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# ON THE UNIQUENESS OF HIGHER ENERGY STATIONARY STATES OF THE SCHRÖDINGER–NEWTON SYSTEM

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## Abstract

On the Uniqueness of Higher Energy Stationary States of the Schrödinger–Newton System

by

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This thesis is a study of the uniqueness of the higher stationary states of the Schrödinger–Newton system under the assumption of spherically symmetric solutions. We begin with a theory of dark matter put foward by Bray [2] involving the Einstein–Klein–Gordon system of equations, and then pose the Schrödinger–Newton system as the low–field nonrelatavistic limit of the Einstein–Klein–Gordon system. From here, by imposing spherical symmetry, we show that the potential term in the Schrödinger–Newton system can be seen as a nonlinear perturbation from the Coulomb potential $\frac{1}{r}$ on the half–line $[0,\infty)$. After proving uniqueness of bound states for the Hydrogen atom on the half–line, we then proceed by defining weighted Banach spaces for which the Schrödinger operator representing the Hydrogen atom on the half–line is Fredholm of index 0. In the last chapter, we detail an iteration scheme involving the implicit function theorem to show a correspondence between bound state solutions of the Hydrogen atom on the half–line and bound state solutions of the full Schrödinger–Newton system to prove the uniqueness result.

To Roger Hingtgen, Beverly Kinne,

& Maxwell.

To lives lived and lives beginning.

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I was very fortunate in my time at UC Santa Cruz to have Jie Qing as my doctoral advisor. I always left our weekly meetings with some newfound appreciation or perspective on a concept or idea I had been struggling with. Perhaps most of all, I am appreciative of his patience. I'm fairly certain in all of our meetings there was never a moment that he did not have complete grasp of the big picture for the problem at hand as well as the path to the solution, but with a careful guiding hand (and a few difficult questions asked of me) Jie always kept me on the right track. And he always somehow managed to get me caught up to where he was in a proof without me ever feeling like I did not eventually get there on my own. That's a neat trick. Hopefully, I will pick it up one day. I am thankful for his mentorship as well as his encouraging words of motivation to maintain momentum towards our goal. A simple thank you seems too little for how much I was given, so I hope my words strike true when I say: Thank you for the guidance and for reminding me that while mathematics can be difficult at times, the understanding of it is always immensely rewarding in the end.

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A strong thanks must be given to Andrew Goetz and Hubert Bray. In the year that I was lucky enough to study with Andrew, I gained deeper understanding of general relativity, was exposed to Bray's elegant and geometrical theory of scalar field dark matter, and had many fruitful discussions with Andrew sharing his own research in this area. The focus of this thesis largely came about by Andrew's direction. It is fair to say that without him, this project likely would not have come to be.

From what feels like a prior lifetime ago in Texas, I must thank the following people: First, I must thank William Cherry for teaching one of best courses I have ever had the pleasure to take, of which I was unaware at the time but now realize, began my deep love of studying functional and harmonic analysis. And a thanks for both pushing me to do and be better as a student of mathematics. Second, a very important thank you to Joseph Iaia, for not giving up on me and helping me rediscover my joy in studying mathematics during a very trying and difficult time in my life, and, of course, for all the great courses he taught me as well. Lastly, I must give my sincere thanks to Dan Mauldin, for playing a large part in my reason and inspiration for becoming a mathematics major all those year ago. Dan is and probably always will be the north star in terms of what I aspire to be as a mathematician and an educator.

Kurt Vonnegut once said ”I urge you to please notice when you are happy, and exclaim or murmur or think at some point, ’If this isn't nice, I don't know what is.’” At this moment I must remark at how thankful and lucky I am for all of the friends, mathematical and otherwise, I have had the pleasure to know in my life. Somehow they tolerate all my fallibilities (of which there are many) and manage to find some reason to keep me around with a smile on their faces to boot. But at this moment I shall trade faux pas for faux pas and not name any names explicitly for fear of missing one. It is the cowards way out, but I hope they will forgive me. Better to be namelessly thanked than the other way around I say. But, with all seriousness, this is me saying a strong thank you to my friends and that ‘this is nice, isn't it?’

Part of my dedication to this thesis is in honor of my Grandpa Rog and Grandma Bev, who, in all honesty, I believe probably would not have been too excited about reading a thesis based entirely on math, but would have been proud of me nonetheless. To be frank, I could not care less about the math in this instance. I just want to let them know all the love and care throughout the years was appreciated greatly.

To my grandparents I extend the same sentiments. Thank you for the care and support throughout the years. And, as someone who routinely wonders about his place in the world or forgets his base identity, I am lucky to still have my grandparents in my life. Within five minutes in a room with them, it all comes flooding back. I do have a heritage and a culture, and it lies somewhere on a spectrum between the sentiment expressed in the song 'Iowa Stubborn' from *The Music Man* and the ethos of midwestern work ethic that states, "the job is only done when it's done right."

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1. Do your best.

2. Never give up.

3. Always remember that Gordie Howe was the greatest player to ever play the game of hockey.

# Chapter 1

# Introduction

One of the earliest accounts of our awareness as a species to what we now call dark matter is attributed to Fritz Zwicky in the 1930s [21] with his study of the Coma cluster by use of the Virial Theorem. The Theorem is an elegant argument suggesting a need for the existence of additional matter, more than the mass-to-luminosity ratio of recorded galaxy clusters at the time would suggest. Over the following eight decades following Zwicky's original work, the astronomically observed evidence for the existence of dark matter has accumulated tremendously. From Rubin & Ford's work in the early 1970s on flat rotation curves of disk galaxies [7] to gravitational lensing observed in collisions of galaxy clusters (such as the Bullet Cluster), the question of whether or not dark matter exists has been definitively answered in the affirmative. For the interested reader, a detailed recreation of Zwicky's original argument can be found in the appendix.

However, while existence is accepted, the precise character of what dark matter is has yet to come to a unambiguous conclusion. For this work, we will primarily focus on a theory of dark matter known as the scalar field dark matter (SFDM) theory, which is also known as the wave dark matter (WDM) theory or Bose–Einstein condensate (BEC) theory. Our work began from a paper [2] by Hubert Bray that gave a Lagrangian theory for a version of SFDM in which the scalar field representing dark matter arose from a geometrical consideration. A more detailed accounting of this fact is presented in the next section of this paper, while the larger details are left to the appendix.

In general, this version of scalar field dark matter is represented by spherically symmetric solutions to the Einstein–Klein–Gordon equation. The current accepted cosmological paradigm is the $\Lambda$CDM model, $\Lambda$ Cold Dark Matter ($\Lambda$ representing the cosmological constant), in which the dark matter is slowly varying or slow moving. As such, understanding the dynamics of this SFDM theory can be done in the nonrelatavistic low-field limit. In this case, the Einstein-Klein-Gordon system of equations reduces to the Schrödinger–Newton system, and the third section of this paper is devoted to this proof.

The Einstein–Klein–Gordon equations have also been used in the study of boson stars [1], and the Schrödinger–Newton system has also been used to describe wave function collapse due to gravitational interaction [14]. In these situations, as well as the one considered in this work, knowledge pertaining to the character of bound (stationary) state solutions to the Einstein–Klein–Gordon & Schrödinger–Newton system is paramount. The existence of a discrete countable family of stationary state solutions to both systems has been known for some time [13] [1], and there has been many results showing that the ground state (lowest energy eigenvalue stationary state) is unique in a myriad of conditions. [11] [5] Numerical models strongly suggest that the higher bound states are, in fact, unique. [14] [10] However, what has eluded rigorous mathematical proof up to this point is the uniqueness of these higher stationary state solutions.

Our goal for this work is to present an argument for showing the uniqueness of the higher bound states of the simpler Schrödinger–Newton system under the assumption of spherically symmetric solutions, with a future project purposed to extending these results to the higher energy spherically symmetric stationary states of the Einstein–Klein–Gordon system. At chapter four, the argument begins in earnest by showing that the potential term in the Schrödinger–Newton system can be viewed as a nonlinear perturbation of the Coulomb potential on the half–line $[0,\infty)$. In the fifth chapter we collect some properties of solutions to the linearization of the Schrödinger–Newton system, which is the Hydrogen atom on the half–line. In particular we show that the eigenvalues of the Hydrogen atom on the half–line are simple (i.e. each bound state in unique). Chapter six is devoted to showing that the operator representing the Hydrogen atom on the half–line is in fact a Fredholm operator of index 0 over a particular weighted Banach space. Concluding in chapter seven, we present a sketch of an iteration scheme involving the implicit function theorem to achieve the desired result.

This thesis is a current work-in-progress towards the result of proving uniqueness of the higher bound states under the assumption of spherically symmetric solutions. The implicit function theorem gives us a smooth extension of the eigenvector/eigenvalue pairs of the Hydrogen atom on a neighborhood of the parameter $\beta = 0$. What remains to be shown is:

- the extension of each eigenvalue/eigenvector pair continues up to $\beta = 1$.
- the Fredholm property on the Frechet derivative at every point, $0 \leq \beta \leq 1$, is maintained.

# Chapter 2

# An Action Functional for Scalar Field Dark Matter

The metric dependent Einstein–Hilbert action is given by

$$L(g) = \int_U (R - 2\Lambda) dV_g,$$

where $U$ is a pre–compact open set of a spacetime $N$, $R$ the scalar curvature, and $\Lambda$ the cosmological constant. For a metric that is a critical point of the Einstein–Hilbert action, one recovers Einstein's equation in a vacuum,

$$G + \Lambda g = 0 \iff \mathrm{Ric} - \frac{1}{2} R g + \Lambda g = 0.$$

This derivation can be found in the appendix for the curious reader. Going further, from the works of Cartan [4] and Weyl [20], Einstein's equation with cosmological constant is the only result one can expect for metrics that are critical points of functionals of the form

$$\int_{\Phi(U)} \mathrm{Quad}_M(M \cup M') dV_{\mathbb{R}^4}, \ \ M = \{g_{ij}\}, \ \ M' = \{g_{ij,k}\}.$$

where $\Phi : \Omega \subset N \to \mathbb{R}^4$ is a coordinate chart of a spacetime $N$, and $\mathrm{Quad}_M(M \cup M')$ is shorthand for

$$\mathrm{Quad}_M(M \cup M') = \sum_{\alpha,\beta} F^{\alpha\beta}(M) m_\alpha m_\beta,$$

a general quadratic expression in elements from the set of metric components and their derivatives and $F^{\alpha\beta}$ functions on $M = \{g_{ij}\}$.

In [2], action funtionals dependent upon a choice of connection $\nabla$ as well as a choice of metric $g$ are considered.

$$F_{\Phi,U}(g,\nabla) = \int_{\Phi(U)} \mathrm{Quad}_M(M' \cup M \cup C' \cup C) dV_{\mathbb{R}^4}, \;\; C = \{\Gamma_{ijk}\}, \;\; C' = \{\Gamma_{ijk,l}\}.$$

In particular, the author of [2], suggests a theory of scalar field dark matter in which the scalar field manifests as the deviation of the choice of connection $\nabla$ on the spacetime from the standard Levi–Civita connection.

To be more specific, let

$$D(X,Y,Z) = \langle \nabla_X Y, Z \rangle_g - \langle \overline{\nabla}_X Y, Z \rangle_g,$$

be the $(0,3)$-tensor representing the difference between a given connection $\nabla$ and the Levi–Civita connection $\overline{\nabla}$. Defining $\gamma_{ijk} = D_{[ijk]}$ to be the antisymmetrization of the tensor $D_{ijk}$, [2] considered a specific action meeting the form of the general action functional above,

$$F_{\Phi,U}(g,\nabla) = \int_U (R - 2\Lambda - c_1 |d\gamma|^2 - c_2 |\gamma|^2) dV_g,$$

where $c_1$ and $c_2$ are constants, and $|\cdot|$ is the norm on $k$–forms given by the Hodge dual, i.e. $|\gamma|^2 dV_g = \gamma \wedge \star\gamma$.

Defining the vector field $v$ by,

$$\gamma = \star(v^*)$$

where $v^*$ is the 1–form dual to $v$ in the metric $g$ and $\star$ is the Hodge star operator, then moving from terms of $\gamma$ to terms of $v$, and leaving the following computations to the appendix, we have the following: $|\gamma|^2 dV_g = -|v|^2 dV_g$ and $|d\gamma|^2 dV_g = -(\nabla \cdot v)^2 dV_g$. Thus, the action functional takes the form,

$$F_{\Phi,U}(g,\nabla) = \int_U (R - 2\Lambda + c_1(\nabla \cdot v)^2 + c_2|v|^2) dV_g.$$

Assuming a pair $(v,g)$ of vector field $v$ and metric $g$ is a critical point of the action functional above leads to the following system of equations

$$\begin{aligned} G + \Lambda g &= c_2(v^* \otimes v^*) - \frac{1}{2}[c_1(\nabla \cdot v)^2 + c_2|v|^2]g \\ \nabla(\nabla \cdot v) &= \frac{c_2}{c_1} v. \end{aligned}$$

If we now define the following scalar function $f$ as,

$$f = \left(\frac{c_1}{c_2}\right)^{1/2} \nabla \cdot v.$$

and introduce the constants $\Upsilon$ and $\mu_0$

$$\frac{c_2}{c_1} = \Upsilon^2, \quad c_2 = 16\pi\mu_0,$$

the system then becomes

$$\begin{aligned} G + \Lambda g &= 8\pi\mu_0 \left[2\frac{df \otimes df}{\Upsilon^2} - \left(\frac{|df|^2}{\Upsilon^2} + f^2\right) g\right] \\ \Box_g f &= \Upsilon^2 f, \end{aligned}$$

which is the Einstein–Klein–Gordon system with a very specific energy–momentum tensor. Note that $\Box_g$ in the formula above denotes the d'Alembertian operator. Thus solutions to the Einstein–Klein–Gordon system with a cosmological constant in geometrized units ($G = c = 1$) arise naturally as critical points of the given action functional. As the components of the connection $\nabla$ can be written as

$$\Gamma_{ijk} = \left(\frac{1}{\Upsilon}\right)(\star df)_{ijk} + \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}),$$

we find the scalar function $f$ encapsulates the deviation of $\nabla$ from the Levi–Civita connection. What we have shown, which is an abridged version of Bray's work in [2], is a theory of scalar field dark matter that arises from deviation from the Levi–Civita connection on a spacetime.

Furthermore, it is conjectured in [2] that the only action functionals for which critical points $(g,\nabla)$ exist must be of the form

$$F_{\Phi,U}(g,\nabla) = \int_U (cR - 2\Lambda - \frac{c_1}{4!}|d\gamma|^2 - \mathrm{Quad}_M(D))dV.$$

Which, if true, would imply that the Einstein–Klein–Gordon system is the only result one could expect from critical points of action functionals of the form

$$F_{\Phi,U}(g,\nabla) = \int_{\Phi(U)} \mathrm{Quad}_M(M' \cup M \cup C' \cup C)dV_{\mathbb{R}^4},$$

# Chapter 3

# From Einstein–Klein–Gordon to Schrödinger–Newton

Here we derive the system of ordinary differential equations for the spherically symmetric static states of wave dark matter. This is an abridged version of what can be found in [3]. The spherically symmetric static states are solutions to the Einstein–Klein–Gordon equations,

$$G = 8\pi\left(\frac{df\otimes d\overline{f}+d\overline{f}\otimes df}{\Upsilon^2} - \left(\frac{|df|^2}{\Upsilon^2}+|f|^2\right)g\right).$$

$$\Box_g f = \Upsilon^2 f.$$

(*Note*: This is not the exact version of the Einstein–Klein–Gordon equation that closed the prior chapter but a complex version of it. The derivation of which can be understood to come from the derivation of the previous chapter on the real and imaginary parts of $f$ respectively.) Recall from the prior section, $f$ is the complex scalar field representing dark matter stemming from the deviation of the given connection on our spacetime from the standard Levi–Civita connection.

To begin, the spherically static metric is given by

$$g = -e^{2V(r)}dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

and define,

$$\Phi(r) = 1 - \frac{2M(r)}{r},$$

so the metric takes the simpler form,

$$g = -e^{2V(r)}dt^2 + \Phi(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

In the low–field limit, the functions $M(r)$ and $V(r)$ regain their typical Newtonian meaning of total mass inside a ball of radius $r$ and gravitational potential at radius $r$ respectively.

To find solutions to the Einstein–Klein–Gordon system, we begin by first collecting the nonzero Christoffel symbols for the metric $g$, and they are

$$\begin{aligned}
\Gamma^t_{tr} &= \Gamma^t_{rt} = V_r & \Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = r^{-1} \\
\Gamma^r_{tt} &= V_r e^{2V}\Phi & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^r_{rr} &= -\frac{1}{2}\Phi^{-1}\Phi_r & \Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = r^{-1} \\
\Gamma^r_{\theta\theta} &= -r\Phi & \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \cot\theta \\
\Gamma^r_{\phi\phi} &= -r\sin^2\theta\Phi.
\end{aligned}$$

Using these equations, and the component formula for the Riemann curvature tensor,

$$R_{\mu\nu\rho}{}^{\alpha} = -\partial_\mu\Gamma^\alpha_{\nu\rho} + \partial_\nu\Gamma^\alpha_{\mu\rho} - \Gamma^\sigma_{\nu\rho}\Gamma^\alpha_{\mu\sigma} + \Gamma^\sigma_{\mu\rho}\Gamma^\alpha_{\nu\sigma}.$$

we find that the nonzero components of the Ricci tensor are precisely the diagonal terms,

$$
\begin{aligned}
\mathrm{Ric}_{tt} &= (V_{rr} + V_r^2 + 2r^{-1}V_r)e^{2V}\Phi + \frac{1}{2}V_r e^{2V}\Phi_r. \\
\mathrm{Ric}_{rr} &= -(V_{rr} + V_r^2) - \frac{1}{2}V_r\Phi^{-1}\Phi_r - r^{-1}\Phi^{-1}\Phi_r \\
\mathrm{Ric}_{\theta\theta} &= 1 - \Phi - \frac{1}{2}r\Phi_r - rV_r\Phi \\
\mathrm{Ric}_{\phi\phi} &= \sin^2\theta\left(1 - \Phi - \frac{1}{2}r\Phi_r - rV_r\Phi\right).
\end{aligned}
$$

Using these formulas we find that the scalar curvature associated to the metric is

$$
R = -2(V_{rr} + V_r^2 + 2r^{-1}V_r)\Phi - V_r\Phi_r + 2r^{-2}(1 - \Phi - r\Phi_r).
$$

Hence the nonzero components of the Einstein tensor are as follows.

$$
\begin{aligned}
G_{tt} &= r^{-2}e^{2V}(1 - \Phi - r\Phi_r) \\
G_{rr} &= -r^{-2}\Phi^{-1}(1 - \Phi - 2rV_r\Phi) \\
G_{\theta\theta} &= r^2\left[(V_{rr} + V_r^2 + r^{-1}V_r)\Phi + \frac{1}{2}r^{-1}\Phi_r + \frac{1}{2}V_r\Phi_r\right] \\
G_{\phi\phi} &= r^2\sin^2\theta\left[(V_{rr} + V_r^2 + r^{-1}V_r)\Phi + \frac{1}{2}r^{-1}\Phi_r + \frac{1}{2}V_r\Phi_r\right]
\end{aligned}
$$

For wave dark matter the energy momentum tensor, $T$, is

$$
T = \frac{df \otimes d\overline{f} + d\overline{f} \otimes df}{\Upsilon^2} - \left(\frac{|df|^2}{\Upsilon^2} + |f|^2\right)g.
$$

Thus the nonzero components of the energy momentum tensor are

$$T_{tt} = e^{2V}|f|^2 + \Upsilon^{-2}|f_t|^2 + \Upsilon^{-2}e^{2V}\Phi|f_r|^2$$

$$T_{tr} = T_{rt} = \Upsilon^{-2}(f_t\overline{f_r} + \overline{f_t}f_r)$$

$$T_{rr} = -\Phi^{-1}|f|^2 + \Upsilon^{-2}e^{-2V}\Phi^{-1}|f_t|^2 + \Upsilon^{-2}|f_r|^2$$

$$T_{\theta\theta} = \Upsilon^{-2}r^2(-\Upsilon^2|f|^2 + e^{-2V}|f_t|^2 - \Phi|f_r|^2)$$

$$T_{\phi\phi} = \Upsilon^{-2}r^2\sin^2\theta(-\Upsilon^2|f|^2 + e^{-2V}|f_t|^2 - \Phi|f_r|^2).$$

Now, solving Einstein's equation, $G = 8\pi T$, is nothing more than equating the components of $G$ and $T$ that have been found. The equations coming from the $\theta\theta$ and $\phi\phi$ components are identical, and thus we only obtain the following four equations.

$$1 - \Phi - r\Phi_r = 8\pi r^2[|f|^2 + \Upsilon^{-2}e^{-2V}|f_t|^2 + \Upsilon^{-2}\Phi|f_r|^2]$$

$$0 = f_t\overline{f_r} + \overline{f_t}f_r$$

$$1 - \Phi - 2rV_r\Phi = 8\pi r^2[|f|^2 - \Upsilon^{-2}e^{-2V}|f_t|^2 - \Upsilon^{-2}\Phi|f_r|^2]$$

$$\Upsilon^2 r^2\left[(V_{rr} + V_r^2 + r^{-1}V_r)\Phi + \frac{1}{2}r^{-1}\Phi_r + \frac{1}{2}V_r\Phi_r\right] = 8\pi r^2(-\Upsilon^{-2}|f|^2 + e^{-2V}|f_t|^2 - \Phi|f_r|^2)$$

From the following well known formula for the coordinate expression of the d'Alembertian,

$$\Box_g f = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|}g^{\lambda\mu}\partial_\mu f).$$

the Klein–Gordon equation becomes,

$$-e^{-2V}f_{tt} + V_r\Phi f_r + \frac{1}{2}\Phi_r f_r + 2r^{-1}\Phi f_r + \Phi f_{rr} = \Upsilon^2 f.$$

The four equations coming from the Einstein equation paired with the Klein–Gordon equation form an overdetermined system of equations. As such, it suffices to solve the following system.

$$1-\Phi-r\Phi_r = 8\pi r^2[|f|^2+\Upsilon^{-2}e^{-2V}|f_t|^2+\Upsilon^{-2}\Phi|f_r|^2]$$

$$1-\Phi-2rV_r\Phi = 8\pi r^2[|f|^2-\Upsilon^{-2}e^{-2V}|f_t|^2-\Upsilon^{-2}\Phi|f_r|^2]$$

$$-e^{-2V}f_{tt}+V_r\Phi f_r+\tfrac{1}{2}\Phi_r f_r+2r^{-1}\Phi f_r+\Phi f_{rr} = \Upsilon^2 f.$$

Lastly, with the added assumption that the complex scalar field $f$ takes the form $f(t,r) = F(r)e^{i\omega t}$ for $F$ a real–valued function and $\omega$ a constant real, the system above reduces to three coupled ordinary differential equations,

$$M_r = \tfrac{4\pi r^2}{\Upsilon^2}\left[(\Upsilon^2+\omega^2 e^{-2V})F^2+\Phi F_r^2\right]$$

$$\Phi V_r = \tfrac{M}{r^2}-\tfrac{4\pi r^2}{\Upsilon^2}\left[(\Upsilon^2-\omega^2 e^{-2V})F^2-\Phi F_r^2\right]$$

$$F_{rr}+\tfrac{2}{r}F_r+V_rF_r+\tfrac{1}{2}\tfrac{\Phi_r}{\Phi}F_r = \Phi^{-1}(\Upsilon^2-\omega^2e^{-2V})F.$$

Of particular to note, the ansatz $f(t,r) = F(r)e^{i\omega t}$, led to the system above being independent of time $t$, hence we are looking at static solutions.

From [3], the low–field limit of the system above comes from imposing the approximations

$$e^{2V}\approx 1,\;\; \Phi\approx 1, \frac{V_r}{\Upsilon\|V\|_\infty}\approx 0,\;\; \frac{\Phi_r}{\Upsilon\|V\|_\infty}\approx 0,$$

in particular, imposing the condition that the metric approximates the Minkowski metric,

$$g\approx -dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2,$$

By further adding the assumption that the group velocities of the wave dark matter are much less than the speed of light, i.e. the system is in the nonrelativistic limit as well, gives the approximations,

$$\frac{\omega}{\Upsilon}\approx 1,\quad \frac{F_r}{\Upsilon\|F\|_\infty}\approx 0.$$

Thus, in the low–field nonrelativistic limit, the coupled system of ordinary differential equations above reduce to [3]

$$M_r = 4\pi r^2 \cdot 2F^2$$

$$V_r = \frac{M}{r^2}$$

$$F_{rr} + \frac{2}{r}F_r = 2\Upsilon^2 VF.$$

The second equation is the standard inverse square law of gravitation from Newtonian theory, while the first equation tells us that $2F^2$ represents the mass-energy density of the dark matter at radius $r$. However, by combining the first two equations above, the system can be written equivalently as

$$(rV)_{rr} = 8\pi rF^2$$

$$(rF)_{rr} = 2\Upsilon^2 rVF.$$

Which is precisely the Schrödinger–Newton system as seen in the next chapter.

Thus, we recover the commonly referenced result [9] that the Schrödinger–Newton system is the low–field nonrelativistic limit of the Einstein–Klein–Gordon equation. We have arrived at the Schrödinger–Newton system via a very particular avenue, but in the next section we will introduce the system in generality. We lose no focus of our goal in doing so, and other applications of this system can come into view with this level of generality. [16]

# Chapter 4

# Character of the Schrödinger–Newton system

For a single particle of mass $m$, the time–independent Schrödinger–Newton equations in $\mathbb{R}^3$ take the form

$$-\frac{\hbar^2}{2m}\Delta\psi + U\psi = \lambda\psi,$$
$$\Delta U = 4\pi G m^2 \|\psi\|^2.$$

where $\psi$ is the wavefunction, $U$ is the gravitational potential energy, $\hbar$ is Planck's constant, $G$ the gravitational constant, and $\lambda$ is the energy eigenvalue. The following change of variable,

$$\psi = \left(\frac{\hbar^2}{8\pi G m^3}\right)^{\frac{1}{2}} S, \qquad \lambda - U = \frac{\hbar^2}{2m}V,$$

gives the Schrödinger–Newton system written in a 'cleaner' form as

$$\Delta S = -SV,$$
$$\Delta V = -S^2.$$

We will consider solutions under the assumption of spherical symmetry ($V$, $S$ only dependent on radial distance from orgin, $r$), thus the system further reduces to

$$(rS)'' = -SV,$$

$$(rV)'' = -rS^2.$$

In finding a solution to the Schrödinger–Newton system, as it is a coupled system of second order differential equations, we require initial data on $S$, $V$, and their first derivatives. To begin, our initial assumption of spherically symmetric solutions requires that $S$ and $V$ are even functions of $r$, i.e. $S(-r) = S(r)$ for $r \in [0,\infty)$ (similar for $V$). Even functions have a vanishing derivative at 0, thus our initial assumption on the character of our solution forces $S'(0) = V'(0) = 0$.

There are also two scale invariances within the system. First, if the triple $(r,S,V)$ is a solution to the Schrödinger–Newton system, then so is the triple

$$(\alpha^{-1}r, \alpha^2 S(\alpha^{-1}r), \alpha^2 V(\alpha^{-1}r))$$

for $\alpha$ a constant real number. At this time, we define the normalization integral of $S$ as

$$I = \int_0^\infty r^2 S^2 dr.$$

It should be noted, that when $I$ is finite, under the scaling $(r,S) \mapsto (\hat{r},\hat{S}) = (\alpha^{-1}r, \alpha^2 S(\alpha^{-1}r))$, that the normalization integral transforms as $I \mapsto \hat{I} = \alpha^4 I$. The second scale invariance is simply that if the triple $(r,S,V)$ is a solution, then so is the triple $(r,-S,V)$.

The second scale invariance allows one to assume that the initial value of $S$, i.e. $S_0 = S(0)$, is nonnegative. The first scale invariance can be used to either set the normalization integral to

1, or to set a specific value for $S_0$ or $V_0$. In [13], Moroz and Tod set $S_0 = 1$, varied possible values for $V_0$, and applied a 'shooting method' to give analytical justification for the following.

- there is a discrete family of finite smooth solutions (bound states) labelled by the positive integers; the $n$th solution $S$ having $n-1$ zeroes;
- for these bound state solutions, $S$ is normalizable, or equivalently, $I$ is finite;
- the energy eigenvalues for each bound state are negative, increasing monotonically with $n$ towards zero.
- the ground state is nondegenerate, i.e. the dimension of the associated eigenspace to the lowest eigenvalue is one–dimensional.

It should be noted that the last bullet point has been studied extensively. In fact, Lieb [11] showed the nondegeneracy or uniqueness of the ground state in $\mathbb{R}^3$ without imposing the assumption of spherically symmetric solutions. And Choquard *et al.* [5] showed the uniqueness of the ground state regardless of the dimension of the ambient space. In [14] Moroz *et al.* found strong numerical evidence that the higher energy bound states were also nondegenerate, but analytical justification of this statement has not yet come to fruition. The goal for the remainder of this discussion will be setting up and posing an argument for this result by way of the linearization of the Schrödinger–Newton system and the implicit function theorem.

## 4.1 $V$ as a perturbation from the Coulomb Potential

For what follows we will consider $\psi = rS$. In this view the Schrödinger–Newton system becomes

$$\begin{cases} \psi'' &= -\psi V \\ (rV)'' &= -\frac{1}{r}\psi^2 \end{cases}$$

Focusing on the second equation $(rV)'' = -\frac{1}{r}\psi^2$, and integrating gives

$$(rV)'(y) = V_0 - \int_0^y \frac{1}{r}\psi^2 dr,$$

as $(rV)'(0) = (V + rV')(0) = V_0$. Integrating again and dividing by $r$ gives an integral formulation for $V$ as

$$V(r) = V_0 - \int_0^r \left(\frac{1}{x} - \frac{1}{r}\right)\psi^2 dx.$$

We use this formulation of $V$ to compute a series for $V$ in inverse powers of $r$, i.e. the Taylor series of $V$ as $r$ approaches $\infty$. The first term in the series

$$\lim_{r\to 0+} V\left(\frac{1}{r}\right) = V_0 - \int_0^\infty \frac{1}{x}\psi^2 dx + \lim_{r\to 0+}\left[r\int_0^{\frac{1}{r}} \psi^2 dx\right]$$

exists for a normalizable solution $\psi$, as $\int_0^\infty \psi^2 dx < \infty$ in this case. Recall the definition of $V$ given as $\lambda - U = \frac{\hbar^2}{2m}V$. Since $U$ is a gravitational potential that decays to 0 as $r \to \infty$, besides a scale given by constants $m$ and $\hbar$, we have an explicit expression for the energy eigenvalue $\lambda$,

$$\lambda = V_0 - \int_0^\infty \frac{1}{x}\psi^2 dx.$$

To find the higher order terms, we will use a change of variable $u = \frac{1}{r}$. Direct computation yields

$$\frac{dV}{du}(0) = \lim_{u\to 0+}\int_0^{\frac{1}{u}} \psi^2 dx = I = 1,$$

where we have exploited the scale invariance to set $I = 1$. Continuing on,

$$\frac{d^2V}{du^u} = -\frac{1}{u^2}\psi^2\left(\frac{1}{u}\right).$$

Let us momentarily pause here to note some other properties $\psi$ has as a normalizable solution to the system. In [13], the following was shown

- For normalizable solutions to exist it must be that $V_0 > 0$.
- $V$ is monotonically decreasing on $[0,\infty)$ and has a zero at some finite value of $r$.
- At a normalizable solution $\psi$, and for $r = b$ with $V(b) = -C^2$, then for $r > b$

$$0 \leq \psi(r) \leq \psi(b)e^{-Cr}.$$

Because of this, for $u$ close enough to 0, we have

$$0 \leq \frac{d^2V}{du^2} \leq \psi^2(b)\frac{e^{-\frac{C}{u}}}{u^2},$$

which implies that $\lim_{u\to 0+}\frac{d^2V}{du^2} = 0$.

For the sake of completeness, we show two remaining computations. First,

$$\frac{d^3V}{du^3} = \frac{2\psi^2(\frac{1}{u})}{u^3} + \frac{2\psi(\frac{1}{u})\psi'(\frac{1}{u})}{u^4}$$

The exponential decay seen in the solution $\psi$ as $u \to 0+$ forces the same property in the derivative, i.e. $\psi' \to 0$ as $u \to 0+$. Thus, $\lim_{u\to 0+}\frac{d^3V}{du^3} = 0$. Second,

$$\frac{d^4V}{du^4} = \frac{4\psi(\frac{1}{u})\psi'(\frac{1}{u})}{u^5} - \frac{6\psi^2(\frac{1}{u})}{u^4} - \frac{2\psi(\frac{1}{u})\psi''(\frac{1}{u})}{u^6} - \frac{2\left[\psi'(\frac{1}{u})\right]^2}{u^6}.$$

All of the terms in the prior formula go to zero as $u \to 0+$ with the possible exception of the last term. While we know that $\psi' \to 0$ as $u \to 0+$, we have not yet given justification that it decays to zero faster than any polynomial. This is no issue however, a quick application of L'Hopital gives

$$\lim_{u\to 0+} \frac{2\left[\psi'(\frac{1}{u})\right]^2}{u^6} = \lim_{u\to 0+} \frac{2\psi(\frac{1}{u})\psi'(\frac{1}{u})V(\frac{1}{u})}{3u^7} = 0.$$

In the above computation $\psi'' = -\psi V$ was used. And so $\lim_{u\to 0+} \frac{d^4V}{du^4} = 0$. From this point on taking further derivatives will have only terms that are polynomial in $\psi(\frac{1}{u})$, $\psi'(\frac{1}{u})$, and $V(\frac{1}{u})$ in the numerator. As $V(\frac{1}{u})$ approaches a finite value as $u \to 0+$ we have that

$$\lim_{u\to 0+} \frac{d^nV}{du^n} = 0 \text{ for } n \geq 5.$$

Because of this the function $F(r) = V(r) - \lambda - \frac{1}{r}$ is a $C^\infty([0,\infty))$ non–analytic function. The integral representation of $F$ is as follows,

$$F(r) = \int_r^\infty \left(\frac{1}{x} - \frac{1}{r}\right)\psi^2 dx$$

In this sense $V$ can be thought of as a nonlinear perturbation of the Coulomb potential, and the equation $\psi'' = -\psi V$ in the Schrödinger–Newton becomes

$$-\psi'' - \frac{1}{r}\psi - \lambda\psi - F(\psi)\psi = 0,$$

where $F(\psi)$ is shorthand for $F(\psi)(r)$ denoting the dependence of the nonlinear perturbation $F$ on $\psi$. Because of this, the character of the solutions to the linearized equation

$$-\psi'' - \frac{1}{r}\psi - \lambda\psi = 0$$

becomes of interest.

# Chapter 5

# The Hydrogen Atom on the Half–Line $[0,\infty)$.

The equation that closed the prior section was precisely the Schrödinger equation with the Coulomb potential on the half–line, which is also called the equation for the Hydrogen Atom on the half–line. From Theorem IX.26 of [18] we have that weak solutions to the equation

$$\left(-\frac{d^2}{dr^2}+V(r)\right)\psi = E\psi,$$

for $E$ a complex number, are $C^\infty$ functions on an open region $\Omega$ if $V(r)$ is equal to a $C^\infty$ function on the same open region $\Omega$. Thus as $V(r) - -\frac{1}{r}$ on $(0,\infty)$, we have that solutions to

$$-\psi'' - \frac{1}{r}\psi - \lambda\psi = 0$$

are $C^\infty$ on $(0,\infty)$. For what follows, let us take $\lambda$ of the form $\lambda = -\beta^2$ for $\beta \geq 0$. Thus we looking for solutions of $\psi'' + (\frac{1}{r} - \beta^2)\psi = 0$.

We expect solutions, $\psi$, of the equation to have a zero at $r = 0$ to counteract the singularity in the Coulomb potential. For large values of $r$, the solution $\psi$ solves the asymptotic equation

$$\psi'' - \beta^2\psi = 0,$$

which has solutions $\psi = e^{\beta r}$, $e^{-\beta r}$. As we expect our solutions to normalizable on the half–line, we see that $\psi$ has the asymptotic behavior of $e^{-\beta r}$. Thus, we 'peel off' this behavior of $\psi$ at $r = 0$ and as $r$ nears infinity by assuming that $\psi$ has the following form: $\psi = re^{-\beta r}\varphi$. Doing so leads to the following

$$0 = \psi'' + \left(\frac{1}{r} - \beta^2\right)\psi$$
$$0 = e^{-\beta r}[r\varphi'' + (2 - 2\beta r)\varphi' + (1 - 2\beta)\varphi]$$

Thus $\varphi$ is a solution to $r\varphi'' + (2 - 2\beta r)\varphi' + (1 - 2\beta)\varphi = 0$. By multiplying by $2\beta$,

$$2\beta r\varphi'' + (2 - 2\beta r)2\beta\varphi' + (1 - 2\beta)2\beta\varphi = 0.$$

Writing $\varphi = F(2\beta r)$, and calling $s = 2\beta r$, we find that $F$ satisfies the following differential equation

$$sF'' + (2 - s)F' + \alpha F = 0, \quad \text{where } \alpha = \frac{1 - 2\beta}{2\beta}.$$

At this point, we employ a power series method of solution to the above problem. If we take $F$ to be of the form $F(s) = \sum_{m=0}^{\infty} a_m s^m$, then we have the following recurrence relation amongst the coefficients of $F$,

$$a_{m+1} = \frac{(m - \alpha)a_m}{(m+1)(m+2)},$$

with $a_0$ being arbitrary. In particular $a_0$ is analoguous to $S_0$ in the original Schrödinger–Newton system, and we have the freedom to specify this value for a particular choice of initial condition or for normalizing our solution.

From this point, we have two cases, either $\alpha \in \mathbb{N}$ and the series for $F$ terminates at some point or $\alpha \notin \mathbb{N}$. We first give an argument for why it must be that $\alpha \in \mathbb{N}$. If this is not the case, i.e. $\alpha \notin \mathbb{N}$, then for large values of $m$, we have that the recurrence relation asymptotically becomes

$$a_{m+1} \approx \frac{a_m}{m+3}.$$

If we call $N \in \mathbb{N}$, the number for which $m > N$ implies the asymptotic relation above holds with sufficient accuracy, then $F$ can be written approximately as

$$F \approx \sum_{m=0}^{N} a_m s^m + 2a_{N+1} s^{N-1}[e^s - s - 1].$$

Which gives an approximation for $\psi$ as

$$\psi \approx re^{-\beta r} g(r) + rf(r)e^{\beta r}$$

for $f(r)$, $g(r)$ polynomials. However, this contradicts our assumption that $\psi$ is normalizable, i.e if this were the case then $\psi \notin L^2([0,\infty))$. Thus, it must be the case that $\alpha \in \mathbb{N}$.

Thus, assume that $\alpha = n$. As $\alpha = \frac{1-2\beta}{2\beta}$, we then have

$$\beta = \frac{1}{2(n+1)}, \quad \text{and} \quad -\beta^2 = -\frac{1}{4(n+1)^2},$$

which is perhaps unsurprising given the standard formula for the energy levels of the Hydrogen atom, $E = -\frac{13.6\text{eV}}{n^2}$. Also, our recurrence relation collapses into the formula

$$a_k = (-1)^k \frac{(n+1)\cdots(k+2)}{k!(n-k)!} \left[\frac{a_0}{n+1}\right], \; 0 \le k \le n,$$

in which the numerator is understood to be equal to 1 if $k+2 > n+1$. This shows that $F$ is actually a generalized Laguerre polynomial.

$$F(s) = \left[\frac{a_0}{n+1}\right] \mathrm{L}_n^1(s),$$

Thus we have found a countable family of solutions to the Hydrogen atom on the half–line given by

$$\psi_n(r) = \left[\frac{a_0}{n+1}\right] re^{-\frac{r}{2(n+1)}} \mathrm{L}_n^1\left(\frac{r}{n+1}\right),$$

with associated eigenvalue $\lambda_n = -\frac{1}{4(n+1)^2}$. In the following sections, arguments that the solutions found above are the only solutions to the Hydrogen atom on the half line will be presented, as well as proof that the eigenvalues are in fact simple.

## 5.1 The zeroes of the generalized Laguerre polynomials

As we saw in the previous section, solutions to the equation

$$sF'' + (2-s)F' + \alpha F = 0$$

are precisely the generalized Laguerre polynomials $\mathrm{L}_n^1(s)$. A quick computation shows that for each $n \in \mathbb{N}$, the generalized Laguerre polynomial $\mathrm{L}_n^1(s)$ also satisfies the equation

$$[s^2 e^{-s} F']' + nse^{-s} F = 0$$

Defining the following

$$p(s) = P(s) = s^2 e^{-s}$$

$$q(s) = nse^{-s}$$

$$Q(s) = (n+1)se^{-s}.$$

Then as $Q(s) \geq q(s)$ we are precisely in the situation to invoke the Sturm comparison theorem. If we assume that $\mathrm{L}_n^1(s)$ has $m$ zeroes and list them $x_1, x_2, \ldots, x_m$, then by Sturm comparison

$\mathrm{L}_{n+1}^1(s)$ has at least a zero in each interval $(x_1,x_2),(x_2,x_3),\ldots,(x_{m-1},x_m)$. Thus, for the moment, at minimum we can guarantee that $\mathrm{L}_{n+1}^1(s)$ has at least $m-1$ zeroes.

The final statement in the previous paragraph relies upon the fact that zeroes of the generalized Laguerre polynomials are of multiplicity one. Clearly this must be the case, otherwise a zero of multiplicity higher than one would be a zero of the generalized Laguerre polynomial as well as its derivative. By viewing the generalized Laguerre polynomial in a Taylor series centered about this zero paired with the fact that the generalized Laguerre polynomial satisfies a second order ordinary differential equation, one would find that the generalized Laguerre polynomial would be identically zero.

Moving forward, the polynomial expansion of $\mathrm{L}_n^1(s)$ is

$$\mathrm{L}_n^1(s) = \sum_{k=0}^{n} \frac{(n+1)n\cdots(k+2)}{k!(n-k)!}(-s)^k,$$

where the coefficient in the sum is understood to be 1 when $k+2 > n+1$. This shows that $\mathrm{L}_n^1(0) = (n+1)$. Use this to define $\varphi_n(s) = \frac{1}{n+1}\mathrm{L}_n^1(s)$, and to define

$$G(s) = \frac{\varphi_{n+1}(s)}{\varphi_n(s)}.$$

Thus $G(0) = 1$. The coefficient on the $x$–term of $\varphi_n(s)$ is $-\frac{n}{2}$, and this shows that

$$G'(0) = \frac{-\frac{n+1}{2} - \left[-\frac{n}{2}\right]}{1^2} = -\frac{1}{2}.$$

And so $G$ is initially decreasing. Since $\varphi_n(s)$ satisfies the same differential equation as $\mathrm{L}_n^1(s)$, we have that $G$ satisfies the following differential equation,

$$xG'' + [2 - x(1 - 2(\ln\varphi_n)')]G' + G = 0.$$

Now let $x_1$ denote the first zero of $\varphi_n(s)$, i.e. $\varphi_n > 0$ on $[0, x_1)$. We aim to show that $\varphi_{n+1}(s)$ has a zero in the interval $[0, x_1)$. Suppose this is not the case, thus $G > 0$ on the entirety of the interval $[0, x_1)$.

At $s = 0$, $G'(0) < 0$, and we aim to show that $G' < 0$ on the whole interval $[0, x_1)$. If this is not the case, i.e. call $y \in [0, x_1)$ a zero of $G'$, then $y$ is either where a local minimum of $G$ occurs ($y$ is a zero of multiplicity one of $G'$) or $G''(y) = 0$ ($y$ is a zero of $G'$ of multiplicity higher than 1). In either case $G''(y) \geq 0$. Then from the differential equation $G$ satisfies, we find

$$yG''(y) + G(y) = 0$$

This is a clear contradiction. Thus $G' < 0$ on the interval $[0, x_1)$. Thus $G(s) < G(0) = 1$ on $[0, x_1)$ which is equivalent to saying $\varphi_{n+1}(s) < \varphi_n(s)$ for all $s \in [0, x_1)$. And this is sufficient to show that $\varphi_{n+1}(s)$ has a zero in the interval $[0, x_1]$. And thus we have shown that $\mathrm{L}^1_{n+1}(s)$ has at least $m$ zeroes.

On the other hand, letting $x_m$ denote the last zero of $\varphi_n(s)$, then without loss of generality we may assume that $\varphi_n(s) > 0$ on $(x_m, \infty)$. From the definition of the Laguerre polynomials

$$\mathrm{L}^1_n(s) = \sum_{k=0}^{n} \frac{(n+1)n\cdots(k+2)}{k!(n-k)!}(-s)^k,$$

we notice the change in parity of the highest degree term in moving from $\mathrm{L}^1_n(s)$ to $\mathrm{L}^1_{n+1}(s)$. In particular, as the end behavior of $\varphi_n(s)$ is $\varphi_n \to \infty$ as $s \to \infty$, we see that $\varphi_{n+1} \to -\infty$ as $s \to \infty$. In the case that $\varphi_{n+1}(s)$ has $m$ zeroes in the intevral $[0, x_m]$, as both $\varphi_n(0) = \varphi_{n+1}(0) = 1$ and the zeroes of both $\varphi_n$ and $\varphi_{n+1}$ are simple, we have that $\varphi_n$ and $\varphi_{n+1}$ must have the same sign initially in the interval $(x_m, \infty)$. Thus, the change in end behavior shows that $\varphi_{n+1}$ must have a zero in $(x_m, \infty)$. Thus we have shown that $\varphi_{n+1}$ has at least $m+1$ zeroes.

It is clear that $\varphi_1(s) = 1 - \frac{1}{2}s$, and thus $\varphi_1$ has one zero. Thus, by an inductive process, and the argument above, we have that $\varphi_n$ has at least $n$ zeroes. And as the degree of $\varphi_n$ is equal to $n$, we have that $\varphi_n$ has exactly $n$ zeroes for each $n \in \mathbb{N}$. In particular, this gives that each solution $\psi_n$ of the Hydrogen atom on the half–line

$$\psi_n(r) = \left[\frac{a_0}{n+1}\right] re^{-\frac{r}{2(n+1)}} \mathrm{L}_n^1\left(\frac{r}{n+1}\right),$$

has exactly $n+1$ zeroes.

## 5.2 The only eigenvalues are $\lambda_n$

For a moment, if we go back and look at the Schrödinger–Newton system

$$(rS)'' = -SV$$

$$(rV)'' = -rS^2.$$

or it's equivalent formulation with $\psi = rS$

$$\begin{cases} \psi'' &= -\psi V \\ (rV)'' &= -\frac{1}{r}\psi^2 \end{cases}$$

We notice a connection with our solutions to the Hydrogen atom on the half–line

$$\psi_n(r) = \left[\frac{a_0}{n+1}\right] re^{-\frac{r}{2(n+1)}} \mathrm{L}_n^1\left(\frac{r}{n+1}\right)$$

namely the zero at $r = 0$. As such we now give rigid definition of exactly what Hilbert space we are working in.

Thus define,

$$H_1 : D(H_1) \subset L^2([0,\infty)) \to L^2([0,\infty)), \quad H_1\varphi = -\varphi'' - \frac{1}{r}\varphi,$$

with the domain of our operator $H_1$ being

$$D(H_1) = \{\varphi \in H^2([0,\infty)) \mid \varphi(0) = 0\},$$

the Sobolev space of functions with square integrable weak derivatives up to second order with a specified value of 0 at $r = 0$. We argue that in solving the eigenvalue problem

$$H_1\varphi = \mu\varphi,$$

that the only eigenvalues are of the form $\lambda_n$.

Thus assume that there is an eigenvalue $\mu$ with $\lambda_n < \mu < \lambda_{n+1}$ and an associated eigenfunction $\varphi$ with $H_1\varphi = \mu\varphi$. But then we have the following

$$\psi_n'' + \left(\frac{1}{r} + \lambda_n\right)\psi_n = 0$$
$$\varphi'' + \left(\frac{1}{r} + \mu\right)\varphi = 0.$$

By taking $p = P = 1$ and $q = \frac{1}{r} + \lambda_n < \frac{1}{r} + \mu = Q$, the Sturm comparison theorem gives that $\varphi$ has at least one zero in each interval $(0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n)$, where $0,x_1,x_2,\ldots,x_n$ are the $n+1$ zeroes of $\psi_n$. As $\varphi \in D(H)$ and therefore $\varphi(0) = 0$, we have that $\varphi$ has at least $n+1$ zeroes.

Next assume to the contrary that $\varphi$ has no zero in the interval $[x_n,\infty)$. Without loss of generality we may assume that $\psi_n$ and $\varphi$ are nonnegative on $[x_n,\infty)$ (if not, just replace $\varphi$ with $-\varphi$, etc.). Now, the Wronskian, $W(\varphi,\psi_n) = \varphi\psi_n' - \varphi'\psi_n$ has as it's derivative

$$\begin{aligned} W'(\varphi,\psi_n) &= \varphi\psi_n'' - \varphi''\psi_n \\ &= (\mu - \lambda_n)\varphi\psi_n \geq 0. \end{aligned}$$

In particular $W'(\varphi, \psi_n)(x_n) = 0$ and $W'(\varphi, \psi_n) > 0$ on $(x_n, \infty)$. And $W(\varphi, \psi_n)(x_n) = \varphi(x_n)\psi_n'(x_n) > 0$ as $\psi_n > 0$ on $(x_n, \infty)$ and zeroes of $\psi_n$ are of multiplicity one. However, this would imply that the Wronskian is bounded away from 0 as $r \to \infty$, which contradicts $\varphi, \psi_n \in D(H_1)$. Thus, $\varphi$ has a zero in the interval $[x_n, \infty)$, and thus has $n+2$ zeroes.

But, as $\mu < \lambda_{n+1}$, this same argument can be performed with $\varphi$ and $\psi_{n+1}$ in place of $\psi_n$ and $\varphi$ respectively. This would show that $\psi_{n+1}$ has at least $n+3$ zeroes, which is a direct contradiction of the fact that $\psi_{n+1}$ has exactly $n+2$ zeroes. Thus, there can not be an eigenvalue $\mu$ between $\lambda_n$ and $\lambda_{n+1}$ for any $n \in \mathbb{N}$.

## 5.3 The simplicity of the eigenvalues $\lambda_n$

Recall our regularity result, i.e. Theorem IX.26 [18], which states that weak solutions to the equation

$$\left(-\frac{d^2}{dr^2} + \frac{1}{r}\right)\psi = E\psi,$$

for $E$ a complex number are $C^\infty$ functions on $(0, \infty)$. If we assume that $\psi$ and $\varphi$ are two eigenfunctions of $H_1$ associated to the eigenvector $\lambda_n$, then we have $\psi, \varphi$ are $C^\infty$ on $(0, \infty)$. Thus it immediately follows that the Wronskian of $\psi$ and $\varphi$ is also $C^\infty$ on $(0, \infty)$.

Our assumption that $\varphi$ and $\psi$ both satisfy $H_1\rho = \lambda_n\rho$ gives that $W'(\varphi, \psi) = 0$, and as such is constant on the interval $(0, \infty)$. As $\varphi, \psi \in D(H)$, and therefore decay to 0 as $r \to \infty$, we must have that $W(\varphi, \psi) = 0$. Thus $\varphi$ and $\psi$ are linearly dependent. Thus, after normalizing each solution, the dimension of the eigenspace associated to $\lambda_n$ is exactly one.

# Chapter 6

# A Fredholm operator of index zero

In this section we define a slight variation on our operator $H_1$. In view of what is to come in chapter 7, it will be of interest that our operator is Fredholm of index 0. The operator $H_1$ does not have this property, but an analogue of $H_1$ defined on weighted Banach spaces does. Thus define

$$H : D(H) \subseteq L^2([0,\infty),(1+r)^2) \to L^2([0,\infty),(1+r)^2)$$

with $D(H) = \{\varphi \in H^2([0,\infty),(1+r)^2) \mid \varphi(0) = 0\}$, and

$$H = -\frac{d^2}{dr^2} - \frac{1}{r} - \lambda_n,$$

where $\lambda_n = -\frac{1}{4(n+1)^2}$.

As we will see in section 6.2, the domain of the operator $H$, $D(H)$, is a Banach space itself. We give the definition of $H$ above as it will suit different needs in the sections to come to consider $H$ as both an operator from $L^2([0,\infty),(1+r)^2)$ to itself and a mapping $H : D(H) \to L^2([0,\infty),(1+r)^2)$. This chapter is primarily devoted to proving the following

**Theorem:** The mapping $H : D(H) \to L^2([0,\infty),(1+r)^2)$ is a Fredholm operator of index zero.

A result that will be proven in parts over the coming sections. We begin by laying some groundwork in the form of inequalities that will play a vital role for the remainder of this chapter as well as the next.

## 6.1 Some important inequalities

For shorthand in what follows, the spaces $L^2_w$ and $H^2_w$ will be used to signify the weighted Hilbert spaces $L^2([0,\infty),(1+r)^2)$ and $H^2([0,\infty),(1+r)^2)$ respectively with weight $w(r) = (1+r)^2$. Also $\|\cdot\|$ and $\|\cdot\|_w$ will be used to represent the $L^2$ norms in the unweighted and weighted spaces respectively. Lastly, common notations appended with $w$ such as $\overline{A}^w$ and $A^{\perp_w}$ will be used to mean closure and orthogonal complement with respect to the weighted spaces.

Let $\{y_k\}$ be a sequence of elements in $\mathrm{Ran}(H)$, hence there is a sequence of elements $\{x_k\} \subset D(H)$ with $y_k = Hx_k$. Thus,

$$y_k = -x_k'' - \frac{1}{r}x_k - \lambda_n x_k.$$

By multiplying by $x_k(1+r)^2$ to both sides and integrating, we have

$$\int_0^\infty x_k y_k (1+r)^2 dr = -\int_0^\infty x_k'' x_k (1+r)^2 dr - \int_0^\infty \frac{1}{r} x_k^2 (1+r)^2 dr - \lambda_n \int_0^\infty x_k^2 (1+r)^2 dr.$$

Integration by parts twice shows that

$$-\int_0^\infty x_k'' x_k (1+r)^2 dr = \|x_k'\|_w^2 + \|x_k\|^2.$$

And the Holder and Young inequality gives

$$\int_0^\infty y_k x_k (1+r)^2 dr \leq \|x_k\|_w \|y_k\|_w \leq \frac{\gamma}{2}\|x_k\|_w^2 + \frac{1}{2\gamma}\|y_k\|_w^2$$

for an arbitrary positive constant $\gamma$. If we take $\gamma = -2\lambda_n$, then

$$\int_0^\infty y_k x_k (1+r)^2 dr \leq -\lambda_n \|x_k\|_w^2 + (n+1)^2 \|y_k\|_w^2.$$

Thus by canceling the $-\lambda_n \|x_k\|_w^2$ terms, we have

$$\|x_k'\|_w^2 + \|x_k\|^2 \leq (n+1)^2 \|y_k\|_w^2 + \int_0^\infty \frac{1}{r} x_k^2 (1+r)^2 dr.$$

Clearly,

$$\int_0^\infty \frac{1}{r} x_k^2 (1+r)^2 dr = \int_0^\infty \frac{1}{r} x_k^2 dr + \int_0^\infty r x_k^2 dr + 2\|x_k\|^2.$$

We will deal with each of these integrals individually.

Beginning again with

$$y_k = -x_k'' - \frac{1}{r} x_k - \lambda_n x_k.$$

And multiplying by $x_k r$ to both sides and integrating, we have

$$\int_0^\infty x_k y_k r dr = -\int_0^\infty x_k'' x_k r dr - \int_0^\infty x_k^2 dr - \lambda_n \int_0^\infty x_k^2 r dr.$$

Integration by parts gives

$$-\int_0^\infty x_k'' x_k r dr = \|\sqrt{r} x_k'\|^2.$$

And use of the Holder and Young inequalities again give

$$\int_0^\infty y_k x_k r dr \leq \frac{-\lambda_n}{2} \|\sqrt{r} x_k\|^2 + 2(n+1)^2 \|\sqrt{r} y_k\|^2.$$

Thus

$$\frac{1}{8(n+1)^2} \|\sqrt{r} x_k\|^2 + \|\sqrt{r} x_k'\|^2 \leq 2(n+1)^2 \|\sqrt{r} y_k\|^2 + \|x_k\|^2.$$

As $\|\sqrt{r}x_k'\|^2 \geq 0$ and $\|\sqrt{r}y_k\|^2 \leq \|y_k\|_w^2$, we have

$$\int_0^\infty rx_k^2 dr \leq 16(n+1)^4\|y_k\|_w^2 + 8(n+1)^2\|x_k\|^2.$$

We now bound the last remaining integral. Let $\varepsilon > 0$ be sufficiently close to 0, and for what follows we will take $\delta > 0$ small enough so that $2\delta\ln\delta + \varepsilon < \frac{1}{2}$ and also so that on the interval $(0,\delta)$ we have $(\ln r)^2 < \frac{\varepsilon}{16r} + 1$. It is clear that

$$\int_0^\infty \frac{1}{r}x_k^2 dr = \int_0^\delta \frac{1}{r}x_k^2 dr + \int_\delta^\infty \frac{1}{r}x_k^2 dr.$$

and

$$\int_\delta^\infty \frac{1}{r}x_k^2 dr \leq \frac{1}{\delta}\|x_k\|^2.$$

Without loss of generality $\delta < 1$, so integration by parts on the first term gives

$$\int_0^\delta \frac{1}{r}x_k^2 dr = [x_k(\delta)]^2\ln\delta + \int_0^\delta 2x_k x_k'|\ln r|dr,$$

where the negative from the integration by parts formula was absorbed by the $\ln r$ term to give $|\ln r|$ in the integrand. For the first term above, Holder's inequality and that $\|\cdot\|_w$ is a stronger norm than $\|\cdot\|$ gives

$$[x_k(\delta)]^2\ln\delta = \left[\int_0^\delta x_k' dr\right]^2 \ln\delta \leq \|x_k'\|^2\delta\ln\delta \leq \|x_k'\|_w^2\delta\ln\delta.$$

The second term via Holder's inequality becomes

$$\int_0^\delta 2x_k x_k'|\ln r|dr \leq \frac{\varepsilon}{2}\|x_k'\|^2 + \frac{8}{\varepsilon}\int_0^\delta x_k^2(\ln r)^2 dr.$$

By the assumption we placed on $\delta$

$$\int_0^\delta 2x_k x_k'|\ln r|dr \leq \frac{\varepsilon}{2}\|x_k'\|_w^2 + \frac{1}{2}\int_0^\delta \frac{1}{r}x_k^2 dr + \frac{8}{\varepsilon}\|x_k\|^2$$

Thus

$$\int_0^\infty \frac{1}{r} x_k^2 dr \leq \|x_k'\|_w^2 [2\delta \ln \delta + \varepsilon] + \left[\frac{16}{\varepsilon} + \frac{1}{\delta}\right] \|x_k\|^2.$$

Putting this all together, we have

$$\int_0^\infty \frac{1}{r} x_k^2 (1+r)^2 dr \leq 16(n+1)^4 \|y_k\|_w^2 + [2\delta \ln \delta + \varepsilon] \|x_k'\|_w^2 + \left[8(n+1)^2 + \frac{16}{\varepsilon} + \frac{1}{\delta} + 2\right] \|x_k\|^2.$$

And so from our assumption that $2\delta \ln \delta + \varepsilon < \frac{1}{2}$, we have

$$\|x_k'\|_w^2 \leq \left[\frac{(n+1)^2 + 16(n+1)^4}{1 - 2\delta \ln \delta - \varepsilon}\right] \|y_k\|_w^2 + \left[\frac{8(n+1)^2 + \frac{16}{\varepsilon} + \frac{1}{\delta} + 1}{1 - 2\delta \ln \delta - \varepsilon}\right] \|x_k\|^2.$$

Our goal now is to find a bound on the $H_w^2$ norm of the $x_k$ in terms of the weighted norm of $y_k$ and the unweighted $L^2$ norm of the $x_k$ terms. By direct computation

$$\|x_k\|_w^2 = \int_0^\infty x_k^2 (1+r)^2 dr = \|x_k\|^2 + 2 \int_0^\infty r x_k^2 dr + \int_0^\infty r^2 x_k^2 dr.$$

The last term $\int_0^\infty r^2 x_k^2 dr$ is the only term we have not found an explicit bound for. By following the same argument for bounding the $\int_0^\infty r x_k^2 dr$ term, we find

$$\frac{1}{8(n+1)^2} \int_0^\infty r^2 x_k^2 dr \leq 2(n+1)^2 \|y_k\|_w^2 + \|x_k\|^2 + \int_0^\infty r x_k^2 dr.$$

Thus for constants $A(n)$ and $B(n)$, dependent on $n \in \mathbb{N}$, we have

$$\|x_k\|_w^2 \leq A(n) \|y_k\|_w^2 + B(n) \|x_k\|^2.$$

For the second derivative term, as

$$-x_k'' = y_k + \frac{1}{r} x_k + \lambda_n x_k,$$

the triangle inequality gives

$$\|x_k''\|_w^2 \leq 3[\|y_k\|_w^2 + \|\frac{1}{r} x_k\|_w^2 + |\lambda_n|^2 \|x_k\|_w^2].$$

The middle term breaks into

$$\|\frac{1}{r}x_k\|_w^2 = \int_0^\infty \frac{1}{r^2}x_k^2 dr + 2\int_0^\infty \frac{1}{r}x_k^2 dr + \|x_k\|^2.$$

Similarly the only term we have yet to bound is $\|\frac{1}{r}x_k\|^2$. Via integration by parts,

$$\int_0^\infty \frac{1}{r^2}x_k^2 dr = 2\int_0^\infty \frac{1}{r}x_k x_k' dr.$$

And application of the Holder and Young inequalities one more time yields

$$\int_0^\infty \frac{1}{r^2}x_k^2 dr \leq \frac{1}{2}\int_0^\infty \frac{1}{r^2}x_k^2 dr + 2\|x_k'\|^2.$$

And thus

$$\int_0^\infty \frac{1}{r^2}x_k^2 dr \leq 4\|x_k'\|_w^2.$$

Using prior inequalities we have found, we know that $\int_0^\infty \frac{1}{r}x_k^2 dr$, $\|x_k\|_w^2$, and $\|x_k'\|_w^2$ can be bounded by $\|y_k\|_w^2$ and $\|x_k\|^2$. Thus, we have a bound on the second derivative $x_k''$. By putting all of these results together, we have

$$\|x_k\|_{H_w^2}^2 \leq A(\varepsilon,\delta,n)\|y_k\|_w^2 + B(\varepsilon,\delta,n)\|x_k\|^2,$$

for $A(\varepsilon,\delta,n)$, $B(\varepsilon,\delta,n)$ constants dependent on $n$ and the choice of $\varepsilon$ and $\delta$.

## 6.2 $H$ is closed as an operator on $L_w^2$, bounded as a map on $H_w^2$.

Recall the domain of $H$ given by $D(H) = \{\varphi \in H_w^2 \mid \varphi(0) = 0\}$. Assume that there is a sequence $\{\psi_n\} \subseteq D(H)$ with $\psi_k \to \psi$ in $L_w^2$, and $y_k = H\psi_k$ with $y_k \to x$ in $L_w^2$. To show that $H$ is closed as an operator on $L_w^2$ we must show that $\psi \in D(H)$ and $x = H\psi$.

As $\|\cdot\|_w$ is a stronger norm than the usual $L^2$ norm, the inquality at the end of section 6.1 gives that $\|\psi_k - \psi\|_{H_w^2} \to 0$ as $k \to \infty$. Thus $\psi \in H_w^2$. And

$$[\psi(0)]^2 = \left[\int_0^\infty \psi' - \psi_k' dr\right]^2 = \left[\int_0^\infty \frac{1}{1+r}(\psi - \psi_k)'(1+r)dr\right]^2$$
$$\leq \|\psi' - \psi_k'\|_w^2 \int_0^\infty \frac{1}{(1+r)^2} dr \leq \|\psi - \psi_k\|_{H_w^2}^2.$$

Thus as the left hand side is independent of $k$, taking the limit of both sides shows that $\psi(0) = 0$, hence $\psi \in D(H)$.

From section 6.1, we have that

$$\int_0^\infty \frac{1}{r}\psi_k^2(1+r)^2 dr \leq A(\varepsilon, \delta, n)\|H\psi_k\|_w^2 + B(\varepsilon, \delta, n)\|\psi_k\|^2,$$

which shows that $\{\frac{1}{r}\psi_k\}$ is Cauchy in $L_w^2$ therefore $\frac{1}{r}\psi_k \to \frac{1}{r}\psi$ in $L_w^2$. But then $y_k \to -\psi - \frac{1}{r}\psi - \lambda_n\psi = H\psi$ as $k \to \infty$. Thus, by the uniqueness of limits, $x = H\psi$. And so, $H$ is closed.

Now, for the boundedness of $H$ on $H_w^2$. For $\varphi \in D(H)$, the triangle inequality gives

$$\|H\varphi\|_w^2 \leq 3[\|\varphi''\|_w^2 + \|\frac{1}{r}\varphi\|_w^2 + |\lambda_n|^2\|\varphi\|_w^2],$$

and as we saw earlier,

$$\int_0^\infty \frac{1}{r}\varphi^2 dr \leq \|\varphi'\|_w^2[2\delta \ln\delta + \varepsilon] + \left[\frac{16}{\varepsilon} + \frac{1}{\delta}\right]\|\varphi\|^2$$
$$\int_0^\infty \frac{1}{r^2}\varphi^2 dr \leq 4\|\varphi'\|_w^2.$$

Thus for a fixed choice of $\varepsilon$ and $\delta$, there exists some positive $M \in \mathbb{R}$ such that

$$\|H\varphi\|_w^2 \leq M\|\varphi\|_{H_w^2}^2,$$

and thus $H$ is bounded.

## 6.3 The Kernel of $H$

As we saw in section 5, for the operator $H_1 : D(H_1) \subset L^2([0,\infty)) \to L^2([0,\infty))$, we found that $\mathrm{Ker}(H_1 - \lambda_n) = \mathrm{span}(\psi_n)$, for $\psi_n$ the eigenvector of $H_1$ with eigenvalue $\lambda_n$. As

$$\psi_n(r) = re^{-\frac{r}{2(n+1)}} \mathrm{L}_n^1\left(\frac{r}{n+1}\right),$$

decays exponentially as $r \to \infty$, we have that $\psi_n \in D(H) = \{\varphi \in H_w^2 \mid \varphi(0) = 0\}$. Thus $\mathrm{Ker}(H) = \mathrm{span}(\psi_n)$ and is clearly of dimension one.

## 6.4 The range of $H$ is closed

We begin this section with a theorem. A theorem reminiscent of the standard Sobolev embedding result for bounded open domains in $\mathbb{R}$. For the sake of completeness the Kondrachov compactness theorem [8] tells us that for any bounded open intevral $[0,R)$ in the half–line that $H^1([0,R))$ compactly embeds in $C([0,R])$. As the supremum norm on continuous functions over a compact set is stronger than the $L^2$ norm over the same compact set, we have that $H^1([0,R))$ compactly embeds in $L^2([0,R])$.

**Theorem:** The Banach space $H_w^1([0,\infty))$ with weight $w(r) = (1+r)^2$ compactly embeds in $L^2([0,\infty))$.

*Proof.* Let $\{\varphi_n\}$ be a sequence in $H_w^1([0,\infty))$, and assume that $\{\varphi_n\}$ is uniformly bounded, i.e. there exists a positive $M \in \mathbb{R}$ such that $\|\varphi_n\|_{H_w^1} \leq M$. As we will see shortly, the introduction of the weight $w(r) = (1+r)^2$ allows one to effectively truncate the sequence $\{\varphi_n\}$ outside of some compact set $[0,R]$. Inside of the compact set $[0,R]$ the standard Kondrachov compactness theorem applies. This can be seen explicitly in the following inequality,

$$\begin{aligned}\|\varphi_n - \varphi_m\|^2 &= \int_0^R (\varphi_n - \varphi_m)^2 dr + \int_R^\infty \frac{(1+r)^2}{(1+r)^2} (\varphi_n - \varphi_m)^2 dr \\ &\leq \int_0^R (\varphi_n - \varphi_m)^2 dr + \frac{1}{(1+R)^2} \|\varphi_n - \varphi_m\|_{H_w^1} \\ &\leq \int_0^R (\varphi_n - \varphi_m)^2 dr + \frac{2M}{(1+R)^2}.\end{aligned}$$

Then there exists $R_1 \in \mathbb{R}$ large enough that $\frac{2M}{(1+R_1)^2} < \frac{1}{2}$, and by the Kondrachov compactness theorem there is a subsequence $\{\varphi_n^1\} \subseteq \{\varphi_n\}$ such that for $N_1 \in \mathbb{N}$ and any $m, n > N_1$,

$$\int_0^{R_1} (\varphi_n^1 - \varphi_m^1)^2 dr < \frac{1}{2}.$$

Thus, for $m, n > N_1$ we have $\|\varphi_n^1 - \varphi_m^1\|^2 < 1$.

Proceeding by induction, we can assume that for each $j \in \mathbb{N}$ there exists $R_j \in \mathbb{R}$ and $N_j \in \mathbb{N}$ with $\{\varphi_n^j\}$ a subsequence $\{\varphi_n^j\} \subseteq \{\varphi_n^{j-1}\}$ and for all $m, n > N_j$, we have $\|\varphi_n^j - \varphi_m^j\|^2 < \frac{1}{j}$. Now, define $y_n$ to be the diagonal sequence $\{\varphi_n^n\}$. Clearly, $\{y_n\} \subseteq \{\varphi_n\}$, and by definition for each $k \geq n$, $\{y_k\}_{k \geq n} \subseteq \{\varphi_k^n\}$. Thus for $\varepsilon > 0$, by the archimidean property there exists some $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. And for $m, n > N_p$,

$$\|y_n - y_m\|^2 \leq \|\varphi_n^p - \varphi_m^p\|^2 < \frac{1}{p} < \varepsilon.$$

Hence $\{y_k\}$ is Cauchy in $L^2$, and so $\{\varphi_n\}$ has a convergent subsequence in $L^2$. $\square$

Momentarily let us return to the operator $H_1 : D(H_1) \subseteq L^2 \to L^2$ defined by $H_1 = -\frac{d^2}{dr^2} - \frac{1}{r}$ with domain $D(H_1) = \{\varphi \in H^2 \mid \varphi(0) = 0\}$. It is clear that $C_0^\infty([0, \infty))$, the space of infinitely differentiable functions vanishing at the boundary, is contained within $D(H_1)$. Thus $H_1$ is densely defined, and so the adjoint $H_1^*$ is well–defined. A quick application of integration by parts shows that $H_1$ is a symmetric operator, thus $H_1 \subseteq H_1^*$. In particular,

$$H_1^* = -\frac{d^2}{dr^2} - \frac{1}{r}.$$

While it is possible that $D(H_1) \subset D(H_1^*)$, Theorem IX.26 of [18] gives that solutions to $H_1^*\varphi = \lambda\varphi$ for $\lambda \in \mathbb{R}$ will be $C^\infty$ on the interval $(0,\infty)$. From here, the arguments of section 5.3 in particular imply that $\mathrm{Ker}(H_1^* - \lambda_n) = \mathrm{span}(\psi_n)$.

As we have remarked before, the weighted norm $\|\cdot\|_w$ is stronger than the usual $L^2$ norm. In particular, for any $\varphi \in D(H)$, we have $\|\varphi\|_{H^2} \leq \|(1+r)^2\varphi\|_{H^2}$, thus $\varphi \in D(H_1)$. In this manner, on $L^2$ the operator $H_1 - \lambda_n$ extends $H$. Thus, it is clear that $\mathrm{Ran}(H) \subset \mathrm{Ran}(H_1 - \lambda_n)$. In what follows, all closures and orthogonal complements will be with respect to the unweighted inner product on $L^2$. We have the following string of containments

$$\overline{\mathrm{Ran}(H)} \subseteq \overline{\mathrm{Ran}(H_1 - \lambda_n)} = [\mathrm{Ker}(H_1^* - \lambda_n)]^\perp = [\mathrm{span}(\psi_n)]^\perp.$$

Now, for the actual argument of the range being closed. Let us take $y \in \overline{\mathrm{Ran}(H)}^w$, where the script $w$ indicates that the closure is being taken in the weighted $L^2_w$ space. Then there exists a sequence $\{y_k\} \in \mathrm{Ran}(H)$ with $y_k \to y$ in $L^2_w$. As the $L^2_w$ norm is stronger it is also true that $y_k \to y$ in $L^2$, thus

$$\overline{\mathrm{Ran}(H)}^w \subseteq \overline{\mathrm{Ran}(H)} \subseteq [\mathrm{span}(\psi_n)]^\perp.$$

(Note that this is not saying that all elements of $\overline{\mathrm{Ran}(H)}^w$ are perpendicular to $\psi_n$ in the weighted inner product $\langle\cdot,\cdot\rangle_w$, but that all elements $\overline{\mathrm{Ran}(H)}^w$ are perpendicular to $\psi_n$ in the unweighted inner product.) Thus it is safe to assume that $y$, $y_k \in [\mathrm{span}(\psi_n)]^\perp$. By the definition of the $y_k$, there exists $x_k \in D(H) \cap [\mathrm{span}(\psi_n)]^\perp$ (as $\mathrm{Ker}(H) = \mathrm{span}(\psi_n)$) with $y_k = Hx_k$. There are now two possibilities for the sequence $\{x_k\}$.

- Either the $\{x_k\}$ are uniformly bounded in $L^2$.
- Or the $\{x_k\}$ are not uniformly bounded in $L^2$.

Let us at first assume that the $\{x_k\}$ are not uniformly bounded in $L^2$. Thus, there is no positive $M \in \mathbb{R}$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$. Stated equivalently, $\lim_{k\to\infty}\|x_k\| = \infty$. Define $\alpha_k = \|x_k\|$ and $z_k = \frac{1}{\alpha_k}x_k$. Thus $Hz_k = \frac{1}{\alpha_k}y_k$ and $\|z_k\| = 1$ for all $k \in \mathbb{N}$. The inequality at the end of section 6.1 gives

$$\|z_k\|^2_{H^2_w} \leq A(\varepsilon,\delta,n)\frac{\|y_k\|^2_w}{\alpha_k^2} + B(\varepsilon,\delta,n)\|z_k\|^2.$$

As $y_k$ converges in $L^2_w$, the terms $\|y_k\|^2_w$ are bounded, and $\lim_{k\to\infty}\frac{1}{\alpha_k}y_k = 0$. Thus, $\|z_k\|_{H^2_w}$ is uniformly bounded, and as the $H^2_w$ norm is stronger than the $H^1_w$ norm the previous theorem gives a subsequence of $\{z_k\}$ that converges in $L^2$. Without loss of generality, let us assume that we have passed to the convergent subsequence and thus assume $z_k$ converges to some $z$ in $L^2$.

The inequality above gives that $z_k \to z$ in $H^2_w$, and $Hz_k \to 0$ in $L^2_w$. As $H$ is a closed operator, we have $Hz = 0$ which implies that $z \in \operatorname{span}(\psi_n)$. Thus $z = \beta\psi_n$. And so, assuming that $\psi_n$ is normalized

$$\beta = \langle z,\psi_n\rangle = \left\langle \lim_{k\to\infty} z_k,\psi_n \right\rangle = \lim_{k\to\infty}\langle z_k,\psi_n\rangle = 0.$$

This shows that $z = 0$, but

$$0 = \|z\| = \lim_{k\to\infty}\|z_k\| = 1.$$

A clear contradiction.

Thus, it must be the case that the $\{x_k\}$ are uniformly bounded. Thus, there is some $M \in \mathbb{R}$

such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$. So, the inequality from section 6.1

$$\|x_k\|_{H_w^2}^2 \leq A(\varepsilon, \delta, n)\|y_k\|_w^2 + B(\varepsilon, \delta, n)\|x_k\|^2,$$

gives us that $\{x_k\}$ is uniformly bounded in $H_w^2$. Thus, by our theorem there is an $L^2$ convergent subsequence. Passing through to this subsequence and assuming $x_k \to x$ in $L^2$, the inequality immediately gives us that $x_k \to x$ in $H_w^2$ (and thus clearly in $L_w^2$). Lastly, $H$ being a closed operator immediately implies that $y = Hx$ for $x \in D(H)$. Thus $y \in \mathrm{Ran}(H)$, and so the range of $H$ is closed in the topology generated by the weighted norm on $L_w^2$.

## 6.5 The Cokernel of $H$

For $H : D(H) \subseteq L_w^2 \to L_w^2$, the domain of the adjoint $H^*$ is defined to be the set of all vectors $\rho$ for which there exists a $\mu \in L_w^2$ such that

$$\langle H\varphi, \rho\rangle_w = \langle \varphi, \mu\rangle_w, \quad \forall \varphi \in D(H),$$

and the adjoint $H^*$ is defined to be $H^*\rho = \mu$. As $H$ is densely defined the adjoint $H^*$ exists (i.e. is well–defined). A quick application of integration by parts shows that

$$\begin{aligned}
0 &= \langle H\varphi, \rho\rangle_w - \langle \varphi, \mu\rangle_w \\
&= \int_0^\infty \varphi \left[\mu + \rho'' + \frac{1}{r}\rho + \lambda_n\rho + \frac{4}{1+4}\rho' + \frac{2}{(1+r)^2}\rho\right](1+r)^2.
\end{aligned}$$

Thus, as $D(H)$ is dense in $L_w^2$, we have

$$H^*\rho = H\rho - \frac{4}{1+r}\rho' - \frac{2}{(1+r)^2}\rho.$$

If we define the following map $F : L^2_w \to L^2$ by $F(\varphi) = (1+r)^2\varphi$, direct computation shows

$$[(1+r)^2\varphi]' = 2(1+r)\varphi + (1+r)^2\varphi'$$
$$[(1+r)^2\varphi]'' = 2\varphi + 4(1+r)\varphi' + (1+r)^2\varphi'',$$

and so $F$ is well–defined as a map from $D(H)$ to $D(H_1)$. It is clear that $F$ is $\mathbb{R}$–linear and injective. A similar computation to the one above shows that for $\rho \in D(H_1)$ that $\frac{1}{(1+r)^2}\rho \in D(H)$. Thus $F(\frac{1}{(1+r)^2}\rho) = \rho$, and so $F$ is surjective, and $F^{-1}$ is defined as division by $(1+r)^2$.

A direct computation shows that $F$ restricts to a bijective linear map, $F : \mathrm{Ker}(H^*) \to \mathrm{Ker}(H)$. As $\mathrm{Ker}(H) = \mathrm{span}(\psi_n)$, we have that $F$ is an isomorphism $F : \mathrm{Ran}(H)^{\perp_w} \to [\mathrm{span}(\psi_n)]$ from the fact that $\mathrm{Ran}(H)^{\perp_w} = \mathrm{Ker}(H^*)$. Thus $H$ has cokernel of dimension one.

Thus we have shown that $H : D(H) \to L^2_w$ is a bounded map with closed range and finite dimensional kernel and cokernel. Thus $H$ is a Fredholm operator with index

$$\mathrm{ind}(H) = \dim(\mathrm{Ker}(H)) - \dim(\mathrm{CoKer}(H)) = 1 - 1 = 0.$$

# Chapter 7

# An argument for the nondegeneracy of each eigenstate

At the end of section 4 we saw that a bound state of the Schrödinger–Newton system must satisfy

$$-\psi'' - \frac{1}{r}\psi - \lambda\psi - F(\psi)\psi = 0,$$

where $F$, given by

$$F(r) = \int_r^\infty \left(\frac{1}{x} - \frac{1}{r}\right)\psi^2 dx$$

was the nonlinear perturbation of the potential $V$ from the Coulomb potential with the energy eigenvalue $\lambda$. Denote the Banach space that is the domain of $H$ as $\mathcal{H} = D(H) = \{\varphi \in H_w^2 \mid \varphi(0) = 0\}$, and define a map

$$G : \mathcal{H} \times \mathbb{R}^2 \to L_w^2 \times \mathbb{R}$$

by

$$G(\varphi,\lambda,\beta)=\left(-\psi''-\frac{1}{r}\psi-\lambda\psi-\beta F(\psi)\psi\ ,\ \|\varphi\|^2-1\right).$$

From our previous work if $\psi_n$ is the normalized (in $L^2$) eigenvector of the Hydrogen atom on the half–line associated to the eigenvalue $\lambda_n$ then

$$G(\psi_n,\lambda_n,0)=(H\psi_n,0)=(0,0).$$

is codifying nothing more than the solutions to the Schrödinger equation with the Coulomb potential found earlier. Perhaps unsurprisingly, the linearization of $G$ involves the operator $H$ from chapter 6 as well. We will show how the Fredholm properties on $H$ extend to the linearization of $G$ about the points $(\psi_n,\lambda_n,0)$ with a goal in mind to invoke the implicit function theorem and extend these solutions in the parameter $\beta$ about a neighborhood of $\beta=0$.

From this point we will argue why this process can be extended until $\beta=1$, i.e. at a solution of the Schrödinger–Newton system, and justify why this solution inherits the same nondegeneracy of each eigenstate that the Schrödinger equation on the half–line enjoys.

## 7.1 The Frechet Derivative of $G$

For computing the Frechet derivative of $G$ it will be easier to denote $G$ as $G=(G_1,G_2)$ with $G_1:\mathcal{H}\times\mathbb{R}\to L^2_w$ and $G_2:\mathcal{H}\times\mathbb{R}^2\to\mathbb{R}$. Let us look at our nonlinear perturbation $F$, for $\varphi\in\mathcal{H}$,

$$F(\varphi)\varphi=\left[\int_r^\infty\left(\frac{1}{x}-\frac{1}{r}\right)\varphi^2dx\right]\varphi(r)=\varphi\int_r^\infty\frac{1}{x}\varphi^2dx-\frac{\varphi}{r}\int_r^\infty\varphi^2dx,$$

the inequalities in section 6.1, namely the bounds on $\int_0^\infty\frac{1}{r}\varphi^2dr$ and $\int_0^\infty\frac{1}{r^2}\varphi^2dr$, show that $F(\cdot)\cdot:\mathcal{H}\to L^2_w$ is well–defined. We first compute the Frechet derivative of $F$ as it will arise in computing the Frechet derivative of $G$. For $\varphi, h \in \mathcal{H}$, computation shows

$$F(\varphi+h)(\varphi+h)-F(\varphi)\varphi = h\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\varphi^2 dx + 2\varphi\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\varphi h dx$$
$$+\varphi\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)h^2 dx + 2h\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\varphi h dx + h\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)h^2 dx.$$

Let us call the first two terms (that are linear in $h$) an operator $B$,

$$Bh = h\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\varphi^2 dx + 2\varphi\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\varphi h dx$$

The inequalities in section 6.1 show that

$$\lim_{h\to 0}\frac{\|F(\varphi+h)(\varphi+h)-F(\varphi)\varphi-Bh\|}{\|h\|_{H_w^2}} = 0$$

and thus $B = \frac{\partial F}{\partial \varphi}(\varphi)$ is the Frechet derivative of $F$ evaluated at $\varphi$. To be more formal

$$\frac{\partial F}{\partial \varphi}\Big|_{(\psi)}(h) = h\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\psi^2 dx + 2\psi\int_r^\infty \left(\frac{1}{x}-\frac{1}{r}\right)\psi h dx$$

Routine computation gives the following,

$$\begin{aligned}
\frac{\partial G_1}{\partial \varphi}\Big|_{(\psi,\lambda,\beta)} &= -\frac{d^2}{dr^2} - \frac{1}{r} - \lambda - \beta\frac{\partial F}{\partial \varphi}\Big|_{(\psi)}.\\
\frac{\partial G_1}{\partial \lambda}\Big|_{(\psi,\lambda,\beta)} &= -\psi\\
\frac{\partial G_1}{\partial \beta}\Big|_{(\psi,\lambda,\beta)} &= -F(\psi)\psi\\
\frac{\partial G_2}{\partial \varphi}\Big|_{(\psi,\lambda,\beta)} &= 2\langle\psi,\cdot\rangle\\
\frac{\partial G_2}{\partial \lambda}\Big|_{(\psi,\lambda,\beta)} &= 0\\
\frac{\partial G_2}{\partial \beta}\Big|_{(\psi,\lambda,\beta)} &= 0.
\end{aligned}$$

With this we can write the Frechet derivative of $G$ in terms of a block matrix,

$$DG|_{(\psi,\lambda,\beta)} = \begin{bmatrix} -\frac{d^2}{dr^2} - \frac{1}{r} - \lambda - \beta\frac{\partial F}{\partial\varphi}\Big|_{\psi} & -\psi & -F(\psi)\psi \\ 2\langle\psi,\cdot\rangle & 0 & 0 \end{bmatrix}$$

At an eigenvalue–eigenvector pair of the Schrödinger equation $(\psi_n, \lambda_n, 0)$, we have

$$DG|_{(\psi_n,\lambda_n,0)} = \begin{bmatrix} H & -\psi_n & -F(\psi_n)\psi_n \\ 2\langle\psi_n,\cdot\rangle & 0 & 0 \end{bmatrix}.$$

In the following sections, we will show how the Fredholm properties of $H$ pass through to the $2\times 2$ block matrix

$$D = \frac{\partial G}{\partial(\varphi,\alpha)}\Big|_{(\psi_n,\lambda_n,0)} = \begin{bmatrix} H & -\psi_n \\ 2\langle\psi_n,\cdot\rangle & 0 \end{bmatrix}$$

and to the whole derivative $DG|_{(\psi_n,\lambda_n,0)}$.

## 7.2 $D$ is Fredholm index 0

As norms on a finite product of Hilbert spaces are equivalent, without loss of generality assume that we are working in the euclidean norm of the product of the Hilbert spaces $\mathcal{H}\times\mathbb{R}^2$ and $L^2_w\times\mathbb{R}$ respectively, i.e.

$$\|(\varphi,\lambda,\beta)\|_{\mathcal{H}\times\mathbb{R}^2} = \sqrt{\|\varphi\|^2_{H^2_w} + |\lambda|^2 + |\beta|^2}$$
$$\|(\varphi,\alpha)\|_{L^2_w\times\mathbb{R}} = \sqrt{\|\varphi\|^2_{L^2_w} + |\alpha|^2}.$$

To show that $D$ is Fredholm index 0, we must check that $D$ is bounded, has closed range, and finite dimensional kernel and cokernel. We will check that these follow from the Fredholm properties of $H$ directly. To show that $D$ is bounded, let $(\varphi,\alpha) \in \mathcal{H} \times \mathbb{R}$, and compute the following,

$$
\begin{aligned}
\|D(\varphi,\alpha)\|^2_{L^2_w \times \mathbb{R}} &= \|(H\varphi - \alpha\psi_n, 2\langle \psi_n, \varphi\rangle)\|^2_{L^2_w \times \mathbb{R}} \\
&= \|H\varphi - \alpha\psi_n\|^2_{L^2_w} + 4|\langle \psi_n, \varphi\rangle|^2 \le [\|H\varphi\|_{L^2_w} + |\alpha|\|\psi_n\|_{L^2_w}]^2 + 4|\langle \psi_n, \varphi\rangle|^2 \\
&\le 2\|H\varphi\|^2_{L^2_w} + 2|\alpha|^2\|\psi_n\|^2_{L^2_w} + 4\|\psi_n\|^2\|\varphi\|^2
\end{aligned}
$$

which comes from the inequality $2ab \le a^2 + b^2$ as well as the Cauchy–Schwarz inequality. As $H$ is bounded, and as the $L^2$ norm is weaker than the $H^2_w$ norm, there exists an $M \ge 0$ such that

$$\|D(\varphi,\alpha)\|^2_{L^2_w \times \mathbb{R}} \le [2M + 4\|\psi_n\|^2]\|\varphi\|^2_{H^2_w} + 2\|\psi_n\|^2_{L^2_w}|\alpha|^2.$$

Thus taking $K^2 = \max\{2M + 4\|\psi_n\|^2, 2\|\psi_n\|^2_{L^2_w}\}$, we have

$$\|D(\varphi,\alpha)\|_{L^2_w \times \mathbb{R}} \le K\|(\varphi,\alpha)\|_{H^2_w \times \mathbb{R}},$$

thus $D$ is bounded.

Now, assume that $(\varphi,\alpha) \in \overline{\mathrm{Ran}(D)}$, then there is a sequence $\{(\varphi_k,\alpha_k)\} \subseteq \mathrm{Ran}(D)$ with $(\varphi_k,\alpha_k) \to (\varphi,\alpha)$ in $L^2_w \times \mathbb{R}$. Thus, there is also a sequence $\{(x_k,c_k)\} \subseteq \mathcal{H} \times \mathbb{R}$ such that $D(x_k,c_k) = (\varphi_k,\alpha_k)$. As $\mathrm{Ker}(H) = \mathrm{span}(\psi_n)$, let us write $x_k = x_{\|,k} + x_{\perp,k}$ with the parallel and perpendicular components taken with respect to $\mathrm{span}(\psi_n)$. Thus, there exists $\{d_k\} \subset \mathbb{R}$ such that $x_k = d_k\psi_n + x_{\perp,k}$. Then

$$\begin{bmatrix} \varphi_k \\ \alpha_k \end{bmatrix} = \begin{bmatrix} H & -\psi_n \\ 2\langle \psi_n, \cdot\rangle & 0 \end{bmatrix} \begin{bmatrix} x_k \\ c_k \end{bmatrix} = \begin{bmatrix} Hx_{\perp,k} - c_k\psi_n \\ 2d_k \end{bmatrix}.$$

It follows from the uniqueness of limits that $d_k \to \frac{\alpha}{2}$ in $\mathbb{R}$. As $\mathrm{Ran}(H) = [\mathrm{span}(\psi_n)]^\perp$, where the orthogonal complement is taken in the unweighted standard $L^2$ inner product, the pythagorean theorem gives us that

$$\|\varphi_k\|^2_{L^2_w} \geq \|\varphi_k\|^2 = \|Hx_{\perp,k}\|^2 + |c_k|^2\|\psi_n\|^2 \geq |c_k|^2$$

(recalling that $\|\psi_n\| = 1$.) Thus $\{c_k\}$ is Cauchy in $\mathbb{R}$ and thus converges, hence $c_k \to c$ for some $c \in \mathbb{R}$. As $Hx_{\perp,k} = \varphi_k + c_k\psi_n$, it is clear that $\{Hx_{\perp,k}\}$ converges in $L^2_w$ to $\varphi + c\psi_n$.

From the inequality

$$\|x_{\perp,k}\|^2_{H^2_w} \leq A(\varepsilon,\delta,n)\|Hx_{\perp,k}\|^2_w + B(\varepsilon,\delta,n)\|x_{\perp,k}\|^2,$$

and an argument very similar to the one at the end of section 6.4, it must be that the sequence $\{x_{\perp,k}\}$ is uniformly bounded in $L^2$. Then the theorem from section 6.4 gives the existence of an $L^2$–convergent subsequence of $\{x_{\perp,k}\}$. Pass through to this subsequence without loss of generality and hence one more use of the inequality above implies that $\{x_{\perp,k}\}$ is Cauchy in $\mathcal{H} = D(H)$. Thus take $x_{k,\perp} \to x_\perp$. Thus $x_k \to \frac{\alpha}{2}\psi_n + x_\perp$ in $\mathcal{H}$. As $H$ is bounded and therefore closed, we have $\frac{\alpha}{2}\psi_n + x_\perp \in D(H) = \mathcal{H}$ and $H(\frac{\alpha}{2}\psi_n + x_\perp) = Hx_\perp = \varphi + c\psi_n$.

Thus $(\frac{\alpha}{2}\psi_n + x_\perp, c) \in \mathcal{H} \times \mathbb{R}$, and

$$D\begin{bmatrix} \frac{\alpha}{2}\psi_n + x_\perp \\ c \end{bmatrix} = \begin{bmatrix} H & -\psi_n \\ 2\langle\psi_n,\cdot\rangle & 0 \end{bmatrix}\begin{bmatrix} \frac{\alpha}{2}\psi_n + x_\perp \\ c \end{bmatrix} = \begin{bmatrix} \varphi \\ \alpha \end{bmatrix}.$$

Thus $(\varphi,\alpha) \in \mathrm{Ran}(D)$, and so the range of $D$ is closed.

Now, assume that $(\varphi,\alpha) \in \mathrm{Ker}(D)$. Write $\varphi = b\psi_n + \varphi_\perp$ using the same decompisition as above, i.e. breaking $\varphi$ into it's parallel and perpendicular components to $\psi_n$ in the inner product on $L^2$. Then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = D\begin{bmatrix} \varphi \\ \alpha \end{bmatrix} = \begin{bmatrix} H & -\psi_n \\ 2\langle\psi_n,\cdot\rangle & 0 \end{bmatrix}\begin{bmatrix} \varphi \\ \alpha \end{bmatrix} = \begin{bmatrix} H\varphi_\perp - \alpha\psi_n \\ b \end{bmatrix}$$

Thus $b = 0$, so $\varphi = \varphi_\perp$. In $L^2$, we have that $\mathrm{Ran}(H) = [\mathrm{span}(\psi_n)]^\perp$, thus

$$\alpha = \langle \alpha\psi_n, \psi_n \rangle = \langle H\varphi_\perp, \psi_n \rangle = 0.$$

And so $H\varphi_\perp = 0$. But then this would imply that $\varphi_\perp \in \mathrm{Ker}(H) = \mathrm{span}(\psi_n)$, which implies that $\varphi_\perp = 0$. Thus, $(\varphi, \alpha) = (0, 0)$. Thus, $\mathrm{Ker}(D) = \{(0, 0)\}$, and is therefore of dimension 0.

For $\varphi \in [\mathrm{span}(\psi_n)]^\perp$ with the orthogonal complement coming from the unweighted inner product, we have

$$D \begin{bmatrix} \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} H\varphi \\ 0 \end{bmatrix}.$$

If $\varphi \in \mathrm{span}(\psi_n)$, i.e. $\varphi = \alpha\psi_n$ for some $\alpha \in \mathbb{R}$, then

$$D \begin{bmatrix} \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\alpha \end{bmatrix}$$

This shows that $D(\mathcal{H} \times \{0\}) = \mathrm{Ran}(H) \times \mathbb{R}$. And

$$D \begin{bmatrix} 0 \\ c \end{bmatrix} = \begin{bmatrix} -c\psi_n \\ 0 \end{bmatrix}$$

shows that $D(\{0\} \times \mathbb{R}) = \mathrm{span}(\psi_n) \times \{0\}$. Thus, as $\mathrm{Ran}(H) = [\mathrm{span}(\psi_n)]^\perp$ in the standard $L^2$ inner product, we have that $D(\mathcal{H} \times \mathbb{R}) = L^2_w \times \mathbb{R}$. Thus $D$ is onto, and therefore the dimension of the cokernel of $D$ is 0. Thus $D$ is also Fredholm index 0.

## 7.3 $DG|_{(\psi_n,\lambda_n,0)}$ is Fredholm index 1

Before moving onto the full Frechet derivative of $G$, let us pause and return to our nonlinear perturbation

$$F(\varphi)\varphi = \varphi\int_r^\infty \frac{1}{x}\varphi^2 dx - \frac{\varphi}{r}\int_r^\infty \varphi^2 dx = A(r)\varphi - B(r)\frac{\varphi}{r}.$$

As we saw in section 6.1

$$\begin{aligned} A(r) &= \int_r^\infty \frac{1}{x}\varphi^2 dx \le \int_0^\infty \frac{1}{r}\varphi^2 dr \le K(\varepsilon,\delta)\|\varphi\|_{H_w^2}^2. \\ B(r) &= \int_r^\infty \varphi^2 dx \le \|\varphi\|^2 \le \|\varphi\|_{H_w^2}^2. \end{aligned}$$

Which gives that

$$|F(\varphi)\varphi|^2 \le K(\varepsilon,\delta)\|\varphi\|_{H_w^2}^4 \left[\varphi^2 + \frac{\varphi^2}{r} + \frac{\varphi^2}{r^2}\right].$$

Thus, the inequalities in section 6.1 state that

$$\|F(\varphi)\varphi\|_{L_w^2} \le K(\varepsilon,\delta)\|\varphi\|_{H_w^2}^3,$$

where in the last computations $K(\varepsilon,\delta)$ represented a perhaps different constant at each step, but a constant nonetheless. This, of course, shows nothing more than the fact that for a fixed $\varphi$, the multiplcation operator $F(\varphi)\varphi : \mathbb{R} \to L_w^2$ given by $[F(\varphi)\varphi](z) = zF(\varphi)\varphi$ is bounded.

It is clear that

$$DG|_{(\psi_n,\lambda_n,0)} \begin{bmatrix} \varphi \\ \alpha \\ z \end{bmatrix} = D \begin{bmatrix} \varphi \\ \alpha \end{bmatrix} + z \begin{bmatrix} -F(\psi_n)\psi_n \\ 0 \end{bmatrix}.$$

As $D$ is bounded, $\|D(\varphi,\alpha)\|_{L^2_w\times\mathbb{R}} \leq M\|(\varphi,\alpha)\|_{H^2_w\times\mathbb{R}}$ for $M$ some positive real number. Then

$$\begin{aligned}\|DG|_{(\psi_n,\lambda_n,0)}(\varphi,\alpha,z)\|^2_{L^2_w\times\mathbb{R}} &\leq \|D(\varphi,\alpha)\|^2_{L^2_w\times\mathbb{R}} + |z|^2\|(-F(\psi_n)\psi_n,0)\|^2_{L^2_w\times\mathbb{R}} \\ &\leq M^2\|(\varphi,\alpha)\|^2_{H^2_w\times\mathbb{R}} + K^2(\varepsilon,\delta)\|\psi_n\|^3_{H^2_w}|z|^2.\end{aligned}$$

If we take $N^2 = \max\{M^2, K^2(\varepsilon,\delta)\|\psi_n\|^3_{H^2_w}\}$, then

$$\|DG|_{(\psi_n,\lambda_n,0)}(\varphi,\alpha,z)\|^2_{L^2_w\times\mathbb{R}} \leq N^2\|(\varphi,\alpha,z)\|^2_{H^2_w\times\mathbb{R}^2}.$$

Thus $DG|_{(\psi_n,\lambda_n,0)}$ is bounded.

It is also clear that

$$DG|_{(\psi_n,\lambda_n,0)}\begin{bmatrix}\varphi\\ \alpha\\ 0\end{bmatrix} = D\begin{bmatrix}\varphi\\ \alpha\end{bmatrix},$$

and so the range of $DG|_{(\psi_n,\lambda_n,0)}$ equals the range of $D$, which was $L^2_w\times\mathbb{R}$. Thus the range of $DG|_{(\psi_n,\lambda_n,0)}$ is closed and the dimension of the cokernel is zero.

For $(\varphi,\alpha,z)\in \mathrm{Ker}(DG|_{(\psi_n,\lambda_n,0)})$, we have

$$\begin{bmatrix}0\\ 0\end{bmatrix} = DG|_{(\psi_n,\lambda_n,0)}\begin{bmatrix}\varphi\\ \alpha\\ z\end{bmatrix} = D\begin{bmatrix}\varphi\\ \alpha\end{bmatrix} + z\begin{bmatrix}-F(\psi_n)\psi_n\\ 0\end{bmatrix}.$$

As $D$ is invertible, we have

$$\begin{bmatrix}\varphi\\ \alpha\end{bmatrix} = zD^{-1}\begin{bmatrix}-F(\psi_n)\psi_n\\ 0\end{bmatrix}.$$

This shows that

$$\mathrm{span}\left\{\left\{D^{-1}\begin{bmatrix}-F(\psi_n)\psi_n\\ 0\end{bmatrix}\right\}\times\{1\}\right\} = \mathrm{Ker}(DG|_{(\psi_n,\lambda_n,0)}).$$

Thus the kernel of $DG|_{(\psi_n,\lambda_n,0)}$ is of dimension one. Thus, $DG|_{(\psi_n,\lambda_n,0)}$ is Fredholm of index 1.

## 7.4 The argument for uniqueness

We now present the argument for our uniqueness result. Essentially the result falls from an application of the implicit function theorem. From section 7.2 we found that the operator

$$D = \frac{\partial G}{\partial(\varphi,\alpha)}\Big|_{(\psi_n,\lambda_n,0)} = \begin{bmatrix} H & -\psi_n \\ 2\langle\psi_n,\cdot\rangle & 0 \end{bmatrix}$$

was Fredholm of index 0. In particular, as $D$ is both invertible and bounded, the closed graph theorem gives that $D^{-1}$ is also bounded. Thus, by the implicit function theorem, there are $C^1$ extensions of $\lambda_n$ and $\psi_n$ in terms of the parameter $\beta$. To be more explicit, there is a neighborhood $[0,\varepsilon)$ of $\beta = 0$ such that

$$G(\psi_n(\beta),\lambda_n(\beta),\beta) = (0,0), \;\; \text{for} \;\; \beta \in [0,\varepsilon).$$

An operator being Fredholm is also an open condition with respect to the operator norm. To be clear, by defining the following extension of $D$ in terms of $\beta$ as

$$D(\beta) = \frac{\partial G}{\partial(\varphi,\alpha)}\Big|_{(\psi_n(\beta),\lambda_n(\beta),\beta)} = \begin{bmatrix} -\frac{d^2}{dr^2} - \frac{1}{r} - \lambda_n(\beta) - \beta\frac{\partial F}{\partial\varphi}\Big|_{\psi_n(\beta)} & -\psi_n(\beta) \\ 2\langle\psi_n(\beta),\cdot\rangle & 0 \end{bmatrix},$$

there is also a neighborhood of $\beta = 0$ such that $D(\beta)$ is Fredholm on this neighborhood. From [6], the Fredholm index is a homotopy invariant, and thus $D(\beta)$ being Fredholm on this neighborhood will in fact imply that $D(\beta)$ is index 0 on the neighborhood as well. It remains to be shown that these $C^1$ extensions of $\psi_n$ and $\lambda_n$ can be defined on the entirety of the interval

$\beta \in [0,1]$ and not just on a neighborhood of $\beta = 0$. The current argument for this, that has yet to be proven in detail, is an iteration scheme involving the implicit function theorem.

To give a sketch, the paragraph above details an initial neighborhood $[0,\varepsilon)$ of $\beta = 0$ for which $C^1$ extensions of $\psi_n$ and $\lambda_n$ are defined and $D(\beta)$ is Fredholm index 0. For $\beta = \varepsilon$ or arbitrarily close to $\varepsilon$ we then use the implicit function theorem again as well as the openness of the Fredholm condition to find a neighborhood of $\varepsilon$ such that the $C^1$ extensions are defined and on which $D(\beta)$ is Fredholm index 0. It remains to be shown that this process can be iterated in a manner such that the radii on the successively defined neighborhoods in the iteration schema do not tend to zero, or in particular, remain bounded away from zero until $\beta = 1$. Once this has been shown, the $C^1$ extensions of $\psi_n$ and $\lambda_n$ are defined for $\beta \in [0,1]$, and furthermore this process details the construction of an open set $A_n$ (open tube) in $\mathcal{H} \times \mathbb{R} \times [0,1]$ containing the $C^1$ extensions.

Returning now to the definition of $G$ given as $G : \mathcal{H} \times \mathbb{R}^2 \to L^2_w \times \mathbb{R}$

$$G(\varphi,\lambda,\beta) = \left( -\psi'' - \frac{1}{r}\psi - \lambda\psi - \beta F(\psi)\psi \; , \; \|\varphi\|^2 - 1 \right),$$

and by restricting the definition of $G$ to $G : A_n \to L^2_w \times \mathbb{R}$ we then have that the point $(0,0) \in L^2_w \times \mathbb{R}$ is a regular value of $G$. This follows as $DG|_{(\psi_n(\beta),\lambda_n(\beta),\beta)}$ is Fredholm of index 1 for all $\beta \in [0,1]$. (as $D(\beta)$ is Fredholm index 0 for $\beta \in [0,1]$ and an argument similar to that in section 7.3) Because of which $DG|_{(\psi_n(\beta),\lambda_n(\beta),\beta)}$ is onto with a bounded right inverse for each $\beta \in [0,1]$. Therefore, again by the implicit function theorem [12], we have that $G^{-1}((0,0))$ is a $C^1$ Banach submanifold of $A_n$ of dimension 1. At this point, the uniqueness result is immediate. Suppose $\psi_n(1)$ and $\varphi$ are two normalized linearly independent eigenvectors associated to $\lambda_n(1)$, then we arrive at an immediate contradiction. This either contradicts the submanifold structure at the point $(\psi_n(1), \lambda_n(1), 1)$, or contradicts the Fredholm index of $DG|_{(\psi_n(1),\lambda_n(1),1)}$.

As a final remark, it should be noted that this will give uniqueness of the higher energy stationary states, but precisely for the states of the Schrödinger–Newton system that arise as the apex ($\beta = 1$) of a stalk originating at an eigenvalue/eigenvector pair of the Hydrogen atom on the half–line ($\beta = 0$). It only now remains to show that all higher bound states of the Schrödinger–Newton system can be reached in this manner. At the current time, there are two possible sketches for a proof of this result

1. Show that the number of zeroes in a solution $\psi_n(\beta)$ is also homotopy invariant.

2. Prove that the argument above can be carried in reverse: Beginning at $\beta = 1$, argue an iterative scheme of $C^1$ extension of solutions down to $\beta = 0$.

Either of these arguments assure that the matchmaking process between bound state solutions of the Hydrogen atom on the half–line and the Schrödinger–Newton system is, in fact, bijective.

# Appendix A

# Zwicky, the Virial Theorem, and Dark Matter

Let $O$ represent and orgin fixed in space, and let $\mathbf{r}_i$ and $\mathbf{v}_i$ denote the time dependent position and velocity vectors of the $i$th particle in a system of $n$ mass particles of masses, $m_i$, respectively. We let $r_{jk} = |\mathbf{r}_j - \mathbf{r}_k|$, and follow the convention that $w = \sqrt{(\mathbf{w} \cdot \mathbf{w})}$ for vectors $\mathbf{w}$. Define the moment of inertia, $I$, by

$$I = \frac{1}{2} \sum_{k=1}^{n} m_k r_k^2 = \frac{1}{2} \sum_{k=1}^{n} m_k (\mathbf{r}_k \cdot \mathbf{r}_k).$$

and the Virial of the system as

$$\text{Vir} = \sum_{k=1}^{n} = \mathbf{r}_k \cdot \mathbf{F}_k = \sum_{k=1}^{n} m_k \mathbf{r}_k \cdot \ddot{\mathbf{r}}_k$$

By differentiating the moment of inertia twice with respect to time, we find

$$\ddot{I} = \sum_{k=1}^{n} m_k v_k^2 + \sum_{k=1}^{n} \mathbf{r}_k \cdot m_k \ddot{\mathbf{r}}_k = 2K_T + \text{Vir},$$

where $K_T$ is the total kinetic energy of the system.

If the system interacts gravitationally with potential given by $U$,

$$U = -\sum_{1 \le j < k \le n} \frac{Gm_j m_k}{r_{jk}}.$$

then by Newton's second law, the $k$th particle satisfies the equation

$$m_k \ddot{\mathbf{r}}_k = \sum_{\substack{j=1 \\ j \ne k}}^{n} \frac{Gm_j m_k}{r_{jk}^3} (\mathbf{r}_j - \mathbf{r}_k).$$

Now, the Virial of the system takes a very particular form,

$$\mathrm{Vir} = \sum_{k=1}^{n} m_k \mathbf{r}_k \cdot \ddot{\mathbf{r}}_k = \sum_{\substack{j,k=1 \\ j \ne k}}^{n} \frac{Gm_j m_k}{r_{jk}^3} (\mathbf{r}_j \cdot \mathbf{r}_k - r_k^2).$$

From the law of cosines we find $\mathbf{r}_j \cdot \mathbf{r}_k - r_k^2 = \frac{1}{2}(r_j^2 - r_k^2 - r_{jk}^2)$. Thus we can split the last sum above into three separate sums,

$$\frac{1}{2} \sum_{\substack{j,k=1 \\ j \ne k}}^{n} \frac{Gm_j m_k}{r_{jk}^2} r_j^2 - \frac{1}{2} \sum_{\substack{j,k=1 \\ j \ne k}}^{n} \frac{Gm_j m_k}{r_{jk}^2} r_k^2 - \frac{1}{2} \sum_{\substack{j,k=1 \\ j \ne k}}^{n} \frac{Gm_j m_k}{r_{jk}}.$$

By the symmetry $r_{jk} = r_{kj}$, it is clear that the first two sums cancel each other, and the third is $2U$. Thus,

$$\mathrm{Vir} = U.$$

Thus

$$\ddot{I} = 2K_T + U = K_T + E = -U + E,$$

for $E$ the total energy of the system via conservation of energy $K_T + U = E$.

Let us now define the two time averages.

$$\overline{K_T} = \lim_{t \to \infty} \frac{1}{t} \int_0^t K_T(\tau) d\tau, \text{ and } \overline{U} = \lim_{t \to \infty} \frac{1}{t} \int_0^t U(\tau) d\tau.$$

and state the Virial theorem.

**Virial Theorem**: The time averages $\overline{K_T}$ and $\overline{U}$ exist and satisfy the equation $2\overline{K_T} = -\overline{U}$ if and only if $\lim_{t\to\infty} t^{-1}\dot{I} = 0$.

*Proof.* Clearly, as $K_T = -U + E$ and $E$ is a constant, if one of $\overline{K_T}$ or $\overline{U}$ exists, then so does the other. This equation also implies that $\overline{K_T} = -\overline{U} + E$, hence the equation $2\overline{K_T} = -\overline{U}$ is equivalent to $\overline{K_T} = -E$.

Now, taking the equation $\ddot{I} = K_T + E$ and integrating once followed by dividing by $t$, gives

$$\frac{\dot{I}}{t} = \frac{1}{t}\int_0^t K_T(\tau)d\tau + E + \frac{K}{t},$$

for some constant $K$. This clearly shows that $\overline{K_T} = -E$ if and only if $\lim_{t\to\infty} t^{-1}\dot{I} = 0$. $\square$

Let us note that the condition $\lim_{t\to\infty} t^{-1}\dot{I} = 0$ is precisely equivalent to saying that the time average of the second time derivative of the moment of inertia is 0,

$$\overline{\frac{d^2I}{dt^2}} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \ddot{I}(\tau)d\tau = \lim_{t\to\infty}[t^{-1}\dot{I}(t) - t^{-1}I(0)] = 0.$$

as $\dot{I}(0)$ is finite.

And now, we may present Zwicky's classic argument. For what follows each vector $\mathbf{r}_k$ now represents the position of a nebula in a cluster of $n$ stars. For a mechanically stationary cluster, the moment of inertia oscillates about a constant value [17], thus the time average of its derivative will vanish. Thus, we are in the precise case in which the Virial theorem can be applied. And so

$$-\overline{U} = 2\overline{K_T} = \sum_{k=1}^{n} m_k\overline{v_k^2}.$$

From Newtonian mechanics it is known that a self–gravitating sphere of constant density $\rho$, mass $\mathcal{M}$, and radius $R$, has potential energy $U$ is given by

$$U = -\frac{3G\mathcal{M}^2}{5R}.$$

Thus assuming a uniform distribution of the cluster total mass $\mathcal{M}$ about a sphere of radius $R$ gives

$$\frac{3G\mathcal{M}^2}{5R} = \sum_{k=1}^{n} m_k \overline{v_k^2}$$

Using a second bar to denote a second average, this time over nebulae velocity, we find

$$\mathcal{M} = \frac{5R\overline{\overline{v^2}}}{3G}.$$

In general, the assumption of uniform distribution will not be fulfilled. But, the actual potential energy $U$ will have a value, that at least in order of magnitude, will be correctly given by the equation above. Zwicky used this to assume that

$$\mathcal{M} > \frac{R\overline{\overline{v^2}}}{5G}$$

is a conservative estimate of the minimum value of the total mass.

By the relation $\overline{\overline{v^2}} = 3\overline{\overline{v_s^2}}$ between the squared average velocity and the squared average velocity along the line of sight, and the observations of the Coma cluster available at the time of Zwicky [21], $\overline{\overline{v_s^2}} = 5 \times 10^{15}\text{cm}^2\ \text{sec}^{-2}$, we find

$$\mathcal{M} > 9 \times 10^{46}\text{gr}.$$

As the Coma cluster contains about one thousand nebulae, the average mass of one of these nebulae is

$$\overline{M} > 9 \times 10^{43}\text{gr} = 4.5 \times 10^{13} M_{\odot},$$

where $M_\odot = 2 \times 10^{33}$gr is the mass of the sun. Given that the luminosity of an average nebula is about that of $8.5 \times 10^7$ suns, the above would imply that the conversion factor, $\gamma$, from luminosity to mass for nebulae in the Coma cluster would be on the order of $\gamma = 500$, as compared with $\gamma' = 3$ for the Kapteyn stellar system (another well studied star system as of Zwicky's time). It was this discrepancy that led Zwicky to believe the existence of dark matter.

# Appendix B

# Derivation of vacuum Einstein Equation with Cosmological constant

We first define the Einstein–Hilbert action

$$L(g) = \int_U (R - 2\Lambda) dV_g.$$

Where $U$ is a pre–compact open subset of $M$, $R$ is the scalar curvature, $\Lambda$ is the cosmological constant, and $dV_g$ is the volume form, which in any right–handed coordinate basis, has the form $\sqrt{|g|}dx^1 \wedge dx^2 \cdots \wedge dx^n$.

Let us see what happens when we suppose a metric $g_{ab}$ is a critical point of the Einstein–Hilbert action $L(g)$. Let $g(s)$ be a variation of $g = g(0)$. Call $h = \dot{g}(0)$. (We will denote differentiation with respect to the parameter of our variation with a dot)

Our supposition is that

$$\frac{d}{ds} L(g(s))\Big|_{s=0} = 0,$$

or

$$\int_U \dot{R} dV_g + (R - 2\Lambda) d\dot{V}_g = 0.$$

**Lemma** For a differentiable non–singular matrix valued function $A(t)$,

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\mathrm{tr}(A'(t)A^{-1}(t)).$$

*Proof.* We write the determinant of $A(t)$ as a multilinear function of its $n$ rows $a_1, \ldots, a_n$. Then

$$\frac{d}{dt}\det(A(t)) = \lim_{h \to 0}\left[\frac{\det(a_1(t+h), \ldots, a_n(t+h)) - \det(a_1, \ldots, a_n)}{h}\right]$$

By adding and subtracting terms, and using the multilinearity,

$$\frac{d}{dt}\det(A(t)) = \lim_{h \to 0}\left[\sum_{k=1}^{n} \det\left(a_1(t), \ldots, a_{k-1}(t), \frac{a_k(t+h) - a_k(t)}{h}, \ldots, a_n(t+h)\right)\right]$$

By the continuity of the determinant function, we may bring the limit inside,

$$\frac{d}{dt}\det(A(t)) = \sum_{k=1}^{n} \det(a_1, \ldots, a'_k, \ldots, a_n).$$

Now, let $C_{ij}$ denote the cofactor matrix relative to $A(t)$, then

$$\det(a_1, \ldots, a'_k, \ldots, a_n) = \sum_{j=1}^{n} a'_{kj} C_{kj}.$$

Recall the definition of the adjugate matrix, $\mathrm{Adj} = C^T$, as it is the transpose of the cofactor matrix. Hence what we just found above states,

$$\det(a_1, \ldots, a'_k, \ldots, a_n) = (A'\mathrm{Adj})_{kk}.$$

Thus, it is clear that $\frac{d}{dt}\det(A(t)) = \mathrm{tr}(A'(t)\mathrm{Adj})$. Lastly, due to the linearity of the trace and that $\mathrm{Adj} = \det(A)A^{-1}$, we have

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\mathrm{tr}(A'(t)A^{-1}(t)).$$

$\square$

We now immediately make use of the lemma to find the variation of $dV_g$. Thus,

$$\begin{aligned} d\dot{V}_g &= \frac{d}{ds}[\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n]\Big|_{s=0} \\ &= \frac{1}{2\sqrt{|g|}}|g|\mathrm{tr}(g^{-1}h)dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}\mathrm{tr}(g^{-1}h)dV_g. \end{aligned}$$

It is not difficult to see that $\mathrm{tr}(g^{-1}h) = \langle h, g\rangle$, using the inner product on tensors. Thus $d\dot{V}_g = \frac{1}{2}\langle h, g\rangle dV_g$.

We now turn to the variation of the scalar curvature. We shall follow the convention that an index following a comma denotes differentiation with respect to some chosen coordinate basis, while an index following a semi–colon denotes covariant differentiation. Einstein summation notation will be used extensively. We will also perform the variation using normal coordinates at a point $p$ for the metric $g = g(0)$. Thus at $p$, assume the coordinate system has the properties that $g_{ij}|_p = \delta_{ij}$, $\Gamma^k_{ij}(p) = 0$, and $\partial_i g_{jk}|_p = 0$. [15]

The Riemann curvature tensor can be expressed in the following manner in a chosen coordinate system,

$$R_{ijk}{}^{l} = \Gamma^l_{ik,j} - \Gamma^l_{jk,i} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}.$$

The scalar curvature $R$ is given by $R = g^{ij}R_{ij} = g^{ij}R_{ikj}{}^{k}$. Thus,

$$R = g^{ij}[\Gamma^k_{ik,j} - \Gamma^k_{jk,i} + \Gamma^m_{ik}\Gamma^k_{jm} - \Gamma^m_{jk}\Gamma^k_{im}]$$

And so,

$$\dot{R} = \dot{g}^{ij}R_{ikj}{}^{k} + g^{ij}[\dot{\Gamma}^k_{ik,j} - \dot{\Gamma}^k_{jk,i} + \dot{\Gamma}^m_{ik}\Gamma^k_{jm} - \Gamma^m_{jk}\Gamma^k_{im}].$$

As $g_{ik}g^{kl} = \delta_{il}$, we have that

$$h_{ik}g^{kl} + g_{ik}\dot{g}^{kl} = 0.$$

So, after some manipulation, we find $\dot{g}^{ml} = -g^{mi}h_{ik}g^{kl}$, which once evaluating at $s = 0$ in normal coordinates where $g^{mi} = \delta_{mi}$, we find,

$$\dot{g}^{ml} = -h_{ml}.$$

When differentiating the terms quadratic in the Christoffel symbols, once evaluated in normal coordinates at $s = 0$,

$$(\Gamma^m_{ij}\dot{\Gamma}^k_{km}) = \dot{\Gamma}^m_{ij}\Gamma^k_{km} + \Gamma^m_{ij}\dot{\Gamma}^k_{km} = 0.$$

Thus, we have

$$\dot{R} = -h_{ij}R_{ij} + g^{ij}[\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{kj,i}].$$

By direct computation, we have

$$\Gamma^k_{ij,m} = \frac{1}{2}g^{kl}{}_{,m}[g_{il,j} + g_{jl,i} - g_{ij,l}] + \frac{1}{2}g^{kl}[g_{il,jm} + g_{jl,im} - g_{ij,lm}]$$

which implies,

$$\begin{aligned}\dot{\Gamma}^k_{ij,m} = &\frac{1}{2}(\dot{g^{kl}}_{,m})[g_{il,j} + g_{jl,i} - g_{ij,l}] + \frac{1}{2}g^{kl}{}_{,m}[h_{il,j} + h_{jl,i} - h_{ij,l}] \\ &+ \frac{1}{2}(\dot{g^{kl}})[g_{il,jm} + g_{jl,im} - g_{ij,lm}] + \frac{1}{2}g^{kl}[h_{il,jm} + h_{jl,im} - h_{ij,lm}]\end{aligned}$$

At $s = 0$, in normal coordinates,

$$\Gamma^k_{ij,m}\Big|_{s=0} = \frac{1}{2}[g_{ik,jm} + g_{jk,im} - g_{ij,km}],$$

thus,

$$\dot{\Gamma}^k_{ij,m} = (\dot{g^{kl}})\Gamma^l_{ij,m}\Big|_{s=0} + \frac{1}{2}[h_{ik,jm} + h_{jk,im} - h_{ij,km}].$$

Using these formulas give

$$\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{kj,i} = h_{ik,jk} - \frac{1}{2}h_{ij,kk} - \frac{1}{2}h_{kk,ji} + (\dot{g^{kl}})\Gamma^l_{ij,k}\Big|_{s=0} - (\dot{g^{kl}})\Gamma^l_{kj,i}\Big|_{s=0}.$$

Thus, evaluating in normal coordinates at $s = 0$,

$$g^{ij}[\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{kj,i}] = h_{ik,ik} - h_{ii,kk} - h_{kl}\Gamma^l_{ii,k} + h_{kl}\Gamma^l_{ki,i}.$$

Now, for a symmetric 2–tensor,

$$h_{ij;k} = h_{ij,k} - \Gamma^m_{ki}h_{mj} - \Gamma^n_{kj}h_{in}.$$

and

$$h_{ij;kl} = h_{ij,kl} - \Gamma^m_{ki,l}h_{mj} - \Gamma^m_{ki}h_{mj,l} - \Gamma^n_{kj,l}h_{in} - \Gamma^n_{kj}h_{in,l},$$

hence in normal coordinates,

$$h_{ij;kl} = h_{ij,kl} - \Gamma^m_{ki,l}h_{mj} - \Gamma^n_{kj,l}h_{in}.$$

Plugging this in above gives

$$g^{ij}[\dot{\Gamma}^k_{ij,k} - \dot{\Gamma}^k_{kj,i}] = h_{ik;ik} - h_{ii;kk}.$$

So,

$$\dot{R} = -h_{ij}R_{ij} + h_{ik;ik} - h_{ii;kk},$$

or in other words,

$$\dot{R} = -\langle h, \mathrm{Ric}\rangle_g + \mathrm{div}_g(\mathrm{div}_g(h)) - \Box_g \mathrm{tr}(h).$$

Thus, we have

$$0 = \int_U [-\langle h, \mathrm{Ric}\rangle_g + \mathrm{div}_g(\mathrm{div}_g(h)) - \Box_g \mathrm{tr}(h) + \frac{1}{2}(R - 2\Lambda)\langle h, g\rangle_g] dV_g.$$

Via Stokes theorem, we have that $\mathrm{div}_g(\mathrm{div}_g(h))$ and $\Box_g \mathrm{tr}(h)$ will only contribute boundary terms. Thus, the assumption that $h$ is compactly supported in $U$ will cause these terms to vanish. And so,

$$0 = \int_U \langle -h, \mathrm{Ric} - \frac{1}{2}Rg + \Lambda g\rangle_g dV_g.$$

As this must hold for all variations, we see that

$$\mathrm{Ric} - \frac{1}{2}Rg + \Lambda g = 0,$$

which is just $G + \Lambda g = 0$.

# Appendix C

# Derivation of the Einstein–Klein–Gordon equations

We begin with the action functional of the form

$$F_{\Phi,U}(g,\nabla) = \int_U (R - 2\Lambda - c_1|d\gamma|^2 - c_2|\gamma|^2)dV,$$

and define the vector field $v$ by,

$$\gamma = \star(v^*)$$

where $v^*$ is the 1–form dual to $v$ and $\star$ is the Hodge star operator. As the metric $g_{ab}$ is a Lorentz metric,

$$\star\gamma = \star\star(v^*) = (-1)^{1(4-1)}(-1)^1 v^* = v^*,$$

which implies the following

$$
\begin{aligned}
|\gamma|^2 dV = \langle \gamma, \gamma \rangle dV = \gamma \wedge \star\gamma \\
&= \star(v^*) \wedge v^* = -(v^* \wedge \star v^*) \\
&= -\langle v^*, v^* \rangle dV = -\langle (v^*)^\sharp, (v^*)^\sharp \rangle dV = -\langle v, v \rangle dV \\
&= -|v|^2 dV.
\end{aligned}
$$

Properties of the Hodge star also give

$$\star\star(d\star(v^*)) = (-1)^{4(0)}(-1)^1 d\star(v^*) = -d\star(v^*).$$

which implies that,

$$
\begin{aligned}
|d\gamma|^2 dV = \langle d\star(v^*), d\star(v^*) \rangle dV = d\star(v^*) \wedge \star d\star(v^*) \\
&= d\star v^* \wedge (\mathrm{div}(v)) = \mathrm{div}(v) d\star v^* \\
&= -\mathrm{div}(v)\star(\star d\star v^*) = -\mathrm{div}(v) \wedge \star(\mathrm{div}(v)) \\
&= -(\nabla \cdot v)^2 dV.
\end{aligned}
$$

where we have used the formula $\nabla \cdot v = \mathrm{div}(v) = \star d\star(v^*)$. Thus, the action functional takes the equivalent form,

$$F_{\Phi,U}(g,\nabla) = \int_U (R - 2\Lambda + c_1(\nabla \cdot v)^2 + c_2|v|^2) dV_g.$$

We next perform variations with respect to the vector field $v$ and the metric $g$ to compute the associated Euler–Lagrange equations. We first consider the variation of the vector field $v$. Letting $w = \dot{v}$, it is clear from $|v|^2 = \langle v, v \rangle$ that $\dot{|v|^2} = 2\langle v, w \rangle$. In any choice of coordinates, the divergence of $v$ takes the form,

$$\nabla \cdot v = v^i{}_{,i} + \Gamma^i_{ij} v^j,$$

which easily shows that

$$\frac{d}{ds}(\nabla\cdot v) = w^i{}_{,i} + \Gamma^i_{ij}w^j = \nabla\cdot w.$$

Thus, in performing the variation, we find,

$$0 = \int_U (2c_1(\nabla\cdot v)(\nabla\cdot w) + 2c_2\langle v, w\rangle)dV.$$

A quick use of Leibnitz's rule gives

$$\begin{aligned}\nabla_b((\nabla\cdot v)w^b) &= (\nabla\cdot v)(\nabla\cdot w) + (\nabla_b(\nabla\cdot v))w^b\\ &= (\nabla\cdot v)(\nabla\cdot w) + \langle\nabla(\nabla\cdot v), w\rangle.\end{aligned}$$

As the left hand side is a divergence term, our assumptions about the variations of $v$ being compactly supported in $U$ imply that the left hand will contribute nothing when integrated, thus

$$0 = \int_U \langle -2c_1\nabla(\nabla\cdot v) + 2c_2 v, w\rangle dV.$$

And as this holds for all variations, $w$, we find,

$$\nabla(\nabla\cdot v) = \frac{c_2}{c_1}v.$$

For performing the variation of the metric, $g$, we recall the following formulas from appendix B.

$$\frac{d}{ds}R = -\langle \mathrm{Ric}, h\rangle + \mathrm{div}(\mathrm{div}(h)) + \Box\mathrm{tr}(h).$$

$$\frac{d}{ds}dV_g = \frac{1}{2}\langle g, h\rangle dV_g.$$

In an arbitrary coordinate system, let $g$ denote the determinant of the components of $g_{ab}$ evaluated in these coordinates. Using another well known formula for the divergence of $v$,

$$\nabla\cdot v = \frac{1}{\sqrt{-g}}\sum_i \frac{\partial}{\partial x^i}(v^i\sqrt{-g}).$$

we find,

$$\frac{d}{ds}(\nabla\cdot v)=\frac{d}{ds}\left(\sum_i\frac{\partial v^i}{\partial x^i}+\frac{1}{\sqrt{-g}}\sum_i v^i\frac{1}{2\sqrt{-g}}\frac{\partial(-g)}{\partial x^i}\right)=\frac{d}{ds}\left(\frac{1}{2g}\sum_i v^i\frac{\partial g}{\partial x^i}\right).$$

Thus,

$$\begin{aligned}\frac{d}{ds}(\nabla\cdot v)&=-\frac{1}{2g^2}g\langle g,h\rangle\sum_i v^i\frac{\partial g}{\partial x^i}+\frac{1}{2g}\sum_i v^i\frac{\partial}{\partial x^i}(g\langle g,h\rangle)\\&=-\frac{\langle g,h\rangle}{2g}\sum_i v^i\frac{\partial g}{\partial x^i}+\frac{\langle g,h\rangle}{2g}\sum_i v^i\frac{\partial g}{\partial x^i}+\frac{1}{2}\sum_i v^i\frac{\partial}{\partial x^i}(\langle g,h\rangle)\\&=\frac{1}{2}\langle v,\nabla\langle g,h\rangle\rangle.\end{aligned}$$

Now, taking $\nu=v^*$ to be the 1–form dual to $v$, i.e. $\nu_a=g_{ab}v^b$, we find

$$\begin{aligned}\frac{d}{ds}|v|^2=\frac{d}{ds}(g_{ab}v^av^b)&=h_{ab}v^av^b\\&=h_{ab}g^{ac}\nu_cg^{bd}\nu_d=h^{cd}\nu_c\nu_d\\&=\langle h,\nu\otimes\nu\rangle.\end{aligned}$$

Let $B$ denote the integrand of the action functional. Performing the variation and ignoring the terms that will only contribute to the boundary, we find

$$0=\int_U[-\langle\mathrm{Ric},h\rangle-c_1(\nabla\cdot v)\langle v,\nabla\langle g,h\rangle\rangle+c_2\langle\nu\otimes\nu,h\rangle]dV_g+\frac{1}{2}B\langle g,h\rangle dV_g.$$

Focusing on the second term in this integral.

$$\begin{aligned}(\nabla\cdot v)\langle v,\nabla\langle g,h\rangle\rangle&=(\nabla_av^a)(v^b\nabla_b\langle g,h\rangle)\\&=(\nabla_av^a)[\nabla_b(v^b\langle g,h\rangle)-(\nabla_cv^c)\langle g,h\rangle]\\&=(\nabla_av^a)\nabla_b(v^b\langle g,h\rangle)-(\nabla_cv^c)^2\langle g,h\rangle\\&=\nabla_b(v^b\nabla_av^a\langle g,h\rangle)-\nabla_c(\nabla_dv^d)v^c\langle g,h\rangle-(\nabla_ev^e)^2\langle g,h\rangle.\end{aligned}$$

And by what we found earlier performing the variation of $v$, $\nabla_a(\nabla \cdot v) = \frac{c_4}{c_3} v_a$, hence

$$c_1(\nabla \cdot v)\langle v, \nabla\langle g,h\rangle\rangle = c_1 \nabla_b (v^b \nabla_a v^a \langle g,h\rangle) - c_2|v|^2\langle g,h\rangle - c_1(\nabla \cdot v)^2\langle g,h\rangle.$$

The first term is a divergence term, and once integrated, will vanish based upon our assumptions on the variation. So,

$$0 = \int_U \langle -\mathrm{Ric} - c_2|v|^2 g - c_1(\nabla \cdot v)^2 g + c_2(\nu \otimes \nu) + \frac{1}{2}Bg, h\rangle dV_g.$$

As this holds for all variations, we find

$$G + \Lambda g = \frac{1}{2}[-c_1(\nabla \cdot v)^2 - c_2|v|^2]g + c_2(\nu \otimes \nu).$$

Thus we have the following system of equations

$$\begin{aligned} G + \Lambda g &= c_2(\nu \otimes \nu) - \frac{1}{2}[c_1(\nabla \cdot v)^2 + c_2|v|^2]g \\ \nabla(\nabla \cdot v) &= \frac{c_2}{c_1} v. \end{aligned}$$

To simplify the understanding of these formulae, we introduce a new function $f$ with the property,

$$f = \left(\frac{c_1}{c_2}\right)^{1/2} \nabla \cdot v.$$

Thus the second formula above implies that

$$v = \left(\frac{c_1}{c_2}\right)^{1/2} \nabla f.$$

As by definition, $\nabla f$ is dual to $df$, we also have $\nu = \left(\frac{c_1}{c_2}\right)^{1/2} df$. Thus, we have the following,

$$c_2(\nu \otimes \nu) = c_1(df \otimes df), \quad c_1(\nabla \cdot v)^2 = c_2 f^2, \quad c_2|v|^2 = c_1|df|^2,$$

which gives an equivalent system of eqautions in terms of $f$,

$$G+\Lambda g = c_1\left[df\otimes df - \frac{1}{2}\left(|df|^2 + \frac{c_2}{c_1}f^2\right)g\right]$$
$$\Box f = \frac{c_2}{c_1}f.$$

And, via the Koszul formula, the connection $\Gamma$ has components,

$$\Gamma_{ijk} = \left(\frac{c_1}{c_2}\right)^{1/2}(\star df)_{ijk} + \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}).$$

We lastly, introduce the constants $\Upsilon$ and $\mu_0$ defined by

$$\frac{c_2}{c_1} = \Upsilon^2, \quad c_4 = 16\pi\mu_0.$$

We thus obtain the Einstein–Klein–Gordon system of equations with a cosmological constant in geometrized units (the gravitational constant and speed of light set to 1),

$$G+\Lambda g = 8\pi\mu_0\left[2\frac{df\otimes df}{\Upsilon^2} - \left(\frac{|df|^2}{\Upsilon^2} + f^2\right)g\right]$$
$$\Box f = \Upsilon^2$$

where $G$ is the Einstein curvature tensor, $f$ is the scalar field representing dark matter, $\Lambda$ is the cosmological constant, and $\Upsilon$ is some new fundamental constant of nature that is yet to be determined. For those who approach wave dark matter from a particle physics viewpoint instead of the geometric viewpoint described here, the fundamental constant $\Upsilon$ is the mass $m$ of the dark matter particle related by $m = \frac{\hbar\Upsilon}{c}$. [3]

# Appendix D

# Scale Invariance for the Schrödinger–Newton system

Let us begin with a triple $(r, S, V)$ thats is a solution to the Schrödinger–Newton system

$$(rS)'' = -SV$$
$$(rV)'' = -rS^2.$$

We aim to find values $\alpha$ and $\beta$ such that the triple

$$(\tilde{r}, \tilde{S}, \tilde{V}) = (\lambda^\alpha r, \lambda^\beta S(\lambda^\alpha r), \lambda^\beta V(\lambda^\alpha r)$$

is also a solution. For what follows, a $\dot{S}$ will denote differentiation with respect to $\tilde{r}$ and $S'$ will denote differentiation with respect to $r$. A quick application of the chain rule gives,

$$\dot{\tilde{S}} = \frac{d}{d\tilde{r}}[\tilde{S}(\tilde{r})] = \frac{d\tilde{S}}{dr}\frac{dr}{d\tilde{r}} = [\lambda^{\beta+\alpha} S'(\lambda^\alpha r)]\lambda^{-\alpha} = \lambda^\beta S'(\lambda^\alpha r).$$

Similarly, one can find $\ddot{\tilde{S}} = \lambda^{\beta} S''(\lambda^{\alpha} r)$. Now,

$$\begin{aligned}
\frac{d^2}{d\tilde{r}^2}(\tilde{r}\tilde{S}) &= 2\dot{\tilde{S}} + \tilde{r}\ddot{\tilde{S}} \\
&= \lambda^{\beta}[2S'(\lambda^{\alpha} r) + \lambda^{\alpha} r S''(\lambda^{\alpha} r)] \\
&= \lambda^{\beta-2\alpha}[\lambda^{2\alpha} S'(\lambda^{\alpha} r) + (\lambda^{2\alpha} r S'(\lambda^{\alpha} r))'] \\
&= \lambda^{\beta-2\alpha}[\lambda^{\alpha} S(\lambda^{\alpha} r) + \lambda^{2\alpha} r S'(\lambda^{\alpha} r)]' \\
&= \lambda^{\beta-2\alpha}[\lambda^{\alpha} r S(\lambda^{\alpha} r)]''.
\end{aligned}$$

At this point we make use of the fact that $(rS)'' = -rSV$, hence

$$\begin{aligned}
\frac{d^2}{d\tilde{r}^2}(\tilde{r}\tilde{S}) &= \lambda^{\beta-2\alpha}[-\lambda^{\alpha} r S(\lambda^{\alpha} r) V(\lambda^{\alpha} r)] \\
&= \lambda^{-\beta-2\alpha}[-\tilde{r}\tilde{S}\tilde{V}].
\end{aligned}$$

Effectively the same computations as above show that

$$\frac{d^2}{d\tilde{r}^2}(\tilde{r}\tilde{V}) = \lambda^{-\beta-2\alpha}[-\tilde{r}\tilde{S}^2].$$

Thus the triple $(\tilde{r}, \tilde{S}, \tilde{V})$ is a solution to the Schrödinger–Newton system when $\beta + 2\alpha = 0$. Thus take $\beta = 2$ and $\alpha = -1$.

# Appendix E

# The Sturm Comparison Theorem

For the sake of completeness, we present a proof of the Sturm comparison theorem.

**Theorem:** (Sturm) Let $p, q, P$ and $Q$ be continuous functions on an interval $[a,b]$. Also assume $(p(t)u')' + q(t)u = 0$ and $(P(t)v')' + Q(t)v = 0$, and assume that $p(t) \geq P(t) > 0$ and $Q(t) \geq q(t)$ on the interval $[a,b]$. If $y_1$ and $y_2$ are succesive zeroes of $u$ then

- either there exists a $x \in (y_1, y_2)$ with $v(x) = 0$.
- or there exists a $\lambda \in \mathbb{R}$ such that $v = \lambda u$.

*Proof.* Assume that $c$,$d$ are consecutive zeroes of the function $u$, and assume that $u > 0$ on the interval $(c,d)$. Let us also assume that $v$ does not have a zero in the interval $[c,d]$. Direct computation yields the following

$$\left[\frac{u}{v}(pu'v - Puv')\right]' = \frac{P(u'v - v'u)^2}{v^2} + (p - P)(u')^2 + (Q - q)u^2$$

Integrating both sides from $c$ to $d$ and making use of the fact that $u(c) = u(d) = 0$ gives that

$$-\int_c^d \frac{P(u'v - v'u)^2}{v^2} = \int_c^d (p - P)(u')^2 + \int_c^d (Q - q)u^2$$

As the right hand side is nonnegative and $P > 0$ it must be that $W(u,v) = u'v - v'u = 0$. Thus there exists $\lambda \in \mathbb{R}$ such that $v = \lambda u$. $\square$

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