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Benjamin–Ono at Low Regularity: An Integrability Approach

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Benjamin–Ono at Low Regularity: An Integrability Approach

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by

Blaine Aaron Foster Talbut

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ABSTRACT OF THE DISSERTATION

Benjamin–Ono at Low Regularity: An Integrability Approach

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2021
Professor Rowan Brett Killip, Co-Chair
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We employ the integrable structure of the Benjamin–Ono equation in order to study its rough solutions. For rough data, our most useful tools are the Lax pair formalism and, as in the inverse scattering transform, the structural information embedded in solutions to the scattering equation. Using these, we prove that Sobolev norms are conserved and locally smoothed for rough initial data. Using the integrable structure, we construct a Hamiltonian that usefully approximates the Benjamin–Ono Hamiltonian. With this we may provide a short new proof that the Cauchy problem is well-posed in L^2.

The dissertation of Blaine Aaron Foster Talbut is approved.

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1 Introduction

We study real-valued solutions to the Benjamin–Ono equation

$$\frac{d}{dt}q = -Hq'' + 2qq' \tag{BO}$$

on the line $\mathbb{R}$ and the circle $\mathbb{R}/\mathbb{Z}$, where the Hilbert transform H is defined in either setting by

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Benjamin [2] introduced this equation to model the propagation of long waves that form at the interface of two fluids of differing densities. He observed (theoretically and empirically) the existence of solitary wave solutions akin to those which were well-attested for the Korteweg–de Vries (KdV) equation [16]. For KdV, these soliton solutions can be explained in the framework of integrability: specifically, an inverse scattering transform. Formally, Benjamin–Ono also admits an IST, although making rigorous sense of the formalism is challenging even for smooth solutions (see [28], [27]).

Killip and Vişan [14] exploit KdV's IST apparatus to prove that the equation is well-posed in the sharpest possible Sobolev space H^{-1}. In particular, the existence of infinitely many commuting flows associated to a sequence of conservation laws of increasing regularity is used to construct a new commuting flow that makes sense for rough data, analogous to the Taylor series expansion of a rational function of negative degree using polynomials of increasing positive degree. Given the similarities between KdV and BO, [1] it is reasonable to expect the same techniques to apply to Benjamin–Ono. This is the line we pursue in this thesis.

[1] This relationship can be stated precisely. The Intermediate Long Wave equation (ILW) models interface waves between two fluids of finite depth. Letting the depth paramater of the ILW tend to infinity, one recovers BO; letting the depth parameter tend to 0, one recovers the surface wave equation KdV. See [24] for a survey of this topic.

In section 2, we review the results of [25], where a one-parameter family of conserved quantities $\alpha(\kappa)$ is constructed, which control the H^s norm of solutions to (BO) for $s > -\frac{1}{2}$, the critical regularity. This quantity is realized as the perturbation determinant associated to the Lax operator for the Benjamin–Ono IST scheme. In section 3, we use the Jost solutions associated to the scattering problem to exhibit a conserved Hamiltonian H_κ with the property that $H_\kappa \propto \partial_\kappa \alpha$. We prove that $H_\kappa \to H_{\text{BO}}$ as $\kappa \to \infty$. In section 4 we use this equipment to provide a short new proof that Benjamin–Ono is well-posed in the space of L^2 initial data. In section 5, we prove Kato-type local smoothing of one half degree for solutions to (BO). Such a result that holds, not for the H_{BO} Hamiltonian flow but for the $H_\kappa - H_{\text{BO}}$ difference flow, is the missing ingredient in extending well-posedness on the line down to the critical regularity $s > -\frac{1}{2}$.

1.1 A Brief History

1.1.1 Integrability

(BO) is a Hamiltonian system; that is to say, it takes the form

$$q_t = \frac{\partial}{\partial x}\frac{\delta}{\delta q} H_{\text{BO}}$$

for the Hamiltonian energy

$$H_{\text{BO}} = \int \frac{1}{3}q^3 - \frac{1}{2}qHq'\,dx.$$

A finite-dimensional Hamiltonian ODE on a symplectic manifold of dimension d can be integrated—the initial value problem solved explicitly by quadrature—if it admits $d/2$ indepdendent, Poisson-commuting conserved quantities (Liouville integrabiliy; see [1], §49–50). However, for Hamiltonian PDE, integrability is an ambiguous concept. (Of course, even in the case of ODE, there is some ambiguity about what we mean by an "explicit" solution when the resultant integrals may not admit closed form solutions.) In analogy with the finite-dimensional case, Liouvile integrability of PDE is sometimes interpreted to mean the construction of infinitely many commuting independent conserved quantities. However, for PDE this is not usually sufficient to explicitly solve the IVP.

It was out of the ambition to do just that the inverse scattering transform (IST) was first realized for KdV on the line by researchers at the Princeton Plasma Physics Laboratory [6]. Their discovery was of a certain subset of eigensolutions and their eigenvalues, the *scattering data*, associated to the stationary Schrödinger equation with potental q:

$$-v_{xx} + qv = \lambda v \tag{1.1}$$

with the properties that

1. the potential q can be recovered from the scattering data by an explicit integral (the *inverse scattering problem* (ISP)), and

2. if the potential q evolves according to the KdV equation, then (a minor miracle) the induced evoluton of the scattering data is linear.

The scattering equation (1.1) can be solved explicitly, and the scattering data may thereby be explicitly computed (the *forward scattering problem* (FSP)). Let $S(t)$ denote the scattering data arising from the time-dependent potential $q(t)$. Then the IST is defined by the following commutative square, in which each arrow is soluble:

$$\begin{tikzcd} S(0) \arrow[r, "\text{linear}"] & S(t) \arrow[d, "\text{ISP}"] \\ u(0) \arrow[u, "\text{FSP}"] \arrow[r, dashed, "\text{IST}"] & u(t) \end{tikzcd}$$

Benjamin–Ono is integrable on the line in that it admits an IST analogous to the above. The IST was first described in [5] and simplified in the case of real potentials in [12]. Our only object here is to see the essential elements of the IST apparatus appear in their own context; therefore we will restrict our treatment to the case of real solutions with $\int_{-\infty}^{\infty} u(x)dx \neq 0$ for which the inverse scattering problem is simplest. The scattering equation that plays the role of the Schrödinger equation for (BO) is

$$iv_x + \lambda v = -(qv)_+ \tag{1.2}$$

where f_+ denotes the restriction to positive frequencies on the Fourier side. In addition to this, it is convenient to consider an auxiliary inhomogeneous modification of (1.2) which admits solutions with non-vanishing boundary data:

$$iv_x + \lambda(v-1) = -(qv)_+. \tag{2-a}$$

Notice that this scattering equation is a nonlocal differential equation, unlike in the case of KdV. (This is not surprising, since the Hilbert transform which appears in (BO) is itself is nonlocal.)

The scattering data consists in:

(i) The set of all bound-state solutions ψ_j, λ_j to (1.2) with $\lambda_j < 0$, $\psi_k \sim 1/x$ for $j = 1, \ldots, J$ for $J \in \mathbb{N} \cup \{\infty\}$

(ii) The Jost solutions $\bar{M}, N$ to (1.2) and $M, \bar{N}$ to (2-a) when $\lambda \geq 0$ with boundary conditions

$$N(x,\lambda) \xrightarrow[x\to\infty]{} e^{i\lambda x}, \quad \bar{M}(x,\lambda) \xrightarrow[x\to-\infty]{} e^{i\lambda x},$$

$$\bar{N}(x,\lambda) \xrightarrow[x\to\infty]{} 1, \quad M(x,\lambda) \xrightarrow[x\to-\infty]{} 1.$$

(iii) The reflection coefficient $\beta(\lambda)$ that satisfies for $\lambda > 0$

$$M(x,\lambda) = \bar{N}(x,\lambda) + \beta(\lambda)N(x,\lambda)$$

and is given explicitly by

$$\beta(\lambda) = i\int_{-\infty}^{\infty} q(y)M(y,\lambda)e^{-i\lambda y}dy.$$

(iv) A set $\{\gamma_j\}_{j=1}^J$ of parameters that relate the asymptotics of ψ_j to those of the Jost solutions. I omit the precise definition except to say that if one knows the ψ_j and any particular Jost solution from (ii) then these can be computed.

The potential q can be recovered from N and the ψ_j by

$$[q(x)]_+ = \frac{1}{2\pi i}\int_0^\infty \beta(\lambda)N(x,\lambda)d\lambda + i\sum_{j=1}^J \psi_j(x). \tag{1.3}$$

The inverse scattering problem consists therefore in recovering N and $\{\psi_j\}$. These can be recovered together by solving a system of equations

$$\begin{cases} (x+\gamma_j)\psi_j(x) + i\displaystyle\sum_{j\neq k}\frac{1}{\lambda_j-\lambda_k}\psi_k(x) - \frac{1}{2\pi i}\int_0^\infty \frac{\beta(\lambda)N(x,\lambda)d\lambda}{\lambda-\lambda_j} = 1, \\ N(x,\lambda)e^{-i\lambda x} = \displaystyle\frac{1}{2\pi}\int_0^\infty \beta(\lambda')N(x,\lambda')w(x;\lambda,\lambda')\frac{d\lambda'}{\lambda'} - \sum_{j=1}^J \frac{\psi_j}{\lambda_j}w(x;\lambda,\lambda_j) \end{cases} \tag{1.4}$$

where

$$w(x;\lambda,\lambda') = \frac{1}{2\pi}\int_0^\lambda \frac{\beta^*(\ell)e^{-i\ell x}d\ell}{\ell-\lambda'-i\varepsilon}.$$

The IST of [5] and [12] can be outlined as follows:

1. We begin with an initial profile $u(x,0)$.

2. We compute the $\{(\psi_j,\lambda_j,\gamma_j)\}_{j=1}^J, M(x;\lambda), \beta(\lambda)$ associated to (1.2), (2-a) at time $t=0$.

3. We propogate $\dot{\lambda}_j = 0, \dot{\gamma}_j = 2\lambda_j, \dot{\beta}(\lambda) = i\lambda^2\beta(\lambda)$.

4. We solve (1.4) at time t to obtain $N(\lambda), \{\psi_j\}$.

5. We solve (1.3) at time t to recover $[q(x,t)]_+$, which when q is real recovers $q = q_+ + (q_+)^*$.

The regularity conditions required to carry out this scheme are not addressed in [5], [12]. For sufficiently smooth and rapidly decaying potentials q, [28] proves that the spectrum of the scattering operator is discrete and finite, justifying scattering the exisence of scattering data (i); and [27] proves that the Jost solutions $N, \bar{M}, \bar{N}, M$ exist. This at least may satisfy us that the scattering data described above exist when $q(t)$ is for all time unimpeachably regular.

There is another interpretation of the IST that is relevant to this paper, the *Lax pair* formalism. The essential elements of the Lax pair for Benjamin–Ono appear in [21], [4], and it is described in the modern form by Wu [28]. Formally, we define operators L, P in terms of a given potential q by

$$Lv = iv_x + (qv)_+$$

$$Pv = iv_{xx} + 2\big[q_x v + qv_x - q_x^+ v\big]_+.$$

The scattering data of the IST can be formulated in terms of the spectrum of the operator L. If $q = q(t)$ varies in time, then q solves (BO) if and only if

$$\frac{d}{dt}L = [P, L]. \tag{1.5}$$

It follows from (1.5) that

$$\frac{d}{dt}L^n = [P, L^n]$$

and therefore

$$\frac{d}{dt}\operatorname{tr} L^n = \operatorname{tr}[P, L^n] = 0.$$

This gives rise to an infinite hierarchy of conservation laws for Benjamin–Ono which Poisson commute (since they arise from powers of the same operator), and hence to integrability in the naive sense. However, this is purely formal; the traces above do not make sense. If we are uncowed by this, we may see that another formal computation expands the resolvent function in a series:

$$R := (L+\kappa)^{-1} = \frac{1}{\kappa}\sum_{\ell=0}^{\infty} \kappa^{-n} L^n$$

from which we can see that $\operatorname{tr} R$ is, formally, a generating function for the formally conserved quantities $\operatorname{tr} L^n$. As we will see, this is an insight of which we can make real use.

It follows also from (1.5) that if v solves the scattering equation

$$Lv + \lambda v = 0$$

then

$$\begin{aligned} \dot{L}v + L\dot{v} + \lambda\dot{v} &= 0 \\ PLv - LPv + L\dot{v} + \lambda\dot{v} &= 0 \\ -(L+\lambda)Pv + (L+\lambda)\dot{v} &= 0 \end{aligned}$$

which implies, if $L+\lambda$ is invertible (which, as we will see later, it is for λ sufficiently large), that

$$\dot{v} = Pv. \tag{1.6}$$

While the IST is purely a phenomenon of the line, the Lax pair makes just as much sense on either the line or circle geometry. The Lax pair and its role in generating a hierarchy of conservation laws are the crucial tool used in [7] to prove that, on the circle, Benjamin–Ono is linearized by global Birkhoff coordinates, and hence enjoys the classical integrable structure of ODE on finite-dimensional symplectic manifolds.

1.1.2 Well-posedness

The first studies of the Cauchy problem for (BO) on the line and the circle showed that the equation is globally well-posed in the space of classical solutions $C_t^0 H_x^3 \cap C_t^1 H_x^1$ (Iorio [11], Saut [23]). Note that a classical $C_t^0 H_x^3$ solution to (BO) is automatically also $C_t^1 H_x^1$. The first result to go below the classical regime appears to be Ponce [22], who supplemented the classical commutator estimates with Kato local smoothing and semilinear Strichartz estimates to obtain local well-posedness in H^s for $s = \frac{3}{2}$. Koch and Tsvetkov [15] lowered the threshold to $s = \frac{5}{2}$ by using Littlewood-Paley theory to control the nonlinearity.

For many years the well-posedness theory for (BO) lagged behind that of similar nonlinear equations such as the quadratic nonlinear Schrodinger equation

$$\frac{d}{dt} q = \pm i q'' + |q|^2.$$

The difficulty can be attributed to the presense of a complete derivative in the nonlinearity of (BO) and the inadequate smoothing of the linearized equation. Molinet, Saut, and Tzvetkov [20] proved that, as a result, the equation is not semilinear in the sense that it cannot be solved by iteration and contraction mapping arguments in H^s must fail. Tao [26] circumvented this difficulty by appling a Gauge transformation to the equation, eliminating the derivative in the most difficult frequency regimes. In this way local (and consequently global) well-posedness was achieved in $H^1(\mathbb{R})$. This approach, together with the mean value theorem and the use of Bourgain spaces, sufficed to prove LWP and GWP on $H^{1/2}(\mathbb{R})$ (Molinet [17]).

The current state of the art was first reached by Ionescu and Kenig [10], who used the Gauge transform of [26] to obtain local well-posedness in $L^2(\mathbb{R})$ via contraction mapping in a complicated modification of an $X^{s,b}$ space. Molinet [18] obtained the same result in the periodic setting. Molinet and Pilod [19] provided a simpler unified proof of well-posedness on both the line and the circle by using Littlewood-Paley theory to control the nonlinearity. Ifrim and Tataru [9] provided yet another proof of LWP in $L^2(\mathbb{R})$ which eliminated the need to work in complicated functional spaces by augmenting the Gauge transform of [26] with a Shatah type normal form correction.

The equation (BO) enjoys a scaling symmetry, to wit

$$q \mapsto \lambda q(\lambda^2 t, \lambda x).$$

This symmetry leaves $\|q\|_{\dot{H}^s}$ invariant for $s = -\frac{1}{2}$; as $\lambda \to 0$ the norm blows up for $s < -\frac{1}{2}$. This indicates that below the critical regularity $s = -\frac{1}{2}$ (BO) is ill-posed. As far as well-posedness in the Sobolev space H^s, there remains a gap between the best known results of $s = 0$ on the line and the scaling-critical regularity $s = -1/2$.

None of these approaches use the integrability of (BO) directly. The story of the well-posedess of the Cauchy problem intersects the story of integrability with Gerard, Kappeler, and Topalov [8], who use the global Birkhoff coordinates constructed in [7] to prove well-posedness in $H^s(\mathbb{T})$ for s down to the critical exponent $s = -\frac{1}{2}$.

In this thesis, I will indicate some ways of bringing the integrability theory for (BO) on both the line and the circle to bear on PDE questions. Our approach is motivated by recent work on the Korteweg–de Vries equation, to which Benjamin–Ono is related as we have mentioned. Using the integrable structure of KdV and in particular the Lax pair, Killip, Vişan, and Zhang [13] obtained conservation laws which govern the H^s norm of the solution for $s \geq -1$. These same conservation laws were employed by Killip and Vişan [14] to obtain global well-posedness of KdV in the space H^{-1}.

1.2 Notation

We write $A \lesssim B$ to mean that $A \leq CB$ for an absolute constant C; if we wish to specify that the value of C depends on further parameters $a, b, \ldots$ then we write $A \lesssim_{a,b,\ldots} B$. We write $A \lesssim B^{\gamma\pm}$ to mean that, for any $\varepsilon > 0$, $A \lesssim_\varepsilon B^{\gamma\pm\varepsilon}$.

In this paper our conventions for the Fourier transform are

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \qquad \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi) d\xi$$

for functions on the line and

$$\hat{f}(\xi) = \int_0^1 e^{-ix\xi} f(x) dx, \qquad \check{f}(x) = \sum_{\xi \in 2\pi\mathbb{Z}} e^{ix\xi} f(\xi)$$

for functions on the circle.

We will occasionally use the Japanese bracket $\langle x \rangle := \sqrt{1 + |x|^2}$, especially in the context of the Sobolev norms defined by

$$\|f\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi, \qquad \|f\|^2_{H^s(\mathbb{R}/\mathbb{Z})} = \sum_{k \in 2\pi\mathbb{Z}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2$$

Let $H^s_0(\mathbb{R}/\mathbb{Z})$ denote the subspace of $H^s(\mathbb{R}/\mathbb{Z})$ functions with $\hat{f}(0) = 0$, i.e. mean zero. We write L^2 for H^0.

If $f \in H^s(D)$ and $g \in H^{-s}(D)$, then the scalar product

$$\langle f, g \rangle = \int_D f(x) \overline{g(x)} dx$$

converges and satisfies Plancherel's theorem $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$, which on the circle we interpret as

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle := \sum_{\xi \in 2\pi\mathbb{Z}} \hat{f}(\xi) \overline{\hat{g}(\xi)}.$$

We will have frequent need to represent the restriction of a function to positive or to negative frequencies; which is to say projected orthogonally onto positive or negative frequency space. Let $C_\pm$ define the Cauchy projections

$$(C_\pm f)^\wedge(\xi) = \hat{f}(\xi)\mathbb{1}(\pm\xi > 0).$$

We will often write $C_\pm f = f_\pm$.

2 The Perturbation Determinant

In this section, we reproduce the results of [25] for the sake of a complete treatment: following the method of [13] to obtain low-regularity conservation laws for (BO). Our principal result is the following:

Theorem. *Let q be a classical solution to* (BO) *on the line or the circle and let $-\frac{1}{2} < s < 0$. Then*

$$(1 + \|q(0)\|_{H^s}^{\frac{2}{1+2s}})^s \sup_{t\in\mathbb{R}} \|q(t)\|_{H^s} \lesssim_{s,r} \|q(0)\|_{H^s} \lesssim_{s,r} (1 + \|q(0)\|_{H^s}^{\frac{2}{(1+2s)^2}})^{-s} \inf_{t\in\mathbb{R}} \|q(t)\|_{H^s}.$$

This will be proved as Theorem 2.6.

Let us review the method of [13] as it applies to our problem. The first thing to note is that (BO) has a Lax pair. We will treat the formalism in more detail here than in section 1.1.1, although our discussion is still formal. We follow [28] in presenting the Lax pair as it decomposes along the Hardy spaces $H^\pm$ of L^2 functions whose Fourier transforms are supported on positive and negative modes, respectively. On the line,

$$L^2(\mathbb{R}) = H^+(\mathbb{R}) \oplus H^-(\mathbb{R}).$$

On the circle we must be more careful, because the zero frequency mode contributes positive mass. However, if we restrict to the space $L^2_0(\mathbb{R}/\mathbb{Z})$ of mean-zero L^2 functions, then

$$L^2_0(\mathbb{R}/\mathbb{Z}) = H^+(\mathbb{R}/\mathbb{Z}) \oplus H^-(\mathbb{R}/\mathbb{Z}).$$

Concordantly, for much of this section we will assume that all our solutions to (BO) on the circle have mean 0. Because the (BO) flow preserves the mean of the data (since its right hand side is a complete derivative), this amounts to requiring the initial data to have mean 0. This assumption will be removed in the end by way of the Galilei transformation (2.15).

The orthogonal Cauchy projections $C_\pm : L^2(\mathbb{R}) \to H^\pm(\mathbb{R})$ and $C_\pm : L^2_0(\mathbb{R}/\mathbb{Z}) \to H^\pm(\mathbb{R}/\mathbb{Z})$ are given by

$$C_\pm f = \tfrac{1}{2}(f \pm iHf).$$

Given a smooth, decaying function $q(t,x)$, we define operators $L_\pm, P_\pm$ by

$$L_\pm(t) = \pm C_\pm \tfrac{1}{i}\nabla - C_\pm q(t) C_\pm,$$

$$P_\pm(t) = \pm \tfrac{1}{i} C_\pm \nabla^2 + 2C_\pm\big((C_\pm q_x(t)) - q_x(t) - q(t)\nabla\big)C_\pm$$

Because these operators leave $H^\pm$ (respectively) invariant, we are free to understand them to act on L^2 or on $H^\pm$. It was shown in [28] that $q(t)$ (mean 0 if on the circle) solves (BO) if and only if

$$\frac{d}{dt}L_\pm = [P_\pm, L_\pm].$$

Let us restrict our attention to the action on H^+. Because of the above Lax pair, the (BO) flow preserves all the spectral properties of $L_+(t)$. Thus, formally, we expect the perturbation determinant (where the determinant is taken over H^+)

$$\det((\kappa + L_+(t))R_0(\kappa) = \det(\mathrm{id} - C_+ q(t) C_+ R_0(\kappa))$$

to be preserved in time if q solves (BO). Here

$$R_0(\kappa) = C_+(\kappa - i\nabla)^{-1}C_+$$

is defined by multiplication on the Fourier side by $\mathbb{1}_{(0,\infty)}(\xi)(\kappa+\xi)^{-1}$. If $\kappa > 0$, this is a positive definite operator on H^+, and hence $\sqrt{R_0}$ makes sense and the symbol of $\sqrt{R_0}$ is the square root of that of R_0. Its inverse $L_0 = R_0^{-1}$ also makes sense, albeit as an unbounded operator.

Taking a logarithm, we find

$$-\log\det((\kappa + L_+(t))R_0(\kappa)) = \sum_{\ell=1}^{\infty} \frac{1}{\ell}\operatorname{tr}\left\{\left(C_+ q(t) C_+ R_0(\kappa)\right)^{\ell}\right\}. \tag{2.7}$$

It will be convenient to reformulate the above in terms of the operator

$$A(\kappa; q) := \sqrt{R_0(\kappa)}C_+ q C_+ \sqrt{R_0(\kappa)} \tag{2.8}$$

given on the Fourier side on $\mathbb{R}$ by

$$\widehat{A(\kappa; q)} f(\xi) = \mathbb{1}_{(0,\infty)}(\xi) \int_0^\infty \frac{1}{\sqrt{\kappa+\xi}} \hat{q}(\xi - \eta) \frac{1}{\sqrt{\kappa+\eta}} \hat{f}(\eta) d\eta$$

and on $\mathbb{R}/\mathbb{Z}$ by

$$\widehat{A(\kappa; q)} f(\xi) = \mathbb{1}_{\{2\pi, 4\pi, \ldots\}}(\xi) \sum_{\eta = 2\pi, 4\pi, \ldots} \frac{1}{\sqrt{\kappa+\xi}} \hat{q}(\xi - \eta) \frac{1}{\sqrt{\kappa+\eta}} \hat{f}(\eta).$$

$A(\kappa; q)$ depends linearly on q and is self-adjoint when q is real. Cycling the trace, we may rewrite (2.7) as

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell}\operatorname{tr}\{A(\kappa; q(t))^{\ell}\}.$$

This quantity almost makes sense; however, $A(\kappa; q)$ is not a trace-class operator, even if q is Schwartz. On the other hand, considered formally (and ignoring the Cauchy projections),

$$\operatorname{tr}\{A(\kappa; q(t))\} = \int_0^\infty \frac{1}{\kappa+\xi} d\xi \cdot \int q\,dx = \infty \cdot \int q\,dx$$

ought to be preserved by the (BO) flow because $\int q\,dx$ is. Thus we may have some confidence in dropping the $\ell = 1$ term to study the quantity

$$\alpha(\kappa; q) := \sum_{\ell=2}^{\infty} \frac{1}{\ell}\operatorname{tr}\{A(\kappa; q)^{\ell}\}.$$

As we shall see, this series makes sense if $q \in H^s$ for any $s > -\frac{1}{2}$ and κ is sufficiently large.

The crux of the method is to show, as the foregoing discussion suggests, that $\alpha(\kappa; q)$ is conserved by the (BO) flow (section 2.2) and that it controls the relevant norm(s) of the solution (section 2.3). In our case and unlike in [13], the main term of $\alpha(\kappa; q)$ is not commensurate with any Sobolev norm of q. Since the H^s norms for $s > -\frac{1}{2}$ are strictly stronger than the norm controlled by $\alpha(\kappa; q)$, the main theorem is recovered in section 2.3 from a kind of persistence of regularity.

2.1 Functional Preliminaries

Because our problem is translation-invariant, we may avoid any functional-analytic subtleties by working entirely on the Fourier side. If T is a linear operator given on the Fourier side by

$$\widehat{T\varphi}(\xi) = \int_{\mathbb{R}} K(\xi, \eta)\hat{\varphi}(\eta)d\eta$$

then we may define the Hilbert-Schmidt norm of T by

$$\|T\|_{\mathfrak{I}_2}^2 = \iint_{\mathbb{R}^2} |K(\xi, \eta)|^2 d\eta d\xi.$$

If $\|T\|_{\mathfrak{I}_2} < \infty$, we say that T is a *Hilbert-Schmidt operator* and write $T \in \mathfrak{I}_2$. If $n \geq 2$ and $T_1, \ldots, T_n$ are Hilbert-Schmidt operators with Fourier kernels $K_1, \ldots, K_n$, then we say $T_1 \cdots T_n$ is trace class and define the trace

$$\operatorname{tr}\{T_1 \cdots T_n\} = \int_{\mathbb{R}^n} K_1(\xi_1, \xi_2) \cdots K_n(\xi_n, \xi_1) d\xi_1 \cdots d\xi_n,$$

which is finite by the Cauchy-Schwarz inequality. In this formulation, cycling the trace amounts to an application of Fubini's theorem.

By the Cauchy-Schwarz inequality, $\alpha(\kappa; q)$ is a sub-geometric series with a common ratio $\lesssim \|A(q)\|_{\mathfrak{I}_2}$. The following lemma gives sufficient conditions for this series to converge and submit to term-by-term differentiation and ensures that $\alpha(\kappa; q)$ is comparable to its first term.

Lemma 2.1. *Let $t \mapsto A(t)$ define a C^1 curve in $\mathfrak{I}_2$. Suppose for some t_0 we have*

$$\|A(t_0)\|_{\mathfrak{I}_2} < \frac{1}{3}.$$

Then there is a closed interval I containing t_0 on which the series

$$\alpha(t) := \sum_{\ell=2}^{\infty} \frac{1}{\ell} \operatorname{tr}\{A(t)^{\ell}\}$$

converges uniformly and defines a C^1 function which can be differentiated term by term:

$$\frac{d}{dt}\alpha(t) = \sum_{\ell=2}^{\infty} \operatorname{tr}\{A(t)^{\ell-1} \frac{d}{dt} A(t)\}.$$

If $A(t)$ is self-adjoint, then

$$\frac{1}{3}\|A(t)\|_{\mathfrak{I}_2}^2 \leq \alpha(t) \leq \frac{2}{3}\|A(t)\|_{\mathfrak{I}_2}^2.$$

For a proof of this lemma, see [13], Lemma 1.5.

2.2 Conservation of the Perturbation Determinant

In light of Lemma 2.1, our first task is to understand $\|A(\kappa; q(t))\|_{\mathfrak{I}_2}$. Our next result is most conveniently formulated in terms of the linear operator T_κ given by the Fourier multiplier

$$\widehat{T_\kappa f}(\xi) = \frac{\log(2 + |\xi|/\kappa)}{\sqrt{\kappa^2 + \xi^2}} \hat{f}(\xi).$$

Theorem 2.2. *If $q \in H^s(\mathbb{R})$ or $q \in H_0^s(\mathbb{R}/\mathbb{Z})$ for $-\frac{1}{2} < s < 0$, then for $\kappa \geq 1$*

$$\|A(\kappa; q)\|_{\mathfrak{I}_2}^2 \sim \langle q, T_\kappa q \rangle \lesssim_s \kappa^{-1-2s} \|q\|_{H^s}^2.$$

Proof. We first consider the case of the line. We compute

$$\begin{aligned}
\|A(\kappa;q)\|_{\mathfrak{I}_2}^2 &= \int_{\xi\geq 0}\int_{\eta\geq 0}(\kappa+\xi)^{-1}(\kappa+\eta)^{-1}|\hat{q}(\xi-\eta)|^2 d\eta d\xi \\
&= \int_{-\infty}^{\infty}\int_{\eta\geq\max(0,-\xi)}(\kappa+\xi+\eta)^{-1}(\kappa+\eta)^{-1}|\hat{q}(\xi)|^2 d\eta d\xi \\
&= \int_0^\infty \frac{1}{\xi}\log\left(1+\frac{\xi}{\kappa}\right)|\hat{q}(\xi)|^2 d\xi - \int_{-\infty}^0 \frac{1}{\xi}\log\left(1-\frac{\xi}{\kappa}\right)|\hat{q}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty}\frac{\log(1+\frac{|\xi|}{\kappa})}{|\xi|}|\hat{q}(\xi)|^2 d\xi \\
&\sim \int_{-\infty}^{\infty}\frac{\log(2+\frac{|\xi|}{\kappa})}{\sqrt{\kappa^2+\xi^2}}|\hat{q}(\xi)|^2 d\xi
\end{aligned}$$

where the implicit constant in the last line is absolute. This proves the first inequality. The second inequality follows from the fact that

$$\log(2+|\xi|/\kappa)(\kappa^2+\xi^2)^{-1/2} \lesssim_s \kappa^{-1}\left(1+\left(\frac{\xi}{\kappa}\right)^2\right)^s \leq \kappa^{-1-2s}(1+\xi^2)^s$$

for any $-\frac{1}{2} < s < 0$, $\kappa \geq 1$.

In the case $q \in H_0^s(\mathbb{R}/\mathbb{Z})$, a similar computation to the above may be repeated, although the analogue of the third equality holds only within the bounds of multiplicative constants, rather than exactly. ☐

Theorem 2.3. *Let q be a $C_t^0 H_x^3 \cap C_t^1 H_x^1$ solution to* (BO) *on the line or the circle, having mean 0 if on the circle. For any $t \in \mathbb{R}$ and $s > -\frac{1}{2}$, there exists a constant $C = C(s)$ such that for all $\kappa \geq 1 + C\|q(t)\|_{H^s}^{\frac{2}{1+2s}}$,*

$$\frac{d}{dt}\alpha(\kappa;q(t)) = 0.$$

Proof. We choose C large enough that Theorem 2.2 ensures that

$$\|A(\kappa;q(t))\|_{\mathfrak{I}_2} < \frac{1}{3}$$

whenever $\kappa \geq 1 + C\|q(t)\|_{H^s}^{\frac{2}{1+2s}}$. We then apply Lemma 2.1 to conclude that $\alpha(\kappa; q)$ converges on a neighborhood of t and

$$\begin{aligned} \frac{d}{dt}\alpha(k; q(t)) &= \sum_{\ell=2}^{\infty} \operatorname{tr}\left\{A(\kappa, q)^{\ell-1}A(\kappa; q_t)\right\} \\ &= \sum_{\ell=2}^{\infty} \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; -Hq'' + 2qq')\right\}. \end{aligned}$$

By Theorem 2.2, $A(\kappa; q)$ is a Hilbert-Schmidt operator, as is $A(\kappa; -Hq'' + 2qq')$ if $q \in H^3$, so we may cycle a copy of $A(\kappa; q)$ in the trace to obtain

$$\begin{aligned} \frac{d}{dt}\alpha(\kappa; q(t)) = -\sum_{\ell=2}^{\infty} \operatorname{tr}\left\{A(\kappa; q)^{\ell-2}A(\kappa; Hq'')A(\kappa; q)\right\} + \\ + \sum_{\ell=2}^{\infty} \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; 2qq')\right\}, \end{aligned}$$

which we rearrange slightly to give a telescoping series:

$$\begin{aligned} \frac{d}{dt}\alpha(\kappa; q(t)) &= -\operatorname{tr}\left\{A(\kappa; q)A(\kappa; Hq'')\right\} + \\ &+ \sum_{\ell=2}^{\infty} \left[2\operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; qq')\right\} - \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; Hq'')A(\kappa; q)\right\}\right]. \end{aligned}$$

Evidently it suffices to show that

$$\operatorname{tr}\left\{A(\kappa; q)A(\kappa; Hq'')\right\} = 0 \tag{2.9}$$

and

$$2\operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; qq')\right\} = \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}A(\kappa; Hq'')A(\kappa; q)\right\} \tag{2.10}$$

for all $\ell \geq 2$.

To see (2.9), we compute the trace directly on the line:

$$\begin{aligned} &\operatorname{tr}\{A(\kappa; q)A(\kappa; Hq'')\} \\ &\quad = -\int_{\xi\geq 0}\int_{\eta\geq 0} (k+\xi)^{-1}\hat{q}(\xi-\eta)(k+\eta)^{-1}\hat{H}(\eta-\xi)(\eta-\xi)^2\hat{q}(\eta-\xi)\,d\eta d\xi \\ &\quad = -i\int_{\xi\geq 0}\int_{\eta\geq 0} \frac{\operatorname{sgn}(\xi-\eta)(\xi-\eta)^2}{(k+\xi)(k+\eta)}|\hat{q}(\xi-\eta)|^2\,d\eta d\xi. \end{aligned}$$

This integral converges absolutely when $q \in H^2$. The integrand is odd with respect to $\xi = \eta$, so the integral evaluates to 0. The computation on the circle is similar.

To reduce the number of derivatives on q in the right hand side of (2.10), we require a Leibniz rule for the derivative operator $L_0 = C_+(\kappa - i\nabla)C_+ = R_0^{-1}$. If $f \in H^2$, we write

$$C_+ f' C_+ = iC_+[C_+(\kappa - i\nabla)C_+, f]C_+ = iC_+[R_0^{-1}, f]C_+$$

and so, commuting C_+ and R_0 as needed,

$$\begin{aligned} A(\kappa;q)A(\kappa;f')A(\kappa;q) &= i\sqrt{R_0}C_+qC_+R_0C_+[R_0^{-1}, f]C_+R_0C_+q\sqrt{R_0} \\ &= i\sqrt{R_0}C_+qC_+\big(fR_0 - R_0f\big)C_+qC_+\sqrt{R_0} \\ &= i\sqrt{R_0}C_+qC_+fC_+\sqrt{R_0}A(\kappa;q) \\ &\qquad - iA(\kappa;q)\sqrt{R_0}C_+fC_+qC_+\sqrt{R_0}. \end{aligned}$$

Because L_0 is an unbounded operator, the first equality above holds only on the domain of L_0, which is a dense subset of H^+. However, $A(\kappa;q) \in \mathfrak{I}_2$ and

$$\sqrt{R_0}C_+fC_+g\sqrt{R_0} \in \mathfrak{I}_2$$

when $f, g \in H^2$. This suffices to conclude

$$\begin{aligned} A(\kappa;q)A(\kappa;f')A(\kappa;q) = i\sqrt{R_0}C_+qC_+fC_+\sqrt{R_0}A(\kappa;q) \\ - iA(\kappa;q)\sqrt{R_0}C_+fC_+qC_+\sqrt{R_0} \end{aligned} \tag{2.11}$$

with equality as operators on H^+.

Now we show (2.10). We write

$$Hq'' = \tfrac{1}{i}q_+'' - \tfrac{1}{i}q_-'' = (\tfrac{1}{i}q_+' - \tfrac{1}{i}q_-')',$$

where $\varphi_\pm$ denotes the projection of φ onto $H^\pm$. Letting $f = \frac{1}{i}q'_+ - \frac{1}{i}q'_-$ in (2.11) , we find

$$\begin{aligned}
&\operatorname{tr}\{A(\kappa;q)^{\ell-1}A(\kappa;Hq'')A(\kappa;q)\} \\
&\qquad = \operatorname{tr}\left\{A(\kappa;q)^{\ell-2}\sqrt{R_0}C_+qC_+q'_+C_+\sqrt{R_0}A(\kappa;q)\right\} \\
&\qquad\quad - \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+q'_+C_+qC_+\sqrt{R_0}\right\} \\
&\qquad\quad - \operatorname{tr}\left\{A(\kappa;q)^{\ell-2}\sqrt{R_0}C_+qC_+q'_-C_+\sqrt{R_0}A(\kappa;q)\right\} \\
&\qquad\quad + \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+q'_-C_+qC_+\sqrt{R_0}\right\} \\
&\qquad = \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+qC_+q'_+C_+\sqrt{R_0}\right\} \\
&\qquad\quad - \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+q'_+C_+qC_+\sqrt{R_0}\right\} \\
&\qquad\quad - \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+qC_+q'_-C_+\sqrt{R_0}\right\} \\
&\qquad\quad + \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+q'_-C_+qC_+\sqrt{R_0}\right\} \\
&\qquad =: A - B - C + D.
\end{aligned}$$

We pass to the penultimate line above by cycling a copy of $A(\kappa;q)$ in two of the trace terms. Adding and subtracting $A + D$ yields

$$\operatorname{tr}\{A(\kappa;q)^{\ell-1}A(\kappa;Hq'')A(\kappa;q)\} = 2(A + D) - A - B - C - D.$$

We exploit some identities of the Cauchy projections in order to simplify the above expressions. If $f \in L^2(\mathbb{R})$ or $f \in L^2_0(\mathbb{R}/\mathbb{Z})$, then $C_+f_+C_+ = f_+C_+$ and $C_+f_-C_+ = C_+f_-$. Thus

$$A = \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+qq'_+C_+\sqrt{R_0}\right\},\ D = \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}\sqrt{R_0}C_+q'_-qC_+\sqrt{R_0}\right\}.$$

Applying the identity $f_+ + f_- = f$, we find

$$A + D = \operatorname{tr}\left\{A(\kappa;q)^{\ell-1}A(\kappa;qq')\right\}.$$

Thus to show (2.10) and complete the proof of the theorem, it suffices to show $A + B + C + D = 0$. By the same identity, we may simplify

$$A + C = \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}\sqrt{R_0}C_+qC_+q'C_+\sqrt{R_0}\right\}$$

and

$$B + D = \operatorname{tr}\left\{A(\kappa; q)^{\ell-1}\sqrt{R_0}C_+q'C_+qC_+\sqrt{R_0}\right\}.$$

When $\ell \geq 3$, we apply the Leibniz identity

$$iC_+[R_0^{-1}, qC_+q]C_+ = C_+q'C_+qC_+ + C_+qC_+q'C_+$$

and cycle a copy of $A(\kappa; q)$ in the trace to find $A + B + C + D = \operatorname{tr}\{X\}$, where

$$\begin{aligned} X &= iA(\kappa; q)^{\ell-2}\sqrt{R_0}C_+[R_0^{-1}, qC_+q]C_+\sqrt{R_0}A(\kappa; q) \\ &= iA(\kappa; q)^{\ell-3}\sqrt{R_0}C_+qC_+qC_+qC_+\sqrt{R_0}A(\kappa; q) \\ &\quad - iA(\kappa; q)^{\ell-2}\sqrt{R_0}C_+qC_+qC_+qC_+\sqrt{R_0}. \end{aligned}$$

Because $\sqrt{R_0}C_+qC_+qC_+qC_+\sqrt{R_0} \in \mathfrak{I}_2$, we may substitute this into the trace and cycle a copy of $A(\kappa; q)$ to obtain

$$A + B + C + D = \operatorname{tr}\{X\} = 0.$$

In the case $\ell = 2$, we do not have two copies of $A(\kappa; q)$ to place around the commutator, so we cannot apply the Leibniz rule as an operator identity. Instead we apply the same idea at the level of the integrals:

$$
\begin{aligned}
&(A+C)+(B+D) \\
&\qquad = \int_{\xi\geq 0}\int_{\eta\geq 0}\int_{\nu\geq 0} \frac{i(\nu-\xi)}{(\kappa+\xi)(\kappa+\eta)}\hat{q}(\xi-\eta)\hat{q}(\eta-\nu)\hat{q}(\nu-\xi)d\nu d\eta d\xi \\
&\qquad\quad + \int_{\xi\geq 0}\int_{\eta\geq 0}\int_{\nu\geq 0} \frac{i(\eta-\nu)}{(\kappa+\xi)(\kappa+\eta)}\hat{q}(\xi-\eta)\hat{q}(\eta-\nu)\hat{q}(\nu-\xi)d\nu d\eta d\xi \\
&\qquad = \int_{\xi\geq 0}\int_{\eta\geq 0}\int_{\nu\geq 0} \frac{i(\eta-\xi)}{(\kappa+\xi)(\kappa+\eta)}\hat{q}(\xi-\eta)\hat{q}(\eta-\nu)\hat{q}(\nu-\xi)d\nu d\eta d\xi \\
&\qquad = i\int_{\xi\geq 0}\int_{\eta\geq 0}\int_{\nu\geq 0} \frac{1}{(\kappa+\xi)}\hat{q}(\xi-\eta)\hat{q}(\eta-\nu)\hat{q}(\nu-\xi)d\nu d\eta d\xi \\
&\qquad\quad - i\int_{\xi\geq 0}\int_{\eta\geq 0}\int_{\nu\geq 0} \frac{1}{(\kappa+\eta)}\hat{q}(\xi-\eta)\hat{q}(\eta-\nu)\hat{q}(\nu-\xi)d\nu d\eta d\xi.
\end{aligned}
$$

The above integrals converge by Cauchy-Schwarz. Cycling the variables $\xi \mapsto \nu \mapsto \eta \mapsto \xi$ in the second integral, we see that the two integrals in the last identity are equal. This completes the proof. □

Because α is comparable to its first term, as a corollary to this result we obtain uniform in time control of $\|A(\kappa; q(t))\|_{\mathfrak{I}_2}$.

Corollary 2.4. *Let $s > -\frac{1}{2}$ and let q be a $C_t^0 H_x^3 \cap C_t^1 H_x^1$ solution to* (BO) *on the line or the circle, having mean 0 if on the circle. Then there exists a constant $C = C(s)$ such that for all $\kappa \geq 1 + C\|q(0)\|_{H^s}^{\frac{2}{1+2s}}$,*

$$\sup_{t\in\mathbb{R}} \|A(\kappa; q(t))\|_{\mathfrak{I}_2}^2 \leq 2\|A(\kappa; q(0))\|_{\mathfrak{I}_2}^2 < \frac{1}{9}$$

and therefore, by Theorem 2.2,

$$\langle q(t), T_\kappa q(t)\rangle \lesssim \langle q(0), T_\kappa q(0)\rangle.$$

Proof. We may choose C sufficiently large that $\|A(\kappa; q(0))\|_{\mathfrak{I}_2}^2 < \frac{1}{18}$. By Lemma 2.1 and Theorem 2.3, there exists a neighborhood I of 0 on which

$$\|A(\kappa; q(t))\|_{\mathfrak{I}_2}^2 \leq 3\alpha(\kappa; q(t)) = 3\alpha(\kappa; q(0)) \leq 2\|A(\kappa; q(0))\|_{\mathfrak{I}_2}^2 < \frac{1}{9}. \tag{2.12}$$

Since $\|A(\kappa;q(t))\|_{\mathfrak{I}_2} < \frac{1}{3}$, Lemma 1.1 implies that (2.12) is an open condition, and the corollary follows by a continuity argument. $\square$

2.3 Conservation of Norms

Because of the logarithmic factor, $\langle q, T_\kappa q\rangle$ is not commensurate with any H^s norm of q; it behaves like $\|q\|^2_{H^{-1/2}}$ at frequencies $\lesssim \kappa$ and like $\|\log(|\nabla|)\langle\nabla\rangle^{-1/2} q\|^2_{L^2}$ at frequencies $\gg \kappa$. This difficulty is avoided if we "build" $\|q\|_{H^s}$ for $-\frac{1}{2} < s < 0$ one frequency scale at a time, using the contribution of $\langle q, T_\kappa q\rangle$ at the frequency scale κ where it behaves like a pure Sobolev norm.

This is naturally expressed in terms of the Besov norms

$$\|f\|_{B^{s,2}_r} = \left(\|\hat{f}(\xi)|\|^r_{L^2(|\xi|\leq 1)} + \sum_{N\geq 1} N^{rs} \|\hat{f}(\xi)\|^r_{L^2(N\leq|\xi|<2N)} \right)^{1/r}$$

where the sum is taken over dyadic $N = 1, 2, 4, \dots$ and with the usual interpretation in the case $r = \infty$. The following lemma (the analogue of Lemma 3.2 in [13]) relates this norm to (the leading term of) $\alpha(k;q)$.

Lemma 2.5. *Fix $-\frac{1}{2} < s < 0$, $1 \leq r \leq \infty$, $\kappa_0 \geq 1$. For any H^2 function f,*

$$\|f\|^r_{B^{s,2}_r} \lesssim \sum_{N\in 2^{\mathbb{N}}} N^{rs} \big(\kappa_0 N \langle f, T_{\kappa_0 N} f\rangle\big)^{r/2} \tag{2.13}$$

and

$$\sum_{N\in 2^{\mathbb{N}}} N^{rs} \big(\kappa_0 N \langle f, T_{\kappa_0 N} f\rangle\big)^{r/2} \lesssim_s \kappa_0^{-rs} \|f\|^r_{B^{s,2}_r}. \tag{2.14}$$

Proof. The inequality (2.13) follows easily from the estimate

$$\|\hat{f}(\xi)\|^2_{L^2(|\xi|\leq N)} \leq \frac{2}{\log 2} \int \frac{\kappa_0 N \log(2 + \frac{|\xi|}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2 + \xi^2}} |\hat{f}(\xi)|^2 d\xi.$$

To control the other direction, we decompose

$$
\begin{aligned}
&\int \frac{\kappa_0 N \log(2+\frac{|\xi|}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2+\xi^2}} |\hat{f}(\xi)|^2 d\xi \\
&\leq \log(3)\|\hat{f}(\xi)\|^2_{L^2(|\xi|\leq 1)} + \sum_{M\in 2^{\mathbb{N}}} \frac{\kappa_0 N \log(2+\frac{2M}{\kappa_0 N})}{\sqrt{\kappa_0 N^2+M^2}} \|\hat{f}(\xi)\|^2_{L^2(M<|\xi|\leq 2M)} \\
&\leq \Bigg(\sqrt{\log(3)}\|\hat{f}(\xi)\|_{L^2(|\xi|\leq 1)} \\
&\qquad + \sum_{M\in 2^{\mathbb{N}}} \left(\frac{\kappa_0 N \log(2+\frac{2M}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2+M^2}} \right)^{1/2} \|\hat{f}(\xi)\|_{L^2(M<|\xi|\leq 2M)} \Bigg)^2 .
\end{aligned}
$$

This shows that the left-hand side of (2.14) is bounded by

$$
\begin{aligned}
&\Bigg\| \sqrt{\log(3)} N^s \|\hat{f}(\xi)\|_{L^2(|\xi|\leq 1)} \\
&\qquad + \sum_{M\in 2^{\mathbb{N}}} \left(\frac{\kappa_0 N^{1+2s} M^{-2s} \log(2+\frac{2M}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2+M^2}} \right)^{1/2} M^s \|\hat{f}(\xi)\|_{L^2(M<|\xi|\leq 2M)} \Bigg\|^r_{\ell^r(N\in 2^{\mathbb{N}})}
\end{aligned}
$$

which reduces our task to estimating the operator norm of a certain $\ell^r \to \ell^r$ matrix. To do this, we apply Schur's test. The row sums of this operator are bounded by

$$
\sqrt{\log(3)} N^s + \sum_{M\in 2^{\mathbb{N}}} \left(\frac{\kappa_0 N^{1+2s} M^{-2s} \log(2+\frac{2M}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2+M^2}} \right)^{1/2} \lesssim_s 1 + \kappa_0^{-s}
$$

uniformly in N, while the column sums are bounded by

$$
\sum_{N\in 2^{\mathbb{N}}} \sqrt{\log(3)} N^s \lesssim_s 1, \qquad \sum_{N\in 2^{\mathbb{N}}} \left(\frac{\kappa_0 N^{1+2s} M^{-2s} \log(2+\frac{2M}{\kappa_0 N})}{\sqrt{\kappa_0^2 N^2+M^2}} \right)^{1/2} \lesssim_s \kappa_0^{-s}
$$

uniformly in M. Note that to make these estimates we require the condition $-\frac{1}{2} < s < 0$. This proves (2.14). $\square$

Our main result now follows easily from the foregoing lemma and Corollary 2.4.

Theorem 2.6. *Let q be a $C_t^0 H_x^3 \cap C_t^1 H_x^1$ solution to* (BO) *on the line or the circle and let $-\frac{1}{2} < s < 0$, $1 \leq r \leq \infty$. Then*

$$(1 + \|q(0)\|_{B_r^{s,2}}^{\frac{2}{1+2s}})^s \sup_{t\in\mathbb{R}} \|q(t)\|_{B_r^{s,2}} \lesssim_{s,r} \|q(0)\|_{B_r^{s,2}}$$

and

$$\|q(0)\|_{B_r^{s,2}} \lesssim_{s,r} (1 + \|q(0)\|_{B_r^{s,2}}^{\frac{2}{(1+2s)^2}})^{-s} \inf_{t\in\mathbb{R}} \|q(t)\|_{B_r^{s,2}}.$$

The particular case of $r = 2$ is equivalent to the conservation of the Sobolev norm:

$$(1 + \|q(0)\|_{H^s}^{\frac{2}{1+2s}})^s \sup_{t\in\mathbb{R}} \|q(t)\|_{H^s} \lesssim_s \|q(0)\|_{H^s} \lesssim_s (1 + \|q(0)\|_{H^s}^{\frac{2}{(1+2s)^2}})^{-s} \inf_{t\in\mathbb{R}} \|q(t)\|_{H^s}.$$

Proof. On the circle, we first assume that q has mean 0. By Hölder's inequality, we have an embedding $B_r^{s_1,2} \hookrightarrow B_2^{s_2,2} = H^{s_2}$ for any $s_2 < s_1$. Let

$$\kappa_0 = 1 + C\|q(0)\|_{B_r^{s,2}}^{\frac{2}{1+2s}} \gtrsim_s 1 + C\|q(0)\|_{H^{s-}}^{\frac{2}{1+2s}}$$

for a sufficiently large constant C, so that we may apply Corollary 2.4. Then, for any time t, Lemma 2.5 implies

$$\begin{aligned}
\|q(t)\|_{B_r^{s,2}}^r &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^{rs}\big(\kappa_0 N\langle q(t), T_{\kappa_0 N} q(t)\rangle\big)^{r/2} \\
&\lesssim \sum_{N\in 2^{\mathbb{N}}} N^{rs}\big(\kappa_0 N\langle q(0), T_{\kappa_0 N} q(0)\rangle\big)^{r/2} \\
&\lesssim_s \kappa_0^{-rs}\|q(0)\|_{B_r^{s,2}}^r \\
&\lesssim_s (1 + \|q(0)\|_{B_r^{s,2}}^{\frac{2}{1+2s}})^{-rs}\|q(0)\|_{B_r^{s,2}}^r.
\end{aligned}$$

This proves the first inequality. By time translation symmetry, we then also obtain

$$\|q(0)\|_{B_r^{s,2}} \lesssim_s (1 + \|q(t)\|_{B_r^{s,2}}^{\frac{2}{1+2s}})^{-s}\|q(t)\|_{B_r^{s,2}}$$

and applying the first inequality to the quantity in parentheses produces the second inequality.

To remove the mean zero assumption on the circle, we employ Galilean invariance: if q solves (BO), then so does

$$\tilde{q}(t,x) = q(t, x + 2\mu t) + \mu. \tag{2.15}$$

The estimate

$$\|\tilde{q}(t)\|^2_{B^{s,2}_r(\mathbb{R}/\mathbb{Z})} \sim \|q(t)\|^2_{B^{s,2}_r(\mathbb{R}/\mathbb{Z})} + \mu^2$$

then implies the general theorem. □

3 The Scattering Data

In section 1.1.1, we discussed the scattering data of the Fokas–Ablowitz IST for Benjamin–Ono in a purely formal way. In this section we discuss those elements of the theory that can withstand a rigorous treatment when the regularity of the potential is low, and we use these to construct a well-behaved approximation H_κ to the Benjamin–Ono Hamiltonian.

3.1 The Jostish functions

For $\kappa > 0$, let $R_0(\kappa)$ be the free resolvent as defined in section 2:

$$(R_0(\kappa)f)^\wedge(\xi) = \frac{1}{\kappa + |\xi|}\hat{f}(\xi).$$

We introduce the Jost-ish function $m_\pm = m_\pm(x, \kappa; q)$ which solves for a given potential function q

$$m_\pm = 1 + R_0(\kappa)(qm_\pm)_\pm,$$

which implies that $m_\pm$ solves a minor modification of (2-a):

$$\mp i m'_\pm + \kappa(m-1) = (qm_\pm)_\pm. \tag{3.16}$$

Formally, $m_\pm - 1$ is given by the series

$$m_\pm - 1 = R_0 q_\pm + R_0(qR_0q_\pm)_\pm + \cdots \tag{3.17}$$

For $q \in H^s$, $s > -\frac{1}{2}$, and for κ sufficiently large, this series converges in H^{1+s}. The proof relies on the easy fact that $\|R_0 f\|_{H^r} \lesssim_\kappa \|f\|_{H^{r-1}}$ and the bilinear estimate

Lemma 3.1 (fR$_0$g Lemma)**.** *Let $\sigma, r, s \geq 0$ such that $r + s - \sigma < \frac{1}{2}$, $\sigma \geq r$, and $r + s \leq 1$. Then*

$$\|fR_0g\|_{H^{-\sigma}} \lesssim \|f\|_{H^{-r}} \|R_0^s g\|_{L^2}$$

Proof. We write

$$\|fR_0g\|^2_{H^{-\sigma}} = \int_{-\infty}^{\infty} \langle\xi\rangle^{-2\sigma} \left| \int_{-\infty}^{\infty} \frac{1}{|\eta| + \kappa} \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta \right|^2 d\xi$$

and split the inner integral into two regimes. On one hand,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \langle\xi\rangle^{-2\sigma} \left| \int_{|\eta|>2|\xi|} \frac{1}{|\eta| + \kappa} \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta \right|^2 d\xi \\
&\qquad \lesssim \int_{-\infty}^{\infty} \langle\xi\rangle^{-2\sigma} (\kappa + |\xi|)^{-2+2r+2s} \cdot \\
&\qquad\qquad \cdot \left| \int_{-\infty}^{\infty} (|\xi - \eta| + \kappa)^{-r} \hat{f}(\xi - \eta)(|\eta| + \kappa)^{-s} \hat{g}(\eta) d\eta \right|^2 d\xi \\
&\qquad \lesssim \int_{-\infty}^{\infty} \langle\xi\rangle^{-2\sigma} (\kappa + |\xi|)^{-2+2r+2s} d\xi \|R_0^{|s|} f\|^2_{L^2} \|R_0^{|s|} g\|^2_{L^2} \\
&\qquad \lesssim \|\langle\xi\rangle^{-1-\sigma+r+s}\|_2^2 \|R_0^{|r|} f\|^2_{L^2} \|R_0^{|s|} g\|^2_{L^2}
\end{aligned}$$

which is acceptable if $r + s - \sigma < \frac{1}{2}$. We use $r + s \leq 1$ to pass to the last line.

On the other hand,

$$
\begin{aligned}
&\left(\int_{-\infty}^{\infty}\langle\xi\rangle^{-2\sigma}\left|\int_{|\eta|<2|\xi|}\frac{1}{|\eta|+\kappa}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right|^2 d\xi\right)^{1/2}\\
&\qquad\lesssim\int_{-\infty}^{\infty}\frac{1}{\kappa+|\eta|}|\hat{g}(\eta)|\left(\int_{2|\xi|>\eta}\langle\xi\rangle^{-2\sigma}|\hat{f}(\xi-\eta)|^2 d\xi\right)^{1/2}d\eta\\
&\qquad\lesssim\int_{-\infty}^{\infty}\frac{1}{\kappa+|\eta|}\langle\eta\rangle^{-\sigma+r}|\hat{g}(\eta)|\left(\int_{2|\xi|>\eta}\langle\xi-\eta\rangle^{-2r}|\hat{f}(\xi-\eta)|^2 d\xi\right)^{1/2}d\eta\\
&\qquad\lesssim\|f\|_{H^{-r}}\|R_0^s g\|_{L^2}\|\langle\eta\rangle^{-1-\sigma+r+s}\|_2
\end{aligned}
$$

which is acceptable if $r+s-\sigma<\frac{1}{2}$. We use $\sigma\geq r$ to pass to the penultimate line and $s\leq 1$ to pass to the last line. $\square$

Let $0<\varepsilon<\frac{1}{2}$, $s=-\frac{1}{2}+\varepsilon$. Then Lemma 3.1 implies

$$
\|fR_0 g\|_{H^s}\lesssim\kappa^{-\varepsilon}\|f\|_{H^s}\|g\|_{H^s}. \tag{3.18}
$$

Using (3.18), we see that

$$
\|R_0(qR_0q_+)_+\|_{H^{1+s}}\lesssim\|qR_0q_+\|_{H^s}\lesssim\kappa^{-\varepsilon}\|q\|_{H^s}^2,
$$

$$
\|R_0(qR_0(qR_0q_+)_+)_+\|_{H^{1+s}}\lesssim\|qR_0(qR_0q_+)_+\|_{H^s}\lesssim\kappa^{-2\varepsilon}\|q\|_{H^s}^3.
$$

and so on. Therefore the right hand side of (3.17) is bounded in H^{1+s} by

$$
\|R_0q_+\|_{H^{1+s}}+\sum_{n=2}^{\infty}(\kappa^{-\varepsilon}\|q\|_{H^s})^n\leq 2\|q\|_{H^s}\lesssim\|q\|_{H^s}
$$

if we choose $\kappa\gg 1$ such that $\kappa^{-\varepsilon}\|q\|_{H^s}<\frac{1}{2}$.

3.2 The Approximate Hamiltonian Flow

For the sake of a cleaner notation, we will write $m=m_+$ for the rest of this paper. It was proved by Kaup and Matsuno [12] that $\int q(m-1)dx$ generates the conserved quantities of the Benjamin–Ono equation. Indeed, we may expand m in a series in powers of κ:

$$m(x;\kappa) = \sum_{n=0}^{\infty} \kappa^{-n} m_n(x)$$

and use the fact that m solves (3.16) to infer that

$$m_0 = 1, \qquad m_{n+1} = im_n' + (qm_n)_+.$$

So $m_1 = q_+, m_2 = iq_+' + (qq_+)_+$, etc. Thus

$$\begin{aligned}\int q(m-1)dx &= \frac{1}{\kappa}\int qq_+dx + \frac{1}{\kappa^2}\int iqq_+' + q(qq_+)_+dx + O(\kappa^{-3}) \\ &= \frac{1}{2\kappa}\int q^2dx + \frac{1}{\kappa^2}\int -\frac{1}{2}qHq' + \frac{1}{3}q^3dx + O(\kappa^{-3}).\end{aligned}$$

From this we see that

$$H_\kappa := \kappa^2\int q(m-1)dx - \frac{\kappa}{2}\int q^2dx = H_{\mathrm{BO}} + O(\kappa^{-1}).$$

We use the series expansion (3.17) of $m_\pm$ to compute

$$\begin{aligned}&\frac{\delta}{\delta q}\int q(m-1)dx \\ &= \frac{\delta}{\delta q}\int qR_0q_+ + qR_0(qR_0q_+)_+ + qR_0(qR_0(qR_0q_+)_+)_+ + \cdots dx \\ &= R_0q_+ + R_0q_- + R_0(qR_+q_+)_+ + R_0q_+R_0q_- + R_0(qR_0q_-)_- + \\ &\qquad + R_0(qR_0(qR_0q_+)_+)_+ + R_0q_-R_0(qR_0q_+)_+ + R_0(qR_0q_-)_-R_0q_+ + \\ &\qquad + R_0(qR_0(qR_0q_-)_-)_- + \cdots \\ &= m_+m_- - 1.\end{aligned}$$

The flow of q under the approximate Hamiltonian H_κ is

$$q_s = \nabla\frac{\delta}{\delta q}H_\kappa = \kappa^2(m_+m_-)' - \kappa q'. \tag{3.19}$$

We expect $q_t - q_s = o_{\kappa\to\infty}(1)$.

The resulting Hamiltonian flow is well-posed:

Theorem 3.2. *Let $\delta > 0, \kappa, \varkappa \geq 1$ and $-\frac{1}{2} < s \leq 0$. Let B_δ denote the δ-ball in H^s. The H_κ flow is globally well-posed on B_δ and preserves $\alpha(\varkappa)$. (It is also well-posed when $s > 0$, but this requires a different proof and is not relevant to our purposes.)*

Proof. Global well-posedness follows from local well-posedness and conservation of $\alpha(1)$. Local well-posedness is proved using Picard iteration applied to the integral equation

$$q(t,x) = q(0, x-\kappa t) + \int_0^t \kappa^2 m(x+\kappa^2(t-s);\kappa, q(s))n(x+\kappa^2(t-s);\kappa,q(s))ds$$

together with the estimate (shortly to be proved)

$$\|m(q)n(q) - m(\tilde{q})n(\tilde{q})\|_{H^s} \lesssim \|q-\tilde{q}\|_{H^s}.$$

We will prove this for the quadratic terms of the series expansion of m, n, and it will be evident that all the terms in the series can be handled by this method. By applying Lemma 3.1, we have

$$\begin{aligned}
&\kappa^2\|R_0q_+R_0q_- - R_0\tilde{q}_+R_0\tilde{q}_-\|_{H^s} \\
&\qquad\lesssim \kappa^2\|R_0q_+R_0(q-\tilde{q})_-\|_{H^s} + \kappa^2\|R_0(q-\tilde{q})_+R_0\tilde{q}_-\|_{H^s} \\
&\qquad\lesssim \kappa^2\|R_0q\|_{H^s}\|\|q-\tilde{q}\|_{H^s} + \kappa^2\|R_0\tilde{q}\|_{H^s}\|q-\tilde{q}\|_{H^s} \\
&\qquad\lesssim \delta\kappa\|q-\tilde{q}\|_{H^s}.
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\kappa^2\|R_0(qR_0q_+)_+ - R_0(\tilde{q}R_0\tilde{q}_+)_+\|_{H^s} &\lesssim \kappa\|qR_0(q-\tilde{q})_+\|_{H^s} + \kappa\|(q-\tilde{q})R_0\tilde{q}_+\|_{H^s} \\
&\lesssim \kappa\delta\|q-\tilde{q}\|_{H^s}.
\end{aligned}$$

We now turn our attention to the conservation of α. We see that α is conserved from the identity (3.20), from which it follows

$$\begin{aligned}\frac{d}{ds}\frac{d}{d\kappa}\alpha &= -\frac{1}{\kappa}\frac{d}{ds}H_\kappa \\ &= -\frac{1}{\kappa}\{H_\kappa, H_\kappa\} = 0\end{aligned}$$

where here the Poisson bracket is

$$\{F, G\} = \int \frac{\delta}{\delta q}F\left(\frac{\delta}{\delta q}G\right)' dx.$$

Let q be any sufficiently regular data; say $q \in H^3$. Then α is continuously differentiable in s, κ by Lemma 2.1, so we may freely interchange the derivatives to find that $\frac{d}{ds}\alpha(q(x,s))$ is constant in κ. As $\kappa \to \infty$, $\alpha \to 0$, so we conclude that $\frac{d}{ds}\alpha = 0$. Since we already have local well-posedness, global well-posedness extends to general data, and the proof is complete.

□

3.3 Relation to α

We proved in Theorem 2.3 that

$$\alpha(q;\kappa) := \sum_{\ell=2}^{\infty} \frac{1}{\ell}\operatorname{tr}\{(\sqrt{R_0}C_+qC_+\sqrt{R_0})^\ell\}$$

is conserved. We compute for $n \geq 2$

$$
\begin{aligned}
&\frac{1}{n}\operatorname{tr}\{(\sqrt{R_0}C_+ qC_+\sqrt{R_0})^n\} \\
&\qquad = \frac{1}{n}\int_{[0,\infty)^n} \frac{1}{\kappa+\xi_1}\hat{q}(\xi_1-\xi_2)\cdots\frac{1}{\kappa+\xi_n}\hat{q}(\xi_n-\xi_1)d\xi_1\cdots d\xi_n \\
&\qquad = \int_{0\leq\xi_1\leq\min(\xi_2,\ldots,\xi_n)} \frac{1}{\kappa+\xi_1}\hat{q}(\xi_1-\xi_2)\cdots\frac{1}{\kappa+\xi_n}\hat{q}(\xi_n-\xi_1)d\xi_1\cdots d\xi_n \\
&\qquad = \int_0^\infty \frac{1}{\kappa+\xi}\int_{\mathbb{R}^{n-1}} \hat{q}(-(\zeta_1+\cdots+\zeta_n))\cdot \\
&\qquad\qquad \cdot\prod_{i=1}^n \hat{R}_+(\kappa+\xi)(\zeta_i+\cdots+\zeta_n)\hat{q}(\zeta_i)d\zeta_1\cdots d\zeta_n d\xi \\
&\qquad = \int_\kappa^\infty \frac{1}{\lambda}\int qR_0(\lambda)(\cdots qR_0(\lambda)q_+)_+ dx d\lambda.
\end{aligned}
$$

Summing in n and recalling (3.17), we find

$$\alpha(q;\kappa) = \int_\kappa^\infty \frac{1}{\lambda}\int q(x)(m-1)(x;\lambda)dxd\lambda$$

or equivalently,

$$-\kappa\frac{d}{d\kappa}\alpha(q;\kappa) = \int q(m-1)dx. \tag{3.20}$$

Since α is conserved by (BO), it follows from the above and from Theorem 3.2 that

Proposition 3.3. *The quantity*

$$\int q(x,t)(m(x;\kappa,q)-1)dx$$

is conserved in t by the Hamiltonian flows of H_{BO} and H_κ.

4 A Short Proof that Benjamin–Ono is Well-Posed in $L^2(\mathbb{R})$

The basic idea behind our proof is to prove that the H_κ flow converges in L^2 to the H_{BO} flow; since the former is well-posed in L^2, so must the latter be. Rather than prove L^2 convergence directly, we will first show that both flows preserve equicontinuity in L^2, whereafter it suffices to prove that the flows converge in some H^s space for $s < 0$ as small as is convenient. In this strategy we follow [14].

4.1 The higher-order terms

Recall that, when κ is sufficiently large, the scattering equation (1.2) yields a convergent series expansion for its solution:

$$m(\kappa; q) = \sum_{\ell=0}^{\infty} (R_0(\kappa) C_+ q)^\ell \cdot 1$$

where q is to be interpreted as the multiplication operator and the power indicates operator composition. Because the resolvent operator R_0 decays like κ^{-1}, the terms in this series are increasingly meager for large κ. In this section, we will repeatedly discard those higher-order terms in this series which contribute negligibly to some limit as $\kappa \to \infty$. We are justified in doing so by the following lemma.

We write

$$E(x; \kappa, n) = (R_0 C_+ q)^n \cdot 1 \in L^2$$

$$F(x; \kappa, n) = (R_0 C_- q)^n \cdot 1 \in L^2$$

where q is to be interpreted as a multiplier operator and the exponent indicates operator composition.

Lemma 4.1. *The following decay estimates hold:*

$$\|qE(\kappa, n)\|_2 + \|qF(\kappa, n)\|_2 \lesssim \kappa^{-n/2} \|q\|_2^{n+1} \tag{4.21}$$

and

$$\|E(\kappa, n)\|_2 + \|F(\kappa, n)\|_2 \lesssim \kappa^{-(n+1)/2} \|q\|_2^{n} \tag{4.22}$$

Proof. We treat $E(\kappa, n)$ with no loss of generality. We have

$$\begin{aligned}
\|qE(\kappa,n)\|_2 &= \|\hat{q} * E(\kappa,n)^\wedge\|_2 \\
&\lesssim \|q\|_2 \|E(\kappa,n)^\wedge\|_1 \\
&\lesssim \|q\|_2 \|\frac{1}{\kappa+|\xi|}\|_2 \|qE(\kappa,n-1)\|_2 \\
&= \kappa^{-1/2}\|q\|_2\|qE(\kappa,n-1)\|_2,
\end{aligned}$$

from which (4.21) follows inductively.

(4.22) follows directly from (4.21):

$$\|E(q,n)\|_2 \lesssim \kappa^{-1}\|qE(q,n-1)\|_2 \lesssim \kappa^{-(n+1)/2}\|q\|_2^n.$$

□

Since $\int q(m-1)dx$ is conserved by the $H_{\mathrm{BO}}, H_\kappa$ flows (Proposition 3.3) and since, as we have already noted, $\int q(m-1)dx$ is a generating function for a hierarchy of conserved quantities, it follows that the H_{BO} and H_κ flows both conserve this hierarchy. For example, the preceding estimate allows us to prove

Proposition 4.2. *Let q solve either* (BO) *or* (3.19). *Then $\|q\|_{L^s}$ is constant in time.*

Proof. Observe that

$$\begin{aligned}
\kappa \int q(m-1)dx &= \kappa \sum_{\ell=1}^{\infty} \int qE(\kappa,\ell)dx \\
&= \kappa \int qR_0q_+dx + \kappa \sum_{\ell=2}^{\infty}\int qE(\kappa,\ell)dx.
\end{aligned}$$

If $q \in L^2$, then as $\kappa \to \infty$.

$$\kappa \int qR_0q_+dx \to \int qq_+dx = \frac{1}{2}\int q^2dx.$$

As for the higher order terms, according to Cauchy-Schwarz and (4.22), we have

$$\kappa \sum_{\ell=2}^{\infty} \int qE(\kappa, \ell) dx \lesssim \sum_{\ell=2}^{\infty} \|q\|_2^{\ell+1} \kappa^{-(\ell-1)/2} \to 0$$

for $q \in L^2$ as $\kappa \to \infty$. In conclusion of the preceding,

$$\kappa \int q(m-1) dx \to \frac{1}{2} \int q^2 dx$$

as $\kappa \to \infty$. Since $\kappa \int q(m-1)dx$ is conserved for any κ sufficiently large, so too must its limit as $\kappa \to \infty$. $\square$

4.2 Equicontinuity

Definition 1. *A subset Q of a metric space X is equicontinuous if*

$$\lim_{h \to 0} \sup_{q \in Q} \|q - q(\cdot + h)\|_X = 0. \tag{4.23}$$

Proposition 4.3. *Fix $\varepsilon > 0$. If Q is a bounded subset of $L^2(\mathbb{R})$, then Q is equicontinuous if and only if*

$$\lim_{\kappa \to \infty} \sup_{q \in Q} \int_{|\xi| > \kappa} |\hat{q}(\xi)|^2 d\xi = 0 \tag{4.24}$$

which is equivalent to

$$\lim_{\kappa \to \infty} \sup_{q \in Q} \|(R_0|\nabla|)^\sigma q\|_{L^2} = \lim_{\kappa \to \infty} \sup_{q \in Q} \int \frac{|\xi|^\sigma}{(\kappa + |\xi|)^\sigma} |\hat{q}(\xi|)|^2 d\xi = 0 \tag{4.25}$$

for any $\sigma > 0$.

Proof. Suppose $Q \in \{\|f\|_{L^2} \leq R\}$. Fix $h \in \mathbb{R}$. Then

$$\begin{aligned}\sup_{f \in Q} \|f - f(\cdot + h)\|_{L^2} &= \sup_{f \in Q} \int |e^{i\xi h} - 1|^2 |\hat{f}(\xi)|^2 d\xi \\ &\lesssim \kappa^2 h^2 \sup_{f \in Q} \int |\hat{f}(\xi)|^2 d\xi + \sup_{f \in Q} \int_{|\xi| > \kappa} |\hat{f}(\xi)|^2 d\xi\end{aligned}$$

Taking $h \to 0$ and $\kappa \to \infty$ shows that (4.24) implies (4.23).

To prove the converse, we compute

$$\int \kappa e^{-2\kappa|h|} dh = 1,$$

$$\int |e^{i\xi h} - 1|^2 \kappa e^{-2\kappa|h|} dh = \frac{2\xi^2}{\xi^2 + 4\kappa^2} \gtrsim 1 - \mathbb{1}_{[-\kappa,\kappa]}(\xi).$$

From these it follows that

$$\begin{aligned}
\sup_{f\in Q} \int_{|\xi|>\kappa} |\hat{f}(\xi)|^2 d\xi &\lesssim \sup_{f\in Q} \int \frac{|\xi|^2}{\kappa^2 + |\xi|^2} |\hat{f}(\xi)|^2 d\xi \\
&\lesssim \sup_{f\in Q} \int \int |e^{i\xi h} - 1|^2 |\hat{f}(\xi)|^2 \kappa e^{-2\kappa|h|} dh d\xi \\
&= \sup_{f\in Q} \int \|f - f(\cdot + h)\|_{L^2} \kappa e^{-2\kappa h} dh \\
&\lesssim \sup_{f\in Q} \sup_{|h|<\delta} \|f - f(\cdot + h)\|_{L^2} + R \int_{|h|>\delta} \kappa e^{-2\kappa|h|} dh
\end{aligned}$$

for any $\delta > 0$. Letting $\delta \to 0$ and then $\kappa \to \infty$ proves that (4.23) implies (4.24) .

Finally, let us prove that (4.25) is equivalent to (4.24). We have

$$\int_{|\xi|>\kappa} |\hat{f}(\xi)|^2 \leq 2^\sigma \int_{|\xi|>\kappa} \frac{|\xi|^\sigma}{(\kappa + |\xi|)^\sigma} |\hat{f}(\xi)|^2 d\xi,$$

which proves the forward direction. Also,

$$\int \frac{|\xi|^\sigma}{(\kappa + |\xi|)^\sigma} |\hat{f}(\xi)|^2 d\xi \lesssim \kappa^{-\sigma/2} \int_{|\xi|<\sqrt{\kappa}} |\hat{f}(\xi)|^2 d\xi + \int_{|\xi|>\sqrt{\kappa}} |\hat{f}(\xi)|^2 d\xi.$$

This proves the reverse direction. □

Theorem 4.4. *Let Q be equicontinuous and bounded in H^s, $s \leq 0$. Then*

$$Q^* = \{e^{tJ\nabla(H_{\mathrm{BO}} - H_\kappa)} q : q \in Q, t \in \mathbb{R}, \kappa \geq 1\}$$

is also equicontinuous in H^s.

Proof. Let us begin with the case $s < 0$, which we will use in Section 5. We introduce a new notation: the operator $\langle \nabla \rangle$ defined on the Fourier side by

$$(\langle \nabla \rangle f)^\wedge(\xi) = \langle \xi \rangle \hat{f}(\xi).$$

From the definition (4.23) we see immediately that a set Q is equicontinuous in H^s, $s < 0$ if and only if the set $\{\langle \nabla \rangle^s q : q \in Q\}$ is equicontinuous in L^2; since Q^* is bounded in H^S according to Theorem 2.6, it then follows from Proposition 4.3 that it suffices to prove that

$$\|R_0^{|s|} q\|_{L^2} = \|(R_0 \langle \nabla \rangle)^{|s|} \langle \nabla \rangle^s q\|_{L^2} \to 0$$

uniformly over $q \in Q$ as $\kappa \to \infty$. This follows from the conservation of α and the estimate

$$-\int_\kappa^\infty \varkappa^{2(s+1)} \frac{\partial \alpha}{\partial \varkappa} \frac{d\varkappa}{\varkappa} \sim_s \|R_0^s q\|_{L^2}. \tag{4.26}$$

Let us prove (4.26) by employing the identity

$$-\kappa \frac{\partial \alpha}{\partial \kappa} = \int q(m-1) dx$$

and the series expansion of m to obtain:

$$\begin{aligned} -\int_\kappa^\infty & \varkappa^{2(s+1)} \frac{\partial \alpha}{\partial \varkappa} \frac{d\varkappa}{\varkappa} \\ &= \int_\kappa^\infty \int \varkappa^{2s+1} q(m-1)(\varkappa; q) dx \frac{d\varkappa}{\varkappa} \\ &= \int_\kappa^\infty \int \varkappa^{2s+1} q R_0(\varkappa) q_+ dx \frac{d\varkappa}{\varkappa} + \sum_{\ell=2}^\infty \int_\kappa^\infty \int \varkappa^{2s+1} q E(\varkappa; \ell) dx \frac{d\varkappa}{\varkappa}. \end{aligned}$$

The $\ell \geq 2$ terms are negligible according to Lemma 4.1. To handle the main term, we observe that

$$\int_\kappa^\infty \varkappa^{2s+1} \frac{1}{\varkappa + |\xi|} \frac{d\varkappa}{\varkappa} \sim_s (\kappa + |\xi|)^{2s}. \tag{4.27}$$

To see this, we compute

$$\int_\kappa^\infty \varkappa^{2s+1} \frac{1}{\varkappa + |\xi|} \frac{d\varkappa}{\varkappa} = \int_\kappa^\infty (\varkappa + |\xi|)^{2s-1} \left(\frac{\varkappa + |\xi|}{\varkappa} \right)^{2|s|} d\varkappa.$$

This shows that the $\gtrsim$ direction of (4.27) is trivial. In the case $\kappa \geq |\xi|$, the factor $(\varkappa + |\xi|)/\varkappa \leq 2$, which proves the $\lesssim$ direction of (4.27). It remains only to show, in the case $\kappa < |\xi|$,

$$\begin{aligned}
\int_{\kappa}^{|\xi|} \varkappa^{2s+1} \frac{1}{\varkappa + |\xi|} \frac{d\varkappa}{\varkappa} &\lesssim \frac{1}{\kappa + |\xi|} \int_0^{|\xi|} \varkappa^{2s} d\varkappa \\
&= \frac{|\xi|^{2s+1}}{\kappa + |\xi|} \\
&\lesssim (\kappa + |\xi|)^{2s}.
\end{aligned}$$

This completes the proof of (4.27) and, by extension, the theorem in the case $s < 0$.

(4.26) fails in the case $s = 0$ because it exhibits a logarithmic divergence. Instead I will use the scattering equation (1.2) satisfied by m to introduce the frequency multiplier that yields $\|(R_0|\nabla|)^{1/2} q\|_{L^2}$. Observe that

$$\begin{aligned}
\frac{1}{2}\|(R_0|\nabla|)^{1/2} q\|_{L^2}^2 &= \int q|\nabla_x| R_0 q_+ dx \\
&= \kappa \int q R_0 q_+ dx - \int q q_+ dx \\
&= \kappa \int q(m-1) dx - \kappa \sum_{\ell=2}^{\infty} \int q(R_0 C_+)^{\ell} \cdot q - \int q q_+ dx \\
&= -\kappa^2 \frac{\partial \alpha}{\partial \kappa} - \frac{1}{2} \int q^2 dx - \kappa \sum_{\ell=2}^{\infty} \int q E(\kappa, \ell)
\end{aligned}$$

The first two terms appearing at the end of this chain of equalities are conserved. The series vanishes uniformly as $\kappa \to \infty$. Indeed, by Cauchy-Schwarz and (4.21) we then find that

$$\begin{aligned}
\kappa \sum_{\ell=2}^{\infty} \int q E(\kappa, \ell) dx &\lesssim \|q\|_2 \sum_{\ell=2}^{\infty} \|q E(\kappa, \ell - 1)\|_2 \\
&\lesssim \|q\|_2 \sum_{\ell=2}^{\infty} \kappa^{-\ell/2} \|q\|_2^{\ell} \\
&= \kappa^{-1} \frac{\|q\|_2^3}{1 + \kappa^{-1/2}\|q\|_2}
\end{aligned}$$

which vanishes as $\kappa \to \infty$, as promised. We have therefore shown that

$$\frac{1}{2}\||\nabla|^{1/2}q\|^2_{H_\kappa^{-1/2}} = -\kappa^2\frac{\partial\alpha}{\partial\kappa} - \frac{1}{2}\int q^2 dx + o_{\kappa\to\infty}(1).$$

Since the explicit terms on the RHS are conserved by both the H_{BO} and the H_κ flows (Theorem 2.3, Proposition 4.2), it follows that for q evolving according to either flow,

$$\frac{1}{2}\|(R_0|\nabla|)^{1/2}q(t)\|^2_{L^2} = \frac{1}{2}\|(R_0|\nabla|)^{1/2}q(0)\|^2_{L^2} + o(1)$$

and the result follows. □

As we mentioned in the introduction to this section, equicontinuity is useful to us because it allows us to obtain convergence in L^2 by proving convergence in a weaker Sobolev norm. This is the content of the following lemma.

Lemma 4.5. *Let $s < 0$ and Q an equicontinuous, uniformly bounded subset of L^2. If a sequence of functions $f_n \in Q$ converges to a limit f in H^s, then $f \in L^2$ and $f_n \to f$ in L^2.*

Proof. We show that $\{f_n\}$ is a Cauchy sequence in L^2. Observe that for any $\kappa > 1$:

$$\|f_n - f_m\|^2_{H^s} \leq (1+\kappa^2)^{-s/2}\int_{|\xi|\leq\kappa}\langle\xi\rangle^{2s}|\hat{f}_n(\xi) - \hat{f}_m(\xi)|^2 d\xi + \int_{|\xi|>\kappa}|\hat{f}_n(\xi) - \hat{f}_m(\xi)|^2 d\xi.$$

If $\kappa \gg 1$ is chosen sufficiently large, then the second integral can be made negligible by equicontinuity and (4.24). Then the first integral vanishes for n, m large (relative to κ) by H^s-convergence of the f_n.

□

4.3 Well-Posedness

In this subsection we give a short proof of

Theorem 4.6. (BO) *is well-posed in the class of $L^2(\mathbb{R})$ initial data on the line.*

Proof. Let Q be equicontinuous and bounded by δ in L^2. By Theorem 4.4, $\{q^{\nabla(tH_{\mathrm{BO}}-sH_\kappa)} : q \in Q, t, s \geq 0\}$ is also equicontinuous and, by conservation of the L^2 norm, bounded by δ. Our first goal is to demonstrate that for $\sigma < 0$ sufficiently negative,

$$\|q_t - q_s\|_{H^\sigma} = o_{\kappa\to\infty}(1). \tag{4.28}$$

For $q \in C^3$, we have

$$\begin{aligned} q_t - q_s &= -Hq'' + 2qq' - \kappa^2(mn)' + kq' \\ &= -R_0 Hq''' + 2qq' - \kappa^2(mn - R_0 q)'. \end{aligned}$$

The lone linear term is bounded by $\delta\kappa^{-1}$ in H^σ for $\sigma < -3$. Let us next focus on the quadratic terms. These are

$$2qq' - \kappa^2 R_0(qR_0q_+)'_+ - \kappa^2(R_0q_+R_0q_-)' - \kappa^2 R_0(qR_0q_-)'_-$$

We can decompose $(q^2)' = (qq_+)'_+ + (q_+q_-)' + (qq_-)'_-$ and identify three terms to control:

$$(qq_+ - \kappa^2 R_0(qR_0q_+))'_+ + (q_+q_- - \kappa^2 R_0q_+R_0q_-)' + (qq_- - \kappa^2 R_0(qR_0q_-))'_-. \tag{4.29}$$

The first and third of these can be handled in the same manner, to wit:

$$\begin{aligned} (qq_+ - \kappa^2 R_0(qR_0q_+))'_+ &= \kappa R_0 H(qR_0q_+)''_+ + \kappa R_0(qR_0Hq'_+)'_+ - R_0(qR_0q'_+)''_+ \\ &= \kappa R_0 H(qR_0q_+)''_+ - H(qR_0q'_+)'_+. \end{aligned}$$

By the fR_0g lemma, Lemma 3.1, we have

$$\|\kappa R_0 H(qR_0q_+)''_+\|_{H^\sigma} \lesssim \|H(qR_0q_+)_+\|_{H^{\sigma+2}} \lesssim o_{\kappa\to\infty}(1)\|q\|^2_{H^s}$$

and

$$\|H(qR_0q_+')_+'\|_{H^\sigma} \lesssim \|qR_0q_+'\|_{H^{\sigma+1}} \lesssim o_{\kappa\to\infty}(1)\|q\|_{H^s}\|q'\|_{H^{s-1}} \lesssim o_{\kappa\to0}(1)\|q\|_{H^s}^2$$

as long as $\sigma < 2s - \frac{3}{2}$ (which is entailed by our earlier condition $\sigma < -3$).

The middle term of (4.29) is also controlled by Lemma 3.1:

$$(q + q_- - \kappa^2 R_0q_+R_0q_-)' = \kappa(R_0Hq_+'R_0q_-)' + \kappa(R_0q_+R_0Hq_-')' + (R_0q_+'R_0q_-')'$$

and

$$\begin{aligned}\|\kappa(R_0Hq_+'R_0q_-)'\|_{H^\sigma} &\lesssim \kappa\|R_0Hq_+'R_0q_-\|_{H^{\sigma+1}} \\ &\lesssim \kappa o_{\kappa\to\infty}(1)\|R_0Hq_+'\|_{H^{s-1}}\|q\|_{H^s} \\ &\lesssim o_{\kappa\to\infty}(1)\|q\|_{H^s}^2\end{aligned}$$

for s as above. Therefore, the quadratic terms are all acceptable.

Now we consider the higher-order terms. The general form of the error terms is

$$\kappa^2 E(\kappa, n)F(\kappa, m)$$

where $n + m \geq 3$. In the case $m = 0$, we may apply (4.21):

$$\begin{aligned}\kappa^2\|E(\kappa, n)\|_{H^\sigma} &\lesssim \kappa\|qE(\kappa, n-1)\|_{H^\sigma} \\ &= \kappa\|\langle\xi\rangle^\sigma \hat{q} * E(\kappa, n-1)^\wedge\|_2 \\ &\lesssim \kappa\|\langle\xi\rangle^\sigma\|_2\|\hat{q} * E(\kappa, n-1)^\wedge\|_\infty \\ &\lesssim \|q\|_2\|qE(\kappa, n-2)\|_2 \\ &\lesssim \kappa^{1-n/2}\|q\|_2^n\end{aligned}$$

which vanishes as $\kappa \to \infty$ since $\sigma < -\frac{1}{2}$ and $n > 2$. In the case $n = 0$ we obtain the same vanishing result. In all other cases we see from (4.22) that

$$\|\kappa^2 E(\kappa, n)F(\kappa, m)\|_{H^\sigma} \lesssim \kappa^2\|\langle\xi\rangle^\sigma\|_2\|E(\kappa, n)\|_2\|F(\kappa, m)\|_2 \lesssim \kappa^{1-(m+n)/2}\|q\|^{m+n}$$

which is acceptable since $\sigma < -\frac{1}{2}$ and $m + n \geq 3$.

We have proven (4.28) for $\sigma < -3$.

Let Q be a bounded, equicontinuous subset of L^2 and let $\{q_n(0)\} \subset Q \cap H^3$. Let $q_n(t)$ and $q_n(s)$ denote the H_{BO} and H_κ Hamiltonian flows, respectively, of $q_n(0)$. Suppose $q_n(0) \to q(0)$ in L^2. Then for $s, t \in [-T, T]$, $n, m \geq K$,

$$\|q_n(t) - q_m(t)\|_{H^\sigma} \lesssim \|q_n(t) - q_n(s)\|_{H^\sigma} + \|q_n(s) - q_m(s)\|_{L^2} + \|q_m(s) - q_m(t)\|_{L^2}$$

$$\begin{aligned} &\lesssim 2 \sup_{q \in Q^*} \int_0^T \|q_t - q_s\|_{L^2} + \sup_{n,m \geq K} \|q_n(s) - q_m(s)\|_{L^2} \\ &= o_{\kappa \to \infty}(1) T + o_{K \to \infty}(1) \end{aligned}$$

where we pass to the last line by (4.28) and the well-posedness of the flow in s. We have proved that $q_n(t)$ converges to something in H^σ, and since Q^* is equicontinuous it follows from Lemma 4.5 that the sequence converges in L^2 for each t. This extends the Benjamin–Ono solution map to L^2; we write $q(t) = \lim_{n\to\infty} q_n(t)$. It follows from the above that the convergence is uniform on compact regions of time and L^2-continuous in the initial data. □

The proof works just as well on the circle as on the line, though we have elected to present the proof on the line only to avoid constant interruptions over notational details (for example, the substitution of series for integrals on the Fourier side).

5 Local Smoothing

In the proof of Theorem 4.2 it can be seen that the higher-order terms (cubic or greater in q) are ultimately controlled by $\kappa^{-1/2}\|q\|_2^3$. The estimates employed here were quite crude; nevertheless, by trading the decay in κ that arises from R_0 for a gain in regularity, the control of the higher-order terms extends without significant modification to data $q \in H^s$ for $s > -\frac{1}{6}$. Well-posedness in this regime would be a novel result.

The barrier lies in the quadratic terms $H(qR_0q'_+)'_+$, $H(qR_0q'_-)'_-$, $(R_0q'_+R_0q'_-)'$, none of which are bounded a priori in any H^σ, no matter how small σ is, unless $q \in L^2$. This obstacle could be overcome by means of a local smoothing estimate for the difference flow as in Propositions 4.6, 4.8 of [3]. Controlling the contribution of the higher-order terms in order to obtain such an estimate is a formidable challenge; however, a sufficiently heroic act of inequality-crunching would suffice to push the frontier of well-posedness for Benjamin–Ono to $s > -\frac{1}{6}$ if not the expected sharp result of $s > -\frac{1}{2}$.

In this section we prove two local smoothing results for rough solutions $s > -\frac{1}{2}$ to the (BO) flow itself. While obtaining similar results for the H_κ flow is more difficult, these represent partial progress toward that goal. The first result is standard Kato–type local smoothing; the second is adapted to high frequency scales $|\xi| \sim \kappa \gg 1$.

5.1 The flow of the Jostish functions

Recall our Lax pair (restricted to the positive Hardy space H^+)

$$L = \frac{1}{i}\nabla - C_+q,$$

$$P = \frac{1}{i}\partial_{xx} - 2C_+(q' + q\nabla - q'_+),$$

$$\dot{L} = [L, P] \text{ if and only if } q \text{ solves (BO).}$$

Since m solves (3.16), which is equivalent to the "scattering equation":

$$Lm = \kappa(m-1),$$

we expect its flow to be given by something like (1.6):

$$\dot{m} = Pm.$$

Let $q \in H^s$ for some $-\frac{1}{2} < s < 0$. On H^+, define the resolvent $R(\kappa) = (L+\kappa)^{-1}$, which is given by the series (recalling the definition (2.8))

$$R(\kappa) = \sum_{\ell=0} \sqrt{R_0} A(\kappa; q)^\ell \sqrt{R_0},$$

which converges in L^2 operator norm according to the estimate

$$\|A(\kappa; q)\|_{\mathrm{op}} \le \|A(\kappa; q)\|_{\mathfrak{I}_2} < \frac{1}{3}$$

when $\kappa \gg 1$ is large according to Lemma 2.2. We can rewrite (3.16) in the form

$$m - 1 = Rq_+. \tag{5.30}$$

We observe

$$\begin{aligned} 0 &= R\frac{d}{dt}(LR) \\ &= R[L, P]R + RL\dot{R} \\ &= PR - RP + \dot{R} \end{aligned}$$

so $\dot{R} = [R, P]$. Taking the time derivative of both sides of (5.30), we find (in the distributional sense)

$$\begin{aligned} \dot{m} &= \dot{R}q_+ + R\dot{q}_+ \\ &= -PRq_+ + R(Pq_+ + \dot{q}_+) \\ &= -P(m-1), \end{aligned} \tag{5.31}$$

since

$$Pq_+ = Hq_+'' - 2(q'q_+ + qq_+' - q_+'q_+)_+ = Hq_+'' - 2(q'q_+ + q_-q')_+ = -\dot{q}_+,$$

again in the distributional sense.

5.2 $\kappa = 1$ Local Smoothing

Theorem 5.1 (Local smoothing)**.** *Let q solve* (BO)*. Then for any $-\frac{1}{2} < s \leq 0$,*

$$\int_0^1 \int_0^1 q^2 dx dt \lesssim_{\|q\|_{H^s}} 1.$$

Proof. Let f be a function such that f' is positive and Schwarz. Fix $\kappa \gg 1$ large enough that $m-1$ can be defined by series and satisfies (5.31). Note that

$$\int f(q(m-1))_+ \lesssim \|f\|_\infty \|q\|_{H^s} \|m-1\|_{H^{-s}} \lesssim \|f\|_\infty \|q\|_{H^s} \|m-1\|_{H^{1+s}} \lesssim_{f, \|q\|_{H^s}} 1.$$

We apply (3.16), (5.31) and compute

$$\begin{aligned}
&\frac{d}{dt} \int f(qm-1))_+ dx \\
&\quad = \frac{d}{dt} \int f \left[im' + \kappa m - \kappa - q \right]_+ dx \\
&\quad = \int -f'''(m-1) - 2if''q(m-1) + i\kappa f''(m-1) - 2\kappa f' q(m-1) + f'' Hq dx \\
&\qquad - \int 2if' q_+'(m-1) - 2\kappa f q_+'(m-1) dx + \int f' q^2 dx.
\end{aligned}$$

All the terms in the first integral are acceptable because f' is Schwartz. For the second integral, we use Plancherel and the fact that q_+ and $m-1$ are each supported on positive frequencies:

$$\begin{aligned}
\langle q_+'(m-1), f \rangle &= \int_0^\infty \hat{f}(-\xi) \int_0^\xi i(\xi - \eta) \hat{q}_+(\xi - \eta)(m-1)^\wedge(\eta) d\eta d\xi \\
&\lesssim \int_0^\infty \langle \xi \rangle^{1+|s|} |\hat{f}(\xi)| \int_0^\xi \langle \xi - \eta \rangle^s |\hat{q}_+(\xi - \eta)| \langle \eta \rangle^{1+s} |(m-1)^\wedge(\eta)| d\eta d\xi \\
&\lesssim_f \|q_+\|_{H^s} \|m-1\|_2 \lesssim \|q\|_{H^s}^2.
\end{aligned}$$

by Cauchy-Schwarz and since f' is Schwartz. We therefore find that

$$\int_0^1 \int_0^1 f' q^2 dx dt =$$
$$= \int_0^1 f\Big(q(t{=}1,x)(m-1)(t{=}1,x) - q(0,x)(m-1)(t{=}0,x)\Big)dx + C(\|q\|_{H^s})$$
$$\lesssim_{f,\kappa,\|q\|_{H^s}} 1.$$

Since f' is positive, fixing any large $\kappa > 1$ implies that

$$\int_0^1 \int_0^1 q^2 dx dt \lesssim_{\|q\|_{H^s}} 1.$$

□

5.3 Local Smoothing Uniformly in κ

Theorem 5.2. *Let q solve* (BO) *and $-\frac{1}{2} < s \leq 0$. Then,*

$$\lim_{\kappa\to\infty} \int_0^1 \int \frac{|\xi|}{\kappa + |\xi|} |(\rho q)^\wedge(\xi)|^2 d\xi = 0.$$

Proof. Let f be a primitive of a Schwarz function, i.e. f' Schwarz. Let κ be large enough that the series defining $m-1$ converges. By Theorem 4.4, $\{q(t) : 0 \leq t \leq 1\}$ is equicontinuous in $H^s(\mathbb{R})$. Therefore, $\|R_0 q\|_{H^{1+s}} = \|R_0 \langle\xi\rangle q\|_{H^s}$ vanishes uniformly as $\kappa \to \infty$. It follows that

$$\sup_t \|(m-1)(\kappa; q(t))\|_{H^{1+s}} \lesssim \sup_t \|R_0(\kappa) q_+(t)\|_{H^{1+s}} + \sum_{n=2}^{\infty} (\kappa^{-\varepsilon}\|q\|_{H^s})^n = o_{\kappa\to\infty}(1)$$

A key identity is the commutator relation

$$fHg + Hfg = 2H(f_+ g_+ + f_- g_-).$$

Terms which resemble $\int fg_+h_+dx$ or $\int fg_-h_-dx$ are negligible and subsumed into a running error total ☺. We compute

$$\begin{aligned}
\frac{d}{dt}\int fq(m-1)dx &= \int f\Big(-Hq''(m-1)+2qq'(m-1)-qP(m-1)\Big)dx\\
&= \int f\Big(q''H(m-1)-qHm''+2qq'(m-1)+2q(q(m-1))'_+\\
&\qquad\qquad -2qq'_+(m-1)\Big)dx + ☺\\
&= \int f\Big((qH(m-1))''-2q'Hm'-2qHm''\\
&\qquad\qquad +2q'_-q(m-1)+2q(q(m-1))'_+dx\Big) + ☺\\
&= -\int f''qH(m-1)dx + 2\int f'qHm'dx\\
&\qquad\qquad +2\int f'q_-(q(m-1))_+dx + ☺.
\end{aligned}$$

The main term is the quadratic part of $2\int f'qHm'dx$. We dispose of the other terms as follows:

$$\int f''qH(m-1)dx \lesssim \|f''\|_{L^2}\|qH(m-1)\|_{L^\infty} \lesssim_f \|q\|_{H^s}\|m-1\|_{H^{1-s}}$$

vanishes uniformly in t, and

$$\begin{aligned}
\int f'q_-(q(m-1))_+dx &\lesssim \|\sqrt{f'}q\|_{L^2}\|\sqrt{f'}q(m-1)\|_{L^2}\\
&\lesssim \|\sqrt{f'}q\|_{L^2}\||\nabla|^\sigma\sqrt{f'}\|_{L^\infty}\|q(m-1)\|_{H^\sigma}\\
&\lesssim_f \|\sqrt{f'}q\|_{L^2}\|q\|_{H^s}\|m-1\|_{H^{1-s}}\\
&\lesssim_{f,\|q\|_{H^s}} \|\sqrt{f'}q\|_{L^2}o_{\kappa\to\infty}(1)
\end{aligned}$$

vanishes uniformly in t after we integrate in time and apply $\kappa=1$ local smoothing

(Theorem 5.1). The higher-order terms obey

$$\begin{aligned}\int f'qH(m-R_0q_+)'dx &= \sum_{\ell=2}^{\infty}\int f'qA(\kappa,\ell)dx\\ &\lesssim \|f'q\|_{L^2}\sum_{\ell=2}^{\infty}\|A(\kappa,\ell)\|_{L^2}\\ &\lesssim \||\nabla|^{1/2}f'\|_{L^\infty}\|q\|_{H^s}\sum_{\ell=2}^{\infty}\kappa^{-(\ell+1)/2}\|q\|_2^{\ell}\\ &\lesssim_f \|q\|_{H^s}^2\kappa^{-1/2}\end{aligned}$$

for κ sufficiently large. Finally, we consider the error term ☺. Using the commutator relation, we see

$$\begin{aligned}☺ &= \int f(-Hq''(m-1)-q''H(m-1))dx+\\ &\quad +2\int f\Big(q'_-(q(m-1))_- + q_+(q(m-1))'_+\Big)dx\\ &= 2\int f\Big(-H(q''_+(m-1)) + q'_-(q(m-1))_- + q_+(q(m-1))'_+\Big)dx.\end{aligned}$$

I will not treat each of these terms in detail, but they are acceptable because of constructive interference between frequencies of the same sign[2]; the fact that justifies this assertion is that for any $\alpha,\beta>0$:

$$\begin{aligned}\int fg_+h_+dx &= \int_0^\infty \hat f(-\xi)\int_0^\xi \hat g(\xi-\eta)\hat h(\eta)d\eta d\xi\\ &\lesssim \int_0^\infty \langle\xi\rangle^{\alpha+\beta}|\hat f(-\xi)|\int_0^\xi \langle\xi-\eta\rangle^{\alpha}|\hat g(\xi-\eta)|\langle\eta\rangle^{\beta}|\hat h(\eta)|d\eta d\xi\\ &\lesssim \|f\|_{H^{\alpha+\beta}}\|g\|_{H^{-\alpha}}\|h\|_{H^{-\beta}}.\end{aligned}$$

This together with the bilinear estimates we have used above suffices to prove:

$$\int_0^1\int f'qR_0Hq'_+dxdt = o(1).$$

[2]Though they may not appear at first glance to be complete derivatives, same-frequency products satisfy $\int g_+h_+dx = 0$, which may be a moral excuse for "shifting the derivatives onto f" here.

Taking the complex adjoint shows that the same identity holds with q_+ replaced by q_- and hence by q:

$$\int_0^1 \int f' q R_0 H q' dx dt = o(1).$$

We wish to rearrange the left hand side so that it is coercive of the local smoothing norm. Let f be a primitive of φ^2, where φ is smooth and supported on some compact set. We observe that

$$\begin{aligned}\varphi R_0 H q' - R_0 H(\varphi q)' &= \kappa R_0(\varphi R_0 H q') - \kappa R_0 H(\varphi R_0 q)' \\ &= -\kappa R_0(\varphi' R_0 q) - 2i R_0(\varphi R_0 q'_+)_- + 2i R_0(\varphi R_0 q'_-)_+.\end{aligned}$$

Thus, since C_+ and C_- are adjoint,

$$\begin{aligned}\int \varphi^2 q R_0 H q' dx = &\int \varphi q R_0 H(\varphi q)' dx \\ &+ \kappa \int 2i(\varphi q)_- R_0(\varphi R_0 q'_-) - 2i(\varphi q)_+ R_0(\varphi R_0 q'_+) - \varphi q R_0(\varphi' R_0 q) dx.\end{aligned}$$

The second integral is an error which decays as $\kappa \to \infty$; the first two terms because of same-signed-frequency interactions and the smoothness of φ, and the third term is trivial. We at last find, after applying Plancherel, that

$$\int_0^T \int \frac{|\xi|}{\kappa + |\xi|} |(\rho q)^\wedge(\xi)|^2 d\xi = o_{\kappa\to\infty}(1).$$

$\square$

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