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Complex and p-adic computations of Chow-Heegner points

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Complex and p-adic Computations of Chow–Heegner Points

by

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Abstract

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In this work we delve into the theory of Chow–Heegner points, establishing some of their basic properties and developing two methods for their explicit computation. Chapters 1 and 2 cover the background material necessary for the later chapters. Chapter 3 gives several definitions of Chow–Heegner points and explores the specific setting of modular abelian varieties. Chapter 4 develops an algorithm for computing Chow–Heegner points via complex analytic methods, and Chapter 5 considers a corresponding algorithm in the p-adic setting.

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Introduction

Let E/K be an elliptic curve, where K is a number field. What is the structure of $E(\mathbb{Q})$? The Mordell–Weil theorem says that $E(\mathbb{Q})$ is a finitely generated abelian group, and thus $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T$, where $r \geq 0$ is the *algebraic rank* of E and T is a finite group. We would like to relate r to other arithmetic information about E. Namely, for each prime $\mathfrak{p}$ of $\mathcal{O}_K$, set $q = |\mathcal{O}_K/\mathfrak{p}|$. Writing $a_\mathfrak{p} = q + 1 - \#E(\mathcal{O}_K/\mathfrak{p})$, Hasse showed that $|a_\mathfrak{p}| \leq 2\sqrt{q}$. We can form the L-function of E as the Euler product

$$L(E,s) = \prod_{\mathfrak{p}\nmid\Delta}(1 - a_\mathfrak{p} q^{-s} - q^{1-2s})^{-1} \prod_{\mathfrak{p}|\Delta}(1 - a_\mathfrak{p} q^{-s})^{-1}$$

where Δ is the discriminant of E. Using Hasse's estimate, it can be shown that $L(E,s)$ converges absolutely in the complex half-plane $\mathrm{Re}(s) > \frac{3}{2}$. The Birch and Swinnerton-Dyer conjecture (BSD) predicts that $L(E,s)$ encodes information about r, more precisely:

Conjecture. $L(E,s)$ *has an analytic continuation to the entire complex plane, and*

$$\mathrm{ord}_{s=1} L(E,s) = r.$$

The most successful approaches to solve BSD emerged from the study of the *modular elliptic curves*. Let $\Gamma_0(N)$ denote the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ given by matrices with lower-left entry divisible by N, and $S_k(\Gamma_0(N))$ the space of holomorphic cusp forms for $\Gamma_0(N)$. Let $f \in S_2(\Gamma_0(N))$ be a simultaneous eigenvector for the Hecke operators T_n, $n \geq 1$. To every such f, Shimura associated an abelian variety $A_f/\mathbb{Q}$, and the modular elliptic curves are the elliptic curves that arise in this manner. The work of Wiles [Wil95], Taylor–Wiles [TW95], and Breuil–Conrad–Diamond–Taylor [BCDT01] shows that all elliptic curves over $\mathbb{Q}$ are modular, and thus when working over $\mathbb{Q}$ there is no loss of generality in studying this class of curves. The analytic properties of modular forms can be used to establish the analytic continuation of $L(E,s)$ for modular elliptic curves predicted by BSD. Additionally, the work of Gross–Zagier [GZ86] and Kolyvagin [Kol88] proves the conjecture for modular elliptic curves E when $\mathrm{ord}_{s=1} L(E,s) \leq 1$.

In the case when $\mathrm{ord}_{s=1} L(E,s) > 1$, very little is known and BSD is much more difficult. So instead we set our sights on a more modest goal: if $L(E,1) = 0$, how can we find a nontorsion point $P \in E(K)$? The first successful approach to systematically producing points of infinite order on elliptic curves came from the theory of *Heegner points*. Let $\mathfrak{H}$

denote the complex upper-half plane and N the conductor of E. Then by modularity, there exists a surjective morphism

$$\pi_f : X_0(N) \to E.$$

The images under π_f of a special collection of points $\tau \in \Gamma_0(N)\backslash\mathfrak{H} = X_0(N)(\mathbb{C})$ are defined over a ray class field H_τ of an imaginary quadratic field K. Under suitable conditions on N and K, $L(E_{H_\tau}, 1) = 0$ and $\pi_f(\tau)$ is nontorsion. Heegner points provided an instrumental role in the work of Gross–Zagier and Kolyvagin, and in fact have a wealth of applications to other areas of number theory such as the special values of p-adic L-functions. For more on Heegner points, the interested reader may consult [DR] for an overview and the references listed therein for a more detailed account.

Recent work of Bertollini, Darmon, Prasanna, Rotger, and Sols ([BDP13], [BDP12], [DRS12], [DR12]) has developed a theory of *Chow–Heegner points* akin to Heegner points by replacing special points on $X_0(N)$ with cycles on higher dimensional varieties. In particular, [DRS12] and [DR12] study the *Gross–Kudla–Schoen modified diagonal cycle* Δ_{GKS}, an algebraic cycle on the triple product $X_0(N) \times X_0(N) \times X_0(N)$. The cycle Δ_{GKS}, first studied in [GK92] and [GS95], gives rise to a collection of points in $E(\mathbb{Q})$ via intersection with suitably chosen cycles. Unfortunately, these points are always torsion points when $\mathrm{ord}_{s=1}L(E, s) > 1$, and so this does not yield to the computation of points in cases not covered by the theory of Heegner points. However, current work in progress by Darmon and Rotger aims to replace $\mathbb{Q}$ with an abelian extension of $\mathbb{Q}$, including settings where nontorsion points have not been found but are expected to exist.

The goal of this monograph is to formally define Chow–Heegner points, study the basic properties of those coming from Δ_{GKS}, and describe two methods for explicitly computing them. The author hopes that these methods will be useful in more general settings, including the aforementioned study of Chow–Heegner points over abelian extensions of $\mathbb{Q}$, and other analogous settings.

Chapter 1

Intersection theory and cohomology

1.1 Algebraic cycles

In this section, we briefly recount some of the basics of intersection theory and the cycle class map on algebraic cycles. For full details and proofs, the reader may consult [Ful98], though our conventions and notation differ.

Algebraic cycles and Chow groups

Let K be a field. By a variety over K we shall mean an integral separated scheme V equipped with a morphism of finite type $V \to \mathrm{Spec}(K)$. If V and V' are varieties over K, then we will write $V \times V'$ for the product over $\mathrm{Spec}(K)$. Let V be a variety over K of dimension d. The following definition of algebraic cycle is taken from [DR] and differs slightly from that in [Ful98].

Definition 1.1.1. An *algebraic cycle* of codimension c on V is a finite formal sum $Z = \sum n_i V_i$, where $n_i \in \mathbb{Z}$ and each V_i is a subvariety of $V_{\bar{K}}$ of codimension c.

If W is a subvariety of V and $f \in \bar{K}(W)$, then $\mathrm{div}(f) = \sum \mathrm{ord}_{W'}(f) W'$ is a divisor on V, where the sum is taken over all codimension–one subvarieties W' of W. This leads us to the notion of rational equivalence.

Definition 1.1.2. Two divisors Z and Z' of codimension c are said to be *rationally equivalent*, written $Z \sim_{\mathrm{rat}} Z'$, if $Z - Z'$ can be expressed as a finite sum $\sum_i \mathrm{div}(f_i)$ for $f \in \bar{K}(W_i)$ and W_i subvarieties of $V_{\bar{K}}$ of codimension $c-1$.

The codimension–c Chow group with values in K is the group of codimension c algebraic cycles Z such that $Z \sim_{\mathrm{rat}} Z^\sigma$ for all $\sigma \in G_K$ modulo rational equivalence. We denote this by $\mathrm{CH}^c(V)(K)$ and we write $[Z]$ for the equivalence class of a cycle Z. For any $0 \leq c, c' \leq d$, there is an intersection product

$$\mathrm{CH}^c(V)(K) \times \mathrm{CH}^{c'}(V)(K) \to \mathrm{CH}^{c+c'}(V)(K), \qquad (Z, Z') \mapsto Z \cdot Z'. \tag{1.1}$$

Let $f : V \to V'$ be a morphism defined over K, where V'/K is a variety of dimension d'. If f is flat, then the pullback map on cycles preserves rational equivalence, and hence induces a map

$$f^* : \mathrm{CH}^c(V')(K) \to \mathrm{CH}^c(V)(K). \tag{1.2}$$

If f is proper, then the pushforward map preserves rational equivalence, and induces a map

$$f_* : \mathrm{CH}^c(V)(K) \to \mathrm{CH}^{c'}(V')(K), \tag{1.3}$$

where $c' = d' - d + c$. Note the convention that if W is a closed subvariety of V and $\dim f(W) < \dim W$, then $f_*([W]) = 0$. Finally, let $\Pi \in \mathrm{CH}^{d+c'-c}(V \times V')(K)$, and denote by $\mathrm{pr}_V : V \times V' \to V$ and $\mathrm{pr}_{V'} : V \times V' \to V'$ the natural projection morphisms. Then define a map

$$\Pi_* : \mathrm{CH}^c(V)(K) \to \mathrm{CH}^{c'}(V')(K), \qquad [Z] \mapsto \mathrm{pr}_{V'*}(\mathrm{pr}_V^*(Z) \cdot \Pi). \tag{1.4}$$

1.2 Betti cohomology

In this section, we recall the basic properties of Betti cohomology that will be used later, without proof. For more details, see [Hat02].

Singular homology and cohomology

Let X be a topological space. For any $n \geq 0$, write Δ^n for the standard n-simplex, and $C_n(X)$ for the free abelian group generated by all continuous maps

$$\sigma_n : \Delta^n \to X.$$

Let Δ_k^{n-1} denote the face of Δ^n obtained by omitting the kth vertex, a face of Δ^n. Define a map $\partial_n : C_n(X) \to C_{n-1}(X)$ by the rule

$$\partial_n(\sigma_n) = \sum_{k=0}^{n} (-1)^k \sigma_n|_{\Delta_k^{n-1}}.$$

Write $C_\bullet(X)$ for the chain complex

$$\cdots \to C_{n+1}(X) \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \to 0,$$

and define the *nth singular homology group (with coefficients in* $\mathbb{Z}$*)* $H_n(X, \mathbb{Z})$ to be the nth homology group of $C_\bullet(X)$. If $f : X \to Y$ is a continuous map, then the map

$$f_* : C_n(X) \to C_n(Y), \qquad \sigma_n \mapsto f \circ \sigma_n$$

induces a group homomorphism $f_* : H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z})$.

Now for any ring R, set $C^\bullet(X,R) = \mathrm{Hom}(C_\bullet(X),R)$. Define the nth singular cohomology group $H^n(X,R)$ with coefficients in R to be the nth cohomology group of the chain complex $C^\bullet(X,R)$. As above, a continuous map $f : X \to Y$ induces an R-module homomorphism $f^* : H^n(Y,R) \to H^n(X,R)$. If $A \subset X$ is a subspace, define the cohomology group $H^n_A(X,R)$ to be the nth cohomology group of the chain complex $C^\bullet(X,R)/C^\bullet(X-A,R)$. It fits into a long exact sequence of R-modules

$$\cdots \to H^n_A(X,R) \to H^n(X,R) \to H^n(X-A,R) \to H^{n+1}_A(X,R) \to \cdots \tag{1.5}$$

Theorem 1.2.1. *(Universal coefficient theorem) The group $H^n(X,R)$ fits into an exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(X,\mathbb{Z}),R) \to H^n(X,R) \to \mathrm{Hom}(H_n(X,\mathbb{Z}),R) \to 0.$$

In particular, if $R = K$ is a field of characteristic 0 and $H_n(X,\mathbb{Z})$ and $H_{n-1}(X,\mathbb{Z})$ are finitely generated, then

$$H^n(X,K) \simeq H^n(X,\mathbb{Z}) \otimes K.$$

Proof. See [Hat02], Theorem 3.2 for the first statement. For the second statement, note that $\mathrm{Ext}(H_{n-1}(X,\mathbb{Z}),K) = 0$ since K is an injective $\mathbb{Z}$-module, and thus when $R = K$ the exact sequence becomes

$$H^n(X,K) \xrightarrow{\sim} \mathrm{Hom}(H_n(X,\mathbb{Z}),K).$$

Since $H_{n-1}(X,\mathbb{Z})$ is finitely generated, $\mathrm{Ext}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z})$ is a torsion $\mathbb{Z}$-module, so letting $R = \mathbb{Z}$ and tensoring with K yields

$$H^n(X,\mathbb{Z}) \otimes K \xrightarrow{\sim} \mathrm{Hom}(H_n(X,\mathbb{Z}),\mathbb{Z}) \otimes K.$$

Finally, the natural map $\mathrm{Hom}(H_n(X,\mathbb{Z}),\mathbb{Z}) \otimes K \to \mathrm{Hom}(H_n(X,\mathbb{Z}),K)$ is an isomorphism since $H_n(X,\mathbb{Z})$ is finitely generated. □

Poincaré Duality

We briefly recall the statement of Poincaré duality for an oriented closed manifold M.

Theorem 1.2.2. *(Poincaré duality) Let R be a ring, and suppose that M is an R-orientable closed manifold of dimension n. Then there are natural isomorphisms*

$$H^k(M,R) \xrightarrow{\sim} H_{n-k}(M,R)$$

for all k.

Proof. See Theorem 3.30 in [Hat02]. □

Betti cohomology

Now suppose that $V/\mathbb{C}$ is a smooth variety. The set of its complex points $V(\mathbb{C})$ can be endowed with the structure of a complex manifold, denoted by V^{an}. This allows us to define the Betti cohomology of V as the singular cohomology of V^{an}.

Definition 1.2.3. Let R be a ring. The *kth Betti cohomology group with coefficients in R* $H^k_B(V, R)$ of V is defined as

$$H^k_B(V, R) := H^k(V^{\mathrm{an}}, R).$$

Theorem 1.2.4. *If X is a closed manifold, then its homology groups $H_n(X, \mathbb{Z})$ are finitely generated. In particular, if V is projective, then the Betti cohomology groups $H^k_B(V, \mathbb{Z})$ are finitely generated.*

Proof. See [Hat02], §3.A. □

1.3 De Rham cohomology and the Hodge filtration

In this section we state some basic facts about de Rham cohomology without proof. For details, the interested reader may consult [Har75] or [dJ]. We only treat the case of smooth varieties, although the references consider more general settings.

De Rham cohomology

Let K be a number field, and V/K a smooth variety of dimension n. Denote by $\Omega^1_{V/K}$ the sheaf of differential forms on V, a locally free sheaf of rank d. For each $i \geq 0$, write $\Omega^i_{V/K} = \wedge^i \Omega^1_{V/K}$. Then the canonical map $d^0 : \mathcal{O}_V \to \Omega^1_{V/K}$ induces maps $d^i : \Omega^i_{V/K} \to \Omega^{i+1}_{V/K}$. Note that d^i is not a map of $\mathcal{O}_V$-modules, though it is a map of sheaves of K-vector spaces. This gives us a complex

$$0 \to \mathcal{O}_V \xrightarrow{d^0} \Omega^1_{V/K} \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Omega^n_{V/K} \to 0.$$

denoted by $\Omega^\bullet_{V/K}$. The kth de Rham cohomology of V is given by the kth hypercohomology group of this complex,

$$H^k_{\mathrm{dR}}(V/K) = R^k\Gamma(V, \Omega^\bullet_{V/K}).$$

We may calculate $H^k_{\mathrm{dR}}(V/K)$ as follows. Choose a finite open affine cover $\{U_j\}$ and denote by $C^q(\Omega^p_{V/K})$ the q-th term in the Čech resolution of Ω^p with respect to $\{U_j\}$. Write $c^p_q : C^q(\Omega^p_{V/K}) \to C^{q+1}(\Omega^p_{V/K})$ for the transition map, and $d^p_q : C^q(\Omega^p_{V/K}) \to C^q(\Omega^{p+1}_{V/K})$ for the map induced by d^p. Finally, let $\mathrm{Tot}^\bullet(C^\bullet(\Omega^\bullet_{V/K}))$ be the total complex of the double complex $C^\bullet(\Omega^\bullet_{V/K})$ with kth transition map $\sum_{p+q=k}(d^p_q + (-1)^p c^p_q)$. Then

$$H^k_{\mathrm{dR}}(V/K) = H^k(\mathrm{Tot}^\bullet(C^\bullet(\Omega^\bullet_{V/K}))).$$

If W/K is a variety, then $H^k_{\mathrm{dR}}(V \times W/K)$ decomposes in accordance with the Künneth formula

$$H^k_{\mathrm{dR}}(V \times W/K) \simeq \bigoplus_{i+j=k} H^i_{\mathrm{dR}}(V/K) \otimes H^j_{\mathrm{dR}}(W/K). \tag{1.6}$$

Let $Z \subset V$ be a smooth closed subscheme of codimension c, and let $\Gamma_Z(V, \mathcal{F})$ denote the group of sections of a sheaf $\mathcal{F}$ with support in Z. Then the long exact of hypercohomology groups

$$\cdots \to R^k\Gamma_Z(V, \Omega^\bullet_{V/K}) \to R^k\Gamma(V, \Omega^\bullet_{V/K}) \to R^k\Gamma(V-Z, \Omega^\bullet_{V/K}|_{V-Z}) \to R^{k+1}\Gamma_Z(V, \Omega^\bullet_{V/K}) \to \cdots$$

gives rise to a long exact sequence

$$\cdots \to H^k_Z(V/K) \to H^k_{\mathrm{dR}}(V/K) \to H^k_{\mathrm{dR}}(V-Z/K) \to H^{k+1}_Z(V/K) \to \cdots \tag{1.7}$$

where we write $H^k_Z(V/K) := R^k\Gamma_Z(V, \Omega^\bullet_{V/K})$.

The Hodge filtration

There is a spectral sequence, called the Hodge to de Rham spectral sequence,

$$E_1^{p,q} = H^q(V, \Omega^p_{V/K}) \Rightarrow H^{p+q}_{\mathrm{dR}}(V/K).$$

When V is proper, this spectral sequence degenerates at the E_1 term. This can be proved using analytic techniques, or by deducing it from a much more general result of Deligne and Illusie [DI87]. In particular, this gives $H^k_{\mathrm{dR}}(V/K)$ a filtration

$$H^k_{\mathrm{dR}}(V/K) = \mathrm{Fil}^0 H^k_{\mathrm{dR}}(V/K) \supset \mathrm{Fil}^1 H^k_{\mathrm{dR}}(V/K) \supset \cdots \supset \mathrm{Fil}^{k+1} H^k_{\mathrm{dR}}(V/K) = 0 \tag{1.8}$$

satisfying $\mathrm{gr}^j H^k_{\mathrm{dR}}(V/K) = H^{k-j}(V, \Omega^j_{V/K})$. We can use this spectral sequence to compute $H^k_{\mathrm{dR}}(\mathbb{P}^n/K)$. By exercise 7.3 in [Har75], we have $H^q(\mathbb{P}^n, \Omega^p) = 0$ if $p \neq q$ and is 1–dimensional if $p = q$. It follows that $H^{2i+1}_{\mathrm{dR}}(\mathbb{P}^n/K) = 0$ and $H^{2i}_{\mathrm{dR}}(\mathbb{P}^n/K)$ is 1–dimensional for $0 \leq i \leq n$, with

$$\mathrm{Fil}^j H^{2i}_{\mathrm{dR}}(\mathbb{P}^n/K) = \begin{cases} H^{2i}_{\mathrm{dR}}(\mathbb{P}^n/K), & j \leq i \\ 0, & j > i. \end{cases}$$

In particular, taking $n = 1$ we can define

$$H^k_{\mathrm{dR}}(V/K)(-m) := H^k_{\mathrm{dR}}(V/K) \otimes H^2_{\mathrm{dR}}(\mathbb{P}^1/K)^{\otimes m}$$

with the convention $M^{-1} = \mathrm{Hom}(M, K)$ for any filtered K-vector space M. Since $H^2_{\mathrm{dR}}(\mathbb{P}^1/K)$ is 1–dimensional with nontrivial graded piece at $j = 1$, $H^k_{\mathrm{dR}}(V/K)(m)$ is simply $H^k_{\mathrm{dR}}(V/K)$ as a K-vector space with its filtration shifted by m. Specifically, we have $\mathrm{Fil}^j H^k_{\mathrm{dR}}(V/K)(m) = \mathrm{Fil}^{j+m} H^k_{\mathrm{dR}}(V/K)$.

Notice that in the Hodge to de Rham spectral sequence we have $E_1^{p,q} = 0$ when $p > d$ or $q > d$. Thus, the only nonzero term on the diagonal $p + q = 2d$ is

$$E_1^{d,d} = H^d(V, \Omega^d_{V/K}) \simeq H^0(V, \mathcal{O}_V)^\vee \simeq K,$$

with the middle isomorphism coming from Serre duality. So $H^{2d}_{\mathrm{dR}}(V/K)$ is 1–dimensional, with filtration given by

$$\mathrm{Fil}^j H^{2d}_{\mathrm{dR}}(V/K) = \begin{cases} H^{2d}_{\mathrm{dR}}(V/K), & j \le d \\ 0, & j > d. \end{cases}$$

It follows that $H^{2d}_{\mathrm{dR}}(V/K)(d) \simeq K$ with the trivial filtration.

The Poincaré Pairing

Let V and W be varieties of dimensions d and d', respectively. If $f : V \to W$ is a morphism, then the pullback map $f^{-1}\Omega^\bullet(W) \to \Omega^\bullet(V)$ induces

$$f^* : H^k_{\mathrm{dR}}(W/K) \to H^k_{\mathrm{dR}}(V/K).$$

Twisting by d and applying this in the case where $W = V \times V$, $f = \Delta$ is the diagonal, and $k = 2d$ yields a map

$$\Delta^* : H^{2d}_{\mathrm{dR}}(X \times X/K)(d) \to H^{2d}_{\mathrm{dR}}(X/K)(d) \simeq F.$$

Decomposing the left-hand side into its Künneth factors using (1.6) results in an alternating non-degenerate pairing for each $0 \le k \le 2d$

$$\langle\ ,\ \rangle : H^k_{\mathrm{dR}}(V/K) \times H^{2d-k}_{\mathrm{dR}}(V/K)(d) \to K$$

called the *Poincaré pairing*. This allows us to define a morphism

$$f_* : H^k_{\mathrm{dR}}(V/K) \to H^{k-2(d-d')}_{\mathrm{dR}}(W/K)(d-d')$$

as the adjoint of f^*. Specifically, f_* is the composition

$$H^k_{\mathrm{dR}}(V/K) \simeq H^{2d-k}_{\mathrm{dR}}(V/K)(d)^\vee \xrightarrow{(f^*)^\vee} H^{2d-k}_{\mathrm{dR}}(W/K)(d)^\vee \simeq H^{k-2(d-d')}_{\mathrm{dR}}(W/K)(d'-d)$$

and satisfies the property

$$\langle f^*\alpha, \beta\rangle_V = \langle \alpha, f_*\beta\rangle_W \tag{1.9}$$

for $\alpha \in H^{2d-k}_{\mathrm{dR}}(W/K)$ and $\beta \in H^k_{\mathrm{dR}}(V/K)$.

More generally, let $\Pi \in \mathrm{CH}^c(V \times W/K)$ be an algebraic cycle of codimension c. Then Π defines a map

$$\Pi^* : H^k_{\mathrm{dR}}(W/K) \to H^{k-2(d'-c)}_{\mathrm{dR}}(V/K)(c-d'),$$

$$\Pi_* : H^k_{\rm dR}(V/K) \to H^{k-2(d-c)}_{\rm dR}(W/K)(c-d).$$

To define Π^* and Π_*, first consider the case where $\Pi = [Z]$ is the class of a closed, irreducible subvariety. Letting $\mathrm{pr}_V : Z \to V$ and $\mathrm{pr}_W : Z \to W$ denote the projection maps, we can set $\Pi^* = \mathrm{pr}_{V*}\mathrm{pr}_W^*$ and $\Pi_* = \mathrm{pr}_{W*}\mathrm{pr}_V^*$. For general Π, we can write it as a sum of irreducible subvarieties and extend the previous definition linearly. It is not hard to check that this definition is independent of the representation of Π. Furthermore, by construction we have

$$\langle \Pi^*\alpha, \beta\rangle_V = \langle \alpha, \Pi_*\beta\rangle_W$$

for $\alpha \in H^{2(d-d'+c)-k}_{\rm dR}(W/K)$ and $\beta \in H^k_{\rm dR}(V/K)$. When we take $\Pi = \Gamma_f$ to be the graph of a morphism f, we recover $\Pi^* = f^*$ and $\Pi_* = f_*$.

Hodge theory and relation with Betti cohomology

Now suppose that V is a smooth variety over $\mathbb{C}$, and let $V^{\rm an}$ denote the corresponding analytic space. Let $\Omega^\bullet_{V^{\rm an}}$ denote the chain complex of regular analytic differentials on $V^{\rm an}$. Then we define the *analytic de Rham cohomlogy* $H^k_{\rm dR}(V^{\rm an})$ of $V^{\rm an}$ as the hypercohomology of this complex. There is a natural map

$$H^k_{\rm dR}(V/\mathbb{C}) \to H^k_{\rm dR}(V^{\rm an})$$

coming from the fact that an algebraic differential form gives rise to an analytic form. This map is an isomorphism by the following theorem of Grothendieck.

Theorem 1.3.1. *The canonical map $H^k_{\rm dR}(V/\mathbb{C}) \to H^k_{\rm dR}(V^{\rm an})$ is an isomorphism.*

Proof. See the corollary to Theorem 2 in [Gro66]. □

This theorem gives us a canonical splitting of the Hodge filtration of $H^k_{\rm dR}(V/\mathbb{C})$. Namely, let $H^{p,q}(V^{\rm an})$ denote the subspace of $H^k_{\rm dR}(V^{\rm an})$ consisting of classes represented by a harmonic form of type (p,q). The main theorem of Hodge theory asserts that if V is projective, then every cohomology class in $H^k_{\rm dR}(V^{\rm an})$ has a unique harmonic representative, and

$$H^k_{\rm dR}(V^{\rm an}) = \bigoplus_{p+q=k} H^{p,q}(V^{\rm an}).$$

Furthermore, it can be shown that

$$H^{p,q}(V^{\rm an}) \simeq H^q(V^{\rm an}, \Omega^p_{V^{\rm an}}),$$

that $H^{p,q}(V^{\rm an}) = \overline{H^{q,p}(V^{\rm an})}$, and that

$$\mathrm{Fil}^j H^k_{\rm dR}(V^{\rm an}) = \bigoplus_{p+q=k, p\geq j} H^{p,q}(V^{\rm an}).$$

For proofs of these statements, see Chapter 0, §§6-7 of [GH94].

Finally, via Grothendieck's isomorphism, we relate the algebraic de Rham cohomology $H^k_{\rm dR}(V/\mathbb{C})$ to the Betti cohomology $H^k_B(V, \mathbb{C})$ of V.

Theorem 1.3.2. *(Analytic de Rham theorem) The natural map* $\underline{\mathbb{C}} \to \Omega^\bullet_{V^{\mathrm{an}}}$ *is a resolution of the constant sheaf* $\underline{\mathbb{C}}$*, inducing an isomorphism*

$$H^k(V^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\sim} H^k_{\mathrm{dR}}(V^{\mathrm{an}}). \tag{1.10}$$

Proof. See Chapter 0 of [GH94]. □

Corollary 1.3.3. *There is a natural isomorphism* $H^k_B(V, \mathbb{C}) \to H^k_{\mathrm{dR}}(V/\mathbb{C})$*.*

A cohomology class $\xi \in H^k_{\mathrm{dR}}(V/\mathbb{C})$ is called a *Hodge class* if $k = 2n$ is even and $\xi \in H^k_B(V, \mathbb{Q}) \cap H^{n,n}(V^{\mathrm{an}})$. Additionally, ξ is said to be *integral* if ξ is in the image of $H^k_B(V, \mathbb{Z})$.

De Rham cohomology of curves

We end this section by specializing to the case where $V = X$ is a smooth proper curve. There is a simple description of $H^1_{\mathrm{dR}}(X/K)$ that will be useful for computations later. Let $P_1, P_2 \in X(K)$ be closed points, and write $U_1 = X - P_1$ and $U_2 = X - P_2$. Then the Čech double complex arising from the covering U_1, U_2 shows that

$$H^1_{\mathrm{dR}}(X) = Z^1(X; U_1, U_2)/B^1(X; U_1, U_2),$$

where

$$Z^1(X; U_1, U_2) = \{(\omega_1, \omega_2, F_{12}) \in \Omega^1_X(U_1) \times \Omega^1_X(U_2) \times \mathcal{O}(U_1 \cap U_2) \;:\; (\omega_1 - \omega_2)|_{U_1 \cap U_2} = dF_{12}\}$$

$$B^1(X; U_1, U_2) = \{(df_1, df_2, (f_1 - f_2)|_{U_1 \cap U_2}) \in \Omega^1_X(U_1) \times \Omega^1_X(U_2) \times \mathcal{O}(U_1 \cap U_2) \;:\; f_i \in \mathcal{O}(U_i)\}.$$

We can simplify this description even more by introducing differentials of the second kind. Let $U \subset X$ be an open set. For any $\omega \in \Omega^1_X(U)$, we say that ω is of the second kind if $\mathrm{res}_D(\omega) = 0$ for all divisors D of X. Then we write $\Omega^1_{II}(X)$ for the space of differentials of the second kind defined on some open set $U \subset X$. Finally, let $K(X)$ be the function field of X. For any $f \in K(X)$, we have $df \in \Omega^1_{II}(X)$.

Proposition 1.3.4. *There is an isomorphism*

$$H^1_{\mathrm{dR}}(X/K) \simeq \frac{\Omega^1_{II}(X/K)}{dK(X)}$$

given by

$$(\omega_1, \omega_2, F_{12}) \mapsto [\omega_1],$$

where $H^1_{dR}(X/K)$ *is identified with* $Z^1(X; U_1, U_2)/B^1(X; U_1, U_2)$*. Furthermore, the Poincaré pairing can be computed on differentials of the second kind via the formula*

$$\langle \omega, \eta \rangle = \mathrm{res}_{P_1}(F_\omega \eta) = -\mathrm{res}_{P_1}(\omega F_\eta). \tag{1.11}$$

Remark 1.3.5. Proposition 1.3.4 also holds for $H^1_{\mathrm{dR}}(X^{\mathrm{an}})$, with $\Omega^1_{II}(X/K)$ replaced by the space of analytic differentials of the second kind, and $K(X)$ with the field of meromorphic functions on X^{an}.

Chapter 2

Abel–Jacobi maps

2.1 The cycle class map

In this section we define the cycle class maps $\mathrm{cl}_B : \mathrm{CH}^c(V)(\mathbb{C}) \to H^{2c}_B(V,\mathbb{Z})(c)$ and $\mathrm{cl}_{\mathrm{dR}} : \mathrm{CH}^c(V)(K) \to H^{2c}_{\mathrm{dR}}(V/K)(c)$, and establish some basic properties. For the remainder of this section, write $H^{2c}(V)(c)$ for either $H^{2c}_B(V,\mathbb{Z})(c)$ or $H^{2c}_{\mathrm{dR}}(V/K)(c)$ and omit the field of definition for $\mathrm{CH}^c(V)$, which is implicitly K in the de Rham case and $\mathbb{C}$ in the Betti case.

Definition of cl_B and $\mathrm{cl}_{\mathrm{dR}}$

Let $Z \in \mathrm{CH}^c(V)$ be a smooth irreducible closed subvariety of V. Then either (1.5) or (1.7) gives us a long exact sequence

$$\cdots \to H^{2c}_Z(V)(c) \to H^{2c}(V)(c) \to H^{2c}(V-Z)(c) \to H^{2c+1}_Z(V)(c) \to \cdots$$

of cohomology groups. We have a natural isomorphism (see See [Mil80], Theorem 16.1):

$$H^{2c}_Z(V)(c) \simeq H^0(Z).$$

Definition 2.1.1. The *cycle class maps*

$$\mathrm{cl}_B : \mathrm{CH}^c(V)(\mathbb{C}) \to H^{2c}_B(V,\mathbb{Z})(c),$$

$$\mathrm{cl}_{\mathrm{dR}} : \mathrm{CH}^c(V)(K) \to H^{2c}_{\mathrm{dR}}(V/K)(c),$$

are given for irreducible closed subvarieties by sending $[Z]$ to the image of 1 under the composition

$$H^0(Z) \simeq H^{2c}_Z(V)(c) \to H^{2c}(V)(c)$$

when specialized to Betti and de Rham cohomologies, respectively, and extended linearly for a general cycle Δ.

Properties of cl_B and cl_{dR}

We now establish a few basic properties that we will need later on. Suppose V and W are varieties of dimensions d and d'. Recall that we have a Künneth decomposition

$$H^n(V \times W) \simeq \bigoplus_{i+j=n} H^i(V) \times H^j(W).$$

Write $\mathrm{pr}_k : H^n(V \times W) \to H^k(V) \otimes H^{n-k}(W)$ for the projection map. Let A and B be finitely generated abelian groups, or finite dimensional vector spaces over a field. Then the natural map

$$A^\vee \otimes B \to \mathrm{Hom}(A, B), \qquad \phi \otimes b \mapsto (a \mapsto \phi(a) \cdot b)$$

is an isomorphism. Combining this with the Poincaré duality isomorphism

$$H^{2d-k}(V)(d)^\vee \simeq H^k(V)$$

yields an isomorphism

$$\mathrm{Hom}(H^{2d-k}(V)(d), H^{2c-k}(W)(c-d)) \xrightarrow{\sim} H^k(V) \otimes H^{2c-k}(W)(c).$$

Proposition 2.1.2. *Let $0 \le k \le d + d'$.*

(1) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{CH}^c(V \times W) & \xrightarrow{\mathrm{cl}} & H^{2c}(V \times W)(c) \\ \downarrow{\scriptstyle \Pi \mapsto \Pi_*} & & \downarrow{\scriptstyle \mathrm{pr}_k} \\ \mathrm{Hom}(H^{2d-k}(V), H^{2c-k}(W)(c-d)) & \xrightarrow{\sim} & H^k(V) \otimes H^{2c-k}(W)(c). \end{array}$$

(2) For any $Z \in \mathrm{CH}^c(V)$ and $Z' \in \mathrm{CH}^{c'}(W)$, we have

$$\mathrm{cl}(Z \times Z') = \mathrm{cl}(Z) \otimes \mathrm{cl}(Z') \in H^{2c}(V)(c) \otimes H^{2c'}(W)(c') \subset H^{2(c+c')}(V \times W)(c + c').$$

Proof. The first statement is a straightforward computation using the cup product on cohomology; see page 1 of [dJ] for the definition of cup product and the ensuing discussion for its properties. The second statement follows from the definition using the fact that the map

$$H^0(Z) \otimes H^0(Z') = H^0(Z \times Z') \to H^{2(c+c')}(V \times W)(c + c')$$

is induced from the maps $H^0(Z) \to H^{2c}(V)(c)$ and $H^0(Z') \to H^{2c'}(W)(c')$. □

Proposition 2.1.3. *Let $f : V \to W$ be a morphism.*

(1) The following diagrams commute:

$$\begin{array}{ccc} \mathrm{CH}^c(V) & \xrightarrow{\mathrm{cl}} & H^{2c}(V)(c) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{CH}^{c-(d-d')}(W) & \xrightarrow{\mathrm{cl}} & H^{2c-2(d-d')}(W)(c+d'-d) \end{array}$$

$$\begin{array}{ccc} \mathrm{CH}^c(W) & \xrightarrow{\mathrm{cl}} & H^{2c}(W)(c) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{CH}^c(V) & \xrightarrow{\mathrm{cl}} & H^{2c}(V)(c) \end{array}$$

when specialized to cl_B *or* $\mathrm{cl}_{\mathrm{dR}}$ *and the corresponding cohomology theory.*

(2) The cycle class map is compatible with the de Rham isomorphism (1.10) in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CH}^c(V)(\mathbb{C}) & \xrightarrow{\mathrm{cl}_B} & H^{2c}_B(V,\mathbb{C})(c) \\ \| & & \downarrow \\ \mathrm{CH}^c(V)(\mathbb{C}) & \xrightarrow{\mathrm{cl}_{\mathrm{dR}}} & H^{2c}_{\mathrm{dR}}(V/\mathbb{C})(c) \end{array}$$

Proof. The first part follows from the definition of cl and the fact that f induces maps f_* and f^* between the long exact sequences (1.5) and (1.7). The second part is true for the same reasons with the de Rham isomorphism replacing f_* and f^*. □

Denote by $\mathrm{CH}^c(V)_0(K)$ the kernel of $\mathrm{cl}_{\mathrm{dR}}$, the group of *cohomologically trivial cycles*. By the proposition, this kernel coincides with the kernel of cl_B when $K = \mathbb{C}$. Furthermore, if $f : V \to W$ is a morphism and $\Pi \in \mathrm{CH}^{d+c'-c}(V \times W)(K)$, then the maps (1.2), (1.3), and (1.4) induce

$$f^* : \mathrm{CH}^c(W)_0(K) \to \mathrm{CH}^c(V)_0(K),$$
$$f_* : \mathrm{CH}^c(V)_0(K) \to \mathrm{CH}^{d'-d+c}(W)_0(K),$$
$$\Pi_* : \mathrm{CH}^c(V)_0(K) \to \mathrm{CH}^{c'}(W)_0(K).$$

2.2 The complex Abel–Jacobi map

Mixed Hodge structures

In this section we define mixed Hodge structure and establish a few basic results we will need to define the complex Abel–Jacobi map.

Definition 2.2.1. Let $k \in \mathbb{Z}$. A *pure Hodge structure of weight* k is a finitely generated $\mathbb{Z}$-module H equipped with a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$

$$H_{\mathbb{C}} \supset \cdots \supset \mathrm{Fil}^p H_{\mathbb{C}} \supset \mathrm{Fil}^{p+1} H_{\mathbb{C}} \supset \cdots, \ p \in \mathbb{Z}$$

called the *Hodge filtration* satisfying the following two properties:

(1) $\mathrm{Fil}^{\bullet}$ is exhaustive and separated, meaning

$$\mathrm{Fil}^p H_{\mathbb{C}} = H_{\mathbb{C}} \text{ for } p \ll 0, \qquad \mathrm{Fil}^p H_{\mathbb{C}} = 0 \text{ for } p \gg 0.$$

(2) For all $p \in \mathbb{Z}$,

$$H_{\mathbb{C}} = \mathrm{Fil}^p(H_{\mathbb{C}}) \oplus \overline{\mathrm{Fil}^{k+1-p}(H_{\mathbb{C}})}.$$

The kth Betti cohomology group $H = H_B^k(V, \mathbb{Z})$ of a smooth proper variety over $\mathbb{C}$ is the example of primary interest of a pure Hodge structure of weight k. By Theorems 1.2.1 and 1.2.4, we have $H_B^k(V, \mathbb{Z}) \otimes \mathbb{C} \simeq H_B^k(V, \mathbb{C})$, and thus $H_{\mathbb{C}}$ has a Hodge filtration coming from Corollary 1.3.3 and the discussion following Theorem 1.3.1.

If H and H' are two pure Hodge structures of different weights, then there are no pure Hodge structure extensions of H by H'. Thus it is beneficial to enlarge the category of pure Hodge structures to create a more robust theory.

Definition 2.2.2. A *mixed Hodge structure* is a triple $(H_{\mathbb{Z}}, \mathrm{Fil}^{\bullet}, W_{\bullet})$, consisting of a finitely generated $\mathbb{Z}$-module H, equipped with

- the *Hodge filtration*, a decreasing exhaustive and separated filtration $\mathrm{Fil}^p H_{\mathbb{C}}$ on $H_{\mathbb{C}}$;
- the *weight filtration*, an increasing exhaustive and separated filtration $W_p H_{\mathbb{Q}}$ on $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$;

such that $\mathrm{Fil}^{\bullet}$ induces a pure Hodge structure of weight k on the kth graded piece $H_k := W_k H_{\mathbb{Q}} / W_{k-1} H_{\mathbb{Q}}$ of the weight filtration.

A morphism of mixed Hodge structures $\rho : A \to B$ is a homomorphism of $\mathbb{Z}$-modules inducing maps $\rho_h : \mathrm{Fil}^p A_{\mathbb{C}} \to \mathrm{Fil}^p B_{\mathbb{C}}$ and $\rho_w : W_p A_{\mathbb{Q}} \to W_p B_{\mathbb{Q}}$ for all $p \in \mathbb{Z}$. A homomorphism is an isomorphism if ρ, ρ_h, and ρ_w are all isomorphisms. A short exact sequence of mixed Hodge structures is a short exact sequence of $\mathbb{Z}$-modules

$$0 \to A \to C \to B \to 0$$

such that the induced sequences

$$0 \to \mathrm{Fil}^p A_{\mathbb{C}} \to \mathrm{Fil}^p C_{\mathbb{C}} \to \mathrm{Fil}^p B_{\mathbb{C}} \to 0$$

$$0 \to W_p A_{\mathbb{Q}} \to W_p C_{\mathbb{Q}} \to W_p B_{\mathbb{Q}} \to 0$$

on Hodge and weight filtrations are short exact for all $p \in \mathbb{Z}$. Write $\operatorname{Ext}^1_{\text{mhs}}(B, A)$ for the set of all such extensions C. We can also form the tensor product $A \otimes B$ by taking

$$\operatorname{Fil}^p(A_{\mathbb{C}} \otimes B_{\mathbb{C}}) = \sum_{p_1+p_2=p} \operatorname{Fil}^{p_1}(A_{\mathbb{C}}) \otimes \operatorname{Fil}^{p_2}(B_{\mathbb{C}}),$$

$$W_p(A_{\mathbb{C}} \otimes B_{\mathbb{C}}) = \sum_{p_1+p_2=p} W_{p_1}(A_{\mathbb{C}}) \otimes W_{p_2}(B_{\mathbb{C}}).$$

For any $j \in \mathbb{Z}$, define a pure Hodge structure $\mathbb{Z}(j)$ of weight $-2j$ by setting the underlying $\mathbb{Z}$-module to be $\mathbb{Z}$, and

$$\operatorname{Fil}^{-j}\mathbb{Z}(j)_{\mathbb{C}} = \mathbb{C},\ \operatorname{Fil}^{-j+1}\mathbb{Z}(j)_{\mathbb{C}} = 0,\ W_{-j}\mathbb{Z}(j)_{\mathbb{Q}} = \mathbb{Q},\ W_{-j+1}\mathbb{Z}(j)_{\mathbb{Q}} = 0.$$

For any mixed Hodge structure H, write $H(j) = H \otimes \mathbb{Z}(j)$. If H is pure of weight k, then $H(j)$ is also pure of weight $k - 2j$. The trivial Hodge structure $\mathbb{Z}(0)$ is simply written as $\mathbb{Z}$.

Now let H be a pure Hodge structure of negative weight. Suppose we have a short exact sequence

$$0 \to H \xrightarrow{i} E \xrightarrow{\rho} \mathbb{Z} \to 0, \tag{2.1}$$

that is, $E \in \operatorname{Ext}^1_{\text{mhs}}(\mathbb{Z}, H)$. Since $\operatorname{wt}(H) < \operatorname{wt}(\mathbb{Z}) = 0$, we must have

$$W_j E_{\mathbb{Q}} = \begin{cases} 0, & j < \operatorname{wt}(H), \\ i(H_{\mathbb{Q}}), & \operatorname{wt}(H) \leq j < 0, \\ E_{\mathbb{Q}}, & j \geq 0. \end{cases} \tag{2.2}$$

Choose elements η_E^{hodge} and η_E^{int} of $\operatorname{Fil}^0 E_{\mathbb{C}}$ and E, respectively, such that

$$\rho_{\mathbb{C}}(\eta_E^{\text{hodge}}) = 1, \qquad \rho(\eta_E^{\text{int}}) = 1.$$

Then the element $\eta_E := \eta_E^{\text{hodge}} - \eta_E^{\text{int}}$ is in the kernel of $\rho_{\mathbb{C}}$, and can be viewed as an element of $H_{\mathbb{C}}$. Furthermore, η_E^{hodge} and η_E^{int} are well-defined modulo $\operatorname{Fil}^0 H_{\mathbb{C}}$ and H, respectively, and thus the class of η_E in $H_{\mathbb{C}}/(\operatorname{Fil}^0 H_{\mathbb{C}} + H)$ does not depend on the choices of η_E^{hodge} and η_E^{int}.

Proposition 2.2.3. *The assignment $E \mapsto \eta_E$ yields an isomorphism*

$$\operatorname{Ext}^1_{\text{mhs}}(\mathbb{Z}, H) \simeq \frac{H_{\mathbb{C}}}{\operatorname{Fil}^0 H_{\mathbb{C}} + H}.$$

Proof. The choice of η_E^{int} determines a splitting of the sequence (2.1), making $M = H \oplus \mathbb{Z}$ with η_E^{int} corresponding to $(0, 1)$. The weight filtration is determined by (2.2), and so all that remains is to determine the Hodge filtration on $E_{\mathbb{C}}$ compatible with i and ρ. This is determined uniquely by the choice of element η_E^{hodge}, via

$$\operatorname{Fil}^j E_{\mathbb{C}} = \begin{cases} i(\operatorname{Fil}^j H_{\mathbb{C}}), & j > 0, \\ i(\operatorname{Fil}^j H_{\mathbb{C}}) + \mathbb{C}\eta_E^{\text{hodge}}, & j \leq 0. \end{cases}$$

Two choices η and η' for η_E^{hodge} induce the same filtration if and only if $\eta - \eta' \in \text{Fil}^0 H$. Additionally, two choices for η_E^{int} differ by an element $\eta \in H$, yielding an isomorphism of extensions

$$H \oplus \mathbb{Z} \xrightarrow{\sim} H \oplus \mathbb{Z}, \qquad (h, n) \mapsto (h + \eta, n).$$

This proves the proposition. □

Corollary 2.2.4. *For any smooth proper variety $V/\mathbb{C}$ of dimension d, we have*

$$\text{Ext}^1_{\text{mhs}}(\mathbb{Z}, H_B^{2c-1}(V/\mathbb{C})(c)) \simeq \frac{\text{Fil}^{d-c+1} H_{\text{dR}}^{2d-2c+1}(V/\mathbb{C})^\vee}{H_{2d-2c+1}(V^{\text{an}}, \mathbb{Z})}.$$

Proof. Apply Proposition 2.2.3 to $H = H_B^{2c-1}(V/\mathbb{C})(c)$ along with the isomorphisms

$$H_B^{2c-1}(V, \mathbb{Z}) \otimes \mathbb{C} \simeq H_B^{2c-1}(V, \mathbb{C}) \simeq H_{\text{dR}}^{2c-1}(V/\mathbb{C})$$

from Theorems 1.2.1 and 1.2.4 and Corollary 1.3.3, and the isomorphism

$$H_B^{2c-1}(V, \mathbb{Z}) \simeq H_{2d-2c+1}(V^{\text{an}}, \mathbb{Z})$$

from Theorem 1.2.2. The Poincaré pairing induces a duality

$$H_{\text{dR}}^{2c-1}(V/\mathbb{C})(c) \times H_{\text{dR}}^{2d-2c+1}(V/\mathbb{C})(d-c) \to \mathbb{C},$$

where $\text{Fil}^0 H_{\text{dR}}^{2c-1}(V/\mathbb{C})(c)$ and $\text{Fil}^{2d-2c+1} H_{\text{dR}}^{2d-2c+1}(V/\mathbb{C})(d-c) = \text{Fil}^{d-c+1} H_{\text{dR}}^{2d-2c+1}(V/\mathbb{C})$ are exact annihilators of each other. Thus

$$\frac{H_{\text{dR}}^{2c-1}(V/\mathbb{C})(c)}{\text{Fil}^0 H_{\text{dR}}^{2c-1}(V/\mathbb{C})(c)} \simeq \text{Fil}^{d-c+1} H_{\text{dR}}^{2d-2c+1}(V/\mathbb{C})^\vee.$$

This establishes the claim. □

The complex Abel–Jacobi map

Let $\Delta \in \text{CH}^c(V)_0(\mathbb{C})$, and denote by $|\Delta|$ the support of Δ. From the long exact sequence (1.5), we have a short exact sequence

$$0 \to H_B^{2c-1}(V, \mathbb{Z})(c) \to H_B^{2c-1}(V - |\Delta|, \mathbb{Z})(c) \to H_{|\Delta|}^{2c}(V, \mathbb{Z})(c)_0 \to 0,$$

where $H_{|\Delta|}^{2c}(V, \mathbb{Z})(c)_0$ is the kernel of the map $H_{|\Delta|}^{2c}(V, \mathbb{Z})(c) \to H_B^{2c}(V, \mathbb{Z})(c)$. By the construction of cl_B in §2.1, $\text{cl}_B(\Delta)$ is the image of a cohomology class from $H_{|\Delta|}^{2c}(V)(c)$, and since $\Delta \in \text{CH}^c(V)_0(\mathbb{C})$ is cohomologically trivial, this class is an element of $H_{|\Delta|}^{2c}(V, \mathbb{Z})(c)_0$. This allows us to form an element $E_\Delta \in \text{Ext}^1_{\text{mhs}}(\mathbb{Z}, H_B^{2c-1}(V, \mathbb{Z})(c))$ by pulling back along cl_B in the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_B^{2c-1}(V, \mathbb{Z})(c) & \longrightarrow & E_\Delta & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow {\scriptstyle \text{cl}_B} & & \\
0 & \longrightarrow & H_B^{2c-1}(V, \mathbb{Z})(c) & \longrightarrow & H_B^{2c-1}(V - |\Delta|, \mathbb{Z})(c) & \longrightarrow & H_{|\Delta|}^{2c}(V, \mathbb{Z})(c)_0 & \longrightarrow & 0.
\end{array}$$

Definition 2.2.5. The *complex Abel–Jacobi map*

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0(\mathbb{C}) \to \mathrm{Ext}^1_{\mathrm{mhs}}(\mathbb{Z}, H_B^{2c-1}(V,\mathbb{Z})(c)) \simeq \frac{\mathrm{Fil}^{d-c+1} H^{2d-2c+1}_{\mathrm{dR}}(V/\mathbb{C})^\vee}{H_{2d-2c+1}(V,\mathbb{Z})}$$

is the map sending Δ to the extension class E_Δ.

Suppose that $W/\mathbb{C}$ is a variety of dimension d', and let $\Pi \in \mathrm{CH}^{d+c'-c}(V \times W)(\mathbb{C})$ for some $0 \le c' \le d'$. As in §1.1 and §1.3, Π induces maps

$$\Pi_* : \mathrm{CH}^c(V)_0(\mathbb{C}) \to \mathrm{CH}^{c'}(W)_0(\mathbb{C}),$$
$$\Pi^* : \mathrm{Fil}^{d'-c'+1} H^{2d'-2c'+1}_{\mathrm{dR}}(W/\mathbb{C}) \to \mathrm{Fil}^{d-c+1} H^{2d-2c+1}_{\mathrm{dR}}(V/\mathbb{C}).$$

The next proposition establishes a property of $\mathrm{AJ}_{\mathbb{C}}$ that will be useful later.

Proposition 2.2.6. *Let* $\omega \in \mathrm{Fil}^{d'-c'+1} H^{2d'-2c'+1}_{\mathrm{dR}}(W/\mathbb{C})$. *Then*

$$\mathrm{AJ}_{\mathbb{C}}(\Pi_*\Delta)(\omega) = \mathrm{AJ}_{\mathbb{C}}(\Delta)(\Pi^*\omega)$$

as elements of $\mathrm{Fil}^{d'-c'+1} H^{2d'-2c'+1}_{\mathrm{dR}}(W/\mathbb{C})^\vee / H_{2d'-2c'+1}(W,\mathbb{Z})$.

Proof. Unwinding Definition 2.2.5 and Corollary 2.2.4 shows that

$$\mathrm{AJ}_{\mathbb{C}}(\Pi_*\Delta)(\omega) = \langle \eta_{E_{\Pi_*\Delta}}, \omega \rangle_W \pmod{H_{2d'-2c'+1}(W,\mathbb{Z})},$$
$$\mathrm{AJ}_{\mathbb{C}}(\Delta)(\Pi^*\omega) = \langle \eta_{E_\Delta}, \Pi^*\omega \rangle_V \pmod{H_{2d-2c+1}(V,\mathbb{Z})}.$$

The adjoint of Π^* is the map from §1.3

$$\Pi_* : H^{2c-1}_{\mathrm{dR}}(V)(c) \to H^{2c'-1}_{\mathrm{dR}}(W)(c'),$$

and hence we can rewrite the second line as

$$\mathrm{AJ}_{\mathbb{C}}(\Delta)(\Pi^*\omega) = \langle \Pi_*\eta_{E_\Delta}, \omega \rangle_W \pmod{H_{2d'-2c'+1}(V,\mathbb{Z})}.$$

Thus, it suffices to show that

$$\Pi_*\eta_{E_\Delta} = \eta_{E_{\Pi_*\Delta}} \pmod{\mathrm{Fil}^0 H^{2c'-1}_{\mathrm{dR}}(W/\mathbb{C})(c') + H_{2c'-1}(W,\mathbb{Z})(c')}.$$

This can be established from the definitions using a diagram chase, and is left to the reader. □

Theorem 2.2.7. *The complex Abel–Jacobi map is given by*

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0(\mathbb{C}) \to \frac{\mathrm{Fil}^{d-c+1} H^{2d-2c+1}_{\mathrm{dR}}(V/\mathbb{C})^\vee}{H_{2d-2c+1}(V,\mathbb{Z})}, \qquad \Delta \mapsto \int_{\partial^{-1}\Delta},$$

where $\partial^{-1}\Delta$ *is a* $2d-2c+1$ *real dimensional piecewise differentiable chain on the real manifold* $V(\mathbb{C})$ *with boundary* Δ.

Proof. See §12.3.3 in [Voi07]. □

Remark. Originally the complex Abel–Jacobi map was *defined* as in Theorem 2.2.7, and then later was it proved to be equivalent to Definition 2.2.5. The reason we are presenting in this manner is to mirror our treatment of the p-adic Abel–Jacobi map in the next section, which was first defined before a suitable p-adic integration theory existed in higher dimensions.

2.3 The p-adic Abel–Jacobi map

Filtered Frobenius modules

Now we turn to filtered Frobenius modules, the p-adic counterpart of mixed Hodge structures. They will be useful in defining the p-adic Abel–Jacobi map. Let p be a prime, F a finite unramified extension of $\mathbb{Q}_p$ of degree n, and V/F a variety extending to a smooth proper model $\mathcal{V}$ over $\mathcal{O}_F$, the ring of integers of F. Denote by σ the Frobenius automorphism of $F/\mathbb{Q}_p$, the unique element of $\mathrm{Gal}\,(F/\mathbb{Q}_p)$ lifting the p-power automorphism of the residue field of $\mathcal{O}_F$.

Definition 2.3.1. A *filtered Frobenius module* over F is a finite-dimensional F-vector space D equipped with

(1) A decreasing exhaustive and separated filtration

$$D \supset \cdots \supset \mathrm{Fil}^q D \supset \mathrm{Fil}^{q+1} D \supset \cdots, \ q \in \mathbb{Z}$$

called the *Hodge filtration.*

(2) An invertible σ-linear operator

$$\phi : D \to D$$

called the *Frobenius morphism.*

We say that a filtered Frobenius module is *pure of weight* k if the eigenvalues of ϕ^n acting on D are Weil numbers with complex absolute value $p^{nk/2}$.

Note that this last definition makes sense, as $\sigma^n = \mathrm{id}$ on F and hence ϕ^n is F-linear. It is a nontrivial fact that $H^k_{\mathrm{dR}}(V/F)$ is a filtered Frobenius module of pure weight k, with the Hodge filtration coming from (1.8) and the Frobenius morphism coming from $H^k_{\mathrm{crys}}(\tilde{V}/\mathcal{O}_F)$ and the comparison isomorphism

$$H^k_{\mathrm{crys}}(\tilde{V}/\mathcal{O}_F) \otimes_{\mathcal{O}_F} F \simeq H^k_{\mathrm{dR}}(V/F),$$

where $\tilde{V}$ is the special fiber of $\mathcal{V}$.

A morphism $\rho : D_1 \to D_2$ of filtered Frobenius modules is a map of filtered vector spaces such that $\rho \circ \phi_1 = \phi_2 \circ \rho$, and ρ is an isomorphism if all the induced maps $\rho : \mathrm{Fil}^q D_1 \to \mathrm{Fil}^q D_2$ are isomorphisms. A sequence

$$0 \to D_1 \to D_3 \to D_2 \to 0$$

is exact if the induced sequences

$$0 \to \mathrm{Fil}^q D_1 \to \mathrm{Fil}^q D_3 \to \mathrm{Fil}^q D_2 \to 0$$

are all exact. The set of isomorphism classes of such D_3 is denoted by $\mathrm{Ext}^1_{\mathrm{ffm}}(D_2, D_1)$. The tensor product $D_1 \otimes D_2$ is defined by setting

$$\phi(d_1 \otimes d_2) = \phi_1(d_1) \otimes \phi_2(d_2),$$

$$\mathrm{Fil}^q(D_1 \otimes D_2) = \sum_{q_1+q_2=q} \mathrm{Fil}^{q_1}(D_1) \otimes \mathrm{Fil}^{q_2}(D_2).$$

For any $j \in \mathbb{Z}$, define a filtered Frobenius module $F(j)$ of pure weight $-2j$ with underlying vector space F by setting

$$\phi(x) = p^{-j}\sigma(x),\ \mathrm{Fil}^{-j}F(j) = F(j),\ \mathrm{Fil}^{-j+1}F(j) = 0.$$

For any filtered Frobenius module D, write $D(j)$ for $D \otimes F(j)$. If D is of pure weight k, then $D(j)$ is pure of weight $k - 2j$. Finally, simply write F for the trivial filtered Frobenius module $F(0)$.

Let D be a filtered Frobenius module of pure negative weight. Suppose we have a short exact sequence

$$0 \to D \xrightarrow{i} E \xrightarrow{\rho} F \to 0, \tag{2.3}$$

that is, $E \in \mathrm{Ext}^1_{\mathrm{ffm}}(F, D)$. Since $\mathrm{wt}(D) < 0$, restriction to the subspace on which ϕ^n acts as the identity yields

$$E^{\phi^n=1} \xrightarrow{\sim} F. \tag{2.4}$$

Choose elements η_E^{hodge} and η_E^{frob} of $\mathrm{Fil}^0 E$ and $E^{\phi^n=1}$, respectively, such that

$$\rho(\eta_E^{\mathrm{hodge}}) = 1, \qquad \rho(\eta_E^{\mathrm{frob}}) = 1.$$

Then the element $\eta_E := \eta_E^{\mathrm{hodge}} - \eta_E^{\mathrm{frob}}$ is in the kernel of ρ, and can be viewed as an element of D. Furthermore, η_E^{hodge} is well-defined modulo $\mathrm{Fil}^0 D$ and η_E^{frob} is uniquely determined by (2.4), and thus the class of η_E in $D/(\mathrm{Fil}^0 D)$ does not depend on the choice of η_E^{hodge}.

Proposition 2.3.2. *The assignment $E \mapsto \eta_E$ yields an isomorphism*

$$\mathrm{Ext}^1_{\mathrm{ffm}}(F, D) \simeq \frac{D}{\mathrm{Fil}^0 D}.$$

Proof. The relation (2.4) gives a splitting of (2.3) as Frobenius modules. Thus, $E \simeq D \oplus F$ with η_E^{frob} corresponding to the element $(0, 1)$, and $\phi_E(d, f) = (\phi_D(d), \sigma(f))$. This determines all of the data for E as a filtered Frobenius module except for the Hodge filtration. The choice of filtration compatible with i and ρ is uniquely determined from the choice of η_E^{hodge} by setting

$$\mathrm{Fil}^j E = \begin{cases} i(\mathrm{Fil}^j D), & j > 0, \\ i(\mathrm{Fil}^j D) + F\eta_E^{\mathrm{hodge}}, & j \leq 0. \end{cases}$$

Two different choices η and η' for η_E^{hodge} yield the same filtration if and only if $\eta - \eta' \in \mathrm{Fil}^0 D$. This proves the proposition. □

Corollary 2.3.3. *For any smooth proper variety V/F of dimension d with good reduction, we have*

$$\mathrm{Ext}^1_{\mathrm{ffm}}(F, H^{2c-1}_{\mathrm{dR}}(V/F)(c)) \simeq \mathrm{Fil}^{d-c+1} H^{2d-2c+1}_{\mathrm{dR}}(V/F)^\vee.$$

Proof. The proof is similar to that of Corollary 2.2.4, only using Proposition 2.3.2 instead of Proposition 2.2.3 and without the extra group of periods. □

The p-adic Abel–Jacobi map

Let $\Delta \in \mathrm{CH}^c(V)_0(F)$, and denote by $|\Delta|$ the support of Δ. From the long exact sequence (1.7), we have a short exact sequence

$$0 \to H^{2c-1}_{\mathrm{dR}}(V/F)(c) \to H^{2c-1}_{\mathrm{dR}}(V - |\Delta|/F)(c) \to H^{2c}_{|\Delta|}(V/F)(c)_0 \to 0,$$

where $H^{2c}_{|\Delta|}(V/F)(c)_0$ is the kernel of the map $H^{2c}_{|\Delta|}(V/F)(c) \to H^{2c}_{B}(V/F)(c)$. By the construction of $\mathrm{cl}_{\mathrm{dR}}$ in §2.1, $\mathrm{cl}_{\mathrm{dR}}(\Delta)$ is the image of a cohomology class from $H^{2c}_{|\Delta|}(V/F)(c)$, and since $\Delta \in \mathrm{CH}^c(V)_0(F)$, this class is an element of $H^{2c}_{|\Delta|}(V/F)(c)_0$. This allows us to form an element $E_\Delta \in \mathrm{Ext}^1_{\mathrm{ffm}}(F, H^{2c-1}_{\mathrm{dR}}(V/F)(c))$ by pulling back along $\mathrm{cl}_{\mathrm{dR}}$ in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(V/F)(c) & \longrightarrow & E_\Delta & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \mathrm{cl}_{\mathrm{dR}} & & \\ 0 & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(V/F)(c) & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(V - |\Delta|/F)(c) & \longrightarrow & H^{2c}_{|\Delta|}(V/F)(c)_0 & \longrightarrow & 0. \end{array}$$

Definition 2.3.4. The *p-adic Abel–Jacobi map*

$$\mathrm{AJ}_p : \mathrm{CH}^c(V)_0(F) \to \mathrm{Ext}^1_{\mathrm{ffm}}(F, H^{2c-1}_{\mathrm{dR}}(V/F)(c)) \simeq \mathrm{Fil}^{d-c+1} H^{2d-2c+1}_{\mathrm{dR}}(V/F)^\vee$$

is the map sending Δ to the extension class E_Δ.

Just as in the complex case, the p-adic Abel–Jacobi map satisfies nice functorial properties. If W/F is another variety of dimension d', $0 \le c' \le d'$, and $\Pi \in \mathrm{CH}^{d+c'-c}(V \times W)(F)$, then we have the following:

Proposition 2.3.5. *Let $\omega \in \mathrm{Fil}^{d'-c'+1} H^{2d'-2c'+1}_{\mathrm{dR}}(W/F)$. Then*

$$\mathrm{AJ}_p(\Pi_* \Delta)(\omega) = \mathrm{AJ}_p(\Delta)(\Pi^* \omega)$$

as elements of $\mathrm{Fil}^{d'-c'+1} H^{2d'-2c'+1}_{\mathrm{dR}}(W/F)^\vee$.

Proof. The proof is identical to that of Proposition 2.2.6, only without the extra period lattice. □

Theorem 2.3.6 ([Bes00], Theorem 1.2)**.** *The p-adic Abel–Jacobi map is given by*

$$\mathrm{AJ}_p : \mathrm{CH}^c(V)_0(F) \to \mathrm{Fil}^{d-c+1} H_{\mathrm{dR}}^{2d-2c+1}(V/F)^\vee, \qquad \Delta \mapsto \int_\Delta,$$

where $\int_\Delta$ is Besser's generalization of Coleman integration.

Proof. See Theorem 1.2 in [Bes00]. □

Remark. We will not need any of the details of Besser's p-adic integration theory, other than that it extends Coleman's original theory [Col85]. In particular, when V is a curve, the p-adic Abel–Jacobi map is given by Coleman integration:

$$\mathrm{AJ}_p : \mathrm{CH}^1(V)_0(F) \to \mathrm{Fil}^1 H_{\mathrm{dR}}^1(V/F)^\vee, \qquad [P] - [Q] \mapsto \int_Q^P .$$

Chapter 3

Diagonal cycles and Chow–Heegner points

3.1 Chow–Heegner points

In this section, we define the notion of a Chow–Heegner point on the Jacobian variety of any smooth proper curve X/K. The general definition is very broad, which allows us to encompass a large class of points. The downside is that very little can be said about the basic properties of these points in this context. Thus we will specialize the construction to the case where the Chow–Heegner points arise from modified diagonal cycles on the triple product of the curve, then further specialize to the case where $X = X_0(N)$ is a modular curve over $\mathbb{Q}$.

General definition

By a curve over K, we shall mean a variety over K of dimension one. Let X/K be a curve and V/K a variety of dimension d, both smooth and proper. For any $0 \leq c \leq d$, given any $\Pi \in \mathrm{CH}^{d+1-c}(V \times X)(K)$, we have the map

$$\Pi_* : \mathrm{CH}^c(V)_0(K) \to \mathrm{CH}^1(X)_0(K)$$

from the end of §2.1.

Definition 3.1.1. Let (V, Π, Δ) be a triple where V and Π are as above and $\Delta \in \mathrm{CH}^c(V)_0(K)$. The *Chow–Heegner point* on the Jacobian J_X of X associated to (V, Π, Δ) is given by $\Pi_*(\Delta) \in \mathrm{CH}^1(X)_0(K) \simeq J_X(K)$. The image of $\Pi_*(\Delta)$ on quotients of J_X are still referred to as Chow–Heegner points.

Example 3.1.2. Let $X_0(N)/\mathbb{Q}$ denote the modular curve and $f \in S_2(N, \chi)$ be a newform. Denote by A_f the abelian variety associated to f, a quotient of $J_0(N)$. Choosing $X = V = X_0(N)$, Π the diagonal cycle on $X \times X$, and $\Delta = ([\tau] - [\infty]) \in \mathrm{CH}^1(X_0(N))_0(H_{\mathcal{O}_\tau})$ a

Heegner divisor defined over the ray class field $H_{\mathcal{O}_\tau}$ of $\mathcal{O}_\tau$, we recover the classical Heegner point $P_\tau \in A_f(H_{\mathcal{O}_\tau})$ from the previous definition.

The modified diagonal cycle

We now specialize the construction in section 3.1 to the case where $V = X^3$ and Δ is the diagonal on X^3, suitably modified to make it cohomologically trivial. This cycle was first studied in [GK92] and [GS95].

Fix a point $o \in X(K)$ and let $J \subset \{1, 2, 3\}$. Write $i_o : X \to X$ for the constant morphism with image o. Denote by i_J the morphism from X to X^3 determined by id_X for the factors X_i with $i \in J$ and by i_o for the factors with $i \notin J$. Finally, write X_J for the image of i_J. We will write X_{123}, X_{12}, $X_1, \ldots$ for $X_{\{1,2,3\}}$, $X_{\{1,2\}}$, $X_{\{1\}}, \ldots$ to simplify notation.

Definition 3.1.3. The *Gross-Kudla-Schoen modified diagonal cycle* Δ_{GKS} is the sum $X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3$.

Each cycle appearing in the definition of Δ_{GKS} has codimension 2. Also, since $o \in X(K)$, Δ_{GKS} is defined over K, and thus $\Delta_{GKS} \in \mathrm{CH}^2(X^3)(K)$. Actually, we have $\Delta_{GKS} \in \mathrm{CH}^2(X^3)_0(K)$, as the next proposition shows.

Proposition 3.1.4. *The cycle Δ_{GKS} is cohomologically trivial.*

Proof. Let $\mathrm{pr}_i : X^3 \to X$ denote the i-th projection map and $\mathrm{pr}_o : X_3 \to X$ the constant map with image o. For each $J \subset \{1, 2, 3\}$, define a map $p_J : X^3 \to X^3$ as the product of the maps pr_i if $i \in J$ and pr_o otherwise for $i = 1, 2, 3$. Define $p_e = \sum_{J \neq \emptyset} (-1)^{|J|+1} p_{J,*}$ as an endomorphism of $\mathrm{CH}^2(X^3)$. Then $p_e(X_{123}) = \Delta_{GKS}$. Now, the Künneth formula (1.6) gives us a decomposition

$$H^4_{\mathrm{dR}}(X^3/K) = \bigoplus_{n_1+n_2+n_3=4} H^{n_1}_{\mathrm{dR}}(X/K) \otimes H^{n_2}_{\mathrm{dR}}(X/K) \otimes H^{n_3}_{\mathrm{dR}}(X/K).$$

As a map on $H^4_{\mathrm{dR}}(X^3/K)$, $p_{J,*}$ acts as the identity on a direct summand if and only if $\sum_{i \in J} n_i = 4 - 2(3 - |J|) = 2|J| - 2$, and otherwise is the zero map. As $n_1 + n_2 + n_3 = 4$, this condition is equivalent to $n_i = 2$ for all $i \notin J$, or $J \supset \{i \mid n_i \neq 2\}$. An easy computation shows that, for any such summand,

$$\sum_{J \supset \{i \mid n_i \neq 2\}} (-1)^{|J|+1} = 0.$$

Therefore, p_e is the zero map on $H^4_{\mathrm{dR}}(X^3/K)$. By Proposition 2.1.3, p_e commutes with the cycle class map, and thus

$$\mathrm{cl}(\Delta_{GKS}) = \mathrm{cl}(p_e(X_{123})) = p_e \mathrm{cl}(X_{123}) = 0$$

since $\mathrm{cl}(X_{123}) \in H^4_{\mathrm{dR}}(X^3/K)(2)$. □

All that remains to define a Chow–Heegner point is to give a cycle $\Pi \in \mathrm{CH}^2(X^4)(K)$. For any cycle $T \in \mathrm{CH}^1(X^2)(K)$, we can choose $\Pi_T = T \times X_{34}$, where X_{34} is the diagonal in the last two factors. The triple $(X^3, \Pi_T, \Delta_{GKS})$ and choice of π_A gives rise to a Chow–Heegner point $P(T, \pi_A, o) \in J_X(K)$, where o emphasizes the dependence of Δ_{GKS} on the point o.

Remark 3.1.5. Rather than choosing $T \in \mathrm{CH}^1(X^2)(K)$, we may in the future use $T \in \mathrm{CH}^1(X^2)(K) \otimes \mathbb{Q}$, and the resulting point lands in $J_X(K) \otimes \mathbb{Q}$. Then there exists a positive integer $n \geq 1$ such that $nT \in \mathrm{CH}^1(X^2)(K)$, and so we define $P(T, \pi_A, o) = P(nT, \pi_A, o) \otimes 1/n \in J_X(K) \otimes \mathbb{Q}$.

We end this section with an alternate definition of the point $P(T, \pi_A, o)$ without reference to the cycle Δ_{GKS}. Continuing the notation from earlier, for any subset $J \subset \{1, 2\}$, let i_J denote the inclusion of X in X^2 as defined before Definition 3.1.3 with respect to the base point o. Set

$$T_{12} = i_{12}^*(T),$$
$$T_1 = i_1^*(T),$$
$$T_2 = i_2^*(T),$$
$$P(T, \pi_A, o) = \pi_A(T_{12} - T_1 - T_2 - \deg(T_{12} - T_1 - T_2)o).$$

Proposition 3.1.6. *The two definitions of $P(T, \pi_A, o)$ agree.*

Proof. Set $\Pi = \Pi_T$. Straight from the definitions, one computes that

$$\Pi_*(X_{123}) = T_{12},$$
$$\Pi_*(X_{12}) = \deg(T_{12})o,$$
$$\Pi_*(X_{13}) = T_1,$$
$$\Pi_*(X_{23}) = T_2,$$
$$\Pi_*(X_1) = \deg(T_1)o,$$
$$\Pi_*(X_2) = \deg(T_2)o,$$
$$\Pi_*(X_3) = 0.$$

This proves the proposition. □

Example 3.1.7. Let T be the divisor $i_{1*}(X)$, the curve X embedded into the first factor of X^2. Then $T_{12} = o$, $T_1 = 0$, and $T_2 = o$, and hence $P(T, \pi_A, o) = 0$. More generally, let $\mathrm{pr}_i : X^2 \to X$ denote the i-th projection. Then $P(T, \pi_A, o) = 0$ for any divisor

$$T \in \mathrm{pr}_1^*\mathrm{CH}^1(X) + \mathrm{pr}_2^*\mathrm{CH}^1(X).$$

Such divisors are called the horizontal and vertical divisors on X^2.

3.2 Chow–Heegner points on modular abelian varieties

For the remainder of this chapter, we will restrict our attention to the construction from §3.1 in the case where $X = X_0(N)$ is the modular curve, $A = A_f$ is the abelian variety associated to a weight 2 cuspidal eigenform f, and T is a correspondence in the g-isotypic component $\mathrm{CH}^1(X \times X)(\mathbb{Q})[g] \otimes \mathbb{Q}$ for some weight 2 cuspidal normalized eigenform g not $G_{\mathbb{Q}}$-conjugate to f.

Cusp forms and Hecke correspondences

For any positive integer N, let $\Gamma_0(N)$ denote the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined in the introduction, and write $X_0(N)$ for the associated modular curve over $\mathbb{Q}$. Let $\mathbb{T} = \mathbb{Z}[\ldots, T_n, \ldots]$ denote the subalgebra of $\mathrm{End}_{\mathbb{Q}}(S_2(\Gamma_0(N)))$ generated by the Hecke correspondences on $X_0(N)$, and write $\mathbb{T}'$ for the subalgebra generated by the T_n with $\gcd(n, N) = 1$. Additionally, write $\mathbb{T}_{\mathbb{Q}} = \mathbb{T} \otimes \mathbb{Q}$ and $\mathbb{T}'_{\mathbb{Q}} = \mathbb{T}' \otimes \mathbb{Q}$. Often we will write $\mathbb{T}_{\mathbb{Z}}$ and $\mathbb{T}'_{\mathbb{Z}}$ in place of $\mathbb{T}$ and $\mathbb{T}'$ for emphasis. The map $T_n \mapsto (a_n(h))$ induces an isomorphism

$$\mathbb{T}'_{\mathbb{Q}} \simeq \bigoplus_h K_h \tag{3.1}$$

where h runs over all $G_{\mathbb{Q}}$-conjugacy classes of newforms $h \in S_2(\Gamma_0(M))$ for all $M \mid N$, and $K_h = \mathbb{Q}(\ldots, a_n(h), \ldots)$. For any eigenform g, denote by e_g idempotent of $\mathbb{T}'_{\mathbb{Q}}$ whose image in the right-hand side of 3.1 is the element with a 1 in the K_g factor and a 0 everywhere else.

Let $J_0(N)$ be the Jacobian of $X_0(N)$, and write $\mathrm{End}^0_{\mathbb{Q}}(J_0(N)) = \mathrm{End}_{\mathbb{Q}}(J_0(N)) \otimes \mathbb{Q}$. As correspondences on $X_0(N)$ give rise to endomorphisms on $J_0(N)$, $\mathrm{End}^0_{\mathbb{Q}}(J_0(N))$ contains $\mathbb{T}_{\mathbb{Q}}$, and thus we can view e_g as an element of $\mathrm{End}^0_{\mathbb{Q}}(J_0(N))$, where it remains an idempotent. We will denote by $\mathrm{End}^0_{\mathbb{Q}}(J_0(N))[g]$ the g-isotypic component $e_g \cdot \mathrm{End}^0_{\mathbb{Q}}(J_0(N))$. There is a natural isomorphism

$$\mathrm{End}^0(J_0(N)) \simeq (\mathrm{CH}^1(X_0(N)^2) \otimes \mathbb{Q})/(\mathrm{pr}_1^*\mathrm{CH}^1(X_0(N)) \otimes \mathbb{Q} + \mathrm{pr}_2^*\mathrm{CH}^1(X_0(N)) \otimes \mathbb{Q}).$$

Define $\mathrm{CH}^1(X_0(N) \times X_0(N))(\mathbb{Q})[g] \otimes \mathbb{Q}$ to be the group of cycles mapping to $\mathrm{End}^0_{\mathbb{Q}}(J_0(N))[g]$ as above modulo vertical and horizontal divisors. For every $T \in \mathrm{End}^0(J_0(N))[g]$, we can associate to it a cycle in $\mathrm{CH}^1(X_0(N) \times X_0(N))(\mathbb{Q})[g] \otimes \mathbb{Q}$, also denoted T by abuse of notation. When $T = e_g$, we denote this cycle by T_g.

Finally, let $f \in S_2(\Gamma_0(N_f))$ be a newform of level $N_f \mid N$ with abelian variety A_f, arising as a quotient of $J_0(N_f)$. We assume that f and g are not $G_{\mathbb{Q}}$-conjugate. Write $\pi_f : J_0(N_f) \to A_f$ for the quotient map with connected kernel. For each $d \mid N/N_f$, there is a degeneration map $\pi_d : X_0(N) \to X_0(N_f)$ such that $\pi_d^* f(q) = df(q^d)$ on q-expansions. This induces a map $J_0(N) \to J_0(N_f)$, which we will also denote π_d by abuse of notation. Denote by π_f^d the composition $\pi_f Z \pi_d : J_0(N) \to A_f$.

Definition 3.2.1. Let f and d be as above and $T \in \mathrm{CH}^1(X_0(N) \times X_0(N))[g] \otimes \mathbb{Q}$ for some $g \neq f^\sigma$ for all $\sigma \in G_{\mathbb{Q}}$. Then the Chow–Heegner point associated to f and T is

$$P_{T,f_d} := P(T, \pi_f^d, \infty) \in A_f(\mathbb{Q}) \otimes \mathbb{Q}$$

where $P(T, \pi_f^d, \infty) \in J_0(N)(\mathbb{Q}) \otimes \mathbb{Q}$ is the point defined in section 3.1 using π_f^d for the quotient map and the cusp ∞ for the choice of base point o. When $T = T_g$ or $d = 1$, we write P_{g,f_d} or $P_{T,f}$, respectively.

Remark 3.2.2. By Example 3.1.7, the point associated to T is well defined modulo vertical and horizontal divisors, justifying our definition of $\mathrm{CH}^1(X_0(N) \times X_0(N))(\mathbb{Q})[g] \otimes \mathbb{Q}$.

Although it is unclear when P_{T,f_d} is nontrivial for a particular T and d, the following theorem gives a criterion for when P_{T,f_d} is nonzero for some $T \in \mathrm{CH}^1(X_0(N)^2)(\mathbb{Q})[g]$ and $d \mid N/N_f$. Let $L(g, g, f, s)$ denote Garrett's triple product L-function; the interested reader may consult [Gar87], most notably Theorem 1.3.

Theorem 3.2.3 ([DRS12], Corollary 1.4)**.** *Suppose $K_f = \mathbb{Q}$, so A_f is an elliptic curve. Assume that the local signs $\varepsilon_p(g, g, f)$ of the L-function $L(g, g, f, s)$ are $+1$ at all primes $p \mid N$. Then the module of points*

$$\underline{P}_{g,f} := \langle P_{T,f_d} : T \in \mathrm{CH}^1(X_0(N)^2)[g] \otimes \mathbb{Q}, d \mid N/N_f \rangle \subset A_f(\mathbb{Q}) \otimes \mathbb{Q}$$

is nonzero (or equivalently a multiple of some $P_{T,f_d} \in A_f(\mathbb{Q})$ is non-torsion) if and only if the following conditions hold:

i. $L(f, 1) = 0$,

ii. $L'(f, 1) \neq 0$, *and*

iii. $L(f \otimes \mathrm{Sym}^2 g^\sigma, 2) \neq 0$ *for all* $\sigma \in G_{\mathbb{Q}}$.

Remark 3.2.4. When $\varepsilon_p(g, g, f) = -1$ for some $p \mid N$, computations suggest that $\underline{P}_{g,f} = 0$. We will prove this in many cases in Theorem 3.3.8.

3.3 Zhang points

We end this chapter with one more definition of a collection of points on an abelian variety, the *Zhang points*. We proceed to establish some basic properties of Zhang points, as well as relate them to the points P_T from §3.1.

Definition of Zhang points

Let X/K be a curve, $o \in X(K)$ a K-rational point, J_X/K the jacobian of X, and $\phi_o : X \to J_X$ the Albanese morphism determined by the point o. Suppose that A and B are abelian variety quotients of J_X satisfying $\mathrm{Hom}_K(B, A) = 0$, and fix (possibly not surjective) maps $\pi_A : J_X \to A$ and $\pi_B : J_X \to B$. Finally, let $\mathcal{L} \in \mathrm{Pic}(B)(K)$ be an invertible sheaf.

Definition 3.3.1. Set $n = \deg(\phi^* \pi_B^* \mathcal{L})$. The *Zhang point* associated to $\mathcal{L}$ and o is defined to be the point

$$P(\mathcal{L}, \pi_A, \pi_B, o) := \pi_A((\phi_o^* \pi_B^* \mathcal{L}) \otimes \mathcal{O}_X([o])^{-n}) \in A(K).$$

Here $\mathcal{O}_X([o])$ is the invertible sheaf on X associated to the divisor $[o]$, and we identify $\mathrm{Pic}^0(X)(K)$ with $J_X(K)$.

Proposition 3.3.2. *For any $\mathcal{L} \in \mathrm{Pic}^0(B)$, we have $P(\mathcal{L}, \pi_A, \pi_B, o) = 0$.*

Proof. From the definition, $P(\mathcal{L}, \pi_A, \pi_B, o)$ is the image of $\mathcal{L}$ under the composition

$$\pi_A \circ \phi_o^* \circ \pi_B^* : \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(J_X) \to \mathrm{Pic}^0(X) = J_X \to A.$$

This morphism gives an element of $\mathrm{Hom}_K(B^\vee, A)$ by identifying $\mathrm{Pic}^0(B)$ with $B^\vee$. Since B is a quotient of a Jacobian variety, it is isogenous to $B^\vee$, and hence $\mathrm{Hom}_K(B^\vee, A) = \mathrm{Hom}_K(B, A) = 0$. The proposition follows. □

Properties of Zhang points

Fix choices for π_A and o. Let $\mathrm{NS}(B)(K) := \mathrm{Pic}(B)(K)/\mathrm{Pic}^0(B)(K)$ denote the Néron-Severi group of B. By Proposition 3.3.2, the association $\mathcal{L} \mapsto P_{\mathcal{L}, \pi_A, \pi_B, o}$ induces a homomorphism

$$\mathrm{NS}(B)(K) \to A(K).$$

Recall from §3.1 the construction of the point $P(T, \pi_A, o)$ from a cycle $T \in \mathrm{CH}^1(X \times X)(K)$. We now relate these two points. To any $\mathcal{L} \in \mathrm{Pic}(B)(K)$ we associate $\wedge\mathcal{L} \in \mathrm{Pic}(B \times B)(K)$ by the formula

$$\wedge\mathcal{L} = m^*(\mathcal{L}) \otimes \mathrm{pr}_1^*(\mathcal{L})^{-1} \otimes \mathrm{pr}_2^*(\mathcal{L})^{-1}, \tag{3.2}$$

where $m : B \times B \to B$ is the group law, and $\mathrm{pr}_i : B \times B \to B$ is the i-th projection. Let $\psi, \rho : J_X \to B$ be two choices of quotient map, and set $\phi_\psi = \psi \circ \phi$ and $\phi_\rho = \rho \circ \phi$. To the triple $(\mathcal{L}, \psi, \rho)$, we associate a cycle

$$T_{\mathcal{L}, \psi, \rho} = (\phi_\psi \times \phi_\rho)^*(\wedge\mathcal{L}) \in \mathrm{Pic}(X \times X)(K) = \mathrm{CH}^1(X \times X)(K). \tag{3.3}$$

Proposition 3.3.3. *For all $\mathcal{L} \in \mathrm{Pic}(B)$, we have*

$$P(T_{\mathcal{L}, \psi, \rho}, \pi_A, o) = P(\mathcal{L}, \pi_A, \psi + \rho, o) - P(\mathcal{L}, \pi_A, \psi, o) - P(\mathcal{L}, \pi_A, \rho, o).$$

In particular, taking $\rho = \psi$ yields

$$P(T_{\mathcal{L}, \psi, \psi}, \pi_A, o) = 2P(\mathcal{L}, \pi_A, \psi, o).$$

Proof. To simplify notation, write T for $T_{\mathcal{L},\psi,\rho}$. Let i_1, i_2 denote the inclusions of X into the first and second factors of $X \times X$ with respect to the base point o, and i_{12} the diagonal. Proposition 3.1.6 shows that

$$P(T, \pi_A, o) = \pi_A(i_{12}^*(T) - i_1^*(T) - i_2^*(T) - \deg(i_{12}^*(T) - i_1^*(T) - i_2^*(T))o).$$

It is straightforward to show that

$$i_{12}^*(T) = \phi_{\psi+\rho}^*\mathcal{L} \otimes \phi_\psi^*\mathcal{L}^{-1} \otimes \phi_\rho^*\mathcal{L}^{-1}$$

and that $i_1^*(T)$ and $i_2^*(T)$ are trivial using the definition of T in (3.3). This proves the first part. For the second part, note that

$$P(\mathcal{L}, \pi_A, 2\psi, o) = P([2]_B^*\mathcal{L}, \pi_A, \psi, o)$$

where $[2]_B$ indicates multiplication by 2 on B. A consequence of the theorem of the cube is

$$[n]^*\mathcal{L} = \mathcal{L}^{\frac{n^2+n}{2}} \otimes [-1]^*\mathcal{L}^{\frac{n^2-n}{2}};$$

see [Mum08] Chapter 2, §6, Corollary 2. Additionally, $\mathcal{L}$ and $[-1]^*\mathcal{L}$ have the same image in $\mathrm{NS}(B)$; see *loc. cit.* for details. Thus, $[2]^*\mathcal{L} = \mathcal{L}^4$ in $\mathrm{NS}(B)(K)$ and so by Proposition 3.3.2,

$$P([2]_B^*\mathcal{L}, \pi_A, \psi, o) = 4P(\mathcal{L}, \pi_A, \psi, o).$$

Combining this with the first statement proves the proposition. □

Now, we turn to the issue of dependence on base point $o \in X(K)$. Let $\kappa \in X(K)$ denote another choice of base point (possibly equal to o), $\phi_o, \phi_\kappa : X \to J_X$ the corresponding maps.

Proposition 3.3.4. *Setting* $n = \deg(\phi_\kappa^*\pi_B^*\mathcal{L})$, *we have*

$$P(\mathcal{L}, \pi_A, \pi_B, \kappa) = P(\mathcal{L}, \pi_A, \pi_B, o) - n \cdot \pi_A(\phi_o(\kappa)).$$

In particular, if $\pi_A(\phi_o(\kappa))$ *is torsion, then* $P(\mathcal{L}, \pi_A, \pi_B, o)$ *is torsion if and only if* $P(\mathcal{L}, \pi_A, \pi_B, \kappa)$ *is torsion.*

Proof. Let $t_o : J_X \to J_X$ denote the translation-by-$\phi_\kappa(o)$ map, and $t_{o,B} : B \to B$ the translation-by-$\pi_B(\phi_\kappa(o))$ map. Then, we have

$$\phi_\kappa = t_o \circ \phi_o,$$

$$\pi_B \circ t_o = t_{o,B} \circ \pi_B.$$

Also, $P(t_{o,B}^*\mathcal{L}, \pi_A, \pi_B, o) = P(\mathcal{L}, \pi_A, \pi_B, o)$ by Proposition 3.3.2 since $\mathcal{L}$ and $t_{o,B}^*\mathcal{L}$ have the same image in $\mathrm{NS}(B)(K)$; see [Mum08]. Putting everything together yields

$$\begin{aligned}
P(\mathcal{L}, \pi_A, \pi_B, \kappa) &= \pi_A((\phi_\kappa^*\pi_B^*\mathcal{L}) \otimes \mathcal{O}_X([\kappa])^{-n}) \\
&= \pi_A((\phi_o^*t_o^*\pi_B^*\mathcal{L}) \otimes \mathcal{O}_X([o])^{-n} \otimes \mathcal{O}_X([o])^n \otimes \mathcal{O}_X([\kappa])^{-n}) \\
&= \pi_A((\phi_o^*\pi_B^*t_{o,B}^*\mathcal{L}) \otimes \mathcal{O}_X([o])^{-n}) + \pi_A(\mathcal{O}_X([o])^n \otimes \mathcal{O}_X([\kappa])^{-n}) \\
&= P(t_{o,B}^*\mathcal{L}, \pi_A, \pi_B, o) - n \cdot \pi_A(\phi_o(\kappa)) \\
&= P(\mathcal{L}, \pi_A, \pi_B, o) - n \cdot \pi_A(\phi_o(\kappa)).
\end{aligned}$$

This proves the proposition. □

Now suppose that Y/K is a curve and let $\alpha, \beta : Y \to X$ be morphisms. Then α and β induce homomorphisms $\alpha_*, \beta_* : J_Y \to J_X$ and $\alpha^*, \beta^* : J_X \to J_Y$. Thus A and B are quotients of J_Y via the morphisms $\pi_{A,\alpha} := \pi_A \circ \alpha_*$ and $\pi_{B,\beta} := \pi_B \circ \beta_*$. Furthermore, assume that $\alpha_*\beta^*(\ker \pi_A) \subset \ker \pi_A$, so that $\alpha_*\beta^*$ induces an endomorphism

$$\begin{array}{ccc} J_X & \xrightarrow{\alpha_*\beta^*} & J_X \\ \big\downarrow{\scriptstyle \pi_A} & & \big\downarrow{\scriptstyle \pi_A} \\ A & \xrightarrow{\theta_{\alpha\beta}} & A. \end{array}$$

Let $\tau \in \beta^{-1}(o)$ and write $\phi_\tau : Y \to J_Y$. Set $n = \deg(\phi_o^*\pi_B^*\mathcal{L})$ and $m = \deg(\beta)$, so that $\deg(\phi_\tau^*\pi_{B,\beta}^*\mathcal{L}) = mn$.

Proposition 3.3.5. *We have the relation*

$$P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\beta}, \tau) = \theta_{\alpha\beta}(P(\mathcal{L}, \pi_A, \pi_B, o)) + \sum_{\kappa \in \beta^{-1}(o)} n \cdot \pi_{A,\alpha}(\phi_\tau(\kappa)).$$

In particular,

$$\sum_{\tau \in \beta^{-1}(o)} P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\beta}, \tau) = m \cdot \theta_{\alpha\beta}(P(\mathcal{L}, \pi_A, \pi_B, o)).$$

Proof. For the first part, we have

$$\begin{aligned} P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\beta}, \tau) &= \pi_{A,\alpha}((\phi_\tau^*\pi_{B,\beta}^*\mathcal{L}) \otimes \mathcal{O}([\tau])^{-mn}) \\ &= \pi_{A,\alpha}((\phi_\tau^*(\beta_*)^*\pi_B^*\mathcal{L}) \otimes \beta^*\mathcal{O}([o])^{-n} \otimes \beta^*\mathcal{O}([o])^{n} \otimes \mathcal{O}([\tau])^{-mn}) \\ &= \pi_{A,\alpha}((\beta^*\phi_o^*\pi_B^*\mathcal{L}) \otimes \beta^*\mathcal{O}([o])^{-n}) + \pi_{A,\alpha}(\beta^*\mathcal{O}([o])^{n} \otimes \mathcal{O}([\tau])^{-mn})) \\ &= \pi_A(\alpha_*\beta^*((\phi_o^*\pi_B^*\mathcal{L}) \otimes \mathcal{O}([o])^{-n})) + \pi_{A,\alpha}((\bigotimes_{\kappa \in \beta^{-1}(o)} \mathcal{O}([\kappa])^{n}) \otimes \mathcal{O}([\tau])^{-mn}) \\ &= \theta_{\alpha\beta}(P(\mathcal{L}, \pi_A, \pi_B, o)) + \sum_{\kappa \in \beta^{-1}(o)} n \cdot \pi_{A,\alpha}(\phi_\tau(\kappa)) \end{aligned}$$

The second statement follows from the first by summing over all $\tau \in \beta^{-1}(o)$ along with the fact that

$$\sum_{\tau, \kappa \in \beta^{-1}(o)} \phi_\tau(\kappa) = 0.$$

This proves the proposition. □

Corollary 3.3.6. *Suppose $\pi_{A,\alpha}(\phi_\tau(\kappa))$ is torsion for all $\tau, \kappa \in \beta^{-1}(o)$. Then if $P(\mathcal{L}, \pi_A, \pi_B, o)$ is torsion, so is $P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\beta}, \tau)$ for any $\tau \in \beta^{-1}(o)$. Conversely, if in addition $\theta_{\alpha\beta}$ is an isogeny, then $P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\beta}, \tau)$ torsion for some $\tau \in \beta^{-1}(o)$ implies $P(\mathcal{L}, \pi_A, \pi_B, o)$ is torsion.*

Proof. This follows from the first part of Proposition 3.3.5 using the fact that $\theta_{\alpha\beta}(P(\mathcal{L}, \pi_A, \pi_B, o))$ is torsion if and only if $P(\mathcal{L}, \pi_A, \pi_B, o)$ is when $\theta_{\alpha\beta}$ is an isogeny. □

The vanishing of $P(\mathcal{L}, \pi_A, \pi_B, o)$

We conclude this chapter with a proof that Zhang points are torsion under suitable conditions, ultimately establishing Remark 3.2.4 in many cases. Let $w : X \to X$ be an automorphism such that $w_*(\ker(\pi_A)) \subset \ker(\pi_A)$ and $w_*(\ker(\pi_B)) \subset \ker(\pi_B)$, and write $\theta_{w,A} : A \to A$ and $\theta_{w,B} : B \to B$ for the induced morphisms. Furthermore, assume that $\theta_{w,A} = [-1]_A$ and $\theta^*_{w,B} : \mathrm{NS}(B)(K) \to \mathrm{NS}(B)(K)$ is the identity. Finally, write $\tau = w(o)$ and assume that $\pi_A(\phi_o(\tau))$ is torsion.

Lemma 3.3.7. *Under the hypotheses of the previous paragraph, $P(\mathcal{L}, \pi_A, \pi_B, o)$ is torsion for all $\mathcal{L} \in \mathrm{Pic}(B)(K)$.*

Proof. First notice that taking $\alpha = \beta$ in Proposition 3.3.5 yields

$$P(\mathcal{L}, \pi_{A,\alpha}, \pi_{B,\alpha}, \tau) = m \cdot P(\mathcal{L}, \pi_A, \pi_B, o) \tag{3.4}$$

where $m = \deg(\beta)$, since in this case $\theta_{\alpha\beta} = [m]_A$ and

$$\alpha_*(\phi_\tau(\kappa)) = \beta_*(\phi_\tau(\kappa)) = \phi_o(\beta(\kappa)) = \phi_o(o) = 0.$$

Applying (3.4) when $Y = X$ and $\alpha = \beta = w$ then gives

$$P(\mathcal{L}, \pi_{A,w_*}, \pi_{B,w_*}, \tau) = P(\mathcal{L}, \pi_A, \pi_B, o) \tag{3.5}$$

An examination of the definition of $P(\mathcal{L}, \pi_{A,w_*}, \pi_{B,w_*}, \tau)$ shows that

$$P(\mathcal{L}, \pi_{A,w_*}, \pi_{B,w_*}, \tau) = \theta_{w,A}(P(\theta^*_{w,B}\mathcal{L}, \pi_A, \pi_B, \tau)).$$

Under our assumptions on $\theta_{w,A}$ and $\theta^*_{w,B}$, we have

$$\theta_{w,A}(P(\theta^*_{w,B}\mathcal{L}, \pi_A, \pi_B, \tau)) = -P(\theta^*_{w,B}\mathcal{L}, \pi_A, \pi_B, \tau) = -P(\mathcal{L}, \pi_A, \pi_B, \tau)$$

by Proposition 3.3.2. Combining this with (3.5) shows that

$$-P(\mathcal{L}, \pi_A, \pi_B, \tau) = P(\mathcal{L}, \pi_A, \pi_B, o).$$

Finally, invoking Proposition 3.3.4 gives

$$-P(\mathcal{L}, \pi_A, \pi_B, o) + n \cdot \pi_A(\phi_o(\kappa)) = P(\mathcal{L}, \pi_A, \pi_B, o),$$

or $2P(\mathcal{L}, \pi_A, \pi_B, o) = n \cdot \pi_A(\phi_o(\kappa))$ is torsion by assumption. Hence $P(\mathcal{L}, \pi_A, \pi_B, o)$ is torsion, proving the lemma. □

Theorem 3.3.8. *Suppose there exists a prime p exactly dividing N_f, N_g, and N such that $\epsilon_p(g, g, f) = -1$. Then P_{T,f_d} is torsion for all $T \in \mathrm{CH}^1(X_0(N) \times X_0(N))[g]$ and any $d \mid N/N_f$.*

Proof. First, note that the map

$$\bigoplus_{d',d''|N/N_g} \mathrm{NS}(A_g)(\mathbb{Q}) \otimes \mathbb{Q} \to \mathrm{CH}^1(X_0(N) \times X_0(N))(\mathbb{Q})[g] \otimes \mathbb{Q}, \ (\mathcal{L})_{d',d''} \mapsto T_{\mathcal{L},\pi_g^{d'},\pi_g^{d''}}$$

with $T_{\mathcal{L},\pi_g^{d'},\pi_g^{d''}}$ defined in (3.3) is an isomorphism; see [Kan08] for instance. Hence, by Proposition 3.3.3, we may replace P_{T,f_d} with the point $P(\mathcal{L}, \pi_f^d, \psi, \infty)$ for $\mathcal{L} \in \mathrm{NS}(A_g)(\mathbb{Q}) \otimes \mathbb{Q}$ and ψ a linear combination of $\pi_g^{d'}$ for $d' \mid N/N_g$. Let $w_p : X_0(N) \to X_0(N)$ be the Atkin-Lehner involution. The result will follow if we can apply Lemma 3.3.7 to $X_0(N)$ along with w_p, so we must establish three properties of w_p:

(a) $\pi_f^d(w_p(\infty))$ is torsion.

(b) $w_{p*}(\ker(\pi_f^d)) \subset \ker(\pi_f^d)$ and $w_{p*}(\ker(\psi)) \subset \ker(\psi)$.

(c) $\theta_{w_p,A_f} = [-1]_{A_f}$ and $\theta^*_{w_p,A_g}$ is the identity on $\mathrm{NS}(A_g)(\mathbb{Q})$.

By the Manin-Drinfeld Theorem [Man72], the difference of two cusps is torsion on $J_0(N)$, so letting 0 denote the cusp $[0:1]$, we have $w_p(\infty) = 0$ and $\phi_\infty(0)$ is torsion on $J_0(N)$, showing (a). Since $p \mid\mid N$, we have the relation

$$U_p + w_p = \pi_1^* \pi_{p*}$$

where $\pi_1, \pi_p : X_0(N) \to X_0(N/p)$ are the degeneration maps, as can be seen by examining the definitions. By assumption, f is p-new, so for any $d \mid N/N_f$, we have $\pi_1^*\pi_{p*}(f_d) = 0$. Hence, we have

$$w_p(f_d) = -U_p(f_d) = -a_p(f) f_d,$$

the last equality holding since $p \nmid N/N_f$. A similar statement holds for g since g is also p-new. Statement (b) is equivalent to w_p stabilizing $\pi_f^{d*}(H^0(A_f, \Omega^1_{A_f}))$ and $\psi^*(H^0(A_g, \Omega^1_{A_g}))$. The former is spanned by $\{f_d^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$, and the latter by $\{\sum_{d'} c_{d'} g_{d'}^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ if $\psi = \sum_{d'} c_{d'} \pi_g^{d'}$. So (b) follows from the fact that f_d and $g_{d'}$ are eigenvectors for w_p as shown above.

If f, g, and h are eigenforms with p exactly dividing N_f, N_g, and N_h, then Gross–Kudla [GK92] proved that $\epsilon_p(h, g, f) = -a_p(h)a_p(g)a_p(h)$. Since $a_p(g) = \pm 1$, the condition $\epsilon_p(g, g, f) = -1$ implies that $a_p(f) = 1$. So $w_p(f_d) = -a_p(f) f_d = -f_d$ as above, and we have the following commutative diagram:

$$\begin{array}{ccc} J_0(N) & \xrightarrow{w_{p*}} & J_0(N) \\ \downarrow{\scriptstyle \pi_f^d} & & \downarrow{\scriptstyle \pi_f^d} \\ A_f & \xrightarrow{[-1]_{A_f}} & A_f \end{array}$$

showing that $\theta_{w_p,A_f} = [-1]_{A_f}$. We have a similar diagram for A_g, only with $[\pm 1]_{A_g}$ on the bottom row, depending on whether $w_p(g_{d'}) = \pm g_{d'}$. In either case, $\theta^*_{w_p,A_g} : \mathrm{NS}(A_g) \to \mathrm{NS}(A_g)$ is the identity, establishing (c). This proves the theorem. □

Remark 3.3.9. It is straightforward to remove the restriction that $p \mid\mid N$ for T of the form $T_{\mathcal{L},\pi_g^{d'},\pi_g^{d''}}$ where $\mathrm{ord}_p(d') = \mathrm{ord}_p(d'')$. Indeed, in this case, we have

$$P(T, \pi_f^d, \infty) = P(\mathcal{L}, \pi_f^d, \pi_g^{d'} + \pi_g^{d''}, \infty) - P(\mathcal{L}, \pi_f^d, \pi_g^{d'}, \infty) - P(\mathcal{L}, \pi_f^d, \pi_g^{d''}, \infty)$$

by Proposition 3.3.3. Set $a = \mathrm{ord}_p(d)$, $b = \mathrm{ord}_p(d') = \mathrm{ord}_p(d'')$, and $c = \mathrm{ord}_p(N)$, and write $\pi_{p^a}, \pi_{p^b} : X_0(N) \to X_0(N/p^{c-1})$ for the degeneration maps. Then we can apply Corollary 3.3.6 with $\alpha = \pi_{p^a}$ and $\beta = \pi_{p^b}$ to reduce to the case where $p \mid\mid N$, which is covered by the theorem.

Chapter 4

Complex computations of Chow–Heegner points

The material in this chapter is joint work with Henri Darmon, Sam Lichtenstein, and Victor Rotger [DDLR11]. It is devoted to computing the points P_{T,f_d} when A_f is an elliptic curve via complex analytic means. Although the definition of P_{T,f_d} is purely algebraic, explicitly computing all of the intersections would require knowing the algebraic equations defining the modular curve $X_0(N)$ and the Hecke correspondences T_n for sufficiently many n. It is much more convenient to work with the description of $X_0(N)(\mathbb{C})^{\mathrm{an}}$ as a quotient of the extended complex upper-half plane $\mathfrak{H}^*$ by the group $\Gamma_0(N)$. To this end, we will use the complex Abel–Jacobi map from §2.2 and employ the theory of iterated path integrals to obtain an explicitly computable expression for P_{T,f_d}.

Recall that $P_{T,f_d} = \pi_f^d(\Pi_*(\Delta_{GKS}))$. Let ω_E be the invariant differential of the elliptic curve E. Then $\pi_f^{d*}(\omega_E) = c_E\omega_{f^{(d)}}$, where c_E is the *Manin constant* of E. Manin conjectured that $c_E = 1$; this holds in all the computations done in Appendix A, and we will assume this throughout the rest of this monograph. By Proposition 2.2.6, we can write

$$\mathrm{AJ}_{\mathbb{C}}(P_{T,f_d})(\omega_E) = \mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})(\Pi_T^*(\omega_{f^{(d)}}))$$

where the source of $\mathrm{AJ}_{\mathbb{C}}$ is $\mathrm{CH}^1(X)_0(\mathbb{C})$ on the left-hand side and $\mathrm{CH}^2(X^3)_0(\mathbb{C})$ on the right-hand side. By Proposition 2.1.2(1), we can determine $\Pi_T^*(\omega_{f^{(d)}})$ by computing $\mathrm{cl}_{\mathrm{dR}}(\Pi_T)$, and by Proposition 2.1.2(2), we have

$$\mathrm{cl}_{\mathrm{dR}}(\Pi_T) = \mathrm{cl}_{\mathrm{dR}}(T \times X_{34}) = \mathrm{cl}_{\mathrm{dR}}(T) \otimes \mathrm{cl}_{\mathrm{dR}}(X_{34}).$$

Now, X_{34} is the diagonal embedded in $X \times X$, so X_{34}^* induces the identity map on cohomology groups, determining $\mathrm{cl}_{\mathrm{dR}}(X_{34})$, again by Proposition 2.1.2(1). Unwinding the isomorphism

$$\mathrm{Hom}(H^1_{\mathrm{dR}}(X), H^3_{\mathrm{dR}}(X^3)(1)) \simeq H^1_{\mathrm{dR}}(X) \otimes H^3_{\mathrm{dR}}(X^3)(2)$$

shows that $\Pi_T^*(\omega_{f^{(d)}}) = \mathrm{cl}_{\mathrm{dR}}(T) \otimes \omega_{f^{(d)}}$. Hence

$$\mathrm{AJ}_{\mathbb{C}}(P_{T,f_d})(\omega_E) = \mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})(\mathrm{cl}_{\mathrm{dR}}(T) \otimes \omega_{f^{(d)}}). \tag{4.1}$$

We will revisit computing $\mathrm{cl}_{\mathrm{dR}}(T)$ later. Otherwise, the only remaining unknown is computing $\mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})$. Although we have defined the complex Abel–Jacobi map for X^3, it is not immediately clear how to do explicit computations from the definition. This can be accomplished using the theory of iterated path integrals, as will be explained in this section. The main reference is [DRS12], though the exposition will follow more closely that in [DDLR11].

4.1 The complex Abel–Jacobi map via iterated integrals

Iterated integrals

We now turn to recalling the definition and basic properties of iterated integrals; see [Che77], [Hai], [Hai87] for more details.

Let $X/\mathbb{Q}$ denote a curve, and fix a point $\infty \in X(\mathbb{Q})$. Set $Y = X - \{\infty\}$, and write X^{an} and Y^{an} for the Riemann surfaces corresponding to $X(\mathbb{C})$ and $Y(\mathbb{C})$. Choose a base point $o \in Y^{\mathrm{an}}$ and denote by $\Gamma := \pi_1(Y^{\mathrm{an}}; o)$ the fundamental group of Y^{an}. We write $I \subset \mathbb{Z}[\Gamma]$ for augmentation ideal of the integral group ring of Γ. Recall that $H_1(X^{\mathrm{an}}, \mathbb{Z}) = H_1(Y^{\mathrm{an}}, \mathbb{Z}) \simeq \Gamma^{\mathrm{ab}}$, as can be seen from the well-known presentation for the fundamental group of a Riemann surface, and that this abelian group is naturally identified with I/I^2.

Definition 4.1.1. The *path space* on Y^{an} based at o, denoted $\mathbf{P}(Y^{\mathrm{an}}; o)$, is the set of piecewise-smooth paths

$$\gamma : [0,1] \longrightarrow Y^{\mathrm{an}}, \qquad \text{with } \gamma(0) = o.$$

Let $\pi : \tilde{Y} \to Y^{\mathrm{an}}$ and $\pi : \tilde{X} \to X^{\mathrm{an}}$ denote the universal covering spaces corresponding to the basepoint o equipped with the natural projection maps, both of which we refer to as π by abuse of notation. The group Γ acts on $\tilde{Y}$ transitively and without fixed points, and the map $\gamma \mapsto \gamma(1)$ identifies the quotient $\tilde{Y}/\Gamma$ with Y^{an}.

Recall that a closed, $\mathbb{C}$-valued smooth 1-form (resp. a meromorphic 1-form of the second kind) η on X^{an} admits a smooth (resp. meromorphic) primitive $F_\eta : \tilde{X} \to \mathbb{C}$, defined by the rule

$$F_\eta(\gamma) := \int_0^1 \gamma^*\eta.$$

Definition 4.1.2. The *basic iterated integral* attached to an ordered n-tuple $(\omega_1, \ldots, \omega_n)$ of smooth 1-forms on Y^{an} is the function $\mathbf{P}(Y; o) \to \mathbb{C}$, denoted $\int \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n$, defined by

$$\gamma \mapsto \int_\gamma \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n := \int_{\Delta^n} (\gamma^*\omega_1)(t_1)(\gamma^*\omega_2)(t_2) \cdots (\gamma^*\omega_n)(t_n),$$

where Δ^n is the simplex in $[0,1]^n$ defined by $0 \le t_n \le t_{n-1} \le \cdots \le t_1 \le 1$. The integer n is called the *length* of this basic iterated integral.

Example 4.1.3. When $n = 2$, the basic iterated integral attached to ω and η can be computed by the formula

$$\int_\gamma \omega \cdot \eta = \int_\gamma \omega F_\eta = \int_0^1 \gamma^*(\omega F_\eta).$$

In the expression in the middle, we abusively use the same notation ω for the differential $\pi^*\omega$ on $\tilde{Y}$. The 1-form ωF_η is to be integrated along a lift of γ to $\tilde{Y}$, which is unique once a lift of o to $\tilde{Y}$ is specified.

Definition 4.1.4. An *iterated integral* is a linear combination of basic iterated integrals, viewed as a function on $\mathbf{P}(Y; o)$. Its *length* is defined to be the maximum of the lengths of its constituent basic iterated integrals. It is said to be *homotopy invariant* if its value on any path γ depends only on the homotopy class of γ.

A homotopy-invariant iterated integral defines a $\mathbb{C}$-valued function on Γ, and by extending linearly induces a homomorphism of abelian groups $\mathbb{Z}[\Gamma] \to \mathbb{C}$. Observe that a homotopy invariant iterated integral of length $\leq n$ vanishes on the $(n+1)^{\text{st}}$ power I^{n+1} of the augmentation ideal in $\mathbb{Z}[\Gamma]$, and hence gives rise to a well-defined element of $\text{Hom}(I/I^{n+1}, \mathbb{C})$. The natural map

$$\{\text{homotopy invariant iterated integrals of length} \leq n\} \longrightarrow \text{Hom}(I/I^{n+1}, \mathbb{C}) \tag{4.2}$$

is an isomorphism; see Theorem 4.6 of [Hai87].

We will be interested in numerically evaluating certain homotopy invariant iterated integrals on Y^{an} of length ≤ 2. Suppose ω and η are two differentials of the second kind on X, regular on Y, representing cohomology classes $\omega, \eta \in H^1_{\text{dR}}(X/\mathbb{C})$ in the manner of §1.3. The basic iterated integral $\int \omega \cdot \eta$ of length 2 is not generally homotopy invariant. But when either ω or η is holomorphic on X — i.e., has no pole at ∞ — a suitable modification of $\int \omega \cdot \eta$ will be homotopy invariant, as we now explain.

Recall that a differential on a Riemann surface is said to have a *logarithmic pole* at a point if its expansion in terms of a local parameter q at this point is of the form $\sum_{n=0}^{\infty} a_n q^n \frac{dq}{q}$. When ω is holomorphic at ∞, we let $\alpha_{\omega,\eta}$ be a meromorphic 1-form on X that is regular on Y and is such that the induced differential $\omega F_\eta - \alpha_{\omega,\eta}$ on $\tilde{X}$ has at worst a logarithmic pole at (any point lying over) ∞. This condition is well posed because the principal part of ωF_η at $\tilde{x} \in \tilde{X}$ depends only on the image x of $\tilde{x}$; see [DRS12, §2]. The form $\alpha_{\omega,\eta}$ exists — and in fact can even be taken to be algebraic and defined over $\mathbb{Q}$ — by Riemann–Roch. If ω is not holomorphic at ∞ but η is, then we define $\alpha_{\omega,\eta} := -\alpha_{\eta,\omega}$.

Lemma 4.1.5. *Let ω and η be as above, and assume that either ω or η is holomorphic at ∞. Then the iterated integral $J_{\omega,\eta} := \int \omega \cdot \eta - \alpha_{\omega,\eta}$ is homotopy invariant.*

Proof. The homotopy invariance of $J_{\omega,\eta}$ follows from the fact that $J_{\omega,\eta}(\gamma) = \int_\gamma \omega F_\eta - \alpha_{\omega,\eta}$, and the 1-form on $\tilde{X}$ in the integrand is holomorphic when restricted to $\tilde{Y}$. □

Remark 4.1.6. Note that if ω and η are both holomorphic at ∞, then we can take $\alpha_{\omega,\eta} = 0$.

A formula for the complex Abel–Jacobi map

Now consider an integral Hodge class $\xi \in H^1(X^{\mathrm{an}}, \mathbb{Z}) \otimes H^1(X^{\mathrm{an}}, \mathbb{Z})$. Since $H^1(X^{\mathrm{an}}, \mathbb{Z})$ is torsion free, we can identify it with a subspace of

$$H^1(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1(X^{\mathrm{an}}, \mathbb{C}).$$

Using the isomorphisms from Theorem 1.3.2 and Proposition 1.3.4

$$H^1(X^{\mathrm{an}}, \mathbb{C}) \simeq H^1_{\mathrm{dR}}(X^{\mathrm{an}}) \simeq \Omega^1_{II}(X^{\mathrm{an}})/dK(X^{\mathrm{an}}),$$

we can thus can represent any element of $H^1(X^{\mathrm{an}}, \mathbb{Z})$ by a differential of the second kind on X. As a consequence of Riemann–Roch, we may even assume that it is holomorphic on Y. The Hodge condition implies that we can choose a basis $\{\omega_i\}$ such that when we write $\xi = \sum c_{i,j}\omega_i \otimes \omega_j$, then either ω_i or ω_j is holomorphic at ∞ whenever $c_{i,j} \neq 0$; see the end of §1.3. By the Lemma 4.1.5, the iterated integral $J_\xi = \sum c_{i,j} J_{\omega_i,\omega_j}$ is homotopy invariant.

Lemma 4.1.7. *Suppose that ξ is an integral Hodge class on $X \times X$ as above. Using* (4.2)*, identify J_ξ with a homomorphism of abelian groups $I/I^3 \to \mathbb{C}$. Then the restriction of J_ξ to I^2/I^3 is $\mathbb{Z}$-valued and agrees with ξ viewed as an element of*

$$\begin{aligned} H^1(X^{\mathrm{an}}, \mathbb{Z}) \otimes H^1(X^{\mathrm{an}}, \mathbb{Z}) &\simeq (H_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes H_1(X^{\mathrm{an}}, \mathbb{Z}))^\vee \\ &= (I/I^2 \otimes I/I^2)^\vee = (I^2/I^3)^\vee. \end{aligned}$$

(Here $A^\vee$ denotes $\mathrm{Hom}(A, \mathbb{Z})$*, for any abelian group A.)*

Proof. See the discussion at the beginning of §2 of [DRS12], and *loc. cit.*, Lemma 1.1(2). □

By Lemma 4.1.7, the map J_ξ induces a homomorphism

$$J_\xi : H_1(X^{\mathrm{an}}, \mathbb{Z}) = I/I^2 \to \mathbb{C}/\mathbb{Z}.$$

The following observation, which is extended in Theorem 4.1.8 below to the entire Jacobian of X, is key in our approach to calculating Chow–Heegner points. Fix any holomorphic 1-form $\rho \in H^{1,0}(X_{\mathbb{C}}) \subset H^1(X^{\mathrm{an}}, \mathbb{C})$ corresponding to an elliptic curve factor E of the Jacobian of X, and denote by Λ the period lattice

$$\Lambda := \left\{ \int_\gamma \rho, \quad \gamma \in H_1(X^{\mathrm{an}}, \mathbb{Z}) \right\} \subset \mathbb{C}$$

attached to ρ. The class $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbb{C})$ that is Poincaré dual to ρ actually belongs to $H_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes \Lambda$. Consequently $J_\xi(\gamma_\rho)$ can be viewed as a well-defined element of $\mathbb{C}/\Lambda$, and hence of $E(\mathbb{C})$.

Let $T \in \mathrm{CH}^1(X^2)$ and write $\mathrm{cl}(T) = \sum c_{i,j}\omega_i \otimes \omega_j$ where either ω_i or ω_j is holomorphic at ∞ whenever $c_{i,j} \neq 0$. The following theorem allows us to compute $\mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})(\mathrm{cl}(T) \otimes \rho)$.

Theorem 4.1.8 ([DRS12], Corollary 3.6). *Suppose that T annihilates ρ under the induced map $T_* : H^1_{\mathrm{dR}}(X^{\mathrm{an}}) \to H^1_{\mathrm{dR}}(X^{\mathrm{an}})$. Then*

$$\mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})(\mathrm{cl}(T) \otimes \rho) = \sum_{i,j} c_{i,j} \int_{\gamma_\rho} (\omega_i \cdot \omega_j - \alpha_{\omega_i,\omega_j}) + \deg(D_T) \int_o^\infty \rho \in \mathbb{C},$$

where $\gamma_\rho \in H_1(X^{\mathrm{an}}, \mathbb{C})$ is Poincaré dual to $\rho \in H^{1,0}(X^{\mathrm{an}}) \subset H^1_{\mathrm{dR}}(X^{\mathrm{an}})$ and

$$D_T = T \cap X_{12} - T \cap X_1 - T \cap X_2.$$

4.2 A complex-analytic algorithm for computing Chow–Heegner points

The goal of this section is to give an algorithm for explicitly computing the submodule

$$\underline{P}_{\mathbb{T}_{\mathbb{Q}}[g],f} := \{P_{T,f} : T \in \mathbb{T}_{\mathbb{Q}}[g]\} = \langle P_{g,f,n} : n \geq 1 \rangle \subseteq \underline{P}_{g,f}$$

of $E(\mathbb{Q}) \otimes \mathbb{Q}$. Although the methods generalize to the full $\underline{P}_{g,f}$, the computation of $\underline{P}_{\mathbb{T}[g],f}$ is simpler, and often sufficient for the purpose of finding a non-torsion point when one exists by Theorem 3.2.3. The interested reader may consult [Kan08] and use Lemma 4.2.2 below for more general T, and simply replace f by $f^{(d)}$ for $d \neq 1$ to compute $\underline{P}_{g,f}$ in its entirety.

Remark 4.2.1. In order to compute all of $\underline{P}_{\mathbb{T}_{\mathbb{Q}}[g],f}$ (resp. $\underline{P}_{g,f}$), it suffices to compute $P_{T_n,f}$ (resp. P_{T,f_d}) for a set of generators of $\mathbb{T}_{\mathbb{Q}}[g]$ (resp. a set of generators of $\mathrm{End}^0(J_0(N))[g]$ and all $d \mid N/N_f$). Then any other point in the module can be computed by simply writing it in terms of the set of generators.

Throughout this section, we will fix an elliptic curve E, distinct newforms f and g such that $E_f = E$ of levels N_f and N_g, and a positive integer N divisible by N_f and N_g. Additionally, we will write X for the modular curve $X_0(N)$. Let $\pi_f : J_0(N_f) \to E$ denote the corresponding modular parametrization of minimal degree, a morphism of abelian varieties defined over $\mathbb{Q}$. Then $\pi_f^*(\omega_E) = c\omega_f$, where ω_E is the invariant differential of E suitably normalized, c is the Manin constant, and $\omega_f = 2\pi i f(z)dz$. We will assume throughout that $\ker \pi_f$ is connected and $c = 1$. In this case, the Néron lattice of E coincides with the period lattice Λ_f of the differential $\omega_f \in \Omega^1(X_0(N_f)^{\mathrm{an}})$ corresponding to f.

The map π_f can be computed on complex points explicitly, using the Abel–Jacobi isomorphism

$$\mathrm{AJ}_{\mathbb{C}} : J_0(N_f)(\mathbb{C}) \simeq \Omega^1(X_0(N_f)^{\mathrm{an}})^\vee / H_1(X_0(N_f)^{\mathrm{an}}, \mathbb{Z}),$$

the Weierstrass uniformization $W : \mathbb{C}/\Lambda_f \simeq E(\mathbb{C})$, and the analytic parametrization

$$\pi_f^{\mathrm{an}} : \Omega^1(X_0(N_f)^{\mathrm{an}})^\vee / H_1(X_0(N_f)^{\mathrm{an}}, \mathbb{Z}) \to \mathbb{C}/\Lambda_f.$$

The map π_f^{an} sends the coset of a functional on $\Omega^1(X_0(N_f)^{\text{an}})$ to the evaluation of that functional at ω_f. Thus for the Chow–Heegner point $P_{g,n} \in J_0(N_f)(\mathbb{C})$ we have

$$\pi_f(P_{g,n}) = W(\pi_f^{\text{an}}(\text{AJ}_{\mathbb{C}}(P_{g,n}))) = W(\text{AJ}_{\mathbb{C}}(P_{g,n})(\omega_f)).$$

Writing $\text{cl}(T_{g,n}) = \sum_{i,j} c_{i,j}^n \omega_i \otimes \omega_j$, then combining the last equation with 4.1 and Theorem 4.1.8 yields

$$P_{g,f,n} = \pi_f(P_{g,n}) = W\left(\sum_{i,j} c_{i,j}^n \int_{\gamma_f} (\omega_i \cdot \omega_j - \alpha_{\omega_i,\omega_j})\right), \tag{4.3}$$

where $\gamma_f := \gamma_{\omega_f}$ is Poincaré dual to ω_f. Note that we omitted the final term $\deg(D_{T_{g,n}}) \int_o^\infty \omega_f$ from Theorem 4.1.8. For the purposes of computation, we will ignore this term; as the point $[\infty] - [0] \in J_0(N)(\mathbb{Q})$ is torsion as discussed in the proof of Theorem 3.3.8, this will not affect $P_{g,f,n}$ as an element of $E(\mathbb{Q}) \otimes \mathbb{Q}$.

The following ingredients are needed to compute $P_{g,f,n}$ using (4.3):

1. The Poincaré dual $\gamma_f \in H_1(X, \mathbb{C})$ of $\omega_f \in H^1_{\text{dR}}(X^{\text{an}}, \mathbb{C})$.
2. A basis $\mathcal{B} = \{\omega_1, \ldots, \omega_{2t}\}$ for $H^1_{\text{dR}}(X/\mathbb{Q})$, $t = g(X)$, consisting of a collection of rational differentials of the second kind regular away from ∞.
3. The coefficients $c_{i,j}^n$ appearing in $\text{cl}(T_{g,n})$ with respect to the basis $\mathcal{B}$.
4. Meromorphic differentials $\alpha_{\omega_i,\omega_j}$ on X, regular on Y, such that $\omega_{g,i} F_{\omega_{g,j}} - \alpha_{\omega_{g,i},\omega_{g,j}}$ has at worst a logarithmic pole at (any point lying over) ∞ for each pair (i,j) such that $c_{i,j}^n \neq 0$.
5. The evaluation of an iterated integral of the form $\int_{\gamma_f} (\omega_{g,i} \cdot \omega_{g,j} - \alpha_{\omega_{g,i},\omega_{g,j}})$.

These data must be "known" in a sufficiently concrete form to evaluate the iterated integrals occuring in (4.3). It is also desirable to know

6. the denominator $d_{g,n}$ of the projector $T_{g,n} \in \mathbb{T}_{\mathbb{Q}}$, that is the smallest positive integer such that $d_{g,n} T_{g,n} \in \mathbb{T}_{\mathbb{Z}}$.

This last item will allow for the computation of a point in $E(\mathbb{Q})$, as opposed to one in $E(\mathbb{Q}) \otimes \mathbb{Q}$ (see Remark 3.1.5). The rest of this section is devoted to methods for computing these six ingredients. These inputs are not independent of each other, and thus we will address them in a different order than they are listed above for the sake of straightforward exposition.

Computing $\mathrm{cl}(\epsilon_o T_{g,n})$

First, we explain how to compute $\mathrm{cl}(\epsilon_o T_{g,n})$, as this will reveal how to optimally choose our basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$. For the sake of computational efficiency, we would like to minimize the number of $c^n_{i,j} \neq 0$, thus requiring the evaluation of fewer iterated integrals. Specifically, we take advantage of the decomposition of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$ into isotypic subspaces via the action of $\mathbb{T}'_{\mathbb{Q}}$. The action of $\mathbb{T}'_{\mathbb{Q}}$ on $S_2(\Gamma_0(N))$ extends to all of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$ by viewing $\mathbb{T}'$ as an algebra of correspondences on X. Under this action, we have

$$H^1_{\mathrm{dR}}(X/\mathbb{Q}) \simeq H^1_{\mathrm{dR}}(X/\mathbb{Q})[h_1] \oplus \cdots \oplus H^1_{\mathrm{dR}}(X/\mathbb{Q})[h_n],$$

indexed by Galois conjugacy classes of newforms of all levels M dividing N. Suppose $\mathcal{B} = \{\omega_{g,1}, \ldots, \omega_{g,2k}\}$ is a collection of differentials of the second kind on X representing a basis for $H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$. Write $T_n \omega_{g,i} = \sum_j a^n_{ij} \omega_{g,j}$, and denote by A_n and B the matrices $(a^n_{ij})_{1 \le i,j \le 2k}$ and $(\langle \omega_{g,i}, \omega_{g,j} \rangle)_{1 \le i,j \le 2k}$, respectively, where $\langle\ ,\ \rangle$ denotes the Poincaré pairing. Then we have the following:

Lemma 4.2.2. $\mathrm{cl}(\epsilon_o T_{g,n}) = \sum_{i,j} c^n_{ij} \omega_{g,i} \otimes \omega_{g,j}$*, where* $(c^n_{ij})_{1 \le i,j \le 2k} = -B^{-1} A_n$*.*

Proof. We invoke Proposition 2.1.2(1). The projector ϵ_o acts on $H^2_{\mathrm{dR}}(X \times X)$ by annihilating the $H^0_{\mathrm{dR}}(X) \otimes H^2_{\mathrm{dR}}(X)$ and $H^2_{\mathrm{dR}}(X) \otimes H^0_{\mathrm{dR}}(X)$ components of the Künneth decomposition, so we have

$$\mathrm{cl}(\epsilon_o T_{g,n}) \in H^1_{\mathrm{dR}}(X) \otimes H^1_{\mathrm{dR}}(X).$$

Note from the definition that $T_{g,n}$ acts on $H^1_{\mathrm{dR}}(X)[h]$ as T_n if $h = g$ and 0 otherwise, so $\mathrm{cl}(\epsilon_o T_{g,n})$ is equal to the image of T_n under the identification:

$$\mathrm{End}(H^1_{\mathrm{dR}}(X)[g]) \simeq H^1_{\mathrm{dR}}(X)[g]^\vee \otimes H^1_{\mathrm{dR}}(X)[g] \simeq H^1_{\mathrm{dR}}(X)[g] \otimes H^1_{\mathrm{dR}}(X)[g].$$

The first map is the canonical isomorphism of finite dimensional vector spaces, and the second is induced from the inverse of the identification $H^1_{\mathrm{dR}}(X)[g] \simeq H^1_{\mathrm{dR}}(X)[g]^\vee$ via the map $v \mapsto (w \mapsto \langle v, w \rangle)$. The remainder of the proof is a straightforward exercise in linear algebra, and is left to the reader. □

Therefore, rather than choosing a basis for the entire space $H^1_{\mathrm{dR}}(X/\mathbb{Q})$, we can minimize the number of nonzero $c^n_{i,j}$ by using a basis for the g-isotypic component. Furthermore, if we choose $\omega_{g,1}, \ldots, \omega_{g,k}$ to lie in the subspace $\Omega^1(X)$, then as this space is stable under T_n, the upper right quadrant of the matrix A_n will be 0. This, in turn, will guarantee that whenever $c^n_{i,j} \neq 0$, either $\omega_{g,i}$ or $\omega_{g,j}$ will be regular at ∞, and thus $\mathrm{cl}(\epsilon_o T_{g,n})$ will satisfy the condition discussed before Theorem 4.1.8.

Remark 4.2.3. To further reduce the number of iterated integrals required to compute $P_{g,f,n}$ when $n = 1$, it is convenient to find a *symplectic* basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$, that is a basis $\{\omega_{g,1}, \ldots, \omega_{g,k}, \eta_{g,1}, \ldots, \eta_{g,k}\}$ satisfying $\langle \omega_{g,i}, \eta_{g,i} \rangle = 1$, $\langle \omega_{g,i}, \eta_{g,j} \rangle = 0$ for $i \neq j$, and

$\langle \omega_{g,i}, \omega_{g,j} \rangle = \langle \eta_{g,i}, \eta_{g,j} \rangle = 0$. Indeed, in this case the matrix B is just the standard symplectic matrix, while A_1 is the identity matrix. Thus, we have

$$P_{g,f} = W \left(\sum_{i=1}^{k} \int_{\gamma_f} (\omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i} - 2\alpha_{\omega_{g,i},\eta_{g,i}}) \right).$$

Once a basis $\{\omega_{g,1}, \ldots, \omega_{g,2k}\}$ and the matrix B for this basis have been computed, then assuming $\omega_{g,1}, \ldots, \omega_{g,k} \in \Omega^1(X)$ it is a matter of linear algebra to modify $\omega_{g,k+1}, \ldots, \omega_{g,2k}$ to obtain a symplectic basis $\{\omega_{g,1}, \ldots, \omega_{g,k}, \eta_{g,1}, \ldots, \eta_{g,k}\}$.

In order to utilize Lemma 4.2.2, we must be able to compute such a basis $\mathcal{B}$ and the accompanying matrices A_n and B. The next few sections will discuss methods for computing $\mathcal{B}$ and the matrix A_n. The matrix B can be computed once $\mathcal{B}$ is known by appealing to formula 1.11 in Proposition 1.3.4.

Calculating a symplectic basis for $H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$

The calculation of a basis for the de Rham cohomology can be carried out by first writing down a modular function u which is regular away from ∞. Such a function exists by Riemann–Roch and a q-expansion for one such function can sometimes be computed explicitly using the Dedekind eta-function, as explained in the next section.

Using a modular symbol algorithm, one can compute q-expansions for a basis of $S_2(\Gamma_0(N))$ consisting of cusp forms with rational Fourier coefficients; *cf.* for example [Ste07]. Write $\omega_1, \ldots, \omega_t$ for the corresponding holomorphic 1-forms on X, where for convenience we denote by $t = \dim_{\mathbb{C}} S_2(\Gamma_0(N))$ the genus of X. Recall that the point $\infty \in X(\mathbb{Q})$ is called a *Weierstrass point* if $\mathrm{ord}_\infty \omega_i \geq t$ for some i.

Define $\eta_i = u\omega_i$, which is a differential of the second kind by the residue theorem, and let $\mathcal{B} = \{\omega_1, ..., \omega_t, \eta_1, ..., \eta_t\} \subset H^1_{\mathrm{dR}}(X/\mathbb{Q})$ be the corresponding set of cohomology classes. A simple application of Riemann–Roch shows the following.

Lemma 4.2.4. *The set $\mathcal{B}$ is basis for $H^1_{\mathrm{dR}}(X/\mathbb{Q})$ whenever ∞ is not a Weierstrass point on X and u has a pole of order $t+1$ (i.e., the smallest possible) at ∞.*

Proof. Since ∞ is not a Weierstrass point on X, we may assume that $\mathrm{ord}_\infty(\omega_i) = i - 1$, and thus $\mathrm{ord}_\infty(\eta_i) = i - t - 2$. For any differential of the second kind ω', we can find a linear combination of $\eta_1, \ldots, \eta_t$ and dh for an appropriate rational function h having the same principal part as ω'. Thus the difference is holomorphic, and lies in the span of $\{\omega_1, \ldots, \omega_t\}$. □

Remark 4.2.5. By a result of Ogg [Ogg78], the cusp ∞ is not a Weierstrass point when the level N is prime, or more generally when $N = pM$ for prime p and an integer $M \geq 1$ such that $X_0(M)$ has genus zero and $p \nmid M$.

When ∞ is a Weierstrass point, there is a rational function with a single pole at ∞ of order $\leq g(X)$. When u is taken to be such a function, then the set $\mathcal{B}$ will never be a basis.

Indeed, since ∞ is a Weierstrass point, there exists a holomorphic differential form ω with order of vanishing $\geq g(X)$ at ∞. Then $u\omega$ is still holomorphic, and thus lies in the span of $\{\omega_1, \ldots, \omega_t\}$. But $u\omega$ also is in the span of $\{\eta_1, \ldots, \eta_t\}$ by definition of the η_i, giving rise to a linear dependence relation. Hence, in order for $\mathcal{B}$ to be a basis, it is necessary for u to have a pole at ∞ of order greater than the order of vanishing at ∞ of any holomorphic differential.

This lemma is not strictly necessary for the computation, but rather serves to guarantee its success in certain cases. Even if the hypotheses of the lemma do not hold (for example, if ∞ is not a Weierstrass point but u has a pole of order $> t+1$), the set $\mathcal{B}$ may still be a basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$, and almost always is at levels < 200. Moreover this can be checked easily in any particular example by computing the matrix for the Poincaré pairing.

Given one basis $\mathcal{B}$ for $H^1_{\mathrm{dR}}(X/\mathbb{Q})$, it is then a matter of linear algebra to produce a better basis that is adapted to the action of the Hecke algebra. Note that the usual formula for the action of the Hecke algebra $\mathbb{T}'$ on holomorphic modular forms in terms of q-expansions extends to weakly holomorphic modular forms, and preserves the subspace of differentials regular on Y. In particular, one can compute the action of $\mathbb{T}'$ on any 1-form representing an element of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$. Using q-series for the elements of the basis $\mathcal{B}$, we can thus write down the matrix $[T_p] \in \mathrm{Mat}_{2t\times 2t}(\mathbb{Q})$ that describes the action of $T_p \in \mathbb{T}'$ with respect to $\mathcal{B}$.

After identifying $H^1_{\mathrm{dR}}(X/\mathbb{Q}) \simeq \mathbb{Q}^{2t}$ via the basis $\mathcal{B}$, by finding the eigenspaces of finitely many such matrices one can write down $\mathbb{Q}$-bases for each isotypic component of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$. As is shown in [Ste07], the Hecke algebra $\mathbb{T}'$ is generated as a $\mathbb{Z}$-module by T_i for $1 \leq i \leq \frac{m}{6} - \frac{m-1}{N}$, where $m = [\Gamma(1) : \Gamma_0(N)]$. This gives an upper bound on the number of matrices needed, although in practice considerably fewer are necessary. Using these it is simple linear algebra to produce the desired basis $\omega_{g,1}, \ldots, \omega_{g,2k}$ for the isotypic component $H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$. With a small amount of extra work, we can even take this basis to be symplectic as discussed in Remark 4.2.3.

Modular units and η-products

The preceding discussion raises the question of how to compute the q-expansion about ∞ of a rational function u used to write down an initial choice of basis $\mathcal{B}$ for $H^1_{\mathrm{dR}}(X/\mathbb{Q})$.

Recall that a *modular unit* on X is a modular function $u \in \mathbb{Q}(X)^\times$ whose associated divisor is supported on the cusps of X. Denote by U the multiplicative group of modular units.

Definition 4.2.6. The *eta group* U_η is the subgroup of $\mathbb{Q}(X)^\times$ of rational functions of the form

$$u(q) = \lambda \prod_{d|N} \eta(q^d)^{r_d},$$

where $\lambda \in \mathbb{Q}^\times$, $\eta(q) = q^{1/24}\prod_{n>0}(1-q^n)$ is the classical eta function, and $\{r_d\}_{d|N}$ is a collection of integers satisfying the following conditions:

i. $\sum_{d|N} r_d = 0$,

ii. $\prod_{d|N} d^{r_d} \in \mathbb{Q}^\times$ is a square,

iii. $(n_d) := A_N \cdot (r_d)$ is a vector of integers divisible by 24, where A_N is the $\sigma(N) \times \sigma(N)$-matrix whose entry indexed by (d, d') is $\frac{N \cdot (d,d')^2}{dd'(d',N/d')}$.

As in Chapter 2 of [Köh11], one can show that functions satisfying the above conditions are modular functions on X; that is, $U_\eta \subset U$. In fact more is true:

Proposition 4.2.7. *$U_\eta \otimes_{\mathbb{Z}} \mathbb{Q} = U \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. The set $\{\frac{a}{d} : d \mid N, a \in (\mathbb{Z}/(d, N/d)\mathbb{Z})^\times\} \subset \mathbb{P}^1(\mathbb{Q})$ is a complete set of representatives of the cusps of X; see, for instance, [DS05] §3.8. The subspace $U_\eta \otimes_{\mathbb{Z}} \mathbb{Q} \subset U \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with $U' \otimes_{\mathbb{Z}} \mathbb{Q}$, where $U' \subset U$ consists of modular units that have the same valuation at any two cusps a/d, a'/d with the same denominator; *cf.* [GR91, Prop. 2]. This implies the proposition in light of the next lemma, since an element $u \in U \subset \mathbb{Q}(X)$ has the same valuation at any two Galois-conjugate cusps. □

Lemma 4.2.8. *Let $d|N$. Then the cusp $1/d$ is rational if and only if $(d, N/d) = 1$. More generally, the Galois orbit of the cusp $1/d$ is $\{\frac{a}{d} : a \in (\mathbb{Z}/(d, N/d)\mathbb{Z})^\times\}$.*

Proof. We prove the first statement using the results of [Ste82, §1.3]. Namely, it is known that the cusps of X are rational over $\mathbb{Q}(\zeta_N)$, and the Galois action of $\mathrm{Gal}\,(\mathbb{Q}(\zeta_N))/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$ can be described explicitly as follows [Ste82, Thm. 1.3.1]: given $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, let τ_b be the automorphism of $\mathbb{Q}(\zeta_N)$ that sends $\zeta_N \mapsto \zeta_N^b$. If $a \in \mathbb{Z}$ is chosen so that $ab \equiv 1 \pmod N$ then τ_b sends the cusp $\frac{1}{d}$ to $\frac{1}{ad}$. Hence the Galois orbit of $\frac{1}{d}$ is

$$\{\tfrac{1}{ad} : a \in (\mathbb{Z}/N\mathbb{Z})^\times\},$$

and it can be shown by an elementary argument that this set of cusps is equal to the image of

$$\{\tfrac{a}{d} \mid a \in (\mathbb{Z}/(d, N/d)\mathbb{Z})^\times\}$$

in $\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$. □

By the Riemann–Roch theorem, there exist nonconstant rational functions on X that are regular away from ∞. The proposition implies that an integer power of such a function belongs to the subgroup $U_\eta \subset U$, which yields the following.

Corollary 4.2.9. *There exists an eta product $u \in U_\eta$ that is regular away from ∞.* □

It is thus possible to compute the rational function u required in the compation of a basis for $H^1_{\mathrm{dR}}(X)$ as an eta product.

A practical approach to finding the vector $(r_d)_{d|N}$ giving rise to the u we seek is to apply a linear programming algorithm: one minimizes the pole order $-n_N$ of u at ∞ subject to the criteria of Newman–Ligozat in Definition 4.2.6 and the condition that the orders n_d of u at other cusps are non-negative.

Evaluating iterated integrals

Let $J = \int \omega \cdot \eta - \alpha_{\omega,\eta}$ be a homotopy-invariant iterated integral of length ≤ 2 on Y, expressed in terms of differentials of the second kind on X, regular on Y. We wish to compute $J(\gamma)$, where $\gamma \in H_1(Y^{\mathrm{an}}, \mathbb{C}) = H_1(X^{\mathrm{an}}, \mathbb{C}) = H_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C}$. Note that $H_1(X^{\mathrm{an}}, \mathbb{Z})$ is the abelianization of the quotient $\pi_1(X^{\mathrm{an}}, o) = \bar{\Gamma}_0(N)$ of $\Gamma_0(N)$ by the smallest normal subgroup containing the elliptic and parabolic elements.

To evaluate $J(\gamma)$ for $\gamma \in H_1(Y^{\mathrm{an}}, \mathbb{C})$, we may simply represent γ as a $\mathbb{C}$-linear combination of elements of $H_1(Y^{\mathrm{an}}, \mathbb{Z})$, reducing the problem to evaluating $J(\gamma)$ for $\gamma \in H_1(Y^{\mathrm{an}}, \mathbb{Z})$. Choose the basepoint o away from the set S of elliptic points and cusps on Y^{an} and lift γ arbitrarily to a path $\tilde{\gamma} \in \pi_1(Y^{\mathrm{an}} \backslash S, o)$. For each elliptic point $x \in S$, let $e_x = |\mathrm{Stab}_{\Gamma_0(N)}(x)/\{\pm 1\}|$ denote the index of x (which is either 2 or 3) and let γ_x be a sufficiently small counterclockwise loop around x. Writing H for the normal subgroup of $\pi_1(Y^{\mathrm{an}} \setminus S, o)$ generated by $\{\gamma_x^{e_x}, x \in S\}$, there is a natural isomorphism $\Gamma_0(N) \simeq \pi_1(Y^{\mathrm{an}} - S; o)/H$.

We may regard then $\tilde{\gamma}$ as an element of $\Gamma_0(N)$; this causes no ambiguity because H lies in the kernel of the natural projection $H_1(Y^{\mathrm{an}} - S, \mathbb{Z}) \to H_1(Y^{\mathrm{an}}, \mathbb{Z})$. The path $\tilde{\gamma}$ can then also be viewed as a path in $\mathfrak{H}$ from τ_0 to $\tilde{\gamma}\tau_0$, where $\tau_0 \in \mathfrak{H}^*$ is a lift of o.

Lemma 4.2.10. *Suppose γ is Poincaré-dual to ρ. As an element of $\mathbb{C}/\Lambda_\rho$, we have*

$$J(\gamma) = \int_{\tau_0}^{\tilde{\gamma}\tau_0} \omega F_\eta - \alpha_{\omega,\eta}$$

where we identify 1-forms on X with their pullbacks to $\mathfrak{H}^* = \mathfrak{H} \cup \{\infty\}$. Moreover, F_η has Laurent expansion about $\infty \in \mathfrak{H}^*$ given by formally integrating the Laurent expansion of η about the cusp $\infty \in X$.

Proof. This follows from the preceding discussion, using the definition of iterated integrals and the homotopy invariance of J. □

Here we have chosen τ_0 to be a lift of the basepoint o, which we originally chose to be the cusp at 0. However, for computational purposes it will be convenient to choose a different basepoint for evaluating the integral $\int_\gamma \omega \cdot \eta - \alpha_{\omega,\eta}$. The following lemma allows us to do exactly that.

Lemma 4.2.11. *Suppose the Poincaré dual $\rho_\gamma \in H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C})$ of γ satisfies $\langle \rho_\gamma, \omega \rangle = \langle \rho_\gamma, \eta \rangle = 0$. Then $\int_\gamma (\omega \cdot \eta - \alpha_{\omega,\eta})$ is independent of choice of basepoint $o \in Y^{\mathrm{an}}$.*

Proof. Choose any loop $\tilde{\gamma}$ representing the homology class γ. Changing the basepoint from o to o' amounts to conjugating $\tilde{\gamma}$ by a path β from o to o'. This does not affect the value of the integral of the meromorphic 1-form $\alpha_{\omega,\eta}$. Additionally, by [Hai, Exer. 8], for any 1-forms ω, η, loop $\tilde{\gamma}$, and path β, we have

$$\int_{\beta\tilde{\gamma}\beta^{-1}} \omega \cdot \eta = \int_{\tilde{\gamma}} \omega \cdot \eta + \begin{vmatrix} \int_{\tilde{\gamma}} \omega & \int_{\tilde{\gamma}} \eta \\ \int_\beta \omega & \int_\beta \eta \end{vmatrix}. \tag{4.4}$$

But we have $\int_{\tilde{\gamma}} \omega = \langle \rho_\gamma, \omega \rangle = 0$, and similarly for $\int_{\tilde{\gamma}} \eta$. Thus the determinant is 0, and

$$\int_{\beta\tilde{\gamma}\beta^{-1}} \omega \cdot \eta = \int_{\tilde{\gamma}} \omega \cdot \eta.$$

□

Note that the extra conditions of the lemma are satisfied in our situation. Indeed, as $\mathrm{cl}(\epsilon_o T_{g,n})$ can be written in terms of differentials $\omega_{g,1}, \ldots, \omega_{g,2k} \in H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$, and γ_f is Poincaré dual to ω_f, we have $\langle \omega_f, \omega_{g,i} \rangle = 0$ for all $1 \le i \le 2k$.

Now that we are free to choose any $\tau_0 \in \mathfrak{H}$, we will optimize this choice for computational efficiency. As we are computing all rational differentials of the second kind on X^{an} in terms of their q-expansions about the cusp ∞, we are able to write

$$\omega F_\eta - \alpha_{\omega,\eta} = \sum a_n q^n \frac{dq}{q},$$

with $q = e^{2\pi i \tau}$. By the fundamental theorem of calculus and Lemma 4.2.10, we have

$$\int_\gamma (\omega \cdot \eta - \alpha_{\omega,\eta}) = \sum \frac{a_n}{n} (e^{2\pi i \tilde{\gamma}\tau_0} - e^{2\pi i \tau_0}).$$

Thus, the integral will converge fastest when the imaginary parts of $\tilde{\gamma}\tau_0$ and τ_0 are as large as possible. As $\mathrm{Im}(\tau_0) \to \infty$, we have $\mathrm{Im}(\tilde{\gamma}\tau_0) \to 0$, so we must choose a value that maximizes the quantity $\min\{\mathrm{Im}(\tau_0), \mathrm{Im}(\tilde{\gamma}\tau_0)\}$. If we write

$$\tilde{\gamma} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

it is a fun exercise to check that the best compromise is taking $\tau_0 = -\frac{d}{c} + \frac{1}{|c|} i$, which also yields $\mathrm{Im}(\tilde{\gamma}\tau_0) = 1/|c|$.

Another issue we need to address is that since $\gamma \in H_1(Y^{\mathrm{an}}, \mathbb{C})$, it may not be representable by a single element of $\Gamma_0(N)$. If this is the case, we must deal with the possibility that the optimal base points for the constituent parts of γ may not coincide. Even though Lemma 4.2.11 shows that $\int_\gamma (\omega \cdot \eta - \alpha_{\omega,\eta})$ is independent of base point, if we write

$$\gamma = \sum \beta_j \gamma_j$$

for a set of generators $\{\gamma_j\}$ of $H_1(Y^{\mathrm{an}}, \mathbb{Z})$, then the individual integrals $\int_{\gamma_j} (\omega \cdot \eta - \alpha_{\omega,\eta})$ will *not* be independent of base point in general. Thus we cannot simply choose the optimal base point for each integral.

To rectify this, let

$$I_{\tau_0}(\lambda; \tilde{\gamma}) := \int_{\tau_0}^{\tilde{\gamma}\tau_0} \lambda$$

for any differential of the second kind λ on $\tilde{X}$ and $\tilde{\gamma} \in \Gamma_0(N)$. For any other choice of base point τ_j (in particular, we may choose τ_j to be optimal with respect to $\tilde{\gamma}_j$), we have $I_{\tau_0}(\alpha_{\omega,\eta}; \tilde{\gamma}_j) = I_{\tau_j}(\alpha_{\omega,\eta}; \tilde{\gamma}_j)$ since $\alpha_{\omega,\eta}$ is defined not only on $\tilde{X}$ but also on X. To deal with $I_{\tau_0}(\omega F_\eta; \tilde{\gamma}_j)$, we appeal to the following lemma.

Lemma 4.2.12. *Let ω, η, τ_0, τ_j, and $\tilde{\gamma}_j$ be as above. Then*

$$I_{\tau_0}(\omega F_\eta; \tilde{\gamma}_j) = I_{\tau_j}(\omega F_\eta; \tilde{\gamma}_j) - I_{\tau_0}(\eta; \tilde{\gamma}_j) \int_{\tau_0}^{\tau_j} \omega.$$

Proof. Since $\lambda = \omega F_\eta$ is a holomorphic 1-form on $\mathfrak{H}$, its integral along a closed contour vanishes. Thus

$$I_{\tau_0}(\lambda; \tilde{\gamma}_j) = I_{\tau_j}(\lambda; \tilde{\gamma}_j) + \int_{\tau_0}^{\tau_j} \lambda - \int_{\tilde{\gamma}_j \tau_0}^{\tilde{\gamma}_j \tau_j} \lambda.$$

To evaluate the second term on the right-hand side, we observe that ω comes from a 1-form on X, so it is $\Gamma_0(N)$-invariant; it thus pulls back to itself along the fractional linear transformation defined by $\tilde{\gamma}_j$. On the other hand,

$$I_\tau(\eta; \tilde{\gamma}_j) = \int_{\tau}^{\tilde{\gamma}_j \tau} \eta = F_\eta(\tilde{\gamma}_j \tau) - F_\eta(\tau), \qquad \text{for all } \tau \in \mathfrak{H}.$$

Hence $(\tilde{\gamma}_j)^* F_\eta = F_\eta + I_\tau(\eta; \tilde{\gamma}_j)$. So $(\tilde{\gamma}_j)^* \lambda = \lambda + I_\tau(\eta; \tilde{\gamma}_j)\omega$, and we find

$$\int_{\tilde{\gamma}_j \tau_0}^{\tilde{\gamma}_j \tau_j} \lambda = \int_{\tau_0}^{\tau_j} (\tilde{\gamma}_j)^* \lambda = \int_{\tau_0}^{\tau_j} \lambda + \int_{\tau_0}^{\tau_j} I_\tau(\eta; \tilde{\gamma}_j)\omega.$$

Finally, note that $I_\tau(\eta; \tilde{\gamma}_j)$ is independent of τ, so we can set $\tau = \tau_0$ and pull it out of the integral, which yields the lemma. □

Observe that every term on the formula from Lemma 4.2.12 can be computed using the fundamental theorem of calculus, evaluating power series only at the points τ_0 and τ_j. In particular, each $\tilde{\gamma}_j$ will have $|c| \geq N$, so taking $\tau_0 = i/N$, each such evaluation converges at least as fast as an evaluation at τ_j since $\mathrm{Im}(\tau_j) = 1/|c| \leq 1/N$, so this formula for the integral is "optimally efficient".

Remark 4.2.13. We warn the reader that possibly $I_{\tau_0}(\omega F_\eta; \tilde{\gamma}_j) \neq \int_{\tilde{\gamma}_j} \omega \cdot \eta$ (regarding $\tilde{\gamma}_j$ as an element of $\pi_1(Y^{\mathrm{an}}; o)$). Indeed, the iterated integral attached to $\omega \cdot \eta$ need not even be homotopy invariant, so $\int_{\tilde{\gamma}_j} \omega \cdot \eta$ is not even well-defined! In particular, one *cannot* relate $I_{\tau_0}(\omega F_\eta; \tilde{\gamma}_j)$ to $I_{\tau_j}(\omega F_\eta; \tilde{\gamma}_j)$ using the change-of-basepoint formula (4.4) for iterated integrals.

Computing the Poincaré dual γ_f of ω_f

To evaluate the integrals in Lemma 4.2.12, we need to express γ_f as a $\mathbb{C}$-linear combination of a basis for $H_1(X^{\mathrm{an}}, \mathbb{Z})$, which can be represented by elements of $\Gamma_0(N)$. As always, we wish to choose elements of $\Gamma_0(N)$ that optimize the efficiency of the algorithm. Recall that our optimal base point for

$$\tilde{\gamma} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\tau_0 = -\frac{d}{c} + \frac{1}{|c|}i$ and we want to minimize $\mathrm{Im}(\tau_0) = 1/|c|$. Thus, when we write

$$\gamma_f = \sum \beta_j \gamma_j,$$

we should like to choose a basis $\{\gamma_j\}$ with representatives $\tilde{\gamma}_j \in \Gamma_0(N)$ with lower-left entries as small as possible. By a brute-force search it is straightforward to find such elements $\tilde{\gamma}_j$ giving rise to a basis in practice. For $N < 200$, one often only needs to consider matrices with $c = N$ or $c = 2N$. Once a set of representatives $\{\tilde{\gamma}_j\}$ has been found, all that remains is calculating the β_j.

For any $m \in H_1(X^{\mathrm{an}}, \mathbb{C})$, write η_m for its Poincaré dual. Conversely, for any differential η of the second kind on X, let $m_\eta \in H_1(X^{\mathrm{an}}, \mathbb{C})$ denote the Poincaré dual of its cohomology class. We normalize the Poincaré duality isomorphism so that it is characterized by the property

$$\langle \eta_m, \eta \rangle = \int_m \eta. \tag{4.5}$$

The vector space $H_1(X^{\mathrm{an}}, \mathbb{C})$ is also equipped an intersection product, which is related to the Poincaré pairing by the formula

$$m \cdot m_\eta = \frac{1}{2\pi i} \langle \eta_m, \eta \rangle. \tag{4.6}$$

The homology of X also admits a natural action of the Hecke algebra, compatible with the action on cohomology via Poincaré duality. For any $m \in H_1(X^{\mathrm{an}}, \mathbb{C})$, write $m^f \in H_1(X^{\mathrm{an}}, \mathbb{C})[f]$ for the projection of m onto the f-isotypic component of homology. Similarly, for $\eta \in H^1_{\mathrm{dR}}(X/\mathbb{Q})$ write η^f for its projection onto the f-isotypic component. We can assume that via the method described above we have computed a symplectic basis

$$\mathcal{S} = \{\omega_{f,1}, \ldots, \omega_{f,\ell}, \eta_{f,1}, \ldots \eta_{f,\ell}\}$$

for $H^1_{\mathrm{dR}}(X/\mathbb{Q})[f]$.

Lemma 4.2.14. *Fix $\tilde{\gamma}_1$, $\tilde{\gamma}_2 \in \Gamma_0(N)$ and let m_1, $m_2 \in H_1(X^{\mathrm{an}}, \mathbb{Z})$ denote the corresponding homology classes on X. Then we have*

$$m_1^f \cdot m_2^f = \frac{1}{2\pi i} \sum_{i=1}^{\ell} I(\omega_{f,i}; m_1) I(\eta_{f,i}; m_2) - I(\omega_{f,i}; m_2) I(\eta_{f,i}; m_1).$$

Proof. Let $\eta_k = \eta_{m_k}$ and write $\eta_k^f = \sum c_i^{(k)}\omega_{f,i} + \sum d_i^{(k)}\eta_{f,i}$. Then we compute

$$\begin{aligned} m_1^f \cdot m_2^f &= \frac{1}{2\pi i}\langle \eta_1^f, \eta_2^f\rangle \\ &= \sum_i \frac{1}{2\pi i}(c_i^{(1)}d_i^{(2)} - c_i^{(2)}d_i^{(1)}) \\ &= \frac{1}{2\pi i}\sum_i (I(\omega_{f,i}; m_1)I(\eta_{f,i}, m_2) - I(\eta_{f,i}; m_1)I(\omega_{f,i}; m_2)). \end{aligned}$$

This proves the lemma. □

Using (4.5), (4.6), and Lemma 4.2.14, we can compute the Poincaré dual γ_f of ω_f. Let $m_1, \ldots, m_{2\ell}$ be modular symbols giving rise to a basis of $H_1(X^{\mathrm{an}}, \mathbb{Z})[f]$, which can be computed using a modular symbols algorithm (*cf.* [Ste07]). In particular, if f is new, then $\ell = 1$. Write M for the matrix $(m_i \cdot m_j)_{1\le i,j\le 2\ell}$, which can be computed using Lemma 4.2.14, and let v be the column vector $(m_i \cdot m_{\omega_f})_{i=1}^{2\ell}$, which can be computed using (4.6) in conjunction with (4.5). Then the vector $M^{-1}v$ gives the coefficients expressing m_{ω_f} as a linear combination of $m_1, \ldots, m_{2\ell}$. Then, using a change-of-basis matrix between $m_1, \ldots, m_{2\ell}$ and $\{\gamma_j\}$, these coefficients can then be converted into the β_j's.

Computing the matrix A_n

The last ingredient we need to apply Lemma 4.2.2 is the matrix A_n giving the action of T_n on the basis $\mathcal{B}$. In section 4.2, we gave a method for computing A_n using the action of T_n on q-expansions. However, this only works if $\gcd(n, N) = 1$, so we must resort to other methods in general. We exploit the fact that the action of T_n on $H_1(X^{\mathrm{an}}, \mathbb{C})[g]$ is readily computable using modular symbols; see [Ste07] for details. Recall that we have a Hecke-equivariant duality

$$H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C})[g] \times H_1(X^{\mathrm{an}}, \mathbb{C})[g] \to \mathbb{C}$$

given by the integration pairing $\langle \omega, \alpha\rangle = \int_\alpha \omega$; here the Hecke-equivariance means that $\langle T_n\omega, \alpha\rangle = \langle \omega, T_n\alpha\rangle$. Using modular symbols and the techniques of [Ste07] one can compute the matrix C_n of T_n acting on $H_1(X^{\mathrm{an}}, \mathbb{C})[g]$ on the left, with respect to a basis $m_1, \ldots, m_{2k}$. Write $D = (\langle \omega_i, m_j\rangle)_{i,j=1}^{2k}$, which can be computed efficiently via the method explained in §4.2 (using an appropriate basis $\{m_j\}$ derived from the generators $\gamma_0^{(j)}$ for $H_1(X^{\mathrm{an}}, \mathbb{Z})$ discussed above). Then it is straightforward linear algebra to show that $A_n = DC_nD^{-1}$.

Computing the adjustments $\int_{\gamma_f} \alpha$

Write the homology class γ_f Poincaré dual to ω_f as

$$\gamma_f = \sum \beta_j \gamma_j$$

for $\beta_j \in \mathbb{C}$ and homology classes γ_j whose lifts to $\Gamma_0(N)$ are the $\tilde{\gamma}_j$ found in §4.2. Let ω and η be differentials of the second kind, at least one of which is regular at ∞. Using the methods described so far, we are already able to compute

$$z_{\omega,\eta} := \sum_j \beta_j \int_{\tau_0}^{\tilde{\gamma}_j \tau_0} \omega F_\eta.$$

We stress that the value of $z_{\omega,\eta}$ depends on τ_0 and the choices we made in representing γ_f. It is simply one part of the iterated integral $J_{\omega,\eta}(\gamma_f) = \int_{\gamma_f} \omega \cdot \eta - \alpha_{\omega,\eta}$, which is independent of these choices. In this section, we describe a method for computing

$$J_{\omega,\eta}(\gamma_f) - z_{\omega,\eta} = -\sum_j \beta_j \int_{\tau_0}^{\tilde{\gamma}_j \tau_0} \alpha_{\omega,\eta}. \tag{4.7}$$

This amounts to computing the q-expansion of $\alpha_{\omega,\eta}$.

Recall that the defining property of $\alpha_{\omega,\eta}$ is that its principal part at ∞ agrees with that of ωF_η on $\tilde{X}$, modulo dq/q, and thus their difference has at worst logarithmic poles. However, note that since $\int_{\gamma_f} \lambda = 0$ for exact 1-forms λ, we may replace $\alpha_{\omega,\eta}$ by any cohomologous 1-form. The cohomology class of $\alpha_{\omega,\eta}$ is determined by the data $\langle \lambda_i, \alpha_{\omega,\eta} \rangle$, where $\lambda_1, \ldots, \lambda_{2t}$ (for t the genus of X) form a basis of $H^1_{\mathrm{dR}}(X/\mathbb{Q})$, so it suffices to compute these values of the Poincaré pairing.

We choose $\lambda_1, \ldots, \lambda_t$ to be holomorphic. In this case, we compute

$$\langle \lambda_i, \alpha_{\omega,\eta} \rangle = \mathrm{res}_\infty(F_{\lambda_i} \cdot \alpha_{\omega,\eta}) = \mathrm{res}_{\tilde{\infty}}(F_{\lambda_i} \cdot F_\eta \cdot \omega),$$

where the second equality holds because $\mathrm{res}_\infty(F_{\lambda_i} \cdot \alpha_{\omega,\eta})$ depends only on

$$\mathrm{pp}_\infty(\alpha_{\omega,\eta}) \mod \frac{dq}{q} = \mathrm{pp}_{\tilde{\infty}}(\omega F_\eta).$$

Lemma 4.2.15. *Let $\lambda_1, \ldots, \lambda_t \in H^0(X, \Omega^1_{X/\mathbb{Q}})$ be a basis of regular 1-forms on X. Then $\alpha \in H^1_{\mathrm{dR}}(X/\mathbb{Q})$ lies in the subspace $H^0(X, \Omega^1_{X/\mathbb{Q}})$ if and only if $\langle \lambda_i, \alpha \rangle = 0$ for all $1 \le i \le t$.*

Proof. The subspace $H^0(X, \Omega^1_{X/\mathbb{Q}}) \subseteq H^1_{\mathrm{dR}}(X)$ is isotropic for the Poincaré pairing because the pairing can be computed using residues. For dimension reasons, it is maximal isotropic, and the lemma follows. □

By the lemma, if $\langle \lambda_i, \alpha \rangle = \langle \lambda_i, \alpha' \rangle$ for $i = 1, \ldots, t$, then $\alpha - \alpha'$ is cohomologous to a regular 1-form. Since $\alpha_{\omega,\eta}$ is only well defined modulo $H^0(X, \Omega^1_{X/\mathbb{Q}})$, it follows that we can choose $\langle \lambda_i, \alpha_{\omega,\eta} \rangle$ for $i = t+1, \ldots, 2t$ arbitrarily. For convenience, we choose $\langle \lambda_i, \alpha_{\omega,\eta} \rangle = 0$ for $i = t+1, \ldots, 2t$. Define the matrix $B = (\langle \lambda_i, \lambda_j \rangle)_{i,j=1}^{2t}$ and the vector

$$w = (\langle \lambda_i, \alpha_{\omega,\eta} \rangle)_{i=1}^{2t} = (\mathrm{res}_{\tilde{\infty}}(F_{\lambda_1} F_\eta \omega), \cdots, \mathrm{res}_{\tilde{\infty}}(F_{\lambda_t} F_\eta \omega), 0, \ldots, 0).$$

It then follows by elementary linear algebra that the vector $B^{-1}w$ yields the coefficients of an expression for $\alpha_{\omega,\eta}$ as a linear combination of $\lambda_1, \ldots, \lambda_{2t}$. Then, using the fundamental theorem of calculus, we can compute $\int_{\tau_0}^{\tilde{\gamma}_j \tau_0} \lambda_i$ for each i, allowing us to compute (4.7).

Computing the denominator $d_{g,n}$

The final ingredient to be computed is the denominator $d_{g,n}$, or the smallest positive integer such that $d_{g,n}T_{g,n} \in \mathbb{T}_{\mathbb{Z}}$. This can be accomplished by computing a $\mathbb{Z}$-basis for the ($\mathbb{Z}$-finite free) Hecke algebra $\mathbb{T}_{\mathbb{Z}}$ as a subring of $M_{2t}(\mathbb{Q})$, where t is the genus of $X_0(N)$, by identifying $\mathbb{T}_{\mathbb{Z}}$ with an algebra of endomorphisms of the $(2t)$-dimensional $\mathbb{Q}$-vector space of cuspidal modular symbols of weight 2 and level N. As $\mathbb{T}_{\mathbb{Z}}$ is generated as an abelian group by T_i for $1 \leq i \leq \frac{m}{6} - \frac{m-1}{N}$ (see [Ste07], Theorem 9.23), where $m = [\Gamma(1) : \Gamma_0(N)]$, this is a finite computation. Once $\mathbb{T}_{\mathbb{Z}}$ has been computed it is a simple matter to find the matrix representation of $T_{g,n}$ and compute the smallest $d_{g,n}$ such that $d_{g,n}T_{g,n} \in \mathbb{T}_{\mathbb{Z}}$.

4.3 Examples of the algorithm

In this section we work through some examples of the algorithm from §4.2.

The elliptic curve 37a1

Take $N = 37$ in the setup of our algorithm. In this setting, the space of regular differentials on $X = X_0(37)$ is spanned by ω_f and ω_g, which are associated to elliptic curves over $\mathbb{Q}$ (labeled **37a1** and **37b1** in Cremona's database) of ranks 1 and 0, respectively. The elliptic curve $37a1$ has minimal Weierstrass equation given by

$$y^2 + y = x^3 - x,$$

and its Mordell-Weil group is generated by the point $(0 : 0 : 1)$.

By computing the periods attached to ω_f and ω_g, it can be checked that the classes of the matrices

$$\tilde{\gamma}_1 = \begin{pmatrix} 2 & -1 \\ 37 & -18 \end{pmatrix}, \tilde{\gamma}_2 = \begin{pmatrix} 3 & -1 \\ 37 & -12 \end{pmatrix}, \tilde{\gamma}_3 = \begin{pmatrix} 5 & 2 \\ 37 & 15 \end{pmatrix}, \tilde{\gamma}_4 = \begin{pmatrix} 14 & 3 \\ 37 & 8 \end{pmatrix}$$

generate the rational homology of X. These are a "nice" basis for the homology in the sense of the first paragraph of §4.2; that is, the lower left entries are all exactly 37. Using this basis, the integral $\int_\tau^{\gamma_i \tau} \lambda$ can be evaluated efficiently for any meromorphic differential 1-form λ on $X_0(37)$ or its universal cover regular away form ∞, by the method of §4.2.

To obtain differentials of the second kind representing classes in the de Rham cohomology, we consider the elements of the form

$$\eta_1 = u \cdot \omega_f, \qquad \eta_2 = u \cdot \omega_g, \qquad u = \eta(q)^2 \eta(q^{37})^{-2} = q^{-3} \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{37n})^{-2},$$

where $\eta(q)$ is the Dedekind eta function. The modular function u is an example of an eta product with its only pole at ∞, as considered in §4.2. It is not hard to check directly by calculating the Poincaré pairing on all pairs of elements that the classes of $\omega_f, \omega_g, \eta_1$ and η_2 generate the de Rham cohomology of X; alternatively one could apply Lemma 4.2.4.

After computing the matrix M of the Hecke operator T_2 acting on $H^1_{\mathrm{dR}}(X_0(37))$ with respect to the basis $\omega_f, \omega_g, \eta_1, \eta_2$, and then determining the eigenspaces of M, one finds that

$$\begin{aligned}\eta_f &= \frac{1}{4}(-37\omega_g + 4\eta_1 - 8\eta_2), \\ \eta_g &= \frac{1}{4}(37\omega_f - 6\eta_1 + 10\eta_2)\end{aligned}$$

are in the f and g isotypic components of the de Rham cohomology respectively. In addition these linear combinations of 1-forms have been chosen so that $\{\omega_f, \eta_f\}$ and $\{\omega_g, \eta_g\}$ form symplectic bases for the components with respect to the Poincaré pairing.

When one computes the Poincaré dual γ_f of ω_f as in §4.2, one finds (with our normalization):

$$\gamma_f = \frac{1}{2\pi i}\left(A\left([\tilde{\gamma}_2] - [\tilde{\gamma}_3] + [\tilde{\gamma}_4]\right) - B\left(-[\tilde{\gamma}_1] + 2[\tilde{\gamma}_2]\right)\right).$$

Here

$$A \approx (2.4513893\ldots)i, \qquad B \approx 2.9934586\ldots$$

are certain linear combinations of the periods of ω_f against a basis of $H_1(X)[f]$; see §4.2 for a more exact description.

The method of §4.2 can be used to compute α_{ω_g,η_g}. However, in this case, it is easy to find α_{ω_g,η_g} by inspection. Working with principal parts, one finds that $\mathrm{pp}_\infty(\omega_g F_{\eta_g}) \equiv \mathrm{pp}_\infty(\frac{1}{4}(\eta_1 - \eta_2)) \mod \frac{dq}{q}$. Thus we may take $\alpha_{\omega_g,\eta_g} = \frac{1}{4}(\eta_1 - \eta_2)$. Integrating this over γ_f yields $\int_{\gamma_f} \alpha_{\omega_g,\eta_g} = -0.4999999\ldots$, likely the rational number $-\frac{1}{2}$.

Since g is a rational newform, then by Remark 4.2.1, we can find all the points $P_{g,f,n}$ by only computing $P_{g,f}$. According to Remark 4.2.3, this amounts to computing the complex number $z_{g,f} := \int_{\gamma_f}(\omega_g \cdot \eta_g - \eta_g \cdot \omega_g - 2\alpha_{\omega_g,\eta_g})$. The method in §4.2, coupled with the previous paragraph, yields

$$z_{g,f} = -0.4093610\ldots + (1.2256946\ldots)i.$$

Let W be the Weierstrass uniformization of E. Then the point $W(z_{g,f}) \in E(\mathbb{C})$ does not necessarily lie in $E(\mathbb{Q})$. This is because T_g is a *rational* combination of cycles, and so $W(z_{g,f})$ is a $\mathbb{Q}$-linear combination of points in $E(\mathbb{Q})$. Thus, the image of $W(z_{g,f})$ in $E(\mathbb{C}) \otimes \mathbb{Q}$ lies in the subspace $E(\mathbb{Q}) \otimes \mathbb{Q}$. So in order to write $P_{g,f}$ as an element of this space, we must compute the "denominator" of T_g. As in §4.2, one can compute using the first few Fourier coefficients of f and g that the idempotent $e = (0,1) \in \mathbb{Q} \times \mathbb{Q} \simeq \mathbb{T}_\mathbb{Q}$ does not belong to $\mathbb{T}_\mathbb{Z} \subset \mathbb{T}_\mathbb{Q}$ but $2e$ does. Here, the isomorphism associates $T_n \otimes 1 \in \mathbb{T}_\mathbb{Q}$ to $(a_n(f), a_n(g)) \in \mathbb{Q} \times \mathbb{Q}$. By definition, T_g corresponds to e as an element of the Hecke algebra, so it has denominator 2. Thus, we can write $P_{g,f} = W(2z_{g,f}) \otimes \frac{1}{2} \in E(\mathbb{Q}) \otimes \mathbb{Q}$. One finds that $W(2z_{g,f})$ agrees with the global point $(\frac{1357}{841} : \frac{28888}{24389} : 1)$ to within 13 digits of accuracy using 350 Fourier coefficients, so we expect that

$$P_{g,f} = \left(\frac{1357}{841} : \frac{28888}{24389} : 1\right) \otimes \frac{1}{2} = 12(0:0:1) \otimes \frac{1}{2} = 6(0:0:1) \in E(\mathbb{Q}) \otimes \mathbb{Q}.$$

The elliptic curve 43a1

Let $N = 43$ and let E be the elliptic curve labeled **43a1** in Cremona's database, with minimal Weierstrass equation given by

$$y^2 + y = x^3 + x^2.$$

The modular curve $X = X_0(43)$ has genus 3. There are two isotypic components of $H^1_{\mathrm{dR}}(X)$, one of dimension 2 corresponding to the modular form f that parametrized E, and another of dimension 4 corresponding to a newform g with Fourier coefficients in $\mathbb{Q}(\sqrt{2})$, associated to an abelian surface quotient of $J_0(43)$.

In this case, the eta-quotient u that is modular for $\Gamma_0(43)$ of weight 0, holomorphic away from the cusp ∞, and with minimal pole order at ∞, must be of the form

$$u = \left(\frac{\eta(q)}{\eta(q^{43})}\right)^n$$

for some n. From Definition 4.2.6, we see that it must be

$$u = \frac{\eta(q)^4}{\eta(q^{43})^4} = q^{-7} - 4q^{-6} + 2q^{-5} + 8q^{-4} - 5q^{-3} - 4q^{-2} - 10q^{-1} + 8 + 9q + 14q^3 + O(q^4).$$

Computing the Poincaré pairing shows that for a basis of cuspforms with rational Fourier coefficients, corresponding to holomorphic 1-forms $\omega_f, \omega_{g,1}, \omega_{g,2}$ on X, the collection

$$\omega_f, \omega_{g,1}, \omega_{g,2}, u\omega_f, u\omega_{g,1}, u\omega_{g,2}$$

forms a basis for $H^1_{\mathrm{dR}}(X/\mathbb{Q})$. By finding the matrices of a few Hecke operators with respect to this basis, one can as in the case $N = 37$ produce symplectic bases

$$\omega_f, \eta_f, \qquad \text{and} \qquad \omega_{g,1}, \omega_{g,2}, \eta_{g,1}, \eta_{g,2}$$

for $H^1_{\mathrm{dR}}(X/\mathbb{Q})[f]$ and $H^1_{\mathrm{dR}}(X/\mathbb{Q})[g]$ respectively.

We can compute the Poincaré dual γ_f and the iterated integrals

$$\int_{\gamma_f} (\omega_{g,i} \cdot \omega_{g,j} - \alpha_{\omega_{g,i},\omega_{g,j}}), \int_{\gamma_f} (\omega_{g,i} \cdot \eta_{g,j} - \alpha_{\omega_{g,i},\eta_{g,j}}), \int_{\gamma_f} (\eta_{g,i} \cdot \omega_{g,j} - \alpha_{\eta_{g,i},\omega_{g,j}})$$

in the same manner as in the case $N = 37$ with one exception. One simply cannot find $\alpha_{\omega_{g,i},\eta_{g,j}}$ by inspection. No linear combination of our chosen basis has the same principal part as $\omega_{g,i}F_{\eta_{g,j}}$. However, some linear combination is cohomologous to such a form. The techniques from §4.2 can be used to find one.

Each $T_{g,n}$ gives rise to an element of $\mathrm{End}(H^1_{\mathrm{dR}}(X)[g]) \otimes \mathbb{Q}$. The collection of elements arising from $T_{g,n}$, $n \geq 1$ generate a subspace of dimension 2, generated by $T_{g,1}$ and $T_{g,2}$. Thus, we can effectively compute $P_{g,f,n}$ for all n simply by computing $P_{g,f}$ and $P_{g,f,2}$. The formula for $P_{g,f}$ is the one given in Remark 4.2.3, so we have

$$\begin{aligned} z_{g,f} &= \int_{\gamma_f} (\omega_{g,1} \cdot \eta_{g,1} - \eta_{g,1} \cdot \omega_{g,1} - 2\alpha_{\omega_{g,1},\eta_{g,1}} + \omega_{g,2} \cdot \eta_{g,2} - \eta_{g,2} \cdot \omega_{g,2} - 2\alpha_{\omega_{g,2},\eta_{g,2}}) \\ &= -2.0768300\ldots + (2.7263648\ldots)i \end{aligned}$$

The Hecke algebra $\mathbb{T}_{\mathbb{Q}}$ can be identified with $\mathbb{Q} \times \mathbb{Q}(\sqrt{2})$ via $T_n \otimes 1 \mapsto (a_n(f), a_n(g))$. Under this identification, $T_{g,1}$ corresponds to $e_1 = (0,1)$, and an examination of the Fourier coefficients of f and g shows that e_1 does not lie in the image of $\mathbb{T}_{\mathbb{Z}}$, but $2e_1$ does. So, we have

$$P_{g,f} = W(2z_{g,f}) \otimes \frac{1}{2} = \left(\frac{11}{49} : -\frac{363}{343} : 1\right) \otimes \frac{1}{2} \in E(\mathbb{Q}) \otimes \mathbb{Q}.$$

Finding $P_{g,f,2}$ is a little more involved, as we must compute the matrix of T_2 acting on $\omega_{g,1}, \omega_{g,2}, \eta_{g,1}, \eta_{g,2}$. Two methods for doing this were discussed in §4.2 and §4.2, and either shows that $T_2\omega_{g,1} = 2\omega_{g,2}$, $T_2\omega_{g,2} = \omega_{g,1}$, $T_2\eta_{g,1} = -\frac{97997}{132319}\omega_{g,2} + \eta_{g,2}$ and $T_2\eta_{g,2} = \frac{97997}{132319}\omega_{g,1} + 2\eta_{g,1}$. So the matrix A_2 is given by

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{97997}{132319} & 0 & 1 \\ \frac{97997}{132319} & 0 & 2 & 0 \end{pmatrix}.$$

Combining this with Lemma 4.2.2 and remembering that $\alpha_{\omega_{g,i},\omega_{g,j}} = 0$ by Remark 4.1.6, we find that

$$\begin{aligned} z_{g,f,2} &= -\frac{97997}{132319}\int_{\gamma_f} (\omega_{g,1} \cdot \omega_{g,2} - \omega_{g,2} \cdot \omega_{g,1}) + \int_{\gamma_f} (\omega_{g,1} \cdot \eta_{g,2} - \eta_{g,2} \cdot \omega_{g,1} - 2\alpha_{\omega_{g,1},\eta_{g,2}}) \\ &\quad + 2\int_{\gamma_f} (\omega_{g,2} \cdot \eta_{g,1} - \eta_{g,1} \cdot \omega_{g,2} - 2\alpha_{\omega_{g,2},\eta_{g,1}}) \\ &= 2.4055874\ldots - (1.0710898\ldots)i. \end{aligned}$$

The cycle $T_{g,2}$ corresponds to the element $e_2 = (0, \sqrt{2})$ in $\mathbb{Q} \times \mathbb{Q}(\sqrt{2})$, which belongs to $\mathbb{T}_{\mathbb{Z}}$ by inspection of the Fourier coefficients of f and g. By evaluating the Weierstrass uniformisation on $z_{g,f,s}$ we find:

$$P_{g,f,2} = W(z_{g,f,2}) \otimes 1 = (-1 : 0 : 1) \otimes 1 \in E(\mathbb{Q}) \otimes \mathbb{Q}.$$

Chapter 5

p-adic computations of Chow–Heegner points

The aim of this section is to perform the same computations done in Chapter 4 p-adically. This is achieved by replacing the complex Abel–Jacobi map with its p-adic counterpart, the p-adic Abel–Jacobi map. We will then appeal to a result of Darmon and Rotger to obtain a formula for AJ_p on X^3. The iterated integrals from Chapter 4 will be replaced by the ordinary projection of a certain p-adic modular form associated to f and g. See §5.2 below or [DR12] for more details.

5.1 Overconvergent and p-adic modular forms

We briefly describe the aspects of theory of overconvergent and p-adic modular forms that we will need later. For a more complete treatment, see [Kat73] or [Gou88].

Overconvergent and p-adic modular forms of weight two

Fix $N \geq 4$ and a prime $p \geq 5$ not dividing N. Let $\mathcal{X}_1(N)$ denote the scheme over $\mathrm{Spec}(\mathbb{Z}[1/N])$ classifying elliptic curves with an N-torsion point. For the remainder of this section, fix the notation

$$\mathcal{X} := \mathcal{X}_1(N) \times_{\mathbb{Z}[1/N]} \mathrm{Spec}(\mathbb{Z}_p),$$
$$X := \mathcal{X} \times_{\mathbb{Z}_p} \mathrm{Spec}(\mathbb{Q}_p),$$
$$\tilde{X} := \mathcal{X} \times_{\mathbb{Z}_p} \mathrm{Spec}(\mathbb{F}_p).$$

Let $\tilde{P}_1, \ldots, \tilde{P}_s \in \tilde{X}(\mathbb{F}_{p^2})$ denote the supersingular points of $\tilde{X}$. They are precisely the zeros of the *Hasse invariant*, a mod p modular form of weight $p-1$. We choose lifts $P_1, \ldots, P_s \in \mathcal{X}(\mathbb{Z}_{p^2})$ as the zeros of E_{p-1}, the weight $p-1$ Eisenstein series lifting the Hasse invariant to characteristic 0. This choice is customary rather than necessary, as the exact choice of lifts is not important.

Since $\mathcal{X}$ is proper, we have a canonical reduction map

$$\mathrm{red} : X(\mathbb{C}_p) \to \tilde{X}(\bar{F}_p).$$

Then the ordinary locus of $X(\mathbb{C}_p)$ is given by $\mathcal{A} := \mathrm{red}^{-1}(\tilde{X}(\bar{\mathbb{F}}_p) - \{\tilde{P}_1, \ldots, \tilde{P}_s\})$. For each $0 < \epsilon < 1$, define a wide open neighborhood of $\mathcal{A}$ by

$$\mathcal{W}_\epsilon := \mathcal{A} \cup \{x \in X(\mathbb{C}_p) \mid \mathrm{ord}_p(E_{p-1}(x)) < \epsilon\}.$$

Here $\mathrm{ord}_p(E_{p-1}(x)) = \mathrm{ord}_p(E_{p-1}(A_x, \omega_x))$, where A_x is the elliptic curve corresponding to x and $\omega_x \in \Omega^1(A_x/\mathbb{C}_p)$ extends to a regular differential in $\Omega^1(A_x/\mathcal{O}_{\mathbb{C}_p})$ if A_x has good reduction, or corresponds to the canonical differential if A_x is the Tate curve. Since any two choices of ω_x differ by an element of $\mathcal{O}_{\mathbb{C}_p}^\times$, it is clear that $\mathrm{ord}_p(E_{p-1}(x))$ is independent of this choice.

Definition 5.1.1. Let F be a subfield of $\mathbb{C}_p$. We define the space of p-adic modular forms $M_2^{(p)}(\Gamma_1(N), F)$ of weight 2 with coefficients in F to be the space of rigid differentials $\Omega^1(\mathcal{A}/F)(\log \mathrm{cusps})$ with logarithmic poles at the cusps defined over F. The space of overconvergent modular forms $M_2^{\mathrm{oc}}(\Gamma_1(N), F, \epsilon)$ with coefficients in F and radius ϵ is given by $\Omega^1(\mathcal{W}_\epsilon, F)(\log \mathrm{cusps})$. The p-adic and overconvergent cusp forms $S_2^{(p)}(\Gamma_1(N), F)$ and $S_2^{\mathrm{oc}}(\Gamma_1(N), F, \epsilon)$ are the subspaces of forms regular at the cusps.

Restriction defines a natural inclusion $S_2^{\mathrm{oc}}(\Gamma_1(N), F, \epsilon) \subset S_2^{(p)}(\Gamma_1(N), F)$, and we say a p-adic modular form is overconvergent if it lies in the subspace

$$S_2^{\mathrm{oc}}(\Gamma_1(N), F) := \bigcup_{\epsilon > 0} S_2^{\mathrm{oc}}(\Gamma_1(N), F, \epsilon).$$

Let $\Gamma := \Gamma_1(N) \cap \Gamma_0(p)$, and write $\mathcal{X}(\Gamma)$ for the scheme over $\mathrm{Spec}(\mathbb{Z}[1/Np])$ parametrizing elliptic curves with an N-torsion point and a p-isogeny. Write $X' := \mathcal{X}(\Gamma) \times_{\mathbb{Z}[1/Np]} \mathbb{Q}_p$. Then Katz showed that the map $X' \to X$ forgetting the p-isogeny has a canonical section defined over $\mathcal{W}_\epsilon$ whenever $\epsilon < \frac{p}{p+1}$. This section gives rise to an inclusion

$$S_2(\Gamma, F) \subset S_2^{\mathrm{oc}}(\Gamma_1(N), F, \epsilon)$$

of classical modular forms of weight 2 with coefficients in F into the space of ϵ-overconvergent forms. In particular, let χ be a primitive Dirichlet character of conductor dividing N and $g \in S_2(N, \chi)$ be an eigenform such that $a_p(g)$ is a p-adic unit; such a form is called *ordinary*. Let α and β denote the roots of the polynomial

$$X^2 - a_p(g)X + \chi(p)p$$

ordered so that $\mathrm{ord}_p\, \alpha = 0$. Then the *p-stabilizations*

$$g_\alpha(z) := g(z) - \beta g(pz),$$

$$g_\beta(z) := g(z) - \alpha g(pz)$$

are classical modular forms for Γ, and thus can be considered as overconvergent modular forms.

De Rham cohomology and the U operator

Write $\mathcal{Y} = \mathcal{X} - \{P_1, \ldots, P_s\}$, and let Y denote the generic fiber of $\mathcal{Y}$. Since Y is affine, then $H^1_{\mathrm{dR}}(Y/F) = \Omega^1(Y/F)/d\mathcal{O}_{Y/F}$. Define

$$H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/F) = \frac{\Omega^1(\mathcal{W}_\epsilon/F)}{\mathcal{O}_{\mathcal{W}_\epsilon/F}}.$$

Restriction of differential forms yields a map $\mathrm{comp}_\epsilon : H^1_{\mathrm{dR}}(Y/F) \to H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/F)$. For each $\tilde{P}_1, \ldots, \tilde{P}_s$, let $\mathcal{V}_j$ be the annulus

$$\mathcal{V}_j = \{x \in \mathrm{red}^{-1}(\tilde{P}_j) \mid \mathrm{ord}_p(E_{p-1}(x)) < \epsilon\}.$$

Then, in addition to the standard residue map $\mathrm{res}_{P_j} : \Omega^1(Y/F) \to F(-1)$, there is an annular residue map

$$\mathrm{res}_{\mathcal{V}_j} : \Omega^1(\mathcal{W}_\epsilon/F) \to F(-1);$$

see [Col89], Lemma 2.1. Here $F(-1)$ refers to the filtered Frobenius module as defined in §2.3. Using isomorphism 1.3.4, we have an exact sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(X/F) \longrightarrow H^1_{\mathrm{dR}}(Y/F) \xrightarrow{\oplus_j \mathrm{res}_{P_j}} F(-1)^s \xrightarrow{\Sigma} F(-1) \longrightarrow 0$$

Proposition 5.1.2. *The map* comp_ϵ *is an isomorphism, inducing a commutative diagram*

$$\begin{array}{ccccccc} H^1_{\mathrm{dR}}(Y/F) & \xrightarrow{\oplus_j \mathrm{res}_{P_j}} & F(-1)^s & \xrightarrow{\Sigma} & F(-1) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{comp}_\epsilon} & & \| & & \| & & \\ H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/F) & \xrightarrow{\oplus_j \mathrm{res}_{\mathcal{V}_j}} & F(-1)^s & \xrightarrow{\Sigma} & F(-1) & \longrightarrow & 0 \end{array}$$

allowing us to identify $H^1_{\mathrm{dR}}(X/F)$ *with subspace of classes in* $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/F)$ *with vanishing annular residues.*

Proof. See Theorem 4.2 of [Col89]. □

We denote by $S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0$ the subspace of $S_2^{\mathrm{oc}}(\Gamma_1(N), F)$ of forms f whose associated differential ω_f has vanishing annular residues. Thus, for any $f \in S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0$, we obtain a cohomology class $[\omega_f] \in H^1_{\mathrm{dR}}(X/F)$ by the previous proposition.

Suppose $\epsilon < \frac{p}{p+1}$. Then Katz showed in [Kat73] that for every $x \in \mathcal{W}_\epsilon$, the corresponding elliptic curve A_x admits a canonical subgroup Z_x of order p, even if A_x is supersingular at p. This allows us to define a canonical lift of the Frobenius morphism $\Phi : \mathcal{W}_{\epsilon/p} \to \mathcal{W}_\epsilon$ by setting $\Phi(x) := A_x/Z_x$, the elliptic curve A_x/Z_x with the N-torsion point coming from A_x. The induced map $\Phi : \Omega^1(\mathcal{W}_\epsilon/F) \to \Omega^1(\mathcal{W}_{\epsilon/p}/F)$ defines a map on $H^1_{\mathrm{dR}}(Y/F)$ via the diagram

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(Y/F) & \xrightarrow{\Phi} & H^1_{\mathrm{dR}}(Y/F) \\ \downarrow{\scriptstyle \mathrm{comp}_\epsilon} & & \downarrow{\scriptstyle \mathrm{comp}_{\epsilon/p}} \\ H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon/F) & \xrightarrow{\Phi} & H^1_{\mathrm{dR}}(\mathcal{W}_{\epsilon/p}/F) \end{array}$$

Since Φ is compatible with annular residues, i.e. $\mathrm{res}_{\mathcal{V}_j}\Phi(\omega) = \Phi(\mathrm{res}_{\mathcal{V}_j}\omega)$, then Φ leaves the subspace $H^1_{\mathrm{dR}}(X/F)$ invariant. On this space, the Poincaré pairing can be described in terms of annular residues. We have the formula

$$\langle \rho_1, \rho_2 \rangle = \sum_j \mathrm{res}_{\mathcal{V}_j}(F^{(j)}_{\omega_1} \cdot \omega_2), \tag{5.1}$$

where $\omega_1, \omega_2 \in \Omega^1(\mathcal{W}_\epsilon/F)$ are representatives for ρ_1, ρ_2 and $F^{(j)}_{\omega_1}$ is a local primitive for ω_1 on $\mathcal{V}_j$. Furthermore,

$$\langle \Phi(\omega_1), \Phi(\omega_2) \rangle = \Phi\langle \omega_1, \omega_2 \rangle = p\langle \omega_1, \omega_2 \rangle. \tag{5.2}$$

Let $H^1_{\mathrm{dR}}(X/F)^{\mathrm{ur}}$ denote the *unit-root* subspace of $H^1_{\mathrm{dR}}(X/F)$ spanned by vectors on which Φ acts via multiplication by a p-adic unit. More generally, let $H^1_{\mathrm{dR}}(X/F)^{\Phi,t}$ denote the *slope-t subspace* spanned by vectors on which Φ acts via multiplication by a scalar with p-adic valuation t. Then the last equality shows that the Poincaré pairing descends to a perfect pairing

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X/F)^{\mathrm{ur}} \times H^1_{\mathrm{dR}}(X/F)^{\Phi,1} \to F(-1). \tag{5.3}$$

The spaces of p-adic and overconvergent modular forms are equipped with actions of two operators, denoted U_p and V_p, whose effect on q-expansions is given by

$$(U_p f)(q) = \sum_{n=1}^{\infty} a_{np}(f) q^n, \qquad (V_p f)(q) = \sum_{n=1}^{\infty} a_n(f) q^{np}.$$

These operators satisfy the relations

$$(U_p V_p f)(q) = f(q), \qquad (V_p U_p f)(q) = \sum_{n=1}^{\infty} a_{np}(f) q^{np},$$

so that

$$f^{[p]}(q) := (1 - V_p U_p) f(q) = \sum_{p \nmid n} a_n(f) q^n,$$

the *p-depletion* of f, is also a p-adic modular form. This notion will be useful later when describing a formula for the p-adic Abel–Jacobi map.

As seen above, U_p and V_p are not quite inverses of each other. However, they are inverses on $H^1_{\mathrm{dR}}(X/F)$. The form $f^{[p]}$ has a rigid analytic primitive,

$$F^{[p]}(q) = \sum_{p \nmid n} \frac{a_n(f)}{n} q^n,$$

an overconvergent modular form of weight 0, and hence an element of $\mathcal{O}_{\mathcal{W}_\epsilon/F}$ for suitable $\epsilon > 0$. Hence, $(1 - V_p U_p) f(q) = 0$ in $H^1_{\mathrm{dR}}(X/F)$, showing that U_p and V_p are inverses of each other. In terms of the Frobenius operator Φ, we have $\Phi(\omega_f) = p\omega_{V_p f}$, where $\omega_f \in \Omega^1(\mathcal{W}_\epsilon/F)$ is the associated differential form. Therefore we have the relation

$$\Phi = pV_p = pU_p^{-1} \tag{5.4}$$

on cohomology.

Denote by $S_2^{\mathrm{oc}}(\Gamma_1(N), F)^{\mathrm{ord}}$ the *ordinary* subspace of $S_2^{\mathrm{oc}}(\Gamma_1(N), F)$, that is, the space spanned by vectors on which U_p acts via multiplication by a p-adic unit. The operator

$$e_{\mathrm{ord}} := \lim_n U_p^{n!}$$

is Hida's projection to the ordinary subspace. Sometimes we will write f^{ord} for $e_{\mathrm{ord}} f$. From 5.4, we can conclude that if $f \in S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0^{\mathrm{ord}}$, then $\omega_f \in H^1_{\mathrm{dR}}(X/F)^{\Phi,1}$. Hence, we have the following proposition.

Proposition 5.1.3. *For any $\eta \in H^1_{\mathrm{dR}}(X/F)^{\mathrm{ur}}$ and $f \in S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0$, we have*

$$\langle \eta, \omega_f \rangle = \langle \eta, \omega_{f^{\mathrm{ord}}} \rangle,$$

and Poincaré duality induces a non-degenerate pairing

$$\langle\ ,\ \rangle : H^1_{\mathrm{dR}}(X/F)^{\mathrm{ur}} \times S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0^{\mathrm{ord}} \to K.$$

Proof. This follows from 5.3, 5.4, and the previous discussion. □

To simplify notation, for any $f \in S_2^{\mathrm{oc}}(\Gamma_1(N), F)_0^{\mathrm{ord}}$, we will write $\langle \eta, \omega_f \rangle$ and $\langle \eta, f \rangle$ interchangeably henceforth.

Katz expansions

In this section, we provide an alternate description of p-adic modular forms that is more amenable to explicit computation. Let $M_k(N, \chi, \mathbb{Z}_p)$ be the space of classical modular forms of level N, character χ, and coefficients in $\mathbb{Z}_p$. For each $i \geq 0$, multiplication by E_{p-1} gives rise to an injective map

$$E_{p-1} : M_{k+i(p-1)}(N, \chi, \mathbb{Z}_p) \to M_{k+(i+1)(p-1)}(N, \chi, \mathbb{Z}_p).$$

Set $\mathbf{W}_0(N, \chi, \mathbb{Z}_p) = M_k(N, \chi, \mathbb{Z}_p)$, and for all $i \geq 0$ choose submodules $\mathbf{W}_i(N, \chi, \mathbb{Z}_p)$ of $M_{k+i(p-1)}(N, \chi, \mathbb{Z}_p)$ such that

$$M_{k+i(p-1)}(N, \chi, \mathbb{Z}_p) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(N, \chi, \mathbb{Z}_p) \oplus \mathbf{W}_i(N, \chi, \mathbb{Z}_p).$$

Such a choice is not canonical. For any finite extension F of $\mathbb{Q}_p$, let B denote the ring of integers of F, and write $\mathbf{W}_i(N, \chi, B) := \mathbf{W}_i(N, \chi, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B$. For any $r \in B$, the ring

$M_k(N, \chi, B, r)$ of r-overconvergent modular forms of weight k is the space of all "Katz expansions"

$$f = \sum_{i \geq 0} r^i \frac{b_i}{E_{p-1}^i}, \qquad b_i \in \mathbf{W}_i(N, \chi, B), \qquad \lim_{i \to \infty} b_i = 0,$$

where $\lim_{i\to\infty} b_i = 0$ means in the sense of the p-adic topology. We define $M_k(N, \chi, F, r) := M_k(N, \chi, B, r) \otimes_B F$. Note that the primary role of the element $r \in B$ is to determine the radius $p^{-\mathrm{ord}_p r}$ of the annulus of overconvergence. Thus it is simpler to adopt the convention used in [Lau12] of defining for any $\alpha \in \mathbb{Q}$ the space of α-overconvergent forms $M_k(N, \chi, \mathbb{Z}_p, \alpha)$ to be Katz expansions of the form

$$f = \sum_{i \geq 0} p^{\lfloor \alpha i \rfloor} \frac{b_i}{E_{p-1}^i}, \qquad b_i \in \mathbf{W}_i(N, \chi, \mathbb{Z}_p), \qquad \lim_{i \to \infty} b_i = 0.$$

5.2 A p-adic formula for Chow–Heegner points

In this section we derive a formula for the points $P_{g,f,n}$ that will be useful for the computations in the next section. Let N_f and N_g denote the levels of f and g, respectively, and let N be a positive integer divisible by N_f and N_g. Let $p \geq 5$ be a prime not dividing N. Denote by K_g the coefficient field of g, and let Ψ denote the set of embeddings $\sigma : K_g \to \bar{\mathbb{Q}}_p$. For each $\sigma \in \Psi$ and any $h \in S_2(\Gamma_0(N), K_g)$, denote by $h^\sigma \in S_2(\Gamma_0(N), \bar{\mathbb{Q}}_p)$ the image of h under σ. Fix a subfield $F \subset \bar{\mathbb{Q}}_p$ containing the images of all $\sigma \in \Psi$. Finally, we assume that f and g are ordinary at p, that is $a_p(f^\sigma)$ and $a_p(g^\sigma)$ are p-adic units for all $\sigma \in \Psi$.

A basis for $H^1_{\mathrm{dR}}(X_0(N)/F)[g]$

Write $N_{\mathrm{rel}} = N/N_g$. For any $d \mid N_{\mathrm{rel}}$, there exists a modular form of level N with q-expansion

$$g_d(q) = d \cdot g(q^d) \in S_2(\Gamma_0(N), K_g).$$

For any $\sigma \in \Psi$, the form g_d^σ is an eigenvector for all Hecke operators T_ℓ such that $\ell \nmid d$ with eigenvalue $a_\ell(g^\sigma)$. The set $\{g_d^\sigma\}_{d \mid N_{\mathrm{rel}}, \sigma \in \Psi}$ forms a basis for $S_2(\Gamma_0(N), F)[g]$, and thus $\{\omega_{g_d^\sigma}\}_{d,\sigma}$ forms a basis for $H^0(X_0(N)_F, \Omega^1_X)[g]$. For simplicity, we will write ω_d^σ for the remainder of the section.

Since g is ordinary, restriction of the exact sequence

$$0 \to H^0(X_0(N)_F, \Omega^1_X)[g] \to H^1_{\mathrm{dR}}(X_0(N)/F)[g] \to H^1(X_0(N)_F, \mathcal{O}_X)[g] \to 0$$

to the unit-root subspace yields an isomorphism

$$H^1_{\mathrm{dR}}(X_0(N)/F)[g]^{\mathrm{ur}} \simeq H^1(X_0(N)_F, \mathcal{O}_X)[g].$$

This can be seen directly from (5.4) and the definitions of ordinary and the unit-root subspace. Thus, any class $\eta \in H^1(X_0(N)_F, \mathcal{O}_X)[g]$ has a *unique* lifting to $H^1_{\mathrm{dR}}(X_0(N)/F)[g]^{\mathrm{ur}}$, denoted by η^{ur}.

Now consider the anti-holomorphic form

$$\eta_{g_d}^{\text{a-h}} := \overline{\omega_{g_d}} \cdot \langle \omega_{g_d}, \overline{\omega_{g_d}} \rangle^{-1} \in \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega_X^1)[g]} \subset H^1_{\text{dR}}(X_0(N)/\mathbb{C})[g].$$

This form is actually defined over K_g (Corollary 2.3, [DR12]), and thus for any $\sigma \in \Psi$ we obtain a form $\eta^{\text{a-h}}_{g_d^\sigma} \in H^1_{\text{dR}}(X/F)[g]$. Its image $\eta_{g_d^\sigma} \in H^1(X_F, \mathcal{O}_X)[g]$ has a unique lifting $\eta^{\text{ur}}_{g_d^\sigma} \in H^1_{\text{dR}}(X/F)[g]^{\text{ur}}$. We shall write η_d^σ for this last element henceforth.

Proposition 5.2.1. *The set* $\{\omega_d^\sigma, \eta_d^\sigma\}_{d|N_{\text{rel}}, \sigma\in\Psi}$ *forms a basis for* $H^1_{\text{dR}}(X/F)[g]$.

Proof. The forms $\eta^{\text{a-h}}_{g_d^\sigma}$ for $\sigma \in \text{Aut}(\mathbb{C})$ are a basis for the anti-holomorphic subspace of $H^1_{\text{dR}}(X_0(N)/\mathbb{C})[g]$, and thus the classes $\eta_{g_d^\sigma}$ form a basis for $H^1(X_0(N)_{K_p}, \mathcal{O}_X)[g]$ since the holomorphic and anit-holomorphic subspaces intersect trivially. Thus, the η_d^σ give a basis for $H^1_{\text{dR}}(X_0(N)/F)[g]^{\text{ur}}$. Finally, since g is ordinary, the subspaces $H^0(X_0(N)_F, \Omega^1_X)[g]$ and $H^1_{\text{dR}}(X_0(N)/F)[g]^{\text{ur}}$ intersect trivially and generate the entire space, so the proposition follows. □

Proposition 5.2.2. *Let* $d_1, d_2 \mid N_{\text{rel}}$ *and* $\sigma_1, \sigma_2 \in \Psi$. *Write* $d = d_1 d_2 / \gcd(d_1, d_2)^2$. *Then we have*

$$\langle \omega_{d_1}^{\sigma_1}, \omega_{d_2}^{\sigma_2} \rangle = 0, \tag{5.5}$$

$$\langle \eta_{d_1}^{\sigma_1}, \eta_{d_2}^{\sigma_2} \rangle = 0, \tag{5.6}$$

$$\langle \omega_{d_1}^{\sigma_1}, \eta_{d_2}^{\sigma_2} \rangle = \begin{cases} \frac{a_d(g^{\sigma_1})}{[\Gamma_0(N_g):\Gamma_0(dN_g)]}, & \sigma_1 = \sigma_2 \\ 0, & \sigma_1 \neq \sigma_2 \end{cases}. \tag{5.7}$$

Proof. The first two statements follow from the fact that $H^0(X_F, \Omega^1_X)$ and $H^1_{\text{dR}}(X/F)^{\text{ur}}$ are isotropic with respect to the Poincaré pairing, as can be deduced from (5.1) and (5.2). Similarly, the third statement when $\sigma_1 \neq \sigma_2$ follows from the fact that Hecke operators T_n are self-adjoint when $\gcd(n, N) = 1$ and $\omega_{d_1}^{\sigma_1}$ and $\eta_{d_2}^{\sigma_2}$ have distinct eigenvalues for some such n when $\sigma_1 \neq \sigma_2$.

All that remains is the $\sigma_1 = \sigma_2$ case, which we will write simply as σ. Let ω_g^σ and η_g^σ denote the classes in $H^0(X_0(N_g)_F, \Omega^1_X)$ and $H^1_{\text{dR}}(X_0(N_g)/F)^{\text{ur}}$ corresponding to the newform g^σ. By construction, we have $\langle \omega_g^\sigma, \eta_g^\sigma \rangle = 1$. We will compute $\langle \omega_{d_1}^\sigma, \eta_{d_2}^\sigma \rangle$ using this relation and property (1.9) of the Poincaré pairing.

For any divisor $t \mid N_{\text{rel}}$, let $\pi_t : X_0(N) \to X_0(N_g)$ denote the morphism of modular curves corresponding to "quotient by level t structure". Then $\omega_{d_1}^\sigma = \pi_{d_1}^* \omega_g^\sigma$. From the definition above, we also have

$$\begin{aligned} \eta^{\text{a-h}}_{g_{d_2}} &= \overline{\omega_{g_{d_2}}} \cdot \langle \omega_{g_{d_2}}, \overline{\omega_{g_{d_2}}} \rangle^{-1} \\ &= \overline{\pi_{d_2}^* \omega_g} \cdot \langle \pi_{d_2}^* \omega_g, \pi_{d_2}^* \overline{\omega_g} \rangle^{-1} \\ &= \pi_{d_2}^*(\overline{\omega_g}) \cdot [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \cdot \langle \omega_g, \overline{\omega_g} \rangle^{-1} \\ &= [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \cdot \pi_{d_2}^* \eta_g^{\text{a-h}}. \end{aligned}$$

The third equality comes from (1.9) via

$$\langle \pi_{d_2}^* \omega_g, \pi_{d_2}^* \overline{\omega_g} \rangle = \langle \pi_{d_2 *} \pi_{d_2}^* \omega_g, \overline{\omega_g} \rangle = \deg(\pi_{d_2}) \langle \omega_g, \overline{\omega_g} \rangle.$$

From this relation, it follows that $\eta_{d_2}^\sigma = [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \cdot \pi_{d_2}^* \eta_g^\sigma$. Again from (1.9), we have

$$\langle \omega_{d_1}^\sigma, \eta_{d_2}^\sigma \rangle = [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \langle \pi_{d_1}^* \omega_g^\sigma, \pi_{d_2}^* \eta_g^\sigma \rangle = [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \langle \pi_{d_2 *} \pi_{d_1}^* \omega_g^\sigma, \eta_g^\sigma \rangle,$$

so it suffices to determine the effect of $\pi_{d_2 *} \pi_{d_1}^*$ on ω_g^σ. Let $m = \gcd(d_1, d_2)$, $d = d_1 d_2 / m^2$ and write $d_i = m n_i$. Then we can factor $\pi_{d_i} = \pi_{n_i} \pi_m$, where $\pi_m : X_0(N) \to X_0(dN_g)$ and $\pi_{n_i} : X_0(dN_g) \to X_0(N_g)$. Thus, we have

$$\begin{aligned} \pi_{d_2 *} \pi_{d_1}^* \omega_g^\sigma &= (\pi_{n_2 *} \pi_{m *})(\pi_m^* \pi_{n_1}^* \omega_g^\sigma) \\ &= \pi_{n_2 *}(\pi_{m *} \pi_m^* (\pi_{n_1}^* \omega_g^\sigma)) \\ &= [\Gamma_0(dN_g) : \Gamma_0(N)] \cdot \pi_{n_2 *} \pi_{n_1}^* \omega_g^\sigma \end{aligned}$$

For any prime power ℓ^k such that $\ell \nmid N_g$, the Hecke operator T_{ℓ^k} is given by the correspondence $\pi_{1*} \pi_{\ell^k}^* = \pi_{\ell^k *} \pi_1^*$, where $\pi_1, \pi_{\ell^k} : X_0(\ell^k N_g) \to X_0(N_g)$. If $\ell \mid N_g$, then the Hecke operator T_{ℓ^k} is still given by $\pi_{\ell^k *} \pi_1^*$, but T_{ℓ^k} is no longer self-adjoint. However, g is a newform, and T_{ℓ^k} is self-adjoint when restricted to the new subspace, so we can identify T_{ℓ^k} with both $\pi_{\ell^k *} \pi_1^*$ and $\pi_{1*} \pi_{\ell^k}^*$ in this case. Observe that $\pi_{n_2 *} \pi_{n_1}^*$ is the composition of the correspondences

$$\begin{aligned} \pi_{\ell_1^{k_1} *} \pi_1^* &= T_{\ell_1^{k_1}}, \\ \pi_{1*} \pi_{\ell_2^{k_2}}^* &= T_{\ell_2^{k_2}}, \end{aligned}$$

where ℓ_i runs over all primes dividing n_i and $\ell_i^{k_i}$ is the largest power of ℓ_i dividing n_i. Since $d = n_1 n_2$, this composition is simply T_d, and thus $\pi_{n_2 *} \pi_{n_1}^* \omega_g^\sigma = T_d \omega_g^\sigma = a_d(g^\sigma) \omega_g^\sigma$. So

$$\pi_{d_2 *} \pi_{d_1}^* \omega_g = [\Gamma_0(dN_g) : \Gamma_0(N)] a_d(g) \omega_g^\sigma,$$

and therefore

$$\begin{aligned} \langle \omega_{d_1}^\sigma, \eta_{d_2}^\sigma \rangle &= [\Gamma_0(N_g) : \Gamma_0(N)]^{-1} \langle \pi_{d_2 *} \pi_{d_1}^* \omega_g^\sigma, \eta_g^\sigma \rangle \\ &= \frac{[\Gamma_0(dN_g) : \Gamma_0(N)]}{[\Gamma_0(N_g) : \Gamma_0(N)]} a_d(g^\sigma) \langle \omega_g^\sigma, \eta_g^\sigma \rangle \\ &= \frac{a_d(g^\sigma)}{[\Gamma_0(N_g) : \Gamma_0(dN_g)]}. \end{aligned}$$

□

An explicit formula for the p-adic Abel–Jacobi map

Let $\alpha_p(g^\sigma)$ and $\beta_p(g^\sigma)$ be the roots in F of the polynomial

$$x^2 - a_p(g^\sigma)x + p = 0,$$

chosen such that $\mathrm{ord}_p(\alpha_p(g^\sigma)) = 0$ and $\mathrm{ord}_p(\beta_p(g^\sigma)) = 1$, which is possible since g is ordinary. Define $\alpha_p(f)$ and $\beta_p(f)$ analogously, noting that they are independent of σ since f has rational q-expansion. Following the notation of [DR12], write

$$\begin{aligned}\mathcal{E}_1(g^\sigma) =& 1 - \beta_p(g^\sigma)^2p^{-2},\\ \mathcal{E}(g^\sigma, g^\sigma, f) =& (1 - \beta_p(g^\sigma)\alpha_p(g^\sigma)\alpha_p(f)p^{-2}) \times (1 - \beta_p(g^\sigma)\beta_p(g^\sigma)\alpha_p(f)p^{-2})\\ &\times (1 - \beta_p(g^\sigma)\alpha_p(g^\sigma)\beta_p(f)p^{-2}) \times (1 - \beta_p(g^\sigma)\beta_p(g^\sigma)\beta_p(f)p^{-2})\\ =& (1 - a_p(f)p^{-1} + p^{-1}) \times (1 - \beta_p(g^\sigma)^2a_p(f)p^{-2} + \beta_p(g^\sigma)^4p^{-3}).\end{aligned}$$

Recall the rigid analytic primitive $G_d^{\sigma,[p]}$ of the p-depletion $g_d^{\sigma,[p]}$ of g_d and the ordinary projector e_{ord} as defined in §5.1. The following theorem of Darmon and Rotger will be essential in determining a precise formula for $P_{g,f,n}$.

Theorem 5.2.3 ([DR12], Theorem 3.8)**.** *Let $\eta_{d_1}^\sigma$, $\omega_{d_2}^\sigma$, ω_f, $\mathcal{E}_1(g^\sigma)$, and $\mathcal{E}(g^\sigma, g^\sigma, f)$ be as above. Then*

$$\mathrm{AJ}_p(\Delta_{GKS})(\eta_{d_1}^\sigma \otimes \omega_{d_2}^\sigma \otimes \omega_f) = \frac{\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)}\langle \eta_{d_1}^\sigma, e_{\mathrm{ord}}(G_{d_2}^{\sigma,[p]} \times f)\rangle.$$

Proof. See [DR12]. □

The formula for $P_{g,f,n}$

We define the p-adic logarithm of E to be the map

$$\log_E : E(F) \to F, \qquad P \mapsto \int_O^P \omega_E,$$

where $\int_O^P \omega_E$ is the Coleman integral of the invariant differential ω_E, suitably normalized in a formal neighborhood of O. Writing red : $E(\mathbb{C}_p) \to E(\bar{\mathbb{F}}_p)$ for the reduction map, if $\mathrm{red}(P) = \tilde{O}$, where $\tilde{O} \in E(\bar{\mathbb{F}}_p)$ is the identity, then this definition agrees with the formal group logarithm of E. Thus, $\log_E(P)$ is explicitly computable for any $P \in E(F)$ as

$$\log_E(P) = \frac{1}{m}\log_E(P^m),$$

where $m = \#E(k)$ with k the residue field of $\mathcal{O}_F$. Extend $\log_E$ to a map $\log_E : E(F)\otimes\mathbb{Q} \to F$ by the rule $\log_E(P \otimes \frac{r}{s}) = \frac{r}{s}\log_E(P)$.

Combining Proposition 5.2.2 with Lemma 4.2.2, we can obtain a formula for $\mathrm{cl}(\epsilon_0 T_{g,n})$, and subsequently a formula for $P_{g,f,n}$ using Theorem 5.2.3. In the notation of Lemma 4.2.2, the matrices A_n and B are of the form

$$\begin{bmatrix} C_n & 0 \\ 0 & C_n \end{bmatrix}, \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

where C_n is the matrix for the action of T_n on $\{\omega_d^\sigma\}_{d|N_{\mathrm{rel}},\sigma\in\Psi}$ and

$$D = (\langle \omega_{d_i}^\sigma, \eta_{d_j}^\sigma \rangle)_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi}.$$

For the first matrix, this is because T_n stabilizes the subspace spanned by the η_d^σ and the action is the same as that on the ω_d. The second matrix is due to Proposition 5.2.2 and the skew-symmetry of the Poincaré pairing. For each $d_i \mid N_{\mathrm{rel}}$ and $\sigma \in \Psi$, write

$$e_{ord}(G_{d_i}^{\sigma,[p]} \times f) = \sum_{d_j|N_{\mathrm{rel}},\sigma\in\Psi} \gamma_{d_i,d_j}^\sigma g_{d_j}^{\sigma,(p)} + \cdots$$

where $g_{d_j}^{\sigma,(p)}$ is the ordinary p-stabilization of $g_{d_j}^\sigma$. Finally, continuing the notation of [DR12], set

$$\mathcal{E}_0(g^\sigma) = 1 - \beta_p(g^\sigma)^2 p^{-1}.$$

Theorem 5.2.4. *In the notation above, we have*

$$\log_E(P_{g,f,n}) = \sum_{\sigma\in\Psi} \frac{2\mathcal{E}_0(g^\sigma)\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \sum_{d_i,d_j|N_{\mathrm{rel}}} m_{ij}^\sigma \gamma_{d_i,d_j}^\sigma,$$

where $M = (m_{ij}^\sigma)_{i,j,\sigma}$ *is the matrix*

$$M = D^{-1} C_n D.$$

In particular, if $\gcd(n, N_{\mathrm{rel}}) = 1$*, then* C_n *commutes with* D*, and thus*

$$\log_E(P_{g,f,n}) = \sum_{\sigma\in\Psi} \frac{2\mathcal{E}_0(g^\sigma)\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \sum_{d|N_{\mathrm{rel}}} a_n(g^\sigma)\gamma_{d,d}^\sigma,$$

Proof. From Theorem 2.3.6 and Proposition 2.3.5, we have

$$\begin{aligned} \log_E(P_{g,f,n}) &= \int_O^{P_{g,f,n}} \omega_E \\ &= \mathrm{AJ}_p(P_{g,f,n})(\omega_E) \\ &= \mathrm{AJ}_p(\Delta_{GKS})(\mathrm{cl}(\epsilon_o T_{g,n}) \otimes \omega_f). \end{aligned}$$

Note the slight abuse of notation: technically, $P_{g,f,n}$ is an element of $\mathrm{CH}^1(E)_0(\mathbb{Q})\otimes\mathbb{Q}$, but in the first two expressions we are identifying it with an element of $E(\mathbb{Q})\otimes\mathbb{Q}$ via its canonical isomorphism with $\mathrm{CH}^1(E)_0(\mathbb{Q})\otimes\mathbb{Q}$.

By Lemma 4.2.2, $\mathrm{cl}(\epsilon_0 T_{g,n})$ is given by the matrix $-B^{-1}A$, or

$$-\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}^{-1}\begin{bmatrix} C_n & 0 \\ 0 & C_n \end{bmatrix} = \begin{bmatrix} 0 & D^{-1} \\ -D^{-1} & 0 \end{bmatrix}\begin{bmatrix} C_n & 0 \\ 0 & C_n \end{bmatrix}$$
$$= \begin{bmatrix} 0 & D^{-1}C_n \\ -D^{-1}C_n & 0 \end{bmatrix}.$$

Write $D^{-1}C_n = (x^\sigma_{ij})_{i,j,\sigma}$. Then,

$$\mathrm{cl}(\epsilon_o T_{g,n}) = \sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}(\omega^\sigma_{d_i}\otimes\eta^\sigma_{d_j} - \eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}).$$

This yields

$$\log_E(P_{g,f,n}) = \sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}\mathrm{AJ}_p(\Delta_{GKS})(\omega^\sigma_{d_i}\otimes\eta^\sigma_{d_j}\otimes\omega_f - \eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f).$$

The cycle Δ_{GKS} is stable under the involution $i: X_0(N)^3 \to X_0(N)^3$ interchanging the first two factors, and thus using Proposition 2.3.5, we can rewrite the last expression as

$$\begin{aligned}
&\sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}(\mathrm{AJ}_p(\Delta_{GKS})(\omega^\sigma_{d_i}\otimes\eta^\sigma_{d_j}\otimes\omega_f) - \mathrm{AJ}_p(\Delta_{GKS})(\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f))\\
=&\sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}(\mathrm{AJ}_p(\Delta_{GKS})(i^*(-\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f)) - \mathrm{AJ}_p(\Delta_{GKS})(\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f))\\
=&\sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}(-\mathrm{AJ}_p(\Delta_{GKS})(\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f) - \mathrm{AJ}_p(\Delta_{GKS})(\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f))\\
=&-2\sum_{d_i,d_j|N_{\mathrm{rel}},\sigma\in\Psi} x^\sigma_{ij}\mathrm{AJ}_p(\Delta_{GKS})(\eta^\sigma_{d_j}\otimes\omega^\sigma_{d_i}\otimes\omega_f).
\end{aligned}$$

By Theorem 5.2.3, this becomes

$$\begin{aligned}
\log_E(P_{g,f,n}) &= -\sum_{\sigma\in\Psi}\frac{2\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma,g^\sigma,f)}\sum_{d_i,d_j|N_{\mathrm{rel}}} x^\sigma_{ij}\langle\eta^\sigma_{d_j}, e_{\mathrm{ord}}(G^{\sigma,[p]}_{d_i}\times f)\rangle\\
&= -\sum_{\sigma\in\Psi}\frac{2\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma,g^\sigma,f)}\sum_{d_i,d_j|N_{\mathrm{rel}}} x^\sigma_{ij}\sum_{d_k|N_{\mathrm{rel}}}\gamma^\sigma_{d_i,d_k}\langle\eta^\sigma_{d_j}, g^{\sigma,(p)}_{d_k}\rangle.
\end{aligned}$$

Now $g^{\sigma,(p)}_{d_k} = g^\sigma_{d_k} - \frac{1}{\alpha_p(g^\sigma)}g^\sigma_{pd_k}$ is ordinary, so $e_{\mathrm{ord}}g^{\sigma,(p)}_{d_k} = g^{\sigma,(p)}_{d_k}$. Similarly, $g^\sigma_{d_k} - \frac{1}{\beta_p(g^\sigma)}g^\sigma_{pd_k}$ has U_p-eigenvalue $\beta_p(g^\sigma)$ of slope 1, so its ordinary projection is 0. From these relations, we can establish

$$e_{\mathrm{ord}}g^\sigma_{d_k} = \frac{\alpha_p(g^\sigma)}{\alpha_p(g^\sigma)-\beta_p(g^\sigma)}g^{\sigma,(p)}_{d_k} = \frac{1}{1-\beta_p(g^\sigma)^2p^{-1}}g^{\sigma,(p)}_{d_k} = \frac{1}{\mathcal{E}_0(g^\sigma)}g^{\sigma,(p)}_{d_k}.$$

Additionally, by Propositions 5.1.3 and 5.2.2 and the skew-symmetry of the Poincaré pairing, we have

$$\langle \eta_{d_j}^\sigma, e_{\mathrm{ord}} g_{d_k}^\sigma \rangle = \langle \eta_{d_j}^\sigma, \omega_{d_k}^\sigma \rangle = -\langle \omega_{d_k}^\sigma, \eta_{d_j}^\sigma \rangle = -\langle \omega_{d_j}^\sigma, \eta_{d_k}^\sigma \rangle.$$

Putting it all together, we find that

$$\begin{aligned}
\log_E(P_{g,f,n}) &= -\sum_{\sigma\in\Psi} \frac{2\mathcal{E}_1(g^\sigma)\mathcal{E}_0(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \sum_{d_i,d_j \mid N_{\mathrm{rel}}} x_{ij}^\sigma \sum_{d_k \mid N_{\mathrm{rel}}} \gamma_{d_i,d_k}^\sigma \langle \eta_{d_j}^\sigma, e_{\mathrm{ord}} g_{d_k}^\sigma \rangle \\
&= \sum_{\sigma\in\Psi} \frac{2\mathcal{E}_1(g^\sigma)\mathcal{E}_0(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \sum_{d_i,d_k \mid N_{\mathrm{rel}}} \gamma_{d_i,d_k}^\sigma \sum_{d_j \mid N_{\mathrm{rel}}} x_{ij}^\sigma \langle \omega_{d_j}^\sigma, \eta_{d_k}^\sigma \rangle \\
&= \sum_{\sigma\in\Psi} \frac{2\mathcal{E}_1(g^\sigma)\mathcal{E}_0(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \sum_{d_i,d_k \mid N_{\mathrm{rel}}} m_{ik}^\sigma \gamma_{d_i,d_k}^\sigma .
\end{aligned}$$

This proves the first assertion. For the second, note that when $\gcd(n, N_{\mathrm{rel}}) = 1$ the classes $\omega_{d_i}^\sigma$ and $\eta_{d_j}^\sigma$ are eigenforms for T_n with eigenvalue $a_n(g^\sigma)$. Let r denote the number of divisors of N_{rel}. Then the matrix C_n diagonal formed by $|\Psi|$ blocks of $r \times r$ scalar matrices with entries $a_n(g^\sigma)$ for $\sigma \in \Psi$. The matrix D is also composed of $r \times r$ blocks along the diagonal, with 0 everywhere else. This is because $\langle \omega_{d_i}^{\sigma_1}, \eta_{d_j}^{\sigma_2} \rangle = 0$ when $\sigma_1 \neq \sigma_2$ by Proposition 5.2.2. From these descriptions, it is obvious that C_n and D commute, and so the second assertion follows from the first. □

5.3 A p-adic algorithm for computing Chow–Heegner points

With Theorem 5.2.4 in hand, we can now devise an algorithm for computing $P_{g,f,n}$. The quantities $\mathcal{E}_1(g^\sigma)$, $\mathcal{E}_0(g^\sigma)$, and $\mathcal{E}(g^\sigma, g^\sigma, f)$ appearing in the formula from the theorem can all be computed from the q-expansions of g and f and the embeddings Ψ. If $\gcd(n, N_{\mathrm{rel}}) = 1$, then the only other ingredients are $a_n(g^\sigma)$ and $\gamma_{d,d}^\sigma$ for $d \mid N_{\mathrm{rel}}$. The former can be computed from the q-expansion of g and the embeddings Ψ, and the latter using the ordinary projection algorithm of Lauder [Lau12]. Even if $\gcd(n, N_{\mathrm{rel}}) > 1$, we can still compute the coefficients m_{ij}^σ from the matrices C_n and D. The matrix C_n is established from the action of the Hecke operator T_n on the forms g_d for $d \mid N_{\mathrm{rel}}$, and D is given by Proposition 5.2.2.

In the next few subsections, we lay out the details for carrying out the computations of these various ingredients. It is convenient, though not indispensable, to assume that p is unramified in K_g. For the sake of computational efficiency, we will always choose the smallest p satisfying these hypotheses for the computations in the tables that can be found in Appendix A. As in Chapter 4.2, we only need to compute the points $P_{g,f,n}$ for Hecke operators T_n forming a basis for $T_{\mathbb{Q}}[g]$. As $p \nmid N$, we can always find a set of generators consisting of T_n with $p \nmid n$, so we will assume this henceforth.

Computing the embeddings Ψ

The first step is computing the coefficient field K_g, a subfield F of $\bar{\mathbb{Q}}_p$ containing the images of all embeddings Ψ, and then finding the embeddings themselves. The first item, the field K_g along with a defining polynomial $h(x)$, can be found easily using the modular forms package in `MAGMA`. Then we proceed by factoring $h(x)$ mod p. If p is unramified in K_g, then we can take as F the unramified extension $\mathbb{Q}_{p^d}$ of degree d over $\mathbb{Q}_p$, where d is the least common multiple of the degrees of all factors of $h(x)$ mod p. Otherwise, we must take a ramified extension of $\mathbb{Q}_{p^d}$. While this poses no problems computationally, it is an uncommon occurrence in practice, and it is often easier to simply choose another prime p that is unramified in K_g.

Once the field F has been determined, it is a simple application of Hensel's lemma to find the roots of $h(x)$ in $\mathcal{O}_F$ mod p^m for any desired precision m. These roots then determine $[K_g : \mathbb{Q}]$ embeddings of K_g into F: since $K_g \simeq \mathbb{Q}[x]/(h(x))$, we can write any $\beta \in K_g$ as $\sum_{i=0}^{\deg(h)-1} a_i x^i$, with $a_i \in \mathbb{Q}$. Then, if α is a root of $h(x)$ in $\mathbb{Q}_{p^d}$, set

$$\sigma_\alpha(\beta) = \sum_{i=0}^{\deg(h)-1} a_i \alpha^i.$$

Then $\Psi = \{\sigma_\alpha \mid \alpha \in \mathbb{Q}_{p^d}, h(\alpha) = 0\}$. In this way we can represent Ψ as a list of approximations of roots of $h(x)$ in F.

Computing $e_{\mathrm{ord}}(G_{d_i}^{\sigma,[p]} \times f)$

Now we set our sights on computing the quantities $\gamma_{d_i,d_j}^{\sigma}$. The first step is to compute $e_{\mathrm{ord}}(G_{d_i}^{\sigma,[p]} \times f)$. We use an algorithm of Lauder to this end, which we briefly describe in this subsection. For more details, the interested reader may consult [Lau12]. The notation in this section is chosen to be consistent with that of *loc. cit.*, and any inconsistencies with other sections should be disregarded.

Let $H \in M_k(N, \chi, F, \frac{1}{p+1})$ be an overconvergent modular form of weight k, coefficient field K, and convergence radius $\frac{1}{p+1}$. For any integer $m \geq 1$, the algorithm computes the image

$$e_{\mathrm{ord}} H \in \mathcal{O}_F[[q]]/(p^m, q^{s(m,p)}),$$

where $s(m,p)$ is an explicit function of m and p to be explained later. The most obvious method for computing $e_{\mathrm{ord}} H$ is to iterate the U_p operator sufficiently many times; this, however, is very expensive computationally, requiring $p^s m$ Fourier coefficients of H to compute the s-fold iterate of U_p. The more efficient approach used by Lauder is to compute (approximations of) a basis of $M_k(N, \chi, F, \frac{1}{p+1})$, the matrix for U_p with respect to this basis, and then iterate this matrix on H sufficiently many times. All computations must be accurate enough to guarantee that the output for $e_{\mathrm{ord}} H$ is correct mod $(p^m, q^{s(m,p)})$.

The first step is to compute Katz expansions. Set $k_0 := k$. Let $n = \lfloor \frac{p+1}{p-1}(m+1) \rfloor$, and for each $i = 0, \ldots, n$, compute d_i, the dimension of the space of classical modular forms of level

N, character χ, and weight $k_0 + i(p-1)$. Set $m_0 := d_0$ and $m_i := d_i - d_{i-1}$ for $i \geq 1$, and $\ell := m_0 + \cdots + m_n = d_n$. To guarantee the output is correct mod p^m, we will be working mod $p^{m'}$, where $m' = m + \lceil \frac{n}{p+1} \rceil$. Also compute the Sturm bound $\ell' \geq \ell$ for the space of classical modular forms of level N, character χ, and weight $k_0 + n(p-1)$.

As in §5.1, choose submodules $\mathbf{W}_i(N, \chi, \mathbb{Z}_p)$ of the spaces $S_{k_0+i(p-1)}(N, \chi, \mathbb{Z}_p)$. Compute the q-expansions of the Eisenstein series E_{p-1} and a basis $b_{i,1}, \ldots, b_{i,m_i}$ of $\mathbf{W}_i(N, \chi, \mathbb{Z}_p)$ in $\mathbb{Z}[[q]]/(p^{m'}, q^{p\ell'})$. Then compute the Katz basis elements

$$e_{i,s} := p^{\lfloor \frac{i}{p+1} \rfloor} E_{p-1}^{-i} b_{i,s}.$$

The next step is to compute the matrix of the U_p operator. First, compute the q-expansions of the elements $t_{i,s} := U_p(e_{i,s})$ in $\mathbb{Z}[[q]]/(p^{m'}, q^{\ell'})$. Then, let T be the $\ell \times \ell'$ matrix with the ℓ' coefficients of the ℓ elements $t_{i,s}$ as entries, and E the $\ell \times \ell'$ matrix with the ℓ' coefficients of the ℓ elements $e_{i,s}$ as entries. Using linear algebra over $\mathbb{Z}/(p^{m'})$, compute the matrix A' such that $T = A'E$. This is the matrix for the U_p operator with respect to the chosen basis of Katz expansions, and by reducing mod p^m we obtain the matrix A that we require.

Now, we are ready to compute $e_{\mathrm{ord}}H$. The first step is to find the q-expansion for $H \in \mathcal{O}_F[[q]]/(p^{m'}, q^{p\ell'})$, and then $U_p(H) \in \mathcal{O}_F[[q]]/(p^{m'}, q^{\ell'})$ to improve overconvergence. This last step ensures that $U_p(H) \in M_k(N, \chi, F, \frac{p}{p+1})$ so that the coefficients in the infinite vector representing $U_p(H)$ with respect to our Katz basis decay p-adically. This allows us to iterate the matrix A on the first ℓ coefficients, as the rest vanish mod p^m. Now find coefficients $\alpha_{i,s} \in \mathcal{O}_F/(p^m)$ such that $U_p(H) \equiv \sum_{i,s} \alpha_{i,s} e_{i,s} \bmod (p^m, q^{\ell'})$. Notice the loss of precision of $m' - m$ in this step, which is why we have been using m' up until now.

Now we wish to iterate the matrix A on the vector $\alpha = (\alpha_{i,s})$ to obtain the ordinary projection. We must choose a power of A that will kill all subspaces of positive slope and fix the ordinary subspace mod p^m. Thus, we compute the integer f such that all unit roots of the reverse characteristic polynomial of A lie in an extension of $\mathbb{Q}_p$ with residue field of degree f over $\mathbb{F}_p$. Then, A^r, where $r := (p^f - 1)p^m$, will be sufficient. Since we have already applied U_p once, we compute A^{r-1} using fast exponentiation, and then set $\gamma := \alpha A^{r-1}$. Finally,

$$e_{\mathrm{ord}}H = \sum_{i,s} \gamma_{i,s} e_{i,s} \in \mathcal{O}_F[[q]]/(p^m, q^{s(m,p)}),$$

where $s(m,p) = p\ell'$.

Computing the coefficients $\gamma^{\sigma}_{d_i,d_j}$

Once we know $e_{\mathrm{ord}}(G^{\sigma,[p]}_{d_i} \times f)$, it remains to determine $\gamma^{\sigma}_{d_i,d_j}$, where

$$e_{ord}(G^{\sigma,[p]}_{d_i} \times f) = \sum_{d_j | N_{\mathrm{rel}}, \sigma \in \Psi} \gamma^{\sigma}_{d_i,d_j} g^{\sigma,(p)}_{d_j} + \cdots .$$

We will do this by computing $e^\sigma_{d_j}$, that is the functional on $M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$ sending $g^{\sigma,(p)}_{d_j}$ to 1 and all other eigenforms to 0. First, we must compute a basis of $M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$. By Theorem 6.1 of [Col96], all such forms are classical, and thus we may compute a basis via classical methods. However, as Np gets large this can be computationally intensive, whereas with a little extra work Lauder's method for computing e_{ord} also produces the desired basis. Namely, if we compute $B := A^r = A^{r-1}A$, then B is the ordinary projector on our Katz basis. Hence, if we let $(B_{i,s})$ be the nonzero rows in the echelon form of B, then the elements $\sum_{i,s} B_{i,s}e_{i,s}$ form a basis of $M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$.

Note that $e^\sigma_{d_j}$ is determined by the properties

$$e^\sigma_{d_j}(g^{\sigma,(p)}_{d_j}) = 1, \qquad e^\sigma_{d_j}(T_n h) = a_n(g^\sigma_{d_j})e^\sigma_{d_j}(h)$$

for all T_n with $p \nmid n$ and $h \in M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$. As we can compute the action of T_n on q-expansions, this gives us a procedure for determining e_{d_j}. Set $d_{\mathrm{ord}} := \dim M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$ and let $h_1,\ldots,h_{d_{\mathrm{ord}}}$ be the basis elements found using Lauder's algorithm. Then there exist indices $m_1,\ldots,m_{d_{\mathrm{ord}}}$ such that we can represent any $h \in M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}}$ as the vector $(a_{m_i}(h))$ of its Fourier coefficients. Then create a matrix Q and a vector v as follows:

- The first row of Q is $(a_{m_i}(g^{\sigma,(p)}_{d_j}))$ and the first entry of v is 1.
- Add rows to Q of the form $(a_{m_i}(T_n h_j) - a_{m_i}(g^{\sigma,(p)}_{d_j})a_{m_i}(h_k))$ for Hecke operators T_n and $h_1,\ldots,h_{d_{\mathrm{ord}}}$ so long as the rank of Q increases. Stop when Q has d_{ord} rows.
- The remaining entries of v are all 0.

Then, writing $(q_i) = Q^{-1}v$, we have $e^\sigma_{d_j}(h) = \sum_i q_i a_{m_i}(h)$. Finally, from $e^\sigma_{d_j}$, we can compute

$$\gamma^\sigma_{d_i,d_j} = e^\sigma_{d_j}(e_{\mathrm{ord}}(G^{\sigma,[p]}_{d_i} \times f)).$$

Computing the coefficients m^σ_{ij}

The final ingredient in the formula for $P_{g,f,n}$ is determining the entries m^σ_{ij} of the matrix M from Theorem 5.2.4. If $\gcd(n, N_{\mathrm{rel}}) = 1$, then $M = C_n$ and $m^\sigma_{ij} = a_n(g^\sigma)$ if $i = j$, and is 0 otherwise, leading to the formula at the end of the theorem. So we may assume $\gcd(n, N_{\mathrm{rel}}) > 1$. Recall that M is given by

$$M = D^{-1}C_nD,$$

where C_n encodes the action of T_n on the basis $\{\omega^\sigma_d\}_{d|N_{\mathrm{rel}},\sigma\in\Psi}$ and

$$D = (\langle\omega^{\sigma_i}_{d_i}, \eta^{\sigma_j}_{d_j}\rangle)_{d_i,d_j|N_{\mathrm{rel}},\sigma_i,\sigma_j\in\Psi}.$$

The matrix D is given explicitly by Proposition 5.2.2, so it suffices to compute C_n, or simply $T_n g^\sigma_d$. If $n = \prod \ell_i^{k_i}$, then $T_n = \prod T_{\ell_i^{k_i}}$. If $\ell \nmid N_{\mathrm{rel}}$, then $T_{\ell^k} g^\sigma_d = a_{\ell^k}(g^\sigma)g^\sigma_d$, so we only need to consider $\ell \mid N_{\mathrm{rel}}$. Since $N_{\mathrm{rel}} \mid N$, then $\ell \mid N$, and for such ℓ we have $T_{\ell^k} = T_\ell^k$, and so we only need to determine the action for T_ℓ. The next lemma gives us this action.

Lemma 5.3.1. *For any* $\ell \mid N_{\mathrm{rel}}$*,*

$$T_\ell g_d^\sigma = \begin{cases} a_\ell(g^\sigma)g_d^\sigma - g_{\ell d}^\sigma, & \ell \nmid d, \ell \nmid N_g, \\ a_\ell(g^\sigma)g_d^\sigma, & \ell \nmid d, \ell \mid N_g, \\ \ell g_{d/\ell}^\sigma, & \ell \mid d. \end{cases}$$

Proof. Before we begin the proof, we gather a few facts that we will need. Recall that $g_d^\sigma = \pi_d^* g^\sigma$, where $\pi_d : X_0(N) \to X_0(N_g)$ is the degeneracy map. Let $\pi_1, \pi_\ell : X_0(N) \to X_0(N/\ell)$ be the degeneracy maps, and $w_\ell : X_0(N) \to X_0(N)$ be the morphism given by "quotient by the ℓ-primary subgroup". Then, we have

$$T_\ell + w_\ell^* = \pi_1^* \pi_{\ell *} \tag{5.8}$$

if $\ell^2 \nmid N$ and

$$T_\ell = \pi_1^* \pi_{\ell *} \tag{5.9}$$

if $\ell^2 \mid N$. Let $\pi'_d, \pi'_{d/\ell} : X_0(N/\ell) \to X_0(N_g)$ be the degeneracy maps, with the second one only defined if $\ell \mid d$. Then, we can write $\pi_d = \pi'_d \pi_1$ if $\ell \nmid d$ and $\pi_d = \pi'_{d/\ell}\pi_\ell$ otherwise. There are two cases:

Case 1: $\ell \mid d$. If $\ell \nmid N_g$, then $\ell^2 \nmid N = N_{\mathrm{rel}} N_g$, and so $\pi_\ell w_\ell = \pi_1$. Then we have

$$w_\ell^* g_d^\sigma = w_\ell^* \pi_d^* g^\sigma = w_\ell^* \pi_\ell^* \pi_{d/\ell}'^* g^\sigma = \pi_1^* \pi_{d/\ell}'^* g^\sigma = \pi_{d/\ell}^* g^\sigma = g_{d/\ell}^\sigma.$$

We compute $\pi_1^* \pi_{\ell *} g_d^\sigma$ as

$$\pi_1^* \pi_{\ell *} g_d^\sigma = \pi_1^* \pi_{\ell *} \pi_d^* g^\sigma = \pi_1^* \pi_{\ell *} \pi_\ell^* \pi_{d/\ell}'^* g^\sigma = \deg(\pi_\ell) \pi_1^* \pi_{d/\ell}^* g^\sigma = \deg(\pi_\ell) g_{d/\ell}^\sigma.$$

We have $\deg(\pi_\ell) = \ell + 1$ if $\ell^2 \nmid N$ and $\deg(\pi_\ell) = \ell$ if $\ell^2 \mid N$. So by equations (5.8) and (5.9), we have

$$T_\ell g_d^\sigma = \ell g_{d/\ell}.$$

Case 2: $\ell \nmid d$. Let $\ell^k \mid\mid N_{\mathrm{rel}}$, where $k \geq 1$ is a positive integer. We will prove this case by induction on k. First suppose that $k = 1$. Much like case 1, if $\ell \nmid N_g$, then $\pi_1 w_\ell = \pi_\ell$. So

$$w_\ell^* g_d^\sigma = w_\ell^* \pi_d^* g^\sigma = w_\ell^* \pi_1^* \pi_d'^* g^\sigma = \pi_\ell^* \pi_d'^* g^\sigma = \pi_{\ell d}^* g^\sigma = g_{\ell d}^\sigma.$$

Now $\pi_d'^* g^\sigma$ is an eigenform for the Hecke operator $T_\ell = \pi_{\ell *} \pi_1^*$ with eigenvalue $a_\ell(g^\sigma)$, so

$$\pi_1^* \pi_{\ell *} g_d^\sigma = \pi_1^* \pi_{\ell *} \pi_d^* g^\sigma = \pi_1^* \pi_{\ell *} \pi_1^* \pi_d'^* g^\sigma = \pi_1^*(a_\ell(g^\sigma) \pi_d'^* g^\sigma) = a_\ell(g^\sigma) \pi_d^* g^\sigma = a_\ell(g^\sigma) g_d^\sigma.$$

Again by equations (5.8) and (5.9), we have

$$T_\ell g_d^\sigma = \begin{cases} a_\ell(g^\sigma) g_d - g_{\ell d}^\sigma, & \ell \nmid N_g \\ a_\ell(g^\sigma) g_d^\sigma, & \ell \mid N_g \end{cases}.$$

This proves it when $k=1$. For general $k>1$, let $g_d'^{\sigma} = \pi_d'^{*} g^{\sigma}$ be the form on $X_0(N/\ell)$. By the inductive hypothesis, we have

$$T_\ell g_d'^{\sigma} = \begin{cases} a_\ell(g^\sigma) g_d'^{\sigma} - g_{\ell d}'^{\sigma}, & \ell \nmid N_g \\ a_\ell(g^\sigma) g_d'^{\sigma}, & \ell \mid N_g \end{cases}.$$

Recalling that $T_\ell = \pi_{\ell *}\pi_1^*$, if $\ell \nmid N_g$ we have

$$\begin{aligned}\pi_1^*\pi_{\ell*} g_d^\sigma &= \pi_1^*\pi_{\ell*}\pi_d^* g^\sigma = \pi_1^*\pi_{\ell*}\pi_1^*\pi_d'^{*} g^\sigma = \pi_1^*\pi_{\ell*}\pi_1^* g_d'^{\sigma} = \pi_1^*(a_\ell(g^\sigma) g_d'^{\sigma} - g_{\ell d}'^{\sigma}) \\ &= a_\ell(g^\sigma) g_d^\sigma - g_{\ell d}^\sigma,\end{aligned}$$

whereas if $\ell \mid N_g$, then

$$\pi_1^*\pi_{\ell*} g_d^\sigma = \pi_1^*\pi_{\ell*}\pi_d^* g^\sigma = \pi_1^*\pi_{\ell*}\pi_1^*\pi_d'^{*} g^\sigma = \pi_1^*\pi_{\ell*}\pi_1^* g_d'^{\sigma} = \pi_1^*(a_\ell(g^\sigma) g_d'^{\sigma}) = a_\ell(g^\sigma) g_d^\sigma.$$

Since $k>1$, then $p^2 \mid N$, and thus by formula (5.9) we have $T_\ell = \pi_1^*\pi_{\ell*}$. This proves the lemma. □

Extracting $P_{g,f,n}$

At this point, we have methods for computing all of the ingredients in Theorem 5.2.4 necessary to find $\log_{E_f}(P_{g,f,n})$, so the final step is to compute $P_{g,f,n}$ itself. The simplest way to do this is to first find a generator $P \in E(\mathbb{Q})$ modulo torsion and then compute $\log_{E_f}(P)$ and compare it to $\log_{E_f}(P_{g,f,n})$. However, we cannot directly compute $\log_{E_f}(P)$, since P is not in the residue disk of the identity $O \in E_f(\mathbb{C}_p)$. Instead, we set $A_p := \#\tilde{E}_f(\mathbb{F}_p) = p+1-a_p(f)$ and compute $\log_{E_f}(A_pP)$, as A_pP is in the proper residue disk. This is a straightforward computation, as explained in Chapter IV of [Sil09]. Then, $\log_{E_f}(A_p d_{g,n} P_{g,f,n})$ is an integral multiple of $\log_{E_f}(A_pP)$, and thus we can determine $P_{g,f,n}$ as an element of $E(\mathbb{Q}) \otimes \mathbb{Q}$ by multiplying $\log_{E_f}(P_{g,f,n})$ by $A_p d_{g,n}$ and comparing it with $\log_{E_f}(A_pP)$.

Precision of the algorithm

In this final section, we analyze the accuracy of the algorithm. For the step supplied by Lauder, we will not go into details as this is discussed in [Lau12]. Hence, we will assume that the output of his computation is $e_{\mathrm{ord}}(G_{d_i}^{\sigma,[p]} \times f)$ and a basis of $M_2^{\mathrm{oc}}(N,\chi,F)^{\mathrm{ord}} \bmod p^m$.

The main obstacle is the loss of precision incurred by computing Q^{-1}. In order to quantify this, let us introduce two types of precision when computing with $\mathbb{Z}_p$ and $\mathbb{Q}_p$. For any $a \in \mathbb{Z}_p$, the *absolute precision* of a is the largest r such that a is known mod p^r. This notion does not exist for elements of $\mathbb{Q}_p$, so instead for $a \in \mathbb{Q}_p$ write $a = p^{\mathrm{ord}_p a}\sum_{n\geq 0} a_n p^n$. Then the *relative precision* of a is the largest r such that $a_0,\ldots,a_{r-1}$ are known.

We will now use these concepts to determine an upper bound on the loss of precision. Let $m_0 = \mathrm{ord}_p \det(Q)$, which we know with absolute precision m along with the entries of Q. If $m_0 = 0$, then Q^{-1} has entries in $\mathbb{Z}_p$ of absolute precision m. However, if $m_0 > 0$ then

Q^{-1} will have entries in $\mathbb{Q}_p$. Consider an entry a_{ij} of Q^{-1}. We know that $\det Q$ has relative precision $m - m_0$, so if $\mathrm{ord}_p a_{ij} \le m_0$, then we also know a_{ij} with relative precision $m - m_0$. On the other hand, if $\mathrm{ord}_p a_{ij} > m_0$, then we only know a_{ij} with relative precision $m - \mathrm{ord}_p a_{ij}$ (setting $\mathrm{ord}_p a_{ij} = m$ if a_{ij} is 0 mod p^m). We do know, however, a_{ij} with absolute precision $m - m_0$.

Now, as the vector (q_i) is simply the first column of Q^{-1}, the same is true for (q_i). Thus, when we compute $e_{g_\alpha^\sigma}(h) = \sum_i q_i a_{m_i}(h)$, the worst possible case is if $\mathrm{ord}_p q_i = -m_0$, q_i has relative precision $m - m_0$, and $\mathrm{ord}_p a_{m_i}(h) = 0$. In this case, the largest coefficient we would know is that of p^{m-2m_0-1}. For every other possibility, we are guaranteed to know the coefficient of p^{m-2m_0-1}, so we know $e_{g_\alpha^\sigma}(h)$ with absolute precision $m - 2m_0$, yielding a loss of $2m_0$. Thus, if we wish to compute $\lambda_{g^\sigma,f}$ with absolute precision m_1, we may need to run Lauder's algorithm with $m = m_1 + 2m_0$. This creates two additional problems: the necessity of computing m_0 beforehand, and the longer computing time from increasing the precision of Lauder's algorithm. The author has discovered that, at least when Np is relatively small, it is more efficient to simply compute a basis for $M_2^{\mathrm{oc}}(N, \chi, F)^{\mathrm{ord}}$ rationally by finding a basis of the space of classical forms, and avoiding the loss of precision entirely.

Finally, the quantity

$$\mathrm{ord}_p \left(-2A_p d_{g,n} \frac{\mathcal{E}_0(g^\sigma)\mathcal{E}_1(g^\sigma)}{\mathcal{E}(g^\sigma, g^\sigma, f)} \right)$$

will result in a further gain or loss of absolute precision depending on whether it is positive or negative. However, this can be computed at the start of the algorithm, and all subsequent computations can take this extra change into account.

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Appendix A

Tables of Chow-Heegner points

We present two tables of Chow-Heegner points computed using the algorithms presented in Chapters 4 and 5. Both tables compute a basis for the module of points $\underline{P}_{\mathbb{T}_{\mathbb{Q}}[g],f}$, where

- f ranges over newforms with level $N_f < 100$, and E_f rank 1,
- g ranges over newforms with $N_g \mid N_f$,
- $N = N_f$.

As one would hope, the two algorithms agree in all cases considered. The algorithm from Chapter 4 is performed using `Sage`, while the algorithm from Chapter 5 is carried out using `MAGMA`, as these are the most efficient programs for each. In both tables, the curve E_f is listed by its label in the Cremona database of elliptic curves, P is a generator of $E_f(\mathbb{Q})$ modulo torsion, n is the positive integer appearing in the definition of the cycle $T_{g,n}$, $d_{g,n}$ is a positive integer such that $d_{g,n}T_{g,n} \in \mathbb{T}$, and $P_{g,f,n} \in E_f(\mathbb{Q}) \otimes \mathbb{Q}$ is the point defined in Chapter 3.

Table A.1 uses the algorithm from Chapter 4 and lists g by its index in the `Sage` routine `ModularSymbols(N).cuspidal_subspace()`. The complex number $z_{g,f,n}$ is the output of the algorithm and should be considered as an element of $\mathbb{C}/\Lambda_f$. Using a slightly different coding of the same algorithm might have the result of modifying $z_{g,f,n}$ by an element of Λ_f. The point $P_{g,f,n}$ can be extracted from $z_{g,f,n}$ by taking the complex elliptic exponential of $d_{g,n}z_{g,n,f}$.

Table A.2 uses the algorithm from Chapter 5 and lists g by its level N_g and its index in the `Newforms(CuspForms(Ng))` routine in `MAGMA`. The prime p is the smallest prime such that f and g are ordinary at p and p is unramified in K_g, and we write A_p for $\#E_f(\mathbb{F}_p) = p+1-a_p(f)$. Finally, the p-adic number $\log_{E_f}(A_p d_{g,n} P_{g,f,n})$ is the output of the algorithm, $\log_{E_f}(P_{g,f,n})$, multiplied by $A_p d_{g,n}$, from which $P_{g,f,n}$ can be extracted by comparison with $\log_{E_f}(A_p P)$.

Table A.1: Chow-Heegner points computed complex analytically on curves of rank 1 and conductor < 100

E_f	P	$g\ (N_g)$	n	$d_{g,n}$	$z_{g,f,n}$	$P_{g,f,n}$
37a1	$(0,-1)$	1 (37)	1	2	$-(0.4093\ldots)+(1.2256\ldots)i$	$-6P$
43a1	$(0,-1)$	1 (43)	1	2	$-(2.0768\ldots)+(2.7263\ldots)i$	$4P$
			2	1	$-(2.4055\ldots)+(0.0000\ldots)i$	$2P$
53a1	$(0,-1)$	1 (53)	1	2	$-(1.2782\ldots)+(7.7029\ldots)i$	$-2P$
			2	2	$-(2.7691\ldots)-(1.5405\ldots)i$	$-8P$
			3	2	$(0.2126\ldots)+(7.7029\ldots)i$	$4P$
57a1	$(2,1)$	1 (57)	1	12	$(0.0407\ldots)+(4.3961\ldots)i$	$\frac{4}{3}P$
		2 (57)	1	3	$-(0.1630\ldots)+(3.5169\ldots)i$	$-\frac{16}{3}P$
		3 (19)	1	2	$-(0.8167\ldots)+(8.1529\ldots)i$	$-4P$
			3	2	$(0.5721\ldots)-(7.6733\ldots)i$	$-4P$
58a1	$(0,-1)$	1 (58)	1	4	$(4.2294\ldots)+(2.2236\ldots)i$	$4P$
		2 (29)	1	2	$(43.7247\ldots)+(8.8944\ldots)i$	0
			2	2	$-(10.8009\ldots)-(4.4472\ldots)i$	$4P$
			3	2	$(21.9926\ldots)+(8.8944\ldots)i$	$4P$
			4	2	$-(10.8009\ldots)-(1.1118\ldots)i$	$4P$
61a1	$(1,-1)$	1 (61)	1	2	$(2.2974\ldots)+(4.4874\ldots)i$	$-2P$
			2	2	$(3.0715\ldots)+(2.4930\ldots)i$	$4P$
			3	1	$-(1.5382\ldots)+(0.9972\ldots)i$	$-4P$
65a1	$(-1,1)$	1 (65)	1	2	$(4.2861\ldots)+(5.0850\ldots)i$	P
			2	2	$-(5.1338\ldots)-(6.3563\ldots)i$	$3P$
		2 (65)	1	2	$(4.2861\ldots)+(5.0850\ldots)i$	P
			2	2	$(0.7469\ldots)-(1.2712\ldots)i$	P
77a1	$(2,3)$	1 (77)	1	20	$(0.0563\ldots)+(0.9796\ldots)i$	$\frac{12}{5}P$
		2 (77)	1	6	$(0.1020\ldots)+(1.2559\ldots)i$	$-\frac{4}{3}P$
		3 (11)	1	6	$-(0.1020\ldots)+(1.7583\ldots)i$	$\frac{4}{3}P$
			7	6	$(1.2777\ldots)-(2.5119\ldots)i$	$\frac{44}{3}P$
		4 (77)	1	10	$-(0.4563\ldots)+(1.6578\ldots)i$	$-\frac{12}{5}P$
			2	2	$(1.9059\ldots)-(0.7535\ldots)i$	$-4P$
79a1	$(0,0)$	1 (79)	1	2	$-(7.0579\ldots)+(8.0526\ldots)i$	$-4P$
			2	2	$(1.8682\ldots)-(2.0131\ldots)i$	$-4P$
			3	2	$-(1.1071\ldots)+(0.0000\ldots)i$	$-4P$
			4	2	$-(2.9754\ldots)+(4.0263\ldots)i$	0
			5	2	$-(11.9016\ldots)+(14.0920\ldots)i$	0

E_f	P	g (N_g)	n	$d_{g,n}$	$z_{g,f,n}$	$P_{g,f,n}$
82a1	$(0,0)$	1 (82)	1	4	$-(10.3779\dots)+(8.9281\dots)i$	0
			3	2	$(1.8759\dots)+(0.0000\dots)i$	$2P$
		2 (41)	1	2	$-(29.2580\dots)+(26.7844\dots)i$	$2P$
			2	2	$(5.1889\dots)-(6.6961\dots)i$	0
			3	2	$(1.8759\dots)-(0.0000\dots)i$	$2P$
			4	2	$(2.5944\dots)-(6.6961\dots)i$	0
			5	2	$(19.3188\dots)-(17.8562\dots)i$	$4P$
			6	2	$(30.4153\dots)-(24.5524\dots)i$	$2P$
83a1	$(0,0)$	1 (83)	1	2	$-(5.9053\dots)+(8.8072\dots)i$	0
			2	2	$(1.6527\dots)+(0.9785\dots)i$	$2P$
			3	2	$(0.7745\dots)-(0.9785\dots)i$	$4P$
			4	2	$-(10.8979\dots)+(8.8072\dots)i$	$-4P$
			5	1	$-(3.3054\dots)+(5.8714\dots)i$	$-4P$
			7	2	$-(4.2180\dots)+(6.8500\dots)i$	0
88a1	$(2,-2)$	1 (88)	1	16	$(0.0000\dots)-(3.7247\dots)i$	0
			3	16	$(0.0000\dots)-(2.0692\dots)i$	0
		2 (44)	1	8	$(0.0000\dots)-(4.1385\dots)i$	0
			2	2	$(1.8916\dots)-(1.6554\dots)i$	$8P$
		3 (11)	1	2	$(0.0000\dots)-(6.6216\dots)i$	0
			2	2	$-(2.3608\dots)+(0.0000\dots)i$	$8P$
			4	1	$-(0.4692\dots)+(1.6554\dots)i$	$16P$
			8	1	$(3.7833\dots)-(9.9325\dots)i$	$16P$
89a1	$(0,-1)$	1 (89)	1	5	$-(0.1632\dots)+(1.6095\dots)i$	$\frac{8}{5}P$
		2 (89)	1	10	$-(6.2875\dots)+(10.4620\dots)i$	$\frac{2}{5}P$
			2	10	$(0.2451\dots)-(1.0347\dots)i$	$\frac{22}{5}P$
			3	10	$(4.4909\dots)-(6.0932\dots)i$	$-\frac{16}{5}P$
			4	10	$-(15.9229\dots)+(24.0282\dots)i$	$-\frac{2}{5}P$
			6	10	$(3.6747\dots)-(9.5423\dots)i$	$\frac{24}{5}P$
91a1	$(0,0)$	1 (91)	1	4	$-(2.9787\dots)+(3.3385\dots)i$	$2P$
		2 (91)	1	4	$-(6.8760\dots)+(6.6770\dots)i$	$2P$
			2	2	$(1.0301\dots)+(1.6692\dots)i$	$-2P$
		3 (91)	1	4	$-(9.8547\dots)+(10.0156\dots)i$	$4P$
			2	2	$-(2.9787\dots)+(3.3385\dots)i$	$2P$
			3	2	$(6.6527\dots)-(6.6770\dots)i$	$6P$

E_f	P	g (N_g)	n	$d_{g,n}$	$z_{g,f,n}$	$P_{g,f,n}$
91b1	$(-1,3)$	0 (91)	1	4	$-(1.0065\ldots)+(3.6260\ldots)i$	0
		2 (91)	1	4	$-(2.0131\ldots)+(5.8016\ldots)i$	0
			2	2	$(0.0000\ldots)-(0.0000\ldots)i$	0
		3 (91)	1	4	$-(6.0394\ldots)+(10.1528\ldots)i$	0
			2	2	$-(1.0065\ldots)+(3.6260\ldots)i$	0
			3	2	$(5.0329\ldots)-(7.9772\ldots)i$	0
92b1	$(1,1)$	1 (92)	1	2	$-(0.0000\ldots)+(0.0000\ldots)i$	0
		2 (46)	1	15	$(0.0000\ldots)-(0.0000\ldots)i$	0
			2	5	$-(0.9414\ldots)+(1.3179\ldots)i$	0
		3 (23)	1	20	$(0.0000\ldots)+(0.0000\ldots)i$	0
			2	5	$(0.9414\ldots)-(3.5145\ldots)i$	0
			3	4	$(0.0000\ldots)+(0.0000\ldots)i$	0
			4	5	$(3.7656\ldots)+(1.3179\ldots)i$	0
			6	1	$-(4.7070\ldots)-(4.3931\ldots)i$	0
			8	5	$(0.9414\ldots)+(3.0752\ldots)i$	0
99a1	$(2,0)$	1 (99)	1	12	$(0.1687\ldots)-(5.1198\ldots)i$	$-\frac{2}{3}P$
		2 (99)	1	12	$-(0.0000\ldots)-(5.5137\ldots)i$	0
		3 (99)	1	6	$-(0.1687\ldots)-(4.3322\ldots)i$	$\frac{2}{3}P$
		4 (33)	1	12	$-(0.1687\ldots)-(9.8459\ldots)i$	$\frac{2}{3}P$
			3	3	$(0.8244\ldots)+(1.5753\ldots)i$	$\frac{8}{3}P$
		5 (11)	1	6	$(0.1687\ldots)-(15.3596\ldots)i$	$-\frac{2}{3}P$
			3	3	$-(2.4553\ldots)-(1.5753\ldots)i$	$-\frac{2}{3}P$
			9	3	$-(0.0179\ldots)+(29.9316\ldots)i$	$-\frac{22}{3}P$

Table A.2: Chow-Heegner points computed p-adically on curves of rank 1 and conductor < 100

E_f	P	g	p	A_p	n	$d_{g,n}$	$\log_{E_f}(A_p d_{g,n} P_{g,f,n})$	$P_{g,f,n}$
37a1	$(0,0)$	37(2)	7	9	1	2	$5\cdot7+5\cdot7^2+4\cdot7^3+O(7^6)$	$6P$
43a1	$(0,0)$	43(2)	5	10	1	2	$5+5^2+5^3+2\cdot5^4+4\cdot5^5+O(5^6)$	$-4P$
					2	1	$4\cdot5+2\cdot5^2+5^3+4\cdot5^4+4\cdot5^5+O(5^6)$	$-2P$
53a1	$(0,0)$	53(2)	7	12	1	2	$6\cdot7+2\cdot7^2+3\cdot7^3+5\cdot7^4+7^5+O(7^6)$	$2P$
					2	2	$3\cdot7+4\cdot7^2+6\cdot7^3+O(7^6)$	$8P$
					3	2	$2\cdot7+7^2+3\cdot7^4+3\cdot7^5+O(7^6)$	$-4P$
57a1	$(2,-2)$	57(2)	5	9	1	3	$2\cdot5+5^3+3\cdot5^4+2\cdot5^5+O(5^6)$	$\frac{16}{3}P$
		57(3)	5	9	1	12	$3\cdot5+4\cdot5^2+3\cdot5^3+5^4+2\cdot5^5+O(5^6)$	$-\frac{4}{3}P$
		19(1)	5	9	1	2	$5+3\cdot5^3+5^4+5^5+O(5^6)$	$4P$
					3	2	$5+3\cdot5^3+5^4+5^5+O(5^6)$	$4P$
58a1	$(0,1)$	58(2)	5	9	1	4	$4\cdot5+2\cdot5^2+3\cdot5^3+5^4+3\cdot5^5+O(5^6)$	$-4P$
		29(1)	5	9	1	2	$O(5^6)$	0
					2	2	$2\cdot5+1\cdot5^2+4\cdot5^3+4\cdot5^5+O(5^6)$	$-4P$
					3	2	$2\cdot5+1\cdot5^2+4\cdot5^3+4\cdot5^5+O(5^6)$	$-4P$
					6	2	$3\cdot5+3\cdot5^2+4\cdot5^4+O(5^6)$	$4P$
61a1	$(1,-1)$	61(2)	5	9	1	2	$2\cdot5^2+3\cdot5^4+O(5^6)$	$-2P$
					2	2	$5^2+4\cdot5^3+3\cdot5^4+3\cdot5^5O(5^6)$	$4P$
					3	1	$2\cdot5^2+3\cdot5^4+O(5^6)$	$-4P$
65a1	$(-1,1)$	65(2)	7	12	1	2	$3\cdot7^2+6\cdot7^5+O(7^6)$	P
					2	2	$3\cdot7+4\cdot7^2+6\cdot7^3+O(7^6)$	$3P$
		65(3)	7	12	1	2	$3\cdot7^2+6\cdot7^5+O(7^6)$	P
					2	2	$3\cdot7^2+6\cdot7^5+O(7^6)$	P
77a1	$(2,3)$	77(2)	5	7	1	6	$3\cdot5+3\cdot5^2+3\cdot5^5+O(5^6)$	$-\frac{4}{3}P$
		77(3)	5	7	1	20	$2\cdot5+3\cdot5^2+4\cdot5^4+5^5+O(5^6)$	$\frac{12}{5}P$
		77(4)	13	18	1	10	$2\cdot13+11\cdot13^2+7\cdot13^3+5\cdot13^4+O(13^5)$	$-\frac{12}{5}P$
					2	2	$5\cdot13+12\cdot13^2+6\cdot13^3+10\cdot13^4+O(13^5)$	$-4P$
		11(1)	5	7	1	6	$2\cdot5+5^2+4\cdot5^3+4\cdot5^4+5^5+O(5^6)$	$\frac{4}{3}P$
					7	6	$2\cdot5+2\cdot5^3+3\cdot5^4+5^5+O(5^5)$	$\frac{44}{3}P$

E_f	P	g	p	A_p	n	$d_{g,n}$	$\log_{E_f}(A_p d_{g,n} P_{g,f,n})$	$P_{g,f,n}$
79a1	$(0,0)$	79(2)	5	9	1	2	$5+4\cdot5^3+3\cdot5^4+2\cdot5^5+O(5^6)$	$-4P$
					2	2	$5+4\cdot5^3+3\cdot5^4+2\cdot5^5+O(5^6)$	$-4P$
					3	2	$5+4\cdot5^3+3\cdot5^4+2\cdot5^5+O(5^6)$	$-4P$
					4	2	$O(5^6)$	0
					5	2	$O(5^6)$	0
82a1	$(-1,1)$	82(2)	5	8	1	4	$O(5^6)$	0
					3	2	$2\cdot5+3\cdot5^2+3\cdot5^5+O(5^6)$	$-2P$
		41(1)	5	8	1	2	$2\cdot5+3\cdot5^2+3\cdot5^5+O(5^6)$	$-2P$
					2	2	$O(5^6)$	0
					3	2	$2\cdot5+3\cdot5^2+3\cdot5^5+O(5^6)$	$-2P$
					5	2	$4\cdot5+5^2+5^3+5^5+O(5^6)$	$-4P$
					6	2	$2\cdot5+3\cdot5^2+3\cdot5^5+O(5^6)$	$-2P$
					10	2	$5+3\cdot5^2+3\cdot5^3+4\cdot5^4+3\cdot5^5+O(5^6)$	$4P$
83a1	$(0,0)$	83(2)	7	11	1	2	$O(7^6)$	0
					2	2	$4\cdot7^2+5\cdot7^3+6\cdot7^4+3\cdot7^5+O(7^6)$	$2P$
					3	2	$7^2+4\cdot7^3+6\cdot7^4+O(7^6)$	$4P$
					4	2	$6\cdot7^2+2\cdot7^3+6\cdot7^5+O(7^6)$	$-4P$
					5	1	$3\cdot7^2+7^3+3\cdot7^5+O(7^6)$	$-4P$
					7	2	$O(7^6)$	0
88a1	$(2,-2)$	88(2)	5	9	1	16	$O(5^6)$	0
					2	16	$O(5^6)$	0
		44(1)	5	9	1	8	$O(5^6)$	0
					2	2	$2\cdot5+5^2+2\cdot5^3+2\cdot5^4+2\cdot5^5+O(5^6)$	$8P$
		11(1)	5	9	1	2	$O(5^6)$	0
					2	2	$2\cdot5+5^2+2\cdot5^3+2\cdot5^4+2\cdot5^5+O(5^6)$	$8P$
					4	1	$2\cdot5+5^2+2\cdot5^3+2\cdot5^4+2\cdot5^5+O(5^6)$	$16P$
					8	1	$2\cdot5+5^2+2\cdot5^3+2\cdot5^4+2\cdot5^5+O(5^6)$	$16P$

E_f	P	g	p	A_p	n	$d_{g,n}$	$\log_{E_f}(A_p d_{g,n} P_{g,f,n})$	$P_{g,f,n}$
89a1	$(0,0)$	89(2)	5	7	1	5	$5+5^2+5^5+O(5^6)$	$-\frac{8}{5}P$
		89(3)	11	14	1	10	$4\cdot 11^2+9\cdot 11^3+4\cdot 11^4+3\cdot 11^5+O(11^6)$	$-\frac{2}{5}P$
					2	10	$4\cdot 11^3+9\cdot 11^4+4\cdot 11^5+O(11^6)$	$-\frac{22}{5}P$
					3	10	$11^2+2\cdot 11^3+5\cdot 11^4+5\cdot 11^5+O(11^6)$	$\frac{16}{5}P$
					4	10	$7\cdot 11^2+11^3+6\cdot 11^4+7\cdot 11^5+O(11^6)$	$\frac{2}{5}P$
					6	10	$4\cdot 11^2+2\cdot 11^3+3\cdot 11^4+8\cdot 11^5+O(11^6)$	$-\frac{24}{5}P$
91a1	$(0,0)$	91(2)	5	9	1	4	$3\cdot 5+3\cdot 5^2+5^3+4\cdot 5^4+O(5^6)$	$2P$
		91(3)	5	9	1	4	$3\cdot 5+3\cdot 5^2+5^3+4\cdot 5^4+O(5^6)$	$2P$
					2	2	$5+3\cdot 5^2+5^3+2\cdot 5^5+O(5^6)$	$-2P$
		91(4)	5	9	1	4	$5+2\cdot 5^2+3\cdot 5^3+3\cdot 5^4+5^5+O(5^6)$	$4P$
					2	2	$4\cdot 5+5^2+3\cdot 5^3+4\cdot 5^4+2\cdot 5^5+O(5^6)$	$2P$
					3	2	$4\cdot 5+3\cdot 5^4+2\cdot 5^5+O(5^6)$	$6P$
91b1	$(-1,3)$	91(1)	5	9	1	4	$O(5^6)$	0
		91(3)	5	9	1	4	$O(5^6)$	0
					2	2	$O(5^6)$	0
		91(4)	5	9	1	4	$O(5^6)$	0
					2	2	$O(5^6)$	0
					3	2	$O(5^6)$	0
92b1	$(1,-1)$	92(1)	7	12	1	2	$O(7^6)$	0
		46(1)	7	12	1	2	$O(7^6)$	0
					2	2	$O(7^6)$	0
		23(1)	7	12	1	20	$O(7^6)$	0
					2	5	$O(7^6)$	0
					3	4	$O(7^6)$	0
					4	5	$O(7^6)$	0
					6	1	$O(7^6)$	0
					12	1	$O(7^6)$	0

E_f	P	g	p	A_p	n	$d_{g,n}$	$\log_{E_f}(A_p d_{g,n} P_{g,f,n})$	$P_{g,f,n}$
99a1	$(0,0)$	99(2)	5	10	1	12	$O(5^6)$	0
		99(3)	5	10	1	12	$5+5^2+4\cdot5^4+3\cdot5^5+O(5^6)$	$\frac{2}{3}P$
		99(4)	5	10	1	6	$2\cdot5+4\cdot5^2+4\cdot5^3+2\cdot5^4+O(5^6)$	$-\frac{2}{3}P$
		33(1)	5	10	1	12	$4\cdot5+3\cdot5^2+4\cdot5^3+5^5+O(5^6)$	$-\frac{2}{3}P$
					3	3	$4\cdot5+3\cdot5^2+4\cdot5^3+5^5+O(5^6)$	$-\frac{8}{3}P$
		11(1)	5	10	1	6	$3\cdot5+2\cdot5^4+4\cdot5^5+O(5^6)$	$\frac{2}{3}P$
					3	3	$4\cdot5+2\cdot5^2+2\cdot5^3+3\cdot5^4+4\cdot5^5+O(5^6)$	$\frac{2}{3}P$
					9	3	$4\cdot5+3\cdot5^3+3\cdot5^4+5^5+O(5^6)$	$\frac{22}{3}P$