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Degree Three Cohomological Invariants and Motivic Cohomology of Reductive Groups

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Doctor of Philosophy in Mathematics

by

Donald Joseph Laackman III

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ABSTRACT OF THE DISSERTATION

Degree Three Cohomological Invariants

and Motivic Cohomology

of Reductive Groups

by

Donald Joseph Laackman III

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2018

Professor Alexander Sergee Merkurjev, Chair

This dissertation is concerned with calculating the group of degree three cohomological invariants of a reductive group over a field of arbitrary characteristic. We prove a formula for the group of degree three cohomological invariants of a split reductive group G with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ over a field F of arbitrary characteristic. As an application, we then use this to define the group of reductive invariants of split semisimple groups, and compute these groups in all (almost) simple cases. We additionally prove the existence of a discrete relative motivic complex for any reductive group, which could be used to compute the degree two and three invariants of arbitrary reductive groups.

The dissertation of Donald Joseph Laackman III is approved.

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CHAPTER 1

Introduction

If A is any sort of algebraic object, a great deal of data about A can be recovered from its automorphism group $Aut(A)$. If, in particular, A is an algebraic object defined over a field, its automorphism group can have the structure of an algebraic group. In this case, there is a correspondence between the twisted forms of A - those objects that become isomorphic to A when passing to a field extension - and the $Aut(A)$-torsors; if you send a twisted form B to the $Aut(A)$-torsor $Iso(B, A)$, you get an bijection between the set of isomorphism classes of twisted form as the set of isomorphism classes of $Aut(A)$-torsors.

One classic example of this correspondence is the relationship between torsors of the Orthogonal group and nondegenerate quadratic forms; since if q is a nondegenerate quadratic form of dimension n, then $Aut(q) \simeq O_n$, and the twisted forms of q are precisely the quadratic forms of dimension n, so we can study all nondegenerate quadratic forms of dimension n by studying O_n-torsors.

Another frequently studied case is that of PGL_n; since $PGL_n \simeq Aut(M_n)$, the torsors of PGL_n classify the twisted forms of the algebra M_n, which are all central simple algebras of degree n.

In order to take advantage of this correspondence, the concept of a cohomological invariant is extremely useful.

Let G be a linear algebraic group over a field F. Consider a functor

$$G\text{-torsors} : Fields_F \to Sets$$

from $Fields_F$, the category of field extensions of F, taking a field K to the set of isomorphism classes of G-torsors over Spec$K$. Let

$$\Phi : Fields_F \to Abelian\ Groups$$

be some other functor. Then a Φ-*invariant* of G is a natural transformation

$$I : G\text{-torsors} \to \Phi$$

with Φ viewed as going to *Sets*. The group of Φ-invariants of G is written $\mathrm{Inv}(G, \Phi)$.

An Invariant $I \in \mathrm{Inv}(G, \Phi)$ is *normalized* if $I(E) = 0$ for all trivial G-torsors E. The normalized invariants, $\mathrm{Inv}(G, \Phi)_{\mathrm{norm}}$, form a subgroup, and

$$\mathrm{Inv}(G, \Phi) \simeq \Phi(F) \oplus \mathrm{Inv}(G, \Phi)_{\mathrm{norm}}$$

An invariant is called a cohomological invariant if the functor Φ is Galois cohomology; when $\Phi = H^n(-, A)$, then the standard notation is $\mathrm{Inv}(G, \Phi) = \mathrm{Inv}^n(G, A)$.

One class of examples of cohomological invariants are the Stiefel-Whitney classes, which are invariants for the orthogonal group of a quadratic form in characteristic other than two with values in $\mathbb{Z}/2\mathbb{Z}$; in dimension 1, this is the discriminant of the quadratic form, and in dimension 2 it is the Hasse-Witt invariant.

Here, we will consider the cohomology functors Φ taking a field K/F to the Galois cohomology $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ and write $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for this group of *cohomological invariants of G of degree n with coefficients in* $\mathbb{Q}/\mathbb{Z}(j)$. These are of particular interest because if a group G has a non-constant unramified invariant with values in $\mathbb{Q}/\mathbb{Z}(j)$, then the classifying space BG is not stably rational; to date, no such examples have been found over algebraically closed fields, and the study of these invariants is one of the most promising avenues of attack.

If G is connected, then $\mathrm{Inv}^1(G, \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{norm}} = 0$. For reductive G, $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} = Pic(G)$, the Picard group of G. [7]

The degree 2 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, and $\mathrm{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ where T is an algebraic torus, were computed by Merkurjev and Blinstein [2]. $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$

was computed by Rost when G is simply connected [7] and by Merkurjev for G an arbitrary semisimple group[18].

In this dissertation, we are interested in computing the group of degree three cohomological invariants of reductive groups. The next chapter contains needed background, including a discussion of torsors, classifying spaces, motivic cohomology, $K-$cohomology, and the structure of reductive groups. In the next chapter, we relate the degree 3 invariants of a split reductive group G to the dual lattice T^* of a split maximal torus $T \subseteq G$ and the dual lattice of a particular finite group C; in particular, $C^* \simeq Pic(G)$, so we are building up the degree three invariants out of the degree 1 invariants and the root data of G:

Theorem Let G be a split reductive group, $T \subset G$ a split maximal torus, W the Weyl group, and C the kernel of the universal cover of the commutator subgroup of G. Then there is an exact sequence

$$0 \to C^* \otimes F^\times \to \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to S^2(T^*)^W/\mathrm{Dec}(G) \to 0$$

where $\mathrm{Dec}(G)$ is the subgroup of decomposable elements in $S^2(T^*)^W$.

In the last chapter, we discuss the relative motivic complex, which has applications both in particular to computing cohomological invariants, and in general to relating the algebraic and geometric structures of torsors of an algebraic group.

CHAPTER 2

Preliminaries

2.1 Algebraic Groups

The following discussion of algebraic groups is informed by their presentation in [12] and [28]. For a field F, let $\mathbf{Alg}_F$ be the category of commuative F-algebras with $F-$algebra homomorphisms as the morphisms.

Definition. *A* Hopf Algebra *over F is a commutative F-algebra, with three extra $F-$algebra homomorphisms,*

$$c : A \to A \otimes_F A \textit{(comultiplication)}$$

$$i : A \to A \textit{(co-inverse)}$$

$$u : f \to A \textit{(co-unit)}$$

such that, if $m : A \otimes_F A \to A$ is the multiplication of A, the following hold:

(1) The diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad c \quad} & A \otimes_F A \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c \otimes Id} \\ A \otimes_F A & \xrightarrow{Id \otimes c} & A \otimes_F A \otimes_F A \end{array}$$

commutes.

(2) The map

$$A \xrightarrow{c} A \otimes_F A \xrightarrow{u \otimes Id} F \otimes_F A = A$$

is equal to the identity map $Id : A \to A$.

(3) The two compositions

$$A \xrightarrow{c} A \otimes_F A \xrightarrow{i \otimes Id} A \otimes_F A \xrightarrow{m} A$$

$$A \xrightarrow{u} F \xrightarrow{\cdot 1} A$$

are the same map.

A *Hopf algebra homomorphism* $f : A \to B$ is an $F-$algebra homomorphism preserving c, i, and u; so, $(f \otimes f) \circ c_A = c_B \circ f$, $f \circ i_A = i_B \circ f$, and $u_A = u_B \circ f$. The kernel of any Hopf algebra homomorphism will be a *Hopf ideal*; an ideal J of A such that

$$C(J) \subseteq J \otimes_F (A) + A \otimes_F J, i(J) \subseteq J, u(J) = 0$$

These conditions mean that if J is a Hopf ideal of A, then the algebra A/J admits the structure of a Hopf algebra, and there is a surjective Hopf algebra homomorphism $A \to A/J$ with kernel J. For example, the kernel of $u : A \to F$ is a Hopf ideal, and $A/ker(u) = F$ is the trivial Hopf algebra.

The additional homomorphisms associated to a Hopf algebra provide precisely the extra structure needed so that when we consider the set of $F-$algebra homomorphisms from a Hopf algebra A to an arbitrary $F-$algebra R, it is endowed with a natural group structure; multiplication in the group is given via comultiplication as $fg = m_R \circ (f \otimes_F g) \circ c$, which is associative by the first Hopf algebra axiom. The identity is precisely the unique composition $A \xrightarrow{u} F \to R$, by the third Hopf algebra axiom. Lastly, $f^{-1} = f \circ c$ by the second Hopf algebra axiom. Indeed, if A is merely an F-algebra with a comultiplication morphism $c : A \to A \otimes_F A$, that is enough to endow $\mathrm{Hom}_{\mathbf{Alg}_F}(A, R)$ with a binary operation, and if it is a group with respect to that operation, A is guaranteed to be a Hopf algebra, with counit and comultiplication uniquely defined.

Definition. *An* affine group scheme G *over* F *is a functor* $G : \mathbf{Alg}_F \to \mathbf{Groups}$ *that is isomorphic to* $Hom_{\mathbf{Alg}_F}(A, -)$ *for some Hopf algebra* A *over* F*.*

By Yoneda's lemma, if G is an affine group scheme, the Hopf algebra A is uniquely determined up to isomorphism; we denote it as $A = F[G]$. They form one of two primary pillars of the study of algebraic groups, alongside abelian varieties - Chevalley's structure theorem tells us that any algebraic group G over a perfect field has a unique normal closed subgroup H such that H is an affine group, and G/H is an abelian variety.

A group scheme homomorphism $\rho : G \to H$ is just a natural transformation from G to H; by Yoneda's lemma, ρ is determined by the corresponding Hopf algebra homomorphism $\rho^* : F[H] \to F[G]$, defined such that, if ρ_R is the group homomorphism given by ρ from $G(R)$ to $H(R)$, $\rho_R(g) = g \circ \rho^*$. This means that the correspondence between Hopf algebras and affine groups schemes is in fact an equivalence of categories.

Some important examples of affine group schemes include the trivial group $\mathbf{1}(R) = 1$, represented by $A = F$, the additive group $G_a(R) = R$, represented by $F[t]$, the multiplicative group $G_m(R) = R^\times$, represented by $F[t, t^{-1}]$, and the general linear group $\mathbf{GL}_n(R) = GL(R^n)$, represented by $F[t_{ij}, \frac{1}{detT}]$ where T is the matrix (t_{ij}). More generally, if A is a unital associative $F-$algebra of dimension N, then $\mathbf{GL}_1(A)$, defined as $\mathbf{GL}_1(A)(R) = (A_R)^\times$, is an affine group scheme. All of these examples are finitely generated as algebras; an affine group scheme G is said to be *algebraic* if the $F-$algebra $F[G]$ is finitely generated.

Note that, if L/F is a field extension, we can define a group scheme G_L over L represented by $G[F] \otimes_F L$; since any L-algebra is also an F-algebra this will satisfy $G_L(R) = G(R)$, since for any $R \in \mathbf{Alg}_L$,

$$G_L(R) = \mathrm{Hom}_{\mathbf{Alg}_L}(F[G] \otimes_F L, R) = \mathrm{Hom}_{\mathbf{Alg}_F}(F[G], R) = G(R)$$

Whenever G is an affine group scheme, $A = F[G]$, and we have a Hopf ideal $J \subset A$, the affine group scheme H represented by A/J has a group scheme homomorphism $\rho : H \to G$ induced by the natural map $A \to A/J$; for any $R \in \mathbf{Alg}_F$, $\rho_R : H(R) \to G(R)$ is injective, and so we can identify $H(R)$ with a subgroup in $G(R)$. When this is the case, we call H a *closed subgroup of* G, and ρ is a *closed embedding*. H is normal if, for every $R \in \mathbf{Alg}_F$, $H(R)$ is normal in $G(R)$.

This definition demonstrates the power of blending the two equivalent concepts of Hopf algebras and affine group schemes; we catalogue subgroups via their contravariant correspondence with Hopf ideals, then check their normality based on the group of R-points.

The subgroup associated to the ideal $ker(u)$ is always the trivial subgroup $\mathbf{1}$, since $F[G]/ker(u) \simeq F$. Given any homomorphism $f : G \to H$ of group schemes, and subgroup H' of H given by the Hopf ideal $J \subset F[H]$, the inverse image $f^{-1}(H')$ is the functor taking an F-algebra R to

$$f^{-1}(H')(R) = \{g \in G(R) | f_R(g) \in H'(R)\}$$

is a subgroup of G associated to the Hopf ideal $f^*(J)F[G]$; in the case where $H' = \mathbf{1}$, we get the kernel of f, $ker(f)$, associated to the hopf ideal $f^*(I)F[G]$, where I is the kernel of the counit in $F[H]$.

We call a group scheme homomorphism f *surjective* if the Hopf algebra homomorphism f^* is injective; note that this doesn't mean that the induced homomorphisms of groups of points will be surjective; for example, the n^{th} power homomorphism $f : \mathbf{G}_m \to \mathbf{G}_m$ is surjective because $f^* : F[t, t^{-1}] \to F[t, t^{-1}]$ is given by $f^*(t) = t^n$, which is injective, but $f_R : R^\times \to R^\times$ is not, in general, surjective.

One very important class of affine group scheme homomorphisms are the characters.

Definition. *A* character *of an affine group scheme G over F is a group scheme homomorphism $\chi : G \to \mathbf{G}_m$. The set of characters of G forms an abelian group, denoted G^*.*

The character group of a group scheme G will be a much easier to work with object than G as a whole, but we will see that it contains a great deal of the same information. Any given character $\chi : G \to \mathbf{G}_m$ is uniquely determined by the element $f = \chi^*(t) \in F[G]^\times$, satisfying $c(f) = f \otimes f$. The set of all elements of $F[G]^\times$ which satisfy this condition form a subgroup, called the subgroup of *group-like elements*, which is isomorphic to G^*.

This correspondence between affine group schemes and abelian groups in fact determines a subcategory of the category of affine group schemes. If we first take an abstract abelian group H, there is a Hopf algebra structure on the group algebra $F < H >$ over F given by $c(h) = h \otimes h$, $i(h) = h^{-1}$, and $u(h) = 1$. The group scheme represented by $F < H >$ is called a *diagonalizable* group scheme, and is written H_{diag}. The group-like elements of H_{diag} are precisely of the form $h \otimes h$ for $h \in H$, so H^*_{diag} is naturally isomorphic to H.

Take a separable closure F_{sep} of F, and let $\Gamma = \text{Gal}(F_{\text{sep}}/F)$. A group scheme G over F is of *multiplicative type* if $G_{\text{sep}} = G_{F_{\text{sep}}}$ is diagonalizable. If H is an abelian group with a continuous Γ-action, then there is a corresponding group of multiplicative type; H_{mult} is represented by the Hopf algebra of Γ-stable eleements in $F_{\text{sep}} < H >$; we can explicitly compute H_{mult} by

$$H_{\text{mult}}(R) = Hom_{\Gamma}(H, (R \otimes_F F_{\text{sep}})^{\times}).$$

$(-)_{\text{mult}}$ is an equivalence of categories between affine group schemes of multiplicative type and abelian groups with continuous Γ-action; the functor going the other direction is simply $(-)^*$, the character group.

Definition. *An* algebraic torus *is an affine group scheme of multiplicative type H_{mult}, where H is a free abelian group of finite rank; a torus is* split *if it is a diagonalizable group scheme.*

Split tori are isomorphic to the group scheme of diagonal matrices in $\mathbf{GL}_n(F)$; for any torus, T_{sep} is split over F_{sep}.

So, when we gain information about a group scheme based on its character group, we are often really considering a sub-torus of the group scheme.

Thus far, we have worked generally, but going forward, there will be one more restriction applied to our affine group schemes; they must be *smooth*, meaning, equivalently, that $F[G]_L$ is reduced for any field extension L/F, or $F[G]_{F_{\text{sep}}}$ is reduced. A smooth group scheme is called an *algebraic group*. In particular, $\mathbf{GL}_1(A)$, is smooth for any central simple F-algebra A, and H_{mult} is smooth if and only if H has no $p-$torsion with $p = \text{char}(F)$.

2.1.1 Linear Representations

If G is an affine group scheme and V is a vector space, a *linear representation* ρ of G in V is a group scheme homomorphism $\rho : G \to GL(V) = \mathbf{GL}_1(\text{End}(V))$; then V is an

$F[G]-$comodule.

Definition. *A* linear algebraic group *over F is an algebraic group which is a closed subgroup of $\mathbf{GL}_n$ for some value of n.*

Equivalently, an algebraic group G is linear if it has a finite-dimensional faithful representation, or if there is a surjective Hopf algebra homomorphism $F[\mathbf{GL}_n] \to F[G]$. Note that all linear algebraic groups are affine, since they are closed subgroups of the affine general linear groups.

Theorem 1. *For any affine algebraic group G, there is a finite-dimensional faithful linear representation; so affine algebraic groups are precisely linear algebraic groups. [28]*

Proof. Let V be a finite-dimensional sub-comodule of $F[G]$ containing a set of generators for A as an F-algebra (this set is finite because we have assumed G to be an algebraic group). Take a basis for V, $\{e_i | 1 \leq \imath \leq n\}$, and write $c(e_i) = \sum_i e_i \otimes a_{ij}$. We can take $r : F[X_{11}, \ldots, X_{nn}, \frac{1}{\det}] \to F[G]$ with $r(X_{ij}) = a_{ij}$, since the v_j form a basis; but $v_j = (u \otimes \mathrm{Id}_{F[G]})c(v_j) = \sum_i u(v_j)a_{ij}$, so v_{ij} is in the image of r, so the image contains V; thus, it must be all of $F[G]$. So r is surjective, meaning that $G \to \mathbf{GL}(V)$ is injective. $\square$

This means that any affine algebraic group can be thought of as a group of matrices (over all extensions of the base field); however, the work building to this point was not wasted. Such a representation is far from unique, and requires a number of choices to be made.

Having defined representations, it is natural to ask whether, as is the case for finite groups, all representations of G are sums of irreducible representations. Consider a maximal connected, normal, solvable subgroup R of G (which exists because the closure of the product of any two normal solvable subgroups is also normal and solvable); this is the *radical* of G. The subgroup U of R of unipotent elements is called the *unipotent radical* of G.

Definition. *An algebraic group G is* semisimple *if its radical R is trivial; G is* reductive *if its unipotent radical is trivial.*

The name reductive comes from the fact that, when $\mathrm{char}(F) = 0$, all representations are sums of irreducibles if and only if G is reductive. This result comes from the fact that it is true for tori in general, and for semisimple groups in the characteristic zero case, and a reductive group is a product of a semisimple subgroup and a subtorus.

This result fails in characteristic p, but a more fundamental result carries through: the classificiation of semisimple groups. Closely related to the classification of semisimple lie algebras, the simple semisimple groups fall into four infinite families and five exceptional types. Within a type, simple groups are distinguished by their root systems; in particular, by T^* as a lattice sitting between the root and weight lattices.

2.1.2 Borel Subgroups

Aside from algebraic tori, there is one other particularly important class of subgroups that we will consider.

Definition. *A* Borel subgroup B *of an algebraic group* G *is a maximal closed and connected solvable algebraic subgroup.*

The classic example of a Borel subgroup is the subgroup of invertible upper triangular matrices inside of $\mathbf{GL}_n$. If G is an algebraic group over an algebraically closed field, all Borel subgroups are conjugate to one another. A Borel subgroup B contains a maximal torus T; B together with the normalizer of T generates all of G. There is a natural map $p : G/T \to G/B$.

Proposition 2. *If G is a connected linear algebraic group, and $H \leq G$ is a closed subgroup, then G/H is projective if and only if H contains a Borel subgroup (such subgroups are called parabolic). [6]*

2.2 Torsors

2.2.1 Galois Cohomology

Again, consider the profinite group $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$, the absolute Galois group of F. If G is an algebraic group over F, Γ acts continuously on the discrete group $G(F_{\mathrm{sep}})$, which means we can define Galois cohomology as the group cohomology $H^i(F, G) = H^i(\Gamma, G(F_{\mathrm{sep}}))$ for $i = 0, 1$. Note $H^0(F, G) = G(F_{\mathrm{sep}})^\Gamma = G(F)$.

The first Galois cohomology set of an algebraic group often classifies a particular type of algebra tied to that algebraic group. In order to achieve this, two tools are used.

First, let G be a group scheme over F, and $\rho : G \to \mathbf{GL}(W)$ with W a finite dimensional vector space. An element $w' \in W_{\mathrm{sep}}$ is a *twisted form* of $w \in W$ if $w' = \rho_{\mathrm{sep}}(g)(w)$ for some $g \in G(F_{\mathrm{sep}})$. Let $A(\rho, w)$ be the groupoid whose objects are the twisted ρ-forms of w inside W, with maps $w' \to w''$ corresponding to $g \in G(F)$ such that $\rho(g)(w') = w''$; so over the separable closure this is a connected groupoid, but in general it may have multiple isomorphism classes; the set of isomorphism classes is $\mathrm{Isom}(A(\rho, w))$.

Let $\mathbf{Aut}_G(w)$ denote the stabilizer of w; it is a subgroup of G.

Proposition 3. *If $H^1(F, G) = 1$, there is a natural bijection of pointed sets*

$$\mathit{Isom}(A(\rho, w)) \simeq H^1(F, \boldsymbol{Aut}_G(w))$$

which maps the isomorphism class of w to the base point of $H^1(F, \boldsymbol{Aut}_G(w))$. [25]

Proof. Let $\tilde{A}(\rho, w)$ have the same objects as $A(\rho, w)$, but with morphisms corresponding instead to $G(F_{\mathrm{sep}})$. Then $\tilde{A}(\rho, w)^\Gamma = A(\rho, w)$. At the same time, $\tilde{A}(\rho, w)$ corresponds to the left cosets of $G(F_{\mathrm{sep}})$ modulo $\mathbf{Aut}_G(w)$. So, taking Galois cohomology, with $C = \mathrm{coker}(\mathbf{Aut}_G(w) \to G)$ we get

$$\mathbf{Aut}_G(w)(F_{\mathrm{sep}})^\Gamma \longrightarrow G(F_{\mathrm{sep}})^\Gamma \longrightarrow C^\Gamma \longrightarrow H^1(F, \mathbf{Aut}_G(w)) \longrightarrow H^1(F, G) = 0$$

So, we get a pointed set bijection between $C^\Gamma / G(F_{\mathrm{sep}})^\Gamma$ and $H^1(F, \mathbf{Aut}_G(w))$; but $C^\Gamma = A(\rho, w)$, and $G(F_{\mathrm{sep}})^\Gamma = G(F)$, so $C^\Gamma / G(F_{\mathrm{sep}})^\Gamma = \mathrm{Isom}(A(\rho, w))$, giving the required bijection, which can be explicitly defined on $w' \in A(\rho, w)$ by choosing $g \in G(F_{\mathrm{sep}})$ sich that

$\rho_{\text{sep}}(w) = w'$, and sending it to the 1-cocycle α with $\alpha_\sigma = g^{-1} \cdot \sigma(g)$; injectivity follows from the cocycle condition, and surjectivity is a result of the assumption $H^1(F, G) = 0$.

$\square$

This proposition, while fairly specialized, finds diverse applications because a wide range of algebraic groups have trivial first Galois cohomology.

Theorem 4. *(*Hilbert's Theorem 90*) For any separable and associative F-algebra A,*

$$H^1(F, \mathbf{GL}_1(A)) = 1$$

In particular, $H^1(F, \mathbf{G}_m) = 1$.

Using proposition 2 and Hilbert's Theorem 90, we can now classify multiple kinds of algebras.

Let A be a finite dimensional algebra over F. Multiplication in A gives a linear $w : A \otimes FA \to A$. Let $W = \text{Hom}_F(A \otimes A, A)$ and $G = \mathbf{GL}(A)$, the general linear group just viewing A as an F-vector space. if $\rho : G \to \mathbf{GL}(W)$ is given by

$$\rho(g)(\phi)(x \otimes y) = g \circ \phi(g^{-1}(x) \otimes g^{-1}(y))$$

for $g \in G$, $\phi \in W$, and $x, y \in A$. A linear map $g \in G$ is an algebra automorphism of A if and only if $\rho(g)(w) = w$, so the group scheme $\mathbf{Aut}_{\text{alg}}(A)$ is the same as $\mathbf{Aut}_G(w)$, and a twisted ρ-form of w is an algebra structure A' on the vector space A such that A'_{sep} and A_{sep} are isomorphic F_{sep}-algebras. So, there is a bijectoin between $H^1(F, \mathbf{Aut}_{\text{alg}}(A))$ and the set of F-isomorphism classes of F-algebras that become isomorphic to A when extending to F_{sep}.

As a particular example, if $A = \mathbf{M}_n(F)$, the matrix algebra of degree n, the twisted forms of A are precisely the central simple F-algebras of degree n. $\mathbf{Aut}_{\text{alg}}(A) = \mathbf{PGL}_n$, so F-isomorphism classes of central simple F-algebras of degree n are counted by the first $\mathbf{PGL}_n$ Galois cohomology set.

The concept of finding algebraic objects whose twisted forms are classified by a particular algebraic group naturally leads to the concept of a torsor.

Definition. *Let G be an algebraic group over a field F. Then a G-torsor over a scheme X over F is a flat, surjective morphism $\pi : P \to X$ of schemes over F with a G-action on P such that the map $P \times_F G \to P \times_X P$ taking (p, g) to $(p, g \cdot p)$ is an isomorphism.*

If F is algebraically closed, then all of the G-torsors over F are isomorphic to $G \to \mathrm{Spec}(F)$. In general, there is an isomorphism of pointed sets from the set of isomorphism classes G-torsors over a field extension K/F (with distinguished point the trivial torsor $G \times_F K \to K$) to the first Galois cohomology set $H^1(K, G) = H^1(\Gamma_K, G(K_{sep}))$.

2.2.2 Classifying Varieties and Versal Torsors

In order to compute cohomological invariants, the following object is of great use.

Definition. *Suppose G/F is an algebraic group; then a* versal G-torsor *is a G-torsor $E \to X$ over a smooth F-variety X such that, for any field extension K/F where K is infinite, given any G-torsor E' over K and nonempty subvariety $U \subseteq X$, there is a point $x \in U(K)$ such that $E' \simeq E_x$, the fiber of E over the point x. When we have this versal torsor, X is called a* classifying variety for G*, while if Spec$(K) \to X$ is the generic point ξ, then $E_\xi \to$ Spec(K) is called a* generic torsor*.*

One can thus study a particular versal G-torsor in order to obtain information about all G-torsors over fields.

Proposition 5. *Versal torsors exist.*

Proof. Since an affine algebraic group is also a linear algebraic group, we can find a faithful representation V of G; then take a G-equivariant open subset $U \subset V$, which has $U \to U/G$ a G-torsor; this is a versal G-torsor. [2]

□

In fact, for many cohomological questions, e.g. stable rationality, any one such classifying variety is stably rational if and only if another is. Properties that hold universally for all classifying varieties U/G (or for all with V at least some particular finite dimension) are said to hold for the classifying space BG; cohomology groups of BG are precisely those cohomology groups of U/G which are independent of our choice of classifying torsor.

2.3 $H^{i+1}(F, \mathbb{Q}/\mathbb{Z}(i))$

The cohomology groups $H^{i+1}(F, \mathbb{Q}/\mathbb{Z}(i))$ and $H^{i+1}(F, \mathbb{Z}/n\mathbb{Z}(i))$ can be defined for any field f, with $d \geq 0$, $n \geq 1$ [7, Appendix A].

First, we decompose them into the p-part for each prime, and assume for the moment that $p \neq \mathrm{char}(F)$

$$H^{i+1}(F, \mathbb{Q}/\mathbb{Z}(i)) = \coprod_{p \text{ prime}} H^{i+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

$$H^{i+1}(F, \mathbb{Z}/n\mathbb{Z}(i)) = \coprod_{p \text{ prime}} H^{i+1}(F, \mathbb{Z}/p^{v_p(n)}\mathbb{Z}(i))$$

The coefficients are Γ-modules; $\mathbb{Z}/p^m\mathbb{Z}(i) = (\mu_{p^m})^{\otimes d}$ and $\mathbb{Q}_p/\mathbb{Z}_p(i) = \lim_{\to_m} \mathbb{Z}/p^m\mathbb{Z}(i)$.

with μ_{p^m} being the group of p^m-th roots of unity in $F_{\mathrm{sep}}^{\times}$; now, we can pull out the colimit to see

$$H^{i+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \lim_{\to_m} H^{i+1}(F, \mathbb{Z}/p^m\mathbb{Z}(i))$$

So, for example,

$$H^1(F, \mathbb{Q}_p/\mathbb{Z}_p(0)) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$H^2(F, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathrm{Br}(F)\{p\}$$

For $i \leq 2$, which are the only cases considered here, the group $H^{i+1}(F, \mathbb{Z}/p^m\mathbb{Z}(i))$ can be identified with the subgroup of $H^{i+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(i))$ consisting of elements of order dividing p^m.

On the other hand, when $p = \mathrm{char}(F)$, set

$$H^{i+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^2(F, K_d(F_{\mathrm{sep}}))\{p\}$$

where K_d is the d^{th} Milnor K-group, to be defined shortly; but in particular, note that this choice is connected to the norm residue homomorphism,

$$K_r(F) \to H^r(F, \mathbb{Z}/p^m\mathbb{Z}(r))$$

With these definition, several computations can be made:

$$H^1(F, \mathbb{Q}/\mathbb{Z}(0)) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z})$$

$$H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(F)$$

2.4 Cohomological Invariants

Let G be an algebraic group defined over a field F, and continue using $H^1(F, G)$ to denote the first Galois cohomology set of G, and for a field extension E/F, $H^1(E, G) = H^1(\mathrm{Gal}(E_{\mathrm{sep}}/E), G(E_{\mathrm{sep}}))$; these sets together give a functor from the category of field extensions over F to the category of sets,

$$H^1(-, G) : \mathbf{Fields}_{/F} \to \mathbf{Sets}, E/F \mapsto H^1(E, G).$$

If there is another functor

$$H : \mathbf{Fields}_{/F} \to \mathbf{Abelian\ Groups}$$

(allowed to be arbitrary, but for our purposes, this will eventually be $H^{i+1}(-, \mathbb{Q}/\mathbb{Z}(i))$)

then we define

Definition. *An* invariant of G with values in H, *also known as an* H-invariant of G, *is a natural transformation (treating H as having values also in **Sets**)*

$$a : H^1(-, G) \to H$$

which is equivalent to a collection of functions $a_E : H^1(E, G) \to H(E)$ for every field extension E/F such that for any field inclusion $i : E \to E'$ over F, the diagram

$$\begin{tikzcd} H^1(E,G) \arrow[r, "a_E"] \arrow[d, "H^1(i,G)"] & H(E) \arrow[d, "H(i)"] \\ H^1(E',G) \arrow[r, "a_{E'}"] & H(E') \end{tikzcd}$$

is commutative.

The set of all invariants of G with values in H is an abelian group, a structure inherited because H lands in the category **Abelian Groups**. If we have an algebraic group homomorphism $f : G \to G'$, there is an induced group homomorphism,

$$\phi^* : \mathrm{Inv}(G', H) \to \mathrm{Inv}(G, H)$$

which means that taking invariants with values in H is a contravariant functor from the category of algebraic groups over F to **Abelian Groups**.

There are certain invariants which we usually will want to hold apart; given any $h \in H(F)$, there is an invariant a^h defined for any $x \in H^1(E, G)$ as $a^h_E(x) = H(i)(h)$ with $i : F \to E$ the inclusion; this invariant, not depending on x, is called a *constant* invariant. The set of constant invariants is a subgroup, $\mathrm{Inv}(G, H)_{\mathrm{const}} \subseteq \mathrm{Inv}(G, H)$, and the set of constant invariants is isomorphic to $H(F)$ via $a^h \mapsto h$.

On the other hand, an invariant with values in H is *normalized* if it takes the distinguished element of the pointed set $H^1(F, G)$ to zero; note that the only constant, normalized invariant is the 0 invariant. Writing the subgroup of normalized invariants as $\mathrm{Inv}(G,H)_{\mathrm{norm}}$, we get that

$$\mathrm{Inv}(G,H) = \mathrm{Inv}(G,H)_{\mathrm{const}} \oplus \mathrm{Inv}(G,H)_{\mathrm{norm}}$$

2.4.1 Cohomological Invariants

In a certain sense, invariants are often relating something fundamentally algebraic (like the first Galois cohomology set, describing the different algebraic structure that can converge when reaching a separable closure) to something geometric - albeit usually still through the lens of algebra. A particularly fruitful choice of such a geometric H is to let it be a cohomology functor; if $H^i(E) = H^i(E,M)$ for some cohomology with coefficients in M, then we write $\mathrm{Inv}^i(G,M)$ for the group of invariants $Inv(G,H^i)$.

Example. *Take H be a semisimple algebraic group over F, with C the kernel of the universal cover $\tilde{H} \to H$. If $\rho \in C^*$, we have a diagram,*

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] \arrow[d, "\rho"] & \tilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "Id"] & 1 \\
1 \arrow[r] & \mathbf{G}_m \arrow[r] & G' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

With G' the pushout of the two arrow from C. Taking cohomology of the bottom sequence, note that the connecting map from degree 1 to degree 2 takes the set $H^1(E,G)$ to $H^2(E,\mathbf{G}_m)$, which means that we can get a group homomoprhism

$$C^* \to Inv^2(G,\mathbb{Q}/\mathbb{Z}(1))$$

Example. *A connected algebraic group S over F is* special *if $H^1(E,S) = 1$ for every field extension E/F. Given a special algebraic group S, it mist be the case that $Inv(S,H)_{norm} = 0$ for every H. In particular, $\mathbf{GL}_n$, $\mathbf{SL}_n$, $\mathbf{Sp}_{2n}$ and any product of them is special. Special groups can be substituted in for $\mathbf{GL}_n$ when defining classifying torsors; instead of a faithful representation from G to $\mathbf{GL}_n$, we can instead take any injective $\rho : G \to S$, and look at $S/\rho(G)$; the elements of $H^1(E,G)$ match up with the orbits of the action of $S(E)$ on*

$(S/\rho(G))(E)$.

2.5 K-Theory, Cycle Modules, and K-Cohomology

In order to compute the cohomological invariants, we will need to understand Milnor $K-$theory, the generalization of Milnor $K-$theory to cycle modules, and $K-$cohomology.

2.5.1 Milnor $K-$theory

For a field F, the calculation of K-groups is notably easier than for rings in general; because all modules over a field are free, $K_0(F) = \mathbb{Z}$, and because F is already commutative, $K_1(F) = F^\times$.

K_2 is harder, but still straightforward; according to Matsumoto,

Theorem 6. *For any field F,*

$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / < a \otimes (1-a) | a \neq 0, 1 >$

This inspired Milnor, leading to [5]

Definition. The n^{th} Milnor $K-$group *of a field F, $K_n(F)$, is defined as the n^{th} graded piece of the tensor ring*

$$K_\cdot(F) = T^\cdot F^\times / (a \otimes (1-a))$$

The Milnor $K-$groups are sometimes written $K_n^M(F)$, to distinguish them from the Quillen $K-$groups, which are no longer isomorphic when $n \geq 3$, but the Milnor $K-$groups are more relevant to the problem at hand, so they receive the distinction of a lack of superscript.

2.5.2 Cycle Modules

Rost's cycle modules are a generalization of Milnor K-theory, introduced in [23].

Definition. *A* cycle module *over a field F is a function on objects $M : Fields_F \to$ Abelian Groups from the category of field extensions over F to the category of abelian groups, and a grading $M = \coprod_n M_n$ such that:*

1. *For any morphism $\phi : E \to K$ in $Fields_F$, there is a degree 0 restriction homomorphism $\phi_* : M(E) \to M(K)$.*

2. *If $[K : E]$ is finite, then there is a degree 0 corestriction homomorphism $\phi^* : M(K) \to M(E)$.*

3. *For every extension E/F, there is a left $K_*(E)$-module structure on $M(E)$ respecting the gradings of both K and M, in the sense that $K_n(E) \cdot M_m(E) \subseteq M_{n+m}(E)$.*

4. *For every extension E/F which has a discrete valuation ν that is trivial on F with residue field κ_ν, there is a degree -1 residue homomorphism $\delta_\nu : M(E) \to M(\kappa_\nu)$.*

Given any morphism $Y \to X$ of varieties over F, a cycle module over X can be restricted to a cycle module over Y. In general, a cycle module M can be viewed as a collection of functors

$$M_i : \mathbf{Fields}_{/F} \to \mathbf{Abelian\ Groups}.$$

This means we can take invariants of algebraic groups with values in cycle modules.

The central example of a cycle module is the Milnor $K-$groups $K_n(E)$ for all field extensions E/F; they form a cycle module over Spec(F). Another example, assuming $p \neq$ char(F), is the rule

$$E \mapsto H^{i+1}(E, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

which we can call a cohomological cycle module; when F is characteristic p, there are no longer residue homomorphisms, hence it is no longer a cycle module.

Cycle modules allow the definition of Chow groups with coefficients; if $F(x)$ is the residue field of a point $x \in X$, then denoting $M(F(x))$ as $M(x)$, the i^{th} homology of the complex $C^\cdot(X, M)$

$$\ldots \to \coprod_{x \in X^{(i-1)}} M_{d-i+1}(F(x)) \to \coprod_{x \in X^{(i)}} M_{d-i}(F(x)) \to \coprod_{x \in X^{(i+1)}} M_{d-i-1}(F(x)) \to \ldots$$

is $A^i(X, M_n)$, the Chow group with coefficients in M_n. This name comes from the fact that, when X is smooth, and our cycle module $M_* = K_*$ is Milnor's K-ring, then the group $A^p(X, K_p)$ is precisely the Chow group $CH^p(X)$ of codimension p cycles on X modulo rational equivalence. The one other case in which Chow groups with coefficients in K_* are straightforward to compute is $A^0(X, K_1) = F[X]^\times$, the group of invertible regular functions on X. Going forward, we will use the fact that Chow groups with coefficients are functorial and homotopy invariant.

2.6 K-cohomology of split tori and split simply connected algebraic groups

Let G be a connected algebraic group over F; remember that the character group G^* can be expressed as the subgroup of "group like elements" inside $F[G]^\times$. Now, by [22, Theorem 3]

$$F[G]^\times = F^\times \oplus G^\times \tag{2.1}$$

which means that every invertible regular function f on G such that $f(1) = 1$ is a character of G. If Y is a trivial $G-$torsor over F (which means Y has an $F-$point), then taking $y \in Y(F)$ and $h \in F[Y]^\times$, if we define

$$f(g) = h(gy) \cdot h(y)^{-1}$$

Given any other $y \in Y(F)$, $y' = g'y$ for some $g' \in G(F)$, and $h(gy') \cdot h(y')^{-1} =$

$h(gg'y) \cdot h(g'y)^{-1} = h(gg'y) \cdot h(y)^{-1} \cdot h(y) \cdot h(g'y)^{-1} = f(gg') \cdot f(g')^{-1} = f(g)$

So this rule gives us a homomorphism $F[Y]^\times \to G^*$, and so by 2.1, there is a split exact sequence,

$$1 \to F^\times \to F[Y]^\times \to G^* \to 0 \tag{2.2}$$

Now, even if Y isn't necessarily trivial, Y_{sep} is, so we get a homomorphism $F_{\mathrm{sep}}[Y_{\mathrm{sep}}]^\times \to G^*_{\mathrm{sep}}$, which is Galois-equivariant, and so taking the Galois invariant elements, we always get a homomorphism $F[Y]^\times \to G^*$, and so by Hilbert's Theorem 90,

Theorem 7. *For every G-torsor Y over F, the sequence* (2.2) *is exact.*

Corollary 8. *If G is a semisimple group, $F[Y]^\times = F^\times$ for every $G-$torsor Y over F.*

Because of their relatively elementary structure, it is straightforward to compute the $K-$cohomology of split tori.

Note that $A^*(X \times \mathbf{G}_m, K_*)$ is a bimodule over $A^*(X, K_*)$, and the projection $f : X \times \mathbf{G}_m \to \mathbf{G}_m$ is an invertible regular function on $X \times \mathbf{G}_m$, meaning it gives an element of $A^0(X \times \mathbf{G}_m, K_1) = F[X \times \mathbf{G}_m]^\times$.

Proposition 9. *The right $A^*(X, K_*)$-module $A^*(X \times \mathbf{G}_m, K_*)$ is free, with basis $\{1, f\}$.*

Proof. Take the closed embedding $i : X \to X \times \mathbb{A}^1_F$ given by $i(x) = (x, 0)$, and let $j : X \times \mathbf{G}_m \to X \times \mathbb{A}^1_F$ be the open embedding; then if we take the exact localization sequence,

$$\cdots \xrightarrow{i_*} A^*(X \times \mathbb{A}^1_F, K_*) \xrightarrow{j^*} A^*(X \times \mathbf{G}_m, K_*) \xrightarrow{\partial} A^*(X, K_{*-1}) \xrightarrow{i_*} \cdots$$

the connecting homomorphism ∂ is split by left multiplication by f [23, Rule R3d], so $i_* = 0$; by homotopy invariance [23, Proposition 8.6], this means that the projection $p : X \times \mathbb{A}^1_F \to X$ induces an isomorphism $p^* : A^*(X, K_*) \to A^*(X \times \mathbb{A}^1_F, K_*)$, so the image of j^* is precisely $1 \cdot A^*(X, K_*)$

□

Since a split torus is a product of copies of $\mathbf{G}_m$, and $A^*(\mathrm{Spec}(F), K_*) = K_*(F)$, induction gives the following:

Corollary 10. *Let $f_1, f_2, \ldots, f_m$ be a basis for T^*. Then*

- *$A^0(T, K_*)$ is a free $K_*(F)$-module with basis consisting of the elements $\{f_{i_1}, f_{i_2}, \ldots, f_{i_q}\} \in K_q(F)$ for each $q = 0, 1, \ldots, m$, and strictly increasing $q-$tuples of indices $i_1 < i_2 < \cdots < i_q$.*

- *$A^p(T, K_*) = 0$ whenever $p > 0$*

This completes our computation of the K-cohomology of split tori; now let's consider the K-cohomology of split simply connected groups.

Proposition 11. *If Y is a torsor for a simply connected semisimple group, then the chow groups $CH^1(Y)$ and $CH^2(Y)$ are always trivial.*

This means that, since $A^i(Y, K_i) = CH^i(Y)$, two of our groups are already computed. However, note that the simply connected assumption is critical here; if the character group of a maximal torus differs from the weight lattice, the Chow group will no longer be trivial.

Let G be a split simply connected semisiple group over F, and let T be a split maximal torus in G; call $X = G/T$, the variety of cosets. Then we have a canonical map $\pi : G \to X$. For any $x \in X(E)$ over an extension E/F, the coset $\pi^{-1}x \subset G$ is a trivial T-torsor over E. Then, if we consider the Rost spectral sequence [23, Sec. 8] associated to π, we have

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} A^q(\pi^{-1}x, K_{n-p}) \Rightarrow A^{p+q}(G, K_n)$$

But since the terms are $K-$cohomology of a trivial T-torsor, $E_1^{p,q} = 0$ whenever $q > 0$. This means that, if $M_n(x) = A^0(\pi^{-1}x, K_n)$ is a cycle module over X, we get an isomorphism $A^p(G, K_n) \simeq A^p(X, M_n)$.

Theorem 12. *[6, Theorem 3.7]*

Let G be a split simply connected group, $T \subset G$ a split maximal torus, and define $X = G/T$. For every $n \geq 0$ there is a spectral sequence

$$E_1^{p,q} = K_{n-q} \otimes CH^{p+q}(X) \otimes \Lambda^{-p} T^* \Rightarrow A^{p+q}(G, K_n)$$

This spectral sequence is trivial outside the triangle determined by $p \leq 0$, $p+q \geq 0$, and $n \geq q$.

This spectral sequence allows computation of $A^1(G, K_2)$.

Corollary 13. *For any split simply connected group G, There is a homomorphism ν_G : $A^1(G, K_2) \to S^2(T^*)$ which factors through $CH^1(X) \otimes T^*$, and which induces an isomorphism on its image, $S^2(T^*)^W$.*

2.7 Cohomological invariants of algebraic groups

Cohomological invariants of an algebraic group G with coefficients in $\mathbb{Q}/\mathbb{Z}(i)$ can be computed by computing motivic cohomology groups:

Theorem 14. *[2, Theorem 4.1] Let X be a smooth variety over F; then there is an exact sequence*

$$0 \to CH^2(X) \to H^4(X, \mathbb{Z}(2)) \to H^0_{Zar}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0$$

We won't define the motivic complex $\mathbb{Z}(2)$; rather, we'll just use several properties that it is known to have in order to compute cohomology with $\mathbb{Z}(2)$ coefficients in terms of K-cohomology.

Theorem 15. *[11, Theorem 1.1] Let G be a linear algebraic group over F (and if F is finite, assume G is connected). Let $E \to X$ be a classifying G-torsor with E a G-rational variety with a point over F. Then there is an isomorphism:*

$$Inv^n(G, \mathbb{Q}/\mathbb{Z}(i)) \simeq H^0_{Zar}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(i)))_{bal}$$

So, if we can understand the degree 4, weight 2 motivic cohomology of a classifying torsor of an algebraic group, we can gain a similar level of understanding of its degree 3, weight 2 cohomological invariants.

Lemma 16. *[2,]*

For an algebraic torus T, let U/T be a classifying torsor, and T° be the dual torus of T - a torus whose character group is the dual group of T^*. There is an exact sequence:

$$0 \to H^1(F, T^\circ) \to \overline{H}^4(U/T, \mathbb{Z}(2))_{bal} \to S^2(T^*_{sep})^\Gamma \to H^2(F, T^\circ)$$

In fact, we can remove $\overline{H}^4(U/T, \mathbb{Z}(2))_{\mathrm{bal}}$ from our reckoning altogether for tori, because T_{sep} is split, meaning that its invariants are trivial.

Lemma 17. *[2, Theorem 4.2]*

For a torus T and classifying torsor U/T, with classifying space BT, there are isomorphisms

$$\overline{H}^4((U/T)_{sep}, \mathbb{Z}(2))_{bal} \simeq CH^2(BT_{sep}) \simeq S^2(T^*_{sep})$$

Combining these facts, we get

Theorem 18. *[2, Theorem 4.3] Let T be an algebraic torus over a field F. Then there is an exact sequence*

$$0 \to CH^2(BT)_{tors} \to H^1(F, T^\circ) \to Inv^3(T, \mathbb{Q}/\mathbb{Z}(2))_{norm} \to S^2(T^*_{sep})^\Gamma/Dec \to H^2(F, T^\circ)$$

Where Dec is the subgroup of $S^2(T^*_{sep})^\Gamma$ of decomposable elements, generated by the image of the quadratic trace map $Qtr : T^\Gamma \to S^2(T)^\Gamma$

The same structure applies to computing the cohomological invariants of semisimple groups. If G is simply connected, and Λ_w is its weight lattice, then $\overline{H}^4(BG, \mathbb{Z}(2)) = Q(G)$, where $Q(G) = (S^2(\Lambda_w)^W)^\Gamma$. This leads to

Theorem 19. *[18, Theorem 3.9] Let G be a semisimple group over F. Let C be the kernel of the universal cover of G. Then there is an exact sequence*

$$0 \to CH^2(BG)_{tors} \to H^1(F, C^*(1)) \to Inv^3(G, \mathbb{Q}\mathbb{Z}(2))_{norm} \to Q(G)/Dec(G) \to H^2(F, C^*(1)).$$

CHAPTER 3

Degree Three Cohomological Invariants of Split Reductive Groups

By comparing cohomology groups of a connected, split reductive group G over a field F with the cohomology of a split maximal torus $T \subseteq G$, a borel subgroup B of G containing T, and the quotients of G by T and B, we can extend results on the invariants of split semisimple groups to split reductive groups. Further notation used in this chapter are W for the Weyl group of G, $H = [G, G]$ the commutator subgroup of G, $Q = G/H$ is a split torus, $\pi : \tilde{H} \to H$ a simply connected cover of H, $C = \mathrm{Ker}(\pi)$, Λ_w the weight lattice of $\tilde{H}$.

3.1 K-Cohomological Background

The kernel of the natural homomorphism $T^* \to \Lambda_w$ (which is 0 in the semisimple case) are the characters in T^* which are trivial when restricting to $\tilde{H}$, so it is isomorphic to Q^*. The cokernel of that natural map is isomorphic to the finite group C^*, just as it is in the semisimple case.

We can relate directly to the semisimple case by considering the smooth projective variety G/B, which is the flag variety for the simply connected group $\tilde{H}$. By [GMS, Part 2, Section 6], there is a natural isomorphism

$$\Lambda_w \to CH^1(G/B) \tag{3.1}$$

This isomorphism extends to a ring homomorphism (with the intersection product on the Chow Ring)

$$S^*(\Lambda_w) \to CH^*(G/B) \tag{3.2}$$

where S^* is the symmetric ring.

The following proposition summarizes existing results on the $K-$cohomology of G and quotients thereof.

Proposition 20. *Let G be a split reductive group, and D either be a split maximal torus $T \subseteq G$ or a Borel subgroup of G containing T. Then if $E \to Y$ is a G-torsor over a smooth variety Y,*

1. *The pull-back homomorphism $A^*(E/B, K_*) \to A^*(E/T, K_*)$ induced by the natural map $E/T \to E/B$ is a ring isomorphism.*
2. *For every smooth variety Z over F, the external product map gives an isomorphism*

$$A^*(Z, K_*) \otimes CH^*(G/D) \to A^*(Z \times (G/D), K_*)$$

3. *There is a natural isomorphism $\Lambda_w \to CH^1(G/D)$.*
4. *The kernel of the surjective homomorphism $S^2(\Lambda_w) \to CH^2(G/D)$ is equal to the group of W-invariant elements $S^2(\Lambda_w)^W$ in $S^2(\Lambda_w)$, so $CH^2(G/D) \simeq S^2(\Lambda_w)/S^2(\Lambda_w)^W$.*

Proof. 1. The fibers of $E \to E/B$ over a field K are B-torsors. B, as a Borel group, is a special group, which means all B-torsors over all fields are trivial. Thus, these fibers, being trivial B-torsors, are split and isomorphic to B_K. This means that the fibers of the natural morphism $E/T \to E/B$ over K are isomorphic to $(B/T)_K$, and so they are affine spaces over K. Then, by [EKM, Theorem 52.13], the Homotopy Invariance Property of K-cohomology says that the pull-back homomorphism is an isomorphism.

2. By part (1), we can treat G/B and G/T the same when it comes to K-cohomology, so we may as well assume $D = B$. In this case, G/B is a cellular variety, and so the statement follows by [6, Proposition 3.7]

3. The proof of this is above.

4. This is Theorem 6.7 and Corollary 6.12 of [7, Part 2]

□

G/T has a natural W-action, which means that $CH^i(G/T)$, and so by the proposition, $CH^i(G/D)$, are naturally W-modules. Moreover, all the maps we have mentioned so far are W-module homomorphisms.

Now, we can relate the K-cohomology of the base of a torsor to that of another variety.

Proposition 21. *Let $E \to Y$ be a G-torsor with Y a smooth variety, $D = B$ or $D = T$, and $f : X = E/D \to Y$ the induced morphism. Then*

1. *The natural homomorphism*

$$A^0(Y, K_2) \to A^0(X, K_2)$$

   is an isomorphism, and

2. *There is a natural complex*

$$0 \to A^1(Y, K_2) \to A^1(X, K_2) \to \Lambda_w \otimes F[Y]^\times \to 0$$

   which is acyclic whenever the torsor E is trivial.

Proof. By proposition 2(1), we can just consider the case when $D = B$. Rost's spectral sequence for the morphism f provides an exact sequence

$$0 \to A^0(X, K_2) \to \coprod_{y \in Y^{(0)}} A^0(X_y, K_2) \to \coprod_{y \in Y^{(1)}} A^0(X_y, K_1)$$

The fiber X_y is a projective homogeneous G-variety over the field $F(y)$, which means that the natural homomorphism $K_i(F(y)) \to A^0(X_y, K_i)$ is an isomorphism by [26, Corollary 5.6]. It follows that $A^0(X, K_2) \simeq A^0(Y, K_2)$. In the next degree, Rost's spectral sequence for F gives the exact sequence

$$A^1(Y, K_2) \to A^1(X, K_2) \to \coprod_{y \in Y^{(0)}} A^1(X_y, K_2) \to \coprod_{y \in Y^{(1)}} A^1(X_y, K_1)$$

Because X_y is a projective homogeneous G-variety over $F(y)$, then the group $A^1(X_y, K_i)$ is canonically identified with a subgroup of $CH^i(G/B) \otimes K_{i-1}(F(y)) = \Lambda_w \otimes K_{i-1}(F(y))$ when $i \leq 2$, which means that there is a natural map from the kernel of $\delta : \coprod_{y \in Y^{(0)}} A^1(X_y, K_2) \to \coprod_{y \in Y^{(1)}} A^1(X_y, K_1)$ to $A^0(Y, \Lambda_w \otimes K_1) = \Lambda_w \otimes A^0(Y, K_1) = \Lambda_w \otimes F[Y]^\times$, which gives our desired map $\alpha : A^1(X, K_2) \to \Lambda_w \otimes F[Y]^\times$.

If E is a trivial torsor, $E \simeq Y \times G$, so $X \simeq Y \times (G/D)$; so by Proposition 2, $A^1(X, K_2) \simeq A^1(Y, K_2) \oplus (\Lambda_w \otimes F[Y]^\times)$.

$\square$

Note that the projection of $A^1(X, K_2)$ onto $\Lambda_w \otimes F[Y]^\times$ above is equal to the map α.

3.1.1 Classifying Spaces

Let G be an algebraic group; the Chow ring $CH^*(BG)$ of the classifying space of G was defined by Totaro in [27], and extended to Chow rings with coefficients by Guillot in [9] as follows. Fix an integer $i \geq 0$ and choose a generically free representation V of G such that there is a $G-$equivariant open subset $U \subset V$ with the property $\mathrm{codim}_V(V \setminus U) \geq i + 1$ and a versal G-torsor $f : U \to U/G$. Then we define $A^i(BG, K_*) := A^i(U/G, K_*)$, and the definition is independent of the choice of V and U.

Because the fibers of U/T and U/B are affine spaces, the homotopy invariance property implies that $A^i(BB, K_j) \simeq A^i(BT, K_j)$, and then by the Kunneth formula, $A^i(BT, K_j) \simeq S^i(T^*) \otimes K_{j-i}(F)$.

3.2 Motivic Cohomology of Weight at Most 2

In order to connect K-cohomology to cohomological invariants, we look at motivic cohomology.

Proposition 22. *Let X be a smooth projective rational variety. Then the natural homomorphism*

$$H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \to H^0_{Zar}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n_{nr}(F(X), \mathbb{Q}/\mathbb{Z}(j))$$

is an isomorphism.

Proof. This statment is well known (see [4, Theorem 4.1.5]) without the p-primary component of $\mathbb{Q}/\mathbb{Z}(j)$ when $\mathrm{char}(F) = p > 0$; here is a proof that works in arbitrary cases.

Proceed by inducting on $dim(X)$. Because $H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ is a birational invariant, we can assume $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$. Take an element $\alpha \in H^0_{\mathrm{Zar}}(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$; pulling back with respect to the morphism $\mathbb{P}^{n-1}_{F(\mathbb{P}^1)} \to X$, we get $\alpha \in H^0_{\mathrm{Zar}}(\mathbb{P}^{n-1}_{F(\mathbb{P}^1)}, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n(F(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}(j))$ by the induction hypothesis. Then the result follows by [2, Proposition 5.1]. □

Because we have this, we can show that the corresponding etale sheaves are trivial.

Proposition 23. *Let $E \to Y$ be a G-torsor and $f : X = E/B \to Y$ be the natural morphism; then the etale sheaf associated with the presheaf on Y*

$$Z \mapsto H^0_{Zar}(f^{-1}(Z), \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$$

is trivial for $n > 0$.

Proof. Because the G-torsor f is locally trivial in the etale topology, we can assume that the torsor E is trivial, so $f^{-1}(Z) \simeq Z \times (G/B)$. Then, the pull-back homomorphism with respect to the morphism $(G/B)_{F(Z)} \to f^{-1}(Z)$ is an injection:

$$H^0_{\mathrm{Zar}}(f^{-1}(Z), \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \to H^0_{\mathrm{Zar}}((G/B)_{F(Z)}, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$$

Then by proposition 22, because G/B is a smooth, projective variety,

$H^0_{\mathrm{Zar}}((G/B)_{F(Z)}, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n(F(Z), \mathbb{Q}/\mathbb{Z}(j))$; but the etale sheaf associated with the presheaf

$$Z \mapsto H^n(F(Z), \mathbb{Q}/\mathbb{Z}(j))$$

is trivial for $n > 0$. $\square$

Now, for an arbitrary smooth variety X over F, consider the *motivic complexes* $\mathbb{Z}_X(j)$ *of weight* j=0, 1, and 2 in the category $D^+\mathrm{Sh}_{\mathrm{et}}(X)$. The complex $\mathbb{Z}_X(0)$ is $\mathbb{Z}$ concentrated in degree 0, and $\mathbb{Z}(1) = \mathbb{G}_m[-1]$. Note that we will write $H^{n,j}(X)$ to mean $H^n_{\mathrm{et}}(X, \mathbb{Z}(j))$. Kahn showed that $H^{0,2}(X) = 0$, $H^{1,2}(X) = K_3(F(X))_{\mathrm{ind}}$, the cokernel of the map from Milnor K_3 to Quillen K_3; $H^{2,2}(X) = A^0(X, K_2)$, and $H^{3,2}(X) = A^1(X, K_2)$.

When $E \to Y$ is a G-torsor with Y a smooth variety, and $f : X = E/D \to Y$ is the induced morphism, then we write $\mathbb{Z}_f(2)$ for the cone of the natural morphism $\mathbb{Z}_Y(2) \to Rf_*(\mathbb{Z}_X(2))$ in the category $D^+\mathrm{Sh}_{\mathrm{et}}(Y)$. We can compute $\mathcal{H}^n(\mathbb{Z}_f(2))$ for small values of n, which connects $\mathbb{Z}_X(2)$ and $\mathbb{Z}_Y(2)$.

Proposition 24. *Let G be a split reductive algebraic group over F, D either a maximal split torus T of G or a Borel subgroup B of G, and Λ_w the weight lattice of the commutator subgroup of G; then if $E \to Y$ is a G-torsor with Y a smooth variety, and $f : X = E/D \to Y$ is the induced morphism, then $H^n(\mathbb{Z}_f(2)) = 0$ for $n \leq 2$, and $H^3(\mathbb{Z}_f(2)) = \Lambda_w \otimes \mathbb{G}_m$. Further, there is an exact sequence of etale sheaves on Y,*

$$0 \to [S^2(\Lambda_w)/S^2(\Lambda_w)^W] \otimes \mathbb{Z}_Y \to \mathcal{H}^4(\mathbb{Z}_f(2)) \to L \to 0$$

where L is the etale sheaf on Y associated to the presheaf

$$Z \mapsto H^0_{Zar}(f^{-1}(Z), \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$$

and L is trivial when $D = B$.

Proof. The complex $\mathbb{Z}(2)$ is supported in degrees 1 and 2, so we immediately get triviality for the cases of $n < 0$.

$\mathbb{Z}_f(2)$ is defined via an exact triangle

$$\mathbb{Z}_Y(2) \to Rf_*(\mathbb{Z}_X(2)) \to \mathbb{Z}_f(2) \to \mathbb{Z}_Y(2)[1] \tag{3.3}$$

which gives an exact sequence in homology, and thus isomorphisms $R^n f_*(\mathbb{Z}_X(2)) \simeq \mathcal{H}^n(\mathbb{Z}_f(2))$ for $n \geq 3$. $\mathcal{H}^1(\mathbb{Z}_Y(2))$ is the etale sheaf associated to the presheaf $Z \mapsto K_3F(Z)_{\text{ind}}$, and $R^1 f_*(\mathbb{Z}_X(2))$ is the etale sheaf associated to the presheaf $Z \mapsto K_3F(f^{-1}Z)_{\text{ind}}$. Now, assume for the moment that the torsor $E \to Y$ is trivial; G/D is rational, so the natural homomorphism $K_3F(Z)_{\text{ind}} \to K_3F(f^{-1}Z)_{\text{ind}}$ is an isomorphism by [MS, Lemma 4.2], because the field extension $F(f^{-1}Z)/F(Z)$ is purely transcendental. Thus the morphism of homology sheaves $\mathcal{H}^1(\mathbb{Z}_Y(2)) \to R^1 f_*(\mathbb{Z}_X(2))$ is an isomorphism, since our torsor is locally trivial in the etale topology.

Now, consider $n = 2$. We have seen that $\mathcal{H}^2(\mathbb{Z}_Y(2))$ is the etale sheaf associated to the presheaf $Z \mapsto A^0(Z, K_2)$, and $R^2 f_*(\mathbb{Z}_X(2))$ is the etale sheaf associated to the presheaf $Z \mapsto A^0(f^{-1}Z, K_2)$. Now, we know $A^0(X, K_2) \simeq A^0(Y, K_2)$, so that means $\mathcal{H}^2(\mathbb{Z}_f(2)) = 0$ when $n \leq 2$.

For $n = 3$, we have seen that $R^3 f_*(\mathbb{Z}_X(2)) \simeq \Lambda_w \otimes \mathbb{G}_m$, so $\mathcal{H}^3(\mathbb{Z}(2)) \simeq \Lambda_w \otimes \mathbb{G}_m$.

Finally, we take $n = 4$. Again, $\mathcal{H}^4(\mathbb{Z}_f(2)) = R^4 f_*(\mathbb{Z}_X(2))$; these are the etale sheaf on Y associated to the presheaf $Z \mapsto H^{4,2}(f^{-1}Z)$.

We'll compare it to a sheaf M on Y associated to the presheaf $Z \mapsto CH^2(f^{-1}Z)$. Let z be a generic point of Z and L an algebraic closure of $F(z)$. The fiber $f^{-1}(z)$ is split over L, which means it is isomorphic to $(G/D)_L$; then we get a morphism from M to the constant sheaf $[S^2(\Lambda_w)/S^2(\Lambda_w)^W] \otimes \mathbb{Z}_Y$ from the composition $CH^2(f^{-1}Z) \to CH^2(f^{-1}(z)) \to CH^2(G/D)_L$.

This morphism is an isomorphism; assume E is trivial over Z, meaning $f^{-1}Z \simeq Z \times (G/D)$; then $CH^2(f^{-1}Z) \simeq CH^2(Z) \oplus (CH^1(Z) \otimes CH^1(G/D)) \oplus (CH^0(Z) \otimes CH^2(G/D))$; projecting onto the last summand is precisely this morphism. But the sheaves associated to the presheaves $Z \mapsto CH^i(Z)$ are trivial for $i > 0$.

Now, M is a subsheaf of $\mathcal{H}^4(\mathbb{Z}_f(2))$ by [11, Theorem 1.1], and the factor sheaf is the sheaf associated to the presheaf $Z \mapsto H^0_{\mathrm{Zar}}(f^{-1}(Z), \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$, which we saw in Proposition 5 is trivial in the case $D = B$. $\square$

Theorem 25. *Let G be a split reductive group over a field F, W its Weyl group, B a Borel subgroup, $E \to Y$ a G-torsor with Y a smooth connected variety, and $f : X = E/B \to Y$ the induced morphism. Then there are exact sequences of W-modules,*

$$0 \to A^1(Y, K_2) \to A^1(X, K_2) \to \Lambda_w \otimes F[Y]^\times \to H^{4,2}(Y) \to H^{4,2}(X) \to H^4(Y, \mathbb{Z}_f(2))$$

and

$$0 \to \Lambda_w \otimes CH^1(Y) \to H^4(Y, \mathbb{Z}_f(2)) \to S^2(\Lambda_w)/S^2(\Lambda_w)^W \to \Lambda_w \otimes Br(Y)$$

Proof. Let D be either B or T, and $g : E/D \to Y$ the induced morphism; since $\mathcal{H}^n(\mathbb{Z}_g(2)) = 0$ for $n \leq 2$, and $\mathcal{H}^3(\mathbb{Z}_g(2)) = \Lambda_w \otimes \mathbb{G}_m$, then we get an exact triangle in $D^+\mathrm{Sh}_{\mathrm{et}}(Y)$,

$$\Lambda_w \otimes \mathbb{G}_m[-3] \to \tau_{\leq 4}\mathbb{Z}_g(2) \to \mathcal{H}^4(\mathbb{Z}_g(2))[-4] \to \Lambda_w \otimes \mathbb{G}_m[-2] \tag{3.4}$$

where $\tau_{\leq 4}$ truncates above degree 4. Applying cohomology to this triangle, we get a diagram with exact rows, and vertical maps induced by the morphism $E/T \to E/B$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_w \otimes CH^1(Y) & \longrightarrow & H^4(Y, \mathbb{Z}_f(2)) & \longrightarrow & H^0(Y, \mathcal{H}^4(\mathbb{Z}_f(2))) & \longrightarrow & \Lambda_w \otimes Br(Y) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda_w \otimes CH^1(Y) & \longrightarrow & H^4(Y, \mathbb{Z}_h(2)) & \longrightarrow & H^0(Y, \mathcal{H}^4(\mathbb{Z}_h(2))) & \longrightarrow & \Lambda_w \otimes Br(Y) \end{array}$$

Where $f = g$ in the case $D = B$, and $h = g$ in the case $D = T$. In this latter case, there is a natural W-action on X such that h is W-equivariant (with W acting trivially on Y), so W acts on the complex $\mathbb{Z}_h(2)$, and so the bottom sequence of the diagram is a sequence of W-module homomorphisms. The third vertical map is injective by Proposition 24, and thus by the 5-Lemma, all the vertical maps are injective; the top row is thus also a sequence of W-equivariant homomorphisms. This gives the second sequence in the statement of the theorem.

Applying the cohomology functor to the defining exact triangle of $\mathbb{Z}_g(2)$, and applying the equalities derived so far, we again get the sequence

$$0 \longrightarrow A^1(Y, K_2) \longrightarrow A^1(E/D, K_2) \longrightarrow \Lambda_w \otimes F[Y]^\times \longrightarrow$$

$$H^{4,2}(Y) \longrightarrow H^{4,2}(E/D) \longrightarrow H^4(Y, \mathbb{Z}_g(2)) \longrightarrow H^{5,2}(Y)$$

When $D = B$, this gives the first exact sequence of the theorem; but if we compare the sequences we get here for $D = T$ and $D = B$ via the morphism $E/T \to E/B$, we get a commutative diagram as above, and we already know the sequence for $D = T$ is a sequence of W-module homomorphisms; thus by the 5-Lemma, so is the sequence for $D = B$.

□

3.3 Cohomology of the classifying space

By approximating it via versal torsors, we can now compute the motivic cohomology of the classifying space BG.

Theorem 26. *Let G be a split reductive group over F, $T \subset G$ a split maximal torus, and C the kernel of the universal cover of the commutator subgroup of G. Then there is an exact sequence*

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(BG) \longrightarrow S^2(T^*)^W \longrightarrow 0$$

Proof. Applying Theorem 7 to the group $D = B$ and the versal G-torsors $U^n \to U^n/G$ for all n, we get exact sequences of W-modules,

$$A^1(U^n/B, K_2) \longrightarrow \Lambda_w \otimes F[U^n/G]^\times \longrightarrow$$

$$H^{4,2}(U^n/G) \longrightarrow H^{4,2}(U^n/B) \longrightarrow H^4(U^n/G, \mathbb{Z}_f(2))$$

Now, $A^1(U^n/B, K_2) = A^1(BB, K_2) = T^* \otimes F^\times$, and note $F^\times \subset F[U^n/G]^\times \subset F[U^n]^\times = F[V^n]^\times = F^\times$ by the codimension requirements of U in V, so $F[U^n/G]^\times = F^\times$. Thus, the cokernel of the first homomorphism in the exact sequence above is isomorphic to $C^* \otimes F^\times$ since $C^\times$ is the cokernel of the natural map $T^* \to \Lambda_w$.

Because B is special, every invariant of B is constant, so $\overline{H}^{4,2}(BG) \simeq \mathrm{CH}^2(BB) \simeq S^2(T^*)$; taking the balanced elements in the exact sequence of cosimplicial groups above, we get a sequence of $W-$module homomorphisms

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(BG) \longrightarrow \overline{H}^{4,2}(BB) \longrightarrow H^4(U/G, \mathbb{Z}_f(2))$$

where f is the map $f : U \to U/G$. This sequence is exact, since the first term is a constant cosimplicial group.

W acts trivially on $\overline{H}^{4,2}(BG)$; then if we take the $W-$invariant elements above, and use the equality $\overline{H}^{4,2}(BB) = S^2(T^*)$, we get the exact sequence

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow \overline{H}^{4,2}(BG) \longrightarrow S^2(T^*)^W \longrightarrow H^4(U/G, \mathbb{Z}_f(2))^W$$

and so if we can show that the last term is trivial, we have proven the theorem.

The second sequence in Theorem 25 is

$$0 \longrightarrow \Lambda_w \otimes Q^* \longrightarrow H^4(U/G, \mathbb{Z}_f(2)) \longrightarrow S^2(T^*)/S^2(T^*)^W$$

with $Q^* = G^* = CH^1(BG)$. Because W acts transitively on Λ_w and trivially on Q^*, $(\Lambda_w \otimes Q^*)^W = 0$; since $(S^2(T^*)/S^2(T^*)^W)^W = 0$, we can conclude $H^4(U/G, \mathbb{Z}_f(2))^W = 0$. $\square$

3.4 Degree 3 Invariants

For a split reductive group G over F, and $T \subseteq G$ a split maximal torus, we have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & C^* \otimes F^\times \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & CH^2(BG) \arrow[r] \arrow[dr, "\gamma"] & \overline{H}^{4,2}(BG) \arrow[r] \arrow[d] & \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] & 0 \\
 & & S^2(T^*)^W \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The image of γ is the subgroup $\mathrm{Dec}(G)$ of "obvious" elements in $S^2(T^*)^W$, generated by all elements of either the form $\sum_{i<j} x_i x_j$ with $\{x_i\}$ the $W-$orbit of a character in T^*, or xy with $x, y \in (T^*)^W = Q^*$.

Theorem 27. *Let G be a split reductive group, $T \subset G$ a split maximal torus and C the kernel of the universal cover of the commutator subgroup of G. Then there is an exact sequence*

$$0 \longrightarrow C^* \otimes F^\times \longrightarrow Inv^3(G, \mathbb{Q}/\mathbb{Z}(2))_{norm} \longrightarrow S^2(T^*)^W/Dec(G) \longrightarrow 0$$

Proof. Exactness at the middle and righthand terms follow by diagram chases. To show that the first homomorphism is injective, let H be the commutator subgroup of G; then as a semisimple group, we get that the composition

$$C^* \otimes F^\times \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))$$

is injective, which gives the injectivity of the first morphism. □

We saw that the group of normalized Brauer invariants of G,

$$\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} = \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$$

is isomorphic to $\mathrm{Pic}(G) = C^*$; the first homomorphism in the exact sequence of the theorem is given by the cup product, and its image consists of the decomposable invariants.

Writing $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ for the factor group of $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ by the subgroup of decomposable invariants, we get a natural isomorphism

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \simeq S^2(T^*)^W/\mathrm{Dec}(G) \tag{3.5}$$

3.5 Reductive Invariants

If G is a split reductive group, and H its commutator subgroup, we can consider the restriction homomorphism

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)).$$

We will also use the *polar* homomorphism,

$$\mathrm{pol} : S^2(\Lambda_w) \to \Lambda_w \otimes \Lambda_w, xy \mapsto x \otimes y + y \otimes x$$

$\mathrm{pol}(S^2(\Lambda_w)^W)$ is contained in $\Lambda_w \otimes \Lambda_r$; the embedding of Λ_r into Λ_w factors through T^*. If α is the composition

$$S^2(\Lambda_w)^W \to (\Lambda_w \otimes \Lambda_r)^W \to (\Lambda_w \otimes T^*)^W$$

and S is a split maximal torus of H contained in T, then there is a commutative diagram,

$$\begin{tikzcd}
S^2(S^*)^W \arrow[rr, "\mathrm{pol}"] \arrow[d, hook] & & S^* \otimes S^* \arrow[d, hook] \arrow[dr, hook] & \\
S^2(\Lambda_w)^W \arrow[r, "\alpha"] & (\Lambda_w \otimes T^*)^W \arrow[r, "\beta"] & \Lambda_w \otimes S^* \arrow[r, hook] & \Lambda_w \otimes \Lambda_w
\end{tikzcd}$$

Then, if $Q = G/H = T/S$, then the kernel of the homomorphism $\Lambda_w \otimes T^* \to \Lambda_w \otimes S^*$ is $\Lambda_w \otimes Q^*$. Since $(\Lambda_w \otimes Q^*)^W = 0$, β is injective, which gives the commutative diagram

$$\begin{tikzcd}
 & S^2(S^*)^W \arrow[dl] \arrow[dr] & \\
S^2(\Lambda_w)^W \arrow[dr, "\alpha"] & (S^* \otimes T^*)^W \arrow[r, hook] \arrow[d, hook] & S^* \otimes S^* \arrow[d, hook] \\
 & (\Lambda_w \otimes T^*)^W \arrow[r, hook] \arrow[d, two heads] & \Lambda_w \otimes S^* \arrow[d, two heads] \\
C^* \otimes Q^* \arrow[r, hook] & C^* \otimes T^* \arrow[r, two heads] & C^* \otimes S^*
\end{tikzcd}$$

with vertical exact sequences; note that C^* is both Λ_w/S^* and the character group of the kernel C of a universal cover $\tilde{H} \to H$.

Lemma 28. *An element $u \in S^2(S^*)^W$ belongs to the image of $S^2(T^*)^W \to S^2(S^*)^W$ if and only if pol(u) belongs to the image of $(S^* \otimes T^*)^W \to S^* \otimes S^*$.*

Proof. Let X be the kernel of the natural homomorphism $S^2(T^*) \to S^2(S^*)$. Mapping via the polar map and then pulling back, we get an exact sequence

$$0 \to S^2(Q^*) \to X \to S^* \otimes Q^* \to 0$$

Since W acts trivially on the first term, $H^1(W, S^2(Q^*)) = 0$. Thus $H^1(W, X) \to H^1(W, S^* \otimes Q^*)$ is injective, which gives the lemma by diagram chase.

□

Then, combining the lemma and the larger diagram, we get the following:

Proposition 29. *The sequence*

$$S^2(T^*)^W \to S^2(S^*)^W \to C^* \otimes Q^*$$

is exact.

The first homomorphism takes $\mathrm{Dec}(G)$ surjectively onto $\mathrm{Dec}(H)$; so the composition is trivial. Then, applying the previous theorem, we get a homomorphism $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2)) \to C^* \otimes Q^*$.

This gives

Theorem 30. *Let G be a split reductive group, $H \subset G$ the commutator subgroup, $Q = G/H$ and C the kernel of the unversal cover $\tilde{H} \to H$. Then the sequence*

$$0 \to Inv^3(G, \mathbb{Q}/\mathbb{Z}(2)) \to Inv^3(H, \mathbb{Q}/\mathbb{Z}(2)) \to C^* \otimes Q^*$$

is exact.

From this theorem, we get

Corollary 31. *If the commutator subgroup H is either simply connected or adjoint, then the restriction homomorphism $Inv^3(G, \mathbb{Q}/\mathbb{Z}(2)) \to Inv^3(H, \mathbb{Q}/\mathbb{Z}(2))$ is an isomorphism.*

This comes because either H is simply connected, and thus $C^* = 0$, or H is adjoint, which means that the surjection $T^* \to S^*$ is split by $\Lambda_r \to T^*$, which means that $S^2(T^*)^W \to S^2(S^*)^W$ is surjective.

Now, if we go the other direction and start with a split semisimple group H, and take G a strict reductive envelope of H, then the image of the injective map

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \to \mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$$

which we'll call $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$, is independent of the choice of G; this is the subgroup of reductive indecomposable invariants of H.

We can make specific quantitative arguments about the group of reductive invariants using the following proposition:

Proposition 32. *Let $q = \sum_j k_j q_j \in S^2(S^*)^W \subseteq S^2(\Lambda_w)^W$ with $k_j \in \mathbb{Z}$. Let I be the element of $Inv^3(H, \mathbb{Q}/\mathbb{Z}(2))_{ind}$ corresponding to q. Then I is a reductive indecomposable invariant if and only if the order of $\overline{w_{ij}}$ in C^* divides $d_{ij}k_j$ for all i and j.*

Proof. The composition $S^2(S^*)^W \to C^* \otimes Q^* \to C^* \otimes T^*$ factors into the composition

$$S^2(S^*)^W \longrightarrow S^2(\Lambda_w)^W \xrightarrow{\text{pol}} \Lambda_w \otimes \Lambda_r \longrightarrow C^* \otimes \Lambda_r \longrightarrow C^* \otimes T^*$$

Because $G$is strict, Λ_r is a direct summand of T^*, so the last map in this sequence is injective; so the sequence

$$S^2(T^*)^W \longrightarrow S^2(S^*)^W \xrightarrow{\theta'} C^* \otimes \Lambda_r$$

is exact. Thus, by theorem 30, I is reductive indecomposable if and only if q belongs to the kernel of θ'; the polar form of q_j is equal to

$$\sum_i d_{ij} w_{ij} \otimes \alpha_{ij} \in \Lambda_w \otimes \Lambda_r$$

The roots α_{ij} form a $\mathbb{Z}$-basis for Λ_r, so q belongs to the kernel of θ' if and only if the order of $\overline{w_{ij}}$ in C^* divides $d_{ij}k_j$ for all i and j.

□

The A_{n-1} Case: the group $G = \mathbf{GL}_n/\mu_m$ is a strict envelope of $H = \mathbf{SL}_n/\mu_m$. A G-torsor over a field K corresponds to a central simple algebra A of degree n over K and exponent dividing m. This means that every reductive indecomposable invariant of H is an invariant of such an algebra. I claim such invariants are all trivial; the argument is by considering the $p-$primary component of $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}}$. Let r be the largest power of p dividing m. The kernel of the natural homomorphism $\mathbf{GL}_n/\mu_r \to \mathbf{GL}_n/\mu_m$ is finite, and of degree prime to p. Then, by [19, Proposition 7.1], the p-primary components of the groups of degree three invariants of H and $\mathbf{sL}_n/\mu_r$ are isomorphic, so we can assume that m is a power of p.

If q is the canonical generator of $S^2(S^*)^W$, then if $\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}} \neq 0$, $mq \in \mathrm{Dec}(H)$. But, if I is a reductive indecomposable invariant of H corrispoinding to a multiple kq of q, k must be divisible by the order of the first fundamental weight in $C^* = \mathbb{Z}/m\mathbb{Z}$, which is m; so m divides k, so $kq \in \mathrm{Dec}(H)$, so I is trivial.

This means that any invariant of A, being decomposable, is equal to $[A] \cup (x) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for some $x \in F^\times$; so central simple algebras of fixed degree and exponent have no nontrivial indecomposable degree three invariants.

The D_n Case: If H is the special orthogonal group $\mathbf{O}^+_{2n}$, then $\mathrm{Inv}^3_{\mathrm{ind}}(H, \mathbb{Q}/\mathbb{Z}(2)) = 0$, so we just need to consider $H = \mathbf{HSpin}_{2n}$, the half-spin group with $n \geq 4$ even. In general,

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = \begin{cases} 0 & \text{if } n \equiv 2 \text{ modulo } 4 \text{ or } n = 4 \\ 2\mathbb{Z}q/4\mathbb{Z}q & \text{if } n \equiv 4 \text{ modulo } 8 \text{ and } n \neq 4 \\ \mathbb{Z}q/4\mathbb{Z}q & \text{if } n \equiv 0 \text{ modulo } 8 \end{cases},$$

where q is the canonical generator of $S^2(\Lambda_w)^W$. The orders of the fundamental weights in $C^* = \mathbb{Z}/2\mathbb{Z}$ are either 1 or 2; so by proposition 14, we get that

$$\mathrm{Inv}^3(H, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{red}} = \begin{cases} 0 & \text{if } n \equiv 2 \text{ modulo } 4 \text{ or } n = 4 \\ 2\mathbb{Z}q/4\mathbb{Z}q & \text{if } n \equiv 0 \text{ modulo } 4 \text{ and } n > 4 \end{cases}.$$

CHAPTER 4

The Relative Motivic Complex

In the previous chapter, we utilized the weight two relative motivic complex $\mathbb{Z}_f(2)$ associated to the map $f : E/B \to Y$, where $E \to Y$ was a G-torsor and B was a split Borel subgroup of G; this allowed us to compare the invariants of split reductive groups with the invariants of semisimple groups. However, there is another relative motivic complex even more closely tied to the motivic cohomology of G, and that is $\mathbb{Z}_f(2)$ where $f : X \to Y$ is a $G-$torsor. This complex is important because, if $f : X \to Y$ is a versal torsor, then there is an isomorphism

$$H^3(Y, \mathbb{Z}_f(2)) \to \overline{H}^4(Y, \mathbb{Z}(2))_{\mathrm{bal}} \tag{4.1}$$

but this latter cohomology group is precisely what is meant by $H^4(BG, \mathbb{Z}(2))$, which is the cohomology group needed in order to compute $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$.

4.1 Motivation

When G is an arbitrary reductive group, the complex $\mathbb{Z}_f(2)$ for a general G-torsor $f : X \to Y$ still has not been computed. However, there is a conjectured value for the truncated complex $\tau_{\leq 3}\mathbb{Z}_f(2)$ which is appealing both because it specializes to the known results in the cases of a torus $G = T$, of a semisimple group $G = H$, and as computed in the previous chapter, of a split reductive group.

Further, when we restrict the torsors under consideration to just torsors $f : X \to K$ over fields, the conjectured value indeed holds; this case, while it cannot be used to compute the motivic cohomology of an arbitrary G-torsor (and thus the motivic cohomology of the classifying space of G), is of independent interest. The lower-weight version, $\tau_{\leq 2}\mathbb{Z}_f(1)$ was defined by Borovoi and van Hamel in [3] in the case of algebraically closed fields of characteristic zero; they called it the *extended Picard complex*, because its first cohomology is the Picard group of X; its zeroth cohomology is $K_{sep}[X]^\times/K_{sep}^\times$. These are each important Galois modules associated to X, and the extended Picard complex contains more information than the two of them separately.

Definition. *Let G be a reductive group with maximal torus T; let H^{sc} be the simply connected universal cover of $H = [G,G]$, and S^{sc} the preimage of $S = T \cap H$ in H^{sc}; it is a maximal torus in H^{sc}. Then the complex $N(G)$ of Galois modules is defined as the induced map*

$$T^*_{sep} \to (H^{sc}_{sep})^*$$

Note that the terms in this complex depend on our choice of T; it turns out that, as a complex in the derived category of Galois modules, $N(G)$ is independent of that choice. I will prove this by showing that $N(G)$ is in fact the extended Picard complex, but one could prove the independence directly, by constructing a quasi-isomorphism with the complex where the Galois group permutes the fundamental weights by the action on the Dynkin diagram of G, also known as the $*-$action. As a result, instead of writing $(H^{sc}_{sep})^*$, we can instead choose the copy in which it is naturally isomorphic to the weight lattice Λ_w.

The complex $N(G)$ is noteworthy, aside from our applications, because it succinctly captures the combinatorial root datum of the group G; it is a natural thing to study if one wants to understand the algebraic structure of G, and simultaneously, its cohomology groups express information about the geometry of G.

There is one other complex of lattices that we need to consider in order to understand the relative motivic complex.

Definition. *Let G be a reductive group with maximal torus T; let H^{sc} be the simply connected universal cover of $H = [G,G]$, and S^{sc} the preimage of $S = T \cap H$ in H^{sc}; it is a maximal torus in H^{sc}. Let $Z = G/T = H^{sc}/S^{sc}$; then $S^2((S^{sc})^*)/S^2((S^{sc})^*)^W \simeq CH^2(Z)$; let $\alpha : T^*_{sep} \otimes (S^{sc}_{sep})^* \to CH^2(Z_{sep})$ be the induced map.*

Then $D(G)$ is the complex of Galois modules

$$\Lambda^2(T^*_{sep}) \to ker(\alpha)$$

obtained by truncating the sequence

$$\Lambda^2(T^*_{sep}) \to T^*_{sep} \otimes (S^{sc}_{sep})^* \to CH^2(Z)$$

Again, this is a complex defined in terms of the combinatorial data of G, but now each of the terms is two dimensional.

4.2 The Extended Picard Complex

While we cannot compute the extended Picard complex $\tau_{\leq 2}\mathbb{Z}_f(1)$ for arbitrary G-torsors, we can when it is $f : X \to \mathrm{Spec}(F)$. This method is new, and the computation is new outside of the algebraically closed characteristic zero case.

4.3 Defining N(G)

Let G be a connected reductive group over F, and $H = [G, G]$ its semisimple commutator subgroup and H^{sc} the simply connected universal cover of H; then

$$\rho : H^{sc} \to H \to G$$

is known as Deligne's homomorphism. Given a maximal torus T of G defined over F, let $S := T \cap H$ be the corresponding maximal torus of H, and let $T^{sc} := \rho^{-1}(T)$ be the corresponding maximal torus of H^{sc}. Then ρ induces a complex

$$T^{sc}_{sep} \xrightarrow{\rho} T_{sep}$$

Taking $Hom(-, \mathbb{G}_m)$ of this complex, we get the complex $N(G_{sep}) = [T^*_{sep} \xrightarrow{\rho^*} (T^{sc}_{sep})^*\rangle$; the individual terms depend on the choice of T, but the complex is determined in the derived category of discrete Galois modules.

For any smooth variety X over F, let $\mathbb{Z}_X(1)$ be the motivic complex of consisting of the étale sheaf over X $\mathbb{G}_m$ in degree 1.

For any $i \geq 0$, the homology of the complex $C^\bullet(X, K_d)$:

$$\ldots \to \coprod_{x \in X^{(i-1)}} K_{d-i+1}(F(x)) \xrightarrow{\delta} \coprod_{x \in X^{(i)}} K_{d-i}(F(x)) \xrightarrow{\delta} \coprod_{x \in X^{(i+1)}} K_{d-i-1}(F(x)) \to \ldots$$

Where $X^{(i)}$ is the codimension i points in X, is denoted $A^i(X, K_d)$, as in [23]; in particular, when $i = d$, this is $CH^i(X)$.

$C^\bullet(X, K_d)$ is a nontrivial in finitely many terms - specifically, when $0 \leq i \leq d$. Additionally, there is a natural complex morphism $K_d(F) \to C^\bullet(X, K_d)$ induced by the map $K_d(F) \to K_d(F(X))$; call the cone of this map $\overline{C}^\bullet(X, K_d)$. Homology of $\overline{C}^\bullet(X, K_d)$ differs from that of $C^\bullet(X, K_d)$ at degree 0; there, it is denoted $\overline{A}^0(X, K_d)$.

Let $f : X \to \mathrm{Spec}(F)$ be a torsor for G; then define the weight 1 relative motivic complex for f to be $\mathbb{Z}_f(1)$, the cone of the natural morphism

$$\mathbb{Z}_F(1) \to Rf_*(\mathbb{Z}_X(1)) \tag{4.2}$$

in the derived category of étale sheaves over F.

4.3.1 Cohomology of $\mathbb{Z}_f(1)$

Since there is an exact triangle

$$\mathbb{Z}_F(1) \to Rf_*(\mathbb{Z}_X(1)) \to \mathbb{Z}_f(1) \to \mathbb{Z}_F(1)[1] \tag{4.3}$$

and $\mathbb{Z}(1) = \mathbb{G}_m[-1]$, we get $\mathcal{H}^i(\mathbb{Z}_f(1)) = 0$ for $i \leq 0$, and an exact sequence

$$0 \to \mathbb{G}_{m,F} \to f_*(\mathbb{G}_{m,X}) \to \mathcal{H}^1(\mathbb{Z}_f(1)) \to 0 \tag{4.4}$$

Thus, $\mathcal{H}(\mathbb{Z}_f(1))$ is the etale sheaf associated with the presheaf $U \mapsto F[f^{-1}U]^\times / F[U]^\times$; by a result of Sansuc [24], $\mathcal{H}^1(\mathbb{Z}_f(1)) \simeq G^*$.

For $q \geq 2$, the exact triangle gives isomorphsims $\mathcal{H}^q(\mathbb{Z}_f(1)) = R^q f_*(\mathbb{Z}_X(1))$, so in particular $H^2(\mathbb{Z}_f(1)) \simeq \mathrm{Pic}(G)$.

4.3.2 Comparing $\mathbb{Z}_f(1)$ with $\overline{C}^\bullet(X_{sep}, K_1)$

Consider the diagram of categories of sheaves

$$\begin{array}{ccccc} Sh_{et}(X) & \xrightarrow{\alpha} & Sh^{\Gamma}_{zar}(X_{sep}) & \xrightarrow{\delta} & Sh_{et}(F) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle \beta} & & \\ Sh_{et}(X_{sep}) & \xrightarrow{\gamma} & Sh_{zar}(X_{sep}) & & \end{array}$$

Where α and γ are change of site, res is restriction, β is the forgetful functor from $Sh^{\Gamma}_{zar}(X_{sep})$, the Γ_F-equivariant Zariski sheaves on X_{sep} to Zariski sheaves on X_{sep}, and δ is the global section functor. Note that $\delta\alpha = f_*$ with $f : X \to SpecF$.

Passing to derived categories, we get

$$\begin{array}{ccccc} D^+Sh_{et}(X) & \xrightarrow{R\alpha} & D^+Sh^{\Gamma}_{zar}(X_{sep}) & \xrightarrow{R\delta} & D^+Sh_{et}(F) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle R\beta} & & \\ D^+Sh_{et}(X_{sep}) & \xrightarrow{R\gamma} & D^+Sh_{zar}(X_{sep}) & & \end{array}$$

Lemma 33. *There is a morphism $\tau_{\leq 2}R\gamma\mathbb{Z}_{X_{sep}}(1) \to \mathcal{K}_{1,X_{sep}}[-1]$ in the derived category of Γ-equivariant Zariski sheaves over X_{sep}.*

Proof: consider the object $\mathbb{Z}_X(1) \in D^+Sh_{et}(X)$. Restriction takes this to $Z_{X_{sep}}(i) \in D^+Sh_{et}(X_{sep})$, and then to $R\gamma\mathbb{Z}_{X_{sep}}(i)$. By [11, 1.4], this has trivial cohomology in degree $i+1$ and cohomology K_i in degree i. So, there is a map of Zariski sheaves $\tau_{\leq 2}R\gamma\mathbb{Z}_{X_{sep}}(1) \to \mathcal{K}_1$. Commutativity of the above diagrams of categories, along with exactness of the forgetful functor β, gives the equalities

$$\beta(\mathcal{H}^j R\alpha\mathbb{Z}_X(1)) = \mathcal{H}^j(R\beta R\alpha\mathbb{Z}_X(1)) = \mathcal{H}^j(R\gamma Rres\mathbb{Z}_X(1)) = \mathcal{H}^j(R\gamma\mathbb{Z}_{X_{sep}}(1))$$

So, in particular $\beta(\mathcal{H}^2 R\alpha\mathbb{Z}_X(1)) = 0$, means that $\mathcal{H}^2 R\alpha\mathbb{Z}_X(1) = 0$, and $\beta(\mathcal{H}^1 R\alpha\mathbb{Z}_X(1)) = \mathcal{K}_{1,X_{sep}}$ means that $\mathcal{H}^1 R\alpha\mathbb{Z}_X(1) = \mathcal{K}_{1,X_{sep}}$. Taken together, these proves the lemma. $\square$

Now, applying $R\delta$ to this morphism, we get

$$\begin{array}{ccc} R\delta(\tau_{\leq i+1}R\alpha\mathbb{Z}_X(i)) & \longrightarrow & R\delta\mathcal{K}_{i,X_{sep}}[-i] \\ \downarrow & & \\ R\delta(R\alpha\mathbb{Z}_X(i)) & & \end{array}$$

Truncating this entire diagram at degree $i+1$, we get

$$\begin{array}{ccc} \tau_{\leq i+1}R\delta(\tau_{\leq i+1}R\alpha\mathbb{Z}_X(i)) & \longrightarrow & \tau_{\leq i+1}R\delta\mathcal{K}_{i,X_{sep}}[-i] \\ \downarrow & & \\ \tau_{\leq i+1}R\delta(R\alpha\mathbb{Z}_X(i)) & & \end{array}$$

The vertical map is an isomorphism in the derived category. To see this, first note that the original map was an isomorphism on homology in degrees up through $i+1$, and came from a map of complexes, the first of which is 0 in degrees above $i+1$. Thus, after passing through $R\delta$ and then truncating at $i+1$, we get an isomorphism. On the other hand, $R\delta(\mathcal{K}_1)$ coincides with the Rost complex $C(X_{sep}, K_1)$ (see [21, §7]).

Theorem 34. *The truncated relative motivic complex $\tau_{\leq 2}\mathbb{Z}_f(1)$ is isomorphic in the derived category to $\tau_{\leq 2}\overline{C}^\bullet(X_{sep}, K_1)[-1]$*

Proof:

we have a diagram:

$$\begin{tikzcd} \tau_{\leq 2}\mathbb{Z}_F(1) \arrow[r] \arrow[d] \arrow[dr] & K_{i,F_{sep}}[-1] \arrow[d] \\ \tau_{\leq 2}Rf_*\mathbb{Z}_X(1) \arrow[r] \arrow[d] & C^\bullet(X_{sep}, K_1)[-1] \arrow[d] \\ \tau_{\leq 2}\mathbb{Z}_f(1) \arrow[r, dashed] & \overline{C}^\bullet(X_{sep}, K_1)[-1] \end{tikzcd}$$

The left column is the truncation of an exact triangle (since $\mathbb{Z}_f(1)$ is defined as the cone of the previous morphism), and the right column is an exact triangle. If we can show that following the arrows from $\tau_{\leq 2}\mathbb{Z}_F(1)$ to $\overline{C}^\bullet(X_{sep}, K_1)$ is trivial, we'll get the existence of a morphism $\tau_{\leq 2}\mathbb{Z}_F(1) \to \overline{C}^\bullet(X_{sep}, K_1)$; but since the top horizontal map is an isomorphism, the composition is trivial.

Now, we have a morphism $\tau_{\leq 2}Rf_*\mathbb{Z}_X(1) \to \overline{C}^\bullet(X_{sep}, K_1)$ that, when precomposed with $\tau_{\leq 2}\mathbb{Z}_F(1) \to \tau_{\leq 2}Rf_*\mathbb{Z}_X(1)$ is trivial. This means that there is some corresponding morphism

$\beta_f : \tau_{\leq 2}\mathbb{Z}_f(1) \to \overline{C}^\bullet(X_{sep}, K_1)$. We still need to see that this morphism is unique; to do this, take $Hom(-, \overline{C}^\bullet(X_{sep}, K_1))$ of the lefthand exact triangle. This gives us:

$$Hom(\tau_{\leq 2}\mathbb{Z}_F(1)[1], \overline{C}^\bullet(X_{sep}, K_1)) \to Hom(\tau_{\leq 2}\mathbb{Z}_f(1), \overline{C}^\bullet(X_{sep}, K_1)) \to$$
$$Hom(\tau_{\leq 2}Rf_*\mathbb{Z}_X(1), \overline{C}^\bullet(X_{sep}, K_1)) \to Hom(\tau_{\leq 2}\mathbb{Z}_F(1), \overline{C}^\bullet(X_{sep}, K_1))$$

So, if we show that $Hom(\tau_{\leq 2}\mathbb{Z}_F(1)[1], \overline{C}^\bullet(X_{sep}, K_1)) = 0$, our morphism $\beta_f : \tau_{\leq 2}\mathbb{Z}_f(1) \to \overline{C}^\bullet(X_{sep}, K_1)$ will be unique. To see this, note that $\overline{C}^\bullet(X_{sep}, K_1)$ is concentrated in degrees 1 and 2.

$Hom(\tau_{\leq 2}\mathbb{Z}_F(1)[1], \overline{C}^\bullet(X_{sep}, K_1))$ is the set of maps maps from a complex concentrated in degree 0 to a complex concentrated in degrees at least 1. It is a more general fact that any morphism in a derived category $A \to B$ where A is concentrated below B is trivial. This is because such a morphism corresponds to a house, $C \to A$ a quasi-isomorphism, and $C \to B$ a morphism of complexes. But, because C is quasi-isomorphic to A, the morphism $\tau_{\leq i-1}C \to C$ is also a quasi-isomorphism; composing we get a morphism of complexes $\tau_{\leq i-1}C \to B$ that is trivial, and quasi-isomorphic to the original morphism under consideration.

Finally, we need to see that this map is an isomorphism. Since the top morphism is an isomorphism, and the next morphism induces isomorphism on homology, the five lemma tells us that β_f also induces isomorphism on homology, and so is a quasi-isomorphism. □

So we have successfully identified the truncated relative motivic complex with the complex of Galois modules $N(G)$.

4.4 The Weight 2 Relative Motivic Complex

Again, we let $f : X \to \mathrm{Spec}(F)$ be a torsor for G; then define the weight 2 relative motivic complex for f to be $\mathbb{Z}_f(2)$, the cone of the natural morphism

$$\mathbb{Z}_F(2) \to Rf_*(\mathbb{Z}_X(2)) \tag{4.5}$$

in the derived category of étale sheaves over F.

Our computation of $\tau_{\leq 3}\mathbb{Z}_f(2)$ will take advantage of Rost's spectral sequence associated to $\pi : X \to X/T$, converging to $A^i(X, K_n)$. We can assume that T is split, since we're going to be computing over a separable closure.

The first page of the spectral sequence is

$E_1^{p,q} = \coprod_{z\in Z^{(p)}} A^q(\pi^{-1}(z), K_{2-p}) \Rightarrow A^{p+q}(G, K_2)$

Because the fibers are all isomorphic to a split torus, $E_1^{p,q} = 0$ when $q > 0$. If we then consider the cycle module M over Z defined by $M_n(z) = A^0(\pi^{-1}z, K_n)$ for all $z \in Z(K)$, K/F a field extension, then we get from the spectral sequence that $A^p(G, K_n) \cong A^p(Z, M_n)$.

Now, this cycle module has a cycle module filtration $0 = M_2^{(1)} \subseteq M_2^{(0)} \subseteq M_2^{(-1)} \subseteq M_2^{(-2)} = M_2$ with factor cycle modules $M_2^{(p/p+1)} \cong \Lambda^{-p}(\hat{T}) \otimes K_{2+p}$.

Theorem 35. *The complex $\overline{C}^{\cdot}(X_{sep}, K_2)$ fits into an exact triangle*

$$N(1) \to \overline{C}^{\cdot}(X_{sep}, K_2) \to D(G) \to N(1)[1]$$

Proof. From the filtration above, we get that $\overline{C}^{\cdot}(X_{sep}, K_2)$ is quasi-isomorphic to $U = (\overline{M_2}(F_{sep}(Z)) \to \coprod_{z\in Z^{(1)}} M_1F_{sep}(z) \to \coprod_{z\in Z^{(2)}} M_0F_{sep}(z))$

$\overline{M_2}(F_{sep}(Z)) \cong \overline{A}^0(T, K_2)$, which has a filtration $F^\times \otimes T^* \subset \overline{A}^0(T, K_2) \to \Lambda^2 T^*$

$M_1F_{sep}(z) \cong A^0(T, K_1)$, which has a filtration $F^\times \to F[T]^\times \to T^*$

$M_0F_{sep}(z) \cong \mathbb{Z}$

Let V be the subcomplex of U that has $V_0 = F^\times \otimes T^*$, and $V_1 = U_1, V_2 = U_2$; let W be the Rost complex for Z; W is a subcomplex of V, with $W_2 = V_2$.

Given the inclusions $W \to V \to U$, we get an exact triangle,

$V/W \to U/W \to U/V \to V/W[1]$.

Note that $U_1 = V_1$ and $U_2 = W_2$; this means we can get a map from U to the cohomology of the above exact triangle;

$H^0(U/V) \to H^1(V/W) \to H^2(W)$

Let δ_i be the morphism $W_i \to W_{i+1}$ and $d_i : V_i \to d_{i+1}$. Finally, note that V/W is the two-term complex $\pi : F(Z)^\times \to \coprod_{z \in Z^{(1)}} T^*$.

Then we have a morphism $U_0 \to U_0/V_0 = H^0(U/V)$, a morphism $V_1 \to coker(\pi) = H^1(V/W)$, and a morphism $W_2 \to coker(\delta_1) = H^2(W)$ - the first and third of these are maps from objects to cokernels of maps to them, while the second is the composition $V_1 \to V_1/W_1 \to coker(\pi)$. So we have a complex map $U \to D$, with $D \cong \Lambda^2 T^* \to T^* \otimes T^{sc*} \to CH^2(Z)$.

We also know what the kernels of these maps are; the kernel of $U_0 \to U_0/V_0$ is $d_0^{-1}(W_1)$ and the kernel of $V_1 \to coker(\pi) = H^1(V/W)$ is $ker(\delta_1)$; finally,

$d_0^{-1}(W_1) \to ker(\delta_1)$ is quasi-isomorphic to $ker(\pi) \to ker(\delta_1)/im(\delta_0)$, which is $N \otimes F_{sep}^\times$.

So, we get our exact triangle,

$N \otimes F^\times \to \overline{C}^{\cdot}(X_{sep}, K_2) \to D(G) \to N \otimes F^\times[1]$

□

Theorem 36. *The truncated relative motivic complex $\tau_{\leq 3}\mathbb{Z}_f(2)$ associated to a G-torsor $f : X \to Spec(F)$, is quasi-isomorphic to the complex $\overline{C}^{\cdot}(X_{sep}, K_2)$.*

Proof. Proof:

we have a diagram:

$$\begin{tikzcd}
{}_{\tau_{\leq 3}}\mathbb{Z}_F(2) \arrow[r] \arrow[d] \arrow[dr] & K_{2,F_{sep}} \arrow[d] \\
{}_{\tau_{\leq 3}}Rf_*\mathbb{Z}_X(2) \arrow[r] \arrow[d] & {}_{\tau_{\leq 3}}R\delta K_{2,X_{sep}} \arrow[d] \\
{}_{\tau_{\leq 3}}\mathbb{Z}_f(2) \arrow[r, dashed] & \overline{C}^{\cdot}(X_{sep}, K_2)
\end{tikzcd}$$

The right column is exact, and the left column is the truncation of an exact triangle (since $\mathbb{Z}_f(2)$ is defined as the cone of the previous morphism). If we can show that following the arrows from ${}_{\tau_{\leq 3}}\mathbb{Z}_F(2)$ to MR_i is trivial, we'll get the existence of a morphism ${}_{\tau_{\leq 3}}\mathbb{Z}_F(2) \to$

$\overline{C}^{\cdot}(X_{sep}, K_2)$.

$\mathbb{Z}(2)$ is supported in degrees 1 and 2. $\mathcal{H}^1(\mathbb{Z}_F(2)) \cong R^1 f_*(\mathbb{Z}_X(2))$, and $\mathcal{H}^2(\mathbb{Z}_F(2)) \cong K_{2,F_{sep}}$, so the composition is trivial.

Now, we have a morphism ${}_{\tau_{\leq 3}}Rf_*\mathbb{Z}_X(2) \to \overline{C}^{\cdot}(X_{sep}, K_2)$ that, when precomposed with ${}_{\tau_{\leq 3}}\mathbb{Z}_F(2) \to_{\tau_{\leq 3}} Rf_*\mathbb{Z}_X(2)$ is trivial. This means that there is some corresponding morphism $\beta_f :_{\tau_{\leq 3}} \mathbb{Z}_f(2) \to \overline{C}^{\cdot}(X_{sep}, K_2)$. We still need to see that this morphism is unique; to do this, take $Hom(-\overline{C}^{\cdot}(X_{sep}, K_2))$ of the lefthand exact triangle. This gives us:

$$Hom({}_{\tau_{\leq 3}}\mathbb{Z}_F(2)[1], \overline{C}^{\cdot}(X_{sep}, K_2)) \to Hom(\tau_{\leq 3}\mathbb{Z}_f(2), \overline{C}^{\cdot}(X_{sep}, K_2)) \to$$
$$Hom({}_{\tau_{\leq 3}}Rf_*\mathbb{Z}_X(2), \overline{C}^{\cdot}(X_{sep}, K_2)) \to Hom(\tau_{\leq 3}\mathbb{Z}_F(2), \overline{C}^{\cdot}(X_{sep}, K_2))$$

So, if we show that $Hom({}_{\tau_{\leq 3}}\mathbb{Z}_F(2)[1], \overline{C}^{\cdot}(X_{sep}, K_2)) = 0$, our morphism $\beta_f :_{\tau_{\leq 3}} \mathbb{Z}_f(2) \to \overline{C}^{\cdot}(X_{sep}, K_2)$ will be unique. To see this, note that $\overline{C}^{\cdot}(X_{sep}, K_2)$ is concentrated in degrees 2 and 3.

When $i = 2$, $\mathbb{Z}_f(2)$can be represented by a complex represented in degrees 1 and 2, so $\mathbb{Z}_f(2)[1]$ is concentrated in degrees 0 and 1.

So, we have $Hom({}_{\tau_{\leq 3}}\mathbb{Z}_F(2)[1], \overline{C}^{\cdot}(X_{sep}, K_2))$ as maps from a complex concentrated in degrees at most 1 to a complex concentrated in degrees at least 2. It is a more general result that any morphism in a derived category $A \to B$ where A is concentrated below B is trivial. This is because such a morphism corresponds to a house, $C \to A$ a quasi-isomorphism, and $C \to B$ a morphism of complexes. But, because C is quasi-isomorphic to A, the morphism ${}_{\tau_{\leq 1}}C \to C$ is also a quasi-isomorphism; composing we get a morphism of complexes ${}_{\tau_{\leq 1}}C \to B$ that is trivial, and quasi-isomorphic to the original morphism under consideration.

Finally, we need to see that this map is an isomorphism. This follows because it induces isomorphism on homology; in degrees higher than $i+1$, both have been truncated, and so are trivial. In degrees below 2, $\overline{C}^{\cdot}(X_{sep}, K_2)$ is trivial, while $\mathbb{Z}_f(2)$ has trivial homology, because the homology sheaves of $\mathbb{Z}_F(2)$ and $Rf_*\mathbb{Z}_X(2)$ are isomorphic. In degree 2, both complexes have homology $\overline{A}^0(X, K_2)$ and in degree 3, both have homology $A^1(X, K_2)$.

$\square$

4.5 Hypothetical Calculations

Using a conjectured exact triangle that the complex $\tau_{\leq 3}\mathbb{Z}$ fits inside, we can predict a relationship between $\mathrm{CH}^2(BG)$, $\overline{H}^4(BG, \mathbb{Z}(2))$, $Inv^3(G, \mathbb{Q}/\mathbb{Z}(2))$, and the complexes $N(G)$ and $D(G)$.

Conjecture 1. *Let G be a reductive group, and let $f : X \to Y$ be an arbitrary G-torsor, and $\mathbb{Z}_f(1)$ the corresponding weight one relative motivic complex. Then the truncation $\tau_{\leq 2}\mathbb{Z}_f(1)$ fits into an exact triangle*

$$T^*_{sep} \to \Lambda_w \to \tau_{\leq 2}\mathbb{Z}_f(1)[2] \to T^*_{sep}[1] \tag{4.6}$$

When $G = T$ is a torus, then $\Lambda_w = 0$; this implies that every normalized Brauer invariant of T is linear, which is indeed the case. On the other hand, when G is semisimple, then $T^*_{sep} \to \Lambda_w$ is injective, with cokernel C^*, which is equal to the group of normalized Brauer invariants of G.

Let $N(1) = N(G) \otimes \mathbb{Z}(1)$; $N(1) = [T^*_{sep} \otimes F^\times_{sep}[-1] \to \Lambda_w \otimes F^\times_{sep}[-1]]$.

Conjecture 2. *Let G be a reductive group, and let $f : X \to Y$ be an arbitrary G-torsor, and $\mathbb{Z}_f(2)$ the corresponding weight two relative motivic complex. Then the truncation $\tau_{\leq 3}\mathbb{Z}_f(2)$ fits into an exact triangle*

$$N(1) \to \tau_{\leq 3}\mathbb{Z}_f(2)[3] \to D(G) \to N(1)[1]. \tag{4.7}$$

Again, we can see that this simplifies to the actual situation when G is either semisimple or a torus; if $G = T$ is a torus, then since $\Lambda_w = 0$, and $D(G) = S^2(T^*_{sep})$, taking cohomology would give us an exact sequence

$$0 \to H^1(Y, T^*_{sep}) \to H^3(Y, \mathbb{Z}_f(2)) \to S^2(T^*_{sep})^\Gamma \to H^2(Y, T^*_{sep})$$

which is precisely what happens. Specializing instead to the case of a semisimple group, $N(1)$ will be quasi-isomorphic to $C^*(1)$, and we'll get an exact sequence

$$0 \to H^1(Y, C^*(1)) \to H^3(Y, \mathbb{Z}_f(2)) \to H^0(Y, D) \to H^2(Y, C^*(1))$$

which is an exact sequence that does in fact hold whenever $f : X \to Y$ is a torsor for a semisimple group.

Given this evidence in favor of the conjectures, it is worth following the line of logic that conjecture 2 provides toward a computation of the degree three cohomological invariants of a reductive group.

In general, if V is a generically free representation of G with an open G-invariant subscheme $U \subseteq V$ and a G-torsor $U \to U/G$ with $U(F) \neq \emptyset$, and $V \backslash U$ is of codimension at least 3, then there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(U^n/G) \longrightarrow \overline{H}^4(U^n/G, \mathbb{Z}(2)) \longrightarrow \overline{H}^0_{\mathrm{Zar}}(U^n/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \longrightarrow 0$$

for every n. The lefthand group is independent of n, so we can write it as $\mathrm{CH}^2(BG)$, and taking the balanced elements (those which agree on both projections $p_i : (U \times U)/G \to U/G$), we get

$$0 \longrightarrow \mathrm{CH}^2(BG) \longrightarrow \overline{H}^4(U^n/G, \mathbb{Z}(2))_{\mathrm{bal}} \longrightarrow \overline{H}^0_{\mathrm{Zar}}(U^n/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\mathrm{bal}} \longrightarrow 0$$

But the righthand group $\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{bal}}$ is canonically isomorphic to $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$, so it is also independent of the choice of V, which means the middle term is as well, and we can write $\overline{H}^4(BG, \mathbb{Z}(2))$ for $\overline{H}^4(U^n/G, \mathbb{Z}(2))_{\mathrm{bal}}$. This gives the exact row of the following diagram. Because $H^3(Y, \mathbb{Z}_f(2)) \simeq \overline{H}^4(Y, \mathbb{Z}(2))_{\mathrm{bal}}$ whenever $f : X \to Y$ is a versal torsor, the exact triangle for $\mathbb{Z}_f(2)$ gives us the exact column of the following diagram.

Theorem 37. *If G is a reductive group, and conjecture 2 holds for any G-torsor, then there is a commutative diagram of the form*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1(F, N(1)) \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & CH^2(BG) \arrow[r] \arrow[dr] & \overline{H}^4(BG, \mathbb{Z}(2)) \arrow[r] \arrow[d] & Inv^3(G, \mathbb{Q}/\mathbb{Z}(2))_{norm} \arrow[r] & 0 \\
 & & H^1(F, D) \arrow[d] & & \\
 & & H^2(F, N(1)) & &
\end{tikzcd}$$

REFERENCES

[1] H. Bermudez and A. Ruozzi, *Degree 3 cohomological invariants of split simple groups that are neither simply connected nor adjoint*, J. Ramanujan Math. Soc. 29 (2014), no. 4, 465-481.

[2] S. Blinstein and A. Merkurjev, *Cohomological invariants of algebraic tori*, Algebra Number Theory, 7:1643-1684

[3] M. Borovoi and J. van Hamel, *Extended Picard complexes and linear algebraic groups*, J. reine angew. Math. 627 (2009), 53-82

[4] J.-L. Colliot-Th'el'ene, *Birational invariants, purity and the Gersten conjecture*, Ktheory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 1-64.

[5] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.

[6] H. Esnault, B. Kahn, M. Levine, and E. Viehweg, *The Arason invariant and mod 2 algebraic cycles*, J. Amer. Math. Soc. 11 (1998), no. 1, 73-118.

[7] R. Garibaldi, A. Merkurjev, and Serre J.-P., *Cohomological Invariants in Galois Cohomology*, American Mathematical Society, Providence, RI, 2003.

[8] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Mem. Soc. Math. France (N.S.) (1985), no. 21, 87.

[9] P. Guillot, *Geometric methods for cohomological invariants*, Doc. Math. 12 (2007),521-545 (electronic).

[10] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. Ecole Norm. Sup. (4) 12 (1979), no. 4, 501-661.

[11] B. Kahn, *Applications of weight-two motivic cohomology*, Doc. Math. 1 (1996), No. 17, 395-416.

[12] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of Involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits

[13] D. Laackman and A. Merkurjev, *Degree three cohomological invariants of reductive groups*, Commentarii Mathematici Helvetici, 91, (2016), no. 3, 493-518.

[14] S. Lichtenbaum, *The construction of weight-two arithmetic cohomology* , Invent. Math. 88 (1987), no. 1, 183-215.

[15] S. Lichtenbaum, *New results on weight-two motivic cohomology*, The Grothendieck Festschrift, Vol. III, Birkhauser Boston, Boston, MA, 1990, pp. 35-55.

[16] A. Merkurjev, *The group H1 (X, K2) for projective homogeneous varieties*, Algebra i Analiz 7 (1995), no. 3, 136-164.

[17] A. Merkurjev and A. A. Suslin, *The group K3 for a field*, newblock Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 3, 522-545.

[18] A. Merkurjev, *Degree three cohomological invariants of semisimple groups*, J. Eur. Math. Soc. (JEMS), 18 (2016), no. 2, 657-680.

[19] A. Merkurjev, *Unramified degree three cohomological invariants of reductive groups*, Adv. Math., 293, (2016), 697–719.

[20] J. S. Milne, *Etale cohomology* , Princeton University Press, Princeton, N.J., 1980.

[21] D. Quillen, *Higher algebraic K-theory. I,* (1973), 85?147. Lecture Notes in Math., Vol. 341.

[22] M. Rosenlicht, *Toroidal algebraic groups*, Proc. Amer. Math. Soc. 12 (1961) 984-988

[23] M. Rost, *Chow groups with coefficients*, Doc. Math. 1 (1996), No. 16, 319-393 (electronic).

[24] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. 327 (1981), 12?80.

[25] J.-P. Serre, *Galois Cohomology*, Springer-Verlag, Berlin, 1997.

[26] A. A. Suslin, *Torsion in K2 of fields*, K-Theory 1 (1987), no. 1, 5-29.

[27] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249-281.

[28] W.C. Waterhouse, *Introduction to affine group schemes*, Springer-Verlag, New York, 1979.