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Foliations, Contact Structures and Finite Group Actions

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Christopher Anthony Carlson

June 2012

Dissertation Committee:

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The Dissertation of Christopher Anthony Carlson is approved:

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ABSTRACT OF THE DISSERTATION

Foliations, Contact Structures and Finite Group Actions

by

Christopher Anthony Carlson

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2012
Dr. Reinhard Schultz , Chairperson

I have considered two main questions in my research. First, which foliations on a manifold are compatible with a particular symmetry group. Second, which contact structures are preserved by which symmetry groups. The existence of both foliations and contact structures have been studied for over a half-century with no consideration for symmetry groups. I have looked thus far at finite symmetry groups. Historically, the study of manifolds and symmetry groups goes back to Riemann's formulation of the manifold concept in his 1854 lecture on the foundations of geometry.

An equivalence class of smooth 1-forms $[\lambda]$ on a manifold M consists of all $h\lambda$ where h is any smooth function $M \rightarrow \mathbb{R} - 0$. Given a smooth finite group action G on M , preserving this equivalence class means that there is a homomorphism $\varepsilon : G \rightarrow \{-1, 1\}$ and a $\lambda' \in [\lambda]$ such that for any $g \in G$, $g^*\lambda' = \varepsilon(g)\lambda'$. In other words, there is a representative form λ' such that $g^*\lambda' = \pm\lambda'$.

There are mostly negative results for orientation reversing group actions. Let M be a smooth oriented $(4n - 1)$ -manifold. If \mathbb{Z}_{2k} , $k > 1$ acts smoothly but not orientation preservingly on M then there is no contact form on M which is compatible with this group action. For foliations, there are no compatible codimension 1 foliations on

S^3 with a group action containing an orientation reversing subgroup that is isomorphic to \mathbb{Z}_{2k} , for $k \geq 2$.

An equivalent condition for a smooth n -manifold M to have a G -invariant codimension 1 foliation is that each connected component of each fixed point set has Euler characteristic zero, where G is an odd order group that acts smoothly on M , with isotropy groups linearly ordered by inclusion.

The last result is constructive. Let M be a closed smooth oriented 3-manifold with a smooth orientation-preserving G -action, where G is a group of prime order p . Then there is a G -invariant contact form θ on M . This form is constructed from an open-book decomposition, and a branched covering.

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Chapter 1

Introduction

Manifolds are vital in many areas of mathematics and physics, especially general relativity. Manifolds are locally Euclidean, but to understand the manifold as a whole, we have to examine the global properties. There are many ways to look more closely at a manifold other than simply zooming in. Embedded submanifolds and immersions can determine a lot about the larger manifold. One such way of breaking a manifold up is a foliation.

1.1 Foliations

Definition 1 A *foliation* \mathcal{F} of an m -dimensional manifold M is a collection of lower n -dimensional disjoint connected submanifolds, $\{L_i\}_{i \in S}$ called *leaves* such that their union is all of M . Locally, they appear as flat n -planes, $\mathbb{R}^n \times \{x\}$, where x is a point with $m - n$ coordinates. The difference $m - n$ is called the **codimension** of the foliation. For any $b \in M$, there is a neighborhood U and local coordinates $x = (x^1, \dots, x^m)$ such that for any $U \cap \{L\}_i$, this set can be described by: $x^{n+1} = c_{n+1}, \dots, x^m = c_m$. These are called local coordinates distinguished by \mathcal{F} . A maximal covering of M in local coordinates

distinguished by \mathcal{F} is called a foliated atlas.

A 3-dimensional manifold with a codimension 1 foliation locally looks like a stack of papers. The union of tangent bundles of the n -dimensional leaves of the foliation $T(\mathcal{F})$ is an n -dimensional subbundle of the tangent space $T(M)$, called an **n -plane distribution**. Locally, a foliation is defined by a family of $(m - n)$ differential 1-forms. Note that these 1-forms have to be nowhere zero, and at each point in M , linearly independent. A codimension 1 foliation of a simply connected manifold can be defined by a single suitable 1-form λ . This λ does so by annihilating the tangent subbundle, $\ker(\lambda) = T(\mathcal{F})$. There is an equivalence class of forms defining a single foliation. Given a form λ that annihilates exactly the tangent subbundle of the foliation, any nowhere zero $f \in C^\infty(M)$ can act on this form by multiplication, and the kernel is preserved $\ker(f\lambda) = \ker(\lambda)$. For simplicity, a representative will be chosen, and referred to as **the foliation form**. A codimension 1 foliation is considered trivial at the boundary of M if the foliation on the boundary extends to a collared neighborhood of the boundary. The new leaves will be $\partial M \times \{t\}$ where $t \in [0, \epsilon]$ is the collar parameter for a suitably small positive ϵ . Which 1-forms on smooth 3-manifolds define a codimension 1 foliation?

Theorem 2 *On a smooth 3-manifold, a nowhere zero one form λ defines a codimension 1 foliation if and only if $\lambda \wedge d\lambda = 0$*

The proof requires the Frobenius [Fro77] theorem, which gives equivalent conditions for a k -plane bundle to define a foliation.

Theorem 3 *(Also the work of A. Clebsch and F. Deahna) Equivalent conditions for a k -plane distribution $E \subset T(M)$ on a smooth manifold M to define a k -dimensional foliation:*

- E is completely integrable, i.e.: $\forall x \in M, \exists L_\alpha$ such that $i_*(T_x(\mathcal{L}_\alpha)) = E_x$
- $I^*(E)$ is a differentiable graded ideal
- $d(I^1(E)) \subset I^2(E)$
- E is involutive (it's a Lie subalgebra)
- There is a C^∞ foliated atlas on M of codimension $q = m - k$, where every plaque is an integral manifold to E

Now, to prove theorem 2, using the Frobenius theorem.

Proof. Define

$$E = \ker(\omega) = \{X \in \Gamma(TM) : \omega(X) = 0\}$$

$$I(E) = \{\omega \in \Lambda^p(T^*M) : \omega(\zeta) = 0, \forall \zeta \in \Lambda_p(E)\}$$

Note: $I(E)$ is a 2 sided graded ideal.

(\implies) Claim: $I(E)$ is closed under exterior derivative. E defines a foliation, so the Frobenius theorem implies $I(E)$ is differentiable. Thus $d\omega = \omega \wedge \theta$. Now, $\omega \wedge d\omega = \omega \wedge (\omega \wedge \theta) = 0$

(\impliedby) Locally, extend ω to a basis of 1-forms: $\{\omega, \alpha, \beta\}$. Then $d\omega = P\alpha \wedge \omega + Q\beta \wedge \omega + R\alpha \wedge \beta$, for functions $P, Q, R \in C^\infty(M)$. Because

$$\begin{aligned} \omega \wedge d\omega &= \omega \wedge (P\alpha \wedge \omega + Q\beta \wedge \omega + R\alpha \wedge \beta) \\ &= 0 + 0 + R\omega \wedge \alpha \wedge \beta \\ &= 0 \end{aligned}$$

R must be zero everywhere. Let $\theta = P\alpha + Q\beta$, then $d\omega = \omega \wedge \theta$, and $d(I^1(E)) \subset I^2(E)$.

■

Definition 4 Let M be a compact manifold. An **open book decomposition** on M is a smooth function $F : M \rightarrow \mathbb{C}$ such that:

1. $0 \in \mathbb{C}$ is a regular value. Let $B = F^{-1}(0)$. B is an oriented link, and has a neighborhood U such that $F|_U$ is a submersion. B is called the **binding**.
2. The function $f = \frac{F}{|F|} : M - B \rightarrow S^1$ is a smooth submersion.

There is a neighborhood U that is diffeomorphic to $B \times D^2$. The fiber $B_\gamma = f^{-1}(e^{i\gamma})$ is called a **page**. The closure of a page $\overline{B_\gamma}$ is a compact manifold with $\partial\overline{B_\gamma} = B$.

Theorem 5 [Woo70] *Codimension 1 foliations exist on all closed oriented 3-manifolds.*

This foliation is created from the open book decomposition that [Ale23] proved exists on any closed oriented 3-manifold. Around each component in the binding C , there is a neighborhood $U \cong B \times D^2$ that is a submersion into \mathbb{C} . Take a codimension 1 foliation on U to be the Reeb foliation. Explicitly, consider the map

$$\begin{aligned} h : U &\rightarrow \mathbb{R} \\ (r, \theta, \phi) &\mapsto (1 - r^2)e^{i\phi} \end{aligned}$$

where ϕ is the coordinate along the binding B , and (r, θ) are coordinates for D^2 . The fibers of this map are the leaves of the Reeb foliation on U . The fibers of strictly positive real numbers are the usual leaves that are sometimes called endless snakes eating themselves. The fiber $h^{-1}(0)$ is the leaf that is diffeomorphic to T^2 , and is trivial at the boundary, $h^{-1}(0) = \partial U$. Some leaves are depicted in Figure 1.1, including the leaf $h^{-1}(0) \cong T^2$.

The foliation on the outside of U is constructed differently. The leaves will be the pages outside a neighborhood $N \supset U$. These pages transversely intersect U , so

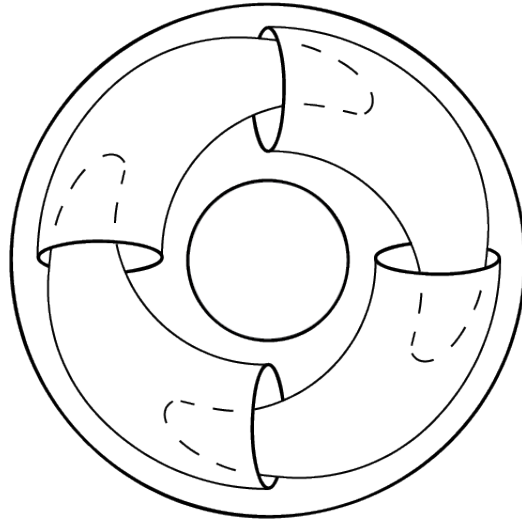


Figure 1.1: Reeb foliation of the torus

to accommodate these leaves, inside N , they are spiraled around ∂U . This is a view where the induced Reeb foliation on D^2 appears as concentric circles, the outermost circle being the leaf that is $\partial U = T^2$ intersected with D^2 .

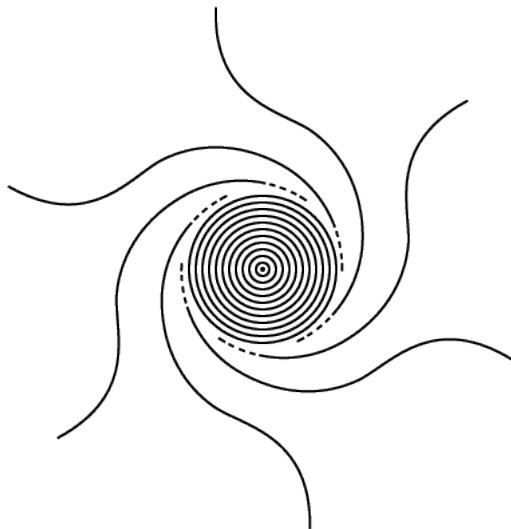


Figure 1.2: A cross-section of the Reeb foliation meeting the page foliation

1.2 Contact Structures

We saw that a foliation 1-form λ had the property that $\lambda \wedge d\lambda = 0$ everywhere on M . The opposite of a foliation form is a **contact structure**. A contact structure corresponds to a 1-form λ on a $(2k + 1)$ -manifold with $\lambda \wedge (d\lambda)^k \neq 0$. Such structures arose in the work of C. Huygens and S. Lie. One particular motivation for studying symmetric contact structures is their relationship to symplectic structures. The latter arise in the Lagrange-Hamiltonian approach to classical mechanics, and during the past two decades symplectic structures have been studied, particularly for 4-manifolds. It is useful to analyze how such structures can be built out of pieces which are manifolds with boundary [Gom95]. These boundaries turn out to have contact structures, and thus compatible symmetric groups of contact structures arise immediately in any study of symmetric symplectic manifolds that can be built out of pieces. Furthermore, there has also been a surge of recent activity on contact manifolds, particularly in dimension 3, and it is also natural to analyze how this progress applies to manifolds with symmetry groups.

A smooth contact form θ annihilates an n -plane bundle, just as a foliation form does. The action of a nowhere-zero function $f : M \rightarrow \mathbb{R} - \{0\}$ by multiplication will preserve the n -plane bundle annihilated by θ because $\ker(\theta) = \ker(f\theta)$. Thus an equivalence class of contact forms will be considered, and a representative will be chosen as needed.

On smooth 3-manifolds, in addition to the difference between a contact structure θ and a foliation form λ mentioned above, $\lambda \wedge d\lambda = 0 \neq \theta \wedge d\theta$, there is another big difference between these two forms. The kernel of a foliation form $\ker(\lambda) = T(\mathcal{F}) \subset T(M)$ is a 2-plane bundle that consists of the union of all tangents to the leaves of a foliation.

Consider a point $x \in L_i \subset M$. Taking any smooth path in $\gamma : (0, 1) \rightarrow M$ passing through x with the property that the tangents along γ are contained in $T(\mathcal{F})$ means that the path must remain on the same leaf L_i . This condition on the foliation form λ is that $\lambda(\gamma'(t)) = 0$ for all $t \in (0, 1)$. This effectively partitions M into disjoint leaves when considering these smooth paths.

A contact structure θ has a kernel $E = \ker(\theta) \subset T(M)$ that is also a 2-plane subbundle of the tangent space of M . Unlike how a foliation form keeps smooth paths on a single leaf, partitioning a manifold into leaves, the kernel of a contact form allows piecewise smooth paths to go from any point to any other point in a 3-manifold, while keeping the tangents of these paths contained in $\ker(\theta)$. When a smooth curve is tangent to the subbundle E in the kernel of a contact form on a 3-manifold, it is called *Legendrian*. The standard contact form on \mathbb{R}^3 is $\theta = -ydx + dz$. To show there is a path from any point to any other point, we will construct a path from the origin to any point, then compose it with the reverse of another path from the origin to the other point. First, begin the path at $(0, 0, 0)$. Notice that we can always travel in the y -direction, at any point. To change the z -coordinate, move in the y -direction $\pm(0, 1, 0)$ to $(0, 1, 0)$ and then travel in the direction $\pm(1, 0, 1)$ until the desired z -coordinate is reached. Travel back to the $y = 0$ -plane in the direction $\pm(0, -1, 0)$. Then travel to the proper x -coordinate along $\pm(1, 0, 0)$. Then finally to the proper y -coordinate along $\pm(0, 1, 0)$. In general, there is a Legendrian path between any 2 points in a connected manifold [EF08], and references cited there.

Proposition 6 *Given a smooth function $h : M \rightarrow \mathbb{R} - \{0\}$ acting on forms by multiplication. If λ is a foliation form a smooth 3-manifold, then $h\lambda$ is also a foliation form on M . If θ is a contact form on a smooth $(2n + 1)$ -manifold M then $h\theta$ is also a contact*

form on M .

Proof. Since $\lambda \neq 0$ and $\theta \neq 0$ on all of M , $h\lambda \neq 0$ and $h\theta \neq 0$ on M .

Computing the foliation form first,

$$\begin{aligned}
h\lambda \wedge d(h\lambda) &= h\lambda \wedge [(dh \wedge \lambda) + (hd\lambda)] \\
&= h\lambda \wedge dh \wedge \lambda + h^2\lambda \wedge d\lambda \\
&= 0 + 0
\end{aligned}$$

we see that $h\lambda \wedge d(h\lambda) = 0$, so $h\lambda$ is a foliation form on a 3-manifold.

As a contact form, θ has the property that $\theta \wedge (d\theta)^n \neq 0$. Computing this for $h\theta$ yields

$$\begin{aligned}
h\theta \wedge [d(h\theta)]^n &= h\theta \wedge d(h\theta) \wedge [d(h\theta)]^{n-1} \\
&= h\theta \wedge (dh \wedge \theta + hd\theta) \wedge [d(h\theta)]^{n-1} \\
&= (h\theta \wedge dh \wedge \theta + h\theta \wedge hd\theta) \wedge [d(h\theta)]^{n-1} \\
&= (0 + h^2\theta \wedge d\theta) \wedge [d(h\theta)]^{n-1} \\
&\quad \vdots \\
&= h^{n+1}\theta \wedge [d\theta]^n \\
&\neq 0
\end{aligned}$$

showing that $h\theta$ is a contact form on M . ■

Frequently, given a metric on the cotangent bundle, the form that will be chosen to represent a foliation form or a contact form will be the one with unit length. This metric is needed, especially in the following section. These results are used through the remainder of the paper, so all manifolds are assumed to be Riemannian.

1.3 Group Actions and Compatible Forms

A smooth group action G on M is said to be compatible with a foliation if it simply permutes the leaves, $g(L_i) = L_j$ for any $g \in G$. This is to say that each leaf maps exactly to itself, or another leaf. Note that points within the leaf may not be fixed, yet the leaf may map to itself. When the foliation is defined by a 1-form, this compatibility with a group action can be examined using just this form. This also works for contact 1-forms. Preserving a smooth 1-form λ amounts to preserving the hyperplane field annihilated by λ . This preservation of the form means that at $x \in M$, the tangent space map $T(g)$ maps the hyperplane E_x at $x \in M$ to the hyperplane, $E_{g(x)}$ at $g(x)$. This condition means that for each $g \in G$, there is a nowhere zero function $h : M \rightarrow \mathbb{R}$ such that $g^*\lambda = h\lambda$, as this preserves the hyperplane field annihilated by λ . When M is connected, h is either always positive or negative. This compatibility condition can also be examined using equivalence classes. A form λ is preserved by a smooth group action G if any $g \in G$ maps λ into its equivalence class. In other words, $G\lambda \subset [\lambda]$.

Proposition 7 *Suppose a finite group G acts smoothly on a smooth connected Riemannian manifold M , and H is a smooth hyperplane field on M which is G -invariant. Let $[\lambda']$ be the equivalence class of nowhere zero forms that annihilate H . Then there is a $\lambda \in [\lambda']$ such that $g^*\lambda = \pm\lambda$, for any $g \in G$. Also there is a homomorphism $\varepsilon : G \rightarrow \{-1, 1\}$ such that $g^*\lambda = \varepsilon(g) \cdot \lambda$.*

Proof. Averaging the Riemannian metric L on the cotangent bundle, define a new metric

$$|\lambda| = \frac{1}{|G|} \sum_{g \in G} L(\lambda, \lambda).$$

This metric is invariant under the action of G on T^*M . There is a form $\lambda \in [\lambda]$ such that $|\lambda| = 1$. One such form is $\frac{\lambda'}{|\lambda'|}$. By construction $|g^*\lambda| = |\lambda| = 1$ for all $g \in G$.

Since G preserves the hyperplane field, we have $G^*\lambda = h_g\lambda$ for some nowhere zero function $h_g : M \rightarrow \mathbb{R}$. Combining this with the preceding calculations,

$$|h_g| = |h_g||\lambda| = |h_g\lambda| = |g^*\lambda| = 1$$

so that $h_g(x) = \pm 1$ for all $x \in M$. Since M is connected, h_g must be constant, call this constant $\varepsilon(g)$.

There is such a constant function ε for each $g \in G$. Let $\varepsilon : G \rightarrow \{-1, 1\}$ be the function associated to g . Claim: this assignment is a homomorphism. Take any $g_1, g_2 \in G$,

$$\begin{aligned} \varepsilon(g_1g_2)\lambda &= (g_1g_2)^*\lambda \\ &= g_2^*(g_1^*\lambda) \\ &= g_2^*(\varepsilon(g_1) \cdot \lambda) \\ &= \varepsilon(g_1)(g_2^*\lambda) \\ &= \varepsilon(g_1)\varepsilon(g_2)\lambda \end{aligned}$$

then $\varepsilon(g_1g_2) = \varepsilon(g_1)\varepsilon(g_2)$, and this assignment is a homomorphism. ■

Contact structures will be defined to be compatible with a group action in a similar fashion to foliation forms.

Definition 8 *An equivalence class $[\theta']$ of smooth contact structures on a smooth manifold M is compatible with a smooth group action G on M if there exists a homomorphism $\varepsilon : G \rightarrow \{-1, 1\}$ such that for some $\theta \in [\theta']$ and any $g \in G$, $g^*\theta = \varepsilon(g)\theta$. An equivalent condition is that $G[\theta'] \subset [\theta']$, where the G -action is a pullback.*

Now we examine a specific foliation preserved by a finite linear group action.

Example 9 A finite group G acts linearly on \mathbb{R}^4 . Let $x, y \in \mathbb{R}^2 \times \mathbb{R}^2$. Define a new inner product:

$$\langle x, y \rangle = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle_{\mathbb{R}^4}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ is the standard inner product on \mathbb{R}^4 . Now, G acts orthonormally. Specifically, G acts as diffeomorphisms on $S^3 \subset \mathbb{R}^4$. Denote $S(X)$ be the unit ball in X .

G acts on $\mathbb{R}^4 = \mathbb{R} \oplus L \oplus W$, in a decomposable way. Let $(x, y) \in S(\mathbb{R}^2 \oplus \mathbb{R}^2)$ such that $|x|^2 + |y|^2 = 1$

$$\begin{aligned} S(\mathbb{R} \oplus L \oplus W) &= \\ S(\mathbb{R} \oplus L) \times D(W) \cup_{S(\mathbb{R} \oplus L) \times S(W)} D(\mathbb{R} \oplus L) \times S(W) &= \\ \{(x, y) : |y| \leq |x| \leq 1\} \cup_{\{(x, y) : |x|=|y|=\frac{1}{\sqrt{2}}\}} \{(x, y) : |x| \leq |y| \leq 1\} &= \end{aligned}$$

The open book decomposition is defined in the following way:

The binding, $B = \{(x, y) : |y| = 0\} = S(\mathbb{R} \oplus L) \times \{0\} = S(\mathbb{R} \oplus L)$ and map,

$$\begin{aligned} \pi : S(\mathbb{R} \oplus L \oplus W) - B &\rightarrow D(\mathbb{R} \oplus L) \times \mathbb{C} \\ (x, y) &\mapsto (x, y) \end{aligned}$$

Where the open book map F is just $(x, y) \mapsto \frac{y}{|y|}$. Now, G preserves the binding, and the boundary torus $S(\mathbb{R} \oplus L) \times S(W)$, since they are defined using only the norms of x and y .

If G is any subgroup of the torus group T^2 acting as described above, then the Reeb foliation \mathcal{F} , of S^3 is compatible with the standard linear action of $G \subset T^2 \subset U_2$ on S^3 . This action preserves the leaf $S(\mathbb{R} \oplus L) \times S(W)$, as it is defined using only the norm. The snake-like Reeb leaves on $S(\mathbb{R} \oplus L) \times D(W)$ are either spun, which preserves them, or permuted. They are permuted when a group element acts on the first coordinate, and

spun when a group element acts on the second coordinate. The group elements acting on both coordinates will both spin and permute these leaves. The 2-sheets that foliate $D(\mathbb{R} \oplus L) \times S(W)$ are spiraled around $S(\mathbb{R} \oplus L) \times S(W)$. If a group element acts on the first coordinate, these sheets are permuted. If a group element acts on the second coordinate, then these sheets are translated to themselves, preserving each of these leaves. And again for group elements that change both coordinates, these leaves are permuted and translated.

Free actions and compatible foliations on smooth closed manifolds are easy to pair up. For a free G -action, any foliation \mathcal{F} on M/G can be lifted back to a foliation on M , with the same codimension. This is a consequence of the inverse function theorem. In the 3-dimensional case, the same is true for contact structures.

Theorem 10 *Let G be a discrete group acting smoothly and freely on a smooth 3-manifold M such that M/G is also a smooth manifold, and $q : M \rightarrow M/G$ is a submersion. A G -compatible codimension 1 foliation exists on M if a codimension 1 foliation exists on M/G . A G -compatible contact form exists on M if a contact form exists on M/G .*

Proof. Let λ be a foliation form on M/G , and let θ be a contact form on M/G . Also let $q : M \rightarrow M/G$ be the quotient map. If a 1-form is nowhere zero on M/G then it is also nowhere zero on M , because the pullback q^* is injective. The pullbacks $q^*\lambda$ and $q^*\theta$ are G -invariant since $g^*q^* = (qg)^* = q^*$. Now we check if the pullbacks are a foliation form and contact form,

$$\begin{aligned} \lambda \wedge d\lambda &= 0 \neq \theta \wedge d\theta \\ q^*(\lambda \wedge d\lambda) &= 0 \neq q^*(\theta \wedge d\theta) \\ q^*(\lambda) \wedge d(q^*(\lambda)) &= 0 \neq q^*(\theta) \wedge d(q^*(\theta)). \end{aligned}$$

Thus, $q^*(\lambda)$ is a G -invariant foliation form on M if λ is a foliation form on M/G . Likewise, $q^*(\theta)$ is a G -invariant contact form on M if θ is a contact form on M/G . ■

Contact structures exist on all closed orientable 3-manifolds. Thurston and Winkelnkemper [Thu75] have a constructive proof where they create the form from an open book decomposition. We will create a contact structure in a much more careful fashion so that it is compatible with the group action. In some cases, smooth group actions have compatible contact structures and in other cases there are no compatible contact structures.

1.4 Local Results on Tangency and Transversality

The finite group G acts smoothly on a smooth 3-manifold M . We saw earlier that if this G -action preserves a codimension 1 foliation, then there is a foliation 1-form λ such that any $g \in G$, $g^*\lambda = \pm\lambda$. We will now consider the implications of this action near a fixed point set. Since we are looking at the action locally, assume that M is connected. The fixed sets considered are of strictly positive dimension, so they have a non-trivial tangent space. Singular fixed point sets are not considered in this paper. Let $N \subset M$ be the fixed point set of G , and let $x \in F$ be any point fixed by G . These hypothesis and variables are used in the next two theorems.

Theorem 11 *If $g^*(\lambda) = \lambda$ for all $g \in G$, then the fixed point set must be transverse to the leaves.*

Proof. Since transversality is a local property, it is enough to examine a fixed point, and restrict to an invariant open neighborhood on which the action is orthogonal. We will also use the isomorphism between forms and vector fields associated to a specified

invariant Riemannian metric. Without loss of generality, the fixed point set is the 0 in the orthogonal representation V .

Consider the image of the vector field $X_\omega \leftrightarrow \omega$ in the tangent space $T(V) \cong V \times V$. The condition $g^*\omega = \omega$ translates into saying that $X(v) = (v, \varphi(v))$ where $\varphi : V \rightarrow V^G$ and $v \in V^G$.

This means that $\varphi(v)$, which is normal to the leaf \mathcal{L}_v through v , is also tangent to the fixed set $V^G \subseteq V$, so that \mathcal{L}_v and V^G meet transversely. ■

Theorem 12 *If $g^*(\lambda) = -\lambda$ for some $g \in G$, then the fixed point set must be tangent to the leaves. This is to say that each connected component will be contained in some leaf.*

Proof. Suppose that there is a connected component C of the fixed point set not contained in a leaf. C is fixed by G thus so is its tangent space $T(C)$. C must intersect some leaf \mathcal{L} . Take $x \in C \cap \mathcal{L}$. Because C is not contained in \mathcal{L} , there must be some $Y \in T_x(C)$ such that $\lambda(Y)_x \neq 0$. Now the pullback is computed using a $g \in G$ such that $g^*(\lambda) = -\lambda$.

$$\begin{aligned} g^*\lambda(Y)_x &= \lambda(g_*Y)_{g(x)} \\ -\lambda(Y)_{g(x)} &= \lambda(Y)_{g(x)} \\ -\lambda(Y)_x &= \lambda(Y)_x \end{aligned}$$

This is a contradiction since these quantities are both non-zero. Therefore, there cannot be fixed connected components that are not contained in a leaf. This means that all connected fixed components are contained in leaves, making the fixed point set tangent to the leaves. ■

Example 13 *Suppose we are given a faithful representation of a finite group, G acting*

on a real 3-dimensional vector space, V . Then it is easy to check that codimension 1 foliations and contact structures exist if and only if V can be written as a direct sum of a 2-dimensional and 1-dimensional representation $W \oplus L$. The leaves are $W \times \{l\}$ for $l \in L$. On the other hand, if G is the symmetry group of a regular polyhedron, such as a cube, tetrahedron, or dodecahedron, then no such structures exist because every faithful representation is irreducible. On the other hand, if G is abelian or dihedral, then they always exist. This is because the action can be split into a 2-dimensional and a 1-dimensional representation.

1.5 General background material

On a Riemannian manifold, M , the metric tensor, $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ induces an isomorphism between 1-forms and vector fields. A nowhere zero vector field corresponds to a nowhere zero 1-form. The existence of one implies the existence of the other, which will frequently be used.

Theorem 14 (*Poincaré-Hopf*) [Mil65] *Let M be a closed connected oriented n -manifold. Let ξ be an oriented n -plane bundle. $\chi(\xi) = 0 \iff \exists$ non-vanishing section of ξ .*

When $\xi = TM$, since ξ is a fiber bundle with contractable fiber, $F = \mathbb{R}^n \cong \text{point}$ thus, $\chi(\mathbb{R}^n) = 1$. Now, $\chi(TM) = \chi(M)\chi(F) = \chi(M)$. So, $\chi(M) = 0 \iff \exists$ non-zero vector field, or form.

The relative version of this theorem will also be used:

Theorem 15 *Relative Poincaré-Hopf* [Jub09]: *$(M, \partial M)$ is compact, and X is a vector field on M such that $X|_{\partial M}$ is inward pointing with respect to some collar, then $\text{index}(X) = \pm\chi(M, \partial M)$*

By rotating this inward pointing vector field near the boundary such that $X|_{\partial M}$ is tangent to ∂M , we have a similar version of this theorem, which will be used extensively.

Given a non-vanishing vector field defined on ∂M , this can be extended to a non-vanishing vector field on M exactly when $\chi(M, \partial M) = 0$. Now, the existence of a non-vanishing vector field on ∂M implies that $\chi(\partial M) = 0$, thus to extend X to all of M , all that is needed is $\chi(M) = 0$.

Lastly, the Euler characteristic will be computed over a covering space, and the following theorem is needed.

Theorem 16 *If $p : M \rightarrow N$ is an n -sheeted covering, then $\chi(M) = n \cdot \chi(N)$.*

Chapter 2

Orientation Reversing Group

Actions

There are no compatible foliations or contact structures for group actions with an orientation reversing subgroup \mathbb{Z}_{2k} on 3-manifolds for $k \geq 2$. Foliations will be examined first, followed by contact structures. Assume all group actions are effective.

2.1 Contact Structures and Orientation Reversing Group Actions

This theorem will only be used in the 3-dimensional case, but it is presented here in it's generality.

Theorem 17 *Let M be a smooth oriented $(4n-1)$ -manifold. If \mathbb{Z}_{2k} , $k \geq 2$ acts smoothly but not orientation preservingly on M then there is no contact form on M which is compatible with this group action.*

Proof.

If M is not connected, reduce the general case to the connected case by considering each connected component separately. For simplicity, call each connected component M .

For effective \mathbb{Z}_{2k} actions, the free orbits are open and dense by local linearity. The proof considers the case when G acts freely. In the non-free case, let $M_0 \subset M$ be the set of points on free orbits. M_0 is open and dense. If there was a compatible contact form on M , it would restrict to a compatible contact form on M_0 , which will be shown to not exist.

Consider the orbit manifold $M^* = M/\mathbb{Z}_{2k}$. M^* is orientable exactly when there exists a nowhere zero $(4n - 1)$ -form. Suppose that there is such an orientation form Ω , a section of $\Lambda^{4n-1}(M^*)$. Its pullback $p^*\Omega \in \Gamma(\Lambda^{4n-1}(M))$ is also nowhere zero. $p^*\Omega$ is fixed for any $g \in \mathbb{Z}_{2k}$ because $g^*p^*\Omega = (pg)^*\Omega = p^*\Omega$. Now let $g_0 \in \mathbb{Z}_{2k}$ be an orientation reversing generator. The space of sections of $\Lambda^{4n-1}(M)$ is isomorphic to $C^\infty(M)$. Since \mathbb{Z}_{2k} acts orientation reversingly, and the action of g_0 on $\Gamma(\Lambda^{4n-1}(M))$ is simply the multiplication of a real valued function $h \in C^\infty(M)$ that is negative everywhere. Consider g_0^* on $p^*(\Omega)$,

$$g_0^*p^*\Omega = h \cdot p^*\Omega$$

$$p^*\Omega = h \cdot p^*\Omega.$$

Since $h < 0$ everywhere, this is impossible, and thus such an orientation form Ω cannot exist on M^* . Therefore, M^* cannot be oriented.

Suppose that there is a contact structure compatible with the action of G . Then there is a contact form θ on M and a homomorphism $\varepsilon : \mathbb{Z}_{2k} \rightarrow \{-1, 1\}$ such that

$g^*\theta = \varepsilon(g)\theta$ for any $g \in \mathbb{Z}_{2k}$.

$$\begin{aligned}
g^*(\theta \wedge (d\theta)^{2n-1}) &= g^*\theta \wedge (dg^*\theta)^{2n-1} \\
&= \varepsilon(g)\theta \wedge (d\varepsilon(g)\theta)^{2n-1} \\
&= (\varepsilon(g))^{2n}\theta \wedge (d\theta)^{2n-1} \\
&= \theta \wedge (d\theta)^{2n-1}
\end{aligned}$$

This form is contact, so the pullback form $g^*(\theta \wedge (d\theta)^{2n-1}) = \theta \wedge (d\theta)^{2n-1}$ is nonzero, and thus is also a contact form, which is invariant under \mathbb{Z}_{2k} .

Claim 18 *Let $p : N' \rightarrow N$ be a smooth regular covering of smooth n -manifolds with deck transformation group D . The map $p^* : \Lambda^*(N) \rightarrow \Lambda^*(N')$ is injective and its image is all forms fixed under g^* , for any $g \in D$.*

Proof.

The map p is a smooth submersion, so p^* is injective. Take any $g \in D$, then $p \circ g = p$ making $g^*(p^*\omega) = (pg)^*\omega = p^*\omega$. Thus $Im(p^*)$ contains only D -invariant forms. Now we show all D -invariant forms are contained in the image.

Take any $\omega \in \Gamma(T^*(N'))$ such that $g^*\omega = \omega$ for all $g \in D$. We will construct a $\theta \in \Gamma(T^*(N))$ such that $p^*\theta = \omega$.

Case I: $N' \rightarrow N$ is evenly covered. Then $N' \cong N \times D$, and we will use this second formulation of N' . Take any $g \in D$, then

$$\begin{aligned}
g : N \times D &\rightarrow N \times D \\
(x, a) &\mapsto (x, ga)
\end{aligned}$$

Let ω_g be the restriction of ω to $N \times \{g\}$, then $\omega_g = g^*(\omega_e)$, where $e \in D$ is the identity. Therefore, $\omega = p^*(\theta)$ where $\theta \in D^*(N)$, such that $\theta = \omega_e$ via the diffeomorphism $N \cong N \times \{e\}$.

Case II: $N' \rightarrow N$ is not evenly covered. Take an open covering $\{U_\alpha\}$ of N such that U_α is diffeomorphic to an open set in \mathbb{R}^n , and U_α is evenly covered for all α . Let ω_α be the restriction of ω to $p^{-1}[U_\alpha]$.

Case I implies that there exists θ_α on U_α such that $p^*(\theta_\alpha) = \omega_\alpha$, because ω_α is D -invariant. There is a θ_α for each α , but they need to fit together properly, so they fit together to make a 1-form that will pullback to ω . This compatibility condition is that $\theta_\alpha|_{U_\alpha \cap U_\beta} = \theta_\beta|_{U_\beta \cap U_\alpha}$. The pullback function p^* is injective, so if these forms pullback to the same form, then they must be equal.

$$p^*(\theta_\alpha|_{U_\alpha \cap U_\beta}) = \omega_\alpha|_{p^{-1}[U_\alpha \cap U_\beta]} = \omega|_{p^{-1}[U_\alpha \cap U_\beta]} = \omega_\beta|_{p^{-1}[U_\alpha \cap U_\beta]} = p^*(\theta_\beta|_{U_\alpha \cap U_\beta})$$

This is indeed that case, so $\theta_\alpha|_{U_\alpha \cap U_\beta} = \theta_\beta|_{U_\beta \cap U_\alpha}$, and we get a well-defined form θ on N . To ensure $p^*(\theta) = \omega$, it suffices to check locally, on each $p^{-1}[U_\alpha]$. By the construction of θ , this is true. ■

From this claim, $\theta \wedge (d\theta)^{2n-1}$ must be a pullback from $\Lambda^{4n-1}(M^*)$. So it is a nowhere $(4n-1)$ -form that is a pullback from an orientation form. There are no orientation forms on M^* , so there can not be any contact forms on M compatible with the \mathbb{Z}_{2k} action. ■

2.2 Foliations and Orientation Reversing Group Actions

Now foliations and orientation reversing group actions are considered.

Remark 19 *Before considering negative results, there are codimension 1 foliations compatible with a smooth \mathbb{Z}_{2k} action. For example, let L and W be nontrivial representations of \mathbb{Z}_{2k} , with dimensions 1 and $4n-2$ respectively. Then, $L \oplus W$, which is homeomorphic to \mathbb{R}^{4n-1} as a manifold, has the product foliation $\{v\} \times W$, where $v \in L$.*

The incompatibility of codimension 3 foliations on S^3 and orientation reversing \mathbb{Z}_{2k} actions will now be constructed.

Example 20 *The Klein bottle, K can be viewed as $S^1 \times S^1 / \mathbb{Z}_2$, where \mathbb{Z}_2 acts freely by:*

$$F(z, w) = (-z, \bar{w}),$$

where K fibers over $S^1 \cong S^1 / \mathbb{Z}_2$, via:

$$\begin{aligned} K &\rightarrow S^1 / \mathbb{Z}_2 = S^1 \\ [z, w] &\mapsto [z]. \end{aligned}$$

Consider the nowhere zero vector field on $S^1 \times S^1$ defined by $\frac{\partial}{\partial \theta}$, where $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ generate the vector fields on $S^1 \times S^1$ respectively. The integral curves of $\frac{\partial}{\partial \theta}$ are of the form $S^1 \times \{c\}$ where $c \in S^1$, and these define a codimension 1 foliation of $S^1 \times S^1$. Look at the image of this foliation in K . It defines a codimension 1 foliation in K that is transverse to the fibers of the circle bundle $K \rightarrow S^1$. If $c \neq \pm 1$, then $S^1 \times \{c\}$ maps 1-1 into a leaf, L such that $L \rightarrow S^1$ has degree 2, and if $c = \pm 1$ then $S^1 \times \{c\}$ maps 2-1 onto a leaf L such that $L \rightarrow S^1$ has degree 1. Visually, this is two copies of the usual foliation on the Mobius strip, joined on their boundaries:

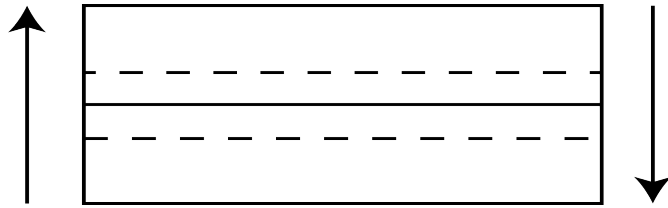


Figure 2.1: A leaf in two mobius strips joined on their boundaries

where the dotted lines are a leaf that wraps around S^1 twice.

Does this foliation extend to the solid Klein bottle? The solid Klein bottle can be viewed as:

$$S^1 \times_{\mathbb{Z}_2} D^2 \cong [S^1 \times D^2 / (z, w) \sim (-z, \bar{w})]$$

Claim 21 *The answer is no.*

Claim 22 *This implies that if \mathbb{Z}_{2k} acts orientation reversingly and linearly on S^3 , then there is no codimension 1 foliation compatible with the action.*

Note that if $k > 1$, then the fixed point set of the group is S^0 , while if $k = 1$ then it is S^2 or S^0 . It is only necessary to consider the S^0 case here because the S^2 case is easily shown to have no compatible foliations. If the fixed point set of \mathbb{Z}_{2k} is S^2 then the foliation \mathcal{F} is either transverse to the fixed point set, or it is tangent. In the transverse case, codimension 1 foliation induces a codimension 1 foliation on S^2 . By the Poincaré-Hopf Theorem, the existence of this foliation implies that $\chi(S^2) = 0$, and this is not true. In the tangent case, S^2 must be contained in a leaf, L . Both S^2 and L are connected 2-manifolds, and S^2 is open in L . Therefore, $S^2 = L$. Let $\mathcal{T} = 2$ -plane bundle of tangents to the leaves of the foliation \mathcal{F} , so \mathcal{T} is a subbundle of $T(S^3)$. Restricting \mathcal{T} to S^2 , $\mathcal{T}|_{S^2} = T(S^2)$. The Euler classes satisfy $\chi(\mathcal{T}) = 0 \in H^2(S^3)$, but $\chi(\mathcal{T}|_{S^2}) \neq 0$. Thus the fixed point set cannot be S^2 .

Theorem 23 *The codimension 1 foliation on the hollow Klein bottle does not extend to the solid Klein bottle. More specifically, the 1-dimensional normal bundle to the leaves of this foliation does not extend to a line bundle on the solid Klein bottle.*

Proof. The normal bundle to the foliation on the Klein bottle can be presented as the image of the subbundle of

$$T(S^1 \times D^2) \cong S^1 \times D^2 \times \mathbb{R}^3 \cong S^1 \times D^2 \times \mathbb{R} \times \mathbb{C}$$

of all elements taking the form: $(z, w; 0, -i\omega t)$ for $t \in \mathbb{R}$. Let ξ be this subbundle. ξ is invariant under:

$$\begin{aligned} F^* : T(S^1 \times D^2) &\rightarrow T(S^1 \times D^2) \\ (z, \omega, s, \zeta) &\mapsto (-z, \bar{\omega}, s, \bar{\zeta}). \end{aligned}$$

Looking at the subbundle: $F^*(z, \omega, 0, -i\omega t) = (-z, \bar{\omega}, 0, \overline{-i\omega t}) = (-z, \bar{\omega}, 0, i\bar{\omega}t)$.

Hence, the normal bundle to the foliation is just $S^1 \times S^1 \times \mathbb{R}$ modulo the identification $(z, \omega, t) \sim (-z, \bar{\omega}, -t)$

We will verify that this line bundle embedding does not extend to a line bundle embedding of the solid Klein bottle: $S^1 \times_{\mathbb{Z}_2} D^2$.

Since $H^1(S^1) \rightarrow H^1(K)$ is a monomorphism with \mathbb{Z}_2 coefficients and the bundle over K extends to $S^1 \times_{\mathbb{Z}_2} D^2$, it follows that there is an *unique* extension to a line bundle over the solid Klein bottle.

Equivalently, we need to check that the \mathbb{Z}_2 equivariant vector bundle embedding of

$$S^1 \times S^1 \times \mathbb{R} \rightarrow S^1 \times D^2 \times \mathbb{R} \times \mathbb{C}$$

does not extend to a \mathbb{Z}_2 -equivariant vector bundle embedding of

$$S^1 \times D^2 \times \mathbb{R} \rightarrow S^1 \times D^2 \times \mathbb{R} \times \mathbb{C}.$$

Suppose such an embedding exists, and look at its restriction to $D_+^1 \times D^2 \times \mathbb{R} \cong [0, \pi] \times D^2 \times \mathbb{R}$.

We then get a map

$$\begin{aligned} \varphi : [0, \pi] \times D^2 \times \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{C} \\ (\theta, z, t) &\mapsto (t, z) \end{aligned}$$

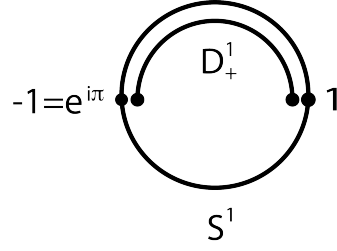


Figure 2.2: Restricted domains of the \mathbb{Z}_2 -equivariant embedding

Also, the restriction to $[0, \pi] \times D^2 \times \{1\}$ is non-zero, and the restrictions to $\{0\} \times D^2 \times \{1\}$ and $\{\pi\} \times D^2 \times \{1\}$ are linear on the fibers, so they are related by the identification condition:

$$\varphi(\theta, \omega, t) = t\varphi(\theta, \omega, 1) = t\varphi_0(\theta, \omega)$$

where

$$\begin{aligned} \varphi_0 : [0, \pi] \times D^2 &\rightarrow \mathbb{R} \times \mathbb{C} \\ (\theta, \omega) &\mapsto \varphi(\theta, \omega, 1). \end{aligned}$$

Now because $(0, \omega, t) \sim (\pi, \bar{\omega}, -t)$, we have $\varphi_0(0, \omega)t \sim -\varphi_0(\pi, \bar{\omega})t$ for all ω, t .

On the other hand, equivariance implies that we have a commutative diagram

$$\begin{array}{ccc} \{0\} \times D^2 \times \mathbb{R} & \xrightarrow{\varphi} & \mathbb{R} \times \mathbb{C} \\ \text{const} \times \text{conj} \times -1 \downarrow & & \text{id} \times \text{conj} \downarrow \\ \{\pi\} \times D^2 \times \mathbb{R} & \xrightarrow{\varphi} & \mathbb{R} \times \mathbb{C} \end{array}$$

so that $-\varphi_0(\pi, \bar{\omega})t = A\varphi_0(0, \omega)t$. *conj* is the conjugate map, and *const* is the constant map, and A is

$$\begin{aligned} A : \mathbb{R} \times \mathbb{C} &\rightarrow \mathbb{R} \times \mathbb{C} \\ (\theta, \omega) &\mapsto (\theta, \bar{\omega}). \end{aligned}$$

Thus $\varphi_0(\pi, \omega) = -A\varphi_0(0, \bar{\omega})$.

Let's look at the restriction of φ_0 to $S^2 \cong \partial([0, \pi] \times D^2)$ more closely. This is homotopic to a map from S^2 to S^2 , and since φ_0 extends to $[0, \pi] \times D^2$, which is the interior of S^2 , its degree must be zero.

On $[0, \pi] \times S^1$ the map sends (s, ω) to $(s, -i\omega)$, so we can deform it so that it sends $[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta] \times S^1$ into a belt around the equator in S^2 in the obvious diffeomorphic fashion for some $\delta > 0$. Visually the map is where the tropical zone is mapped diffeomorphically, as in Figure 2.3.

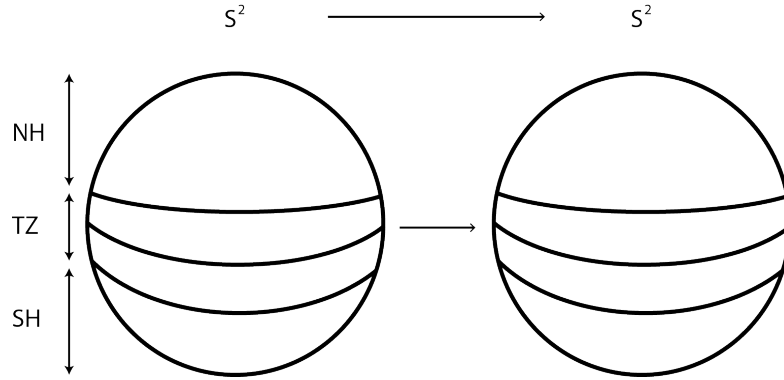


Figure 2.3: Restriction of φ to S^2

The maps on the top and bottom disks are equivariant by the formula $\varphi_0(\pi, \omega) = -A\varphi_0(0, \bar{\omega})$. In other words, we have diffeomorphisms α and β of D^2 and S^2 such that $g = \beta\alpha f$, where $f = \varphi_0|_{\partial([0, \pi] \times D_+^2)}$ and $g = \varphi_0|_{\partial([0, \pi] \times D_-^2)}$. In other words, f is the restriction of φ_0 on the northern hemisphere, and g is the restriction of φ_0 to the southern hemisphere. Over the tropical zone, there is a single preimage of this regular value.

Claim 24 *f and g make equal contributions to the mod 2 degree.*

Proof. View $S^2 \cong \partial([0, \pi] \times D^2)$ as the unit sphere in $\mathbb{R} \oplus \mathbb{C}$. The equator is $\{0\} \times S(\mathbb{C})$, the north and south poles are $\{\pm 1, 0, 0\}$. To simplify notation, let NH be the northern hemisphere with $x_1 > 0$, SH be the southern hemisphere with $x_1 < 0$,

and TZ be the tropical zone with $-\delta < x_1 < \delta$. Let h be φ_0 smoothly deformed so that $h|_{TZ}$ is a diffeomorphism.

Define $Y = h^{-1}[\{y\}]$. Then $Y \cap TZ = \{pt\}$ that lies on the equator, and because the tropical zone is mapped diffeomorphically,

$$(Y \cap NH) \cap (Y \cap TZ) = \emptyset = (Y \cap SH) \cap (Y \cap TZ).$$

Also $(Y \cap NH) \cap (Y \cap SH) \subset NH \cap SH = \emptyset$. Thus, the three sets are pairwise disjoint, open in Y , and their union is Y . Hence, they are also closed, but $Y \subset S^2$ is compact, so each of these closed subsets are also compact. Thus, $Y := h^{-1}[\{y\}]$ is a union of pairwise disjoint open-closed subsets $Y \cap NH$, $Y \cap SH$ and $Y \cap TZ$. Therefore, Y is a union of three pairwise disjoint open-closed compact subsets $Y \cap NH$, $Y \cap SH$ and $Y \cap TZ$.

The local degrees of the restrictions $h|_{NH}$, $h|_{SH}$ and $h|_{TZ}$ at $y \in \{0\} \times S^1$ are definable. Using the additive properties of local degrees [Dol80] we have

$$\begin{aligned} \deg(h) \bmod 2 &= \text{loc deg}_y(h) \bmod 2 \\ &= \text{loc deg}_y(h|_{NH}) + \text{loc deg}_y(h|_{SH}) + \text{loc deg}_y(h|_{TZ}) \bmod 2 \end{aligned}$$

Since $h|_{TZ}$ is a diffeomorphism on its image, the second term must be ± 1 . Thus, mod 2 it equals 1.

It will suffice to show that $\text{loc deg}_y(h|_{NH}) \bmod 2 = \text{loc deg}_y(h|_{SH}) \bmod 2$ for some y on the equator.

The restrictions of h to NH and SH are related by the identity $\varphi_0(\pi, w) - A\varphi_0(0, \bar{w})$. This implies $\text{loc deg}_y(h|_{SH}) \bmod 2 = \text{loc deg}_{-Ay}(h|_{NH}) \bmod 2$. Therefore, the mod 2 local degrees are equal if $y = -Ay$. By construction $-Ay = -\bar{y}$, so a y is needed so that $y = -\bar{y}$. This is true for $y = \pm i$, and for this choice, the local degrees will

be the same *mod 2*. When they are added, $\text{loc deg}_y(h|_{NH}) + \text{loc deg}_y(h|_{SH}) \text{ mod } 2 = 0$.

Therefore

$$\text{deg}(h) \text{ mod } 2 = 0 + \text{loc deg}_y(h|_{TZ}) \text{ mod } 2 = 1 \text{ mod } 2$$

h and thus φ_0 on the northern hemisphere and the southern hemisphere contribute equally to the degree *mod 2*.

■

Hence, the local degree is odd, and the degree of the map is odd. The degree was also shown to be even, so this is a contradiction. Therefore, the line bundle embedding over K does *not* extend to the solid Klein bottle, $S^1 \times_{\mathbb{Z}_2} D^2$.

■

Conclusion 25 *Application to Foliations*

Suppose we have a codimension 1 foliation on S^3 that is compatible with a linear, orientation reversing \mathbb{Z}_{2k} -action.

Display the action of \mathbb{Z}_{2k} on S^3 via some representation $\mathbb{R} \oplus L \oplus W$, where \mathbb{Z}_{2k} acts trivially on \mathbb{R} , non-trivially on L , with dimension 1, and by rotation on the 2-dimensional vector space W . Take the usual splitting

$$S(\mathbb{R} \oplus L \oplus W) = (S(\mathbb{R} \times L) \times D(W)) \cup_{\partial} (D(\mathbb{R} \times L) \times S(W))$$

where each piece on the right hand side is a solid torus, and the orbit space of the second piece is a solid Klein bottle. For convenience, call the first torus: $S(\mathbb{R} \times L) \times D(W)$ the *inner torus*, and the second torus, $D(\mathbb{R} \times L) \times S(W)$ the *outer torus*.

Theorem 26 *Let G be a finite group acting smoothly on M^n , and let \mathcal{F} be a compatible codimension 1 foliation on M^n . Suppose that $g : V^k \rightarrow M^n$ is a smooth equivariant*

embedding which is transverse to the leaves of \mathcal{F} , and that $g[V]$ is closed. Then there is a closed tubular neighborhood E of V^k such that:

1. ∂E is transverse to the leaves of \mathcal{F} such that \mathcal{F} determines a pair of codimension k foliations on $(E, \partial E)$
2. the leaves of $(E, \partial E)$ are the fibers of the projections of E and ∂E to V .

Proof. Let $T\mathcal{L}(M) \subset T(M)$ be the subbundle of tangents along the leaves of M , and take a suitable G -compatible spray on $T(M)$. Near $g[V]$, define the metric on this spray to be an orthogonal direct sum of a metric on $T(V)$ and a metric on $T\mathcal{L}(M)$. Near $g[V]$, this metric makes the fibers of the tubular neighborhood totally geodesic submanifolds.

From the hypothesis, the normal bundle to $g[V]$ is equal to $T\mathcal{L}(M)|_{g[V]}$. By construction, an invariant tubular neighborhood is G -diffeomorphic to an invariant neighborhood of the 0-section and the diffeomorphism is given by the spray's exponential map, exp . The exponential map sends a vector $w \in T\mathcal{L}(M)|_{g[V]}$, which projects to $v \in V$, into a point w' such that w' and $g(v)$ are joined by a geodesic whose initial condition is $w \in T\mathcal{L}(M)|_{g[V]}$. Since the fibers are totally geodesic and $T\mathcal{L}(M)$ is the bundle of tangents along the fibers of E and along the leaves of \mathcal{F} , it follows that w' lies in the leaf of \mathcal{F} containing $g(v)$.

By the invariance of domain and the inverse functions theorem, it follows that the image of exp contains an open neighborhood of $g(v)$. Hence locally, the points in E_v correspond to points in $\mathcal{L}_{g(v)}$ and conversely.

Transversality to the boundary implies that ∂E will be transverse to the leaves if chosen to be sufficiently close to the zero section.

■

By the preceding construction, the foliation can be isotoped so that it is a product foliation given by $\{pt\} \times D(W)$ on the inner torus. So on the boundary, $S(\mathbb{R} \oplus L) \times S(W)$ the foliation is $\{pt\} \times S(W)$. This is also the boundary of the outer torus.

Let $\mathbb{Z}_2 = G/H$ be the quotient of G by all of the orientation preserving elements. The foliation passes to the orbit space, $S(\mathbb{R} \times L) \times S(W)/\mathbb{Z}_{2k} \cong S^1 \times S^1/\mathbb{Z}_{2k} \cong S^1 \times S^1/\mathbb{Z}_2$ which is a Klein bottle. The leaves of this foliation are just as in the foliation of the Klein bottle at the start of this chapter. The image of this foliation on the orbit space of the outer torus, $S^1 \times D^2/\mathbb{Z}_2$ cannot exist. This is a contradiction, thus the following theorem is proved,

Theorem 27 *There are no compatible codimension 1 foliations on S^3 with a group action containing an orientation reversing subgroup that is isomorphic to \mathbb{Z}_{2k} , for $k > 1$.*

Chapter 3

Increasing Isotropy Subgroups and Foliations

The odd order finite group G acts on a smooth manifold M . Assume that all of the subgroups H_i of G are linearly ordered and normal. Let $H \subseteq G$ be the maximal isotropy subgroup. Let M^H , the fixed point set of H , which has a free G/H -action because $H \triangleleft G$.

Theorem 28 *Let G be an odd order abelian group acting on a smooth n -manifold, with isotropy groups linearly ordered by inclusion. If there is a compatible codimension 1 foliation, then this foliation must be transverse to any 1-dimensional fixed point sets.*

Proof. Let C be a 1-dimensional fixed point set of G . For any $p \in C$, the tangent space $T_p(C)$ must be fixed by any g_* , with $g \in G$.

From representation theory of finite groups, every irreducible real representation of the odd order group G comes from a complex representation and hence is even-dimensional [FH91]. Every real representation of H is a direct sum of irreducible representations, so $T_p(M)$ is a direct sum of a trivial representation, and even-dimensional

irreducible representations. Thus, the fixed subspace S_p of $T_p(M)$ must be 1-dimensional and lie in $T_p(C)$. Considering the dimensions, $\dim(T_p(C)) = 1 = \dim(S_p)$, the spaces must then be equal.

Let $\lambda \in \Lambda^1(M)$ be a unit length 1-form defining the invariant foliation. G being odd order implies that $g^*(\lambda) = \lambda$. In other words, $\varepsilon(g) = 1$ for all $g \in G$. Using a riemannian metric, let ω be the dual of λ . G preserves λ , and thus also ω . Therefore, $\omega \in T_p(C)$ and the foliation must be transverse to the fixed point set C . ■

Theorem 29 *Suppose G is an odd order group that acts smoothly on a smooth n -manifold M , with isotropy groups linearly ordered by inclusion. Each connected component of each fixed point set of each isotropy group has Euler characteristic zero if and only if M has a G -invariant codimension 1 foliation.*

In the case of a 3-manifold, the fixed point sets must be a link L , and $\chi(L) = 0$ automatically.

Suppose there is a line bundle over a smooth manifold C which may or may not be trivial. Line bundles are classified by elements of $H^1(C; \mathbb{Z}_2)$ or equivalently by homomorphisms $\pi_1(C) \rightarrow \mathbb{Z}_2$. If the line bundle is trivial then it is the image of a nowhere zero vector field and thus $\chi = 0$ by the Poincaré-Hopf theorem. If it is not trivial, let C' be the 2-sheeted covering associated to the kernel of the homomorphism $\pi_1(C) \rightarrow \mathbb{Z}_2$. Then the pullback of the line bundle to C' is trivial and hence $\chi(C') = 0$ by the Poincaré-Hopf theorem. Since $\chi(C') = 2\chi(C)$, we again get $\chi(C) = 0$ even if the line bundle is nontrivial.

Proof. (\Leftarrow)

Let $p \in M$ with a *non-minimal* isotropy subgroup H . The component C of M^H containing p is a proper smooth submanifold. Suppose that on some invariant

neighborhood of p the foliation is defined by a suitable 1-form λ . G acts linearly on $T_p(M)$, and the fixed point set of this action on the tangent space is $T_p(C)$. The subspace $S_p \subset T_p(M)$ spanned by a vector field dual to $\lambda(p)$, with respect to some invariant metric, will be H -invariant, and 1-dimensional.

This means that the leaves of the foliation must be transverse to C , and hence define a codimension 1 foliation of C . The normal bundle to the leaves of this restricted foliation on C must then define a 1-dimensional vector subbundle of $T(C)$. The existence of this subbundle implies that $\chi(C) = 0$, by the preceding discussion.

(\Rightarrow)

This is an inductive argument. The base case is very similar to the inductive step, so they will both be presented together. All fixed point sets are connected and the isotropy subgroups are all normal and linearly ordered by inclusion. For each isotropy subgroup H , let $\sigma(M^H) \subset M^H$ be the singular set of all points whose isotropy subgroups strictly contain H .

Base case: Let H be the maximal isotropy subgroup. Since there are no points whose isotropy subgroups strictly contain H , $\sigma(M^H)$ will then be empty, and have the empty foliation which is trivially compatible with any group action. Note, if M^G is nonempty, then the maximal isotropy subgroup is the entire group, G .

Inductive step: Let H be a non-maximal isotropy subgroup. The inductive hypothesis is that there is a codimension 1 foliation compatible with the G -action on $\sigma(M^H)$. In either case, the foliation on $\sigma(M^H)$ will then be extended to a compatible foliation on M^H .

$\sigma(M^H)$ is a connected G -invariant smooth submanifold of M^H , so it has an invariant closed tubular neighborhood N with boundary ∂N .

Assume there is a compatible codimension 1 foliation on $\sigma(M^H)$ such that the bundle of normals to the leaves is the trivial line bundle. Using the submersion $N \rightarrow \sigma(M^H)$, pull this foliation back to N . Now we have a codimension 1 foliation on N that is invariant under H . This foliation extends to a foliation on an open tubular neighborhood $W \supset N$. Call this foliation \mathcal{F} . By construction, the normal bundle to the leaves will also be trivial.

Let P be obtained from M^H by cutting out the interior of N , so that $\partial P = \partial N$. Then G acts on P with a single isotropy subgroup, H and

$$G/H \rightarrow P \rightarrow P/G$$

is a principle bundle. The compatible foliation yields a codimension 1 foliation on an invariant collar neighborhood of the orbit manifold $\partial N/G$, and again the normal bundle to the leaves will be trivial, so that it corresponds to a nowhere zero vector field X on $\partial P/G$. In order to proceed, this foliation must be extended to all of P/G . It suffices to extend the normal vector field to the leaves on a neighborhood of $\partial P/G = \partial N/G$ to a nowhere zero vector field over all of P/G .

The Law of Vector Fields due to M. Morse [Mor29], allows X to be extended to all of P/G , if $\chi(P/G) = 0$. Since P is a finite covering of P/G ,

$$\chi(P) = \chi(P/G)|G|$$

so it suffices to show that $\chi(P) = 0$.

If $\sigma(M^H)$ is empty, then $\chi(\sigma(M^H)) = 0$. If $\sigma(M^H)$ is not empty, then $\sigma(M^H) = M^K$ for some isotropy subgroup K properly containing H . So, $\sigma(M^H)$ is a closed manifold that has a nowhere zero vector field because $\partial N \rightarrow \sigma(M^H)$ is a submersion. Therefore we have

$$0 = \chi(\sigma(M^H)) = \chi(\partial N)$$

On the other hand, $\chi(M^H) = 0$. Using excision

$$\begin{aligned}
 \chi(P, \partial N) &= \chi(P, \partial P) = \chi(M^H, \sigma(M^H)) \\
 &= \chi(M^H) - \chi(\sigma(M^H)) \\
 &= 0 - 0.
 \end{aligned}$$

These calculations imply

$$\begin{aligned}
 \chi(P) &= \chi(P, \partial N) + \chi(\partial N) \\
 &= 0 + 0 \\
 &= 0.
 \end{aligned}$$

Both $\chi(P)$ and $\chi(P/G)$ are zero, so the foliation $\mathcal{F}|_{\partial P/G}$ extends to all of P/G .

Now that there is a nowhere zero vector field on P/G , there is a foliation on P/G extending the pulled-back foliation near the boundary of P/G . Pull this foliation back to P , and glue it to \mathcal{F} . This is now a foliation of M^H . This concludes the inductive step.

The foliation is fully constructed when H is the minimal isotopy subgroup, the trivial group $\{e\}$. At this point, the G -compatible codimension 1 foliation has been extended to $M^{\{e\}} = M$. ■

Chapter 4

Invariant Open Book

Decompositions and Contact

Structures

Contact structures have been constructed on smooth closed orientable 3-manifolds by Thurston and Winkelnkemper [TW75]. They can be constructed using open book decompositions, where they are carefully defined on pages, and then across the binding link. We will construct a contact form in a similar fashion, except it will be constructed on the quotient space, and pulled back. This will yield an invariant contact form on the manifold. In the case of fixed point sets, a branched covering will be examined, and a form constructed there so the the pullback will be a contact form, and match up with the form as already defined.

One correspondence that may suggest that this is the correct path to take is a theorem by Giroux,

Theorem 30 [Gir02] *If M is a closed oriented 3-manifold then there is a one-to-one*

correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilization.

We can see that open book decompositions correspond to a class of contact structures.

An open book decomposition exists for every closed smooth orientable 3-manifold.

Theorem 31 (Alexander) [Ale23] *Every closed, smooth, orientable 3-manifold M^3 is diffeomorphic to $C \times D^2 \cup_{id} X$, where D^2 is a 2-disc and X an orientable 2-manifold with boundary that is also a mapping torus.*

To prove the following theorem, a smooth function with certain properties will be created. This can be done under these conditions:

Lemma 32 *Function interpolation construction*

Given C^∞ functions f and g defined near 0 and $b \in \mathbb{R}$ respectively such that

$$f(0) = f'(0) = 0, \text{ and } f''(0) \geq 0$$

$$g(b) > 0, \text{ and } g'(b) > 0.$$

Then there is a C^∞ function defined on an open neighborhood of $[0, b]$ such that

$$h = f \text{ near } 0$$

$$h = g \text{ near } 1$$

$$h' > 0 \text{ on } (0, b]$$

The following proposition is used. It is a common result in immersion theory.

Proposition 33 *Suppose a closed 1-manifold Γ is embedded as $\Gamma \hookrightarrow W \xrightarrow{F} S^1$, where F is a smooth submersion, and W is a smooth manifold with $\dim(W) \geq 3$. Then the embedding can be isotoped so that this new composition is 1-1 on each fiber.*

Proof. There is a vector bundle splitting $TW \cong \alpha \oplus F^*(TS^1)$ such that the derivative map of tangent spaces $F_* : TW \rightarrow TS^1$ is zero on α and an isomorphism on $F^*(TS^1)$. Note that TS^1 is trivial.

Locally the submersion F looks like

$$\begin{aligned} \mathbb{R}^{m-1} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, y) &\mapsto y \end{aligned}$$

so by standard chart-by-chart approximation techniques, and compactness of Γ , we need to prove the following local result:

Let $\gamma : (-3, 3) \rightarrow \mathbb{R}^m$ be a regular smooth curve where $m \geq 3$. Then there is a C^1 approximation to γ by a curve β such that

- $\gamma = \beta$ for $|t| \geq 2$
- the last coordinate of β' is positive if $|t| \leq 1$.

Note: Since close enough approximations to γ are smoothly isotopic to γ keeping $\gamma = \beta$ for $|t| \geq 2$, then the isotopy will follow.

Let K be the set of all vectors and their negatives in S^{m-1} of the form $\frac{\gamma'(t)}{|\gamma'(t)|}$ for $|t| < 3$. Since $m - 1 \geq 2$, this set has measure zero by Sard's theorem. So there is some $q \in S^{m-1}$ such that $q \notin K$ and the last coordinate of q is positive. Let $\varphi(t)$ be a smooth bump function on $(-3, 3)$ which is 0 if $|t| \geq 2$ and 1 if $|t| \leq 1$. Define

$$\beta = \gamma(t) + t \cdot \varphi(t) \cdot h \cdot q$$

where $h > 0$ is a small constant.

If h is small enough, then β is a good C^1 approximation to γ .

$$|\beta(t) - \gamma(t)| = |t| \cdot |\varphi(t)| \cdot h \leq 3h$$

and

$$\begin{aligned}
|\beta'(t) - \gamma'(t)| &= |\varphi(t) + t\varphi'(t)| \cdot h \\
&\leq (|\varphi(t)| + |t| \cdot |\varphi'(t)|) \cdot h \\
&\leq (1 + 3 \cdot \max|\varphi'(t)|) \cdot h
\end{aligned}$$

so both $|\beta - \gamma|$ and $|\beta' - \gamma'|$ can be made arbitrarily small for a suitable choice of h .

Lastly to check that β is regular. $\gamma'(t)$ and q are always linearly independent by definition, so

$$\beta'(t) = \gamma'(t) + [\varphi(t) + t \cdot \varphi'(t)] \cdot q \cdot h$$

is never zero. ■

Now we are ready to consider a smooth \mathbb{Z}_p -action on a manifold.

Theorem 34 *If M is a closed smooth oriented 3-manifold with a smooth orientation-preserving G -action, where G is a group of prime order p , then there is a G -invariant contact form θ on M .*

Proof.

If G acts freely, then M/G is a manifold, and a contact form on M/G can be pulled back to a contact form on M . This form will be invariant under G .

This contact form exists if M/G is a closed orientable 3-manifold [TW75]. For this reason, assume that M^G is non-empty, and M is connected. If M consists of multiple connected components, then an invariant contact form can be constructed on each component separately, and thus an invariant contact form will exist across the entire manifold.

Let M^G be the fixed point set of M by G . For $x \in M^G$, G maps the component containing x into itself. By local linearity, the components of M^G are 1-dimensional.

We will begin by constructing an open book decomposition.

Theorem 35 [Sch82] *There is a smooth structure on the orbit space M/G , which is a topological 3-manifold, such that the orbit space projection $M \rightarrow M/G$ is a regular branched covering.*

Begin by taking any open book decomposition of M/G and let $\Sigma = q(M^G)$ where $q : M \rightarrow M/G$ is the quotient map. Let B be the diffeomorphic image of the binding of the open book decomposition in M/G .

Because $\dim(\Sigma) = 1$ and $\dim(M) = 3$, the inclusion $\Sigma \hookrightarrow M$ is isotopic to an embedding whose image is disjoint from B , and transverse to the pages of the open book. Specifically if $\Sigma \cap B = \emptyset$ then we have a submersion $(M/G - B) \rightarrow S^1$, and by immersion theory we can isotop the embedding $\Sigma \hookrightarrow (M/G - B)$ so that the composite

$$\Sigma \hookrightarrow (M/G - B) \xrightarrow[\text{fibering}]{F} S^1$$

is a submersion, $(M/G - B)$ is the disjoint union of the pages, and F is the open book map described in definition 4. This is due to proposition 33.

The regular branched coverings along this embedding of Σ , and Σ are smoothly equivariant, so the original group action on M is equivalent to an action for which the orbit space projection $M \rightarrow M/G$ is branched at this embedding, instead of being branched at Σ . This equivalent action will be considered henceforth. The open book decomposition of M is obtained by taking the inverse image of the pages and the binding in M/G . Also, Σ will refer to this embedding of the original Σ .

The pages are examined first. We will be looking at pages in M/G . A page is $B_\gamma = F^{-1}(e^{i\gamma})$, where $e^{i\gamma} \in S^1$. Σ is transverse to the pages of the open book decomposition of M/G , and disjoint from the binding B , which implies that the branched

covering

$$\begin{array}{ccc} (M - q^{-1}(B)) & \xrightarrow{q} & (M/G - B) \\ M^G & \xrightarrow{q} & \Sigma \end{array}$$

is sheet preserving, and the second map is the restricted map of branched sets. The composite with the fibering

$$(M - q^{-1}(B)) \xrightarrow{q} (M/G - B) \xrightarrow{F} S^1$$

is also a fibering. On each page B_γ this composite is a branched covering of surfaces along the finite sets $\Sigma \cap B_\gamma$. On the other hand, near the binding the orbit space projection q is an ordinary unbranched covering space projection.

Let $B' = q^{-1}(B)$, which is the binding in M . Since the fixed set M^G is disjoint from the binding, and transverse to the pages, there is an invariant closed tubular neighborhood $N \supset M^G$ in M such that $N \cap B' = \emptyset$ and the fibers of N are contained in pages of the open book. N is diffeomorphic to $M^G \times D^2$ such that G acts trivially on M^G and by rotation on D^2 . Also $N/G \cong \Sigma \times D^2/G$.

Now that the open book decomposition has been created, we will construct a contact form. We will begin by creating a general volume form on D^2 . Using the coordinates $u = r\cos(\theta)$ and $v = r\sin(\theta)$, let

$$\alpha = \frac{1}{2}(udv - vdu) = \frac{1}{2}r^2d\theta$$

polar *cartesian*

Applying the exterior derivative:

$$\begin{aligned}
d\alpha &= \frac{1}{2}d(udv - vdu) = \frac{1}{2}d(r^2d\theta) \\
&= \frac{1}{2}(du \wedge dv - dv \wedge du) = rdrd\theta \\
&= du \wedge dv
\end{aligned}$$

we see that this is the standard volume form on D^2 .

The binding B is a link in M/G , and it has a fiberwise tubular neighborhood U such that $U \cong B \times D^2$, and each $\{b\} \times D^2$ is contained in a single page. Also since the binding B is disjoint from N , U can be shrunk so that $U \cap N = \emptyset$. Let $X = (M/G) - U$, the complement of U in M/G . Note that $U \subset \text{Int}(X)$. The restricted map $F : X \rightarrow S^1$ is a fiber bundle, with fiber P ,

$$P \hookrightarrow X \xrightarrow{F|_X} S^1.$$

This fiber is a page intersected with X . We can also look at the fiber bundle restricted to ∂X . Note the new fibers are isolated points, one for each connected component of B ,

$$\pi_0(B) \hookrightarrow \partial X \xrightarrow{F|_{\partial X}} S^1.$$

The boundary of X is $\partial X \cong B \times S^1$. The entire manifold M/G is diffeomorphic to $(B \times D^2) \cup_{\partial} X$.

Let $X_0 = (X - \text{Int}(\Sigma \times D^2))$. A single page intersected with X_0 is $P_0 = P \cap X_0 = P - \text{Int}(\bigsqcup_{\pi_0(\Sigma)} D^2)$, where $\bigsqcup_{\pi_0(L)} D^2$ is the finite disjoint union of the intersection of $\Sigma \times D^2$ with P . There is one point of intersection for each connected component of the link Σ . Then there is a fiber bundle, $P_0 \hookrightarrow X_0 \xrightarrow{F|_{X_0}} S^1$.

Now we create the 1-form on a page of this new bundle P_0 , then we will extend this form to X_0 . The boundary of P_0 is $\partial P_0 = \partial P \cup (\bigsqcup_{\pi_0(\Sigma)} \partial D^2) = \partial P \cup (\bigsqcup_{\pi_0(\Sigma)} S^1)$. The page P is an oriented manifold thus P_0 is also orientable. Let $d\psi$ be a volume form on

∂P_0 . Let α_0 be any 1-form on P_0 that is $td\psi$ near ∂P_0 , with t being a collar parameter of $\partial P_0 \times [1, 1 + \delta] \subset P_0$ such that $t = 1$ on ∂P_0 , and δ is a small positive number.

$$\int_{P_0} d\alpha_0 = \int_{\partial P_0} \alpha_0 = \int_{\partial P_0} td\psi = \int_{\partial P_0} d\psi = 1$$

This means $d\alpha_0$ is a volume form on P_0 .

Take any volume form Ω on P_0 such that $\Omega = dt \wedge d\phi$ near ∂P_0 . Since P_0 is a 2-manifold, $d(\Omega - d\alpha_0) = 0$. P_0 is non-compact, so $H^2(P_0) = 0$. By de Rham's theorem there is a volume form β on P_0 such that $d\beta = \Omega - d\alpha_0$, and from the way β is defined, $\beta = 0$ near ∂P_0 . Let $\alpha' = \alpha_0 + \beta$, then

$$\int_{P_0} d\alpha' = \int_{P_0} d(\alpha_0 + \beta) = \int_{P_0} d\alpha_0 + \int_{P_0} \Omega - \int_{P_0} d\alpha_0 = 1 + 1 - 1 = 1.$$

Now α' satisfies (1) α' is a volume form on P_0 , and (2) $\alpha' = td\psi$ near ∂P_0 .

The form α' was constructed on a single fiber of P_0 . It will now be extended across X_0 . The 1-forms satisfying (1) and (2) are a convex set, so there is a 1-form α on X_0 such that α restricted to any fiber P_0 satisfies (1) and (2). ψ and t are defined as a parameter of a one dimensional boundary, and a collar neighborhood parameter respectively. On $(\partial P_0 \times [1, 1 + \delta]) \times S^1 \subset X_0$, property (2) is true, so on a neighborhood of ∂X_0 , $\alpha = td\psi$.

Let $\omega = \alpha + Kd\phi$, where K is a positive constant, and $d\phi$ is the pull back of a volume form over S^1 via $F|_{X_0}$. The form $d\phi \wedge d\alpha$ is a volume form since it is the wedge of complementary volume forms. X_0 is compact, so $\alpha \wedge d\alpha$ is bounded. Then

$$\omega \wedge d\omega = (\alpha + Kd\phi) \wedge d\alpha = \alpha \wedge d\alpha + Kd\phi \wedge d\alpha$$

will be nonzero when K is sufficiently large. Therefore, ω is a contact form on X_0 . Near the boundary of X_0 , $\omega = td\psi + Kd\phi$.

There are two components of the boundary of X_0 and each will be treated differently, $\partial X_0 = \partial X \sqcup \Sigma \times S^1$. On $\Sigma \times S^1$, a 1-form will be created that is compatible with the group action, and the fixed point set $\Sigma \times \{0\} \subset \Sigma \times D^2$. On the boundary $\partial X = B \times S^1$, a form will be created that is compatible with the component containing the binding, $B \times D^2$. To be compatible with ω , these extensions need to be $td\psi + Kd\phi$ near the boundary. We will begin by extending ω on this second component $B \times D^2$.

On the boundary of X , which is $\partial X = B \times S^1$, the contact form is $\omega = td\psi + Kd\phi = rd\psi + Kd\phi$ where r and ψ are polar coordinates for D^2 . Here $M/G - \text{Int}(X) = U = \partial X \times D^2$, and ϕ is still the coordinate for ∂X . Over U , r ranges from 0 to 1. The form ω will be changed so that $\omega \wedge d\omega$ is the standard volume form near $B \times \{0\}$:

$$rdr \wedge d\psi \wedge d\phi = du \wedge dv \wedge d\phi$$

in polar and rectangular coordinates respectively. To achieve this, ω needs to equal $-d\psi + \frac{1}{2}r^2d\phi$ when $r \in [0, \epsilon)$, for a small positive ϵ .

Now near $r = 0$,

$$\begin{aligned} \omega \wedge d\omega &= (-d\psi + \frac{1}{2}r^2d\phi) \wedge d(-d\psi + \frac{1}{2}r^2d\phi) \\ &= (-d\psi + \frac{1}{2}r^2d\phi) \wedge (rdr \wedge d\phi) \\ &= -rd\psi \wedge dr \wedge d\phi \\ &= rdr \wedge d\psi \wedge d\phi \end{aligned}$$

which is a volume form, making ω a contact form near $r = 0$. ω is defined for $r \in [0, \epsilon) \cup (1 - \epsilon, 1]$, so it needs to be defined for $r \in [\epsilon, 1 - \epsilon]$, where here ϵ may need to be a smaller positive number. We will create two smooth functions $h_0, h_1 : [0, 1] \rightarrow \mathbb{R}$ such that $\omega = h_0(r)d\psi + h_1(r)d\phi$. These functions need to have certain properties. First, for $r \in (1 - \epsilon, 1]$, the functions $(h_0(r), h_1(r)) = (r, K)$, and for $r \in [0, \epsilon)$, the

functions $(h_0(r), h_1(r)) = (-1, \frac{1}{2}r^2)$. The form so defined is a contact form for all r except $r \in [\epsilon, 1 - \epsilon]$. For r in this range, $d\omega = h'_0 dr \wedge d\psi + h'_1 dr \wedge d\phi$, then

$$\begin{aligned} \omega \wedge d\omega &= (h_0(r)d\psi + h_1(r)d\phi) \wedge (h'_0 dr \wedge d\psi + h'_1 dr \wedge d\phi) \\ &= h_0 h'_1 d\psi \wedge dr \wedge d\phi + h_1 h'_0 d\phi \wedge dr \wedge d\psi \\ &= (h_1 h'_0 - h_0 h'_1) dr \wedge d\psi \wedge d\phi \end{aligned}$$

we can see that ω will be a contact form on all of U when $h_1 h'_0 - h_0 h'_1 > 0$. This is to say that the position and tangent vector of the curve $(h_0(r), h_1(r))$ in \mathbb{R}^2 must be linearly independent. One such curve is depicted below.

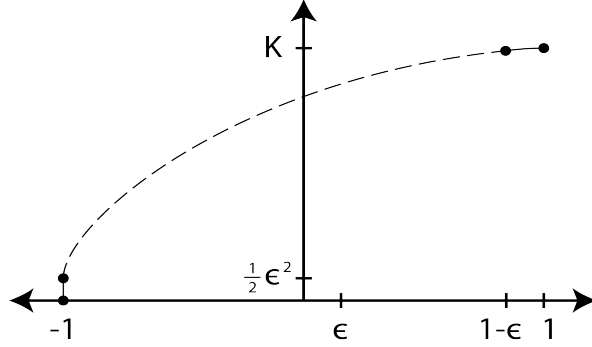


Figure 4.1: One possible smooth parameterized function (h_0, h_1)

To summarize, this curve must be smooth, and

$$(h_0(r), h_1(r)) = \begin{cases} (-1, \frac{1}{2}r^2) & \text{if } 0 \leq r < \epsilon \\ \text{the above graph} & \text{if } \epsilon \leq r < 1 - \epsilon \\ (r, K) & \text{if } 1 - \epsilon \leq r \leq 1. \end{cases}$$

This form ω is a contact form on X_0 , that is equal to $td\phi + Kd\psi$ near the boundary. Pull this contact form back over $F|_{X_0}$ to get a G -invariant contact form on $F^{-1}(X_0)$.

In a similar fashion, we must extend ω across the last remaining piece, $\Sigma \times D^2$ of M/G . Suppose that $b : N \rightarrow N/G$ is a smooth branched covering of degree $p > 1$ with branch set Σ , and N a smooth 3-manifold. Composing with an automorphism of \mathbb{Z}_p if necessary, there are smooth injections $\Sigma \times D^2 \hookrightarrow N$ and $\Sigma \times D^2 \hookrightarrow N/G$ such that this diagram commutes.

$$\begin{array}{ccc} N & \xrightarrow{b} & N/G \\ \uparrow & & \uparrow \\ \Sigma \times D^2 & \longrightarrow & \Sigma \times D^2 \end{array}$$

Where the induced map $\Sigma \times D^2 \rightarrow \Sigma \times D^2$ is given in rectangular and polar coordinates by

$$\begin{aligned} (x, z) &\mapsto (x, z^p) \\ (x, (r, \psi)) &\mapsto (x, (r^p, p\psi)). \end{aligned}$$

Since $b|_{N-\Sigma \times \{0\}}$ is a smooth submersion, if ω is a contact form on N/G then $b^*\omega$ will be a contact form on $N - \Sigma \times \{0\}$. Now we will construct a contact form on $\Sigma \times D^2$ which extends $b^*\omega$ near $\Sigma \times S^1$, where r is close to 1.

Using the form created previously $\omega = \alpha + Kd\phi$, its pullback under $b|_{\Sigma \times D^2}$ would be

$$b^*\omega|_{\Sigma \times D^2} = \frac{p}{2}r^p d\psi + Kd\phi.$$

This pullback is not a contact form because $db^*\omega = \frac{p^2}{2}r^{p-1}dr \wedge d\psi$, and so

$$b^*\omega \wedge db^*\omega = K \frac{p^2}{2} r^{p-1} d\phi \wedge dr \wedge d\psi$$

is zero when $r = 0$ which happens exactly on $\Sigma \times \{0\}$. To make ω into a contact form, it must be modified near $\Sigma \times \{0\}$. As discussed earlier, the standard contact form in a neighborhood of $\Sigma \times \{0\}$ is $\frac{1}{2}r^2 d\psi + Kd\phi$. A new smooth function $H : \Sigma \times D^2 \rightarrow \mathbb{R}$ is

needed such that $\mu = Hd\psi + Kd\phi$ will be a contact form on all of $\Sigma \times D^2$, and this μ will be a smooth extension of ω defined on all of X . H will be defined so that it depends only on r . To ensure the extension is smooth from X_0 to $\Sigma \times D^2$, when $r \in (1 - \epsilon, 1]$, $H(r) = \frac{p}{2}r^p$. For μ to be a contact form near $\Sigma \times \{0\}$, when $r \in [0, \epsilon)$, $H(r)$ needs to equal $\frac{1}{2}r^2$. Now we have to connect these two definitions of H , and ensure that μ is still a contact form between them. On $\Sigma \times D^2$, $d\mu = d(Hd\psi + Kd\phi) = H'dr \wedge d\psi$. Now,

$$\mu \wedge d\mu = KH'dr \wedge d\psi \wedge d\phi$$

and we can see that $H'(r) > 0$ for all $0 \leq r \leq 1$ means that μ is a contact form. The Interpolation Lemma (29) yields a smooth function $H(r)$ with the properties described.

Another way to define H by constructing its derivative, $h = H'$. We know that $H(0) = 0$, so this integral condition will have a constant of integration of zero. Let

$$h(r) = \begin{cases} r & \text{if } 0 \leq r < \epsilon \\ \text{the following graph} & \text{if } \epsilon \leq r < 1 - \epsilon \\ \frac{p^2}{2}r^{p-1} - \frac{p^2}{2} + 1 & \text{if } 1 - \epsilon \leq r. \end{cases}$$

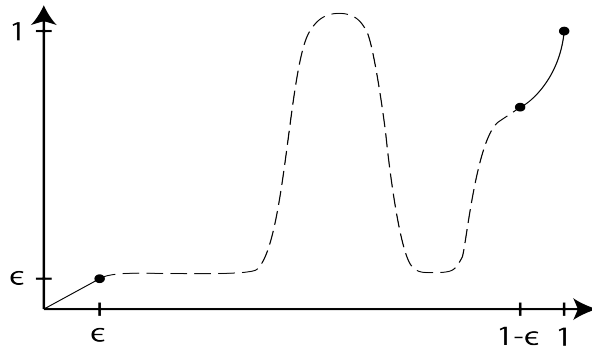


Figure 4.2: One possible smooth function h , with a bump than can be made arbitrarily high

Then, $H(r) = \int_{s=0}^{s=r} h(s)ds$. The constant ϵ can be chosen small enough so

that $H(b) = \int_{s=0}^{s=b} h(s)ds < \frac{p^2}{2}b^{p-1}$ for $b < 1 - \epsilon$. Also, the height of the bump can be arbitrarily high, so $H(1) = \int_{s=0}^{s=1} h(s)ds = \frac{p^2}{2} \geq 1$. This allows us to smoothly define h with the properties mentioned above. To summarize,

$$H(r) = \int_{s=0}^{s=r} h(s)ds = \begin{cases} \frac{1}{2}r^2 & \text{if } 0 \leq r < \epsilon \\ \text{smooth with } H'(r) > 0 & \text{if } \epsilon \leq r < 1 - \epsilon \\ \frac{p}{2}r^p & \text{if } 1 - \epsilon \leq r \end{cases}$$

Now, the pullback $F^*(\omega)$ is extended by $b^*\mu$ to all of $F^{-1}(\Sigma \times D^2)$. This extended form is a G -invariant contact form on all of M .

■

Bibliography

- [Ale23] J. W. Alexander. A Lemma on Systems of Knotted Curves. *Proc. Nat. Acad. Sci.*, 9:93–95, 1923.
- [Bie74] E. Bierstone. Equivariant Gromov Theory. *Topology*, 13:327–345, 1974.
- [Dol80] A. Dold. *Lectures on Algebraic Topology*. Springer-Verlag, 1980.
- [EF08] Y. Eliashberg and M. Fraser. Topologically Trivial Legendrian Knots. *ArXiv e-prints*, January 2008.
- [FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 1991.
- [Fro77] G. Frobenius. Uber das Pfaffsche Problem. *J. fr Reine und Agnew. Math.*, 82:230–315, 1877.
- [Gir02] E. Giroux. Géométrie de Contact: de la Dimension Trois vers les Dimensions supérieures. *Proceedings of the International Congress of Mathematicians, II*: 405-414, 2002.
- [Gom95] R.E. Gompf. A New Construction of Symplectic Manifolds. *The Annals of Mathematics*, 142:527–595, 1995.
- [Hae73] A. Haeflinger. Sur les Classes Caractristiques des Feuilletages. *Lecture Notes in Math.*, 317:239–260, 1973.
- [Hir59] Morris W. Hirsch. Immersions of Manifolds. *Transactions of the American Mathematical Society*, 93(2):pp. 242–276, 1959.
- [Isa76] I.M. Isaacs. *Character Theory of Finite Groups*. Number v. 69 in Pure and Applied Mathematics. Academic Press, 1976.
- [Jub09] B. Jubin. A generalized Poincaré-Hopf index theorem. *ArXiv e-prints*, pages 1–2, March 2009.
- [Law74] H.B. Lawson. Foliations. *Bull. Amer. Math. Soc.*, 80(3):369–418, 1974.
- [Mil65] J.W. Milnor. *Topology from the differentiable viewpoint*. Univ. Virginia Press, 1965.
- [Mor29] M. Morse. Singular points of vector fields under general boundary conditions. *American Journal of Mathematics*, 51(2):165–178, 1929.

- [Sch82] R. Schultz. Exotic spheres admitting circle actions with codimension four stationary sets. *Proceedings of the Northwestern Homotopy Theory Conference*, 19:339–368, 1982.
- [Thu76] W.P. Thurston. Existence of Codimension-One Foliations. *The Annals of Mathematics*, *Second Series*, 104(2):249–268, Sep 1976.
- [TW75] W. P. Thurston and H. E. Winkelnkemper. On the Existence of Contact Forms. *Proceeding of the AMS*, 52:345–347, Oct 1975.
- [Woo70] J. Wood. Foliations of Codimension one. *Bull. Amer. Math. Soc.*, 76(5):1107–1111, 1970.