} \quad L_g(h) = gh$$

and this induces a map of the tangent bundle to itself: $\left(L_g\right)_* : T_hG \longrightarrow T_{gh}G$. A left-invariant vector field is a section X of $\mathrm{T}G$ such that

$$\left(L_g\right)_* X = X \quad \forall g \in G$$

The **Maurer-Cartan form** ω is a g-valued one-form on G defined on vectors $v \in \mathrm{T}_gG$ by the formula

$$\omega_g(v) = \left(L_{g^{-1}}\right)_* v$$

Remark 1.1.45. If X is a left-invariant vector field on G, then $\omega(X)$ is constant on G. Furthermore, if X and Y are both left-invariant, then

$$\omega([X, Y]) = [\omega(X), \omega(Y)]$$

Moreover, we have **Maurer-Cartan equation**

$$d\omega(X, Y) + [\omega(X), \omega(Y)] = 0$$

for any $X, Y \in TG$.

Remark 1.1.46. Since every Lie algebra has a bilinear Lie bracket operation, the wedge product of two Lie algebra-valued forms can be composed with the bracket operation to obtain another Lie algebra-valued form. This operation, denoted by $[\omega \wedge \eta]$, is given by: for $\mathfrak{g}$-valued p-form ω and $\mathfrak{g}$-valued q -form η. Then, we can define

$$\begin{aligned}&[\omega \wedge \eta]\left(v_1, \cdots, v_{p+q}\right)\\=&\frac{1}{(p+q)!} \sum_\sigma \operatorname{sgn}(\sigma)\left[\omega\left(v_{\sigma(1)}, \cdots, v_{\sigma(p)}\right), \eta\left(v_{\sigma(p+1)}, \cdots, v_{\sigma(p+q)}\right)\right].\end{aligned}$$

Then, the Maurer-Cartan equation can also be written as

$$d\omega + \frac{1}{2}\omega \wedge \omega = 0$$

Moreover, by the Jacobi identity of Lie algebra, we have

$$\omega \wedge \omega \wedge \omega = 0$$

Definition 1.1.47 ([12])**. (Principal connection)** Let $\pi : P \longrightarrow M$ be a smooth principal K-bundle over a smooth manifold M. Then a **principal K-connection** on P is a differential 1-form $\theta \in \Omega^1(P, \mathfrak{l}) \cong C^\infty\left(P, T^*P \otimes \mathfrak{l}\right)$ on P with values in the Lie algebra $\mathfrak{l}$ of K satisfying that

1. $\mathrm{Ad}_g\left(R_g^*\theta\right) = \theta$ where R_g denotes right multiplication by g and Ad_g is the adjoint representation on $\mathfrak{l}$ (explicitity, $\mathrm{Ad}_g X = \frac{d}{dt} g \exp(tX) g^{-1}\big|_{t=0}$);

2. if $\xi \in \mathfrak{l}$ and X_ξ is the vector field on P associated to ξ by differentiating the G action on P, then $\theta\left(X_\xi\right) = \xi$ (identically on P).

Remark 1.1.48. A principal connection induces a connection on every associative vector bundle $E = G \times_\rho E_0$ by the following way. Let $f \in C^\infty(M, E)$ be a section of E

and consider its lift $\tilde{f}$. Then we set for any $v \in T_pM$. Then,

$$\nabla^{\tilde{E}}_v f = \left(d\tilde{f}\,(v') + \rho_*\theta\,(v')\,\tilde{f}\right)$$

where $v' \in T_{p'}G$ is any vector projectiong to v, i.e. $\pi(p') = p$ and $d\pi(v') = v$.

Now, for symmetric space $M = G/K$ with Lie algebra of G, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$. Then, G can be thought of as a principal bundle of M with structure group K. Consider the Maurer-Cartan form on G ω. Let $X \in \mathfrak{g}$ be a Killing field. Then

$$\omega(X(g)) = Ad(g^{-1})(X)|_{id}$$

It is straightforward to check that

$$\theta(X(g)) = \omega(X(g))|_{\mathfrak{l}}$$

is a principal connection.

Let $f \in C^\infty(M, E)$ be a section of a homogeneous vector bundle with lift $\tilde{f} \in C^\infty(G, G \times E_0)$ and $v \in \mathfrak{g}$. And we can define the covariant derivative (connection) as following

$$\nabla_{\widetilde{(d\pi_e)_e(v)|_p}} f = d\tilde{f}(d\pi_e(v))|_g + \rho_{0*}(Ad(g^{-1})(v)|_{\mathfrak{l}})(\tilde{f})$$

where g is an arbitrary element in G such that $g(p_0) = p$ and $|_{\mathfrak{l}}$ means orthogonal projection on $\mathfrak{l}$ with respect to Killing form on $\mathfrak{g}$. For the definition of $d\pi_e$ and $(d\pi_{p_0})_e$, see 1.3.4 (1). (See details in Kobayashi and SS.Chern, Foundation of Differential Geometry.)

1.1.5 Laplacian operator on the spherical coordinate

Next, we will write down the Laplacian operator on the symmetric space. Let $v = k_i \in \mathfrak{l}$ and $p = \exp(k_i t)\exp(rx_0)(p_0) \in M$ where $r \geq 0$ and $x_0 \in \mathfrak{a}$ ($\mathfrak{a}$ is the Cartan subalgebra). Then

$$\nabla_{\widetilde{d\pi_e(k_i)|_p}} f = \frac{d}{dt}\tilde{f}(\exp(k_i t)\exp(rx_0)) + \rho_{0*}(ch(ad(-rx_0))Ad(\exp(-k_i t))k_i)(\tilde{f})$$
$$= \frac{d}{dt}\tilde{f}(\exp(k_i t)\exp(rx_0)) + \rho_{0*}(ch(ad(-rx_0))k_i)(\tilde{f})$$

Therefore,

$$\nabla_{d\pi_e\widetilde{(k_i)|_p \nabla_{d\pi_e(k_i)}}} f = \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0))$$
$$+ 2\rho_{0*}(ch(ad(-rx_0))k_i)(\frac{d}{dt}\tilde{f})$$
$$+ \rho_{0*}^2(ch(ad(-rx_0))k_i)(\tilde{f})$$

Now, let $k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$, where x_i is the positive root and $y_i = \sigma(x_i)$ is the negative root. (See detail in the section of root system.)

$$\nabla_{d\pi_e\widetilde{(k_i)|_p \nabla_{d\pi_e(k_i)}}} f = \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0}$$
$$+ 2\rho_0(ch(-r\alpha_i(x_0))k_i)(\frac{d}{dt}\tilde{f})$$
$$+ \rho_0^2(ch(-r\alpha_i(x_0))k_i)(\tilde{f})$$

On the other hand,

$$\widetilde{d\pi_e(k_i)}(\exp(k_i t)\exp(rx_0)) = (d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)$$
$$= (d\pi_{p_0})_e(sh(ad(-rx_0))k_i) \in T_{p_0}M$$

Therefore,

$$\nabla_{\widetilde{\pi_*(k_i)|_p \pi_*}}(k_i) = \frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)$$
$$+ \rho_0(ch(ad(-rx_0))Ad(\exp(k_i t))k_i)[\pi_{**}(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)]$$

at $t = 0$,

$$\begin{aligned}\nabla_{\widetilde{d\pi_e(k_i)|_p d\pi_e}}(k_i) &= \frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))k_i)\\ &+ \rho_{0*}(ch(ad(-rx_0))k_i)[(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)]\\ =&[sh(ad(-rx_0))k_i, ch(ad(-rx_0))k_i]\\ =&[sh(-\alpha_i(x_0)r)\mathfrak{p}_i, ch(-\alpha_i(x_0)r)k_i]\\ =&sh(-\alpha_i(x_0)r)ch(-\alpha_i(x_0)r)(d\pi_{p_0})_e([p_i, k_i])\end{aligned}$$

where $\mathfrak{p}_i = \frac{1}{\sqrt{2}}(x_i - y_i)$ and $[\mathfrak{p}_i, k_i] = [x_i, y_i] = \sum_{j=1}^{r} \alpha_i(p_j)p_j$. ($r$ is the rank of the symmetric space and p_j is the basis of the maximal abelian subalgebra. See details in the section of root system). Therefore, for $p = \exp(rx_0)(p_0)$, we have

$$\begin{aligned}
\widetilde{\Delta f}|_p &= \sum_{j=1}^{r} \nabla_{d\pi_e\widetilde{(p_j)}\nabla_{d\pi_e}(p_j)} f \\
&\quad + \frac{1}{sh^2(-\alpha_i(x_0)r)}\Big[\sum_{i=1}^{n-r} \nabla_{d\pi_e\widetilde{(k_i)|_p}\nabla_{d\pi_e}(k_i)} f - \sum_{i=1}^{n-r} \nabla_{\widetilde{\nabla_{d\pi_e(k_i)|_p} d\pi_e(k_i)}} f\Big] \\
&= \sum_{j=1}^{r} \frac{d^2}{dt^2}\widetilde{f}(\exp(tp_j)\exp(rx_0))|_{t=0} \\
&\quad - \sum_{i=1}^{n-r} \frac{ch(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \sum_{j=1}^{r} \alpha_i(p_j)\frac{d}{dt}\widetilde{f}(\exp(tp_j)\exp(rx_0)) \\
&\quad + \sum_{i=1}^{n-r} \frac{1}{sh^2(-\alpha_i(x_0)r)}\frac{d^2}{dt^2}\widetilde{f}(\exp(k_it)\exp(rx_0))|_{t=0} \\
&\quad + \sum_{i=1}^{n-r} \frac{2coth(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)}\rho_{0*}(k_i)\frac{d}{dt}\widetilde{f}(\exp(k_it)\exp(rx_0))|_{t=0} \\
&\quad + \sum_{i=1}^{n-r} \frac{ch^2(-\alpha_i(x_0)r)}{sh^2(-\alpha_i(x_0)r)}\rho_{0*}^2(k_i)\widetilde{f}
\end{aligned}$$

Remark 1.1.49. We can see the above formula for Laplacian operator is only for the point $p = \exp(k_it)\exp(x_0r)(p_0)$. For the other point, we will use the spherical invariance of Laplacian to get it.

Remark 1.1.50. We see the above discussions do not rely on the choice of the representation ρ_0. Therefore, the above results also holds for the general associative vector bundle of the principal bundle $G \to G/K$

Here, we use the same notation with the section of root system. Let $\mathfrak{m}_0$ be $\mathfrak{l}_0^{\perp}$ in $\mathfrak{l}$. That is to say $\mathfrak{l} \cong \mathfrak{m}_0 \oplus \mathfrak{l}_0$. (Moreover, we have $[\mathfrak{m}_0, \mathfrak{m}_0] \in \mathfrak{l}_0$ and $[\mathfrak{l}_0, \mathfrak{l}_0] \in \mathfrak{l}_0$) The **Casmir operator** is defined as follow

$$C(\mathfrak{m}_0, \rho_0) = -\sum_{i=1}^{n-r} \rho_{0*}^2(k_i)$$

where $\{k_i\}_{i=1}^n$ is orthonormal basis of $\mathfrak{m}_0$ with respect to the Killing form $< .,. >$. This operator does not depend on the choices of the basis $\{k_i\}_{i=1}^n$.

Remark 1.1.51. One of important properties of the Casmir operator is that for $v \in \mathfrak{l}_0$, $[\rho_{0*}(v), C(\mathfrak{m}_0, \rho_{0*})] = 0$. In fact, let $[v, k_i] = \sum_{j=1}^{n-r} a_{ij} k_j$. Then

$$< [v, k_i], k_l > + < k_i, [v, k_l] >= 0$$

implies that $a_{il} + a_{li} = 0$. Moreover, we have

$$\begin{aligned}[\rho_{0*}(v), C(\mathfrak{m}_0, \rho_{0*})] &= \sum_{i}^{n-r} [\rho_{0*}(v), \rho_{0*}(k_i)]\rho_{0*}(k_i) + \rho_{0*}(k_i)[\rho_{0*}(v), \rho_{0*}(k_i)] \\ &= \sum_{i=1}^{n-r}\sum_{j=1}^{n-r} a_{ij}\rho_{0*}(k_j)\rho_{0*}(k_i) + a_{ij}\rho_{0*}(k_i)\rho_{0*}(k_j) = 0\end{aligned}$$

Let $f \in C^\infty(M, E)$ be a section of homogeneous vector bundle $E = G \times_\rho E_0$ on noncompact (M, g) with lift $\widetilde{f} \in C^\infty(G, T_{p_0}M)$. This G is the isometric group of M and K is the isotropic group for a fixed point p_0. f is called the **spherically invariant vector field** if $\widetilde{f}(h \exp(x_0)) = A(\exp(x_0))\rho_0(h^{-1})v$, where $v \in E_0$ is a fixed vector, $h \in K$, $x_0 \in \alpha$ (α is the maximal abelian subalgebra) and $A(\exp(x_0))$ is a linear transformation of E_0. Therefore, the Laplacian operator on the spherical invariant vector field is as following

$$\begin{aligned}\widetilde{\Delta f}(\exp(rx_0))|_p =& \sum_{j=1}^{r} \frac{d^2}{dt^2}(A(\exp(tp_j)\exp(rx_0))v) \\ &+ \sum_{i=1}^{n-r} \frac{ch(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \sum_{j=1}^{r} \alpha_i(p_j)\frac{d}{dt}(A(\exp(tp_j)\exp(rx_0))v) \\ &+ \sum_{i=1}^{n}[\frac{1}{sh^2(-\alpha_i(x_0)r)} A(\exp(rx_0))\rho_{0*}^2(k_i)v \\ &- 2\frac{ch(-\alpha_i(x_0)r)}{sh^2(-\alpha_i(x_0)r)}\rho_{0*}(k_i)A(\exp(rx_0))\rho_{0*}(k_i)v \\ &+ coth^2(-\alpha_i(x_0)r)\rho_{0*}^2(k_i)A(\exp(rx_0))v]\end{aligned}$$

where $p = \exp(rx_0)(p_0)$. Moreover, we can rewrite the above formula as following

$$\begin{aligned}\widetilde{\Delta f}(\exp(rx_0)) =& [\sum_{j=1}^{r} \frac{d^2}{dt^2}A(\exp(tp_i)\exp(x_0)) + \sum_{i=1}^{n-r}\sum_{j=1}^{r} \alpha_i(p_j)\frac{d}{dt}A(\exp(tp_j)\exp(rx_0)) \\ &- C(\mathfrak{m}_0, \rho_0)A(\exp(rx_0))]v + B(\exp(rx_0))v\end{aligned}$$

where $B(\exp(rx_0))$ is a higher order term with respect to $||rx_0||$.

1.1.6 The rank one cases

In this section, we are going to find a good coordinate of symmetric space which can easy to see the underlying structure of it.

Real hyperbolic space

We see the hyperbolic space can be thought of as a quotient space $\mathbb{H}^{n+1} = \mathrm{SO}(1, n+1)/\mathrm{SO}(n)$. Therefore, consider the Minkowski space $(\mathbb{R}^{n+2}, g)$ with the pseudo Riemannian metric

$$g = -2dx_0dx_{n+1} + |d\mathbf{x}|^2$$

where $x_0,\ x_{n+1} \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $|\mathbf{x}|^2 = \sum_{i=1}^n dx_i^2$. Then it is easily to see that $\mathbb{H}^{n+1} = \{(x_0, \mathbf{x}, x_{n+1}) \in \mathbb{R}^{n+2} | -2x_0x_{n+1} + |\mathbf{x}|^2 = -1\}$ with the induced metric of g.

Therefore, we can consider the coordinate change

$$(x_0, \mathbf{x}, x_{n+1}) \to (\alpha, \mathbf{x}', \rho) \quad with \quad \alpha = \sqrt{2x_0x_{n+1} - |\mathbf{x}|^2},\ \rho = \alpha/x_0,\ \mathbf{x}' = \mathbf{x}/x_0$$

Therefore, we have the inverse change

$$x_0 = \frac{\alpha}{\rho},\ \mathbf{x} = \frac{\alpha}{\rho}\mathbf{x}',\ x_{n+1} = \frac{1}{2}\rho\alpha + \frac{1}{2}\frac{\alpha}{\rho}|\mathbf{x}'|^2$$

Then, under the new coordinate $(\alpha, \mathbf{x}, \rho)$,

$$g = -d\alpha^2 + \frac{\alpha^2}{\rho^2}[d\rho^2 + |d\mathbf{x}'|^2]$$

Therefore, as $\alpha = 1$,

$$g = \frac{1}{\rho^2}[d\rho^2 + |d\mathbf{x}'|^2]$$

Complex hyperbolic space

Let $\mathbb{C}^{n+1} = \{(z_0, \mathbf{z}, z_{n+1}) | z_0, z_{n+1} \in \mathbb{C} \ and \ \mathbf{z} \in \mathbb{C}^n\}$ be a Hermitian manifold with natural complex structure and Hermitian form

$$H = -|dz_0|^2 + |d\mathbf{z}|^2 + |dz_{n+1}|^2$$

Obviously, $\mathrm{U}(1, n+1)$ is the group action which keeps this Hermitian form.

‘

Let $\mathbb{CP}^{n+1}$ be the projective space with homogeneous coordinate $\{[z_0, \mathbf{z}, z_{n+1}]\}$. It is easy to see that for $A \in U(1, n+1)$, $A \in \mathrm{U}(1, n+1)$ can induce a transformation of

$\mathbb{CP}^{n+1}$ by the following way

$$A : \mathbb{CP}^{n+1} \longrightarrow \mathbb{CP}^{n+1} \quad A([z_0, \mathbf{z}, z_{n+1}]) = [A(z_0, \mathbf{z}, z_{n+1})]$$

Moreover, this action is transitive on $\mathbb{CP}^{n+1}$

We just want to find a proper Hermitian form on $\mathbb{CP}^{n+1}$ such that the group action of $U(1, n+1)$ keep this Hermitian form.

The idea is fix the Hermitian form on $[1, 0, 0]$ and use the group action to translate this Hermitian form to every point. We are going to realize this idea step by step.

Step1: (Coordinate) Let $U = \{[z_0, \mathbf{z}, z_{n+1}] \in \mathbb{CP}^{n+1} | z_0 \neq 0\}$ be a coordinate chart on $\mathbb{CP}^{n+1}$ with coordinate function

$$\varphi : U \longrightarrow \mathbb{C}^{n+1} \quad \varphi([z_0, \mathbf{z}, z_{n+1}]) = (\frac{\mathbf{z}}{z_0}, \frac{z_{n+1}}{z_0})$$

Suppose that at $[1, \mathbf{0}, 0]$ the Hermitian form is $H([1, \mathbf{0}, 0]) = |d\mathbf{z}|^2 + |dz_{n+1}|^2$ under the coordinate (U, φ).

Step2: (Group action) Consider the following matrix

$$A = \begin{bmatrix} cosh(|t|) & 0 & sinh(|t|)\frac{|t|}{\bar{t}} \\ 0 & I & 0 \\ sinh(|t|)\frac{|t|}{t} & 0 & cosh(|t|) \end{bmatrix}$$

where $t \in \mathbb{C}$ and I is an $n \times n$ identity matrix. It is straightforward to check that

$A \in U(1, n+1)$. Therefore A can act on $\mathbb{CP}^{n+1}$ by $A([z_0, \mathbf{z}, z_{n+1}]) = [A(z_0, \mathbf{z}, z_{n+1})]$. Furthermore, in the coordinate (U, φ), we have

$$\varphi \circ A \circ \varphi^{-1} : U \to U \quad (\mathbf{z}, z_{n+1}) \to (\mathbf{z}', z'_{n+1})$$

where

$$\mathbf{z}' = \frac{\mathbf{z}}{cosh(|t|) + z_{n+1} sinh(|t|)\frac{|t|}{\bar{t}}}$$
$$z'_{n+1} = \frac{sinh(|t|)\frac{|t|}{t} + z_{n+1} cosh(|t|)}{cosh(|t|) + z_{n+1} sinh(|t|)\frac{|t|}{\bar{t}}}$$

Then, at $A([1, \mathbf{0}, 0]) = [cosh(|t|), \mathbf{0}, sinh(|t|)\frac{|t|}{\bar{t}}]$, we have

$$d\mathbf{z}' = \frac{1}{cosh(|t|)} d\mathbf{z}$$
$$dz'_{n+1} = \frac{1}{cosh^2(|t|)} dz_{n+1}$$

Step3: (Hermitian form) Since at $[1, \mathbf{0}, 0]$, the Hermitian form is $H([1, \mathbf{0}, 0]) = |d\mathbf{z}|^2 + |dz_{n+1}|^2$ in the coordinate of (U, φ),

$$H([cosh(|t|), \mathbf{0}, sinh(|t|)\frac{|t|}{\bar{t}}]) = cosh^4(|t|)|dz'_{n+1}|^2 + cosh^2(|t|)|d\mathbf{z}'|^2$$

in the coordinate of (U, φ). Now, since

$$[cosh(|t|), \mathbf{0}, sinh(|t|)\frac{|t|}{\bar{t}}] = [1, \mathbf{0}, \frac{sinh(|t|)}{cosh(|t|)\frac{|t|}{t}}]$$

Therefore,

$$\varphi[cosh(|t|), \mathbf{0}, sinh(|t|)\frac{|t|}{\bar{t}}] = (\mathbf{0}, \frac{sinh(|t|)}{cosh(|t|)\frac{|t|}{t}})$$

which implies that $z'_{n+1} = \frac{sinh(|t|)}{cosh(|t|)\frac{|t|}{t}}$. Therefore

$$cosh(|t|) = \frac{1}{\sqrt{1 - |z'_{n+1}|}}$$

We have

$$
\begin{aligned}
&H([cosh(|t|), \mathbf{0}, sinh(|t|)\frac{|t|}{\bar{t}}]) = cosh^4(|t|)|dz'_{n+1}|^2 + cosh^2(|t|)|d\mathbf{z}'|^2 \\
=&\frac{1}{(1 - |z'_{n+1}|^2)^2}|dz'_{n+1}| + \frac{1}{1 - |z'_{n+1}|^2}|d\mathbf{z}'|^2 \\
=&\frac{1}{(1 - |z'_{n+1}|^2)^2}\big((1 - |z'_{n+1}|^2)(|d\mathbf{z}'|^2 + |dz'_{n+1}|^2) + |z'_{n+1}|^2|dz'_{n+1}|^2\big) \\
=&\frac{1}{(1 - |z'_{n+1}|^2)^2}\big((1 - |z'_{n+1}|^2)(|d\mathbf{z}'|^2 + |dz'_{n+1}|^2) + (\bar{z}'_{n+1}dz'_{n+1}) \cdot (z'_{n+1}d\bar{z}'_{n+1})\big)
\end{aligned}
$$

Step4: General Hermitian form Consider the following matrix

$$C = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$$

where $B \in \mathrm{U}(n+1)$. Therefore, $C \in U(1, n+1)$, we see that

$$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ z'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ z'' \\ z''_{n+1} \end{bmatrix}$$

We see that

$$1 - |z'_{n+1}|^2 = 1 - |d\mathbf{z}''|^2 - |dz''_{n+1}|^2$$

$$|d\mathbf{z}'|^2 + |dz'_{n+1}|^2 = |d\mathbf{z}''|^2 + |dz''_{n+1}|^2$$

$$(\bar{z}'_{n+1}dz'_{n+1}) = (\bar{z}'')dz'' + (\bar{z}''_{n+1}dz''_{n+1})$$

We get the general Hermitian form in the coordinate (U, φ)

$$H = \frac{-1}{< Z, Z >^2} det \begin{bmatrix} < Z, Z > & < Z, dZ > \\ < Z, dZ > & < dZ, dZ > \end{bmatrix}$$

where $Z = (1, \mathbf{z}, z_{n+1})$ and $< . >$ is a Hermitian form on $\mathbb{C}^{n+2}$ such that the invariant group of this Hermitian form is $U(1, n+1)$. This Hermitian form also called the Bergman metric on (U, φ). □

Consider the Bergman metric on $\mathbb{CH}^{n+1}$,

$$g = \frac{-4}{< Z, Z >^2} det \begin{bmatrix} < Z, Z > & < Z, dZ > \\ < Z, dZ > & < dZ, dZ > \end{bmatrix}$$

where $< ., . >= -dz_0 d\bar{z}_{n+1} - dz_{n+1} d\bar{z}_0 + |d\mathbf{z}|^2$

Now, take $z_0 = 1$ and let $\rho^2 = z_{n+1} + \bar{z}_{n+1} - |\mathbf{z}|^2$ and $v = \mathrm{Im}(z_{n+1})$. Then we can solve

$$z_{n+1} = \rho^2 + |\mathbf{z}|^2 + iv$$

Now, write down the Bergman metric under the coordinate $(\mathbf{z}, \rho, v)$. It is easy to get

$$g = \frac{4|d\mathbf{z}|^2}{\rho^2} + \frac{1}{\rho^4}(4\rho^2(d\rho)^2 + 4(dv + \mathrm{Im}(\mathbf{z}d\mathbf{z}))^2)$$

which can see the contact form $dv + \mathrm{Im}(\mathbf{z}d\mathbf{z})$.

Then, we make use of the root system to find a frame of the symmetric space. And then under this frame, we can write down the Laplacian operator in a beautiful form especially for the spherically invariant section. The reason we emphasis the spherically invariant vector bundle is that the green function and heat kernel of the Laplacian operator on the symmetric space is spherically invariant. Let $(M, g) = G/K$ be a simply connected noncompact symmetric space of rank 1. This G is the isometric group under the metric g. K is the isotropic group fixing a fixed point p_0. Use the same notations in the section of root system. we can do the following computation.

- Maximal Abelian subalgebra $\alpha = Span\{x_0\}$ with $< x_0, x_0 >= 1$
- $< [x_i, y_i], x_0 >= \alpha_i(x_0) = \lambda_i$. Therefore, $[x_i, y_i] = \lambda_i x_0$
- $[x_0, k_i] = \frac{1}{\sqrt{2}}([x_0, x_i] + [x_0, y_i]) = -\lambda_i \mathfrak{p}_i$
- $[x_0, \mathfrak{p}_i] = \frac{1}{\sqrt{2}}([x_0, x_i] - [x_0, y_i]) = -\lambda_i k_i$

Therefore, we have

$$sh(ad(-rx_0))k_i = sh(-r\lambda_i)\mathfrak{p}_i$$
$$ch(ad(-rx_0))k_i = ch(-r\lambda_i)k_i$$
$$[\mathfrak{p}_i, k_i] = [\frac{x_i + y_i}{\sqrt{2}}, \frac{x_i - y_i}{\sqrt{2}}] = [x_i, y_i] = \lambda_i x_0$$

where $\lambda_i = \alpha_i(x_0)$.

Let (ρ, E_0) be a representation of K on a finite linear space E_0. Let $E = G \times_\rho E_0$ be the corresponding homogeneous vector bundle.

(1)(Covariant derivative) Let $f \in C^\infty(M, E)$ be a section on homogeneous vector bundle E with lift $\tilde{f} \in C^\infty(G, G \times E_0)$ and $v \in \mathfrak{g}$. And we can define the covariant derivative as following

$$\widetilde{\nabla_{d\pi_e(v)|_p}} f = v|_d \tilde{f}(v)|_g + \rho_{0*}(Ad(g^{-1})(v)|_{\mathfrak{l}})(\tilde{f})$$

where g is an arbitrary element in G such that $g(p_0) = p$ and $|_{\mathfrak{l}}$ means orthogonal projection on $\mathfrak{l}$ with respect to Killing form on $\mathfrak{g}$. For the definition of $d\pi_e$ and $(d\pi_{p_0})_e$, see the 1.3.4 (1).

(2)(Laplacian Operator on the spherical coordinate) Now, let $v = k_i \in \mathfrak{l}$ and $p = \exp(k_i t)\exp(rx_0)(p_0) \in M$ where $r \leq 0$ and $x_0 \in \mathfrak{a}$ is a normal vector with respect to Killing form ($\mathfrak{a}$ is the maximal abelian subalgebra). Then

$$\widetilde{\nabla_{d\pi_e(k_i)|_p}} f = \frac{d}{dt}\tilde{f}(\exp(k_i t)\exp(rx_0)) + \rho_{0*}(ch(ad(-rx_0))Ad(\exp(-k_i t))k_i)(\tilde{f})$$

Therefore,

$$\begin{aligned}\widetilde{\nabla_{d\pi_e(k_i)|_p}\nabla_{d\pi_e(k_i)}} f =& \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0)) \\ &+ 2\rho_{0*}(ch(ad(-rx_0))Ad(\exp(-k_i t))k_i)(\frac{d}{dt}\tilde{f}) \\ &+ \rho_{0*}^2(ch(ad(-rx_0))Ad(\exp(-k_i t))k_i)(\tilde{f})\end{aligned}$$

At $t = 0$

$$\begin{aligned}\widetilde{\nabla_{d\pi_e(k_i)|_p}\nabla_{d\pi_e(k_i)}} f =& \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\ &+ 2\rho_{0*}(ch(-r\lambda_i))k_i)(\frac{d}{dt}\tilde{f}) \\ &+ \rho_{0*}^2(ch(-r\lambda_i)k_i)(\tilde{f})\end{aligned}$$

On the other hand,

$$\widetilde{d\pi_e(k_i)}(\exp(k_i t)\exp(rx_0)) = (d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i) \in T_{p_0}M.$$

Therefore,

$$\nabla_{d\pi_e(k_i)|_p} d\pi_e(k_i) = \frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)$$
$$+ \rho'_{0*}(ch(ad(-rx_0))Ad(\exp(k_i t))k_i)[\pi_{**}(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)]$$

where ρ'_{0*} is the representation corresponding to the tangent bundle. At $t = 0$,

$$\begin{aligned}\nabla_{d\pi_e(k_i)|_p} d\pi_e(k_i) =& \frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))k_i) \\ &+ \rho'_{0*}(ch(ad(-rx_0))k_i)[\pi_{**}(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)] \\ =&[sh(ad(-rx_0))k_i, ch(ad(-rx_0))k_i] \\ =&[sh(-\lambda_i r)p_i, ch(-\lambda_i r)k_i] \\ =&sh(-\lambda_i r)ch(-\lambda_i r)\lambda_i\pi_{**}(x_0)\end{aligned}$$

Therefore, for $p = \exp(rx_0)(p_0)$, we have

$$\begin{aligned}
\widetilde{\Delta f}|_p =& \widetilde{\nabla_{d\pi_e(x_0)}\nabla_{d\pi_e(x_0)}f} + \frac{1}{sh^2(-\lambda_i r)}\Big[\sum_{i=1}^{n}\widetilde{\nabla_{d\pi_e(k_i)|_p}\nabla_{d\pi_e(k_i)}f} - \sum_{i=1}^{n}\widetilde{\nabla_{\nabla_{d\pi_e(k_i)|_p}d\pi_e(k_i)}f}\Big] \\
=& \frac{d^2}{dr^2}\widetilde{f}(\exp(rx_0)) - \sum_{i=1}^{n}\frac{ch(-\lambda_i r)}{sh(-\lambda_i r)}\lambda_i\frac{d}{dr}\widetilde{f}(\exp(rx_0)) \\
&+ \sum_{i=1}^{n}\frac{1}{sh^2(-\lambda_i r)}\frac{d^2}{dt^2}\widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\
&+ \sum_{i=1}^{n}\frac{2coth(-\lambda_i r)}{sh(-\lambda_i r)}\rho_{0*}(k_i)\frac{d}{dt}\widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\
&+ \sum_{i=1}^{n}\frac{ch^2(-\lambda_i r)}{sh^2(-\lambda_i r)}\rho_{0*}^2(k_i)\widetilde{f}
\end{aligned}$$

Remark 1.1.52. We can see the above formula for Laplacian operator is only for the point $p = \exp(k_i t)\exp(x_0 r)(p_0)$. For the other point, we will use the spherical invariance of Laplacian to get it. (See)

Remark 1.1.53. We see the above discussions do not rely on the choice of the representation ρ_0. Therefore, the above results also holds for the general associative vector bundle of the principal bundle $G \to G/K$

(3)(Spherically invariant section) Let $f \in C^\infty(M, E)$ and $\widetilde{f} \in C^\infty(G, G \times E_0)$. f is called the **spherically invariant vector field** if $\widetilde{f}(h\exp(rx_0)) = A(r)\rho_0(h^{-1})v$ where $v \in T_{p_0}M$ is a fixed vector, $h \in K$ and $A(r)$ is a linear transformation of $T_{p_0}M$. Therefore, the Laplacian operator on the spherical invariant vector field is as following

$$\begin{aligned}\widetilde{\Delta f}(\exp(rx_0)) =&\partial_r^2(A(r)v)+\mathcal{H}(r)(\partial_r(A(r)v)\\&+\sum_{i=1}^{n}[\frac{1}{sh^2(-\lambda_i r)}A(r)\rho_{0*}^2(k_i)v-2\frac{ch(-\lambda_i r)}{sh^2(-\lambda_i r)}\rho_{0*}(k_i)A(r)\rho_{0*}(k_i)v\\&+coth^2(-\lambda_i r)\rho_{0*}^2(k_i)A(r)v]\end{aligned}$$

where $p = \exp(rx_0)$ and $\mathcal{H}(r) = \sum_{i=1}^{n}\frac{ch(\lambda_i r)}{sh(\lambda_i r)}\lambda_i$. Moreover, we can rewrite the above formula as following

$$\widetilde{\Delta f}(\exp(rx_0)) =[\partial_r^2 A(r)+\mathcal{H}\partial_r A(r)-C(\mathfrak{m}_0,\rho_0)A(r)]v+B(r)v$$

where $\mathcal{H} = \sum_{i=1}^{n}\lambda_i$ and

$$\begin{aligned}B(r) =&[\mathcal{H}(r)-\mathcal{H}]\partial_r A(r)+\sum_{i=1}^{n}[\frac{1}{sh^2(-\lambda_i r)}A(r)\rho_{0*}^2(k_i)v\\&-2\frac{ch(-\lambda_i r)}{sh^2(-\lambda_i r)}\rho_{0*}(k_i)A(r)\rho_{0*}(k_i)v+(coth^2(-\lambda_i r)\rho_{0*}^2(k_i)-1)A(r)v]\end{aligned}$$

Therefore, $|B(r)| = O(\exp(-r))[|A(r)|+|\partial_r A(r)|]$

§ 1.2 Parabolic geometries and asymptotically symmetric metrics

In this section, we introduce the parabolic geometry. The reason we introduce this geometry is that the boundary geometry of rank 1 noncompact type symmetric space is still a homogeneous space and can be thought of as the model space of a parabolic geometry (Iwassawa Decomposition). Then, we can define the general parabolic geometry with this type. And it turns out that this kind of parabolic geometry is equivalent to the infinitesimal flag structure. In particular, for the hyperbolic case, the boundary parabolic geometry is conformal geometry and for the complex hyperbolic case, the boundary parabolic geometry is strictly pseudoconvex partial integrable almost CR geometry.

In section 1.2.1, we define the boundary geometry of noncompact type of symmetric space.

In section 1.2.2, we introduce the basic definition of Cartan geometry which is the general case of parabolic geometry and define the parabolic geometry and show how this geometry is characterized by the flag structure on the tangent space.

1.2.1 Geodesic compactifications of symmetric spaces

In this section, we will define the boundary of the noncompact symmetric space. We will identity the boundary of the noncompact symmetric space with the geodesic classes. Let M be a noncompact symmetric space.

- **The asymptotic ray** [18] Two (unit speed) geodesics ray $\sigma, \tau : [0, +\infty) \longrightarrow M$ are called asymptotic if the function $t \mapsto d(\sigma(t), \tau(t))$ is bounded.
- **Martin boundary** [18] The boundary at infinity $\partial_\infty M$ of M is the set of equivalence classes of rays for the equivalence relation "being asymptotic". The equivalence class of a ray σ will be denoted $\sigma(\infty)$.
- **The topology of $\partial_\infty M$** [18] Let $U_xM \subseteq T_xM$ be the unit ball in T_xM. Then the

map $\Phi_x : U_x M \longrightarrow \partial_\infty M$ is bijective. Thus we can induce the topology of $U_x M$ onto $\partial_\infty M$.

Now, we see that the isometric group of the noncompact symmetric space can also be acted on the Martin boundary of it. Furthermore, this action is transitive. Therefore, the Martin boundary can also be thought of as a homogeneous space. And the isotropic group is the so called Borel group.

- **The group action on '$\partial_\infty M$'** The isometric group of M, G, can transitively act on the $\partial_\infty M$.
- **The geometry of the boundary** Let $\xi \in \partial_\infty M$ and G_ξ is the isotropic group at ξ. Then $\partial_\infty M = G/G_\xi$.
- **Borel group** [25] Borel group is a subgroup in G, consist of AN^+L_0, where A, N^+, L_0 is the Lie group corresponding to $\mathfrak{a}$, $\mathfrak{n}^+$ and $\mathfrak{l}_0$ respectively. (This decomposition actually is Iwasawa decomposition)
  - Borel group fix the Weyl chamber at infinity ∂_∞.
  - In particularly, for the rank 1 noncompact symmetric space, the Borel group is the isotropic group of the infinity.

We will see from the point of the Cartan geometry, the boundary geometry is, in fact, the model space of the parabolic geometry and its corresponding curvature. Moreover, we will introduce the Liouville theorem, which says that the vanish of the curvature implies that the geometry is locally model space(homogeneous space).

Definition 1.2.1 ([12], p.71)**.** Let $H \subseteq G$ be a Lie subgroup in a Lie group G, and let $\mathfrak{g}$ be the Lie algebra of G. A Cartan geometry of type (G, H) on a manifold M is a principal fiber bundle $p : \mathcal{P} \to M$ with structure group H, which is endowed with a $\mathfrak{g}-$valued one-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the Cartan connection, which means

$$\begin{gathered} \left(r^h\right)^* \omega = \mathrm{Ad}\left(h^{-1}\right) \circ \omega \text{ for all } h \in H \\ \omega\left(\zeta_X(u)\right) = X \text{ for each } X \in \mathfrak{h} \\ \omega(u) : T_u\mathcal{P} \to \mathfrak{g} \text{ is a linear isomorphism for all } u \in \mathcal{P} \end{gathered} \tag{1.2.1}$$

Actually, all the Cartan geometries will form a Category.

Definition 1.2.2 ([12], p.73)**.** A morphism between two Cartan geometries $(\mathcal{P} \to M, \omega)$ and $(\mathcal{P}' \to M', \omega')$ of type (G, H) is a principal bundle morphism $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi^*\omega' = \omega$.

Now, we can define the so-called model Cartan geometry. The **homogeneous model** for Cartan geometries of type (G, H) is the canonical bundle $p : G \to G/H$ endowed with the left **Maurer-Cartan form** $\omega_G \in \Omega^1(G, \mathfrak{g})$, which can be thought of as the flat case of Cartan geometry. We see that $d\omega_G + [\omega_G, \omega_G] = 0$, which enlightens us to define the general curvatureof the Cartan geometry.

Definition 1.2.3 ([12],p.71)**.** The curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{P} \to M, \omega)$ is defined by the srtructure equation $K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$

The most important things for the curvature is the Liouville Theorem which basically says that the curvature of a Cartan geometry $(P \to M, \omega)$ vanishes identically if and only if any point $x \in M$ has an open neighborhood U such that the restriction $\left(p^{-1}(U) \to U, \omega\right)$ is isomorphic to the restriction of the homogeneous model $(G \to G/H, \omega_G)$ to an open neighborhood of o.

Theorem 1.2.4 (Liouville Theorem, [12], p.73)**.** *Suppose that G/H is connected. Then* *any isomorphism between two restrictions of* $(G \to G/H, \omega_G)$ *to connected open subsets of* G/H *uniquely globalizes to an automorphism of the homogeneous model.*

1.2.2 Parabolic geometries on the boundary

Now, we will define a special Cartan geometry, parabolic geometry. We will see that the boundary homogeneous space of the symmetric space of the noncompact type is such model space.

- **The graded Lie algebra** A Lie algebra $\mathfrak{g}$ is graded Lie algebra if there exist a decomposition of $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \in \mathfrak{g}_{i+j}$ ($\mathfrak{g}_i = 0$ if $|i| > k$) and the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ can be generated by $\mathfrak{g}_{-1}$
- **The parabolic subgroup** If G is the Lie group with the graded Lie algebra, then $K < G$ is the parabolic subgroup if the Lie algebra of K is the $\oplus_{i=0}^{k} \mathfrak{g}_i$
- **The parabolic geometry** If G is semisimple Lie algebra and K is the parabolic subgroup of G. Then the Cartan geomerty $(P \to M, \omega)$ of the type (G, K) is called parabolic geometry.
  - In particular, the Borel group is exactly the parabolic subgroup of G. Thus, for the rank 1 case of symmetric space M, $\partial_\infty M = G/P$ is the homogeneous of parabolic geometry.

We see that the Borel group can not act on the tangent space. However, we see that the subgroup of Borel group, Levi subgroup, can act on the tangent space. If G is a Lie group with the graded Lie algebra, then $G_0 < G$ is the **Levi subgroup** if the Lie algebra of G_0 is $\mathfrak{g}_0$. By this Levi group, we can define the infinitesimal flag structure which will be an alternative of the regular parabolic geometry.

Definition 1.2.5 (The infinitesimal flag structure, [12], Definition 3.1.6)**.** An infinitesimal flag structure of type (G, P) on a smooth manifold M is given by:

(1) A filtration $TM = T^{-k}M \supseteq \cdots \supseteq T^{-1}M$ of the tangent bundle of M such that the rank of T^iM equals the dimension of $\mathfrak{g}^i/\mathfrak{p}$ for all $i = -k, \ldots, -1$.

(2) A principal G_0-bundle $p : E \to M$.

(3) A collection $\theta = (\theta_{-k}, \ldots, \theta_{-1})$ of smooth sections $\theta_i \in \Gamma\left(L\left(T^iE, g_i\right)\right)$ which are G_0-equivariant in the sense that $(r^g)^*\,\theta_i = \mathrm{Ad}\left(g^{-1}\right) \circ \theta_i$ for all $g \in G_0$, and such that for each $u \in E$ and $i = -k, \ldots, -1$ the kernel of $\theta_i(u) : T^i_uE \to \mathfrak{g}_i$ is $T^{i+1}_uE \subseteq T^i_uE$. ($\mathfrak{g}^i = \oplus_{j=i}^k \mathfrak{g}_j$ and $T^iE = dp^{-1}(T^iM)$ $i < 0$)

The infinitesimal flag structures also forms a category by the following definition. We will see that if we add a regular condition of this infinitesimal flag structure, this category can be corresponded to the normal regular parabolic geometry category in some sense.

Definition 1.2.6 ([12], Definition 3.1.6)**.** Let M and $\tilde{M}$ be smooth manifolds endowed with infinitesimal flag structrues $\left(\left\{T^iM\right\}, p : E \to M, \theta\right)$ and $\left(\left\{T^i\tilde{M}\right\}, \tilde{p} : \tilde{E} \to \tilde{M}, \tilde{\theta}\right)$ of type (G, P). Then a morphism of infinitesimal flag structures is a principal bundle homomorphism $\Phi : E \to \tilde{E}$ which covers a local diffeomorphism $f : M \to \tilde{M}$ such that Tf is filtration preserving and $\Phi^*\tilde{\theta}_i = \theta_i$ for all $i = -k, \ldots, -1$.

Next, we will define the regular condition for the infinitesimal flag structure. Before that, we need to introduce the filtered manifold. A **filtered manifold** is a smooth manifold M together with a filtration $TM = T^{-k}M \supseteq \cdots \supseteq T^{-1}M$ of its tangent bundle by smooth subbundles, which is compatible with the Lie bracket in the sense that $[\xi, \eta] \in \Gamma\left(T^{i+j}M\right)$.

Definition 1.2.7 (The regular infinitesimal flag structure, [12], Proposition 3.1.7)**.** Let $\left(\left\{T^iM\right\}, p : E \to M, \theta\right)$ be an infinitesimal flag structure such that $\left(M, \left\{T^iM\right\}\right)$ is a filtered manifold. Then the structure is regular if for all $i, j < 0$ such that $i + j \geq -k$ and all sections $\xi \in \Gamma\left(T^iE\right)$ and $\eta \in \Gamma\left(T^jE\right)$ we have

$$\theta_{i+j}([\xi, \eta]) = \left[\theta_i(\xi), \theta_j(\eta)\right]$$

- Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be $|k|-$ graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$, $P \subseteq G$ a parabolic subgroup corresponding to the grading and $G_0 \subseteq P$ the Levi subgroup. Then any regular infinitesimal flag structure of type (G, P) on a smooth manifold M is induced by a normal parabolic geometry of type (G, P).

By the [Theorem 3.1.14] [12], the regular infinitesimal flag structure can be identified as a normal regular parabolic geometry under some assumptions of the cohomology space for the type (G, P).

1.2.3 The conformal geometry and asymptotically hyperbolic metrics

In this section, We will first review the traditional definition of the conformal manifolds and show that this is, actually, a special parabolic geometry whose model space is the boundary geometry of the hyperbolic space.

Then, we will review the asymptotically hyperbolic manifolds whose boundary is, in fact, a conformal manifold. Finally, we will introduce some basic properties of asymptotically hyperbolic manifolds.

Traditionally, a conformal manifold is a Riemannian manifold equipped with an equivalence class of metric tensors, in which two metrics g and h are equivalent if and only if

$$h = \lambda^2 g$$

where λ is a real-valued non zero function.

Then, we will show that there exists an infinitesimal flag structure corresponding to this conformal manifold. First, it is easy to show that there exists a subbundle of a Frame bundle $p : E \longrightarrow M$ such that the structure group is $A \times \mathrm{SO}(n)$ where A is the set of all the non-zero scalar matrices. Now consider the

$$\theta : TE \longrightarrow \mathfrak{g}_{-1}, \quad (p, A, v_1, v_2) \longrightarrow A^{-1} v_1$$

where $p \in M, A \in A \times \mathrm{SO}(n), v_1 \in T_p M, v_2 \in \mathrm{Lie}(A \times \mathrm{SO}(n))$ and the $\mathfrak{g}_{-1}$ is an Abelian Lie algebra. Moreover, we see that $\mathfrak{so}(1, n+1) \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$. By the Proposition 3.1.14 [12], the conformal manifold corresponds to a uniquely parabolic geometry.

Next, we will define the asymptotically hyperbolic manifold and identify its boundary as a conformal manifold. Suppose that X^{n+1} is a smooth manifold of dimension $n+1$ with smooth boundary $\partial X = M^n$. A defining function for the boundary M^n in X^{n+1} is a function x on $\bar{X}^{n+1}$ such that

$$\begin{cases} x > 0 \text{ in X} \\ x = 0 \text{ on M} \\ dx \neq 0 \text{ on M} \end{cases}$$

A Riemannian metric g_+ on X^{n+1} is conformally compact if $(\bar{X}^{n+1}, x^2 g_+)$ is a compact Riemannian manifold with boundary M^n for a defining function x. Conformally compact manifold (X^{n+1}, g_+) carries a well-defined conformal structure on the boundary M^n, where each metric $\hat{g}$ in the class is induced from $\bar{g} = x^2 g^+$ for a defining function x. We call $(\mathbf{M}^n, [\hat{g}])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g). It can be computed that, given a defining function x,

$$R_{\alpha\beta\delta\gamma}[g] = -|dx|^2_{\bar{g}} \left(g_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\gamma} g_{\beta\delta}\right) + O\left(x^2\right)$$

in coordinate $(0, \epsilon) \times \mathrm{M}^n \subseteq \mathrm{X}^{n+1}$ Therefore, if we assume that g is also asymptotically locally hyperbolic, i.e. its sectional curvatures approach -1 at the infinity, then

$$|dx|^2_{\bar{g}}\Big|_{\mathrm{M}} = 1$$

for any defining function x. Therefore an asymptotically hyperbolic (AH) manifold is a conformally compact manifold in addition to being asymptotically local hyperbolic.

Definition 1.2.8. [31] For a smooth manifold X^{n+1} with boundary $\partial X^{n+1} = M^n$, a Riemannian metric g_+ is said to be asymptotically hyperbolic Einstein (AHE) if it is AH and it is Einstein

$$\mathrm{Ric}\left[g^+\right] = -n g^+$$

Given an AH manifold (X^{n+1}, g_+) and a representative $\hat{g}$ in $[\hat{g}]$ of the conformal infinity Mn, there is a uniquely defining function x such that, on $M \times (0, \epsilon)$ in X, g_+ has the normal form

$$g = x^{-2}\left(dx^2 + g_x\right)$$

where g_x is a 1-parameter family of metrics on M. This is because

Lemma 1.2.9. *(Geodesic Defining function) Suppose that*(X^{n+1}, g_+)*is an AH manifold with the conformal infinity* $(M, [\hat{g}])$. *Then, for any* $\hat{g} \in [\hat{g}]$, *there exists a unique defining function* x *such that*

$$|dx|^2_{x^2 g} = 1$$

in a neighborhood of the boundary $[0, \epsilon) \times M$ *for some* $\epsilon > 0$ *and*

$$x^2 g^+\big|_{\mathrm{M}} = \hat{g}$$

Proof: Let x_0 be any defining function for M^n in X^{n+1} and $\bar{g}^0 = x_0^2 g$. Then set $x = e^{\omega} x_0$. Hence

$$|dx|^2_{x^2 g} = |dx_0 + x_0 dw|^2_{\bar{g}^0} = |dx_0|^2_{\bar{g}^0} + 2x_0 dw\left(\nabla_{\bar{g}^0} x_0\right) + x_0^2 |dw|^2_{\bar{g}^0}$$

Therefore the equation is

$$|dx|^2_{\bar{g}} = 1$$

$$2dw\left(\nabla_{\bar{g}^0} x_0\right) + x_0 |dw|^2_{\bar{g}^0} = \frac{1 - |dx_0|_{\bar{g}^0}}{x_0}$$

This is a non-characteristic nonlinear first order partial differential equation for ω. Thus there exists a unique solution ω at least near the boundary M^n with the given boundary condition

$$e^{2w} x_0^2 g = \hat{g} \quad \text{ on } M^n$$

Given an AHE manifold (X^{n+1}, g_+), in the local coordinate $(0, \epsilon) \times M^n$ near the boundary where the metric takes the normal form (2.1), the Einstein equations turn into a second order ordinary differential equations point-wisely on Mn with $x = 0$ as a regular singular point. We have, as an improvement of Theorem 1.6.1, from (1.26) (1.27) (1.28) the following expansions of the metric.

Theorem 1.2.10. [20] *Suppose that (X^{n+1}, g) is a conformally compact Einstein manifold with the conformal infinity $(M^n, [\hat{g}])$. And suppose that x is the geodesic defining function associated with a metric $\hat{g} \in [\hat{g}]$. Then*

$$\begin{aligned} g_x = \hat{g} + g^{(2)}x^2 + (\textit{even powers of } x) \\ + g^{(n-1)}x^{n-1} + g^{(n)}x^n + \cdots \end{aligned}$$

when n is odd, and

$$\begin{aligned} g_x = \hat{g} + g^{(2)}x^2 + (\textit{even powers of } x) \\ + g^{(n)}x^n + hx^n \log x + \cdots \end{aligned}$$

a) $g^{(2i)}$ are determined by $\hat{g}$ for $2i \leq n$;

b) $g^{(n)}$ is traceless when n is odd;

c) the trace part of $g^{(n)}$ is determined by $\hat{g}$ and h is traceless and determined by $\hat{g}$ when n is even;

d) the traceless part of $g^{(n)}$ is divergence free;

e) the trace-free part of $g^{(n)}$ is non-local and determined by the $g^{+}$and $\hat{g}$;

f) the rest of power series is determined by $\hat{g}$ and $g^{(n)}$ when n is odd and the rest in the powers of r and $\log r$ is determined by $\hat{g}$ and $g^{(n)}$ when n is even.

1.2.4 The CR geometry and asymptotically complex hyperbolic metrics

In this section, we will first introduce the traditional definition of the pseudoconvex partially integrable CR-structure and show that this structure is, actually, a special parabolic geometry whose model space is the boundary geometry of the complex hyperbolic space.

Then, we will introduce the asymptotically complex hyperbolic metric whose boundary is, in fact, a pseudocovex partially integrable CR-structure.

Definition 1.2.11 (Partially integrable CR manifold, [13], Definition 1.7)**.** Let M^{2n+1} be a smooth compact orientable manifold endowed with a $2n$ dimensional distribution $H \subseteq TM$ and an almost complex structure J on H. Then, (M^{2n+1}, H^{2n}, J) is said to be an partially integrable CR manifold if

1) H is maximal non-integrable. (For any vector field $v_1 \neq v_2 \in H$, $[v_1, v_2] \notin H$)

2) $[H^{1,0}, H^{1,0}] \in \mathbb{C} \otimes H$, where $H^{1,0}$ is the holomorphic vector field of H under the almost complex structure J and $\mathbb{C} \otimes H$ is the complexification of the distribution H.

From the definition of partially integrable CR structure, we see there exists a conformal class of one form $[\eta]$ ($ker\eta = H$) such that for arbitrary $v_1 \in H$ and $v_2 \in H$, $d[\eta](J(v_1), J(v_2)) = d[\eta](v_1, v_2)$ and $d[\eta](v_1, J(v_2)) = d[\eta](v_2, J(v_1))$ and A partially integrable CR manifold is called **pseudoconvex** if $-d\eta(., J.)$ is definite.

It is easy to construct a

Definition 1.2.12. (Asymptotically Complex Hyperbolic Metric) Let $\bar{M}^{2n+2}$ be a manifold with boundary ∂M^{2n+1}. $(\partial M, H, J)$ is a partially integrable CR contact structure on ∂M. (M, g_+) is a complete Riemannian metric in the interior of M.

Then g_+ is said to be a asymptotically complex hyperbolic metric, if there exist a defining function ρ $(i.e\ \rho > 0$ in M, $\rho = 0$ and $d\rho \neq 0$ on $\partial M)$, an extension of η, $\tilde{\eta}$ $(i.e\ \eta|_{T\partial M} = \eta)$ and one forms $\{\tilde{\omega}^i\}_{i=1}^{2n}$ such that

1) $\rho^4 g_+ \in C^{k,\alpha}(\bar{M})$ and $\rho^4 g_+|_{T\partial M} = \eta^2$

2) $\rho^2 g_+|_{\bar{M},ker\tilde{\eta}} \in C^{k,\alpha}(\bar{M})$ and $\rho^2 g_+|_{(\partial M,ker\eta)} = \frac{1}{4}d\eta(.,J.)$

3) $|\frac{d\rho}{\rho}|_{g_+} \in C^{k,\alpha}(\bar{M})$ and $|\frac{d\rho}{\rho}|_{g_+}|_{\partial M} = 1$

4) $\{d\rho, \tilde{\omega}^i, \tilde{\eta}\}_{i=1}^{2n}$ be a basis of $T^*\bar{M}$ near the boundary.

   And the metric g_+ can be written as follow

   $$g_+ = a_{00}\frac{(d\rho)^2}{\rho^2} + a_{0i}\frac{d\rho\otimes\tilde{\omega}^i}{\rho^2} + a_{i0}\frac{\tilde{\omega}^i\otimes d\rho}{\rho^2} + a_{ij}\frac{\tilde{\omega}^i\otimes\tilde{\omega}^j}{\rho^2} + a_{i(2n+1)}\frac{\tilde{\omega}^i\otimes\tilde{\eta}}{\rho^3} + a_{(2n+1)i}\frac{\tilde{\eta}\otimes\tilde{\omega}^i}{\rho^3} + a_{(2n+1)(2n+1)}\frac{(\tilde{\eta})^2}{\rho^4} + a_{0(2n+1)}\frac{d\rho\otimes\tilde{\eta}}{\rho^3} + a_{(2n+1)0}\frac{\tilde{\eta}\otimes d\rho}{\rho^3}$$

   where the matrix $A = \{a_{ij}\}_{i=0}^{2n+1} \in C^{k,\alpha}(\bar{M})$ is a positive definite matrix.

Remark 1.2.13. Actually, the definition of the ACH metric does not depend on the choice of the defining function, the extension of η in the interior and the choice of the $\{\tilde{\omega}^j\}_{j=1}^{2n}$. The following lemma will show this fact.

Lemma 1.2.14. *Let*

$$d\rho_1 = f d\rho + \rho df$$
$$\beta^i = A_0^i d\rho + \sum_{j=1}^{2n} A_j^i \tilde{\omega}^j + A_i^{2n+1} \tilde{\eta}$$
$$\tilde{\eta_1} = A_0^{2n+1} d\rho + \sum_{j=1}^{2n} A_j^{2n+1} \tilde{\omega}^j + f\tilde{\eta}$$

form a new basis of T^*M near the boundary with $A_0^{2n+1}|_{\partial M} = A_j^{2n+1}|_{\partial M} = 0$ and $f \in C^\infty(\bar{M})$ is a positive function with $f = 1$ on ∂M. Then, (M, g_+) is still an ACH manifold for the new defining function $\rho_1 = f\rho$ and the new one-form basis $\{d\rho_1, \beta_j, \tilde{\eta_1}\}_{j=1}^{2n}$

Proof. Since $A^0_{2n+1}|_{\partial M} = A^j_{2n+1}|_{\partial M} = 0$, $ker\tilde{\eta_1}|_{\partial M} = ker\tilde{\eta}|_{\partial M}$. Therefore, item 1) and item 2) of the definition of ACH metric is satisfied for ρ_1 and $\tilde{\eta}_1$.

For the item 3), in fact,

$$|\frac{d\rho_1}{\rho_1}|^2_{g_+^*} = |\frac{d(f\rho)}{f\rho}|^2_{g_+^*} = |\frac{(fd\rho) + (\rho df)}{f\rho}|^2_{g_+^*} = |\frac{d\rho}{\rho}|^2_{g_+^*} + 2 < d\rho, df >_{g_+^*} + \frac{1}{f^2}|df|^2_{g_+^2}$$

By the lemma 1.1, $< d\rho, df >_{g_+^*} |_{\partial M} = \frac{1}{f^2}|df|^2_{g_+^*}|_{\partial M} = 0$ on the boundary. Therefore, the item 3) holds.

Suppose that $\{\partial_{\rho_1}, u_i, \tilde{T}_1\}_{i=1}^{2n}$ is the dual basis of $\{d\rho_1, \beta_j, \tilde{\eta_1}\}_{j=1}^{2n}$. Then, we see

$$\begin{bmatrix} d\rho_1 \\ \beta^i \\ \tilde{\eta_1} \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ A_0^i & A_j^i & A_i^{2n+1} \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} d\rho \\ \tilde{\omega}^j \\ \tilde{\eta} \end{bmatrix} \tag{1.2.2}$$

on the boundary. Therefore,

$$\begin{bmatrix} \partial_{\rho_1} \\ u_i \\ \tilde{T}_1 \end{bmatrix} = \begin{bmatrix} f^{-1} & B_0^j & 0 \\ 0 & B_i^j & 0 \\ 0 & B_{2n+1}^j & f^{-1} \end{bmatrix} \begin{bmatrix} \partial_\rho \\ \tilde{e}_i \\ \tilde{T} \end{bmatrix} \tag{1.2.3}$$

where

$$\begin{bmatrix} f^{-1} & B_0^j & 0 \\ 0 & B_i^j & 0 \\ 0 & B_{2n+1}^j & f^{-1} \end{bmatrix}^T = \begin{bmatrix} f & 0 & 0 \\ A_0^i & A_j^i & A_i^{2n+1} \\ 0 & 0 & f \end{bmatrix}^{-1}$$

Let $A' = \{a'_{ij}\}_{i=0,j=0}^{2n+1}$, where $a'_{00} = \rho^2 g_+(\partial_\rho, \partial_{\rho_1})$, $a'_{0i} = \rho^2 g_+(\partial_{\rho_1}, u_i)$, $a'_{0(2n+1)} = \rho^3 g_+(\partial_{\rho_1}, \tilde{T}_1)$, $a'_{ij} = \rho^2 g_+(u_i, u_j)$, $a'_{i(2n+1)} = \rho^2 g_+(u_i, \tilde{T}_1)$ and $a'_{(2n+1)(2n+1)} = \rho^4 g_+(\tilde{T}_1, \tilde{T}_1)$.

Therefore, A' satisfies the item 4) in the definition of the ACH.

□

□

1.2.5 Asymptotically symmetric manifolds and parabolic geometries

§ 1.3 Microlocal analysis

In this section, we will show the meromorphic continuation of the Laplacian operator for the AH manifolds M^{n+1} which is the key for us to relate the resolvent of the Laplacian operator to the heat operator. The meromorphic continuation theorem relies on some priori knowledge of the spectrum of the Laplacian operator on the AH manifold.

We will first introducee some classic result of the spectrum for the Laplacian operator on the AH manifold. It follows from the H.P.Mckean [38] that the essential spectrum of Laplacian operator on M^{n+1} is $[\frac{n^2}{4}, +\infty)$ and follows from Mazzeo [36] that there is no embedded eigenvalues on the essential spectrum. And by the Mazzeo-Melrose [33], there are at most finite eigenvalues in $(0, \frac{n^2}{4})$, each with finite multiplicity. Therefore, the resolvent $(\Delta_g - \xi(n-\xi))^{-1}$ is holomorphic for $\mathrm{Re}\xi > \frac{n}{2}$ except for a finite number of poles in $(\frac{n}{2}, n)$. The classic work of Mazzeo and Melrose [33] show that this resolvent meromorphically continues to the whole complex plane with some discrete real points. Later, C.Guillarmou [22], improve their result to show that for even AH manifold, the resolvent $(\Delta_g - \xi(n-\xi))^{-1}$ can meromorphically continues to the whold complex plane without exceptional points. Later, Vasy [48][47] use a different way to show a similar meromorphic continuation result which is easier to be generalized into the tensor case [10], since it is independent with the explicit formula for the resolvent of Laplacian on the standard hyperbolic space $\mathbb{H}^{n+1}$.

1.3.1 The analytical Fredholm theorem

In this subsection, we will introduce the analytical Fredholem theorem which just tells us that in order to show the meromorphical continuation of the resolvent it is sufficient to show that (1) the original operator is Fredholm; (2) the original operator is invertible at some point. First, let us start with the definition of the Fredholm operator.

Definition 1.3.1 (Fredholm operator, [14], Definition C.2)**.** (i) A bounded linear operator $P : X_1 \longrightarrow X_2$ is called a Fredholm operator if the kernel of P,

$$\ker P := \{u \in X_1 \mid Pu = 0\}$$

and the cokernel of P,

$$\text{coker}\, P := X_2/PX_1, \text{ where } PX_1 := \{Pu \mid u \in X_1\}$$

are both finite dimensional. Here the cokernel of P is defined algebraically, that is a vector space of cosets, $u + PX_1, u \in X_2$. (ii) The index of a Fredholm operator is $\text{ind}\, P := \dim \ker P - \dim \text{coker}\, P$.

Then, we will define the meromorphic family of a family of operators.

Definition 1.3.2 ([14], Definition C.6)**.** Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If X and Y are Banach spaces then, $z \mapsto B(z) \in \mathcal{L}(X, Y)$* is holomorphic in Ω if for any $x \in X$ and $y^* \in Y^*$ (the dual of Y) , $z \mapsto y^*(B(z)x)$ is a holomorphic function in Ω.

Definition 1.3.3 ([14], Definition C.7)**.** We say that $z \mapsto B(z)$ is a meromorphic family of operators in Ω if for any $z_0 \in \Omega$ there exist operators $B_j, 1 \le j \le J$, of finite rank** and a family of operators $z \mapsto B_0(z)$, holomorphic near z, such that

$$B(z) = B_0(z) + \frac{B_1}{z - z_0} + \cdots \frac{B_J}{(z - z_0)^J}, \qquad \text{near } z_0$$

We say that $B(z)$ is a meromorphic family of Fredholm operators if for every z_0, $B_0(z)$ is a Fredholm operator for z near z_0. For nonsingular z_0, $B_0(z) = B(z)$

The following theorem is key to the A.Vasy method, which relate the Fredholm property to the meromorphic extension of the inverse operator.

Theorem 1.3.4 ([14], Theorem C.8)**.** *Suppose* $\Omega \subseteq \mathbb{C}$ *is a connected open set and* $\{A(z)\}_{z \in \Omega}$ *is a holomorphic family of Fredholm operators. If* $A(z_0)^{-1}$ *exists at some*

* $\mathcal{L}(X, Y)$ is the set of the bounded operators from X to Y.

** A finite rank operator is a bounded operator between two Banach spaces with finite-dimensional range *point $z_0 \in \Omega$, then the family $z \mapsto A(z)^{-1}$, $z \in \Omega$, is a meromorphic family of operators with poles of finite rank.*

1.3.2 The meromorphic continuation theorems of Mazzeo-Melrose and Vasy

In this section, we will first introduce the meromorphic continuation theorms of Mazzeo-Melrose. Then, we will introduce the C.Guillarmou‘s improvement for their result. Finally, we will introduce the Vasy‘version of the meromorphic continuation theorem. In the original version of the results of Mazzeo and Melrose [33], their result is as following

Theorem 1.3.5 ([33], Theorem 7.1)**.** *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold, Δ its Laplacian acting on functions and ρ a boundary defining function on $\bar{M}$. The modified resolvent*

$$R(\xi) := \left(\Delta - \frac{n^2}{4} - \xi^2\right)^{-1} \in \mathcal{M}er_f\left(\mathcal{O}_0, \mathcal{L}\left(L^2(M)\right)\right)$$

with poles at points $\xi \in \mathcal{O}_0$ such that $(\frac{n^2}{4}+\xi^2) \in \sigma_{pp}(P)$, extends to a finite-meromorphic family

$$R(\xi) \in \mathcal{M}er_f\left(\mathcal{O}_N \backslash \left(Z_+^1 \cup Z_+^2\right), \mathcal{L}\left(\rho^N L^2(M), \rho^{-N} L^2(M)\right)\right), \quad \forall N \geq 0$$

where

$$\mathcal{O}_N := \{\xi \in \mathbb{C}; \Im(\xi) < N\}, \quad Z_\pm^k := \pm\imath\left(\frac{k}{2} + \mathbb{N}_0\right) \subseteq \mathbb{C}$$

Then, C.Guillarmou [22] their result in setting of the even AH manifold. we will first define the even asymptotically hyperbolic manifold. Suppose that $\bar{M}$ is a compact manifold with boundary $\partial\bar{M} \neq \emptyset$ of dimenison $n+1$. We denote M by the interior of $\bar{M}$. The Riemannian manifold (M, g) is even asymptotically hyperbolic if there exists functions $\rho \in C^\infty(\bar{M})$ and $\rho \in C^\infty(\bar{M}, (0, \infty))$, $\rho|_{\partial\bar{M}} = 0$, $d\rho|_{\partial\bar{M}} \neq 0$, such that there exists a diffeomorphism

$$\rho^{-1}([0,1]) \longrightarrow [0,1] \times \partial\bar{M}, \quad p \mapsto (\rho(p), i(p)) \quad \text{where } i(p) \in \partial\bar{M} \tag{1.3.1}$$

where

$$i : M \longrightarrow \partial M$$

is a smooth map. And near $\partial\bar{M}$, the metric has the form

$$g|_{\rho\le 1} = \frac{1}{\rho^2}(d\rho^2 + h(\rho^2)) \tag{1.3.2}$$

where $[0,1] \ni t \mapsto h(t)$, is a smooth family of Riemannian metrics on $\partial\bar{M}$.

Let $\Delta \geq 0$ be the Laplacian-Beltrami operator for the metric g. Since the spectrum is contain in $[0, \infty)$ the operator $\Delta - (\frac{n^2}{4} + \zeta^2)$ is invertible from $H^2(M, dvol_g)$ to $L^2(M, dvol_g)$ for $\mathrm{Im}(\zeta) < -\frac{n}{2}$. Hence we can define

$$R(\zeta) := (\Delta - (\frac{n^2}{4} + \zeta^2))^{-1} : L^2(M, dvol_g) \longrightarrow H^2(M, dvol_g), \quad \mathrm{Im}(\zeta) < -\frac{n}{2}$$

We note that elliptic regularity shows that $R(\zeta) : C_c^\infty(M) \longrightarrow C^\infty(M)$, $\mathrm{Im}(\zeta) < -\frac{n}{2}$.

Theorem 1.3.6 ([33], [22])**.** *Let (M^{n+1}, g_+) be an even asymptotically hyperbolic manifold with Δ its Laplacian acting on functions and ρ a boundary defining function on $\bar{M}$.*

The modified resolvent

$$R(\zeta) := \left(\Delta - \frac{n^2}{4} - \zeta^2\right)^{-1} \in \mathcal{M}er_f\left(\mathcal{O}_0, \mathcal{L}\left(L^2(M)\right)\right)$$

with poles at points $\xi \in \mathcal{O}_0$ *such that* $(\frac{n^2}{4}+\zeta^2) \in \sigma_{pp}(P)$*, extends to a finite-meromorphic family*

$$R(\zeta) \in \mathcal{M}er_f\left(\mathcal{O}_N, \mathcal{L}\left(\rho^N L^2(M), \rho^{-N} L^2(M)\right)\right), \quad \forall N \geq 0$$

where

$$\mathcal{O}_N := \{\zeta \in \mathbb{C}; \mathrm{Im}(\zeta) < N\}$$

Actually, Vasy‘version of the above result is a little bit weaker, it only shows the following result.

Theorem 1.3.7. *Let* $\left(M^{n+1}, g_+\right)$ *be even asymptotically hyperbolic manifold. Then the inverse of*

$$P(\zeta) := \Delta - \frac{n^2}{4} - \zeta^2 \text{ acting on } L^2(M)$$

written $R(\zeta)$ *has a meromorphic continuation from* $\mathrm{Im}(\xi) \ll -\frac{n}{2}$ *to* $\mathbb{C}$*,*

$$R(\zeta) : C_c^\infty(M) \to \rho^{\imath\zeta+\frac{n}{2}} C_{even}^\infty\left(\bar{M}\right)$$

with finite rank poles.

we can show this result can be induced from theorem 1.3.6.

1.3.3 Vasy‘s approach

In this section, we will present the main ideal of the theorem 1.3.7 by the method of Vasy [48] [47]. First, we will make use of the eveness to convert the noncompact problem into a compact problem by defining new manifold and new operator. Then, the meromorphic continuation of the orginal operator is equivalent to the meromorphic continuation of the new operator. Next, we will show the new operator is Fredholm operator in some space and it is invertible at some point. Then, by the analytical Fredholm theorem, the resolvent of this new operator can be meromorphic continue to the whole complex plane.

We will first introduce some basic result of the spectrum of the Laplacian operator for tensor which we will use in the proof of the meromorphic continuation theorem for the spectrum of the tensor case. For function case and the differential form case, we have the result of the [36]. For the symmetric two tensor case, we have the following results

Theorem 1.3.8 ([15] and [31])**.** *On an n+1-dimensional asymptotically hyperbolic manifold with $n > 1$, the essential spectrum of the Lichnerowicz Laplacian acting on trace free symmetric covariant two tensors is the ray*

$$\left[\frac{n(n-8)}{4}, +\infty\right]$$

For the hyperbolic space, this is the spectrum.

Theorem 1.3.9 ([16])**.** *For $n \geq 1$, let us consider $(N, \widehat{g})$ an $n+1$-dimensional compact Einstein manifold. Let $M = (0, +\infty) \times N$ equipped with an asymptotically hyperbolic metric $g = dr^2 + f^2(r)\widehat{g}$. Then there are no L^2 TT-eigentensors of the Lichnerowicz Laplacian Δ_L with eigenvalue embedded in the essential spectrum. For the real hyperbolic space, there are no L^2 eigentensors of Δ_L*

Remark 1.3.10. TT-eigentensors refers to trace free and divergent free symmetric two tensor.

Next, we will define the new operator.

Let M be an even asymptotically hyperbolic manifold with defining function ρ and $\bar{U} \subseteq \partial\bar{M}$ be an open subset on $\partial\bar{M}$. Let

$$\varphi : \bar{U} \to \mathbb{R}^n, \quad \hat{p} \mapsto (\theta^1(\hat{p}), \cdots, \theta^n(\hat{p}))$$

be a local coordinate of the open set $\bar{U}$. Then, by the diffeomrophism of (1.3.1), we have a boundary coordinate of $\bar{M}$ as

$$\rho^{-1}([0,1]) \cap i^{-1}(\bar{U}) \to [0,1] \times \mathbb{R}^n, \quad p \mapsto (\rho(p), \theta^1(i(p)), \theta^n(i(p))) \tag{1.3.3}$$

Moreover, the asymptotically hyperbolic metric is

$$g|_{\rho \geq 1} = \frac{1}{\rho^2}(d\rho^2 + h_{ij}(\rho^2, \theta)d\theta^i d\theta^j)$$

where $h_{ij}(\rho^2, \theta)$ is a family of the metric on $\partial\bar{M}$. Then, in this coordinate, the Laplacian operator Δ can be written as

$$\Delta = -\rho^2\partial_\rho^2 + (n-1)\rho\partial_\rho - \frac{1}{2}\rho^2(\partial_\rho(\gamma(\rho^2, \theta)) \cdot \partial_\rho + \rho^2\Delta_h$$

where $\gamma(\rho^2, \theta) = \ln(\sqrt{\det(h(\rho^2, \theta))})$.

In the following section, we will show that the unique L^2 solutions to the equation

$$(\Delta - \frac{n^2}{4} - \zeta^2) = f \in C_c^\infty(M), \quad \text{Im}(\zeta) < -\frac{n}{2}$$

satisfies

$$u = \rho^{\iota\zeta+\frac{n}{2}} C^\infty(\bar{M}) \quad \text{and} \quad \rho^{-\iota\zeta-\frac{n}{2}} u|_{\rho<1} = F(\rho^2, \theta), \quad F \in C^\infty([0,1] \times \partial\bar{M}) \tag{1.3.4}$$

Eventually we will show that the meromorphc continuation of the resolvent provides solutions of this form for all $\zeta \in \mathbb{C}$ except the poles.

This suggests two things:

- To reduce the investigation to the study of smooth solutions we should conjugate $(\Delta - \frac{n^2}{4} - \zeta^2)$ by the weight $\rho^{\iota\zeta}$.
- The desired smoothness properties should be stronger in the sense that the functions should be smooth in (ρ^2, θ)

Let $P(\zeta) = \Delta - \frac{n^2}{4} - \zeta^2$. Then, by direct computation, we have that

$$\begin{aligned} P_2(\zeta) := \rho^{-\iota\zeta-\frac{n}{2}} P(\zeta) \rho^{\iota\zeta+\frac{n}{2}} =& -\rho^2\partial_\rho^2 + (n-1)\rho\partial_\rho - \frac{1}{2}\rho^2 h^{ij}(\partial_\rho\gamma)\partial_\rho + \rho^2\Delta_h \\ & - 2(\iota\zeta + \frac{n}{2})\rho\partial_\rho f - \frac{1}{2}(\iota\zeta + \frac{n}{2})\rho(\partial_\rho\gamma) \end{aligned}$$

where $\gamma = \partial_\rho(\ln(\sqrt{\det(h)}))$.

Now, by the (1.3.4), let $y = \rho^2$. Then, by direct computation, we have that

$$P_2(\zeta) = y(4yD_y^2 - 4(\iota - \zeta)D_y - \iota\gamma(y)(\zeta - \iota\frac{n}{2} + 2yD_y) + \Delta_h)$$

where $\gamma(y) = \partial_y(\ln(\sqrt{\det(h)}))$ and $D_y = -\iota\partial_y$. Now, let

$$P_1(\zeta) := 4yD_y^2 - 4(\iota - \zeta)D_y - \iota\gamma(y)(\zeta - \iota\frac{n}{2} + 2yD_y) + \Delta_h \tag{1.3.5}$$

The relation of the three operators $P(\zeta)$, $P_1(\zeta)$ and $P_2(\zeta)$ is

$$\begin{aligned} P_1(\zeta) =&\rho^{-\iota\zeta-\frac{n}{2}-2}P(\zeta)\rho^{\iota\zeta+\frac{n}{2}} = y^{-\iota\frac{\zeta}{2}-\frac{n}{4}-1}P(\zeta)y^{\iota\frac{\zeta}{2}+\frac{n}{4}} \\ P_2(\zeta) =&\rho^{-\iota\zeta-\frac{n}{2}}P(\zeta)\rho^{\iota\zeta+\frac{n}{2}} = y^{-\iota\frac{\zeta}{2}-\frac{n}{4}}P(\zeta)y^{\iota\frac{\zeta}{2}+\frac{n}{4}} \end{aligned} \tag{1.3.6}$$

To define the operator $P_1(\zeta)$ geometrically we introduce a new manifold using coordinate of the (1.3.3), $(\rho, \theta^1, \cdots, \theta^n)$. Let $y = \rho^2$ and consider the following manifold

$$X := ([-1, 1]_y \times \partial M) \sqcup (M \backslash \rho^{-1}((0, 1)))$$

Then, we can extend $y \mapsto h(y)_{ij}$ to a family of smooth non-degenerate metric on $[-1, 1]_y$ which provide a natural extension of $\gamma(y, \theta) = \partial_\rho(\ln(\sqrt{\det(h)}))$ and the operator Δ_h. We notice that the asymptotically hyperbolic space (M, g) is diffeomorphic to $X_1 := X \cap \{y > 0\}$. However, $\bar{X}_1$ and $\bar{M}$ have different smooth structure. They have different boundary coordinate which are not diffeomorphic. In fact, the boundary coordinate for $\bar{X}_1$ is

$$(y, \theta^1, \cdots, \theta^n) \tag{1.3.7}$$

while the boundary coordinate for $\bar{M}$ is

$$(\rho, \theta^1, \cdots, \theta^n). \tag{1.3.8}$$

They are not diffeomorphic, since $y = \rho^2$. The operator $P_1(\zeta)$ defined in the (1.3.5) can be naturally extended into an operator on X. Consider the following the volume form on X

$$d\mu = \sqrt{\det(h)}dy \wedge d\theta^1 \wedge \cdots \wedge d\theta^n$$

Then, we can show that

$$P_1(\zeta)^* = P_1(\bar{\zeta})$$

We can now define spaces on which $P(\zeta)$ is a Fredholm operator. For that, we denote $\bar{H}^s(X^\circ)$ the space of restrictions of elements of H^s on an extension of X across the boundary to the interior of X- See Section B.2 in [26]. In fact, for a smooth compact manifold, X, with boundary, we follow the section B.2 in [26] and define Sobolev spaces of extendible distributions, $\bar{H}^s(\bar{X})$ and of supported distributions $\dot{H}^s(X)$. Here $X = X^\circ \cup \partial X$ and X° is the interior of X. These are modeled on the case of $X = \bar{\mathbb{R}}^n_+$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \ y > 0\}$ in which case

$$\bar{H}^s(\mathbb{R}^n_+) := \{u : \ \exists\, U \in H^s(\mathbb{R}^n),\ u = U|_{y>0}\} \tag{1.3.9}$$

$$\dot{H}^s(\bar{\mathbb{R}}^n_+) := \{u \in H^s(\mathbb{R}^n) : \ \mathrm{supp}(u) \subseteq \bar{\mathbb{R}}^n_+\} \tag{1.3.10}$$

Then, put

$$\mathscr{Y}^s := \bar{H}^s\,(X)\,, \quad \mathscr{X}^s := \left\{u \in \mathscr{Y}^{s+1} : P_1(0)u \in \mathscr{Y}^s\right\}.$$

The norm on $\mathscr{X}^s$ is defined as the graph norm, that is

$$\|u\|_G := \|u\|_{\bar{H}^{s+1}(X)} + \|P_1(0)u\|_{\bar{H}^s(X)}$$

Since the dependence on ζ in $P(\zeta)$ occurs only in lower order terms we can replace $P(0)$ by $P(\zeta)$ in the definition of $\mathscr{X}$. Then, we will show that Then, we can show that

Then, we can show that

Theorem 1.3.11 (Theorem 2, [50])**.** *For* $-\mathrm{Im}(\zeta) > -s - \frac{1}{2}$ *the operator*

$$P_1(\zeta) : (\mathcal{X}^s, \|.\|_G) \to (\mathcal{Y}^s, \|.\|_{\bar{H}^s(X)})$$

has the Fredholm property, that is

$$\dim\{u \in \mathcal{X}^s : P(\zeta)u = 0\} < \infty, \quad \dim(\mathcal{Y}^s \backslash P(\zeta)\mathcal{X}^s) < \infty$$

and $P_1(\zeta)\mathcal{X}^s$ *is closed.*

Then, we can show that

Theorem 1.3.12 (Theorem 3, [50])**.** *For* $\mathrm{Im}(\zeta) < 0$, $\zeta^2 + (\frac{n}{2})^2) \notin \mathrm{Spec}(\Delta_g)$ *and* $s > \mathrm{Im}(\zeta) - \frac{1}{2}$,

$$P_1(\zeta) : (\mathcal{X}^s, \|.\|_G) \to (\mathcal{Y}^s, \|.\|_{\bar{H}^s(X)})$$

is invertible. Hence, for $s \in \mathbb{R}$ *and* $-\mathrm{Im}(\zeta) > -s - \frac{1}{2}$,

$$\zeta \mapsto P_1(\zeta)^{-1} : (\mathcal{Y}^s, \|.\|_{\bar{H}^s(X)}) \to (\mathcal{X}^s, \|.\|_G)$$

is meromorphic family of operators with poles of finite rank.

Now, fix $s \in \mathbb{R}$ and let $-\mathrm{Im}(\zeta) > -s - \frac{1}{2}$. By the above theorem we can define

$$R_1(\zeta) := P_1(\zeta)^{-1} : (\mathcal{Y}^s, \|.\|_{\bar{H}^s(X)}) \to (\mathcal{X}^s, \|.\|_G)$$

which is a meromorphic family of operators with poles of finite rank. Then, we restrict the operator $R_1(\zeta)$ onto $C_c^\infty(X_1)$. We can define

$$R_1'(\zeta) : C_c^\infty(X_1) \to C^\infty(\bar{X}_1), \quad f \mapsto R_1(\zeta)(f)|_{X_1}$$

Now, by the (1.3.6), we see that

$$f = P_1(\zeta)R_1'(\zeta)f = y^{-\iota\frac{\zeta}{2}-\frac{n}{2}-1}P(\zeta)y^{\iota\frac{\zeta}{2}+\frac{n}{2}}R_1(\zeta)f, \quad \text{for any } f \in C_c^\infty(X_1)$$

which implies that

$$y^{\iota\frac{\zeta}{2}+\frac{n}{2}+1}f = P(\zeta)y^{\iota\frac{\zeta}{2}+\frac{n}{2}}R_1(\zeta)y^{-\iota\frac{\zeta}{2}-\frac{n}{2}-1}y^{\iota\frac{\zeta}{2}+\frac{n}{2}+1}f, \quad \text{for any } f \in C_c^\infty(X_1)$$

Therefore, we define

$$R(\zeta) := y^{\iota\frac{\zeta}{2}+\frac{n}{2}}R_1(\zeta)y^{-\iota\frac{\zeta}{2}-\frac{n}{2}-1} : C_c^\infty(X_1) \to y^{\iota\frac{\zeta}{2}+\frac{n}{2}}C^\infty(\bar{X}_1)$$

The Theorem 1.3.7 follows.

§ 1.4 Linear and non-linear functional analysis

In this section, we will review the spectrum theory for the self-adjoint operator.

1.4.1 From the resolvent to the heat operator

In this section we will introuce the spectrum theorem which can relate the heat operator to the resolvent of the Laplacian operator. Therefore, we can relate the parabolic problem to the elliptic problem and make use of the estimate of Schwartz kernel of the resolvent to estimate the heat kernel. Once we have the estimate for the heat kernel, we can use it to show the stability of the solution to this heat equation.

Theorem 1.4.1 (Spectrum Theorem, Theorem A.14, [7]). *For A an self-adjoint operator on a separable Hilbert space $\mathcal{H}$, there exists a measure space (Ω, μ), where Ω is a union of copies of $\mathbb{R}$, and a unitary map $W : L^2(\Omega, d\mu) \to \mathcal{H}$, and a real-valued measurable* *function a on Ω such that*

$$W^{-1}AWf(x) = a(x)f(x)$$

for $f \in W^{-1}\mathcal{D}(A)$, which is equivalent to the condition that $af \in L^2(\Omega, d\mu)$

Remark 1.4.2. The spectral theorem gives us a functional calculus for operators: given a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$, we can define

$$h(A) := Wh(a(x))W^{-1}$$

This functional calculus admits an explicit formulation in terms of the resolvent.

Theorem 1.4.3 (Resolvent Functional Calculus, Corollary A.15 [7])**.** *If A is a self-adjoint operator on $\mathcal{H}$, then for $h : \mathbb{R} \to \mathbb{C}$ bounded and continuous,*

$$h(A) = \int_{-\infty}^{\infty} h(\lambda) d\Pi(\lambda)$$

where dII is the operator-valued measure on $\mathbb{R}$ given by

$$d\Pi(\lambda) := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} \right] d\lambda$$

with the limit taken in the operator topology.

Definition 1.4.4. Define the projection operator $P(S) = W\chi_S(a(x))W^{-1}$ where χ_S is a characteristic function of $S \subseteq \mathbb{R}$.

Theorem 1.4.5 (Stone's Formula, Corollary A.16, [7])**.** *The spectral projectors associated with a self-adjoint operator A are expressed in terms of the resolvent by,*

$$\frac{1}{2} \left(P_{[\alpha,\beta]} + P_{(\alpha,\beta)} \right) = \int_{\alpha}^{\beta} d\Pi(\lambda)$$

where $d\Pi$ is the operator valued measure in corollary 1.4.3.

Corollary 1.4.6. *Let A be a self-adjoint operator on $\mathcal{H}$. And let $h : \mathbb{R} \to \mathbb{C}$ bounded and continuous. Then*

$$\frac{1}{2}\left(\chi_{[a,b]}(A) \circ h(A) + \chi_{(a,b)}(A) \circ h(A)\right) = \int_a^b h(\lambda) d\Pi(\lambda)$$

where dII is the operator-valued measure on $\mathbb{R}$ given by

$$d\Pi(\lambda) := \frac{1}{2\pi i} \lim_{\varepsilon\to 0} \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] d\lambda$$

with the limit taken in the operator topology.

Now, Let (M^{n+1}, g) be an asymptotically hyperbolic manifold. If A is a operator on $\mathrm{Sym}^2(T^*M)$ with all its spectrum in $[\frac{n^2}{4}, \infty)$ and no eigenvalue at $\frac{n^2}{4}$. Then, consider the operator $B = A - \frac{n^2}{4}$. And take $h(x) = e^{-tx}$. Then consider the following operator

$$h(B) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} h(\lambda) d\Pi(\lambda) = \lim_{\varepsilon\to 0} \int_0^{\infty} e^{-t\lambda} d\Pi(\lambda)$$

where

$$d\Pi(\lambda) := \frac{1}{2\pi i} \lim_{\varepsilon\to 0} \left[(B-\lambda-i\varepsilon)^{-1} - (B-\lambda+i\varepsilon)^{-1}\right] d\lambda$$

Let $\lambda = s^2$ and $s = a + bi$

$$d\Pi\left(a^2\right) = d\Pi\left(\mathrm{Re}\left(s^2\right)\right) = \frac{1}{2\pi i} \lim_{b\to 0} \left[\left(B - s^2\right)^{-1} - \left(B - \bar{s}^2\right)^{-1}\right] 2a da$$

$$d\Pi\left(a^2\right) = \frac{1}{2\pi i} \lim_{b\to 0} \left[R(s) - R(\bar{s})\right] 2a da = \frac{1}{2\pi i} \lim_{b\to 0} \left[R(a+bi) - R(a-bi)\right] 2a da$$

where $R(s) = (B - s^2)^{-1}$. Therefore,

$$\begin{aligned}
h(B) &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} h(\lambda)d\Pi(\lambda) = \lim_{\varepsilon\to 0}\int_{0}^{\infty} e^{-t\lambda}d\Pi(\lambda)\\
&= \lim_{b\to 0^+}\int_0^\infty e^{-ta^2}d\Pi(a^2)\\
&= \lim_{b\to 0^+}\frac{1}{2\pi i}\int_0^\infty e^{-ta^2}\left[R(a+bi)-R(a-bi)\right]2ada\\
&= \lim_{b\to 0^+}\frac{1}{2\pi i}\int_0^\infty e^{-ta^2}\left[R(a+bi)-R(a-bi)\right]2ada\\
&= \lim_{b\to 0^+}-\frac{1}{2\pi i}\int_{-\infty}^\infty e^{-ta^2}\left[R(a-bi)\right]2ada
\end{aligned}$$

Therefore,

$$\begin{aligned}
h(A) =& h(B+\frac{n^2}{4}) = \lim_{\varepsilon\to 0}\int_0^\infty e^{-t(\lambda+\frac{n^2}{4})}d\Pi(\lambda) = \lim_{\varepsilon\to 0} e^{-t\frac{n^2}{4}}\int_0^\infty e^{-t\lambda}d\Pi(\lambda) = e^{-t\frac{n^2}{4}}h(B)\\
=& e^{\frac{-n^2}{4}t}\lim_{b\to 0^+}-\frac{1}{2\pi i}\int_{-\infty}^\infty e^{-ta^2}\left[R(a-bi)\right]2ada
\end{aligned}$$

Moreover, we can show that

$$\frac{d}{dt}h(A)(u) = Ah(A)u$$

It is sufficient to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}W(e^{-(t+\varepsilon)a(x)}+a(x)e^{-ta(x)})W^{-1}u \to 0 \quad \text{in the sense of } L^2(\Omega, d\mu)$$

where $x\in\Omega$. In fact,

$$\|\frac{1}{\varepsilon}W(e^{-(t+\varepsilon)a(x)}+a(x)e^{-ta(x)})W^{-1}u\| \le |\frac{1}{\varepsilon(e^{-(t+\varepsilon)a(x)}-a(x)e^{ta(x)})}|\cdot\|W^{-1}u\|_{L^2(\Omega,d\mu)}$$

It is $L^2(\Omega, d\mu)$ convergent.

1.4.2 The semi-group theory

In this section, we will introduce some basic concept of the semigroup and its generator, which is another way to relate the parabolic problem and the elliptic problem which is not as explicit as the previous section. For more detail, refers to [31].

Definition 1.4.7 (Semigroup)**.** $S(t)$ is called the semi-group if it satisfies that

- $\{S(t)\}_{t\geq 0}$ is a family of bounded linear mapping from the Banach space X to X
- $S(0) = Id_X$
- $S(t+s) = S(t)S(s) = S(s)S(t)$
- $t \mapsto S(t)u$ is continuous from $[0, \infty)$ to X

Definition 1.4.8 (Generator of semigroup)**.** Write

$$D(A) := \left\{ u \in X | \lim_{t\to 0+} \frac{S(t)u - u}{t} \text{ exists in } X \right\}$$

and

$$Au := \lim_{t\to 0+} \frac{S(t)u - u}{t} \quad (u \in D(A))$$

We call $A : D(A) \to X$ the (infinitesimal) generator of the semigroup $\{S(t)\}_{t\geq 0}$; $D(A)$ is the domain of A.

There are some basic properties about the semigroup and its generator.

Theorem 1.4.9. *Assume* $u \in D(A)$*. Then*

1) $S(t)u \in D(A)$ *for each* $t \geq 0$.

2) $AS(t)u = S(t)Au$ *for each* $t \geq 0$.

3) The mapping $t \mapsto S(t)u$ is differentiable for each $t > 0$.

4) $\frac{d}{dt}S(t)u = AS(t)u$ $(t > 0)$.

Proof : See 7.4.1 Theorem 1 in [17]. □

Definition 1.4.10 (Resolvent set)**.** We say a real number λ belongs to $\rho(A)$, the resolvent set of A, provided the operator

$$\lambda I - A :\to X$$

is on to one and onto. And if $\lambda \in \rho(A)$, the resolvent operator $R_\lambda : X \to X$ is defined by $R_\lambda u := (\lambda I - A)^{-1}u$

Remark 1.4.11. According to the Closed Graph Theorem, $R_\lambda : X \to D(A) \subseteq X$ is bounded linear operator.

Theorem 1.4.12 (Hille-Yosida-Phillips)**.** *Let A be a closed, densely defined linear operator on X. Then A is the generator of a semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$(c, \infty) \subseteq \rho(A) \quad \text{and} \quad ||R_\lambda|| \leq \frac{1}{\lambda - c} \text{ for } \lambda > 0$$

Moreover, we have $||S(t)|| \leq e^{-ct}$

Proof : See 7.4.2 Theorem 4 in [17]. □

Now, let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold and take

$$X = C^{0,\alpha}_\delta(Sym^2T^*M^{n+1})$$

with $\delta \in (0, n)$ and trivial L^2 kernel of P on $Sym^2T^*M^{n+1}$. By the lemma 3.7 of [31], the $P = \Delta_L + 2nId$ is an isomorphism from $C^{2,\alpha}_\delta$ to $C^{0,\alpha}_\delta$. Then we have

$$||Pu||_{C^{0,\alpha}_\delta} \geq c||u||_{C^{0,\alpha}_\delta}$$

where $c > 0$. And for $c \geq -\lambda$, we have

$$||Pu + \lambda u||_{C^{0,\alpha}_\delta} \geq (\lambda + c)||u||_{C^{0,\alpha}_\delta}$$

Therefore,

$$(-c, \infty) \subseteq \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda + c} \text{ for } \lambda > 0$$

Therefore, P is a generator of a semigroup $S(t)$ with $|S(t)| \leq e^{-ct}$

1.4.3 The Newton's method for non-linear equations

The next theorem is a way to convert the existence of the solution to the nonlinear problem into the invertibility of a linear operator.

Theorem 1.4.13 ([4], Lemma I.4.13)**.** *LEMMA I.4.13. Let $\Phi : E \longrightarrow F$ be aC^2 mapping between Banach spaces, such that $\Phi(0) = 0$ and $d_0\Phi$ is invertible. If ζ and ε are chosen such that*

$$\zeta \left\|(d_0\Phi)^{-1}\right\| \sup_{B_\zeta} \left\|d^2\Phi\right\| < \frac{1}{2}$$
$$\varepsilon \left\|(d_0\Phi)^{-1}\right\| < \frac{\zeta}{2}$$
$$(4.9)$$

then for $y \in F$ satisfying $\|y \,|< \varepsilon$, the equation $\Phi(x) = y$ has a unique solution with $\|x\| < \zeta$.

Chapter 2

The existences of the asymptotically symmetric Einstein metrics

In this chapter, we shall introduce two methods to show the existence of the AH Einstein manifolds with a given conformal boundary which is sufficiently close to a conformal boundary of an AH Einstein manifold. There are two ways to deal with this problem. One is the elliptic way and another is the parabolic way. Furthermore, we shall introduce the O.Biquard‘s work [4] about the existence of the Asymptotically symmetric Einstein metric.

For the elliptic way, it is developed by the J. Lee and R. Graham [30] and [21]. In [21], C.Graham and J.Lee show that every conformal structure on $\mathbb{S}^n$ sufficiently close to that of the round metric is the conformal infinity of an Einstein metric close to the hyperbolic metric. Later, J.Lee generalize this result into nondegenerate asymptotically hyperbolic Einstein manifold case ([Theorem A] [30]). The idea is to show that the existence of the corresponding Einstein equation. The Einstein equation is a non-linear equation. By the newton‘s method, the existence of the solution for a non-linear equation is pretty much equivalent to construct the a initial metric such that the initial metric has the required conformal infinity, is closed to the Einstein metric and the linearization of the Einstein equation at this initial metric is invertible in the weighted space $C^{2,\alpha}_{\delta}$ for $\delta > 2$. (The reason for $C^{2,\alpha}_{\delta}$ is that we hope the solution has an infinite-order asymptotic expansion in powers of ρ and $\log \rho$ and in fact it is smooth when n is odd. See [9]).

It is straightforward to verify that the linearization of the Einstein equation is not a strictly elliptic operator. In order to make use of the knowledge about elliptic operators, we need to consider the gauge Einstein equation whose linearization is the so called Lichnerowicz operator plus a translation which is strictly elliptic. This consideration is reasonable, since the existence of the solution for gauge Einstein equation is equivalent to the existence of the solution for the Einstein equation in our settings.

Furthermore, by the Fredholm theorem, we can show that the invertibility of the translated Lichnorwicz operator on the weighted space $C^{k,\alpha}_{\delta}$ for $|\delta - n/2| < R$ is invertible if and only if this operator has a trivial L^2 kernel. This indicate that for the invertibility, we only need to care about the L^2 space.

The idea to construct the required initial metric is to glue the boundary part and interior part together. For the boundary part, we need to make use of the result of C.Fefferman and C.R.Graham about the expansions of the asymptotically hyperbolic metric at boundary ([20]) which can ensure that the boundary part metric is close to the given asymptotically hyperbolic Einstein metric in C^2 and close to Einstein metric ($\|Ric - (n-1)g + \text{gauge term}\|_{C^{2,\alpha}_{\delta}}$ small enough) in $C^{2,\alpha}_{\delta}$. For the interior part, we directly use the interior of the given asymptotically hyperbolic metric. Glue the interior and boundary parts together by an cut-off function. Then, we can show this metric satisfies the requirement of the Newton‘s method.

For the proof of the Fredholm theorem, there are two ways to prove it. One is from J.

Lee [30] another is from O.Biquard [4]. Both of them first show the invertibility of the translated operator is actually invertibility on standard hyperbolic space and then, use the Mobius coordinates to show the Fredholm theorem on the asymptotically hyperbolic manifold. Their difference is the way they show the invertibility on hyperbolic space. For the J.Lee‘s method, he first make use of the N.Koiso‘s eigenvalue estimate to show the existence of the Green function. Then, he make use of the result of regular-singular equation to find the decay of the Green function. Then, by an interesting inequality (Lemma 5.4 [30]), he get the invertiblity of the Laplacian operator on weighted space. For the method of O.Biquard, he make use of the expression of the Laplacian operator on the spherical coordinate to first convert the equation satisfied by the Green function into sort of ordinary equation. Then, he make use of the result about ordinary equation to show the decay of the Green function. Then, he can construct a translated scalar Laplacian operator such that Green function for the translated scalar Laplacian operator has the same exponential decay rate with the tensor Laplacian operator, which make it is possible to use the scalar Green function to control the tensor Green function. Therefore, we only need to show that the invertiblity of the translated scalar Laplacian operator, which depends on the L^2 eigenvalue estimate (Lemma I.2.3 [4]). By the method O.Biquard, we can skip the the inequality [4], which seems to make this way more easier to apply on the the general Symmetric space.

For the parabolic method, it is originally from the J.Qing, Y.Shi and J.Wu [42]. They recover the existence result [31] (Theorem A) for dimension $n \geq 5$. Basically, they make use of the fact that the if the normalized Ricci flow converge, then the limit metric should be an Einstein metric. Therefore, the key is to show the long time existence, convergence of the normalized Ricci flow and the flow does not change the conformal infinity. See more detail in the section

This chapter is organized as following. In the section 2.1, we will first introduce the definition of Mobius coordinates, weighted spaces. Then, we will show the invertibility of the Lichnorwicz operator on the standard hyperbolic space. Next, we will make use of the invertibility of the Lichnorwicz operator on the standard hyperbolic space together with the Mobius coordinates to induce the Fredholm theorem. In the section 2.2, we will first introduce the gauge Einstein equation and then make use of the Newton's method to get the existence of the conformally compact Einstein with given boudnary. In the section 2.3, we will introduce the O.Biquard's work [4] on the Asymptotically symmetric metrics where he show the existence for the asymptotically symmetric Einstein metrics of rank 1 cases.

§ 2.1 The analysis on AH manifolds and AH Einstein metrics

In this section, we will first introduce the definition of Mobius coordinates, weighted spaces. Then, we will show the invertibility of the Lichnorwicz operator on the standard hyperbolic space. Next, we will make use of the invertibility of the Lichnorwicz operator on the standard hyperbolic space together with the Mobius coordinates to induce the Fredholm theorem.

2.1.1 Mobius coordinates

In this section, we will introduce the asymptotically hyperbolic manifolds and Mobius chart. In the Mobius chart of asymptotically hyperbolic manifolds, the metric can be uniformly bounded (See Lemma 2.1.5) and approaching the standard hyperbolic metric as approaching the boundary. Therefore, we can get a pretty good globally elliptic and parabolic estimate. Most content of this section is from [30].

In order to define the asymptotically hyperbolic manifolds, we need to first introduce the conformally compact manifold. Defining function is the key in these concepts.

Definition 2.1.1 (Defining function)**.** Let $\bar{M}$ be a smooth, compact, $(n+1)$ -dimensional manifold-with-boundary, $n \geq 1$, and M its interior. A defining function will mean a function $\rho : \bar{M} \to \mathbb{R}$ of class at least C^1 that is positive in M, vanishes on ∂M, and has nonvanishing differential everywhere on ∂M.

Definition 2.1.2 (Conformal compactness)**.** A Riemannian metric g on M is said to be conformally compact of class $C^{l,\beta}$ for a nonnegative integer l and $0 \leq \beta < 1$ if for any smooth defining function ρ, the conformally rescaled metric $\rho^2 g$ has a $C^{l,\beta}$ extension, denoted by $\bar{g}$, to a positive definite tensor field on $\bar{M}$.

Remark 2.1.3. For such a metric g, the induced boundary metric $\widehat{g} := \bar{g}|_{T\partial M}$ is a $C^{l,\beta}$ Riemannian metric on ∂M whose conformal class $[\widehat{g}]$ is independent of the choice of smooth defining function ρ; this conformal class is called the **conformal infinity** of g.

Definition 2.1.4 (Asymptotically hyperbolic manifolds)**.** If g is conformally compact of class $C^{l,\beta}$ with $l \geq 2$, and $|d\rho|^2_{\bar{g}} = 1$ on ∂M, we say g is **asymptotically hyperbolic of class** $C^{l,\beta}$ and the corresponding manifold is called **asymptotically hyperbolic manifold**.

We begin by choosing a covering of a neighborhood of ∂M in $\bar{M}$ by finitely many smooth coordinate charts (Ω, Θ), where each coordinate map Θ is of the form $\Theta = (\theta, \rho) = \left(\theta^1, \ldots, \theta^n, \rho\right)$ and extends to a neighborhood of $\bar{\Omega}$ in $\bar{M}$. Throughout this monograph, we will use the Einstein summation convention, with Roman indices $i, j, k, \ldots$ running from 1 to $n+1$ and Greek indices $\alpha, \beta, \gamma, \ldots$ running from 1 to n. Therefore, we can write $\left(\theta^1, \ldots, \theta^n, \rho\right)$ as θ^i if we think of ρ as θ^{n+1}.

We fix once and for all finitely many such charts covering a neighborhood W of ∂M in $\bar{M}$. We will call any of these charts "**background coordinates**" for $\bar{M}$. Take a local background coordinate (θ, ρ). Define $H_c(p)$ as the following set

$$Z_c(p) \triangleq \{(\theta, \rho) : |\theta - \theta(p)| < c, 0 \leq \rho < c\}$$

And define the set A_c as following

$$A_c \triangleq \{p \in W : \exists\ backgroud\ local\ coordinate\ chart\ (U, \theta^i)\ such\ that\ Z_c(p) \subseteq U\}$$

We see that for $c_1 \leq c_2$, we have $A_{c_2} \subseteq A_{c_1}$. And by the compactness $\bar{M}$, there exist c_0 such that A_{c_0} forms a neighborhood of ∂M. Now, we will define the Mobius charts based on these background coordinates and the standard coordinate of hyperbolic space.

In the upper half-space model, we regard hyperbolic space as the open upper half-space

$$\mathbb{H} = \mathbb{H}^{n+1} \triangleq \{(x^1, \cdots, x^n, y) \subseteq \mathbb{R}^{n+1} : y > 0\}$$

endowed with the hyperbolic metric $\breve{g} = y^{-2} \sum_i \left(dx^i\right)^2$.

For any $r > 0$, we let $B_r \subseteq \mathbb{H}$ denote the hyperbolic geodesic ball of radius r about the point$(x, y) = (0, 1)$

$$B_r = \left\{(x, y) \in \mathbb{H} : d_{\breve{g}}((x, y), (0, 1)) < r\right\}$$

Then

$$B_r \subseteq \{(x, y) : |x| < \sinh r, e^{-r} < y < e^r\}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$.

If p_0 is any point in $A_{c_0/8}$, choose such a background chart containing p_0, and $\{(\theta, \rho) : |\theta - \theta(p_0)| \leq c_0, 0 < \rho < c_0\}$ and define a map $\Phi_{p_0} : B_2 \to M$, called a **Möbius chart** centered at p_0, by

$$(\theta, \rho) = \Phi_{p_0}(x, y) = (\theta_0 + \rho_0 x, \rho_0 y)$$

where (θ_0, ρ_0) are the background coordinates of p_0. Therefore, we see that

$$|\theta - \theta_0| \leq \rho_0 x \leq \rho_0 \sinh(2) \leq 4\rho_0 \qquad \rho \leq \rho_0 e^2 \leq 8\rho_0$$

Since $p_0 \in A_{c_0/8}$, $\rho_0 \leq c_0/8$. Therefore,

$$\Phi(B_2) \subseteq \{(\theta, \rho) : |\theta - \theta(p)| \leq c_0, 0 < \rho < c_0\}$$

is still contained in the same background local coordinate.

We also choose finitely many smooth coordinate charts $\Phi_i : B_2 \to M$ such that the sets $\{\Phi_i (B_2)\}$ cover a neighborhood of $M \backslash A_{c_0/8}$,. For consistency, we will also call these "**Mobius charts**." Therefore, we have a Mobius charts covering

$$\{\Phi_i(B_2), \Phi_i\}_{i=1}^N \cup \{\Phi_{p_0}(B_2), \Phi_{p_0}\}_{p_0 \in A_{c_0/8}}$$

For simplicity, we just write is as

$$\{\Phi_{p_i}(B_2), \Phi_{p_i}\}_{p_i \in M}$$

where $\Phi_{p_i}(\mathbf{0}, 1) = p_i$.

The following lemma shows the uniformly bounded of the Mobius coordinate.

Lemma 2.1.5 (Lemma 2.1 [30])**.** *There exists a constant $C > 0$ such that if $\Phi_{p_0} : B_2 \to M$ is any y, Möbius chart,*

$$\left\|\Phi_{p_0}^* g - \breve{g}\right\|_{C^{l,\beta}(B_2)} \leq C$$
$$\sup_{B_2} \left|\left(\Phi_{p_0}^* g\right)^{-1} \breve{g}\right| \leq C$$

(The Hölder and sup norms in this estimate are the usual norms applied to the components of a tensor in coordinates; since $\bar{B}_2$ is compact, these are equivalent to the intrinsic Hölder and sup norms on tensors with respect to the hyperbolic metric.

2.1.2 Weighted spaces

In this section, we will define the weighted Holder space on the asymptotically hyperbolic manifolds by the Mobius coordinate. Most of the content of this section is from [31].

Throughout this section, we assume $\bar{M}$ is a connected smooth $(n+1)$ -manifold, g is a metric on M that is asymptotically hyperbolic of class $C^{l,\beta}$, with $l \geq 2$ and $0 \leq \beta < 1$, and ρ is a fixed smooth defining function for ∂M. (It is easy to verify that choosing another smooth defining function will replace the norms we define below by equivalent ones, and will leave the function spaces unchanged.)

A **geometric tensor bundle** over $\bar{M}$ is a subbundle E of some tensor bundle $T^{r_1}_{r_2}\bar{M}$ (tensors of covariant rank r_1 and contravariant rank r_2) associated to a direct summand (not necessarily irreducible) of the standard representation of $\mathrm{O}(n+1)$ (or $\mathrm{SO}(n+1)$ if M is oriented) on tensors of type $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ over $\mathbb{R}^{n+1}$. We will also use the same symbol E to denote the restriction of this bundle to M.

Definition 2.1.6 (Holder space)**.** Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l \geq 2$. Let α be a real number such that $0 \leq \alpha < 1$, and let k be a nonnegative integer such that $k + \alpha \leq l + \beta$. For any tensor field u with locally $C^{k,\alpha}$ coefficients, define the norm $\|u\|_{k,\alpha}$ by

$$\|u\|_{k,\alpha} := \sup_{\Phi} \|\Phi^* u\|_{C^{k,\alpha}(B_2)}$$

where $\|v\|_{C^{k,\alpha}(B_2)}$ is just the usual Euclidean Hölder norm of the components of v on $B_2 \subseteq \mathbb{H}$, and the supremum is over all Möbius charts defined on B_2. Let $C^{k,\alpha}(M;E)$ be the space of sections of E for which this norm is finite. This space is called **Holder space**.

Definition 2.1.7 (Weighted Holder spaces)**.** The **Weighted Hölder spaces** are defined for $\delta \in \mathbb{R}$ by

$$C^{k,\alpha}_{\delta}(M;E) := \rho^{\delta} C^{k,\alpha}(M;E) = \left\{\rho^{\delta} u : u \in C^{k,\alpha}(M;E)\right\}$$

with norms

$$\|u\|_{k,\alpha,\delta} := \left\|\rho^{-\delta}u\right\|_{k,\alpha}$$

Remark 2.1.8. If $U \subseteq M$ is a subset, the restricted norms are denoted by $\|\cdot\|_{k,\alpha,\delta;U}$, and the space $C^{k,\alpha}_{\delta}(U;E)$ are the spaces of sections over U for which these norms are finite.

The following lemma just show that the above Holder norm actually is equivalent to the usual intrinsic C^k norm $\sum_{0\le i\le k}\sup_M |\nabla^i u|$ for $0 \le k \le l$.

Lemma 2.1.9 (Lemma 3.4 [30])**.** *Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l \ge 2$. Let u be a locally integrable section of a tensor bundle E over an open subset $U \subseteq M$ If $0 \le \alpha < 1$ and $0 < k+\alpha \le l+\beta$, $u \in C^{k,\alpha}_{\delta}(U;E)$ if and only if $\rho^{-\delta}\nabla^j u \in C^{0,\alpha}\left(U; E\otimes T^j M\right)$ for $0 \le j \le k$, and the $C^{k,\alpha}_{\delta}$ norm is equivalent to*

$$\sum_{0\le j\le k}\sup_U\left|\rho^{-\delta}\nabla^j u\right| + \left\|\rho^{-\delta}\nabla^k u\right\|_{0,\alpha;U}$$

Given a Mobius charts $\{\Phi_{p_i}(B_2), \Phi_{p_i}\}_{p_i\in M}$, we will see the transition function and its derivative is uniformly bounded.

Lemma 2.1.10. *Let (M^{n+1}, g) be an asymptotically hyperbolic manifold with boundary regularity $C^{l,\beta}$, $l \ge 2$. Given a Mobius charts covering $\{\Phi_{p_i}(B_2), \Phi_{p_i}\}_{p_i\in M}$, there exists a constant C such that*

$$\|\Phi_{p_j}^{-1}\circ\Phi_{p_i}\|_{C^{l,\beta}(U)} \le C$$

where $U = B_2 - \Phi_{p_i}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2))$.

Proof : The transition map can be written down as

$$\Phi_{p_j}^{-1}\circ\Phi_{p_i} : \Phi_{p_j}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2)) \longrightarrow \Phi_{p_j}^{-1}(\Phi_{p_i}(B_2)\cap\Phi_{p_j}(B_2))$$

$$\mathbf{x} \mapsto \mathbf{y}$$

where $\mathbf{x}, \mathbf{y} \in B_2 \subseteq \mathbb{H}^{n+1}$. We can thought this as

$$\Phi_{p_j}^{-1} \circ \Phi_{p_i} : \Gamma(U, TM) \longrightarrow \Gamma(U, TM)$$

Where $\Gamma(U, TM)$ is the section of the tangent bundle on U. Then we have

$$\Phi_{p_j}^{-1} \circ \Phi_{p_i} = \sum_{t=1}^{n+1} \frac{\partial}{\partial x^t} \otimes dx^i \in TM \otimes T^*M$$

Moreover

$$\|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\| = k + 1 \quad and \quad \nabla \Phi_{p_j}^{-1} \circ \Phi_{p_i} = 0$$

By Lemma 2.1.9, we have

$$\|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\|_{C^{l,\beta}(U)} \leq C$$

□

Lemma 2.1.11 (Lemma 3.5 [30])**.** *Let* (M^{n+1}, g) *be an asymptotically hyperbolic manifold with boundary regularity* $C^{l,\beta}$, $l \geq 2$. *Let* u *be a global section of a tensor bundle* E *and* $u \in C_\delta^{k,\alpha}(M; E)$ *with* $0 \leq \alpha < 1$ *and* $0 < k + \alpha \leq l + \beta$. *Fix arbitrary* $0 \leq \epsilon \leq 2$. *Suppose that* $\{\Phi_{p_i}(B_2), \Phi_{p_i}\}$ *is a Mobius charts covering of* M *satisfying that*

$$\cup_{p_i} \Phi_{p_i}(B_r) = M \quad for\ arbitrary\ \epsilon \leq r \leq 2$$

Then we have the following norm equivalence

$$C^{-1} \sup_i \rho\,(p_i)^{-\delta} \left\|\Phi_i^* u\right\|_{k,\alpha;B_r} \leq \|u\|_{k,\alpha,\delta} \leq C \sup_i \rho\,(p_i)^{-\delta} \left\|\Phi_i^* u\right\|_{k,\alpha;B_r}.$$

Proof : Then first inequality is obvious. Because the $\|.\|_{k,\alpha,\delta}$ is defined in the Mobius chart in B_2. For the second inequality, we can make use of Lemma 2.1.9 to show it. In fact, we only need to show that

$$\|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2)} \leq C \sup_j \|\Phi_{p_j}^* u\|_{C^{k,\alpha}(B_r)}$$

Consider all the p_j such that $\Phi_{p_i}(B_2) \cap \Phi_{p_j}(B_r) \neq \emptyset$. Then from lemma 2.1.9, we have

$$\|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\|_{C^{l,\beta}(U_j)} \leq C$$

where $U_j = B_2 - \Phi_{p_i}^{-1}(\Phi_{p_i}(B_2) \cap \Phi_{p_j}(B_r))$. Then

$$\begin{aligned} \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2)} &\leq \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_r)} + \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_2 - B_r)} \\ &\leq \|\Phi_{p_i}^* u\|_{C^{k,\alpha}(B_r)} + \sup_{p_j} \|\Phi_{p_j}^{-1} \circ \Phi_{p_i}\|_{C^{k,\alpha}(U_j)} \times \|\Phi_{p_j}^* u\|_{C^{k,\alpha}(B_r)} \end{aligned}$$

□

2.1.3 The elliptic estimates

In this section, we will mainly talk about the invertibility of the Lichnerowicz operator on the standard Hyperbolic space. By the estimate of the Green function and a special inequality, we can easily get this kind of invertibility.

For the purposes of this section, we will use the Poincaré ball model, identifying hyperbolic space with the unit ball $\mathbb{B} \subseteq \mathbb{R}^{n+1}$, with coordinates $\left(\xi^1, \ldots, \xi^{n+1}\right)$, and with the hyperbolic metric $\breve{g} = 4(1 - |\xi|)^{-2} \sum_i \left(d\xi^i\right)^2$. The hyperbolic distance function can be written in terms of the Euclidean norm and dot product as

$$d_{\breve{g}}(\xi, \eta) = \cosh^{-1} \frac{\left(1 + |\xi|^2\right)\left(1 + |\eta|^2\right) - 4\xi \cdot \eta}{\left(1 - |\xi|^2\right)\left(1 - |\eta|^2\right)}$$

It will be convenient to use

$$\rho(\xi) = \frac{1}{\cosh d_{\breve{g}}(\xi, 0)} = \frac{1 - |\xi|^2}{1 + |\xi|^2}$$

as a defining function for the ball, where $0 = (0, \ldots, 0)$ denotes the origin in $\mathbb{B} \subseteq \mathbb{R}^{n+1}$.

Throughout this chapter, E will be a geometric tensor bundle of weight r over $\mathbb{B}$, and $P : C^\infty(\mathbb{B}; E) \to C^\infty(\mathbb{B}; E)$ will be a formally self-adjoint geometric elliptic operator of order m. The fact that P is geometric implies that it is isometry invariant: If φ is any orientation-preserving hyperbolic isometry and u is any section of E, then

$$\varphi^*(Pu) = P(\varphi^* u)$$

We will assume that P satisfies (1.4). Then by Lemma 4.10, $P : H^{m,2}(\mathbb{B}; E) \to L^2(\mathbb{B}; E)$ is Fredholm. The next lemma shows that this is equivalent to being an isomorphism.

Proposition 2.1.12 (Proposition 5.2, [31])**.** *PROPOSITION 5.2. Let $P : C^\infty(\mathbb{B}; E) \to C^\infty(\mathbb{B}; E)$ be a formally self-adjoint geometric elliptic operator of order m satisfying* (1.4)*. Then P has positive indicial radius R, and for any $\varepsilon > 0$ there is a constant C such that*

$$|K(\xi, \eta)| \leq C\rho(\xi, \eta)^{n/2+R-\varepsilon}$$

whenever $d_{\breve{g}}(\xi, \eta) \geq 1$. (The norm here is the pointwise operator norm on $\operatorname{Hom}(E_\eta, E_\xi)$ *with respect to the hyperbolic metric.)*

In order to do the convolution, we need the following inequality.

Proposition 2.1.13 (LEMMA 5.4, [31])**.** *Suppose a and b are real numbers such that $a + b > n$ and $a > b$. There exists a constant C depending only on n, a, b such that the* *following estimate holds for all $\xi, \zeta \in \mathbb{B}$:*

$$\int_{\mathbb{B}} \rho(\xi,\eta)^a \rho(\eta,\zeta)^b dV_{\breve{g}}(\eta) \leq C\rho(\xi,\zeta)^b.$$

By the above two results, we have that

Proposition 2.1.14 (Proposition 5.6, [31]). *If $1 < p < \infty$, $k \geq m$, and $|\delta + n/p - n/2| < R$, then there exists a constant C such that*

$$\|u\|_{k,p,\delta} \leq C\|Pu\|_{k-m,p,\delta}$$

for all $u \in H^{k,p}_{\delta}(\mathbb{B}; E)$.

PROOF. Using Lemma 4.8, it suffices to prove that

$$\|u\|_{0,p,\delta} \leq C\|Pu\|_{0,p,\delta}$$

for all $u \in H^{k,p}_{\delta}(\mathbb{B}; E)$. Because $C^{\infty}_{c}(\mathbb{B}; E)$ is dense in $H^{k,p}_{\delta}(\mathbb{B}; E)$, it suffices to prove this inequality for $u \in C^{\infty}_{c}(\mathbb{B}; E)$. Since $u = P^{-1}(Pu)$ in that case, it suffices to prove the estimate

$$\left\|P^{-1}f\right\|_{0,p,\delta} \leq C\|f\|_{0,p,\delta} \text{ for all } f \in C^{\infty}_{c}(\mathbb{B}; E).$$

Put

$$p^* = \frac{p}{p-1}$$
$$a = \frac{1}{p^*}\left(\delta + \frac{n}{p}\right)$$

so that

$$\frac{n}{2} - R < ap^* < \frac{n}{2} + R$$
$$\frac{n}{2} - R < ap - \delta p < \frac{n}{2} + R.$$

By Hölder's inequality and Lemma 5.5, we estimate

$$
\begin{aligned}
\left|P^{-1} f(\xi)\right|_{\breve{g}} &\leq \int_{\mathbb{B}} |K(\xi,\eta)||f(\eta)|_{\breve{g}} dV_{\breve{g}}(\eta) \\
&= \int_{\mathbb{B}} \left(|K(\xi,\eta)|^{1/p} \rho(\eta)^{-a} |f(\eta)|_{\breve{g}}\right) \left(|K(\xi,\eta)|^{1/p^*} \rho(\eta)^{a}\right) dV_{\breve{g}}(\eta) \\
&\leq \left(\int_{\mathbb{B}} |K(\xi,\eta)|\rho(\eta)^{-ap}|f(\eta)|_{\breve{g}}^p dV_{\breve{g}}(\eta)\right)^{1/p} \times \\
&\left(\int_{\mathbb{B}} |K(\xi,\eta)|\rho(\eta)^{ap^*} dV_{\breve{g}}(\eta)\right)^{1/p^*} \\
&\leq C\rho(\xi)^a \left(\int_{\mathbb{B}} |K(\xi,\eta)|\rho(\eta)^{-ap}|f(\eta)|_{\breve{g}}^p dV_{\breve{g}}(\eta)\right)^{1/p}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|P^{-1} f\right\|_{0,p,\delta}^p &= \int_{\mathbb{B}} \rho(\xi)^{-\delta p} \left|P^{-1} f(\xi)\right|_{\breve{g}}^p dV_{\breve{g}}(\xi) \\
&\leq C^p \int_{\mathbb{B}} \int_{\mathbb{B}} \rho(\xi)^{ap-\delta p} |K(\xi,\eta)|\rho(\eta)^{-ap}|f(\eta)|_{\breve{g}}^p dV_{\breve{g}}(\eta) dV_{\breve{g}}(\xi).
\end{aligned}
$$

By Lemma 5.5 again, we can evaluate the ξ integral first to obtain

$$
\begin{aligned}
\left\|P^{-1} f\right\|_{0,p,\delta}^p &\leq C' \int_{\mathbb{B}} \rho(\eta)^{ap-\delta p} \rho(\eta)^{-ap} |f(\eta)|_{g}^p dV_{\breve{g}}(\eta) \\
&= C' \|f\|_{0,p,\delta}^p
\end{aligned}
$$

□

Theorem 2.1.15 (THEOREM 5.7, [31])**.** *Let $P : C^\infty(\mathbb{B}; E) \to C^\infty(\mathbb{B}; E)$ be a formally self-adjoint geometric elliptic operator of order m satisfying* (1.4)*. If $k \geq m$, $1 < p < \infty$, and $|\delta + n/p - n/2| < R$, then the natural extension $P : H_\delta^{k,p}(\mathbb{B}; E) \to H_\delta^{k-m,p}(\mathbb{B}; E)$ is an isomorphism.*

Proposition 2.1.16 (Proposition 5.8, [31])**.** *If $|\delta - n/2| < R$, there exists a constant C such that $\left\|P^{-1} f\right\|_{0,0,\delta} \leq C\|f\|_{0,0,\delta}$ (5.18) for all $f \in C_\delta^{0,0}(\mathbb{B}; E)$.*

PROOF. By Lemma 2.1.11,

$$\begin{aligned}\left|P^{-1}f(\xi)\right|_{\breve{g}} &\leq \int_{\mathbb{B}} |K(\xi,\eta)||f(\eta)|_{\breve{g}} dV_{\breve{g}}(\eta)\\ &\leq C\int_{\mathbb{B}} |K(\xi,\eta)|\rho(\eta)^{\delta}\|f\|_{0,0,\delta} dV_{\breve{g}}(\eta)\\ &\leq C'\rho(\xi)^{\delta}\|f\|_{0,0,\delta}\end{aligned}$$

which implies

$$\left\|P^{-1}f\right\|_{0,0,\delta} = \sup_{\xi\in\mathbb{B}}\left(\rho(\xi)^{-\delta}\left|P^{-1}f(\xi)\right|\breve{g}\right) \leq C'\|f\|_{0,0,\delta}.$$

Theorem 2.1.17 (THEOREM 5.9, [31])**.** *Let $P : C^{\infty}(\mathbb{B}; E) \to C^{\infty}(\mathbb{B}; E)$ be a formally self-adjoint geometric elliptic operator of order m satisfying (1.4). If $0 < \alpha < 1, k \geq m$, and $|\delta - n/2| < R$, then the natural extension $P : C^{k,\alpha}_{\delta}(\mathbb{B}; E) \to C^{k-m,\alpha}_{\delta}(\mathbb{B}; E)$ is an isomorphism.*

2.1.4 The Fredholm theory

In this section, we will patch the interior and the boundary for AH manifold together. We will construct a parametrix for the Lichnerowicz operator and show that its error term is in fact a compact operator. To make use of the discussions in the previous section on the analysis of elliptic operator on hyperbolic space we like to introduce the boundary Möbius coordinate charts. The boundary Möbius coordinate chart is built based on the half space model for hyperbolic space again. For a point $p \in \partial\mathbf{X}^{n+1} = \mathbf{M}^n$, consider a boundary coordinate chart $\left(x^1, x^2, \cdots, x^n, x^{n+1}\right)$ for $\overline{\mathrm{X}^{n+1}}$ around p so that it is a normal coordinate with respect to the compact metric $\bar{g}$ for a geodesic defining function $x = x^{n+1}$, where

$$\phi_p : \bar{U} \subseteq \overline{\mathbb{R}^{n+1}_+} \to \overline{\mathrm{X}^{n+1}}$$

Let

$$\mathbb{Y}^{\mathbb{H}} = \left\{(z, z_{n+1}) \in \mathbb{R}^{n+1}_{+} : |z| < 1 \text{ and } z_{n+1} \in (0,1)\right\}$$

and $(x, x_{n+1}) = (sz, sz_{n+1})$, where

$$\psi_{p,s} : \mathbb{Y}^{\mathbb{H}} \to \mathbb{Z}_{p,s} \subseteq \mathbf{X}^{n+1}$$

is said to be a boundary Möbius coordinate chart.

Lemma 2.1.18. *Suppose that $\psi_{p,s}$ is a boundary Möbius coordinate chart around a boundary point $p \in \mathbf{M}^n$ of an AH manifold $\left(\mathbf{X}^{n+1}, g^+\right)$of regularity $C^{l,\beta}$. Then*

$$\left\|\psi^*_{p,s} g^+ - g_{\mathbb{H}}\right\|_{l,\beta,\mathbb{Y}^{\mathbb{H}}} \leq Cs.$$

Furthermore in a boundary Möbius chart one can easily verify that

Lemma 2.1.19. *Suppose that $\psi_{p,s}$ is a boundary Möbius coordinate chart around a boundary point $p \in \mathbf{M}^n$ of an AH manifold $\left(\mathbf{X}^{n+1}, g^+\right)$. Then, there is a constant C such that*

$$C^{-1} s^{-\delta} \left\|\psi^*_{p,s} u\right\|_{k,\alpha,\delta,\mathbb{Y}^{\mathbb{H}}} \leq \|u\|_{k,\alpha,\delta,\mathbb{Z}_{p,s}} \leq C s^{-\delta} \left\|\psi^*_{p,s} u\right\|_{k,\alpha,\delta,\mathbb{Y}^{\mathbb{H}}}$$
$$C^{-1} s^{-\delta} \left\|\psi^*_{p,s} u\right\|_{k,p,\delta,\mathbb{Y}^{\mathbb{H}}} \leq \|u\|_{k,p,\delta,\mathbb{Z}_{p,s}} \leq C s^{-\delta} \left\|\psi^*_{p,s} u\right\|_{k,p,\delta,\mathbb{Y}^{H}}$$

Suppose that $\{\phi\}$ is a partition of unity associated with the above covering. Le.

$$G_s(u) = \sum \left(\psi^{-1}_{p,s}\right)^* \left(P^{\mathbb{H}}\right)^{-1} \psi^*_{p,s} \left(\phi_{p,s} u\right)$$
$$S_s(u) = \sum \left(\psi^{-1}_{p,s}\right)^* \left(P^{\mathbb{H}}\right)^{-1} \left(P_{p,s} - P^{\mathbb{H}}\right) \psi^*_{p,s} \left(\phi_{p,s} u\right)$$

and

$$C_s(u) = \sum \left(\psi^{-1}_{p,s}\right)^* \left(P^{\mathbb{H}}\right)^{-1} \psi^*_{p,s} \left(\left[\phi_{p,s}, P\right] u\right).$$

This is because we have

$$\begin{aligned}
G_s P u &= \sum \left(\psi_{p,s}^{-1}\right)^* \left(P^{\mathbb{H}}\right)^{-1} \psi_{p,s}^* \left(\phi_{p,s} P u\right) \\
&= \sum \left(\psi_{p,s}^{-1}\right)^* \left(P^{\mathbb{H}}\right)^{-1} \psi_{p,s}^* \left(P \phi_{p,s} u\right) + C_s(u) \\
&= \sum \left(\psi_{p,s}^{-1}\right)^* \left(P^{\mathbb{H}}\right)^{-1} P^{\mathbb{H}} \psi_{p,s}^* \left(\phi_{p,s} u\right) + S_s(u) + C_s(u) \\
&= u + S_s(u) + C_s(u).
\end{aligned}$$

Equivalently we may write

$$(\mathrm{Id} + S_s)^{-1} G_s P u = u + (\mathrm{Id} + S_s)^{-1} C_s(u)$$

when $\mathrm{Id} + S_s$ is invertible. More precisely

Theorem 2.1.20. *Suppose that P is a formally self-adjoint geometric differential operator of order m on a tensor bundle of hyperbolic space $\mathbb{H}^{n+1}$ and that*

$$P^{\mathbb{H}} : H^{m,2}\left(\mathbb{H}^{n+1}, \mathrm{E}\right) \to L^2\left(\mathbb{H}^{n+1}, \mathrm{E}\right)$$

is Fredholm. Let the indicial radius of P be $R > 0$. Then, for $p \in (1, \infty)$ and $\left|\delta + \frac{n}{p} - \frac{n}{2}\right| < R$,

$$\begin{aligned}
G_s : H_\delta^{k,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) &\to H_\delta^{k+m,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \\
S_s : H_\delta^{k,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) &\to H_\delta^{k,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)
\end{aligned}$$

and

$$C_s : H_\delta^{k,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to H_\theta^{k+1,p}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$$

are all bounded, for any $\theta \in (\delta, \delta+1]$ and $\left|\theta - \frac{n}{2} + \frac{n}{p}\right| < R$. Moreover there is a constant C such that

$$\|S_s(u)\|_{k,p,\delta} \le Cs\|u\|_{k,p,\delta}.$$

Analogously, for $\alpha \in (0,1)$ *and* $\left|\delta - \frac{n}{2}\right| < R$,

$$G_s : C^{k,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to C^{k+m,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$$

$$S_s : C^{k,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to C^{k,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$$

and

$$C_s : C^{k,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to C^{k+1,\alpha}_{\theta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$$

are all bounded, for any $\theta \in (\delta, \delta+1]$ *and* $\left|\theta - \frac{n}{2} + \frac{n}{p}\right| < R$. *Moreover there is a constant* C *such that*

$$\|S_s(u)\|_{k,\alpha,\delta} \leq Cs\|u\|_{k,\alpha,\delta}.$$

Theorem 2.1.21 ([31])**.** *Suppose that* P *is a formally self-adjoint geometric differential operator of order* m *on a tensor bundle of an AH manifold* $\left(\mathrm{X}^{n+1}, g^{+}\right)$*and that*

$$P^{\mathbb{H}} : H^{m,2}\left(\mathbb{H}^{n+1}, \mathrm{E}\right) \to L^2\left(\mathbb{H}^{n+1}, \mathrm{E}\right)$$

is Fredholm. And suppose that the so-called L^2*-kernel* Z *is trivial. Assume that the indicial radius is* $R > 0$. *Then -* $P : H^{k,p}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to H^{k-m,p}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$ *is an isomorphism for all* $\left|\delta - \frac{n}{2} + \frac{n}{p}\right| < R$ *and -* $P : C^{k,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right) \to C^{k-m,\alpha}_{\delta}\left(\mathrm{X}^{n+1}, \mathrm{E}\right)$ *is an isomorphism for all* $\left|\delta - \frac{n}{2}\right| < R$.

Corollary 2.1.22. *Suppose that* $\left(\mathrm{X}^{n+1}, g^{+}\right)$*is conformally compact Einstein manifold and that* $\mathbb{P}_{\mathrm{E}}$ *is the linearization of Einstein equations. Assume that* $\mathbb{P}_{\mathrm{E}}$ *has no* L^2*-kernel on* $\left(\mathrm{X}^{n+1}, g^{+}\right)$ *.Then*

$$\mathbb{P}_{\mathrm{E}} : C^{2,\alpha}_{\delta}\left(\mathbf{X}^{n+1}, \mathbb{S}^2\right) \to C^{0,\alpha}_{\delta}\left(\mathbf{X}^{n+1}, \mathbb{S}^2\right)$$

is an isomorphism for $\delta \in (0,n)$, *where* $\mathbb{S}^2$ *is the bundle of symmetric 2-tensors.*

§ 2.2 The existence of the AHE

In this subsection we will first formulate the appropriate analytic problem for solving conformally compact Einstein metrics. Then, based on the previous discussions we build the functional analysis framework and a version of implicit function theorem (Lemma 3.6.3). To apply the linear theory we have developed we will need to construct asymptotic solution to Einstein equations on a given manifold X^{n+1} with a prescribed conformal infinity $(\mathrm{M}^n, [\hat{g}])$ near the conformal infinity $(\mathbf{M}^n, [\hat{g}_0])$ of a given conformally compact Einstein metric g_0^+.

2.2.1 The gauge Einstein equation

Let us first work out the appropriate analytic problem. To eliminate the obvious degeneracy of diffeomorphism one needs to choose a gauge (coordinate) to set the Einstein Equations. Hence we recall the so-call De Turck's trick first. For a symmetric 2-tensor t on a Riemannian manifold $\left(\mathrm{X}^{n+1}, g\right)$ we define

$$\left(G_g t\right)_{ij} \triangleq t_{ij} - \frac{1}{2} t_{kl} g^{kl} g_{ij}$$

and the divergence

$$\left(\delta_g t\right)_i = -t_{ij,k} g^{jk}.$$

The formal adjoint operator takes an 1-form ω to a symmetric 2-tensor as follows:

$$\left(\delta_g^* \omega\right)_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right).$$

Due to Bianchi identity we know that

$$\delta_g G_g Ric[g] = 0$$

for any Riemannian metric g. Recall the deformation of Ricci curvature at g

$$(DRic)h = \frac{1}{2}\left(\nabla^*\nabla + 2\overset{\circ}{Rc} - Rm^\circ - 2\delta^*\delta G\right)h,$$

where the last term represents the degeneracy. To cancel the last term, by De Turck's trick, we add a term and instead consider

$$Q(g,t) = \mathrm{Ric}[g] + ng - \delta^* g t^{-1}\delta G(t)$$

for a metric g and a symmetric 2 -tensor t. In fact, when t is also a metric, one may verify that $\tau = gt^{-1}\delta G(t)$ is exactly the dual form of the so-called tension field of the identity map Id : $\left(\mathrm{X}^{n+1}, g\right) \to \left(\mathrm{X}^{n+1}, t\right)$ with respect to the usual Dirichlet energy, that is, Id is a harmonic map from $\left(\mathbf{X}^{n+1}, g\right)$ to $\left(\mathbf{X}^{n+1}, t\right)$ when $\tau = 0$. Recall, for a map $u : \left(\mathbf{X}^{n+1}, g\right) \to \left(\mathbf{X}^{n+1}, t\right)$, the dual tension field in local coordinates is $\tau_\theta = -g_{\theta\gamma}g^{\alpha\beta}u^\gamma_{\alpha,\beta} = -g_{\theta\gamma}g^{\alpha\beta}\left(\partial_\beta u^\gamma_\alpha - u^\gamma_\xi\left(\Gamma_g\right)^\xi_{\alpha\beta} + u^\zeta_\alpha\left(\Gamma_t\right)^\gamma_{\zeta\xi}u^\xi_\beta\right)$. Hence, for the identity map, $u^\gamma_\alpha = \delta^\gamma_\alpha$, we have

$$\tau_\theta = g_{\theta\gamma}g^{\alpha\beta}\left(\left(\Gamma_g\right)^\gamma_{\alpha\beta} - \left(\Gamma_t\right)^\gamma_{\alpha\beta}\right).$$

It is easier to verify that $\tau = gt^{-1}\delta G(t)$ if one uses the normal coordinate at each arbitrary given point. The De Turck's trick consists of the following two steps. First one has

Lemma 2.2.1. *Suppose that* $\left(\mathrm{X}^{n+1}, g^+\right)$*is an AH manifold that satisfies* $Q\left(g^+, t\right) = 0$ *for a given AH metric* t*. And suppose that* g^+*has a strictly negative Ricci curvature. Then the identity map Id :* $\left(\mathrm{X}^{n+1}, g^+\right) \to \left(\mathrm{X}^{n+1}, t\right)$ *is harmonic and* g^+*is Einstein.*

Lemma 2.2.2. *Suppose that* $\left(X^{n+1}, g^{+}\right)$*is conformally compact Einstein manifold. Then the linearization with respect to the first variable (metric) of* Q *at* $g = t = g^{+}$*is given as follows:*

$$D_g Q\left(g^{+}, g^{+}\right) = D_g Q(g, t)\Big|_{g=t=g^{+}} = \frac{1}{2}\mathbb{P}_{\mathrm{E}}.$$

2.2.2 A general perturbational existence theorem

Let us first work out the appropriate analytic problem. To eliminate the obvious degeneracy of diffeomorphism one needs to choose a gauge (coordinate) to set the Einstein Equations. Hence we recall the so-call De Turck's trick first. For a symmetric 2-tensor t on a Riemannian manifold $\left(X^{n+1}, g\right)$ we define

$$\left(G_g t\right)_{ij} \triangleq t_{ij} - \frac{1}{2} t_{kl} g^{kl} g_{ij}$$

and the divergence

$$\left(\delta_g t\right)_i = -t_{ij,k} g^{jk}.$$

The formal adjoint operator takes an 1-form ω to a symmetric 2-tensor as follows:

$$\left(\delta_g^{*}\omega\right)_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right).$$

Due to Bianchi identity we know that

$$\delta_g G_g Ric[g] = 0$$

for any Riemannian metric g. Recall the deformation of Ricci curvature at g

$$(DRic)h = \frac{1}{2}\left(\nabla^{*}\nabla + 2\overset{\circ}{Rc} - Rm^{\circ} - 2\delta^{*}\delta G\right)h,$$

where the last term represents the degeneracy. To cancel the last term, by De Turck's trick, we add a term and instead consider

$$Q(g,t)=\mathrm{Ric}[g]+ng-\delta^{*}gt^{-1}\delta G(t)$$

for a metric g and a symmetric 2 -tensor t. In fact, when t is also a metric, one may verify that $\tau=gt^{-1}\delta G(t)$ is exactly the dual form of the so-called tension field of the identity map Id : $\left(\mathrm{X}^{n+1},g\right)\rightarrow\left(\mathrm{X}^{n+1},t\right)$ with respect to the usual Dirichlet energy, that is, Id is a harmonic map from $\left(\mathbf{X}^{n+1},g\right)$ to $\left(\mathbf{X}^{n+1},t\right)$ when $\tau=0$. Recall, for a map $u:\left(\mathbf{X}^{n+1},g\right)\rightarrow\left(\mathbf{X}^{n+1},t\right)$, the dual tension field in local coordinates is $\tau_{\theta}=-g_{\theta\gamma}g^{\alpha\beta}u_{\alpha,\beta}^{\gamma}=-g_{\theta\gamma}g^{\alpha\beta}\left(\partial_{\beta}u_{\alpha}^{\gamma}-u_{\xi}^{\gamma}\left(\Gamma_{g}\right)_{\alpha\beta}^{\xi}+u_{\alpha}^{\zeta}\left(\Gamma_{t}\right)_{\zeta\xi}^{\gamma}u_{\beta}^{\xi}\right)$. Hence, for the identity map, $u_{\alpha}^{\gamma}=\delta_{\alpha}^{\gamma}$, we have

$$\tau_{\theta}=g_{\theta\gamma}g^{\alpha\beta}\left(\left(\Gamma_{g}\right)_{\alpha\beta}^{\gamma}-\left(\Gamma_{t}\right)_{\alpha\beta}^{\gamma}\right).$$

It is easier to verify that $\tau=gt^{-1}\delta G(t)$ if one uses the normal coordinate at each arbitrary given point. The De Turck's trick consists of the following two steps. First one has

Lemma 2.2.3. *Suppose that* $\left(\mathrm{X}^{n+1},g^{+}\right)$*is an AH manifold that satisfies* $Q\left(g^{+},t\right)=0$ *for a given AH metric* t*. And suppose that* g^{+}*has a strictly negative Ricci curvature. Then the identity map Id :* $\left(\mathrm{X}^{n+1},g^{+}\right)\rightarrow\left(\mathrm{X}^{n+1},t\right)$ *is harmonic and* g^{+}*is Einstein.*

Lemma 2.2.4. *Suppose that* $\left(\mathrm{X}^{n+1},g^{+}\right)$*is conformally compact Einstein manifold. Then the linearization with respect to the first variable (metric) of* Q *at* $g=t=g^{+}$*is given as follows:*

$$D_{g}Q\left(g^{+},g^{+}\right)=D_{g}Q(g,t)\big|_{g=t=g^{+}}=\frac{1}{2}\mathbb{P}_{\mathrm{E}}.$$

Then, by the Newton's method, we can get that

Theorem 2.2.5 ([31])**.** *Suppose that* $\left(\mathrm{X}^{n+1},g^{+}\right)$*is a conformally compact Einstein man-*

ifold with the conformal infinity $(M^n, [\hat{g}])$. *And suppose that the linearized operator* $\mathbb{P}_E$ *of gauged Einstein equations has no nontrivial* L^2*-kernel. Then for any conformal class* $[\hat{g}_\epsilon]$ *that is sufficiently close to* $[\hat{g}]$ *there exists a unique polyhomogeneously smooth conformally compact Einstein metric* g^+_ϵ*whose conformal infinity is* $[\hat{g}_\epsilon]$.

2.2.3 Examples

§ 2.3 Asymptotically symmetric Einstein metrics of rank 1 cases

In this section, we will introduce the generalization result of the last section into the setting of the asymptotically symmetric Einstein metrics. The ideas are pretty much same and from O.Biquard [7]. In order to shw the existence of the ASE metric on the setting of the boundary perturbation. We need to consider the the gauge Einstein equation. And its linearization is again a Lichnerowicz operator which can be shown to be invertible on the standard symmetric space. And there also exists the Mobius coordinate covering which satisfies the same result as it in the AH case. Therefore, we can still show the Fredholm properties in the AS metrics. Finally, by the newton's method, we can show the existence of the ASE metric.

However, there are something which is different from the AH metric. First of all, the Green function estimate is different. Second, we do not have the inequality.

2.3.1 Definitions

First, let us introduce the definition of AS manifold, which is pretty much derive from the cartan geometry for the model geometry on the boundary of the standard symmetric space.

Definition 2.3.1 (Definition A, [7])**.** Let $H = U_{m-1}, Sp_{m-1}Sp_1$ or Spin $_7$, corresponding to the complex, quaternionic or octonionic cases, respectively. Let S^{n-1} be a manifold with a contact 1-form η with values in $\mathbb{R}, \mathbb{R}^3$ or $\mathbb{R}^7$, respectively, and let $V = \ker \eta$. A Carnot-Carathéodory H-metric compatible with $d\eta$ is defined to be a metric γ on V such that

- in the complex case, the restriction to V of $d\eta$ is a symplectic form compatible with g (that is, $d\eta(\cdot, \cdot) = \gamma(I\cdot, \cdot)$ where I is an almost complex structure on V);
- in the quaternionic case, the three 2-forms $(d\eta_1, d\eta_2, d\eta_3)$ on V provide a quaternionic structure compatible with γ (that is, $d\eta_i(\cdot, \cdot) = \gamma\ (I_i, \cdot)$ for almost complex structures I_i satisfying the quaternionic commutation relations);
- in the octonionic case, the seven 2-forms $(d\eta_1, \dots, d\eta_7)$ on V provide a Spinn structure compatible with γ (that is, $d\eta_i(\cdot, \cdot) = \gamma\ (I_i\cdot, \cdot)$ for almost complex structures I_i satisfying the octonionic commutation relations).

2.3.2 The perturbation existences

In this section, we will introduce the mainly perturbation result for the ASE metric from O.biquard.

Theorem 2.3.2 (THEOREM I.4.14. [7])**.** *Suppose $\mathbb{K} = \mathbb{H}$ or $\mathbb{O}$ and $H = Sp_{m-1}Sp_1$ or Spin $_7$; suppose the manifold M is of dimension $4m$ or 16 , with boundary S, and has an asymptotically symmetric Einstein metric g_0 such that $L^2\boldsymbol{H}^1(g_0) = 0$; suppose g_V is a Carnot-Carathéodory H-metric on S with $C^{2,\alpha}$ regularity, close to the conformal infinity of g_0, and let g be an asymptotically symmetric metric associated via (4.7), then there exists a metric h such that*

$$\mathrm{Ric}^h = -\lambda h; \quad h - g \in C_1^{2,\alpha}.$$

Locally, the metric h is unique modulo the action of diffeomorphisms inducing the identity on the boundary.

First, we need to introduce the precise definition of the weighted space on asymptotically symmetric manifolds. Just like what we did for the hyperbolic space. First, we introduce the definition of asymptotically symmetric manifold

Definition 2.3.3. An asymptotically symmetric metric is a metric which can be written as $g + h$ near infinity, where g is defined by (3.1), with g_V having $C^{2,\alpha}$ local regularity, and $h \in C_1^{2,\alpha}$.

Then, we need to introduce the Holder space. Hölder norms. For a metric such as g, the geometry is uniform at infinity, which is translated, in particular, by the fact that the sectional curvature is bounded above and the injectivity radius r_{inj} is bounded below. Consequently, following proposition 6.4.6 of [BK81], there exists a radius r_{cony} such that in a ball of radius r_{conv} , two points x and y are linked by a unique minimizing geodesic.

Let us verify that the C^α tensor norm, used earlier, has a natural definition. Let us fix a parameter $\rho < r_{\text{conv}}$, and for x, y such that $d(x,y) < \rho$, let $p_{x\to y}$ denote the parallel transportation along the minimizing geodesic joining x to y. The Hölder norm is defined by

$$\|u\|_{C^\alpha} = \sup|u| + \sup_{d(x,y)<\rho} \frac{\left|p_{x\to y}(u(x)) - u(y)\right|}{d(x,y)^\alpha}$$

The $C^{k,\alpha}$ norm is then naturally defined by

$$\|u\|_{C^{k,\alpha}(g)} = \sum_0^{k-1} \sup\left|\nabla^i u\right| + \left\|\nabla^k u\right\|_{C^\alpha}$$

and the Hölder weighted norm by

$$\|u\|_{C^{k,\alpha}_\delta(g)} = \left\|\cosh(r)^\delta u\right\|_{C^{k,\alpha}(g)}$$

Harmonic coordinates. In order to deal with problems relating to local elliptic regularity, we need coordinates on our balls for which the metric is controlled. In principle, this does not raise a problem because of the uniform geometry at infinity, but there are a number of subtleties associated with the use of metrics with $C^{2,\alpha}$ regularity.

If the metric g_V on S is smooth, the curvature of g is completely controlled by the calculation (1.7) and, in geodesic coordinates, the metric g will be uniformly close to the flat metric, for example in C^2, α. Adding a perturbation tending to 0 at infinity, $h \in C^{2,\alpha}_\delta$, changes nothing. The same is true for a smooth asymptotically symmetric metric. In this situation, Proposition I. 3.2 below is of no use.

On the other hand, if the metric g_V has only $C^{2,\alpha}$ regularity, the above reasoning does not work, since the geodesic coordinates lead to loss of the regularity. To remedy this problem, we shall instead use the existence of harmonic coordinates, which are of maximum regularity. The following proposition will suffice for us.

Proposition 2.3.4. *Let $Q > 1$, if $\|\mathrm{Ric}^g\|_{C^{k-2,\alpha}} \leq C$ and $r_{inj} \geq i$, then there exists $r_{harm,}$ depending only on Q, C, i and $k+\alpha$, such that any ball $B_{r_{harm}}(x)$ has harmonic coordinates in which the coefficients g_{ij} of the metric satisfy*

$$Q^{-1}\left(\delta_{ij}\right) \leq \left(g_{ij}\right) \leq Q\left(\delta_{ij}\right)$$
$$\sum_{1\leq|\beta|\leq k} r_{harm}^{|\beta|} \sup\left|\partial_\beta g_{ij}\right| + \sum_{|\beta|=k} r_{harm}^{k+\alpha} \sup \frac{\left|\partial_\beta g_{ij}(y) - \partial_\beta g_{ij}(x)\right|}{|x-y|^\alpha} \leq Q$$

2.3.3 Main ideals of the proofs

In this section, the manifold is always the rank 1 symmetric space and the Laplacian operator refers to the Laplacian operator with respect to the standard metric on symmetric space. We refer [30] to show that the Laplacian operator is a Fredholm operator on so called weight space with zero index.

The infinity behavior of the Green function

(1)(Geometric invariance for the Laplacian operator) Geometric invariance for the Laplacian operator refers to the fact that Laplacian operator satisfies the following commutative diagram.

$$\begin{array}{ccc} C^\infty(M,TM) & \xrightarrow{\Delta} & C^\infty(M,TM) \\ \downarrow h_* & & \downarrow h_* \\ C^\infty(M,TM) & \xrightarrow{\Delta} & C^\infty(M,TM) \end{array}$$

where $h \in G \cong ISO(M)$. Therefore,

$$(\Delta f)(h(p)) = h_*(\Delta(h_*^{-1}(f))(p))$$

(2)(The definition of Green function) The Green function $G_{\xi_{p_0}}(x) \in \mathrm{Hom}(E_{p_0}, E_x)$

is section satisfying that

$$\Delta G_{\xi_{p_0}} = \delta_{p_0}\xi_{p_0}$$

where δ_{p_0} is the Dirac function at $p_0 \in M$ and $\xi_{p_0} \in E_{p_0}$.

(3)(The linearality of the Green function)

$$aG_{\xi_{p_0}} + bG_{\eta_{p_0}} = G_{(a\xi_{p_0}+\eta_{p_0})}$$

where $\xi_{p_0}, \eta_{p_0} \in T_{p_0}M$ and a, b are constants. Then, the green function can be thought of as a linear map as following

$$\begin{aligned} G(p) : T_{p_0}M &\longrightarrow T_pM \\ \xi_{p_0} &\mapsto G_{\xi_{p_0}}(p) \end{aligned}$$

The lift of $\widetilde{G}_{\xi_{p_0}}$ can be also thought of as a linear map as following

$$\begin{aligned} \widetilde{G}(g) : T_{p_0}M &\longrightarrow T_{p_0}M \\ \xi_{p_0} &\mapsto \widetilde{G}_{\xi_{p_0}}(g) \end{aligned}$$

where $g \in G$.

(4)(The geometric invariance for the Green function) The geometric invariance for the Green function refers to

$$h_*(G_{\xi_{p_0}}) = G_{h_*\xi_{p_0}}$$

where $h \in K \subseteq G$. In fact, by the geometric invariance for the Laplacian operator, we have

$$\Delta(h_* G_{\xi_{p_0}}) = h_*(\Delta G_{\xi_{p_0}}) = \delta_x h_*(\xi_{p_0})$$

(5)(The Green function is a spherically invariant vector field) From 1.2.1 (3) (6), we have $\widetilde{G}_{h_*\xi_{p_0}}(g) = \widetilde{h_* G_{\xi_{p_0}}}(g) = \widetilde{G}_{\xi_{p_0}}(h^{-1}g)$. Suppose that $\widetilde{G}(\exp(rx_0)) = A(r)$ is the transformation of $T_{p_0}M$, then

$$\widetilde{G}_{\xi_{p_0}}(\exp(rx_0)) = A(r)\xi_{p_0}$$

And from 1.2.2 (3),

$$\widetilde{G}_{\xi_{p_0}}(h\exp(rx_0)) = \widetilde{G}_{h_*^{-1}\xi_{p_0}}(\exp(rx_0)) = A(r)\xi_{p_0} = A(r)h_*^{-1}(\xi_{p_0}) = A(r)\rho_0(h^{-1})(\xi_{p_0})$$

where $h \in G$. By the definition of spherically invariant vector field in 1.2.1 (11), $\widetilde{G}_{\xi_{p_0}}$ is spherically invariant vector field.

(6)

Lemma 2.3.5. *For the above linear transformation $A(r)$, we have*

$$\rho_0(h)A(r)\rho_0(h^{-1}) = A(r)$$

where $h = \exp(vt)$ with $v \in \mathrm{I}_0$ and $t > 0$.

Proof: Since $h = \exp(\mathrm{I}_0 t)$, $h(\exp(\mathrm{I}_0 t)\exp(rx_0)(p_0)) = \exp(rx_0)(p_0)$. Therefore, we have $\widetilde{h_*(f)}(\exp(rx_0)) = \rho_0(h)(\widetilde{f}(\exp(rx_0)))$. Therefore,

$$\widetilde{h_* G_{\xi_{p_0}}}(\exp(rx_0)) = \rho_0(h)(\widetilde{G_{\xi_{p_0}}}(\exp(rx_0))) = \rho_0(h)A(r)(\xi_{p_0})$$

On the other hand, from 1.2.2 (4), (5) $\widetilde{h_*(G_{\xi_{p_0}})}(\exp(rx_0)) = \widetilde{G_{h_*\xi_{p_0}}}(\exp(rx_0)) =$

$A(r)\rho_0(h)(\xi_{p_0})$. Therefore, we have

$$\rho_0(h)A(r) = A(r)\rho_0(h)$$

(7) We can decompose the $T_{p_0}M$ into the irreducible invariant subspace $E_1 \oplus \cdots \oplus E_l$ for the group K_0 which is generated by the lie algebra $\mathfrak{l}_0$. Since $\rho_0(h)A(r) = A(r)\rho_0(h)$, $A(r)|_{E_i} = f_i(r) id_{E_i}$. From 1.2.1 (10), $C(\mathfrak{m}_0, \rho_0)|_{E_i} = \mu_i id_{E_i}$. Therefore, for $v \in E_i$, as in the 1.2.1 (12), we have

$$\widetilde{\Delta G_v}(\exp(rx_0)) = \partial_r^2 f_i(r)v + \mathcal{H}\partial_r f_i(r)v - \mu_i f_i(r)v + B(r)v$$

(8)(The equivalence of the norm) Since

$$\widetilde{G_v}(\exp(rx_0)) = A(r)v \quad \widetilde{G_v}(h\exp(rx_0)) = A(r)\rho_0(h^{-1})v$$

for $h \in K \subseteq G \cong ISO(M)$ and $v \in T_{p_0}M$, then we can easily get

$$|G_v| = |\widetilde{G_v}(\exp(rx_0))| \leq C_1|A(r)| \cdot |v|$$

$$|G_v| = |\widetilde{G_v}(\exp(rx_0))| \geq C_2|A(r)| \cdot |v|$$

Therefore, $|\partial_r^m G_v|$ is equivalent to $|\partial_r^m A(r)| \cdot |v|$ for $m \geq 0$. Moreover, we have

$$||\partial_r^m G_v||_{L^2} = \int_0^{+\infty} < \partial_r^m A(r)v, \partial_r^m A(r)v > \exp(\mathcal{H}r)dr$$

(9)(The elliptic estimate) For the elliptic equation $\Delta G_v = 0$, if $G_v \in L^2(M)$, then by the elliptic regularity,

$$\partial_r G_v \in L^2(M) \quad \partial_r^2 G_v \in L^2(M)$$

Then, we have

$$
\begin{aligned}
|\widetilde{G_v}(\exp(rx_0))|^2 &= < A(r)v, A(r)v > \\
&= -\int_r^{+\infty} \exp(-\mathcal{H}\tau) 2 < \partial_\tau A(\tau)v, A(\tau)v > \exp(\mathcal{H}\tau) d\tau \\
&\leq \exp(-\mathcal{H}r)[\int_M |\partial_\tau A(\tau)v|^2 dV + \int_M |A(\tau)v|^2 dV]
\end{aligned}
$$

Therefore, we have

$$|\widetilde{G_v}(\exp(rx_0))| \leq O(\exp(-\frac{\mathcal{H}}{2}r)) \cdot |v|$$

Similarly, we have

$$|\partial_r \widetilde{G_v}(\exp(rx_0))| \leq O(\exp(-\frac{\mathcal{H}}{2}r)) \cdot |v|$$

Therefore, from the 1.2.2 (8), we have

$$|A(r)| \leq O(\exp(-\frac{\mathcal{H}}{2}r)) \quad |\partial_r A(r)| \leq O(\exp(-\frac{\mathcal{H}}{2}r))$$

Remark 2.3.6. We will explain why $G \in L^2(M)$. First, $G(x, y)$ satisfying the equation

$$\Delta_x G(x, y) = 0 \quad \text{if } x \neq y$$

Then, fix $y \in M$

$$\|G(x, y)\|_{L^2(M)} = \|G(x, y)\|_{L^2(M - B_\delta(y))} + \|G(x, y)\|_{L^2(B_\delta(y))} \tag{2.3.1}$$

For the first part, consider the following function

$$G_1(x, y) = \begin{cases} G(x, y) & \text{if } x \in M - B_\delta(y) \\ f(x, y) & \text{if } x \in B_\delta(y) \end{cases}$$

where $f \in C^\infty(\overline{B_\delta(y)})$ such that the second derivative of G_1 existence. Then, by the elliptic estimate on Hyperbolic space (Lemma 4.8 in [30]), we have that

$$\|G_1(x,y)\|_{L^2(M)} \leq C\|\Delta_x f(x,y)\|_{L^2(M)} \leq C.$$

While for the secon part of (2.3.1), it can be controlled by the green function in $\mathbb{R}^{n+1}$ (Newtonian potential), which implies that

$$\|G(x,y)\|_{L^2(B_\delta(y))} \leq C.$$

Therefore, $G(x,y) \in L^2(M)$

(10) For the equation in 1.2.2 (7)

$$\widetilde{\Delta G_v}(\exp(rx_0)) = \partial_r^2 f_i(r)v + \mathcal{H}\partial_r f_i(r)v - \mu_i f_i(r)v + B(r)v$$

From 1.2.1 (12) and 1.2.2 (9), we have

$$|B(r)| = O(\exp(-r))[|A(r)| + |\partial_r A(r)|] \leq O(\exp(-\frac{\mathcal{H}}{2}r - r))$$

(11)(A result of ODE)

Lemma 2.3.7. *For constant coefficients ODE,*

$$\partial_r^2 f + a_1\partial_r f + a_0 f = g(r) \tag{2.3.2}$$

with $|g(r)| \sim \exp(kr)$ *as* r *goes to infinity and* a_1, a_0 *constant. The solution,* $f(r)$*, of the above ODE satisfies*

$$|f(r)| \leq C_1\exp(kr) + C_2\exp(\lambda_1 r) + C_3\exp(\lambda_2 r) \tag{2.3.3}$$

where C_1*,* C_2 *and* C_3 *are constants and* λ_1 *and* λ_2 *are two distinct solution of the characteristic equation* $\lambda^2 + a_1\lambda + a_0 = 0$

Proof. By the variation of the constant method, the generation of the equation (2.3.2), is

$$G(r) = C_1 G_1(r) + C_2 G_2(r) + \int_{r_0}^{r} \frac{G_1(\tau)G_2(r) - G_1(r)G_2(\tau)}{G_1(\tau)G_2'(\tau) - G_1'(\tau)G_2(\tau)} g(\tau)d\tau$$

where $G_1(r)$ and $G(r)$ are two linearly independent solution of the corresponding homogeneous solution. Therefore,

$$G_1(r) = e^{\lambda_1 r} \quad \text{and} \quad G_2(r) = e^{\lambda_2 r}$$

Therefore, we have

$$G(r) = C_1 e^{\lambda_1 r} + C_2 e^{\lambda_2 r} + \frac{e^{\lambda_2 r}}{\lambda_2 - \lambda_1} \int_{r_0}^{r} e^{-\lambda_2 \tau} g(\tau)d\tau - \frac{e^{\lambda_1 r}}{\lambda_2 - \lambda_1} \int_{r_0}^{r} e^{-\lambda_1 \tau} g(\tau)d\tau$$

Then, the (2.3.3) follows. □

(12)(The infinity behavior of the Green function)

Theorem 2.3.8. *As r goes to infinity, the Green function G_v for the Laplacian operator satisfies*

$$|G_v| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu})r)) \cdot |v|$$

where μ is the smallest eigenvalue of the Casmir operator $C(\mathfrak{m}_0, \rho_0)$

Proof: From 1.2.2 (8), we only need to show that

$$|A(r)| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu})r))$$

Step1 Assume that

$$|A(r)| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r))$$

where $0 \le \lambda < \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$. Then, by the interior Holder estimate, we have

$$|\partial_r A(r)| \le O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r))$$

Therefore, we have

$$|B(r)| \le O(\exp(-(\frac{\mathcal{H}}{2} + \lambda + 1)r))$$

Therefore, if we consider the equation

$$\partial_r^2 A(r)v + \mathcal{H}\partial_r A(r)v - C(\mathfrak{m}_0, \rho_0)A(r)v = -B(r)v$$

in each irreduciable subspace of $T_{p_0}M$, then the above will induce some ODEs as in 1.2.2 (7)

$$\partial_r^2 f_i(r)v + \mathcal{H}\partial_r f_i(r)v - \mu_i f_i(r)v = -B(r)v$$

Then, Combining the result of ODE in 1.2.2 (11), we have

$$|A(r)| \le O(\exp(-(\frac{\mathcal{H}}{2} + \lambda + 1)r)) + O(\exp(-(\frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu})r))$$

which contradict to our assumption.

Step2 Assume that the λ in step1 is

$$\lambda > \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Again, by the Holder interior estimate, we have

$$|\partial_r^m A(r)| \le O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r)) \quad for\ m \ge 0 \tag{2.3.4}$$

and

$$|B(r)| \leq O(\exp(-(\frac{\mathcal{H}}{2} + \lambda + 1)r))$$

Suppose that the $f(r)$ is the eigenvalue of $A(r)$ such that the corresponding eigenspace E_i corresponds to μ. Then, we have

$$f(r) \sim O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r))$$

Suppose that

$$f(r) = g(r)\exp(-(\frac{\mathcal{H}}{2} + \lambda)r),$$

Then,

$$|g(r)| \sim O(1).$$

And by the $H\ddot{o}lder$ estimate (2.3.4), we have

$$|\partial_r^m f(r)| = |\sum_{i=0}^{m} C_i \partial_r^i g(r) \cdot \left(\partial_r^{m-i} \exp(-(\frac{\mathcal{H}}{2} + \lambda)r)\right)| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r)),$$

which implies that

$$|\partial_r^i g(r)| \leq O(1)$$

Then,

$$\begin{aligned}
&\partial_r^2 f(f) + \mathcal{H}\partial_r f(r) - \mu f(r) \\
=& \left(\partial_r^2 g(r) - (\frac{\mathcal{H}}{2} + \lambda)\partial_r g(r) + \mathcal{H}\partial_r g(r)\right)\exp(-(\frac{\mathcal{H}}{2} + \lambda)r) \\
&+ \left((\frac{\mathcal{H}}{2} + \lambda)^2 - \mathcal{H}(\frac{\mathcal{H}}{2} + \lambda) - \mu\right)\exp(-(\frac{\mathcal{H}}{2} + \lambda)r) \\
=& \left(\partial_r^2 g(r) - (\frac{\mathcal{H}}{2} + \lambda)\partial_r g(r) + \mathcal{H}\partial_r g(r) + (\frac{\mathcal{H}}{2} + \lambda)^2 - \mathcal{H}(\frac{\mathcal{H}}{2} + \lambda) - \mu\right)\exp(-(\frac{\mathcal{H}}{2} + \lambda)r)
\end{aligned}$$

On the other hand, we have

$$|\partial_r^2 f(r) + \mathcal{H}\partial_r f(r) - \mu f(r)| = | - B(r)| \sim O(\exp(-(\frac{\mathcal{H}}{2} + \lambda + 1)r))$$

Therefore,

$$\partial_r^2 g(r) - (\frac{\mathcal{H}}{2} + \lambda)\partial_r g(r) + \mathcal{H}\partial_r g(r) + (\frac{\mathcal{H}}{2} + \lambda)^2 - \mathcal{H}(\frac{\mathcal{H}}{2} + \lambda) - \mu = 0,$$

which implies that

$$\partial_r^2 f(r) + \mathcal{H}\partial_r f(r) - \mu f(r) = 0$$

Therefore,

$$f(r) = C\exp(-(\frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu})),$$

which contradicts that

$$f(r) \sim O(\exp(-(\frac{\mathcal{H}}{2} + \lambda)r))$$

Therefore, $\lambda = \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$

The eigenvalue estimate for scalar Laplacian operator

(1)(The weighted space)

Definition 2.3.9. We define the weighted Holder space $C^{k,\alpha}_{\delta}$ for the sections f of an associative vector bundle of the principal bundle $G \longrightarrow G/K$ by requiring that

$$(\cosh r)^{\delta}\sigma \in C^{k,\alpha}$$

where $C^{k,\alpha}_{\delta}$ refers to the local regularity of the section.

We see that these are then sections with local $C^{k,\alpha}_{\delta}$ regularity which decrease at infinity in $\exp(-\delta r)$.

Definition 2.3.10. We define the weighted Soblve space for the sections f of an associative vector bundle of the principal bundle $G \longrightarrow G/K$ by

$$W^{k,p}_{\delta} = \{f| \exp((\delta - \frac{\mathcal{H}}{p})r)\partial^k f \in L^p(M)|\}$$

(2)(The eigenvalue estimate for scalar Laplacian operator on the weighted space) Consider the operator

$$P = \Delta_0 - \mu$$

where Δ_0 stands for the scalar Laplacian operator and μ is the smallest eigenvalue of the Casmir operator $C(\mathfrak{m}_0, \rho_0)$.

Lemma 2.3.11. *For compact supported smooth function f, we have*

$$\|e^{-\gamma r} P f\|_{L^2(M)} \geq \left(\frac{\mathcal{H}^2}{4} - \gamma^2 + \mu\right) \|e^{-\gamma r} f\|_{L^2(M)}$$

Proof: For compact supported smooth function $f \in C_c^\infty(M)$,

$$\begin{aligned}\int_{B_R} < d^*df, f > e^{-2\gamma r} &= \int_{B_R} |df|^2 e^{-2\gamma r} - 2\gamma f(\partial_r f) e^{-2\gamma r} \\ &\geq \int_{B_R} (\partial_r f)^2 e^{-2\gamma r} - 2\gamma f(\partial_r f) e^{-2\gamma r}\end{aligned} \tag{2.3.5}$$

where $supp(f) \subseteq B_R$ and B_R is a geodesic ball with radius R and center p_0. And since

$$-f \cdot (\partial_r f) \leq \frac{1}{2}(a^{-1}(\partial_r f)^2 + af^2)$$

together with (2.3.5), we have

$$\begin{aligned}\int_{B_R} < d^*df, f > e^{-2\gamma r} &\geq \int_{B_R} (\partial_r f)^2 e^{-2\gamma r} - 2\gamma f(\partial_r f) e^{-2\gamma r} \\ &\geq \int_{B_R} (-2af \cdot (\partial_r f) - a^2 f^2 - 2\gamma \cdot (\partial_r f) \cdot f) e^{-2\gamma r} \\ &= \int_{B_R} (-2(a+\gamma) f \cdot \partial_r f - a^2 f^2) e^{-2\gamma r}\end{aligned} \tag{2.3.6}$$

If we take a spherical coordinate in B_R, then we have

$$\int_{B_R} -f \cdot (\partial_r f) \cdot e^{-2\gamma r} = \int_{\partial B_R} \int_0^R -f \cdot (\partial_r f) \cdot e^{-2\gamma r} \omega_n(r) dr \tag{2.3.7}$$

where $\omega_n(r)$ is the volume of the n-dimensional unit sphere. Then, by the integration by parts, we have

$$\begin{aligned}&\int_{\partial B_R} \int_0^R -f \cdot (\partial_r f) \cdot e^{-2\gamma r} \omega_n(r) dr \geq \frac{1}{2} \int_{\partial B_R} \int_0^R f^2 \cdot e^{-2\gamma r} \cdot (\partial_r \omega_n(r)) dr \\ &- \int_{\partial B_R} \int_0^R \gamma \cdot f^2 \cdot e^{-2\gamma r} \omega_n(r) dr = \frac{1}{2} \int_{\partial B_R} \int_0^R f^2 \cdot e^{-2\gamma r} \cdot \frac{\partial_r \omega(r)}{\omega_n(r)} \omega_n(r) dr \\ &- \int_{\partial B_R} \int_0^R \gamma \cdot f^2 \cdot e^{-2\gamma r} \omega_n(r) dr\end{aligned} \tag{2.3.8}$$

Since $\mathcal{H} = \frac{\partial_r \omega(r)}{\omega_n(r)}$,

$$\int_{B_R} -f \cdot (\partial_r f) \cdot e^{-2\gamma r} \geq \int_{B_R} (\frac{\mathcal{H}(r)}{2} - \gamma) \cdot f^2 \cdot e^{-2\gamma r} \tag{2.3.9}$$

Then, combine (2) and (5), we have

$$\int_{B_R} < d^* df, f > e^{-2\gamma r} \geq \int_{B_R} [2(a + \gamma)(\frac{\mathcal{H}(r)}{2} - \gamma) - a^2] f^2 \cdot e^{-2\gamma r} \tag{2.3.10}$$

Take $a = \frac{\mathcal{H}(r)}{2} - \gamma$. Then we have

$$\int_{B_R} < -\Delta f, f > e^{-2\gamma r} \geq (\frac{\mathcal{H}^2}{4} - \gamma^2) \int_{B_R} f^2 \cdot e^{-2\gamma r}$$

Therefore,

$$\int_{B_R} < -\Delta f + \mu f, f > e^{-2\gamma r} \geq (\frac{\mathcal{H}^2}{4} - \gamma^2 + \mu) \int_{B_R} f^2 \cdot e^{-2\gamma r} \tag{2.3.11}$$

By the Cauchy-Schwartz inequality, we have

$$\|e^{-\gamma r} P f\|_{L^2(M)} \geq \left(\frac{\mathcal{H}^2}{4} - \gamma^2 + \mu \right) \|e^{-\gamma r} f\|_{L^2(M)}$$

Isomorphism theorems

Theorem 2.3.12. *The scalar operator*

$$P = \Delta_0 - \mu$$

where Δ_0 stands for the scalar Laplacian operator and μ is the smallest eigenvalue of the Casmir operator $C(\mathfrak{m}_0, \rho_0)$, is an isomorphism, provided that

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Proof: Obviously, P is an injective by the lemma 1.3. We only need to show that P is surjective. Since we have

$$\int_{B_R} < -\Delta f + \mu f, f > e^{-2\gamma r} \geq (\frac{\mathcal{H}^2}{4} - \gamma^2 + \mu) \int_{B_R} f^2 \cdot e^{-2\gamma r}$$

then, by the Lax-Milgram theorem, we can get the existence of the weak solution for the equation

$$Pu = f$$

for $f \in H^k_\delta$. Then by the interior Soblve elliptic estimate

$$||u||_{H^2_\delta} \leq C(||u||_{L^2_\delta} + ||f||_{L^2_\delta})$$

We see this weak solution $u \in L^{2,k+2}_\delta$. Therefore, P is also a surjective. □

Lemma 2.3.13. *If* $u \in L^p_\delta(M)$*, then*

1) $||u||_{L^2_{\delta_1}} \leq C(p, \delta, \delta_1)||u||_{L^p_\delta}$ *for* $2 < p < +\infty$ *and* $\delta_1 < \delta$*;*

2) $||u||_{L^2_{\delta_1}} \leq C(p, \delta, \delta_1)||u||_{L^p_\delta}$ *for* $1 \leq p < 2$ *and* $\delta_1 > \delta$*;*

Proof: $u \in L^p_\delta$ implies that

$$e^{(\delta - \frac{\mathcal{H}}{p})r} u \in L^p$$

Therefore, by the Holder inequality, we have

$$\begin{aligned} ||u||^2_{L^2_{\delta_1}} &= \int_M (e^{2(\delta_1 - \frac{\mathcal{H}}{2})r} u^2) = \int_M e^{2(\delta_1 - \delta)r} \cdot e^{2\delta r} u^2 e^{-\mathcal{H}r} dV_M \\ &\leq \Big(\int_M e^{2(\delta_1 - \delta)r q_*} e^{-\mathcal{H}r} dV_M\Big)^{\frac{1}{q_*}} \Big(\int_M (e^{\delta r} u)^{2q} e^{-\mathcal{H}r} dV_M\Big)^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{q_*} + \frac{1}{q} = 1$. Then, take $q = \frac{p}{2}$ and $q_* = \frac{p}{p-2}$. Then, we have

$$||u||^2_{L^2_{\delta_1}} \leq \Big(\int_M e^{\frac{2p}{p-2}(\delta_1 - \delta) r q_*} e^{-\mathcal{H} r} dV_M \Big)^{\frac{p-2}{p}} \cdot ||u||^2_{L^p_\delta}$$

Then, we can get the result. □

Lemma 2.3.14. *The scalar operator* $P : W^{2,p}_\delta \longrightarrow L^p$ *(which is another extension of the operator in Theorem 1.2) is an isomorphism, provided*

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Proof: For injective, if $Pu = 0$ and $u \in W^{2,p}_\delta$. Then by Lemma 1.4, there exists a δ_1 which can be as close to δ as you want, such that $||u||_{L^2_{\delta_1}} \leq C(p, \delta, \delta_1)||u||_{L^p_\delta}$. Therefore,

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta_1 < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

From the lemma 1.2, we see that $u \in L^2_{\delta_1}$. Therefore, $u \equiv 0$. For surjective, consider the transpose operator of $P : W^{2,p}_\delta \longrightarrow L^p$, $P^* : W^{2,p_*}_{\delta^*} \longrightarrow L^{p_*}_{\delta_*}$. Since $\delta_* = \mathcal{H} - \delta$ and $\frac{1}{p_*} + \frac{1}{p} = 1$, we have

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta_* < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Therefore, $P^* : W^{2,p_*}_{\delta^*} \longrightarrow L^{p_*}_{\delta_*}$ is also injective which implies that cokernel of P is trivial. Therefore, P is surjective. □

Theorem 2.3.15. *The scalar operator* $P : \Delta_0 + \mu$ *is an isomorphism as the map* $P : C^{2,\alpha}_\delta \longrightarrow C^{0,\alpha}_\delta$

Proof: For injective, by the Holder interior estimate, we have

$$||u||_{C^{2,\alpha}_\delta(B_1)} \leq C(||u||_{C^0_\delta(B_2)} + ||Pu||_{C^{0,\alpha}_\delta(B_2)})$$

where C is a constant which does not depends on the center of the concentric ball B_1 and B_2. Therefore, we have the globally Holder estimate,

$$||u||_{C^{2,\alpha}_\delta(M)} \leq C(||u||_{C^0_\delta((M)} + ||Pu||_{C^{0,\alpha}_\delta(M)})$$

Therefore, if $Pu \equiv 0$, then we have

$$||u||_{C^{2,\alpha}_\delta(M)} \leq C||u||_{C^0_\delta(M)}$$

Since $||u||_{L^p_{\delta_1}(M)} \leq C(p, \delta_1, \delta)||u||_{C^0_\delta(M)}$ for $\delta_1 \leq \delta$, $u \in L_{\delta_1}$. And moreover, we can always take

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta_1 < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Therefore, from Theorem 1.5, we have $u \equiv 0$ which implies that $P : C^{2,\alpha}_\delta(M) \to C^{0,\alpha}_\delta(M)$ is injective.

For surjective, let $f \in C^{0,\alpha}_\delta(M)$, we shall show that there exists a function $u \in C^{2,\alpha}_\delta(M)$ such that $Pu = f$. Since $\|f\|_{L^p_{\delta_1}(M)} \leq C\,(p, \delta_1, \delta)\,\|f\|_{C^0_\delta}$, $f \in L^p_{\delta_1}(M)$. Then, by theorem 1.5, there exists $u \in L^p_{\delta_1}(M)$ such that $Pu = f$ and

$$||u||_{W^{2,p}_{\delta_1}(M)} \leq C(||f||_{L^p_{\delta_1}(M)})$$

By Morrey inequality, we have

$$||e^{(\delta_1 - \frac{\mathcal{H}}{p})r}u||_{C^0(B_1)} \leq C||u||_{W^{2,p}_{\delta_1}(B_1)}$$

where, by the homogenity of the symmetric space, the constant C does not reply on the center of the ball B_1. Therefore, we have

$$||u||_{C^0_{\delta_1-\frac{\mathcal{H}}{p}}(M)} \leq C||u||_{W^{2,p}_{\delta_1}(M)}$$

Then, by the Holder interior estimate,

$$\begin{aligned}
||u||_{C^{2,\alpha}_{\delta_1-\frac{\mathcal{H}}{p}}(M)} \leq & C(||u||_{C^0_{\delta_1-\frac{\mathcal{H}}{p}}(M)} + ||Pu||_{C^{0,\alpha}_{\delta_1-\frac{\mathcal{H}}{p}}(M)}) \\
\leq & C(||u||_{W^{2,p}_{\delta_1}(M)} + ||Pu||_{C^{0,\alpha}_{\delta_1-\frac{\mathcal{H}}{p}}(M)}) \\
\leq & C(||Pu||_{L^p_{\delta_1}(M)} + ||Pu||_{C^{0,\alpha}_{\delta_1-\frac{\mathcal{H}}{p}}(M)}) \\
\leq & C||Pu||_{C^{0,\alpha}_{\delta}(M)} \leq C||f||_{C^{0,\alpha}_{\delta}(M)}
\end{aligned}$$

which implies that $u \in C^{2,\alpha}_{\delta_1-\frac{\mathcal{H}}{p}}(M)$.

We shall improve the weight $\delta_1 - \frac{\mathcal{H}}{p}$ to δ by maximal principal. Let $\Phi = e^{\delta r}u$. Then, we have the following equation

$$e^{\delta r}(Pu) = \Delta\Phi - 2\delta(\partial_r\Phi) + (\delta^2 - \mathcal{H}(r)\delta - \mu)\Phi$$

Take R sufficiently large such that for $r > R$, we always have $(\delta^2 - \mathcal{H}(r)\delta - \mu) \leq \frac{1}{2}(\delta^2 - \mathcal{H}\delta - \mu) < 0$

Then, by maximal principal, we have

$$\begin{aligned}
\sup_{r\geq R}\Phi \leq & C\sup\left(\sup_{r=R}\Phi, 2(\delta^2 - \mathcal{H}\delta - \mu)^{-1}\sup_{r\geq R}(e^{\delta r}Pu)\right) \\
\min_{r\geq R}\Phi \leq & C\min\left(\min_{r=R}\Phi, 2(\delta^2 - \mathcal{H}\delta - \mu)^{-1}\min_{r\geq R}(e^{\delta r}Pu)\right)
\end{aligned}$$

Therefore, we have

$$||\Phi||_{C^0(M-B_R)} \leq C(||\Phi||_{C^0(\partial B_R)} + ||Pu||_{C^{0,\alpha}_{\delta}(M)})$$

Moreover, we have

$$||\Phi||_{C^0(B_R)} = ||e^{\delta r}u||_{C^0(B_R)} \leq C(R,p_0)||u||_{C^{2,\alpha}_{\delta_1 - \frac{\mathcal{H}}{p}}(M)} \leq C(R,p_0)||f||_{C^{0,\alpha}_{\delta}(M)}$$

Therefore, we have

$$||u||_{C^0_\delta(M)} \leq C||Pu||_{C^{0,\alpha}_\delta(M)} \leq C||f||_{C^{0,\alpha}_\delta(M)}$$

Then, by the Holder estimate, we have $u \in C^{2,\alpha}_\delta(M)$ which implies that $P : C^{2,\alpha}_\delta \to C^{0,\alpha}_\delta$ is surjective. □

Theorem 2.3.16. *If* $\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$*, the operator* P *on scalars is an isomorphism* $C^{k+2,\alpha}_\delta \to C^{k,\alpha}_\delta$.

Proof: Because the $ker(P)$ does not rely the choice of the weighted space. Therefore, this $P : C^{k+2,\alpha}_\delta \to C^{k,\alpha}_\delta$ is also injective. We just need to show that $P : C^{k+2,\alpha}_\delta \to C^{k,\alpha}_\delta$ is surjective.

Theorem 2.3.17. *For the Green function of* G_0 *for the scalar operator* $PG_0 = \delta_{p_0}$ *satisfies the following properties*

$$\begin{aligned} &G_0(r) \sim \ constant \ \cdot e^{-(\mathscr{H}+\sqrt{\mathscr{H}^2+\mu})r}, \quad r \to \infty \\ &G_0(r) \sim \tfrac{\omega_n}{r^{n-2}}, \quad r \to 0 \\ &G_0(r) > 0 \end{aligned} \tag{2.3.12}$$

Therefore, the green function of homogeneous vector bundle, $G_{\xi_{p_0}}$, can be controlled in the following way,

$$|G_{\xi_{p_0}}| \leq CG_0$$

Theorem 2.3.18. *Let* E *be a homogeneous bundle associated with the representation* ρ_0*,* $P = \Delta$ *which satisfies that* $(Pu,u) \geq c(u,u)$*. If* μ *is the smallest eigenvalue of*

$C(\mathfrak{m}_0, \rho_0)$, then

$$P : C_\delta^{k+2,\alpha} \to C_\delta^{k,\alpha}$$

is an isomorphism, provided that

$$\frac{\mathcal{H}}{2} - \sqrt{\frac{\mathcal{H}^2}{4} + \mu} < \delta < \frac{\mathcal{H}}{2} + \sqrt{\frac{\mathcal{H}^2}{4} + \mu}$$

Proof: Let us consider the equation $Pf = g$, hence

$$f(x) = \int G(x, y) g(y) dy$$

$$|f(x)| \leq c \int G_0(x, y)|g(y)| dy = c f_0(x)$$

Therefore, by the theorem 1.3 we can get the result. □

Remark 2.3.19. For the Lichnerowicz operator $\Delta_L - 2(n)$ on hyperbolic space. $\delta \in (0, n)$.

2.3.4 Remark

For the minimal regularity that an asymptotically symmetric should have to ensure the its metric is polyhomogeneous near the boundary, there are some results about real and complex case.

Theorem 2.3.20 ([9])**.** *Let g be a Riemannian metric on M. Suppose that* $\dim M = n+1 \geq 3$; *g is Einstein with Ric $(g) = -ng$; g is conformally compact of class C^2; and the representative $\gamma = \rho^2 g\big|_{\partial M}$ of the conformal infinity of g is smooth. Let $\widetilde{\gamma}$ be any smooth representative of the conformal class $[\gamma]$. Then for any $0 < \lambda < 1$, there exists $R > 0$ and a $C^{1,\lambda}$ collar diffeomorphism $\Phi : \bar{M}_R \to \bar{M}$ such that $\Phi^* g$ can be written in the form (1.1) $\Phi^* g = \rho^{-2}\left(d\rho^2 + G(\rho)\right)$, where $\{G(\rho) : 0 < \rho \leq R\}$ is a one-parameter* *family of smooth Riemannian metrics on* Y, $d\rho^2 + G(\rho)$ *has a continuous extension to* $\bar{M}_R$ *with* $G(0) = \widetilde{\gamma}$, *and has the following regularity: (a) If* $\dim M$ *is even or equal to 3 , then* $d\rho^2 + G(\rho)$ *extends smoothly to* $\bar{M}_R$, *so* $\Phi^* g$ *is conformally compact of class* C^∞. *(b) If* $\dim M$ *is odd and greater than 3 , then* G *can be written in the form*

$$G(\rho) = \varphi\left(\rho, \rho^n \log \rho\right)$$

with $\varphi(\rho, z)$ *a two-parameter family of Riemannian metrics on* Y *that is smooth in all of its arguments as a function on* $Y \times [0, R] \times [R^n \log R, 0]$. *Furthermore,* $\Phi^* g$ *is smoothly conformally compact if and only if* $\partial_z \varphi(0, 0)$ *vanishes identically on* ∂M.

and

Theorem 2.3.21 ([6])**.** *Either half a ball*

$$M = \begin{cases} \left\{x_1^2 + \cdots + x_n^2 < 1, x_1 > 0\right\} & \text{in the real case,} \\ \left\{x_1^2 + x_2^2 + \left(\frac{x_3^2 + \cdots + x_n^2}{2}\right)^2 < 1, x_1 > 0\right\} & \text{in the complex case} \end{cases}$$

and γ *a smooth metric on* $\partial_\infty M = \{x_1 = 0\} \cap \bar{M}$, *resp.* η *a smooth contact structure and* J *a smooth almost complex structure in* $H = Ker\,\eta$ *on* $\partial_\infty M$ *such that* $\gamma = d\eta(., J.)$ *is defined positive.*

If g *is an asymptotically hyperbolic Einstein metric, resp. complex, in the sense that* $g - g_0(\gamma) \in C^{1,\alpha}_\varepsilon$ *for* $\alpha \in [0, 1]$ *and* $\epsilon > 0$, *then there is a half ball* N *included in* M, *a diffeomorphism* Φ *of* $\bar{N}$ *inducing identity on* $\partial N \cup \partial_\infty N$, *a sequence of asymptotically hyperbolic metrics* g_k *on* N, *with finite polyhomogeneous development, and a couple* (δ, η) *of strictly positive reals such that*

$$\forall k \in \mathbb{N}, \quad \Phi^* g - g_k \in C^\infty_{\mu_+ + a_k + \delta, \eta}(N)$$

In addition, the same estimate is valid for all the transverse derivatives.

Chapter 3

The stability of AHE manifolds under Ricci flow

In this chapter, we shall mainly talk about the stability of the AHE manifold with some addition restrictions on its conformal boundary. In the section 3.1, we shall review the stability for compact quotients of hyperbolic space following the paper of R.Ye [49] and that for the hyperbolic spaces following the paper [29] of H.Li and H.Yin, the paper [43] of OC.Schnürer, F.Schulze, and M.Simon and the paper [2] of R.Bamler. In the section 3.2, we shall review the stability result for the hyperbolic space following the paper [2] of R.Bamler to see the relations between the heat kernel estimate and the stability of Einstein manifolds. In fact, based on [2], in order to obtain the stability result for more general AH Einstein manifolds, it suffices to give the corresponding heat kernel estimate (See lemma 3.2.2). In the section 3.4, we shall talk about this kind of heat kernel estimate following the method of X.Chen and A.Hassell [10], by which they obtain the scalar heat kernel estimate on more general AH manifolds. In the light of their result, we can see that the corresponding heat kernel estimate is determined by the meromorphic continuation of the resolvent of the modified Laplacian (the result of

R.Mazzeo and R.Melrose in [33] or the result of A.Vasy in [48]) and the high energy resolvent estimates in strips around the real axis via a parametrix construction (the result of AS.Barreto, A.Vasy and R.Melrose in [44]). Therefore, in order to obtain a similar heat kernel estimate for the symmetric two tensor, we need to generalize the above two scalar results into tensor case. C.Hadfield generalized the meromorphic continuation result into the tensor case in [24]. The high energy estimate for the tensor case is still open.

§ 3.1 The stability for hyperbolic spaces

Let $\bar{M}$ be a $(n+1)-$dimensional compact manifold with boundary ∂M. Suppose that there is a complete Riemannian metric $\tilde{g}$ in the interior of $\bar{M}$ denoted it M, and there is a defining function ρ on $\bar{M}$, (i.e. $\rho > 0$ on M; $\rho = 0$ on ∂M; $d\rho \neq 0$ on ∂M) such that $\rho^2\tilde{g}$ can be extended into a smooth Riemannian metric on $\bar{M}$. Since for different defining functions ρ_1 and ρ_2 on $\bar{M}$, there exists a positive function f such that $\rho_1 = f\rho_2$, the interior Riemannian metric $\tilde{g}$ uniquely determine a conformal structure on boundary ∂M. We call $(M^{n+1}, \tilde{g})$ an asymptotically hyperbolic manifold with conformal boundary $\rho^2\tilde{g}|_{\partial M}$.

Moreover, we call asymptotically hyperbolic manifold $(M, \tilde{g})$ asymptotically hyperbolic Einstein manifold if $(M, \tilde{g})$ is also an Einstein manifold. The asymptotically hyperbolic Einstein manifold $(M, \tilde{g})$ is a fixed point of the normalized Ricci flow equation

$$\partial_t g_t = -2\mathrm{Ric}_{g_t} + 2\lambda g_t \tag{3.1.1}$$

where λ is the Einstein constant of $(M, \tilde{g})$, (i.e. $\mathrm{Ric} = \lambda g$). We can always normalize the equation into the standard case $\lambda = -n$.

In this notes, we want to prove stabality result for asymptotically hyperbolic Einstein metric $\tilde{g}$, (i.e. we will show that every sufficiently small perturbation $g_0 = \tilde{g} + h$ flow back to $\tilde{g}$ under the normalized Ricci flow as $t \to \infty$).

If $(M, \tilde{g})$ is an hyperbolic space, then we have the result of R. Bamler [2]

Theorem 3.1.1 ([2]). *Let* $(M, \tilde{g})$ $\mathbb{H}^{n+1}$ *for* $n \geq 2$*, choose a basepoint* $x_0 \in M$ *and let* $r = d(\cdot, x_0)$ *denote the radial distance function.*

There is an $\varepsilon_1 > 0$ *and for every* $q < \infty$ *an* $\varepsilon_2 = \varepsilon_2(q) > 0$ *such that the following holds: If* $g_0 = \tilde{g} + h$ *and* $h = h_1 + h_2$ *satisfies*

$$|h_1| < \frac{\varepsilon_1}{r+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q \, dx \right)^{1/q} < \varepsilon_2$$

then the normalized Ricci flow exists for all time and we have convergence $g_t \longrightarrow \bar{g}$ *in the pointed Cheeger-Gromov sense.*

We just want to generalize the above result into the case of asymptotically hyperbolic Einstein manifold in the interior of $n+1$ dimensional ball such that the conformal boundary of this asymptotically hyperbolic Einstein manifold is a perturbation of the conformal structure on the standard n dimensional sphere.

Theorem 3.1.2 (Our Goal)**.** *Let $M = B^{n+1}$ be a ball wiht $n \geq 3$ and $\hat{h}$ the standard metric on the sphere S^n and $g_{\mathbb{H}}$ be the standard hyperbolic metric on B^{n+1}. For any asymptotically hyperbolic Einstein manifold (M, g) with nonpositive sectional curvature and a defining function ρ such that $\hat{g} = \rho^2 g|_{\partial M}$ is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$ norm, for some $0 < \alpha < 1$ and g is sufficiently close to $g_{\mathbb{H}}$ in the sense of C^0. And choose a basepoint $x_0 \in M$ and let $r = d(., x_0)$ denote the radial distance function.*

There is an $\epsilon > 0$ such that the following holds: If $g_0 = g + h$ satisfies

$$|h| < \frac{\epsilon}{r+1}$$

then the normalized Ricci flow exists for all time and we have convergence $g_t \to g$ in the pointed Cheeger-Gromov sense.

The existence of such asymptotically hyperbolic manifolds is by the result of R. Graham and J. Lee (Theorem A. [21]).

§ 3.2 The heat kernel estimate and stability of asymptotically hyperbolic Einstein manifolds

In order to prove the long time existence of the Ricci flow, we need some estimates for heat kernel

$$\partial_t k_t = \Delta k_t + R(k_t) \quad and \quad k_t \to \delta_{p_0} id_{E_{p_0}} \quad as\ t \to 0 \tag{3.2.1}$$

where $(k_t)_{0<t<T} \in C^\infty(M;E) \otimes E_{p_0}$ and $E = Sym^2T^*M$ and $R(h)_{il} = \tilde{g}^{jk_1}\tilde{g}^{i_1i_2}\tilde{R}_{ii_2lj}h_{k_1i_1}$.

Lemma 3.2.1. *Let (M, g) be an asymptotically hyperbolic Einstein manifold of theorem 3.1.2. Choose a basepoint $x_0 \in M$ and consider the radius distance function $r = d(\cdot, x_0)$. If the heat kernel k_t defined by (3.2.1) satisfying that: For all $x_1 \in M$ and $r_1 = r(x_1)$, $t \geq 0$*

$$\int_M |k_t|(x_1, x)\,|h|(x)dx < \frac{C(\omega)}{(r_1 + 1 + a + t)^w}$$

provided that $h \in C^\infty\left(M; \mathrm{Sym}_2 T^*M\right)$ and that

$$|h|(x) < \frac{1}{(r(x) + 1 + a)^w}$$

for some $a \geq 0$. The the result of theorem 3.1.2 holds.

By the argument of lemma 6.3, lemma 6.4 in [2], the result of lemma 3.2.1 is determined by

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C\exp(\lambda_B t) \qquad \text{for } t > 0$$

Lemma 3.2.2. *Let (M, g) be an asymptotically hyperbolic Einstein manifold of theorem 3.1.2. If the heat kernel satisfies that*

$$\|k_t\|_{L^1(M)} \leq C \quad \|k_t\|_{L^2(M)} \leq C\exp(\lambda_B t) \qquad \textit{for } t > 0$$

Then the assumption of lemma 3.2.1 holds.

3.2.1 Ingredients of the proof of the heat kernel estimate

In [2], R. Bamler have given the L^1 boundedness and L^2 decay of the heat kernel on $\mathbb{H}^{n+1}$ for $n \geq 3$ and $\mathbb{CH}^{2(n+1)}$ for $n \geq 2$. In this section, we will derive the same result on $\mathbb{H}^{n+1}$ directly from the equivalent formula of the result of Davies-Mandouvalos.

Theorem 3.2.3 (L^1 Integration)**.** *Let $\mathbb{H}^{n+1}$ be a hyperbolic space. Then the heat kernel of the Laplacian operator on $\mathbb{H}^{n+1}$, $H(t,z,z')$, satisfies that*

$$\|H\|_{L^1(\mathbb{H}^{n+1})} \le C \quad for\ t > 0$$

where $C = C(n)$.

Proof. From the result of Davies and Mandouvalos, we only need to show the following integral is uniformly bounded.

$$\int_0^\infty t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} - \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)\sinh^n(r)dr$$

By the fact that

$$\sinh^n(r) \to \exp(nr) \text{ as } r \to \infty \quad and \quad \sinh^n(r) \to r^n \text{ as } r \to 0$$

we only need to show that

$$\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} + \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)dr < +\infty \tag{3.2.2}$$

and

$$\int_0^1 t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} - \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)r^n dr < +\infty \tag{3.2.3}$$

Then, we can discuss the case that $0 < t \le 2$ and the case that $t > 2$ respectively.

Case 1.1 ($n \ge 2$) : If $0 < t \le 2$, then for (1)

$$
\begin{aligned}
&\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{n^2 t}{4} - \frac{r^2}{4t} + \frac{nr}{2}\right) \cdot (1+r+t)^{n/2-1}(1+r)dr \\
\leq &\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{4t} + \frac{nr}{2}\right) \cdot (3+r)^{n/2-1}(1+r)dr \\
\leq &C \int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{4t} + \frac{nr}{2}\right) r^{n/2} dr
\end{aligned}
$$

where we use the fact that $(a+r)^k \leq C(a)(1+r^k)$. Then, we have

$$
\begin{aligned}
&\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{4t} + \frac{nr}{2}\right) r^{n/2} dr \\
= &\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t} - [\frac{r^2}{8t} - \frac{nr}{2}]\right) r^{n/2} dr \\
\leq &\int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t} + \frac{n^2 t}{2}\right) r^{n/2} dr \\
\leq &\exp\left(\frac{n^2 t}{2}\right) \int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) r^{n/2} dr \\
\leq &\exp\left(\frac{n^2 t}{2}\right) \int_1^\infty t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) r^{n} dr \\
\leq &\exp\left(\frac{n^2 t}{2}\right) \frac{8^{\frac{n}{2}+\frac{1}{2}}}{2} \int_1^\infty \exp\left(-\frac{r^2}{8t}\right) \left(\frac{r^2}{8t}\right)^{\frac{n}{2}-\frac{1}{2}} d(\frac{r^2}{8t}) \\
\leq &\exp\left(\frac{n^2 t}{2}\right) \frac{8^{\frac{n}{2}+\frac{1}{2}}}{2} \int_{\frac{1}{8t}}^\infty \exp(-x)\,(x)^{\frac{n}{2}-\frac{1}{2}}\, dx \\
\leq &\exp\left(\frac{n^2 t}{2}\right) \frac{8^{\frac{n}{2}+\frac{1}{2}}}{2} \int_0^\infty \exp(-x)\,(x)^{\frac{n}{2}-\frac{1}{2}}\, dx \leq \exp\left(\frac{n^2 t}{2}\right) \frac{8^{\frac{n}{2}+\frac{1}{2}}}{2} \Gamma(\frac{n}{2}+\frac{1}{2}) \leq +\infty
\end{aligned}
$$

And for (2),

$$\begin{aligned}
&\int_0^1 t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}-\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)r^n dr\\
\leq& C\int_0^1 t^{-(n+1)/2}\exp\left(-\frac{r^2}{4t}\right)r^n dr\\
\leq& C2^n\int_0^1 \exp\left(-\frac{r^2}{4t}\right)\left(\frac{r^2}{4t}\right)^{\frac{n}{2}-\frac{1}{2}} d\left(\frac{r^2}{4t}\right)\\
\leq& C2^n\int_0^{\frac{1}{8t}}\exp(-x)(x)^{\frac{n}{2}-\frac{1}{2}}dx \leq C2^n\Gamma(\frac{n}{2}+\frac{1}{2}) \leq +\infty
\end{aligned}$$

Case 1.2 ($n \geq 2$) : If $t > 2$, for (1), we have

$$\begin{aligned}
&\int_1^\infty t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}+\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)dr\\
=&\left[\int_1^t+\int_t^\infty\right]t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}+\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)dr
\end{aligned}$$

Then, for the first part, we have

$$\begin{aligned}
&\int_1^t t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}+\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)dr\\
\leq& \exp\left(-\frac{(n^2-2n)t}{4}\right)(1+2t)^{(n/2)-1}(1+t)\int_1^t t^{-(n+1)/2}\exp\left(-\frac{r^2}{4t}\right)dr\\
\leq& \exp\left(-\frac{(n^2-2n)t-\varepsilon}{4}\right)\int_1^t t^{-(n+1)/2}\exp\left(-\frac{r^2}{4t}\right)dr\\
\leq& 2^n\exp\left(-\frac{(n^2-2n)t-\varepsilon}{4}\right)\int_1^t \exp\left(-\frac{r^2}{4t}\right)\left(\frac{r^2}{4t}\right)^{\frac{n}{2}-\frac{1}{2}} d\left(\frac{r^2}{4t}\right)\\
\leq& 2^n\exp\left(-\frac{(n^2-2n)t-\varepsilon}{4}\right)\int_{\frac{1}{4t}}^{4t}\exp(-x)(x)^{\frac{n}{2}-\frac{1}{2}}dx\\
\leq& 2^n\exp\left(-\frac{(n^2-2n)t-\varepsilon}{4}\right)\Gamma(\frac{n}{2}+\frac{1}{2}).
\end{aligned}$$

and for the second part, we have

$$
\begin{aligned}
&\int_t^\infty t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}+\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)dr\\
\leq& C\exp\left(-\frac{n^2t}{4}\right)t^{-(n+1)/2}\int_t^\infty \exp\left(-\frac{r^2}{4t}+\frac{nr}{2}\right)\cdot r^{n/2}dr\\
\leq& C\exp\left(-\frac{n^2t}{4}\right)t^{-(n+1)/2}\int_t^\infty \exp\left(-\frac{1}{4t}(r-nt)^2+\frac{n^2t}{4}\right)\cdot r^{n/2}dr\\
\leq& Ct^{-(n+1)/2}\int_t^\infty \exp\left(-\frac{1}{4t}(r-nt)^2\right)\cdot r^{n/2}dr\\
\leq& Ct^{-(n+1)/2}\left[\int_t^{nt} \exp\left(-\frac{1}{4t}(r-nt)^2\right)\cdot r^{n/2}dr+\int_{nt}^\infty \exp\left(-\frac{1}{4t}(r-nt)^2\right)\cdot r^{n/2}dr\right]\\
\leq& Ct^{-(n+1)/2}\left[\int_0^{(n-1)t} \exp\left(-\frac{x^2}{4t}\right)\cdot(nt-x)^{n/2}dx+\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)\cdot(x+nt)^{n/2}dx\right]\\
\leq& C2t^{-(n+1)/2}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)\cdot(x+nt)^{n/2}dx\\
\leq& C2n^{n/2}t^{-(n+1)/2}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)(\sqrt{x}+\sqrt{nt})^n dx
\end{aligned}
$$

$$
\begin{aligned}
\leq& C2n^{n/2}t^{-(n+1)/2}\sum_{k=0}^{n}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)\sqrt{x}^k\sqrt{nt}^{n-k}dx\\
\leq& 2C\sum_{k=0}^{n} n^{n-\frac{k}{2}}t^{-\frac{k+1}{2}}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)x^{k/2}dx\\
\leq& 2C\sum_{k=0}^{n} n^{n-\frac{k}{2}}t^{-\frac{k+1}{2}}t^{\frac{k}{4}+\frac{1}{2}}2^{k/2}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)\left(\frac{x^2}{4t}\right)^{k/4}d\left(\frac{x^2}{4t}\right)\\
\leq& 2^{k/2+1}C\sum_{k=0}^{n} n^{n-\frac{k}{2}}t^{-\frac{k}{4}}\Gamma(\frac{k}{4}+1)\leq+\infty
\end{aligned}
$$

Remark 3.2.4. Actually, we can not expect the integral (1) decay as t goes to infinity. The above estimate is sharp.

And for (2), we have

$$\begin{aligned}
&\int_0^1 t^{-(n+1)/2}\exp\left(-\frac{n^2t}{4}-\frac{r^2}{4t}-\frac{nr}{2}\right)\cdot(1+r+t)^{n/2-1}(1+r)r^n dr\\
\leq &Ct^{-3/2}\int_0^1 \exp\left(-\frac{r^2}{4t}\right)dr \leq Ct^{-1}
\end{aligned}$$

Case 2.1 ($n = 1$) : If $0 < t \leq 2$, for (1), we have

$$\begin{aligned}
&\int_1^\infty t^{-1}\exp\left(-\frac{t}{4}-\frac{r^2}{4t}+\frac{r}{2}\right)\cdot(1+r+t)^{-1/2}(1+r)dr\\
\leq &\int_1^\infty t^{-1}\exp\left(-\frac{r^2}{4t}+\frac{r}{2}\right)\cdot(1+r)^{-1/2}(1+r)dr\\
\leq &C\int_1^\infty t^{-1}\exp\left(-\frac{r^2}{4t}+\frac{r}{2}\right)r^{1/2}dr\\
\leq &\exp\left(\frac{t}{2}\right)\int_1^\infty t^{-1}\exp\left(-\frac{r^2}{8t}\right)r^{1/2}dr\\
\leq &\exp\left(\frac{t}{2}\right)\int_1^\infty t^{-1}\exp\left(-\frac{r^2}{8t}\right)rdr\\
\leq &\exp\left(\frac{t}{2}\right)4\int_1^\infty \exp\left(-\frac{r^2}{8t}\right)d(\frac{r^2}{8t})\\
\leq &\exp\left(\frac{t}{2}\right)4\int_{\frac{1}{8t}}^\infty \exp(-x)\,dx \leq \exp\left(\frac{t}{2}\right)4\Gamma(1) \leq +\infty
\end{aligned}$$

and for (2)

$$\begin{aligned}
&\int_0^1 t^{-1}\exp\left(-\frac{t}{4}-\frac{r^2}{4t}-\frac{r}{2}\right)\cdot(1+r+t)^{-1/2}(1+r)r^n dr\\
\leq &C\int_0^1 t^{-1}\exp\left(-\frac{r^2}{4t}\right)rdr\\
\leq &C2\int_0^1 \exp\left(-\frac{r^2}{4t}\right)d\left(\frac{r^2}{4t}\right)\\
\leq &C2^n\int_0^{\frac{1}{8t}} \exp(-x)dx \leq C2\Gamma(1) \leq +\infty
\end{aligned}$$

Case 2.2 ($n = 1$) **:** If $n = 1$ and $t > 2$, then for (1), we have

$$\begin{aligned}
&\int_1^{\infty} t^{-1} \exp\left(-\frac{t}{4} - \frac{r^2}{4t} + \frac{r}{2}\right) \cdot (1+r+t)^{-1/2}(1+r)dr \\
=&\left[\int_1^{t} + \int_t^{\infty}\right] t^{-1} \exp\left(-\frac{t}{4} - \frac{r^2}{4t} + \frac{r}{2}\right) \cdot (1+r+t)^{-1/2}(1+r)dr
\end{aligned}$$

Then, for the first part, we have

$$\begin{aligned}
&\int_1^{t} t^{-1} \exp\left(-\frac{t}{4} - \frac{r^2}{4t} + \frac{r}{2}\right) \cdot (1+r+t)^{-1/2}(1+r)dr \\
\leq& \exp\left(-\frac{t}{4}\right) \frac{C}{t^{3/2}} \int_1^{t} \exp\left(-\frac{r^2}{4t} + \frac{r}{2}\right) rdr \\
\leq& \exp\left(-\frac{t}{4}\right) \frac{C}{t^{3/2}} \int_1^{t} \exp\left(-\frac{1}{4t}(r-t)^2 + \frac{t}{4}\right) rdr \\
\leq& \frac{C}{t^{3/2}} \int_1^{t} \exp\left(-\frac{1}{4t}(r-t)^2\right) rdr \\
\leq& \frac{C}{t^{3/2}} \int_0^{t-1} \exp\left(-\frac{y^2}{4t}\right)(t-y)dy \\
\leq& \frac{C}{t^{3/2}} \left[t \int_0^{t-1} \exp\left(-\frac{y^2}{4t}\right) dy - \int_0^{t-1} \exp\left(-\frac{y^2}{4t}\right) ydy\right] \\
\leq& C\left[C\Gamma(\frac{1}{2}) + t^{-1/2}\Gamma(1)\right] \leq +\infty
\end{aligned}$$

and for the second part, we have

$$\begin{aligned}
&\int_t^\infty t^{-1}\exp\left(-\frac{t}{4}-\frac{r^2}{4t}+\frac{r}{2}\right)\cdot(1+r+t)^{-\frac{1}{2}}(1+r)dr\\
\leq&\int_t^\infty t^{-1}\exp\left(-\frac{1}{4t}(r-t)^2\right)\cdot(1+r+t)^{-\frac{1}{2}}(1+r)dr\\
\leq&\int_t^\infty t^{-\frac{3}{2}}\exp\left(-\frac{1}{4t}(r-t)^2\right)\cdot(1+r)dr\\
\leq& t^{-\frac{3}{2}}\int_t^\infty \exp\left(-\frac{1}{4t}(r-t)^2\right)\cdot rdr\\
\leq& t^{-3/2}\int_0^\infty \exp\left(-\frac{x^2}{4t}\right)\cdot(x+t)dx\\
\leq& t^{-3/2}\left[\frac{2}{t}\int_0^\infty e^{-x}dx+t^{3/2}\int_0^\infty\exp\left(-x^2\right)dx\right]\leq+\infty
\end{aligned}$$

And for (2), we have

$$\begin{aligned}
&\int_0^1 t^{-1}\exp\left(-\frac{t}{4}-\frac{r^2}{4t}-\frac{r}{2}\right)\cdot(1+r+t)^{-\frac{1}{2}}(1+r)rdr\\
\leq& Ct^{-3/2}\int_0^1\exp\left(-\frac{r^2}{4t}\right)dr\leq Ct^{-1}
\end{aligned}$$

□

Theorem 3.2.5 (L^2 Integrability)**.** *Let $\mathbb{H}^{n+1}$ be a hyperbolic space with $n \geq 2$. Then the heat kernel of the Laplacian operator on $\mathbb{H}^{n+1}$, $H(t,z,z')$, satisfies that*

$$\|H\|_{L^2(\mathrm{H}^{n+1})}\leq C\exp(\lambda_B t)\quad \text{for } t>0$$

where $C = C(n)$

Proof. From the result of Davies and Mandouvalos, we only need to show the following integral is uniformly bounded.

$$\int_0^\infty t^{-(n+1)}\exp\left(-\frac{n^2t}{2}-\frac{r^2}{2t}-nr\right)\cdot(1+r+t)^{n-2}(1+r)^2\sinh^n(r)dr$$

By the fact that $\sinh^n(r) \to \exp(nr)$ as $r \to \infty$ and $\sinh^n(r) \to r^n$ as $r \to 0$ we only need to show that

$$\int_1^\infty t^{-(n+1)} \exp\left(-\frac{n^2 t}{2} - \frac{r^2}{2t}\right) \cdot (1+r+t)^{n-2}(1+r)^2 dr < +\infty$$

and

$$\int_0^1 t^{-(n+1)} \exp\left(-\frac{n^2 t}{2} - \frac{r^2}{2t} - nr\right) \cdot (1+r+t)^{n-2}(1+r)^2 r^n dr < +\infty$$

$$\begin{aligned}
&\int_1^\infty t^{-(n+1)} \exp\left(-\frac{n^2 t}{2} - \frac{r^2}{2t}\right) \cdot (1+r+t)^{n-2}(1+r)^2 dr \\
\leq &\exp\left(-\frac{n^2 t}{2}\right) \int_1^\infty t^{-(n+1)} \exp\left(-\frac{r^2}{2t}\right) \cdot (1+r+t)^{n-2}(1+r)^2 dr \\
\leq &C(n) \sum_{k=0}^{n-2} \exp\left(-\frac{n^2 t}{2}\right) \int_1^\infty t^{-(n+1)} \exp\left(-\frac{r^2}{2t}\right) \cdot r^{n-k} t^k dr \\
\leq &C(n) \sum_{k=0}^{n-2} \exp\left(-\frac{n^2 t}{2}\right) \int_1^\infty t^{-(n+1-k)} \exp\left(-\frac{r^2}{2t}\right) \cdot r^{2(n-k)+1} dr \\
\leq &C(n) \sum_{k=0}^{n-2} \exp\left(-\frac{n^2 t}{2}\right) \Gamma(n+1-k) \\
\leq &C(n) \exp\left(-\frac{n^2 t}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{-(n+1)} \exp\left(-\frac{n^2 t}{2} - \frac{r^2}{2t} - nr\right) \cdot (1+r+t)^{n-2}(1+r)^2 r^n dr \\
\leq &\exp\left(-\frac{n^2 t}{2}\right) \int_0^1 t^{-(n+1)} \exp\left(-\frac{r^2}{2t}\right) \cdot (1+r+t)^{n-2}(1+r)^2 dr \\
\leq &C(n) \exp\left(-\frac{n^2 t}{2}\right)
\end{aligned}$$

The last step is from the $\int_1^\infty$.

Remark 3.2.6. For $n = 1$, actually the heat kernel of $\|H\|_{L^2(M)}(t) \to \infty$ as $t \to 0$.

Case 2 : (n=1) Then, we have

$$
\begin{aligned}
&\int_1^{\infty} t^{-2} \exp\left(-\frac{t}{2} - \frac{r^2}{2t}\right) \cdot (1+r+t)^{-1}(1+r)^2 dr \\
\leq& C \int_1^{\infty} t^{-3} \exp\left(-\frac{t}{2} - \frac{r^2}{2t}\right) \cdot r^4 dr \\
\leq& C \exp\left(-\frac{t}{2}\right) \int_1^{\infty} t^{-3} \exp\left(-\frac{r^2}{2t}\right) \cdot r^4 dr \\
\leq& C \exp\left(-\frac{t}{2}\right) \Gamma(3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{1} t^{-2} \exp\left(-\frac{t}{2} - \frac{r^2}{2t} - r\right) \cdot (1+r+t)^{-1}(1+r)^2 r dr \\
\geq& C \frac{\exp\left(-\frac{t}{2}\right)}{(2+t)t^2} \int_0^1 \exp\left(-\frac{r^2}{2t}\right) r dr \\
\geq& C \frac{\exp\left(-\frac{t}{2}\right)}{(2+t)t^2} \int_0^{\frac{1}{2t}} \exp(-y^2) y^{-\frac{1}{2}} \sqrt{t} dy \to +\infty \quad as \;\; t \to 0
\end{aligned}
$$

□

Theorem 3.2.7 (Short time convolution)**.** *Let $\mathbb{H}^{n+1}$ be a hyperbolic space with $n \geq 2$. And let $H(t, z, z')$ be the heat kernel of the Laplacian operator on $\mathbb{H}^{n+1}$. Then, for every $0 \leq t \leq T$, we have*

$$
\int_M H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz \leq C(n, T, \omega) \frac{1}{[r(z', z_0) + 1 + a]^{\omega}}
$$

where $a \geq 0$, $\omega \geq 0$ and $C(n, T, \omega)$ is a constant only depending on n, T and ω.

Proof. First, by the heat kernel estimate of [11] in the complete manifold and the symmetry of the hyperbolic space, we have

$$H\left(z, z', t\right) \leq C(T, n) t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right)$$

Therefore,

$$\begin{aligned}
&\int_M H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz \\
=&\int_M t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz
\end{aligned}$$

Let $d = r(z', z_0)$. Then, we have

$$\begin{aligned}
&\int_M t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz \\
=&\left[\int_{M \setminus B_d(z')} + \int_{B_d(z')}\right] t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz
\end{aligned}$$

For the first part, by the triangle inequality, we have

$$\begin{aligned}
&\int_{M \setminus B_d(z')} t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz && (3.2.4)\\
\leq&\int_{M \setminus B_d(z')} t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z') - d + 1 + a]^{\omega}} dz && (3.2.5)\\
\leq&\int_d^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r - d + 1 + a]^{\omega}} \sinh^n(r) dr && (3.2.6)
\end{aligned}$$

For the second part, by the triangle inequality, we have

$$\begin{aligned}
&\int_{B_d(z')} t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[r(z, z_0) + 1 + a]^{\omega}} dz && (3.2.7)\\
\leq&\int_{B_d(z')} t^{-(n+1)/2} \exp\left(-\frac{r^2\left(z, z'\right)}{8t}\right) \frac{1}{[d - r(z, z') + 1 + a]^{\omega}} dz && (3.2.8)\\
\leq&\int_0^{d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d - r + 1 + a]^{\omega}} \sinh^n(r) dr && (3.2.9)
\end{aligned}$$

Case 1 : $(d \geq 2)$ For the (5), we have

$$\int_d^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \sinh^n(r) dr \tag{3.2.10}$$

$$\leq C \int_d^{2d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \exp(nr) dr \tag{3.2.11}$$

$$+ C \int_{2d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \exp(nr) dr \tag{3.2.12}$$

Then, for (10), we have

$$\begin{aligned}
&\int_d^{2d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \exp(nr) dr \\
\leq& C \exp(4n^2 t) \frac{1}{[1+a]^{\omega}} \int_d^{2d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) dr \\
=& C \exp(4n^2 t) d \frac{1}{[1+a]^{\omega}} \int_1^{2} t^{-(n+1)/2} \exp\left(-\frac{d^2x^2}{16t}\right) dx \\
=& C \exp(4n^2 t) d \frac{1}{[1+a]^{\omega}} T^{\beta/2} \int_1^{2} t^{-(n+\beta+1)/2} \exp\left(-\frac{d^2x^2}{16t}\right) dx \\
\leq& C \exp(4n^2 t) d T^{\beta/2} \frac{1}{[1+a]^{\omega}} \int_1^{2} \exp\left(-\frac{d^2x^2}{16t}\right) \cdot \left(\frac{x^2}{t}\right)^{\frac{(n+\beta-1)}{2}} d\left(\frac{x^2}{t}\right) \\
\leq& C \exp(4n^2 t) T^{\beta/2} \frac{1}{[1+a]^{\omega}} \frac{4^{(n+\beta+1)}}{d^{(n+\beta)}} \Gamma(\frac{n+\beta+1}{2}) \leq C(n,T,\beta) \frac{1}{[1+a]^{\omega}} \frac{1}{d^{n+\beta}} \\
\leq& \frac{C(n,T,\omega)}{[r(z',z_0)+1+a]^{\omega}}
\end{aligned}$$

Remark 3.2.8. Here, we just use the fact that

$$\frac{(d+1+a)^{\omega}}{(1+a)^{\omega} d^{\omega}} \leq C \quad and \quad for\ d \geq 2,\ a > 0$$

where $C = C(\omega)$ is a constant only relying on ω.

Then, for (11), we have

$$\begin{aligned}
&\int_{2d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \exp(nr) dr \\
\leq & \frac{1}{[d+1+a]^{\omega}} \int_{2d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t} + nr\right) dr \\
\leq & \frac{C\exp(4n^2t)}{[d+1+a]^{\omega}} \int_{2d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) dr \\
\leq & \frac{C\exp(4n^2t)}{[d+1+a]^{\omega}} \int_{2d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) r^n dr \\
\leq & \frac{C\exp(4n^2t)4^{n+1}}{[d+1+a]^{\omega}} \int_{2d}^{\infty} \exp\left(-\frac{r^2}{16t}\right) \cdot \left(\frac{r^2}{16t}\right)^{\frac{(n-1)}{2}} d\left(\frac{r^2}{16t}\right) \\
\leq & \frac{C\exp(4n^2t)4^{n+1}}{[d+1+a]^{\omega}} \Gamma(\frac{n+1}{2}) \leq \frac{C(n,T,\omega)}{[r(z',z_0)+1+a]^{\omega}}
\end{aligned}$$

For (8), we have

$$\int_0^d t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^{\omega}} \sinh^n(r) dr \tag{3.2.13}$$

$$\leq C \int_0^1 t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^{\omega}} r^n dr \tag{3.2.14}$$

$$+ C \int_1^d t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^{\omega}} \exp(nr) dr \tag{3.2.15}$$

For (13), we have

$$\begin{aligned}
&\int_0^1 t^{-(n+1)/2}\exp\left(-\frac{r^2}{8t}\right)\frac{1}{[d-r+1+a]^\omega}r^n dr\\
\leq&\frac{1}{[d+a]^\omega}\int_0^1 t^{-(n+1)/2}\exp\left(-\frac{r^2}{8t}\right)r^n dr\\
\leq&\frac{1}{[d+a]^\omega}\int_0^1 \exp\left(-\frac{r^2}{8t}\right)\cdot\left(\frac{r^2}{t}\right)^{\frac{n-1}{2}} d\left(\frac{r^2}{t}\right)\\
\leq&\frac{8^{\frac{n+1}{2}}}{[d+a]^\omega}\int_0^1 \exp\left(-\frac{r^2}{8t}\right)\cdot\left(\frac{r^2}{8t}\right)^{\frac{n-1}{2}} d\left(\frac{r^2}{8t}\right)\\
\leq&\frac{8^{\frac{n+1}{2}}}{[d+a]^\omega}\Gamma(\frac{n+1}{2})\leq C(n,\omega)\frac{1}{[d+1+a]^\omega}\leq C(n,\omega)\frac{1}{[r(z',z_0)+1+a]^\omega}
\end{aligned}$$

Then for (14), we have

$$\begin{aligned}
&\int_1^d t^{-(n+1)/2}\exp\left(-\frac{r^2}{8t}\right)\frac{1}{[d-r+1+a]^\omega}\exp(nr)dr\\
=&\left[\int_1^{\frac{d}{2}}+\int_{\frac{d}{2}}^d\right]t^{-(n+1)/2}\exp\left(-\frac{r^2}{8t}\right)\frac{1}{[d-r+1+a]^\omega}\exp(nr)dr
\end{aligned}$$

$$
\begin{aligned}
&\int_1^{\frac{d}{2}} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^\omega} \exp(nr)dr \\
\leq &\frac{1}{[\frac{d}{2}+1+a]^\omega} \int_1^{\frac{d}{2}} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}+nr\right) dr \\
\leq &\frac{C(\omega)\exp(4n^2t)}{[d+1+a]^\omega} \int_1^{\frac{d}{2}} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) dr \\
\leq &\frac{C(\omega)\exp(4n^2t)}{[d+1+a]^\omega} \int_1^{\frac{d}{2}} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) r^n dr \\
\leq &\frac{C(\omega)\exp(4n^2t)4^{n+1}}{[d+1+a]^\omega} \int_1^{\frac{d}{2}} \exp\left(-\frac{r^2}{16t}\right)\left(\frac{r^2}{16t}\right)^{\frac{n-1}{2}} d\left(\frac{r^2}{16t}\right) \\
\leq &\frac{C(\omega)\exp(4n^2t)4^{n+1}}{[d+1+a]^\omega}\Gamma(\frac{n+1}{2}) \leq C(n,T,\omega)\frac{1}{[r(z',z_0)+1+a]^\omega}
\end{aligned}
$$

Remark 3.2.9. Here, we just use the fact that

$$
\frac{(d+1+a)^\omega}{(\frac{d}{2}+1+a)^\omega} \leq 2^\omega \quad for\ d \geq 2,\ a > 0.
$$

$$
\begin{aligned}
&\int_{\frac{d}{2}}^{d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^{\omega}} \exp(nr)dr \\
\leq& \frac{1}{[1+a]^{\omega}} \int_{\frac{d}{2}}^{d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}+nr\right) dr \\
\leq& \frac{C\exp(4n^2t)}{[1+a]^{\omega}} \int_{\frac{d}{2}}^{d} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) dr \\
\leq& \frac{C\exp(4n^2t)T^{\beta/2}}{[1+a]^{\omega}} \int_{\frac{d}{2}}^{d} t^{-(n+\beta+1)/2} \exp\left(-\frac{r^2}{16t}\right) \left(\frac{r}{d}\right)^{n+\beta} dr \\
\leq& \frac{C\exp(4n^2t)T^{\beta/2}}{[1+a]^{\omega}} \frac{1}{d^{n+\beta}} \int_{\frac{d}{2}}^{d} t^{-(n+\beta+1)/2} \exp\left(-\frac{r^2}{16t}\right) r^{n+\beta} dr \\
\leq& \frac{C\exp(4n^2t)T^{\beta/2}4^{\frac{n+\beta+1}{2}}}{[1+a]^{\omega}} \frac{1}{d^{n+\beta}} \Gamma(\frac{n+\beta+1}{2}) \\
\leq& C(n,T,\omega)\frac{1}{[r(z',z_0)+1+a]^{\omega}}
\end{aligned}
$$

Case 2 : $(1 \leq d < 2)$ We only need to show that (5) and (8) are both bounded. For (5), we have

$$
\begin{aligned}
&\int_{d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[r-d+1+a]^{\omega}} \sinh^n(r)dr \\
\leq& \frac{C(\omega)}{[1+a]^{\omega}} \int_{d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}+nr\right) dr \\
\leq& \frac{C(\omega)\exp(4n^2t)}{[1+a]^{\omega}} \int_{d}^{\infty} t^{-(n+1)/2} \exp\left(-\frac{r^2}{16t}\right) r^n dr \\
\leq& \frac{C(\omega)\exp(4n^2t)4^{n+1}}{[1+a]^{\omega}} \Gamma(\frac{n+1}{2}) \leq C(n,T,\omega)
\end{aligned}
$$

For (8), we have

$$\int_0^d t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^\omega} \sinh^n(r) dr$$
$$= \left[\int_0^1 + \int_1^d\right] t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^\omega} \sinh^n(r) dr$$

$$\int_0^1 t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^\omega} r^n dr$$
$$\leq \frac{C(\omega) 8^{\frac{n+1}{2}}}{[1+a]^\omega} \Gamma(\frac{n+1}{2}) \leq C(n, T, \omega)$$

$$\int_1^d t^{-(n+1)/2} \exp\left(-\frac{r^2}{8t}\right) \frac{1}{[d-r+1+a]^\omega} r^n dr$$
$$\leq \frac{C(\omega) 8^{\frac{n+1}{2}}}{[1+a]^\omega} \Gamma(\frac{n+1}{2}) \leq C(n, T, \omega)$$

Case 3 : $(0 \leq d < 1)$ This case is almost same with the case 2. We just omit it. □

3.2.2 The spectrum results

Let (M^{n+1}, g) be an asymptotically hyperbolic manifolds with defining function ρ. In the following sections, we will talk about some results about the spectrum of the Laplacian operator of (M^{n+1}, g) and the behavior of its modified resolvent after meromorphic continuation.

The first result is about the spectrum of the asymptotically hyperbolic manifold. The q-form result is originally from the R. Mazzeo [35]. And later, it is reproved by the J. Lee [30].

Theorem 3.2.10 (Theorem 1.3, [35]; Proposition F, [30])**.** *Let (M^{n+1}, g) be an asymptotically hyperbolic space. The Laplace-Beltrami operator on M acting on q - forms* *satisfies that*

(a) When $0 \leq q < n/2$*, with* $R = n/2 - q$*.*

(b) When $n/2 + 1 < q \leq n + 1$*, with* $R = q - n/2 - 1$*.*

In each case, the essential L^2 *spectrum of* Δ *is* $\left[R^2, \infty\right)$*.*

For the symmetric two tensor result is from J.Lee [30] and E.Delay [15].

Theorem 3.2.11 (Main theorem, [15], Proposition D, [30])**.** *Let* (M^{n+1}, g) *be an asymptotically hyperbolic manifold. The Lichnerowicz operator* Δ_{L} *acting on symmetric 2 -tensors has the essential* L^2 *spectrum,* Δ_{L} *is* $\left[n^2/4 - 2n, \infty\right)$ *.*

Then, R. Mazzeo [36] show that there is no embedded eigenvalue into the essential spectrum.

Theorem 3.2.12 (Theorem 16, [36])**.** *Let* (M^{n+1}, g) *be an asymptotically hyperbolic manifold. The Laplace-Beltrami operator on* M *acting on function satisfies that there is no eigenvalue embedded into the essential spectrum* $[\frac{n^2}{4} - 2n, \infty)$*.*

Then, E.Delay prove a corresponding symmetric two tensor case.

Theorem 3.2.13 ([16])**.** *For* $n \geq 2$*, let us consider* $(N, \hat{g})$ *an* $n-$*dimensional compact Einstein manifold. Let* $M = (0, \infty) \times N$ *equipped with an asymptotically hyperbolic metric*

$$g = (dr)^2 + f^2(r)\hat{g}$$

Then there are no L^2 *TT-eigentensors (Trace-free and divergent free symmetric two tensor) of the Lichnerowicz Laplacian* Δ_L *with eigenvalue embedded in the essential spectrum. For the real hyperbolic space, there are no* L^2 *eigentensors of* Δ_L*.*

Moreover, we call asymptotically hyperbolic manifold $(M, \tilde{g})$ asymptotically hyperbolic Einstein manifold if $(M, \tilde{g})$ is also an Einstein manifold. The asymptotically hyperbolic Einstein manifold $(M, \tilde{g})$ is a fixed point of the normalized Ricci flow equation

$$\partial_t g_t = -2\mathrm{Ric}_{g_t} + 2\lambda g_t \tag{3.2.16}$$

where λ is the Einstein constant of $(M, \tilde{g})$, (i.e. $\mathrm{Ric} = \lambda g$). We can always normalize the equation into the standard case $\lambda = -n$.

§ 3.3 Heat kernel estimates of R.Bamler

Theorem 3.3.1 (Long time convolution)**.** *Let $\mathbb{H}^{n+1}$ be a hypebolic space with $n \geq 2$. And let $H(t, z, z')$ be the heat kernel of the Laplacian operator on this hyperbolic space. Then, there exists a constant $C(n, \omega)$, such that*

$$\int_M H\left(t, z, z'\right) \frac{1}{\left[r\left(z, z_0\right) + 1 + a\right]^{\omega}} dz \leq C(n, \omega) \frac{1}{\left[r\left(z', z_0\right) + t + 1 + a\right]^{\omega}}$$

where $a \leq 0$ and $\omega > 0$.

Before we proving this theorem, first we need to introduce a property hyperbolic space which have been used in the [2] (Lemma 6.3).

Lemma 3.3.2. *Let $M = \mathbb{H}^{n+1}$ or $\mathbb{CH}^{2(n+1)}$. There are constants $C < \infty$ and $\mu > 0$ such that:*

Consider two distinct points $z_0, z' \in M$ *and let* $r_0 > 0, 0 < \alpha < \frac{\pi}{2}$. *Let* $v \in T_{z'}M$ *be the vector pointing towards* z_0 *and define the sector*

$$S_{v,\alpha} = \left\{\exp_{z'}(u) : u \in T_{z'}M, <_{z'} (u, v) \leq \alpha\right\}$$

Then for $d = d(z_0, z') - r_0$ *we have*

$$\operatorname{vol}\left(B_{r_0}(z_0) \backslash S_{v,\alpha}\right) \leq Ce^{-\mu d}\alpha^{-2(n-1)}.$$

Proof. By rescaling we can assume that the sectional curvature are less than -1. By the graph, we see that

$$B_{r_0}(z_0) \setminus S_{v,\alpha} \subseteq B_a(z')$$

We know the law of cosines in $\mathbb{H}^2$ is

$$\cosh(r_0) = \cosh(r_0 + d)\cosh(a) - \sinh(r_0 + d)\sinh(a)\cos\alpha$$

By the Rauch comparison theorem, in the $\mathbb{H}^{n+1}$ or $\mathbb{CH}^{2(n+1)}$, we have

$$\cosh(r_0) \geq \cosh(r_0 + d)\cosh(a) - \sinh(r_0 + d)\sinh(a)\cos\alpha$$

Since

$$\cosh(r_0) = \cosh(r_0 + d)\cosh(d) - \sinh(r_0 + d)\sinh(d),$$

we have

$$\begin{aligned}
&\cosh(r_0+d)\cosh(d)-\sinh(r_0+d)\sinh(d)\\
\geq\ &\cosh(r_0+d)\cosh(a)-\sinh(r_0+d)\sinh(a)\cos\alpha\\
&\sinh(r_0+d)\sinh(a)-\sinh(r_0+d)\sinh(d)\\
\geq\ &\cosh(r_0+d)\cosh(a)-\cos\alpha\cosh(r_0+d)\cosh(d)\\
&\sinh(r_0+d)[\sinh(a)\cos\alpha-\sinh(d)]\geq\cosh(r_0+d)[\cosh(a)-\cosh(d)]
\end{aligned}$$

and

$$\begin{aligned}
\sinh(a)\cos\alpha-\sinh(d)&\geq\cosh(a)-\cosh(d)\\
\cosh(d)-\sinh(d)&\geq\cosh(a)-\sinh(a)\cos\alpha\\
e^{-d}&\geq\sinh(a)(1-\cos\alpha)
\end{aligned}$$

Therefore, we have

$$\sinh(a)\leq e^{-d}(1-\cos\alpha)^{-1}$$

Then, we have

$$\operatorname{vol}\left(B_{r_0}(z_0)\setminus S_{\upsilon,\alpha}\right)\leq\operatorname{vol}\left(B_a(z')\right)\leq C\sinh^n(a)\leq Ce^{-nd}(1-\cos\alpha)^{-n}\leq Ce^{-nd}\alpha^{-2n}$$

□

Proof of Theorem 3.3.1. For small times $t>1$, the estimate follows with the help of the Theorem 3.2.7. So we assume that $t>1$. Let $r_2,k_1,\cdots,k_6,k_7$ be some positive constants to be determined.

Case 1 : $(r_1 + k_1 t \leq 1 + a)$ By the L^1 boundedness of $H(t, z, z')$, we have

$$\int_M H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^\omega} dz \leq \frac{C(n, \omega)}{[1 + a]^\omega} \leq \frac{C(n, \omega, k_1)}{[r_1 + 1 + a + t]^\omega}.$$

Case 2 : $(r_1 + k_1 t > 1 + a)$ Then we have

$$\begin{aligned} &\int_M H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^\omega} dz \\ = &\left[\int_{M \setminus B_{r_2}(z_0)} + \int_{B_{r_2}(z_0)} \right] H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^\omega} dz \end{aligned}$$

For the first part, we have

$$\int_{M \setminus B_{r_2}(z_0)} H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^\omega} dz \leq \frac{C(k_1)}{[r_2 + 1 + a]^\omega} \tag{3.3.1}$$

For the second part, we have

$$\int_{B_{r_2}(z_0)} H(t, z, z') \frac{1}{[r(z, z_0) + 1 + a]^\omega} dz \leq \frac{1}{[1 + a]^\omega} \Big[\int_{B_{r_2}(z_0) \setminus S_{v,\alpha}} + \int_{B_{r_2}(z_0) \cap S_{v,\alpha}} \Big] \tag{3.3.2}$$

By the L^1 integrability and the L^2 exponential decay of the heat kernel

$$\|H\|_{L^1(M)}(t) \leq C(n) \quad \textit{and} \quad \|H\|_{L^2}(t) \leq C(n) \exp(-\lambda_B t),$$

together with Lemma 3.3.2, we have

$$\int_{B_{r_2}(z_0) \setminus S_{v,\alpha}} \leq C(n) \exp(n(r_2 - r_1) - \lambda_B t) \alpha^{-2n}$$

and

$$\int_{B_{r_2} \cap S_{v,\alpha}} \leq C(n) \alpha^n \quad \text{This is because that the } L^1 \text{ integrability of } H$$

Therefore, we just need to find some proper α and r_2 such that

$$\exp(n(r_2 - r_1) - \lambda_B t)\alpha^{-2n} \leq \frac{C(n,\omega)}{[r_1 + 1 + a + t]^\omega} \quad and \quad \alpha^n \leq \frac{C(n,\omega)}{[r_1 + 1 + a + t]^\omega}$$

Let $r_2 - r_1 = -k_2 r_1 + k_3 t + k_4(1 + a)$. In order to guarantee that $r_2 > 0$, we need

$$(1 - k_2)r_1 + k_3 t + k_4(1 + a) > 0$$

By the assumption that $r_1 + k_1 t \geq a + 1$, we can just simply set

$$r_2 := (1 - k_2)(r_1 + k_1 t - (1 + a)) \quad 0 < k_2 < 1$$

On the other hand, in order to guarantee that

$$\alpha^n \leq \frac{C(n,\omega)}{[r_1 + 1 + a + t]^\omega},$$

we set

$$\alpha := \exp(-k_5 r_1 - k_6 t - k_7(1 + a)).$$

Together with the requirement

$$\exp(n(r_2 - r_1) - \lambda_B t)\alpha^{-2n} \leq \frac{C(n,\omega)}{[r_1 + 1 + a + t]^\omega},$$

we need to require that

$$\begin{aligned}
&\exp(n(r_2 - r_1) - \lambda_B t)\alpha^{-2n} \\
=&\exp(-nk_2 r_1 + n(1 - k_2)k_1 t - n(1 - k_2)(1 + a) - \lambda_B t + 2nk_5 r_1 + 2nk_6 t + 2nk_7(1 + a)) \\
=&\exp(-n(k_2 - 2k_5)r_1 - (\lambda_B - n(1 - k_2)k_1 - 2nk_6)t - n(1 - k_2 - 2k_7)(1 + a))
\end{aligned}$$

Take proper $k_1, \cdots, k_7$ such that the coefficients of the above formula are all positive.

We can take $k_1 = 1, k_2 = \frac{1}{2}, k_5 = \frac{1}{8}, k_6 = \frac{\lambda_B}{n} - \frac{3}{4}$ and $k_7 = \frac{1}{8}$. Then we have $(k_2 - 2k_5) = \frac{1}{4}$, $(\lambda_B - n(1-k_2)k_1 - 2nk_6) = \frac{1}{4}$ and $(1 - k_2 - 2k_7) = \frac{1}{4}$ which satisfies the requirement.

□

3.3.1 The spherical coordinates and the Casimir operators

Now, let $v = k_i \in \mathfrak{l}$ and $p = \exp(k_i t)\exp(rx_0)(p_0) \in M$ where $r \geq 0$ and $x_0 \in \mathfrak{a}$ ($\mathfrak{a}$ is the Cartan subalgebra). Then

$$\begin{aligned}\widetilde{\nabla_{d\pi_e(k_i)|_p} f} =& \frac{d}{dt}\tilde{f}(\exp(k_i t)\exp(rx_0)) + \rho_{0*}(ch(ad(-rx_0))Ad(\exp(-k_i t))k_i)(\tilde{f}) \\ =& \frac{d}{dt}\tilde{f}(\exp(k_i t)\exp(rx_0)) + \rho_{0*}(ch(ad(-rx_0))k_i)(\tilde{f})\end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{\nabla_{d\pi_e(k_i)|_p}\nabla_{d\pi_e(k_i)} f} =& \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0)) \\ &+ 2\rho_{0*}(ch(ad(-rx_0))k_i)(\frac{d}{dt}\tilde{f}) \\ &+ \rho_{0*}^2(ch(ad(-rx_0))k_i)(\tilde{f})\end{aligned}$$

Now, let $k_i = \frac{1}{\sqrt{2}}(x_i + y_i)$, where x_i is the positive root and $y_i = \sigma(x_i)$ is the negative root. (See detail in the section of root system.)

$$\begin{aligned}\widetilde{\nabla_{d\pi_e(k_i)|_p}\nabla_{d\pi_e(k_i)} f} =& \frac{d^2}{dt^2}\tilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\ &+ 2\rho_0(ch(-r\alpha_i(x_0))k_i)(\frac{d}{dt}\tilde{f}) \\ &+ \rho_0^2(ch(-r\alpha_i(x_0))k_i)(\tilde{f})\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{d\pi_e(k_i)}(\exp(k_i t)\exp(rx_0)) =&(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)\\ =&(d\pi_{p_0})_e(sh(ad(-rx_0))k_i) \in T_{p_0}M\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_{\pi_*(k_i)|_p}\widetilde{\pi_*}(k_i) =&\frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)\\ &+ \rho_0(ch(ad(-rx_0))Ad(\exp(k_i t))k_i)[\pi_{**}(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)]\end{aligned}$$

At $t = 0$,

$$\begin{aligned}\nabla_{d\pi_e(k_i)|_p}\widetilde{d\pi_e}(k_i) =&\frac{d}{dt}(d\pi_{p_0})_e(sh(ad(-rx_0))k_i)\\ &+ \rho_{0*}(ch(ad(-rx_0))k_i)[(d\pi_{p_0})_e(sh(ad(-rx_0))Ad(\exp(-k_i t))k_i)]\\ =&[sh(ad(-rx_0))k_i, ch(ad(-rx_0))k_i]\\ =&[sh(-\alpha_i(x_0)r)\mathfrak{p}_i, ch(-\alpha_i(x_0)r)k_i]\\ =&sh(-\alpha_i(x_0)r)ch(-\alpha_i(x_0)r)(d\pi_{p_0})_e([p_i, k_i])\end{aligned}$$

where $\mathfrak{p}_i = \frac{1}{\sqrt{2}}(x_i - y_i)$ and $[\mathfrak{p}_i, k_i] = [x_i, y_i] = \sum_{j=1}^{r}\alpha_i(p_j)p_j$. ($r$ is the rank of the symmetric space and p_j is the basis of the maximal abelian subalgebra. See details in the section of root system). Therefore, for $p = \exp(rx_0)(p_0)$, we have

$$
\begin{aligned}
\widetilde{\Delta f}|_p &= \sum_{j=1}^{r} \nabla_{d\pi_e(p_j)} \widetilde{\nabla_{d\pi_e(p_j)} f} \\
&\quad + \frac{1}{sh^2(-\alpha_i(x_0)r)} \Big[\sum_{i=1}^{n-r} \nabla_{d\pi_e(k_i)|_p} \widetilde{\nabla_{d\pi_e(k_i)} f} - \sum_{i=1}^{n-r} \nabla_{\widetilde{\nabla_{d\pi_e(k_i)|_p} d\pi_e(k_i)}} f \Big] \\
&= \sum_{j=1}^{r} \frac{d^2}{dt^2} \widetilde{f}(\exp(tp_j)\exp(rx_0))|_{t=0} \\
&\quad - \sum_{i=1}^{n-r} \frac{ch(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \sum_{j=1}^{r} \alpha_i(p_j) \frac{d}{dt} \widetilde{f}(\exp(tp_j)\exp(rx_0)) \\
&\quad + \sum_{i=1}^{n} \frac{1}{sh^2(-\alpha_i(x_0)r)} \frac{d^2}{dt^2} \widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\
&\quad + \sum_{i=1}^{n} \frac{2coth(-\alpha_i(x_0)r)}{sh(-\alpha_i(x_0)r)} \rho_{0*}(k_i) \frac{d}{dt} \widetilde{f}(\exp(k_i t)\exp(rx_0))|_{t=0} \\
&\quad + \sum_{i=1}^{n} \frac{ch^2(-\alpha_i(x_0)r)}{sh^2(-\alpha_i(x_0)r)} \rho_{0*}^2(k_i) \widetilde{f}
\end{aligned}
$$

Remark 3.3.3. We can see the above formula for Laplacian operator is only for the point $p = \exp(k_i t)\exp(x_0 r)(p_0)$. For the other point, we will use the spherical invariance of Laplacian to get it. (See)

Remark 3.3.4. We see the above discussions do not rely on the choice of the representation ρ_0. Therefore, the above results also holds for the general associative vector bundle of the principal bundle $G \to G/K$

3.3.2 The maximal principle

Lemma 3.3.5. *For every $t > 0$ and $v \in \mathfrak{a}$ denote by $K_t(v)(\min)$ resp. $K_t(v)(\max)$ the minimal resp. maximal eigenvalue of the endomorphism $K_t(v)$. Then*

$$0 < K_t(v)(\min) \leq K_t(v)(\max) \leq K_t^\circ(v)$$

Moreover, $K_t(v)(\max)$ is a subsolution to the heat operator $\partial_t + L^\circ$ in the following sense: If $(G_t)_{t\geq t_0} \in C^\infty(\mathfrak{a})$ is a solution to the equation $\partial_t G_t = -L^\circ G_t$ and a spherical model at all times, then $K_{t_0}(v)(\max) \leq G_{t_0}$ implies $K_t(v)(\max) \leq G_t$ for all $t \geq t_0$.

Proof. The proof makes use of the maximum principle. We will first establish the bound $K_t(v)(\min) > 0$. If the inequality was not true then, by the local behavior of the heat kernel for small times, we would find some $\varepsilon > 0$ and some first time $t' > 0$ such that there are $v' \in \mathfrak{a}$ and $e' \in E_0$ with $|e'| = 1$ such that

$$\langle K_t(v)e, e\rangle \geq -\varepsilon$$

holds for all $t \leq t', v \in \mathfrak{a}$ and $e \in E_0$ with $|e| = 1$ with equality for $t = t', v = v'$ and $e = e'$. This implies $K_{t'}(v')\, e' = K_{t'}(v')\,(\min) e' = -\varepsilon e'$ and

$$\langle \partial_t K_{t'}(v')\, e', e'\rangle \leq 0$$

as well as $\langle \partial_u K_{t'}(v')\, e', e'\rangle = 0$ for any direction $u \in \mathfrak{a}$ and $\langle \triangle K_{t'}(v')\, e', e'\rangle \geq 0$. As for the zero order terms we compute (note that due to the invariance of $\langle \cdot, \cdot\rangle$ on E_0 under the action of K, we have $\langle k_i \cdot e_1, e_2\rangle + \langle e_1, k_i \cdot e_2\rangle = 0$ for any $i = 1, \ldots, n-r$ and $e_1, e_2 \in E_0$)

$$\langle K_{t'}(v')\,(k_i \cdot k_i \cdot e'), e'\rangle = \langle K_{t'}(v')\,(e'), k_i \cdot k_i \cdot e'\rangle$$

$$= K_{t'}(v')\,(\min)\,\langle k_i \cdot k_i \cdot e', e'\rangle = \varepsilon\,\langle k_i \cdot e', k_i \cdot e'\rangle$$

and

$$\langle k_i \cdot (K_{t'}(v')(k_i \cdot e')), e'\rangle = -\langle K_{t'}(v')(k_i \cdot e'), k_i \cdot e'\rangle \leq \varepsilon\,\langle k_i \cdot e', k_i \cdot e'\rangle$$

So

$$\begin{aligned}\langle \partial_t K_t(v')\,e', e'\rangle \geq & \sum_{i=1}^{n-r}\Big\langle \frac{1}{\mathrm{sh}^2\alpha_i(v')} K_{t'}(v')(k_i \cdot k_i \cdot e') + \frac{\mathrm{ch}^2\alpha_i(v')}{\mathrm{sh}^2\alpha_i(v')} k_i \cdot k_i \cdot K_{t'}(v')(e') \\ & -2\frac{\mathrm{ch}\,\alpha_i(v')}{\mathrm{sh}^2\alpha_i(v')} k_i \cdot (K_{t'}(v')(k_i \cdot e')), e'\Big\rangle \\ \geq & \varepsilon \sum_{i=1}^{n-r}\left(\frac{1-\mathrm{ch}\,\alpha_i(v')}{\mathrm{sh}\,\alpha_i(v')}\right)^2 |k_i \cdot e'|^2 > 0\end{aligned}$$

3.3.3 A remark about the result of O.Biquard and R.Bamler

Combine the result of R.Bamler and O.Biquard for the quaternion hyperbolic space and octonions hyperbolic space, we can deduce a very strange result.

Definition 3.3.6 (Definition A, [7])**.** Let $H = U_{m-1}, Sp_{m-1}Sp_1$ or Spin $_7$, corresponding to the complex, quaternionic or octonionic cases, respectively. Let S^{n-1} be a manifold with a contact 1-form η with values in $\mathbb{R}, \mathbb{R}^3$ or $\mathbb{R}^7$, respectively, and let $V = \ker\eta$. A Carnot-Carathéodory H-metric compatible with $d\eta$ is defined to be a metric γ on V such that

- in the complex case, the restriction to V of $d\eta$ is a symplectic form compatible with g (that is, $d\eta(\cdot,\cdot) = \gamma(I\cdot,\cdot)$ where I is an almost complex structure on V);
- in the quaternionic case, the three 2-forms $(d\eta_1, d\eta_2, d\eta_3)$ on V provide a quaternionic structure compatible with γ (that is, $d\eta_i(\cdot,\cdot) = \gamma(I_i,\cdot)$ for almost complex

structures I_i satisfying the quaternionic commutation relations);

- in the octonionic case, the seven 2-forms $(d\eta_1, \ldots, d\eta_7)$ on V provide a Spinn structure compatible with γ (that is, $d\eta_i(\cdot,\cdot) = \gamma(I_i\cdot,\cdot)$ for almost complex structures I_i satisfying the octonionic commutation relations).

Theorem 3.3.7 ([2]). *Let $(M, \bar{g})$ be either $\mathbb{H}^n$ for $n \geq 3$ or $\mathbb{CH}^{2n}$ for $n \geq 2$, choose a basepoint $x_0 \in M$ and let $r = d(\cdot, x_0)$ denote the radial distance function. There is an $\varepsilon_1 > 0$ and for every $q < \infty$ an $\varepsilon_2 = \varepsilon_2(q) > 0$ such that the following holds: If $g_0 = \bar{g} + h$ and $h = h_1 + h_2$ satisfies*

$$|h_1| < \frac{\varepsilon_1}{r+1} \quad \text{and} \quad \sup_M |h_2| + \left(\int_M |h_2|^q \, dx\right)^{1/q} < \varepsilon_2,$$

then Ricci flow (1.1) exists for all time and we have convergence $g_t \longrightarrow \bar{g}$ in the pointed Cheeger-Gromov sense.

§ 3.4 Heat kernel estimates of X.Chen and A.Hassell

he basic strategy for analyzing the heat kernel is to express it in terms of the spectral measure

$$e^{-t(\Delta_X)} = e^{-tn^2/4} e^{-t\left(\Delta_X - n^2/4\right)} = e^{-tn^2/4} \int_0^\infty e^{-t\sigma} dE_{\left(\Delta_X - n^2/4\right)}(\sigma) d\sigma$$

and then, via Stone's formula, in terms of the resolvent:

$$e^{-t(\Delta_X)} = \frac{\imath}{2\pi} e^{-tn^2/4} \int_{-\infty}^{\infty} e^{-t\lambda^2} R(\lambda - \imath 0) 2\lambda d\lambda, \quad \sigma = \lambda^2$$

Theorem 3.4.1 ([10]). *Suppose (X, g) is an $n+1$ -dimensional asymptotically hyperbolic CartanHadamard manifold with no resonance at the bottom of the continuous spectrum* *and denote the operator* $\sqrt{\left(\Delta_X - \mathrm{n}^2/4\right)_+}$ *by* P. *The Schwartz kernel of the spectral measure* $dE_P(\lambda)$ *satisfies bounds*

$$|dE_P(\lambda)\,(z,z')| \leq \begin{cases} C\lambda^2, & \text{if} \quad \lambda \leq 1 \\ C\lambda^n & \text{if} \quad \lambda \geq 1 \end{cases}$$

Theorem 3.4.2. *Assume that* (X,g) *is an asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Let r denote geodesic distance on* $X \times X$. *Then the resolvent,* $R(\lambda) := \left(\Delta_X - n^2/4 - \lambda^2\right)^{-1}$ *is analytic in a neighbourhood of the closed lower half plane* $\operatorname{Im}\lambda \leq 0$, *and satisfies in this region of the* λ*-plane and for* $r(1+|\lambda|) \geq 1$ *(the 'off-digaonal regime') (2.7)*

$$R(\lambda)\,(z,z') = e^{-i\lambda r} R_{od}(\lambda)\,(z,z')\,, \quad r = d\,(z,z')$$

where

- *for* $|\lambda| \leq 1, R_{od}(\lambda)$*is an element of* $(\rho_L\rho_R)^{n/2}\,\mathcal{A}^0\left(X_0^2\right)$

- *for* $|\lambda| \geq 1, R_{od}\,(\lambda)$ *is of the form*

$$\rho_{\mathcal{L}}^{n/2}\rho_{\mathcal{R}}^{n/2}\rho_{\mathcal{A}}^{-n/2+1}\rho_{\mathcal{S}}^{-n+1}\mathcal{A}^0\left(X_0^2 \times_1 [0,1)_h\right)$$

In particular, $\mathrm{R}_{\mathrm{od}}(\lambda)$ *is a kernel bounded pointwise by a multiple of*

$$(r(1+|\lambda|))^{n/2-1} r^{-n+1} = r^{-n/2}(1+|\lambda|)^{n/2-1}$$

for $r \leq C$, *and*

$$e^{-nr/2}(1+|\lambda|)^{n/2-1}$$

for $r \geq C$.

This result is from the result of Melrose-Barreto-Vasy [44] [40]. [10] obtain a semiclassical resolvent

$$\tilde{R}(h,\sigma) = \left(h^2\Delta_X - h^2n^2/4 - \sigma^2\right)^{-1} \quad \text{with } |\sigma| = 1, \operatorname{Im}\sigma \le 0 \text{ and } h \in [0,1)$$

through the parametrix $G(h,\sigma)$ constructed by Melrose, Sà Barreto and Vasy. Then the properties of the resolvent $R(\lambda)$ in Theorem 2.2 follow from the counterparts for $\tilde{R}(h,\sigma)$. Their idea is from [36] [33]. Let (X,g) be a $n+1$ dimensional asymptotically hyperbolic manifold. They construct $0-$double space (i.e. blow up of double space X^2). Then, introduce the $0-$calculus. By this way, they construct the resolvent for Laplacian operator.

3.4.1 The heat kernel estimate for function case

In this section, we will introdue the idea of X. Chen and A. Hassell to do the heat kernel estimate. Basically, first they use the spectrum theorem to covert the heat kernel estimate to the Schwartz kernel estimate of Laplacian operator. Then they make use of the high frequecy estimate for Schwartz on Asymptotically hyperbolic manifold to get the corresponding heat kernel estimate.

Theorem 3.4.3 ([10])**.** *Let X be an $n+1$-dimensional asymptotically hyperbolic Cartan-Hadamard manifold with no eigenvalues and no resonance at the bottom of the spectrum. Then the heat kernel obeys*

$$e^{-t\Delta_X}(z,z') \le Ct^{-(n+1)/2}e^{-n^2t/4-r^2/(4t)-nr/2}(1+r+t)^{n/2-1}(1+r)$$

where r is the geodesic distance between z and z'

We see the heat kernel can be write like

$$h(t,z,z') = \lim_{\varepsilon\to 0} e^{-\frac{n^2}{4}t}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}R(\lambda-i\varepsilon)2\lambda d\lambda$$

By the theorem 3.4.2, we see that if $r(1+|\lambda|^2+\varepsilon^2)\geq 1$, we have

$$R(\lambda)(z,z') = e^{-i\lambda r}R_{od}(\lambda)(z,z')$$

Now, suppose that $r\geq 1$. Then, Obviously, $r(1+|\lambda|^2+\varepsilon^2)\geq 1$. Therefore,

$$\begin{aligned} h(t,z,z') &= \lim_{\varepsilon\to 0} e^{-\frac{n^2}{4}t}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}R(\lambda-i\varepsilon)2\lambda d\lambda \\ &= \lim_{\varepsilon\to 0} e^{-\frac{n^2}{4}t}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}e^{-i(\lambda-i\varepsilon)r}R_{od}(\lambda-i\varepsilon)2\lambda d\lambda \\ &= \lim_{\varepsilon\to 0} e^{-\frac{n^2}{4}t}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}e^{-i\lambda r}e^{-\varepsilon r}R_{od}(\lambda-i\varepsilon)2\lambda d\lambda \end{aligned}$$

Since

$$e^{-t\lambda^2}e^{-i\lambda r} = e^{-t(\lambda^2+i\frac{\lambda r}{t}-\frac{r^2}{4t^2}+\frac{r^2}{4t^2})} = e^{-t(\lambda+\frac{ir}{2\lambda})^2-\frac{r^2}{4t}}$$

Therefore

$$h(t,z,z') = \lim_{\varepsilon\to 0} e^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}e^{-\varepsilon r}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-t(\lambda+\frac{ir}{2t})^2}R_{od}(\lambda-i\varepsilon)2\lambda d\lambda$$

Therefore, we have that

$$h(t,r(z,z')) = e^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\frac{i}{2\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{t(\lambda+\frac{ir}{2t})^2}R_{od}(\lambda-i\varepsilon)2\lambda d\lambda$$

Now, let $\omega = \lambda+\frac{ir}{2t}$. Then, we have

$$h(t,r(z,z')) = e^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\frac{i}{2\pi}\lim_{\varepsilon\to 0}\int_{\Im(\omega)=\frac{r}{2t}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega$$

Again, by the theorem 3.4.2, we have the kernel of $R_{od}(\lambda)$ bounded pointwisely by multiple of

$$e^{-nr/2}(1+|\lambda|)^{n/2-1}$$

Then, we have

$$|h(t,r(z,z'))| \leq e^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\frac{i}{2\pi}\int_{\Im(\omega)=\frac{r}{2t}} e^{-t\omega^2}e^{-\frac{nr}{2}}(1+|\omega-\frac{ir}{2t}-i\varepsilon|)^{\frac{n}{2}-1}2(\omega-\frac{ir}{2t})d\omega$$

The shift of the integral contour: We want to show that

$$\int_{\Im(\omega)=\frac{t}{2r}} e^{t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega = \int_{\mathbb{R}} e^{t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega$$

Consider the following integral

$$\int_{P_3} = \int_{\Im(\omega)=\frac{t}{2r}} e^{t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega$$

Now, define

$$\int_{P_1} = \int_{-R_1}^{R_2} e^{t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega$$

and

$$\int_{P_2} = \int_{\Re(\omega)=R_2,\ 0\leq\Im(\omega)\leq\frac{r}{2t}} e^{t\omega^2}R_{od}(\omega-\frac{ir}{2t}-i\varepsilon)2(\omega-\frac{ir}{2t})d\omega$$

Let $\omega = R_2 + ib$. Then

$$\int_{P_2} = \int_0^{\frac{r}{2t}} e^{-t(R_2+ib)^2}R_{od}(R_2+ib-\frac{ir}{2t}-i\varepsilon)2(R_2+ib-\frac{ir}{2t})idb$$

Then, similarly, define

$$\int_{P_4} = \int_{\Re(\omega)=-R_1,\ 0\leq\Im(\omega)\leq\frac{r}{2t}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t} - i\varepsilon)2(\omega - \frac{ir}{2t})d\omega.$$

By letting $\omega = -R_1 + ib$, we can write $\int_{P_4}$ as

$$\int_{P_4} = \int_0^{\frac{r}{2t}} e^{-t(-R_1+ib)^2} R_{od}(-R_1 + ib - \frac{ir}{2t} - i\varepsilon)2(-R_1 + ib - \frac{ir}{2t})idb$$

Then, since R_{od} is analytic in the domain

$$\Im(\omega - \frac{ir}{2t} - i\varepsilon) \leq 0 \Rightarrow \Im(\omega) \leq \frac{r}{2t} + \varepsilon$$

Therefore, we have that

$$\int_{P_3} = \lim_{R_1\to\infty\ R_2\to\infty} \int_{P_2} + \int_{P_1} - \int_{P_4}$$

For $\int_{P_2}$, by the theorem 3.4.2, we have that

$$|\int_{P_2}| \leq C\int_0^{\frac{r}{2t}} e^{-tR_2^2} e^{-2itbR_2} e^{tb^2} e^{-\frac{nr}{2}} (1 + \sqrt{R_2^2 + (b - \frac{r}{2t} - \varepsilon)^2})^{\frac{n}{2}-1} (\sqrt{R_2^2 + (b - \frac{r}{2t})^2})db$$

By the existence of the term $e^{-tR_2^2}$, we can see that

$$\lim_{R_2\to\infty} \int_{P_2} = 0$$

Similarly, we can get that

$$\lim_{R_1\to\infty} = 0$$

Therefore, we have that

$$\int_{P_3} = \lim_{R_1\to\infty,\ R_2\to\infty} \int_{P_1}$$

Therefore,

$$\int_{\Im(\omega)=\frac{t}{2r}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t} - i\varepsilon) 2(\omega - \frac{ir}{2t}) d\omega$$
$$= \int_{\mathbb{R}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t} - i\varepsilon) 2(\omega - \frac{ir}{2t}) d\omega$$

Exchange the order of the limit and integral: We are going to show that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t} - i\varepsilon) 2(\omega - \frac{ir}{2t}) d\omega$$
$$= \int_{\mathbb{R}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t}) 2(\omega - \frac{ir}{2t}) d\omega$$

First, we change the variables. Let $\omega = \eta + i\varepsilon$. Then

$$\int_{\mathbb{R}} e^{t\omega^2} R_{od}(\omega - \frac{ir}{2t} - i\varepsilon) 2(\omega - \frac{ir}{2t}) d\omega \tag{3.4.1}$$
$$= \int_{\Im(\eta)=-\varepsilon} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) 2(\eta - \frac{ir}{2t} + i\varepsilon) d\eta \tag{3.4.2}$$
$$= \int_{\Im(\eta)=-\varepsilon} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) 2(\eta - \frac{ir}{2t}) d\eta \tag{3.4.3}$$
$$+ 2i\varepsilon \int_{\Im(\eta)=-\varepsilon} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) d\eta \tag{3.4.4}$$

First, by the previous section, we see that

$$(3.4.3) = \int_{\mathbb{R}} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) 2(\eta - \frac{ir}{2t}) d\eta$$
$$(3.4.4) = 2i\varepsilon \int_{\mathbb{R}} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) d\eta$$

For (3.4.4), by the theorem 3.4.2, we have

$$|(3.4.4)| \le 2\varepsilon \int_{\mathbb{R}} e^{t\eta^2} e^{t\varepsilon^2} |e^{-2it\eta\varepsilon}| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} d\eta$$

The integral is bounded by the existence of the term $e^{t\eta^2}$. Therefore, (3.4.4)$\to$ 0 as

$\varepsilon \to 0$.

For (3.4.3), consider the difference

$$
\begin{aligned}
&|(3.4.3) - \int_{\mathbb{R}} e^{-t(\eta+i\varepsilon)^2} R_{od}(\eta - \frac{ir}{2t}) 2(\eta - \frac{ir}{2t}) d\eta| \\
\leq& 2\int_{\mathbb{R}} |e^{t(\eta+i\varepsilon)^2} - e^{t\eta^2}| \cdot |R_{od}(\eta - \frac{ir}{2t})| \cdot |\eta - \frac{ir}{2t}| d\eta \\
\leq& 2\int_{\mathbb{R}} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot |R_{od}(\eta - \frac{ir}{2t})| cdot |\eta - \frac{ir}{2t}| d\eta
\end{aligned}
$$

Then, by the theorem 3.4.2, we have

$$
\begin{aligned}
& 2\int_{\mathbb{R}} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot |R_{od}(\eta - \frac{ir}{2t})| \cdot |\eta - \frac{ir}{2t}| d\eta \\
\leq& 2\int_{\mathbb{R}} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta \\
=& 2\int_{|\eta|\leq\delta} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta \\
&+ 2\int_{|\eta|>\delta} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta.
\end{aligned}
$$

And

$$
\begin{aligned}
& 2\int_{|\eta|>\delta} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta \\
\leq& 4\int_{|\eta|>\delta} e^{-t\eta^2} \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta
\end{aligned}
$$

Therefore, take large enough δ, this term is going be small enough. And

$$
\begin{aligned}
& 2\int_{|\eta|\leq\delta} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta \\
\leq& 2\int_{|\eta|\leq\delta} e^{-t\eta^2} |e^{t\varepsilon^2} e^{-2it\eta\varepsilon} - 1| \cdot e^{-\frac{nr}{2}} (1 + |\eta - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\eta - \frac{ir}{2t}| d\eta
\end{aligned}
$$

fix δ, we see that

$$|e^{t\varepsilon^2}e^{-2it\eta\varepsilon} - 1| \to 0$$

uniformly, as $\varepsilon \to 0$.

The estimate of the heat kernel: Now, we have that the heat kernel

$$h(t, r(z, z')) = e^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\frac{i}{2\pi}\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega - \frac{ir}{2t})2(\omega - \frac{ir}{2t})d\omega.$$

Then, by the theorem 3.4.2, for $r \geq 1$ we have that

$$|R_{od}(\lambda)| \leq Ce^{-\frac{nr}{2}}(1 + |\lambda|)^{\frac{n}{2}-1}$$

Therefore, we have that

$$\begin{aligned}
|h(t, r(z, z'))| \leq& Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\int_{\mathbb{R}} e^{-t\omega^2}|R_{od}(\omega - \frac{ir}{2t})| \cdot |\omega - \frac{ir}{2t}|d\omega \\
\leq& Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}e^{-\frac{nr}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1 + |\omega - \frac{ir}{2t}|)^{\frac{n}{2}-1} \cdot |\omega - \frac{ir}{2t}|d\omega \\
\leq& Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}e^{-\frac{nr}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1 + \sqrt{\omega^2 + \frac{r^2}{4t^2}})^{\frac{n}{2}-1} \cdot \sqrt{\omega^2 + \frac{r^2}{4t^2}}d\omega
\end{aligned}$$

Then, we will make use the following formula

$$C_1(|a| + |b|) \leq \sqrt{a^2 + b^2} \leq |a| + |b|$$

$$C_3(|a|^k + |b|^k) \leq (a + b)^k \leq C_4(|a|^k + |b|^k)$$

for $k \geq 0$. By the above inequalities, we have that

$$\begin{aligned} |h(t, r(z, z'))| \leq & Ce^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} e^{-\frac{nr}{2}} \int_{\mathbb{R}} e^{-t\omega^2} (1 + \sqrt{\omega^2 + \frac{r^2}{4t^2}})^{\frac{n}{2}-1} \cdot \sqrt{\omega^2 + \frac{r^2}{4t^2}} d\omega \\ \sim & Ce^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} e^{-\frac{nr}{2}} \int_{\mathbb{R}} e^{-t\omega^2} (1 + (|\omega| + \frac{r}{2t})^{\frac{n}{2}-1})(|\omega| + \frac{r}{2t}) d\omega \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\frac{n^2}{4}t} e^{\frac{r^2}{4t}} e^{\frac{nr}{2}} |h(t, r(z, z'))| &\leq \int_{\mathbb{R}} e^{-t\omega^2} (1 + (|\omega| + \frac{r}{2t})^{\frac{n}{2}-1})(|\omega| + \frac{r}{2t}) d\omega \\ &\leq \int_{\mathbb{R}} e^{-t\omega^2} [(|\omega| + \frac{r}{2t}) + (|\omega| + \frac{r}{2t})^{\frac{n}{2}}] d\omega \\ &\leq \int_{\mathbb{R}} e^{-t\omega^2} [(|\omega| + \frac{r}{2t}) + (|\omega|^{\frac{1}{2}} + (\frac{r}{2t})^{\frac{1}{2}})^n] d\omega \\ &\leq \int_{\mathbb{R}} e^{-t\omega^2} [|\omega| + \frac{r}{2t} + |\omega|^{\frac{n}{2}} + (\frac{r}{2t})^{\frac{n}{2}}] d\omega \end{aligned}$$

Then, by the fact that

$$\int_{\mathbb{R}} e^{-t\omega^2} |\omega|^n = 2 \int_0^\infty e^{-y} \frac{1}{t^{\frac{n+1}{2}}} y^{\frac{n-1}{2}} \frac{1}{2} dy = \frac{1}{t^{\frac{n+1}{2}}} \Gamma(\frac{n+1}{2}),$$

Then, we have that

$$\begin{aligned} e^{\frac{n^2}{4}t} e^{\frac{r^2}{4t}} e^{\frac{nr}{2}} |h(t, r(z, z'))| &\leq \int_{\mathbb{R}} e^{-t\omega^2} [|\omega| + \frac{r}{2t} + |\omega|^{\frac{n}{2}} + (\frac{r}{2t})^{\frac{n}{2}}] d\omega & (3.4.5) \\ &\sim \frac{1}{t} + \frac{r}{2t^{\frac{3}{2}}} + \frac{1}{t^{\frac{n+2}{4}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} & (3.4.6) \end{aligned}$$

On the other hand, for the heat kernel of hyperbolic space, we have that

$$e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}e^{\frac{nr}{2}}h_{\mathbb{H}^{n+1}}(t, r(z, z')) \tag{3.4.7}$$

$$\sim\frac{1}{t^{\frac{n+1}{2}}}(1+r+t)^{\frac{n}{2}-1}(1+r) \tag{3.4.8}$$

$$\sim\frac{1}{t^{\frac{n+1}{2}}}(1+(r+t)^{\frac{n}{2}-1})(1+r)* \tag{3.4.9}$$

$$\sim\frac{1}{t^{\frac{n+1}{2}}}(1+(\sqrt{r}+\sqrt{t})^{n-2})(1+r) \tag{3.4.10}$$

$$\sim\frac{1}{t^{\frac{n+1}{2}}}(1+r^{\frac{n-2}{2}}+t^{\frac{n-2}{2}})(1+r) \tag{3.4.11}$$

$$\sim\frac{r}{t^{\frac{n+1}{2}}}+\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}+\frac{r}{t^{\frac{3}{2}}} \tag{3.4.12}$$

$$\sim\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}+\frac{r}{t^{\frac{3}{2}}} \tag{3.4.13}$$

The last step is because that for $r \geq 0$,

$$\sum_{i=1}^{n} r^i \sim r^n.$$

We see that if we fix r, then, as $t \to \infty$, the formula (3.4.6) can not be controlled by the formula (3.4.13). (The bad term is $\frac{1}{t}$)

(1) (The region for $2 \leq 2\sqrt{t} \leq r$**)** However, if we require that $1 \leq \sqrt{t} \leq r$, then,

$$\begin{aligned}(3.4.13) \sim&\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}+\frac{r}{2t^{\frac{3}{2}}}+\frac{1}{2t}\\ \geq& C[\frac{1}{t}+\frac{r}{2t^{\frac{3}{2}}}+\frac{1}{t^{\frac{n+2}{4}}}+\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}] \sim (3.4.6)\end{aligned}$$

Moreover, if $0 \leq t \leq 1$, we see that

$$(3.4.6) = \frac{1}{t}+\frac{r}{2t^{\frac{3}{2}}}+\frac{1}{t^{\frac{n+2}{4}}}+\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} \sim \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}+\frac{r}{t^{\frac{3}{2}}} = (3.4.13)$$

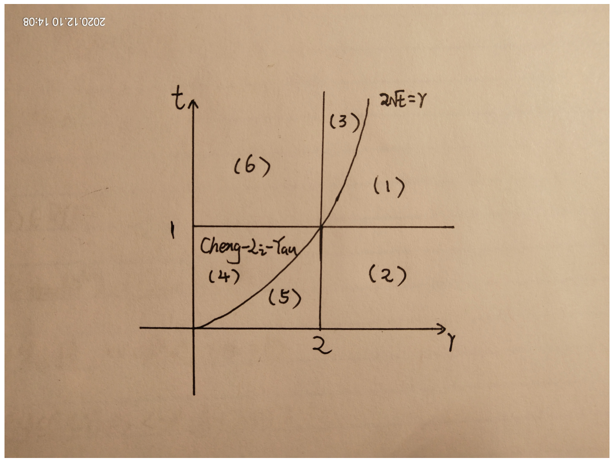

Figure 3.1: Regions for the proof of the upper bound

(2) (The region for $0 \le t \le 1,\ 0 \le r \le 2$) We have

$$(3.4.6) = \frac{1}{t} + \frac{r}{2t^{\frac{3}{2}}} + \frac{1}{t^{\frac{n+2}{4}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} \sim \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}$$

and

$$(3.4.13) = \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} + \frac{r}{t^{\frac{3}{2}}} \sim \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}$$

Therefore, there exists a constant C, such that

$$(3.4.6) \le C(3.4.13)$$

(3) The region for $1 \le r \le 2\sqrt{t}$ We start with the formula for heat kernel

$$h(t, r(z, z')) = e^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} \frac{i}{2\pi} \int_{\mathbb{R}} e^{-t\omega^2} R_{od}(\omega - \frac{ir}{2t}) 2(\omega - \frac{ir}{2t}) d\omega.$$

And for the standard hyperbolic space, from (3.4.13), its heat kernel is

$$e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}e^{\frac{nr}{2}}h_{\mathbb{H}^{n+1}}(t,r(z,z')) \tag{3.4.14}$$

$$\sim e^{-\frac{nr}{2}}(\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}+\frac{r}{t^{\frac{3}{2}}}) \tag{3.4.15}$$

Then, we consider

$$e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}h(t,r(z,z')) = \frac{i}{\pi}\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})(\omega-\frac{ir}{2t})d\omega \tag{3.4.16}$$

$$=\frac{i}{\pi}\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\omega d\omega - \frac{ir}{2t}\frac{i}{\pi}\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})d\omega \tag{3.4.17}$$

The second term of the (3.4.17): By the theorem 3.4.2, we have

$$|\frac{ir}{2t}\frac{i}{\pi}\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})d\omega| \le C\frac{r}{2t}\int_{\mathbb{R}} e^{t\omega^2}e^{-\frac{nr}{2}}(1+|\omega+\frac{ir}{2t}|)^{\frac{n}{2}-1}$$
$$\le C\frac{r}{2t}e^{-\frac{nr}{2}}\left(\frac{1}{t^{\frac{1}{2}}}+\frac{1}{t^{\frac{n}{4}}}+\frac{r^{\frac{n}{2}-1}}{t^{\frac{n-1}{2}}}\right)$$
$$\le Ce^{-\frac{nr}{2}}\left(\frac{r}{t^{\frac{3}{2}}}+\frac{r}{t^{\frac{n+4}{4}}}+\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}\right)$$
$$\sim Ce^{-\frac{nr}{2}}\left(\frac{r}{t^{\frac{3}{2}}}+\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}}\right) \sim (3.4.15)$$

The first term of the (3.4.17):

$$\int_{\mathbb{R}} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\omega d\omega \tag{3.4.18}$$

$$=\int_{|\omega|\le\delta} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\omega d\omega + \int_{|\omega|>\delta} e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\omega d\omega \tag{3.4.19}$$

Then, for the second term of (3.4.19), by the theorem 3.4.2, we have

$$
\begin{aligned}
&|\int_{|\omega|>\delta} e^{-t\omega^2} R_{od}(\omega - \frac{ir}{2t})\omega d\omega| \le \int_{|\omega|>\delta} e^{-t\omega^2} e^{-\frac{nr}{2}} (1 + |\omega + \frac{ir}{2t}|)^{\frac{n}{2}-1} |\omega| d\omega \\
\sim & e^{-\frac{nr}{2}} \int_{|\omega|>\delta} e^{-t\omega^2} (1 + |\omega|^{\frac{n}{2}-1} + (\frac{r}{2t})^{\frac{n}{2}-1}) |\omega| d\omega \\
\le & e^{-\frac{nr}{2}} C(\delta, \varepsilon) e^{-t(\delta^2-\varepsilon)} (1 + r^{\frac{n}{2}-1}) \le (3.4.15)
\end{aligned}
$$

where ε can be any number in $(0, \delta^2)$. The last inequality above needs the following remark.

Remark 3.4.4. In this remark, we are going to show that

$$
\int_{|\omega|>\delta} e^{-t\omega^2} |\omega|^n \le C(\delta) e^{-t(\delta^2-\varepsilon)}
$$

First, we have

$$
\int_{|\omega|>\delta} e^{-t\omega^2} |\omega|^n = 2 \int_{\delta}^{\infty} e^{-t\omega^2} \omega^n d\omega
$$

Let $y = t\omega^2$ and $\omega = \sqrt{\frac{y}{t}}$. Then, we have that

$$
\int_{|\omega|>\delta} e^{-t\omega^2} |\omega|^n = 2 \int_{\delta}^{\infty} e^{-t\omega^2} \omega^n d\omega = \int_{t\delta^2}^{\infty} e^{-y} \frac{1}{t^{\frac{n+1}{2}}} y^{\frac{n-1}{2}} dy
$$

Let $z = y - t\delta^2$. Then, we have that

$$
\begin{aligned}
&\int_{t\delta^2}^{\infty} e^{-y} \frac{1}{t^{\frac{n+1}{2}}} y^{\frac{n-1}{2}} dy = \frac{1}{t^{\frac{n+1}{2}}} \int_0^{\infty} e^{-t\delta^2} e^{-z} (z + t\delta^2)^{\frac{n-1}{2}} dz \\
\sim & \frac{1}{t^{\frac{n+1}{2}}} \int_0^{\infty} e^{-t\delta^2} e^{-z} (z^{\frac{n-1}{2}} + (t\delta^2)^{\frac{n-1}{2}}) dz \\
= & \frac{e^{-t\delta^2}}{t^{\frac{n+1}{2}}} \int_0^{\infty} e^{-z} z^{\frac{n-1}{2}} dz + \frac{e^{-t\delta^2} \delta^{n-1}}{t} \int_0^{\infty} e^{-z} dz \le C(\delta, \varepsilon) e^{-t(\delta^2-\varepsilon)}
\end{aligned}
$$

where ε can be any number in $(0, \delta^2)$.

For the first term of the (3.4.19), we have

$$\int_{|\omega|\le\delta} e^{-t\omega^2} R_{od}(\omega-\frac{ir}{2t})\omega d\omega = \int_{|\omega|\le\delta} -\frac{1}{2t}(\partial_\omega e^{-t\omega^2})R_{od}(\omega-\frac{ir}{2t})d\omega$$

By the integration by parts, we have that

$$\int_{|\omega|\le\delta} -\frac{1}{2t}(\partial_\omega e^{-t\omega^2})R_{od}(\omega-\frac{ir}{2t})d\omega \tag{3.4.20}$$

$$=-\frac{1}{2t}\int_{|\omega|\le\delta}\partial_\omega\left(e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\right)d\omega - \frac{1}{2t}\int_{|\omega|\le\delta} e^{-t\omega^2}\partial_\omega(R_{od}(\omega-\frac{ir}{2t}))d\omega \tag{3.4.21}$$

For the first term of the (3.4.21), we have that

$$|\frac{1}{2t}\int_{|\omega|\le\delta}\partial_\omega\left(e^{-t\omega^2}R_{od}(\omega-\frac{ir}{2t})\right)d\omega| \le \frac{1}{2t}e^{-t\delta^2}|R_{od}(\delta-\frac{ir}{2t}) - R_{od}(-\delta-\frac{ir}{2t})|$$

Then, by the theorem 3.4.2, we have that

$$\begin{aligned}
&\frac{1}{2t}e^{-t\delta^2}\left(R_{od}(\delta-\frac{ir}{2t}) - R_{od}(-\delta-\frac{ir}{2t})\right)\\
\le&\frac{C}{t}e^{-t\delta^2}e^{-\frac{nr}{2}}|(1+|\delta-\frac{ir}{2t}|)^{\frac{n}{2}-1} - (1+|-\delta-\frac{ir}{2t}|)^{\frac{n}{2}-1}|\\
\sim&\frac{1}{t}e^{-t\delta^2}e^{-\frac{nr}{2}}(1+\delta^{\frac{n}{2}-1}+(\frac{r}{2t})^{\frac{n}{2}-1})\\
\sim&e^{-\frac{nr}{2}}\left(\frac{e^{-t\delta^2}}{t}+\frac{e^{-t\delta^2}r^{\frac{n}{2}-1}}{t^{\frac{n}{2}-1}}\right) \le e^{-\frac{nr}{2}}C(\delta,\epsilon)e^{-t(\delta^2-\epsilon)}(1+r^{\frac{n}{2}-1}) \le (3.4.15)
\end{aligned}$$

where ϵ can be any number in $(0,\delta^2)$.

For the second term of the (3.4.21),

$$\frac{1}{2t}\int_{|\omega|\le\delta} e^{-t\omega^2}\partial_\omega(R_{od}(\omega-\frac{ir}{2t}))d\omega, \tag{3.4.22}$$

we need to estimate the term

$$\partial_\omega R_{od}(\omega - \frac{ir}{2t})$$

We see that $R_{od}(\lambda)$ is analytic in a neighborhood of $\lambda = 0$. Therefore, we can take small enough δ, such that the Ball, $B_\delta(\omega - \frac{ir}{2t}) \subseteq \mathbb{C}$, is always in the analytic domain of R_{od}, provided $|\omega| \leq \delta$. Then we make use of the Cauchy integral formula:

$$\partial_\lambda R_{od}(\lambda)|_{\lambda=a} = \frac{1}{2\pi i}\int_{\partial B_\delta(a)} \frac{R_{od}(\lambda)}{(\lambda - a)^2} d\lambda$$

where $a = \omega - \frac{ir}{2t}$ Then we have that

$$|\partial_\omega R_{od}(\omega - \frac{ir}{2t})| \leq \frac{C}{\delta} \sup_{\lambda\in\partial B_\delta(a)} |R_{od}(\lambda)|$$

By the theorem 3.4.2, we have that

$$|\partial_\omega R_{od}(\omega - \frac{ir}{2t})| \leq \frac{C}{\delta} \sup_{\lambda\in\partial B_\delta(a)} |R_{od}(\lambda)| \leq \frac{C}{\delta} e^{-\frac{nr}{2}} (1 + |\lambda|)^{\frac{n}{2}-1} \tag{3.4.23}$$

$$\leq \frac{C}{\delta} e^{-\frac{nr}{2}} (1 + |\omega| + \frac{r}{2t} + \delta)^{\frac{n}{2}-1} \tag{3.4.24}$$

Then, plug (3.4.24) into (3.4.22), we have that

$$
\begin{aligned}
&|\frac{1}{2t}\int_{|\omega|\leq\delta} e^{-t\omega^2}\partial_\omega(R_{od}(\omega-\frac{ir}{2t}))d\omega| \\
\leq&\frac{1}{2t}\int_{|\omega|\leq\delta} e^{-t\omega^2}\frac{C}{\delta}e^{-\frac{nr}{2}}(1+|\omega|+\frac{r}{2t}+\delta)^{\frac{n}{2}-1} \\
\sim&\frac{C\delta^{-1}}{t}e^{-\frac{nr}{2}}\int_{|\omega|\leq\delta} e^{-t\omega^2}(1+|\omega|^{\frac{n}{2}-1}+(\frac{r}{2t})^{\frac{n}{2}-1}+\delta^{\frac{n}{2}-1}) \\
\sim&\frac{\delta^{-1}}{t}e^{-\frac{nr}{2}}\left(\frac{1}{t^{\frac{1}{2}}}+\frac{1}{t^{\frac{n}{4}}}+\frac{r^{\frac{n}{2}-1}}{t^{\frac{n-1}{2}}}\right) \\
\sim&\delta^{-1}e^{-\frac{nr}{2}}\left(\frac{1}{t^{\frac{3}{2}}}+\frac{1}{t^{\frac{n+4}{4}}}+\frac{r^{\frac{n}{2}-1}}{t^{\frac{n+1}{2}}}\right) \\
\sim&\delta^{-1}e^{-\frac{nr}{2}}\left(\frac{1}{t^{\frac{3}{2}}}+\frac{r^{\frac{n}{2}-1}}{t^{\frac{n+1}{2}}}\right)\leq(3.4.15)
\end{aligned}
$$

Therefore, in this region, we have that

$$
h(t,r)\leq h_{\mathbb{H}}(t,r).
$$

(4) (The region for $0\leq r\leq 2\sqrt{t}\leq 2$**)** We will make use of the following theorem

Theorem 3.4.5 ((Cheng-Li-Yau)**.** *Let M be a complete non-compact Riemannian manifold whose sectional curvature is bounded from below and above. For any constant $C>4$, there exists C_1 depending on $C, T, z\in M$, the bounds of the curvature of M so that for all $t\in[0,T]$ the heat kernel $H(t,z,z')$ obeys*

$$
h\left(t,r(z,z')\right)\leq\frac{C_1(C,T,z)}{\left|B_{\sqrt{t}}(z)\right|}\exp\left(-\frac{r^2(z,z')}{Ct}\right)
$$

where $r(z,z')$ is the geodesic distance on M.

Form the above theorem, we see that in our asymptotically hyperbolic manifold

(X^{n+1}, g_+), its heat kernel, in this region, has the following estimate

$$h(t, r(z, z')) \leq \frac{C}{t^{\frac{n+1}{2}}} \leq h_{\mathbb{H}^{n+1}}(t, r)$$

where

$$h_{\mathbb{H}^{n+1}}(t, r) \sim \frac{1}{t^{\frac{n+1}{2}}} e^{-\frac{n^2}{4}t} e^{-\frac{nr}{2}} e^{\frac{r^2}{4t}} (1 + r + t)^{\frac{n}{2}-1}(1 + r)$$

(5) (The region for $0 \leq 2\sqrt{t} \leq r \leq 2$**)** We go back to the original formula for the heat kernel

$$h(t, r) = \lim_{\varepsilon \to 0} e^{-\frac{n^2}{4}t} \frac{i}{\pi} \int_{\mathbb{R}} e^{-t\lambda^2} R(\lambda - i\varepsilon) d\lambda$$

If we can shift the integral contour to $\Im(\lambda) = \frac{-ir}{2t}$, then,

$$\begin{aligned} h(t, r) &= \lim_{\varepsilon \to 0} e^{-\frac{n^2}{4}t} \frac{i}{\pi} \int_{\mathbb{R}} e^{-t(\omega - \frac{ir}{2t})^2} R(\lambda - i\varepsilon)\lambda d\lambda \\ &= \lim_{\varepsilon \to 0} e^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} \frac{i}{\pi} \int_{\mathbb{R}} e^{-t\omega^2} e^{i\lambda r} R(\lambda - i\varepsilon)\lambda d\lambda. \end{aligned}$$

At this time we see that

$$|\lambda| \geq \frac{r}{2t} \Rightarrow r|\lambda| \geq \frac{r^2}{2t} \geq 2 \Rightarrow r(1 + |\lambda|) > 1.$$

Therefore, we can make use of the theorem 3.4.2. We have that

$$h(t, r) = \lim_{\varepsilon \to 0} e^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} \frac{i}{\pi} \int_{\mathbb{R}} e^{-t\omega^2} R_{od}(\lambda - i\varepsilon)\lambda d\lambda$$

By the section of the exchange the order of the limit and integral, we see that

$$h(t, r) = e^{-\frac{n^2}{4}t} e^{-\frac{r^2}{4t}} \frac{i}{\pi} \int_{\mathbb{R}} e^{-t\omega^2} R_{od}(\lambda - i\varepsilon)\lambda d\lambda$$

Then, by the theorem 3.4.2,

$$|R_{od}(\lambda)| \leq r^{-\frac{n}{2}}(1+\lambda)^{\frac{n}{2}-1}$$

Therefore, we have that

$$\begin{aligned}|h(t,r(z,z'))| \leq & Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}\int_{\mathbb{R}} e^{-t\omega^2}|R_{od}(\omega-\frac{ir}{2t})|\cdot|\omega-\frac{ir}{2t}|d\omega \\ \leq & Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}r^{-\frac{n}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1+|\omega-\frac{ir}{2t}|)^{\frac{n}{2}-1}\cdot|\omega-\frac{ir}{2t}|d\omega \\ \leq & Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}r^{-\frac{n}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1+\sqrt{\omega^2+\frac{r^2}{4t^2}})^{\frac{n}{2}-1}\cdot\sqrt{\omega^2+\frac{r^2}{4t^2}}d\omega\end{aligned}$$

Then, we will make use the following formula

$$C_1(|a|+|b|) \leq \sqrt{a^2+b^2} \leq |a|+|b|$$

$$C_3(|a|^k+|b|^k) \leq (a+b)^k \leq C_4(|a|^k+|b|^k)$$

for $k \geq 0$. By the above inequalities, we have that

$$\begin{aligned}|h(t,r(z,z'))| \leq & Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}r^{-\frac{n}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1+\sqrt{\omega^2+\frac{r^2}{4t^2}})^{\frac{n}{2}-1}\cdot\sqrt{\omega^2+\frac{r^2}{4t^2}}d\omega \\ \sim & Ce^{-\frac{n^2}{4}t}e^{-\frac{r^2}{4t}}r^{-\frac{n}{2}}\int_{\mathbb{R}} e^{-t\omega^2}(1+(|\omega|+\frac{r}{2t})^{\frac{n}{2}-1})(|\omega|+\frac{r}{2t})d\omega\end{aligned}$$

Therefore,

$$\begin{aligned}e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}r^{\frac{n}{2}}|h(t,r(z,z'))| \leq & \int_{\mathbb{R}} e^{-t\omega^2}(1+(|\omega|+\frac{r}{2t})^{\frac{n}{2}-1})(|\omega|+\frac{r}{2t})d\omega \\ \leq & \int_{\mathbb{R}} e^{-t\omega^2}[(|\omega|+\frac{r}{2t})+(|\omega|+\frac{r}{2t})^{\frac{n}{2}}]d\omega \\ \leq & \int_{\mathbb{R}} e^{-t\omega^2}[(|\omega|+\frac{r}{2t})+(|\omega|^{\frac{1}{2}}+(\frac{r}{2t})^{\frac{1}{2}})^{n}]d\omega \\ \leq & \int_{\mathbb{R}} e^{-t\omega^2}[|\omega|+\frac{r}{2t}+|\omega|^{\frac{n}{2}}+(\frac{r}{2t})^{\frac{n}{2}}]d\omega\end{aligned}$$

Then, by the fact that

$$\int_{\mathbb{R}} e^{-t\omega^2}|\omega|^n = 2\int_0^{\infty} e^{-y}\frac{1}{t^{\frac{n+1}{2}}}y^{\frac{n-1}{2}}\frac{1}{2}dy = \frac{1}{t^{\frac{n+1}{2}}}\Gamma(\frac{n+1}{2}),$$

Then, we have that

$$e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}r^{-\frac{n}{2}}|h(t,r(z,z'))| \le \int_{\mathbb{R}} e^{-t\omega^2}[|\omega| + \frac{r}{2t} + |\omega|^{\frac{n}{2}} + (\frac{r}{2t})^{\frac{n}{2}}]d\omega \tag{3.4.25}$$

$$\sim\frac{1}{t} + \frac{r}{2t^{\frac{3}{2}}} + \frac{1}{t^{\frac{n+2}{4}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} \tag{3.4.26}$$

Therefore,

$$\begin{aligned} e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}|h(t,r(z,z'))| &\sim \frac{r^{\frac{n}{2}+1}}{t} + \frac{r^{\frac{n}{2}+1}}{2t^{\frac{3}{2}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+2}{4}}} + \frac{r^n}{t^{\frac{n+1}{2}}} \\ &\sim \frac{r^{\frac{n}{2}+1}}{2t^{\frac{3}{2}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+2}{4}}} + \frac{r^n}{t^{\frac{n+1}{2}}} \le C\frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} \end{aligned}$$

On the other hand, for the heat kernel of hyperbolic space, we have that

$$\begin{aligned} & e^{\frac{n^2}{4}t}e^{\frac{r^2}{4t}}h_{\mathbb{H}^{n+1}}(t,r(z,z')) \\ \sim & \frac{1}{t^{\frac{n+1}{2}}}(1+r+t)^{\frac{n}{2}-1}(1+r) \\ \sim & \frac{1}{t^{\frac{n+1}{2}}}(1+(r+t)^{\frac{n}{2}-1})(1+r)^{*} \\ \sim & \frac{1}{t^{\frac{n+1}{2}}}(1+(\sqrt{r}+\sqrt{t})^{n-2})(1+r) \\ \sim & \frac{1}{t^{\frac{n+1}{2}}}(1+r^{\frac{n-2}{2}}+t^{\frac{n-2}{2}})(1+r) \\ \sim & \frac{r}{t^{\frac{n+1}{2}}} + \frac{r^{\frac{n}{2}}}{t^{\frac{n+1}{2}}} + \frac{r}{t^{\frac{3}{2}}} \end{aligned}$$

Therefore, in this region, we have that

$$h(t,r) \le Ch_{\mathbb{H}^{n+1}}(t,r)$$

(6) (The region for $t \geq 1,\ 0 \leq r \leq 2$) We start with the formula for the heat kernel

$$h(t,r) = e^{-t(\Delta_X - \frac{n^2}{4})} = e^{\frac{n^2}{4}t}\int_0^\infty e^{-t\lambda}E_{\Delta_X-\frac{n^2}{4}}(d\lambda)$$

where $E_{\Delta_X-\frac{n^2}{4}}$ is the spectrum for the operator $\Delta_X - \frac{n^2}{4}$. By the fact that

$$E_{\Delta_X-\frac{n^2}{4}} = E_{\sqrt{\Delta_X-\frac{n^2}{4}}},$$

we have that

$$h(t,r) = e^{-t(\sqrt{\Delta_X-\frac{n^2}{4}})^2} = e^{\frac{n^2}{4}t}\int_0^\infty e^{-t\lambda^2}E_{\sqrt{\Delta_X-\frac{n^2}{4}}}(d\lambda)$$

By the theorem 3.4.1, we have that

$$\int_0^\infty e^{-t\lambda^2}dE_{\sqrt{\Delta_X-n^2/4}}(\lambda) \leq C\int_0^1 e^{-t\lambda^2}\lambda^2 d\lambda + C\int_1^\infty e^{-t\lambda^2}\lambda^n d\lambda \leq Ct^{-3/2}$$

On the other hand, in this region,

$$h_{\mathbb{H}^{n+1}}(t,r) \sim \frac{1}{t^{\frac{n+1}{2}}}e^{-\frac{n^2}{4}t}e^{-\frac{nr}{2}}e^{\frac{r^2}{4t}}(1+r+t)^{\frac{n}{2}-1}(1+r) \sim e^{-\frac{n^2}{4}t}\frac{1}{t^{\frac{3}{2}}}$$

Therefore, we have that

$$|h(t,r)| \leq C|h_{\mathbb{H}^{n+1}}(t,r)|$$

Therefore, we finish the estimate for the upper bound of the heat kernel.

3.4.2 The Vasy's approach for tensor cases

Let (M^{n+1}, g_+) be an asymptotically hyperbolic space with the conformal boundary ∂M and defining function ρ. The resolvent (Definition 6.1 [7]) of Δ is defined for $Re(s) > \frac{n}{2}$,

$s \notin [\frac{n}{2}, n]$, by

$$R_M(s) := (\Delta - s(n-s))^{-1}$$

Let $U \subseteq \mathbb{C}$ be a open subset.

$$\mathcal{M}er(U, \mathcal{L}(\mathcal{H}))$$

stands for the set of meromorphic functions on U with values in $\mathcal{L}(\mathcal{H})$ where $\mathcal{L}(\mathcal{H})$ is the bounded linear operator on $\mathcal{H}$.

Our goal is to finitely meromorphically extend the resolvent $R_M(s)$ from the above region to the complex plane $\mathbb{C}$. For function we have the classical result of Melrose and Mazzeo which is as following

Theorem 3.4.6 (Theorem 7.1 in [33])**.** *Theorem 1.1. Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold, Δ its Laplacian acting on functions and ρ a boundary defining function on $\bar{M}$. The modified resolvent*

$$R(s) := (\Delta - s(n-s))^{-1} \in \mathcal{M}er_f\left(\mathcal{O}_0, \mathcal{L}\left(L^2(M)\right)\right)$$

with poles at points $\lambda \in \mathcal{O}_0$ such that $\lambda(n-\lambda) \in \sigma_{pp}(P)$, extends to a finite-meromorphic family

$$R(s) \in \mathcal{M}er_f\left(\mathcal{O}_N \backslash \left(Z_-^1 \cup Z_-^2\right), \mathcal{L}\left(\rho^N L^2(M), \rho^{-N} L^2(M)\right)\right), \quad \forall N \geq 0$$

where

$$\mathcal{O}_N := \left\{\lambda \in \mathbb{C}; Re(\lambda) > \frac{n}{2} - N\right\}, \quad Z_\pm^k := \frac{n}{2} \pm \left(\frac{k}{2} + \mathbb{N}_0\right) \subseteq \mathbb{C}$$

While Colin Guillarmou modified the above result by adding an extra condition-evenness

Definition 3.4.7 (Definition 1.2 in [22]). Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold and $k \in \mathbb{N} \cup \{\infty\}$. We say that g_+ is even modulo $O\left(\rho^{2k+1}\right)$ if there exists $\epsilon > 0$, a boundary defining function ρ and some tensors$(h_{2i})_{i=0,\ldots,k}$ on ∂M such that

$$\phi^*\left(\rho^2 g\right) = dt^2 + \sum_{i=0}^{k} h_{2i} t^{2i} + O\left(t^{2k+1}\right)$$

where ϕ is the diffeomorphism induced by the flow ϕ_t of the gradient $\mathrm{grad}_{\rho^2 g}(\rho)$

Theorem 3.4.8 (Proposition 1.3 in [22]). *Under the assumptions of Theorem 3.4.6, the modified resolvent extends to a finite-meromorphic family*

$$R(s) \in \mathcal{M}er_f\left(\mathcal{O}_N \backslash Z^1_-, \mathcal{L}\left(\rho^N L^2(M), \rho^{-N} L^2(M)\right)\right), \quad \forall N \geq 0$$

and if g is even modulo $O\left(x^{2k+1}\right)$, this extension satisfies

$$R(s) \in \mathcal{M}er\left(\mathcal{O}_N, \mathcal{L}\left(\rho^N L^2(M), \rho^{-N} L^2(M)\right)\right), \quad \forall N \in \left[0, k + \frac{1}{2}\right) \qquad (*)$$

Conversely if $(*)$ holds true for $k \geq 2$ then g is even modulo $O\left(\rho^{2k-1}\right)$.

3.4.3 The High Frequency Result for the tensor

In this section, we will introduce the High Frequency result of the tensor from the [24].

Theorem 3.4.9 ([24]). *Let $\left(X^{n+1}, g\right)$ be even asymptotically hyperbolic and Einstein. Then the inverse of $\Delta - \frac{n(n-8)}{4} + \lambda^2$ acting on $L^2\left(X; \mathcal{E}^{(2)}\right) \cap \ker \Lambda \cap \ker \delta$ written $\mathcal{R}_\lambda$ has a meromorphic continuation from $\mathrm{Re}\,\lambda \gg 1$ to $\mathbb{C}$,*

$$\mathcal{R}_\lambda : C_c^\infty\left(X; \mathcal{E}^{(2)}\right) \cap \ker \Lambda \cap \ker \delta \to \rho^{\lambda + \frac{n}{2} - 2} C^\infty_{\text{even}}\left(\bar{X}; \mathcal{E}^{(2)}\right) \cap \ker \Lambda \cap \ker \delta$$

with finite rank poles.

Theorem 3.4.10 ([24]). *Suppose that X is an even asymptotically hyperbolic manifold which is non-trapping. Then the meromorphic continuation, written Q_λ^{-1} of the inverse of Q_λ initially acting on $L^2_s(X;\mathcal{E})$ has non-trapping estimates holding in every strip $|\operatorname{Re}\lambda| < C, |\operatorname{Im}\lambda| \gg 0$: for $s > \frac{1}{2} + C$*

$$\left\|\rho^{-\lambda-\frac{n}{2}+m} Q_\lambda^{-1} f\right\|_{H^s_{|\lambda|-1}(X;\mathcal{E})} \leq C|\lambda|^{-1} \left\|\rho^{-\lambda-\frac{n}{2}+m-2} f\right\|_{H^{s-1}_{|\lambda|-1}(X;\mathcal{E})}$$

If X is furthermore Einstein, then restricting to symmetric 2-cotensors, the meromorphic continuation $\mathcal{R}_\lambda$ of the inverse of

$$\Delta - \frac{n(n-8)}{4} + \lambda^2$$

initially acting on $L^2\left(X;\mathcal{E}^{(2)}\right) \cap \ker\Lambda \cap \ker\delta$ has non-trapping estimates holding in every strip $|\operatorname{Re}\lambda| < C, |\operatorname{Im}\lambda| \gg 0$: for $s > \frac{1}{2} + C$

$$\left\|\rho^{-\lambda-\frac{n}{2}+2} \mathcal{R}_\lambda f\right\|_{H^s_{|\lambda|-1}\left(X;\mathcal{E}^{(2)}\right)} \leq C|\lambda|^{-1} \left\|\rho^{-\lambda-\frac{n}{2}} f\right\|_{H^{s-1}_{|\lambda||-1}\left(X;\mathcal{E}^{(2)}\right)}.$$

Remark 3.4.11. The above theorem is not enough to prove the heat kernel estimate for the Lichnerowicz operator. We need a more precise high frequency like theorem 3.4.2. I am still working on this.

Chapter 4

Main results

In this section, I will introduce my research about the long time existence and convergence of the normalized Ricci flow starting with an asymptotically hyperbolic manifolds. Our study demonstrates that the normalized Ricci flow exists globally and converges to an Einstein metric, provided that the initial metric is non-degenerate and sufficiently Ricci pinched. Notably, this result holds under weaker conditions compared to the corresponding outcome in [42]. Subsequently, we obtain the corresponding stability result of the conformally compact Einstein metric under the normalized Ricci DeTurck flow. Furthermore, the normalized Ricci flow enables us to partially recover the existence results in [21], [30], and [3]. Specifically, we consider conformally compact Einstein metrics with conformal infinities, which are perturbations of the given non-degenerate conformally compact Einstein metric.

§ 4.1 Introduction

In this section, we study the normalized Ricci flows on asymptotically hyperbolic manifolds and use normalized Ricci flows to construct conformally compact Einstein metrics. We recall that Ricci flow starting from a metric g_0 on a manifold M^n is a family of metrics $g(t)$ that satisfies the following:

$$\begin{cases} \frac{d}{dt}g(t) = -2\,\mathrm{Ric}_{g(t)} \\ g(0) = g_0 \end{cases}$$

We then consider the normalized Ricci flow as follows:

$$\begin{cases} \frac{d}{dt}g(t) = -2\left(\mathrm{Ric}_{g(t)} + ng(t)\right) \\ g(0) = g_0 \end{cases}$$

It is easily seen that the above two equations are equivalent. In fact explicitly

$$g^N(t) = e^{-2nt} g\left(\frac{1}{2n}\left(e^{2nt} - 1\right)\right)$$

solves the second equation if and only if g(t) solves the first equation.

Naturally one initial step is to study normalized Ricci flows starting from metrics that are close to be Einstein. Such questions on compact manifolds were studied in [49], where it was observed that the normalized Ricci flow exists globally and converges exponentially to an Einstein metric if the initial metric g_0 is sufficiently close to Einstein metric and non-degenerate.

To be more precise, suppose that (M^{n+1}, g) is a Riemannian manifold. We say a metric g on M is ε-Einstein if

$$\|h_g\|_{C^0} \leq \varepsilon$$

on M, where $h_g = Ric_g + ng$ is called **Ricci pinching curvature**. The **non-degeneracy** of the metric g is defined to be the first L^2 eigenvalue of the linearization of the Ricci pinching curvature tensor h_g as follows:

$$\lambda = \inf \frac{\int_{\mathcal{M}} \left\langle \left(\Delta_L + 2(n-1)\right) u_{ij}, u_{ij} \right\rangle}{\int_{\mathcal{M}} |u|^2}$$

where the infimum is taken among symmetric 2 -tensors u such that

$$\int_{\mathcal{M}} \left(|\nabla u|^2 + |u|^2\right) dv < \infty$$

and Δ_L is Lichnerowicz Laplacian on symmetric 2-tensors.

Theorem 4.1.1 ([49])**.** *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with non-degeneracy $\lambda > 0$ and the pinching condition*

$$\int_M |h_g|^2 dv < \varepsilon(n, \|Rm\|_{C^0}, d);$$

for some positive number ε depending on the C^0 norm Riemannian curvature , $\|Rm\|_{C^0}$, the diameter d and the dimension, n. Then g can be deformed to an Einstein metric through the normalized Ricci flow. In particular, M supports Einstein metrics.

In the primary literature of [49], the definition of the Ricci flow varies slightly from our own. Nevertheless, the outcome and approach remain same.

There are also several works in the non-compact cases. In [29], the stability of the hyperbolic space under the normalized Ricci flow was established. This stability result on the hyperbolic space in [29] later is improved and extended in [2] [1] [43] [45]. In the light of [29], [42] gives a more general long time existence result about the normalized Ricci flow on asymptotically hyperbolic manifolds.

Prior to presenting the findings of [42], it is imperative to first provide an introduction to fundamental concepts. Suppose that $M^{(n+1)}$ is a smooth manifold with boundary ∂M^n. A **defining function** of the boundary is a smooth function $x : \overline{M} \to R^+$ such that, 1) $x > 0$ in M; 2) $x = 0$ on ∂M; 3) $dx \neq 0$ on ∂M. A metric g on M is said to be **conformally compact** if $x^2 g$ is a Riemannian metric on $\overline{M}$ for a defining function x. The metric g is said to be **conformally compact of regularity** $C^{k,\alpha}$ if $x^2 g$ is a $C^{k,\alpha}$ metric on $\overline{\mathcal{M}}$. The metric $\bar{g} = x^2 g$ induces a conformal class of metric $[\hat{g}]$ on the boundary $\partial\mathcal{M}$ when defining functions vary. The conformal manifold $(\partial M, [\hat{g}])$ is called the **conformal infinity** of the conformally compact manifold (M, g). Furthermore, (M, g) is said to be asymptotically hyperbolic if it is conformally compact and the sectional curvature of g goes to -1 approaching the boundary at the infinity and (M, g) is said to be a **conformally compact Einstein manifold** if $Ric_g = -ng$.

In addition, we say a metric g on a manifold M^{n+1} is ϵ**-Einstein of order** δ if

$$|h_g|(x) \leq \epsilon e^{\delta d(x_0, x)}$$

on M^{n+1}, where $d(x, x_0)$ is the distance to a fixed point $x_0 \in M^{n+1}$.

Theorem 4.1.2 ([42]). *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold with non-degeneracy $\lambda > 0$, $n \geq 2$, $\|\nabla Rm\|_{C^0} < k_1$ for some positive number k_1 and pinching* *condition*

$$|h_{g_+}|(x) < \varepsilon e^{-\gamma d(x,x_0)}$$

for a fixed point $x_0 \in M$, some $\gamma > 0$ satifying

$$\gamma + \sqrt{\lambda} > \frac{n}{2}$$

and ε depending on n, λ, k_1, γ, $\|Rm\|_{C^0}$, υ_0, and C_0, where

$$\upsilon_0 = \inf_{x \in M} (vol(B_{g_+}(x, 1))$$

and

$$C_0 = \sup_{x_0 \in M} (\int_M exp(-nd(x, x_0))dx).$$

Then, the normalized Ricci flow starting from a metric g_+ exists for all the time and converges exponentially to an Einstein metric in the sense of C^∞.

It is apparent that the prerequisites for the results presented in [42] and [49] share a significant resemblance. However, the key difference lies in the pinching conditions. Specifically, [49] stipulates that

$$\int_M |h_g|^2 d\upsilon_g < \varepsilon.$$

allowing for the exponential decay of $\|h_g\|_{L^2(M)}$ in the time direction to be easily obtained using non-degeneracy. Once the exponential decay of $\|h_g\|_{L^2(M)}$ with respect to time is established, the exponential decay of $\|h_g\|_{C^0}$ with respect to time can be derived by the De Giorgi-Nash-Moser theorem. A contradiction argument can then be employed to achieve the desired long-term existence and convergence outcome.

While, [42] requires that

$$|h_g|(x) < \varepsilon e^{-\gamma d(x,x_0)}$$

It is worth noting that when $\gamma \leq n/2$, such a condition precludes the possibility of achieving exponential decay of $\|h_g\|_{L^2(M)}$ in relation to time. Consequently, the author of [42] utilizes an auxiliary function constructed in [29] to acquire local L^2 exponential decay of h_g, which still suffices to achieve the exponential decay of $|h_g|_{C^0}$ with respect to time through the De Giorgi-Nash-Moser theorem. Specifically, the authors consider the following auxiliary function

$$\xi(x,y,t,s) = -\frac{d_0^2(x,y)}{(2+C_0\varepsilon)(t-s)}$$

where $d_0(x,y)$ is the distance from $y \in M$ to the geodesic ball $B_0(\sqrt{r/2},x))$ with respect to the initial metric and C_0 is chosen so that

$$\xi_s + \frac{1}{2}|\nabla\xi|^2 \leq 0$$

Subsequently, they set

$$J(x,t,s) = \int_M \exp(\xi(x,y,t,s)) \cdot |h_g|^2(y,s)dy$$

It is evident that

$$J(x,t,s) \leq \exp(-2(\lambda-\varepsilon)s)J(x,t,0)$$

and

$$\|h_g\|^2_{L^2(B_0(\sqrt{r/2},x)))}(s) \leq J(x,t,s)$$

Thus, through the De Giorgi-Nash-Moser theorem

$$\|h_h\|^2_{C^0((t-r/2)\times B_0(\sqrt{r/2},x))} \leq C(n, v_0, k_1, r)\int_{t-r}^{t}\int_{B_0(\sqrt{r/2},x))} |h_g|^2(y,s)dyds,$$

the following local estimate can be derived

$$\begin{aligned}
&\|h_h\|^2_{C^0((t-r/2)\times B_0(\sqrt{r/2}}, x) \leq C(n, v_0, k_1, r)\exp(-2(\lambda-\varepsilon)t)J(x,t,0)\\
=&C\exp(-2(\lambda-\varepsilon)t)\int_M \exp(-\frac{d_0^2(x,y)}{(2+C_0\varepsilon)t})\cdot|h_g|^2(y,0)dy\\
\leq&C\exp(-2(\lambda-\lambda_0-\varepsilon)t)\int_M \exp(-2\sqrt{\frac{2\lambda_0}{2+C_0\varepsilon}}d_0(x,y))\cdot\varepsilon exp(-2\gamma d_0(y,x_0))dy
\end{aligned}$$

To obtain a uniformly bounded last integral, the additional constraint

$$\gamma+\sqrt{\lambda}>\frac{n}{2},$$

is imposed by the following lemma

Lemma 4.1.3 (lemma 6.1 in [30])**.** *Let* (M^{n+1}, g_+) *be an asymptotically hyperbolic manifold. Then*

$$\int_M \exp(-\alpha d(x,x_0))dvol_{g_+} \leq C$$

for any constant $\alpha > n$*, where* C *is independent of* $x_0 \in M$*.*

Then, by applying the same contradiction argument as in [49], [42] still achieve the long-time existence and convergence theorem. It is worth noting that for hyperbolic space $\mathbb{H}^{n+1}$, $\lambda = \frac{n^2}{4}$ and the above additional condition always holds if $\gamma > 0$.

Drawing on the method presented in [49] and [42] and taking into account the observation that the Rayleigh quotient tends towards $\frac{n^2}{4}$ as the function's support approaches infinity,

Lemma 4.1.4 (Lemma 7.16 in [30]). *The following asymptotic estimate holds for any smooth, compactly supported, trace-free symmetric 2-tensor* u *:*

$$(u, (\Delta_L + 2n)u) \gtrsim \frac{n^2}{4}\|u\|^2$$

we derive the ensuing enhanced global existence and convergence theorem. This theorem effectively demonstrates that the aforementioned supplementary constraint is unnecessary.

Theorem 4.1.5. *Let* (M^{n+1}, g_+) *be an asymptotically hyperbolic manifold with non-degeneracy* $\lambda > 0$, $n \geq 3$, $\|Rm\|_{C^5} < k$ *for some positive number* $k > 0$. *Suppose the metric* g_+ *also satisfies the pinching condition*

$$|h_{g_+}|(x) < \varepsilon e^{-\gamma d(x,x_0)}$$

for a fixed point $x_0 \in M$, *some* $\gamma > 0$ *and* ε *depending on* n, λ, k_l, γ, *and* C_0, *where*

$$v_0 = \inf_{x\in M}(vol(B_{g_+}(x, 1))$$

and

$$C_0 = \sup_{x_0\in M}(\int_M exp(-nd(x, x_0))dx.$$

Then, the normalized Ricci flow starting from a metric g_+ *exists for all the time and converges exponentially to an Einstein metric in the sense of* C^∞.

After obtaining the aforementioned improved long time existence result, we can apply theorem 4.5 in [42] to obtain the following theorem, which states that the normalized

Ricci flow also preserves conformal infinity as time goes to infinity:

Theorem 4.1.6. *Let (M^{n+1}, g_+) be an asymptotically hyperbolic manifold of regularity C^2 with non-degeneracy $\lambda > 0$, $n \geq 3$ and $\|Rm\|_{C^5} < k$ for some positive number $k > 0$ and the pinching condition*

$$|h_{g_+}|(x) < \varepsilon e^{-\gamma d(x,x_0)},\ |\nabla h_{g_+}|(x) < Ce^{-\gamma d(x,x_0)}$$

for a fixed point $x_0 \in M$, some $C > 0$, some $\gamma > 0$ satisfying

$$\gamma \in (\frac{n}{2} - \sqrt{\frac{n^2}{4} - 2}, \frac{n}{2} + \sqrt{\frac{n^2}{4} - 2})$$

and ε depending on n, λ, k, γ, and C_0, where

$$v_0 = \inf_{x \in M} (vol(B_{g_+}(x, 1))$$

and

$$C_0 = \sup_{x_0 \in M} (\int_M exp(-nd(x, x_0))dx.$$

Then, the normalized Ricci flow $g(t)$ starting from a metric g_+ exists for all the time and converges exponentially to an Einstein metric g_∞ in the following sense

$$\lim_{t \to \infty} \|e^{\gamma d_0(x,x_0)}(g(t) - g_\infty)\|_{C^0} = 0$$

In particular, g_∞ is an asymptotically hyperbolic Einstein metric if $\gamma > 2$

It is worth noting that if $n = 3$, the γ can not reach 2.

The theorem 4.1.6 can be regarded as a generalization of the theorem 4.1 in the paper

[42]. In the latter reference, the condition is imposed on the weight γ as follows:

$$\gamma \in \left(\frac{n}{2} - \min\left\{\sqrt{\lambda}, \sqrt{\frac{n^2}{4} - 2}\right\}, \frac{n}{2} + \sqrt{\frac{n^2}{4} - 2}\right)$$

where λ is the non-degeneracy of g_+. However, with the aid of the theorem 4.1.5, the above condition can be weakened to the following

$$\gamma \in \left(\frac{n}{2} - \sqrt{\frac{n^2}{4} - 2}, \frac{n}{2} + \sqrt{\frac{n^2}{4} - 2}\right)$$

Moreover, once we have the long time existence and convergence of normalized Ricci flow, we can derive the following stability theorem of asymptotically hyperbolic manifolds.

Theorem 4.1.7. *Let (M^{n+1}, g_+) be an asymptotically hyperbolic Einstein manifold with nondegeneracy $\lambda > 0$, regularity $C^{2,\alpha}$ and $n \geq 4$. Let g be another asymptotically hyperbolic metric on M^{n+1}. Then, for any $\gamma > 0$, there exists $\epsilon_0(\lambda, \gamma) > 0$, such that if $|g - g_+|_{g_+}(x) \leq \epsilon_0 e^{-\gamma d(x_0, x)}$, Then the normalized Ricci DeTurck flow with the initial g has the long time existence and*

$$\lim_{t \to \infty} \|e^{\gamma d(x, x_0)}(g(t) - g_+)\|_{C^0} = 0$$

For the stability result of hyperbolic space $M^{n+1} = \mathbb{H}^{n+1}$, Schulze, Schnurer and Simon ([43]) have shown stability of $n \geq 3$ for every perturbation $|g - g_{\mathbb{H}^{n+1}}|_{L^\infty}$ is bounded by a small constant depending on $\|g - g_{\mathbb{H}^{n+1}}\|_{L^2}$.

While Li and Yin ([29]) have shown a stability result of $n \geq 2$ if

$$\|g_+ - g_{\mathbb{H}^{n+1}}\|_{C^0} \leq \varepsilon \quad |Ric_{g_+} + ng_+|(x) \leq \varepsilon e^{\gamma d(x, x_0)}$$

for small enough ε.

Furthermore, Bamler ([2]) have shown stability of $n \geq 2$ for the perturbation $|g - g_{\mathbb{H}^{n+1}}| = h_1 + h_2$ for which

$$|h_1|(x) \leq \frac{\epsilon_1}{d(x_0, x) + 1} \quad and \quad \sup_M |h_2| + \left(\int_M |h_2|^q\right)^{\frac{1}{q}} \leq \epsilon_2$$

for every $q < \infty$.

It easy to see that the stability result of [2] just implies that the stability result of [43].

For the theorem 4.1.7, if we take g_+ is the standard hyperbolic metric, then this stability result is implied by the stability result of [2].

In addition, by the theorem 4.1.6, we can partially recover the perturbation existence results in [21] [30] [3]. The idea is to construct an asymptotically hyperbolic metric with prescribed boundary satisfying the condition of theorem 4.1.6.

Theorem 4.1.8. *Let (M^{n+1}, g_+), be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial M, [\hat{g}])$ and $n \geq 4$. And suppose that the non-degeneracy of g satisfies*

$$\lambda > 0$$

Then, for any smooth metric $\hat{h}$ on ∂M, which is sufficiently $C^{2,\alpha}$ close to some $\hat{g} \in [\hat{g}]$ for any $\alpha \in (0, 1)$, there is a conformally compact Einstein metric on M which is of C^2 regularity and with the conformal infinity $[\hat{h}]$.

4.1.1 Curvature flow and its linearization

Let $g(t)$ be a family of metrics on the same manifolds M^{n+1} satisfying the normalized Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\left(\mathrm{Ric}_{g(t)} + n g(t)\right) \\ g(0) = g_+ \end{cases}$$

Let $h(t) = Ric_{g(t)} + ng(t)$. Then, we can get the evolution equation of $h(t)$ as following

$$\frac{\partial}{\partial t} h_{il} = \Delta_L h_{il} - 2n h_{il}$$

where Δ_L is the **Lichnerowicz Laplacian** operator defining as following

$$\Delta_L h_{il} = \Delta h_{il} - g^{jk_1} R_{lj} h_{k_1 i} - g^{jk_1} R_{ij} h_{k_1 l} + 2 g^{jk_1} g^{i_1 i_2} R_{i i_2 l j} h_{k_1 i_1}$$

Moreover, we can also write the above as

$$\frac{\partial}{\partial t} h_{il} = \Delta_{L(g(t-l))} h_{il} - 2n h_{il} + Q$$

where

$$\begin{aligned} Q =& [\Delta_{L(g(t))} - \Delta_{L(g(t-l))}] h_{il} \\ =& g(t) * g(t-l) * [\tilde{\nabla} g(t) * \tilde{\nabla} g(t) + g(t) * (\tilde{\nabla}^2 g(t) + R(g(t-l)))] * h \\ & + g(t) * g(t-l) * [\tilde{\nabla} g(t)] * \tilde{\nabla} h \end{aligned}$$

where $\tilde{\nabla}$ is with respect to $g(t-l)$.

The following metric flow is called the **normalized Ricci-DeTurck flow**

$$\frac{\partial}{\partial t} g_{ij} = -2 R_{ij}(g(t)) + \nabla_i W_j + \nabla_j W_i - 2(n-1) g_{ij}$$

where $W_j = g^{ll_1} g_{jk}(\Gamma^k_{ll_1}(g(t)) - \Gamma^k_{ll_1}(g(0)))$ and ∇ is the covariant derivative with respect to $g(t)$.

Linearization : Let $h_{ij}(t,x) = g_{ij}(t,x) - g_{ij}(0,x)$. Then the Ricci-DeTurck flow is equivalent to the following flow

$$\frac{\partial}{\partial t} h_{ij} = \tilde{\Delta}_L h_{ij} - 2(n-1)h_{ij} - 2(\tilde{R}_{ij} + (n-1)\tilde{g}_{ij}) + Q'_{ij}(t,x)$$

where $\tilde{\Delta}_L$ and $\tilde{R}$ are the **Lichnerowicz** Laplacian operator and Ricci curvature with respect to $g(0) = \tilde{g}$ and the high order term Q is

$$Q'_{ij}(t,x) = g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla} h * \tilde{\nabla} h + g * g^{-1} * \tilde{g} * \tilde{g}^{-1} * \tilde{\nabla}^2 h * h$$

§ 4.2 Long time existences

In the ensuing section, we shall present a formal proof of Theorem 4.1.5. Firstly, we will provide a comprehensive review of the proof of the long time existence of the normalized Ricci flow for the compact case in section 3.1, as expounded in [49]. Subsequently, in section 3.2, we will examine the approach adopted in [42] for the asymptotically hyperbolic manifolds. Finally, we will proffer a proof of Theorem 4.1.5, utilizing the principles outlined in [49] and [42].

4.2.1 The compact case

The idea of [49] comprises three steps. The first step involves demonstrating that the norm of $h_{g(t)}$ in the $L^2(M)$ space exhibits exponential decay over time, subject to certain specified conditions pertaining to the flow $g(t)$. The second step employs the De Giorgi-Nash-Moser estimate to establish that $\|h_{g(t)}\|_{C^2}$ also undergoes exponential decay over time, given the aforementioned conditions about the flow $g(t)$. Finally, Ye's argument in [49] relies on a contradiction approach to deduce the existence of a solution over a long time interval. Let us start with the L^2 estimate.

Lemma 4.2.1 (L^2 estimate)**.** *For any $T > 0$, let $(M^n, g(t))$, $t \in [0,T]$, be a normalized Ricci flow of closed Riemannian manifolds with $n \geq 2$ satisfying that*

1) $\|g(t) - g(0)\|_{C^0(M,g(0))} < \varepsilon$;

2) the nondegeneracy $\lambda(g(t)) > \lambda_0$ for some $\lambda_0 > 0$;

3) $\|Rm_{g(t)}\|(t)_{C^0} \leq k_0$ for some $k_0 > 0$;

4) the diameter $d(g(t)) \leq C_0$ for some $C_0 > 0$

where ε depends on n, λ_0, k_0 and C_0. Then, for any $(x,t) \in M \times [0,T]$, we have

$$\int_M |h_g(t)|^2(x,t)dx \leq Ce^{-(2\lambda_0 - C\varepsilon)t} \int_M |h_g|(x,0)dx$$

where C_1 is a postive constant depending on n, λ_0, k_0 and C_0.

The proof of the lemma 4.2.1 is straightforward. The above exponential decay is derived by the equation of $\int_M |h_g(t)|^2(x,t)dx$,

$$\begin{aligned}\partial_t \int_M |h_g(t)|^2(x,t)dx &\leq -2\int_M ((\Delta_L + 2n)h_{g(t)}, h_{g(t)}) + C\varepsilon \int_M |h_{g(t)}|^2(x,t)dx \\ &\leq -(2\lambda_0 - C\varepsilon)\int_M |h_{g(t)}|^2(x,t)dx\end{aligned}$$

Then, by the De Giorgi-Nash-Moser estimate, the following C^2 estimate is derived.

Lemma 4.2.2 (C^2 estimate)**.** *For any $T > 0$, let $(M^n, g(t))$, $t \in [0,T]$, be a normalized Ricci flow of closed Riemannian manifolds with $n \geq 2$ satisfying that*

1) $\|g(t) - g(0)\|_{C^0((M,g(0)))} < \varepsilon$*;*

2) the nondegeneracy $\lambda(g(t)) > \lambda_0$ *for some* $\lambda_0 > 0$*;*

3) $\|Rm_{g(t)}\|(t)_{C^0} \leq k_0$ *for some* $k_0 > 0$*;*

4) the diameter $d(g(t)) \leq C_0$ *for some* $C_0 > 0$

where ε *depends on* n*,* λ_0*,* k_0 *and* C_0*. Then for any* $(x,t) \in M \times [\tau, T]$*, we have*

$$\|h_g\|_{C^0(M)} \leq C_1 e^{-(2\lambda_0 - C_1\varepsilon)t} \int_M |h_g|^2(x,0)dx$$

and

$$\|\nabla h_g\|_{C^0(M)} + \|\nabla^2 h_g\|_{C^0(M)} \leq C_2 e^{-(2\lambda_0 - C_2\varepsilon)t} \int_M |h_g|^2(x,0)dx$$

where C_1 *is a postive constant depending on* n*,* λ_0*,* k_0 *and* C_0*, and* C_2 *is a positive constant depending on* n*,* λ_0*,* k_0*,* C_0 *and* τ*.*

Then, by the following short time existence of the normalized Ricci flow [23] on the compact manifolds, we can see that there exists a time T such that the conditions in the above lemma hold if the non-degeneracy of the initial metric, $\lambda > 0$.

Theorem 4.2.3 ([23])**.** *Let* (M^n, g) *be a closed Riemannian manifold. Then, there exists* $T > 0$ *depending on* n *and* $\|Rm_g\|_{C^0(M)}$*, such that the normalized Ricci flow starting with* g*,* $g(t)$ *exists on the time interval* $[0,T]$*. Moreover, the following estimates hold at any time* $t \in [0,T]$*:*

$$\|\nabla^k Rm_{g(t)}\|_{C^0(M)} \leq \frac{c(k, n, \|Rm_g\|_{C^0(M)})}{t^{k/2}}, \ k = 0, 1, \cdots$$

where $c(k,n)$ *depends only* k*,* n *and* $\|Rm_g\|_{C^0(M)}$*.*

Then, we will review the contradiction argument in [49] to sketch the proof of the theorem 4.1.1.

Step1 (The choice of ε in lemma 4.2.2): Take λ_0, k_0 and C_0 in lemma 4.2.2 as $\lambda_0 = \lambda/2$, $k_0 = 2\|Rm_g\|_{C^0(M)}$ and $C_0 = 2d(g)$ respectively, where λ, Rm and $d(g)$ are the nondegeneracy, Riemannian curvature and diameter of the initial metric g respectively. Then, we can find a ε in lemma 4.2.2.

Step2 (Short time existence): By the theorem 4.2.3, we can find $T > 0$ such that the conditions in the lemma 4.2.2 holds for above ε, λ_0, k_0 and C_0. Then, take the maximal value of the all the above available T, T_{max}.

Step3 (Long time existence): The long time existence is shown by contradiction. Assume that $T_{max} < +\infty$. Then, by the lemma 4.2.2, we can alway take

$$\int_M |h_g|(x,0)dx$$

small enough such that in $[0, T_{max}]$

1) $\|g(t) - g(0)\|_{g(0)} < \frac{3}{4}\varepsilon$;

2) the nondegeneracy $\lambda(g(t)) > \frac{3}{4}\lambda \geq \lambda_0$;

3) $\|Rm_{g(t)}\|(t)_{C^0} \leq \frac{3}{2}\|Rm_g\|_{C^0(M)} \leq k_0$ for some $k_0 > 0$;

4) the diameter $d(g(t)) \leq \frac{3}{2}d \leq C_0$

which contradict the choice of $T_{max} < +\infty$.

4.2.2 The non-compact case

The idea of [42] is pretty similar with that of [49]. The key difference is that in [42], they make use of the auxiliary function constructed by [29] to obtain the local L^2 estimate of

h_g. First, we will review the local L^2 estimate in [42].

Consider the following auxiliary function

$$\xi(x, y, t, s) = -\frac{d_0^2(x, y)}{(2 - C\varepsilon)(t - s)} \tag{4.2.1}$$

where $d_0(x, y)$ is the distance from $y \in M$ to the geodesic ball $B_0(\sqrt{r/2}, x)$ with respect to the initial metric and C is chosen so that

$$\xi_s + \frac{1}{2}|\nabla\xi|^2 \leq 0$$

Subsequently, set

$$J(x, t, s) = \int_M \exp(\xi(x, y, t, s)) \cdot |h_g|^2(y, s)dy$$

It is evident that

$$\|h_g\|^2_{L^2(B_0(\sqrt{r/2}, x)))}(s) \leq J(x, t, s)$$

Thus, the local L^2 estimate of h_g is related to the control of $J(x, t, s)$.

Lemma 4.2.4 (Local L^2 estimate)**.** *For any $T > 0$, let $(M^{n+1}, g(t))$, $t \in [0, T]$ be a normalized Ricci flow of non-compact complete Riemannian manifolds with $n \geq 3$ satisfying that*

1) the non-degeneracy $\lambda(g(t)) > \lambda_0$ for some $\lambda_0 > 0$;

2)

$$\|h_{g(t)}\|_{C^0([0,T]\times M)} \leq \varepsilon, \quad \|g(t) - g(0)\|_{C^0((M,g(0)))} \leq \varepsilon$$

where $\varepsilon > 0$ depends on n and λ_0;

3) for the initial metric $g(0)$

$$\int_M \exp(\alpha d(x,x_0))dvol_{g(0)} \quad \text{for some } \alpha > 0$$

where $d(x, x_0)$ *is the distance function to a fixed point* $x_0 \in M$ *with respect to* $g(0)$.
Then, we have

$$J(x,t,s) \le e^{-(2\lambda_0 - C\varepsilon)s} J(x,t,0)$$

where $C > 0$ *is a constant depending on* ε, n *and* λ_0.

The above lemma is derived by the following inequality.

$$\begin{aligned}\partial_s J(x,t,s) &\le -2\int_M ((\Delta_L + 2n)(e^{\frac{\xi}{2}} h_{g(t)}), e^{\frac{\xi}{2}} h_{g(t)}) + C\varepsilon J(x,t,x) \\ &\le -(2\lambda_0 - C\varepsilon) J(x,t,s)\end{aligned}$$

Lemma 4.2.5 (C^2 estimate)**.** *For any* $T > 0$*, let* $(M^{n+1}, g(t))$*,* $t \in [0,T]$*, be a normalized Ricci flow of non-compact complete Riemannian manifold with the dimension* $n \ge 3$ *satisfying that*

1) the non-degeneracy $\lambda(g(t)) > \lambda_0$ *for some* $\lambda_0 > 0$*;*

2)

$$\|h_{g(t)}\|_{C^0([0,T]\times M)} \le \varepsilon, \quad \|g(t) - g(0)\|_{C^0((M,g(0)))} \le \varepsilon$$

where $\varepsilon > 0$ *is a constant depending on* n *and* λ_0*;*

3) $\|Rm_{g(t)}\|_{C^0(M)} \le k_0$ *for some* $k_0 > 0$*;*

4)

$$\sup_{(t,x)\in[0,T]\times M} vol_{g(t)}(B(x,1)) > v_0 \ \text{ for some } v_0 > 0$$

5) for the initial metric $g(0)$

$$|h_{g(0)}|(x) \leq \varepsilon_0 e^{\gamma d(x,x_0)}, \quad \int_M \exp(\alpha d(x,x_0))dx \leq C_0$$

for some ε_0 $\gamma, \alpha > 0$ satisfying that

$$\sqrt{\lambda} + \gamma > \frac{\alpha}{2} \ \text{ and } \ \varepsilon_0 \leq \varepsilon$$

where $d(x,x_0)$ is the distance function to a fixed point $x_0 \in M$ with respect to the initial metric $g(0)$.

Then for any $(x,t) \in M \times [\tau, T]$, we have

$$\|h_g\|_{C^0(M)} \leq C_1 \varepsilon_0 e^{-(2\lambda_0 - C_1\varepsilon)t}$$

and

$$\|\nabla h_g\|_{C^0(M)} + \|\nabla^2 h_g\|_{C^0(M)} \leq C_2 \varepsilon_0 e^{-(2\lambda_0 - C_2\varepsilon)t}$$

where C_1 is a postive constant depending on n, λ_0, k_0, v_0, α, γ and ε, and C_2 is a positive constant depending on n, λ_0, k_0, v_0, α, γ, ε and τ.

Through the De Giorgi-Nash-Moser theorem

$$\|h_h\|^2_{C^0((t-r/2)\times B_0(\sqrt{r/2},x))} \leq C(n, v_0, k_1, r) \int_{t-r}^{t} \int_{B_0(\sqrt{r/2},x))} |h_g|^2(y,s)dyds,$$

the following local estimate can be derived

$$\begin{aligned}
&\|h_h\|^2_{C^0((t-r/2)\times B_0(\sqrt{r/2}}, x) \leq C(n, v_0, k_1, r) \exp(-2(\lambda - \varepsilon)t) J(x,t,0) \\
&= C \exp(-2(\lambda - \varepsilon)t) \int_M \exp(-\frac{d_0^2(x,y)}{(2 + C_0\varepsilon)t}) \cdot |h_g|^2(y,0)dy \\
&\leq C \exp(-2(\lambda - \lambda_0 - \varepsilon)t) \int_M \exp(-2\sqrt{\frac{2\lambda_0}{2 + C_0\varepsilon}} d_0(x,y)) \cdot \varepsilon exp(-\gamma d_0(y,x_0))dy
\end{aligned}$$

To obtain a uniformly bounded last integral, the additional constraint

$$\gamma + \sqrt{\lambda} > \frac{\alpha}{2},$$

is imposed. Then, by applying the same contradiction argument as in [49], [42] still achieve the long-time existence and convergence theorem. It is worth noting that for asymptotically hyperbolic manifolds, $\alpha = n$. And in particularly, for hyperbolic space $\mathbb{H}^{n+1}$, $\lambda = \frac{n^2}{4}$. Therefore, for any $\gamma > 0$, we can away find a metric which is closed enough to the standard hyperbolic space, such that

$$\gamma + \sqrt{\lambda} > \frac{n}{2}$$

4.2.3 The proof of the theorem 4.1.5

For arbitrary asymptotically hyperbolic manifolds, the above condition

$$\gamma + \sqrt{\lambda} > \frac{n}{2}$$

might not be satisfied. However, by the lemma 7.13 in [30], we see that if (M^{n+1}, g_+) is an asymptotically hyperbolic manifold, then for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq M$, such that

$$((\Delta_L + 2n)u, u) > (\frac{n^2}{4} - \varepsilon)(u, u)$$

whenever u is smooth and compactly supported in $M \backslash K_\varepsilon$. This observation reminds us of that for asymptotically hyperbolic manifolds case, the above additional constraint might not be necessary, since the γ only works for the part of a function which is closed to the infinity.

Drawing on the method presented in [49] and [42] and taking into the above observation that the Rayleigh quotient tends towards $\frac{n^2}{4}$ as the function's support approaches infinity, we will derive the enhanced global existence and convergence theorem 4.1.5. This theorem effectively demonstrates that the aforementioned supplementary constraint is unnecessary.

The key is to prove the following lemma which is an asymptotically hyperbolic manifolds version of lemma 4.2.5 without the constraint

$$\gamma + \sqrt{\lambda} > \frac{n}{2}$$

Lemma 4.2.6. *For any $T > 0$, let $(M^{n+1}, g(t))$, $t \in [0, T]$, be a normalized Ricci flow of non-compact complete Riemannian manifolds with the dimension $n \geq 3$ satisfying that*

1) the non-degeneracy $\lambda(g(t)) > \lambda_0$ for some $\lambda_0 > 0$;

2)

$$\|h_{g(t)}\|_{C^0([0,T]\times M)} \leq \varepsilon, \quad \|g(t) - g(0)\|_{C^2((M,g(0)))} \leq \varepsilon$$

where $\varepsilon > 0$ is a constant depending on n and λ_0;

3) $\|Rm_{g(t)}\|_{C^5(M)} \leq k$ for some $k > 0$;

4)

$$\sup_{(t,x)\in[0,T]\times M} vol_{g(t)}(B(x, 1)) > v_0 \;\; \text{for some } v_0 > 0$$

5) $g(0)$ is an asymptotically hyperbolic metric with

$$|h_{g(0)}|^2(x) \leq \varepsilon_0 e^{\gamma d(x,x_0)}, \quad \int_M \exp(nd(x, x_0))dx \leq C_0$$

for some ε_0, $\gamma > 0$ satisfying that

$$\varepsilon_0 \leq \varepsilon$$

where $\mathrm{d}(x, x_0)$ *is the distance function to a fixed point* $x_0 \in M$ *with respect to the initial metric* $g(0)$.
Then for any $(x, t) \in M \times [\tau, T]$, *we have*

$$\|h_g\|_{C^0(M)} \leq C_1 \varepsilon_0 e^{-(2\lambda_0 - C_1\varepsilon)t}$$

and

$$\|\nabla h_g\|_{C^0(M)} + \|\nabla^2 h_g\|_{C^0(M)} \leq C_2 \varepsilon_0 e^{-(2\lambda_0 - C_2\varepsilon)t}$$

where C_1 *is a postive constant depending on* n, λ_0, k, υ_0, α, γ *and* ε, *and* C_2 *is a positive constant depending on* n, λ_0, k, υ_0, α, γ, ε *and* τ.

Once we have the above lemma, we can utilize the contradiction argument of [49] to obtain the theorem 4.1.5 as in previous section.

We will sketch the proof of the above lemma step by step.

Step1: (non-linear equation to linear equation) Since $g(t)$, a normalized Ricci flow, is fixed for $t \in [0, T]$, we can think of the Lichnerowicz operator, Δ_L is a operator depending on t, then the equation

$$\partial_t u(t, x) = -(\Delta_L + 2n)u(t, x)$$

as a Linear equation. Once the geometry of $g(t)$ is controlled, the Lichnerowicz operator is controlled.

Step2 : (Cut-off on the boundary) Since the Rayleigh quotient will approcach to $n^2/4$ only if the support of the related function approach to infinity, we need to seperate the interior and the boundary of the given asymptotically hyperbolic manifold. Therefore, we consider the following equations:

$$\begin{cases} \partial_t h_i(t,x) = -\left(\Delta_L + 2n\right) h_i(t,x) \\ h_i((i-1)st, x) = \varphi(x) \cdot h_{i-1}((i-1)st, x) \end{cases}$$

and

$$\begin{cases} \partial_t L_i(t,x) = -(\Delta_L + 2n) L_i(t,x) \\ L_i((i-1)st, x) = (1-\varphi) h_{i-1}((i-1)st, x) \end{cases}$$

for $(i-1)st \le t \le T$, where Δ_L is the Lichnerowicz operator determined by the metric $g(t)$, $st > 0$ is a small time step which we will determine later, φ is a cut-off function on the boundary of $\overline{M}$ which we will determine later as well and $h_0(0,x) = h_{g(0)}(x)$. Let

$$\tilde{h}(t,x) = h_l(t,x)$$

where $l = [t/st] + 1$. It is straightforward to show that

$$h_{g(t)}(x) = \tilde{h}(t,x) + \sum_{i=1}^{l} L_i(t,x)$$

for any $0 < st < T$. Therefore, we just need to show that

$$\|\tilde{h}(t,x)\|_{C^0(M)} \le Ce^{-\tilde{\lambda}t} \quad \text{and} \quad \|\sum_{i=0}^{l} L_i(t,x)\|_{C^0(M)} \le Ce^{-\tilde{\lambda}t} \tag{4.2.2}$$

where C does not reply on T. Once we have the C^0 estimate of $h_{g(t)}$, by the standard parabolic estimate, we easily get the estimate of $\nabla h_{g(t)}$ and $\nabla^2 h_{g(t)}$.

Step3: (Boundary Estimate) we will find out $0 \le st \le T$, such that

$$\|\tilde{h}(t,x)\|_{C^0(M)} \le Ce^{-\tilde{\lambda}t}$$

It is turned out that the following lemma playing an important role.

Lemma 4.2.7. *For any $\delta, a, k, v_0, C_0, C_1, \gamma, t' > 0$, and $n \ge 3$, there exists $\varepsilon(\delta, n) > 0$, $D(\delta, v_0, C_0, C_1, \gamma, n) > 0$ and $st(\delta, n, \varepsilon, v_0, C_0, C_1, \gamma, t', n) > 0$ such that for any normalized Ricci flow of non-compact complete manifolds, $(M^{n+1}, g(t))$, $t \in [0, t']$ satisfying that*

1)

$$\|h_{g(t)}\|_{C^0(M)} \le \varepsilon, \quad \|g(t) - g(0)\|_{C^2((M,g(0)))} \le \varepsilon$$

for any $t \in [0, [0, t']]$;

2) $\|\nabla^l Rm_{g(t)}\|_{C^5(M)} \le k$

3)

$$\sup_{(t,x)\in[0,t']\times M} vol_{g(t)}(B(x,1)) > v_0 \text{ for some } v_0 > 0$$

4) $g(0)$ is an asymptotically hyperbolic metric of regularity $C^{2,\alpha}$ with

$$|h_{g(0)}|^2(x) \le \varepsilon_0 e^{\gamma d(x,x_0)}, \quad \int_M \exp(nd(x,x_0))dx \le C_0$$

for some $\varepsilon_0 \le \varepsilon$, where $d(x, x_0)$ is the distance function to a fixed point $x_0 \in M$ with respect to the initial metric $g(0)$,

and for any cut-off function $\varphi \in C^\infty(\overline{M})$ satisfying that

1) $\varphi(\partial\overline{M}) \equiv 1$;

2) $supp(\varphi) \subseteq \{x \in M \mid d(x, x_0) > D\}$;

3) $\|\varphi\|_{C^5(M)} < C_1$,

we have that

$$\int_M ((\Delta_L + 2n)u_i(t, x, T + a, y), u_i(t, x, t' + a, y))dx$$
$$\geq (\frac{n^2}{4} - \delta) \int_M (u_i(t, x, t' + a, y), u_i(t, x, t' + a, y))dx$$

for any $(i-1) \cdot st \leq t \leq i \cdot st$ *and* $y \in M$, *where*

$$u_i(t, x, t' + a, y) = e^{\xi(x,y,T+a,t)} h_i(t, x, t' + a, y)$$

for ξ *and* h_i *defined as (4.2.1) and (4.2.3) respectively.*

Once we have this lemma, by the same method as what is shown in the previous section, we can get that

$$\|\tilde{h}(t', x)\|_{C^0(M)} \leq Ce^{-\tilde{\lambda}t'}$$

for any $t' \in [0, T]$. Specifically, consider the following auxiliary function

$$J_i(y, t' + a, t) = \int_M (u_i(t, x, t' + a, y), u_i(t, x, t' + a, y))dx$$

for $(i-1) \cdot st \leq t \leq i \cdot st$. Since J_i satisfies the following inequality

$$\partial_t J_u(y, t' + a, t) \leq -2 \int_M ((\Delta_L + 2n - \varepsilon)u_i(t, x, t' + a, y), u_i(t, x, t' + a, y))dx$$
$$\leq -2(\frac{n^2}{4} - \delta) J_i(y, t' + a, t),$$

for any $t \in [(i-1) \cdot st, i \cdot st]$, then

$$J_i(y, t' + a, t) \leq \exp(-2(\frac{n^2}{4} - \delta)(t - (i-1) \cdot st)) J_i(y, t' + a, (i-1) \cdot st).$$

Then, by the fact that

$$J_i(y, t' + a, (i-1) \cdot st) \leq J_{i-1}(y, t' + a, (i-1) \cdot st),$$

we have that

$$\begin{aligned} J_i(y, t' + a, t) &\leq \exp(-2(\frac{n^2}{4} - \delta)t) J_0(y, t' + a, 0) \\ &\leq \exp(-2(\frac{n^2}{4} - \delta)t) \int_M \exp(-\frac{d_0^2(z, y)}{(2 + C\delta)(t' + a)}) |h_{g(0)}(z)|^2 dz \end{aligned}$$

for any $t \in [(i-1) \cdot st, i \cdot st]$. Then, by the De Giorgi-Nash-Moser estimate, we have that

$$\begin{aligned} \|h_i\|_{C^0([t' - \frac{r}{2}, t'] \times B_0(y, \sqrt{r/2}))} &\leq C(n, k, r) \int_{t'-r}^{t'} \int_{B_0(y, \sqrt{r/2})} |h_i(s, z)|^2 dz \\ \leq & C(n, k, r) \int_{t'-r}^{t'} J_{([s/st]+1)}(z, T + a, s) ds \\ \leq & C(n, k, r, a) \exp(-2(\frac{n^2}{4} - \delta - \lambda_0)T) \\ & \cdot \int_M \exp(-\frac{d_0^2(z, y)}{(2 + C\delta)(t' + a)} - 2\lambda_0(t' + a)) |h_{g(0)}(z)|^2 dz \\ \leq & C(n, k, r, a) \exp(-2(\frac{n^2}{4} - \delta - \lambda_0)t') \\ & \cdot \int_M \exp(-2\sqrt{\frac{2}{2 + C\delta}\lambda_0} \cdot d_0(z, y)) \varepsilon_0 \exp(-2\gamma d_0(z, x_0)) dz \end{aligned}$$

We see by the above lemma, for any $\gamma > 0$, there exists a small δ, such that we can find out a proper λ_0 satisfying that

$$(\frac{n^2}{4} - \delta - \lambda_0) > 0 \quad \text{and} \quad \sqrt{\frac{2}{2 + C\delta}\lambda_0} + \gamma > \frac{n}{2}$$

Therefore, by the theorem 5.4 of [30], we can get that

$$\|h_i\|_{C^0(M)}(t') \leq C\exp(-2(\frac{n^2}{4} - \delta - \lambda_0)t')$$

for any $t' \in [0, T]$.

Step4: (Interior Estimate) In this step, we will estimate

$$\|\sum_{i=0}^{l} L_i(t,x)\|_{C^0(M)} \leq Ce^{-\tilde{\lambda}t} \quad \text{for } l = [t/st] + 1$$

in (4.2.2). First, we will get the L^2 estimate of L_i. Then, by the De Giorgi-Nash-Moser estimate, we can have the corresponding C^0 estimate. Since $\|L_i((i-1)\cdot st, x)\|_{L^2(M)} \leq \infty$, by the equation of $L_i(t,x)$. we can get that

$$\begin{aligned}
\|L_i(t,x)\|_{L^2(M)} &\leq \exp(-2(\lambda_1 - \delta)(t - (i-1)\cdot st)\|L_i((i-1)\cdot st, x)\|_{L^2(M)} \\
&\leq C(D)\exp(-2(\lambda_1 - \delta)(t - (i-1)\cdot st)\cdot\|L_i((i-1)\cdot st, x)\|_{C^0(M)} \\
&\leq C(D)\exp(-2(\lambda_1 - \delta)(t - (i-1)\cdot st)\|h_{i-1}((i-1)\cdot st, x)\|_{C^0(M)} \\
&\leq C(n,k,\alpha,a)\varepsilon_0\exp(-2(\lambda_1 - \delta)(t - (i-1)\cdot st)\exp(-\tilde{\lambda}(i-1)\cdot st) \\
&\leq C(n,k,\alpha,a)\varepsilon_0\exp(-2(\lambda_1 - \delta)t)\cdot\exp(-(\tilde{\lambda} - 2(\lambda_1 - \delta))(i-1))
\end{aligned}$$

for any $\lambda_1 \in (0, \lambda]$. Therefore, we can always take small enough λ_1 such that

$$\tilde{\lambda} - 2(\lambda_1 - \delta) > 0.$$

Therefore, we have that

$$\begin{aligned} \| \sum_{i=0}^{l} L_i(t,x) \|_{C^0(M)} &\le C \sum_{i=0}^{l} \|L_i(t,x)\|_{L^2(M)} \\ &\le C\varepsilon_0 \exp(-2(\lambda_1 - \delta)t) \sum_{i=0}^{l} \exp(-(\tilde{\lambda} - 2(\lambda_1 - \delta))(i-1)) \\ &\le C\varepsilon_0 \exp(-(\tilde{\lambda} - 2(\lambda_1 - \delta))(i-1)) \end{aligned}$$

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