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Three Essays on Applied Microeconomic Theory

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

Yu-Jing Xu

2014

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2014

ABSTRACT OF THE DISSERTATION

Three Essays on Applied Microeconomic Theory

by

Yu-Jing Xu

Doctor of Philosophy in Economics

University of California, Los Angeles, 2014

Professor Hugo Hopenhayn, Chair

This dissertation consists of three essays on Applied Microeconomic Theory.

The first chapter studies investment incentives in a dynamic random search environment where a seller can make unobservable and selfish investments to reduce his production cost before searching for buyers. In the unique steady state equilibrium, although sellers make positive investments, equilibrium payoffs and the social welfare are a) constant given any search friction and b) equal to the values that would be created if there were no investment. These results hold even in the limit, with the investment strategy converges to the first best and the stationary investment distribution converges to a point mass at no investment.

The second chapter demonstrates how sorting conditions change in search frictions, in a dynamic random matching environment where heterogeneous buyers have private types and heterogeneous sellers make take-it-or-leave-it offers. We first establish the existence of steady-state equilibrium and then characterize sorting conditions under two extreme search frictions. Positive (negative) assortative matching requires the production function to be log-supermodular (log-submodular) with maximal search frictions. When search frictions vanish, the condition for positive (negative) sorting returns to supermodularity (submodularity).

The last chapter examines how patents influence firms' allocation of resources over a portfolio of projects with different non-obviousness. We consider the situation where two identical firms have the replicas of two projects. One is known to be good and the other is risky but is more innovative conditional on being good. Compared to the allocation of resources without any patent, the total amount of resources allocated on the risky project is more efficient. However, the competition for the patent of the safe project also induces firms to inefficiently delay experimenting the risky project. Overall, the welfare effect of patents might be negative.

The dissertation of Yu-Jing Xu is approved.

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2014

To my parents and my husband

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CHAPTER 1

Unobservable Investments and Search Frictions

1.1 Introduction

In many situations, investments are not relation-specific and the non-investing party cannot perfectly observe the level of investments. An investor is therefore not held up as defined in the classic holdup literature, in the sense that after making investment, he could walk away from an aggressive price offer and still be able to utilize his investment in the future. The current article tries to answer the following two questions. Are the positive continuation payoff and the rents created from the unobservability enough to incentivize agents to invest more efficiently? If so, is there any social welfare generated from positive investments? In the environment considered in this article, the answer to the first question is yes. Quite surprisingly, the answer to the second one is no.

With these questions in mind, I propose a discrete time infinite horizon random search model, where agents can make pre-entry, unobservable and selfish investments, and focus on steady state of the equilibrium. The two sides of the market are called “supplier” and “retailer”. At the beginning of each period, there are one unit mass of ex ante identical newly born suppliers and retailers entering the market. A supplier entrant is endowed with a technology to produce one unit of output at cost x_0 and he can invest to reduce his production cost before entry. Denote the cost resulting from the efficient investment by x^* . Meanwhile, a retailer entrant is endowed with the ability to sell one unit of a supplier’s output at price

$y_0 > x_0$. After the investment stage, all agents (entrants and incumbents, i.e., agents who did not leave the market in the previous period) are randomly paired and the retailer in each pair makes a take-it-or-leave-it offer, without observing the investment. If the offer is accepted, then production takes place, the monetary transfer is made and both agents leave the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. ¹

In the unique steady state equilibrium, suppliers use mixed strategy over investments between zero and the efficient investment, and retailers play mixed strategy over these suppliers' reserve prices.² There are two interesting implications of this result. First of all, suppliers using mixed strategy implies that investments are positive even if sellers have no bargaining power. Secondly, mixed pricing strategy implies that price dispersion emerges in equilibrium even with all identical agents.

This article then characterizes three properties of the steady state equilibrium. First of all, although investments are always positive, agents' payoffs as well as the social welfare are constant given any search friction and they are the same as if there were no investment. To understand this result, first note that suppliers get zero ex ante payoff, following the same logic as in Diamond (1971). Therefore, the social welfare is equal to a retailer's ex-ante payoff. For a retailer who offers a price lower than the reserve price of a type x_0 supplier, there is strictly positive probability that the profitable trade is not conducted. Therefore, the welfare that could be generated from the positive investment is burnt due to the delay in trade and no welfare gain is realized in equilibrium.

Secondly, as search frictions diminish, which is modelled as the shrinking time

¹The setting reflects a broad set of applications. Besides the vertical relationships between suppliers and retailers, the modelled environment might also represent, for instance, product markets where consumers invest in complementary goods before searching stores.

²In a Coasian setting where the buyer's investment determines his valuation, Gul (2001) shows that the investment strategy must be a mixed strategy and that the seller mixes over prices in the first round. The current result demonstrates that the same intuition carries through to a random search environment.

interval time interval between periods, more underinvested suppliers congest the market and retailers price more aggressively. To be more precise, the stationary cost distribution, i.e. the cost distribution of incumbents, converges in distribution to a point mass at the initial cost x_0 and the price offer distribution converges in distribution to a point mass at the reserve price of a type x^* supplier. Because the cost distribution of incumbents is less efficient, it takes more periods in expectation for a retailer to meet a supplier whose cost is low enough to be willing to accept his price. This result shows that trading inefficiency is more severe as search frictions diminish.

The third property is that the investment strategy becomes efficient as search frictions vanish, shifting towards the opposite direction as the stationary cost distribution does. As shown in the second property, retailers price extremely aggressively in the limit. As a result, underinvested suppliers leave the market with almost zero probability each period. Therefore the proportion of underinvested exits converges to zero, despite the fact that the proportion of underinvested suppliers on the market converges to 1. In steady state, the cost distribution of entrants are the same as that of exits. Consequently, the investment strategy also converges in distribution to a point mass at x^* .

Combining the second and the third properties, we can better understand why the social welfare is constant across search frictions. As search frictions diminish, on the one hand investments are more efficient but on the other hand the trading inefficiency is more severe. The net effect of the two is zero.

If we revisit the two questions raised at the beginning, in the current setting where investments are unobservable and not relation-specific, although investments become efficient as search frictions vanish, the social welfare is not improved.

The baseline model is extended along two dimensions to incorporate the possibilities of two-sided investments and two-sided offers. In the first extension,

retailer entrants could also invest before entry to raise the revenue from selling one unit of the output. The results of the baseline model continue to hold. In particular, agents' payoffs and the social welfare equal the values that would be created if investments were observable. As search frictions vanish, the investment strategy becomes efficient while the stationary cost distribution converges to a point mass at x_0 . Moreover, we have a set of new results regarding the retailer's investment strategy. First of all, although retailers have all the bargaining power, they still underinvest and use a mixed strategy over an interval of investments given any search friction. Secondly, the retailers' minimum investment level rises as search frictions diminish but the investment strategy converges in distribution to a point mass at the minimum level.

The second extension concerns the situation where suppliers also make take-it-or-leave-it offers occasionally. The result of welfare equivalence between observable and unobservable investments still holds. The least efficient supplier invests positive amount, since he is a residual claimant when he makes the offer. This minimum investment level converges to the efficient level as search frictions vanish or as the suppliers' bargaining power goes to 1. Hence the investment strategy as well as the social welfare converges to the first best.

Related Literature

The most related work is Gul (2001), which examines the holdup problem in a Coasian setting where a buyer's valuation is determined by his unobservable investment prior to the bargaining game. Similar to our model, equilibrium investment strategy must be a mixed strategy and it becomes efficient as the time between each offer shrinks to zero. The difference is that in the setting of Gul (2001) the social welfare converges to the first best, resulting from the fact that, as emphasized in Lau (2008), there is no "bargaining delay" in the limit. In our model, although the investment strategy is efficient, almost all suppliers on the market are the underinvested type. The per-period trading probability of any

price other than the highest price is therefore bounded away from 1. This delay in trade erodes any welfare gain that could be generated from the efficient investment.

Compared to the literature, we made some simplification assumptions to keep the model tractable. We assume that investments are purely selfish, whereas other works investigate investment incentive with cooperative investments (Che and Hausch (1999), Mailath Postlewaite and Samuelson (2012), Hermalin (2013), etc). Moreover, investments are assumed to be unobservable. In reality, the non-investing party might get some information about investments or the outcome of investments (Rogerson (1992), Lau (2008), Hermalin and Katz (2009), etc). Finally, in stead of assuming free entry (Acemoglu (1996), Davis (2001), Acemoglu and Shimer (1999), etc), we assume that there are fixed measure of entrants in each period.

This article also links to the literature on price (wage) dispersion. The non-existence of a single wage equilibrium with heterogeneous workers in a dynamic search framework is well-known in the literature (Albrecht and Vroman (1992) and their succeeding papers). We demonstrate that price dispersion is also a reason for heterogeneity if it results from agents' costly investments: price dispersion generates a distribution of probability of trading and it in turn makes workers indifferent among investments. To my best knowledge, the current study is the first attempt to examine the possibility of the joint emergence of participants' heterogeneity and the price dispersion in a random search environment.

The searching stage of our model is similar to settings of voluminous works on search and bargaining games with asymmetric information (for instance, Rubinstein and Wolinsky (1990), Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2010a), Lauermann (2012) (2013), etc). The participants in these models are assumed to born with heterogeneous types. One of the central topics in this literature is to understand the efficiency impact of information frictions and search

production cost to $x \geq 0$. We assume $c(x_0) = c'(x_0) = 0$ and $c'(0) < -1$, and for any $x < x_0$, $c(x)$ is strictly decreasing and strictly convex. All entrants on both sides then join the incumbents on the market who did not exit in the last period. The market sizes on both sides are assumed to be the same. In each period, one retailer is randomly matched with one supplier and vice versa. The retailer in each pair makes a take-it-or-leave-it offer p , which is a monetary transfer from the retailer to the supplier, and the supplier decides whether or not to accept it.

Therefore, a supplier's strategy consists of two components: a investment strategy CDF $F_e(x)$ and a reserve price function $r_S(x)$, where $F_e(x)$ measures the probability that the investment is strictly larger than $c(x)$ and $r_S(x)$ is the lowest price that a supplier with cost x is willing to accept. A retailer's strategy is a price offer CDF $H(p)$, where $H(p)$ equals the probability of offering a price weakly lower than p .

Preference: If the offer is accepted, one unit of output is produced and sold, which leaves the supplier payoff $p - x$ and the retailer payoff $y_0 - p$. Both agents exit the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. All agents have the same discount factor $\beta \in [0, 1)$, which is the parametrization of search frictions in the current setting.

Information: A crucial assumption is that retailers have no information about investments. In addition, the matching is anonymous.

We will end this section with remarks on the simplification assumption of selfish investment. This assumption excludes any signalling and moral hazard problem. While investments in many applications are (partially) cooperative, for instance, a supplier's investment improves the quality of his output which increases a retailer's revenue directly, the current setting can serve as a benchmark model for future work with more general assumptions on investments. Discussions on the robustness of other assumptions is postponed to section 1.7.1.

1.2.2 Benchmark Specifications

The First Best

We first characterize the efficient allocation, which consists of both efficient investment and efficient trade.

At the search stage, a social planner would find it optimal to conduct trade between any pair of agents given any cost distribution, since the surplus from trade is always positive and postponing trading is costly due to discounting.

Given that trades take place immediately at the search stage, if a supplier invests to reduce his cost to x , he increases the social welfare by $x_0 - x$ with investment $c(x)$. A social planner will thus choose $x^* > 0$ defined by $c'(x^*) = -1$ so that the marginal cost of investment equals the marginal benefit.

Observable Investment

Next consider the situation where investments are observable. Following the same logic in Diamond (1971), because buyers have all the bargaining power, a supplier with any cost x gets zero search stage payoff. To be more precise, observing the production cost, the retailer in the current match and in all future matches will offer the supplier's discounted continuation payoff. Because the discount factor is strictly less than 1, this infinitely repeated discounting leads to zero search stage payoff. The suppliers therefore have no incentive to invest.

Therefore, in the unique equilibrium, no supplier invests and all retailers offer price x_0 . Investments are inefficient while trades are efficient.

1.3 The Steady State Equilibrium

Let us now solve the steady-state equilibrium in the decentralized market. A steady state equilibrium consists of a supplier's investment strategy CDF $F_e(x)$ when he is an entrant and reserve price function $r_S(x)$ when he is an incumbent,

a retailer's price offer distribution CDF $H(p)$ and a stationary cost distribution $F(x)$.

1.3.1 The Supplier's Problem

At the search stage, a supplier with cost x chooses the lowest price he is willing to accept, i.e. the reserve price $r_S(x)$. Given a price offer distribution $H(p)$, his trading probability is $1 - H(r_S(x)) + Pr(\tilde{p} = r_S(x))$, which is decreasing in $r_S(x)$. The maximization problem of a type x supplier can be summarized as follows,

$$U(x) = \max_r \{ (E(\tilde{p} | \tilde{p} \geq r) - x)(1 - H(r) + Pr(\tilde{p} = r)) + (H(r) - Pr(\tilde{p} = r))\beta U(x) \} \quad (1.1)$$

Solving the above problem, the reserve price $r_S(x)$ is specified as follows:

$$r_S(x) = x + \beta U(x) \quad (1.2)$$

The reserve price function can be interpreted as usual. The first term x reflects the fact that a more efficient supplier (with a smaller x) has stronger incentive to trade and is willing to accept lower prices. In addition, the opportunity to search in the future raises the reserve price, the magnitude of which is positively associated with the discount factor and the equilibrium continuation payoff, as captured in the second term.

It is intuitive to infer that a more efficient supplier has higher search stage payoff $U(x)$ and is willing to accept lower price offers. Denote the highest cost on the support of $F(x)$ by \bar{x} . The following lemma confirms this conjecture.

Lemma 1.1. *In any steady-state equilibrium, $U(x)$ is strictly decreasing and continuous in x , with $U(\bar{x}) = 0$. $r_S(x)$ is strictly increasing and continuous in x .*

Otherwise mentioned, all proofs of this article are provided in the appendix.

Lemma 1.1 implies that $\hat{x}(p)$, the inverse function of $r_S(x)$ is well-defined, continuous and strictly increasing. Function $\hat{x}(p)$ specifies the highest type of a supplier who is willing to accept price p . Intuitively, $\hat{x}(p)$ strictly increases in p since a lower price will be accepted only by a supplier with lower production cost.

In addition, lemma 1.1 shows that the search stage payoff of the least efficient supplier is zero. This is the case because the same logic as in Diamond (1971) holds for this supplier. In equilibrium, no retailer would offer a price higher than this supplier's reserve price. Therefore, the supplier can trade only when he meets a retailer who offers his reserve price, which leaves him his discounted continuation payoff. Then by the same infinitely repeated discounting argument, the least efficient supplier must get zero search stage payoff.

Lemma 1.1 also implies that the highest cost equals the initial cost, i.e. $\bar{x} = x_0$. A supplier with the highest cost gets zero search stage payoff and therefore has no incentive to invest at all. His ex-ante payoff therefore equals 0. In equilibrium, ex-ante identical suppliers must be indifferent over any x on the support of the investment strategy $F_e(x)$, and weakly prefer these x to any other x that is not on the support. In other words, $H(p)$ must be such that

$$U(x) - c(x) = 0, \text{ for any } x \text{ on the support of } F_e(x) \quad (1.3)$$

$$U(x) - c(x) \leq 0, \text{ for any } x \text{ not on the support of } F_e(x) \quad (1.4)$$

1.3.2 The Retailer's Problem

A retailer's strategy is a price offer CDF $H(p)$. We know from the last section that if p is offered, any supplier with cost lower than $\hat{x}(p)$ will agree to trade. Recall that the probability of meeting a retailer with cost weakly lower than x is $F(x)$. The probability of trade therefore equals $F(\hat{x}(p))$.

The equilibrium $F(x)$ must be such that it makes a retailer indifferent over any

p on the support of $H(p)$. That is, p solves the following maximization problem,

$$\pi = \max_p \{(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi\} \quad (1.5)$$

In addition, offering any p' that is not on the support, a retailer must get a weakly lower profit, i.e.,

$$\pi' = (y_0 - p')F(\hat{x}(p')) + (1 - F(\hat{x}(p')))\beta\pi' \leq \pi \quad (1.6)$$

1.3.3 The Supplier's Investment Strategy

The last piece of the model is the entrant cost distribution, which equals the supplier's investment strategy³. In steady state, the measure of outflow of any type must equal the measure of inflow of the same type. A supplier with type x leaves the market if he gets an offer that is weakly higher than $r_S(x)$ (which happens with probability $1 - H(r_S(x))$). Meanwhile, the measure of entrants with type lower than x is $F_e(x)$. Therefore, the steady-state equilibrium require that for any x on the support,

$$F_e(x) = \frac{F(x) - \int_x^x H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{x_0}^x H(r_S(\tilde{x}))dF(\tilde{x})} \quad (1.7)$$

Notice that we wrote down the above equation as if the support of $F(x)$ is an interval. This is merely for the sake of shortening the expression. We will prove that it is indeed an interval in the next section.

1.3.4 Equilibrium Existence and Uniqueness

Let us first summarize the dynamic of a steady-state equilibrium. The price offer strategy $H(p)$ is such that it makes a supplier indifferent across any x on the support of $F_e(x)$. The investment strategy $F_e(x)$ together with the trading

³Since there are one unit mass of entrants on both sides each period, we get the equivalence between the entrant cost distribution and the investment strategy as we abuse the law of large number as usual.

strategy determine the stationary cost distribution $F(x)$, which makes a retailer indifferent across any price on the support of $H(p)$.

Proposition 1.1. *In any steady state equilibrium, the price offer distribution $H(p)$, the supplier's investment strategy $F_e(x)$ and the stationary cost distribution $F(x)$ have the following properties,*

1. $F(x)$ and $F_e(x)$ have support $[x^*, x_0]$ with the unique point mass at x^* .
2. $H(p)$ has support $[r_S(x^*), r_S(x_0)]$ and is atomless;

Although all agents are identical ex-ante, proposition 1.1 shows that the stationary price offer distribution and the investment strategy are non-degenerate. The unobservability of investments is the key behind this result. Suppose all suppliers choose the same investment level. As a result, all retailers will offer the same price, which makes a supplier a residual claimant. Therefore, depending on the price offered, a supplier will find it optimal to invest either efficiently or nothing. In the case where the supplier invests efficiently, retailers will best response to offer x^* and extract the entire surplus. Hence, suppliers are fully held up and have no incentive to invest. In the case where the supplier invests zero, retailers will best response to offer x_0 . The supplier will then deviate to the efficient investment. This cycle can never be an equilibrium.

We would also speculate that there might be a gap on the support of the price distribution and the stationary cost distributions. For instance, it is possible that prices on an interval (p_1, p_2) are not offered, because suppliers with types $x \in (\hat{x}(p_1), \hat{x}(p_2))$ are not on the market in equilibrium and hence there is no gain from offering their reserve prices. Meanwhile, no suppliers choose to become type $x \in (\hat{x}(p_1), \hat{x}(p_2))$ because their reserve prices are not offered. Unfortunately, this intuition neglects one equilibrium restriction: a supplier must be indifferent between $\hat{x}(p_1)$ and $\hat{x}(p_2)$. Because no price between p_1 and p_2 is offered, two suppliers having these two costs trade with the same probability. Therefore, $U(x)$ is linear

on the interval $(\hat{x}(p_1), \hat{x}(p_2))$. On the other hand, the investment cost function $c(x)$ is strictly convex. Hence, a supplier could never be indifferent between these two investment levels.

Proposition 1.1 establishes that there is no point mass on the price distribution either. Any point mass will result in a jump in the probability of trade, which in turn leads to a kink in $U(x)$. However, the investment cost function $c(x)$ is smooth everywhere. This again contradicts the indifference condition.

This proposition also shows that the lowest production cost on the market is the efficient cost x^* , since a supplier with the lowest cost trades in the first period with probability one: any price offer on the market is weakly higher than his reserve price. Since his investment is unobservable to the retailer, this supplier becomes the residual claimant and hence invests efficiently.

Finally, the measure of type x^* suppliers must be positive, since a retailer who offers $r_S(x^*)$ must get positive payoff. Moreover, there is no other point mass, because any other point mass would lead to a jump in the retailer's payoff as a function of p . This contradicts the retailer's indifference condition.

Notice that the above results and intuitions hold even when $\beta = 0$, in which case the retailer in each pair is a monopolist. The monopolist is indifferent over an interval of prices because in the current setting the stationary cost distribution is adjusted through investment and trading strategies so that the elasticity of the demand function is always 1 over the interval of prices.

In the rest of this section, we will first solve $H(p)$ and $F(x)$, and then show the existence and the uniqueness of the steady state equilibrium.

$H(p)$ can be solved from the envelope condition of $U(x)$. We are allowed to take the derivative of $U(x)$ because we have proved that the support is an interval, and that $U(x) - c(x) = 0$ for any x on the support. The smoothness of $c(x)$ thus

implies the smoothness of $U(x)$. The envelope condition is

$$U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x) \quad (1.8)$$

Using the equilibrium restriction that $U'(x) = c'(x)$, $H(p)$ can be solved,

$$H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \quad (1.9)$$

$F(x)$ is solved from the retailer's indifference condition. If a retailer offers $r_S(x_0) = x_0$, he can trade with probability 1. Therefore $\pi = y_0 - x_0$.

Any other price on the support must yield the same expected profit. In other words,

$$(y_0 - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\pi = y_0 - x_0$$

Therefore, the stationary cost distribution $F(x)$ equals,

$$F(x) = \begin{cases} 0, & \text{if } x \in (-\infty, x^*), \\ \frac{y_0 - \beta\pi - x_0}{y_0 - \beta\pi - x - \beta c(x)}, & \text{if } x \in [x^*, x_0], \\ 1, & \text{if } x \in (x_0, +\infty). \end{cases} \quad (1.10)$$

We summarize this section with the following proposition.

Proposition 1.2. *Steady state equilibrium exists and is unique.*

1.4 Equilibrium Characterization

1.4.1 Constant Payoffs and Social Welfare

As shown in the previous section, suppliers always invest with positive probability given any search friction. We would expect that the social welfare is higher than what we would get in the benchmark case with observable investment, where suppliers have no incentive to invest. However, the following theorem shows that this is unfortunately not the case.

Theorem 1.1. *For any $\beta \in [0, 1)$, the supplier's ex-ante payoff equals 0, the retailer's ex-ante payoff and the social welfare equal $y_0 - x_0$.*

Proof. we know $v = U(x) - c(x) = 0$ and $\pi = y_0 - x_0$. The social welfare therefore equals $v + \pi = y_0 - x_0$. □

Comparing agents' payoffs and the social welfare with those of the benchmark case with observable investment, we can easily verify that values are the same. This is because the unobservability, which incentivizes investments, also causes trading inefficiency. The welfare gain generated from investments could be realized fully only if supplier entrants and retailer entrants are paired with each other and agree to trade in the first period after they enter the market. However, this is impossible with the presence of information frictions and search frictions: because of the unobservability, both the cost distribution and the price distribution are non-degenerate in equilibrium. Profitable trades are therefore conducted only probabilistically. In other words, there is expected delay in trade for any retailer whose price offer is strictly lower than $r_S(x_0)$. The welfare loss due to the delay in trade exactly offsets the welfare gain from the more efficient investments.

From section 3.1, we already know that the ex ante payoff of suppliers is zero, since the least efficient supplier gets zero ex ante payoff and all suppliers are indifferent across investment levels. Given that the supplier's payoff as well as the social welfare are the same in both scenarios with observable and unobservable investments, the division of the surplus must also be the same.

In the next section, we will examine how search frictions affect investment and trading efficiency. The result shows that as β increases, on the one hand more entrants invest efficiently but on the other hand, it takes more periods for a retailer offering a given price to trade. The fact that investment efficiency and trading efficiency moving towards the opposite direction explains why the social welfare is constant given any search friction.

1.4.2 Comparative Statics and the Limiting Case

Consider any supplier with cost $x \in (x^*, x_0)$, if the price distribution remains constant as meetings become more frequent, the supplier trades with higher probability per unit of time. Consequently, the marginal benefit of investment, which strictly increases in the probability of trade, will be strictly larger than the marginal cost of investment. To keep a supplier indifferent across investments with smaller search frictions, retailers must price more aggressively. That is, the per-period trading probability $1 - H(r_S(x))$ must strictly decrease in β . Indeed,

$$\frac{\partial(1 - H(r_S(x)))}{\partial\beta} = \frac{(1 + c'(x))c'(x)}{(1 + \beta c'(x))^2} < 0, \text{ for any } x \in (x^*, x_0)$$

As search frictions vanish, the probability of trade $1 - H(r_S(x))$ must converge to 0 for any $x \in (x^*, x_0)$. Or equivalently, retailers price extremely aggressively in the limit: the price offer distribution must converge in distribution to a point mass at $r_S(x^*)$. Otherwise, any supplier trades for sure within any small amount of time if the trading probability is bounded away from zero as the time between successive meetings shrinks to zero. The marginal benefit of investment therefore becomes 1 and a supplier cannot be indifferent across investment levels. This is a contradiction. We can also verify this intuition from equation (1.9). As β converges to 1, the probability of trade $1 - H(r_S(x))$ converges to 0 for any $x \in (x^*, x_0)$.

Next we show that the stationary cost distribution $F(x)$ is less efficient as β increases. If $F(x)$ stays constant when meetings get more frequent, a retailer who is originally indifferent over price offers would strictly prefer to offer the lowest price $r_S(x^*)$. Therefore, the probability of trade of a retailer offering a price lower than $r_S(x_0)$ must decrease in β . That is, the stationary cost distribution $F(x)$ with a larger β first order stochastically dominates an $F(x)$ with a smaller β . Indeed,

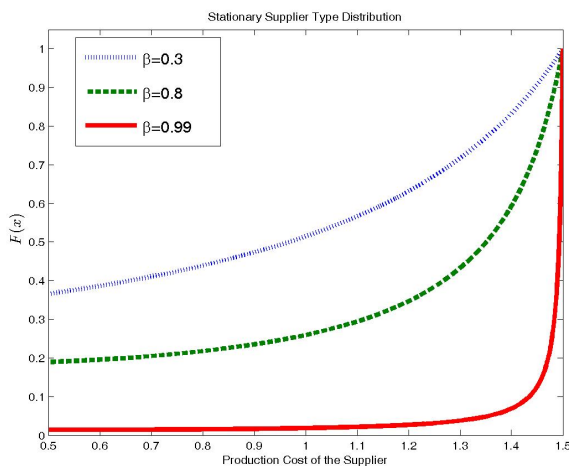


Figure 1.2: Stationary Supplier Type Distribution

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

the derivative of $F(x)$ with respect to β confirms this conjecture,

$$\frac{\partial F(x)}{\partial \beta} = \frac{(y_0 - x_0)(x - x_0 + c(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2} < 0$$

In the limit, it is straight forward to verify that $F(x)$ converges in distribution to a point mass at x_0 from equation (1.10). That is, incumbents compose almost entirely of suppliers who invested zero amount. This result is shown graphically in Figure 1.2. Intuitively, if $F(x)$ is bounded away from 0 in the limit for some $x < x_0$, then the per-period trading probability of a retailer offering $r_S(x)$ is strictly positive. Again, as the time between successive meetings shrinks to zero, it is as if the retailer could trade immediately. Hence, a retailer who offers $r_S(x_0)$ would find it optimal to lower the price offer to $r_S(x)$ without lowering trading probability, leading to a contradiction.

In steady state, the cost distribution of entrants is the same as that of exits to keep the stationary cost distribution constant over periods. Since the cost distribution of incumbents becomes less efficient as β increases, we would expect that exits and hence entrants also consist more of underinvested suppliers as search frictions diminish. That is, the investment strategy $F_e(x)$ becomes less efficient

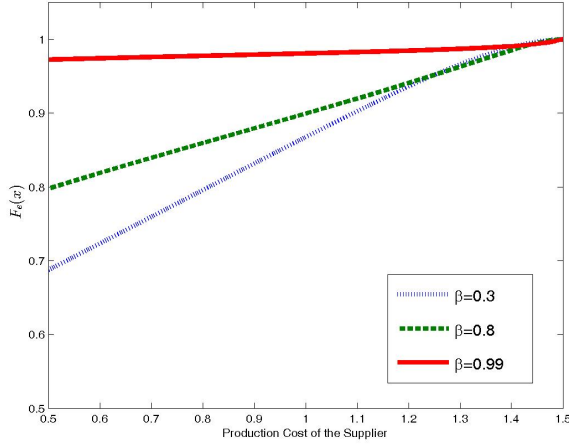


Figure 1.3: Supplier's Investment Strategy

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

as β increases.

It turns out that the opposite result holds: $F_e(x^*)$ strictly increases in β , i.e., an entrant invests efficiently with strictly larger probability. In the limit, the investment strategy becomes efficient. The investment strategy of the same example is plotted in Figure 1.3.

To understand this result, we only need to figure out why exits consist of more efficient suppliers as β increases by the steady-state equilibrium restriction (1.7). We know that a supplier who invests efficiently always exits the market immediately independent of β . In addition, we have three observations from the previous analysis: as β increases, 1) larger proportion of incumbents are underinvested type; 2) per-period trading probabilities of underinvested suppliers strictly decrease; 3) the stationary cost distribution has larger mass on high costs and hence the average cost of underinvested suppliers strictly increases, which implies that a supplier with the average cost exits with lower probability. The first effect raises the proportion of underinvested exits, as captured by the previous intuition. The rest two effects explain why the proportion of underinvested exits decreases

in β : any underinvested supplier exits less often and the composition of the underinvested suppliers becomes more inefficient. These three effects change in β at the same rate. Therefore, the net effect is that the investment strategy converges to a point mass at x^* as β goes to 1.

The above comparative statics and the limiting results are summarized in the following propositions.

Theorem 1.2. *As β increases to 1, in the steady state equilibrium*

1. *$H(r_S(x))$ strictly increases for any $x \in (x^*, x_0)$ and converges in distribution to a point mass at $r_S(x^*)$;*
2. *$F(x)$ strictly decreases for any $x \in [x^*, x_0)$ and converges in distribution to a point mass at x_0 ;*
3. *$F_e(x^*)$ strictly increases and $F_e(x)$ converges in distribution to a point mass at x^* .*

From this proposition, we can better understand the mechanism behind the constant social welfare result. As β increases, new entrants invest more efficiently and this could potentially generate additional social welfare if the trading efficiency remains constant. Unfortunately at the same time trades become more inefficient. As β increases, the incumbent cost distribution has more mass on high costs. Consequently, for a retailer who offers a given price, it takes more periods in expectation to find a supplier whose cost is low enough so that he is willing to accept the price. The retailer's payoff is independent of the magnitude of search friction, because the expected time to trade is constant: although the time between two successive periods shrinks, it also takes more periods on average to trade. We already know that suppliers always get zero ex ante payoff. The constant retailer's ex ante payoff is equivalent to the constant social welfare.

We will end this section with some comments on empirical implications of the results. The fact that the cost distributions of entrants and of incumbents move in the opposite directions suggests that cross sectional data may not be a satisfactory measure of investment efficiency. According to the current model, the incumbent cost distribution is affected by not only the investment decision, but also the trading strategy. For instance, consider the limiting case where $\beta \rightarrow 1$. The cross sectional data would suggest that almost all suppliers invest little, whereas in fact almost all entrants invest efficiently. Therefore, to measure investment efficiency, we need to look at the time series data and identify the cost distribution of entrants.

We can also derive several testable implications from Theorem 1.2. For instance, the frequency of meeting is 1) negatively correlated with entrant/market ratio; 2) positively correlated with investment efficiency; 3) negatively correlated with the efficiency of stationary cost distribution and 4) positively correlated with the aggressiveness of price distribution.

The rest of the article extends the baseline model along two directions. Section 1.5 considers the situation where a retailer could also invest to raise the revenue. Section 1.6 examines a two-sided offering case where a supplier makes a take-it-or-leave-it offer with positive probability.

1.5 Two-Sided Investments

We often observe that both sides of the market make investments before searching for trading partners. For instance, a retailer could also decide how much resource to spend on advertising, which could potentially raise the revenue from selling one unit of the product.

Suppose before entering, a retailer can invest to increase the revenue from y_0 to y with investment $e(y)$, where $e(y_0) = e'(y_0) = 0$ and $e(y)$ is strictly increasing

and strictly convex for any $y > y_0$. The surplus from trade between a type x supplier and a type y retailer is assumed to be $y - x$. Notice that this assumption implies that there is no complementarity between a supplier's and a retailer's investments and this feature simplifies analysis a lot⁴. Finally, the observability of retailer's investment could be arbitrary. Results of this section do not depend on the assumption about the observability of y , since there is no complementarity by assumption and retailers have all the bargaining power.

We again consider the benchmarks of a) the first best, b) observable supplier's investment. In the first best benchmark, since there is no complementarity, socially optimal trading strategy requires all agents to trade upon their first meetings. Given this trading strategy, a social planner would invest efficiently on both sides, i.e., all suppliers invest to reduce the production cost to x^* and all retailers invest to raise the profit to y^* , where y^* is defined implicitly by $e'(y^*) = 1$.

In the benchmark case with observable investments, all suppliers invest zero and all retailers offer x_0 and invest efficiently. In equilibrium, a supplier gets payoff 0 and a retailer gets payoff $y^* - x_0 - e(y^*)$, which is also the social welfare.

In the rest of this section, we will first derive optimality conditions of the steady state equilibrium and then characterize the equilibrium.

1.5.1 The Steady State Equilibrium

Since there is no complementarity, retailers' investments do not benefit suppliers directly and suppliers only care about the price distribution. Therefore, the supplier's problem is exactly the same as in the baseline model and previous equilibrium restrictions for suppliers must continue to hold with two-sided investments.

⁴In the case with one-sided investment, we could impose this assumption without loss of generality, since all retailers are identical. In the two-sided investment case, however, this assumption excludes some interesting scenarios that we could observe in many industries. For instance, the surplus from trade could be supermodular, that is, a higher supplier's investment level leads to a larger marginal benefit of the retailer's investment.

As a result, this section will focus on the retailer's problem. A retailer's strategy consists of his investment strategy CDF $G_e(y)$ and the price offer $p(y)$. $p(y)$ maximizes the search stage payoff of a type y retailer, which is denoted as $\Pi(y)$,

$$\Pi(y) = \max_p \{(y - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\Pi(y)\} \quad (1.11)$$

Moreover, the following indifference conditions must hold, which essentially require a retailer to be indifferent across any y on the support of the investment strategy $G_e(y)$ and weakly prefer those y to any other y that is not on the support,

$$\Pi(y) - e(y) = \pi \geq 0, \text{ for any } y \text{ on the support of } G_e(y) \quad (1.12)$$

$$\Pi(y) - e(y) \leq \pi, \text{ for any } y \text{ not on the support of } G_e(y) \quad (1.13)$$

Proposition 1.3. *In any steady state equilibrium with two-sided investments, $p(y)$ is single-valued and increases in y for any y on the support of $G_e(y)$.*

Proposition 1.3 shows that price offer increases in retailer's type. Intuitively, it costs more for a retailer who has higher revenue to delay trade. Therefore, he is willing to offer a higher price to ensure trade. Consequently, a retailer with the highest revenue \bar{y} on the support of $G_e(y)$ will offer the highest price, which will be accepted by any supplier. In equilibrium, a type x_0 supplier will be exactly indifferent between accepting and rejecting the offer. In other words, $p(\bar{y}) = x_0$ and $F(\hat{x}(p(\bar{y}))) = 1$. Therefore, $\Pi(\bar{y}) = \bar{y} - x_0$ and $\pi = \Pi(\bar{y}) - e(\bar{y}) = \bar{y} - x_0 - e(\bar{y}) > 0$.

Proposition 1.3 also demonstrates that given his ex-ante investments, a retailer will never play mixed pricing strategy at the search stage. Suppose that there are two retailers investing the same $e(y)$ but offer different prices. In particular, retailer 1 offers price p_1 and retailer 2 offers price p_2 , with $p_1 > p_2$. Given the non-degenerate cost distribution, retailer 1 trades faster in expectation and therefore has larger marginal benefit of investment. On the other hand, since they choose

the same investment level, the marginal cost of investments are the same for both retailers. Then the optimality condition of investment, which equates the marginal benefit and the marginal cost, cannot hold simultaneously for these two retailers. We have a contradiction.

More interestingly, the pure pricing strategy implies that retailers, although they have all the bargaining power, will play mixed investment strategy and hence underinvest with strictly positive probability. We already know from the baseline model that the price offer distribution $H(p)$ is non-degenerate when supplier's investments are unobservable. Retailers therefore must offer different prices. A higher price induces stronger incentive to invest, since a retailer can trade faster in expectation with a higher price. In fact, as shown in the following proposition, retailers will play mixed investment strategy over a convex interval.

Proposition 1.4. *In any steady state equilibrium with two-sided investments, the supplier's investment $F_e(x)$, the stationary cost distribution $F(x)$, the retailer investment $G_e(y)$ and the stationary revenue distribution $G(y)$ have the following properties,*

1. $F_e(x)$ and $F(x)$ have support $[x^*, x_0]$ with the unique point mass at x^* .
2. $G_e(y)$ and $G(y)$ have support $[\underline{y}, y^*]$ and is atomless, where \underline{y} is uniquely determined by

$$y^* - x_0 - e(y^*) = [\underline{y} - x^* - \beta c(x^*)]e'(\underline{y}) - e(\underline{y}) \quad (1.14)$$

The last component of the model is the steady-state revenue distribution constraint. That is, the investment strategy $G_e(y)$ must equal the revenue distribution of retailers who exit the market. A retailer exits when his offer is accepted, which happens with probability $F(\hat{x}(p(y)))$. Combined with $G(y)$, the distribution of exits is determined. Equating the entrant and exit distributions, we have the

following equilibrium restriction.

$$G_e(y) = \frac{\int_{\underline{y}}^y F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^y F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} \quad (1.15)$$

We are now ready to solve the equilibrium. The convex supports and the indifference conditions imply that both $U(x)$ and $\Pi(y)$ are differentiable for x and y on the support. We can therefore use envelope conditions to solve for the stationary distributions and the price offer function $p(y)$. The derivation also proves the existence and uniqueness of steady-state equilibrium. We summarize the results in the following proposition.

Proposition 1.5. *The steady state equilibrium with two-sided investment exists. The stationary cost distribution CDF $F(x)$ is defined by (1.17), the suppliers' investment strategy CDF $F_e(x)$ is defined by (1.7), the reserve price $r_S(x)$ is defined by (1.2), the stationary revenue distribution CDF $G(y)$ is defined by (1.18), the retailers' investment strategy CDF is defined by (1.15) and the price offer $p(y)$ is defined by (1.16).*

Moreover, the steady state equilibrium is unique.

$$p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \quad (1.16)$$

$$F(x) = \begin{cases} 0, & \text{if } x \in (-\infty, x^*), \\ \frac{(1-\beta)e'(\hat{y}(r_S(x)))}{1-\beta e'(\hat{y}(r_S(x)))}, & \text{if } x \in [x^*, x_0], \text{ where } \hat{y}(p) \text{ is the inverse of } p(y) \\ 1, & \text{if } x \in (x_0, +\infty). \end{cases} \quad (1.17)$$

$$G(y) = \begin{cases} 0, & \text{if } y \in (-\infty, \underline{y}) \\ \frac{1+c'(\hat{x}(p(y)))}{1+\beta c'(\hat{x}(p(y)))}, & \text{if } y \in [\underline{y}, y^*], \\ 1, & \text{if } x \in (y^*, +\infty). \end{cases} \quad (1.18)$$

In the baseline model, we demonstrated that the equilibrium payoffs and the social welfare are the same as if investments were observable. This result still holds in the two-sided investments extension.

Theorem 1.3. *In the steady state equilibrium with two-sided investment, the supplier's ex-ante payoff v equals 0, the retailer's ex-ante payoff π and the social welfare equals $y^* - x_0 - e(y^*)$.*

1.5.2 Comparative Statics and the Limiting Case

The comparative statics and the limiting results regarding $H(p)$, $F(x)$ and $F_e(x)$ in theorem 1.2 can be extended here. We will therefore devote this section to analyzing how the retailers' investment strategy $G_e(y)$ and the stationary revenue distribution $G(y)$ change in search frictions.

First of all, the the lower bound \underline{y} as defined in condition (1.14) strictly increases in β . To understand this result, we know that a type \underline{y} retailer offers the price which equals a type x^* supplier's reserve price $x^* + \beta c(x^*)$. Therefore, his price offer strictly increases in β . In addition, the trading probability is strictly lower, as $F(x^*)$ strictly decreases in β . Meanwhile, we know that a retailer's ex ante payoff is independent of β . Therefore, \underline{y} must strictly increases in β to keep the retailer's ex ante payoff constant.

Furthermore, the limit of \underline{y} as $\beta \rightarrow 1$ is strictly less than y^* . In other words, the underinvesting result holds even when search frictions vanish. Intuitively, as long as the price distribution is non-degenerate, which is the case with any $\beta \in [0, 1)$, the investment strategy $G_e(y)$ is non-degenerate.

The stationary revenue distribution $G(y)$ also adjusts as β changes so that the resulting price distribution makes a supplier indifferent across investment levels. We know from the baseline model that price distribution shifts towards lower reserve prices as β increases. Combined with the fact that the price offer $p(y)$ strictly increases in y , there must be larger mass of retailers who invest small amount and offer low prices. $G(y)$ therefore converges in distribution to a point mass at \underline{y} in the limit. Figure 1.4 graphically shows the above two results using a

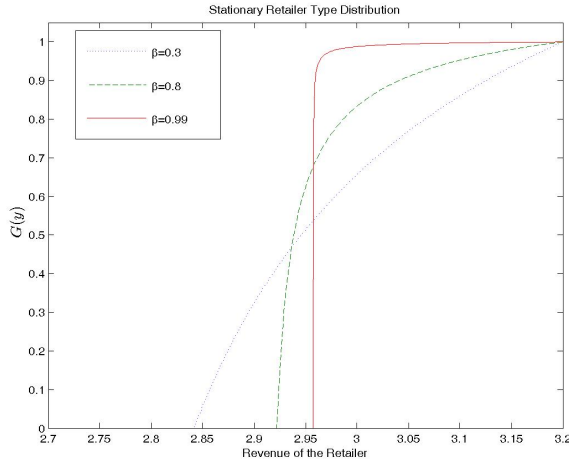


Figure 1.4: Stationary Retailer Type Distribution

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

numerical example.

The retailers' investment strategy $G_e(y)$ also converges in distribution to a point mass at \underline{y} as β goes to 1. We know that $G(y)$ converges to a point mass at \underline{y} . In addition, the per-period trading probability of a retailer with any type $y < y^*$ goes to zero in the limit. The revenue distribution of retailers who exit therefore converges to a point mass at \underline{y} . Entrants who replace these exits hence also have the same limiting revenue distribution.⁵ The investment strategy with the same set of parameters is plotted in Figure 1.5.⁶

The above discussions are summarized in proposition (1.6).

Proposition 1.6. *In the steady-state equilibrium with two-sided investment, as β increases to 1,*

⁵Notice that the above argument will not hold if there is a point mass at y^* , like what we had for the supplier's limiting investment strategy. The reason is that, when there is positive measure of agents who exit the market with probability 1, the average exit type might be higher than \underline{y} or even approaches y^* .

⁶To clarify, although the investment strategy has more mass on lower types, we cannot conclude that the investment strategy becomes less efficient, because the lowest revenue \underline{y} strictly increases in β .

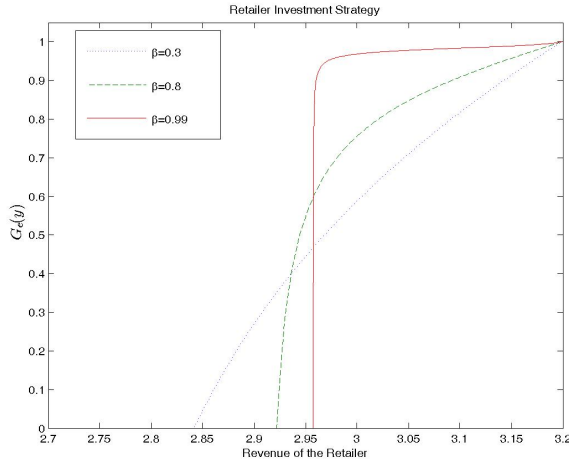


Figure 1.5: Retailer's Investment Strategy

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

1. (i) The lowest revenue \underline{y} strictly increases; (ii) A retailer with revenue that equals the $t * 100$ th percentile of $G(y)$ offers the reserve price of a more efficient supplier, for any $t \in (0, 1)$.

In the limit, (i) \underline{y} is still bounded away from y^* , i.e., $\lim_{\beta \rightarrow 1} \underline{y} < y^*$; (ii) $G_e(y)$ and $G(y)$ converge in distribution to a point mass at \underline{y} .

2. (i) $F(x)$ strictly decreases for $x \in [x^*, x_0)$ and converges in distribution to a point mass at x_0 ; (ii) $F_e(x^*)$ strictly increases and $F_e(x)$ converges in distribution to a point mass at x^* .

1.6 Two-Sided Offers

In some situations a supplier also has the opportunity to make offers occasionally. This section extends the baseline model to incorporate such possibility.

Specifically, in each meeting, nature randomly selects the supplier to make a take-it-or-leave-it offer with probability $\alpha \in (0, 1)$ and selects the retailer with the complementary probability. Therefore, a supplier's strategy now also includes

price offer $p_S(x)$ and a retailer's strategy also includes reserve price r_R .

It is not hard to see that $r_R = y_0 - \beta\pi$, that is, a retailer is willing to pay the price if it leaves him more than his discounted continuation payoff. Therefore, a supplier with any cost will propose $p_S(x) = r_R$ if $r_R - x$ is weakly larger than $\beta U(x)$.

1.6.1 Benchmark: Observable Investment

As a benchmark, let us first characterize the steady-state equilibrium with observable investment. It is easy to verify that conditional on investing positive amount, a supplier will choose \bar{x} that solves the following maximization problem.

$$\max_x \left\{ \frac{\alpha(y_0 - x - \beta\pi)}{1 - \beta(1 - \alpha)} - c(x) \right\}$$

Therefore, \bar{x} is implicitly defined by,

$$c'(\bar{x}) = \frac{-\alpha}{1 - \beta(1 - \alpha)} \quad (1.19)$$

Depending on parameters, one of the following two equilibria will arise. The detailed analysis of the above two equilibrium and is included in the appendix.

Equilibrium 1: Market size > Entrant size. In situations with $\alpha(y_0 - \bar{x}) < c(\bar{x})$, in the steady-state equilibrium, a supplier's ex ante payoff $v = 0$, a retailer's ex ante payoff $\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}$, entrant cost distribution $F_e(x)$ is a point mass at \bar{x} , incumbent cost distribution $F(x)$ has two point mass at x_0 and \bar{x} , with $F(\bar{x}) = \frac{(1 - \beta)\pi}{(1 - \alpha)(y_0 - \bar{x} - \beta c(\bar{x}) - \beta\pi)}$. In equilibrium, trade takes place only when a retailer meets an invested supplier.

Equilibrium 2: Market size = Entrant size. In situations with $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$, in the steady-state equilibrium, a supplier's ex ante payoff $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$, a retailer's ex ante payoff $\pi = (1 - \alpha)(y_0 - \bar{x})$, both $F_e(x)$ and $F(x)$ are a point mass at \bar{x} . In equilibrium, agents trade in their first meetings.

Proposition 1.7. *In the two-sided offer case with observable investment, as β converges to 1, $F_e(\bar{x})$ converges in distribution to a point mass at x^* and the social welfare converges to the first best.*

Proposition 1.7 shows that even if investments are observable, when suppliers have some positive bargaining power, investments become efficient in the limit. To understand this result, note that with a continuum of suppliers and retailers, a retailer's reserve price is independent of his opponent's investments. Hence a supplier is the residual claimant of his investments when he makes the offer. As the time between two meetings shrinks to zero, any positive α implies that the supplier has the chance to make the offer almost immediately after entry. Therefore he becomes the full residual claimant and will invest efficiently, although he only gets α share of the total surplus from trade.

Moreover, the social welfare also convergence to the first best, since not only investments but also trades are efficient. This is apparent with equilibrium 2. In equilibrium 1, the proportion of invested type is bounded away from zero in the limit. Hence, a retailer can find an invested supplier almost immediately when meetings become extremely frequent. Trades are therefore efficient in the limit.

To simplify the analysis, in the rest of the section, I will focus on situations where the condition in equilibrium 2 is satisfied, i.e., $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$. The analysis of the other case is similar and is available upon request.

1.6.2 Steady-State Equilibrium

Let us turn to the steady-state equilibrium. We can first show that for any x on the support of $F_e(x)$, trade always takes place if a type x supplier is selected to make the offer. The detailed proof is provided in the appendix. This claim is true if the surplus from trade $y_0 - x - \beta\pi - \beta U(x)$ is non-negative for any x on the support. Note that the surplus being negative implies that a type x supplier can

never trade and must get zero search stage payoff. However, investment is costly and this leads to a contradiction.

We can follow the same logic as in the baseline model to verify that in steady-state, the support of $F_e(x)$ and $F(x)$ is $[x^*, \bar{x}]$ and retailers play mixed pricing strategy over $[r_S(x^*), r_S(\bar{x})]$.

In equilibrium, a supplier's ex-ante payoff v and a retailer's ex-ante payoff π must be non-negative. Using indifference conditions, to solve for v and π , we only need to focus on a supplier with cost \bar{x} and a retailer who offers $r_S(\bar{x})$. Combining their value functions,

$$U(\bar{x}) = \frac{\alpha(y_0 - \bar{x} - \beta\pi)}{1 + \alpha\beta - \beta} = c(\bar{x}) + v$$

$$\frac{1 - \alpha\beta}{1 - \alpha}\pi = y_0 - \bar{x} - \beta c(\bar{x}) - \beta v$$

We can solve v and π :

$$v = \alpha(y_0 - \bar{x}) - c(\bar{x}) \tag{1.20}$$

$$\pi = (1 - \alpha)(y_0 - \bar{x}) \tag{1.21}$$

By the assumption $\alpha(y_0 - \bar{x}) - c(\bar{x}) \geq 0$, both v and π are positive. Moreover, comparing these equilibrium payoffs and the social welfare with those values in the observable investment benchmark, we can conclude that the equivalent result still holds in this extension.

1.6.3 Comparative Statics and the Limiting Case

From (1.19) we know that the highest cost on the market \bar{x} is determined by $c'(\bar{x}) = \frac{-\alpha}{1-\beta(1-\alpha)}$. It is straight forward to verify that \bar{x} strictly decreases in α and β , i.e., the highest cost on the support is closer to the efficient cost when a supplier has larger bargaining power or when meetings become more frequent. A supplier

with cost \bar{x} is the residual claimant only when he makes the offer, the probability of which in one unit of time strictly increases in α and β . This supplier therefore has stronger incentive to invest if α or β is larger.

In the limit as $\beta \rightarrow 1$ or as $\alpha \rightarrow 1$, \bar{x} converges to x^* and hence the investment strategy becomes efficient. Although in the baseline model the investment strategy also converges to the first best, the mechanism behind the result is quite different.

In addition, the two-sided offer case has different prediction on the welfare consequences of diminishing search frictions. While it is still true that any investment above the minimum level does not generate any welfare gain due to delay in trade, since the minimum level itself converges to the efficient level, the equilibrium social welfare converges to the first best in the limit.

For situations where $\alpha(y_0 - \bar{x}) < c(\bar{x})$, we can also show that the limiting investment strategy is efficient and that equilibrium social welfare is equal to the first best.

We summarize the above discussion in the following proposition.

Proposition 1.8. *In the unique steady-state equilibrium with two-sided offers, the highest production cost \bar{x} strictly decreases in β and α . Moreover, the investment strategy converges in distribution to a point mass at x^* as $\beta \rightarrow 1$ or $\alpha \rightarrow 1$.*

The equilibrium social welfare strictly increases in α and β , and converges to the first best as $\beta \rightarrow 1$ or $\alpha \rightarrow 1$.

1.7 Final Remarks

1.7.1 Robustness

In this section, we will check the robustness of results of the baseline model to some alternative assumptions.

Uneven Sizes and (or) General Matching Technology. In the baseline

model, I assume that the measures of incumbent suppliers and retailers are the same and that each player is paired with one player from the other side for sure in each period. The main results are robust if instead we have uneven sizes on two sides of the market and (or) general matching technology so that the probability of not being paired in one period is positive.

To be more precise, the equilibrium investment strategy $F_e(x)$ is still non-degenerate with convex support $[\underline{x}, x_0]$ and a retailer plays mixed pricing strategy over reserve prices of these types. The difference is that \underline{x} is larger than the efficient cost if a supplier cannot be paired with probability 1 each period. Moreover, because the indifference conditions still hold, the equilibrium payoffs and the social welfare equal the values generated with observable investments. Finally, as β converges to 1, \underline{x} converges to x^* : as meetings become more frequent, it is as if the most efficient suppliers could trade immediately and hence they will invest efficiently. The convergence results of $F(x)$ and $F_e(x)$ also extend to a more general market condition, with different rates of convergence that depend on the sizes of two sides and the matching technology.

Exogenous Death Shock. Suppose instead, each player experiences an exogenous death shock with positive probability in each unit of time. For the most part of the analysis, this is equivalent to redefining a smaller discount factor which also converges to 1 in the limit. The only complication is that now the group of exit suppliers also include those who have death shock so we need to rewrite the stationary distribution condition. But this again only affects the rate of convergence of $F_e(x)$ and hence the original results still hold qualitatively.

1.7.2 Other Possible Extensions

Investments are Observable with Positive Probability. As the previous analysis shows, the social welfare that could be generated from positive investment

is completely eroded by the inefficiency of trade. If in the search stage investments could be observed with positive probability q , then profitable trades could be conducted with no delay when investments are observable. We can show that given any search friction, partial information yields strictly higher social welfare than no information (and full information). The optimal q trades off the discouraged investment incentive and the increased trading efficiency.

This idea is also emphasized in Lau (2008), where the same result holds in a Coasian setting with one round of price offer. The difference is that in her setting, no information is better when the bargaining has infinite rounds and the time between two successive rounds shrinks to zero. On the contrary, in the current setting the trading inefficiency is the most severe in the limit and hence partial information could improve social welfare over all search friction levels.

Two-Sided Investments and Offers. It is also possible to extend the model to incorporate two-sided investments and two-sided offers. With both sides investing, it is unclear ex-ante what the socially optimal bargaining power would be.

Unfortunately, a model with continuous investment technology is no longer tractable. Instead, I consider a one-period, binary-investment-choice and symmetric version of the baseline model, where both sides can make price offer with positive probability and can invest. The preliminary result shows that in non-trivial cases where the cost of investment is large so that it is impossible to incentivize both sides to invest, the social welfare is maximized when one side has all the bargaining power.

1.8 Conclusion

This article examines the investment incentive and its welfare consequences in an infinite horizon random search and bargaining game with unobservable and selfish investments.

We demonstrated that in the unique steady state equilibrium, both the investment strategy and the price offer distribution are non-degenerate with convex supports. Unobservability generates rent for high investment and therefore incentivizes investment even if suppliers have no bargaining power.

However, positive investments fail to generate any social welfare for any search friction. Trading inefficiency caused by unobservability erodes the welfare gain that could be created.

Moreover, we showed that as meetings become more frequent, quite strikingly, the investment distributions of incumbents and entrants shift in the opposite directions: incumbent investment distribution converges to a point mass at no investment whereas an entrant's investment becomes efficient.

1.9 Appendices

1.9.1 Proof of Lemma 1.1

Step 1: $U(x)$ Strictly Decreasing. Since type x supplier can always choose the reserve price of type $x + \epsilon$, for some $\epsilon > 0$, $U(x)$ must be strictly decreasing in x .

Step 2: $r_S(x)$ Strictly Increasing. Multiply both sides of the supplier's value function by β and add x , we get the following equation after rearranging,

$$\begin{aligned} r_S(x) - \beta[E(\tilde{p} \mid \tilde{p} \geq r_S(x))(1 - H(r_S(x)) + Pr(\tilde{p} = r_S(x))) \\ + r_S(x)(H(r_S(x)) - Pr(\tilde{p} = r_S(x)))] = (1 - \beta)x \end{aligned} \quad (1.22)$$

The left hand side strictly increases in $r_S(x)$ while the right hand side strictly increases in x . Therefore, $r_S(x)$ must be strictly increasing in x .

Step 3: Continuity. From the last step, we know that the $r_S(\bar{x})$ is the highest reserve price on the market. Clearly, no retailer will offer a price that is higher than that. Therefore, for type $x \geq \bar{x}$, $H(r_S(x)) = 1$ and $U(x) = 0$.

For $x < \bar{x}$, since $U(x)$ decreases in x , $U(x)$ can only have downward jump. Suppose $U(x)$ jumps down at point x . Because $r_S(x) = x + \beta U(x)$, $r_S(x)$ must also jump downwards. However, this contradicts $r_S(x)$ being strictly increasing.

Therefore, for any x , $U(x)$ is continuous. $r_S(x)$ is also continuous as a result.

Step 4: $U(\bar{x}) = 0$. Since $r(\bar{x})$ is the highest price that a retailer is willing to offer, equation (1.1) for $x = \bar{x}$ becomes $U(\bar{x}) = \beta U(\bar{x})$. Therefore $U(\bar{x}) = 0$.

1.9.2 Proof of Proposition 1.1

Step 1: supports of $F(x)$, $F_e(x)$ and $H(p)$.

The closeness is obtained from the assumption that suppliers and retailers will trade in the case of indifference.

Before showing the convexity, it is worth noticing that a price offer p being on the support of $H(p)$ implies $\hat{x}(p)$ being on the support. Otherwise, the retailer offering price p is not optimizing because he can lower the price to $r_S(x')$ without affecting the probability of trade, where x' is the highest supplier type that is smaller than x and on the support.

Now suppose that there exist p_1, p_2 on the support of $H(p)$, such that any $p \in (p_1, p_2)$ is not on the support. Since p_1 and p_2 are on the support, there exist worker type x_1 and x_2 on the support such that $r_S(x_i) = p_i$, $i = 1, 2$. For any $x \in (x_2, x_1)$, $U'(x)$ is a constant since

$$U'(x) = \frac{-1 + H(p_2)}{1 - \beta H(p_2)}$$

On the other hand, $c'(x)$ strictly increases in x . Together with the continuity of $U(x)$, $U(x_1) - c(x_1) < U(x_2) - c(x_2)$. This is a contradiction. Therefore, the support of $H(p)$ is convex. By the continuity of $r_S(x)$, the support of $F(x)$ and $F_e(x)$ is also convex.

The lowest price offer in equilibrium is never lower than the reserve price of

the most efficient suppliers, i.e., $H(r_S(\underline{x})) = 0$. Plugging it into the envelope condition, $U'(\underline{x}) = c'(\underline{x}) = -1$. This implies that $\underline{x} = x^*$.

Hence, the support of $F(x)$ and $F_e(x)$ is $[x^*, x_0]$ and the support of $H(p)$ is $[r_S(x^*), r_S(x_0)]$.

Step 2: $H(p)$ has no point mass.

We first show $Pr(\tilde{p} = r_S(x_0)) = 0$. Suppose $Pr(\tilde{p} = r_S(x_0)) = q > 0$, then $U'(x_0-) = \frac{-q}{1-\beta+\beta q} < c'(x_0)$. This is a contradiction.

Next, we solve $H(p)$. Notice that $U(x)$ is differentiable for any x on the support, because the support is convex, $c(x)$ is differentiable and $U(x) - c(x) = 0$. This also implies that $r_S(x)$ and $\hat{x}(p)$ are differentiable for any x and p on the support. Hence, $H(p)$ can be solved from the equilibrium condition that $U'(x) = c'(x)$,

$$H(r_S(x)) = \frac{1 + c'(x)}{1 + \beta c'(x)} \Rightarrow H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \quad (1.23)$$

It is straightforward to verify that $H(p)$ has no atom.

Step 3: Point mass at x^* and no other point mass.

We already know that $r_S(x^*)$ is on the support of $H(p)$. For a retailer who offers this price, he would get zero payoff if $F(x^*) = 0$. In this case, the retailer would find it profitable to deviate to offering $r_S(x_0) = x_0$ and get strictly positive payoff.

Suppose that there is a point mass at $x \in (x^*, x_0]$, then there exist ϵ , such that retailers would find it profitable not to offer price $p \in (r_S(x - \epsilon), r_S(x))$. This contradicts the convexity property of the support.

1.9.3 Proof of Theorem 1.2

We have shown the first two parts of the theorem.

By equation (1.7), the proportion of type x^* entrants equals,

$$\begin{aligned} F_e(x^*) &= \frac{F(x^*)}{1 - \int_{x^*}^{x_0} H(r_S(x))f(x)dx} \\ &= [1 + \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx]^{-1} \\ &= [1 + A]^{-1} \end{aligned}$$

By Median Value Theorem, there exist a $\tilde{x} \in (x^*, x_0)$ such that,

$$\begin{aligned} A &= \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx \\ &= \frac{1 - F(x^*)}{F(x^*)} [1 - H(r_S(\tilde{x}))] \\ &= \frac{-c'(\tilde{x})(x_0 - x^* - \beta c(x^*))}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))} \end{aligned}$$

Take derivative with respect to β ,

$$\frac{\partial A}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta} (x_0 - x^* - \beta c(x^*)) + c'(\tilde{x}) [c(x^*) + c'(\tilde{x})(x_0 - x^*)]}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))^2}$$

Therefore, the sufficient conditions for the derivative to be negative are

$$\frac{\partial \tilde{x}}{\partial \beta} > 0 \text{ and } c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0$$

We first show that the first condition is satisfied. \tilde{x} is defined by,

$$\int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx = \frac{-c'(\tilde{x})}{1 + \beta c'(\tilde{x})} \quad (1.24)$$

From the previous analysis, as β increases $1 - H(r_S(x))$ strictly decreases for any $x \in (x^*, x_0)$. As a result, the left hand side of (1.24) strictly decreases in β . On the other hand,

$$\frac{\partial \frac{-c'(\tilde{x})}{1 + \beta c'(\tilde{x})}}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta} + [c'(\tilde{x})]^2}{(1 + \beta c'(\tilde{x}))^2}$$

Since the derivative has to be negative so that (1.24) holds, $\frac{\partial \tilde{x}}{\partial \beta} > 0$ must be satisfied.

This in turn implies we only need to check the second sufficient condition for $\beta = 0$, because $c'(\tilde{x})$ is the smallest when $\beta = 0$. It is straight forward to check that the density of the stationary type distribution strictly increases in x , since,

$$f(x) = \frac{(y_0 - x_0 - \beta\pi)(1 + \beta c'(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2}$$

Therefore,

$$\begin{aligned} -c'(\tilde{x}) &= \int_{x^*}^{x_0} -c'(x) \frac{f(x)}{1 - F(x^*)} dx < \int_{x^*}^{x_0} -c'(x) \frac{1}{x_0 - x^*} dx = \frac{c(x^*)}{x_0 - x^*} \\ &\Rightarrow c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0 \end{aligned}$$

We have proved that those two sufficient conditions both hold and thus

$$\frac{\partial F_e(x^*)}{\partial \beta} > 0$$

Next, we will prove that in the limit $F_e(x^*) \rightarrow 1$, which is equivalent to $\tilde{x} \rightarrow x_0$, where \tilde{x} is defined by,

$$\int_{x^*}^{x_0} (1 - H(r_S(x)))f(x)dx = [1 - H(r_S(\tilde{x}))][1 - F(x^*)] \quad (1.25)$$

For any $\epsilon \in (0, x_0 - x^*)$, we can rewrite the left hand side of (1.25) as

$$\begin{aligned} &\int_{x^*}^{x_0 - \epsilon} (1 - H(r_S(x)))f(x)dx + \int_{x_0 - \epsilon}^{x_0} (1 - H(r_S(x)))f(x)dx \\ &= [1 - H(r_S(x_1))][F(x_0 - \epsilon) - F(x^*)] + [1 - H(r_S(x_2))][1 - F(x_0 - \epsilon)] \quad (1.26) \end{aligned}$$

where $x_1 \in (x^*, x_0 - \epsilon)$ and $x_2 \in (x_0 - \epsilon, x_0)$.

Combining (1.25) and (1.26) we have the following equation,

$$[H(r_S(\tilde{x})) - H(r_S(x_2))][1 - F(x^*)] = [H(r_S(x_1)) - H(r_S(x_2))][F(x_0 - \epsilon) - F(x^*)] \quad (1.27)$$

Here $F(x_0 - \epsilon) - F(x^*)$ can be rewritten as,

$$(1 - \beta)(y_0 - x_0) \frac{-x^* - \beta c(x^*) + (x_0 - \epsilon) + \beta c(x_0 - \epsilon)}{(y_0 - \beta\pi - x_0 + \epsilon - \beta c(x_0 - \epsilon))(y_0 - \beta\pi - x^* - \beta c(x^*))}$$

We can verify that for any ϵ and ϕ , there exist an η , such that when $1 - \beta < \eta$, $F(x_0 - \epsilon) - F(x^*) < \phi$. Therefore, the right hand side of (1.27) converges to 0 in the limit. This implies \tilde{x} converges to x_2 in the limit. Since $x_2 \in (x_0 - \epsilon, x_0)$, for small enough ϵ , \tilde{x} converges to x_0 .

1.9.4 Proof of Proposition 1.3

Step 1: $p(y)$ increasing.

If the support of $G(y)$ is degenerate, then we have nothing to prove.

Otherwise, consider any y_1 and y_2 on the support with $y_1 > y_2$. Denote $p_1 = p(y_1)$ and $p_2 = p(y_2)$. Since p_1 (p_2) solves the maximization problem of type y_1 (y_2), the following inequality must hold,

$$\begin{aligned} (y_1 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))] \beta \Pi(y_1) \\ \geq (y_1 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))] \beta \Pi(y_1) \\ (y_2 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))] \beta \Pi(y_2) \\ \geq (y_2 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))] \beta \Pi(y_2) \end{aligned}$$

Adding two equation, we have

$$(y_1 - y_2)[F(\hat{x}(p_1)) - F(\hat{x}(p_2))] \geq [F(\hat{x}(p_1)) - F(\hat{x}(p_2))][\beta \Pi(y_1) - \beta \Pi(y_2)]$$

Since $y_1 - y_2 > \beta[\Pi(y_1) - \Pi(y_2)]$, the above inequality implies $F(\hat{x}(p_1)) \geq F(\hat{x}(p_2))$. This proves $p(y)$ increases in y .

Step 2: Single-Valued.

We can apply the same proof to show that the support of $F(x)$ is $[x^*, x_0]$. Therefore, $F(x)$ is a strictly increasing function of x .

Suppose there exist y that is on the support of $G_e(y)$, such that $p_1 > p_2$ are both optimal price offers of a type y retailer. His search stage payoff can be

rewritten as,

$$\Pi_i(y) = (y - p_i)F(\hat{x}(p_i)) + [1 - F(\hat{x}(p_i))]\beta\Pi_i(y), \quad i = 1, 2$$

It is easy to verify that $\Pi'_1(y) > \Pi'_2(y)$. This contradicts the equilibrium constraint $\Pi(y) - c(y) = \pi$.

1.9.5 Proof of Proposition 1.4

Step 1: We can apply the same proof to show that the support of $F_e(x)$ and $F(x)$ is $[x^*, x_0]$.

Step 2: $F(\hat{x}(p(y))) > 0$ follows from $\Pi(y) = \pi + e(y) > 0$. Together with previous results that $p(y)$ increases in y , $x(p(y)) = x^*$. Hence $F(x^*) > 0$ and by (1.7) $F_e(x^*) > 0$, i.e., there is a point mass at x^* .

Next we show that x^* is the only point mass. Suppose $F(x)$ jump upwards at point $\tilde{x} > x^*$. Denote the corresponding reserve price $\tilde{x} + \beta U(\tilde{x})$ by \tilde{p} . Then for any retailer type, offering \tilde{p} is strictly better than offering any $p \in [\tilde{p} - \epsilon, \tilde{p}]$ for some small ϵ . Therefore, no price $p \in [\tilde{p} - \epsilon, \tilde{p}]$ will be offered. This contradicts the previous result that the support of $H(p)$ is convex.

Step 3: The compactness of the support of $G_e(y)$ and $G(y)$ comes from the fact that y^* is finite and that all agents choose to trade when indifferent.

To see convexity, suppose there exist y_1, y_2 on the support and any $y \in (y_1, y_2)$ is not. Then $p(y_1) < p(y_2)$, otherwise $\Pi'(y)$ will be constant in the interval and the indifference condition at point y_1 and y_2 cannot be satisfied. However, combined with the monotonicity of $p(y)$, this implies any $p \in (p(y_1), p(y_2))$ is not on the support of $H(p)$. We have a contradiction.

Because the support of $G(y)$ is convex, we can use the envelope condition and indifference condition to determine \bar{y} and \underline{y} . That is,

$$e'(\bar{y}) = \Pi'(\bar{y}) = 1 \quad \text{and} \quad e'(\underline{y}) = \Pi'(\underline{y}) = \frac{F(x^*)}{1 - \beta(1 - F(x^*))}$$

Therefore $\bar{y} = y^*$. We know $\pi = y^* - x_0 - e(y^*)$. Using indifference condition, \underline{y} is pinned down by

$$y^* - x_0 - e(y^*) = (\underline{y} - x^* - \beta c(x^*))e'(\underline{y}) - e(\underline{y})$$

It is straightforward to verify that $\underline{y} < y^*$. The right hand side strictly increases in \underline{y} and equals zero when $\underline{y} = y_0$. In addition, it is strictly larger than the left hand side when $\underline{y} = y^*$. Therefore, \underline{y} is uniquely determined by equation (1.14) and $\underline{y} < y^*$.

Finally, no point mass result comes from the fact that no point mass on the support of $H(p)$ is permitted in equilibrium.

1.9.6 Proof of Proposition 1.5

Combining the envelope condition of $\Pi(y)$ and the indifference condition that $\Pi'(y) = e'(y)$ for any y on the support,

$$F(\hat{x}(p(y))) = \frac{(1 - \beta)e'(y)}{1 - \beta e'(y)} \text{ for any } y \text{ on the support}$$

We can use the equilibrium condition $\Pi(y) - e(y) = \pi$ to solve $p(y)$.

$$\begin{aligned} (y - p(y))e'(y) - e(y) &= y^* - x_0 - e(y^*) \\ \Rightarrow p(y) &= y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \end{aligned}$$

We can easily verify that $p(y)$ is continuous and strictly increases in y . Therefore, the inverse function $y(p)$ exists. We can therefore define $F(x)$ and $G(y)$.

From the previous discussion, we can see that $F(x)$, $F_e(x)$, $G(y)$ and $G_e(y)$ defined above are the only distributions that satisfies equilibrium restrictions. Hence, the steady state equilibrium is unique.

1.9.7 Proof of Proposition 1.6

Step 1: Take derivative of equation (1.14) with respect to β , we get

$$[\underline{y} - x^* - \beta c(x^*)]e''(\underline{y})\frac{\partial \underline{y}}{\partial \beta} = c(x^*)e'(\underline{y})$$

Therefore, $\frac{\partial \underline{y}}{\partial \beta}$ is strictly positive.

Denote the $t * 100$ th percentile of $G(y)$ with β as $y_{t,\beta}$, i.e.,

$$\begin{aligned} G(y_{t,\beta}) &= \frac{1 + c'(\hat{x}(p(y_{t,\beta})))}{1 + \beta c'(\hat{x}(p(y_{t,\beta})))} = t \\ \Rightarrow 1 - t &= -(1 - t\beta)c'(\hat{x}(p(y_{t,\beta}))) \end{aligned}$$

Therefore, when β increases, $\hat{x}(p(y_{t,\beta}))$ strictly decreases.

Step 2: $\lim_{\beta \rightarrow 1} \underline{y} < y^*$ can be shown by plugging in $\beta = 1$ and $\underline{y} = y^*$ into equation (1.14). It is easy to check that the left hand side is strictly smaller the right hand side.

Consider any $y > \underline{y}$. Since $\hat{x}(p(y)) > x^*$, $1 + c'(\hat{x}(p(y))) > 0$. Then it is straightforward to verify that $\beta \rightarrow 1$, $G(y) \rightarrow 1$ for any $y > \underline{y}$.

For any $y > \underline{y}$, there exist \check{y} and \tilde{y} , such that

$$\begin{aligned} G_e(y) &= \frac{\int_{\underline{y}}^y F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} = \frac{F(\hat{x}(p(\check{y})))G(y)}{F(\hat{x}(p(\tilde{y})))} \\ &= \frac{e'(\check{y})(1 - \beta e'(\check{y}))}{e'(\tilde{y})(1 - \beta e'(\tilde{y}))}G(y) \end{aligned}$$

When $\beta \rightarrow 1$, both \check{y} and \tilde{y} approaches \underline{y} following the same argument as in the proof for proposition 1.2, and $G(y)$ approaches 1 for any $y > \underline{y}$. Therefore, $G_e(y) \rightarrow 1$ for any $y > \underline{y}$.

1.9.8 Two Equilibria with Observable Investment and Proof of Proposition 1.7

Equilibrium 1. Assume $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

If a supplier invests, he will invest to reduce his production cost to \bar{x} . Given π , his ex ante payoff $v = \frac{\alpha}{1+\alpha\beta-\beta}(y_0 - \bar{x} - \beta\pi) - c(\bar{x})$. To make sure that suppliers are indifferent between \bar{x} and x_0 , v must equal 0. Therefore,

$$\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta} \quad (1.28)$$

If $F(\bar{x}) = q$, we can also solve π as follows,

$$\pi = \alpha\beta\pi + (1 - \alpha)[q(y_0 - \bar{x} - \beta c(\bar{x})) + (1 - q)\beta\pi]$$

Equating two π , q can be solved,

$$q = \frac{(1 - \beta)\pi}{(1 - \alpha)(y_0 - \bar{x} - \beta c(\bar{x}) - \beta\pi)}$$

q is always positive. The condition that q is less than 1 is equivalent to $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

The last equilibrium condition we need to verify is that $y_0 - x_0 - \beta\pi \leq 0$. After plugging in π , this condition is equivalent to $c(\bar{x}) \leq \frac{\alpha}{1+\alpha\beta-\beta}(y_0 - \bar{x})$. This condition is satisfied since $c'(\bar{x}) = \frac{-\alpha}{1+\alpha\beta-\beta}$ and $c(x)$ is strictly convex.

Equilibrium 2. Assume $\alpha(y_0 - \bar{x}) > c(\bar{x})$.

All supplier entrants invest to reduce their production cost to \bar{x} . Agents' ex ante payoffs can be solved: $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$ and $\pi = (1 - \alpha)(y_0 - \bar{x})$. By the assumption, both are positive.

Proof of proposition 1.7.

As $\beta \rightarrow 1$, the right hand side of equation 1.19 converges to -1 , which implies $\bar{x} \rightarrow x^*$. Since $F_e(x)$ is a point mass at \bar{x} in both equilibrium, the limiting distribution converges to a point mass at x^* .

In equilibrium 2, the social welfare equals $y_0 - \bar{x} - c(\bar{x})$. Hence the social welfare converges to the first best.

In equilibrium 1, the social welfare equals π defined in equation (1.28). We can also verify that as $\beta \rightarrow 1$, $\pi \rightarrow (y_0 - x^* - c(x^*))$.

1.9.9 Proof: Non-Negative Surplus From Trade

If $y_0 - x_0 \geq \beta U(x_0) + \beta\pi$, by the strictly monotonicity of $r_S(x)$, $y_0 - x > \beta U(x) + \beta\pi$ for any $x < x_0$.

Now suppose $y_0 - \tilde{x} < \beta U(\tilde{x}) + \beta\pi$ and $\tilde{x} < x_0$, where \tilde{x} is the upper bound of the support. Since $r_S(\tilde{x}) = \tilde{x} + \beta U(\tilde{x})$, this implies $y_0 - r_S(x) < \beta\pi$. In this case, the supplier with type \tilde{x} can never trade on the market: when the supplier makes the offer, $y_0 - \tilde{x} < \beta U(\tilde{x}) + \beta\pi$, i.e., the surplus of trade is not enough to compensate the forgone discounted continuation values; when the retailer makes the offer, $r_S(\tilde{x}) > y_0 - \beta\pi$, i.e., the retailer is unwilling to offer the reserve price. As a result, $U(\tilde{x}) = 0$, which implies $U(\tilde{x}) - c(\tilde{x}) < 0$. This contradicts \tilde{x} being on the support.

Next, suppose $y_0 - x_0 < \beta U(x_0) + \beta\pi$ and $\tilde{x} = x_0$. Then there exist some $x_1 < x_0$ such that for any $x \in (x_1, x_0)$, $y_0 - x < \beta U(x) + \beta\pi$. By the argument from last step, any $x \in (x_1, x_0)$ is not on the support. We then show that $F(x_0) = F_e(x_0) = 1$ so that it is without loss of generality to assume $\tilde{x} = x_1$. For any type $x \in (x_1, x_0]$ supplier's value function is

$$U(x) = \frac{\alpha[y_0 - x - \delta\pi] + (1 - \alpha)(r_S(x_0) - x)Pr(\tilde{p} = r_S(x_0))}{1 - (1 - \alpha)\beta[1 - Pr(\tilde{p} = r_S(x_0))]}$$

This implies that for any $x \in (x_1, x_0]$

$$U'(x) = \frac{-\alpha - (1 - \alpha)Pr(\tilde{p} = r_S(x_0))}{1 - (1 - \alpha)\beta[1 - Pr(\tilde{p} = r_S(x_0))]}$$

To satisfy the restriction that $U(x) - c(x) = v$ for any x on the support, α and $Pr(\tilde{p} = r_S(x_0))$ must be zero. This further implies that the probability of a type x_0 supplier exiting the market is zero. Therefore, $F_e(x_0)$ must be one. Otherwise the mass of supplier type x_0 will blow up. Without loss of generality, we can thus redefine \tilde{x} as x_1 . Then by the same argument as before, $y_0 - \tilde{x} \geq \beta U(\tilde{x}) + \beta\pi$.

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CHAPTER 2

Search with Private Information: Sorting and Price Formation (joint with Kenneth Mirkin)

2.1 Introduction

We investigate the sorting of heterogeneous agents in a two-sided market where the value of a traded good depends on both buyer and seller types, focusing on a setting in which trade is hindered by search frictions and private information. Becker (1973) serves as a benchmark characterization of sorting—in a “frictionless” world, Positive Assortative Matching (henceforth PAM) arises when output is supermodular in types, while Negative Assortative Matching (NAM) occurs with submodular output. We depart from Becker’s setting by restricting the interactions of agents in two ways. First, it is difficult to meet potential trading partners, as each buyer encounters only a random seller in each period (and vice versa). Second, in each meeting of potential trade partners, the buyer is privately informed about her type.

More precisely, our environment is a repeated, bilateral matching market. Buyers and sellers are heterogeneous and have persistent types. These buyers and sellers are randomly matched pairwise in each period, and if the pair agree to trade, the buyer receives the joint output, which is an increasing function of both agents’ types. In each match, the seller’s type can be jointly observed, but only the buyer knows her own type (and thus knows the joint output). In turn, the seller has all the bargaining power and can make a take-it-or-leave-it offer.

This setting reflects a broad set of applications—it is straightforward to imagine a market for vertically-differentiated products, where consumers differ in preferences across goods. In the housing market, for instance, homes for sale vary in kitchen appliances, and sellers are visited by buyers with different marginal return to better appliances. Our model can be seen as the case in which the buyer’s preference is her private information. Similarly, this might represent the hiring process in a labor market where heterogeneous workers and firms produce joint output. For consistency, we describe the model’s agents as buyers and sellers throughout the analysis to follow, but where appropriate, it will be insightful to draw motivation from other such applications.

Our study has two primary objectives—we want to characterize: (i) the sorting patterns that arise in the environment described above and (ii) the patterns in the price distribution that give rise to this sorting. Both tasks require us to understand how each seller determines her reserve price and how this decision changes over seller types. At its most basic level, optimal seller behavior is governed by the following intuition: Because output rises in type, a higher type seller faces a choice of how best to use this extra output—she can simply keep it in each of the trades she would have had, or alternatively, she can give it to buyers to incentivize more of them to trade with her. If prices rise in seller type by the precise amount to generate no sorting at all (meaning that different sellers match with the same set of buyers), then these sellers are clearly choosing the former. If prices are constant across types, sellers are choosing the latter. Inbetween these two outcomes—if sorting is positive, but prices are rising in seller type—sellers are choosing a combination of the two.

Unfortunately, the simplicity of this intuition masks the underlying complexity of equilibrium decisions and interactions; in contrast to related models in the existing literature (e.g. - Shimer and Smith, 2000; Smith, 2006; Atakan, 2006), both prices *and* matching sets arise endogenously in our model. As a result, neither

of the aforementioned objectives is straightforward to achieve, so we will take an indirect approach.¹ We will see that, given prices, matching sets are intimately related to continuation values, and we will attempt to use this relationship to overcome the difficulty of characterizing equilibrium. Specifically, we will investigate how the patterns of interest depend on the factor by which agents discount the future (β), focusing especially on the extreme cases in which $\beta = 0$ (the static case) and $\beta \rightarrow 1$ (search frictions become insignificant).²

In the static case, our analysis is simplified because the discounted continuation value is zero. We find that the direction of sorting depends on the log-supermodularity of output—log-supermodular production functions give rise to PAM, while log-submodularity leads to NAM. This is a stronger condition than the supermodularity that governs sorting in a frictionless market. Intuitively, the buyer’s private information generates an additional tradeoff for the seller between terms of trade and probability of trade. Higher types have a greater incentive to take advantage of their higher type-specific output by increasing the probability of trade (i.e. - by relaxing the terms of trade to induce more types to participate). This incentive pushes equilibrium sorting towards negative assortative even with a supermodular production function. Positive sorting therefore requires an output function with stronger complementarity. It is also worth noting that stronger log-concavity in buyer type of output weakens sorting in either direction, because this decreases the benefits to the seller of trying to trade with more types (lowering the price will induce less movement in the marginal, indifferent buyer under this condition).

¹We also have a characterization of sorting and prices for the general, dynamic setting with discount factors in $(0, 1)$, but this is quite messy analytically, and the conditions are difficult to link coherently to economic intuition. Of course, these findings are available from the authors upon request.

²Throughout the analysis, the distribution of buyer types entering the market in each period is assumed to satisfy the common increasing hazard rate property. Output is assumed to be everywhere positive, log-concave in buyer type, and strictly increasing in both buyer and seller types.

We also characterize how prices move with seller type: we find that prices increase in seller type if output exhibits a sufficiently strong combination of supermodularity and log-concavity in buyer type. For instance—if output is concave in the buyer’s type, then it need only be supermodular for prices to rise with type. If output is merely log-concave in buyer type, however, then it must satisfy the more stringent condition of log-supermodularity to induce prices to rise in type.

At the other extreme, we can think of the case with increasingly patient agents as a setting with disappearing search frictions. We find that, when $\beta \rightarrow 1$, the inefficiency associated with private information disappears, and matching sets reduce to the unique stable matching of a frictionless market (when it exists). Thus, the standard supermodularity (submodularity) reemerges as the condition for PAM (NAM). In this case, we can neatly characterize prices analytically, and in equilibrium, each agent obtains her marginal contribution to total surplus up to a constant.

This frictionless limit is particularly informative regarding sorting and how it depends on the interplay between private information and search. Our analysis suggests generally that, when buyers have private information, agents are less likely to capitalize on productive complementarities by sorting positively. This resistance to sorting disappears, however, when we reduce search frictions. By removing the time preference of agents, we weaken the monopolistic aspect of bilateral trade, which allows agents to appropriate their marginal contributions. In this sense, increasing market competition can help agents sort efficiently even in the presence of private information.

This chapter proceeds as follows: Section 2 frames our findings in the context of related literature. Section 3 introduces the model and establishes the existence of a search equilibrium. Section 4 characterizes sorting and price formation when the model is effectively static ($\beta = 0$), and Section 5 offers results for the other extreme case as agents become patient. Finally, Section 6 draws connections

between these limiting cases and concludes.

2.2 Related Literature

As we will discuss below, our analysis builds upon previous theoretical work related to assortative matching. Studies in this area often attempt to answer two primary questions: (1) What types of agents will match with each other? (2) What types of agents would match with each other to optimize overall welfare, and if these matching patterns differ, why? To properly place our analysis in the context of this literature, we will compare our setting to several that have been previously studied, both through objective sorting characterizations and through insights regarding the relationship between sorting, search, and asymmetric information.

2.2.1 Sorting and Frictions

The Frictionless Matching Benchmark

A standard benchmark used for comparison is the frictionless, "Walrasian" setting studied by Becker (1973) and Rosen (1974). In this environment, there is full information regarding prices and types, and meeting trade partners is fully costless for buyers and sellers. Becker famously demonstrated that supermodular production functions give rise to PAM in this environment.

Beyond this, though, recent studies have taken a renewed interest in sorting, trying to understand how it is impacted by departures from the frictionless benchmark. Among these frictional extensions, the setting we study is especially well-suited for comparison to those involving two particular classes of frictions—the bilateral monopoly which arises in random search (Shimer and Smith, 2000; Smith, 2006; Atakan, 2006) and the coordination frictions in directed search (Eeckhout and Kircher, 2010). We elaborate upon these connections below.

Random Search

There are obvious connections between our study and a series of papers on sorting with random search (Shimer and Smith (2000), Smith (2006) and Atakan (2006)).

Our study differs from these primarily in its incorporation of buyer private information—in the studies mentioned above, there is full information and an exogenously given sharing rule (under transferable utility) in each meeting of potential partners.

These studies focus exclusively on the impact on sorting of random search frictions. Competition is modeled in a reduced form way (Nash Bargaining). In contrast, we focus on the discriminatory behavior that accompanies one-sided private information. Random search is not itself our ultimate, but rather the channel through which we vary the strength of bilateral monopoly power (via the discount factor).

In Shimer and Smith (2000), search is itself the reason that PAM requires more stringent conditions than in the frictionless setting. In our case, private information is behind our stricter conditions for PAM—log-supermodularity is required even in our static environment. Repeated search—specifically the future’s value—actually weakens the requirements for PAM, and the standard supermodularity condition again becomes sufficient in the frictionless limit. As the monopoly power in each match vanishes, the conditions for sorting (and the equilibrium itself) become identical regardless of whether the buyer’s type is private information.

Directed Search

Eeckhout and Kircher (2010) study a static, one-shot setting with buyer private information and directed search. The friction therefore comes from coordination frictions. In contrast, we ignore coordination frictions entirely—searching agent is matched bilaterally to a potential trade partner in each period. This enables us to

focus on the interplay between private information and random search frictions, free of interference from coordination friction-based mechanisms.

Note that, when viewed through the lens of timing, the two settings appear to be much more closely related. Eeckhout and Kircher’s sellers use posted prices to sort agents prior to meeting, while our sellers sort only after the buyer has arrived. In simplest terms, our sellers used *ex post* sorting, while Eeckhout and Kircher’s sellers sort *ex ante*. Intuitively, we focus on settings in which the search process is too imprecise for agents to be guided toward specific trading partners.³

2.2.2 Efficiency

Coarse Matching

Our results can also be thought of as related to coarse matching (Chao and Wilson, 1987; McAfee, 2002; Hoppe et. al., 2011; Shao, 2011), a type of sorting equilibrium in which agents can sort into groups/locations, but these groups cannot fully separate types, and matching is probabilistic within these groups. A key insight of this literature is that the failure of complete sorting need not induce large efficiency losses, and this can be interpreted as a justification for the applied relevance of sorting, in light of the critique that the precision and complexity of theoretical sorting patterns precludes their occurrence in reality.

In some sense, our search equilibrium entails coarse matching—with any search frictions, matching will not be one-to-one, so each agents will have a nontrivial matching set. In the context of the coarse matching literature, our findings can be interpreted as demonstrating the role of prices as a potential mechanism through which coarse matching can arise.⁴

³In some sense, our analysis can be viewed as a bridge between such studies of private information in directed search and the literature considering full information, random search.

⁴The connection to coarse matching may become stronger in future versions of this article, where we hope to characterize the efficiency properties of equilibrium in our model and how the magnitude of search frictions impacts these properties.

2.3 The Model

We consider a discrete-time, dynamic model with heterogeneous buyers and sellers. There are equal measures of buyers and sellers each period in the equilibrium of the steady state economy, where buyer types x are distributed according to cdf $F_B(x)$, and seller types y are distributed according to cdf $F_S(y)$.⁵ Each buyer is randomly matched with one seller (and vice versa) in each period. In each pair, seller types become observable, while buyer types remain private information. Sellers make take-it-or-leave-it offers $P(y)$ to buyers.

If trade occurs, output $z(x, y)$ is produced and both parties leave the market permanently with utility $z(x, y) - P(y)$ for the buyer and $P(y)$ for the seller. Those who do not trade experience an exogenous exit shock with probability δ , in which case they leave the market. Otherwise, remaining buyers and sellers play the same game in the following period, along with exogenous measures of newly entering buyers and sellers, Γ_B and Γ_S . These new entrants are drawn from fixed distributions $\gamma_B(x)$ and $\gamma_S(y)$ over the bounded intervals $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$. The "effective" discount factor for all players is $\beta = \delta \times \beta'$, where β' is the discount factor.

A type x buyer is willing to accept price $P(y)$ if $P(y) \leq z(x, y) - \beta V_0(x)$, where $V_0(x)$ is the buyer's equilibrium payoff. Unlike in a frictionless market, equilibrium will not entail a deterministic, one-to-one matching. Rather, each type will match probabilistically with one agent from a range of "acceptable" types on the opposite side of the market. We therefore denote a type x buyer's surplus from trading with a type y seller as $s(x, y) = z(x, y) - \beta V_0(x) - P(y)$. Given $P(y)$, we call the set of sellers whose price will be accepted by a type x buyer as the buyer x 's *matching set* and denote the equilibrium matching set as $\mathbb{M}_B(x)$.

$$\mathbb{M}_B(x) = \{y : s(x, y) \geq 0\}$$

⁵As will soon be obvious, these distributions have bounded supports in equilibrium.

We can therefore express $V_0(x)$ as

$$V_0(x) = \frac{\int_{\mathbb{M}_B(x)} (z(x, y) - P(y)) dF_S(y)}{1 - \beta + \beta \int_{\mathbb{M}_B(x)} dF_S(y)}$$

$\mathbb{M}_S(y)$ in turn defines a type y seller's *matching set* $\mathbb{M}_S(y) = \{x : s(x, y) \geq 0\}$.

Obviously, $y \in \mathbb{M}_B(x)$ if and only if $x \in \mathbb{M}_S(y)$.

Given $F_B(x)$ and $V_0(x)$, seller y chooses $P(y)$ to solve

$$\Pi(y) = \max_P \left\{ P \int_{\mathbb{M}_S(y; P, V_0(x))} dF_B(x) + (1 - \int_{\mathbb{M}_S(y; P, V_0(x))} dF_B(x)) \beta \Pi(y) \right\}$$

The last equilibrium condition is the steady-state condition: the measure of outflow of any type must equal the measure of inflow of the same type. The pdf of the type distribution f and pdf of the entrant type distribution γ therefore must satisfy,

$$\hat{f}_B(x) = \frac{\gamma_B(x)}{\delta + (1 - \delta) \frac{\int_{\mathbb{M}_S(x)} \hat{f}_S(y) dy}{\int \hat{f}_S(y) dy}}$$

$$\hat{f}_S(y) = \frac{\gamma_S(y)}{\delta + (1 - \delta) \frac{\int_{\mathbb{M}_B(y)} \hat{f}_B(x) dx}{\int \hat{f}_B(x) dx}}$$

$$\text{where } f_i (i = S, B) \text{ satisfy } f_B(x) = \frac{\hat{f}_B(x)}{\int \hat{f}_B(x) dx} \text{ and } f_S(y) = \frac{\hat{f}_S(y)}{\int \hat{f}_S(y) dy}$$

For subsequent analysis, we impose the following assumptions on $z(x, y)$:

Assumption 2.1. *The output function $z(x, y)$ is twice continuously differentiable, strictly increasing in both arguments, and log-concave in x .*

We devote the remainder of this section to technical preliminaries for the sorting analysis. First, we offer a fairly general existence proof for an equilibrium in which prices are continuous in seller type. Following Shimer and Smith (2000), we then define PAM and NAM with non-degenerate matching set. Finally, we close the section by providing conditions that will allow us to tractably characterize

sorting. In particular, we provide sufficient conditions on $z(\cdot, \cdot)$ to ensure that convex seller matching sets will arise.

Existence

We will focus on equilibrium where price is continuous in seller type. Therefore, when proving the existence, we first assume that $P(y, V_0)$ is continuous. At the end of the existence proof, we then verify that the equilibrium price is indeed continuous.

We use Schauder fixed point theorem to prove the existence. In particular, we will show that the mapping from the continuation payoff $V_0(x)$ to itself defined by equilibrium conditions is well-defined and continuous. We will approach the problem by first providing some preliminary results, which is lemma 2.1, 2.2 and 2.3, and then show that the mapping is continuous in proposition 2.1.

Throughout this chapter, we assume that the output function is either supermodular or submodular.

Assumption 2.2. (SUP) *The output function $z(x, y)$ is supermodular.*

(SUB) *The output function $z(x, y)$ is submodular.*

Lemma 2.1. *Given (A1), a type x buyer's outside option function $V_0(x)$ satisfies*

$$V_0(x) \geq \frac{1}{1-\beta} \int_M (z(x, y) - P(y) - \beta V_0(x)) f_S(y) dy$$

for any $M \subseteq [y, \bar{y}]$. In addition, $V_0(x)$ is non-negative, increasing in x and Lipschitz continuous in equilibrium. In addition, if price is continuous in seller type and either A2-Sup or A2-Sub holds, $V_0(x)$ is differentiable in equilibrium, with

$$V_0'(x) = \frac{\int_{\mathbb{M}(x)} z_1(x, y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x)} f_S(y) dy}$$

Unless otherwise mentioned, all proofs are provided in the appendix. Define the indicator function $d(x, y)$. $d(x, y) = 1$ if and only if $s(x, y) \geq 0$ and $d(x, y) = 0$ otherwise.

Lemma 2.2. *Given A1, A2-Sup or A2-Sub, any Borel measurable mapping $V_0 \rightarrow d(x, y)$ from outside option functions to match indicator functions is continuous.*

For the existence proof, we also need to show that the endogenous distribution is continuous in the indicator function $d(x, y)$. To do this, we need to assume that the pdf of entrant distributions are bounded and that the measures of entrants are the same on both sides.

Assumption 2.3. $\gamma_B(x) \in (0, \infty)$ and $\gamma_S(y) \in (0, \infty)$ for any x and y , and $\int \gamma_B(x)dx = \int \gamma_S(y)dy$.

Lemma 2.3. *The mapping $d(x, y) \rightarrow (f_B(x), f_S(y))$ is well-defined and continuous.*

With the above preliminary results, we are now ready to show the existence of the equilibrium.

Proposition 2.1. *Given A1 and A2-Sup or A2-Sub, there exists equilibrium with prices continuous in seller types.*

Definition of Sorting

As we can see, matching sets in an environment with search frictions are normally non-degenerate. A natural definition of sorting for such conditions is that provided in Shimer and Smith (2000)—for PAM, they require that the set of mutually agreeable matches form a lattice. More explicitly:

Definition 2.1. *Take $x_1 < x_2$ and $y_1 < y_2$.*

PAM: *There is PAM if $y_1 \in \mathbb{M}_B(x_1)$ and $y_2 \in \mathbb{M}_B(x_2)$ whenever $y_1 \in \mathbb{M}_B(x_2)$ and $y_2 \in \mathbb{M}_B(x_1)$.*

NAM: *There is NAM if $y_1 \in \mathbb{M}_B(x_2)$ and $y_2 \in \mathbb{M}_B(x_1)$ whenever $y_1 \in \mathbb{M}_B(x_1)$ and $y_2 \in \mathbb{M}_B(x_2)$.*

Note that convex matching sets for both buyers and sellers are necessary conditions for either PAM or NAM.

Convexity of Seller Matching Set

For the earlier existence result, it was unnecessary to place restrictions on agents' matching sets. Obviously, though, the definitions of PAM and NAM above will be economically meaningful only if matching sets are convex. We therefore provide sufficient conditions on $z(x, y)$ for this to always be the case.

Assumption 2.4. $z_1(x, y)$ is log-supermodular.

Assumption 2.5. $z_{12}(x, y)$ is log-supermodular.

Proposition 2.2. *Given A4 and A5, the seller's matching set $\mathbb{M}_S(y)$ is convex for any y .*

2.4 Limits of Search Frictions: $\beta = 0$ (One-Shot Bilateral Monopoly)

In this section, we consider the case in which agents do not value the future at all ($\beta = 0$) – the one-period game. In other words, all agents experience death shock after one period.

If a seller y chooses the price that equals the output with a type x buyer, i.e, $P(y) = z(x, y)$, then any buyer with type above x will accept the price. Therefore, choosing $P(y)$ is equivalent to selecting the marginal type $x^*(y)$ to maximize the expected profit. That is,

$$\Pi(y) = \max_{\tilde{x}} \{z(\tilde{x}, y)(1 - F_B(\tilde{x}))\}$$

Theorem 2.1. *When $\beta = 0$, $P(y) = z(x^*(y), y)$ where the marginal type $x^*(y)$ is determined by:*

$$z_1(x^*(y), y)(1 - F_B(x^*(y))) = z(x^*(y), y)f_B(x^*(y)) \quad (2.1)$$

Further, both $P(y)$ and $x^*(y)$ are unique for any y .

As usual, x^* is chosen so that marginal revenue equals marginal cost. The left hand side of equation (2.1) is the marginal revenue of raising the marginal type: the price is raised by the amount $z_1(x^*(y), y)$ and the seller can collect this increment with probability $1 - F_B(x^*(y))$, which is essentially the trading probability. The right hand side is the marginal cost of raising the marginal type: the seller can no longer sell to type $x^*(y)$ buyers and therefore the loss equals the price ($z(x^*(y), y)$) times the probability of meeting a type $x^*(y)$ buyer ($f_B(x^*(y))$).

Let us now characterize sorting. Under the threshold rule and the assumptions that ensure differentiability, the definition of PAM (NAM) reduces to the condition that the derivative of the marginal type is positive (negative). That is, sorting is positive if $\frac{\partial x^*(y)}{\partial y} \geq 0$ and is negative if $\frac{\partial x^*(y)}{\partial y} \leq 0$. So we only need to do a comparative static exercise to find out under what conditions is the derivative positive.

Theorem 2.2. *Sorting is positive (PAM) if the output function $z(x, y)$ is log-supermodular and sorting is negative (NAM) if $z(x, y)$ is log-submodular.*

Proof. From the equilibrium condition that determines $x^*(y)$, it is easy to see that

$$\begin{aligned} \frac{\partial x^*}{\partial y} &= \frac{z_{12}[1 - F(x^*) - z_2 f(x^*)]}{2z_1 f(x^*) + z f'(x^*) - z_{11}[1 - F(x^*)]} \\ &= \frac{z_{12}z - z_1 z_2}{(z_1)^2} \times \frac{1}{2 + \frac{z f'}{z_1} - \frac{z_{11}z}{(z_1)^2}} \end{aligned} \quad (2.2)$$

From the first line to the second line, we plugged in the equilibrium condition and rearranged terms.

We also know that $\frac{1-F}{f}$ decreases in x . This implies

$$\begin{aligned} \frac{\partial}{\partial x} \frac{1-F}{f} &= \frac{-f^2 - (1-F)f'}{f^2} < 0 \\ \Rightarrow 1 + \frac{1-F}{f} \frac{f'}{f} &> 0 \Rightarrow \frac{z f'}{z_1 f} > -1 \end{aligned}$$

Therefore,

$$2 + \frac{z}{z_1} \frac{f'}{f} - \frac{z_{11}z}{(z_1)^2} > 2 - 1 - 1 \geq 0$$

The sorting is positive if $z_{12}z - z_1z_2 \geq 0$, i.e., $z(x, y)$ is log-supermodular; It is negative if $z_{12}z - z_1z_2 \leq 0$, i.e., $z(x, y)$ is log-submodular. \square

To avoid repetition, we will only focus on PAM case in the following discussion, since the intuition for NAM is symmetric. Notice that log-supermodularity is a stronger condition than supermodularity, because z_{12} has to be larger than $\frac{z_1z_2}{z}$, which is strictly positive. Hence with search and information frictions, we need stronger complementarity to ensure positive sorting.

To see the intuition behind this result, consider two sellers, one with a higher type y_1 and the other one with a lower type y_2 . Suppose currently they choose the same marginal type and each of them is deciding whether or not to raise the marginal type, facing the trade-off between price and probability of trade.

If they raise the marginal type by one unit, because of supermodularity, the type y_1 seller can enjoy larger price increment. This means that a higher type seller has stronger incentive to raise his marginal type. On the other hand, the loss of giving up the marginal type buyer is the current selling price times the probability of meeting the marginal type. Since the price of y_1 is strictly higher, he loses more from a reduced trading probability. Hence a higher type seller also has stronger incentive to increase his trading probability which is equivalent to lower his marginal type.

Recall that PAM requires a higher type seller to choose higher marginal type. Therefore, the first effect must be large enough to outweigh the second one. In other words, supermodularity is not sufficient. Positive sorting then requires an output function with stronger complementarity, in particular log-supermodularity.

In Figure 2.1, we plot the marginal type function $x^*(y)$ with parameter spec-

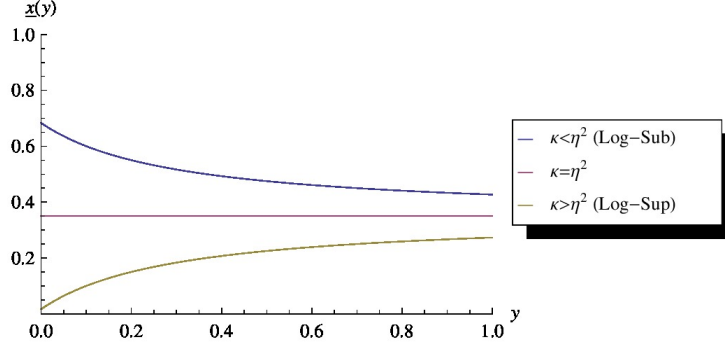


Figure 2.1: Seller Marginal Types

$$\Gamma_B(x) \sim U(0, 1); z(x, y) = \eta(x + y) + xy + \kappa, \eta \in (0, 1), \kappa \in (0, \eta)$$

ifications as shown beneath the figure. From this example, we can easily verify two conclusions we had. First of all, the log-supermodular condition is stronger than supermodular: the output function $z(x, y)$ is always supermodular, but it is log-supermodular if and only if $\kappa > \eta^2$. Secondly, the sorting is positive if and only if $z(x, y)$ is log-supermodular.

When sorting is positive, a higher type seller finds it optimal to scarifies the probability of trade for a higher price.

Lemma 2.4. *If sorting is positive, $P(y)$ is increasing in y and the trading probability decreases in y .*

Proof. Because the trading probability equals $1 - F_B(x^*(y))$, as $x^*(y)$ increases in y , the trading probability decreases. To show $P(y)$ increases in y , notice that,

$$\frac{\partial P(y)}{\partial y} = z_1 \frac{\partial x^*}{\partial y} + z_2 > 0$$

□

2.5 Limits of Search Frictions: $\beta \rightarrow 1$ (Frictionless Limit)

In this section, we consider the special case where the length of each period shrinks to zero. That is, $\beta' \rightarrow 1$ and $\delta \rightarrow 0$. Therefore the actual discount factor $\beta \rightarrow 1$.

Theorem 2.3. *Given A2-SUP or A2-SUB, for any $\xi > 0$, there exists $\epsilon > 0$ such that for any $\beta > 1 - \epsilon$,*

1. $d(x, y) = 1$ if and only if $s(x, y) \in [0, \xi)$;
2. $\mu(\mathbb{M}_B(x)) \in [0, \xi)$ and $\mu(\mathbb{M}_S(y)) \in [0, \xi)$;
3. *The matching set converges to the perfect positive assortative matching if $z(x, y)$ is supermodular, i.e., there exist a strictly increasing function $m(x)$ defined on $[\underline{x}, \bar{x}]$ such that 1) $m(\underline{x}) = \underline{y}$ and $m(\bar{x}) = \bar{y}$, 2) for any (x, y) with $d(x, y) = 1$, $|x - m^{-1}(y)| < \xi$ and $|y - m(x)| < \xi$.*
4. *The matching set converges to the perfect negative assortative matching if $z(x, y)$ is submodular, i.e., there exist a strictly decreasing function $m(x)$ defined on $[\underline{x}, \bar{x}]$ such that 1) $m(\underline{x}) = \bar{y}$ and $m(\bar{x}) = \underline{y}$, 2) for any (x, y) with $d(x, y) = 1$, $|x - m^{-1}(y)| < \xi$ and $|y - m(x)| < \xi$.*

As search frictions vanish, we find that Becker's result can be restored even if information frictions remain, that is, supermodular (submodular) condition is sufficient to ensure positive (negative) sorting in the limit. To understand this result, note that although sellers still face the trade-off between price and trading probability per period, they care less and less about the latter as they meet buyers more and more often, because however small their matching sets are, they can almost for sure sell before they experience death shock. Therefore, they will keep raising the price as long as their matching set is non-empty.

Recall the intuition we had in the static case. A higher type seller has both stronger incentive to secure trade and stronger incentive to raise marginal type.

The direction of sorting depends on the relative strength of these to incentives. As argued in the last paragraph, the first incentive grows inconsequential as we approach to the frictionless limit. Therefore, supermodularity is sufficient to ensure positive sorting.

Based on the function $m(x)$, we can further derive the equilibrium price and thus the division of surplus in equilibrium.

Theorem 2.4. *The equilibrium price pointwise converges to price function $P^*(y)$, where*

$$P^*(y) = z(\underline{x}, \underline{y}) + \int_{\underline{y}}^y z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}, \text{ if } z(x, y) \text{ is supermodular};$$

$$P^*(y) = z(\bar{x}, \underline{y}) + \int_{\underline{y}}^y z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}, \text{ if } z(x, y) \text{ is submodular}.$$

The above theorem shows that besides the equilibrium matching set, the equilibrium price also approaches Walrasian: each player gets her marginal contribution in the limit. Because the buyer in a match can meet another trading partner almost immediately and almost for sure, the seller faces competitions from other sellers. The price increment of a higher type seller thus equals the seller's marginal contribution.

2.6 Conclusion

The presence of buyer private information does impede sorting, and we have highlighted the relationship between the strength of this effect and the degree of competition in the market. At one extreme, when there is bilateral monopoly power in each buyer-seller meeting, PAM requires a log-supermodular production function, which is of course a stronger condition than standard supermodularity. Higher types also have higher opportunity costs of failing to trade, so the added incentives to ensure trade takes place are in conflict with sorting.

These incentives remain relevant in a dynamic frictional setting, but they grow inconsequential as we approach the frictionless limit. Thus, as search frictions vanish, the sorting consequences of private information do as well, and the standard supermodularity condition is sufficient to generate positive sorting.

2.7 Appendices

2.7.1 Proof of Lemma 2.1

Step 1: Inequality and non-negative. First of all,

$$\begin{aligned} V_0(x) &= \int_{\mathbb{M}_B(x)} (z(x, y) - P(y)) f_S(y) dy + \beta [1 - \int_{\mathbb{M}_B(x)} f_S(y) dy] V_0(x) \\ \Rightarrow V_0(x) &= \frac{1}{1 - \beta} \int_{\mathbb{M}_B(x)} (z(x, y) - P(y) - \beta V_0(x)) f_S(y) dy \end{aligned}$$

For any $M \neq \mathbb{M}_B(x)$, it either exclude $y \in \mathbb{M}_B(x)$, in which case $z(x, y) - P(y) - \beta V_0(x) > 0$, or include $y \notin \mathbb{M}_B(x)$, in which case $z(x, y) - P(y) - \beta V_0(x) < 0$.

The inequality thus follows.

Clearly, $V_0(x)$ is non-negative.

Step 2: Increasing in x . Consider $x_2 \geq x_1$,

$$\begin{aligned} &(1 - \beta)[V_0(x_2) - V_0(x_1)] \\ &= \int_{\mathbb{M}_B(x_2)} (z(x_2, y) - P(y) - \beta V_0(x_2)) f_S(y) dy \\ &\quad - \int_{\mathbb{M}_B(x_1)} (z(x_1, y) - P(y) - \beta V_0(x_1)) f_S(y) dy \\ &\geq \int_{\mathbb{M}_B(x_1)} [z(x_2, y) - z(x_1, y) - \beta(V_0(x_2) - V_0(x_1))] f_S(y) dy \\ \Rightarrow V_0(x_2) - V_0(x_1) &\geq \frac{\int_{\mathbb{M}_B(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y) dy} \geq 0 \end{aligned}$$

Step 3: Lipschitz. Following the same steps,

$$V_0(x_2) - V_0(x_1) \leq \frac{\int_{\mathbb{M}_B(x_2)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y) dy}$$

Combine the two inequalities and use the fact that $|z(x_2, y) - z(x_1, y)| \leq \kappa(x_2 - x_1)$,

$$\frac{-\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_1)} f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y) dy} \leq V_0(x_2) - V_0(x_1) \leq \frac{\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_2)} f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y) dy}$$

$|V_0(x_2) - V_0(x_1)| \leq \kappa(x_2 - x_1)$ follows and thus $V_0(x)$ is L-continuous.

Step 4: Differentiability.

Step 4.1: $\mathbb{M}_B(x)$ is continuous.

First, we show $\mathbb{M}_B(x)$ is u.h.c. Take any sequence $(x_n, y_n) \rightarrow (x, y)$ with $y_n \in \mathbb{M}_B(x_n)$ for any n . Therefore $z(x_n, y_n) - \beta V_0(x_n) - P(y_n) \geq 0$ for all n . In the limit, $z(x, y) - \beta V_0(x) - P(y) \geq 0$ because $z(x, y)$, $V_0(x)$ and $P(y)$ are continuous. This implies $y \in \mathbb{M}(x)$.

Next, we show that surplus function is rarely constant in one variable.

Define $N_s(x) = \{y : s(x, y) = 0\}$, $N_s(y) = \{x : s(x, y) = 0\}$ and $N_s = \{(x, y) : s(x, y) = 0\}$. Pick $x \neq x'$ and $y \neq y'$, such that $s(x, y) = s(x', y) = s(x, y') = 0$. If A2-Sup or A2-Sub is satisfied, it must be true that $s(x', y') \neq 0$. To see that, notice

$$\begin{aligned} s(x, y) - s(x', y) &= z(x, y) - z(x', y) - \beta(V_0(x) - V_0(x')) \\ s(x, y') - s(x', y') &= z(x, y') - z(x', y') - \beta(V_0(x) - V_0(x')) \end{aligned}$$

Because $z(x, y) - z(x', y) \neq z(x, y') - z(x', y')$, $0 = s(x, y) - s(x', y) \neq s(x, y') - s(x', y')$, which implies $s(x', y') \neq 0$.

Following a similar approach to that in Appendix B of Shimer and Smith (2000), we show that the following measures are zero almost everywhere: $\mu(N_s(x)) = 0$ for a.e. x , $\mu(N_s(y)) = 0$ for a.e. y and $\mu(N_s) = 0$ a.e.

To show a.e. l.h.c. take any sequence $x_n \rightarrow x$ and any $y \in \mathbb{M}_B(x)$. If there exists a subsequence x_m of x_n , such that for any x_m ,

$$\sup_{\hat{y}} \{z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y})\} \geq z(x, y) - \beta V_0(x) - P(y) \geq \inf_{\hat{y}} \{z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y})\}$$

By the continuity of $z(x, y)$ and $P(y)$, there exists at least one y_m that satisfies,

$$z(x_m, y_m) - \beta V_0(x_m) - P(y_m) = z(x, y) - \beta V_0(x) - P(y)$$

If there are multiple solutions, pick the one that is the closest to y . This defines a sequence $\{y_m\}$. Clearly, $y_m \in \mathbb{M}_B(x_m)$. We only need to show that for almost all x , there exist a subsequence y_k of y_m , such that $y_k \rightarrow y$.

First of all, a convergent subsequence always exists because $\{y_m\}$ is bounded. Suppose $y_k \rightarrow \tilde{y}$.

Case 1. We first consider the case where there exist $\epsilon > 0$ and $K > 0$ such that for any $k > K$ and any $\hat{y} \in [y - \epsilon, y + \epsilon]$, $\hat{y} \notin \mathbb{M}_B(x_k)$. In this scenario, it is possible that $\tilde{y} \neq y$. However, the measure of $\{x, y\}$ that satisfies this condition is 0.

Case 2. Next, we consider all complement scenarios, that is, for any $\delta > 0$ and $K > 0$, there exist $k > K$ and $\hat{y} \in [y - \epsilon, y + \epsilon]$, such that $\hat{y} \in \mathbb{M}_B(x_k)$.

Suppose $y_k \rightarrow \tilde{y} \neq y$. Since we have excluded the case 1, there exists $\hat{\epsilon} > 0$ and $K_1 > 0$, such that for any $k > K_1$,

$$|z(x_k, y) - z(x, y) - \beta(V_0(x_k) - V_0(x))| > 2\hat{\epsilon}$$

On the other hand, by continuity of z and V_0 , there exist K_2 , such that for all $k > K_2$,

$$\begin{aligned} & |z(x_k, y) - z(x, y) - \beta(V_0(x_k) - V_0(x))| \\ & \leq |z(x_k, y) - z(x, y)| + \beta |V_0(x_k) - V_0(x)| < 2\hat{\epsilon} \end{aligned}$$

We can pick $\hat{K} = \max\{K_1, K_2\}$. The above two inequalities hold at the same time. This is a contradiction.

Finally consider the case where subsequence x_m does not exist, i.e., for any subsequence x_m , either $\sup_{\hat{y}} \{z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y})\} < z(x, y) - \beta V_0(x) - P(y)$

or $\inf_{\hat{y}}\{z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y})\} > z(x, y) - \beta V_0(x) - P(y)$. Here we just show the proof for the first case since the second case is similar.

Define $y_m \in \operatorname{argmax}\{z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y})\}$. If there are more than one argmax , pick the one that is closer to y . By constructing y_m this way, we can then follow the same prove of the previous case to show that any convergence subsequence of y_m must converge to y .

Step 4.2: Decomposition of the slope of outside option.

Take any sequence $x_n \rightarrow x$, for each n ,

$$(1 - \beta) \frac{V_0(x_n) - V_0(x)}{x_n - x} = \int_{\mathbb{M}_B(x_n) - \mathbb{M}_B(x)} \frac{z(x_n, y) - P(y) - \beta V_0(x_n)}{x_n - x} f_S(y) dy \\ + \int_{\mathbb{M}_B(x)} \left[\frac{z(x_n, y) - z(x, y)}{x_n - x} - \beta \frac{V_0(x_n) - V_0(x)}{x_n - x} \right] f_S(y) dy$$

Take limit $n \rightarrow \infty$, the first integral vanishes because 1) $\mathbb{M}_B(x)$ is continuous a.e. and when it is not continuous, the measures of the limiting set and the set in the limit are the same, and 2) buyers participate optimally. Rearranging terms we get the proposed derivative.

2.7.2 Proof of lemma 2.2

We have proved that the surplus function is rarely constant in one variable in lemma 2.1. Define set $\sum_s(\eta) = \{(x, y) : |s(x, y)| \in [0, \eta]\}$. This set shrinks monotonically to $\cap_{k=1}^{\infty} \sum_s(1/k) = N_s$.

$$\lim_{\eta \rightarrow 0} (\mu \times \mu)(\sum_s(\eta)) = (\mu \times \mu)(\cap_{k=1}^{\infty} \sum_s(1/k)) = (\mu \times \mu)(N_s) = 0$$

Let V_0^1 and V_0^2 be two outside option functions, and d^1 and d^2 be the corresponding match indicator functions.

Since $P(y, V_0)$ is continuous in V_0 , for any $\epsilon > 0$, there exist $\eta' > 0$, such that

$$\beta \| V_0^1(x) - V_0^2(x) \| < \eta' \Rightarrow |P(y, V_0^1) - P(y, V_0^2)| < \epsilon, \text{ for any } y$$

In words, we can always pick close enough outside option functions such that the price functions are close. Let $\eta = 2 \max\{\eta', \epsilon\}$.

If $s^1(x, y) = z(x, y) - \beta V_0^1(x) - P(y, V_0^1) > \eta$, $s^2(x, y) = z(x, y) - \beta V_0^2(x) - P(y, V_0^2) > 0$. So $d^1(x, y) = d^2(x, y) = 1$. By the same logic, If $s^1(x, y) < -\eta$, $s^1(x, y) < 0$. So $d^1(x, y) = d^2(x, y) = 0$. As a result, $\{(x, y) : d^1(x, y) \neq d^2(x, y)\} \subseteq \sum_{s^1}(\eta)$. The Lebesgue measure of $\sum_{s^1}(\eta)$ vanishes as $\eta \rightarrow 0$. The continuity is thus established $\lim_{\|V_0^1(x) - V_0^2(x)\| \rightarrow 0} \|d^1(x, y) - d^2(x, y)\|_{\mathcal{L}^1} = 0$.

2.7.3 Proof of Lemma 2.3

Step 1: The mapping is well-defined. Given entrant $\gamma_B(x)$ and $\gamma_S(y)$, the mapping is well defined if there exist unique f_B and f_S solves the following system of equations,

$$f_B(x) = \frac{\gamma_B(x)}{\delta + (1 - \delta) \frac{\int d(x, y) f_S(y) dy}{\int f_S(y) dy}}$$

$$f_S(y) = \frac{\gamma_S(y)}{\delta + (1 - \delta) \frac{\int d(x, y) f_B(x) dx}{\int f_B(x) dx}}$$

From those conditions, we know that $f_B(x) \in [\gamma_B(x), \gamma_B(x)/\delta]$ and $f_S(y) \in [\gamma_S(y), \gamma_S(y)/\delta]$.

One can apply similar log transformation method as in Shimer and Smith (2000) and rewrite the problem into a fixed-point problem.

$$\Phi_B(h) = \log \frac{\gamma_B(x)}{\delta + (1 - \delta) \frac{\int d(x, y) e^{h_S(y)} dy}{\int e^{h_S(y)} dy}}$$

$$\Phi_S(h) = \log \frac{\gamma_S(y)}{\delta + (1 - \delta) \frac{\int d(x, y) e^{h_B(x)} dx}{\int e^{h_B(x)} dx}}$$

where $h_B(x) = \log f_B(x)$, $h_S(y) = \log f_S(y)$, $h = (h_B, h_S)'$. The mapping is well defined if $\Phi(h) = h$ has a unique fixed point. We prove it using Contraction

Mapping Theorem. Consider h^1 and h^2 ,

$$\begin{aligned}
\Phi_B(h^2) - \Phi_B(h^1) &= \log \frac{\delta + (1 - \delta) \frac{\int d(x,y) e^{h_S^1(y)} dy}{\int e^{h_S^1(y)} dy}}{\delta + (1 - \delta) \frac{\int d(x,y) e^{h_S^2(y)} dy}{\int e^{h_S^2(y)} dy}} \\
&\leq \log \frac{\delta + (1 - \delta) e^{\|h_S^1 - h_S^2\|} \frac{\int d(x,y) e^{h_S^2(y)} dy}{\int e^{h_S^2(y)} dy}}{\delta + (1 - \delta) \frac{\int d(x,y) e^{h_S^2(y)} dy}{\int e^{h_S^2(y)} dy}} \\
&\leq \log \frac{\delta + (1 - \delta) e^{\|h_S^1 - h_S^2\|}}{\delta + (1 - \delta)} \\
&= \log[\delta + (1 - \delta) e^{\|h_S^1 - h_S^2\|}]
\end{aligned}$$

The first inequality follows because $e^{\|h_S^1 - h_S^2\|} > e^{h_S^1(y) - h_S^2(y)}$ for any y . We thus have

$$\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\|h_S^1 - h_S^2\|} \leq \frac{\log[\delta + (1 - \delta) e^{\|h_S^1 - h_S^2\|}]}{\|h_S^1 - h_S^2\|}$$

In addition, we know that $h_S(y) \in [\log(\gamma_S(y)), \log(\gamma_S(y)) - \log \delta]$, which implies $\|h_S^1 - h_S^2\| \in [0, -\log \delta]$. Since the right hand side of the above inequality increases in $\|h_S^1 - h_S^2\|$,

$$\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\|h_S^1 - h_S^2\|} \leq \frac{\log[\delta + \frac{1-\delta}{\delta}]}{\log \frac{1}{\delta}} = \chi \in (0, 1)$$

The same argument applies to the other direction and one thus get

$$\frac{\|\Phi_B(h^1) - \Phi_B(h^2)\|}{\|h_S^1 - h_S^2\|} \leq \chi$$

We have the symmetric inequality for y . Denote $\Phi(h) = (\Phi_B(h), \Phi_S(h))'$, combining the two inequalities,

$$\|\Phi(h^1) - \Phi(h^2)\| \leq A \|h^1 - h^2\|$$

where A is a matrix with $|A| = -\chi^2 \in (-1, 1)$. We thus proved that it is a contraction mapping.

Step 2: Continuity.

Define $G_B(d, f)(x) = f_B(x)[\delta + (1-\delta) \int d(x, y) \frac{f_S(y)}{\int f_S(y) dy} dy] - \gamma_B(x)$ and $G_S(d, f)(y) = f_S(y)[\delta + (1-\delta) \int d(x, y) \frac{f_B(x)}{\int f_B(x) dx}] - \gamma_S(y)$, $G(d, f) = (G_B(d, f), G_S(d, f))$. In equilibrium, $G(d, f) = 0$

Suppose that there exist d^1 and d^2 with $\|d^1 - d^2\|_{\mathcal{L}^1} \rightarrow 0$, such that $\|f^1 - f^2\|_{\mathcal{L}^1} \not\rightarrow 0$. Then $\|G(d^1, f^2)\|_{\mathcal{L}^1} > \hat{\epsilon}$. WOLG, assume $\|G_B(d^1, f^2)\|_{\mathcal{L}^1} > \epsilon$.

On the other hand,

$$\begin{aligned} \|G_B(d^1, f^2)\|_{\mathcal{L}^1} &= \|G_B(d^1, f^2) - G_B(d^2, f^2)\|_{\mathcal{L}^1} \\ &= \|f_B^2(x)(1-\delta) \int (d^1(x, y) - d^2(x, y)) \frac{f_S^2(y)}{\int f_S^2(y) dy} dy\|_{\mathcal{L}^1} < \epsilon \end{aligned}$$

The last line follows since $\frac{f_S^2(y)}{\int f_S^2(y) dy}$ and $f_B^2(x)$ are bounded for any x and y . This leads to a contradiction.

2.7.4 Proof of Proposition 2.1

Step 1: Equilibrium exists if $T(V_0) = V_0$ has unique fixed point, where,

$$T(V_0) = \int \max\{z(x, y) - P(y, V_0), \beta V_0(x)\} f_S^{V_0}(y) dy$$

Step 2: Following Schauder Fixed Point Theorem, we need a nonempty, closed, bounded and convex domain ψ such that,

1. $T : \psi \rightarrow \psi$.
2. $T(\psi)$ is an equicontinuous family.
3. T is a continuous operator.

Let ψ be the space of L-continuous functions V_0 on $[\underline{x}, \bar{x}]$, with lower bound 0 and upper bound $\sup_{x,y} z(x, y)$. Clearly, ψ is nonempty, closed, bounded and convex.

Step 3: $T : \psi \rightarrow \psi$ **and** $T(\psi)$ **is an equicontinuous.** Take $x_2 \neq x_1$,

$$\begin{aligned}
& | TV_0(x_2) - TV_0(x_1) | \\
& \leq \int | \max\{z(x_2, y) - P(y, V_0), \beta V_0(x_2)\} \\
& \quad - \max\{z(x_1, y) - P(y, V_0), \beta V_0(x_1)\} | f_S^{V_0}(y) dy \\
& \leq \int | \max\{z(x_2, y) - z(x_1, y), \beta(V_0(x_2) - V_0(x_1))\} | f_S^{V_0}(y) dy
\end{aligned}$$

Since both $z(x, y)$ and $V_0(x)$ are L-continuous, $T(\psi)$ is L-continuous, which implies equicontinuous. This also establishes T is a mapping from ψ to ψ .

Step 4: T **is continuous.** Take $V_0^2 \neq V_0^1$ in ψ , for any x ,

$$\begin{aligned}
& | TV_0^2(x) - TV_0^1(x) | \\
& = | \int \max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} f_S^{V_0^2}(y) dy \\
& \quad - \int \max\{z(x, y) - P(y, V_0^1), \beta V_0^1(x)\} f_S^{V_0^1}(y) dy | \\
& \leq | \int \max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} f_S^{V_0^2}(y) dy \\
& \quad - \int \max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} f_S^{V_0^1}(y) dy | \\
& \quad + | \int (\max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} \\
& \quad - \max\{z(x, y) - P(y, V_0^1), \beta V_0^1(x)\}) f_S^{V_0^1}(y) dy | \\
& = D_1(x) + D_2(x)
\end{aligned}$$

For $D_1(x)$

$$\begin{aligned}
D_1(x) & \leq \int \max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} | f_S^{V_0^2}(y) - f_S^{V_0^1}(y) | dy \\
& \leq \sup_{x,y} [\max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\}] \int | f_S^{V_0^2}(y) - f_S^{V_0^1}(y) | dy
\end{aligned}$$

Since $f_S(y)$ is continuous in V_0 , as $\| V_0^2 - V_0^1 \| \rightarrow 0$, $D_1(x) \rightarrow 0$.

For $D_2(x)$,

$$\begin{aligned}
D_2(x) &\leq \int |\max\{z(x, y) - P(y, V_0^2), \beta V_0^2(x)\} \\
&\quad - \max\{z(x, y) - P(y, V_0^1), \beta V_0^1(x)\} | f_S^{V_0^1}(y) dy \\
&\leq \int |\max\{P(y, V_0^1) - P(y, V_0^2), \beta V_0^2(x) - \beta V_0^1(x)\} | f_S^{V_0^1}(y) dy
\end{aligned}$$

Since $P(y, V_0)$ is continuous in V_0 , as $\|V_0^2 - V_0^1\| \rightarrow 0$, $D_2(x) \rightarrow 0$.

Step 5: Verify that there exists at least one price function that is continuous in y and V_0 in equilibrium.

If it does not cause any confusion, we will abuse the notation and use $P(y, V_0)$ to also denote the set of optimal prices of a seller with type y given V_0 . Seller's problem is $\max_p \Omega(y, p, V_0)$. $\Omega(y, p, V_0)$ is continuous in those three arguments because the matching set $\mathbb{M}_S(p; y, V_0)$ is almost everywhere continuous⁶. In addition, $p \in [0, \sup_{x,y}\{z(x, y)\}]$, which is compact valued. By Maximum Theorem, $P(y, V_0)$ is u.h.c. in y and V_0 .

Next, we can show that $P(y, V_0)$ is also l.h.c. in y and V_0 . If we decompose the slope of Ω along price,

$$\begin{aligned}
&(1 - \beta) \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p} \\
&= [p_n - \beta \Omega(y, p_n, V_0)] \frac{\int_{M(y, p_n, V_0) - M(y, p, V_0)} f_B(x) dx}{p_n - p} \\
&\quad + [1 - \beta \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p}] \int_{M(y, p, V_0)} f_B(x) dx \\
&= [p_n - \beta \Omega(y, p_n, V_0)] \frac{\sum_{i=1}^{K_1} \int_{\underline{x}_i(y, p_n, V_0)}^{\bar{x}_i(y, p_n, V_0)} f_B(x) dx - \sum_{j=1}^{K_2} \int_{\underline{x}_j(y, p, V_0)}^{\bar{x}_j(y, p, V_0)} f_B(x) dx}{p_n - p} \\
&\quad + [1 - \beta \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p}] \int_{M(y, p, V_0)} f_B(x) dx
\end{aligned}$$

K_1 and K_2 in the first term might be infinite, but there are always countable

⁶The proof of the a.e. continuity of $\mathbb{M}(p, y, V_0)$ is similar to the proof of the a.e. continuity of $\mathbb{M}(x)$ and thus is skipped here. The proof is available upon request.

number of bounds in the matching set. Take $p_n \rightarrow p$, we get the derivative,

$$\Omega_2(y, p, V_0) = \frac{\int_{M(y,p,V_0)} f_B(x) dx + D(y, p, V_0)}{1 - \beta + \beta \int_{M(y,p,V_0)} f_B(x) dx}$$

where, $D(y, p, V_0) = \sum_i [f_B(\bar{x}_i(y, p, V_0)) \frac{\partial \bar{x}_i(y, p, V_0)}{\partial p} - f_B(\underline{x}_i(y, p, V_0)) \frac{\partial \underline{x}_i(y, p, V_0)}{\partial p}]$

$$\frac{\partial \bar{x}_i(y, p, V_0)}{\partial p} = \frac{1}{z_1(\bar{x}_i(y, p, V_0), y) - \beta V_0'(\bar{x}_i(y, p, V_0))}$$

$$\frac{\partial \underline{x}_i(y, p, V_0)}{\partial p} = \frac{1}{z_1(\underline{x}_i(y, p, V_0), y) - \beta V_0'(\underline{x}_i(y, p, V_0))}$$

The last two lines follow from the boundary condition and Implicit Function Theorem.

Take any sequence $(y^n, V_0^n) \rightarrow (y, V_0)$. Consider any $p \in \operatorname{argmax}_{\hat{p}} \Omega(y, \hat{p}, V_0)$. For interior p , p solves the first order condition $\Omega_2(y, p, V_0) = 0$. To show $P(y, V_0)$ is l.h.c., we need to construct a sequence p_n , such that

1. $p_n \in P(y^n, V_0^n)$ and,
2. $p_n \rightarrow p$.

Pick sequence p_n such that it solves $\Omega_2(y^n, p_n, V_0^n) = 0$ and it is the solution that is closest to p . (Without loss of generality, assume interior solutions exist along the sequence.) The first condition is satisfied by construction. Since the sequence is bounded, there must exist a convergent sequence p_k . Suppose $p_k \rightarrow p' \neq p$. Then there exists $\epsilon > 0$ and $K_1 > 0$, such that for any $k > K_1$,

$$|\Omega_2(y^k, p, V_0^k)| > \epsilon$$

On the other hand, if we can show that $\Omega_2(y, p, V_0)$ is continuous in y and V_0 , then for any $\epsilon > 0$, there exist $K_2 > 0$, such that for any $k > K_2$,

$$|\Omega_2(y^k, p, V_0^k) - \Omega_2(y, p, V_0)| = |\Omega_2(y^k, p, V_0^k)| < \epsilon$$

Define $K = \max\{K_1, K_2\}$, the above two inequalities hold simultaneously for ϵ and any $k > K$. It is a contradiction.

The only step left is to show the continuity of $\Omega_2(y, p, V_0)$, which is equivalent to showing the continuity of $D(y, p, V_0)$.

First we claim that $f_B(x)$ is continuous in x for any given V_0 . Notice that

$$\begin{aligned} 0 &\leq | \log f_B(x_n) - \log f_B(x) | \\ &= | \log \gamma_B(x_n) - \log \gamma_B(x) + \log \frac{\delta + (1 - \delta) \int_{\mathbb{M}_B(x)} f_S(y) dy}{\delta + (1 - \delta) \int_{\mathbb{M}_B(x_n)} f_S(y) dy} | \\ &\leq | \log \gamma_B(x_n) - \log \gamma_B(x) | + | \log \frac{\delta + (1 - \delta) \int_{\mathbb{M}_B(x)} f_S(y) dy}{\delta + (1 - \delta) \int_{\mathbb{M}_B(x_n)} f_S(y) dy} | \end{aligned}$$

Both absolute values go to zero. Therefore, $\log f_B(x)$ is continuous and so is $f_B(x)$.

Next we show that $\| (V_0^n)' - V_0' \| \rightarrow 0$ as $\| V_0^n - V_0 \| \rightarrow 0$.

$$\begin{aligned} (1 - \beta)(V_0^n)' &= \int_{\mathbb{M}_B(x, V_0^n)} (z_1(x, y) - \beta(V_0^n)') f_S^{V_0^n}(y) dy \\ \Rightarrow (V_0^n)' - V_0' &= \\ &= \frac{1}{1 - \beta + \beta \int_{\mathbb{M}_B(x, V_0)} f_S^{V_0}(y) dy} \left\{ \int_{\mathbb{M}_B(x, V_0^n) - \mathbb{M}_B(x, V_0)} [z_1(x, y) - \beta(V_0^n)'] f_S^{V_0^n}(y) dy \right. \\ &\quad \left. + \int_{\mathbb{M}_B(x, V_0)} [z_1(x, y) - \beta V_0'] [f_S^{V_0^n}(y) - f_S^{V_0}(y)] dy \right\} \\ \Rightarrow \| (V_0^n)' - V_0' \| &\rightarrow 0 \text{ as } \| V_0^n - V_0 \| \rightarrow 0 \end{aligned}$$

As a result, $\frac{\partial \bar{x}_i(y, p, V_0)}{\partial p}$ and $\frac{\partial \underline{x}_i(y, p, V_0)}{\partial p}$ are continuous in y and V_0 . Combining with the continuity of \underline{x}_i , \bar{x}_i and $f_B(x)$, $D(y, p, V_0)$ is continuous in y and V_0 .

In sum, $P(y, V_0)$ is u.h.c. and l.h.c., and thus continuous. Therefore, there exists a price function $P(y, V_0)$ that is continuous in y and V_0 .

2.7.5 Proof of Proposition 2.2

A sufficient condition for $\mathbb{M}(y)$ to be convex is $s(x, y)$ being quasi-concave in x . Since $s(x, y)$ is continuous and differentiable in x , by definition, we only need show that $s(x_1, y) < s(x_2, y)$ for $\underline{x} < x_1 < x_2$ implies $s_1(x_1, y) \geq 0$.

Fix any $\underline{x} < x_1 < x_2 < \bar{x}$.

Step 1: $s_1(x, y) \geq 0$ for all $y \geq \hat{y}$ (where \hat{y} is defined below).

In the spirit of Diamond and Stiglitz (1974), Shimer and Smith (2000) adopt a Single Crossing Property (SCP) for gambles in their analysis, and the usefulness of this property extends to our setting. Since $z_{12}(x, y)$ is log-supermodular, there exists \hat{y} such that

$$z_1(x_1, \hat{y}) = \frac{\int_{\mathbb{M}(x_1)} z_1(x_1, y) f_S(y) dy}{\int_{\mathbb{M}(x_1)} f_S(y) dy}$$

Therefore,

$$\begin{aligned} \beta V_0'(x_1) &= \frac{\beta \int_{\mathbb{M}(x_1)} z_1(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \\ &= \frac{\beta z_1(x_1, \hat{y}) \int_{\mathbb{M}(x_1)} f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \\ &\leq z_1(x_1, \hat{y}) \end{aligned}$$

This implies $s_1(x_1, \hat{y}) = z_1(x_1, \hat{y}) - \beta V_0'(x_1) \geq 0$. By the supermodularity of $z(x, y)$, for any $y \geq \hat{y}$, $s_1(x_1, y) \geq s_1(x_1, \hat{y}) \geq 0$.

Step 2: $V_0(x_1) < V_0(x_2)$ and $z(x_1, y) < z(x_2, y)$ whenever $s(x_1, y) < s(x_2, y)$ and $y < \hat{y}$.

If $s(x_1, y) \geq s(x_2, y)$ at x_2 , there is nothing to verify.

If $s(x_1, y) < s(x_2, y)$ at x_2 ,

$$\begin{aligned} z(x_2, y) - z(x_1, y) &> \beta V_0(x_2) - \beta V_0(x_1) \\ &= \frac{\beta \int_{\mathbb{M}(x_2)} z(x_2, y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_2)} f_S(y) dy} - \frac{\beta \int_{\mathbb{M}(x_1)} z(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \\ &\geq \frac{\beta \int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \end{aligned}$$

By the SCP for gambles, for all $x' > x_1$,

$$\frac{\int_{\mathbb{M}(x_1)} z_1(x', y) f_S(y) dy}{\int_{\mathbb{M}(x_1)} f_S(y) dy} \geq z_1(x', \hat{y})$$

Integrate over $x' \in [x_1, x_2]$,

$$\frac{\int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{\int_{\mathbb{M}(x_1)} f_S(y) dy} \geq z(x_2, \hat{y}) - z(x_1, \hat{y})$$

By strict supermodularity of $z(x, y)$, $z(x_2, \hat{y}) - z(x_1, \hat{y}) > z(x_2, y) - z(x_1, y)$.

Combining all inequalities,

$$\begin{aligned} \frac{\int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{\int_{\mathbb{M}(x_1)} f_S(y) dy} &> \frac{\beta \int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \\ \Rightarrow \int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy &> 0 \\ \Rightarrow z(x_2, y) > z(x_1, y) \text{ and } V_0(x_2) > V_0(x_1) \end{aligned}$$

Step 3: $s_1(x_1, y) \geq 0$ whenever $s(x_1, y) < s(x_2, y)$ and $y < \hat{y}$.

If $s(x_1, y) \geq s(x_2, y)$ at x_2 , there is nothing to verify.

If $s(x_1, y) < s(x_2, y)$ at x_2 , we know

$$\begin{cases} V_0'(x_1) = \frac{\int_{\mathbb{M}(x_1)} z_1(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \\ V_0(x_2) - V_0(x_1) \geq \frac{\int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x_1)} f_S(y) dy} \end{cases}$$

$$\begin{aligned} \Rightarrow \frac{V_0'(x_1)}{V_0(x_2) - V_0(x_1)} &\leq \frac{\int_{\mathbb{M}(x_1)} z_1(x_1, y) f_S(y) dy}{\int_{\mathbb{M}(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy} \\ &\leq \frac{z_1(x_1, \hat{y})}{z(x_2, \hat{y}) - z(x_1, \hat{y})} \end{aligned}$$

By the log-supermodularity of $z_1(x, y)$, for $y < \hat{y}$,

$$\begin{aligned}
& z_1(x_1, \hat{y})[z(x_2, y) - z(x_1, y)] \leq z_1(x_1, y)[z(x_2, \hat{y}) - z(x_1, \hat{y})] \\
& \Rightarrow \frac{z_1(x_1, \hat{y})}{z(x_2, \hat{y}) - z(x_1, \hat{y})} \leq \frac{z_1(x_1, y)}{z(x_2, y) - z(x_1, y)} \\
& \Rightarrow \frac{V_0'(x_1)}{V_0(x_2) - V_0(x_1)} \leq \frac{z_1(x_1, y)}{z(x_2, y) - z(x_1, y)} \\
& \Rightarrow \beta V_0'(x_1) \leq z_1(x_1, y) \frac{\beta V_0(x_2) - \beta V_0(x_1)}{z(x_2, y) - z(x_1, y)} < z_1(x_1, y)
\end{aligned}$$

The second inequality in the last line follows because $z(x_2, y) - z(x_1, y) > \beta V_0(x_2) - \beta V_0(x_1)$.

Therefore, $s_1(x_1, y) = z_1(x_1, y) - \beta V_0'(x_1) > 0$.

2.7.6 Proof of Theorem 2.3

It is easy to see that seller y 's profit $\Pi(y) \rightarrow P(y)\mathbf{1}\{\mathbb{M}(y) \neq \emptyset\}$ when $\beta \rightarrow 1$.

Part 1: We first show $d(x, y) = 1 \iff s(x, y) = z(x, y) - P(y) - \beta V_0(x) \in [0, \xi]$ for any $\xi > 0$.

Direction " \Leftarrow " follows from the construction of function $d(x, y)$.

To see the other direction, notice that $d(x, y) = 1$ implies $z(x, y) - P(y) - \beta V_0(x) \geq 0$. Suppose there exist $\tilde{\xi} > 0$, x and y such that $z(x, y) - P(y) - \beta V_0(x) > \tilde{\xi}$ in equilibrium. Then the seller y can raise $\Pi(y)$ by increasing $P(y)$. This leads to a contradiction.

Part 2: Suppose there exist $\tilde{\xi} > 0$ and y such that $\mu(\mathbb{M}_S(y)) > \tilde{\xi}$. In the proof of lemma 2.1, we have proved that the surplus function $s(x, y)$ is rarely constant. Therefore, for a.e., $\mu(\mathbb{M}_S(y)) > \tilde{\xi}$ implies $s(x, y) > \hat{\xi}$ for some x and $\hat{\xi}$. Contradict the conclusion of the last step.

Following the same reasoning, $\mu(\mathbb{M}_B(x)) \in [0, \xi]$ for any $\xi > 0$ a.e.

Next, suppose $s(x, y)$ is non-negative and constant when $x = \tilde{x}$ and $y \in [\tilde{y}_1, \tilde{y}_2]$. We claim that $\mu([\tilde{y}_1, \tilde{y}_2])$ converges to 0 when $\beta \rightarrow 1$. Otherwise, type x buyers

will trade and exit the market faster than other types, which makes type y sellers' ($y \in [\tilde{y}_1, \tilde{y}_2]$) expected profit converge to zero. This is a contraction.

Finally, suppose $s(x, y)$ is non-negative and constant when $y = \tilde{y}$ and $x \in [\tilde{x}_1, \tilde{x}_2]$. We can again show that $\mu([\tilde{x}_1, \tilde{x}_2])$ converges to 0 when $\beta \rightarrow 1$. Suppose otherwise. Then type \tilde{y} sellers will trade and exit the market faster than other types, which makes type x buyers' ($x \in [\tilde{x}_1, \tilde{x}_2]$) payoff $V_0(x)$ converge to 0. Then $s(x, \tilde{y})$ converges to $z(x, \tilde{y}) - P(\tilde{y})$. Since $z(x, y)$ strictly increases in x , this contradicts $s(x, \tilde{y})$ being constant.

Part 3: Pick any x, x', y, y' such that $d(x, y) = d(x', y') = 1$. We first show that $x > x'$ implies $y \geq y'$ under the assumption of supermodularity. Suppose this is not the case, then supermodularity implies,

$$\begin{aligned} z(x', y) + z(x, y') - P(y) - P(y') - \beta V_0(x) - \beta V_0(x') &> \\ z(x, y) + z(x', y') - P(y) - P(y') - \beta V_0(x) - \beta V_0(x') &= 0 \\ \Rightarrow s(x', y) + s(x, y') &> 0 \end{aligned}$$

On the other hand, we know that $s(x', y) \leq 0$ and $s(x, y') \leq 0$ for sufficiently large β from the last step. This leads to a contradiction. Therefore, $x > x'$ implies $y \geq y'$ if the output function is supermodular. Similarly, $y > y'$ implies $x \geq x'$ if supermodular.

Next, we claim that the matching set of any type of buyer or seller is non-empty if there exist at least one lower type with non-empty matching set. Suppose otherwise, say $\mathbb{M}(\hat{x}) = \emptyset$. Define \hat{x}^* as any element in the set $\{x : x < \hat{x} \text{ and } \mathbb{M}(x) \neq \emptyset\}$ and pick any $\hat{y}^* \in \mathbb{M}(\hat{x}^*)$. It follows that $V_0(\hat{x}) = 0$ and $V_0(\hat{x}^*) \geq 0$, contradiction.

Finally, suppose \tilde{x} is the highest buyer type such that $\mathbb{M}(\tilde{x}) = \emptyset$. We will show that for any $\xi > 0$, there exist ϵ , such that $\tilde{x} - \underline{x} < \xi$ for any $\beta > 1 - \epsilon$. Suppose otherwise. Then it must be the case that $f_B(x_1)/f_B(x_2) = \infty$ for any $x_1 \in [\underline{x}, \tilde{x})$ and any $x_2 \in [\tilde{x}, \bar{x}]$. Sellers with type y then have incentive to lower price by a small amount and increase their trading probability.

To sum up, the matching approaches to the perfect positive assortative under the assumption of supermodularity.

The proof of perfect negative assortative matching under submodularity is essentially the same and thus is skipped here.

2.7.7 Proof of Theorem 2.4

We will only show the proof with supermodular output function $z(x, y)$. The proof with a submodular $z(x, y)$ is essentially the same and is available upon request. When $\beta \rightarrow 1$, for any x , the derivative of buyer's value function

$$V_0'(x) \rightarrow z_1(x, m(x))$$

This implies

$$\begin{aligned} V_0(x) &\rightarrow V_0(\underline{x}) + \int_{\underline{x}}^x z_1(\tilde{x}, m(\tilde{x})) d\tilde{x} \\ &= V_0(\underline{x}) + \int_{\underline{x}}^x dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y}) d\tilde{y} \end{aligned}$$

Since $z(x, m(x)) - P(m(x)) - \beta V_0(x) \rightarrow 0$, the price of $y = m(x)$ type seller can be computed,

$$\begin{aligned} P(y) &\rightarrow z(m^{-1}(y), y) - V_0(m^{-1}(y)) \\ &\rightarrow z(\underline{x}, \underline{y}) + \int_{\underline{x}}^x dz(\tilde{x}, m(\tilde{x})) \\ &\quad - [(V_0(\underline{x}) + \int_{\underline{x}}^x dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y}) d\tilde{y})] \\ &= z(\underline{x}, \underline{y}) - V_0(\underline{x}) + \int_{\underline{y}}^y z_2(m^{-1}(\tilde{y}), \tilde{y}) d\tilde{y} \end{aligned}$$

Here, $V_0(\underline{x}) = 0$ following the same argument as in Diamond (1971).

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CHAPTER 3

Patents and Allocation of Resources over Innovation Projects

3.1 Introduction

The patent system is designed to encourage innovation. The standard view is that it encourages innovation by providing the inventor exclusive access to the new technology. How much the society could benefit from the patent system depends on the way it influences firms' innovation activities. One observation is that firms usually have a portfolio of projects to choose from, some being more innovative but riskier. Then an important aspect of the innovation activity is the allocation of resources over projects.

The aim of this chapter is therefore to understand how patents influence firms' allocation of resources when they have a portfolio of innovation projects. Furthermore, we would like to examine whether the patent system improves the social welfare given the environment described in this article.

To be more precise, the model considered in this chapter is based on the two-armed Poisson bandit model with two players. We assume that there are two identical firms, each endowing with the replica of two new technologies, technology R and technology S. Technology S is known to be good, which means that it can deliver outputs after exponentially distributed random times. Technology R is riskier in the sense that it is good only with some probability and no output will be delivered if it is bad. At the same time, the outputs of R have higher value, and

in that way it is more innovative. Each firm has one unit of resources per unit of time and need to decide the proportion spent on R. Notice that using technology R reveals information and therefore it is also a learning process. We will hereafter follow the literature and use experimentation to refer to the behavior of using technology R.

Moreover, we assume that the patent system is present. The firm that first delivers outputs using certain technology, or equivalently, has a breakthrough in certain technology, is granted the patent for that technology. The other firm has to pay license fees that are exogenously determined at the market of patents to use that technology.

As a benchmark, the allocation of resources with no patent has been studied in Keller, Rady and Cripps (2005)¹. They showed that the unique symmetric Markov equilibrium can be summarized by two posteriors. Technology R is used exclusively for beliefs above the higher posterior. Part of resources are allocated on R when the belief decreases to posteriors in between. Finally, technology S is used exclusively when the belief drops to the lower posterior. The equilibrium allocation of resources is inefficient along two dimensions: the total amount of experimentation is insufficient and there is delay in experimentation.

Using the above results as benchmark, this chapter will focus on the effect of patents. Intuitively, when firms compete for patents, their allocation of resources is not only influenced by their belief about R being good, but also the patent status of two technologies. We will first show how patents influence allocation of resources, and then analyse the welfare change due to patents.

We will show in section 3.4 that with patents, the experimentation intensity along the equilibrium path is non-monotonic in the posterior in the absence of a breakthrough in R. At the beginning of the game when no patent has been

¹In Keller, Rady and Cripps(2005), technology S yields constant flow payoff. This difference, though, does not change any of the results.

claimed, using certain technology has additional benefit due to the patent system, which is the license fee of that technology if a breakthrough occurs. Firms will use technology R exclusively when the posterior is high and will start to mix when the posterior drops. Since technology S is good, one firm will have a breakthrough in S for sure after they start to mix and the patent for S will be claimed. Following the breakthrough in S, the additional benefit of using S no longer exist but the additional benefit of using R is still present (although the license fee for R might be different). As a result, the experimentation intensity jumps upwards and is higher than the intensity we would have without patents, and the total amount of experimentation is larger as well.

The equilibrium allocation of resources described above implies that patents have two opposing welfare effects. Patents on the one hand increases the total amount of experimentation closer to the socially efficient level. On the other hand, the patent system introduces competition for the patent of the less innovative technology. This competition induces firms to inefficiently allocate too much resources on S too early and causes delay in experimentation, as we can see from the jump of experimentation intensity at the moment the patent of S is claimed.

Since the magnitude of the above two opposing effects depend on the prior, the net welfare effect of patents depends on the prior as well. Furthermore, the net effect may even be negative when the prior falls in certain range. In section 3.5, we construct a numerical example where we are worse off with patents if the priors is close to 1.

This model can be applied to many industries and is related to discussions about whether and by how much patents can encourage innovation, which have always drawn people's attention. For instance, over the past years, there are many discussions in the media on the 'innovation crisis' that the pharmaceutical industry is facing. One evidence supporting the existence of such crisis is the fact that most new drugs are merely minor modifications of existing treatments (so

called ‘me-too’ drugs). Some blame the patent system. Peter Lansbury, Professor of Neurology in Harvard Medical School wrote in an article,

“... This system (the patent system) was created to reward the innovator, but today, it rewards the imitator. That’s because the requirements for patenting a particular drug have not significantly changed for 50 years. The science of drug discovery, though, has changed dramatically ... Drugs that mimic the mechanism of action of Viagra, Vioxx or Lipitor may be patented even though they are not innovative, do not serve an unmet medical need, and are often no more effective than the ‘trailblazers’ on which they are based...”(Peter Lansbury, **The Washington Post**. Nov 16, 2003. pg. B.02)

The relation between patents and innovation has been examined vastly. Patents encourage innovation by granting innovators ex-post monopoly power (Arrow (1962), Tirole (1988), Scotchmer (2004) etc). Patents at the same time create competition. In the literature of patent races, a central result is that R&D investment is usually excessive due to this competition (Loury (1979), Lee and Wilde(1980), Dasgupta and Stiglitz (1980), Reinganum (1981), Fudenberg et al. (1983), Grossman and Shapiro (1987), etc). There are also insights about how best to encourage innovation (Gilbert and Shapiro (1990), Klemperer (1990), Hopenhayn and Mitchell (2001), Hopenhayn, Llobet, and Mitchell (2006), etc). While this chapter is inspired by previous works in many aspects, the nuance of this article is that we focus on the situation where firms have a portfolio of projects and that we study how patents influence their allocation of resources among them.

This chapter is also related to strategic experimentation literature (Bolton and Harris (1999), Keller, Rady and Cripps (2005), etc). The two-armed bandit model has been applied to study innovation decisions in several recent papers. Acemoglu, Bimpikis and Ozdaglar (2011) demonstrates that appropriately designed patents can eliminate delay in experimentation which would occur when each firm has different risky project. Dosis et al. (2013) applies the model to study a patent

race game. They show that when a firm can only acquire a patent after the risky research phase and the development phase, the free-riding effect might dominate the tragedy-of-the-commons effect. The differences between these papers and ours are that firms in their models have essentially one project and that they mainly focus on the learning aspect of the multi-armed bandit model.

The rest of the chapter is organized as follows. We formally describe the model in section 2, and derive the socially optimal allocation of resource as well as the allocation of resources that would emerge in an equilibrium without patents in section 3. The unique symmetric Markov equilibrium is solved and analyzed in section 4. We construct two numerical examples in section 5 to show that we might be worse off with patents. We then demonstrate in section 6 that imposing more stringent requirements of patentability can improve the welfare. Section 7 concludes this chapter.

3.2 The Model

There are two homogeneous firms competing in an industry². Time is continuous $t \in [0, \infty)$ and the discount rate is $r > 0$. Each firm is endowed with replicas of two technologies, technology ‘R’ and technology ‘S’, and one unit of perfectly divisible resources per unit of time. A technology enables a firm to produce using its resources. Technology S is new but known to be good. It produces one unit of regular products after some exponentially distributed random times. Technology R, on the other hand, can be either ‘good’ or ‘bad’. If it is bad, it never produces independent of the amount of resources allocated to it. If it is good, it produces one unit of superior products after some exponentially distributed random times. At $t = 0$, it is common knowledge that project R is good with probability $p_0 \in (0, 1)$.

The arrivals of successful productions are independent across two firms. Over

²The model can be easily extended to N identical firms.

the time period $[t, t + dt]$, if a firm devotes resources $k_t \in [0, 1]$ on R, then its technology R produces with probability $k_t \lambda dt$ conditional on being good and its technology S produces with probability $(1 - k_t) \lambda dt$, where the constant $\lambda > 0$ is known to all players. We will call the first successful production using a certain technology among two firms a ‘breakthrough’ in that technology. We will also follow the literature and call k_t , the resources allocated on R, the ‘experimentation intensity’ throughout this chapter.

At any time t , consumers’ valuation of one unit of the regular products is $s < 1$ and their valuation of one unit of the superior products is 1. Moreover, we assume that there are more than two units of demand for each type of product at any time.

Finally, a firm’s allocation of resources and production outcomes are instantly observed by the other firm.

Patents

The firm that has a breakthrough in a certain technology is granted the patent for that technology. Without permission from the patent owner, the other firm is excluded from using any part of that technology. Here we assume that as an outsider of the industry, the authority (the patent office) does not know the non-obviousness of projects and will grant patents for all of them.

The owner of a patent can license it to the other firm, i.e., permit the other firm to use the patented parts of the technology in exchange for license fees. We assume that the license fee is a lump-sum transfer at the time when the patent is granted. Depending on the order of breakthroughs, there are three relevant license fees. It is possible that the first breakthrough is with technology R. We denote the licence fee of this patent by R_r . Since technology S is inferior, no patent will be granted afterwards. It is also possible that the first breakthrough is with technology S. We denote the license fee of technology S by R_s . If there

is a breakthrough in technology R followed, then the non-overlapping parts of technology R will be granted a patent, with the license fee R_a .

The main purpose of this chapter is to understand how rewards from patents influence firms' allocation of resources. To simplify the analysis, we will not model the market for patents explicitly. Rather, we take R_s , R_r and R_a as exogenously given, assuming that 1) R_s , R_r and R_a are positive; 2) $R_s + R_a - R_r \geq 0$ and 3) firms are willing to pay these fees.

3.3 The First-Best and Benchmark Case Without Patents

As a comparison, we first examine the following two cases. In the first case, two firms work cooperatively and we achieve the first best. In the second case, we consider the situation where there is no patent and two firms behave strategically.

From the assumptions on the demand side, we know that a firm will sell one unit of the regular product at the price s and the superior product at the price 1. The parallel case has been studied by Keller Rady and Cripps (2005) (hereafter KRC). We therefore skip detailed analysis and rewrite their corresponding propositions.

Proposition 3.1 (Proposition 3.1 in KRC). *When two firms work cooperatively, there is a cut-off belief p^* given by*

$$p^* = \frac{s}{s + (1 + 2\lambda/r)(1 - s)} \quad (3.1)$$

such that when $p < p^$ it is socially optimal for both firms to use technology S exclusively and when $p > p^*$ it is socially optimal for both to use technology R exclusively.*

To simplify the expression, we use $\Omega(p)$ to denote $\frac{1-p}{p}$.

Proposition 3.2 (Proposition 5.1 in KRC). *The experimentation game without patents has a unique symmetric Markov equilibrium with the common posterior*

belief as the state variable. In this equilibrium, there are two thresholds $p_b^l < p_b^h$ where p_b^l is given by,

$$p_b^l = \frac{s}{1 + \lambda(1 - s)/r} \quad (3.2)$$

and p_b^h solves the following equation,

$$1 + \frac{\lambda}{r} - s - \frac{2\lambda s}{r} + \frac{p_b^h \lambda}{r} - \frac{(1 - p_b^h)s}{p_b^l} + (1 - p_b^h)s \ln \frac{\Omega(p_b^h)}{\Omega(p_b^l)} = 0$$

Technology S is used exclusively when $p < p_b^l$ and technology R is used exclusively when $p > p_b^h$. For $p \in [p_b^l, p_b^h]$, the experimentation intensity is

$$k_b(p) = \frac{rZ(p)/\lambda - s}{s - p}$$

where for $p \in [p_b^l, p_b^h]$,

$$Z(p) = (1 - p)s \left(\ln \frac{\Omega(p)}{\Omega(p_b^l)} - \frac{1}{p_b^l} \right) + 1 + \frac{\lambda}{r} - s.$$

Comparing this equilibrium allocation of resources with the socially optimal experimentation intensity, we notice that both the total amount of experimentation and the experimentation intensity after p is lower than p_b^h is inefficient. To see the former, the posterior at which the firms abandon technology R is higher than what is socially optimal, i.e. $p_b^l > p^*$. This implies that the total amount of resources allocated on technology R is inefficient. To see the later, firms allocate only proportion of resources on technology R in equilibrium when their posterior drops below p_b^h , while using technology R exclusively at those posteriors is socially optimal. This procrastination in experimentation also causes welfare loss.

To intuitively understand why insufficient experimentation and procrastination emerge in equilibrium, notice that using technology R before a breakthrough generates information about R and therefore creates positive externality. Strategic firms will not internalize such externality and hence will under-provide experimentation in terms of both the total amount and the intensity at each instant of time.

It is natural to consider patent system as a candidate to alleviate this problem, for it enables a firm to benefit from the information it generates. To be more precise, once its experimentation leads to a breakthrough, the other firm cannot use the information for free and therefore the free-riding incentives are reduced. However, at the same time the patent system introduces competition for the inferior technology, which will distract firms from experimenting on R. As shown in later sections, the resulting efficiency loss might even be larger the efficiency gain in some circumstances and we are better off without the patent system.

In the rest of the chapter, we will often compare the equilibrium outcomes with and without patents. We will therefore refer the second case examined in this section, i.e. the situation where firms act strategically without patents, as the benchmark case.

3.4 Symmetric Markov Equilibrium with Patents

We now turn to the allocation of resources that arises when firms behave strategically under patent system. We will focus on symmetric Markov Equilibrium. Compared to the benchmark case, we now need an additional state variable to indicate whether or not there is a breakthrough in S, since the allocation of resources in principle could depend on it. Therefore, we have two state variables $(p, \mathbb{1})$, where the indicator function $\mathbb{1}$ equals 1 if there is a breakthrough in technology S.

All combinations of state variables can be classified into the following three categories.

H-r: the set of state variables with $p = 1$, i.e., the set of histories following a breakthrough in R.

H-s: the set of state variables with $p < 1$ and $\mathbb{1} = 1$, i.e., the set of histories following a breakthrough in S and no breakthrough in R.

H-n: the set of state variables with $p < 1$ and $\mathbb{1} = 0$, i.e., the set of histories following no breakthrough in either technologies.

To abbreviate notations, although there are two state variables, we will use $W(p)$ to denote the value function with state variables in H-s and use $U(p)$ to denote the value function with state variables in H-n.

3.4.1 Equilibrium Allocation of Resources with H-r

Since when technology R is good it is more efficient than S, both firms will use R exclusively after a breakthrough in R. The discounted present value for each firm equals $\frac{\lambda}{r}$.

3.4.2 Equilibrium Allocation of Resources with H-s

Let us now consider the situation where the combination of state variables is an element in H-s. If firm i has a breakthrough in R, the other firm will transfer R_a to i and the combination of state variables jumps to H-r. Otherwise, state variables remain in H-s with a lower posterior. Denote the combined experimentation intensity by the two firms at time t by K_t . The change of posteriors between time t and time $t + dt$ is

$$p_{t+dt} - p_t = dp_t = -\lambda K_t p_t (1 - p_t) dt \quad (3.3)$$

A firm chooses $k_s(p) \in [0, 1]$ to maximize its payoff $W(p)$,

$$\begin{aligned} W(p) = \max_{k \in [0,1]} \{ & \lambda k p dt (1 + R_a) + \lambda \hat{k}_s(p) p dt (-R_a) + \lambda (k + \hat{k}_s(p)) p dt (1 - r dt) \frac{\lambda}{r} \\ & + \lambda (1 - k) dt s + \lambda (2 - k - \hat{k}_s(p)) dt (1 - r dt) W(p + dp) \\ & + (1 - \lambda (k + \hat{k}_s(p)) p dt - \lambda (2 - k - \hat{k}_s(p)) dt) (1 - r dt) W(p + dp) \} \end{aligned}$$

Here $\hat{k}_s(p)$ denotes the other firm's allocation of resources when the common posterior is p . Apply first order expansion to $W(p + dp)$, plug in dp from (3.3)

and neglect all second order terms, we can rearrange the above equation as,

$$\begin{aligned} \frac{r}{\lambda}W(p) = \max_{k \in [0,1]} k \{ & b_s(p, W) - c_s(p, W) \} \\ & + \hat{k}_s(p) [p(-R_a + \frac{\lambda}{r} - W(p)) - p(1-p)W'(p)] + s \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} b_s(p, W) &= p[1 + R_a + \frac{\lambda}{r} - W(p)], \\ c_s(p, W) &= s + p(1-p)W'(p). \end{aligned}$$

$\lambda b_s(p, W)$ is the expected benefit of using technology R. A firm could benefit from using R only if it leads to a breakthrough, which happens with probability λp and increases the total payoff by $1 + R_a + \frac{\lambda}{r} - W(p)$. $\lambda c_s(p, W)$ is the opportunity cost of using technology R. It consists of two parts. The first part λs is the foregone expected lump-sum payoffs from S. The second part $\lambda p(1-p)W'(p)$ is the reduction of the value function, because no good news reduces the posterior.

It is easy to see that the Bellman equation (3.4) is linear in k . Therefore, the best response function is

$$k_s(p) \begin{cases} = 0, & \text{if } c_s(p, W) > b_s(p, W), \\ \in [0, 1], & \text{if } c_s(p, W) = b_s(p, W), \\ = 1, & \text{if } c_s(p, W) < b_s(p, W). \end{cases} \quad (3.5)$$

Combining the Bellman equation and the above best response function (3.5), we can use the following alternative way to represent the best response function,

$$k_s(p) \begin{cases} = 0, & \text{if } \frac{r}{\lambda}W(p) < s + \hat{k}_s(p) [s - p(1 + 2R_a)], \\ \in [0, 1], & \text{if } \frac{r}{\lambda}W(p) = s + \hat{k}_s(p) [s - p(1 + 2R_a)], \\ = 1, & \text{if } \frac{r}{\lambda}W(p) > s + \hat{k}_s(p) [s - p(1 + 2R_a)]. \end{cases}$$

Therefore, the equilibrium allocation of resources depends on whether $\frac{r}{\lambda}W(p)$ is larger than, equal to or smaller than $s + \hat{k}_s(p) [s - p(1 + 2R_a)]$ for a given

$\hat{k}_s(p)$. Meanwhile, $W(p)$ must satisfy the following O.D.E. after plugging in $k_s(p) = \hat{k}_s(p)$,

$$\text{if } k_s(p) = 1, \quad W'(p) + \frac{2p + r/\lambda}{2p(1-p)}W(p) = \frac{2\lambda + r}{2(1-p)r} \quad (3.6)$$

$$\text{if } k_s(p) \in [0, 1], \quad W'(p) + \frac{1}{1-p}W(p) = \frac{1 + R_a + r/\lambda}{1-p} - \frac{s}{p(1-p)} \quad (3.7)$$

$$\text{if } k_s(p) = 0, \quad W(p) = \frac{\lambda}{r}s \quad (3.8)$$

We can solve the symmetric equilibrium allocation of resources when state variables are in H-s, which is summarized in the following proposition.

Proposition 3.3. *The symmetric equilibrium allocation of resources when state variables are in H-s is unique. Technology S is used exclusively when $p < p_s^l$ and technology R is used exclusively when $p > p_s^h$. p_s^l equals*

$$p_s^l = \frac{s}{1 + \frac{\lambda(1-s)}{r} + R_a} \quad (3.9)$$

and p_s^h is uniquely defined by

$$\frac{r}{\lambda}W(p_s^h) = 2s - p_s^h(1 + 2R_a) \quad (3.10)$$

where for $p \in [p_s^l, p_s^h]$,

$$W(p) = -(1-p)s \left(\ln \frac{\Omega(p_s^l)}{\Omega(p)} + \frac{1}{p_s^l} \right) + 1 + \frac{\lambda}{r} + R_a - s \quad (3.11)$$

For $p \in [p_s^l, p_s^h]$, the fraction of resources allocated to technology R is

$$k_s(p) = \frac{rW(p)/\lambda - s}{s - p - 2pR_a}$$

Proof. For a given \hat{k} , we can define a line

$$D_{\hat{k}} = \left\{ (W, p) \in \mathbb{R}_+ \times (0, 1) : W = \frac{\lambda}{r} \left[s + \hat{k}(s - p(1 + 2R_a)) \right] \right\}$$

For $\hat{k} = 0$, the line is horizontal with value $\frac{rs}{\lambda}$. As \hat{k} increases, the line rotates clock-wise around the point $(W, p) = \left(\frac{rs}{\lambda}, \frac{s}{1+2R_a} \right)$ until $\hat{k} = 1$. Therefore, in a

symmetric equilibrium there must be two posteriors p_s^h and p_s^l such that 1) p_s^h is the largest posterior where $W(p)$ intersects the line $W = \frac{r}{\lambda} [s + s - p(1 + 2R_a)]$ and 2) p_s^l is the smallest posterior where $W(p)$ equals $\frac{rs}{\lambda}$.

We can solve O.D.E. (3.7) with p_s^l , p_s^h and C_s^m to be determined,

$$\text{for } p \in [p_s^l, p_s^h], W(p) = (1 - p)C_s^m + 1 + \frac{\lambda}{r} + R_a - s + (1 - p)s \ln \Omega(p)$$

p_s^l and C_s^m are then determined using smooth-pasting condition and value-matching condition at p_s^l , i.e., $W'(p_s^l) = 0$ and $W(p_s^l) = \frac{\lambda s}{r}$.

For $p \in [p_s^l, p_s^h]$, the experimentation intensity $k_s(p)$ is solved by plugging in $\hat{k} = k_s(p)$ into $D_{\hat{k}}$. Finally, we solve p_s^h from the condition that $k_s(p_s^h) = 1$.

□

If we compare the benefit of using R here and in the benchmark case, a firm expects to get the additional payoff $\lambda p R_a$ when patents could be granted. This incentivizes the firm to devote more resources on technology R given any posterior. This observation suggests that the firm will start to use mixed resource allocation later and will abandon technology R only if they are more pessimistic about it. This intuition is verified in the following proposition.

Proposition 3.4. *Assume $R_a \in (0, \frac{\lambda(1-s)}{r})$, the symmetric equilibrium allocation of resources when state variables are in H-s is more efficient compared to the benchmark case:*

1. $p^* < p_s^l < p_b^l$ and $p_s^h < p_b^h$,
2. $k^*(p) \geq k_s(p) \geq k_b(p)$ for any $p \in (0, 1)$ and $k_s(p) > k_b(p)$ for any $p \in (p_s^l, p_b^h)$.

Proof. Please refer to Appendix 3.8.1.

□

As shown in KRC, for a given prior, the total amount of experimentation is completely determined by the posterior at which the firms stop using R (Lemma 3.1 in KRC). In the next section, we will formally demonstrate that when we start with state variables in H-n, the set of state variables will eventually become an element in H-s if there is no breakthrough in R. Combining these two facts, we can conclude that in the absence of a breakthrough in R, the total amount of experimentation is closer to the socially optimal amount with patents.

3.4.3 Equilibrium Allocation of Resources with H-n

We now turn to symmetric equilibrium allocation of resources when the state variables are in H-n, i.e. neither of the two technologies has a breakthrough. If a firm has a breakthrough with technology R (or S), it will receive monetary transfer R_r (or R_s) from the other firm and the state variables become an element in H-r (or H-s). Otherwise, the state variables are still in H-n with a lower posterior.

A firm chooses $k_n(p) \in [0, 1]$ to maximize its payoff $U(p)$,

$$\begin{aligned}
U(p) = \max_{k \in [0,1]} \{ & \lambda k p dt (1 + R_r) + \lambda \hat{k}_n(p) p dt (-R_r) + \lambda (k + \hat{k}_n(p)) p dt (1 - r dt) \frac{\lambda}{r} \\
& + \lambda (1 - k) (s + R_s) dt + \lambda (1 - \hat{k}_n(p)) (-R_s) dt \\
& + \lambda (2 - k - \hat{k}_n(p)) dt (1 - r dt) W(p + dp) \\
& + (1 - \lambda (k + \hat{k}_n(p)) p dt - \lambda (2 - k - \hat{k}_n(p)) dt) (1 - r dt) U(p + dp) \}
\end{aligned}$$

It can be rearranged as

$$\begin{aligned}
\frac{2\lambda + r}{\lambda} U(p) = \max_{k \in [0,1]} k \{ & b_n(p, U) - c_n(p, U, W) \} + \hat{k}_n(p) [p(-R_r + \frac{\lambda}{r} - U(p)) \\
& + R_s - W(p) + U(p) - p(1 - p)U'(p)] + s + W(p) \quad (3.12)
\end{aligned}$$

where

$$b_n(p, U) = p[1 + R_r + \frac{\lambda}{r} - U(p)],$$

$$c_n(p, U, W) = [s + R_s + W(p) - U(p)] + p(1 - p)W'(p).$$

Like before, $\lambda b_n(p, U)$ and $\lambda c_n(p, U, W)$ are the benefit and the opportunity cost of using technology R, respectively. A firm gets positive payoffs from R only if there is a breakthrough, which happens with probability λp and increases the overall payoff by $1 + R_r + \frac{\lambda}{r} - U(p)$. The opportunity cost of R consists of two parts, the forgone expected incremental $\lambda[s + R_s + W(p) - U(p)]$ and the reduction in payoff $\lambda p(1 - p)W'(p)$ due to lower posterior.

The best response function $k_n(p)$ is similar to (3.5), with threshold $c_n(p, U, W) = b_n(p, U)$. Combining the Bellman equation, we can write the best response function in the following way,

$$k_n(p) \begin{cases} = 0, & \text{if } \frac{2\lambda+r}{\lambda}U(p) < s + W(p) + \hat{k}_n(p) [s + 2R_s - p(1 + 2R_r)], \\ \in [0, 1], & \text{if } \frac{2\lambda+r}{\lambda}U(p) = s + W(p) + \hat{k}_n(p) [s + 2R_s - p(1 + 2R_r)], \\ = 1, & \text{if } \frac{2\lambda+r}{\lambda}U(p) > s + W(p) + \hat{k}_n(p) [s + 2R_s - p(1 + 2R_r)]. \end{cases} \quad (3.13)$$

In symmetric equilibrium, there are again two thresholds p_n^l and p_n^h such that $c_n(p, U, W) > b_n(p, W)$ for any $p < p_n^l$, $c_n(p, U, W) < b_n(p, W)$ for any $p > p_n^h$ and $c_n(p, U, W) = b_n(p, W)$ for any $p \in [p_n^l, p_n^h]$. This claim will be rigorously proven when we show proposition 3.7. Here we will proceed the analysis given this result.

Plugging the symmetric allocation $\hat{k}_n(p) = k_n(p)$ into the Bellman equation (3.12), $k_n(p)$ for $p \in (p_n^l, p_n^h)$ is solved as

$$k_n(p) = \frac{(2 + \frac{r}{\lambda})U(p) - s - 2W(p)}{-p(1 + 2R_r) + s + 2R_s} \quad (3.14)$$

For $p \in (0, p_n^l]$, both firms will switch³ to technology S in symmetric equilibrium. This implies that for $p \in (0, p_n^l]$,

$$U(p) = \frac{\lambda}{2\lambda + r} [s + 2W(p)] \quad (3.15)$$

³Here we use the word ‘switch’ instead of ‘abandon’. This is because, as will be shown later, firms will return to technology R after a breakthrough in S.

For other posteriors, the value function $U(p)$ must satisfy a set of differentiation equations in symmetric equilibrium. For $p \in (p_n^l, p_n^h)$, $b_n(p, w) = c_n(p, U, W)$, which is equivalent to

$$U'(p) - \frac{U(p)}{p} = \frac{1 + R_r + \frac{\lambda}{r}}{1 - p} - \frac{s + R_s + W(p)}{p(1 - p)} \quad (3.16)$$

For $p \in [p_n^h, 1)$, $U(p)$ satisfies the following O.D.E.

$$U'(p) + \frac{2p + \frac{r}{\lambda}}{2p(1 - p)}U(p) = p\left(1 + \frac{2\lambda}{r}\right) \quad (3.17)$$

With a constant C_n^r yet to be determined, the solution to (3.17) is

$$U(p) = \frac{\lambda}{r}p + C_n^r(1 - p)\Omega(p)^{\frac{r}{2\lambda}} \quad (3.18)$$

In the rest of the section, we will solve the symmetric equilibrium allocation of resources when state variables are in H-n. We can first show that the firms switch to S when they are more optimistic comparing to the threshold with H-s, i.e., $p_n^l \geq p_s^l$. Intuitively, this is because now there are also rewards for a breakthrough in S. The competition for the patent of S incentivizes the firms to delay the experimentation on R.

After switching to S, the posterior p stays at p_n^l because there is no further information generated on R. Moreover, since the arrival rate $\lambda > 0$, a breakthrough in S occurs with probability 1. This implies that the set of state variables will become an element in H-s for sure. Combining with the fact that $p_n^l \geq p_s^l$, the firms will return to R at least partially after a breakthrough in S.

Proposition 3.5. *The largest posterior below which a firm uses S exclusively is higher with H-n compared to that with H-s, i.e. $p_n^l \geq p_s^l$. In the absence of a breakthrough in R before p drops to p_n^l , the firms will resume using R after a breakthrough in S.*

Proof. Suppose $p_n^l < p_s^l$. Then there exist $\delta > 0$, such that for any $p \in (p_n^l, p_n^l + \delta)$, $W(p) = \frac{\lambda s}{r}$ and $k(p) \in (0, 1)$.

For these p , the value function satisfies the O.D.E (3.16). After plugging in $W(p)$, $U(p)$ can be solved as

$$U(p) = pC + s\left(1 + \frac{\lambda}{r}\right) + R_s - p \ln \Omega(p) \left[\left(1 + \frac{\lambda}{r}\right)(1-s) + R_r - R_s \right]$$

The constant C and p_n^l are then pinned down from value-matching condition and smooth-pasting condition:

$$\text{(value-matching)} \quad U(p_n^l) = \frac{\lambda s}{2\lambda + r} \left(1 + \frac{2\lambda}{r}\right)$$

$$\text{(smooth-pasting)} \quad U'(p_n^l) = 0$$

p_n^l therefore takes the following solution,

$$p_n^l = \frac{s + R_s}{1 + \frac{\lambda}{r}(1-s) + R_r}$$

We already know $p_s^l = \frac{s}{1 + \frac{\lambda}{r}(1-s) + R_a}$. $p_s^l > p_n^l$ is then equivalent to

$$s(R_a + R_s - R_r) + R_s \left[\left(1 + \frac{\lambda}{r}\right)(1-s) + R_a \right] < 0$$

The above condition is never satisfied under our assumptions about licence fees. We have reached a contradiction. Therefore, $p_n^l \geq p_s^l$. \square

Depending on the parameters, p_n^l could be smaller or larger than p_s^h . In particular, define a function of parameters T ,

$$T = s \ln \frac{\Omega(p_s^h)}{\Omega(p_s^l)} + \frac{2\lambda + r}{r} \left(R_s + \frac{R_a + R_s - R_r}{\Omega(p_s^h)} \right)$$

When $T \geq 0$, $p_n^l \geq p_s^h$. Otherwise, $p_n^l < p_s^h$. We will solve each case as follows.

Case 1: $p_n^l \geq p_s^h$.

We first focus on the situation where $T \geq 0$. From the last section, we can solve the value function $W(p)$ for $p \in [p_n^l, 1)$

$$W(p) = \frac{\lambda}{r} p + C_s^r (1-p) \Omega(p)^{\frac{r}{2\lambda}} \tag{3.19}$$

$$\text{where, } C_s^r = \frac{2\lambda}{r} \frac{s - p_s^h - R_a p_s^h}{(1 - p_s^h) \Omega(p_s^h)^{\frac{r}{2\lambda}}}$$

Plug $W(p)$ into O.D.E. (3.16), $U(p)$ ($p \in [p_n^l, p_n^h]$) can be solved as,

$$U(p) = pC_n^m + s + R_s - p(1 - s + R_r - R_s) \ln \Omega(p) + \frac{2\lambda}{r + 2\lambda} C_s^r (1 - p) \Omega(p)^{\frac{r}{2\lambda}}$$

p_n^l and C_n^m are then uniquely determined from the value-matching condition and smooth-pasting condition at p_n^l .

$$p_n^l = \frac{R_s + \frac{\lambda+r}{2\lambda+r}s}{R_r + 1 - \frac{\lambda}{2\lambda+r}s} \quad (3.20)$$

$$C_n^m = (1 - s + R_r - R_s) \ln \Omega(p_n^l) - \frac{1 - s + R_r - R_s}{1 - p_n^l} + \frac{2\lambda^2}{r(2\lambda + r)} \quad (3.21)$$

We have derived $k_n(p)$ for $p \in [p_n^l, p_n^h]$ in (3.14). Using the fact that $k_n(p_n^h) = 1$, p_n^h is implicitly determined by

$$(2 + \frac{r}{\lambda})U(p_n^h) - 2W(p_n^h) = -p_n^h(1 + 2R_r) + 2s + 2R_s \quad (3.22)$$

Finally, when $p \geq p_n^h$, we know that $U(p)$ takes the form (3.18). The constant C_n^r is determined by value-matching condition at p_n^h :

$$\frac{\lambda}{r}p_n^h + C_n^r(1 - p_n^h)\Omega(p_n^h)^{\frac{r}{2\lambda}} = \frac{\lambda}{2\lambda + r}[2W(p_n^h) + 2s - p_n^h - 2p_n^h R_r + 2R_s]$$

Case 2: $p_n^l < p_s^h$.

This is the case when $T < 0$. For p close to p_n^l , $W(p)$ is represented by (3.11). Plug it into O.D.E. (3.16), $U(p)$ for $p \in [p_n^l, p_s^h]$ can be solved,

$$U(p) = pC_n^l + 1 + R_a + R_s - s - s \ln \Omega(p_s^l) - \frac{s}{p_s^l} + \frac{\lambda}{r} + [s(1 - p) + p(R_a + R_s - R_r)] \ln \Omega(p) \quad (3.23)$$

Combining the value-matching condition and the smooth-pasting condition at p_n^l ,

$$C_n^l = \frac{2\lambda s}{2\lambda + r} [\ln \Omega(p_s^l) + \frac{1}{p_s^l}] + \frac{rs}{2\lambda + r} [\ln \Omega(p_n^l) + \frac{1}{p_n^l}] - (R_a + R_s - R_r) [\ln \Omega(p_n^l) - \frac{1}{1 - p_n^l}]$$

and p_n^l is uniquely determined from

$$s \ln \frac{\Omega(p_n^l)}{\Omega(p_s^l)} + \frac{2\lambda + r}{r} \left(R_s + \frac{R_a + R_s - R_r}{\Omega(p_n^l)} \right) = 0 \quad (3.24)$$

The next step is to solve p_n^h . We can first show that the firms will start to allocate resources to technology S when they are more optimistic if the patent of S has not been granted, i.e. $p_n^h > p_s^h$. Notice that this claim holds trivially in case 1, since $p_n^h > p_n^l \geq p_s^h$.

Proposition 3.6. *The smallest posterior above which a firm uses R exclusively is higher with H-n compared to that with H-s, i.e. $p_n^h > p_s^h$.*

Proof. Please refer to appendix 3.8.2. □

Because $p_n^h > p_s^h$, for $p \in [p_s^h, p_n^h]$ $W(p)$ is represented by (3.19). Plug it into O.D.E. (3.16), $U(p)$ is solved as,

$$U(p) = pC_n^h + s + R_s - p(1 - s + R_r - R_s) \ln \Omega(p) + pC_s^r \frac{2\lambda}{2\lambda + r} \Omega(p)^{\frac{r+2\lambda}{2\lambda}}$$

The constant C_n^h is pinned down from smooth-pasting condition at p_s^h ,

$$\begin{aligned} C_n^h = & (1 - s + R_r - R_s) \left(\ln \Omega(p_s^h) - \frac{1}{1 - p_s^h} \right) - C_s^r \Omega(p_s^h)^{\frac{r}{2\lambda}} \left(\frac{2\lambda}{2\lambda + r} \Omega(p_s^h) - \frac{1}{p_s^h} \right) \\ & + C_n^l - s \left(\ln \Omega(p_s^h) + \frac{1}{p_s^h} \right) \end{aligned} \quad (3.25)$$

We know $k_n(p_n^h) = 1$. Use this condition, p_n^h is defined explicitly from the following equation,

$$\begin{aligned} \left(\frac{2\lambda + r}{\lambda} C_n^h - \frac{2\lambda}{r} + 1 + 2R_r \right) p_n^h - \frac{2\lambda + r}{\lambda} (1 - s + R_r - R_s) p_n^h \ln \Omega(p_n^h) \\ + \frac{r}{\lambda} (s + R_s) = 0 \end{aligned} \quad (3.26)$$

The symmetric equilibrium allocation of resources with H-n is summarized in the following proposition.

Proposition 3.7. *The symmetric allocation of resources that emerges in Markov equilibrium with H-n is unique. Technology S is used exclusively when $p \leq p_n^l$ and technology R is used exclusively when $p \geq p_n^h$. For $p \in [p_n^l, p_n^h]$, the experimentation intensity $k_n(p)$ is expressed in (3.14) with the corresponding $U(p)$ and $W(p)$.*

1. *When $T \geq 0$, $p_n^l \geq p_s^h$. p_n^l is expressed in (3.20) and p_n^h is implicitly determined by (3.22).*
2. *When $T < 0$, $p_n^l < p_s^h$. p_n^l is implicitly defined in (3.24) and p_n^h is implicitly determined by (3.26).*

Proof. The above discussions have already proved this proposition except for the claim that there exist two thresholds p_n^l and p_n^h . Let us define a curve $\hat{D}_{\hat{k}}$ for a given \hat{k} based on the best response function (3.13),

$$\hat{D}_{\hat{k}} = \left\{ (U, p) \in \mathbb{R}_+ \times (0, 1) : U = \frac{\lambda}{2\lambda + r} \left(s + W(p) + \hat{k} [s + 2R_s - p(1 + 2R_r)] \right) \right\}$$

If $\hat{k} = 0$, this curve coincides with the curve $\frac{\lambda}{2\lambda + r} (s + W(p))$. As \hat{k} increases, The curve rotates clock-wise around point $(U, p) = \left(\frac{\lambda}{2\lambda + r} (s + W(\frac{s+2R_s}{1+2R_r})), \frac{s+2R_s}{1+2R_r} \right)$ until $\hat{k} = 1$. Therefore, in a symmetric equilibrium there must be two posteriors p_n^h and p_n^l such that 1) p_n^h is the largest posterior where $U(p)$ intersects the line $U = \frac{\lambda}{2\lambda + r} (s + W(p) + s + 2R_s - p(1 + 2R_r))$ and 2) p_n^l is the smallest posterior where $U(p)$ equals $\frac{\lambda}{2\lambda + r} (s + W(p))$. \square

We have shown in proposition 3.5 and 3.6 that because of the competition for the patent of S, the firms start to use S and completely switch to S with more optimistic posteriors, comparing to the situation when state variables are in H-s. In fact, not only the thresholds are less efficient, the experimentation intensity is lower given any posterior as the following proposition shows.

Proposition 3.8. *$k_n(p) \leq k_s(p)$ for any p and $k_n(p) < k_s(p)$ for any $p \in (p_s^l, p_n^h)$.*

Proof. Please refer to Appendix 3.8.3. \square

3.4.4 Equilibrium Allocation with No Breakthrough in R

Let us summarize this section by illustrating the equilibrium allocation of resources, assuming that there is no breakthrough in R along the way. If there is a breakthrough in R, we know that both firms will devote all resources on R afterwards.

Suppose we start with a prior $p_0 > p_n^h$. The Firms use R exclusively before the posterior drops to p_n^h . After that, the firms start to allocate part of their resources on S, $[1 - k_n(p)]$ to be precise. The proportion of resources allocated on S increases as the posterior decreases and reaches 1 when the posterior drops to p_n^l . In the meantime, one firm must have a breakthrough in S at some posterior $p \in [p_n^l, p_n^h]$. At the moment of the breakthrough in S, the experimentation intensity jumps upwards from $k_n(p)$ to $k_s(p)$. Finally, the experimentation intensity decreases to 0 as the posterior drops to p_s^l , and the firms use S exclusively from then on.

It is worth noticing that in equilibrium, the relationship between experimentation intensity and time (or equivalently, posterior) is non-monotonic. It is piecewise non-increasing but has an upward jump at the time (the posterior) of the breakthrough in S.

3.5 Welfare Effect of the Patent System: an Example

This section will discuss the equilibrium welfare outcomes with the patent system and compare it to the ones without patents. From the analysis in the above section, we know that the patent system has two opposing effects on social welfare.

On the one hand, the patent system could improve the social welfare because the total amount of experimentation is closer to the socially optimal amount. This can be seen from the fact that $p_s^l < p_b^l$.

On the other hand, the competition for the patent of S also leads to delay

in experimentation, which causes efficiency loss. In equilibrium, $p_n^h > p_s^h$ and $p_n^l > p_s^l$.

Summing up these two opposing effects, the overall welfare effect of the patent system might even be negative. Moreover, the magnitudes of these two effects depend on the prior p_0 . For $p_0 \in [p_s^l, p_b^l]$, the second effect is absent because firms stop using R in the benchmark case as well. For larger posteriors, the negative effect arises and it causes the largest welfare loss if we start with the prior $p_0 = p_n^h$. When the prior increases to 1, the difference between the welfare with and without the patent system decreases to 0. Intuitively, when technology R are almost certain to be good, it is very likely that a breakthrough in technology R will happen immediately. Therefore neither the total amount of experimentation nor the delay of experimentation matters much.

To illustrate the fact that we might be better-off without patents and the way the welfare effect of the patent system depends on priors, we construct the following example.

Example 3.1. *We make the following assumptions on parameters:*

1. *preference:* $r = 0.1$;
2. *technology:* $s = 0.6$, $\lambda = 0.2$;
3. *license fees:* $R_s = 0.2 \times \frac{\lambda s}{r} = 0.24$, $R_a = 0.2 \times \frac{\lambda(1-s)}{r} = 0.16$, $R_r = R_s + R_a = 0.4$.

In figure 3.1, we plot allocation of resources as well as the difference between welfare with and without patents.

In this plot, we can see how equilibrium allocation of resources evolves over time in the absence of a breakthrough in R. Starting with prior $p_0 > p_n^l$, the experimentation intensity moves along $k_n(p)$ curve. Eventually there will be a

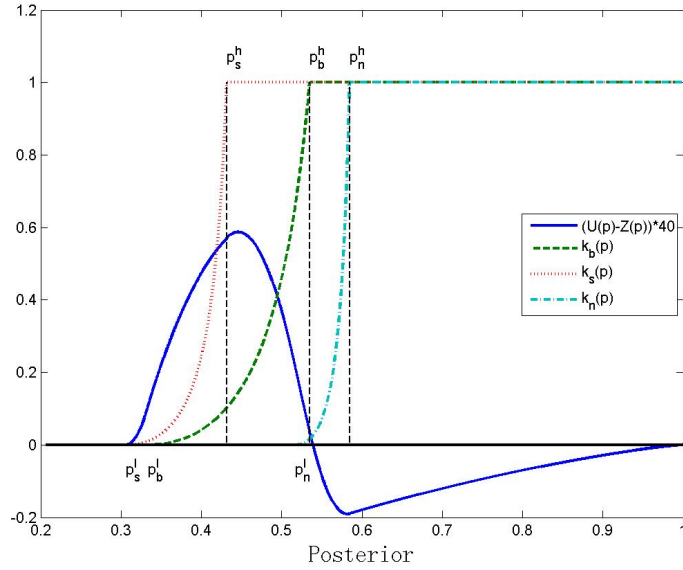


Figure 3.1: Welfare Difference and Resource Allocations in Example 3.1

breakthrough in S and the experimentation intensity will then jump upwards to the curve $k_s(p)$. Moreover, comparing to the benchmark case, the total amount of experimentation is larger. However, the allocation of resources with patents also features new form of delay. In particular, the firms will spend part of or even all of their resources on S when the posterior is between p_n^l and p_n^h .

Because of the additional delay in experimentation, in this example the welfare is lower with patents if the prior is near p_n^h , the belief at which the additional delay matters the most.

Of course, there are situations where the negative effect never outweigh the positive effect given any prior and hence we are strictly better-off with patents. In the next example, we increase R_a and decrease R_s by the same amount. As shown in figure 3.2, the equilibrium payoffs with patents are always larger.

Example 3.2. *All parameter assumptions are the same as in example 3.1 except for R_s and R_a . In this example, we assume that $R_s = 0.184$ and $R_a = 0.216$.*

From the above discussion, we can conclude that the welfare effect of patents

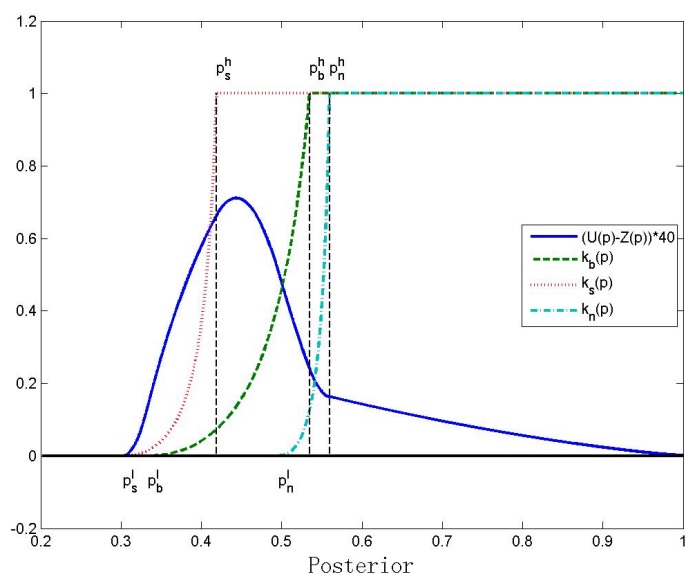


Figure 3.2: Welfare Difference and Resource Allocations in Example 3.2

depend on where the prior is. The patent system is more effective if firms believe that their risky technology is unlikely to be good to begin with. Without the rewards provided by the patent of R in the presence of a breakthrough, firms would experiment with lower intensities or completely abandon technology R when it is still worth trying socially. On the other hand, the patent system is less effective or might even cause net welfare loss when the prior is large. If we take into account costs of maintaining the patent system and enforcing patents, the patent system is less desirable when the prior is large.

3.6 More Stringent Requirements of Patentability

As illustrated in previous sections, the inefficiency of the patent system results from the fact that the patent of technology S distracts the firms from experimenting technology R. A natural candidate to alleviate this problem is to impose more stringent requirements of patentability, which reduces the rewards for the patent

of technology S.

In this section, we examine this intuition using a comparative static analysis. In particular, we will focus on how equilibrium outcomes alter as we change R_s and R_r by the same amount. These two license fees move by the same amount if, for example, the authority changes the probability of identifying a less innovative technology (technology S in this case) when an accused infringer challenges the validity of the patent.⁴ R_s would decrease if such probability increases. R_r would be lower as well, since the court will more likely decide that the value added from using R is not as high.

Denote the increment in R_s and R_r by δ . A negative δ is then equivalent to a decrease in R_s and R_r .

Proposition 3.9. $\frac{\partial k_n(p)}{\partial \delta} \Big|_{\delta=0} \geq 0$ for any p .

Proof. Please refer to Appendix 3.8.4. □

This proposition demonstrates that if the authority strengthens the requirements of patentability in a way that lowers both R_s and R_r , then the procrastination caused by patents can be reduced. Notice that $k_s(p)$ is independent of R_s and R_r . Hence, the proposition also implies that the overall equilibrium experimentation intensity is closer to the first best resource allocation. The society will therefore be strictly better off.

3.7 Conclusion

The patent system is generally considered as a mechanism to encourage innovation. Based on a two-player two-armed Poisson bandit model, we studied how patents influence firms' allocation of resources across two projects: technology R

⁴Typically in a lawsuits of infringement, it is the right of the accused infringer to challenge the validity of the patent. A patent can be found invalid based on information collected and requirements of patentability.

is more innovative and technology S is safer. In an economy without patents, firm's allocation of resources is inefficient along two dimensions, i.e., the total amount of experimentation in the absence of a breakthrough is insufficient and the experimentation intensity over time features procrastination.

In the unique symmetric Markov equilibrium, we showed that patents on the one hand increase the total amount of experimentation in the absence of a breakthrough in R, but on the other hand introduce new form of procrastination. To be more precise, firms will not abandon technology R till they are more pessimistic and hence the total amount of resources allocated on R is closer to the socially optimal amount. However, patents also introduce competition for the patent of S and it tempts the firms to allocate resources on S too early and too much.

As a result, the sign of the net welfare effect of patents depends on the prior the game starts with. For median priors, the negative effect resulting from the additional delay is insignificant and hence the social welfare is strictly higher with patents. For large priors, the welfare loss from delay in experimentation might dominate the welfare gain from larger total amount of experimentation. Therefore, the net welfare gain for large priors is either very small or negative. These results imply that the patent system is more effective when the prior is in median range.

Finally, we showed that the equilibrium outcomes with patents could be improved if the authority raises the probability of identifying a technology that is known to be good.

3.8 Appendices

3.8.1 Proof of Proposition 3.4

Step 1: Compare p_b^l as expressed in (3.2) and p_s^l as expressed in (3.9), it is easy to verify that as long as $R_a > 0$, $p_s^l < p_b^l$. In addition, if we compare p_s^l and p^* as

expressed in (3.1), $p_s^l > p^*$ from the assumption that $R_a < \frac{\lambda(1-s)}{r}$.

Next we show $p_s^h < p_b^h$. Define the following two functions,

$$\begin{aligned} F_b^h(p) &= -s\left(1 + \frac{\lambda}{r}\right) + p\left(1 + \frac{2\lambda}{r} - \frac{\lambda s}{r}\right) + (1-p)s \ln \frac{\Omega(p)}{\Omega(p_b^l)} \\ F_s^h(p) &= -s\left(1 + \frac{\lambda}{r}\right) + p\left(1 + \frac{2\lambda}{r} - \frac{\lambda s}{r} + \frac{2\lambda+r}{r}R_a\right) + (1-p)s \ln \frac{\Omega(p)}{\Omega(p_s^l)} \end{aligned}$$

p_b^h and p_s^h are implicitly determined by $F_b^h(p_b^h) = 0$ and $F_s^h(p_s^h) = 0$, respectively. We have following observations about $F_b^h(p)$ and $F_s^h(p)$.

$$\begin{aligned} F_b^h(p) - F_s^h(p) &= -p \left[\frac{2\lambda+r}{r}R_a + \Omega(p)s \ln \frac{\Omega(p_b^l)}{\Omega(p_s^l)} \right] \tag{3.27} \\ (F_b^h)'(p) &= s \left[\frac{1}{p_b^l} - \frac{1}{p} + \ln \frac{\Omega(p_b^l)}{\Omega(p)} \right] > 0, \text{ for } p > p_b^l \\ (F_s^h)'(p) &= s \left[\frac{1}{p_s^l} - \frac{1}{p} + \ln \frac{\Omega(p_s^l)}{\Omega(p)} \right] + \frac{2\lambda}{r}R_a > 0, \text{ for } p > p_s^l \\ (F_b^h)'(p) - (F_s^h)'(p) &= s \left[\frac{1}{p_b^l} - \frac{1}{p_s^l} + \ln \frac{\Omega(p_b^l)}{\Omega(p_s^l)} \right] - \frac{2\lambda}{r}R_a < 0 \end{aligned}$$

Suppose $p_b^h \leq p_s^h$. Then it must be the case that $F_b^h(p_b^l) - F_s^h(p_b^l) > 0$. Plug $p = p_b^l$ into (3.27), the condition that $F_b^h(p_b^l) - F_s^h(p_b^l) > 0$ is equivalent to

$$\ln \left[1 + \frac{R_a}{(1+\lambda/r)(1-s)} \right] > \frac{2\lambda+r}{r} \frac{R_a}{(1+\lambda/r)(1-s)}$$

Denote $x = \frac{R_a}{(1+\lambda/r)(1-s)} > 0$. The above inequality then becomes

$$\ln(1+x) > \frac{2\lambda+r}{r}x$$

We know when $x = 0$, $\ln(1+x) = \frac{2\lambda+r}{r}x$ and when $x > 0$, $\frac{d\ln(1+x)}{dx} = \frac{1}{1+x} < 1 < \frac{2\lambda+r}{r} = \frac{d((2\lambda+r)x/r)}{dx}$. Therefore the above inequality can never hold. We have a contradiction. Hence, $p_b^h > p_s^h$.

Step 2: It is easy to see that for any $p \leq p_s^l$ and $p \geq p_b^h$, $k_b(p) = k_s(p)$. If $p_s^h \leq p_b^l$, then $k_b(p) < k_s(p)$ holds trivially for any $p \in (p_s^l, p_b^h)$ and we have proved the statement.

Let us consider the case where $p_s^h > p_b^l$. For any $p \in (p_s^l, p_b^l]$ and $p \in [p_s^h, p_b^h)$, it is straightforward to verify that $k_b(p) < k_s(p)$. The final step is to check that the inequality also holds for $p \in (p_b^l, p_s^h)$. For those posteriors, $k_b(p)$ and $k_s(p)$ take following values,

$$k_b(p) = \frac{rZ(p)/\lambda - s}{s - p}$$

$$k_s(p) = \frac{rW(p)/\lambda - s}{s - p - 2pR_a}$$

Therefore, the sufficient condition for the inequality to hold is $W(p) - Z(p) \geq 0$ for any $p \in (p_b^l, p_s^h)$. This can be verified below.

$$W(p) - Z(p) = p \left[\Omega(p)s \ln \left(1 + \frac{R_a}{(1 + \lambda/r)(1 - s)} \right)^{-1} + R_a \right] > 0$$

3.8.2 Proof of Proposition 3.6

When $T \geq 0$, $p_n^h > p_n^l \geq p_s^h$.

When $T < 0$, suppose $p_n^h \leq p_s^h$. Then from (3.14), p_n^h is determined by,

$$\left(2 + \frac{\lambda}{r}\right)U(p_n^h) - 2W(p_n^h) = 2s - p_n^h(1 + 2R_r) + 2R_s$$

where U takes the form of (3.23) and W takes the form of (3.11). Plugging in $U(p_n^h)$ and $W(p_n^h)$, the above condition becomes

$$1 + R_a + \frac{\lambda}{r} + R_s - \frac{2\lambda + r}{r}s + \frac{\lambda}{r}p_n^h(1 + 2R_r) - s \left[\ln \frac{\Omega(p_s^l)}{\Omega(p_n^h)} + \frac{1}{p_s^l} \right]$$

$$+ p_n^h s \left[\ln \frac{\Omega(p_n^l)}{\Omega(p_n^h)} + \frac{1}{p_n^l} \right] + \frac{2\lambda + r}{r}p_n^h(R_a + R_s - R_r) \left[\ln \frac{\Omega(p_n^h)}{\Omega(p_n^l)} + \frac{1}{1 - p_n^l} \right] = 0 \quad (3.28)$$

We know p_s^h is determined by (3.10), which is equivalent to,

$$1 + R_a + \frac{\lambda}{r} - \frac{2\lambda + r}{r}s + \frac{\lambda}{r}p_s^h(1 + 2R_a) + s \left[\ln \frac{\Omega(p_s^h)}{\Omega(p_s^l)} - \frac{1}{p_s^l} \right] (1 - p_s^h) = 0 \quad (3.29)$$

Define the following two functions based on the left hand side of (3.28) and

(3.29).

$$F_n(p) = 1 + R_a + \frac{\lambda}{r} + R_s - \frac{2\lambda + r}{r}s + \frac{\lambda}{r}p(1 + 2R_r) - s\left[\ln \frac{\Omega(p_s^l)}{\Omega(p)} + \frac{1}{p_s^l}\right] \\ + ps\left[\ln \frac{\Omega(p_n^l)}{\Omega(p)} + \frac{1}{p_n^l}\right] + \frac{2\lambda + r}{r}p(R_a + R_s - R_r)\left[\ln \frac{\Omega(p)}{\Omega(p_n^l)} + \frac{1}{1 - p_n^l}\right]$$

$$F_s(p) = 1 + R_a + \frac{\lambda}{r} - \frac{2\lambda + r}{r}s + \frac{\lambda}{r}p(1 + 2R_a) + s\left[\ln \frac{\Omega(p)}{\Omega(p_s^l)} - \frac{1}{p_s^l}\right](1 - p)$$

It is straight forward to compute the difference between the slope of $F_n(p)$ and $F_s(p)$ at any $p \in [p_n^l, 1)$,

$$F_n'(p) - F_s'(p) = -R_s + s\left(\frac{1}{p_n^l} - \frac{1}{p_s^l}\right) + \frac{2\lambda}{r}(R_r - R_a - R_s) \\ + \frac{2\lambda + r}{r}(R_a + R_s - R_r)\left[\ln \frac{\Omega(p)}{\Omega(p_n^l)} + \frac{1}{1 - p_n^l} - \frac{1}{1 - p}\right]$$

We can verify

$$F_n'(p) - F_s'(p) < 0, \text{ for any } p \in (p_n^l, 1)$$

Combining with the assumption $p_n^h \leq p_s^h$, it must be true that $F_s(p_n^l) - F_n(p_n^l) < 0$. The difference equals

$$F_s(p_n^l) - F_n(p_n^l) = p_n^l s \left\{ -\left(\frac{r}{2\lambda + r} \ln \frac{\Omega(p_s^l)}{\Omega(p_n^l)} + 1\right) \Omega(p_n^l) + \Omega(p_s^l) \right\}$$

The condition $F_s(p_n^l) - F_n(p_n^l) < 0$ is therefore equivalent to

$$\left(\frac{\Omega(p_s^l)}{\Omega(p_n^l)}\right)^{\frac{r}{2\lambda+r}} > e^{\frac{\Omega(p_s^l)}{\Omega(p_n^l)} - 1}$$

We know $\frac{\Omega(p_s^l)}{\Omega(p_n^l)} \geq 1$ because $p_s^l \leq p_n^l$. As a result,

$$\left(\frac{\Omega(p_s^l)}{\Omega(p_n^l)}\right)^{\frac{r}{2\lambda+r}} \leq e^{\frac{\Omega(p_s^l)}{\Omega(p_n^l)} - 1}$$

We have a contradiction. Therefore, it must be the case that $p_n^h > p_s^h$ when $T < 0$.

3.8.3 Proof of Proposition 3.8

In case 1, the proposition holds trivially since $p_s^h < p_n^l$.

In case 2, it is easy to see that the claim holds for any $p \leq p_n^l$ and $p \geq p_s^h$. Therefore, we only need to check if $k_n(p) < k_s(p)$ for any $p \in (p_n^l, p_s^h)$.

Recall that $k_n(p)$ equals

$$\begin{aligned} k_n(p) &= \frac{\frac{2\lambda+r}{\lambda}U(p) - 2W(p) - s}{-p(1 + 2R_r) + s + 2R_s} \\ &= \frac{\frac{r}{\lambda}W(p) - s + \frac{2\lambda+r}{\lambda}[U(p) - W(p)]}{s - p - 2pR_a + 2(R_s - pR_r + pR_a)} \end{aligned}$$

Notice that $R_s - pR_r + pR_a > 0$ and that,

$$k_s(p) = \frac{\frac{r}{\lambda}W(p) - s}{s - p - 2pR_a}$$

Hence, a sufficient condition for $k_n(p) < k_s(p)$ is that $U(p) - W(p) < 0$ for any $p \in (p_n^l, p_s^h)$. We can verify this condition as follows. For $p \in (p_n^l, p_s^h)$,

$$\begin{aligned} &U(p) - W(p) \\ &= \frac{r}{2\lambda + r}ps \left[\ln \frac{\Omega(p_n^l)}{\Omega(p_s^l)} + \frac{1}{p_n^l} - \frac{1}{p_s^l} \right] + p(R_a + R_s - R_r) \left[\ln \frac{\Omega(p)}{\Omega(p_n^l)} + \frac{1}{1 - p_n^l} \right] + R_s \end{aligned}$$

$$\begin{aligned} &U'(p) - W'(p) \\ &= \frac{r}{2\lambda + r}s \left[\ln \frac{\Omega(p_n^l)}{\Omega(p_s^l)} + \frac{1}{p_n^l} - \frac{1}{p_s^l} \right] + (R_a + R_s - R_r) \left[\ln \frac{\Omega(p)}{\Omega(p_n^l)} + \frac{1}{1 - p_n^l} - \frac{1}{1 - p} \right] < 0 \end{aligned}$$

The inequality follows since $p > p_n^l$. Finally, plug in $p = p_n^l$ into $U(p) - W(p)$,

$$U(p_n^l) - W(p_n^l) = \frac{r}{2\lambda + r}p_n^l s \left[\frac{1}{p_n^l} - \frac{1}{p_s^l} \right] < 0$$

Therefore, $U(p) - W(p) < 0$ for any $p \in (p_n^l, p_s^h)$.

3.8.4 Proof of Proposition 3.9

We can first verify that T increases in δ , which implies that we move from the more efficient case 2 to the less efficient case 1. As shown earlier,

$$T = s \ln \frac{\Omega(p_s^h)}{\Omega(p_s^l)} + \frac{2\lambda + r}{r} \left(R_s + \frac{R_a + R_s - R_r}{\Omega(p_s^h)} \right)$$

Next we will show that for a given T , the inequality always hold for both case 1 and case 2. Recall that for $p \in [p_n^l, p_n^h]$, $k_n(p)$ equals

$$k_n(p) = \frac{\frac{2\lambda+r}{\lambda}U(p) - 2W(p) - s}{-p(1 + 2R_r) + s + 2R_s}$$

It is straightforward to verify that the denominator strictly increases in δ . Therefore, $\frac{\partial k_n(p)}{\partial \delta} |_{\delta=0} \geq 0$ is equivalent to the condition that $\frac{2\lambda+r}{\lambda}U(p) - 2W(p)$ weakly decreases in δ for corresponding $U(p)$ and $W(p)$.

Let us first consider case 1 where $p_n^l \geq p_s^h$. In this scenario,

$$\frac{2\lambda + r}{\lambda}U(p) - 2W(p) = \frac{2\lambda + r}{\lambda} [pC_n^m + s + R_s - p(1 - s + R_r - R_s) \ln \Omega(p)] - \frac{2\lambda}{r}p$$

where

$$\begin{aligned} C_n^m &= (1 - s + R_r - R_s) \ln \Omega(p_n^l) - \frac{1 - s + R_r - R_s}{1 - p_n^l} + \frac{2\lambda^2}{r(2\lambda + r)} \\ \Rightarrow \frac{\partial C_n^m}{\partial \delta} &= -\frac{1 - s + R_r - R_s}{(1 - p_n^l)^2 p_n^l} \frac{\partial p_n^l}{\partial \delta} \end{aligned}$$

Therefore

$$\frac{\partial \left[\frac{2\lambda+r}{\lambda}U(p) - 2W(p) \right]}{\partial \delta} = \frac{2\lambda + r}{\lambda} \left[1 - \frac{p(1 - s + R_r - R_s)}{p_n^l(1 - p_n^l)^2} \frac{\partial p_n^l}{\partial \delta} \right]$$

Plug in

$$p_n^l = \frac{R_s + \frac{\lambda+r}{2\lambda+r}s}{R_r + 1 - \frac{\lambda}{2\lambda+r}s}$$

and

$$\frac{\partial p_n^l}{\partial \delta} = \frac{1 - s + R_r - R_s}{\left[1 + R_r - \frac{\lambda}{2\lambda+r}s \right]^2}$$

The above derivative becomes,

$$\frac{\partial \left[\frac{2\lambda+r}{\lambda} U(p) - 2W(p) \right]}{\partial \delta} = \frac{2\lambda+r}{\lambda} \left[1 - \frac{p}{p_n^l} \right] \leq 0$$

Hence, $k_n(p)$ decreases in δ in case 1. In case 2, we can again first compute $\frac{2\lambda+r}{\lambda} U(p) - 2W(p)$ and show that it decreases in δ .

For $p \in [p_n^l, p_s^h]$,

$$\begin{aligned} & \frac{2\lambda+r}{\lambda} U(p) - 2W(p) \\ = & \frac{r}{\lambda} \left\{ ps \left[\ln \Omega(p_n^l) + \frac{1}{p_n^l} \right] + \left[-s \ln \Omega(p_s^l) - \frac{s}{p_s^l} + 1 + R_a + \frac{\lambda}{r} + (1-p)s \ln \Omega(p) - s \right] \right\} \\ & + \frac{2\lambda+r}{\lambda} \left\{ R_s - p(R_a + R_s - R_r) \left[\ln \Omega(p_n^l) - \frac{1}{1-p_n^l} \right] \right\} \end{aligned} \quad (3.30)$$

In the above expression, p_s^l is independent of δ and p_n^l is implicitly determined by the following equation,

$$s \ln \frac{\Omega(p_n^l)}{\Omega(p_s^l)} + \frac{2\lambda+r}{r} \left[R_s + \frac{1}{\Omega(p_n^l)} (R_a + R_s - R_r) \right] = 0$$

By the implicit function theorem,

$$\frac{\partial p_n^l}{\partial \delta} = -\frac{2\lambda+r}{r} (1-p_n^l) \left[-\frac{s}{p_n^l} + \frac{2\lambda+r}{r(1-p_n^l)} (R_a + R_s - R_r) \right]^{-1}$$

Take derivative of (3.30),

$$\begin{aligned} & \frac{\partial \left[\frac{2\lambda+r}{\lambda} U(p) - 2W(p) \right]}{\partial \delta} \\ = & \frac{p}{(1-p_n^l)p_n^l} \left[-\frac{rs}{\lambda p_n^l} + \frac{(2\lambda+r)(R_a + R_s - R_r)}{\lambda(1-p_n^l)} \right] \frac{\partial p_n^l}{\partial \delta} + \frac{2\lambda+r}{\lambda} \\ = & \frac{2\lambda+r}{\lambda} \left[1 - \frac{p}{p_n^l} \right] \leq 0 \end{aligned}$$

The last line follows after plugging in $\frac{\partial p_n^l}{\partial \delta}$. Next, we discuss the case when $p \in [p_s^h, p_n^h]$. With these beliefs,

$$\frac{2\lambda+r}{\lambda} U(p) - 2W(p) = \frac{2\lambda+r}{\lambda} \left[pC_n^h + s + R_s - p(1-s+R_r-R_s) \ln \Omega(p) \right] - \frac{2\lambda}{r} p$$

where C_n^h is given by (3.25) and hence $\frac{\partial C_n^h}{\partial \delta} = -\frac{1}{p_n^l}$.

Therefore

$$\frac{\partial \left[\frac{2\lambda+r}{\lambda} U(p) - 2W(p) \right]}{\partial \delta} = \frac{2\lambda+r}{\lambda} \left[1 - \frac{p}{p_n^l} \right] \leq 0$$

3.9 Bibliography

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