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Stability of Asymptotically Hyperbolic Einstein Manifolds

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STABILITY OF ASYMPTOTICALLY HYPERBOLIC EINSTEIN MANIFOLDS

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Table of Contents

Abstract

Stability of Asymptotically Hyperbolic Einstein Manifolds

by

Yucheng Lu

In this thesis we study the stability of the Ricci flow. The stability problem of Ricci flow in different settings have been considered by Ye [16], Li-Yin [17], Schnürer-Schulze-Simon [20] and Bamler [28] etc. We consider a more general case and extend the results to the general case, that is, in the setting of asymptotically hyperbolic Einstein (AHE) manifolds with rough initial data. First we introduce the background of the problem and results on the long time behavior of Ricci flow in detail. Then we compare the difference in methodology of theses results and extend to the AHE case. We consider the normalized Ricci flow on a AHE manifold with initial metrics which are perturbations of a non-degenerate AHE metric h_0. The key step is to obtain exponential decay of certain geometric quantities. Then we prove that the normalized Ricci flow converges exponentially fast to h_0, if the perturbation is L^2-bounded and C^0-small.

To Hong Lu

To the memory of

Jin Liu

Jiugu Xiao

& Liangcheng Lu

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Chapter 1

Riemannian Geometry

1.1 Riemannian Metrics and Levi-Civita Connections

Through out this paper, we follow the Einstein summation convention over repeated indices.

Let M be a manifold and p be a point of M. T_pM denotes the tangent space and T_p^*M the cotangent space at p. Then $TM = \bigcup_{p \in M} T_pM$ denotes the tangent bundle and $T^*M = \bigcup_{p \in M} T_p^*M$ the cotangent bundle.

Definition 1.1.1. *Let M^n be an n-dimensional smooth manifold. A Riemannian metric g on M is a smoothly varying inner product on the tangent spaces. Or equivalently, a smooth section of $T^*M \otimes T^*M$ defining a positive-definite symmetric bilinear form on T_pM for each $p \in M$. (M^n, g) is a Riemannian manifold.*

Let $\{x^i\}_{i=1}^n$ be local coordinates in a neighborhood U of some point $p \in M$. In U the vector fields $\{\partial_i\}_{i=1}^n$ form a local basis for TM and the 1-forms $\{dx^i\}_{i=1}^n$ form a dual basis for T^*M. Then the metric can be written as

$$g = g_{ij} dx^i \otimes dx^j$$

where $g_{ij} \doteqdot g(\partial_i, \partial_j)$.

Definition 1.1.2. *The Levi-Civita connection ∇ is the unique connection on TM that is compatible with the metric and is torsion-free:*

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ \nabla_X Y - \nabla_Y X &= [X,Y] \end{aligned}$$

In local coordinates, the Levi-Civita connection is given by $\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k$, where the Christoffel symbols Γ_{ij}^k are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{1.1.1}$$

1.2 Curvatures

Let (M, g) be a Riemannian manifold. The Riemannian curvature $(1,3)$-tensor is defined by

$$\mathrm{Rm}(X,Y)Z \doteqdot \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In local coordinates, the components of the Riemannian curvature tensor Rm is defined by

$$\mathrm{Rm}(\partial_i, \partial_j)\partial_k \doteqdot R_{ijk}^l \partial_l$$

where

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^p \Gamma_{ip}^l - \Gamma_{ik}^p \Gamma_{jp}^l. \tag{1.2.1}$$

And Rm can be transformed into a $(0,4)$-tensor: $\mathrm{Rm}(X,Y,Z,W) \doteqdot g(\mathrm{Rm}(X,Y)Z,W)$.

In local coordinates, $\mathrm{Rm}(\partial_i,\partial_j,\partial_k,\partial_l) \doteqdot R_{ijkl} = g_{lm}R_{ijk}^m$.

Using (1.1.1), the Riemannian curvature tensor can be expressed in terms of the first and second derivatives of the metric. In local normal coordinates about point p where $g_{ij}(p) = \delta_{ij}$ and the first derivatives of g vanish at p,

$$R_{ijkl} = \frac{1}{2}(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}).$$

If $P \subset T_pM$ is a 2-plane, then the sectional curvature of P is defined by

$$K(P) \doteqdot g(\mathrm{Rm}(e_1,e_2)e_2,e_1)$$

where $\{e_1,e_2\}$ is an orthonormal basis of P. In general, if X and Y are any two vectors spanning P, then

$$K(P) = \frac{g(\mathrm{Rm}(X,Y)Y,X)}{|X|^2|Y|^2 - (g(X,Y))^2}.$$

The Ricci curvature $(0,2)$-tensor Rc is the trace of of the Riemann curvature tensor:

$$\mathrm{Rc}(Y,Z) \doteqdot \mathrm{tr}(X \mapsto \mathrm{Rm}(X,Y)Z).$$

In local coordinates, its components $R_{ij} \doteqdot \mathrm{Rc}(\partial_i,\partial_j)$ are given by

$$R_{ij} = R_{kij}^k.$$

In local normal coordinates,

$$R_{ij} = -\frac{1}{2}g^{kl}(\partial_i\partial_j g_{kl} + \partial_k\partial_l g_{ij} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}). \tag{1.2.2}$$

The Ricci curvature of a line $L \subset T_pM$ is defined by $\mathrm{Rc}(L) \doteqdot \mathrm{Rc}(e_1,e_1)$ where $e_1 \in T_pM$ is a unit vector spanning L.

The scalar curvature is the trace of the Ricci tensor:

$$R \doteqdot \mathrm{tr}(\mathrm{Rc}(\cdot,\cdot)) = \sum_{i=1}^{n} \mathrm{Rc}(e_i, e_i)$$

In local coordinates, $R = g^{ij} R_{ij}$ where $(g^{ij}) \doteqdot (g_{ij})^{-1}$ is the inverse matrix.

Chapter 2

Ricci Flow

2.1 Introduction

Geometric flows, as a class of important geometric partial differential equations, have been highlighted in many fields of theoretical research and practical applications. They have been around at least since Mullin's paper [1] in 1956, which proposed the curve shortening flow to model the motion of idealized grain boundaries. In 1964 Eells and Sampson [2] introduced the harmonic map heat flow and used it to prove the existence of harmonic maps into targets with non-positive sectional curvature. Motivated by the work of Eells and Sampson, in 1975 Hamilton [3] continued the study of harmonic map heat flow by considering manifolds with boundary.

In the late seventies, Thurston suggested a classification of three-dimensional manifolds, which became known as the geometrization conjecture. Thurston's conjecture stated that any compact 3-manifold can be decomposed into one or more types (out of 8) of components with homogeneous geometry, including a spherical type, and thus implied the Poincaré conjecture. In 1982 Thurston won a Fields Medal for his contributions to topology. That year Hamilton [4] introduced the Ricci flow, which he suspected could be relevant for solving Thurston's conjecture. The Ricci flow equation has been called the heat equation for metrics, due to its property of making metrics "better". A large number of innovations that originated in Hamilton's 1982 and subsequent papers have had a profound influence on modern geometric analysis. The most recent result is Perelman's [7][8][9] proof of Thurston's conjecture.

The Ricci flow (RF) is a way of evolving the metric of a Riemannian manifold (M^n, g_0). More specifically, it is a geometric evolution equation (RF) defined on M^n:

$$\text{(RF)} \begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\mathrm{Rc}_{g(t)} \\ g(0) = g_0 \end{cases} \tag{2.1.1}$$

A solution to this equation (RF) is a one-parameter family of metrics $g(t)$ on M, defined on a time interval $I \subseteq \mathbb{R}^1$. We will use the name "Ricci flow" for both the equation (RF) and the solution $g(t)$.

Proposition 2.1.1. *In local harmonic coordinates around a point p, the Ricci tensor at p is*

$$R_{ij} = \mathrm{Rc}(\partial_i, \partial_j) = -\frac{1}{2}\Delta(g_{ij}) + Q(g^{-1}, \partial g)$$

where Q is a quadratic form thus a lower order term in the derivatives of g. As a result, Ricci flow resembles a system of nonlinear heat equations.

Harmonic coordinates are a coordinate system $\{x^i\}_{i=1}^n$ *where each coordinate function is harmonic:* $0 = \Delta x^i = g^{jk}(\partial_j \partial_k - \Gamma^l_{jk}\partial_l)x^i = -g^{jk}\Gamma^i_{jk} = 0.$

Proof. From (1.2.2) we have

$$\begin{aligned} R_{ij} &= -\frac{1}{2}g^{kl}(\partial_i\partial_j g_{kl} + \partial_k\partial_l g_{ij} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}) \\ &= -\frac{1}{2}\Delta g_{ij} + \frac{1}{2}g^{kl}(\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_j g_{kl}). \end{aligned} \tag{2.1.2}$$

And we have the following equation from harmonic coordinates

$$\begin{aligned} 0 &= g^{kl}\Gamma^p_{kl} \\ &= \frac{1}{2}g^{kl}g^{pq}(\partial_k g_{lq} + \partial_l g_{kq} - \partial_q g_{kl}). \end{aligned} \tag{2.1.3}$$

Take derivative ∂_i of both sides and contract with g_{jp}:

$$\begin{aligned} 0 &= g_{jp}\partial_i\Big(\frac{1}{2}g^{kl}g^{pq}(\partial_k g_{lq} + \partial_l g_{kq} - \partial_q g_{kl})\Big) \\ &= \frac{1}{2}g_{jp}g^{kl}g^{pq}(\partial_i\partial_k g_{lq} + \partial_i\partial_l g_{kq} - \partial_i\partial_q g_{kl}) + \text{l.o.t} \\ &= \frac{1}{2}g^{kl}(\partial_i\partial_k g_{jl} + \partial_i\partial_l g_{jk} - \partial_i\partial_j g_{kl}) + \text{l.o.t} \\ &= \frac{1}{2}g^{kl}(2\partial_i\partial_k g_{jl} - \partial_i\partial_j g_{kl}) + \text{l.o.t.} \end{aligned} \tag{2.1.4}$$

Swap indices i and j and we have

$$0 = \frac{1}{2}g^{kl}(2\partial_j\partial_k g_{il} - \partial_i\partial_j g_{kl}) + \text{l.o.t}$$

Add the two equations above together and we get that the tail term in (2.1.2) is 0.

□

Example 2.1.1. *If (M,g) is Einstein, i.e.* $\mathrm{Rc}_g = \lambda g$, *then $g(t) = (1-2\lambda t)g_0$ is a solution to* (RF).

And $\mathrm{Rm}_{g(t)} = \frac{1}{1-2\lambda t}\mathrm{Rm}_{g_0}$, $\mathrm{Rc}_{g(t)} = \mathrm{Rc}_{g_0}$.

If $\lambda > 0$, the solution is shrinking and only exists up to $T_{max} = \frac{1}{2\lambda}$.

If $\lambda = 0$, the solution stays static and exists for all time.

If $\lambda < 0$, the solution is expanding and exists for all positive time.

Remark 2.1.2 (Invariance under diffeomorphisms)**.** *If $\Phi : M \to M$ is a diffeomorphism and $g(t)$ is a Ricci flow, then $\Phi^* g(t)$ is also a Ricci flow.*

2.2 Normalized Ricci Flow

2.2.1 Compact Case

Example 2.1.1 shows that we can not expect long time existence of the Ricci flow. However it's convenient to consider the normalized Ricci flow (NRF) which preserves the volume:

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Rc}_g + \dfrac{2}{n}\sigma_g g \\ g(0) = g_0 \end{cases} \tag{2.2.1}$$

where σ_g is the average scalar curvature $\sigma_g = \dfrac{\int_M R_g \, \mathrm{d}\mu_g}{\int_M \mathrm{d}\mu_g}$.

Remark 2.2.1. *The normalized Ricci flow* (NRF) *is equivalent to the original Ricci flow* (RF) *by reparametrizing in time t and rescaling the metric by a function of t:*

Given a solution $g(t)$ to the Ricci flow, let $\tilde{g}(\tilde{t}) = c(t)g(t)$ where

$$c(t) \doteqdot \exp\left(\frac{2}{n}\int_0^t \sigma_{g(\tau)} \mathrm{d}\tau\right), \quad \tilde{t}(t) \doteqdot \int_0^t c(\tau)\mathrm{d}\tau.$$

And $\tilde{g}(\tilde{t})$ is a solution to the normalized Ricci flow.

2.2.2 Noncompact Case

When the manifold is noncompact, we consider the normalized Ricci flow (NRF):

$$\begin{cases} \frac{\partial}{\partial t} g = -2(\mathrm{Rc}_g + (n-1)g) \\ g(0) = g_0 \end{cases} \tag{2.2.2}$$

Again, the normalized Ricci flow (NRF) in this case is equivalent to the original Ricci flow (RF) by reparametrizing:

Given a solution $g(t)$ to the Ricci flow, let $\tilde{g}(\tilde{t}) = c(t)g(t)$ where

$$c(t) \doteqdot e^{-2(n-1)t}, \quad \tilde{t}(t) \doteqdot \frac{e^{2(n-1)t} - 1}{2(n-1)}.$$

And $\tilde{g}(\tilde{t})$ is a solution to the normalized Ricci flow.

2.3 Short Time Existence

2.3.1 DeTurck's Trick

By computing in normal coordinates, we can establish the evolution of various geometric quantities.

Lemma 2.3.1. *If $g(s)$ is a 1-parameter family of metrics with $\partial_s g_{ij} = v_{ij}$, then*

$$\begin{aligned}
\partial_s \Gamma_{ij}^k &= \tfrac{1}{2} g^{kl} (\nabla_i v_{jl} + \nabla_J v_{il} - \nabla_l v_{ij}) \\
\partial_s R_{ijk}^l &= \tfrac{1}{2} g^{lp} (\nabla_i \nabla_k v_{jp} + \nabla_j \nabla_p v_{ik} - \nabla_i \nabla_p v_{jk} - \nabla_j \nabla_k v_{ip} - R_{ijk}^q v_{pq} - R_{ijp}^q v_{kq}) \\
\partial_s R_{ij} &= \tfrac{1}{2} g^{pq} (\nabla_q \nabla_i v_{jp} + \nabla_q \nabla_j v_{ip} - \nabla_q \nabla_p v_{ij} - \nabla_i \nabla_j v_{qp}) \\
&= -\tfrac{1}{2} \Big(\Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i (\delta v)_j - \nabla_j (\delta v)_i \Big)
\end{aligned}$$

where $V = \mathrm{tr}_g(v) = g^{ij}v_{ij}$*, the divergence of a 2-tensor is* $(\delta v)_j = (\mathrm{div}\ v)_j = g^{ik}\nabla_k v_{ij}$*, and* Δ_L *is the Lichnerowicz Laplacian:*

$$\Delta_L v_{ij} \doteqdot \Delta v_{ij} + 2R_{kijl}v^{kl} - R_{ik}v_j^k - R_{jk}v_i^k$$

Regard the Ricci tensor $\mathrm{Rc}(g)$ as a nonlinear partial differential operator on the metric g, its linearization is

$$[D(\mathrm{Rc}_g)(v)]_{ij} = \frac{1}{2}g^{pq}(\nabla_q\nabla_i v_{jp} + \nabla_q\nabla_j v_{ip} - \nabla_q\nabla_p v_{ij} - \nabla_i\nabla_j v_{qp}) \tag{2.3.1}$$

The principal symbol of the Ricci tensor is

$$[\hat{\sigma}[D(\mathrm{Rc}_g)](\xi)(v)]_{ij} = \frac{1}{2}g^{pq}(\xi_q\xi_i v_{jp} + \xi_q\xi_j v_{ip} - \xi_q\xi_p v_{ij} - \xi_i\xi_j v_{qp})$$

The diffeomorphism invariance of the Ricci tensor implies that the principal symbol above $\hat{\sigma}[D(\mathrm{Rc}_g)](\xi)$ has a nontrivial kernel. As a result the Ricci flow equation is only weakly parabolic and the short time existence does not follow directly from the standard parabolic theory. Hamilton's original proof relied on the sophisticated machinery of the Nash-Moser inverse function theorem. Shortly thereafter, DeTurck [10][11] proposed a simplified proof by showing that the Ricci flow is equivalent to a strictly parabolic system.

Take $\tilde{\Gamma}$ to be the Levi-Civita connection of a fixed background metric $\tilde{g}$. Define a vector field $W = W(g, \tilde{\Gamma})$ by

$$W^i = g^{jk}(\Gamma_{jk}^i - \tilde{\Gamma}_{jk}^i) \tag{2.3.2}$$

And the 1-form g-dual to W is

$$W_i = g_{ij}g^{pq}(\Gamma_{pq}^j - \tilde{\Gamma}_{pq}^j)$$

Now we define the Ricci-DeTurck flow:

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i \\ g(0) &= g_0 \end{aligned} \tag{2.3.3}$$

The extra term is exactly $\mathcal{L}_W g$:

$$P(g)_{ij} = \nabla_i W_j + \nabla_j W_i = \nabla_i g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}) + \nabla_j g_{ik} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$$

And its linearization is

$$\begin{aligned} [DP(g)(v)]_{ij} &= \tfrac{1}{2} g_{jk} g^{pq} \nabla_i [g^{kl} (\nabla_p v_{lq} + \nabla_q v_{pl} - \nabla_l v_{pq})] \\ &\quad + \tfrac{1}{2} g_{ik} g^{pq} \nabla_j [g^{kl} (\nabla_p v_{lq} + \nabla_q v_{pl} - \nabla_l v_{pq})] \\ &\quad + (\text{lower order derivatives of } v) \\ &= g^{pq} (\nabla_i \nabla_p v_{jq} + \nabla_j \nabla_p v_{iq} - \nabla_i \nabla_j v_{pq}) + \text{l.o.t} \end{aligned} \tag{2.3.4}$$

Comparing to the linearized Ricci tensor, we can see that the principal symbol $-2\text{Rc} + P$ is

$$[\hat{\sigma}[D(-2\text{Rc} + P)](\xi)(v)]_{ij} = g^{pq} \xi_p \xi_q v_{ij} = \|\xi\|_g^2 v_{ij}.$$

Thus $2\text{Rc} + P$ is elliptic and the Ricci-DeTurck flow is a strictly parabolic system of partial differential equations.

2.3.2 Compact Case - Hamilton's Existence Result

In Hamilton's [4] original work, he proved that any compact 3-manifold with positive Ricci curvature also admits a metric of constant postive curvature. The method of his proof is by introducing the Ricci flow - start with any metric g_{ij} of strictly positive Ricci curvature R_{ij} and improve it by the Ricci flow equation. His set-up became the foundation to resolve Thurston's Geometrization Conjecture for closed 3-manifolds.

Theorem 2.3.2 (Hamilton, '82)**.** *If M^n is a closed Riemannian manifold with a C^∞ Riemannian metric g_0, then there exists a unique smooth solution $g(t)$ to the Ricci flow defined on some time interval $[0,\varepsilon), \varepsilon > 0$ with $g(0) = g_0$.*

DeTurck's proof.

Step 1. Consider the Ricci-DeTurck flow (2.3.3). Since it is a strictly parabolic system of partial differential equations, it follows from the standard parabolic theory [11] [12, Thm 7.1] that for any smooth initial metric g_0, there exists $\varepsilon > 0$ depending on g_0 such that a smooth solution $g(t)$ to (2.3.3) exists for a short time $t \in [0,\varepsilon)$.

Step 2. The one-parameter family of vector fields $W(t)$ defined by (2.3.2) exist as long as the solution $g(t)$ exists. Then define a 1-parameter family of maps $\phi_t : M^n \to M^n$ by solving the ODE:

$$\begin{aligned} \frac{\partial}{\partial t}\phi_t(p) &= -W(\phi_t(p),t) \\ \phi_0 &= \mathrm{id}_M \end{aligned}$$

Suppose $g(t)$ exists for $t \in [0,\varepsilon)$. Assume $\phi_t(p)$ exists for $t \in [0,T]$ where $0 \le T < \varepsilon$. Fix any $T_1 \in (T,\varepsilon)$. At any given point $p \in M$, the equation above is equivalent to a nonlinear ordinary differential equation. Thus $\phi_t(p)$ always exists for $t \in [T, T+\varepsilon_0)$ for some $\varepsilon_0 > 0$.

Since the manifold M^n is compact, the vector fields $W(\cdot,t)$ are uniformly bounded on $M \times [T,T_1]$. Then there exists an $\bar{\varepsilon}$ independ of $p \in M$ such that ϕ_t exists for $t \in [T, T+\bar{\varepsilon}]$.

The same argument holds for the flow starting from $T+\bar{\varepsilon}$. Iteration yields that ϕ_t exists for $t \in [T,T_1]$. Since T_1 is arbitrary, then ϕ_t exists for $t \in [0,\varepsilon)$.

Step 3. Pulling back $g(t)$ and we obtain $\bar{g}(t) \doteqdot \phi_t^* g(t)$ $(0 \le t < \varepsilon)$. Then $\bar{g}(0) = g(0) = g_0$

and we compute that

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{g}(t) &= \frac{\partial}{\partial t}(\phi_t^* g(t)) = \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t+s)) \\
&= \phi_t^*\left(\frac{\partial}{\partial t}g(t)\right) + \frac{\partial}{\partial s}\Big|_{s=0}(\phi_{t+s}^* g(t)) \\
&= \phi_t^*\big(-2\mathrm{Rc}(g(t)) + \mathcal{L}_{W(t)}g(t)\big) + \frac{\partial}{\partial s}\Big|_{s=0}\big((\phi_t^{-1}\circ\phi_{t+s})^*\phi_t^* g(t)\big) \qquad (2.3.5) \\
&= -2\mathrm{Rc}(\phi_t^* g(t)) + \phi_t^*\big(\mathcal{L}_{W(t)}g(t)\big) - \mathcal{L}_{[(\phi_t^{-1})_* W(t)]}\big(\phi_t^* g(t)\big) \\
&= -2\mathrm{Rc}(\phi_t^* g(t)) = -2\mathrm{Rc}(\bar{g}(t))
\end{aligned}$$

Hence $\bar{g}(t)$ is a solution of the Ricci flow for $t \in [0,\varepsilon)$. □

2.3.3 Noncompact Case - Shi's Existence Result

In the case where M is a noncompact complete Riemannian manifold, one can not expect the short time existence of the Ricci flow equation for an arbitrary initial metric g_0. For example, take $S^2 \times \mathbb{R}$ as the underlying manifold and endow it with a warped product metric so that metrically it consists of an infinite chain of 3-spheres connected by thinner and thinner (and longer and longer) necks. Then for any given $\varepsilon > 0$, we can pick a neck that is sufficiently thin and long, then the Ricci flow will be inclined to pinch it within time ε.

Therefore to get the short time existence, one has to make some assumptions on the curvature of (M, g_0).

Shi proved the short time existence of the Ricci flow on noncompact manifolds in [13]. The general idea of Shi's proof is to use a family of compact sets D_k to exhaust the noncompact manifold M and consider the short time existence problem on each compact set D_k. Then he could get some uniform curvature estimates to obtain the short time existence on M.

Theorem 2.3.3 (Shi - Short time existence)**.** *Let* (M,g) *be an n-dimensional complete noncompact Riemannian manifold with Riemannian curvature tensor satisfying*

$$\left\|R_{ijkl}\right\|^2 \leq k_0$$

where k_0 *is a positive constant. Then there exists a constant* $T(n,k_0) > 0$ *such that the Ricci flow equation (2.1.1) has a smooth solution* $g(x,t)$ *for a short time* $0 \leq t \leq T(n,k_0)$.

Sketch of the proof.

Step 1 (Solving the Dirichlet boundary problem on a compact set)

Assume $D \subset M$ is a domain whose closure $\overline{D}$ is a compact subset of M. The boundary ∂D is a compact C^∞ $(n-1)$-dimensional submanifold of M. Consider the Ricci-DeTurck flow on this domain

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i \quad x \in D \\ g(x,0) &= g_0(x) \quad x \in D \\ g(x,t) &= g_0(x) \quad x \in \partial D,\ t \in [0,T] \end{aligned} \tag{2.3.6}$$

Again one can get the following existence theorem from the standard parabolic theory [12, Thm 7.1]:

Theorem 2.3.4. *There exists a constant* $T(n,k_0)$ *such that the Dirichlet boundary problem (3.2.5) has a unique solution* $g_{ij}(x,t)$ *for* $t \in [0,T(n,k_0)]$.

Step 2 (Local estimates) To get a solution on the whole manifold M by letting ∂D go to infinity on M, one needs to estimate $g_{ij}(x,t)$ locally. That means to control the derivatives of $g_{ij}(x,t)$ only in terms of $(g_0)_{ij}(x)$ and independent of D.

Choose a coordinate system $\{x_i\}$ such that at one point $((g_0)_{ij}) = I_n$ and $(g_{ij}) = diag\{\lambda_1, \cdots, \lambda_n\}$. Applying the maximal principle to the evolution equation of a well-crafted auxiliary function, one can obtain the upper bound for the derivative $\left\|\nabla_0 g_{ij}(x,t)\right\|_{g_0}^2 \leq C$. By induction one can obtain the upper bound for higher derivatives $\left\|\nabla_0^m g_{ij}(x,t)\right\|_{g_0}^2$.

Step 3 (Exhaustion) Fix a point $x_0 \in M$ and a family of domains $\{D_k\}_{k=1}^{\infty}$ such that

(1) ∂D_k is a compact C^∞ $(n-1)$-dim submanifold of M,

(2) $\overline{D_k} = D_k \cup \partial D_k$ is a compact subset of M,

(3) $B(x_0,k) \subset D_k$ where $B(x_0,k)$ is the geodesic ball of radius k with respect to the initial metric g_0.

The existence result in step 1 yields a unique solution to the Dirichlet boundary problem on D_k for $t \in [0, T(n,k_0)]$. The local estimates in step 2 show that the derivatives of $g_{ij}(k,x,t)$ are uniformly bounded on any compact subset of M.

Following the diagonal argument and passing to a subsequence one obtains the solution $g_{ij}(x,t)$ to the Ricci-DeTurck flow on M. And one can get a solution to the original Ricci flow as in DeTurck's proof for the compact case. □

2.3.4 Noncompact Case with Rough Initial Data - Simon's Existence Result

In the two cases above, the initial metrics are always smooth. To deal with the non-smooth initial data, Simon [14] considered a more general version of the Ricci-DeTurck flow. Recall that one fixes the background metric $\tilde{g} = g_0$. While Simon chose a smooth fixed background metric $\tilde{g} = h$ instead. Then define the 1-form as before

$$W_i = g_{ij} g^{pq} (\Gamma_{pq}^j - \tilde{\Gamma}_{pq}^j)$$

And the h-flow is

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i \\ \\ g(0) = g_0 \end{cases} \tag{2.3.7}$$

Though h is not g_0, one still needs to choose h which is close to the initial data g_0 to find a sensible solution to the h-flow.

Definition 2.3.1. *Let M be a complete manifold and g be a C^0 metric. Given a constant $\delta \in [1, \infty)$. A metric h is said to be δ-fair to g, if h is C^∞ and satisfies*

$$\sup_{p \in M} \|\mathrm{Rm}(h)(p)\|_h = k_0 < \infty$$

$$\frac{1}{\delta} h(p) \le g(p) \le \delta h(p) \ for\ all\ p \in M$$

Theorem 2.3.5 (Simon)**.** *Let g_0 be a complete metric on M and h a complete metric which is $(1+\varepsilon(n))$-fair to g_0. Then there exists $T(n, k_0) > 0$ and $g(t) \in C^\infty(M \times (0, T])$ which solves h-flow and h is $(1+2\varepsilon)$-fair to $g(\cdot, t)$ for $t \in (0, T]$. Further more, $\lim_{t \to 0} \sup_{x \in \Omega'} {}^h\|g(\cdot, t) - g_0(\cdot)\| = 0$ and*

$$\sup_{x \in M} \left\| {}^h\nabla^m g \right\|^2 \le \frac{C_m(n, k_0, \cdots, k_m)}{t^m} = 0, \quad m \in \mathbb{N}$$

where $\Omega' \subset\subset \Omega$ is any open set and Ω is any open set on which g_0 is continuous.

Sketch of the proof.

Let $\{{}^\alpha g_0\}_{\alpha \in \mathbb{N}}$ be a sequence of smooth metrics which converges to g_0 uniformly in C^0 norm. Then h is $(1 + \frac{\varepsilon}{2})$-fair to ${}^\alpha g_0$ for all $\alpha \ge N$ for some $N \in \mathbb{N}$.

Using Shi's result above, we can solve the Dirichlet boundary problem for each metric ${}^\alpha g_0$:

Theorem 2.3.6. *Let g_0 be a smooth metric on a compact domain $D \subset M$ and h a metric $(1+\varepsilon(n))$-fair to g_0 with $h|_{\partial D} = g_0|_{\partial D}$. Then there exists $S = S(n, k_0, \varepsilon) > 0$ and a family of metrics $g(t)$ $(t \in [0,S])$ which solve the h-flow, and h is $(1+2\varepsilon(n))$-fair to $g(t)$ for all $t \in [0,S]$ and $g|_{\partial D}(\cdot,t) = g_0(\cdot,t)$, $g(0) = g_0$.*

Then we can flow each metric ${}^{\alpha}g_0$ by h-flow and obtain a family of metrics ${}^{\alpha}g(\cdot,t)$, $t \in [0,T]$ which satisfy

$$\left\| {}^{h}\nabla^{j}\left({}^{\alpha}g(\cdot,t)\right)\right\|^{2} \leq \frac{C_j}{t^j},\ t \in (0,T].$$

Note that the $T = T(n,k_0)$ and C_j's are independent of α. By the Theorem of Arzela-Ascoli, we obtain a limit solution $g(x,t) = \lim_{\alpha\to\infty} {}^{\alpha}g(x,t)$ defined on $(0,T)$.

It remains to show that $g(\cdot,t)$ approaches $g_0(\cdot)$ uniformly as $t \to 0$ on Ω' as $t \to 0$.

First consider the special case when g_0 is smooth. Fix $x_0 \in \Omega'$ and a coordinate chart around x_0, $\phi : U \to M$, $x_0 \in U \subset\subset \Omega$. Define the $(0,2)$-tensor l by

$$l(V,W)(x) \doteqdot V_i(x)W_j(x)h_{qp}(x_0)g_0^{qi}(x_0)h^{pj}(x)$$

Note that $l^{ij}(x_0) = g_0^{ij}(x_0)$.

We can bound $\left\|g_0^{ij}(x) - l^{ij}(x)\right\|_h$ by continuity of g_0 and h. Using a well-chosen cut-off function and maximal principle we can bound $\left\|g^{ij}(x) - l^{ij}(x)\right\|_h$. As a result we can get that for any $\varepsilon > 0$, there exist $r > 0$ and $T > 0$ such that $\left\|g^{ij}(x) - g_0^{ij}(x)\right\|_h \leq \varepsilon$ for all $x \in B_h(x_0,r)$ and $t \leq T$.

Then we consider the general case when g_0 is only complete. Apply the argument above to each solution $\{{}^{\alpha}g(\cdot,t)\}$: for any $\varepsilon > 0$, there exist $r_\alpha > 0$ and $T > 0$ such that $\left\|{}^{\alpha}g^{ij}(x) - {}^{\alpha}g_0^{ij}(x)\right\|_h \leq \varepsilon$ for all $x \in B_h(x_0,r_\alpha)$ and $t \leq T_(n,h,U,r_\alpha,\varepsilon)$.

However, r_α is chosen so that $\left\| {}^{\alpha}g_0^{ij}(x) - {}^{\alpha}l^{ij}(x) \right\|_h \leq \frac{\varepsilon}{2}$ for all $x \in B_h(x_0, r_\alpha)$.

For any $x \in B_h(x_0, r_\alpha)$ and $\beta > \alpha$ we have

$$\left\| {}^{\beta}g_0^{ij}(x) - {}^{\beta}l^{ij}(x) \right\|_h \leq \left\| {}^{\beta}g_0^{ij}(x) - {}^{\alpha}g_0^{ij}(x) \right\|_h + \left\| {}^{\alpha}g_0^{ij}(x) - {}^{\alpha}l^{ij}(x) \right\|_h + \left\| {}^{\alpha}l^{ij}(x) - {}^{\beta}l^{ij}(x) \right\|_h \leq 3\varepsilon$$

As a result, if α and β are chosen large enough, we can choose r independent of α. Then by arguing similarly to Shi, we have the convergence as $t \to 0$.

□

2.4 Remark on Development of Singularity in Finite Time

Let $g(x,t)$ be a solution to the Ricci flow on $M \times [0,T)$ where $0 \leq T \leq \infty$. Suppose $[0,T)$ is the maximal time interval. If $T < \infty$, then the short time existence theorem tells us the curvature of the solution becomes unbounded as $t \to T$. We then say the solution develops a singularity as $t \to T$.

Theorem 2.4.1 (Hamilton)**.** *The Ricci flow equation*

$$\partial_t g_{ij} = -2R_{ij}$$

has a unique solution on a maximal time interval $0 \leq t < T \leq \infty$*. If* $T < \infty$ *then*

$$\max_M \left\| R_{ijkl} \right\| \to \infty \quad \text{as } t \to T.$$

Sketch of the proof. One can show that if $T < \infty$ and $\left\| R_{ijkl} \right\| \leq C$, then the metric g_{ij} converges as $t \to T$ to a limit metric, and all the derivatives converge also, showing the limit metric is smooth. Then the short time existence implies that T is not maximal.

□

Theorem 2.4.2 (Šešum)**.** *Let $g(t)$ be the solution to the Ricci flow equation*

$$\partial_t g_{ij} = -2R_{ij},\ t \in [0,T)$$

on a compact manifold M with $T \leq \infty$, and with uniformly bounded Ricci curvatures along the flow. Then the curvature tensor stays uniformly bounded along the flow.

In other words, Hamilton showed that if the curvature operator is uniformly bounded under the flow for all times $t \in [0,T)$, then the solution can be extended beyond time T. And Šešum [15] showed that if the Ricci curvature is uniformly bounded under the flow for all times $t \in [0,T)$, then the solution can be extended beyond time T. People then conjectured the same result for the bounded scalar curvature case and it is still open.

Understanding the development of finite-time singularities is an important goal in Ricci flow. In dimensions 2 and 3, finite-time singularities are reasonably well understood. In these dimensions, the maximum of the scalar curvature diverges at a singular time (see [4]) and the geometry of the singularity can be analyzed by a blow-up procedure. More specifically, after normalizing the scalar curvature at a sequence of base points via parabolic rescaling, the flow subsequentially converges to a smooth singularity model. In dimension 2, Hamilton and Chow showed (see [5] [6]) that the only such singularity models are the round sphere and projective space, which is equivalent to saying that the flow becomes asymptotically round at a finite-time singularity. In dimension 3, Perelman proved (see [8]) that the singularity models are κ-solutions, which he then classified in a qualitative way. This classification was the basis of the construction of Ricci flows with surgery, which led to the resolution of the Poincaré and Geometrization Conjectures (see also [9]).

Chapter 3

Long Time Behavior of Ricci Flow

3.1 Long Time Behavior of Ricci Flow from almost Einstein Metrics on Compact Manifolds

We introduce Ye's result [16] here. His purpose is to construct Einstein metrics through the Ricci flow under an Einstein or Ricci pinching condition. Roughly speaking, for a given stable metric g, if the L^2-norm of the traceless Ricci tensor $T \doteqdot Ric - \frac{R}{n}g$ is small relatively to suitable geometric quantities, then one can deform g to an Einstein metric through the Ricci flow. The concept "stability" is defined as follows.

Let M be a compact Riemannian manifold of dimension $n \geq 3$. For a given metric g on M, define the Einstein form Q to be the quadratic form on symmetric 2-tensors $h = h_{ij}dx^i dx^j$:

$$Q(h) = \int |\nabla h|^2 - 2\int \mathrm{Rm}(h) \cdot h + \int T(h) \cdot h \qquad (3.1.1)$$

where $\mathrm{Rm}(h)_{ij} = R_{ipjq}h^{pq}$ and $T(h)_{ij} = T_{ik}h^k_j$.

Definition 3.1.1. *The Einstein eigenvalues of g are the eigenvalues of the operator L acting on traceless symmetric 2-tensors:*

$$L(h) = -\Delta h - 2\mathrm{Rm}(h) + T(h)$$

Denote the minimum Einstein eigenvalue by λ_e. We say g is stable if $\lambda_e > 0$.

Remark 3.1.1. *The operator L is the Euler-Lagrange operator of Q.*

And $\lambda_e = \inf \frac{Q(h)}{\int |h|^2}$ over all traceless symmetric 2-tensors h.

Theorem 3.1.2 (Ye). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$ satisfying $d^2 \|\mathrm{Rm}\|_{C^0} \leq \Lambda$, $d^2\lambda_e \geq \sigma$ and the pinching condition either*

$$\fint |T|^2 \leq \varepsilon_1(n, \Lambda, \sigma)\frac{1}{d^4}$$

or

$$\fint |T|^2 \leq \varepsilon_2(n, \Lambda, \sigma)\, \|\mathrm{Rm}\|_{C^0}^2$$

for some positive numbers Λ, σ, where $d =$ diameter. Then g can be deformed to an Einstein metric through the Ricci flow. In particular, M supports Einstein metrics.

The approach here is to get C^0 estimates from L^2 estimates of curvature quantities by the weak maximum principle of Moser. And hence the convergence of the Ricci flow is reduced to establishing L^2 decay of the traceless Ricci tensor T which can be derived from stability. The most important part is to prove that the stability is preserved along the flow. And Ye used the following delicate argument to prove the stability and convergence at the same time.

For a given metric g on M and $A \geq 1$, the Sobolev A-constant $C_S^{(A)}$ of g is defined to be the smallest positive number for which the following Sobolev inequality holds:

$$\left(\int |f|^{2n/(n-2)} \right)^{(n-2)/n} \leq C_S^{(A)} \int |\nabla f|^2 + AV^{-2/n} \int f^2, \quad f \in C^\infty(M)$$

Lemma 3.1.3 (Moser's weak maximum principle)**.** *Let $g(t),\ t \in [0,T]$ be a smooth family of metrics, b a nonnegative constant, and f a nonnegative function on $M \times [0,T]$ satisfying*

$$\frac{\partial}{\partial t} f \leq \Delta f + bf \text{ on } M \times [0,T]$$

Then for any $(x,t) \in M \times (0,T]$ and $A \geq 1$,

$$|f| \leq c\left(b + l + \frac{1}{t}\right)^{1/2} \left(C_S^{(A)} \left(b + l + \frac{1}{t}\right) + AV^{-2/n} \right)^{n/4} e^{cbt} \|f_0\|_{L^2}$$

where $l = \max_t |\frac{\partial}{\partial t}(dv)/dv$, $C_S^{(A)} = \max_t C_S^{(A)}(g(t))$, $V = \min_t V(g(t))$ and $d = \max_t \operatorname{diam}(g(t))$.

Sketch of the proof. Step 1 (short time existence and a priori estimates)

Consider a Riemannian manifold (M,g) satisfying the assumptions and the second pinching condition. Dilating the metric so that $\|\mathrm{Rm}\|_{C^0} = 1$, $d \leq \sqrt{\Lambda}$ and $\lambda_e \geq \sigma/\Lambda$. It suffices to show that the dilated metric g_0 can be deformed to an Einstein metric through the normalized Ricci flow(2.2.1).

We have the following evolution equations by computation:

Lemma 3.1.4.

$$\begin{aligned} \frac{\partial}{\partial t}|T|^2 &= \Delta|T|^2 - 2|\nabla T|^2 + 4\mathrm{Rm}(T)\cdot T + \frac{4}{n}\delta R|T|^2, \\ \frac{\partial}{\partial t}\left(\int |T|^2 \right) &= -2\int |\nabla T|^2 + 4\int \mathrm{Rm}(T)\cdot T + \frac{4}{n}\int \delta R|T|^2. \end{aligned} \tag{3.1.2}$$

where $\delta R = R - \overline{R}$ and $\overline{R} = V^{-1}\int Rdg$ is the average scalar curvature.

Recall the short time existence result of Hamilton[4] and Shi[13]:

Lemma 3.1.5. *The normalized Ricci flow exists on the time interval* $[0,\tau(n)/\Lambda]$ *where* $\Lambda = \|\mathrm{Rm}\|_{C^0}$*. Moreover, the following estimates hold for* $t \in (0,\tau(n)/\Lambda]$*:*

$$\left\|\nabla^k \mathrm{Rm}\right\|_{C^0} \leq \frac{C(n,k)}{t^{k/2}}.$$

As a result, we have the following comparison between metrics g and g_0 when $t \in [0,\tau(n)]$:

$$\frac{1}{C_1} g_0 \leq g \leq C_1 g_0, \quad \|g - g_0\|_{C^0(M_0)} \leq C_2 t. \tag{3.1.3}$$

Step 2 (exponential decay)

Definition 3.1.2. *Let g be a metric. For any positive number* β *we define the* β*-value* λ_β *of g to be*

$$\lambda_\beta \doteqdot \inf\left\{ \frac{Q(h)}{\int |h|^2} \middle| h \text{ is a traceless symmetric } 2-\text{tensor with } \|h\|_{C^0}^2 \leq \frac{1}{\beta}\int |h|^2 \right\}.$$

Lemma 3.1.6. *Assume that the initial metric* g_0 *satisfies* $\lambda_e > 0$ *and* $\|\mathrm{Rm}\|_{C^0} \leq 1$*. Set* $\beta_0 \doteqdot \int_{M_0} |T|^2$*. Then there are numbers* $0 < a(n) < \min(1,\tau(n))$ *and* $c(n) > 1$ *such that* $\lambda_{\beta_0} \geq \gamma_0$ *for* $t \in [0,\tau_0]$*, where* $\tau_0 = \dfrac{a(n)\lambda_e^2}{(1+\lambda_e^2)}$ *and* $\gamma_0 = \dfrac{\lambda_e}{c(n)(1+\lambda_e)}$*.*

From the lemmas and (3.1.3) above, we have $\|\mathrm{Rm}\|_{C^0} \leq 2$, $\lambda_\beta \geq \gamma_0$ $(\beta = 4\beta_0)$ and $\mathrm{diam} M_t \leq C_3 d_0$ when $t \in [0,\tau_0]$. We also have $\int_{M_t} |T|^2 \leq C_4 \int_{M_0} |T|^2$. Together with the evolution equation we have $\int_{M_t} |T|^2 \leq C_5 e^{-\gamma_0/4} \int_{M_0} |T|^2$.

Step 3 (convergence)

We say that a time $\tau \geq \tau_0$ satisfies Condition $B(b_0,b_1,\gamma)$ for $b_0 \geq 1$, $b_1 > 0$ and $\gamma \in (0,\gamma_0/4]$, if the followings are true:

(1) $\lambda_\beta > \gamma_0/2$ on $[0,\tau]$, where $\beta = 4\beta_0 = 4\int_{M_0}|T|^2$;

(2) $\|\mathrm{Rm}\|_{C^0} < 10$ on $[0,\tau]$;

(3) $\int_{M_t}|T|^2 < b_0 e^{-\gamma t}\int_{M_0}|T|^2$ for $t \in [0,\tau]$;

(4) $\mathrm{diam}(M_t) < b_1 d_0$ for $t \in [0,\tau]$, where $d_0 = \mathrm{diam} M_0$.

Then the τ_0 from step 2 satisfies Condition $B(2C_5, 2\sqrt{C_3}, \gamma_0/4)$.

Checking carefully, we know that the estimates in Condition $B(b_0, b_1, \gamma)$ can produce a better Condtion $B(b_0', b_1', \gamma')$ along the flow.

Finally, by showing that Condition B is an open as well as closed condition, we obtain the convergence together with L^2 decay of traceless Ricci tensor. Take $t \to \infty$ and the limit metric is Einstein. □

3.2 Stability of Hyperbolic Space under Ricci Flow

3.2.1 Stability Result of Li and Yin

We introduce the result of H. Li and H. Yin [17] here. First they proved a noncompact version of Ye's result, that is, the metric can be deformed to an Einstein metric through the Ricci flow. Then with an additional assumption on the dimension, they proved the stability of the hyperbolic space under Ricci flow.

Let $\mathbb{H}^n$ be the hyperbolic space with constant sectional curvature -1. Denote the hyperbolic metric by $g_{\mathbb{H}}$.

Definition 3.2.1. *A metric g on $\mathbb{H}^n$ is said to be ε-hyperbolic of order δ for some $\varepsilon > 0$ and $\delta > 0$ if*

$$(1-\varepsilon)g_{\mathbb{H}} \leq g \leq (1+\varepsilon)g_{\mathbb{H}} \tag{3.2.1}$$

$$|e^{\delta d(x,x_0)}(K(x,\sigma)+1)| \leq \varepsilon \tag{3.2.2}$$

where $K(x,\sigma)$ is the sectional curvature of tangent plane σ at $x \in \mathbb{H}^n$ and $d(x,x_0)$ is the distance from x to some fixed point x_0 with respect to the metric g.

Theorem 3.2.1. *For any $n \geq 3$ and $\delta > 0$, there exists $\varepsilon > 0$ such that the normalized Ricci flow (2.2.2) starting from any ε-hyperbolic metric g of order δ on $\mathbb{H}^n$ exists for all time and converges exponentially fast to some Einstein metric.*

Theorem 3.2.2. *For any $n > 5$ and $\delta > 2$, there exists $\varepsilon > 0$ such that the normalized Ricci flow (2.2.2) starting from any ε-hyperbolic metric g of order δ on $\mathbb{H}^n$ exists for all time and converges exponentially fast to the hyperbolic metric $g_{\mathbb{H}}$.*

Proof of Thm.3.2.1. The idea is similar to Ye's. It's about the two intertwining facts: first, as long as the solution remains close to the hyperbolic space, the spectrum radius of $g(t)$ resembles those of the hyperbolic space; second, as long as the spectrum radius is bounded from below, the curvature quantity $|R_{ij}+(n-1)g_{ij}|$ from the right hand side of the normalized Ricci flow equation decays exponentially so that the solution remains close to the hyperbolic space.

Step 1 (Some basic analysis properties [18] about ε-hyperbolic metrics)

Lemma 3.2.3 (Lee)**.** *Let g be an ε-hyperbolic metric on $\mathbb{H}^n$. For a, $b \in \mathbb{R}$, $a > b$ and $a+b >$*

$n-1$, *there exists a constant* $C(n,a,b)$ *such that for all x, y* $\in \mathbb{H}^n$,

$$\int_{\mathbb{H}^n} e^{-ad(x,z)} e^{-bd(y,z)} dV_z \leq Ce^{-bd(x,y)}.$$

Lemma 3.2.4 (Lee)**.** *Let g be an* ε*-hyperbolic metric on* $\mathbb{H}^n$*. There exists a constant* $C(n,\varepsilon)$ *such that for any* $\lambda \leq \frac{(n-1)^2}{4} - C_1$*, the following inequality is true for function f with compact support*

$$\int_{\mathbb{H}^n} |\nabla f|^2 dV_g \geq \lambda \int_{\mathbb{H}^n} |f|^2 dV_g.$$

Lemma 3.2.5 (Lee)**.** *Let g be an* ε*-hyperbolic metric on* $\mathbb{H}^n$*. There exists a constant* $C(n,\varepsilon)$ *such that for any* $\lambda \leq \frac{(n-1)^2}{4} + 2 - C_2$*, the following inequality is true for traceless symmetric 2-tensor* ξ *with compact support*

$$\int_{\mathbb{H}^n} |\nabla \xi|^2 dV_g \geq \lambda \int_{\mathbb{H}^n} |\xi|^2 dV_g.$$

Step 2 (Pointwise estimate of $|R_{ij} + (n-1)g_{ij}|$)

For simplicity, set $h_{ij} \doteqdot R_{ij} + (n-1)g_{ij}$.

Assume that the solution $g(t)$ is ε-hyperbolic when $t \in [0,T]$ where $T > \eta$ for some positive constant η.

We compute the evolution equations for h_{ij} and $|h_{ij}|^2$:

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} - 2R_{ipjq}h_{pq} - 2h_{ip}h_{pj} \\ \frac{\partial}{\partial t}|h_{ij}|^2 &= \Delta |h_{ij}|^2 - 2|h_{ij,k}|^2_{pq} - 4R_{ipjq}h_{ij}h_{pq} - 4h_{ip}h_{pj}h_{ij} \end{aligned} \tag{3.2.3}$$

Since $g(t)$ is ε-hyperbolic, we have

$$\frac{\partial}{\partial t}|h_{ij}|^2 \leq \Delta |h_{ij}|^2 + c|h_{ij}|^2$$

Consider the parabolic ball $B_0(x,\sqrt{\eta}) \times [t-\eta, t]$, we have uniform Sobolev inequality since $g(t)$ is ε-hyperbolic. The standard Moser iteration gives

$$|h_{ij}|^2 \le C(\eta) \int_{t-\eta/2}^{t} \int_{B_0(x,\sqrt{\eta}/2)} |h_{ij}|^2(y,s) dyds$$

However, the quantity $|h_{ij}|^2$ may not be integrable on the noncompact M. Thus we consider an auxiliary function as follows.

We have the following lemma from computation and the fact that $g(t)$ is ε-hyperbolic.

Lemma 3.2.6. *There exists a constant $C_1(\varepsilon)$ such that*

$$\xi(y,s) \doteq= -\frac{d_0^2(y)}{(2+C_3)(t-s)}$$

satisfies

$$\xi_s + \frac{1}{2}|\nabla \xi|^2 \le 0$$

where $d_0(y)$ is a distance function with respect to $g(0)$ and the norm and ∇ are with repect to $g(s)$.

Let $d_0(y)$ be the distance to $B_0(x,\sqrt{\eta}/2)$ with respect to $g(0)$. Define

$$I(s) \doteq \int_{\mathbb{H}^n} e^{\xi} |h_{ij}|^2(y,s)\, dyds.$$

By calculating carefully (where we have to consider the trace and traceless part of $e^{\xi/2}h_{ij}$ separately), we have

$$\frac{dI}{ds} \le -2\lambda I(s) + C_4(\varepsilon) I(s).$$

By Grönwall's inequality, we have

$$I(s) \le e^{-(2\lambda - C_4)t} I(0)$$

Since $\xi(y,s) \equiv 0$ for $y \in B_0(x, \sqrt{\eta}/2)$, we have

$$|h_{ij}|^2(x,t) \le C(\eta) e^{-(2\lambda - C_4)t} I(0)$$

As a result, we have the pointwise estimate

$$\begin{aligned} |h_{ij}|^2(x,t) &\le C(\eta) \int_{\mathbb{H}^n} |h_{ij}|^2(y,0) \exp\Big(-\frac{d_0^2(y)}{2+C_3(\varepsilon)} - (2\lambda - C_4(\varepsilon))t \Big) dy \\ &\le C(\eta) \exp(-C_5(\varepsilon)t) \int_{\mathbb{H}^n} |h_{ij}|^2(y,0) \exp\Big(-(2\sqrt{\lambda} - C_6(\varepsilon))t \Big) dy \end{aligned}$$

By triangle inequality, $d_0(y) > d_0(y,x) - \sqrt{\eta}/2$. Thus we can replace $d_0(y)$ with $d_0(y,x)$. Integrating over time from η to T and we have

$$\int_\eta^T |h_{ij}|(x,t) \le C(\eta,\varepsilon) \Big(\int_{\mathbb{H}^n} |h_{ij}|^2(y,0) \exp\Big(-(2\sqrt{\lambda} - C_6(\varepsilon))t \Big) dy \Big)^{1/2} \tag{3.2.4}$$

Step 3 (exponential decay)

Now consider the initial metric which is ε-hyperbolic of order δ. Given this δ, choose $\varepsilon_1 > 0$ such that

$$2\sqrt{\frac{(n-1)^2}{4} - max\{C_1(\varepsilon), C_2(\varepsilon)\}} - C_6(\varepsilon) + 2\delta > n-1.$$

Let $g(0)$ be any $\frac{1}{10}\varepsilon_1$-hyperbolic initial metric. By continuity and Shi's short time existence result, there exists some $\tau > 0$ such that for any $t \in [0,\tau]$, $g(t)$ is $\frac{1}{2}\varepsilon_1$-hyperbolic. Let T be the maximum time such that $g(t)$ remains ε_1-hyperbolic. Again by the short time existence result of Shi's, there exist uniform bounds $C(k,\tau)$ for k-th derivatives of curvature tensor of $g(t)$ when $t \in [\tau/2, T]$.

Since $g(0)$ is ε-hyperbolic of order δ, we have

$$|h_{ij}(y,0)| \le C\varepsilon e^{-\delta d_0(y,x_0)}.$$

Now apply (3.2.4) to the case $\eta = \tau/2$, we have

$$\int_{\tau/2}^{T} |h_{ij}|(x,t) \le C(\varepsilon_1)\varepsilon\Big(\int_{\mathbb{H}^n} \exp(-2\delta d_0(y,x_0))\exp\Big(-(2\sqrt{\lambda}-C_6(\varepsilon)t\Big)dy\Big)^{1/2}$$

By the choice of ε_1 above, set $\lambda \doteq \frac{(n-1)^2}{4} - max\{C_1(\varepsilon), C_2(\varepsilon)\}$ and the integral in the right hand side above is finite. Hence

$$\int_{\tau/2}^{T} |h_{ij}|(x,t) \le C\varepsilon.$$

Since $g(\tau/2)$ is $\frac{1}{2}\varepsilon_1$-hyperbolic, we can choose ε small enough so that $g(t)$ remains $\frac{3}{4}\varepsilon_1$-hyperbolic when $t \in [\tau/2, T]$.

Step 4 (long time existence and convergence)

Consider the parabolic ball $B_0(x,\sqrt{\tau/2}) \times [t-\tau/2, t]$, since $g(t)$ is ε_1-hyperbolic and the derivatives of the curvature tensor are uniformly bounded for $t \in [\tau/2, T]$, one can choose a local coodinate system on $B_0(x,\sqrt{\tau/2})$, for example the harmonic coordinates with respect to $g(t-\tau/2)$, such that $g_{ij}(t)$ with its derivatives are bounded in the parabolic ball. Then the standard parabolic interior estimates yield that

$$\Big|\frac{\partial h_{ij}}{\partial x_k}\Big|(x,t),\ \Big|\frac{\partial^2 h_{ij}}{\partial x_k \partial x_l}\Big|(x,t) \le C \max_{B_0(x,\sqrt{\tau/2})\times[t-\tau/2,t]} \sum_{p,q} |h_{pq}|.$$

Consider the quantity $|R_{ipjq} - (g_{ij}g_{pq} - g_{iq}g_{jp})|$.

$$\begin{aligned}\frac{\partial}{\partial t}|R_{ipjq} - (g_{ij}g_{pq} - g_{iq}g_{jp})| &\le C\Big(\Big|\frac{\partial}{\partial t}R_{ipjq}\Big| + \Big|\frac{\partial}{\partial t}g_{ij}\Big|\Big)\\ &\le C \max_{B_0(x,\sqrt{\tau/2})\times[t-\tau/2,t]} \sum_{p,q} |h_{pq}|.\end{aligned}$$

Therefore

$$\int_{\tau}^{T} \frac{\partial}{\partial t}|R_{ipjq} - (g_{ij}g_{pq} - g_{iq}g_{jp})|(x,s)\, ds \le C(\varepsilon_1)\varepsilon.$$

Then $|K_{g(t)}(x,\sigma)+1| \le \frac{3}{4}\varepsilon_1$ if we choose ε small enough. Hence $T=\infty$ and the solution remains ε_1-hyperbolic. Take the limit as $t\to\infty$ and the limit metric is Einstein. □

Proof of Thm.3.2.2. Here is a result of Shi and Tian [19] on the rigidity of hyperbolic space.

Theorem 3.2.7. *Suppose that* (X^n, g)*,* $n \ge 3$ *and* $n \ne 4$ *is an ALH manifold of order* $\alpha(\alpha > 2)$*,* $K \le 0$ *and* $\mathrm{Rc}(g) \ge -(n-1)g$*, then* (X^n, g) *is isometric to* $(\mathbb{H}^n, g_{\mathbb{H}^n})$*.*

A complete noncompact Riemannian manifold is called by Shi and Tian an ALH manifold of order α if $|K(x,\sigma)+1| = O(e^{-\alpha d_g(x,p)})$ for some fixed point p.

Now suppose $n > 5$ and $\delta > 2$. By Thm. 3.2.1, there exists some $\varepsilon > 0$ such that the normalized Ricci flow from an ε-hyperbolic of order δ converges to an Einstein metric g_∞. And from the proof, we can see that the solution $g(t)$ remains ε_1-hyperbolic. Hence the curvature conditions $K_{g_\infty} \le 0$ and $\mathrm{Rc}g_\infty \ge -(n-1)g_\infty$ are satisfied. We consider $|R_{ipjq} - (g_{ij}g_{pq} - g_{iq}g_{jp})|$ again, by a similar argument to that in the step 4 above, we can get that $|K_{g_\infty}(x,\sigma)+1| = O(e^{-\alpha d_{g_\infty}(x,p)})$ for some $\alpha > 2$. Then the rigidity theorem of Shi and Tian implies that $g_\infty = g_{\mathbb{H}}$.

□

3.2.2 Stability Result of Schnürer, Schulze and Simon

We introduce the result of O. C. Schnürer, F. Schulze and M. Simon [20] here. They also proved the stability of the hyperbolic space under the Ricci flow, with assumptions on the initial metric different from those of Li and Yin.

Definition 3.2.2. *A metric* g *on* $\mathbb{H}^n$ *is said to be* ε*-close to the standard hyperbolic metric* h *if*

$$(1+\varepsilon)^{-1}h \le g \le (1+\varepsilon)h$$

Theorem 3.2.8. *For any $n \geq 4$ and $K > 0$, there exists $\varepsilon > 0$ such that the following holds. Let g_0 be a C^0 metric on $\mathbb{H}^n$ satisfying*

$$\begin{aligned} \int_{\mathbb{H}^n} |g_0 - h|^2 dV_g &\leq K \\ \sup_{\mathbb{H}^n} |g_0 - h| &\leq \varepsilon \end{aligned}$$

Then the normalized Ricci-DeTurck flow exists for all time and the solution $g(t)$ satisfies

$$\sup_{\mathbb{H}^n} |g(t) - h| \leq C(n,K) e^{-\frac{1}{4(n+2)}t}.$$

Moreover, $g(t)$ converges to h exponentially in C^k as $t \to \infty$ for all $k \in \mathbb{N}$.

Sketch of the proof. Step 1 (short time existence)

Using the same techniques as in [13] and [14] we have the short time existence.

Step 2 (exponential decay)

To prove the long time existence and convergence, the key ingredient is to show that the L^2-norm of $g(t) - h$ is decaying exponentially in time.

Similar to the proof of Li and Yin[17], use the analysis property that the first eigenvalue is close to $\frac{(n-1)^2}{4}$ and calculate the evolution equation of the L^2-norm of $g(t) - h$ on a compact domain, we can get

Theorem 3.2.9. *Let $n \geq 4$. There exists $\delta_0 = \delta_0(n)$ such that the following holds. Let $g(t)$ be a smooth solution to the Dirichlet boundary problem*

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} - 2(n-1)g_{ij} + \nabla_i W_j + \nabla_j W_i, \quad x \in B_R(0) \\ g(x,t) &= h(x), \quad x \in \partial B_R(0),\ t \in [0,T) \end{aligned} \tag{3.2.5}$$

satisfying $\sup_{B_R(0)\times[0,T)} |g - h| \leq \delta_0$. *Then*

$$\int_{B_R(0)} |g(t) - h|^2 dV_h \leq e^{-\alpha t} \int_{B_R(0)} |g(0) - h|^2 dV_h$$

for $\alpha = \alpha(n) = [2(n-1)^2 - 17]/4 \geq \frac{1}{4}$.

Let $R \to \infty$ and we have

Corollary 3.2.10. *Let* $n \geq 4$ *and* $T > 0$*. Assume* g_0 *is a smooth metric on* $\mathbb{H}^n$ *satisfying* $\|g_0 - h\|_{L^2(\mathbb{H}^n)} < \infty$*. There exists* $\varepsilon_0 = \varepsilon_0(n,T)$ *such that the following holds. If* $\sup_{\mathbb{H}^n} |g_0 - h| \leq \varepsilon_0$*, then there exists a solution* $g(t)$ *to the Dirichlet boundary problem (3.2.5) satisfying* $\sup_{\mathbb{H}^n \times [0,T)} |g - h| \leq \delta_0$*, where* δ_0 *is as in Thm 3.2.9. Furthermore,*

$$\|g(t) - h\|_{L^2(\mathbb{H}^n)} \leq e^{-\alpha t} \|g_0 - h\|_{L^2(\mathbb{H}^n)}$$

for all $t \in [0,T)$*, where* $\alpha = \alpha(n) \geq \frac{1}{4}$ *as in Thm 3.2.9.*

Using the interior gradient estimate, we can see that the exponential convergence of the L^2 norm of $|g - h|$ also implies exponential convergence of its pointwise sup-norm.

Theorem 3.2.11. *Let* $n \geq 4$*. Assume* $g(t)$ $(t \in [0,T))$ *is a solution to the Dirichlet boundary problem (3.2.5) with* $\|g_0 - h\|_{L^2(\mathbb{H}^n)} \doteqdot K < \infty$*,* $\sup_{\mathbb{H}^n \times [0,T)} |g - h| \leq \delta_0$ *and*

$$\|g(t) - h\|_{L^2(\mathbb{H}^n)} \leq e^{-\alpha t} \|g_0 - h\|_{L^2(\mathbb{H}^n)}$$

where δ_0 *and* $\alpha(n)$ *are as in Thm 3.2.9. Then*

$$\sup_{\mathbb{H}^n} |g(t) - h| \leq e^{-\beta t} \sup_{\mathbb{H}^n} |g_0 - h|$$

where $\beta = \frac{\alpha}{n+2} > 0$.

Step 3 (long time existence and convergence)

The short time existence in Step 1 and a priori estimates in Step 2 together imply the long time existence. Furthermore, by interpolation, the exponential decay extends to higher derivatives of the solution metric as follows.

Theorem 3.2.12. *Let $n \geq 4$. Let g_0 and $g(t)$ be as in Thm 3.2.11. Then*

$$\sup_{\mathbb{H}^n} \left| {}^h\nabla^j g(t) \right| \leq C(n,K,j) e^{-\beta_j t}$$

where $0 < \beta_j < \beta(n)$, $\beta(n)$ as in Thm 3.2.11.

Since the decay as $t \to \infty$ above does not depend on the smoothness of g_0, we can approximate g_0 using smooth metrics and pass to the limit to obtain the result in Thm 3.2.8.

□

3.2.3 Stability Result of Bamler

We state the result of R. Bamler [28] here. He proved that every finite volume hyperbolic manifold of dimension greater or equal to 3 is stable under renomalized Ricci flow, i.e. that every small perturbation of the finite volume hyperbolic metric flows back to the hyperbolic metric.

Theorem 3.2.13. *For any complete hyperbolic manifold $(M^n, \bar{g})$ of finite volume and dimension $n \geq 3$, there is an $\varepsilon > 0$ such that the following holds: If g_0 is another smooth metric on M with $(1-\varepsilon)g \leq g_0 \leq (1+\varepsilon)g$, then there is a solution $(g_t)_{t\in[0,\infty)}$ to the renormalized Ricci flow equation*

$$\frac{\partial}{\partial t} g = 2(\mathrm{Rc}_g + (n-1)g)$$

starting from g_0 which exists for all time, and as $t \to \infty$ we have convergence $g(t) \to g$ in the pointed smooth Cheeger-Gromov sense, i.e. there is a family of diffeomorphisms Ψ_t of M such *that* $\Psi_t^* g(t) \to \bar{g}$ *in the smooth sense on every compact subset of* M*. Moreover,* ε *can be chosen so that it only depends on an upper volume bound on* M *for* $n \geq 4$ *resp. an upper diameter bound on the compact part* M_{cpt} *of* M *for* $n = 3$*.*

3.2.4 Comparison

All the results above are on the stability of some special metrics under the normalized Ricci flow, where Ye considered the compact case, while Li-Yin, Schnürer-Schulze-Simon and Bamler considered noncompact cases.

Due to the similarity of the problems, the general ideas of all the proofs are similar - to obtain exponential decay from the evolution equations of certain geometric quantities in order to get the long time existence and convergence.

However there are significant differences too. The quantities Ye and Li-Yin considered are from the curvature, while the ones considered by Schnürer-Schulze-Simon and Bamler are from the metric.

As a result, the assumptions of the theorems are also different. Besides the closeness of the initial metric to the special one (Einstein metric in the compact case, and hyperbolic metric in the noncompact cases), Ye and Li-Yin required decay conditions on the curvature, while Schnürer-Schulze-Simon and Bamler required L^2-boundedness of $|g_0 - h|$.

3.3 Ricci Flow Approach to the Existence of AHE Metrics

We introduce the result of J. Qing, Y. Shi and J. Wu [21] here. They investigated the behavior of the normalized Ricci flow on aymptotically hyperbolic manifolds and proved that the flow exists globally and converges to an Einstein metric when starting from a non-degenerate and Ricci pinched metric.

Furthermore, they established the regularity of the conformal compactness along the normalized Ricci flow including that of the limit metric at time infinity. Therefore they recovered the existence results in [22] [18] [23] of conformally compact Einstein metrics with conformal infinities which are perturbations of that of given non-degenerate conformally compact Einstein metrics.

3.3.1 NRF on Non-degenerate and Ricci-pinched Metrics

Definition 3.3.1. *A metric g on a manifold M^n is ε-Einstein of order γ if*

$$\|h_g\|(x) \leq \varepsilon e^{-\gamma d(x,x_0)}$$

where $h_g \doteqdot \mathrm{Rc}_g + (n-1)g$ is the Ricci pinching curvature, and $d(x,x_0)$ is the distance from x to some fixed point x_0 with respect to the metric g.

Definition 3.3.2. *The non-degeneracy of a metric g on M^n is defined to be the first L^2 eigenvalue of the linearization of the curvature tensor h as follows*

$$\lambda \doteqdot \inf \frac{\int_M \langle (\Delta_L + 2(n-1))u_{ij}, u_{ij}\rangle}{\int_M \|u\|^2}$$

where the infimum is taken among symmetric 2-tensors u such that $\int_M (|\nabla u|^2 + |u|^2) < \infty$ *and* Δ_L *is the Lichnerowicz Laplacian on symmetric 2-tensors.*

Theorem 3.3.1. *For any* $n \geq 3$ *and positive constants* k_0, k_1, v_0, λ_0 *and* α*. let* $\gamma > \frac{1}{2}\alpha - \sqrt{\lambda_0}$*. Then there exists* $\varepsilon = \varepsilon(n, k_0, k_1, v_0, \lambda_0, \alpha, C_0, \gamma) > 0$ *such that the normalized Ricci flow starting from a metric* g_0 *exists for all time and converges exponentially to an Einstein metric, provided that*

(1) $\|\mathrm{Rm}_{g_0}\| \leq k_0$, $\|\nabla \mathrm{Rm}_{g_0}\| \leq k_1$,

(2) *the volume bound* $\mathrm{vol}(B_{g_0}(x,1)) \geq v_0$ *for all* $x \in M^n$,

(3) g_0 *is with non-degeneracy* λ_0,

(4) $\int_{M^n} \exp(-\alpha d(x, x_0)) dv_{g_0} < C_0$ *where* C_0 *is independent of* x_0,

(5) g_0 *is* ε*-Einstein of order* γ.

Sketch of the proof.

Step 1 (short time existence)

The short time existence and some curvature estimates can be proved as in [13]:

Lemma 3.3.2. *Let* g_0 *be a Riemannian metric on* M^n *satisfying* $\|\mathrm{Rm}\|_{C^0(M)} \leq k_0$ *and* $\|\nabla \mathrm{Rm}\|_{C^0(M)} \leq k_1$*. Let* $g(t)$ $(t \in [0, T])$ *be the solution to the normalized Ricci flow (2.2.2) obtained in Thm 1.1 in [13]. Then the following estimates hold for any* $t \in (0, T]$*:*

$$\begin{aligned} \|\nabla \mathrm{Rm}\|_{C^0(M)}(t) &\leq C_1 \\ \left\|\nabla^2 \mathrm{Rm}\right\|_{C^0(M)}(t) &\leq \frac{C_2}{\sqrt{t}} \end{aligned}$$

where $C_i = C_i(n, k_0, k_1, T)$ $(i = 1, 2)$ *are constants.*

Step 2 (exponential decay of $\|h\|$)

The strategy here is still to obtain L^2 decay estimates by the heat kernel estimates of Grigor'yan [24] considered in Li-Yin [17]. And then the C^0 decay estimates follow from Moser iteration.

Step 3 (long time existence and convergence)

The approach here is similar to Ye's [16]. First show that NRF starting from a non-degenerate metric with a sufficiently pinched Ricci curvature remains so. Then obtain the long time existence and convergence from the exponential decay from Step 2. The key here is to investigate how the curvature bound, the volume $\mathrm{vol}(B(x,1))$ lower bound and the non-degeneracy evolve along NRF.

□

3.3.2 NRF on AHE Manifolds

Using the theorem above, consider the normalized Ricci flow on asymptotically hyperbolic manifolds.

Definition 3.3.3. *Let M^n be a smooth manifold with boundary ∂M^{n-1}. A defining function of the boundary is a smooth function $x : \overline{M} \to R^+$ such that*

1) $x > 0$ in M;   2) $x = 0$ on ∂M;   3) $dx \neq 0$ on ∂M.

A metric g on M is said to be conformally compact of regularity $C^{k,\alpha}$ if $x^2 g$ is a $C^{k,\alpha}$ metric on $\overline{M}$. The metric $\bar{g} = x^2 g$ induces a metric $\hat{g}$ on the boundary ∂M. And the metric g induces a conformal class of metrics $[\hat{g}]$ on the boundary ∂M when defining functions vary.

The conformal manifold $(\partial M, [\hat{g}])$ is called the conformal infinity of the conformally compact manifold (M, g).

Definition 3.3.4. (M,g) *is said to be asymptotically hyperbolic (AH) if it is conformally compact and the sectional curvature of g tends to* -1 *when approaching the boundary at infinity.*

Remark 3.3.3. *Let* (M,g) *be an AH manifold. For any representative* $\hat{g} \in [\hat{g}]$*, there exists a unique so-called geodesic defining fucntion r such that there is a coordinate neighborhood of the infinity* $(0,r_0)\times\partial M \subset M$ *where the metric g is in the following normal form*

$$g = r^2(dr^2 + g_r)$$

where g_r *is a one-parameter family of metrics on* ∂M *and* $(g_r)|_{r=0} = \hat{g}$*.*

Our goal is to consider Theorem 3.3.1 in the context of asymptotically hyperbolic manifolds. For that purpose we need a basic property of asymptotically hyperbolic manifolds using the boundary Möbius charts introduced in [18].

Lemma 3.3.4. *Let* (M^n,g) *be an AH manifold. Then*

$$\int_M \exp(-\alpha\, d(x,x_0))dV_g \le C$$

for any constant $\alpha > n-1$*, where C is independent of* x_0*.*

In [25], Ecker and Huisken adapted a method from [26] and extended the maximum principle on noncompact manifolds in [26] to allowing the metrics to be time dependent. For our purpose we will need a variant of Theorem 4.3 in [25].

Lemma 3.3.5. *Let* $g(t)$ *be a smooth family of complete Riemannian metrics on the manifold* M^n *with boundary* ∂M *for* $t \in [0,T]$*. Let u be a function on* $M\times[0,T]$ *which is smooth on* $\mathring{M}\times(0,T]$ *and continuous on* $M\times[0,T]$*. Assume that* $g(t)$ *and u satisfy* *(i)* $\frac{\partial}{\partial t}u - \Delta_{g(t)}u \leq \boldsymbol{a}\cdot\nabla u + bu$ *with* $\sup\limits_{M\times[0,T]} |\boldsymbol{a}|, |b| \leq \alpha_1$ *for some* $\alpha_1 < \infty$,

(ii) $\sup\limits_{M} u(x,0) \leq 0$ *and* $\sup\limits_{\partial M\times[0,T]} u(x,t) \leq 0$,

(iii) $\int_0^T \int_M \exp[-\alpha_2 d^t(y,p)]^2 u_+^2(y)dv_t(y)dt < \infty$ *for some positive* α_2 *where* $u_+ = \max u, 0$,

(iv) $\sup\limits_{M\times[0,T]} \left|\frac{\partial}{\partial t}g(x,t)\right| \leq \alpha_3$ *for some* $\alpha_3 < \infty$.

Then $u \leq 0$ *on* $M\times[0,T]$.

Applying the maximal principle above and we have

Theorem 3.3.6. *For any* $n \geq 5$ *and positive constants* k_0, k_1, v_0 *and* λ_0, *let*

$$\gamma \in \left(\frac{n-1}{2} - \min\left\{\sqrt{\lambda_0},\ \sqrt{\frac{(n-1)^2}{4} - 2}\right\}, \frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4} - 2}\right).$$

Suppose that (M^n, g_0) *is an asymptotically hyperbolic manifold of* C^2 *regularity, where* g_0 *satisfies the conditions* $B(k_0, v_0, \lambda_0)$ *and* $\|\nabla \mathrm{Rm}\| \leq k_1$. *Then there exists a positive number* $\delta = \delta(n, k_0, k_1, v_0, \lambda_0) > 0$ *such that NRF on* M^n *starting from* g_0 *exists for all time if the initial metric* g_0 *is* δ*-Einstein of order* γ *and* $\|\nabla h\| \leq Cr^\gamma$. *Moreover NRF converges exponentially to a non-degenerate Einstein metric* g_∞ *in* C^∞ *as* $t \to \infty$, *where* g_∞ *is* C^2*-conformally compact with the same conformal infinity as the initial metric* g_0 *if* $\gamma > 2$.

3.3.3 Perturbation Existence of AHE Metrics

We apply Theorem 3.3.6 above to recapture some results in [22] [18] [23]. Our approach is to construct a good initial metric nearby a given non-degenerate conformally compact Einstein metric The construction is based on the metric expansions of Fefferman and Graham [27]. We glue together the given metric away from the boundary and the metric on collar neighborhood of the boundary from the expansion of Fefferman and Graham. The issues are to make sure the glued metrics satisfy the assumptions on the initial metric in Theorem 3.3.6.

Theorem 3.3.7 (Fefferman - Graham)**.** *Let (M^n, g) be a conformally compact Einstein manifold with the conformal infinity $(\partial M, [\hat{g}]$. Let r be the geodesic defining function associated with the conformal representative $\hat{g} \in [\hat{g}]$ on ∂M. Then the metric expansion is given as follows*

$$\begin{aligned} g_r &= \hat{g} + g^{(2)} r^2 + \cdots + g^{(n-3)} r^{n-3} + h r^{n-1} \log r + g^{(n-1)} r^{n-1} + \cdots \\ &= \hat{g} + g^{(2)} r^2 + \cdots + g^{(k)} r^k + t^{(k)}[g] \end{aligned}$$

for $0 \le k \le n-3$ when $n-1$ is even,

$$\begin{aligned} g_r &= \hat{g} + g^{(2)} r^2 + \cdots + g^{(n-2)} r^{n-2} + g^{(n-1)} r^{n-1} + \cdots \\ &= \hat{g} + g^{(2)} r^2 + \cdots + g^{(k)} r^k + t^{(k)}[g] \end{aligned}$$

for $0 \le k \le n-2$ when $n-1$ is odd, where

$g^{(2i)}$ for $2i < n-1$ are local invariants of $(\partial M^{n-1}, \hat{g})$,

h and $\operatorname{tr} g^{(n-1)}$ (when $n-1$ is even) are local invariants of $(\partial M^{n-1}, \hat{g})$,

h and $g^{(n-1)}$ (when $n-1$ is odd) are trace-free,

$g^{(n-1)}$ (when $n-1$ is odd) and the trace-free part of $g^{(n-1)}$ (when $n-1$ is even) are nonlocal.

To construct a suitable initial metric whose conformal infinity is a perturbation of that of a given conformally compact Einstein metric g, set

$$g_r^{k,\nu} \doteqdot \hat{g}_\nu + g_\nu^{(2)} r^2 + \cdots + g_\nu^{(k)} r^k + t^{(k)}[g]$$

where g_ν is a perturbation of $\hat{g}$ and $g_\nu^{(2i)}$ $(2i \le k)$ are the corresponding curvature terms of $\hat{g}_\nu$ as given in the metric expansion above. Let ϕ be a cut-off function of the variable r such that

$\phi = 0$ when $r \geq \nu_2$ and $\phi = 1$ when $r \leq \nu_1$, where $\nu_1 < \nu_2$ are chosen later. Now consider the following initial metric

$$g^{\phi}_{k,\nu} \doteq r^{-2}(dr^2 + (1-\phi)g_r + \phi g_r^{k,\nu}).$$

From calculation we get that the metric above satisfy the conditions in Theorem 3.3.6. Thus we recover some of the results in [18] [23]:

Theorem 3.3.8. *Let (M^n, g) be a conformally compact Einstein manifold of regularity C^2 with a smooth conformal infinity $(\partial M, [\hat{g}])$. Assume that g is of the non-degeneracy λ_0. Suppose that* $\max\{2, \frac{n-1}{2} - \sqrt{\lambda_0}\} < k+2$ *for some even $k \leq n-1$. Then*

given any smooth metric $[\hat{g}_\nu]$ which is a sufficiently small C^{k+2}-perturbation of $[\hat{g}]$, there exists a C^2-conformally compact Einstein metric on M whose conformal infinity is $[\hat{g}_\nu]$.

Chapter 4

Stability of AHE Manifolds with Rough Initial Data

4.1 Introduction/Main Result

As in Chapter 3, Li-Yin, Schnürer-Schulze-Simon and Bamler proved the stability of the hyperbolic metric with different assumptions. Li-Yin required some decay condition on the curvature, while Schnürer-Schulze-Simon and Bamler required L^2-boundedness of the difference between the initial metric and the hyperbolic metric.

Here we extend the stability result to asymptotically hyperbolic Einstein manifolds. Our method is based on the method of Schnürer-Schulze-Simon. However, we need to be more careful because we are not able to take advantage of the properties of the standard hyperbolic metric any more. Our main result is

Theorem 4.1.1. *For any $n \geq 3$ and positive constants k_0, k_1, v_0 and $\lambda_0 > 2$. Let h_0 be a complete AHE metric on M^n satisfying*

(1) $\|\mathrm{Rm}_{h_0}\| \leq k_0$, $\|\nabla \mathrm{Rm}_{h_0}\| \leq k_1$.

(2) *the volume bound* $\mathrm{vol}(B_{h_0}(x,1)) \geq v_0$ *for all* $x \in M^n$,

(3) h_0 *is with non-degeneracy* λ_0.

Then for any $K > 0$, there exists $\varepsilon = \varepsilon(n, k_0, k_1, v_0, \lambda_0, K) > 0$ such that the normalized Ricci flow starting from any complete metric g_0 exists for all time and converges exponentially to h_0, provided that

(1) $(1-\varepsilon)h_0 \leq g_0 \leq (1+\varepsilon)h_0$,

(1) $\int_{M^n} |g_0 - h_0|^2 dV_g \leq K$,

4.2 Preliminaries

We prove the following property of asymptotically hyperbolic manifolds using the boundary Möbius charts.

Lemma 4.2.1. *Let (M^n, g) be an AH manifold. Then*

$$\int_M \exp(-\alpha\, d(x, x_0)) dV_g \leq C$$

for any constant $\alpha > n-1$, where C is independent of x_0.

Let (M, g) be a conformally compact manifold of regularity $C^{k,\beta}$ and r be its geodesic defining function. For any $\varepsilon > 0$, let A_ε denote the open subset where $0 < r < \varepsilon$. Choose a covering of a neighborhood of ∂M in $\overline{M}$ by finitely many smooth coordinate charts (Ω, Θ), where each coordinate map $\Theta = (r, \theta) = (r, \theta^1, \cdots, \theta^n)$ and extends to a neighborhood of $\overline{\Omega}$ in $\overline{M}$.

Fix finitely many such charts covering a neighborhood W of ∂M in M and call these charts background coordinates. By compactness, there exists a positive number c such that $A_c \subset W$ and each point $p \in A_c$ is contained in a background coordinate chart containing a set of the form $\{(r, \theta) : |\theta - \theta(p)| < c, 0 \leq r < c\}$.

Definition 4.2.1. *For any point $p_0 \in A_{c/8}$, choose such a background chart containing p_0, and define a Möbius chart centered at p_0 to be a map $\Phi_{p_0} : B_2 \to M$ with $(r, \theta) = \Phi_{p_0}(x, y) = (\theta_0 + r_0 x, r_0 y)$ where B_2 is the ball with radius 2 centered at $(0, 1) \in \mathbb{H}$ (upper half space model) and (r_0, θ_0) are the background coordinates of p_0. Note Φ_{p_0} maps $(0, 1)$ to p_0.*

For any point $\hat{p} \in \partial M$, choose some neighborhood Ω on which background coordinates (r, θ) are defined on a set with the following form $\{(r, \theta) : |\theta - \theta(p)| < c, 0 \leq r < c\}$.
Let $\omega^1, \cdots, \omega^n$ be 1-forms chosen so that $(dr, \omega^1, \cdots, \omega^n)$ is an orthonormal coframe for $\bar{g}$ at each point of $\partial M \cap \Omega$. Let the coefficients of ω^α at $\hat{p}$ be $\omega^\alpha_{\hat{p}} = A^\alpha_\beta d\theta^\beta_{\hat{p}} + B^\alpha dr_{\hat{p}}$.
Then define functions on Ω by $\tilde{\theta}^\alpha = A^\alpha_\beta \theta^\beta + B^\alpha r$.
Then $(r, \tilde{\theta}^1, \cdots, \tilde{\theta}^n)$ form new coordinates on Ω where $\bar{g}$ has the form δ_{ij} at $\hat{p}$.
For $0 < \rho < c$, define open sets $Y_1 \doteqdot \{(x, y) \in \mathbb{H} : |x| < 1, \quad 0 < y < 1\}$
and $Z_\rho(\hat{p}) \doteqdot \{(r, \tilde{\theta}) \in \Omega : |\tilde{\theta}| < \rho, \quad 0 < r < \rho\}$

Definition 4.2.2. *For any boundary point $\hat{p}$, define a boundary Möbius chart centered at $\hat{p}$ of radius ρ to be the map $\Psi_{\hat{p},\rho} : Y_1 \to Z_\rho(\hat{p})$ with $(r, \tilde{\theta}) = \Psi_{\hat{p},\rho}(x, y) = (\rho x, \rho y)$.*

Proof. First, for the special case of hyperbolic space form, the lemma is true because the isometry group of the hyperbolic space form is transitive.

Now we consider the general case. For any given $x_0 \in M$, $\int_M \exp(-\alpha\, d(x,x_0))dV_g < \infty$ for any number $\alpha > n-1$. In order to get a uniform bound C, we only need to consider the case where x_0 is near the boundary at the infinity. In other words, we may assume x_0 is in boundary Möbius charts $U_1 = \{(r,\theta) : 0 < r < r_1,\ |\theta| < r_1\} \subset U_2 = \{(r,\theta) : 0 < r < 2r_1,\ |\theta| < 2r_1\}$.

From Lem 6.1 in [18] we have $\int_{U_2} \exp(-\alpha\, d(x,x_0))dV_g \leq C$ for some constant C independent of x_0. Hence it suffices to show $\int_{M\setminus U_2} \exp(-\alpha\, d(x,x_0))dV_g \leq C$ for some constant C independent of x_0.

Consider another set $U_3 = \{(r,\theta) : 0 < r < r_1\}$. We have $\int_{M\setminus U_3} \exp(-\alpha\, d(x,x_0))dV_g \leq vol(M \setminus U_3)$. For any $x \in U_3 \setminus U_2$, $d(x,x_0) \geq \frac{r_1}{r}$ because x_0 is inside U_1 while x is outside of U_2. Hence $\int_{U_3\setminus U_2} \exp(-\alpha\, d(x,x_0))dV_g \leq C \int_0^{r_1} \int_{\partial M} r^{-n} \exp\big(-\alpha \frac{r_1}{r}\big) d\sigma dr \leq C$. Add all three integral inequalities above and we obtain the uniform bound C. □

4.3 Proof of the Main Result

4.3.1 Step 1: Short Time Existence

Recall that for the Ricci-DeTurck flow, the initial metric is chosen to be the background metric. But here we need to consider a modified version as follows. Choose $\tilde{g} = h_0$ instead as the background metric. Then we can similarly define the 1-forms

$$W_i = g_{ij} g^{pq} (\Gamma^j_{pq} - \tilde{\Gamma}^j_{pq})$$

and the corresponding h_0-flow is

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i \\ \\ g(0) = g_0 \end{cases} \tag{4.3.1}$$

Since the initial metric g_0 is ε-close to h_0, we get the short time existence by applying Simon's result (Thm 2.3.5) directly.

4.3.2 Step 2: Exponential Decay

In order to prove the long time existence as well as the convergence, we first derive the exponential decay of the L^2-norm of $g(t) - h_0$.

In the following computations we always assume that in appropriate coordinates, at a fixed point and a fixed time, we have $h_{ij} = \delta_{ij}$ and $(g_{ij}) = diag(\lambda_1, \cdots, \lambda_n)$.

From the h_0-flow equation (4.3.1), we get

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b g_{ij} - g^{kl} g_{ip} R_{jkql}(h_0) - g^{kl} g_{jp} R_{ikql}(h_0) - 2(n-1) g_{ij} \\ &\quad + \frac{1}{2} g^{ab} g^{pq} (\tilde{\nabla}_i g_{pa} \tilde{\nabla}_j g_{qb} + 2\tilde{\nabla}_a g_{jp} \tilde{\nabla}_q g_{ib} - 2\tilde{\nabla}_a g_{jp} \tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_j g_{pa} \tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_i g_{pa} \tilde{\nabla}_b g_{jq}) \end{aligned} \tag{4.3.2}$$

Consider $u(t)$ - the difference between the solution to the normalized Ricci flow $g(t)$ and the given AHE metric h_0. We denote h_0 by h, $\tilde{\nabla}$ by ∇, and $C\varepsilon$ by ε here for simplicity, so $u(t) \doteqdot g(t) - h_0 = g(t) - h$.

The equation (4.3.2) yields that

$$\begin{aligned}
\frac{\partial}{\partial t}|u|^2 &= \frac{\partial}{\partial t}|g-h|^2 = 2\sum_i (g_{ii}-h_{ii})\Big(\frac{\partial}{\partial t}g_{ii}\Big) \\
&= g^{ij}\nabla_i\nabla_j|g-h|^2 - (2-\varepsilon)|\nabla g|^2 + \sum_i (g_{ii}-h_{ii})(\nabla g * \nabla g)_{ii} \\
&\quad +2\sum_i (g_{ii}-h_{ii})\big[2(g_{ii}-h_{ii}) - 2g_{ii}\sum_k \big(g^{kk}(g_{kk}-h_{kk})\big)\big] \qquad (4.3.3) \\
&\leq g^{ij}\nabla_i\nabla_j|u|^2 - (2-\varepsilon)|\nabla u|^2 \\
&\quad +4\sum_i (g_{ii}-h_{ii})\big[(g_{ii}-h_{ii}) - g_{ii}\sum_k \big(g^{kk}(g_{kk}-h_{kk})\big)\big]
\end{aligned}$$

By Cauchy-Schwarz inequality, the last term above satisfies

$$\begin{aligned}
& 4\sum_i (g_{ii}-h_{ii})\big[(g_{ii}-h_{ii}) - g_{ii}\sum_k \big(g^{kk}(g_{kk}-h_{kk})\big)\big] \\
=\ & 4\sum_i (\lambda_i - 1)^2 - 4\sum_i \lambda_i(\lambda_i - 1)\sum_k \big(1 - \frac{1}{\lambda_k}\big) \\
\leq\ & (4+\varepsilon)|u|^2 - 4\Big(\sum_i (\lambda_i - 1)\Big)^2 \\
\leq\ & (4+\varepsilon)|u|^2
\end{aligned}$$

As a result, we arrive at the following lemma.

Lemma 4.3.1. *Let g be a solution to (4.3.1) on the time interval $[0,T)$. If $\varepsilon > 0$ is sufficiently small, then*

$$\frac{\partial}{\partial t}|u|^2 \leq g^{ij}\nabla_i\nabla_j|u|^2 - (2-\varepsilon)|\nabla u|^2 + (4+\varepsilon)|u|^2$$

Next we compute the evolution equation of the L^2-norm of u.

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{B_R(0)}|u|^2dV_h &\leq \int_{B_R(0)} g^{ij}\nabla_i\nabla_j|u|^2-(2-\varepsilon)|\nabla u|^2+(4+\varepsilon)|u|^2dV_h\\
&= \int_{\partial B_R(0)}\nu_i g^{ij}\nabla_j|u|^2-\int_{B_R(0)}\nabla_i g^{ij}\nabla_j|u|^2dV_h\\
&\quad +\int_{B_R(0)}-(2-\varepsilon)|\nabla u|^2+(4+\varepsilon)|u|^2dV_h\\
&\leq \int_{B_R(0)}-(2-\varepsilon)|\nabla|u||^2+(4+\varepsilon)|u|^2dV_h\\
&\leq \int_{B_R(0)}-(2-\varepsilon)\lambda_0|u|^2+(4+\varepsilon)|u|^2dV_h\\
&\leq \int_{B_R(0)}(4-2\lambda_0)|u|^2dV_h
\end{aligned}$$

where ν is the outer unit normal vector, also we used Kato's inequality and the fact that h_0 is asymptotically hyperbolic Einstein with non-degeneracy $\lambda_0>2$ here. Hence we arrive at the following lemma by Grönwall's inequality.

Lemma 4.3.2. *Let $n\geq 3$. There exists $\delta_0=\delta_0(n)$ such that the following holds. Let $g(t)$ be a smooth solution to the Dirichlet boundary problem*

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= -2R_{ij}-2(n-1)g_{ij}+\nabla_iW_j+\nabla_jW_i, \quad x\in B_R(0)\\
g(x,t) &= h(x), \quad x\in\partial B_R(0),\ t\in[0,T)
\end{aligned} \tag{4.3.4}$$

satisfying $\sup_{B_R(0)\times[0,T)}|g-h|\leq\delta_0$. *Then*

$$\int_{B_R(0)}|g(t)-h_0|^2dV_{h_0}\leq e^{-\alpha t}\int_{B_R(0)}|g(0)-h_0|^2dV_{h_0}$$

for $\alpha=\frac{\lambda_0-2}{2}$.

Let $R\to\infty$ and we have

Corollary 4.3.3. *Let $n\geq 3$ and $T>0$. Assume g_0 is a smooth metric on M^n satisfying $\|g_0-h_0\|_{L^2}<\infty$. There exists $\varepsilon_0=\varepsilon_0(n,T)$ such that the following holds. If $\sup_{M^n}|g_0-h_0|\leq\varepsilon_0$, then there ex-*

ists a solution $g(t)$ to the Dirichlet boundary problem (4.3.4) satisfying $\sup_{M^n\times[0,T)}|g-h_0|\leq\delta_0$*, where* δ_0 *is as in Lem 4.3.2. Furthermore,*

$$\|g(t)-h_0\|_{L^2}\leq e^{-\alpha t}\|g_0-h_0\|_{L^2}$$

for all $t\in[0,T)$*, where* $\alpha=\frac{\lambda_0-2}{2}$.

4.3.3 Step 3: Long Time Existence and Convergence

First we obtain exponential decay of pointwise sup-norm of $g(t)-h_0$ from the results we got in step 2.

In Cor 4.3.3, without loss of generality we can assume that $\delta_0<1$. Then we can choose a positive time $\tau\doteqdot-\dfrac{n}{\alpha}\ln\delta_0>0$. This means for $t\in[0,\tau)$ and $\beta\doteqdot\dfrac{\alpha}{n+1}$, we have

$$\sup_{M^n\times[0,\tau)}|g(t)-h_0|\leq e^{-\beta t}.$$

By the interior estimates of the form $|\tilde{\nabla}g(t)|\leq Ct^{-1/2}$, there exists a constant $C'=C'(n)$ such that $|\tilde{\nabla}g(\cdot,t)|_{h_0}\leq C'$ for $t\in[\tau,T)$.

Fix such a time $t_0\in[\tau,T)$ and let $\gamma\doteqdot\sup_{M^n}|g(t_0)-h_0|$. By the definition of supremum, we can choose a point $p_0\in M^n$ such that $|g(p_0,t_0)-h_0|>\dfrac{1}{2}\gamma$.

By the gradient estimate, we have $|g(\cdot,t_0)-h_0|>\dfrac{1}{4}\gamma$ on the geodesic ball $B_{\gamma/(4C')}(p_0)$, which implies

$$\begin{aligned}\|g(t_0)-h_0\|_{L^2(M)} &\geq \|g(t_0)-h_0\|_{L^2(B_{\gamma/(4C')}(p_0))}\\ &\geq \omega_n\left(\frac{\gamma}{4C'}\right)^n\left(\frac{1}{4}\gamma\right)^2\\ &= \omega_n(C')^{-n}\left(\frac{\gamma}{4}\right)^{n+2}\end{aligned}$$

.

This yields $\gamma \leq 4(C')^{n/(n+2)} \left(\frac{K^2}{\omega_n}\right)^{-(n+2)} e^{-\alpha t/(n+2)}$. Hence we arrive at the following lemma.

Lemma 4.3.4. *Let $n \geq 3$. Assume $g(t)$ $(t \in [0,T))$ is a solution to the Dirichlet boundary problem (4.3.4) with $\|g_0 - h_0\|_{L^2} \doteqdot K < \infty$, $\sup_{M^n \times [0,T)} |g_0 - h_0| \leq \delta_0$ and*

$$\|g(t) - h_0\|_{L^2} \leq e^{-\alpha t} \|g_0 - h_0\|_{L^2}$$

where δ_0 and α are as in Lem 4.3.2. Then

$$\sup_{M^n} |g(t) - h_0| \leq e^{-\beta t} \sup_{M^n} |g_0 - h_0|$$

where $\beta = \frac{\alpha}{n+2} > 0$.

The exponential decay above allows us to construct a solution which exists for all time. For any g_0 with bounded $\|g_0 - h_0\|_{L^2(M)}$ and sufficiently small $\sup |g_0 - h_0|$, by the short time existence, there exists $T > 0$ such that the normalized Ricci flow exists on $[0,T)$, that is, there exists a solution $g(t)$ $t \in [0,T)$.

Assume there is a maximal finite time interval $[0, T_{max})$ when the solution exists. The exponential decay implies that as time $t \to T_{max}$, the metric $g(t)$ still satisfies the two conditions and the short time existence tells us that $g(t)$ can flow past T_{max}, contradition! Hence we arrive at the following lemma.

Lemma 4.3.5. *Let $n \geq 3$ and h_0 as before. For all $K > 0$, there exists $\varepsilon_1 = \varepsilon_1(n,K) > 0$ such that the following holds. If g_0 satisfies $\|g_0 - h_0\|_{L^2} \leq K$ and $\sup_{M^n \times [0,T)} |g_0 - h_0| \leq \varepsilon_1$, then there exists a solution $g(t)$ $t \in [0,\infty)$ such that*

$$\sup_{M^n} |g(t) - h_0| \leq C(n,K) e^{-\beta t}$$

where $\beta = \beta(n)$ *as in Lem 4.3.4.*

Next, the exponential decay extends to higher derivatives of the evolving metric by interpolation.

Lemma 4.3.6. *Let* $n \geq 3$*. Let* g_0 *and* $g(t)$ *be as in Lem 4.3.4. Then*

$$\sup_{M^n} \left| {}^h\nabla^j g(t) \right| \leq C(n,K,j)e^{-\beta_j t}$$

where $0 < \beta_j < \beta(n)$*,* $\beta(n)$ *as in Lem 4.3.4.*

Finally, as the decay as $t \to \infty$ above does not depend on the smoothness of g_0, we can approximate g_0 using smooth metrics and pass to the limit to obtain the long time existence and convergence in our main theorem.

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