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Exploring the Relationships between Emergent Mathematical Practices, Individuals' Ways of Reasoning, and Meanings Constructed through Discourse

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Exploring the Relationships between Emergent Mathematical Practices, Individuals' Ways of Reasoning, and Meanings Constructed through Discourse

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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The Dissertation of John David Gruver is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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2016

Dedication

This Dissertation is dedicated to my wonderful wife for supporting me through this process.

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Abstract of the Dissertation

Exploring the Relationships between Emergent Mathematical Practices, Individuals' Ways of Reasoning, and Meanings Constructed through Discourse

by

John Gruver

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2016
San Diego State University, 2016

Professor Joanne Lobato, Chair

The emergent perspective (Cobb & Yackel, 1996) is a way for researchers to conceptualize teaching and learning interactions that gives equal analytic focus to the social environment and individual cognition. According to this theory, individual students' conceptions and activities give rise to ways of reasoning that become accepted in the class community, called emergent mathematical practices. Students' participation in these practices then affects their personal conceptions and activities. In this study, I further contribute to researchers' understanding of the nature of this relationship by documenting the mathematical practices established in a class community and investigating a subset of individuals' subsequent reasoning in a clinical interview. In contrast to previous work, I found the majority of students interviewed reasoned in ways that were qualitatively different from the established practice. I then developed a partial explanation for how students could participate in class activities yet continue to reason in ways that differed from the established practice by examining the mathematical meanings constructed through the classroom discourse.

Chapter 1: Rationale

In the late eighties, mathematics education research took what Lerman (2000) called a “social turn” (p. 19). This meant that researchers began to more seriously consider the social nature of knowing. This is not to say that social interactions were ignored previous to this time. For example, Piaget acknowledged the contributions of the social world to individuals’ construction of knowledge (M. Cole & Wertsch, 1996). However, after the social turn researchers began to conceive of knowledge as inseparable from the social context in which that knowledge was developed, to explore the semiotic and cultural mediation of thought, and investigate learning as enculturation into practice (Brown, Collins, & Duguid, 1989; Wenger, 1998; Wertsch, 1991). Thus, mathematics educators began to expand the unit of analysis beyond the individual to explore the collective mathematical development of a community and to understand regularities in patterns of communication among participants in a classroom.

However, as Lerman (2000) pointed out, this expansion was not without challenges.

> A major challenge for theories from the social turn is to account for individual cognition and difference, and to incorporate the substantial body of research on mathematical cognition, as products of social activity (p. 27).

Thus, as social theories came into prominence, educators began to better understand the nature of social interactions in their classrooms. However, with the expanded unit of analysis nuances in individual cognition were lost. This leaves open the question of what meanings are students making from these social interactions? Also, how will participating in classrooms, perhaps especially classrooms where powerful mathematical ideas are developed, affect how students reason in the future?

The problem of coordinating individual cognition with social interaction is critical in mathematics education research, because no matter how well researchers are able to conceptualize the social environment, if this conceptualization is divorced from individuals' learning it is less likely to impact educators' practice in a way that will positively affect student learning. In order to impact student learning by changing the nature of the interactions in which those students engage, it is necessary not only to understand the nature of classroom interactions, but also their relationships to student learning. This means researchers need a way to understand the learning process as it occurs in social environments in such a way that neither the environment nor the cognition of individual is studied at the expense of the other.

One way this challenge was met was by the development of the *emergent perspective* (Cobb & Yackel, 1996). This theory combines aspects of symbolic interactionism (Bauersfeld, Krummheuer, & Voight, 1988) and constructivism (von Glasersfeld, 1984, 1992) to coordinate social aspects of the classroom microculture with psychological features of the individuals who participate in the classroom activities. In this approach, the social and individual planes have equal weight, in contrast to theories in which the individual plane has primacy (and the social nature of knowing is downplayed) or the social plane has primacy (and the interpretive nature of knowing is downplayed). The emergent perspective has been utilized by mathematics educators around the world to inform a variety of research efforts (Hershkowitz & Jaworski, 2012; Hershkowitz & Schwarz, 1999; Kazemi & Stipek, 2001; Rasmussen, Wawro, & Zandieh, 2015; Roy, 2008; Stephan & Rasmussen, 2002; Voigt, 1995; Wawro, 2011).

The emergent perspective outlines three social aspects of the classroom—social norms, socio-mathematical norms, and classroom mathematical practices—and their individual psychological correlates. *Social norms* are accepted and expected ways of participating in the classroom. Similarly, *socio-mathematical norms* are expected ways of participating that are specific to how students engage with the mathematics (e.g. ways of giving valid mathematical arguments). *Classroom mathematical practices* are mathematical ways of reasoning and operating that become taken-as-shared. This means that in the classroom community, participants assume that other participants are familiar with and understand the way of operating. Researchers use the phrase taken-as-shared rather than shared to describe these ways for reasoning and operating to emphasize that they are not claiming that all students' reason in exactly the same way. Rather, they only claim the mathematical practice is treated as if it is understood and accepted in the community. In this way, the researchers are able to identify a phenomenon at that exists at the classroom level, but not make assumptions about what any one individual participant understands.

Each of these three social aspects also has a psychological correlate (see Table 1.1). The correlate of social norms is students' beliefs about their own role, others' roles, and the general nature of mathematical activity. Essentially, this is the students' personal view on what the expectations of the norm are and how they should participate in them. The correlate of socio-mathematical norms is students' mathematical beliefs and values. This refers to students' view of mathematics in general as well as their view on mathematical practices, including how they see themselves engaging in them. For example, if one of the socio-mathematical norms that exists in a classroom defines the criteria that guides what constitutes a mathematically significant difference in two explanations, individuals' personal judgments about whether or not their solution is significantly different than the solution being presented influence how they interpret and engage with the socio-mathematical norm. Finally, the individual correlate of classroom mathematical practices is students' own mathematical conceptions and activity. This can include students' ways of reasoning about a topic as well as images of mathematical concepts.

Table 1.1: The emergent perspective's interpretive framework (Cobb & Yackel, 1996).

Social Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Socio-mathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

According to Cobb and Yackel (1996) the relationship between individuals' conceptions and mathematical practices is indirect and reflexive. This means that individuals' ideas gives rise to classroom mathematical practices as individuals share and negotiate ideas. Then, as ways of reasoning become accepted in the community, they influence, but do not determine, students' further reasoning and conceptions. Because participation in emergent practices is not deterministic of further ways of reasoning, classroom participants may not share identical conceptions. This diversity of student ideas is acknowledged through use of the metaphor that students participate differentially in classroom mathematical practices.

However, despite the promise of the theory to coordinate individual and social constructs, most of the research conducted by those who have worked from the emergent perspective has focused on fleshing out and investigating the social constructs of social norms, socio-mathematical norms, and classroom math practices (e.g. Kazemi & Stipek, 2001; Pang, 2000; Yackel, 2001). This is understandable, given the need to operationalize these constructs in classroom-based mathematics education research. Indeed, the emergent perspective provides a powerful way to conceptualize social aspects of the learning environment generally, and in particular the mathematical progress of the class at the collective level. However, analyses that focus solely on social aspects are limited in their meaningfulness for individual students' learning unless researchers provide an elaboration of the relationships between social interactions and individuals' mathematical development. Cobb himself (1999) called for the "the need to clarify the relation between individual students' reasoning and the collective practices in which they participate in" (p. 33).

Although understanding the complex relationship between individual interpretations and normative ways of reasoning is in its infancy, a few relevant studies have conducted. Most recently, Rasmussen and his colleagues have elaborated the ways individuals contribute to emergent practices with two studies. In the first, Tabach, Hershkowits, Rasmussen, and Dreyfus (2014) combined two methodological tools, one of which is a lens on the mathematical progress of the individual and one of which is a lens on the mathematical progress of class community, to track the movement of ideas from small group to whole class and vice versa. In another study, Rasmussen, Wawro, and Zandieh (2015) extended notions of what constitutes collective and individual mathematical progress. In particular, they examined not only what conceptions individuals bring to bear when contributing ideas to the classroom community, but also the roles they take on when expressing ideas. For example, they distinguished between and kept track of whether individuals were contributing their own ideas or relaying an idea that was originally brought up by another person. They also expanded notions of collective progress by not only tracking the emergence of mathematical practices, but also recognizing when students engaged in external practices that are central to the work of mathematicians, which they termed disciplinary practices. In sum, Rasmussen and his colleagues' work helps to develop researchers' understanding of the relationships between individual engagement and collective progress by giving ways to conceptualize the movement of ideas and expanding conceptualizations of individual activity and collective progress. Additional work is needed to understand the nature of individual students' ways of reasoning that are qualitatively different from established mathematical practices and their relationships to those practices.

The most comprehensive investigation of the relationship between individuals' ways of reasoning and emergent practices was preformed by Stephan, Cobb, and Gravemeijer (2003). They outlined in narrative form how two first grade students, Nancy and Meagan, participated in the emergence of the classroom mathematical practices as well as how their subsequent ways of reasoning, which the researchers determined by analyzing their utterances in class, related to the mathematical practices. They found that while the students generally reasoned in ways that were consistent with the established practices, there were a few instances where Meagan's mathematical conceptions were qualitatively different than established practices. However, Meagan reorganized her knowledge as the class progressed.

This example suggests that one possible way to describe the relationship between individuals' activity and emergent mathematical practices is that qualitative differences may exist temporarily, but unproductive ways of reasoning typically become problematized as the course continues, at which point the individual will reorganize his or her knowledge so that it is mathematically consonant with the practice. This summation is consistent with the findings of Bowers et al. (1999). While their study focused on documenting the emergent mathematical practices in a third grade teaching experiment, the researchers also conducted pre- and post-interviews with all the students to provide evidence that learning had occurred for individual students. The interview findings revealed that the vast majority of students moved from unproductive strategies to sophisticated strategies on both a conceptually oriented task about place value and on a procedural two-digit subtraction task that required exchanging a group of ten for ten ones.

While these two studies have encouraging results in that they suggest students develop powerful ways of reasoning when they engage in conceptually oriented emergent mathematical practices, researchers' understanding of that relationship is still limited because there are only a few studies that address the topic. While the few studies that exist imply that students with qualitatively different ways of reasoning eventually reorganize their knowledge, is it possible for these ways of reasoning to persist or do they always get resolved? If unproductive ways of reasoning do persist, under what circumstances does this occur? Furthermore, what conditions allow for differing ways of reasoning occur in the first place?

To answer this last question of how differing ways of reasoning arise, it may be efficacious to examine the ways mathematical meanings are constructed in the classrooms. When examining the emergence of mathematical practices, researchers focus on the forms of arguments students give. Cobb and Yackel (1996) originally described a math practice as a way of reasoning that initially needed justification, but whose justifications eventually dropped off. Therefore an analysis to determine whether or not a way of reasoning is taken as shared, examines the structure of arguments that use that way of reasoning. In particular, the analysis examines whether or not the way of reasoning required justification. Other criteria to determine whether or not a way of reasoning is normative in the classroom have since been added (R. Cole et al., 2012; Rasmussen & Stephan, 2008), but these also focus on the changing forms of arguments given by students. An analysis of this type reveals the status of a way of reasoning in the community (i.e. whether or not it is accepted), but does not reveal the meanings associated with that way of reasoning as constructed in the class. This is an important distinction as Meagan's variance from established ways of reasoning centered on her unique interpretation of a particular normative way of reasoning (Stephan et al., 2003).

Thematic analysis (Herbel-Eisenmann & Otten, 2011; Lemke, 1990) provides one way to investigate emergent mathematical meanings in social situations. This technique is rooted in a perspective on language called Systemic Functional Linguistics (SFL; Halliday, 1978; Halliday & Hasan, 1985). According to the assumptions of thematic analysis, semantic relationships, the relationships between words expressed through language, constrain, but do not determine meanings of those words. Thus, by carefully tracking semantic relationships researchers can determined what potential meanings particular words have in the classroom community. For example, through thematic analysis Herbel-Eisenmann and Otten (2011) found there were subtle ambiguities in the ways participants spoke about the content. When explaining relationships among geometric figures, one of the teachers they studied repeatedly said, “rectangles are parallelograms.” While proficient users of mathematics may interpret this to mean rectangles are a type of parallelogram, the word “are” does not establish this semantic relationship unambiguously. For example, “are” can also communicate a synonymous relationship as in the example, “equilateral triangles are triangles whose sides are all equal length.” This ambiguity in the statement “rectangles are parallelograms” may have been missed without careful attention the semantic relationships. If there were students in that classroom who had trouble navigating this ambiguity, one could imagine a scenario in which the idea that “rectangles are parallelograms” became taken-as-shared, while individual students reasoned about rectangles in qualitatively different ways. Thematic analysis could help reveal this ambiguity and help explain why that difference arose.

This example shows how the method to determine established practices, which focuses on argumentation, and the method to determine mathematical meanings, which focuses on semantic relationships, are complementary. Focusing on the structure of arguments is a more macro approach and reveals the progression of ideas as they unfolded in the classroom. This can help educators understand productive ways to develop ideas. Cobb called this “domain specific instructional theory” (Cobb, 1999, p. 6). Thematic analysis, on the other hand, is a more fine-grained approach, which has the power to reveal subtleties in the way meaning is constructed in the community by examining semantic relationships.

In summary, mathematics education took a “social turn” in the late eighties, which enriched the field by moving analytic foci beyond the individual to include the social environments in which learning is embedded. However, a challenge of this development was to account for the social nature of learning, while at the same time not losing sight of what individuals are learning by participating in these social settings. The emergent perspective promised a coordination of the social environment and individuals’ learning, but the nature of this relationship is still inadequately understood. In this dissertation study I will contribute to this field of study by answering the following two research questions.

> Research Question 1: How are individuals’ ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

The purpose of this research question is to document what mathematical progress was made at both the classroom level and at the individual level and investigate the nature and extent of individual variation in students’ conceptions from established classroom practices. To do this I identified the ways of reasoning that became normative in the classroom community by analyzing the way arguments were made in the class and how the form of those arguments changed over time. By analyzing how particular ideas were used in arguments I investigated the status of these ideas in the class community and documented which ideas became accepted in the community.

After establishing the normative ways of reasoning, I investigated individuals’ ways of reasoning by analyzing individual clinical interviews administered after instruction to a subset of students (the focus students). Analyzing these individual ways of reasoning revealed the nature and extent of individual variation in students' conceptions from the normative ways of reasoning.

I then sought to develop a partial explanation for the variation from the emergent practices by investigating the mathematically meanings established through discourse in the classroom. This differed from the analysis for Research Question 1, which examined the status of ways of reasoning, in that it provided insights into the ways students may have potentially interpreted those ways of reasoning. This investigation gave insights into Research Question 2.

> Research Question 2: What mathematical connections exist between the focus students' ways of reasoning in the post interviews and the discursive interactions between them and other students and the teacher in both whole class and small group settings? Furthermore, how might the nature of these discursive interactions give plausible explanations for students' differing conceptions?

To answer this question I analyzed sematic relationships expressed in the class discourse. By comparing various networks of semantic relationships I was able to identify when particular ideas were left implicit or ambiguous in the talk. This helped explain how students could engage in the discourse, yet still have potentially interpreted classroom events in various ways.

Defining the Scope of this Dissertation Study

This study will admittedly only provide a partial accounting of the nature of the relationship between established practices and individuals' ways of reasoning. No one study can determine this relationship completely; only after many studies have investigated this relationship will scholars begin to understand its nature.

Furthermore, this study will only provide a partial explanation of why individuals reason in ways that vary from established practices. In this study I only examined the meanings constructed through discourse to explain this variance. However, there are likely other issues at play. For instance, some scholars have argued convincingly that the discontinuities between home and school practices impact the nature of students' learning (e.g. Heath, 1982; Labov, 1972; Mejía-Arauz, Rogoff, Dexter, & Najafi, 2007). These discontinuities may help explain why there is variance in individuals' ways of reasoning from accepted classroom practices. Thus, it will likely take the coordination of several studies taking several approaches to fully explicate and explain the relationships between individuals' ways of reasoning and those that are developed and accepted in classroom settings. This study contributes to that understanding, but in no way is meant to be a comprehensive explanation.

Significance of this Study

This study has both theoretical and practical significance. It contributes to the theoretical knowledge of the field by elaborating Cobb and Yackel's (1996) emergent perspective. Part of the power of this perspective is that it fully embraces the social turn of mathematics education research, yet still emphasizes the interpretive nature of individual knowledge construction by positing a reflexive relationship between the establishment of collective mathematical practices and individuals' mathematical activity and conceptions. However, this relationship needs further empirical investigation. While this research has begun (e.g. Bowers et al., 1999; Cobb, 1999; Rasmussen et al., 2015; Stephan et al., 2003; Tabach et al., 2014), questions still remain. Rasmussen and his colleagues' work (Rasmussen et al., 2015; Tabach et al., 2014) has helped develop researchers' understanding of how students contribute to emergent practices and how ideas from those practices flow back into small groups. However, Cobb (1999) illustrated that students can have qualitatively different interpretations of math practices, which Rasmussen's work does not directly address. Stephan et al. (2003) provided some insight into this issue; their work suggests that qualitative differences can exist during the course of instruction, but often get resolved by end of the unit. While this idea seems to be supported by the findings of Bowers et al. (1999), neither study was meant to provide a definitive answer as to the nature of the relationship between students' participation in emergent practices and their subsequent reasoning. This dissertation study contributes to this understanding by expanding on the findings of these previously conducted studies.

This expansion is not only done in answering Research Question 1, but also in the investigation for Research Question 2. In this study I not only investigated ways of reasoning that became accepted, but also investigated how mathematical meanings were constructed in that classroom. This gave insights into the ways that students in the classroom could have potentially interpreted accepted ways of reasoning, which may help explain how they could reasonably engage in ways of reasoning that were qualitatively different than those established in class. This coordination of these two types of discourse analysis, one that examines the structure of arguments to document when ways of reasoning became accepted and one that examines semantic relationships to determine the meanings that were constructed in the class community, may be an important analytic approach as scholars begin to understand the relationship between individuals' ways of reasoning and those that become normative in a class community.

Elaborating this theoretical relationship will have practical implications for teaching. As researchers begin to explain why students may reason in qualitatively different ways form those that are established, teachers will gain insights into how to structure classroom interactions that lead to productive interpretations of and ways of participating in emergent practices. For example, in this study, I examined how meanings created through discourse can potentially influence students' interpretations. These insights may have important implications for how teachers orchestrate classroom discourse so that students can engage productively in class discussions.

The National Council of Teachers of Mathematics' view on productive discourse underscores the need for greater understanding of how classroom discourse relates to individuals' ways of reasoning. In their recently released book *Principles to Actions: Ensuring Mathematical Success for All* (2014) they describe eight mathematics teaching practices, which represent the organization's vision for high quality teaching. One of these teaching practices is facilitate meaningful discourse, which is described in the following way.

> Effective teaching of mathematics facilitates discourse among students to build *shared understanding* of mathematical ideas by

> analyzing and comparing student approaches and arguments (p. 10, emphasis added).

Here NCTM claims that as students intellectually engage with each other's ideas, they will begin to develop common conceptions. This claim is at least somewhat dubious given Cobb's (1999) documentation of students reasoning in qualitatively different ways from established practices in a classroom where productive mathematical discourse was the norm. This is not to say I doubt the efficacy of facilitating meaningful discourse for students. Rather, I believe that if practitioners are to effectively engage students in meaningful discourse, researchers need to develop greater theoretical understanding of how the nature of classroom discourse, including both the status of mathematical ideas in the discourse and their meanings, is related to individuals' ways of reasoning. Through this greater theoretical understanding educators will get more precise definitions of what meaningful discourse is for the students in their classrooms.

Chapter 2: Literature Review

This dissertation study focuses on the relationship between the individual and the collective. First, I will examine the relationship between individual ways of reasoning and ways of reasoning that are accepted by the collective. I will then examine classroom interactions in finer detail to determine the mathematical meanings that were constructed through discourse by classroom participants. As such, in this literature review I examine approaches to documenting mathematical progress at both the individual and collective level, including studies that explicitly try to coordinate collective and individual analyses. I then examine research relevant to understanding classroom discourse.

Approaches to Documenting Mathematical Progress

There are two major approaches to the study of students' mathematical progress in mathematics education: one that focuses on the learning of an individual and one that focuses on the mathematical progress of a classroom community. The work that focuses on the learning of individuals identifies increasingly sophisticated states of knowing (Battista, 2004; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Burger & Shaughnessy, 1986; Hackenberg, 2010; Mitchelmore & White, 2000; Norton, 2008; Olive, 1999; Saenz-Ludlow, 1994; Steffe, 2004; Tillema, 2013). Conversely, other scholars track the mathematical progress of a classroom, thereby outlining opportunities students had to learn (Bowers, Cobb, & McClain, 1999; Clements, Wilson, & Sarama, 2004; Cobb, McClain, & Gravemeijer, 2003; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Stephan & Akyuz, 2012). In this review on documenting mathematical progress, I first illustrate researchers' approaches to documenting increasingly sophisticated states of knowing for individuals and then turn to the work on the mathematical progress of the classroom.

Individual Cognitive Milestones

One approach to documenting mathematical progress is to identify individuals' increasingly sophisticated states of knowing, called cognitive milestones (Battista, 2004; Battista et al., 1998; Burger & Shaughnessy, 1986; Hackenberg, 2010; Mitchelmore & White, 2000; Norton, 2008; Olive, 1999; Saenz-Ludlow, 1994; Steffe, 2004; Tillema, 2013). This work helps educators understand the milestones children will pass by as they increase in sophistication in their reasoning about a particular topic. For example, Battista (2004) outlined the cognitive levels students attain as they learn about area and volume. Central to understanding area is the ability to mentally enumerate squares that are structured in rectangular arrays. To do this, the student needs to first see the structure of rows and columns in the array and be able locate particular squares in the array in terms of the row and column it resides in. When Battista (2004) outlined the cognitive levels he talked about them in terms of these competences. For example, at the first cognitive level he identified, the student does not see the structure of rows and columns or locate squares in that structure. When a student at this level was asked to predict how many squares would cover a rectangle, she drew in small squares to cover the rectangle, but these squares were not the uniformly sized and were not arranged in rows and columns. This demonstrated her lack of awareness of the structure. This is in contrast to a more advanced cognitive level, level 4, in which the student is able to mentally fuse squares together to form rows and columns, but is not able to locate squares in terms of the rows and columns. For example, when a student at this level was shown that five squares fit across a rectangle and seven squares fit down the middle of the rectangle, the student was able to count by fives as he covered the rectangle with rows of squares, but did not realize that there should be seven rows. This showed that he was able to recognize the structure of rows, but was not able to coordinate those rows with the squares in the columns. Similarly, other researchers have identified cognitive levels for other topics such as fractions (Hackenberg, 2010; Norton, 2008; Olive, 1999; Saenz-Ludlow, 1994; Steffe, 2004), the development of a power meaning for multiplication (Tillema, 2013), geometry (Burger & Shaughnessy, 1986), and angles (Mitchelmore & White, 2000).

While this work gives insights into the variability in thinking that exists in students' thinking among a general population, it does not give insights into the variation that exists among students who participated in a particular classroom. This is reflected in the methods they used. Some researchers used a cross-sectional approach in which the researchers analyzed how a sampling of students from the population of interest performed on tasks (Battista, 2004; Battista et al., 1998; Burger & Shaughnessy, 1986; Mitchelmore & White, 2000). The researchers then stratified these performances and inferred conceptual milestones. Since the students they studied often experienced a variety of instructional approaches, the variation was not constrained by the type of instruction the students received.

Rather, the variation is supposed to be reflective of the natural variation that exists among all students in the given population.

Other researchers have documented detailed accounts of student thinking as students have learned through instruction in constructivist teaching experiments (Steffe & Thompson, 2000). In such a teaching experiment the researcher interacts with a small number of students to understand their conceptions and then poses a problem in an attempt to advance their thinking. While the method has the word teaching in it, teaching is not the focus of the approach. "In this methodology, researchers use teaching as a tool to understand and explain how students operate mathematically and how their ways of operating change" (Hackenberg, 2010, p. 397). By tracking the changes in student thinking, researchers gained insight into how students learned particular content by leveraging prior conceptions.

This focus on individual student learning was purposeful. Steffe and Thompson (2000) explained the historical context. At the time of the emergence of the method in the United States, the type of study that dominated educational research were ones that assessed teaching effectiveness by looking for statistically significant differences in outcomes on exams. The teaching experiment was a departure from this type of work; turning away from effective teaching moves or environments to taking seriously the nature of students' conceptions and students' learning processes as they engaged with particular content. This broadened the research literature in several ways. It not only redefined teaching from something that acted on students to something that interacted with the knowledge students' brought to the classroom, but also redefined learning from a score on a test to the progression of children's conceptions.

While the level of detail would be sufficient to see variability in the nature of students' conceptions, typically these studies only had one or two students, limiting the amount of variability that could be captured. However, some researchers were able to document variability in student thinking through the use of multiple teaching experiments with students starting at different cognitive levels (Hackenberg, 2010, 2014; Steffe, 1992; Steffe & Olive, 2010). They have argued that the way the student leverages prior knowledge to make mathematical progress depends on the cognitive level he is starting at. This begins to investigate the relationship between students' individual ways of reasoning and their mathematical progress, but stops short of investigating how that variability affects their learning in social situations.

Other scholars have investigated the relationship between participation in classrooms and individual mathematical progress by examining tasks and teacher moves. For example, Ellis, Ozgur, Kulow, Williams, and Amidon (2012) showed how particular tasks encouraged the shifts in students' thinking by encouraging them to explore particular mathematical relationships (e.g. the relationship between two quantities, which may have helped them coordinate the two quantities as they developed proportional reasoning). Tzur (2004) also explored how tasks can support learning. In his study, he identified general categories for the tasks he gave and articulated their function. One type of task, reflective, encouraged students to notice regularities among the results of activities to establish relationships between the activities and their results. Barrett and Clements (2003) took note of teaching moves that encouraged progression on the learning trajectory they advanced for abstracting linear measurement. However, instead of general task categories, like reflective tasks, their moves were topic specific. This type of work explores connections between the social environment and students' learning, but does not explicitly investigate variability from established ways of reasoning.

Documenting the Progress of a Classroom

In order to understand the relationship between individual ways of reasoning and ways of reasoning accepted by a classroom community, scholars need a way to conceive of the mathematical progress of a classroom as a whole. This can be found in work by researchers who did classroom-based research. This research was similar to constructivist teaching experiments in that the researchers carefully considered students' conceptions and how to advance them, but were situated in larger classrooms (see Bowers et al., 1999; Cobb et al., 2003; Simon, 1995).

One example of this type of work was done by Bowers, et al. (1999), which was briefly described in Chapter 1. They reported the mathematical development of a 3rd grade classroom as students learned about place value. In this class, students reasoned about quantities of candies packed in boxes of 100 and rolls of 10. Students were tasked with determining if two arrangements of boxes, rolls, and pieces contained the same amount of candy. One way of doing this that became accepted in the classroom community was transforming both arrangements into the same canonical arrangement. Later, a more sophisticated practice became institutionalized in the micro-culture, in which students could transform one arrangement into the other.

This work was done from the emergent perspective (Cobb & Yackel, 1996), which was described in Chapter 1 of this dissertation. Researchers can use the emergent perspective to track the mathematical progress of a classroom community by identifying *emergent mathematical practices*, which are ways of operating that have become accepted or taken-as-shared in the class community. Taken-as-shared means participants assume that other participants are familiar with and understand the way of operating. This phrase is used to emphasize that the researchers do not claim that all students share identical ways of operating. In fact, the claim is not about individuals' understandings at all. Rather, the researchers claim that the way of operating has a particular status in the classroom community. Namely, that people assume that others understand that way of operating. The evidence for this status is found in the way people use the way of reasoning. In particular, the way of reasoning is thought to be taken-as-shared when students use it without fully explaining or justifying it. This means that math practices are not simply the conceptions held by the majority of students. Rather these are ways of operating that are accepted by the community as a whole. As such, the community as whole should be thought of as its own entity with its own characteristics that exist outside of the characteristics of its individual members.

Rasmussen and Stephan (2008) elaborated the analytic techniques used to establish the practices that emerge in classrooms. As they did so, they made modifications to Cobb and Yackel's (1996) original work. For example, they talked about normative ways of reasoning that function as if shared rather than ones that are taken-as-shared. This highlights the particular methodology used to identify them, which focuses on the function utterances play in an argument. Another difference is that the math practices identified by those following Cobb were usually one normative way of acting, whereas Rasmussen and Stephan's (2008) techniques reveal several conceptually related normative ways of reasoning, which they group together as one practice.

As other researchers have analyzed emergent mathematical practices, the nature of the practices documented has changed. Originally, practices were observable behaviors, activities, or strategies. For example, Bowers et al. (1999) talked about how students transformed the packing arrangement of candies. However, scholars have begun to use more cognitive terms by talking about *ideas* that are taken-as-shared. For example, in a study about negative numbers, Stephan and Akyuz (2012) reported that the idea that "a minus sign is different from a negative sign" (p. 458) became normative in the class. This deals with the meaning of a symbol, not an observable strategy.

This fruitful way of tracking the mathematical progress of the classroom is compelling, as teachers tend to experience classroom interactions as interacting with a collective rather than a collection of individual students. The constructs of taken-as-shared and function as if shared are ways to rigorously identify ways of reasoning that feel as if the class as a whole has accepted them. However, since these interactions reveal proprieties of the collective rather than the individual, a natural question is how does participating in these interactions affect students' personal ways of reasoning? The researchers care in defining properties of the collective without making inferences about individual participants honors the interpretive nature of knowledge, which in turn underscores the necessity of examining individuals' ways of reasoning. While these studies provide compelling images of how educators might advance the mathematical agenda in their classrooms, they ultimately want students to advance their personal ways of reasoning as a result of participation in these classrooms. As such, researchers need to better understand the relationship between individual ways of reasoning and ways of reasoning that became normative in the classroom.

Coordination of Emergent Practices and Individuals' Participation

As the emergent perspective developed, the relationships between emergent practices and individuals' ways of participating in those practices have begun to be studied. For example, shortly after the emergent perspective was put forth (Cobb & Yackel, 1996), Cobb himself (1999) reported how students could participate differently in emergent practices. His example was situated in a ten-week teaching experiment with 29 twelve-year-old students studying ways to reason about the distribution of a data set. As students explored various data sets, the first mathematical practice to emerge was students describing qualitative features of frequency plots (e.g. the data were “bunched up”). As the second mathematical practice emerged, students began to be able to describe these features in more quantitative terms (e.g. by describing how much of the data lay in a certain range). This allowed them to talk about features of the distribution itself, such as the median. Students could then compare data sets by directly reasoning from the distributions without regard to the values of any one particular data point, for example, by comparing medians. However, Cobb noted that as the second practice emerged many students had difficulty in understanding students’ explanations. As students argued using features of the data distribution, other students asked about the actual values of the data points. While these students may have eventually understood the specific arguments, it seems that they struggled to conceive of the distribution itself as an object of study beyond a collection of data points. As such there were significant qualitative differences in the ways the students participated in the practice. Cobb said that this was not unique to this study and highlighted “the need to clarify the relation between individual students' reasoning and the collective practices in which they participate is therefore a pressing one” (p. 33).

The most in-depth response to this Cobb’s call came from Stephan, Cobb, and Gravemeijer (2003), as described briefly in Chapter 1. They explored two first grade students’, Nancy and Meagan’s, participation in the development of collective mathematical practices around measurement. The purpose of this study was to develop an image of how students participated in mathematical practices and thereby illustrate how individual learning can occurs through participation in emergent practices. This is related to, but not the same as, characterizing the relationships between individuals' ways of reasoning and the math practices. However, the narrative is detailed enough that one can see relationships and even begin to hypothesize why those relationships exist. In general, the two students reasoned in ways that were consistent with the emergent mathematical practices, but at times one of the students, Meagan, would reorganize her way of reasoning to be consistent with the practice after it had already been established in the classroom. This was most clearly seen in the first mathematical practice.

Students in this class were asked to engage in a fictional world in which the length of the king's foot was a unit of measure. The first mathematical practice emerged as students debated how to use this unit of measure. Some students did not count the first foot, while other did. The researchers argued that the students who were counting the first foot were able to conceive of the foot as measuring out a certain length, meaning that as they counted paces they were counting the number of times the length of the foot fit into length of the object they were measuring. The other students were essentially counting the number of steps. After teacher intervention, the students decided to count the first step and this way of counting became an established practice. However, analysis of Meagan's way of participating revealed that she had not necessarily reorganized her knowledge as she seemed to still be counting paces. This is reasonable since students could potentially be involved in the debate, but miss the mathematical difference between the two approaches. This could lead to an acceptance of counting the first step, without a reorganization of knowledge. However, since many of the other students seemed to have a similar problem, it came up in class and was explored further. Through this discussion Meagan seemed to reorganize her knowledge and begin conceiving of the pace as covering a length.

Other scholars have also contributed to understanding the relationship between emergent practices and individuals' ways of participating in those practices, by not only comparing individuals ways of reasoning with the practices, but also examining how individuals contribute to and learn from the practices. For example, Rasmussen, Wawro, and Zandieh (2015) expanded the last row of Cobb and Yackel's (1996) interpretive framework by adding new constructs. Instead of thinking of the social perspective as just emergent mathematical practices, they added disciplinary practices. This means that in addition to tracking the emergence of practices specific to local community, they also considered how the classroom participants were engaging in practices central to the work of mathematicians. They also expanded the individual perspective from individual conceptions and activity to now include two constructions, participation in mathematical activity and mathematical conceptions. The researchers conceived of participation in terms of the roles students took on as they contributed ideas to whole class and small group discussions, based on the work of Krummheuer (2007, 2011). They described individual conceptions as the images and ideas they brought to bear in their work. They then used the four constructs in this expanded framework to correspond with four analytic passes as they investigated the ways students participated in and leveraged their conceptions as they developed sophisticated emergent and disciplinary practices.

Tabach, Hershkowitz, Rasmussen, and Dreyfus (2014) also contributed to researchers' understanding of the relationship between individuals' ways of reasoning and those that function as if shared by examining how ideas flowed in classroom from small group to whole class and vice versa. To do this they combined two methodological tools, abstraction in context (AiC), which explains the process by which individual knowledge arises and develops in social contexts, and documenting collective activity (DCA), which explains how knowledge develops at the collective level. Their analysis was fruitful in that is showed the importance of attending to the role of small groups in the advancement of mathematical progress at the classroom level, especially in relation to how ideas were developed in whole class discussion. However, this study did not examine students' individual ways of reasoning after instruction. Rather, the focus was on the relationship between individuals and the collective in the process of the establishment of mathematical practices rather than the results of that participation.

One study that explored students' conceptions after instruction was Bowers et al. (1999). The researchers did this by interviewing the students after instruction. However, the purpose of this study to examine how increasingly sophisticated mathematical practices emerged, not to explore the relationship between mathematical practices and individuals ways of reasoning. As such, the purpose of the interview was to show that the instruction had been successful. However, because the mathematical practices and the interview results were both reported in some detail, relationships can be explored.

In Bowers et al.'s study, the researchers reported the development of five emergent practices as students reasoned about inventory of a candy story. By engaging with problems set in this context, students developed understandings of place values rooted in the context of packing candies. First, they decided to pack candies in packages of 1, 10, and 100 (MP1). Then they explored how they could pack the same amount of candies in different ways (MPs 2 and 3). Then they solved addition and subtraction problems set in the context of keeping track of inventory of candy (MP4). Lastly, they moved to solving addition and subtractions problems symbolically (MP5).

After instruction had concluded, the researchers interviewed the students, asking them to solve two problems. For the first problem, students were shown a bag filled with 360 crayons and asked how many bags 10 crayons could be made with the crayons. In the second task students were asked to solve the problem 42-18, which was given symbolically. In general, students did strikingly well on these problems, suggesting the instruction was successful.

From the interview results, it appears that most students reasoned in ways that were consistent with the math practices that emerged in the classroom. The first task on the interview is related to the first three math practices, in that it was about reorganizing the packing of candies. The second task was most closely connected the last math practice, in that both were ways of reasoning about symbolic problems. While this makes the study compelling, in that the instructional sequence seemed to be efficacious in supporting students in building productive ways of reasoning about addition and subtraction problems, part of the power of the instructional sequence seems to be that the symbolic reasoning arose out of reasoning about a real-world context, packing candies. In this sense, MP4 seems a critical transition from reasoning about candies to reasoning about addition and subtraction of numbers without context. Leaving individual interpretations of this math practice unexplored leaves questions about what students make of these critical transition periods. This could be important as it may have implications for whether or not their symbolic reasoning is procedural or conceptual in nature.

My answer to Research Question 1 will contribute to this area of research. Cobb and Yackel (1996) originally posited that the relationship between individuals' ways of reasoning and emergent practices was reflexive, but not direct. This has been confirmed in empirical studies with Rasmussen and his colleagues (Rasmussen et al., 2015; Tabach et al., 2014) demonstrating the reflexivity of the relationships and Cobb (1999) demonstrating its indirectness. However, despite Cobb's demonstration that students can have personal ways of reasoning that are qualitatively different than established practices, other research seems to suggest that students eventually develop productive ways of reasoning (Bowers et al., 1999; Stephan et al., 2003). Thus, more research that directly examines this relationship, especially those that relate emergent practices to students' subsequent ways of reasoning, is needed. In this study, I examined this relationship as I documented the math practices that emerged in the classroom I studied (see Chapter 4) and then investigated students' personal ways of reasoning and explored the relationship between those ways of reasoning and one of the emergent mathematical practices (see Chapter 5). Furthermore, these results will be expanded upon in Chapter 6 as I investigate the mathematical meanings that were constructed through the classroom discourse to help explain how students may have been interpreting the emergent practices. This may help explain some of the variability in students' ways of reasoning from the practices.

Mathematical Discourse

In this section I consider the research on classroom mathematical discourse. Examining discourse can give powerful insights into a variety of phenomenon. In this review, I chose to focus on studies that examined the teaching and learning of particular mathematical content (as opposed to say, how social relationships are reified through discourse) as the topics of these studies are more closely related to this dissertation study. The studies I considered roughly fell into four categories: (a) those that explored theoretical claims about the relationship between discourse and learning, (b) those that focused on discourse moves and their effects on the learning process, (c) those that used conversation analysis to characterize teacher's practice, and (d) those that used discourse analysis techniques rooted in Systemic Functional Linguistics (SFL). In the following sections, I detail these four categories.

Theoretical Claims of Discourse's Central Role in Learning

As Vygotskian perspectives gained prominence in the thinking of educational researchers, scholars began to look to discourse to gain further understanding of the learning process. Vygotsky claimed that higher mental functioning begins in the social plane and is slowly internalized by students (Vygotsky, 1978). Specifically, as a student and a more knowledgeable other participate in reasoning and problem solving activities, the more knowledgeable other starts by leading the student through the process and then the student slowly begins to gain competence and to take the lead the process, with the more knowledgeable other guiding only when necessary. As such, the learning process is inherently social and co-constructed by both the learner and more knowledgeable other (Ash & Levitt, 2003; John-Steiner & Mahn, 1996). This process is called internalization and is said to occur through the transformation of communicative language into thinking (Enyedy, 2003). This means that thinking in inherently mediated by tools (e.g. language, counting systems, diagrams; Goos, 2004).

Complementing the idea of internalization is the theoretical construct of the *zone of proximal development* (ZPD), which is the space in which internalization takes place. The ZPD is defined as the difference between what a student can do on her own versus what she can do with the help of a more knowledgeable other (Enyedy, 2003; Goos, 2004). Thus, scholars who study teaching and learning interactions from a Vygotskian perspective, study this zone (Goos, 2004; Goos, Galbraith, & Renshaw, 2002; Lau, Singh, & Hwa, 2009). In doing so, they often elaborate the construct. For example, Goos (2004) described three different types of

ZPDs. The first interaction is between a teacher and a student, which she termed scaffolding. In the scaffolding ZPD the teacher and student mutually appropriate each other's actions as the teacher helps the student engage in more sophisticated reasoning. The second type of ZPD is an interaction between two peers, which she termed collaboration. This ZPD differs from the first, in that the two students are more equal status as opposed to a student interacting with a more knowledgeable teacher. Nevertheless, students working together can be more capable than either would be on their own. This means the social interaction expands their capabilities and thus acts as a ZPD. Finally, the third type of interaction is one in which everyday concepts are leveraged to develop more formal ways of reasoning. She calls this ZPD interweaving, as in interweaving everyday and formal ways of reasoning. Using these three types of ZPDs as a lens when investigating the data, Goos investigated teacher actions to support productive engagement in classroom interactions. For example, the teacher asked students to explain and justify their ideas to each other, which supported productive engagement in the collaboration ZPD. Also, the teacher made connections between everyday words and technical terms to promote productive engagement in the interweaving ZPD. In this way, the Goos elaborated the construct of ZPD by not only defining different types, but also by helping to characterize ways teachers can support effective engagement in the ZPD.

Another way in which Vygotsky's work has been extended is through the work of Anna Sfard. She agreed with Vygotsky that participation in discourse is central to learning, but made the stronger claim that thought and communication are a single phenomenon (Sfard, 2007, 2008). Her claim is that thinking is dialogical, in essence self-communication, in which the thinker argues with himself, asks himself questions, and answers himself. She conceives of mathematics as a particular type of discourse, which like all discourses, has its own rules. According to her theory, each discourse, including mathematics, has four characteristics: the way its participants use words, the way they use visual mediators, the narratives that are endorsed, and routines that are used. Visual mediators describe appropriate ways of interacting with visual objects in the discourse. For example, in mathematics, graphs are visual objects that mediate interaction. Narratives are texts that describe objects and relationships. Endorsed narratives are ones the discourse treats as true. For example, in mathematics axioms and theorems are endorsed narratives. Routines are patterns in the discourse. Learning mathematics is then an *individualization* of the discourse. This refers to the process through which individuals come to participate, both with others and with oneself, in the discourse. As they do so, they can reason about things they could not before (like negative numbers, see Sfard, 2008).

As students learn the discourse of mathematics, they learn the rules that govern that discourse. These can be object level rules, which define how objects are related to each other (e.g. acceptable use of a particular word). Object level rules are about the content of the discourse. Meta level rules, sometimes just called meta-rules, on the other hand, are rules about how the discourse should operate (e.g. the rules of proving or defining). Meta-rules is a quite broad term and can refer to phenomenon that might also be referred to as a social norm, like raising one's hand before speaking (Xu & Clarke, 2012). If one can participate in that discourse, by following the rules, he or she is communicating (with others or with self) mathematically. Students learn these rules via breakages in the discourse when they try to participate in it. These are when different participants seem to be abiding by different rules, a situation which Sfard calls commogonitive conflict (2007, 2008). As such, learning mathematics is the same as adopting the discourse of mathematics. This entails learning the vocabulary, routines, and endorsed narratives, as well as learning the rules of how the discourse operates (e.g. the rules regarding what makes an appropriate proof).

Scholars have elaborated Sfard's theory by using her framework to analyze teaching and learning interactions. This is often done by looking at the data through the lens of the four characteristics of discourse (Caspi & Sfard, 2012; Güçler, 2012; Sfard & Lavie, 2005). For example, Güçler (2012) explored the discourse on limits in a beginning-level undergraduate calculus classroom with these four characteristics acting as the four axes on which she coded classroom interactions. Her analysis of word use provided interesting insight into explaining students' difficulties with limits. She argued that the teachers' shifts in his use of the word limit corresponded with student difficulties. In particular, the teacher usually referred to the limit as a distinct mathematical object (over 82% of his references to limits), however, when talking about the informal definitions of limits or how to compute them, he would shift between talking about the limit as number and the limit as a process. Students seemed to struggle navigating this change as they endorsed the narrative that limits were a process in their speech rather than limits being a number. I now detail the second category of literature.

Discourse Moves and their Effects

Some scholars use discourse to understanding the learning process in a different way. The studies presented in the previous paragraphs tightly connect learning to participating in discourse. Therefore, when they study discourse they are studying the nature of the learning process directly. Other scholars conceive of discourse as affecting the learning process through the creation of particular learning environments rather than being the synonymous with learning. For example, scholars have examined how the teacher's or (less frequently analyzed) the students' discourse moves influence students' engagement with the learning environment. The result of this type of research is a description of a particular way of speaking or interacting (the discourse move) along with evidence of its effects. For example, O'Connor and Michaels (1993) illustrated how a particular reaction to expressed student thinking, revoicing, can encourage mathematical argumentation. Revoicing occurs when a teacher summarizes something a student has said, perhaps adding details to or rewording the utterance. When a teacher revoices she can create argumentative positions, thereby fostering mathematical argumentation. Martino and Maher (1999) similarly illustrated how questions can serve different functions in the classroom, such as encouraging students to justify, generalize, make connections, or draw attention other students' solutions. Similarly, Franke et al.

(2009) and Webb et al. (2008) found that when teachers probed student responses by asking clarifying questions and asking students to elaborate their responses, students were more likely to give correct and complete explanations. This may help explain Pierson's (2008) finding that teacher follow-ups that built on the student's idea were correlated with achievement. Responsive follow-ups were particularly important for students who entered instruction with low levels of prior knowledge. Researcher have also found that teacher follow-ups can also constrain opportunities for reasoning. Bieda (2010) found teachers often shut down opportunities for students to engage in proving activities by sanctioning conjectures or putting them to a class vote. Together, these studies show that the teacher's discourse moves have a large effect on shaping the nature of the intellectual work required of students and their subsequent reasoning. Through their discourse moves, teachers can encourage students to extend their mathematical thinking and engage in mathematical argumentation.

Most of the research that investigates discourse moves examines how these moves create opportunities for further reasoning. In these studies, the reasoning is normally talked about in a general way, as in opportunities to prove or engage in mathematical argumentation. However, there also exists a small and emerging type of research that seeks to make mathematical connections between discourse practices and students' reasoning about a specific mathematical topic. For example, Rasmussen and Marrongelle (2006) described a of way of noting ideas, called a transformational record, that advanced the mathematical agenda in the classroom they studied. In this case, the teacher wrote down students' thinking in a way that created a model from which students could later use to reason. The authors illustrated the idea of a transformational record by describing how a teacher recorded students' reasoning population growth in way that an expert would recognize as a tangent vector field. Students then used this record to reason about the shape of the solution. Notice that while the idea of a transformational record could be employed in any mathematical topic, any particular instantiation of the record is highly connected to specific topic being discussed. In particular, the practice is used to advance the students' thinking about a particular mathematical topic.

Similarly, Lobato, Hohensee, and Rhodehamel (2013) also analyzed topic-specific features of discourse practices. In their analysis they created conceptual connections between discursive practices and what students noticed. They did so by comparing two classrooms. In the first class the majority of students were able to coordinate two quantities as they reasoned about a linear situation, while most of the students in the second class inappropriately relied on various forms of non-multiplicative reasoning. Upon investigation of the lessons the students received, they found that in the first class the teacher pressed for the meaning of quantities, a discursive interaction they termed quantitative dialogue, while in the second class the teacher used a discourse routine which emphasized additive growth in a single quantity. The authors identified two other types of discursive interactions that were also important for making sense of their data: highlighting and renaming. A person is said to highlight when he or she draws attention to a particular aspect of a representation by visibly interacting with it, for example, by drawing on it. Renaming is when a person uses a label from mathematical practice to change the name an existing idea. Using these three types of discursive interactions, Lobato et al. (2013) were able to account for the emergence of shifts in students' attention, thereby providing insight into how discourse practices can affect the nature of students' reasoning.

Using Conversation Analysis to Characterize a Teacher's Practice

In the previous two sections, discourse was related to learning either directly (as being born out of and inseparable from discursive interactions) or indirectly (as being influenced by learning environments). Other research uses discourse analysis as a lens on the nature of interactions rather than a description of the mechanisms of learning. Specifically, researchers can investigate discourse to characterize the nature of a teacher's interaction with her students (Blanton, Berenson, & Norwood, 2001; Forman, Mccormick, & Donato, 1998; Nathan & Knuth, 2003). This is different from the previous category in that the analysis is meant to provide a window into the teacher's practice and not meant to describe how the teacher's practice affects the learning environment. This is often done to see if a teacher's practice is in line with her goals (e.g. the amount of student participation or whether or not she is sharing authority). For example, Nathan and Knuth (2003) analyzed who talked to whom (teacher to student, teacher to class, student to teacher, student to student) to investigate the centrality of the teacher in the discussion. They found that very few mathematical statements were student-to-student despite the teacher characterizing lessons as having good student participation. Similarly, Forman, McCormick, and Donato (1998) examined when the teacher overlapped her speech with students (essentially cutting them off) to reveal the extent to which she allowed her students to determine the validity of their peers' solutions. They found that the teacher met her goal of sharing responsibility to give explanations, but not her goals to share authority for evaluating those explanations. Finally, in characterizing a student teacher's evolving practice by examining her talk, Blanton et al. (2001) found that she initially asked leading questions and gave hints to funnel students to a particular strategy. Later however, she used questions less for instructional purposes and more to investigate student behavior.

Just as these analyses are useful to the research community in characterizing the teacher's practice, some scholars suggest that careful attention to language can help teachers understand their own practice. For example, Nathan and Knuth (2003) pointed out to the teachers in their study how the information was flowing in their classrooms. As the one teacher realized her practice was inconsistent with her goals, she changed how she participated in the discourse, which changed how the information flowed. Similarly, as the teacher in the study by Blanton et al. (2001) focused on the language in her classroom, she also changed her practice.

The studies in each of these three sections are important in their own ways, but none of the studies offer systematic ways of examining the construction of meaning in classrooms. For example, the first two categories have powerful implications for educators. By examining the discourse with a particular theoretical lens, scholars can gain insights into the role of discourse in learning the process. These insights normally have implications for teachers. For example, Güçler's (2012) study suggests that teachers should become aware of the way they use words when describing mathematical content as the shifting of use may be difficult for students to navigate. The work on discourse moves has even more direct implications for teachers. For example, revoicing can create argumentative positions or asking students to elaborate can help them articulate complete and correct solutions.

This third category is closer to providing a way to investigating meaning in that it uses discourse as a way to characterize something. However, the thing that it is characterizing is teachers' practice, not mathematical meaning. Systemic Functional Linguistics (SFL; Halliday, 1978; Halliday & Hasan, 1985) is a broad theory meant to describe how English works in general. As such, it offers a systematic way of analyzing how meaning is created through discourse.

Systemic Functional Linguistics

Unlike the other approaches described, which focus specifically on teaching and learning, SFL is a general theory meant to describe how meaning is made through language. Rather than thinking of words as having meaning in and of themselves, words' meanings are created through their function in the text. This meaning is created with the three metafunctions of language: ideational, interpersonal, and textual. At its core, ideational meaning centers on the content of the discourse, while the interpersonal meaning centers on the relationship between the conversation partners and their respective roles. This can include the speakers' expressed attitudes toward the content and towards one another. These meanings are organized by the textual metafunction of language.

Different educational scholars focus on different metafunctions to illuminate various aspects of teaching practice. This was theory was largely introduced to math education through science education's use of it when Lemke examined classroom talk from this perspective in *Talking Science* (Lemke, 1990). In his examination he attended to all three metafunctions of language, which revealed patterns of interaction in the classroom, how the content was talked about, how science was positioned in the classroom, and how students began to adopt ways of talking that are considered scientific. The breadth of his findings attests to the comprehensiveness of the theory. As such, scholars usually need to focus their analysis is some way. This can either be done by attending to how the three metafunctions of language create a particular type of meaning (Atweh, Bleicher, & Cooper, 1998; Morgan, 2005) or by focusing on a particular metafunction of language (Chapman, 1995; Herbel-Eisenmann, 2007; Herbel-Eisenmann & Otten, 2011; Herbel-Eisenmann, Wagner, & Cortes, 2010; Mesa & Chang, 2010).

For example, Morgan (2005) studied the nature of definitions and how the differed when used in a mathematics texts for 15-16 year old students and in an academic research paper. To do this analysis she used Halliday's ideational metafunction to examine what process are being talked about in definitions, the interpersonal metafunction to examine the roles of the actors that are at play in definitions, and textual metafuctinon to examine how is the status of definitions were established. She found that in more advanced texts, definitions were constructed for creative purposes. In texts for less advanced students, there was more of a one-to-one word-concept relationship. This raises the concern of access for these students to not only more advanced mathematical content, but also more authentic mathematical practices.

Other scholars have focused on one metafunction of language in their analyses. For example, Mesa and Chang (2010) and Herbel-Eisenmann (2007) both focused on the interpersonal metafunction in their analyses. Mesa and Chang (2010) explored the differences between two classrooms. Through the coding of the teachers' talk, the researchers found that in one classroom the teacher maintained a more authoritarian position, despite both having high levels of student participation. Herbel-Eisenmann (2007) also examined the interpersonal metafunction, but instead of analyzing classroom interactions, she analyzed a middle school mathematics textbook. She examined the construction of the roles of the authors and readers and their relationships. She found that even though the authors were committed to shifting the locus of authority to students, the relationships constructed through the text were often hegemonic. She suggested this might be due to the traditional forms of discourse in mathematics that are often filled with imperatives that leave little room for the reader to preserve their own agency.

Conversely, Herbel-Eisenmann and Otten (2011) focused on the ideational metafunction of language. These researchers also compared two classrooms, but instead of analyzing the roles of the teacher relative to the students, they analyzed the sematic relationships between key concepts in the lesson. This revealed subtleties in how the mathematical content was being talked about and the meanings that were constructed. One of these subtleties was the ambiguity in phrases like "rectangles are parallelograms" that was mentioned in Chapter 1. Other subtleties found dealt more with shifting meanings in the classrooms. For example, in both classrooms the teachers shifted the meaning of the mathematical terms base and height. They both talked about these words as geometric objects and as quantities. This might be expected since the lessons required discussion of both geometric representations and algebraic formulas. However, the shifts in word meanings were left implicit and it is not clear that all students were able to navigate the change in meanings.

This type of detailed analysis is necessary to answer Research Question 2. This question seeks to establish a plausible, although partial, explanation for how students could participate in emergent practices while potentially reasoning ways that are qualitatively different from the practice. As such, I need to carefully document the mathematical meanings that were being established in the classroom. By systematically looking at the relationships expressed between words SFL techniques can reveal subtleties in the meaning construction that could be missed with coarser grained approaches.

Since this analysis dealt with the mathematical meanings about exponential and logarithmic relationships, I now turn to the somewhat limited literature on teaching and learning exponents and logarithms.

Teaching and Learning of Exponential and Logarithmic Relationships

In my analysis I considered the normative ways of reasoning and individuals' ways of reasoning about a topic that is conceptually rich, conducive to a variety of ways of thinking, and mathematically important—the exploration of exponential and logarithmic relationships. Despite their historical significance and importance for modeling real world phenomena, the teaching and learning of exponential relationships in general, and logarithms in particular, has been understudied.

When logarithms were invented, they dramatically changed the way computations were done. At this time, calculations had to be done by hand. The time-intensive nature of vast computations slowed astronomers, architect, merchants, and bankers in their work. Building on Stifel's ideas, Napier invented logarithms (Villarreal-Calderon, 2008) as he explored ways to compute more efficiently. The logarithm, along with its tables, provided a way for professionals to turn multiplication problems into addition problems, which were much easier to solve by hand. By greatly reducing the time required to compute, Napier's invention immediately changed these professionals' work (Bakst, 1967; Gladstone-Millar, 2003).

Logarithms are not only historically important, but continue to have applications in mathematical modeling. Wood (2005) explained that logarithms are a helpful way to model phenomena when the range of possible values is particularly large, as in the case of decibels and the Richter scale. Liang and Wood (2005) mentioned astronomy and pH level as other applications, while Bakst (1967) reminds us of the application to the brightness of stars. In order to appropriately model and reason about values that can fall on a large scale, logarithms are essential.

Even though logarithms are an important topic, not much is known about how students' think and learn about them. With the notable exception of the work of Confrey and Smith (Confrey, 1994; Confrey & Smith, 1994, 1995) and Kastberg (2002), research on student thinking about logarithms consists of noting calculational mistakes students make (Barnes, 2006; Hoon, Singh, & Ayop, 2010; Liang & Wood, 2005; Nogueira de Lima & Tall, 2006) or the misapplication of linear reasoning (Berezovski, 2004; De Bock, van Dooren, Janssens, & Verschaffel, 2002). As such, this work focuses on the conceptions or skills students lack. For example, many scholars suggest that students may not be conceptualizing a logarithm as number (Berezovski, 2004; Liang & Wood, 2005; Wood, 2005). This research is limited in that it does not reveal what conceptualizations, images, and skills students have that could be useful in the advancing their reasoning about logarithms.

Similar to the work on logarithms, many scholars who study students' understanding of exponential relationships detail the procedural errors individuals make when reasoning (Alagic & Palenz, 2006; Cangelosi, Madrid, Cooper, Olson, &

Hartter, 2013; Davis, 2009). These errors include the over application of linear reasoning (Alagic & Palenz, 2006) and reasoning about negative exponents incorrectly (Cangelosi et al., 2013). These errors exist even in mathematics teachers (Alagic & Palenz, 2006; Davis, 2009). Strom (2006) also discovered teachers found reasoning about non-integer exponents especially difficult.

As mentioned above, one major exception to casting students' knowledge in terms of skills they lack or errors they make is the work of Confrey and Smith (Confrey, 1994; Confrey & Smith, 1994, 1995). They suggested students have separate ways of thinking about additive and multiplicative situations. They introduced a construct, splitting, which describes a one-to-many, multiplicative action. Instead of conceiving of multiplication as repeated addition, they suggested students naturally have the capacity to the think of a simultaneous duplication action, splitting, and claim it is cognitively distinct form repeated addition. Exponential and logarithmic relationships link the additive and splitting worlds, by linking a variable that grows additively with a variable that grows multiplicatively.

The work of Kasterberg (2002) is the other major exception in the work on logarithms. While much of her work focused on students' lack of understanding, she was also able to find cognitive resources that may be valuable to a teacher. In particular, she found that students tried to make sense of the relationships in the numbers by looking for patterns when presented with tables of values. This helped them find common ratios and difference. This provides evidence that they attended to both geometric and arithmetic patterns.

Similar to work on logarithms, only a few studies focus on individuals' ways of thinking about exponential functions rather than just procedural errors. One notable study was a teaching experiment of three 8th grade students conducted by Ellis et al. (2013). They compared the reasoning of two students, one who focused on the covariational relationship between the variables and one who focused on a correspondence relationship. A covariation perspective means that the student coordinates how the dependent variable changes as the independent variable changes, while a correspondence view emphasizes a static relationship between individual x-y pairs (Smith, 2003; Smith & Confrey, 1994). Ellis, Özgür, Kulow, Williams, and Amidon (2015) found the student who focused on the covariation perspective was able to reason more powerfully as she had less trouble shifting between a covariation and correspondence perspective. Furthermore, her understanding of the relationship between the two varying quantities enriched her correspondence view.

In another body of scholarly writings about exponential and logarithmic relationships, researchers give pedagogical recommendations for teaching. Webb, Kooij, & Geist (2011) recommended starting with informal explorations of exponential situations and then dropping context in the style of realistic mathematics education (RME). Others (Katz, 1986; Van Maanen, 1997) recommended a curricular sequence that mirrors the historical development of logarithms. Finally, Weber (2002) developed a hypothetical learning trajectory based on APOS theory (Dubinsky & Mcdonald, 2001), complete with activities to help students progress through the learning trajectory. These recommendations are helpful starting points, but none were implemented with students. In this way this study contributes to the literature on teaching logarithms by not only developing a hypothetical learning trajectory, but also observing its implementation and carefully documenting the learning of the students.

Research Questions Revisited

I now revisit the research questions and elaborate them using ideas from the literature review.

> Research Question 1: How are individuals' ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

One way scholars have documented the mathematical progress of students is to identify increasingly sophisticated ways of reasoning that function as if shared in the classroom community (e.g. Bowers et al., 1999; Stephan & Rasmussen, 2002). This is important because it gives teachers an image of how complex ways of reasoning might develop in their classrooms. However, it is important to note that while these emergent ways of reasoning likely shape students' individual ways of reasoning, they do not determine what students learn. Scholars working from the emergent perspective have always maintained that the relationship between normative ways of reasoning and students' ways of reasoning is indirect and reflexive (Cobb et al., 2003; Rasmussen & Stephan, 2008). This means that as students participate in the social processes involved in creating emergent mathematical practices, this affects their personal conceptions, but a researcher should not assume that all students construct identical conceptions as a result of participation in the classroom.

While the emergent perspective outlines the general nature of this relationship between students' ways of reasoning and normative ways of reasoning, the details of this relationship have been left understudied. Early on, Cobb (1999) provided evidence that students may reason in ways that are qualitatively different from established practices. However, the few studies that have documented students' subsequent reasoning after participating in emergent practices seem to imply that if significant differences exist in students' ways of reasoning the students tend to reorganize their ways of thinking by continuing to participate in classroom discussions (Bowers et al., 1999; Stephan et al., 2003). Because this has only been examined in a few studies, more research is needed for scholars to understand how typical these results are, if qualitative differences can persist beyond instruction, and under what circumstances students reorganize their knowledge to become more consistent with productive emergent practices. In answering this Research Question 1, I will provide more insight into the nature of this relationship. Research Question 2 will then building on the findings of Research Question 1.

> Research Question 2: What mathematical connections exist between the focus students' ways of reasoning in the post interviews and the discursive interactions between them and other students and the teacher in both whole class and small group settings? Furthermore, how might the nature of these discursive interactions give plausible explanations for students' differing conceptions?

The answer to Research Question 1 will describe students' ways of reasoning after they have participated in emergent mathematical practices. While this gives insight into the nature and extent of individual variation from established practices, it did not elaborate why differences may have existed. Answering Research Question 2 will give insight into this. By analyzing semantic relationships among words I will document the mathematical meanings established in class. Similar work has proven to elucidate subtleties in the ways meanings were constructed that revealed things like ambiguities in the talk (e.g. "rectangles are parallelograms") or shifting meanings (e.g. talking about a base as both a geometric object and a quantity). These types of results may prove fruitful in explaining how different participants could have different interpretations of established practices (perhaps by revealing ambiguities) or why may have had a difficult time engaging in certain practices (perhaps because of shifting meanings). While the results of the analysis for Research Question 2 may have similarities in the findings of other researchers using an SFL approach to uncover mathematical meanings in a classroom, this work will expand upon previous results as the meanings will be situated in normative ways of reasoning and help explain documented ways of reasoning students use that differ from emergent practices.

Chapter 3: Research Methods

In this chapter, I present the methods I used to answer the following two research questions:

> Research Question 1: How are individuals' ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?
>
> Research Question 2: What mathematical connections exist between the focus students' ways of reasoning in the post interviews and the discursive interactions between them and other students and the teacher in both whole class and small group settings? Furthermore, how might the nature of these discursive interactions give plausible explanations for students' differing conceptions?

I first outline the connection between the research questions and the research methods and then describe the data collection and analysis methods in depth.

Overview of Methods for Research Question 1

This study was performed in an undergraduate mathematics class designed for prospective secondary teachers. In this course secondary mathematics content was explored in a conceptually oriented way. This meant to deepen the prospective teachers' knowledge of both mathematics content and students' thinking about that content. The class I studied consisted of 26 mathematics majors.

To answer Research Question 1, I first documented all the ways of reasoning that functioned as if shared in the classroom community. I then grouped these normative ways of reasoning into five mathematical practices (see Chapter 4). I then focused on the second emergent mathematical practice, which consisted of two normative ways of reasoning and investigated students' related individual ways of reasoning.

To perform this analysis I needed to collect specific kinds of data. First, I used the *documenting collective activity* method (DCA; Rasmussen & Stephan, 2008), described in more detail in the data analysis section, to infer the normative ways of reasoning. The DCA method was designed to study classrooms where genuine argumentation occurs. Thus, I needed to study a class where students regularly presented and discussed solutions to problems.

Second, I administered task-based clinical interviews (Ginsburg, 1997) after instruction was completed, to infer individuals' conceptions related to the normative ways of reasoning for a sample of students from the class. To infer relevant conceptions, I needed to engage the students in tasks that are related to these ways of reasoning. Because I was not be able to perform the analysis to determine these ways of reasoning before I conducted the interviews (shortly after the class had ended), I constructed tasks that reveal students' ideas about ways of reasoning I had hypothesized will become normative in the classroom community.

In Chapters 4 and 5, I present the findings from this analysis. In Chapter 4 I provide evidence for each of the normative ways of reasoning that occurred in the class. With each normative way of reasoning I provide an elaboration of the way of reasoning with examples of how the arguments shifted in the class once the way of reasoning became established. In Chapter 5 I present categories of conceptions that relate to Math Practice 2. In the Discussion section of Chapter 5, I address Research

Question 1 directly by articulating the relationships between individuals' ways of reasoning and ways of reasoning that functioned as if shared in the class community.

Overview of Methods for Research Question 2

While Research Question 1 focuses on the relationship between the emergent mathematical practices and students' individual ways of reasoning that arose from participating in those practices, Research Question 2 focuses on explaining the variation in student thinking from established practices. As such, I needed to understand how students could reasonably engage in mathematical conversations, which lead to the establishment of Math Practice 2, yet still hold differing conceptions. To do this I investigated the mathematical meanings created in these conversations by investigating the semantic relationships using the technique of thematic analysis from Systemic Functional Linguistics (Herbel-Eisenmann & Otten, 2011; Lemke, 1990) This technique will be described in more detail in the data analysis section of this chapter.

To perform this analysis I needed video of whole class and small group discussion to capture students' utterances, diagrams, and gestures. These video allowed me to determine the semantic relationships expressed. The analysis of these semantic relationships is presented in Chapter 6.

Data Collection

Setting

To answer the research questions, the classroom environment needed meet several criteria. First, to establish normative ways of reasoning using the DCA method, students in the class needed to routinely give arguments and explain their thinking. Second, this study is intended to investigate the discourse of a classroom in which students are building conceptual understanding of a topic. Thus, the treatment of the topic needed to be conceptual. Finally, this research has the potential to give insight into how conceptual understanding can develop in a classroom setting. This means the study will potentially be more useful if it contributes to educators understanding of a topic for which little is known about the teaching and learning of that topic.

A site that met all these criteria was a capstone course for prospective secondary teachers at a large southwestern university, during a 3 week unit on logarithms (consisting of 7.5 hours of instruction across 6 sessions). This class was selected in part because a mathematics education researcher taught it, which increased the chance that she would attend to conceptual issues in the content and encourage students to explain their thinking and give arguments. That turned out to be the case. Finally, the unit I studied focused on the meaning of exponents and logarithms, a topic for which issues of teaching and learning are not well understood by mathematics education researchers.

Because this work deals with the learning of prospective teachers, a critical reader may worry that it focuses on the learning of subject matter knowledge rather than pedagogical content knowledge (PCK), which focuses not only on the mathematics content itself, but also issues of how to teach the particular content (Ball, Thames, & Phelps, 2008; Shulman, 1986). However, it is important to note that mathematical knowledge for teaching, as defined by Ball, Thames, and Phelps (2008), includes common content knowledge and specialized content knowledge, both of which are components of subject matter knowledge. These elements are crucial for teaching and were the focus of this study.

Furthermore, while pedagogical knowledge is also important, there is limited research that illuminates pedagogical content knowledge for logarithms. For example, little is known about student thinking on logarithms (for exceptions see Berezovski, 2004; Kastberg, 2002). While some student errors have been reported, these are more procedural in nature than conceptual (Hoon, Singh, & Ayop, 2010; Liang & Wood, 2005). Similarly, fruitful representations have not been established in the literature. While common instructional sequences and tasks could be found with a textbook analysis, their usefulness is limited because of the lack of research on student thinking, making it difficult to judge how productive these instructional sequences and tasks are. This severely limits what empirically based PCK could be taught in a capstone course. Consequently, the course focuses on subject matter knowledge related to the understanding of logarithms, rather than student thinking.

Participants

In this study there were three types of participants: the teacher, the students in the class, and a subgroup of those students referred to as focus students.

Students. All students in the capstone course participated in this study in that they contributed to whole class interactions, from which I inferred the emergent mathematical practices. There were 26 undergraduates enrolled in the course. Many of these students had at least one mathematics course taught in a non-traditional manner. One class in particular, an undergraduate math course in which they reasoned about spherical geometry taught by a math education researcher, had been taken by many of the students. In this geometry course students worked on open-ended problems, presented their thinking, and engaged in argumentation. Using this format, the instructor used students' ideas to advance the mathematical agenda rather than solely relying on exposition. This format was similar to the one used in the capstone course which served as the setting for this dissertation study.

Since these students were undergraduate mathematics majors, they had likely previously been exposed to logarithms (e.g. in a calculus course in college and briefly in high school). However, given their comments in class it seems many had not fully explored the relationship between the additive and multiplicative relationships inherent in exponential and logarithmic reasoning.

Focus Students. To infer individuals' ways of reasoning, I collected additional data from seven focus students. These seven students were selected purely based on their willingness to volunteer. Originally eight students volunteered, but one did was not able to complete the post-interview.

Teacher. The teacher of this course is an experienced mathematics education researcher. She had taught this specific capstone course 11 times prior to this study and taken notes on her experience. She began including a unit on logarithms recently and had taught it two times previously. Her experiences as a mathematics education researcher and her experiences teaching this class make her an ideal candidate for study. Because of her experience as a researcher, she was aware of the literature on teaching and learning, which she actively drew from as she taught. As an experienced teacher, she had refined the unit on logarithms and had been previously exposed to the types of thinking students typically brought to this course and how prior knowledge could be leveraged. The knowledge she had gained from the literature and her personal teaching experiences maximized the chance there would be rich discourse and conceptual gains in the classroom. This turned out to be the case.

Whole Class Level Data

The main source of whole class data was a video recording of students' presentations to the class and all whole-class discussion. Video of students' presentations captured the student, their gestures, and any inscriptions they created. During whole-class discussion the video recorded the person currently speaking and their gestures. The video was captured by another doctoral student, situated in the back of the room using wide angle shot to avoid panning, as much as possible. Figure 3.1 shows an example shot. During these class interactions I was also in the class observing and noting events I felt were important.

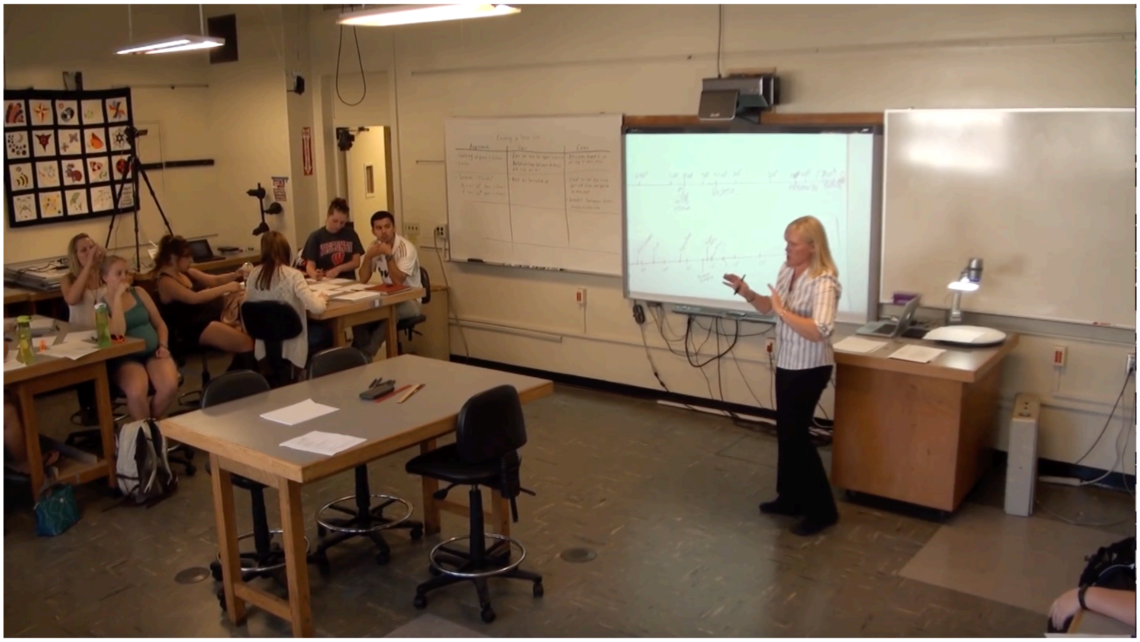


Figure 3.1: The camera capturing whole class discussion.

Small Group Level Data

The seven focus students were distributed among two small groups of four. There interactions were captured by two cameras, one situated on a tripod placed on an adjacent desk and one hung from the ceiling. The camera on the tripod captured students' utterances and gestures. Figure 3.2 shows an example shot. I will videotape and audio-record these four groups during the unit. When coordinated with records of student work, the camera hung from the ceiling was useful in identifying what students were drawing or pointing to on their papers. Figure 3.3 shows and example shot.

Figure 3.2. The camera on a tripod capturing small group interactions.

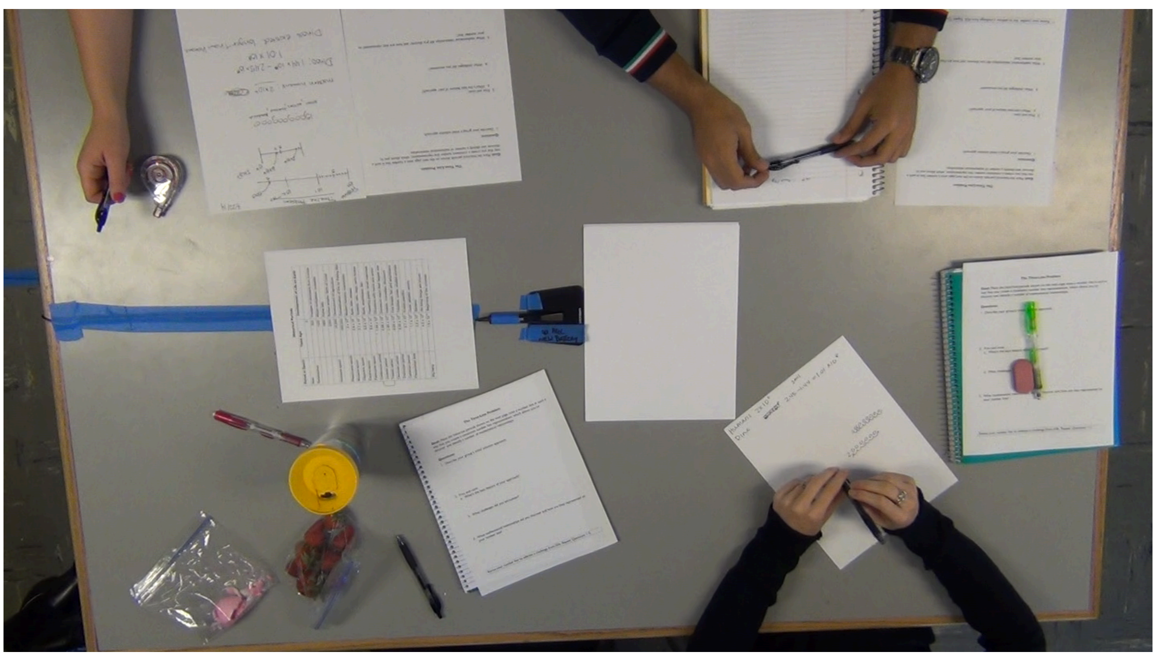


Figure 3.3: The hanging camera capturing small group members' writings.

Individual Level Data

The main sources of individual data were responses to task-based clinical interviews and written work produced in class.

Clinical Interview. I performed an individual task-based after instruction with each of the seven focus students (Ginsburg, 1997). Two features of a clinical interview made it an appropriate choice for the study. First, a clinical interview is characterized by its open-ended questions. The questions I required non-algorithmic thinking, probed conceptual topics with connections to other mathematics, and had a variety of entry points (Zazkis & Hazzan, 1999). I used tasks with these characteristics to maximize the possibility that the student would reveal how he or she actually thought about exponential and logarithmic relationships and related mathematics.

The second feature of clinical interviews that makes it an appropriate choice is what Ginsburg (1997) calls hypothesis testing. As the student engaged in the tasks, I built a model of his or her conceptions. I then generated follow up questions and tasks in the moment designed to test my hypothetical model. This meant that while all students received the same initial prompts, there were variations in follow up questions and tasks to test the individualized models I formed during the interview. As I tested my hypotheses I listened closely to the student's response (Confrey, 1993) for confirming or disconfirming evidence. While I drew on previous research and my own personal understandings of the interview topic to form hypotheses, I was willing to drop my hypotheses at any moment. In this way, I will constantly tested and formed new hypotheses as the interview proceeded (c.f.

Simon, 1995). In this way, I was able to uncover students' individual ways of reasoning.

The clinical interview lasted between 1 and 1.5 hours for each of the seven focus student and was performed after instruction to explore students' ways of reasoning after they had participated in class interactions. The clinical interview was videotaped with two cameras, one focused on the student and interviewer (myself) and one focused on the student's inscriptions. The camera focused on the student and the interviewer captured our gestures and facial expressions (see Figure 3.4 for an example shot). The camera focused on the student's inscriptions (see Figure 3.5 for an example shot) helped me follow her work.

Figure 3.4: Shot from camera capturing interaction of interviewer and student.

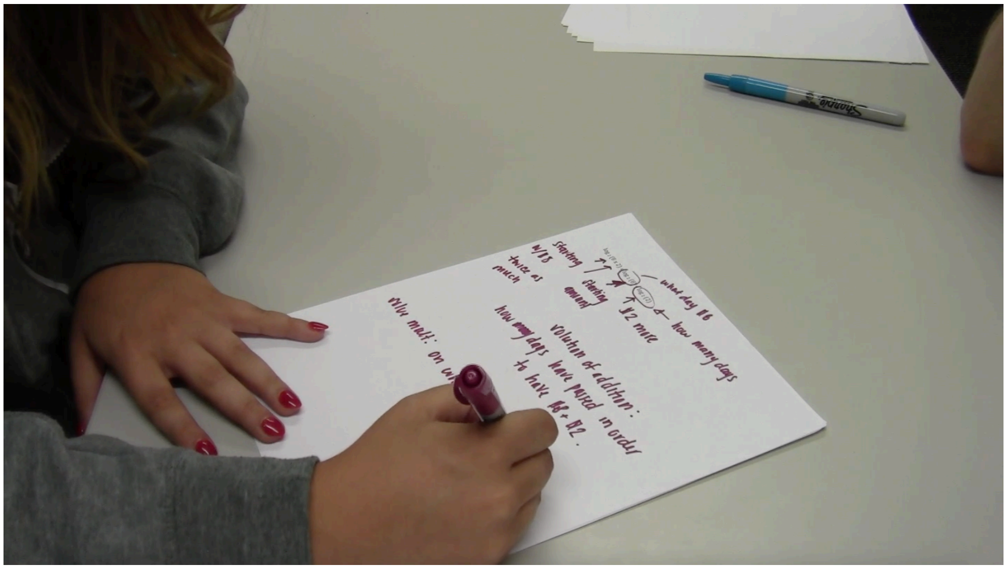


Figure 3.5: Interviewee's inscriptions.

Tasks for Clinical Interviews. The tasks for the interview were designed to uncover conceptions related to the ways of reasoning that I had hypothesized were going to become normative in the classroom. This was necessary because the aim of Research Question 1 was to make connections between normative ways of reasoning and students' conceptions. To maximize the likelihood that I would uncover conceptions related to the normative ways of reasoning, I based my interview tasks on my hypotheses, which will be elaborated later in this chapter.

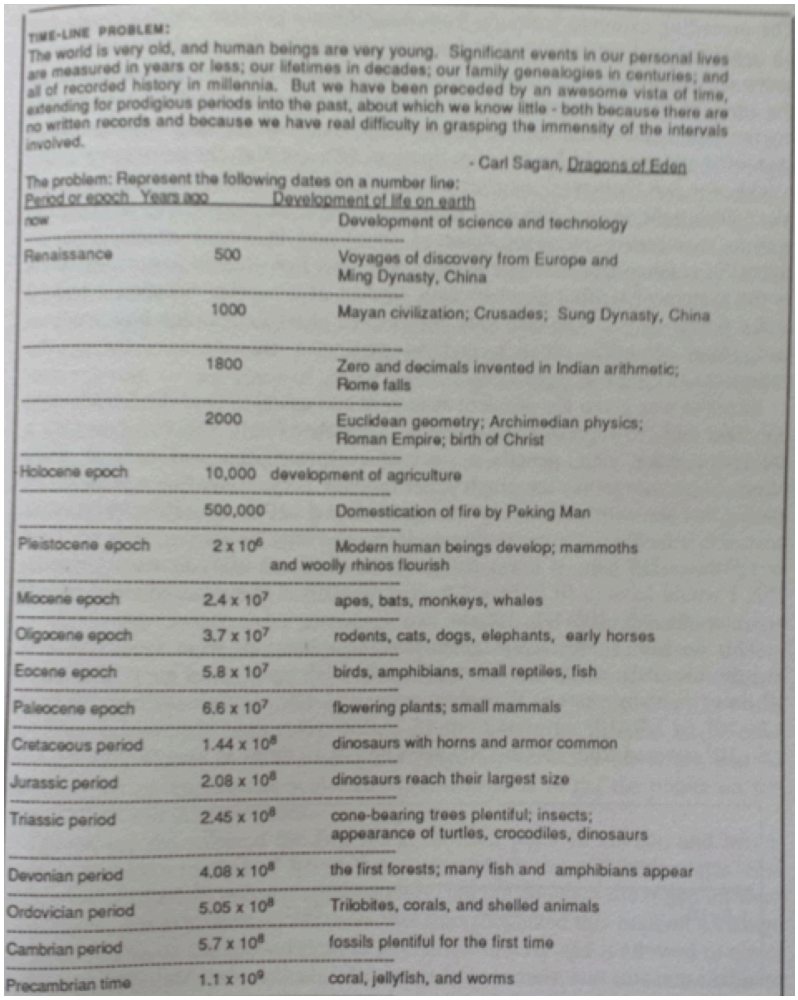
Written Work. I also scanned the written work of the focus students produced in class. At times, students would talk about and gesture to their written work in small group interactions. The scans helped me understand the content of these conversations. I also collected scans of the focus students' homework. This helped me understand students' engagement with tasks outside of class.

Additional Data. Other data sources were collected from the focus students, but were not needed to answer the research questions. These included a clinical interview before instruction to explore students' prior knowledge, an interview after instruction where students reflected on classroom interactions, and a short survey students filled out after each class period where they very briefly reflected on the events of the day. These data were collected to help explain individual variation in student thinking from established practices. The interview conducted after instruction and survey were meant to reveal students' interpretations of classroom events, which may have helped explain variability. Data on prior knowledge was meant to help explain students' interpretations. While I still believe these are important lines of inquiry, when designing the study I did not fully anticipate the complexity of students' knowledge development. Because individual students' ways of reasoning typically underwent several shifts during the course of instruction, it was difficult to account for the differences in ways of reasoning for each of the students as individuals. It was more feasible to explain trends in individual ways of reasoning. In order to explain the relationships between classroom specific classroom interactions and individual student's way of reasoning, a scholar would likely need a more fine grained analysis of particular students' reasoning and how that reasoning changes throughout the course of instruction. This may require more interviewing that occurred during the course of instruction (instead of just before and after).

Conjectured Normative Ways of Reasoning

Since the interview data needed to reveal the connections between individual ways of reasoning and normative ways of reasoning, I needed to anticipate what ways of reasoning would become normative as I developed the interview questions. In consultation with the teacher about the planned classroom activities, we hypothesized that four ways of reasoning are likely to become normative in the class: (a) creating a uniform geometric number line, (b) developing a meaningful interpretation of fractional exponents, (c) coordinating geometric and arithmetic sequences, and (d) treating a logarithm as a number. The hypothesized development was close enough to the actual development that the interview tasks generated useful data. Below I present the hypothesized development. This was written before the unit was taught, though the verb tenses have been changed.

Creating A Uniform Geometric Number Line. Students were given a task (Confrey, 1991), in which they were to create a time line that shows the history of the world (Figure 3.6). In the past instantiations of the course, this had been difficult for the teacher's students. In the past, they had started with arithmetic approaches, but then realized that this either makes the majority of events too close together on the line, or makes the line too long. For example, consider the number line in Figure 3.7. If we take these numbers to represent the number of years in the past from now, everything but the big bang fits on this number line. However, to scale this line so that it fits on a single page, the very first tic mark corresponds to a time when coral, jellyfish, and worms ruled the earth. Seeing the problems with an arithmetic approach, students then move to a hybrid number line, where the powers of 10 are all equally spaced, but have arithmetic subsections (Figure 3.8). So, for example, the halfway point between 1 and 100 would be 50.5, whereas the halfway point between 100 and 10,000 would be 5,050 (Figure 3.8). This means that the hybrid number line functions as several arithmetic number lines of different scales pasted together. Finally, they move to a fully geometric number line (Figure 3.9). This means that any distance on the number line corresponds to a particular multiplicative relationship between the two points. Thus, if two inches corresponds to multiplication by 100, then one inch must correspond to multiplication by 10 because repeating this multiplication twice must be equivalent to multiplying by 100. This means that the halfway point between any two numbers on the line is the geometric mean of those two points. This is in contrast to an arithmetic number line in which the distance between two points is the arithmetic mean.



TIME-LINE PROBLEM:
The world is very old, and human beings are very young. Significant events in our personal lives are measured in years or less; our lifetimes in decades; our family genealogies in centuries; and all of recorded history in millennia. But we have been preceded by an awesome vista of time, extending for prodigious periods into the past, about which we know little - both because there are no written records and because we have real difficulty in grasping the immensity of the intervals involved.

- Carl Sagan, Dragons of Eden

The problem: Represent the following dates on a number line:

Period or epoch	Years ago	Development of life on earth
now		Development of science and technology
Renaissance	500	Voyages of discovery from Europe and Ming Dynasty, China
	1000	Mayan civilization; Crusades; Sung Dynasty, China
	1800	Zero and decimals invented in Indian arithmetic; Rome falls
	2000	Euclidean geometry; Archimedian physics; Roman Empire; birth of Christ
Holocene epoch	10,000	development of agriculture
	500,000	Domestication of fire by Peking Man
Pleistocene epoch	2×10^6	Modern human beings develop; mammoths and woolly rhinos flourish
Miocene epoch	2.4×10^7	apes, bats, monkeys, whales
Oligocene epoch	3.7×10^7	rodents, cats, dogs, elephants, early horses
Eocene epoch	5.8×10^7	birds, amphibians, small reptiles, fish
Paleocene epoch	6.6×10^7	flowering plants; small mammals
Cretaceous period	1.44×10^8	dinosaurs with horns and armor common
Jurassic period	2.08×10^8	dinosaurs reach their largest size
Triassic period	2.45×10^8	cone-bearing trees plentiful; insects; appearance of turtles, crocodiles, dinosaurs
Devonian period	4.08×10^8	the first forests; many fish and amphibians appear
Ordovician period	5.05×10^8	Trilobites, corals, and shelled animals
Cambrian period	5.7×10^8	fossils plentiful for the first time
Precambrian time	1.1×10^9	coral, jellyfish, and worms

Figure 3.6: Timeline task.

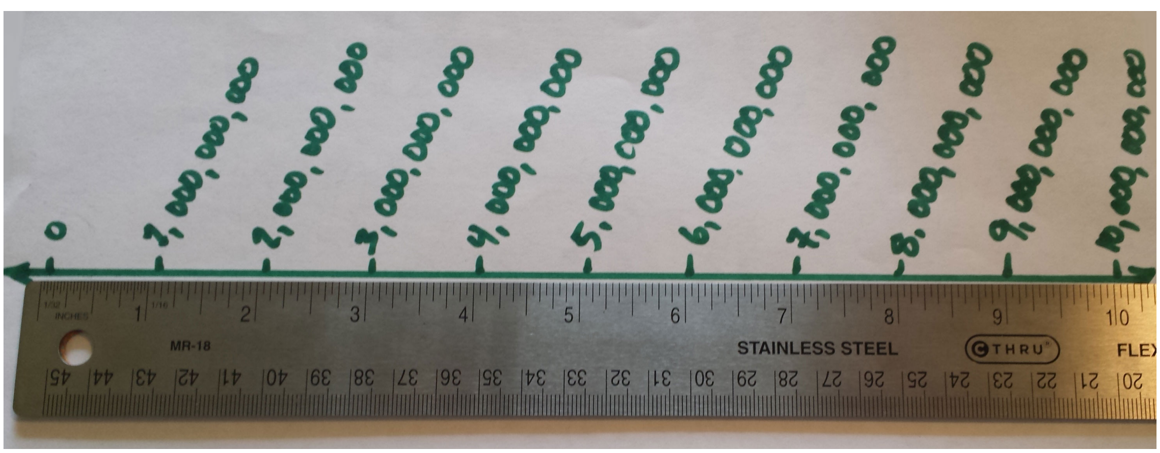


Figure 3.7: This is an arithmetic number line.

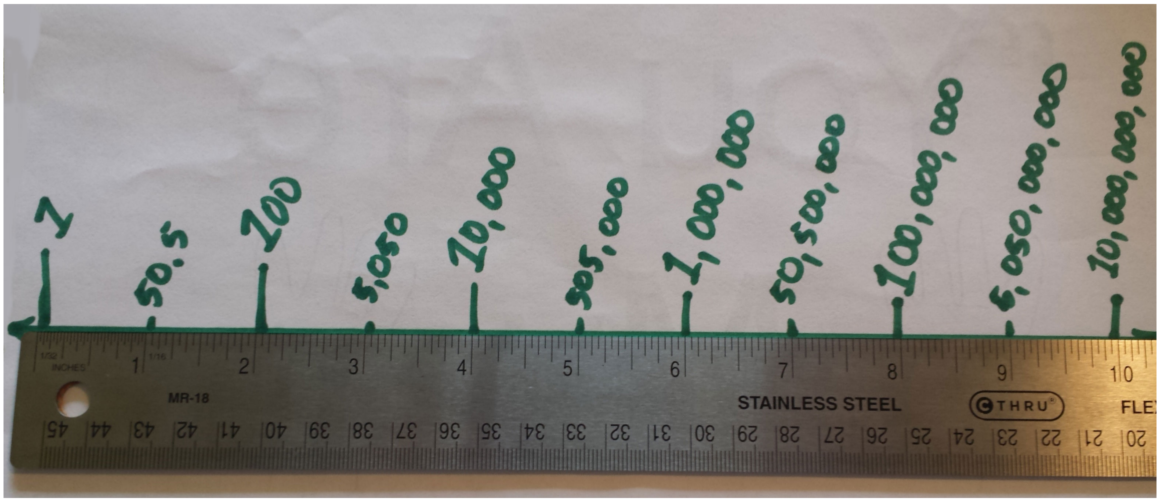


Figure 3.8: This is a hybrid number line.

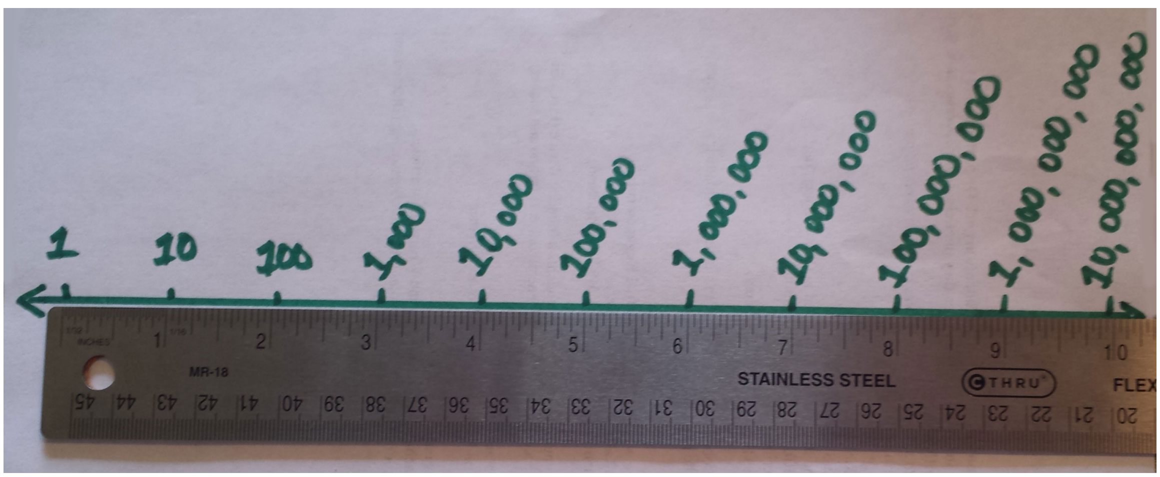


Figure 3.9: This is a uniformly geometric number line

Developing A Meaningful Interpretation of Fractional Exponents. We expected most students would know that raising a number to the ½ power is equivalent to taking the square root of that number. However, we did not expect many students to be able to explain why that must be before instruction or be able to link fractional exponents with subsections of their timelines. In the past, when students began to work on creating a fully geometric number line, they have struggled with how to construct the subsections. They may split the section between 10^7 and 10^8 into 10 sections, but at the beginning they do not know how to label these subsections. They know that they must multiply by a common factor, but they do not anticipate that they should multiply by the tenth root of 10. Instead, they use a guess and check approach, trying out different factors, multiplying 10^7 by that factor ten times in hopes that the product will end up being 10^8. Eventually, however, they come to anticipate that the factor is the tenth root of ten, since they need to multiply that number by itself ten times to get 10. They then realize that since the exponents in the number line are growing arithmetically, the exponents of the subsections should grow arithmetically as well, meaning 10^7 times the tenth root of 10 should be written as $10^{7.1}$.

Coordinating Geometric and Arithmetic Sequences. When dealing with logarithms, one has to coordinate a multiplicative relationship with an additive one. This is in fact how Napier discovered logarithms. As Napier explored the mathematics, he drew two lines and imagined a particle moving on each of the lines. One of the particles moved at a constant velocity, while the other particle moved with a velocity proportional to its position. This means that the relationship between the two particles is logarithmic. Napier defined the logarithm of the position of the second particle as the position of the first particle (Katz, 1986; Villarreal-Calderon, 2008). As this historical note illustrates, coordinating these two sequences is at the conceptual heart of logarithms.

In the classroom, students must also be able to coordinate these sequences. For example, when operating with the number line, a student must know that each time she progresses by one tic mark (an additive relationship), she is multiplying by 10. Or if student was reasoning about a petri dish of bacteria that grows by a factor of 9 each hour, she would need to coordinate the number of hours (an additive relationship) with the total number of bacteria (a multiplicative relationship). I expected that as students talked in class they would begin to explicitly mention both the additively growing quantity and multiplicatively growing quantity and their relationship.

Treating a Logarithm as a Number. I expected that at the beginning of the course, some students may have only thought of a logarithm as an operator that undoes exponentiation. If this was the case, I would expect these student to request justification when a student talks about a symbol like $\log_5 25$ as the number 2. I expected the participants in the course to eventually accept talk of a symbol like $\log_5 25$ as a number as being legitimate. If students can only think of a logarithm as an operator that undoes exponentiation, they will not be able to operate on logarithms (e.g. multiply them by a number, subtract them, or divide them). Furthermore, students need to conceptualize a logarithm as a number to make sense of expressions like $\log_4 8$, which is equivalent to a (non-integer) fraction (Berezovski, 2004; Weber, 2002).

Data Analysis

Analysis for Research Question 1

> Research Question 1: How are individuals' ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

To answer this question, I needed to establish the progression of increasingly sophisticated ways of reasoning that functioned as if shared and individuals' related conceptions. To accomplish this goal, I used the three-phase method outlined by Rasmussen and Stephan (2008). In the first phase, I transcribed whole class interactions using Transana (Woods & Fassnacht, 2015). I then identified arguments in the transcript. I defined an argument as an episode where a participant makes a mathematical assertion and supports that assertion with evidence. I then created an argumentation log, which recorded the arguments identified and their structure, in terms of Toulmin's (1969) scheme. This means that for each argument I identified the function (usually warrant, claim, or data) of each element of the argument. One element of the argument is the claim. This is what the argument attempts to establish, in other words, it is the conclusion of the argument. Another element of the argument is the data. This is the evidence used in the argument. Finally, the warrant is the reasoning that connects the data to the claim.

Tabach, Hershkowitz, Rasmussen, and Dreyfus (2014) illustrated how to parse an argument made in differential equations class according to this scheme. In the class, they were debating whether or not the growth rate of a population of rabbits was constant. Several students contributed to an argument that the rate was not constant. In essence, the students argued that since the entire population is reproducing, the amount of rabbits increases, which means there is more reproduction. Thus, the rate of reproduction is increasing. In this case the claim is that the rate is not constant. The data is that the entire population is reproducing and the warrant is that more rabbits leads to more reproduction.

It is important to note that these arguments are not constructed by talk alone. Rather it is through the coordination of talk, gestures, and inscriptions that mathematical meaning is communicated. Thus, as I performed this analysis I considered all three of these aspects of the classroom discourse. In fact Rasmussen, Stephan, and Allen (2004) found that when ideas shift function in an argument, this can be coordinated with the shift of particular gestures.

After I parsed the arguments to create argumentation log, I moved to the second phase of analysis. In this phase, I used the argumentation log as data to identify normative ways of reasoning. To identify these ways of reasoning I used three criteria. First, if backings or warrants were initially needed to establish a claim, but later become no longer necessary, the way of reasoning was considered normative. Second, a way of reasoning was established as normative if a piece of information shifts the function it plays in the argument (e.g. claim to data). Lastly, R. Cole et al., (2012) added a third way to establish normative ways of reasoning, which is the repeated use of an idea, as data or warrant.

In the third phase of analysis, I grouped mathematically related normative ways of reasoning into mathematical practices. The way these normative ways of reasoning were grouped was a researcher decision.

Research Question 1 also required that I establish individuals' ways of reasoning about the math practices. To infer students' individual ways of reasoning, I analyzed the videos of interviews. Analyzing all the interview data was not tenable, so I needed to reduce the data. This data reduction was informed by the results of the analysis of the normative ways of reasoning using the DCA method. In particular, I focused on a particular math practice, Math Practice 2, which I termed *Subdividing the Segments*. I focused on this math practice both because I knew I had data from the interviews that related to this math practice and because the practice was conceptually rich and seemed to be borne out of significant struggle in the class. Thus, I hoped to see variety in students' ways of reasoning. With this focus, I analyzed students' responses to one task in the interview which elicited students' ways of reasoning related to this math practice.

When analyzing the interview data, I used grounded theory (Strauss & Corbin, 1994, 1998) to develop categories that described the students' ways of reasoning. I first transcribed the reduced data set using Transana. Then I created a descriptive, non-inferential narrative of the reduced data set (Miles & Huberman, 1994). This helped establish what happened in the interview. Then, I engaged in open coding from grounded theory (Strauss & Corbin, 1990). This involved first breaking up the data into smaller episodes, where a student is expressing an idea or making use of a strategy. I then grouped similar episodes to form a category. This category was named so that the category could be discussed as a particular way of reasoning. As I inferred these categories, I made use of the constant comparison method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1990, 1994). This is the comparison of different pieces of data to create and refine categories. As I began to establish categories, I compared the episodes in the category to other episodes in the interviews, both within and between subjects. The purpose of these comparisons is to bring into greater relief the similarities and differences in categories. This was an iterative process. This means that as categories were refined, episodes that have been coded earlier were revisited in light of the new categories (Strauss & Corbin, 1990).

I then compared the normative ways of reasoning in Math Practice 2 to the categories of individual ways of reasoning. In particular, I looked for differences between the math practice and the individual ways of reasoning.

In summary, to address Research Question 1, I will first report the progression of increasingly sophisticated normative ways of reasoning in Chapter 4. Then, I will report categories of individuals' related ways of reasoning and elaborate how these ideas were similar to or different from Math Practice 2 in Chapter 5.

Analysis for Research Question 2

> Research Question 2: What mathematical connections exist between the focus students' ways of reasoning in the post interviews and the discursive interactions between them and other students and the teacher in both whole class and small group settings? Furthermore, how might the nature of these discursive interactions give plausible explanations for students' differing conceptions?

The heart of this research question is to understand how various students could participate in the same interactions where particular ways of reasoning are negotiated and accepted, yet subsequently reason in different ways. One possible explanation is that students are interpreting these interactions differently. Thus, I found it useful to examine how the meanings associated with these explanations were constructed in the class community. This may help explain how various students could make sense of the explanations in different ways.

Thematic analysis (Lemke, 1990; Herbel-Eisenmann, 2011), a systemic functional linguistics (SFL) approach (Halliday, 1978; Halliday & Hasan, 1985; Halliday & Martin, 1993; Halliday & Matthiessen, 2004), is useful for determining meanings as they are constructed in use. Central to this method is the assumption that words derive their meaning from their relationships to other words. For example, if I said "I played xyz yesterday," one could infer the potential meanings for "xyz"—it could be a sport, a game, an instrument, but not a soft drink—by examining the relationship between "xyz" and played. In particular, "xyz" needs to be something that is "playable." As this example suggests, a particular utterance constrains, but does not determine meaning of a word. Thus, the constructed meanings are sometimes referred to as meaning potentials (Herbel-Eisenmann & Otten, 2011), to indicate that they may not by identical to the personal meanings individual participants hold. As such, it is important for researchers to look for patterns over time in the expressed relationships.

To perform the thematic analysis, I first reduced my data to episodes where students were giving explanations that related to Math Practice 2. I then used an adapted version of Herbel-Eisenmann and Otten's (2011) method on the reduced data set. Their method had four phases. First, they created a lexical chain by making a table with columns that are the central ideas expressed in the transcript. In those columns, they placed portions of the transcript where that idea was being talked about. The sections of transcript can be placed in multiple columns when several ideas are being related (see Table 3.1). To determine what column or columns the section of text should go in they asked questions like, "What is the bigger mathematical point of this segment of text? What is the mathematical gist of this section of text" (Herbal-Eisenmann & Otten, 2011, p. 458)?

Table 3.1: An example of the Lexical Chain developed by Herbel-Eisenmann and Otten (2011, p. 459).

Base	Height	Forming a new rectangle	Triangle–rectangle comparison	. . .
		Stacey: . . . where would the rectangle be drawn? . . . to show the same idea of what that group was doing . . . where would you draw that rectangle?		
		Charlie: . . . cut the first three off the right of it [the base of E, which is the top]. . . . you take it [the cut-off section] down and put it on the bottom right there . . .		
			Stacey: . . . compare the dimensions of the triangle to the dimensions of the rectangle . . .	
			Student: Heights stay the same.	
			Student: It [the base] splits in half.	

In the next three phases of analysis they determined sematic relationships between the ideas, or thematic items, identified in phase one. These relationships express how two thematic items are associated. For example, if a student said, "The length of a segment is ten," that student is expressing a relationship between length and segment. Namely, the length is a measure of the segment. This relationship is an EXTENT/ENTITY relationship, where the EXTENT is the thing that is being measured, in this case the segment, and the ENTITY is the measure, in this case the length. See Table 3.2, generated by Herbel-Esienmann and Otten (2011, p. 461) that describes a sample of these relationships, with descriptions and examples (also see Lemke, 1990, p. 221 for a list of several common relationships).

Table 3.2: A Sample of Semantic Relations used by Herbel-Eisenmann and Otten (2011, p. 461).

Linguistic terms	Description	Example
HYPONYM / HYPERNYM	Subset of a set	A square is a parallelogram
MERONYM / HOLONYM	Part of a whole	A side of a polygon
EXTENT / ENTITY	Measure of space associated with an object	The length of a segment
LOCATED / LOCATION	Spatial relationship	A point inside a circle
SYNONYM / SYNONYM	Equivalence relationship	A diamond is a rhombus
PROCESS	Action or operation	Constructing an angle

In phase two they created a canonical map, a two dimensional drawing that represents the relationships between the thematic elements that is faithful to mathematics register (see Figure 3.10). To create this map they drew on curricular materials and other formal mathematical writings as well as their own understanding of the subject.

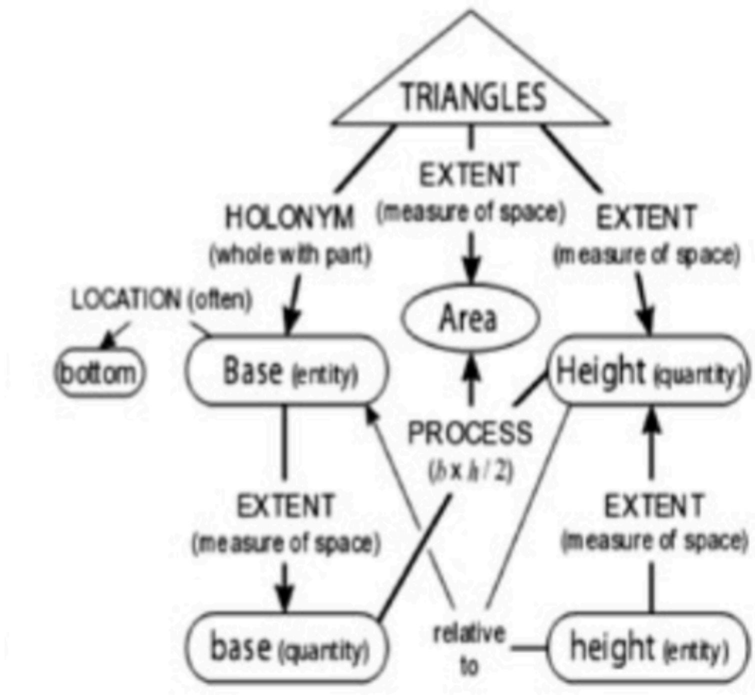


Figure 3.10: Herbal-Eisenmann and Otten's (2011, p. 463) canonical map.

In phase three, they created an analysis document from classroom transcripts that noted when semantic relationships were expressed and what those relationships were. In phase four, they took the relationships they found in phase three and represented them in a map, analogous to one created in phase two, but faithful to the relationships as expressed in class (see Figure 3.11).

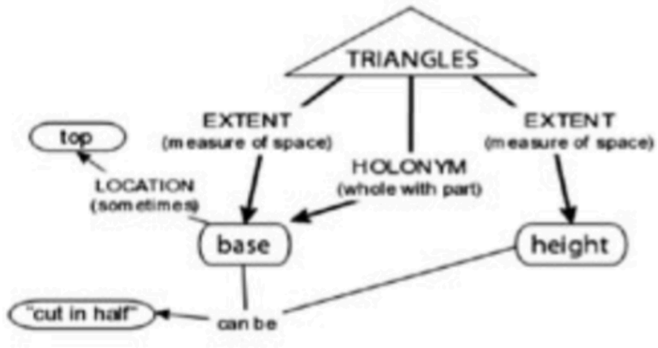


Figure 3.11: One of Herbal-Eisenmann and Otten's (2011, p. 467) classroom maps.

I used a similar method, but with some adaptations. First, because I was interested in the meanings constructed through discourse associated with an inscription—the number line—I could not focus solely on spoken words. As I created a lexical chain, I also included gestures over the number line to determine thematic elements. See Table 3.3 below for an example of the lexical chain I developed for my analysis, which also includes a column that describes the relationships between the lexical items in the utterance. Also, see Table 6.4 for definitions of the sematic relationships I used. Second, in the creation of the canonical map, I drew mostly from my own mathematical understanding as an exponential number line is not typically included in curricular materials. Third, because words have different meanings when expressed in the various ways of reasoning I created different maps for each of the three ways of reasoning as opposed to one map for the whole unit. Fourth, I compared the maps I created. While Herbel-Eisenmann and Otten (2011) do not list this as a stage of analysis, they did this as well. In fact, they join with their intellectual forbearers in describing the comparison of different networks of semantic relationships as a strength of thematic analysis (Herbel-Eisenmann & Otten, 2011, Lemke, 1990, Martin, 2009b; Martin & Rose, 2003). Finally, I also examined episodes where students talked about the methods themselves. In this phase of analysis, students tended to develop semantic relationships in a different way than when they talked about the thematic elements involved in subdividing a segment. They tended to use equivalence and contrast strategies (see Appendix D in Lemke, 1990, p. 226) to show whether they thought two strategies were the same or different. These strategies will be described in more detail in Chapter 6.

Table 3.3: A Sample of the Lexical Chain Used for Analysis Table

Location of Renaissance	Half way	add	100 (Endpoint)	Nine hundred years	Four hundred fifty	Interval	Relationships
R: I'm still confused as to why, like my gut says it wants to put it half way.	R: I'm still confused as to why, like my gut says it wants to put it half way.						wants to put it [location of Ren] - LOCATED/LOCATION- half way
	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.			T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	nine hundred years - LABEL/LABLED- in that interval [puts both hands up like an interval], the half way point - LOCATION/LOCATED- four fifty
R: So it's past half way. If you're going this way [gestures with pen from 10^2 to 10^3]	R: So it's past half way. If you're going this way [gestures with pen from 10^2 to 10^3]						it's [location of Ren] - LOCATED/LCOATION-past -PREPOSITION/OBJECT- half way.
S: Past half? [Questioning them]	S: Past half? [Questioning them]						
K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100				It's [location of Ren] - LOCATED/LOCATION- less than - PREPOSITION/OBJECT- half[way]

Table 3.3: Semantic Relationships Used in Analysis

Linguistic Term	Description	Example
Process/Target	The process is an action that is being carried out. The target is what is being operated on.	Dividing (process) the segment (target) up.
Process/Result*[1]	The result is the outcome of the process.	I divided (process) 500 by 2 and got 250 (result).
Process/Reason*	The reason is why the process is occurring.	I added 200 (process) because that's our starting point.
Entity/Extent	The measure of a physical space	I found the length (extent) of the segment (entity).
Location/Located	Where an object is located.	500 is at the midpoint.
Token/Type	An example of a class of objects.	450 (token) is an amount of elapsed years (type).
Representation/Represented*	The representation is a depiction of something and the represented is what is being depicted	Same-sized segments (representation) represent multiplication by a constant factor (represented)
Label/Labeled*	This is an objects is called something. This can be done verbally or through an inscription.	A student might put a bracket over a segment (labeled) and write "x10" (label). The segment (labeled) is 500 (label).
Preposition/Object	A word that expresses a physical or temporal relationship to another word.	Place the tick to the right (preposition) of the middle (object).
Synonym	When the two words mean the same, or nearly the same, thing	Ten squared is (synonym) one hundred.
Agent/Process	The agent is the person or object that preforms the process.	I (agent) divided (process) the segment.

Reliability and Validity

To address the issue of reliability in coding and interpretation of data, I used peer review (Confrey & Lachance, 2000). This means I made presentations to other

[1] An asterisk denotes the relationship was not reported in *Talking Science* (Lemke, 1990).

researchers in which I showed them how I had coded and interpreted data. In this presentation I sought feedback as to the strength of the evidence provided by the data for my interpretation. Specifically, as I started to parse arguments using Toulmin's scheme I met with Dr. Lobato and Dr. Rasmussen, an expert in the DCA method. In the meeting we discussed how to identify an argument as well as how to code that argument. I presented some of my analyses and Dr. Rasmussen gave examples of how he coded data from his own work as well. As I developed claims from my data that particular ways of reasoning were functioning as if shared in the class community, I met with other doctoral students during semester long class (MSE 830) to get impressions from others as to the validity of my claims. I also continued to meet with Dr. Lobato during this time to discuss the validity of these claims. Then, as I developed categories of meaning from the post-instruction interview, I sought feedback from Dr. Lobato. I presented detailed evidence for my claims for particular categories of ways of reasoning. Finally, I met with Dr. Lemke several times as I examined the classroom discourse. We discussed the classroom discourse and he helped me identify resources and methodological tools that would help reveal subtleties in students' meaning making.

To address the issue of validity I will use the notion of fit from grounded theory (Glaser, 1978). This means I considered the degree to which my interpretations were faithful to the data. To do this, as I developed plausible explanations I constantly compared the emerging explanation and interpretation to the data (Strauss & Corbin, 1994). This required several analytic passes over the data, in which I actively sought disconfirming evidence.

Chapter 4: Collective Mathematical Progress

In this and the subsequent chapter I will answer Research Question 1, which follows.

> **Research Question 1:** How are individuals' ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

To answer this question I need to establish the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom, establish the individuals' ways of reasoning, and then explore their relationship. In this chapter I present the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom. In the next chapter I present the analysis of individuals' ways of reasoning for a sample of students from the class as they worked on math tasks in an interview conducted after instruction had ended. In particular, I explore the nature and extent of individual variation in students' ways of reasoning from the collective mathematical practices that emerged in the classroom. This will answer Research Question 1.

Review of Methods

To determine which practices became accepted in the class community I used the documenting collective activity method (DCA; Rasmussen & Stephan, 2008; Cole et al., 2012). I first created an argumentation log, which catalogued all the public arguments given in class. This log contained the transcript of the arguments as well as the coding of the constituent pieces of the argument as data, claim, or warrant (Toulmin, 1969). I then used the argumentation log as data to look for changes in the structure of arguments over time. If the arguments changed in one of three ways, (a) warrants or backings that were once needed to justify claims dropped off in later arguments, (b) ideas shifted position in the Toulmin scheme (e.g. from claim to data), or (c) ideas functioned repeatedly as data or warrant in arguments, the idea was considered to have begun functioning as if shared. Arguments changing in these ways represent an acceptance by the class community. Criterion 1 shows that ideas are accepted in that they no longer need to be justified. Criterion 2 shows that ideas are accepted in that they can be used to support new ideas under consideration. Criterion 3 shows that a way of reasoning has become a standard way of reasoning in the class. If an idea meets one of these three criteria I labeled it a normative way of reasoning (NWR). Following Rasmussen and Stephan (2008) and others who have followed their method (Cole et al., 2012; Stephan & Akyuz, 2012), I then grouped related normative ways of reasoning. These groups are called collective mathematical practices.

This method was intended to be used in a classroom where multiple people are actively participating in advancing, developing, and evaluating mathematical arguments, as opposed to just the teacher or one dominant student being the author of mathematical ideas and arbiter of their validity. If multiple students are participating, the ideas that take hold are more likely to represent a broader swath of students' personal ways of thinking. This is especially true if students feel comfortable not only in advancing ideas, but also intellectually engaging with others' idea. This was the case in the classroom that I studied. Students routinely questioned each other's ideas (for example, see the reaction to David in the discussion of Argument 2.2.2 that follows for an example) or pressed for more explanation (see students' reaction to Rachel in Argument 1.1.2 an example).

Instructional Context

I studied a logarithm unit in a class for prospective teachers. This unit focused on developing students' understanding of exponential and logarithmic relationship. To develop these ideas, the teacher first asked students to create a representation of exponential relationships, an exponential number line, and then use that representation to explore the meanings of fractional exponents and logarithms.

On the first day of the unit, the students were asked to create a timeline that represented the earth's history—from 15 billion years ago until today. The students were asked to place several historic events (see Figure 4.1) on the line. After students had time to develop these timelines in groups, three distinct approaches were presented in class. The first approach was a linear timeline. In this representation one endpoint was labeled "big bang" and marked 1.5×10^{10}, while the other endpoint was labeled "now" and marked 0. The student presenting the approach explained they placed times on the line by successively halving the line. They first placed 7.5 billion years at the halfway point and then placed 3.25 billion at the halfway point between 0 and 7.5 billion (see Figure 4.2).

TIME-LINE PROBLEM:

The world is very old, and human beings are very young. Significant events in our personal lives are measured in years or less; our lifetimes in decades; our family genealogies in centuries; and all of recorded history in millennia. But we have been preceded by an awesome vista of time, extending for prodigious periods into the past, about which we know little - both because there are no written records and because we have real difficulty in grasping the immensity of the intervals involved.

- Carl Sagan, Dragons of Eden

The problem: Represent the following dates on a number line:

Period or epoch	Years ago	Development of life on earth
now		Development of science and technology
Renaissance	500	Voyages of discovery from Europe and Ming Dynasty, China
	1000	Mayan civilization; Crusades; Sung Dynasty, China
	1800	Zero and decimals invented in Indian arithmetic; Rome falls
	2000	Euclidean geometry; Archimedean physics; Roman Empire; birth of Christ
Holocene epoch	10,000	development of agriculture
	500,000	Domestication of fire by Peking Man
Pleistocene epoch	2×10^6	Modern human beings develop; mammoths and woolly rhinos flourish
Miocene epoch	2.4×10^7	apes, bats, monkeys, whales
Oligocene epoch	3.7×10^7	rodents, cats, dogs, elephants, early horses
Eocene epoch	5.8×10^7	birds, amphibians, small reptiles, fish
Paleocene epoch	6.6×10^7	flowering plants; small mammals
Cretaceous period	1.44×10^8	dinosaurs with horns and armor common
Jurassic period	2.08×10^8	dinosaurs reach their largest size
Triassic period	2.45×10^8	cone-bearing trees plentiful; insects; appearance of turtles, crocodiles, dinosaurs
Devonian period	4.08×10^8	the first forests; many fish and amphibians appear
Ordovician period	5.05×10^8	Trilobites, corals, and shelled animals
Cambrian period	5.7×10^8	fossils plentiful for the first time
Precambrian time	1.1×10^9	coral, jellyfish, and worms

Figure 4.1: The timeline task.

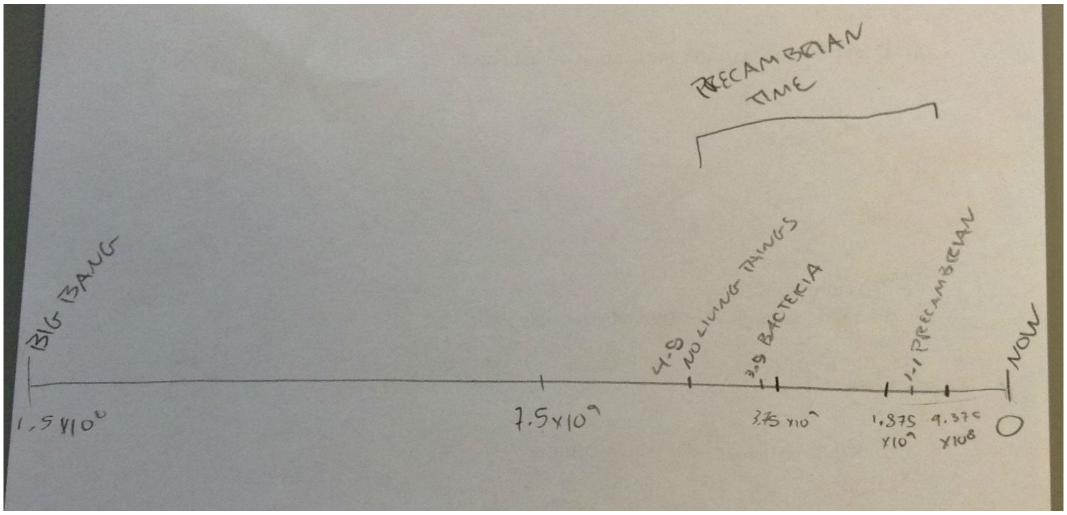


Figure 4.2: The linear time line.

The class named the second approach that was presented the “several chunks approach.” A group of students developed this timeline by chunking the table of events into several groups (see Figure 4.3), each of which used a different linear scale. They grouped events by first converting all the times into scientific notation and then categorizing by powers of 10. For example, 1.44 x 10^8 and 2.08 x 10^8 were in the same group, because they were written as some number times 10^8, while 3.5 x 10^9 was in a different group, because it was written as some number times 10^9. Each group was then given its own scale. For example, in the 10^9 group 1/2 cm represented 1 x 10^9, while in the 10^8 group 1 cm represented 1 x 10^8. They then found the amount of elapsed time between events and plotted them according to the scales they developed. This means that unlike the first approach, in this approach the number line was built up. Students did not start with a line with defined endpoints. Rather, they plotted a point on the line, found how much further away the next point should be, plotted it, and then moved on to the next point.

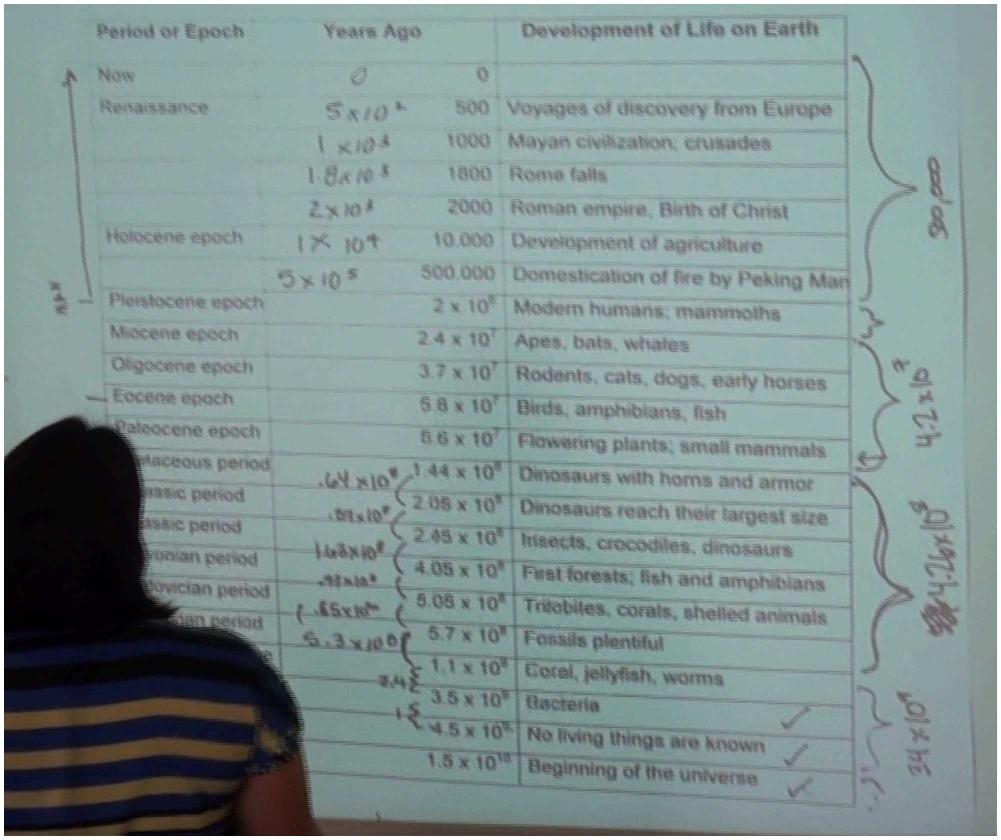


Figure 4.3: Table of events split into several chunks.

Another group of students developed a third approach. This group started by plotting successive powers of 10 on the line, each one-inch apart. They said they then broke up each of the one-inch segments into 10 subsections, but only the halfway points were labeled in the line that was presented (see Figure 4.4). This subdivision seemed to occur in a linear fashion. This approach was originally called exponential, but after extensive investigation of and reflection on the line, it began to be referred to as having an exponential structure at the macro level (among the powers of 10), but a linear structure in between tick marks.

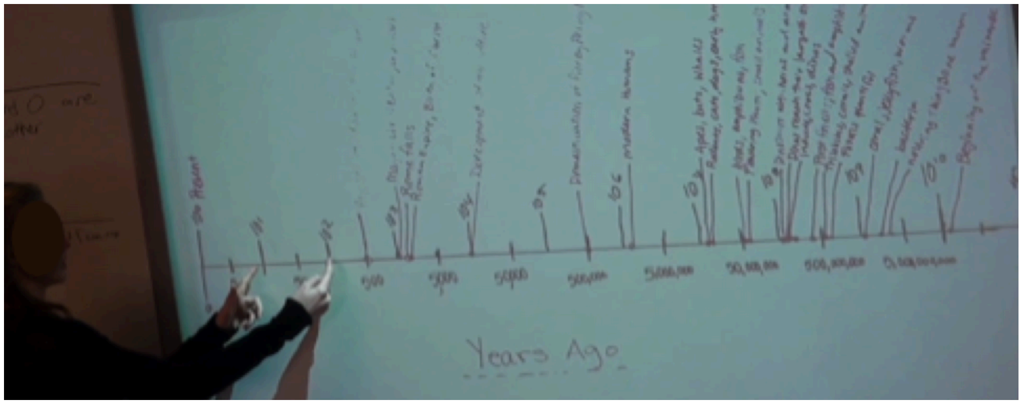


Figure 4.4: The Third Approach

Overview of Claims

In results section that follows, I present evidence for the claim that five mathematical practices emerged in the classroom. These practices are summarized in Table 4.1. Math Practices 1, 2, and 3 centered on creating a fully exponential number line. *Math Practice 1: Developing the Macro Multiplicative Structure* deals with noticing that in an exponential number line a multiplicative structure exists between labeled tick marks. The students then leveraged this multiplicative pattern to reason about subdividing the segments between these tick marks, which led to the establishment of *Math Practice 2: Subdividing the Segments. Math Practice 3: Finding Fractional Exponents* dealt with the methods students used to place events written in the form $10^{a/b}$. *Math Practice 4: Reasoning about Sequences* was the establishment of a definition for additive and multiplicative sequences. Finally, *Math Practice 5: Interpreting logarithms* dealt with making sense of and interpreting logarithms.

As I present the ways of reasoning that began to function as if shared, i.e. the normative ways of reasoning (NWRs), I have used a particular numbering scheme. I have labeled them as NWR a.b. The NWR stands for normative way of reasoning, the *a* corresponds to the Math Practice that is part of and the *b* indexes the ways of reasoning. For example, NWR 1.2 is the second normative way of reasoning (the 2 in 1.2) that is part of Math Practice 1 (the 1 in 1.2). In contrast, NWR 2.1 is the first way of reasoning that is part of Math Practice 2. I extend this labeling scheme to index the arguments that I will present as evidence for the development of the normative ways of reasoning. I label arguments as Argument a.b.c. The a.b tells the reader which normative way of reasoning the argument is acting as evidence for and the c indexes the argument. So for example Argument 1.2.3 would be the third argument (the 3 in 1.2.3) that I am using to support my claim that NWR 1.2 (the 1.2 in 1.2.3) became normative. There is one instance where one argument is used as evidence to support the development of two different NWRs, in which case the way of reasoning was labeled so that it would correspond to the first NWR presented. Also, There were three NWRs that did fit within any math practice, which are labeled NWR 0.x. Each of these ways of reasoning dealt with content that was tangential to the goal of the unit, which was to develop exponential and logarithmic reasoning.

Table 4.1: The emergent mathematical practices.

NWR 0.1: In a linear time line "each distance is going to be the same amount of years"
Math Practice 1: Developing the Macro Multiplicative Structure
NWR 1.1: Multiply the Previous Term by 10 to get the Next Term.
NWR 1.2: Finding Factors Over Several Segments
Math Practice 2: Subdividing the Segments
NWR 2.1: Subdividing Segments by Reasoning Linearly About Exponents
NWR 2.2: Preserving the Multiplicative Relationship within the Segments
Math Practice 3: Finding Fractional Exponents
NWR 3.1: Subdividing Extents that Span Multiple Segments
Math Practice 4: Reasoning about Sequences
NWR 4.1: An Exponential Sequence is one that has a Constant Multiple
NWR 4.2: An Additive Sequence is one that has a Constant Sum
Math Practice 5: Interpreting Logarithms
NWR 5.1: Logarithms are Exponents
NWR 5.2: The "On What Day" Interpretation of Logarithms
NWR 5.3: The Factor Interpretation of Logarithms
Foundational NWRs: Fluently Translating Among Various Notations
NWR 0.2 Translating Between Scientific and Standard Notation
NWR 0.3 Fractional Powers as Roots

Results

Math Practice 0

There were three normative ways of reasoning that were not grouped into math practices because they did focus on the main goal of the unit, which was to develop exponential reasoning. Instead, these normative ways of reasoning dealt with topics that could be viewed as foundational for engaging in the tasks of the unit. The first of these three is more conceptual in nature, as it deals with the definition and meaning of a linear timeline. Since students' engagement with this idea set the stage for their development of an exponential timeline, I give evidence for its establishment first. The other two normative ways of reasoning focus on notation. Since this is more peripheral to the goal of the unit, the evidence for the development of these normative ways of reasoning will be saved for the end of the chapter.

NWR 0.1: In a linear timeline "each distance is going to be the same amount of years." This NWR deals with students' coordination of distances on a timeline with elapsed times. Specifically, they noticed that on a timeline with a linear scale, same sized segments represented the same amount of elapsed time. This feature of the timeline began to function as the definition of a linear timeline as the class continued.

Overview of the Development of NWR 0.1. This normative way of reasoning is a characterization of a linear timeline; same-sized pieces represent the same amount of elapsed time. This idea was initially put forth by Nathan[2] and then elaborated by Danna on the Day 1 (Argument 0.1.1). When Danna introduced the idea, the warrant was fully articulated. On the next day, Day 2, another student, Kathy, argued that a particular line was not linear (in Argument 0.1.2). In her argument, she drew on the definition as an implicit warrant, but did not articulate it. This dropping off of the warrant in arguments fulfills Criterion 1 of the DCA method and provides evidence that the idea was functioning as if shared in the classroom.

Background to Argument 0.1.1. This idea first came up while discussing the first approach on Day 1. Natalie had presented a linear timeline and the class had been discussing what they noticed about the line. The initial comments centered on how the group successively halved the segments, until Nathan made a comment in

[2] All names are gender-preserving pseudonyms

which he pointed out that both the years and the line itself were being halved. In this comment, he introduced the idea of coordinating the length of segments on the line with the elapsed years.

> Nathan: Well, more to add on to that…I like the idea of halving, but you're not just halving the years, … you're also halving the distance so that keeps everything consistent in the absolute differences, so you're halving two things at once.

Danna then built upon the idea of coordinating the years and distance as she named the approach linear and defined this term by saying that a particular amount of elapsed distance on the line represented the a particular amount of years (see Table 4.2).

Overview of Argument 0.1.1. In this argument Dana claims that the timeline generated using the first approach should be characterized as linear. She uses as a warrant that specific lengths, which she calls distances, correspond to specific amounts of elapsed years. This warrant is then expressed more clearly by the teacher. The argument draws on the accepted data that two particular segments in this timeline each represent an elapsed time of 7.5 billion years. The transcript excerpt follows in Table 4.2, along with gestural and other physical actions, and the analytic codes from Toulmin's scheme.

Table 4.2: Coding of Argument 0.1.1

Participant	Speech	Actions	Code
Danna	It's kind of like more linear.		Claim
Teacher	More linear. ... What do you mean by linear?		
Danna	Well like, each distance is going to be the same amount of years.	Makes three chunking gestures in succession in the air by holding forefinger and thumb apart, as if to bound some distance [See Figure 4.5]	Warrant
Teacher	Each distance is going to represent the same amount of years. Can you point to a distance, Natalie, and tell us how many years it is?		
Natalie	Seven point five billion.	Points to the midpoint labeled 7.5×10^9	Data
Teacher	And can you show us the segment that is that much.		
Natalie	This segment right here* and this segment**.	*Makes a chunking gesture over her paper to identify the segment from 0 to 7.5×10^9 [See Figure 4.6] **Similar chunking gestures over 7.5×10^9 to 1.5×10^{10} segment [See Figure 4.7]	Data
Teacher	Each of those segments represents how many years?		
Natalie	Seven point five billion.		Data
Teacher	Seven point five, seven and half billion. Okay. So that's what she meant by linear, the same chunk of distance meant the same amount of time		Claim Warrant

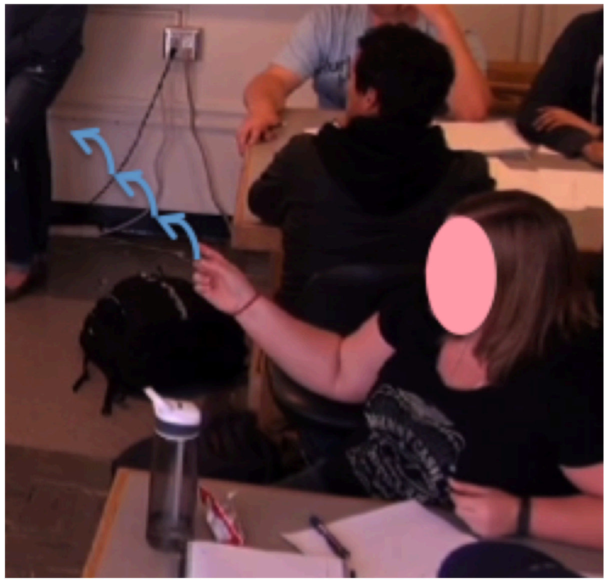


Figure 4.5: Danna Showing the Distances Represent the Same Amount of Years

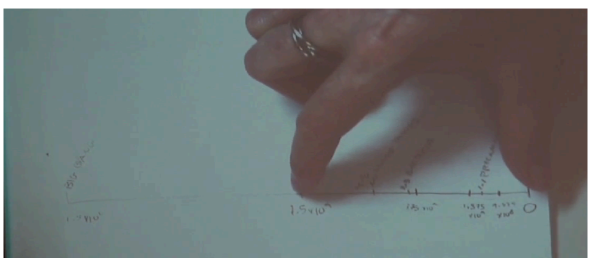


Figure 4.6: Natalie chunking 0 to 7.5 x 10^9

Figure 4.7: Natalie chunking 7.5 x 10^9 to 1.5 x 10^{10}

In this exchange, Danna, Natalie, and the teacher co-constructed an argument that the timeline should be called linear because particular distances represent particular amounts of elapsed time. Natalie provided the data by pointing to two segments that were the same length and represented the same amount of elapsed time and Danna connected the data to the claim by providing the warrant that this feature is a characteristic of a linear timeline. The teacher structured this argument by requesting data when she asked Natalie to point to two segments that represented the same amount of time and when she concisely summarized and more clearly articulated the whole argument.

In fact, a critical reader might suggest that this episode mainly represents the teacher's reasoning. However, it was Danna who originally provided both the claim and the warrant. In the first three turns of talk Danna called the approach linear and explained that linear meant, "each distance is going to be the same amount of years." While this is in response to the teacher asking Danna for further justification, the teacher is not the one providing the justification. The bulk of the rest of the argument is illustrating what this means by having Natalie point to an example of two segments that represent the same amount of elapsed time. In the teachers' final comment, she does add to the argument by articulating the idea more clearly. This revoicing may also serve to legitimize or emphasize Danna's contribution. However, in terms of content, Danna's original statement is quite similar to the teacher's (see Table 4.3).

Table 4.3: Comparison of Two Warrants

Danna's Original Warrant		Teacher's Refined Warrant	
Teacher	What do you mean by linear?	**Teacher**	So that's what she meant by linear, the same chunk of distance meant the same amount of time
Danna	Well like, each distance is going to be the same amount of years.		

Background to Argument 0.1.2. This characterization of linear time lines was then used implicitly the next day when a student argued that a particular time line was not linear. At the beginning of class the teacher asked a group to present two methods for creating a number line, as a way to review the methods presented the previous day. Jaime started by presenting the third approach presented on Day 1 (macro exponential with linear subdivision) and his group mate, Jose, presented the linear approach. Then the teacher asked the class why the linear method was linear.

Danna explained, “I called it linear just because you're associating the distance and the time. They're the same amount each time. So one inch is going to be the same amount of time at the beginning of the time line as it is at the end.” The teacher asked about a name or description of the other approach (macro exponential) and Danna suggested exponential, because it had exponents. “Well just 'cause you're looking clearly at the exponents, ten to the eighth, ten to the ninth, so it's growing exponentially.”

Overview of Argument 0.1.2. The teacher then asked if anyone would like to add anything and Kathy claimed that the approach was not linear. To support this claim Farah pointed to the data that, in this instance, same-sized segments represented different amounts of elapsed time.

Here Kathy claimed that a particular timeline should not be called linear, by implicitly using the, now established, definition of linear in the class. She did not need to fully articulate the warrant by pointing out that the line does not meet the definition of a linear timeline because two same sized segments represent different amounts of elapsed times. Instead she simply provided the data for the claim, “each section doesn’t represent the same amount.”

Table 4.4: Coding of Argument 0.1.2

Participant	Speech	Actions	Code
Teacher	Would anyone else like to add anything under approach? Either an alternative name, or another description of what's happening? [Pause for 10 seconds]. So is everyone satisfied calling it exponential and we all know what we mean?		
Kathy	Exponentially not linearly. [Laughter].		Claim
Teacher	How is it, how is it different from linear…?		
Farah	Each section doesn't represent the same amount.		Data

Summary of NWR 0.1. This normative way of reasoning is a definition of a linear timeline. It centers on the idea that in a linear time line same-sized segments represent the same amount of elapsed time. This idea originated on Day 1 from Nathan and Danna. By Day 2, this idea was accepted as evidenced by Kathy and Farah using it implicitly to argue that a line was not linear. Since the warrant had dropped off in this argument, Criterion 1 of the DCA method is satisfied.

Math Practice 1: Developing the Macro Multiplicative Structure

Math Practice 1 describes how students recognized the macro multiplicative structure of the third approach. It involves several multiplicative patterns. First, students articulated the idea that each of the segments on the line represented an increase by a factor of 10 (NWR 1.1). They then articulated that this meant that a section of the timeline that consisted of multiple segments represented an increase by various factors of 10 (e.g. two segments together represented an increase by a factor of 100, three segments, an increase of 1,000; see NWR 1.2). This practice emerged over Days 2 and 3 of class.

NWR 1.1: Multiply the Previous Term by 10 to get the Next Term. The impetus for the development of this normative way of reasoning was a claim that this multiplicative pattern existed. Specifically, on Day 2 Erin claimed that a "times 10" pattern existed among the tick marks in an exponential line. By the next day this idea had become normative according to Criterion 2 of the DCA method. On Day 3 an argument was co-constructed by two students and the teacher that used this times 10 pattern as data.

Since the times 10 pattern was used as data in the second argument, it may not be the most salient idea of the argument. More salient may be the actual claim of the argument, which was that the macro level multiplication pattern should be extended to guide how segments should be subdivided. This was a major shift in how students were talking about the subdivision and was preceded by significant exploration and discussion. However, the details of the process by which the students arrived at this claim will be saved until the discussion of Math Practice 2. For NWR 1.1, the aspect of the argument that I will focus on is the data, the macro level times 10 pattern.

Background for Argument 1.1.1. Near the beginning of Day 2, the teacher posed a fairly open ended task that generated discussion. She showed an exponential time line (Figure 4.8) and simply asked what mathematical relationships, patterns, or ideas students noticed. Erin responded by noting the existence of a "times 10" pattern.

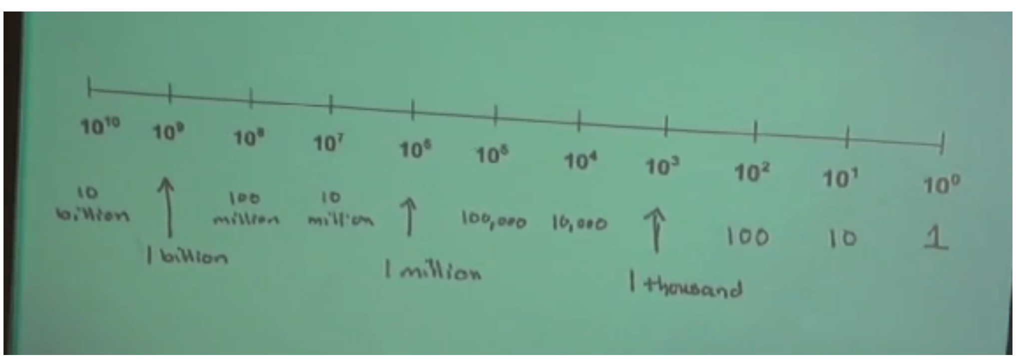


Figure 4.8: The exponential time line

Overview of Argument 1.1.1. Erin appeared to argue for the claim that the number of years ago, represented by consecutive tick marks on the time line from right to left, increases by a factor of ten from each tick mark to the next. She provided data for the claim by inserting a multiplication sign ("x") in two places (i.e., to show that 1 x 10 = 10 and that 10 x 10 = 100; see Figure 4.9), and by inserting "x 10" in three places on the time line (e.g., to express that 10 x 100 = 1 thousand).

Table 4.5: Coding for Argument 1.1.1

Participant	Speech	Actions	Code
Erin	It's increasing by multiples of ten.		Claim
Teacher	Could you mark that? So she says it's increasing by multiples of 10.		
Erin		marked the time line with "x10" in several places Figure 4.4.	Data
Teacher	Can anyone put this pattern in their own words? What am I multiplying by ten?		Claim
Several	The previous term.		Claim

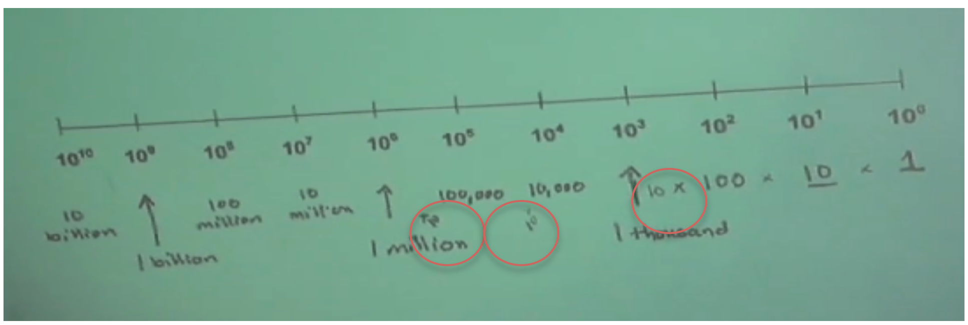


Figure 4.9: Erin's Markings, overlaid with researcher's annotations (red circles)

Background for Argument 1.1.2. On the next day, Day 3, the teacher asked what number went halfway in between 10^2 and 10^3. Lacey came up and suggested that it should be $10^{2.5}$. While Lacey's label was not disputed, a significant discussion ensued about the justification of the point. There were two methods students used for subdividing the segment from 10^2 to 10^3, both of which lend justification to Lacey's claim. These will be further explored when I present the evidence for Math Practice 2. In this discussion several ideas came up, including how the how the multiplicative pattern at the macro level relates to the subdivision of segments.

Overview of Argument 1.1.2. In Argument 1.1.2, Rachel, Kathy, and the teacher co-constructed an argument about how to subdivide segments. They claimed that the halfway point between 10^2 and 10^3 should be $10^{2.5}$ using as a warrant they idea that the multiplicative pattern that existed at macro level should be extended within the subsections. The introduction of this idea that the macro pattern should be extended was a turning point in the class and was preceded by significant mathematical struggle. This may be the most salient thing about this argument. However, I will not explore her method of subdivision until the discussion of Math Practice 2. Rather, I will focus on the data of this argument, the times 10 pattern. This shift of the times 10 pattern from claim (as in Argument 1.1.1) to data (as in this argument, Argument 1.1.2) satisfies Criterion 2 of the DCA method.

Table 4.6: Coding of Argument 1.1.2

Participant	Speech	Actions	Code
Rachel	All I was going to say was, what we discussed last time, was that to get from each next one, so like ten to the one, ten to the two, ten to the three, ten to the four, we're multiplying by 10.		Data
Teacher	So on this one. Right, I'll put up the one we had the other day. Ten to the zero, ten to the one, I believe this was Erin who first put this up and she said what? That 10 to the zero...	Draws a number line on the board with tick marks labeled 10^0, 10^1, 10^2, 10^3, 10^4. See Figure 4.10	
Rachel	You are just timesing by ten, which is the same thing as ten to the one.		Data
Teacher	Okay		
Rachel	Which is the same thing as square root of ten times square root of ten.		
Kathy	You didn't say it enough.		
David	Yeah, you could do it.		
Nathan	Almost there!		
Kathy	Which is the same thing as...		
Teacher	So can someone kind of finish this off? Kathy?		
Kathy	Uh... I don't know if I can finish it off		
Teacher	Okay, Whatever you want to say.		
Kathy	I was going to say it makes sense to me because when we were doing it like half exponential half linear we were adding the two halves, but now we need to have like the first half times the second half give us 10^3. Before we were doing like 500 plus 500 needs to give us a thousand, but that's linear; and we need to do something this half times this half needs to give us 10 to the third.		Warrant
Teacher	That sounds like a breakthrough to me.		
Kathy	Oh, thank you.		
Teacher	I think what she's saying is when we were doing it linear we were adding these chunks, but what you really want to do is continue this pattern[1], it's this times something is this times something is this times something is this[2]. So now we have ten squared times square root of ten is ten to the two point five[3], times square root of ten is ten to the third.		[1]Warrant [2]Data [3]Claim
Kathy	Yep		

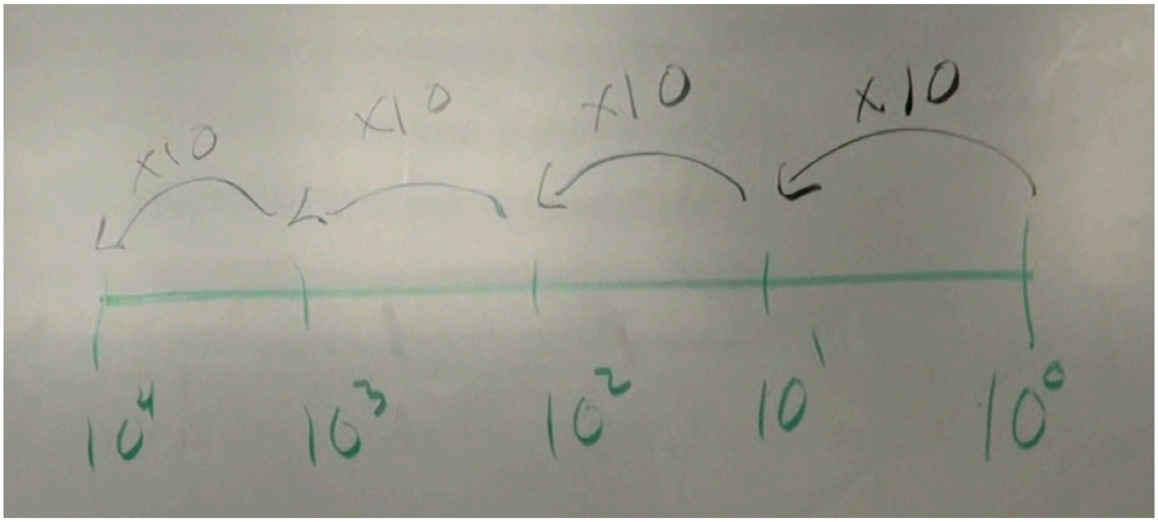


Figure 4.10: The Macro "times 10" Pattern

Summary of NWR 1.1. In this normative way of reasoning students noticed the macro level times 10 pattern. This first came up when Erin said she noticed it. She provided data for this pattern by labeling "x10" among various tick marks (Argument 1.1.1). Later in class, Rachel, Kathy, and the teacher justified the claimed that the halfway point between 10^2 and 10^3 should be $10^{2.5}$ by saying that they needed to continue the pattern that existed at the macro level (Argument 1.1.2). This line of reasoning treats the pattern at the macro level, the "times 10" pattern, as data. By this idea shifting from a claim in Argument 1.1.1 to data in Argument 1.1.2, Criterion 2 of the DCA method is satisfied.

NWR 1.2: Finding Factors Over Several Segments. This normative way of reasoning is similar to NWR 1.1, except that students now looked across several segments to find multiplicative patterns. So instead of simply noticing that the amount of years increased by a factor of 10 over one segment, the times ten pattern, they noticed that, for example, the years increased by a factor of 100 across two segments.

Overview of the Development of NWR 1.2. This idea first arose on Day 2 when students used the idea that going across two segments increases the amount of elapsed years by a factor of 100 as data to support an argument about how one can see 10,000 in the number line. Several days later, on Day 6, the same idea was used as a warrant when reasoning about a different exponential number line. This repeated use of the idea provides evidence that this way of reasoning functioned as if shared, as it had become a standard way to reason about the exponential lines. In particular, this satisfies Criterion 3 of the DCA method, since the idea appeared over several days, as data and then as warrant.

Background for Argument 1.2.1. This argument was advanced on Day 2 when students were reasoning about mathematical patterns they saw in the timeline. The students had been annotating a line that was on the document camera with various patterns (see Figure 4.11). While they were discussing the times 10 pattern, the teacher asked about patterns across sections of the timeline that span several tick marks. David started the conversation by drawing attention to the idea that 10,000 can be thought of as 100 x 100. Samantha then helped articulate how the 100 x 100 can be seen in the timeline.

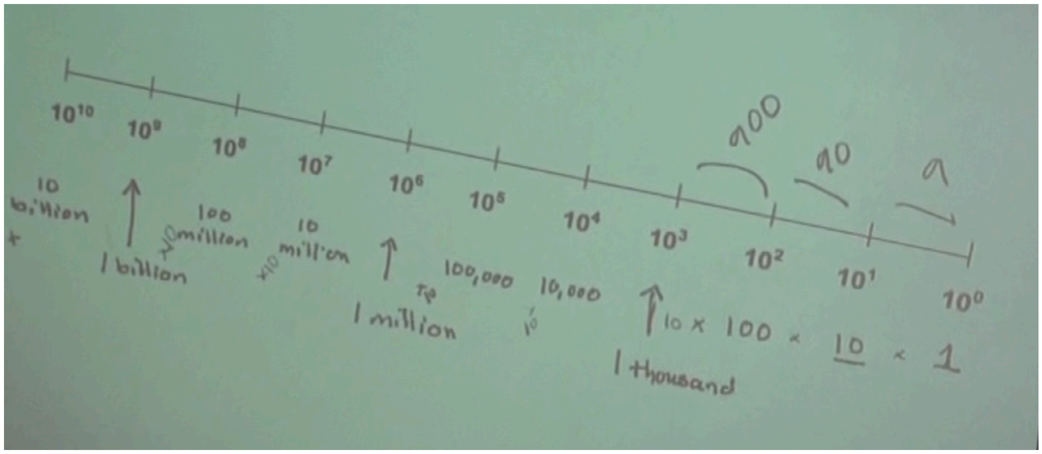


Figure 4.11: Annotating the number line with mathematical patterns.

Overview of Argument 1.2.1. In this argument Samantha claimed that one can see 10,000 in the timeline by looking across two sections of the timeline that each represent 1,000 years (the section from 10^0 to 10^2 and the one from 10^2 to 10^4). In this argument, she uses the idea that the sections each represent 1,000 years as data. Then for the warrant, she said that the two 1,000s should be multiplied.

Table 4.7: Coding for Argument 1.2.1

Participant	Speech	Actions	Code
David	I'm going to go like hundred times hundred*, you'll get the ten thousand**.	*writes 100 x 100 under the line **draws a vertical line under the 10,000 label See Figure 4.12	
	…		
Teacher	I think he has a great idea. Where does the first hundred come from? Someone go up and point to it.		
David	It's from the ten squared.		Data
Teacher	Ten squared.		
David	Yeah.		
Teacher	What's another way to think about how we got it? Samantha?		
Samantha	Well, the ten thousand is ten to the fourth, so it's ten squared times ten squared.		Data
Teacher	Can you write that? …		
Samantha		Samantha goes up to the document camera and writes $10^4 = 10^2$ □ 10^2	Data
Teacher	So Samantha, go ahead and explain your thinking.		
Samantha	So this here is ten to the second* and then ten to the fourth is just double that**[1]. So it's two [sections of the timeline], so you just multiply them[2] and then it gives you ten thousand[3].	*traces out a circle over the section from 10^0 to 10^2 **traces out a circle over the section from 10^0 to 10^2 and then over the section from 10^2 to 10^4 See Figure 4.13	[1]Data [2]Warrant [3]Claim

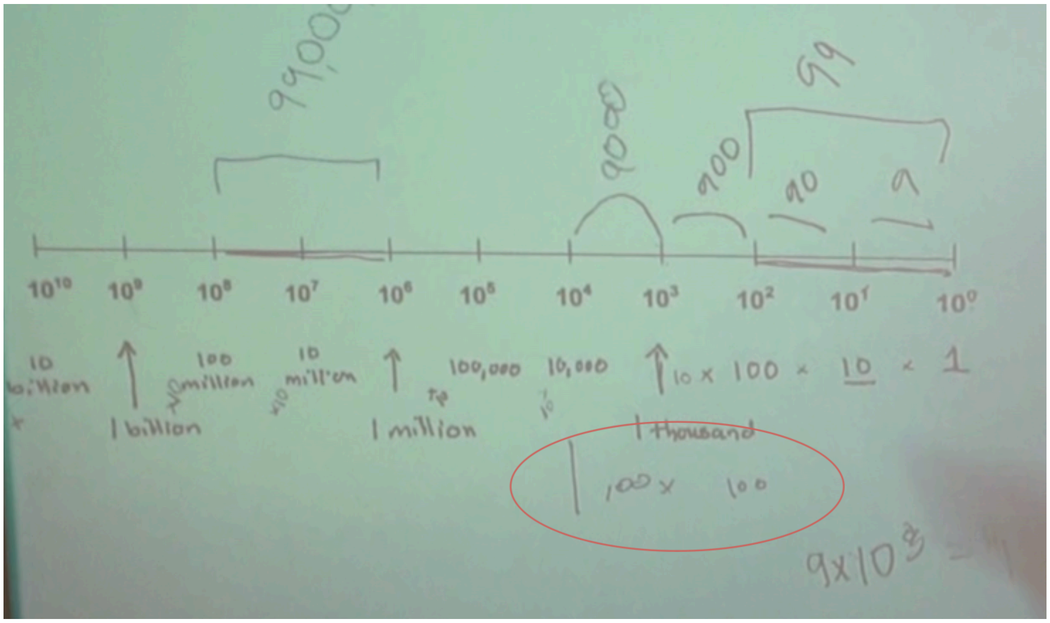


Figure 4.12: David's 100 x 100 Pattern

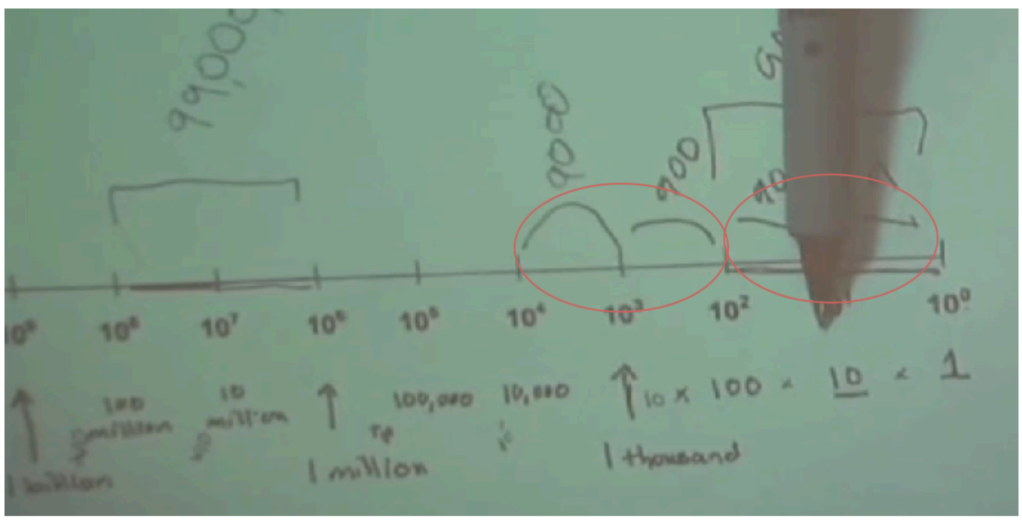


Figure 4.13: Samantha's Gestures

Background for Argument 1.2.2. This normative way of reasoning was established much later in the class, on Day 6, after other mathematical progress had been made. On the previous day, Day 5, the students had been asked to create an exponential number line that represented the accumulation of money in a magic bank where the money tripled each day. At the beginning of Day 6, the teacher gave them a summary sheet, which had the number line they developed the previous day (see Figure 4.14). She then displayed the following prompt on the document camera.

> Suppose you look at your bank account and record how much money you have. The next time you look, you have 81 times as much money. How much time passed between the two observation points?

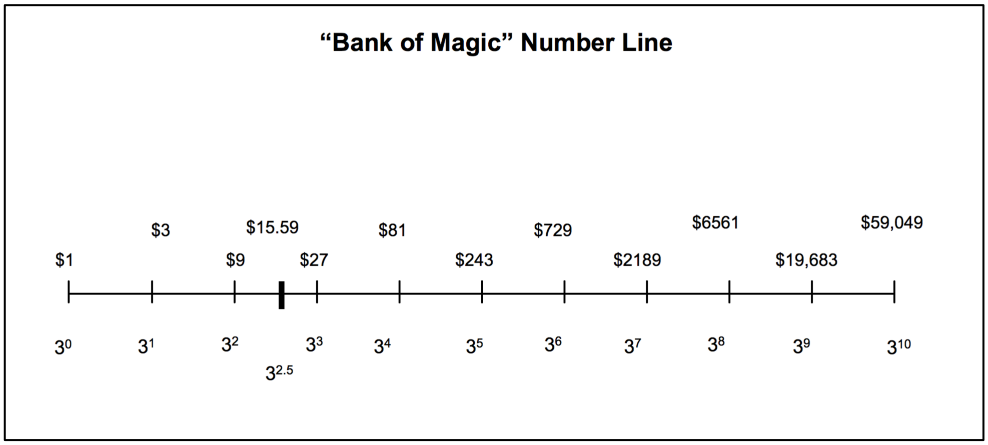


Figure 4.14: The "Bank of Magic" number line.

After some time working on the problem, Kaitlyn claimed the answer was 4 days. Erin supported this answer by saying that it takes 4 days to get 81 dollars, which is 81 times the starting amount of 1. Here several students stated the idea that between any four days, the money increases by a factor of eighty-one. Farah then explained why this true in Argument 1.2.2

Overview of Argument 1.2.2. In this Argument David claimed that over four days, the amount of money increases by a factor of 81. Farah provided the warrant for this claim by pointing out that each day represents an increase by a factor of three, so over four days you get an increase by a factor of 3x3x3x3, in other words by a factor of 81.

Table 4.8: Coding for Argument 1.2.2

Participant	Speech	Actions	Code
Teacher	Okay. Another one? Please raise your hand if you got another one? Okay. David? So what's yours David?		
David	From day four to day eight you're going to have eighty-one times more money.		
Teacher	How do you know?		
David	Because, I don't know, 'cause in the factors of four. I'm just looking at the fours. Four days from when I start, wherever, it's going to be eighty-one times more money.		Claim
Teacher	It's true. Can someone explain why that is? Can someone explain why that is? Lacey?		
Lacey	Well, if you have eighty-one times more you're going to multiply eighty-one by whatever number the day you started at. So in his example.		Claim
Teacher	Go point.		
Lacey	Eighty-one times eighty-one, so this is the day* you're starting to look at it if you multiply Eighty-one times eighty-one you get this number**	*points to day 4, with $81 **points to 6561	Claim
Teacher	How do you know? How do you know that was times a factor of eighty-one? Can someone come point to something?		
Farah	So for every day that goes by you're multiplying by three so you multiply three* times three** times three*** times three**** and that's eighty-one.	*sweeps over the section between day 4 and day 5 ** sweeps over the section between day 5 and day 6 *** sweeps over the section between day 6 and day 7 **** sweeps over the section between day 7 and day 8 See Figure 4.15	Warrant

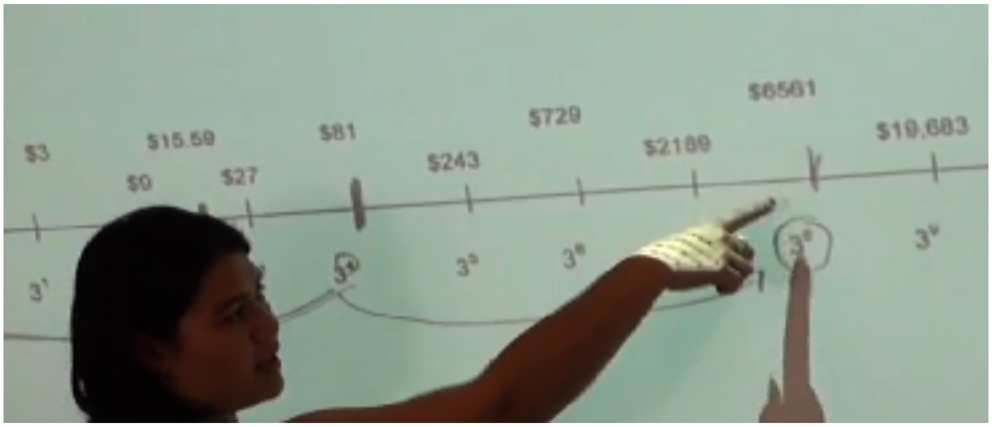


Figure 4.15: Farah's sweeping gestures to illustrate the factor of 3x3x3x3

Summary of NWR 1.2. This normative way of reasoning deals with students reasoning about sections of the timeline that span more than one segment between tick marks. The students had already established the constant multiplicative relationship between tick marks when they noticed the times ten pattern when reasoning about the timeline task (see NWR 1.1). When they reasoned about sections of the timeline that spanned several tick marks they repeatedly used a similar line of reasoning. Namely, that the larger section of the timeline represented multiplication by a factor that was equal to the product of the factors represented by each those smaller segments (e.g. if a large segment spanned two smaller times ten segments it represented a factor of 10x10=100 or if an large segment spanned four smaller times three segments it represented a factor of 3x3x3x3=81). The repeated use of this line of reasoning as data (as in Argument 1.2.1) or warrant (as in Argument 1.2.2) satisfies Criterion 3 of the DCA method.

Summary of Math Practice 1. This math practice had to do with the macro-level multiplicative patterns that exist in the number line. Students first noticed that the segments between tick marks represented an increase by a factor of ten (NWR

1.1). They then began to look across segments to recognize other patterns, such as times 100 (NWR 1.2). These multiplicative patterns are foundational to how students later subdivided segments, which will be addressed in NWR 2.2 of Math Practice 2.

Math Practice 2: Subdividing Segments

This Math Practice deals with ways of reasoning about how to subdivide segments. Two normative ways of reasoning co-developed over the class period on Day 3. In the first way of reasoning, students focused on the linear pattern in the exponents (NWR 2.1). In the second way of reasoning, students generalized the idea that a segment of the line represented an increase by a constant factor from the macro structure of the time line (see Math Practice 1) to subdivided segments. Specifically, they reasoned that each subsection should represent an increase by a particular factor (NWR 2.2).

NWR 2.1: Subdividing Segments by Reasoning Linearly About Exponents. Subdividing linearly means there is a one to one correspondence between how the segment is divided and how the exponent is divided. This means that if the students divided the segment where the exponent increased by 1 into two pieces, they reasoned the exponent increased by ½ over each subsection. Similarly, if they divided the segment into ten subsections, they reasoned the exponent increases by 1/10th over each subsection.

Overview of the Development of NWR 2.1. This normative way of reasoning developed during Day 3. Three arguments were given that all used the same warrant. The first occurred at the very beginning of class, when Lacey reasoned that the midpoint between 10^2 and 10^3 was $10^{2.5}$. After this, the class started to subdivide in a different way, a way consistent with NWR 2.2. However, they returned to reasoning linearly with exponents at the end of class when they needed to place the bow and arrow ($10^{4.5}$ years ago) and the Ordovician period ($10^{8.7}$ years ago). Since the same warrant was used to reason about the placement of three different times, $10^{2.5}$, $10^{4.5}$, and $10^{8.7}$, Criterion 3 of the DCA method is satisfied.

Background for Argument 2.1.1. This argument came at the very beginning of Day 3, when the teacher started the class by putting up a number line and asking the students what the midpoint of the 10^2 and 10^3 should be labeled (see Figure 4.16). Lacey argued that the point should be labeled $10^{2.5}$ and no one challenged her. This was somewhat surprising because previous to this, the only way students had been subdividing segments of the number line was reasoning linearly about the values. In other words, they would have found the difference between 1,000 and 100, which is 900, divide that by 2 to get 450, and add that 100. This linear method of subdivision became problematized at the end of Day 2. On Day 2, the teacher had asked the students to place the Renaissance, which occurred 500 years ago on a number line where the tick marks increased by multiples of ten. They placed this by linearly subdividing the segment from 100 to 1,000. The teacher then asked them to use the same method to place the Renaissance, but to use 1 and 1,000 as the endpoints. The students did this to find the midpoint of 1 and 1,000 to be 500.5. This made it clear that the Renaissance would move depending on the endpoints you choose.

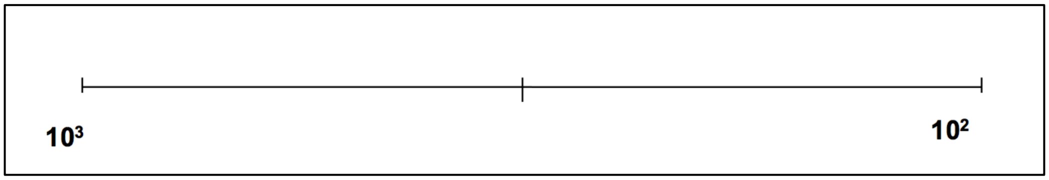


Figure 4.16: The midpoint task.

During the discussion of why this was problematic, Danna argued that the problem was their method of subdivision. She then started to present an alternative idea. She briefly flashed an image of her work on the document camera (see Figure 4.17). In this image, she had a segment of a number line, from 10^2 to 10^3, divided into 10 subsections with each subsection labeled with powers of 10 whose exponents successively increased by a tenth. Even though the image was only on the document camera for about 3 seconds, it is possible that other students noticed the linear pattern in the exponents.

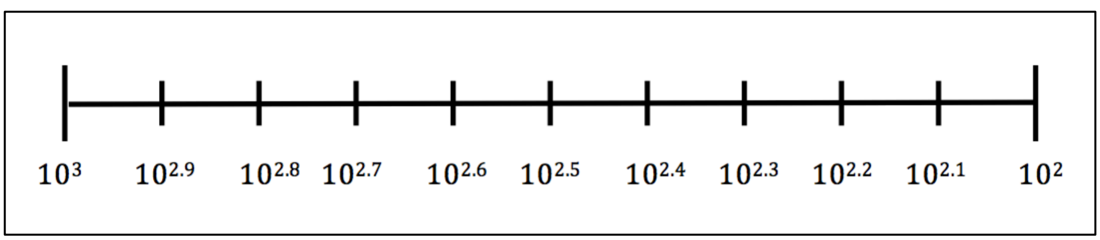


Figure 4.17: Author's recreation of Danna's Work

More ideas about how to place events in between tick marks may have emerged for students as they worked on homework between Day 2 and Day 3. This homework encouraged students to explore the relationship between the multiplicative patterns that exist at the macro level and additive pattern in the exponents. If students were affected by Danna's image, it may be possible that they attended more to linear patterns in the exponents on the homework rather than coordinating linear reasoning with multiplicative patterns, as was intended. This may help explain students' acceptance of Lacey's linear argument.

Overview of Argument 2.1.1. Lacey appeared to argue that the midway point between 10^2 and 10^3 is $10^{2.5}$ (claim) because from 10^2 to 10^3 is a factor of 10^1 (data), and since the segment on the number line is divided in half, you divide the exponent of 1 in half to get 0.5 (warrant). The .5 is then added to the exponent of the endpoint of the segment, which was 2.

Table 4.9: Coding for Argument 2.1.1

Participant	Speech	Action	Code
Lacey	I got ten to the two point five[1]. 'Cause thinking about it, this whole thing* is a factor of ten to the one[2], so then if you're going to divide it in half it's going to be point five of that, so then, you just add the point five to the two to make this a factor of point five** and this a factor of point five***[3].	*sweeps finger from 10^2 to 10^3 **gestures to the segment from 10^2 to midpoint by placing her thumb on 10^2 and index finger on midpoint ***similarly gestures to the segment from the midpoint to 10^3 See Figure 4.18	[1]Claim [2]Data [3]Warrant

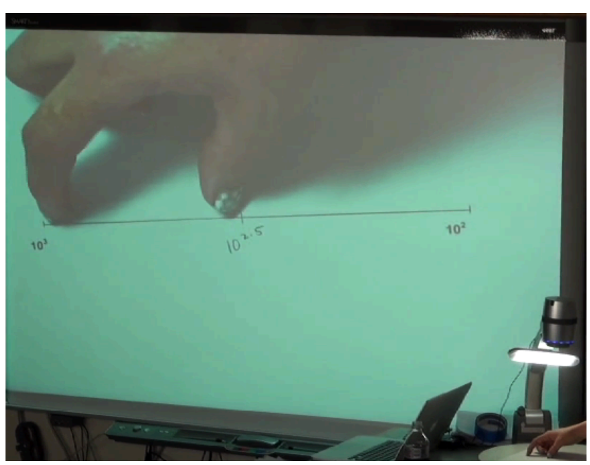


Figure 4.18: Lacey Gesturing to Part a Subsection

I interpret her argument to be reasoning linearly with the exponents and not multiplicative reasoning (as will come in NWR 2.2). Saying "you're going to divided it in half," focuses on the halving, which implies linear reasoning. While she did use some multiplicatively language, calling both the 10^1 and the two sections representing $10^{.5}$ "factors," she did not talk about the factors being multiplied by anything. In fact, she talked about adding the .5 to 2. This suggests that she may simply be using the word factor as a label for the segment and two subsections. Furthermore, this was not taken up as multiplicative reasoning in the class. Mallory elaborated this argument, focusing on the linear relationship between the exponents.

> Mallory: Well I just kinda ignored the ten and just looked at the exponents. So, ten to three and ten to the two, so I just did three minus two so, equals one, so that one so if you put the ten back in there that's the whole thing and then just do one divided by two which is point five so you know each little section is two point five.

This clearly focuses on operating on the exponents, specifically dividing the exponent of one by two.

Background Argument 2.1.2. Toward the end of class on Day 3, the students revisited this way of reasoning as they worked on a worksheet that asked them to place several time periods. The first of these was to place the bow and arrow. The task asked the following.

> 1. The bow and arrow first appeared about 31,600 years ago. Suppose you know that is the same as $10^{4.5}$ years ago. Place this time period on the number line below. Describe your method.

Figure 4.19: The bow and arrow task.

Overview of Argument 2.1.2. After they had time to work in small group, Yessica presented her solution to the whole class, claiming that $10^{4.5}$ should be placed halfway between 10^4 and 10^5. Her warrant focuses on the fact that she has divided segment from 10^4 to 10^5 in half and that each half represents $10^{.5}$. Then, in the follow up discussion the fact that she divided the exponent in half was further emphasized.

Table 4.10: Coding for Argument 2.1.2

Participant	Speech	Actions	Code
Teacher	So bow and arrow appeared thirty one thousand six hundred years ago, and that is ten to the four point five, where should that go and why?		
Yessica	Well it's the same similar to the one we already did. So it's from here to here is going to be ten to the one*[1]. So if we want to get the sorry, so this is going to be, hold on**. Okay.	*draws a bracket from 10^5 to 10^4 and labels it 10^1 **Writes "= $10^{1/2}$ x $10^{1/2}$" to get 10^1 = $10^{1/2}$ x $10^{1/2}$ and then draws a brace from 10^5 to the midpoint and from the midpoint to 10^4 and labels them each $10^{1/2}$. Writes $10^{4.5}$ at the midpoint of the larger extant[2]. See Figure 4.20	[1]Data [2]Claim
Teacher	So can you explain your labels?		
Yessica	Yes, this is going to be the same as ten to the one half* and ten to the one half**, so for here to here, it's ten to the one half***, and the same for here****, ten to the one half, so if we want to get the one, ten to the four point five, so we have to add just the half from here*****, so it's ten to the four and the one half so it's going to be ten to the four point five.	*points to first label of $10^{1/2}$ **points to second label of $10^{1/2}$ ***sweeps over the subsection from 10^5 to the midpoint. ****sweeps over the subsection from the midpoint to 10^4. *****points to the exponent in $10^{1/2}$ that labels the subsection from 10^4 to the midpoint.	Warrant

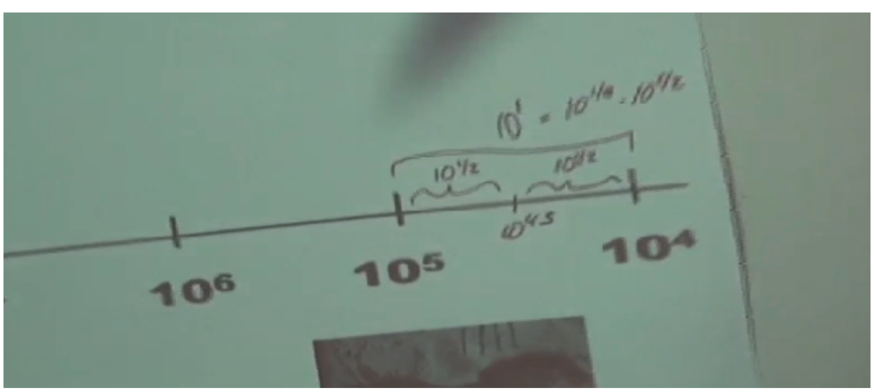


Figure 4.20: Yessica's Drawing

The teacher then asked for questions about Yessica's work. David asked about Yessica's notation—why she wrote 4.5 instead of $4\frac{1}{2}$. Kathy responded to the question and in her response emphasized that Yessica divided the exponent in two.

> Kathy: So what I think is why she didn't do one half is because, she was just splitting the ten to the one so that would just be ten to the one half not, not ten to the four and a half and then she takes that ten to the one half and multiplies it to ten to the four so then you get four and half.

The halving of the exponents was further emphasized in a teacher's follow up question. She asked, "Why is it working to halve when it did not work when you were using a halving linear method for the Renaissance last Thursday?" In response, Samantha explained, "Last time we were solving like ten to the fourth and ten to the fifth and we were halving the, what the answers were to it…rather than take half of five and four." Similarly, when the teacher asked, "What are you taking half of here?" Several students responded, "The exponents." The teacher then reiterated what they were saying, "The exponents, you're not taking half of the one thousand or half of ten thousand or a hundred thousand, you're taking half of the exponent."

Background Argument 2.1.3. This linear method continued as students reasoned about the next task, which was to place the Ordovician period (see Figure 4.16). Jacqueline presented her way of placing this period, which was also linear.

2.	The Ordovician period (shelled animals appear) occurred 5.05 x 10^8 years ago. Suppose you know that 5.05 is about $10^{0.7}$ (meaning the Ordovician period occurred $10^{8.7}$ years ago). Place this time period on the number line below. Describe your method.

Figure 4.21: The Ordovician period task.

Overview of Argument 2.1.3. Jacqueline's claim was the placement of $10^{8.7}$ (see Figure 4.17). Her warrant justifying her placement was dividing the segment between 10^8 and 10^9 into ten subsections. She then placed $10^{8.7}$ on the tick mark for the seventh subsection.

Table 4.11: Coding of Argument 2.1.3

Participant	Speech	Action	Code
Teacher	… Could you tell us how you did the second one [placing $10^{8.7}$]? …		
Jacqueline	Yeah, so for this one, since it says it's ten to the eight point seven and then so I just divide the whole thing to ten pieces and then I just, yeah, and here is the middle* which is eight point five and then I just add two so it's going to be ten to the eight point seven.	*points to the tick mark labeled $10^{8.5}$ Figure 4.22	Warrant
Teacher	… Any different way of thinking about it? How would you describe this method in general? What are you doing? Tanya.		
Tanya	We're splitting the interval into tenths. … What I did was the same thing, but just saw it, the point seven as seven tenths, so divide it into tens and then go seven tenths.		Warrant

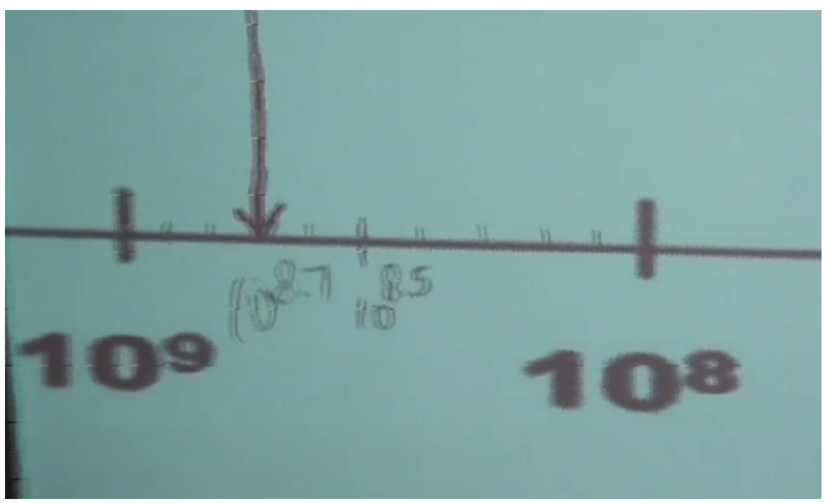


Figure 4.22: Jacqueline's Placement of $10^{8.7}$

Summary of NWR 2.1. Students reasoned linearly with the exponents to justify the placements of three different times in Arguments 2.1.1, 2.2.2, and 2.2.3. This repeated use of a warrant fulfills criterion 3 of the DCA method. Giving further evidence that this way of reasoning functioned as if shared in the classroom, they concluded their discussion by saying that to place 20,000 they would first need to write it as some power of 10 “because you can just plot that.” This reiterates the focus on the exponents when plotting events.

NWR 2.2: Preserving the Multiplicative Relationship within the Segments. This way of reasoning also deals with subdividing the segments of the timeline, but instead of focusing on the additive relationships between the exponents, in this line of reasoning, students focused on the multiplicative relationships among dates represented on the time line.

Overview of the development of NWR 2.2. This normative way of reasoning was established according to Criterion 3 of the DCA method. As I mentioned in the discussion of NWR 2.1, Day 3 began with Lacey’s argument that to $10^{2.5}$ was the midpoint between 10^2 and 10^3. She justified her claim by appealing to a linear pattern in the exponents. However, as the students probed this placement more deeply, they began to talk about multiplicative relationships. During this discussion two different arguments came up where students used multiplicative patterns as the warrant for their subdivisions of segments of the time line. One of these was Argument 1.1.2, whose data I used as evidence for the establishment of NWR 1.1. There I focused on how the macro multiplicative pattern was used as data. Now I focus on the warrant, how this multiplicative pattern was extended to the subsections. Students later used the same warrant to place $10^{1/7}$. This repeated use of a warrant satisfies Criterion 3 of the DCA method.

Revisiting of Argument 1.1.2. Kathy's breakthrough came on Day 3 when Rachel, the teacher, and she co-constructed an argument in which she compared the class's initial linear way of reasoning about subsections, with an alternative, finding a constant factor. As mentioned previously, they appeared to have claimed that the halfway point between 10^2 and 10^3 should be $10^{2.5}$ using as a warrant the idea that the multiplicative pattern that existed at macro level should be extended within the subsections. However, when I presented this argument before, the focus was on data. Now, for NWR 2.2 I focus on the conceptual breakthrough in this argument, the warrant.

Table 4.12: Revisiting of the Coding of Argument 1.1.2

Participant	Speech	Action	Code
Rachel	You are just timesing by ten, which is the same thing as ten to the one.		Data
	…		
Kathy	I was going to say it makes sense to me because when we were doing it like half exponential half linear we were adding the two halves, but now we need to have like the first half times the second half give us 10^3. Before we were doing like 500 plus 500 needs to give us a thousand, but that's linear; and we need to do something this half times this half needs to give us 10 to the third.		Warrant
Teacher	That sounds like a breakthrough to me.		
Kathy	Oh, thank you.		
Teacher	I think what she's saying is when we were doing it linear we were adding these chunks, but what you really want to do is continue this pattern[1], it's this times something is this times something is this times something is this[2]. So now we have ten squared times square root of ten is ten to the two point five[3], times square root of ten is ten to the third.		[1]Warrant [2]Data [3]Claim

Background for Argument 2.2.2. After Kathy had voiced her argument, she pointed out that this line of reasoning could be used with other roots. For example, by splitting a segment into three pieces, one would reason he or she needs a number that when multiplied by itself three times yields ten. That means the subsection would need to represent a factor of the cube root of ten. To follow up on this observation, the teacher drew a segment from 10^2 to 10^3 on the document camera and divided it into seven sections. She asked how the students thought about the relationship between the first subsection and the whole segment (see Figure 4.23).

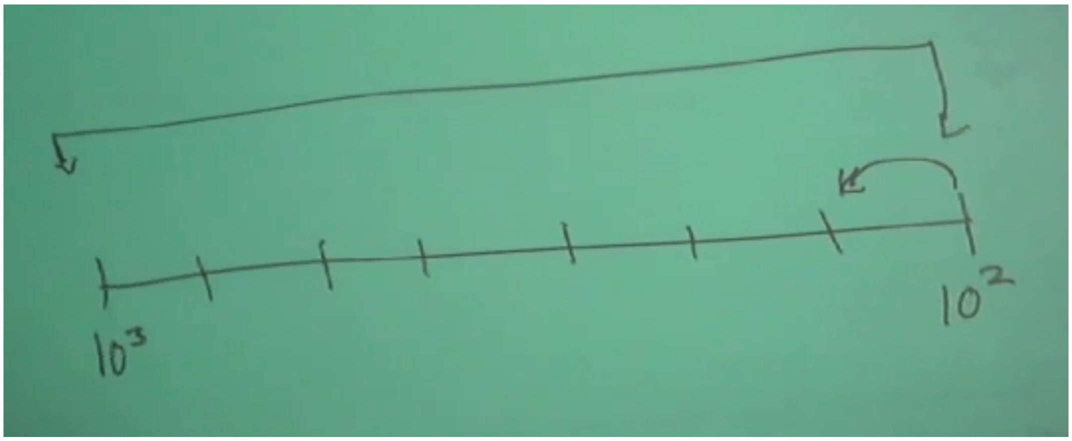


Figure 4.23: The relationship between a segment and its subsections.

Overview of Argument 2.2.2. Eventually, Jade argued that this subsection must represent $10^{1/7}$. Her reasoning that multiplication of $10^{1/7}$ by itself seven times yields 10 acted as a warrant and supported this claim. The data in this argument was simply that the segment was divided into seven subsections. After Jade's argument, Mallory added that thinking of the subsections in this way is crucial to keeping the line exponential.

Table 4.13: Coding for Argument 2.2.2

Participant	Speech	Action	Code
Teacher	… Okay. I've got it in seven sections. Before anybody calculates anything, I want you to tell me, how do you think about the relationship between these two factors**? So I want you to think a minute. How do you think about the relationship between the factor that you multiply ten squared to get ten cubed and the factor you multiply ten squared to get that next tick mark? … Okay. Jade, how do you think about it?	*Draws a line and splits into seven sections **Draws a double sided arrow between 10^3 and 10^2 and another between 10^2 and the first tick mark? See Figure 4.18	
Jade	So going from ten squared to ten to the third you have to multiply ten squared times ten to get to ten to the third.		Data
Teacher	*Is that what you're thinking?	*Labels top arrow "x10"	
Jade	Yeah, yeah. And since you have seven subsections[1], what I did to the exponent of ten, since it's ten to the one, I divided that exponent by seven[2].		[1]Data [2]Warrant
Teacher	I think I'm going to have to have you write.		
Jade	So this whole thing is ten, you multiply ten squared times ten to get ten to the third, or ten to the first[1], and since we have seven subsections[1], each subsection would be ten to the one seventh*[2]. So this is…	She comes up to document camera *She labels the subsection $10^{1/7}$.	[1]Data [2]Claim
Teacher	Why? Why is that? And either Jade can answer or someone else can answer. Can you answer it? Why is it ten to the one seventh?		
Jade	Because if we multiply by ten to the seven ten times [sic.] it's going to give us ten to the one.		Warrant
Teacher	I want you to do that. …		
Mallory	There's seven times so you have to have seven sections.		Warrant
Teacher	Someone else revoice what Mallory just said. Could you all hear her? Say it one more time.		
Mallory	Since there's seven sections you have to multiply the same number seven times, so that's why it's divided by seven.		Warrant

The feature of this argument relevant to the establishment of NWR 2.2 is the use of multiplicative reasoning when determining the placement of dates on subdivided segments. Jade used multiplicative reasoning as a warrant when she said, "because if we multiply by ten to seven ten times, it's going to give us ten to the one." This was in response to the teacher asking why the first tick mark should represent ten to the one seventh, which provides evidence that this idea was being used as a warrant. The warrant was then articulated even more clearly by Mallory when she said, “Since there's seven sections you have to multiply the same number seven times.”

The fact that multiplicative relationship should be preserved was further emphasized when David got confused about what factor two subsections should represent. While talking about this problem, the class had re-expressed $10^{1/7}$ as 1.39 and were discussing what factor two of these one-seventh subsections represented. David said that two subsections should represent an increase by a factor of 2 times 1.39, but this was universally rejected.

Summary of NWR 2.2. The idea of subsections representing multiplication by a constant factor rather than addition by a constant difference was first introduced as a warrant in Kathy’s argument in Day 2. This warrant was also used to reason about how to place $10^{1/7}$. This repeated use of a warrant satisfies Criterion 3 of the DCA method and provides evidence the idea was functioning as if shared in the classroom. This is corroborated by the students’ rejection of David’s suggestion that they should repeatedly add 1.39 in the subsections.

Summary of Math Practice 2. This math practice describes students’ ways of reasoning as they subdivided segments of the exponential number line. There were two ways of doing this. In NWR 2.1, students recognized a linear pattern in the exponents and continued that pattern. In NWR 2.2, students extended the macro-level exponential pattern to the subsections.

Math Practice 3: Finding Fractional Exponents

This math practice deals with students placing numbers of the form $10^{a/b}$ on the number line. This math practice is unique in that it only consists of one normative way of reasoning, NWR 3.1. In this way of reasoning the students rewrote $10^{a/b}$ as $(10^{a})^{(1/b)}$, found the segment from 10^{0} to 10^{a} and subdivided it into *b* subsections. There was a second way of reasoning that was used to place numbers of this form as well, but it did not become normative because students only used this way of reasoning once in whole class discussion (though it was present on homework and on the final exam). In this way of reasoning, students rewrote $10^{a/b}$ as $(10^{1/b})^{a}$, found the segment from 10^{0} to 10^{1} and subdivided it into *b* subsections and then placed the point at the end of the a^{th} subsection. This way of reasoning is similar to NWR 2.2, where students reasoned about how to subdivide a segment by preserving the multiplicative pattern that existed at the macro level within the subsections. Because of these similarities, the participants may have felt it unnecessary to continue exploring this method of placing points of the form $10^{a/b}$. This may be why the way of reasoning was only used once and was not established as normative.

NWR 3.1: Subdividing Extents that Span Multiple Segments. This idea first arose as students divided the section of the timeline that represented 10^{5} into two equal sections to find $10^{(5/2)}$. The idea then became normative when the same warrant was used to reason about $10^{(3/4)}$ as a $1/4^{th}$ segment of 10^{3}. Again, the repeated use of the warrant provides evidence this was a standard way of reasoning in the class and satisfies Criterion 3 of the DCA method.

Background for Argument 3.1.1. On Day 3, Nathan made an argument for placing $10^{2.5}$ at the midpoint between 10^2 and 10^3 by saying that the subsections should form a multiplicative pattern (see NWR 2.2). After he had given his argument the teacher asked students to discuss it in small group. After they had discussed it, the teacher asked Farah to expand on his idea in a whole class discussion. Instead, she said she did it a different way and presented the seemingly unrelated fact that $10^{(5/2)}$ can be rewritten as $(10^5)^{(1/2)}$ or as $\sqrt{10^5}$. She then went to the board, drew a number line with 10^0 and 10^5 as endpoints and marked $10^{2.5}$ at the midpoint (See Figure 4.19). She then explained that one could see $10^{5(1/2)}$ in the number line by thinking of 10^5 as a single section of the timeline, which means $10^{5(1/2)}$ would be the halfway point of that section. At that point Danna said she could connect Farah's idea to Nathan's and gave the following argument

Overview for Argument 3.1.1. Danna claimed that the halfway point was the square root of ten to the fifth, using as a warrant the idea that the square root of a number is the same as the half the distance of a section of the number line. In this case, the square root of 10^5 would be the halfway point of section of the number line from 10^0 to 10^5.

Table 4.14: Coding for Argument 3.1.1

Participant	Speech	Action	Code
Danna	So if you're looking at ten to the fifth, halfway is square root of ten to the fifth, which is also happens to be ten to the two point five, that's all I'm saying. So that square root is half the distance and we know ten to the two point five is half.	Draws a number line with 10^0 and 10^5 at the ends with $10^{2.5}$ and $\sqrt{(10^5)}$ marked in the middle See Figure 4.24	Claim
Chris	But that's between zero and five and we're working with...		
Kathy	with two and three.		
Danna	I know, but it's the same thing so there's your three, there's your two. So this is, the square root's giving you half the distance, well it's half the distance of this*[1]. That's where the five comes in. 'Cause two point five is half of five[2].	*points to segment from 10^0 to 10^5	[1]Warrant [2]Data
Rachel	Because we took our original five dived by two.		
Danna	Yeah. So even though we're looking at this small section*, this shows that it continues throughout the whole time line.	*sweeps over the segment between 10^2 and 10^3	
Kathy	Oh, interesting.		
Danna	Which is what we're trying to show.		

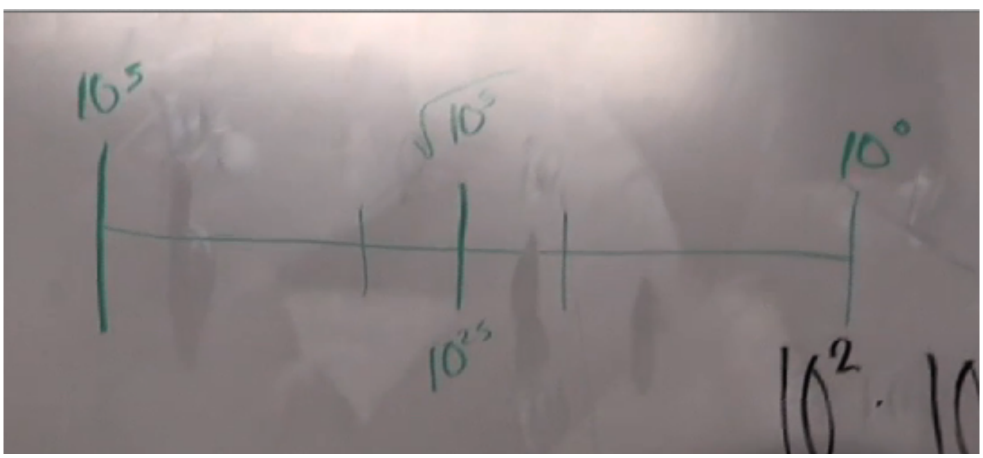


Figure 4.24: Danna's way of finding $10^{5/2}$

Background for Argument 3.1.2. This way of subdividing these larger sections of the timeline was reiterated on the next day, Day 4. The teacher gave a task that asked the students to express $10^{3/4}$ in three ways and to show each expression on the number line. At this point in this episode, the students had already expressed $10^{3/4}$ as $(10^{1/4})^3$ and represented it on the number line. Now they were exploring how to represent $(10^3)^{(1/4)}$ on the number line. Here we see Farah adopting Danna's warrant from the previous day.

Overview for Argument 3.1.2. In this argument Farah claims that if one takes the section of the timeline from 10^0 to 10^3 and divide it into four subsections, the first subsection will be $(10^3)^{1/4}$. The warrant that supports this idea is that dividing the section from 10^0 to 10^3 into fourths is the same thing as raising 10^3 to the one-fourth power.

Table 4.15: Coding for Argument 3.1.2

Participant	Speech	Action	Code
Teacher	Let's get a different model up here of um... Did anybody work with a number line that has the ten cubed on it? So we've got Farah and Danna. Let's start with Farah. …?		
	…		
Farah	So the way I have it is that it's ten cubed as your endpoint, instead of ten to the one. And then you take a fourth root of that[1]. So it's ten cubed to the fourth, which is ten to the three fourths[2]. And then I just did it over again to reiterate what I was doing.	Puts up a timeline Figure 4.25	[1]Warrant [2]Claim
Student	Can you repeat that?		
Farah	So it's ten cubed as a whole and then you take a fourth root of that whole.		
Teacher	Explain what that means to take the fourth root of that whole. What are you looking at?		
Farah	Well, you're looking at that you can take ten, ten cubed to the fourth, three, four times over and get the whole ten cubed.		
	…		
Teacher	… Okay, now I'm going to revoice Farah. Thank you, but I'm going to keep yours for a second. Here's what I heard her say. She extended this to ten to the third, she now thought about taking the fourth root of ten to the third, which is saying I need a number that times itself four times gives me what?		Warrant
Student	Ten to the third		
Teacher	Ten to the third. And what is ten to the third?		
Many	A thousand		

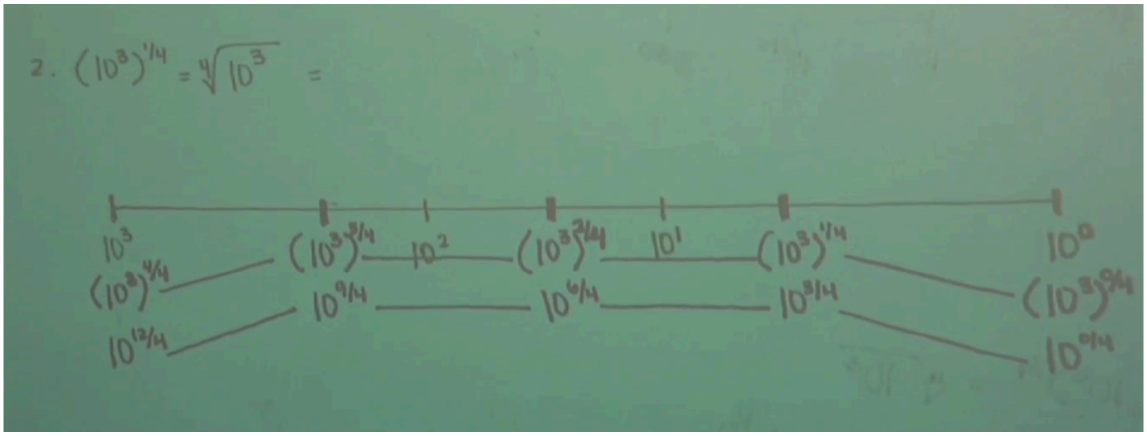


Figure 4.25: Farah's model of $(10^3)^{1/4}$

Summary of NWR 3.1. This normative way of reasoning deals with how students can find a number in the form $10^{a/b}$ on the number line. Students did this by rewriting the number in the form $(10^a)^{1/b}$, and then subdivide the section from 10^0 to 10^a into b subsections. $10^{a/b}$ was the first of these subsections. This was established by Criteria 3 of the DCA method, as students repeatedly used this idea as a warrant.

Another Way of Reasoning about Fractional Exponents. As was mentioned in the background to NWR 3.1, on Day 4 the students were asked to express $10^{(3/4)}$ in multiple ways and to show each expression on the number line. As previously discussed, Farah's representation of $10^{(3/4)}$ as $(10^3)^{(1/4)}$ gave rise to Argument 3.1.2, which helped established NWR 3.1. However, students also represented $10^{(3/4)}$ as $(10^{(1/4)})^3$. This gave rise an alternative way of reasoning about fractional exponents. In this way of reasoning, students partitioned the segment from 10^0 to 10^1 into four pieces to find $10^{(1/4)}$ and then iterated this segment three times to find $(10^{(1/4)})^3$. This way of reasoning technically did not become normative because this was the only instance in which it was used in whole class. However, there are similarities between this way of reasoning and NWR 2.2, in which students subdivided segments by preserving the multiplicative pattern within segments. This made this way of reasoning with fractional exponents feel familiar in the class. This combined with the fact that this was clearly presented as another way of reasoning about $10^{(3/4)}$ in whole class, on homework, and on the final exam warrants a discussion of this way of reasoning under this math practice.

Background to the Argument. The students were tasked with representing $10^{(3/4)}$ in several ways and showing those representations on the number line. Their discussion of these representations began with Santiago suggesting it could be written as $(10^{(1/4)})^3$. They then discussed how they could find this on the number line.

Overview of Argument. The claim in this argument is the location of $10^{3/4}$. The warrant supporting this placement is that subdividing the segment from 10^0 to 10^1 into four pieces yields $10^{1/4}$. Then, if one takes the third subdivision, she gets $10^{1/4}$ times itself three times, which is $(10^{1/4})^3$, which is $10^{3/4}$. This warrant was co-constructed by Samantha and Kathy.

Table 4.16: Coding for Argument

Participant	Speech	Actions	Code
Teacher	...What does the ten to the one-quarter mean? What am I doing with that? Samantha.		
Samantha	You're pretty much breaking it into four parts where you can multiply the same amount four times.		
Teacher	Can you show us with your hand, just come up and gesture, what you broke apart and what you would be multiplying*? So we're just trying to get at the meaning of ten to the one quarter.	*Samantha goes up	
Samantha	So this* you would break it in four parts**, where you would multiply the same amount, which is ten to the one fourth, four times it would give you ten to the one.	*Points to the 10^1 **Points to each of the four sections	Warrant
Teacher	And what are you multiplying the ten to the one fourth by?		
	[Inaudible]		
Rachel	Itself.		
Teacher	Itself. Let's write some expressions here to get that multiplication. So can anybody write above this what am I multiplying the ten and a one fourth by here*? What am I multiplying it by here**? What am I multiplying it by here***? Someone add that as labeling to this?	*Points to first subsection. **Points to second subsection ***Points to third subsection Figure 4.26	
Rachel	Do it.		
Teacher	Kathy, come on up….	Kathy comes to the document camera.	
	…		
Kathy	Can I write it on the bottom?		
Teacher	You can write it wherever you want.		
Kathy		Writes $(10^{1/4})^1$, $(10^{1/4})^2$, $(10^{1/4})^3$, $(10^{1/4})^4$ Figure 4.27.	
Teacher	Can you explain what you wrote?		
Kathy	… Okay, because we're multiplying the one fourth every time, I'm taking the power of ten to the one fourth. So for my like first nitch I took the power, well I'll do this one, like ten to the one fourth, I only multiplied it like one time so I took it to the power of one. Now to get here I need to take ten to the one fourth times ten to the one fourth to get these two parts, so I took ten to the one fourth squared, which gives me ten to the two fourths. Which is like our nitch that we had up here. And then you keep doing[1]... Oh okay so here's like my third pieces so I needed to do it three times so I took ten to the one fourth three times and we get our point ten to the three fourths[2] and then I did it four times.		[1]Warrant [2]Claim

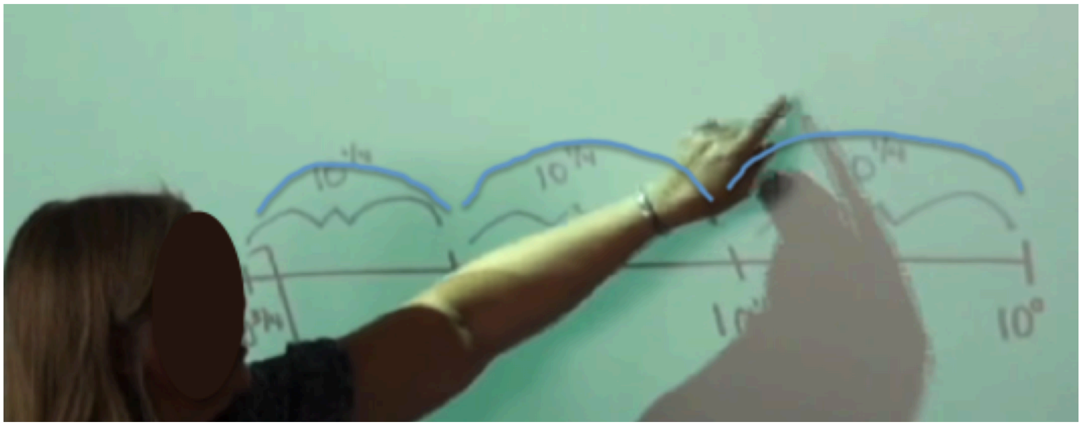


Figure 4.26: The teacher's gestures.

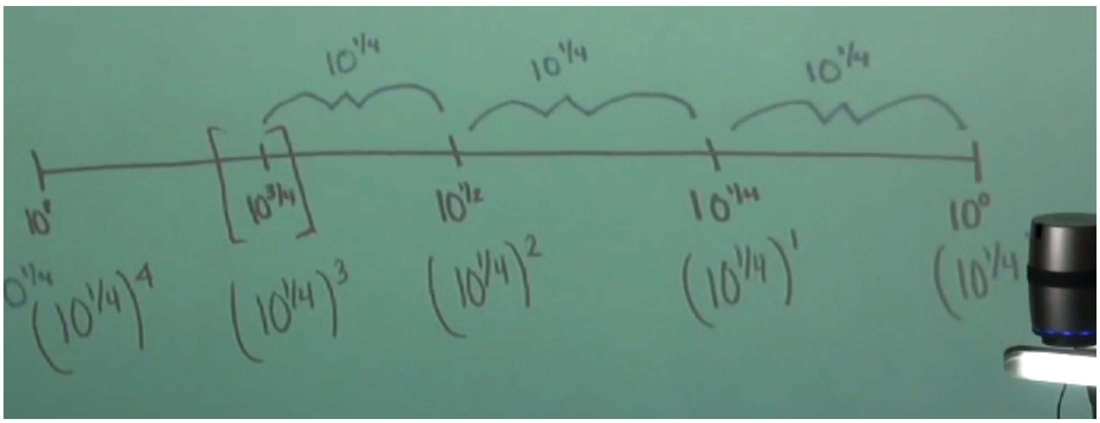


Figure 4.27: Kathy's annotations

Summary of Way of Reasoning. This way of reasoning came up once in whole class, during a discussion of $10^{3/4}$. Students rewrote this as $(10^{1/4})^3$ and found this point on the line by taking the segment from 10^0 to 10^1, subdividing it into four subsection, each representing multiplication by $10^{1/4}$, and then marking the end of the third subsection.

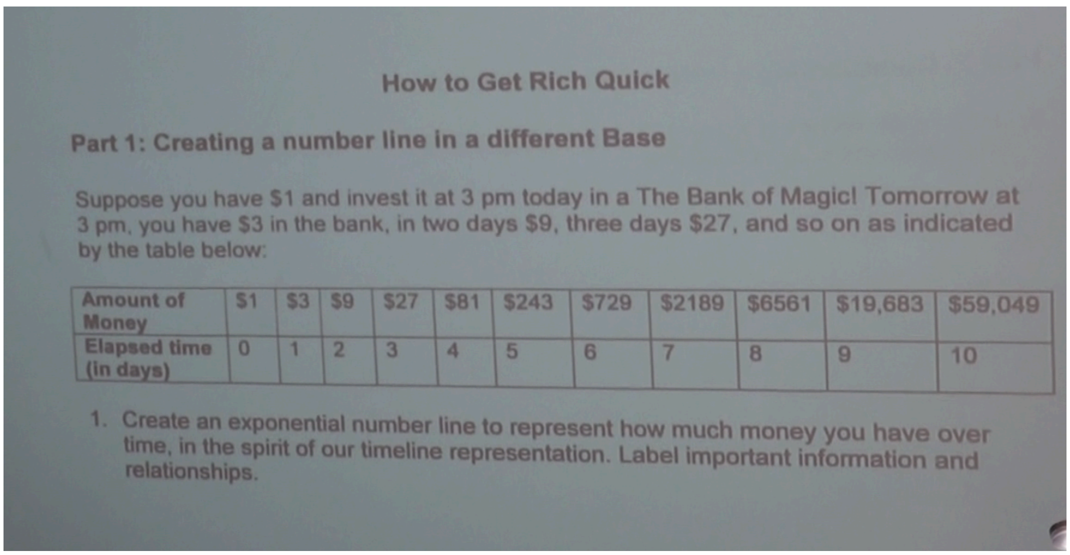
Summary of Math Practice 3. This math practice deals with students placing the numbers of the form $10^{a/b}$ on the number line. They did this in two ways, though only one became normative. In the first way, NWR 3.1, the students rewrote $10^{a/b}$ as $(10^a)^{(1/b)}$, found the segment from 10^0 to 10^a and subdivided it into *b* subsections. In the second way of reasoning, that did not become normative, the students rewrote $10^{a/b}$ as $(10^{1/b})^a$, found the segment from 10^0 to 10^1 and subdivided it into *b* subsections and then placed the point at the end of the a^{th} subsection.

Math Practice 4: Reasoning about Sequences

Math Practice 4 deals with how students defined exponential and additive sequences. This practice consists of two normative ways of reasoning. First, the students defined an exponential sequence as one that had a constant multiple (NWR 4.1). Second, the students defined an additive sequence as one that had a constant sum (NWR 4.2).

NWR 4.1: An Exponential Sequence is one that has a Constant Multiple. The definition of an exponential sequence as one with a constant factor was established as a normative way of reasoning using Criterion 3 of the DCA method. Erin used this definition to claim that 1, 2, 4, 8 was an exponential sequence and Samantha used the definition to claim 1^0, 2^1, 3^2, 4^3 was not an exponential sequence.

Background to Argument 4.1.1. Day 5 began with the teacher introducing the “Get Rich Quick Task.” In this task, students were asked to create a number line that modeled the growth of money in a magic bank. In this bank, one’s money tripled every day. The teacher put the task on the document camera to launch it (Figure 4.28). Question 1 asked students to create an exponential number line to represent the growth of the money in the bank over time. Question 2 asked the students to find a multiplicative sequence in the number line. The students had been engaging with the idea of a multiplicative sequence when they were subdividing segments while developing their exponential number line for the timeline task on Days 3 and 4, but it was formally defined in homework as a sequence that had a constant factor between any two consecutive terms (See Figure 4.29). After they had created a number line and were starting to engage with Question 2 on the “Get Rich Quick Task,” the teacher broadened Question 2. She simply asked for any multiplicative sequence. Erin responded with Argument 4.1.1.


How to Get Rich Quick

Part 1: Creating a number line in a different Base

Suppose you have $1 and invest it at 3 pm today in a The Bank of Magic! Tomorrow at 3 pm, you have $3 in the bank, in two days $9, three days $27, and so on as indicated by the table below:

Amount of Money	$1	$3	$9	$27	$81	$243	$729	$2189	$6561	$19,683	$59,049
Elapsed time (in days)	0	1	2	3	4	5	6	7	8	9	10

1. Create an exponential number line to represent how much money you have over time, in the spirit of our timeline representation. Label important information and relationships.

Figure 4.28: The “Get Rich Quick Task”

Definitions. We have been using the terms **additive** or **linear** in class to describe one type of pattern in a sequence of numbers and the terms **multiplicative** or **exponential** to describe a different type of pattern. To be clear, a sequence like 2, 5, 8, 11, etc. can be described as being additive (or linear) because there is a constant sum or difference between any two consecutive terms (e.g., $2 + 3 = 5$ and $5 + 3 = 8$, and so on). In contrast, a sequence like 1, 2, 4, 8, 16 is multiplicative (or exponential) because there is a constant factor or quotient relating any two consecutive terms (e.g., $1 \times 2 = 2$, $2 \times 2 = 4$, $4 \times 2 = 8$, and so on). Note that because of the constant factor, the sequence can also be expressed using exponents: 2^0, 2^1, 2^2, 2^3, 2^4, etc.

Figure 4.29: Definitions of linear and exponential sequences.

Overview of Argument 4.1.1. In this argument, Erin claimed that 1, 2, 4, 8 was a multiplicative sequence. To justify this claim, she used the warrant that a multiplicative sequence has a common factor. She illustrated the common factors using the data that one times two is two, two times two is four, etc.

Table 4.17: Coding for Argument 4.1.1

Participant	Speech	Action	Code
Teacher	Next, I'm asking you to identify one multiplicative sequence in your number line. Can someone remind us from the definition I gave you in homework twelve? What's a multiplicative sequence? What makes it multiplicative? Or you can give me an example of one.		
Samantha	Multiplying by a constant factor.		Warrant
Teacher	You're multiplying by a constant factor, in a multiplicative sequence. Do people agree with her? One nod head. Two nod heads. Thumbs up? Can someone just write down a multiplicative sequence for us? Not necessarily for this bank of magic, but just any example. And put it on the board. Can you raise your head, hand if you have one? Do you have one Erin? Awesome		Warrant
Erin		Writes: 1, 2, 4, 8. Then underneath 1x2, 2x2, 4x2	Claim
Teacher	So can you explain your thinking?		
Ericka	Well, when we have this order of numbers* you start with one. One times two is two, then we go to that number two** times two is four, four times two is eight, and so on.	*points to 1, 2, 4, 8 **points to 2	Data
Teacher	So a multiplicative sequence has a common factor. This one has what common factor here?		Warrant
Many	Two.		
Teacher	Two.		

Background to Argument 4.1.2. As students continued to discuss what an exponential sequence was, the teacher asked students if the following statement was true, "An exponential sequence is one in which each term is expressed using exponents or scientific notation." After talking about it small groups, the teacher found everyone appeared to disagree with the statement. She then asked someone to put up a counter example. Samantha provided one in Argument 4.1.2.

Overview of Argument 4.1.2. In this argument, Samantha claimed that 1^0, 2^1, 3^2, 4^3 was not an exponential sequence. Danna helped explain why. She argued that it was not an exponential sequence since it did not have a common factor. While the fact that a common factor was not present was treated as data, the warrant connecting the data and claim was the definition of an exponential sequence, namely a sequence that has a common factor.

Table 4.18: Coding for Argument 4.1.2

Participant	Speech	Action	Code
Teacher	Can anybody express a counterexample with exponents? In other words, I want a sequence that can be written with exponents, but it is not an exponential sequence? Get what I'm saying? It can be written with exponents, but it's additive, not multiplicative. Okay, your group seemed like you had something. Samantha, do you want to share?		
Samantha	Do I write it down?		
Teacher	You can put it on the board, or you can put it on the document cam. So she's producing a sequence that's actually additive or linear, arithmetic, not exponential that she's going to use... So, tell us about that.		
Samantha	Well it's increasing by one and so are the exponents, but it's not exponential, it's not an exponential sequence.	Writes on the board 1^0, 2^1, 3^2, 4^3	Claim
Teacher	It's not exponential. ... Do other people agree that it's not exponential? Can I get someone's reaction? Danna? What do you think?		
Danna	Well, I mean you go from one to the second one, or ... one to two, so you multiply it by two, but the to the next one it's to nine, so you're not multiplying by the same number.		Data
Teacher	You're not multiplying by the same number. What do you say Kathy?		Data
Kathy	... Can't we say it is exponential because things are growing exponentially? If I had a chart and I changed those into whole numbers to me like things are growing exponentially, just not at a constant rate, constant factor.		
Danna	But that's how we're classifying exponentially, is it has to have a constant factor. So while they have exponents, they're not an exponential sequence. And that's what we're saying.		Warrant

Summary for NWR 4.1. Students reasoned about exponential sequences using the definition that an exponential sequence is one in which the terms differ by a constant factor. This definition was used to reason about several sequences. Students used it as a warrant to justify the claim that 1, 2, 4, 8 was an exponential sequence and that 1^0, 2^1, 3^2, 4^3 was not an exponential sequence. These instances of the use of this definition as a warrant satisfy Criterion 3 of the DCA method.

NWR 4.2: An Additive Sequence is one that has a Constant Sum. The definition of an additive sequence as one with a constant sum was introduced as a warrant. However, as the class progressed, these arguments became more truncated, with the warrant becoming implicit. Thus, Criterion 1 of the DCA method was satisfied.

Background to Argument 4.2.1. On day 5, right after the students had defined a multiplicative sequence, the teacher asked the students what an additive sequence was. This had been defined on the homework (See Figure 4.24), and Tanya reminded the class of the definition. The teacher then asked for an example, which Tanya provided.

Overview of Argument 4.2.1. Tanya gave the example of 1, 2, 3, 4. Since this was in response to the teacher's request for an additive sequence, Tanya was claiming that 1, 2, 3, 4 is an additive sequence. The data supporting this claim is that constant sum is one and the warrant connecting the data to the claim is the definition of an additive sequence, one that has a constant sum.

Table 4.19: Coding for Argument 4.2.1

Participant	Speech	Action	Code
Teacher	....Can someone remind us what an additive sequence is? If we could get a couple hands up, can someone tell me what an additive sequence is. I'm going to wait until a few people who haven't responded yet at all put their hands up. I want to try to get good participation today. Can I hear from someone I haven't heard from yet today? Tanya, what's an additive sequence?		
Tanya	When you add the same number.		Warrant
Teacher	When you add the same number as you go. Can you give us an example? Can you come write it?		
Tanya		writes 1, 2, 3, 4; and then under 1+1, 2+1, 3+1	Claim
Teacher	So as she writing this I want you to think of, she's saying there's a constant sum, what is the constant sum?		
Many	One.		Data
Teacher	One. Adding one to each success term.		

Background Argument 4.2.2. After students reacted to the statement "An exponential sequence is one in which each term is expressed using exponents or scientific notation," they returned to the "Get Rich Quick Task." Question 3 asked students to find an example of an additive sequence on the number line had been working on. The teacher asked Jade to find one. She came to board pointed to the writing was still on the board from their discussion of exponential sequences, "3^0 x 3, 3^1 x 3, 3^2 x 3, 3^3 x 3" (see Figure 4.30), and explained that the powers of 3 form an additive sequence.

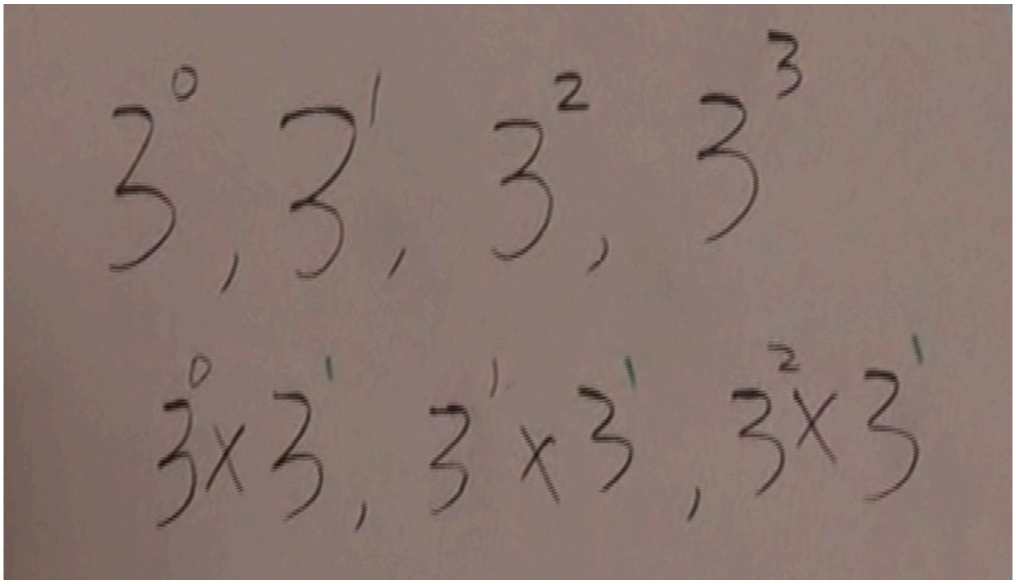


Figure 4.30: The exponents form an additive sequence.

Overview of Argument 4.2.2. In this argument Jade claimed that 1, 2, 3 form an additive sequence. Lacey explains why this is an additive sequence, by pointing out that there is a constant sum of one. The existence of a constant sum serves as data in this argument. The warrant would be the definition of an arithmetic sequence as one that has a constant sum, however this is never articulated explicitly.

Table 4.20: Coding for Argument 4.2.2

Speaker	Speech	Action	Code
Teacher	… An additive sequence, number three. So, can somebody write, raise your hand if you think you can identify an additive sequence on Kathy's number line? … Jade. Can you write yours up on the board?		
Jade	Can I use this?	Points to 3^0 x 3, 3^1 x 3, 3^2 x 3, 3^3x 3 on the board. Figure 4.25.	
Teacher	Yep, you want, what do you want to point out?		
Jade	So, the additive sequence, the constant sum, would be the power of one on the three*. So we add the power, zero plus one would give me two**, wait, wait, sorry***, I'm like, I'm nervous, sorry...	Puts an exponent of one on all the x3s *points to the x3 **writes 0+1 = 2 ***Changes it to 0+1 = 1	
Teacher	Well we're glad you got up there. You're brave.		
Jade	And then one plus, I'm talking about the powers here, one plus one equals two, then two plus one equals three.	Writes 1 + 1 =2; 2+1 = 3	Data
Teacher	Jade, Can you circle each member in the sequence that you're seeing.		
	[Jade circled the exponents in the x3^1, but then Chris helped her identify the exponents 1, 2, 3, etc.]		
Jade	Oh, sorry.	Circles the exponents of 0, 1, 2, 3	Claim
Teacher	What do you think about Chris' suggestion, Jade?		
Jade	I think that's what I meant. Yeah. This would be the one over here.		
Teacher	So do people agree that that forms an arithmetic sequence?		Claim
Students	Yes		
Teacher	Kay, what's the constant sum? [inaudible] Alright, how would you describe that one? How would you describe that sequence? Does anyone have a different way of saying it? Thank you Jade. Lacey?		
Lacey	I was just looking at the exponents and ignoring the base and using that as an additive sequence[1], where you go from zero to one, and then one to two, you're adding one each time[2].		[1]Claim [2]Data
Teacher	Okay, so it's important to say the exponents themselves are forming the additive sequence. We are not saying that this [circles the whole sequence with her finger] forms an additive sequence. Just the exponents. This was actually really important as you probably saw in that youtube video. This is a huge insight that Napier had historically, fifteen, sixteen hundreds. That you had in a number line like this both arithmetic and geometric sequences or we can say it exponential and linear or additive and multiplicative, however, whichever one you want to use. And um coordinating those was very important thing, which we're going to work on a little bit more in [??].		

Summary of NWR 2.2. In this normative way of reasoning, students established the definition of an additive sequence as one that had a constant sum. When determining whether or not a sequence was arithmetic, students looked for a constant sum. In early arguments, the constant sum was explicitly mentioned as the definition of an arithmetic sequence. However, reference to the definition quickly dropped off in subsequent arguments satisfying Criterion 1 of the DCA method.

Summary of Math Practice 4. Math Practice 4 dealt with two ways of reasoning about sequences. In NWR 4.1 students used the definition of an exponential sequence as one in which there is a constant multiplicative factor to determine whether or not particular sequences were exponential. Similarly, in NWR 4.2 students used the definition of an additive sequence as one in which there is a constant sum to argue particular sequences were additive.

Math Practice 5: Interpreting Logarithms

This math practice deals with the three ways of interpreting the word *logarithm* that were established in the class community. The word logarithms was first introduced in the timeline context. When deciding how to place times that were not written as a power of ten, students suggested using logarithms. This lead to the first accepted meaning in the class, which was a logarithm is an exponent (NWR 5.1). The students then used this definition to interpret logarithmic statements in the magic bank context. Two interpretations became accepted in the class. First, a logarithm could be interpreted as the day on which a person had accumulated a given amount of money in the bank (NWR 5.2). The second interpretation was that the logarithm was the number of days it took to increase one's fortune by a particular factor (NWR 5.3).

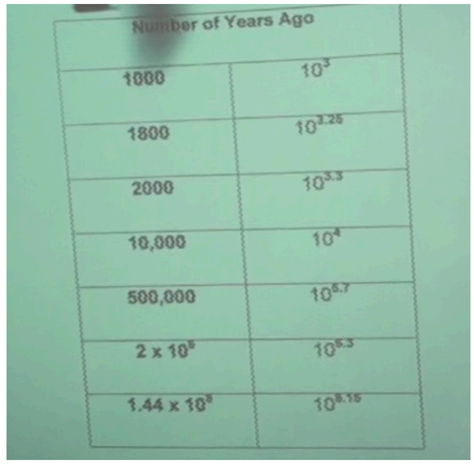
NWR 5.1: Logarithms are Exponents. It is important to mention that it is possible that different students interpreted the statement "a logarithm is an exponent" differently as at times students talked about the number of times one needs to multiply a number by itself (e.g., Argument 5.1.1), while at other times students manipulated symbols to find the appropriate exponent (e.g., Argument 5.1.3). In the class community, these two interpretations were treated as consistent with each other and with the definition of a logarithm as an exponent. In other words, while the meaning of an exponent was never explicitly talked about in the class, it seemed to be taken-as-shared from the beginning that the idea of repeated multiplication is consistent with the idea of exponentiation. Therefore, I treat both types of warrants as support for the idea of "A logarithm is an exponent" as functioning as if shared in the class.

Overview of the Development of NWR 5.1. The definition of a logarithm as an exponent began to function as if shared in the class community when the warrant dropped off, fulfilling Criterion 1 of the DCA method. In the beginning, students justified their calculations of logarithms by appealing to the definition one as an exponent. Eventually, the explicit mention of the definition was no longer necessary. Rather, students implicitly used the definition to reason about logarithms, but did not mention the definition in their arguments.

Background to Argument 5.1.1. This argument was advanced at the end of Day 3. Students had been working on a worksheet that asked them to place the bow and arrow (see Argument 2.1.2) and the Ordovician period (see Argument 2.1.3). The last question on this worksheet asked them to consider how they would place the cave paintings (See Figure 4.31). The students suggested it would be helpful to write 20,000, the number of years ago cave paintings appeared, as a power of ten. The teacher then put up a table of several numbers written in both standard form and scientific notation and as a power of 10 (see Figure 4.32) and asked how they thought she got those numbers. Farah answered, "logarithms." The teacher asked the students to talk in their small groups about what they remembered about a logarithm. As they shared what their groups talked about, Farah suggested that a log gives you an exponent. The teacher then recorded this idea on the board by writing "A logarithm is an exponent". After writing this on the board, the teacher encouraged the students to use the idea to reason about the expression $\log_3 81$.

3. To place when the cave paintings in France were created 20,000 years ago, what would be helpful to know?

Figure 4.31: Cave Paintings Task.



Number of Years Ago	
1000	10^3
1800	$10^{3.25}$
2000	$10^{3.3}$
10,000	10^4
500,000	$10^{5.7}$
2×10^6	$10^{6.3}$
1.44×10^8	$10^{8.15}$

Figure 4.32: Table of times.

Overview of Argument 5.1.1. In this argument Julia claimed that $\log_3 81$ was equal to four. She justified her argument by pointing out that three multiplied by itself four times is eighty-one, which functioned as data in her argument. She seemed to think of this an explanation in and of itself, however, the teacher made explicit the warrant by pointing out the exponent is the logarithm.

Table 4.21: Coding for Argument 5.1.1

Speaker	Speech	Action	Code
Teacher	… The logarithms are the exponents. 'K, now just keep that in mind, a logarithm is an exponent. I'm going to give a problem, don't enter anything into a calculator, just think about it. … Okay. These are just thinking problems with the idea that a logarithm is an exponent. … What do you think the log of eight-one in base three is? What is the log of eighty-one in base three*? Just think it's an exponent. … Okay, Julia, what do you think the logarithm of eighty-one is?	*writes $\log_3 81$ on the board.	
Julia	Four.		Claim
Teacher	Four. Why?		
Julia	Because you multiply three times itself four times to get eighty-one.		Data
Teacher	Because you multiply three times itself four times that is the logarithm*. That exponent is the logarithm of eighty-one in base three.	*Writes $3 \bullet 3 \bullet 3 \bullet 3$ and 3^4 on the board, circles the 4 and points to it. Figure 4.33.	Warrant

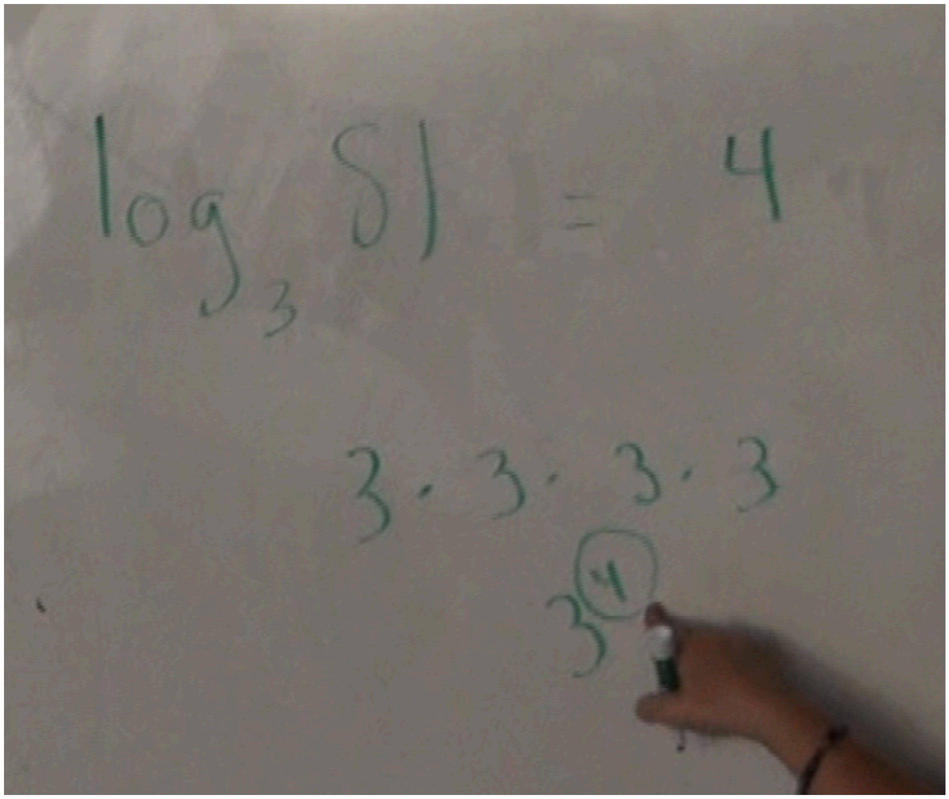


Figure 4.33: "That exponent is the logarithm."

Background Argument 5.1.2. The teacher continued to pose logarithm problems where students needed to use the definition to reason about them. However, as they did more and more problems that increased in complexity, the explicit mention of the definition dropped off. For example, one of the questions the teacher asked was which of $\log_3 30$ and $\log_5 30$ was bigger. After some talk in their groups Danna gave an argument for why $\log_3 30$ was larger.

Overview of Argument 5.1.2. Danna claimed that $\log_3 30$ was bigger. She then pointed out that 3^3 was 27 and 5^2 was 25, but did not explicitly say why that data supported her claim. Rachel then gave a more general argument saying that since the base was smaller you would need to multiply it by itself more times. However, she also did not offer any specific warrant.

Table 4.22: Coding for Argument 5.1.2

Participant	Speech	Action	Code
Teacher	Are we ready? Let's hear some reasoning. I'm going to call on a couple people? Danna?		
Danna	Well you have to multiply three by itself more times to get to thirty than you do with five. And the way I thought about it is...		
Teacher	Which one's bigger? This one or this one? This one?		
Danna	Yeah. Base three.		Claim
Teacher	Why?	Circles $\log_3 30$	
	…	Danna comes to the board	
Danna	Just for getting close to it without going over, three to the three is twenty-seven* so you know it's bigger than three to the third. Or so you know it's bigger than three** and this one is, you know five squared gives you twenty-five***, so it's bigger than two****, but you know it's not bigger than three.	*Writes $3^3 = 27$ on the board. **Writes 3 on the board. ***Writes $5^2 = 25$. ****Writes >2. Figure 4.34	Data
Teacher	That's one way to think about it. Is there a different way to think about it? Without even calculating anything? In fact...		
Rachel	The base is smaller, I mean it's the same process, just without writing anything down, but the base is smaller with the three so you need a bigger exponent to get to the thirty than you would with five with a larger base.		
Teacher	So you don't really have to calculate anything. You need a bigger number because the base is smaller.		

Figure 4.34: Danna's work to argue $\log_3 30$ is bigger.

Another Argument without the Warrant. There were other arguments in which students used the definition, but did not explicate it. On Day 4, after students' had engaged with the fractional exponents tasks (see Math Practice 3), the teacher returned to logarithm problems. One of the more difficult problems they worked on was calculating $\log_4 8$ using the definition of a logarithm. After talking in their small groups for a few minutes, Samantha suggested the answer was 3/2. She said, "I did four to the x equals eighty." She then preformed algebraic manipulations to arrive at x = 3/2 (see Figure 4.35). Notice that as in Argument 5.1.2, there is no explanation that the logarithm is the exponent. Rather, this is simply assumed as Samantha writes $4^x = 8$.

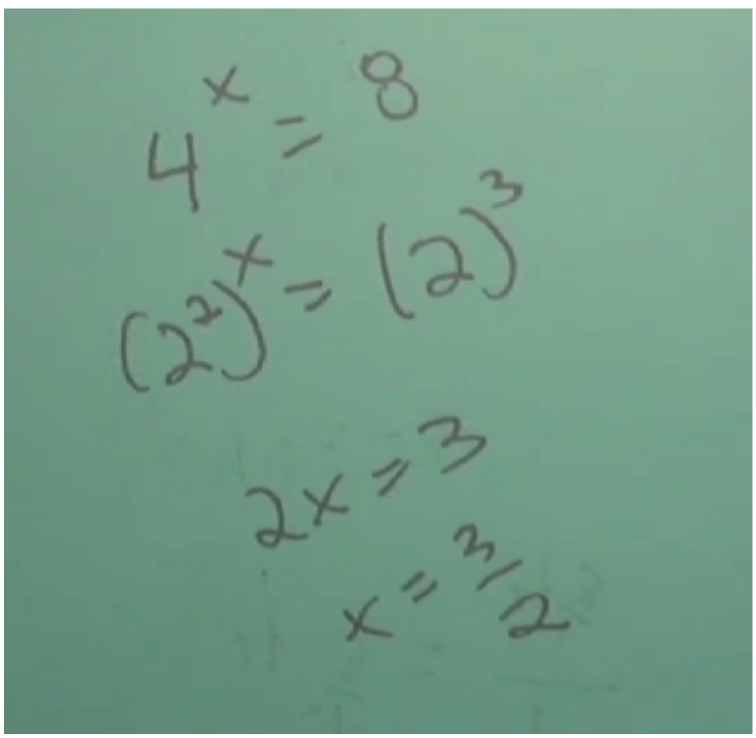


Figure 4.35: Samantha's work.

Summary of NWR 5.1. The definition of logarithm as an exponent was first advanced on Day 3. Students then used this definition to calculate and reason about various logarithms on both Days 3 and 4. Initially when students reasoned about these logarithm problems, they justified their arguments by talking about the interpretation of logarithms as exponents. However, as the problems became more complex they no longer appealed to the definition in their justifications. Since these warrants dropped off as time passed, Criterion 1 of the DCA method is satisfied and there is sufficient evidence that the way of reasoning was normative in the community.

NWR 5.2: The "On What Day" Interpretation of Logarithms. During the "Get Rich Quick" task students leveraged their definition of a logarithm to develop two meanings for a logarithm in the context of the problem. In the first meaning, the argument of the logarithm represented an amount of money while the logarithm itself represented the day you had that amount of money. For example, $\log_3 (9) = 2$ would be the day you had $9, which would be day 2. In the second meaning, the logarithm represented an amount of elapsed time, while the argument represented the factor by which the money increased over that time. Using this interpretation $\log_3 (9) = 2$ would mean it takes 2 days for your money to increased by a factor of 9. The first of these interpretations is NWR 5.2 and the second is NWR 5.3. Both of these interpretations were then used to make sense of the product rule.

Overview for the development of NWR 5.2. In this normative way of reasoning, students interpreted $\log_3 (x)$ as the day on which one has x dollars in the bank. This was established using Criterion 2 of the DCA method. Julia began Day 5 by arguing that this interpretation was valid. In her argument the interpretation functioned as the claim. Then on Day 6, the students used this interpretation to make sense of the product rule log (ab) = log (a) + log (b). In this argument, the interpretation functioned as data. This shift satisfies Criterion 2.

Background for Argument 5.2.1. On Day 5, the students were working on the "Get Rich Quick" task. In this task they were asked to create a number line that represented the growth of their money in a magic bank and find exponential and arithmetic sequences in their line (see Math Practice 4). After finding the arithmetic and exponential sequences, students were supposed to write an expression that gave the day on which one would become a millionaire using this bank (see Figure 4.36). David suggested the expression $\log_3 (10^6)$, because a million is 10^6. The teacher then asked the students to work in their small groups and write two or three more logarithmic statements. After they had time to do this Julia presented her group's work.

> 4. Write an expression using **logarithms** to capture the following question: *In how many days will you be a millionaire?* Explain the meaning of your expression but you do not need to solve.

Figure 4.36: Question 4 in the "Get Rich Quick" Task

Overview for Argument 5.2.1. In this argument Julia used as data the interpretation of a logarithm as an exponent. She wrote $\log_3 81$ and referred to logarithm as "The power that we raise three to to get eighty-one." She then use that fact to support her claim that this logarithm, the exponent, should represent a particular day.

Table 4.23: Coding for Argument 5.2.1

Participant	Speech	Action	Code
Teacher	So Julia's going to write one for us and tell us what it means…		
Julia	So we want to know three to what power gives us eighty-one[1] and that power is going to represent what day that person has eighty-one dollars[2].	Writes $\log_3 81$	[1]Data [2]Claim
Teacher	What do you guys think of that?		
Kathy	Nice.		
Teacher	Nice.		
Student	Can you say that again?		
Teacher	Say it again.		
Julia	The power that we raise three to to get eighty-one represents the day that that person will have eighty-one dollars.		Claim
Teacher	And what should it be? What should the answer be here?		
Many	Four		

Background for Argument 5.2.2. NWR 5.2 became accepted on Day 6, when students began to make sense of the product rule. Day 6 began with a short review after which the class discussed the following problem: "Suppose you look at your bank account and record how much money you have. The next time you look, you have 81 times as much money. How much time passed between the two observation points?" This gave rise to the idea of looking at a logarithm as the number of elapsed days, with the argument of the logarithm being the factor of increase, rather than an amount of money (as in NWR 5.2). With this interpretation, they were well positioned to make sense of the product rule using both interpretations. This rule came up when the teacher asked them what $\log_3 (3 \bullet 27)$ meant in the banking context. After some time discussing in small group, Erin wrote the following series of equations on the board: $\log_3 (3x27) = \log_3 (3) + \log_3 (27) = 1 + 3 = 4$. The teacher then asked the class to make sense of each of logarithm statements.

Argument 5.2.2. In this argument 5.2.2 students claimed that $\log_3(3)$ is the day on which one has $3. It is important to note that here that the claim is not that a logarithm can be interpreted as the day on which you have a particular amount of money as was the case in Argument 5.5.1, but which of the two interpretations should be used for this specific logarithm. Thus, the particular interpretation is used as data.

Table 4.24: Coding for Argument 5.2.2

Participant	Speaker	Action	Code
Teacher	What does each of these things mean? Is this* a dollar amount, a factor, a number of days or something else?	*traces a circle around the second 3 in $\log_3$ **3**.	
	…		
Several	It's a dollar.		
Teacher	So you think this one's a dollar?		
Nathan	It depends on how you look at it		Data
Teacher	It depends on how you look at it.		
Nathan	One of them's probably a dollar.		
Teacher	So there are multiple ways for us to interpret this. Let's start here and see what sense we make of it. So, if this is a dollar amount, what question is this* posing?	Draws a square around $\log_3 3$	Data
Multiple	On what day will you have three dollars?		Claim
Teacher	So what's, what's the answer to that?	Writes "On what day do I have 3 dollars?"	
Several	One		
Teacher	So that one means day one.	Labels the 1 in 1 + 3 as "day 1."	Claim

Summary of NWR 5.2. In this normative way of reasoning, students interpreted $\log_3(x)$ as the day on which one has x dollars in the bank. This was established as students first argued that one could interpret a logarithm in this way and then argued that a particular logarithmic expression should be interpreted in this way. In this way, the interpretation moved from a claim to data, fulfilling Criterion 2 of the DCA method.

NWR 5.3: The Factor Interpretation of Logarithms. NWR 5.3 is another interpretation of logarithmic statements in the banking context—that the $\log_3(x)$ could represent the number of days it takes to increase one's fortune by a factor of *x*. This contrasts to the interpretation in NWR 5.2 where students interpreted *x* as an amount of money and the logarithm as a particular day.

Overview of the Development of NWR 5.3. This normative way of reasoning developed in a similar way to NWR 5.2. First, students argued for this definition on Day 6. Then, when interpreted the product rule they took this interpretation as data and argued which interpretation corresponded to various log statements in the equation $\log_3(3x27) = \log_3(3) + \log_3(27)$. Again, this shift in the interpretation's function, from claim to data fulfills Criterion 2 of the DCA method.

Background to Argument 5.3.1. The foundation for this normative way of reasoning was laid at the end of Day 5 when as students explored the question of how much more a person would have on Day 9 of their investment than they did on Day 6. This led to students thinking about the factor by which money increased over a period of three days. Students found the answer to this, that the money increased by a factor of 27, and also noticed that 27 was 3^3. This led to the noticing of relationships among the exponents, particularly that the exponent of 3 in 3^3 is the difference in the exponents of 3^9 and 3^6, the amounts of money one has on days nine and six. However, at this point the students did not connect this to logarithms.

The next day, Day 6, began with the teacher giving them a summary sheet, which had the number line they had developed the day before. The teacher then gave them following prompt.

> Suppose you look at your bank account and you record how much money you have. The next time you look, you have 81 times as much money. How much time has passed between the two observation points?

The students claimed the answer was four and articulated several different four-day spans of time where the money increased by a factor of eighty-one. They also annotated the number line on the summary sheet to show how these passages of time were represented on the line (see Figure 4.37). The teacher then asked them to explore a particular passage of four days from Day 6 to Day 10. This question led to Argument 5.3.1. In this argument, the teacher asked the students to write a logarithmic statement and interpret it. This led to the interpretation of a logarithm as a number of elapsed days corresponding to an increase by a particular factor.

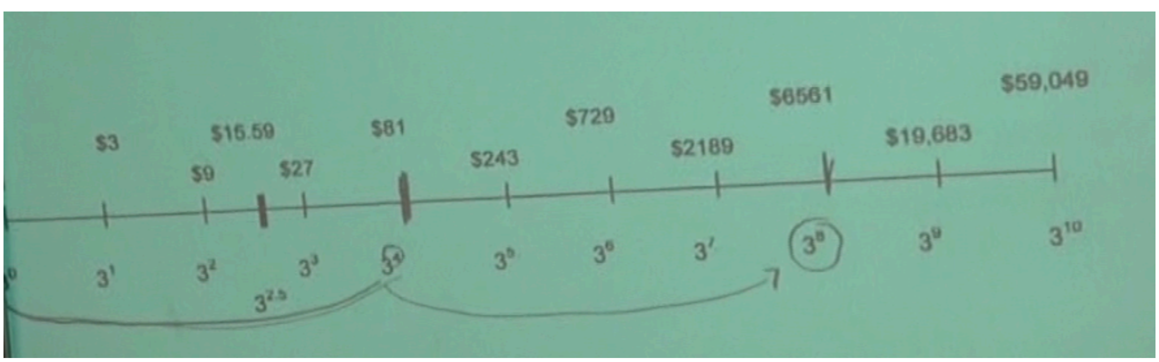


Figure 4.37: Annotations on the summary sheet showing passages of four days.

Overview for Argument 5.3.1. Similar to Argument 5.2.1, the claim of this argument centers on which interpretation should be used. Several students suggest that log the logarithm should be viewed as the number of elapsed days it takes increase one's fortune by a factor of eighty-one. The actual calculation of the logarithm, that $\log_3 81$ is 4, serves as data in this argument.

Table 4.25: Coding for Argument 5.3.1

Participant	Speech	Action	Code
Teacher	…Before we go there, can we just write a statement right in here*? What if I wrote log of eighty-one in base three? What would that be?	Gestures to the segment from 729 to 59,046	
Several	Four		Data
Teacher	And what would that mean here*?	Pointing to the 3^6 x 81 stuff	
Kathy	Number of days that have passed.		Claim
Teacher	Number of days that have passed for what?		
Rachel	Every four days you get eighty-one times as more money as you had.		Claim
Teacher	Every four days you get eighty-one times much as money as you had. Now notice that that's different than here. On Tuesday you said the log of eighty-one in base three meant on what day you'll have eighty-one dollars in the bank. Now let's get down another meaning. I want someone to dictate to me. Could someone rephrase what she said? I'm not going to put the four in there, I'm just going to pose it as a question. Now what would it mean?		
Kathy	Wait, repeat.		
Teacher	Before we were writing log statements that linked these amounts with these days. You had three dollars after one day, nine dollars after two days, right? Log of eighty-one meant you had eighty-one dollars after four days. But, now we're looking at the eighty-one as a factor, not as a total amount of money. So what would this statement mean, log of eighty-one in base three? What question could I write here, just like, a question here? Chris, you've got an idea?		
Chris	Maybe, how many days does it take for you to increase your money by eighty-one times?		Claim
Kathy	That's awesome.		
Teacher	Do agree or disagree with Chris? That it's the meaning of this log statement with this eighty-one? Kathy did she said, "That's awesome."	Writes "How many days does it take for you to increase your money by eighty-one times?"	
	…		
Teacher	What are you seeing is the difference between these two meanings? They're both legitimate. What's the difference you're seeing in these meanings? Rachel?		
Rachel	I feel like the difference is that one, what day will have eighty-one dollars, is assuming you're starting at day one. And the one we just did it doesn't matter what day you start. It's just talking about eighty-one times as more, as much.		

Background to Argument 5.3.2. This argument occurred during the same exchange as in which Argument 5.2.2 occurred, when students were interpreting the product rule. This exchange occurred directly after the students talked about factor interpretation of logarithms. The students were explaining which of the two interpretations, the day on which one has a certain amount of money or the number of days it takes to increase by a certain factor, were appropriate for each of the logarithms in the following equation $\log_3 (3 \times 27) = \log_3 (3) + \log_3 (27)$. Argument 5.2.2 established that $\log_3 (3)$ gave the day on which one has $3 in the bank. In the following argument, the students decided $\log_3 (27)$ gives the number of days it takes to increase by a factor of 27.

Overview of Argument 5.3.2. In this argument, the students claim that $\log_3 (27)$ gives the number of days it takes to increase by a factor of 27. Again, in this discussion the students are not trying to establish the validity of this interpretation in general, but rather decide if it makes sense to interpret a particular expression in that way. In this sense, the fact that there are multiple interpretations for logarithmic expressions, one of which is the number of days it takes to increase one's fortune by a particular factor, is treated as data.

Table 4.26: Coding for Argument 5.3.2

Participant	Speaker	Action	Code
Teacher	What does each of these things mean? …		
	…		
Nathan	It depends on how you look at it		Data
Teacher	It depends on how you look at it.		
Nathan	One of them's probably a dollar.		
Teacher	So there are multiple ways for us to interpret this. Let's start here and see what sense we make of it.	Draws a square around $\log_3 3$	Data
	…		
Teacher	… So this is saying the log of three dollars* is one**. That means I have three dollars* in the bank after one day**. But now what's the next one mean? Is this [the 27] a dollar, a factor, number of days, or something else?	*points to $3 **points to 1 in 3^1.	
Chris	It could be a dollar.		Data
Teacher	Farah, what do you think?		
Farah	I think it's a factor.		Claim
Teacher	Why?		
Farah	… I think it's a factor because if you start with money and you say on what day do we have three, and then you multiply by twenty-seven in the original problem, you're saying, well, on what day do we have twenty-seven more than we do on day one.		
	…		
Teacher	Kay. What do other people think? …We need a meaning for this*. So does anyone … have an idea for what the log of twenty-seven means then? Danna?	*Draws an arrow that points to $\log_3$ (27).	
	…		
Danna	… Here you have a day*, which is day one, and then day three**, or three days more, so this is really how many days later.	*Points to second three in $\log_3 3$ **Points to 27 in $\log_3 27$	
Kathy	Three days more.		
Teacher	So she just said something important I think, is this, I heard two things, is this day three or three days more*?	*Draws an arrow to $\log_3 27$, writes both day 3 and 3 days more	
Several	Three days more.		Claim

In this argument the students used the two interpretations of logarithms already discussed in class to interpret and make meaning of the multiplication rule. Immediately after this, they did the same thing again, but with the statement $\log_3 (27 \times 3) = \log_3 (27) + \log_3 (3)$.

Summary of NWR 5.3. This is another interpretation of a logarithm in the banking context. In this interpretation $\log_3(x)$ yields the number of days it takes to increase one's fortune by a factor of x. This contrasts with the interpretation associated with NWR 5.1, in which $\log_3(x)$ would mean the day on which one has x number of dollars in the bank. This was established in the same way as NWR 5.1 using Criterion 2 of the DCA method. Students first argued whether or not the interpretation was valid and then argued if it should be used to interpret particular logarithmic expressions.

Summary of Math Practice 5. This math practice deals with ways of interpreting the word logarithm that were accepted in the class community and consisted of three normative ways of reasoning. The first way (NWR 5.1) was reasoning about logarithms as exponents. The last two NWRs were ways of interpreting logarithms in a specific context, the banking context. Students were able to interpret logarithms as both a particular day (NWR 5.2) and as an elapsed time (NWR 5.3).

Math Practice 0: Fluently Translating Among Various Notations

Two other normative ways of reasoning were also established in the class. However, these two NWRs were not focused on developing exponential reasoning. Rather, these were about notations. This review was necessary for students to be able to engage in the tasks. Since these ideas were not the focus of the unit, I only briefly present evidence for their establishment.

NWR 0.2 Translating Between Scientific and Standard Notation. On Day 1, when the timeline task was introduced, the teacher asked them what they noticed about the time periods in the task. Students pointed out that some of the times were written in scientific notation. This led to a discussion about how to convert scientific notation to standard notation. In this discussion several students used the same method for translating between scientific and standard notation—moving the decimal the same number of spaces as the exponent. One of these will be presented in Argument 0.2.1. Eventually explicit mention of the method was no longer needed and students fluently translated between the two notations. This dropping off of the warrant fulfills Criterion 1 of the DCA method.

Argument 0.2.1. In this first example, Kaitlyn claimed that 2×10^6, the date for the Pleistocene period, was two million. Her warrant for the equivalence of the two ways to write the number was moving the decimal point six places.

Table 4.27: Coding for Argument 0.2.1

Participant	Speech	Actions	Code
Kaitlyn	…For the Pleistocene, you would move the … decimal … six times to the right.		Warrant
Teacher	So I'm hearing you say that, move it six times*. And what number do you get?	*Writes 2.000000 and then draws a bump under the first three zeros and then under the last three.	Warrant
Many	Two million.		Claim

Argument 0.2.2. The following argument came later in class when Natalie presented a linear number line. She placed her number line on the document camera, which had 1.5 x 10^{10} at the very left edge and 7.5 x 10^{9} in the middle. Natalie explained that the halfway point on the number line represented an amount of elapsed time that is half the total amount of elapsed time. Samantha called this into question when she asked, "Is that times ten to the ninth," presumably referring to the 9 in 7.5 x 10^{9} and said, "So that isn't technically half."

Danna then claimed that 7.5 x 10^{9} was, in fact, half of 1.5 x 10^{10}. In this argument, she used as data the fact that 1.5 x 10^{10} was fifteen billion and 7.5 x 10^{9} was seven point five billion.

Table 4.28: Coding for Argument 0.2.2

Participant	Speech	Actions	Code
Samantha	I have a question.		
Teacher	What's your question?		
Samantha	Is that times ten to the ninth?		
Natalie	Yeah.		
Samantha	So that isn't technically half.		Claim
Natalie	Oh.		
Danna	Because it's fifteen billion to seven and a half billion[1]. So half of fifteen is seven point five billion[2]. So since you're going from tens to just billions, you take off one. So that's why it goes from ten to the tenth to ten to the ninth.		[1]Data [2]Warrant
Teacher	Does that make sense?		
Samantha	Yeah		

NWR 0.3 Fractional Powers as Roots. This next normative way of reasoning deals with how students translated between numbers written as a number raised to a fractional power and as the n^{th} root of a number (e.g. $10^{1/2}$ and $\sqrt{10}$). In the beginning of the unit, students needed to give justification for the translation between these two notations. However, later these translations were used as data to support more complex arguments. This satisfies Criterion 2 of the DCA method.

Argument 0.3.1. The teacher had asked the students if they could express $10^{3/4}$ as an n^{th} root and Santiago claimed that $10^{3/4}$ is the same as $\sqrt[4]{10^3}$. He then explained that the denominator of the faction determines the type of root (in this case the fourth root) and the numerator determines the exponent. This served as the warrant for his claim.

Table 4.29: Coding for Argument 0.3.1

Participant	Speech	Action	Code
Santiago		Writes it $\sqrt[4]{10^3}$.	Claim
Students	Yeah, yeah, yeah. Right. Yes. Yes.		
Teacher	I hear some resounding agreement. How many agree with Santiago?* Anyone disagree? Let's just take a couple explanations that, since we have some agreement, why is that Santiago, how did you figure that out?	*Many students raise their hands.	
Santiago	Well, I know when an exponent is a fraction if we were to write it as an *nth* root. And we have the denominator of the fraction of the exponent to be inside here* and then we take the root of that and then since we're taking that three times, took it, that whole thing to the power of three.	*Points to the crook of the square root sign.	Warrant
Teacher	Kay. Someone else put it in their own words. Thank you. Can I get one more revoicing or explanation? Rachel.		
Rachel	So we're taking … so ten to the one fourth can be represented as the fourth root of ten. …Then just that whole quantity cubed.		

Argument 0.3.2. This argument occurred when the students were reasoning about $\log_4 8$. Samantha had already given her argument (see Argument 3.1.3) that $\log_4 8 = 3/2$. Danna then offered another way to think about it. Keeping the same claim as Samantha, that $\log_4 8 = 3/2$, she argued that since $\sqrt{4} = 4^{1/2}$ is two, and two cubed equal eight, the logarithm must be 3/2. In this compact argument, she treated as data $\sqrt{4} = 4^{1/2}$. Since she used this translation as data, this fulfills Criterion 1 of the DCA method.

Table 4.30: Coding for Argument 0.3.2

Participant	Speech	Actions	Code
Danna	It's kinda the same		Claim
Teacher	Try it.		
Danna	I thought about it differently.		
Teacher	There's space up there.		
Danna	… I just knew four didn't go into eight, but I knew that the square root of four did, which I knew is the same as four to the one half and then since that's two, I knew two cubed equals eight, so that would be four to the three halves.	Writes $\sqrt{4} = 4^{1/2}$ $2^3 \Rightarrow 4^{3/2}$ Figure 4.33.	Data

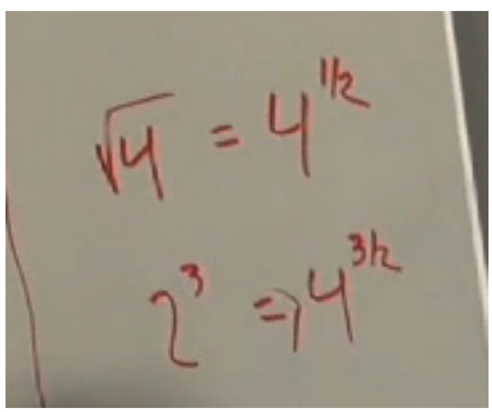


Figure 4.38: Danna's symbolic manipulation

Discussion

The emergence of these five mathematical practices show the development of the exponential and logarithmic relationships as it occurred in the public space in this classroom. Students first built up a fully exponential number line by noticing multiplicative patterns at the macro level (MP1) and then extending those relationships to subdivide segments (MP2). Students then used this number line to make sense of fractional exponents (MP3), arithmetic and exponential sequences (MP4), and logarithms and the product rule for logarithms (MP5).

These results are significant in several ways. First, the research on students' thinking about logarithmic and exponential relationships suggests that students are prone to making calculational mistakes (Barnes, 2006; Hoon, Singh, & Ayop, 2010; Liang & Wood, 2005; Nogueira de Lima & Tall, 2006). This suggests students typically have a procedural understanding these relationships. Scholars have given pedagogical suggestions for developing more conceptual understanding (e.g. Katz, 1986; Van Maanen, 1997; Webb, Kooij, & Geist, 2011; Weber, 2002), but these suggestions are hypothetical in nature as they had not been tested with students. This study, on the other hand, gives an image of how the ideas might productively unfold in a classroom environment.

This includes providing an image of how a conceptually oriented tool for reasoning, an exponential number line, can be developed out of reasoning about how to create a representation of the history of the earth. Giving students such a tool such may help them develop powerful images and ways of reasoning that go beyond a procedural understanding. This is consistent with the work of other scholars using the emergent perspective to study collective development in which a model was developed by reasoning about a real world concept and then leveraged to reason about more sophisticated mathematics (e.g. Bowers, Cobb, & McClain, 1999; Stephan & Akyuz, 2012).

This is particularly significant because the trajectory presents a way for students to transition from linear ways of reasoning to exponential ways of reasoning, which has been shown to be difficult for students (Alagic & Palenz, 2006; Berezovski, 2004; De Bock, van Dooren, Janssens, & Verschaffel, 2002). This was accomplished as students developed the exponential number line. They first subdivided segments linearly, but this was problematized through the Renaissance task (see the background to Argument 2.1.1). Students were then able to develop ways to exponentially subdivide segments (see Math Practice 2). Because of the importance of this transition, I investigate students' personal ways of reasoning about subdivision, as determined by a clinical interview, in the next Chapter. In that chapter I will also explore in relationship between the emergent practice and students' individual ways of reasoning to which will address Research Question 1.

Chapter 5: Individuals' Ways of Reasoning

This chapter, together with Chapter 4, addresses Research Question 1:

> How are individuals' ways of reasoning related to the progression of increasingly sophisticated ways of reasoning that function as if shared in the classroom?

This purpose of this question is to examine the nature of the relationship between individuals' ways of reasoning and the ways of reasoning that were established in the classroom. In particular, it examines the students' ways of reasoning after participation in the classroom. This will help researchers understand the nature and extent of individual variation from established practices.

In the previous chapter, I addressed the last part of Research Question 1, namely the identification of increasingly sophisticated ways of reasoning that function as if shared by the collective community. Specifically, I documented five math practices that emerged in the classroom. This was done using the Documenting Collective Activity (Rasmussen & Stephan, 2008) method, in which Toulmin's (1969) scheme was used to analyze how arguments changed over time. If the arguments changed in particular ways they were said to have begun to have functioned as if shared, or equivalently to have become a normative ways of reasoning (NWR). I then grouped related normative ways of reasoning into math practices.

In this chapter, I investigate how individuals' ways of reasoning relate to Math Practice 2: Subdividing the Segments. This practice was the keystone to students making the transition from linear ways of reasoning to fully exponential ways of reasoning. Since this transition can be difficult for students (Alagic & Palenz, 2006; Berezovski, 2004; De Bock, van Dooren, Janssens, & Verschaffel, 2002), I wanted to better understand how students' were personally reasoning about this important mathematical practice.

Previous work that examines the relationship between individuals' ways of reasoning and emergent practices suggests that while students can reason in ways that are qualitatively different from established practices (Cobb, 1999), students eventually reorganize their knowledge to reason in ways that are more productive (Bowers, Cobb, & McClain, 1999; Stephan, Cobb, & Gravemeijer, 2003). In contrast, I argue that the individual participants in this study maintained meaningful differences in their ways of reasoning even after the unit had concluded.

The bulk of this chapter is thus devoted to supporting the claim that there was variation in the nature of individual student reasoning on the post interview question that was related to Math Practice 2. Specifically, their reasoning fell into the following three categories (a) *multiplicative reasoning coordinated with reasoning linearly with the exponents*, (b) *reasoning linearly with the exponents*, and (c) *elements of reasoning linearly*. The descriptions of these categories will be given later in the chapter. In the discussion section of this chapter I use the results of the analysis presented in Chapter 4, specifically the normative ways of reasoning for MP2, and the results section of this chapter to explore the relationship between individual and normative ways of reasoning to answer Research Question 1.

Review of Method

To determine students' individual ways of reasoning about how to subdivide the segment, I analyzed their responses to a task in which an interviewer asked them to label the midpoints of a segment on an exponential number line during an individual clinical interview (Ginsburg, 1997). I analyzed this task because I wanted to investigate students' individual ways of reasoning about the subdivisions on a number line. This task asked students to engage in such reasoning (see Figure 5.1).

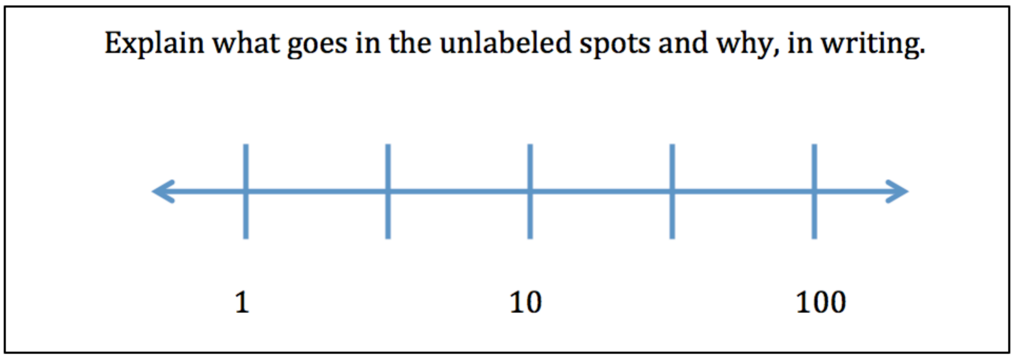


Figure 5.1: The interview task.

When analyzing their responses I used open coding from grounded theory (Strauss & Corbin, 1994, 1998) to develop categories that described the students' ways of reasoning. This involved developing categories for ways of reasoning through iterative cycles of analysis. I started by giving the focus students' ways of reasoning descriptive names. Then, using the constant comparison method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1990, 1994), I grouped ways of reasoning into categories, adjusting the names of categories as needed. This comparison was done on the basis of the features of the students' responses (e.g. what mathematical relationships were they attending to and what justification did they give for their label). As I compared the features of different arguments, their difference came into greater prominence. As categories became established, I revisited the data from previously analyzed interviews and made adjustments as necessary to either the description of the category or the categorization of the way of reasoning.

Overview of Claims

In this section, I provide evidence to support the claim that students reasoned in one of three ways in the interview task. The three of ways of reasoning are (a) *multiplicative reasoning coordinated with reasoning linearly with the exponents*, (b) *reasoning linearly with the exponents*, and (c) *elements of reasoning linearly* (see Table 5.1).

Table 5.1: The three categories for individuals' ways of reasoning.

Code	Characterization
Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents	Recognizing the subsections are associated with multiplication by the square root of ten in addition to using the linear pattern in the exponents to determine placements.
Reasoning Linearly with the Exponents	Finding the midpoints by dividing increases in the exponent by two.
Elements of Linear Reasoning	Determining the first midpoint was five by taking half of ten.

The first way of reasoning, *multiplicative reasoning coordinated with reasoning linearly with the exponents*, is characterized by students recognizing the fact that there was a multiplicative relationship between the subsections generated by subdividing a segment. At a minimum, this means students would reference the fact that the square root of ten times the square root of ten is ten and somehow connect that fact to their reasoning about the subsections. The students in this category also reasoned linearly with the exponents, meaning they used the linear pattern in the exponents to determine placements, but this linear reasoning was accompanied by talk of multiplicative patterns. The evidence will demonstrate that three students, Tanya, Kathy, and Rachel, all reasoned in this way. The second way of reasoning, *reasoning linearly with the exponents*, was characterized by students talking about halving the exponent of 10^1 to find that the midpoints should be represent an increase of .5 in the exponent. This differs from the first category in that the linear pattern used in these explanations was not elaborated by multiplicative reasoning. It is important to note as students reasoned in this way they may have said the word "factor." Simply uttering this word did not automatically mean that they were reasoning multiplicatively. At times students would call 10^1 a factor, but still reasoned solely about the exponent—dividing it in a linear way. In order to be coded as *multiplicative reasoning coordinated with reasoning linearly with the exponents*, students needed to go beyond simply calling something a factor and explain that the factor is being multiplied by something. I will give evidence that Farah and Brittany employed *reasoning linearly with the exponents* and did not accompany it with any multiplicative reasoning. Finally, *elements of linear reasoning* means that at some point the student claimed the midpoint was five, presumably because five is half of ten and the midpoint is halfway. Both Santiago and Lacey made this claim. During the course of the interview Santiago changed his answer as he began to *reasoning linearly with the exponents*. Lacey did not change her answer during the interview.

Results

Multiplicative Reasoning Coordinated With Reasoning Linearly With The Exponents

In this section, I will provide evidence that Tanya, Kathy, and Rachel all expressed or recognized a multiplicative pattern of $\sqrt{10}$ times $\sqrt{10}$ within segments where the values increase by a factor of 10. I will do this by first describing their way of reasoning in the interview and then discuss the aspects of their way of reasoning that lead me to code it as *multiplicative reasoning coordinated with reasoning linearly with the exponents*.

Kathy. Kathy began the task by immediately labeling the empty spots as $10^{.5}$ and $10^{1.5}$ without any explanation. She started to write a few words, but then said, "Well first, this is exponential," and wrote, "This is exponential" at the top the top of her page. She then wrote her actual explanation, which follows.

> We must multiply 1 by 10 to get 10 so between the tick mark is 10^1. If we find the midpoint we are finding half of 10^1 or $10^{.5}$ or $\sqrt{10}$. The same process occurs between 10 and 100. We have to multiply 10 by 10 to get 100. So half is $10^{.5}$ but we now add that to 10^1 so we get $10^{1.5}$

After she finished writing, the interviewer asked what she meant when she wrote, "This is exponential." She responded with the following.

> So, there's equal amount of space between one and ten and ten and a hundred [sweeps pen in an arc from the tick mark labeled 1 to the tick mark labeled 10 and then to the tick mark labeled 100 tick, see Figure 5.2]. So it can't be linear because there's nine, like nine whole values here [makes a sweeping motion with the pen back and forth between

> 1 and 10], like nine, ninety here [sweeping motion over the segment from 10 to 100]. So for it be linear, the space between would need to be much larger.

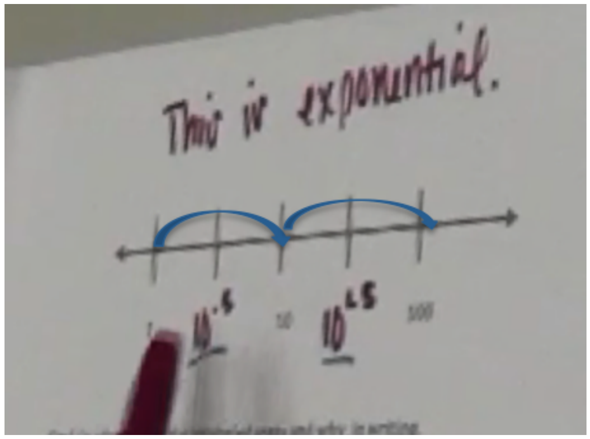


Figure 5.2: Kathy gesturing (represented with arrows) over intervals.

When asked what it meant to be exponential, Kathy said that something needed to be doubling or tripling. When the interviewer asked what was doubling in the line, she said that the numbers were not doubling, but "tens-ing." She explained that this was still exponential. It would not be exponential if ten were added each time.

Kathy then explained that because the line was exponential, you needed to find the midpoint "exponentially." She explained that this meant that one takes half the exponent, not half the ten, as that would be linear. She said, "the value here [traces a circle with her pen in the air over the segment from 1 to 10] represents a multiplication of ten to the one. So to find half of ten to the one, or half of what's in between here [circles over the segment again], I took half of the exponent of one, so it's ten to the point five." She continued, "To get to ten to the one exponentially, we need ten here, times ten point five here, to give us ten to the one." As she said this, she labeled the two subsections as $10^{.5}$ (see Figure 5.3) and explained that they are multiplied to find 10^1. When asked why she was multiplying, she said, "Because everything is multiplication when it's exponential. If it were addition, it would be linear."

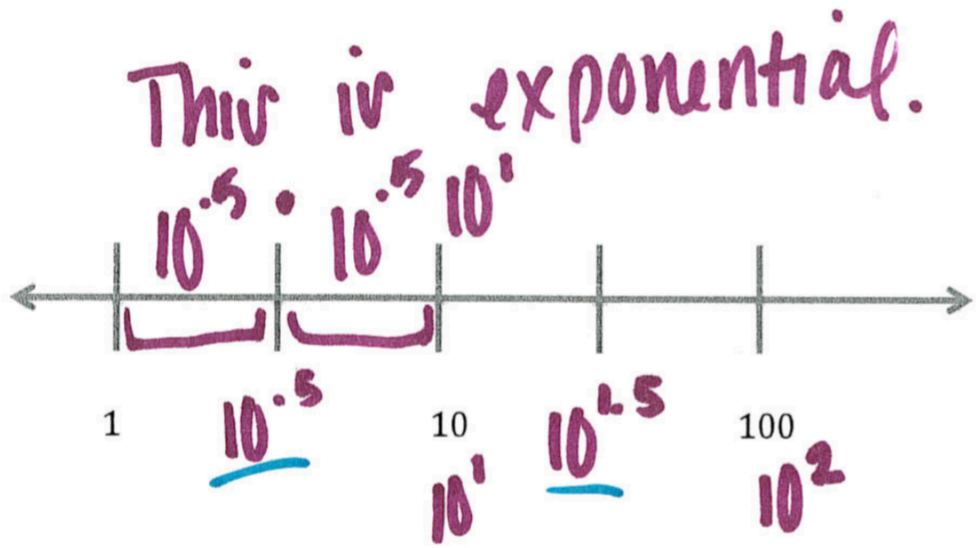


Figure 5.3: Kathy's labels.

Kathy reiterated her process succinctly as she explained her label of $10^{1.5}$ for the next midpoint. She said, "Between these two [points to tick mark labeled 10 and tick mark labeled 100] is ten to the one. … So you take … ten to the one, to find the midpoint, and divide it by a half. But then since you're not starting at zero, you're starting at ten to the one, you have to multiply ten to the point five to ten to the one."

Kathy used multiplication in her explanation in several ways. First, she drew on the multiplicative pattern at the macro level when she explained what it meant to be exponential. As she discussed this she explained that there was an equal amount of space between the tick marks, but the differences in the values were not the same (i.e. one was 9 and the other was 90). Even though this discussion was mainly about the macro level, it appeared to inform how she thought about the relationship between the subsections. Before she felt she could write her explanation of how to subdivide, she first had to establish the multiplicative pattern at the macro level, which hinged on the fact that the segments represented an increase by a factor of ten rather than a difference of ten. While she stopped just short of making an explicit connection between the multiplication at the macro level and how the segments were subdivided, multiplication was still present in her descriptions of how she subdivided. She labeled the first two subsections $10^{.5}$ while she said, "To get to ten to the one exponentially we need ten here times ten point five here, to give us ten to the one." Even though she said "we need ten here," she said this as she labeled the spot $10^{.5}$. As such, I believe she simply dropped the "to the point five" in her utterance. In other words, I believe she tried to communicate that each of the subsections needs to represent multiplication by $10^{.5}$. This is corroborated by her phrase "get to ten to the one exponentially," given her definition of exponential as constant multiplication. Furthermore, when she was asked why she multiplied the subsections, she again talked about the exponential nature of the line. Overall, her reasoning seemed to be that because the line was exponential, since it had a multiplicative pattern at the macro level, the subsections should also represent multiplication.

Even though she appeared to calculate based on linear patterns in the exponents, she seemed to coordinate the linear pattern with multiplication. She said that to find the midpoint exponentially you take half of the exponent, but she also talked about multiplying the subsections. This coordination between additive exponents and multiplicative relationships between the values was explicitly referenced when she said, "You add the exponents. So you have ten to the half times ten to the half gives you ten to the one."

It is important to note that in her coordination of additive and multiplicative relationships, she said something that was mathematically inaccurate. She said, "To find half of ten to the one…I took half the exponent of one," which is technically incorrect. Taking half the exponent does not yield half of ten to the one. She also made this mistake in her written explanation. She wrote, "If we find the midpoint we are finding half of 10^1 or $10^{.5}$ or $\sqrt{10}$," which is inaccurate since half of 10^1 is not $10^{.5}$ or $\sqrt{10}$. While this could indicate that she saw 10^1 only as an exponent of 1 and not as a multiplicative factor, I do not think this is the case. When she talked about finding half of ten to the one, she traced out a circle with her pen over the section of the line from 1 to 10. I interpret this statement to refer to halving the segment (as if 10^1 was a name for the segment), not the value of 10^1. In other words, she seemed to coordinate halving the segment with halving the exponent and with multiplying $10^{.5}$ by $10^{.5}$.

Tanya. Tanya began the interview by claiming the first spot should be labeled $10^{.5}$. She said, "Since this is increasing by a factor of ten to the one, then half of it would be ten to the one half." She then labeled 1 as 10^0, 10 as 10^1, and 100 as 10^2, as well as marking in a brace over the segment from 1 to 10, which she labeled "x10^1" and a brace over the subsection from 1 to the midpoint, which she labeled "x$10^{1/2}$" (see Figure 5.4).

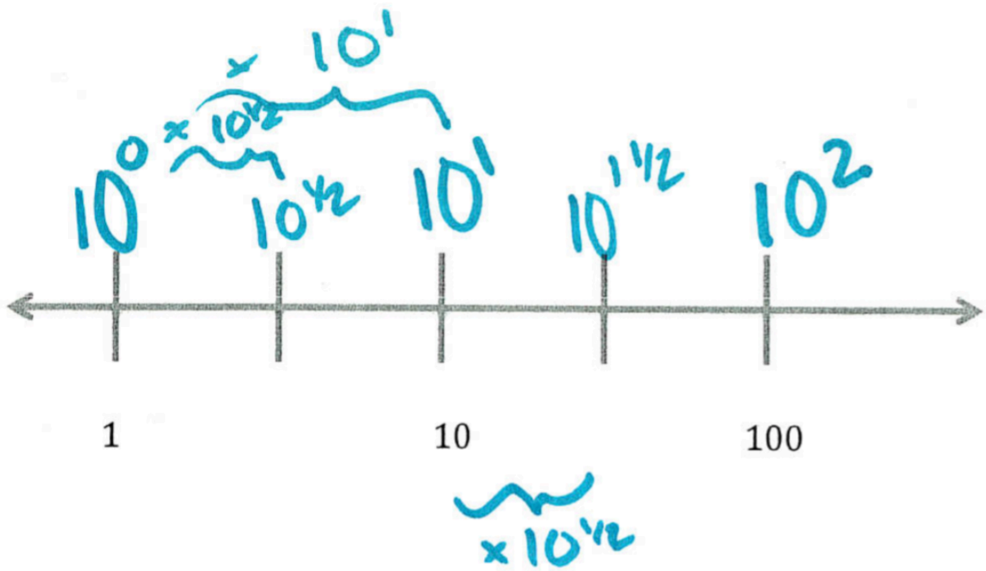


Figure 5.4: Tanya labeled the multiplicative factors.

She then wrote her explanation.

> From 10^0 to 10^1 we increase by a factor of 10^1 ($10^0 \bullet 10^1 = 10^1$). We cut this increment of 10^1 in half, so we half the exponent of 10^1 as well to get $10^{1/2}$. Check by $10^{1/2} \bullet 10^{1/2} = 10^{2/2} = 10^1$. Multiply the previous term by $10^{1/2}$ to obtain the next tick mark, from 10^0 we get the next by $10^0 \bullet 10^{1/2} = 10^{1/2}$, then $10^{1/2} \bullet 10^{1/2} = 10^1$.

When the interviewer asked what she meant by, "We cut this increment of 10^1 in half," she responded with the following.

> Tanya: Since this whole [traces over the segment from 1 to 10], … these increments were ten to the one [points to tick marks labeled 1 and 10 simultaneously] and we only wanted to do halve the distance [points to tick marks labeled 1 and the midpoint of 1 and 10], we don't halve ten, because that just doesn't make sense. So we halve the exponent, so instead of moving by a factor of ten to

the one, we're moving by a factor of ten to the half. So we're halving the exponent.

Interviewer: How do you know to halve the exponent?

Tanya: When we were first trying to figure it out it didn't really make sense to … halve the ten. … We would multiply ten to the zero times ten to the one to get ten to the one and we only want to go half the way and so we wouldn't multiply by half of ten, we wouldn't multiply it by five, so we would halve the exponent.

As with Kathy, Tanya seemed to calculate based on linear patterns, but this was coordinated with multiplication. Multiplication came up several times in Tanya's argument. First, she immediately marked in the multiplicative factors of "x 10^1" and "x $10^{1/2}$" (see Figure 5.1). Importantly, these labels included the multiplication symbol "x," suggesting she did not seem them solely as exponents. Multiplication was also present in her written explanation. She wrote, "Multiply the previous term by $10^{1/2}$ to obtain the next tick mark, from 10^0 we get the next by $10^0 \bullet 10^{1/2} = 10^{1/2}$, then $10^{1/2} \bullet 10^{1/2} = 10^1$." Finally, multiplication was present as she responded to the interviewer's question, "How do you know to halve the exponent?" In response, she explained that her group in class first tried to halve the ten, but that was inconsistent with the macro-level multiplication. She said, "We wouldn't multiply by half of ten, we wouldn't multiply it by five, so we would halve the exponent." Notice that implicit in her comment is the assumption that the relationship should be multiplicative—what needs to be decided is whether the multiplication should be by 5 or by $10^{.5}$. In summary, even though Tanya talked about halving the exponent in her explanation, this was often coordinated with a recognition of the multiplicative nature of the subsections.

Rachel. Rachel started by drawing an arrow from the tick mark labeled 1 to the tick mark labeled 10 and another from the tick marked labeled 10 to the tick mark labeled 100. She labeled these arrows “x 10” and drew two lines, one from 10 to the midpoint of the segment from 10 to 100 and one from the midpoint to 100 (see Figure 5.5). She then drew arrows to the unlabeled spots and wrote calculations, while she verbalized them, to explain how the spots should be labeled (see Figure 5.5). She then explained, “This distance you times by the square root of ten and this distance you times by the square root of 10 [labels two subsections “x $\sqrt{10}$” as shown in Figure 5.5]….You needed to break that up [the segment from 10 to 100] evenly into two of the same, which is ten to the one half times ten to the one half, because the one half and the one half gets you one.”

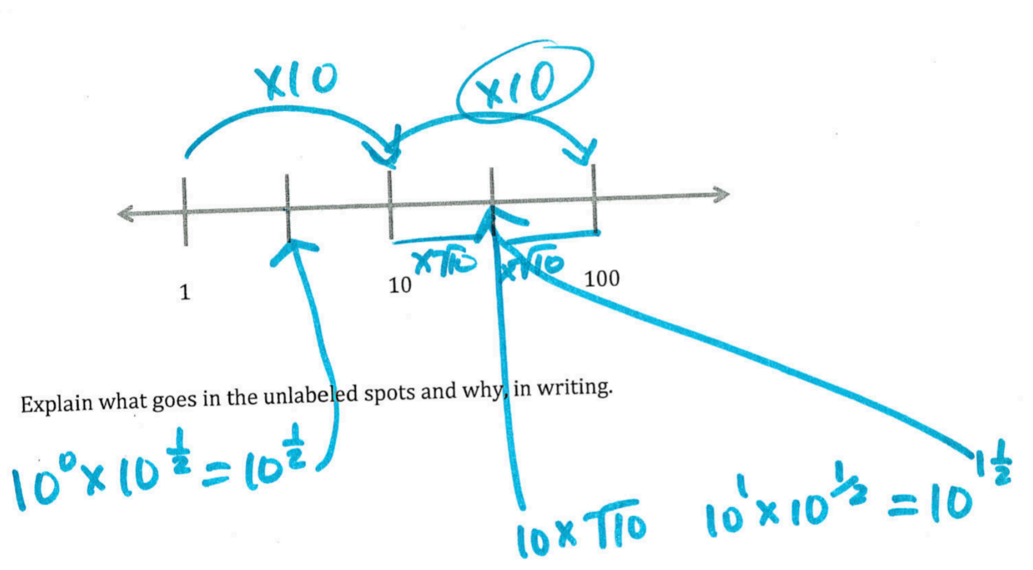


Figure 5.5: Rachel’s labels.

She then wrote her response, which follows.

> From 1 (10^0) to 10 (10^1), we have a multiple of 10^1. To break this in half we need a number that when multiplied twice gives 1: 1/2 x 1/2 = 1. So 1 x $10^{1/2}$ gives us our 1st unknown.

The interviewer pointed out that ½ x ½ does not equal 1, and she said, "That's not what I meant to do." She then changed the words "when multiplied twice" to "multiplied by two" and explained, "Because you need to do it once and then twice" as she pointed to the two subsections. She then explained that if you had three segments you'd divide the exponent into three.

The interviewer asked why the unlabeled spot was not 5.5. She responded in the following way.

> I feel like that would be taking it into linear perspective. So we have to keep everything multiplication, only. Because this is times ten [sweeps her finger across the segment from 1 to 10] that's times ten [sweeps her finger across the segment from 10 and 100]. Like, the next one would be a thousand [sweeps her finger from 100 to the right]; times ten [repeats the last gesture]. The next one would not be two hundred. Because you have to keep a common, to keep it an exponential line.

Like Kathy and Tanya, Rachel had elements of both multiplicative reasoning and reasoning linearly with the exponents present in her justification. She began by highlighting the multiplicative relationship at the macro level by drawing two "x 10" arrows, which seemed to be connected to the way she subdivided the exponents, since she also drew lines that were labeled "x $\sqrt{10}$." This was coordinated with linear reasoning with the exponents, when she said, "You needed to break that up [the segment from 1 to 10] evenly into two of the same, which is ten to the one half times ten to the one half, because the one half and the one half gets you one." She then got a bit confused and went to more linear reasoning in the exponents, but went back to more multiplicative reasoning when the interviewer asked his follow up question.

When the interviewer asked why the unlabeled spot was not 5.5. She responded, “We have to keep everything multiplication, only… you have to keep a common, to keep it an exponential line.” In her elaboration, she focused mostly on the multiplication at the macro level. While it is possible that she saw multiplication being more connected with the macro pattern than with the subdivision of segments, it seems that there was at least some connection to the subdivision of segments since she brought up the macro level pattern in response to a question about subdivision. As with Kathy, the constant multiplicative relationship between values seems to have been a defining characteristic of an exponential line for Rachel.

Reasoning Linearly with the Exponents

Like the first three students, Brittany and Farah drew on the linear pattern in the exponents to reason about the unlabeled tick marks should be labeled. However, unlike Kathy, Tanya, and Rachel, the two students drew exclusively on this pattern and did not talk about multiplicative relationships within the subsections.

Brittany. Brittany began by relabeling the points 1, 10, and 100 as 10^0, 10^1, and 10^2 respectively. She then labeled the first unlabeled spot as $10^{.5}$, said that it was the square root of ten, and then labeled the second marked spot as $10^{1.5}$ (see Figure 5.6). After some talk about what the square root of 10 means, she wrote her explanation.

> Half way between 10^0 and 10^1 is $10^{.5} = \sqrt{10}$ because the factor it takes to get from 10^0 -> 10^1 is 10^1 and half of 1 is ½.

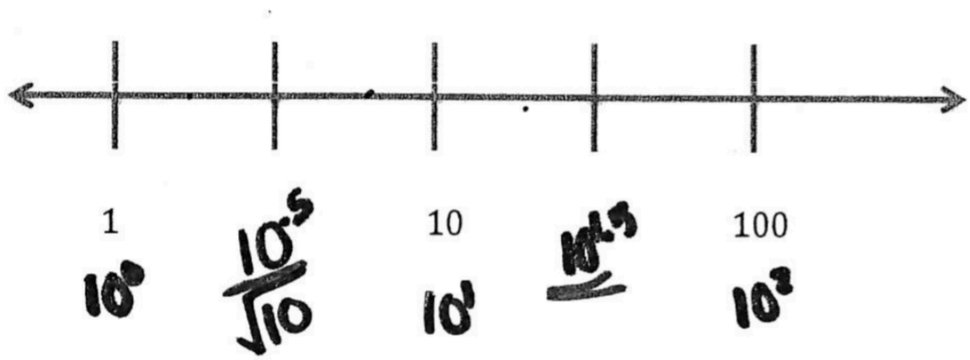


Figure 5.6: Brittany's labels.

After she wrote her explanation, she said, "I don't know if it makes logical sense." When asked what she was questioning, Brittany explained her concern further.

Brittany: I know that to get from one to ten, you have to multiply by ten. So to get halfway between that [points to the tick mark halfway between 1 and 10 with her pen], it's not, I mean, it's five, but not when you're looking at the whole timeline [sweeps pen back and forth over the whole timeline, stops on tick mark marked 10]. Because to get to this [moves pen from 10 to 100] you have multiply by ten as well, but halfway between ten and a hundred is not going to be the same as you find for here [points to the midpoint between 1 and 10]. So we're looking at the exponential values.

Interviewer: So you said something that, I'm not sure I understood what you meant. You said something like, it's five, but not if you look at the whole timeline. Can you say more about that, what you're meaning there?

Brittany: 'Cause this is labeled exponentially, not linearly.

Interviewer: And does that mean, it's exponential, not linear?

Brittany: It's going by a factor of ten [points to 1, 10, and 100 in quick succession], instead of like adding something on. So like, these portions [points to 1, 10, and 100 in quick succession] are not the same values. Linearly, you would add five [points to the midpoint of 1 and 10] and add five [points to 10], but here [traces out a circle over the section between 10 and 100] it would change. So I

> can’t label that five. It's half of the exponential values it's going by. So ten to the one half.

She then explained that for the second unlabeled spot “you just add that half to the one [the exponent of 1 in 10^1], because it’s half way between one and two [the exponents of ten].” She ended by adding on to her written justification, “where the 1 is the exponent of 10. So half the exponent of the multiplicative factor needs to be added to 0 to get halfway between 2 numbers".

When Brittany reasoned about the subsections, she talked exclusively about the linear pattern in the exponents. In her explanation she wrote, “Half way between 10^0 and 10^1 is $10^{.5} = \sqrt{10}$ because the factor it takes to get from 10^0 -> 10^1 is 10^1 and half of 1 is ½”. In her explanation she justified the placement of $10^{.5}$ by pointing out “half of 1 is ½.” The 1 in the explanation likely refers to the exponent of 10^1, given her previous phrase “the factor … is 10^1.” This means that her justification is based solely on halving the exponent. This reasoning continued later in her verbal explanation as well. She said, “It's half of the exponential values it's going by. So ten to the one half.” I interpret the phrase “exponential values” to mean the exponents. This interpretation is corroborated by what she added to the end of her explanation, “where the 1 is the exponent of 10. So half the exponent of the multiplicative factor needs to be added to 0 to get halfway between 2 numbers." This means, again, the explanation if focused on halving exponents. She then explained that for the second unlabeled spot “you just add that half to the one, because it’s half way between one and two.” Again, the focus seems to be on the exponents of one half, one, and two.

While there were instances of multiplicative talk, these all occurred when discussing the pattern at the macro level. For example, in her written explanation, she used the word “factor,” but it was in reference to the “x 10” pattern and she did not extend this pattern to the subsections. She wrote, “the factor it takes to get from 10^0 -> 10^1 is 10^1.” This is how multiplication came up in her verbal explanation as well. She said, “I know that to get from one to ten, you have to multiply by ten” and “It's going by a factor of ten [points to 1, 10, and 100 in quick succession], instead of like adding something on.” These are all establishing the fact that the line was exponential, not linear. During her explanation she never made reference to the idea that the subsection represented multiplication by the square root of ten or that multiplication by the square root of ten times the square root of ten was the same as multiplication by ten. In this way she differed from the previous three students.

Farah. Farah began by labeling the spot $10^{.5}$, and relabeling 1 as 10^0 and 10 as 10^1 (see Figure 5.7). She then wrote the following.

> Because this is an exponential line each label must be representable in exponential form. Each labeled tick mark represents 10^0, 10^1, 10^2 respectively. The halfway marks can not be represented in whole numbers dependent upon endpoints because that will force the value to move depending on the endpoints given. The halfway mark is the half of the exponent of the larger endpoint. 1/2 of 1 = 1/2 [three dots in a triangular pattern to mean therefore] $10^{0.5}$.

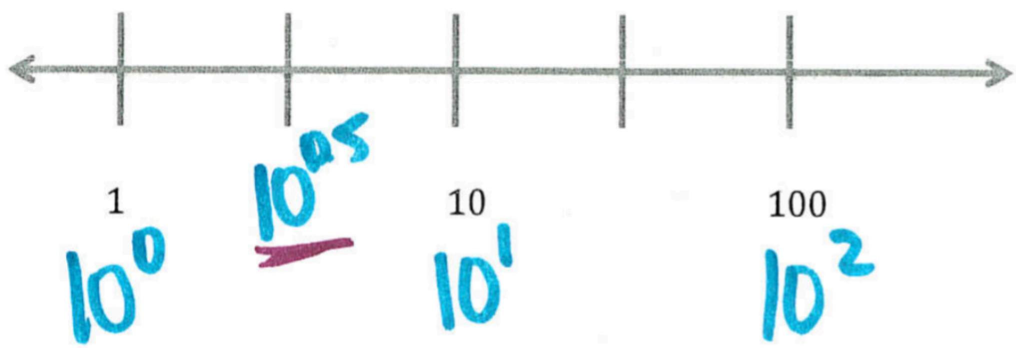


Figure 5.7: Farah's labels.

Here Farah is clear that she was operating on the exponents. She wrote, "The halfway mark is the half of the exponent of the larger endpoint. 1/2 of 1 = ½." She did not mention anything about multiplication at the macro level or within the segments. Furthermore, in her explanation she focused on the form the numbers were written in, which may suggest a focus on the exponents. She wrote, "The halfway marks can not be represented in whole numbers."

Something that might be confusing is her statement, "that will force the value to move depending on the endpoints given." In the interview, she explained that this was in reference to an activity in class where students subdivided segments linearly using two different pairs of endpoints, which resulted in two different placements for the Renaissance (see the background to Argument 2.1.1 in Chapter 4). However, in the class activity, it was not the fact that the endpoints were written in whole numbers that caused the location of the Renaissance to be dependent on the endpoints given, it was the mixed nature of the half exponential half linear number line they were reasoning about. In other words, this had less to do about the notation and more about the reasoning about how to subdivide.

Elements of Reasoning Linearly Among the Values

Unlike the other students, Santiago and Lacey both had elements of reasoning linearly about the actual values on the line. Santiago began by reasoning linearly, but eventually corrected this error, while Lacey reasoned linearly and left the mistake uncorrected.

Lacey. Lacey began by saying, "I think it'd [the first unlabeled spot would] be five" and labeled the first unlabeled spot 5. She then wrote her justification. As she wrote her justification, she labeled the second spot 50 (see Figure 5.8)

> 5 gets in first spot because it looks like ½ distance between 1 and 10.
> 50 in second spot because 50 is ½ of 100.

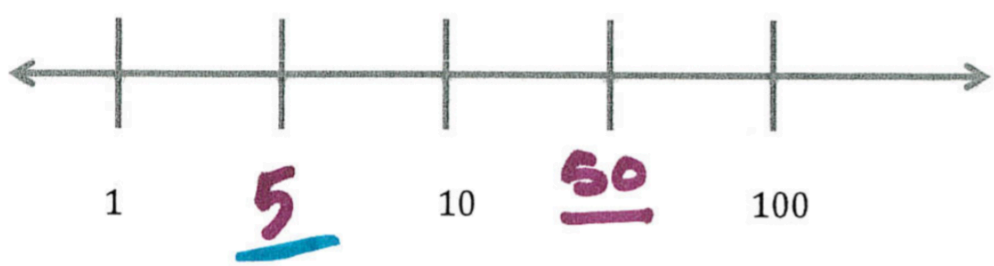


Figure 5.8: Lacey's labels.

After she wrote this, she checked that 5 lay between its surrounding tick marks, 1 and 10. She then said, "This one's going to be fifty because it's half of a hundred and it's still between ten and a hundred."

Lacey's way of reasoning differed from the previous five students' ways of reasoning in that she reasoned linearly on the values. In particular, she halved 10 to get 5 and halved 100 to get 50. This halving resulted in a number that when added twice would give the endpoint (5+5) instead of a number that when multiplied twice would give the endpoint $\left(\sqrt{10}\cdot\ \sqrt{10}\right)$. This linear reasoning is similar to the reasoning in category two in that both require halving, except here it was applied to the actual value of the endpoint (10) instead of the exponent of factor by which it values increased (10^1). This difference is crucial because reasoning linearly with the values does not yield the correct label for the midpoint whereas reasoning linearly with the exponents does.

Santiago. Santiago also thought the first unlabeled spot should be marked 5, although he later revised his thinking. He started by saying the first unlabeled spot should be marked five and labeled it as such. He then moved to reasoning about the second unlabeled spot. He said, "just by looking at half of it [makes a chopping motion over the halfway point between 10 and 100, followed by a sweeping motion from 10 to 100], it's not going to help" and then relabeled 10 as 10^1 and 100 as 10^2. He then said, "exponentially the distance is just adding one [sweeping motion from 10 to 100]." The interviewer then asked what the point should be labeled to which he replied, "ten to the one point five." After some discussion of the meaning of $10^{1.5}$, he then returned to reasoning about the first unlabeled spot. He first relabeled 1 as 10^0 and then relabeled the spot $10^{0.5}$ (see Figure 5.9). The interviewer then asked if he was happier with 5 or $10^{.5}$ as a label for the midpoint between 1 and 10. After some deliberation, he said it should be $10^{.5}$. He then explained his mistake saying, “I looked at it just linearly, half of it, five, instead of exponentially. … ‘Cause usually with lines I think of it as linear. I never think of lines, timelines, or whatever this is, as exponential ones.” He then justified his claim that the spot should be labeled $10^{.5}$ by saying, “Just looking at the exponents, and then half between those exponents [points to 0 and 1] is now point five.”

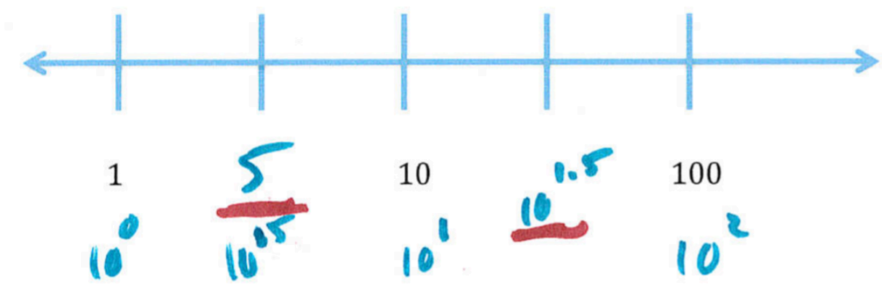


Figure 5.9: Santiago’s labels.

Santiago eventually determined that the midpoint between 1 and 10 should be $10^{.5}$, by reasoning about the exponents. However, I am placing his response to this task in this category because he initially said that the midpoint of 1 and 10 should be labeled 5 because half of ten is 5. He even explicitly said he “looked at it … linearly.” While it would be reasonable to categorize students’ ways of reasoning based on what they eventually decided to do, I felt it important to capture in the results what students were still struggling with after instruction. This struggle was especially striking given the amount of time spent in class talking about linear subdivision and contrasting it with exponential subdivision.

Discussion

Relationships between the Categories of Individuals' Reasoning and the Class NRWs

In Chapter 4 I established the emergence of Math Practice 2. This Math Practice consists of two normative ways of reasoning that describe accepted ways of subdividing segments. The first of these, NWR 2.1: Subdividing Segments by Reasoning Linearly About Exponents, is characterized by students focusing on the exponents. The students would write the endpoints of a segment in the form a^b and then essentially ignore the base and reason linearly with the exponents. In the second way of reasoning, NWR 2.2: Preserving the Multiplicative Relationship within the Segments, students reasoned that since a multiplicative pattern exited among the macro level tick marks (see MP1), a multiplicative pattern should also exist among the subsections. This means that if a segment that represents an increase by a factor of 10 is divided into n subsections, one needs a number that when multiplied by itself n times yields 10. This number is the n^{th} root of ten. This means both ways of reasoning give the same answer for the subdivision. However, they are not redundant, as they highlight different mathematical relationships. Thus, reasoning that is fully consistent with Math Practice would include being able to reason in both ways and recognizing the how the ways of reasoning are related. In this section, I argue that the students whose reasoning was categorized in the first category was fully consistent with Math Practice 2. In contrast, the students whose reasoning was categorized in the second category was consistent solely with NWR

2.1. Finally, the students whose reasoning was categorized in the third category was consistent with a way of reasoning that did not become normative, but was present in classroom discussions, reasoning linearly to subdivide (see Table 5.2). This means that individuals' ways of reasoning, even those that persist, can be qualitatively different from the established practice, though variations observed in this study were rooted in ideas presented in class.

Table 5.2: The relationship between ways of reasoning that appeared in class and those that appeared in the interview.

Ways of Reasoning in Class	Ways of Reasoning in Interviews
Math Practice 2	Category 1: Multiplicative Reasoning Coordinated with Reasoning Linearly with the Exponents
NWR 2.1 Subdividing Segments by Reasoning Linearly About Exponents	Category 2: Reasoning Linearly with the Exponents
Reasoning Linearly to Subdivide	Category 3: Elements of Reasoning Linearly Among the Values

Math Practice 2 and Category 1. Math Practice 2 consists of two normative ways of reasoning. Both of these ways of reasoning yield the same answer, but have different affordances in their use. NWR 2.1, which focuses on the linear pattern in the exponents, is an efficient way to determine the values that should be placed at the endpoints of various subdivisions. NWR 2.2 elaborates NWR 2.1, in that it focuses on a different mathematical relationship, namely the multiplicative relationship between the values rather than the linear one in the exponents.

This analysis of the relationship between NWR 2.2. and 2.1 is consistent with how the ways of reasoning developed in class. NWR 2.1 was first brought up by Lacey at the beginning of Day 3 (see Argument 2.1.1). She reasoned linearly with the exponents to determine that $10^{2.5}$ was the midpoint of the segment from 10^2 to 10^3. In the subsequent exploration of her argument students started talking about multiplicative relationships. It was in this discussion when Kathy had her breakthrough, when she extended the multiplication pattern at the macro level to the subsections (see Overview of the development of NWR 2.2 and Revisiting of Argument 1.1.2 in the previous chapter). In this way, Kathy was elaborating Lacey's reasoning. Furthermore, in Argument 2.2.2 Jade originally reasoned that the subsection that was one seventh of the segment from 10^2 to 10^3 should be $10^{1/7}$ by appealing to the linear pattern in the exponents. Only when the teacher pressed further, asking, "why is it ten to the one seventh," did Jade start to talk about multiplicative relationships within the subsections. In this way, the multiplicative relationships served to explain the answer, which was obtained by reasoning linearly about the exponents.

Using multiplicative relationships to justify the placements of numbers that were determined by reasoning linearly about the exponents is consistent with the reasoning in Category 1. For example, Kathy explained that she found half the exponent (which is linear reasoning), but went on to explain that this gave the square root of ten. She then seemed to indicate that these subsections were being multiplied and connected that multiplication to the macro pattern by positioning both as a consequence of the definition of an exponential line. Similarly, Tanya and Rachel talked about dividing the exponent in half, but also talked about how the factors were being multiplied together.

NWR 2.1 and Category 2. Similar to the students whose reasoning was placed in Category 1, the two students whose reasoning was placed in Category 2, Farah and Brittany, also used linear reasoning in response to the interview task. In this way, their reasoning was consistent with NWR 2.1. However, unlike the first three students, Farah and Brittany did not elaborate their labeling of the subdivisions by drawing on multiplicative relationships, despite probes from the interviewer. In this way, instead of fully engaging with Math Practice 2, these students seemed to engage only with a consistent piece of the emergent practice, NWR 2.1.

An Early Way of Reasoning in Class and Category 3. The previous two ways of reasoning in the interview corresponded to ways of reasoning in class that were eventually accepted. The last way of reasoning that occurred in the interview, *elements of linear reasoning,* also corresponded to a way of reasoning that was expressed in class, but did not become normative, *reasoning linearly to subdivide*. Linear reasoning came up in two ways during students' explorations of how to build a timeline. The first of these was a fully linearly line, which was characterized by equal length segments representing equal elapsed times. This characterization was accepted in class (NWR 0.1). The second approach, which employed *reasoning linearly to subdivide*, was not accepted in class. In this mixed approach the students had a macro exponential "x 10" pattern, but within the segments they would reason linearly. This did not become a normative way of reasoning because it was overturned in favor of fully exponential reasoning (initially at the macro level in MP1 and eventually within segments in MP2).

Answering Research Question 1

In this chapter I have argued that the three categories of individual reasoning correspond to ways of reasoning that were expressed in class. Category 1 corresponded to Math Practice 2, Category 2 corresponded with NWR 2.1, and Category 3 corresponded an early way of reasoning in class that was rejected. This means that individual ways of reasoning that are qualitatively different than the emergent practice (Categories 2 and 3) can persist after instruction. Category 3 was clearly different than established math practice as it resulted in differences in the placement of times. Category 2 was consistent with the math practice in that it provides the same answers, however, it is still significantly different in that fully participation in practice requires an ability to reason using both NWR 2.1 and NWR 2.2 and see the relationships between them.

The fact that ways of reasoning that were qualitatively different from the established practice persisted after instruction may be somewhat surprising given the results of the few studies that examined the relationship between emergent practices and students' subsequent reasoning. These studies seem to indicate that students eventually reorganize their knowledge to be consistent with emergent practices (Bowers et al., 1999; Stephan et al., 2003). Given the discrepancy in results, it is important to examine the impetus for the reorganization of knowledge of the students in these studies. This is clearer in results of the study performed by

Stephan et al. (2003). In that study, one of the students, Meagan, reasoned in ways that were qualitatively different from emergent practices temporarily, but eventually her ways of reasoning became problematized as they became more inconsistent with class activities.

One hypothesis to explain the observed variation in ways of reasoning reported here from the established practice might be that students whose reasoning was placed in Categories 2 and 3 simply stopped intellectually engaging in the class. However, this was not case. For example, Farah contributed to the emergence of NWRs 3.1 (Subdividing Extents that Span Multiple Segments) and 5.1 (Logarithms are Exponents) and Lacey contributed to the emergence of NWRs 2.1 (Subdividing Segments by Reasoning Linearly About Exponents) and 4.2 (An Additive Sequence is one that has a Constant Sum). This implies that the students continued to participate in class discussions and intellectually engage with the materials.

Given the two assumptions that (a) students in this study continued to engage with class activities and (b) continued participation was the impetus for Meagan's reorganization, why did the participants in this study fail to reorganize their knowledge? This question may be best answered separately for Categories 2 and 3. Students whose reasoning was placed in Category 3 were still struggling with the transition from linear reasoning to exponential reasoning. This was surprising given the amount of time spent in class on developing exponential ways of reasoning and the explicit discussion about why reasoning linearly was problematic during the Renaissance activity. However, it should be noted that both students whose reasoning was placed in this category, at some point, correctly placed points by reasoning linearly with the exponents. Thus, the claim here is not that students were not capable of placing points in way that was consistent with a fully exponential line or were completely unaware of the problems with linear reasoning. Rather, it seems that these students were still struggling with knowing when place points linearly and when to place them exponentially. This is consistent with the research on exponential and logarithmic thinking, which shows this is a common problem for students (Alagic & Palenz, 2006; Berezovski, 2004; De Bock et al., 2002). As such, these two students likely just needed more opportunities to reason about when to use which type of reasoning.

Perhaps more puzzling is the question of why did students not adopt multiplicative ways of reasoning? Over half of the students in the post interview reasoned about subdivisions without drawing on multiplicative patterns. This makes it seem less likely that students were able to reason in this way and simply did not in the interview, rather it seems as though many of the focus students failed to make the shift entirely. This may be due to the nature of the NWRs with respect to the Math Practice 2. This practice consisted of two ways of reasoning that both gave the correct placements. As such, students could reason linearly with the exponents with no understanding of multiplicative patterns and still get correct placements. This means that this way of reasoning was powerful enough to allow students to participate in future class activities, without reorganizing their knowledge. This differs from Meagan's experience where her ways of reasoning that were qualitatively different than established practices yielded different answers, thus making her way of reasoning more problematic for continued participation in class activities.

This analysis helps explain how the students in this study could participate in subsequent class activities yet reason in ways that were qualitatively different from established practices. However, this leaves open the question of why students did not reorganize their knowledge as the practice was being established. In other words, given the considerable class time spent on developing multiplicative ways of reasoning, how could students reasonably intellectually engage in those discussions, yet not advance their thinking? This is examined in the next chapter where I analyze the mathematical content of the class discussions held as multiplicative ways of reasoning were being developed.

Chapter 6: Thematic Analysis of Classroom Discourse

In the previous two chapters I examined the relationship between individuals'' ways of reasoning on in an interview administered after instruction and Math Practice 2. I found that in contrast to the findings of other scholars, the focus students in this study still reasoned in ways that were qualitatively different from the established practice, even after instruction had ended. However, these ways of reasoning were all similar to ways of reasoning that were expressed in class. In the first way of reasoning, students coordinated reasoning linearly with the exponents with multiplicative reasoning. This is consistent with Math Practice 2 as it was established in class, which includes two normative ways of reasoning, NWR 2.1: Subdividing Segments by Reasoning Linearly About Exponents and NWR 2.2: Preserving the Multiplicative Relationship within the Segments. In class, NWR 2.1 was positioned as a way to efficiently determine the value of subdivisions and NWR 2.2 was positioned as a way to explain why reasoning linearly with the exponents makes sense. This is consistent with the way students reasoned in post interview whose reasoning was placed in the first category. In the second category were ways of reasoning that solely used the linear pattern in the exponents to determine placements. This is consistent with NWR 2.1, but differs from Math Practice 2 as a whole because it did not include multiplicative reasoning. Finally, ways of reasoning in Category 3 assumed a linear relationship among the values. This is inconsistent with Math Practice 2, but is consistent with the mixed approach that was discussed on Days 1 and 2 in whole class and eventually rejected.

Since over half the focus students did not include multiplicative reasoning in their interview responses, observers are left with the question of how could the students intellectually engage in class discussions, but not personally adopt ways of reasoning consistent with the emergent practice? A partial explanation to this question will be developed in this chapter. In particular, I will focus on the nature of the discourse as multiplicative reasoning was developed in this class. Examining the semantic relationships expressed in the classroom discourse will help illuminate the mathematical meanings established in class. Understanding the multi-faceted nature of these meanings will help provide a plausible account of how students could legitimately engage in class discussions, but not shift their ways of reasoning to include reasoning multiplicatively. This analysis will answer Research Question 2.

> Research Question 2. What mathematical connections exist between the focus students' ways of reasoning in the post interviews and the discursive interactions between them and other students and the teacher in both whole class and small group settings? Furthermore, how might the nature of these discursive interactions give plausible explanations for students' differing conceptions?

It is important to note that the point of this analysis is to explain how students could participate the class without using ways of reasoning on the interview question that mirror the emergent math practice developed in the class community. To do this I will focus on how the discourse allowed for students to continue reasoning linearly with the exponents and not shift to include multiplicative ways of reasoning. As such, the analysis may at times feel like a critique of the instruction, but this is not the intent of this chapter. It is important to recognize that, overall, the instruction could be considered very successful. All of the focus students were at some point able to reason correctly about the exponential number line to make accurate placements. However, there were subtle differences in students' conceptions that have the potential to compound into bigger differences over time, if not addressed. Thus, it is worth examining how the discursive environment was related to these differences to reveal subtle changes that could be made to support a greater number of students developing deep conceptual understanding of this exponential number line. By examining this case, I hope to better understand how discourse can foster multiple interpretations and allow students to participate in emergent mathematical practices without being able to reason on their own in the same the way. As researchers gain more understanding of this process, they may be better able to support teachers in fostering discourse that makes less sophisticated reasoning more problematic for students as they participate in emergent practices. This will encourage students' development of more sophisticated reasoning.

Review of Methods

To determine mathematical meanings as they were constructed in I used a modified version of Herbel-Eisenmann and Otten's (2011) method for thematic analysis (Lemke, 1990; Herbal-Eisenmann, 2011), a systemic functional linguistics (SFL) approach (Halliday, 1978; Halliday & Hasan, 1985; Halliday & Martin, 1993; Halliday & Matthiessen, 2004). The first step was to reduce the data to episodes where students were explaining how to subdivide a segment or referencing a method of subdivision. I first analyzed the episodes where students explained how to subdivide. To do this I created a lexical chain for each episode. This was a way of formatting the transcript to reveal central ideas in the text. The rows were the turns of talk and the columns were the mathematical ideas that were expressed in turn. Since multiple ideas could be expressed in one turn, the utterance could be placed in multiple columns. I also included descriptions of gestures over the line (see Table 6.1 for an example portion of a lexical chain). I then augmented the chain by adding a column to the table in which I recorded the semantic relationships expressed in each turn. Many of the semantic relationships I used are described in *Talking Science* (Lemke, 1990), but I also found it necessary to define new ones (see Table 6.2 for the list or relations I used and their definitions).

In the next stages I created two-dimensional drawings that represented the semantic relationships that were expressed. I made such a map for each way of reasoning expressed in the class. In addition, I also made a "canonical map," a map that is faithful to the mathematics register. To create this map I drew on my personal understanding of the topic. I did this by creating arguments for how to subdivide the line and then analyzed those arguments for the semantic relationships expressed, just as I did for the classroom discourse. I then compared the various maps. This revealed differences between the ways of reasoning expressed in class and between them and the canonical arguments.

Finally, I examined episodes where students referenced the methods themselves. In these episodes students tended to develop semantic relationships in a different way than when they were explaining the methods. In these episodes they tended to use equivalence and contrast strategies (see Appendix D in Lemke, 1990, p. 226) to show whether they thought two strategies were the same or different. These strategies will be described in more detail later in the chapter.

Table 6.1: A Sample of the Lexical Chain Used for Analysis

Location of Renaissance	Half way	add	100 (Endpoint)	Nine hundred years	Four hundred fifty	Interval	Relationships
R: I'm still confused as to why, like my gut says it wants to put it half way.	R: I'm still confused as to why, like my gut says it wants to put it half way.						wants to put it [location of Ren] -LOCATED/LOCATION-half way
	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.			T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	T: Right, so if you're thinking it's nine hundred years in that interval [puts both hands up like an interval], the half way point is four fifty.	nine hundred years -LABEL/LABLED- in that interval [puts both hands up like an interval], the half way point -LOCATION/LOCATED-four fifty
R: So it's past half way. If you're going this way [gestures with pen from 10^2 to 10^3]	R: So it's past half way. If you're going this way [gestures with pen from 10^2 to 10^3]						it's [location of Ren] -LOCATED/LCOATION-past -PREPOSITION/OBJECT- half way.
S: Past half? [Questioning them]	S: Past half? [Questioning them]						
K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100	K: It's less than half....Because you're adding 100				It's [location of Ren] -LOCATED/LOCATION- less than -PREPOSITION/OBJECT-half[way]

Table 6.2: Semantic Relationships used in Analysis

Linguistic Term	Description	Example
Process/Target	The process is an action that is being carried out. The target is what is being operated on.	Dividing (process) the segment (target) up.
Process/Result*	The result is the outcome of the process.	I divided (process) 500 by 2 and got 250 (result).
Process/Reason*	The reason is why the process is occurring.	I added 200 (process) because that's our starting point.
Entity/Extent	The measure of a physical space	I found the length (extent) of the segment (entity).
Location/Located	Where an object is located.	500 is at the midpoint.
Token/Type	An example of a class of objects.	450 (token) is an amount of elapsed years (type).
Representation/Represented*	The representation is a depiction of something and the represented is what is being depicted	Same-sized segments (representation) represent multiplication by a constant factor (represented)
Label/Labeled*	This is an objects is called something. This can be done verbally or through an inscription.	A student might put a bracket over a segment (labeled) and write "x10" (label). The segment (labeled) is 500 (label).
Preposition/Object	A word that expresses a physical or temporal relationship to another word.	Place the tick to the right (preposition) of the middle (object).
Synonym	When the two words mean the same, or nearly the same, thing	Ten squared is (synonym) one hundred.
Agent/Process	The agent is the person or object that preforms the process.	I (agent) divided (process) the segment.

Overview of Claims

In the results section, I present six maps that show semantic relationships between lexical items along with the evidence supporting the claim that these relationships exist. By comparing these maps I give evidence for my four main claims. This is organized as follows. First, I give a conceptual analysis of a traditional number line, one in which there is a consistent linear scale, and an exponential number line. With these analyses as a backdrop, I give an example of how one might reason about them to determine the value of the midpoint between 10^2 and 10^3. Using these example arguments I created the canonical maps for the two number lines. I then compared the two canonical maps to each other. This provides evidence for *Claim 1: The length of segments is an important feature of the number lines that students need to attend to make the transition to multiplicative reasoning*.

I then give evidence from classroom interactions for the semantic relationships I claim were expressed as students used linear reasoning to place events and present these relationships in a map. I compare this map to the canonical map for linear reasoning to provide evidence for *Claim 2: As students subdivided linearly, the terms "distance" and "difference" were used somewhat interchangeably and referred to amounts of elapsed years, the length of segments, and the result of subtraction*.

Next, I present the evidence and maps that illustrate how students talked as they reasoned linearly with the exponents and multiplicatively. I then compare these two maps to support *Claim 3: Students used a term that suggests multiplicative reasoning—"factor"—to refer to segments as they described additive patterns in the exponents*.

Finally, I examine the ways the students talked about the methods themselves. This gives rise to *Claim 4: Students distinguished in their talk between linear and exponential ways of reasoning, but did not distinguish between reasoning linearly with the exponents and multiplicative ways of reasoning. In fact, students seemed to think of both of these methods as the same.*

Together these four claims suggest that the classroom discourse may not have supported students in disambiguating between reasoning linearly with the exponents and reasoning multiplicatively. An outsider may think that these ways of reasoning are obviously different since they focus on different mathematical relationships and have a hard time imagining the students thinking of them as the same. However, there were several aspects of the discourse that may have allowed students to think of the strategies as the same or at least equivalent. If students did not see these methods as different, there may have been little intellectual motivation to fully understand NWR 2.2. This may help explain why students did not adopt this way of reasoning.

Results

Claim 1: The Importance of Attending to Length

In this section, I first present a conceptual analysis of two number lines, one with a consistent linear scale, which I call a traditional number line, and one exponential. These analyses are based on my personal expertise, analysis, and reflection. Using these conceptual analyses as a base, I then present an argument for which number should be placed at the midpoint between 10^2 and 10^3 on a traditional number line and then present an argument for which number should be placed at the same midpoint on an exponential number line. I chose the midpoint of 10^2 and 10^3 because students gave both exponential and linear arguments for the placement of this point in class. This makes the networks of semantic relationships expressed in the canonical arguments more easily compared to the classroom arguments. I then created maps that show the semantic relationships presented in these arguments. By comparing these maps I provide evidence for *Claim 1: The length of segments is an important feature of the number lines that students need to attend to make the transition to multiplicative reasoning*.

Conceptual Analyses. In a traditional number line, numbers that differ by one are the same distance[3] apart on the number line (e.g., one inch, see Figure 6.1). This means that the distance between zero and one determines the distance between one and two, the distance between two and three, etc. A logical consequence of this is that numbers that differ by two will be the same distance apart—double the distance from zero to one (e.g., 2 inches, see Figure 6.2).

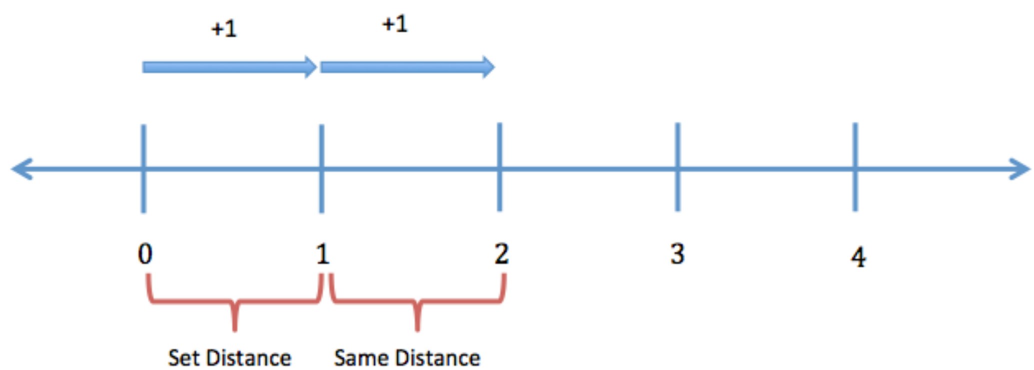


Figure 6.1: Numbers that differ by one are the same distance apart

[3] In this analysis I talk both about the distance between two points and the length of segments. These are really two ways to describe the same thing, but I use both phrases because sometimes it is more natural to refer to the endpoints (and the distance between them) and other times it is more natural to talk about the segment (and its length).

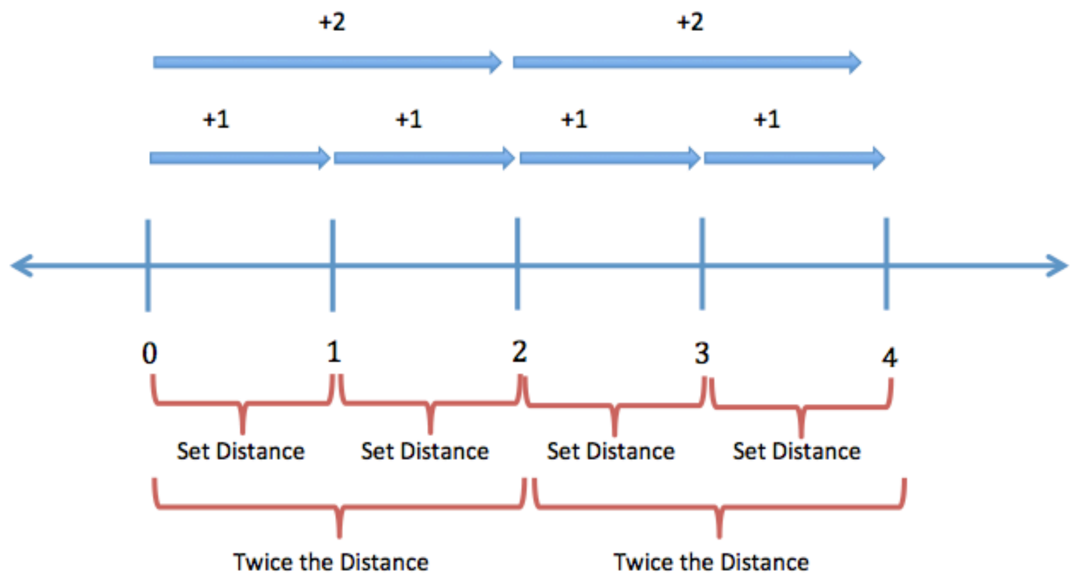


Figure 6.2: Numbers that differ by two are twice the distance apart

In fact, a logical consequence of this is that any pair of integers on the line with the same difference will be the same distance apart (e.g., 96 and 99 will be the same distance apart on the line as 1,544 and 1,547). This relationship can be generalized and one could say that *any* pair of numbers with the same difference will be the same distance apart (e.g. 2 and 3.5 will be the same distance apart on the line as 101 and 102.5). Given the assumption that larger numbers are to the right of smaller numbers, this rule determines the locations of all the numbers on the line once any two numbers are placed on the line. For example, to place ½ one would reason that it would need to be the same distance away from zero as it is from 1, since ½ - 0 = 1 – ½. This means it would need to be halfway between zero and one. Put succinctly, in a traditional number line, segments of the same length represent addition by a particular difference.

An exponential number line can be built up in the same way, except one needs to interpret same size segments differently—as representing multiplication by a particular factor rather than addition by a particular difference. For example, any two numbers that when divided yield a quotient of ten will be the same distance apart (see Figure 6.3).

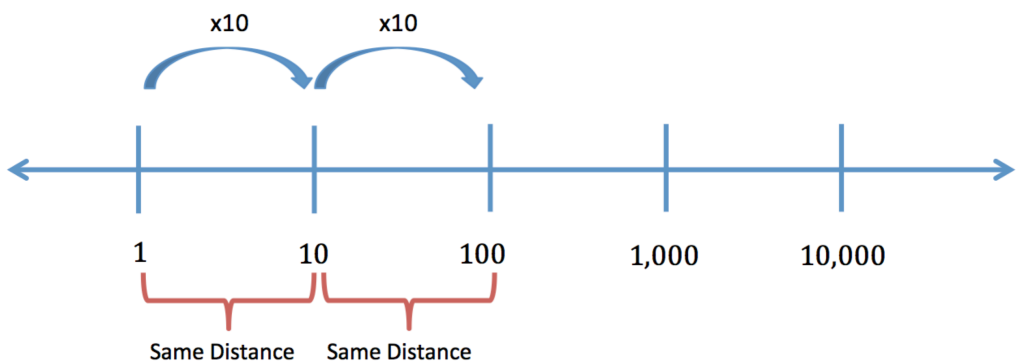


Figure 6.3: The same distance apart means an increase by a constant factor

This means that if two numbers whose quotient is ten are one inch apart on an exponential number line, then numbers that are two inches apart on that number line will have a quotient of one hundred (see Figure 6.4). In general, if $a \div b = q$ and the distance between a and b is x inches and $c \div d = q^2$ the distance between c and d will be $2x$ inches.

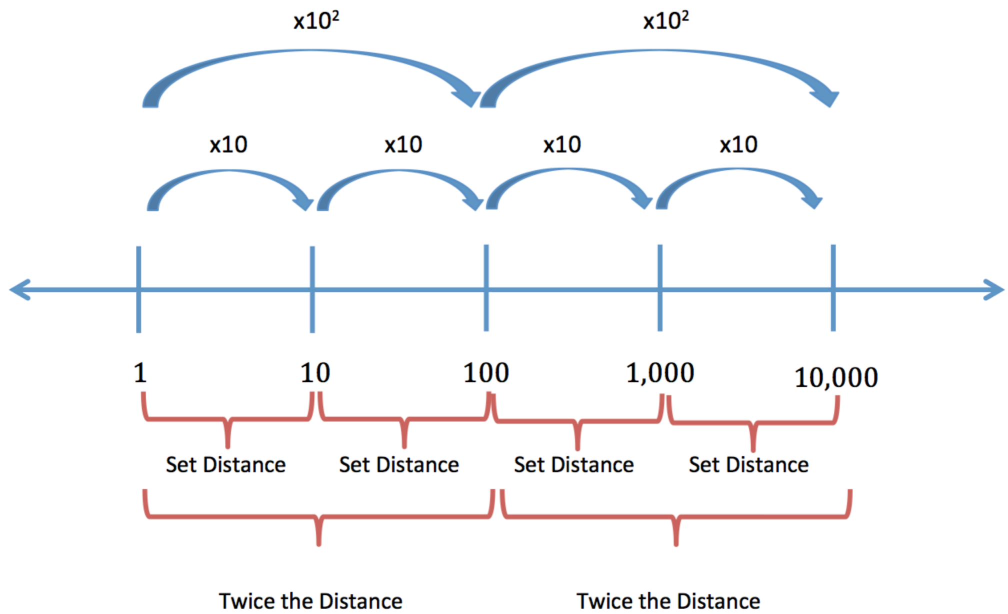


Figure 6.4: Twice the distance means an increase by the square of the factor

This means that once any two numbers are placed, the placement of all of the other numbers is determined. For example, to determine what number should be placed halfway between the tick mark labeled 1 and the tick mark labeled 10 in the example above, call it y, we would recognize that the half way point is equidistant from 1 and 10. That means that 1 times an unknown factor, call it m, would yield y, and y times m would yield 10. So we have $1m = y$ and $ym = 10$ so $1mm = 10$. That means $m = y = \sqrt{10}$.

Canonical Argument for Labeling Point halfway between 100 and 1,000 on a Traditional Number Line. Now I present an argument for why the midpoint of 100 and 1,000 on a traditional number line would represent 550. Then I will discuss the semantic relationships present in the argument (following Herbel-Eisenmann & Otten, 2011; Lemke, 1990). As I do so, I will bold lexical items and write semantic relationships in all caps. These conventions will continue in subsequent sections when I analyze classroom dialogue.

> The **halfway point** between **100** and **1,000 divides** the **segment from 100 to 1,000** into **two subsections of equal length**. Since this is a traditional number line, **same length segments** must represent **addition by the same amount** so each of these **two subsections** must represent **addition by the same number**. Since the **length** of **the whole segment** represents **addition by 900**, the **subsections** must both represent **addition by 450** as **adding 450 twice** is equivalent to **adding 900**. Since the location of the leftmost **endpoint** represents a value of **100**, the **midpoint** must represent a value of **100** plus **450**, or **500**.

In this statement I express several semantic relationships. I describe the halfway point as an AGENT performing a PROCESS, dividing, whose TARGET is the segment. This RESULTS in two subsections of equal lengths. I then specify that these segments REPRESENT addition by 450. This is an IMPLICATION of the fact that same size segments REPRESENT addition by the same number. I then deduce that because the whole segment REPRESENTS addition by 900, each subsection must REPRESENT 450, since 450 plus 450 is 900. Then since the endpoint REPRESENTS 100, this is added to 450, a PROCESS which RESULTS in 550.

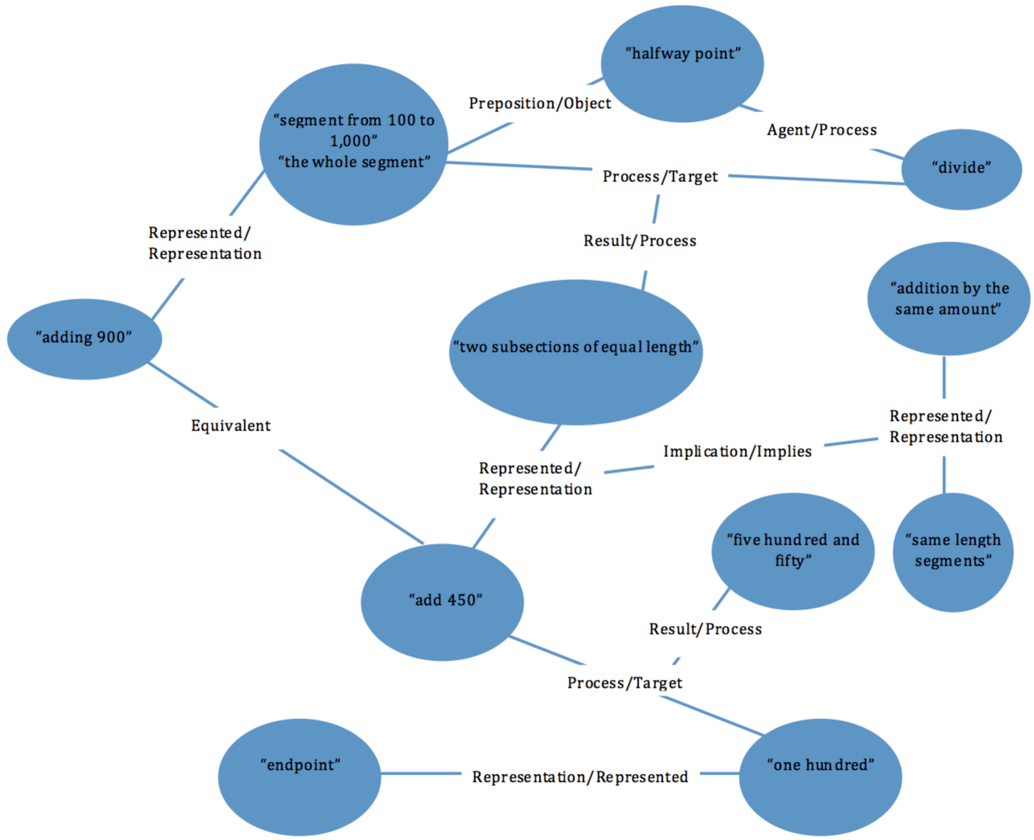


Figure 6.5: Canonical Map for Traditional Number Line

Canonical Argument for Labeling the Halfway Point between 100 and 1,000 on an Exponential Number Line. One can give an argument that has a very similar structure for determining what goes halfway between 100 and 1,000 on an exponential number line if one reinterprets the length of a segment as representing multiplication instead of addition. Such an argument follows.

> The **halfway point between 100 and 1,000 divides** the **segment from 100 to 1,000** into **two subsections of equal length**. Since this is an exponential number line, **same length segments** must represent **multiplication by the same amount** so each of these **two subsections** must represent **multiplication by the same number**. Since the **length** of the **whole segment** represents **multiplication by ten**, the **subsections** must both represent **multiplication by $\sqrt{10}$** as **multiplying by $\sqrt{10}$ twice** is equivalent to **multiplying by ten**. Since the location of the leftmost

endpoint represents a value of **100**, the **midpoint** must represent a value of **100** times $\sqrt{\mathbf{10}}$, about **316**.

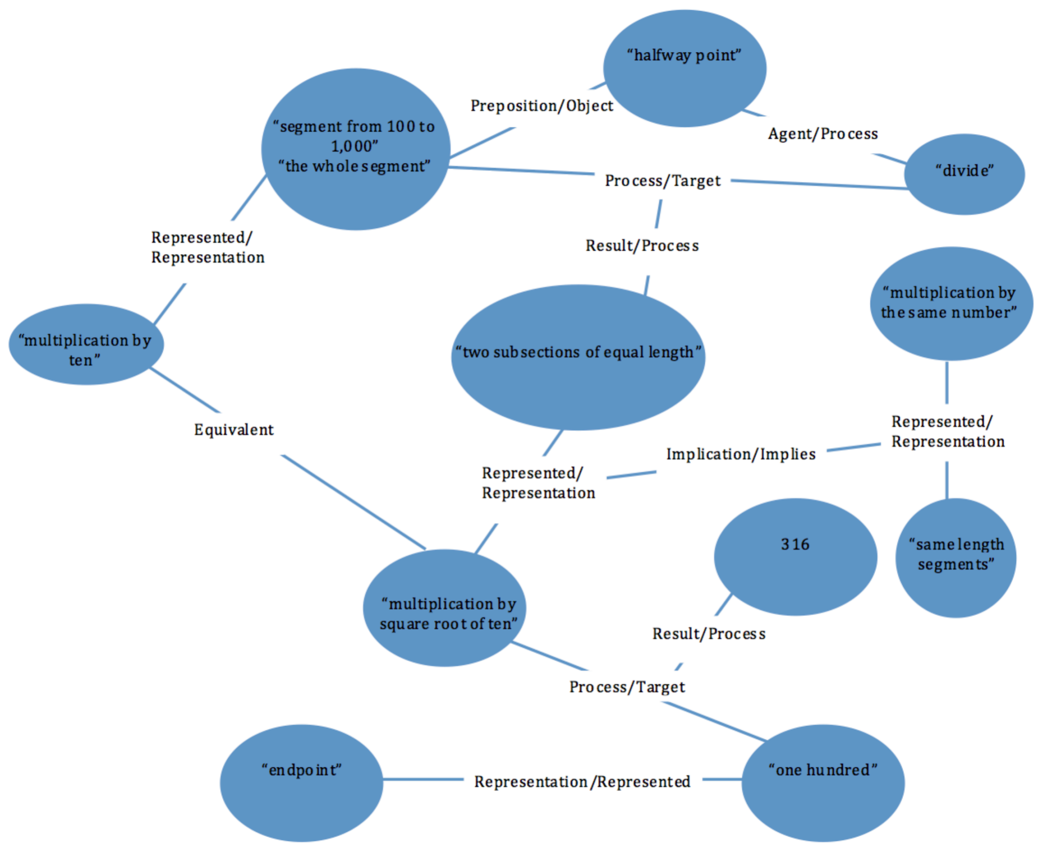


Figure 6.6: Canonical Map for Exponential Number Line

Comparison of Canonical Maps. These arguments are nearly identical except for the interpretation of what the length of a segment, or the distance between two points, represents. On a traditional number line the length represents addition by a particular difference. On the exponential number line the length represents multiplication by a particular factor. This may mean that this shift in interpretation of length is key to understanding the exponential number line.

Of course, this shift is not necessary if students write all numbers on the exponential number line as ten to some power. Written in this form, a numeric pattern emerges. Segments that represent multiplication by ten also represent an increase of one in the exponent because of the way things are notated. So, if one inch represented multiplication by ten, it also would represent addition by one in the exponent. For example, one inch away from 10^0 is 10^1 and one inch away from 10^1 is (10^1 x 10) or 10^{1+1} or 10^2. This extends to subdividing segments as well. Continuing the example, one half an inch would represent multiplication by the square root of ten. So one half inch away from 10^0 would represent 10^0 x $\sqrt{10}$, which can also be written as 10^0 x $10^{.5}$ or as $10^{0+.5}$. This shows that one half inch also represents addition by ½ in the exponent. Put more generally, in an exponential number line, same size segments not only represent multiplication by a particular factor, but also addition by a particular number to the exponent.

The fact that the exponential number line works exactly like a traditional number line, if the students just ignore the base, may help explain why reasoning linearly with the exponents was compelling for the students. Given the fact that students found this way of reasoning appealing, one might say that it should be avoided altogether so students must reason multiplicatively. Alternatively, given the apparent complexity for students, one might wonder if it is worth the time to get students to shift their interpretation of length. Why not just have them reason linearly with the exponents? In response I would argue that the number line becomes a powerful tool when students can interpret the length of segments as representing both multiplication and addition in the exponent. When students can interpret lengths of segments in both ways, students can gain insights into and build intuition for rules that exist in exponential and logarithmic situations. In other words, if students understand that the length of a segment can represent either addition in the exponent by a set number or multiplication by a set factor, reasoning about the number line can build intuition about rules such as $10^{1/2} = \sqrt{10}$.

Similarly, understanding what a length of a segment can represent lays the groundwork for reasoning about logarithms. One way to reason about logarithms using the number line is to interpret $\log_b(x)$ as the length of the segment from 1 to x. This is because the length represents multiplication by a factor of x as well as the increase in the exponent from b^0 to b^t where $b^t = x$. For example, suppose we have a number line where each factor of eight is one inch apart. This means that 8^t will be located t inches away from $8^0 = 1$ and that $\log_8(8^t) = t$.

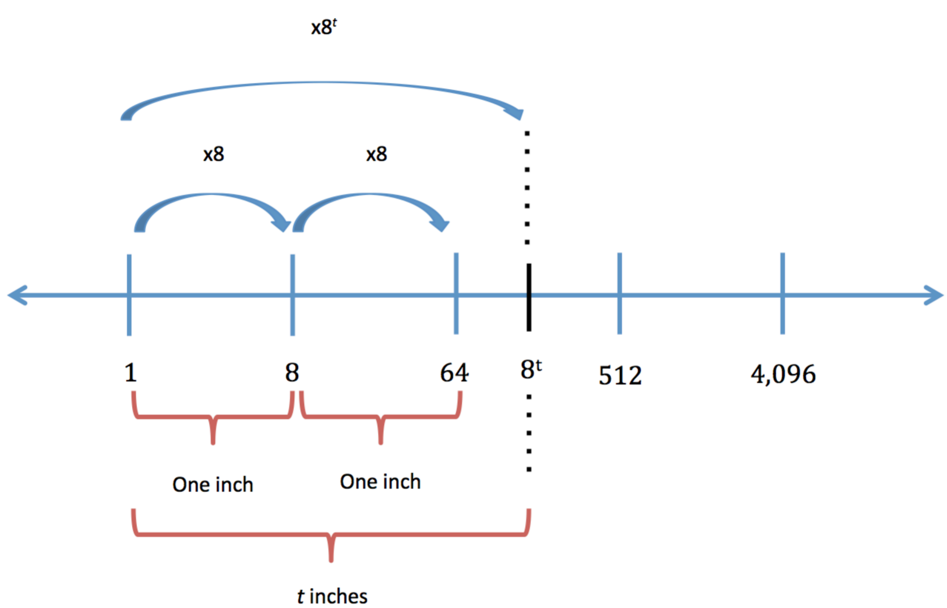


Figure 6.7: Finding "x8t" on the exponential number line.

Understanding this in depth has the potential to build intuitive understanding complex logarithm rules such as the change of base formula that says

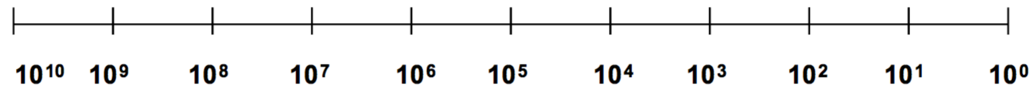
$\log_a x = \frac{\log_b x}{\log_b a}$. If we say a = 2 and x = 8 this rule says $\frac{\log_b 8}{\log_b 2} = \log_2 8 = 3$. In terms of the number line, this equation is simply pointing out that the length of a segment that represents multiplication by a factor of 8 is three times as long as a segment that represents multiplication by a factor of 2. This is intuitively true if your interpretation of the number line is that same sized segments represent multiplication by the same factor, since three segments that each represent multiplication by two would necessarily be a segment that represented multiplication by eight. The interested reader could explore more complicated interpretations of this rule that get into issues of scale, as when b = 2 and a = 8.

Summary. In the first two subsections I have provided conceptual analyses of both the traditional and exponential number line. I then gave arguments for the labeling of the midpoint of 100 and 1,000 on both linear and exponential number lines. From these arguments I developed canonical maps for linear and exponential lines. By comparing these maps I provided evidence for *Claim 1: the length of segments is an important feature of the number lines that students need to attend to make the transition to multiplicative reasoning*.

Claim 2: Subtle Ambiguity in Referring to Lengths

In this section I will provide evidence for *Claim 2: As students subdivided linearly, the terms distance and difference were used somewhat interchangeably and referred to amounts of elapsed years, the length of segments, and the result of subtraction*. As I argued in the previous two sections, this is an important feature of the timeline, especially as students shift from linear to exponential reasoning. As such, this claim may help explain how the discourse failed to support this shift. To support this claim, I will first develop the classroom map that shows the semantic relationships students expressed as they reasoned linearly about subdivisions. I will first analyze episodes where students expressed linear reasoning and present the resultant map. I will then compare the class map to the canonical map to support Claim 2.

Background. On Day 2, the teacher passed out a worksheet that had three problems that encouraged students to think about how to subdivide segments on a timeline. The first asked students to place the Renaissance (500 years ago) on the timeline. The second asked them to place the Oligocene period (3.7 x 10^7 or 37 million years ago). The third was not stated on the worksheet (see Figure 6.8. Note: This is the exact wording of the task, but not the exact formatting). Students had time to work on the first two problems in small group, after which the first problem was discussed in whole class. The teacher then posed the third problem orally, which was to place the Renaissance using the same method they used in the first task, but with different endpoints. The students again had time to work on this in small group and then they discussed it in whole class. On all three of these tasks in both small group and whole class, students subdivided linearly.



1. Place the Renaissance (500 years ago) on the time line above. Describe your reasoning.
2. IF YOU FINISH EARLY: Place the Oligocene epoch (3.7 x 10^7, or 37 million, years ago) on the time line above. Describe your reasoning.
3. (*Question will be posed during class discussion*)

Figure 6.8: Task as Worded in Class: Placing the Renaissance and the Oligocene Epoch

Placing the Renaissance and Oligocene Epoch in Group 1. Rachel, Kathy, Tanya, and Santiago quickly placed the Renaissance without much justification, only saying it goes in the middle of 10^0 and 10^1. Before the task was even passed out, Kathy said, "It's just in the **middle** of **ten to the zero** and **ten to the one**". After they had placed the Renaissance, they moved onto the Oligocene period. It was only when the teacher visited the group and asked them to explain their reasoning did they reflect on their method. As they reflected they realized that 500 would go between 10^2 and 10^3, not between 10^0 and 10^1. They also noticed that it would not go exactly in the middle of the segment. Rachel said, "Oh, that's not right. Because…this is between **a thousand** [points to **10^3** with pen] and **a hundred** [points to **10^2** with pen], so the **difference** in here is **nine hundred** [traces a circle around **the extant between 10^2 and 10^3** with pen]. So **it's [the Renaissance]** not smack dab in the **middle** [points to the **middle**, the current placement of the **Renaissance**]". After this realization, they worked to determine how the midpoint should be labeled. As they worked on this, Tanya, Rachel, and Kathy negotiated whether the Renaissance should be placed to the right or the left of the midpoint, presumably because they were still grappling with whether the midpoint should be labeled 450 or 550 years.

Tanya: Right, so if you're thinking it's **nine hundred years** in that **interval** [puts both hands up like an interval, see Figure 6.9], the **half way point** is **four fifty**.

Rachel: So **it's** [**the placement of the Renaissance**] **past half way.** If you're going this way [gestures with pen from 10^2 to 10^3]

Kathy: **It**'s **less than half** ... because you're **adding one hundred** [to four fifty].

Figure 6.9: Tanya gesturing like an interval

Kathy's idea that the midpoint is 550 because it is 450 plus 100 was accepted and rearticulated later. For example, Rachel said, "**The [midway] point** is not **four fifty**, if it were zero to nine hundred it would be **four fifty**, but we're **starting at 100**, so we need to **add a hundred**."

In this short episode students expressed some of the semantic relationships depicted in the classroom map (Figure 6.11). Rachel LABELED the extant from 10^2 to 10^3 as both nine hundred and a difference when she said, “so the **difference** in here is **nine hundred”** as she circled the extant. Then, Tanya, Rachel, and Kathy expressed semantic relationships relevant to location of the Renaissance. They first established that the midpoint is not 450, because the end of interval is 100, not 0—implying that the midpoint should be labeled 550. This means that 500, the year associated with the Renaissance, would be shifted from the midpoint so its LOCATION was closer to the 10^2 tick mark.

Placing the Renaissance in Group 2. Similar talk occurred in Group 2, consisting of Farah, Samantha, Brittany, and Lacey. As they began working on placing the Renaissance, Brittany began the conversation by saying the midpoint would be 550. She said, “If the **difference** between **ten to the second** and **ten to the third** [has one finger on **10^2** and one on **10^3**] is **nine hundred** [**traces with pen back and forth between two endpoints**], **halfway in between** is going to be like, **five fifty.**” This is consistent with the students’ reasoning in Group 1, although Brittany did not articulate all the steps she took to arrive at 550.

One way in which Group 2’s work differed from Group 1’s, however, is that the students in Group 2 also used the word *distance*, seemingly interchangeably with *difference*. While they were developing their method, Samantha pointed out that “**half** the **distance**” was 450.

Samantha: I found the **distance** [**traces the segment from 10^2 to 10^3**] and I got **nine hundred** [points to a label of **900**

> she has written over a brace she drew over the **segment from 10^2 to 10^3**] and I took **half** of it [traces out an up and down sweeping motion over the vertical line at halfway point with her pen], which is **four fifty**. So that's going to be **half** this **distance** here [points to the halfway point with pen]".

Brittany then clarified that the halfway point would represent 550.

In this quote Samantha expressed semantic relationships involving distance. She talked about the distance as a SYNONYM for the segment from 10^2 to 10^3 by saying "distance" while tracing out the segment and at the same time LABELED the distance 900. She also said she halved the distance, a PROCESS with a RESULT of 450.

Placing the Renaissance in Whole Class. Students continued to use lexical items in ways that were consistent with the relationships they expressed in their small groups as they discussed where the Renaissance should be placed in whole class. This discussion began with a presentation by Ashley and Julia. They showed a vertical number line and Ashley explained that 500 would be the midpoint of 10^2 and 10^3. Lacey immediately challenged her, pointing out the endpoint was 100, not zero. She said the following,

> Lacey: I think it's [the location of the Renaissance] going to be a little bit less than half, because you're not starting at zero years, your starting at a hundred years. So because the distance between is nine hundred, you'll have to add that extra hundred, so your halfway point would actually be five hundred and fifty years.

When the teacher asked her to elaborate at the document camera she continued.

Lacey: I’m thinking it’s actually going to be closer towards the ten to the two tick mark [points to label of 102]. So maybe like here [points slightly above the halfway point, see Figure 6.10]. … Since the distance between ten to the two [points to 102] and ten to the three [points to 103] is nine hundred years [sweeps finger from 102 down the line to 103], half that is four fifty [points to about the midpoint], so then since you're starting at a hundred [points to 102] you have to add that extra hundred [points to 102], to make it, the halfway, five fifty [flops her hand around] so it would be, it’d be fifty less than the halfway point [points slightly above the midpoint].”

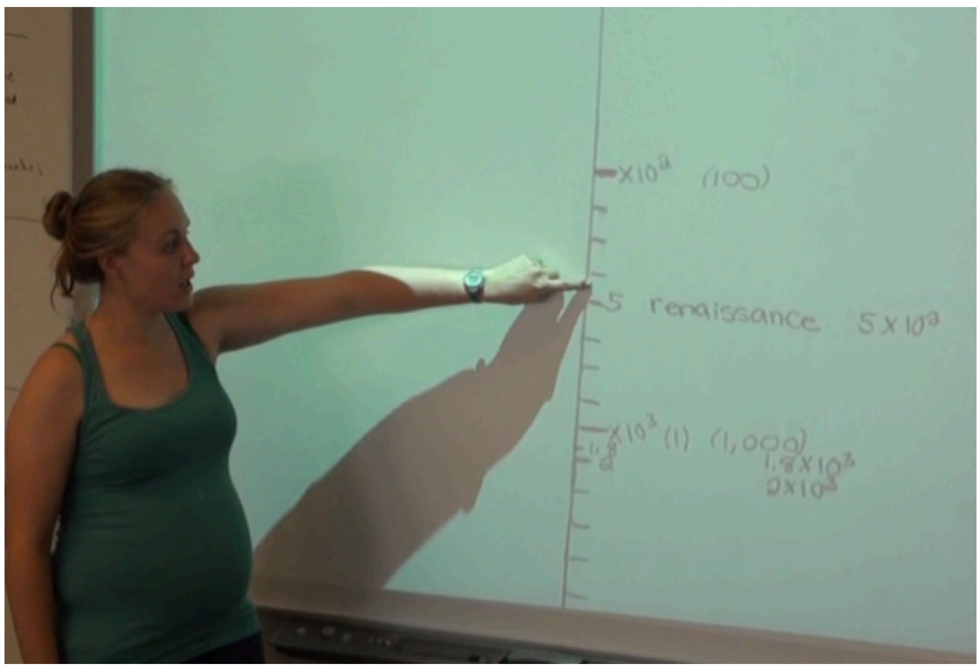


Figure 6.10: Lacey Arguing the Location of the Renaissance Should be Moved.

In her argument Lacey expressed several semantic relationships. For example, she said that the Renaissance should be LOCATED closer to the 10^2 tick mark and said that one needs to add one hundred, a PROCESS, to the 450 to find that 550 is LOCATED at the midpoint. She also referred to the nine hundred years as a distance, which suggests an ENTITY/EXTENT relationship. However, it’s not clear if she is talking about the 900 years as a measure of the segment or as a numerical difference between 1,000 and 100, which would suggest a PROCESS/RESULT relationship. Because of this ambiguity, I have used the more general relationship LABEL/LABELED, to describe the relationship between distance and nine hundred.

Placing the Renaissance Again, but with Different Endpoints. In both groups, the students started by negotiating what they were supposed to do. They determined that the teacher was essentially asking them to ignore the tick marks 10^1 and 10^2. Both groups were able to successfully use the same method as before, reasoning linearly using 1 and 1,000 as the endpoints. They calculated the difference between 1 and 1,000 was 999, divided their result in two to find a quotient of 499.5, and then added that to 1 to find the midpoint represented 500.5 years. They both noticed that this implied that this would shift the Renaissance to about the middle of the segment from 1 to 1,000 rather than near the midpoint of the smaller segment from 100 to 1,000, where it was placed before. They then went through this method interactively whole class. In both whole class and small group, the relationships they expressed were consistent with the relationships they expressed when reasoning about subdividing the segment from 100 to 1,000. For example, consider the following exchange in Group 1.

Rachel:	What's **half** of **nine hundred ninety-nine**?
Santiago:	**Four hundred forty four point five.**
Kathy	**Divide by two**? **Four ninety-nine point five**. But now, since we're starting at **one**, we just need to **add one**, right?
Rachel:	That's what I'm thinking. So we just add one.
Kathy:	So what does that number mean to us though? **Five hundred point five**.
Tanya:	So it would just be to the **left** of the **middle**.
Rachel:	So we need to just go to the…

Tanya:	Teenzy bit **left.**
	…
Rachel:	I didn't plot **nine hundred ninety nine**. I said the **whole thing** is **nine hundred ninety nine** [pulls her hands away from each other in the air].
Kathy:	Okay, what is **this line** [the midpoint]? **Five hundred**?
Tanya:	Well no, the **line** was your **center**, right? [holds flat hand up, perpendicular to the floor]
Rachel:	Which is **five hundred and a half. Five hundred point five**.

Similarly, the relationships expressed in whole class were consistent with those previously expressed.

Teacher:	… What's the **difference** between **one** and **one thousand**?
Several:	**Nine hundred ninety nine**.
Teacher:	**Nine hundred ninety nine** years. Kay. … How do I think about it next? Kaitlyn, how did you think about it next?
Kaitlyn:	We **divided by two**, right?
Teacher:	**Divided** this **distance,** this **number of years** by two and what did you get?
	…
Students:	**Four ninety-nine and a half**.
	…
Teacher:	… I've got **four ninety-nine point five years** that passed, so how many years ago is **right here** [draws an arrow to the midpoint].
Kathy:	**Five hundred point five**.
	…
Teacher:	So where is the Renaissance? Is it to the right or to the left of there?
Several:	To the right.

In both of these instances we see relationships being expressed that are consistent with those expressed in the previous episodes. These relationships are represented in the classroom map (Figure 6.11). There are a few things to note about this map. In some of the blue discs, which represent the lexical items, there are multiple words. There are two reasons why multiple words can be placed in the same disc. First, students may have referred to the same object in different ways. For example, students talked about placing 500 and placing the Renaissance interchangeably. I could have legitimately represented these two items as separate synonymous items, but I chose to simply place them in same disc since students never explicitly expressed this relationship. Second, there are times when different objects functioned the same way in the discourse. For example, both 10^2 and 10^3, and later one and one thousand, had the same role in the arguments that were presented; they functioned as endpoints to a segment. However, I did not simply put the term “endpoint” in the disc because the students did not refer to those points as endpoints.

Also, it worth noting that there were semantic relationships that were expressed, but were later determined to be inappropriate as well as terms that were debated. For example, at one point, students thought that the Renaissance should be LOCATED at the midpoint and then later determined it was close to, but not exactly at the midpoint. I have indicated this overturned relationship with a dotted line. Similarly, students debated whether the Renaissance was to left or to the right of the middle. Eventually, they decided it was to the right of the middle. To indicate this, I colored the disc containing the lexical term “right” orange instead of blue.

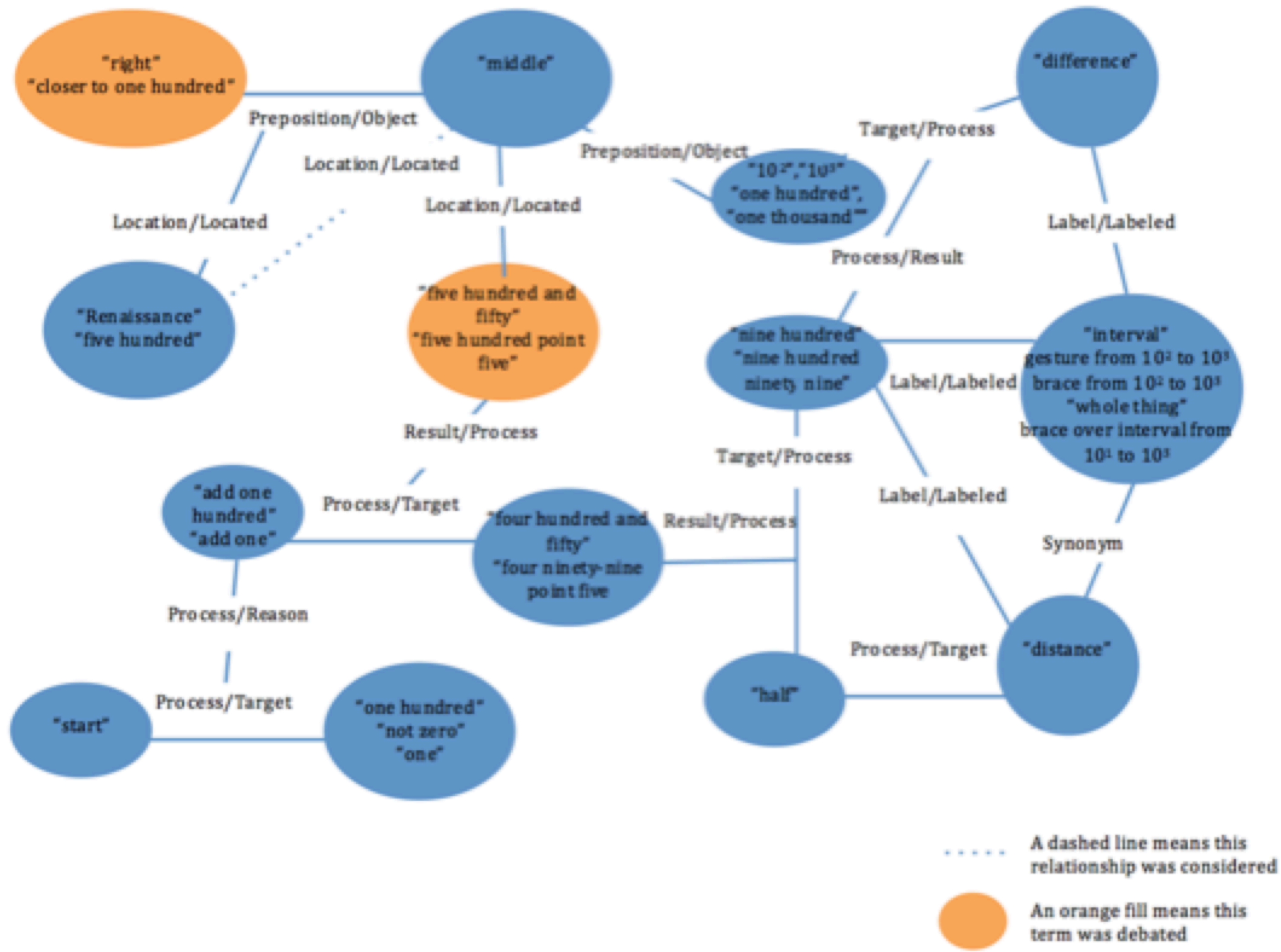


Figure 6.11: Class Map of Linear Method

Comparison of Canonical and Class Maps for Linear Reasoning. In this subsection I will discuss the differences between the canonical and classroom maps which will give evidence for *Claim 2: As students subdivided linearly, the terms distance and difference were used somewhat interchangeably and referred to amounts of elapsed years, the length of segments, and the result of subtraction*. In order to talk about the differences in the maps, I will first introduce some terminology.

Thompson and his colleagues (Smith & Thompson, 2008; Thompson, 1990, 1994) introduced the idea of quantities and values. A quantity is one's conception of a measurable attribute of a situation. It can have an associated numerical value or be conceived without one. For example, in the number line task, the number of years ago the Renaissance occurred is a quantity that has an associated value of 500. Another example of a quantity would be the amount of elapsed years between the invention of zero (1,800 years ago) and the Mayan civilization (1,000 years ago), which has an associated value of 800 years. These examples are quantities that measure attributes of the story context. There are also quantities that measure attributes of the physical number line itself. Examples of these quantities are locations on the line, such as two inches away from the origin, and the length of segments, such as one inch.

As the students presented arguments in class, they did not explicitly discuss the relationship between quantities that measure attributes of the line and those that measure attributes in the problem situation (e.g., number of years ago). In particular, the way students talked about an interval did not distinguish between attributes of the line and attributes of the problem situation. In the class map, the interval was labeled as a "difference" and "nine hundred." Also, the word "distance" was treated as a synonym for the interval. Furthermore, "nine hundred" was also referred to as a "distance". In this way, the terms distance and difference, the numerical difference (e.g., 900), and the segment were all used somewhat interchangeably. This differs from the canonical arguments where relationships are clearly articulated. In the canonical argument the interval is said to represent adding by 900. Furthermore, the length of a segment is clearly articulated as representing "adding by the same amount." In this way, distance (or its equivalent in the canonical argument—length) and differences are distinguished and their relationship is clearly articulated in the canonical argument, but treated somewhat synonymously in the class arguments.

Students may have assumed a natural connection between distance and difference, which could explain why they used both terms to refer to both elapsed years and length and why neither were named explicitly. If the only way to interpret length is as the addition of elapsed years, there is no reason to distinguish it from elapsed years. Thus, students' assumption of linearity supported imprecision in the language in terms distinguishing between length and elapsed years. Reflexively, the imprecision in language perhaps inhibited the consideration of alternative interpretations lengths of segments because it obscured that physical attribute of the line.

It is important to note that I am not necessarily claiming that the students did not understand the relationship between lengths and elapsed years, just that they did not articulate it. In terms of their understanding of the relationship, it is likely that this varied among students and evolved over Days 2 and 3. There was quite a bit of discussion about whether or not the midpoint between 10^2 and 10^3 was 450 or 550, suggesting that for some students this relationship was not completely clear, at least in the beginning. However, even if all students eventually got to the point where they could use linear reasoning accurately to determine the midpoint of segments, this does not mean that they were attending to the length of segments or could articulate what that length represented. In fact, it is unlikely that many students could do this, even at the end, because the discourse supported a conflating of the quantities length and elapsed years since differences and distances were used as synonyms to refer to both quantities.

The other major difference between the two maps is that in the classroom map justification was given for why one adds one hundred. This difference indicates that it would be an oversimplification to say the canonical arguments use greater precision than the class arguments. It is more appropriate to say the arguments differ in where the interlocutors are more precise. This likely has to do with what the participants think need to be justified. As I argued previously, distinguishing between the length of the segment and what the length represents, elapsed years, may be important for students. If students are more aware of their assumptions of what the length of segments represent, that may free them to reassign the meaning.

Exceptions to Claim 2. While in general the relationship between the length of a segment and the elapsed years it represented was implicit, there were times when students at least referred to them as if they were different quantities. For example, during small group Farah **drew a line under the segment from 10^2 to 10^3** and said, "So **this** would be a representation of **nine hundred years**." While Farah still did not specify the attribute of the segment that represents the 900 years or say that the nine hundred years is an elapsed time, she positions the segment as a representation, rather than speaking about the segment and elapsed years interchangeably. However, other students did not distinguish between these two quantities in their speech in future exchanges. Here Farah was talking to Samantha and neither student commented on this shift, nor did it change how Samantha was speaking. In fact, the previous example of Samantha using the word distance to refer to the difference was in response to what Farah said.

A more explicit call to the relationship occurred when Lacey responded to Ashley's presentation. She talked about what the spaces between tick marks represented. She was working with Ashley's representation, which had the segment from 10^2 to 10^3 divided into eight subsections. While explaining, Lacey said, "Each tick mark [**gestures to extant** between small tick marks with two fingers] represents **a hundred years**. Or each space between this tick mark [gestures to 10^2] and this tick mark [points to first small tick mark] is **a hundred years**." So here Lacey specifically referenced the space, an attribute of the line, and said that it represented an elapsed time, 100 years. However, as with Farah, this shift was not commented on nor was it taken up by other students. Instead students continued to focus more on whether the midpoint represented 450 or 550.

During this discussion, the relationships between the lengths of segments and the quantities they represented continued to only be implied, not expressed. For example, consider the following exchange between the teacher and several students.

Teacher:	… And then the **midpoint** [between 10^2 and 10^3] is what?
Many:	**Four-fifty**.
Teacher:	**Four-fifty** what?
Many:	Years

Samantha then clarified that the midpoint does not represent a time of 450 years.

Samantha:	The **four-fifty** is not **the halfway point**; it's half the **distance**.
	…
Teacher:	So she said, who can revoice what she just said? She said **four-fifty** is not the **midway point**, what is it?
Many:	**Half** the **distance**

While Samantha clarified that the midpoint was not 450, presumably because she had a strong understanding of how the four-fifty was represented in the model, she did not articulate how one could see 450 in the model, beyond saying it was half the distance. It is unclear whether she is referring to a literal distance here, referring to the difference of 900, or both at once. This lack of clarity further underscores the point that Lacey's way of speaking did not change the way people spoke in the classroom.

This discussion ended with perhaps the clearest discrimination between lengths and elapsed years when Kathy said, “Isn't there **two midpoints**? Like a distance midpoint, of like where's the **midpoint literally of the distance** [puts two hands up to chunk out a segment] and then there's the **year midpoint** [makes a chopping motion]?" Instead of talking about two thematic items with a particular semantic relationship, Kathy takes a more reflective stance and uses a contrast strategy (see Appendix D of Lemke, 1990) to establish that two thematic elements, the midpoint of the distance and the year midpoint, are in fact separate. Her gesture suggests that by "midpoint literally of the distance" she means the halfway point between the two endpoints. By year midpoint, I assume she is referring to the quantity half the elapsed years. This may have been a turning point for Kathy. On Day 3, she was one of the leaders in the class in putting forth multiplicative ways of reasoning when she said, "we were adding the two halves, but now we need to have like the first half [gestures with thumb and forefinger in segment shape] times the second half [bounces hand with fingers in same position] give us ten to the third [bounces again]." However, it is not clear what sense other students made of her statement.

Claim 3: Using "Factor" to Talk about Addition

In this section I will first provide evidence for the semantic relationships that students expressed as they were using the second method for placing points on the number line, reasoning linearly with the exponents (NWR 2.1). I will then provide the map for the next way of reasoning, multiplicative reasoning (NWR 2.2). After both of these maps are established, I will then compare them to support *Claim 3: Students used a term that suggests multiplicative reasoning, factor, to refer to* *segments as they described additive patterns in the exponents*. The use of the term factor to refer to an additive pattern is significant in that it may help explain how students could listen to arguments that established NWR 2.2, but not realize this was a significantly different way of reasoning than reasoning linearly with the exponents. This may help explain why some students did not shift to thinking multiplicatively.

Background for Episodes of Reasoning Linearly with the Exponents. Linear reasoning with the exponents (NWR 2.1) surfaced at the end of Day 2. After students had placed the Renaissance by linearly subdividing the segment between 10^2 and 10^3, the teacher asked them to use the same method, but use 10^0 and 10^3 as endpoints. Using this reasoning caused the Renaissance to shift position. When students were discussing whether or not it was problematic that the Renaissance moved, which occurred at the very end Day 2 (Thursday), Danna briefly showed an image with a noticeable linear pattern in the exponents (Figure 6.12).

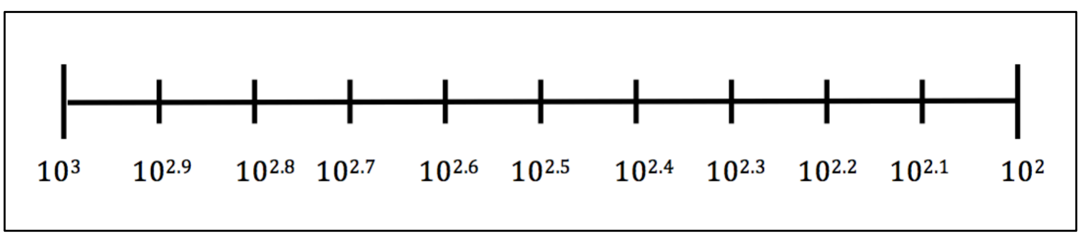


Figure 6.12: Author's recreation of Danna's Work

As discussed in Chapter 4, Danna's explanation seemed impactful for the students as they used this reasoning on their homework over the weekend. The students' homework asked them to explore the relationship between the constant factors shown on the exponential line and additive patterns in the exponents, but students tended to mainly reason about the additive patterns in the exponents and not engage with multiplicative thinking.

This line of reasoning continued through Day 3. At the beginning of Day 3 the teacher showed a number line with endpoints 10^3 and 10^2 and an unlabeled midpoint. The teacher asked for a volunteer to label the midpoint and explain her reasoning. This elicited an explanation from Laruen who used linear reasoning with the exponents to support her claim that it should be $10^{2.5}$. As the day progressed, the class developed multiplicative reasoning, but then returned to linear reasoning when solving the last two problems in class, placing the bow and arrow and placing the Ordovician period. Students' responses to these three activities, labeling the midpoint, placing the bow and arrow, and placing the Ordovician period will be analyzed here to determine the map of semantic relationships for reasoning linearly with the exponents.

Labeling the Midpoint of 10^2 and 10^3. Lacey began by explaining that she found the "**factor**" from 10^2 to 10^3 was 10^1 and "half of that **factor**" was $10^{.5}$. She then added the point five to the two in 10^2 to determine the midpoint should be labeled $10^{2.5}$. She said the following.

Lacey: I got **ten to the two point five**. 'Cause thinking about it, this **whole thing** is a **factor** of **ten to the one** [**points to whole line**], so then if you're going to **divide it in half** it's going to be **point five of that**, so then, you just add the **point five** to the **two** to make this a **factor** of **point five** and this a **factor** of **point five** [points to **the subsection from 10^2 to the midpoint** and then to the **subsection from the midpoint to the 10^3**].

At the teacher's request she marked in the factors she was seeing in the diagram (see Figure 6.13). At this point the teacher asked if others got $10^{2.5}$ and several students raised their hands. During the discussion of her work another student, Mallory, said, "I just ignored the ten".

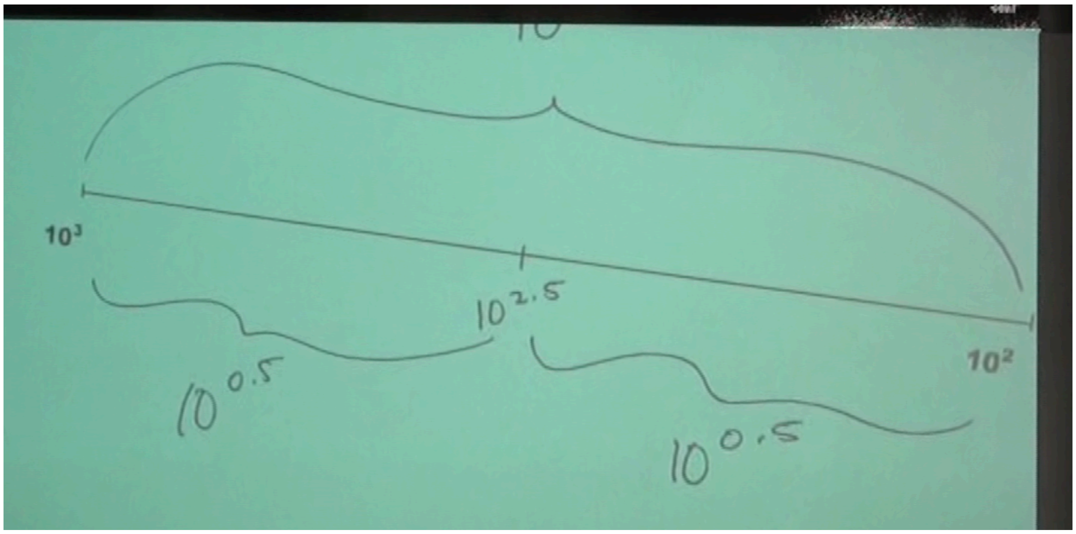


Figure 6.13: Lacey's Labeling of the Factors

Through her pointing and her later drawings of brackets, Lacey LABELED the segment from 10^2 to 10^3 as 10^1 both through inscription and in her speech. Similarly, she also literally and verbally LABELED the subsections as $10^{.5}$. She also talked about the PROCESS of dividing "it" in half, which RESULTED in point five. She then talked about another PROCESS, adding the exponents of point five and two. While she talked about factors in this instance, she seemed to mostly be referring to the exponent. The exponent is what she operated on (i.e. the exponents are what she divided and added) and she even referred to the subsections as "factors of point five," not factors of *ten to* the point five. This focus on the exponents is made even more explicit by Mallory who said she ignored the ten.

Placing the bow and arrow and placing the Ordovician period. The next two tasks in which students used linear reasoning on the exponents came at the end of class, as students placed the bow and arrow and the Ordovician period. Yessica presented her solution to the first problem, placing the bow and arrow. Just like Lacey, she began her explanation by talking about the relationship between the endpoints of the segment. She said, "**From here [10^4] to here [10^5]** is going to be **ten to the one**." She then labeled the segment between 10^5 and 10^4 as "10^1" and wrote "$= 10^{1/2} \cdot 10^{1/2}$." She then divided that segment into two pieces and labeled each of them $10^{1/2}$ and wrote $10^{4.5}$ at the midpoint of the larger segment (see Figure 6.14). The teacher then asked her to explain her labels and she said the following.

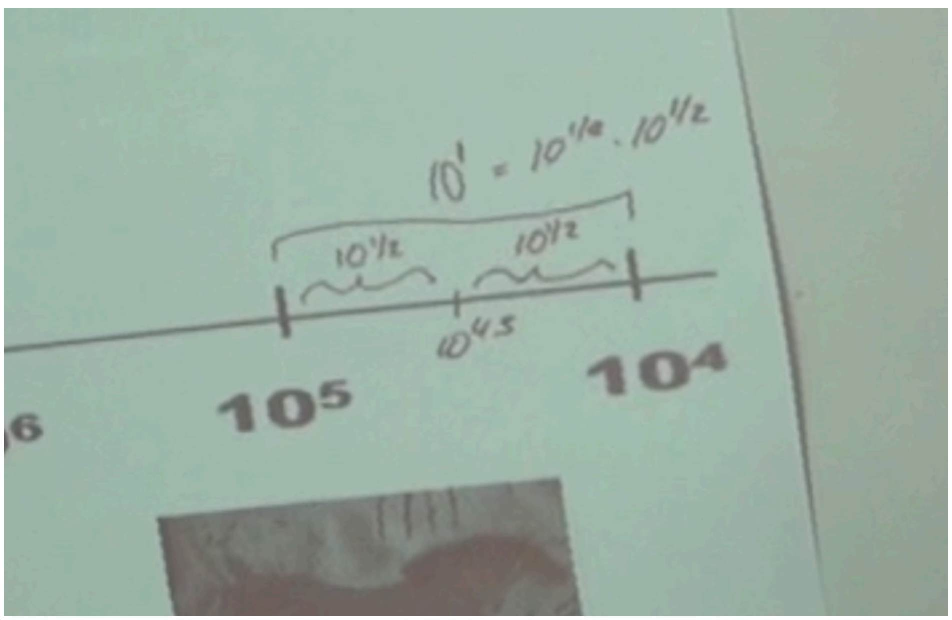


Figure 6.14: Yessica Placing the $10^{4.5}$

Yessica: **This** [points to her label of **10^1**] is going to be the same as **ten to the one half** and **ten to the one half**, so for **here** [points to **10^4**] to **here** [the **midpoint**], it's **ten to the one half**, and the same for here [points to the **subsection from the midpoint to 10^5**], **ten to the one half**, so if we want to get…**ten to the four point five**, so

> we have to **add** just the **one half** from **here** [$\mathbf{10^4}$], so it's **ten to the four** and **the one half** so it's going to be **ten to the four point five**.

Here Yessica expressed very similar semantic relationships as Lacey did during her explanation. Yessica LABELED the segment from 10^4 to 10^5 as 10^1 and each subsection as "**ten to the one half**." She then talked about the PROCESS of adding the point five to the four.

This way of reasoning continued on the last problem, placing the Ordovician period, which occurred $10^{8.7}$ years ago. Jacqueline presented her work at the document camera (See Figure 6.15) and explained how she placed $10^{8.7}$ saying, "Since this **eight point seven**, **I divide the spaces into tenth**, so here's the **middle**, **point five** and then **add two** more….I saw it as **seven tenths**. **Divide it into tenths** and then go **seven tenths**."

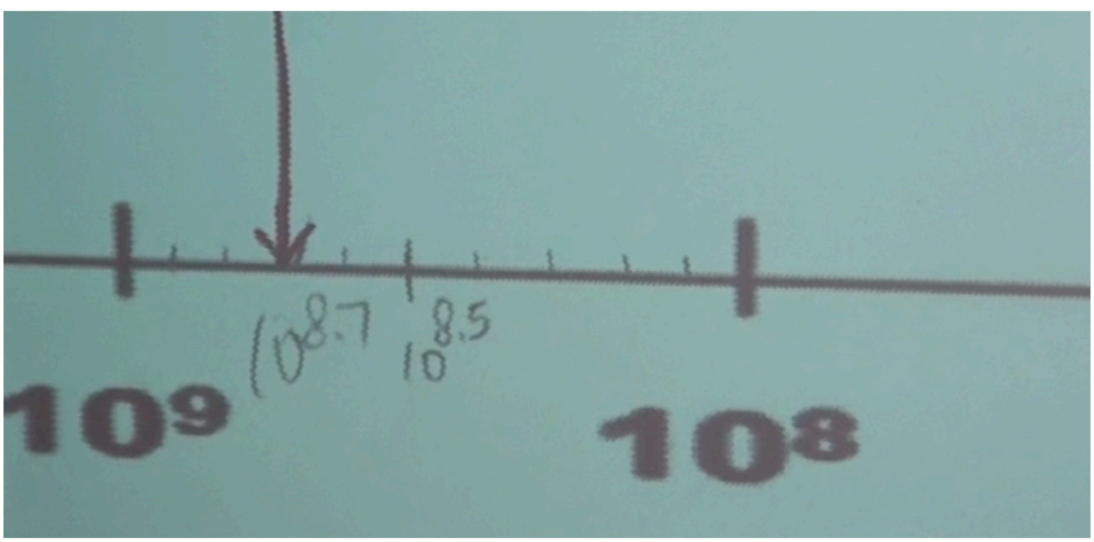


Figure 6.15: Jacqueline Placing $10^{8.7}$

Here, Jacqueline expressed relationships consistent with those expressed by Yessica and Lacey. Like the other two students, she focused on the exponent, talked about the PROCESS of dividing the segment, which seemed to, at the same time, divide the exponent. While Jacqueline did not label the factors or talk about adding the exponents, her labeling of the subdivisions are consistent with the way Yessica and Lacey spoke.

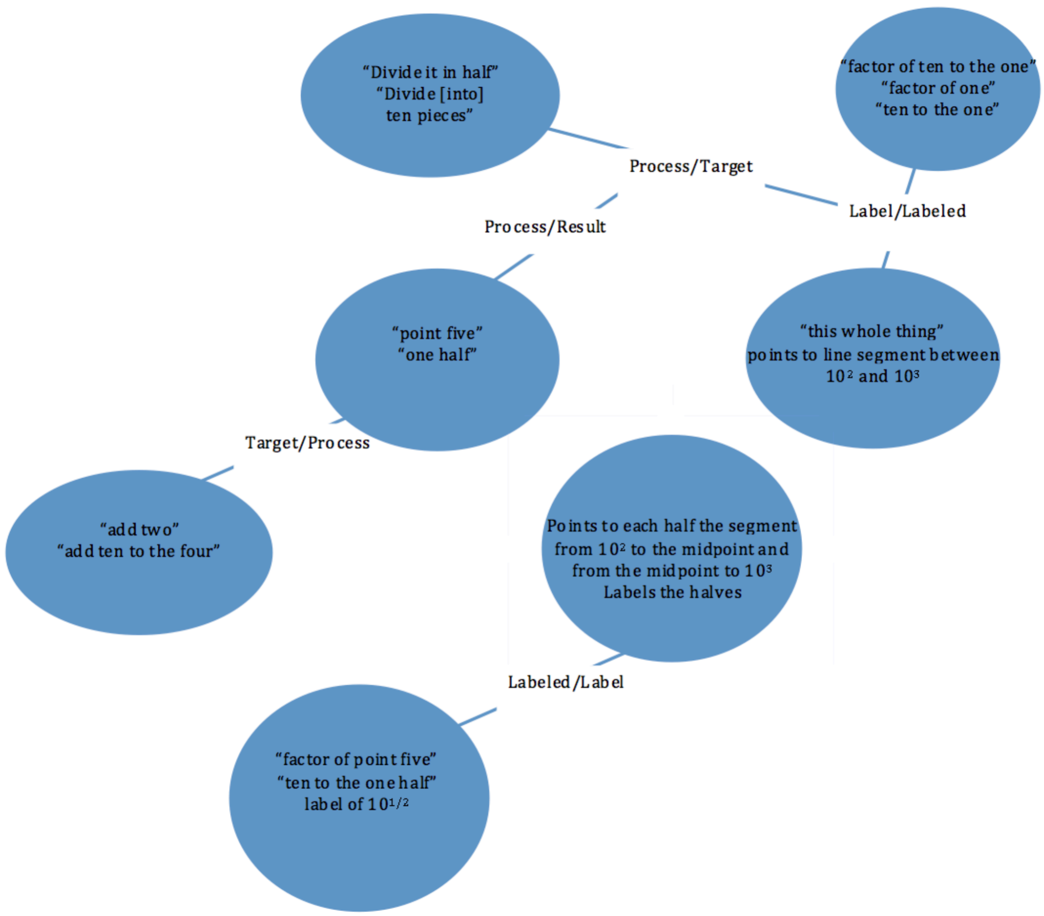


Figure 6.16: The Linear Reasoning with the Exponents Map

There are a few interesting features of this map (Figure 6.16). First, there are two discs that are unconnected to the rest of the lexical items. The disc that represents "point five" and the disc that represents a "factor of point," could be considered the same lexical item. However, I distinguished between these two because of the multiplicative language that is present in "factor of point five," but absent from simply "point five." This suggests that the justification of what the subsections represent are a product of the arithmetic operation of the values of the exponents (i.e. how point five was arrived at) and although it is called a factor, it is not justified by any type of multiplicative thinking.

Overview of Classroom Analysis of Multiplicative Reasoning. In the following subsections I will analyze the three explanations in which students used multiplicative reasoning, describing the semantic relationships they express. From this I will present a map that shows these relationships. Following this section I will present a comparison of the classroom maps that show the relationships expressed when students were reasoning linearly with the exponents and the map developed in this section. This will support *Claim 3: Students used a term that suggests multiplicative reasoning, factor, to refer to segments as they described additive patterns in the exponents*.

Background for Episodes of Reasoning Multiplicatively. The three times multiplicative ways of reasoning were advanced all occurred on Day 3. The first of these instances occurred during the discussion of Lacey's justification for placing $10^{2.5}$ at the midpoint of 10^2 and 10^3. In contrast to Lacey's explanation, which relied on a linear pattern in the exponents, Nathan talked about how to see the factors of the square root of ten as a continuation of the multiplicative pattern that existed at the macro level. The second episode occurred later that day when Kathy also talked about multiplicative relationships as she compared how their timeline had changed from when they were reasoning linearly. Kathy's observation led to an impromptu task, where the teacher asked the students to subdivide a segment of the number line, from 10^2 to 10^3, into seven subsections. Here the students also reasoned multiplicatively.

Nathan's Explanation. Nathan was the first person to bring up multiplicative reasoning. This came during the discussion of Lacey's explanation after Mallory had made her point about ignoring the ten. Unlike Lacey who focused on the exponents, Nathan focused on the "times 10" pattern that existed at the macro level and extended the relationship to the subdivision of segments.

> Nathan: Well, the way I did this one was I was looking at it where, in the more general sense, each **tick** was, each, **each thing apart** on the bigger one is the same **distance**…**multiplicatively** apart, so we're going to do the same thing here. We have two so, we have **two sections** that when **multiplied** all together are **ten**. So, each side we'd had better have the **square root of ten**, because that's the only thing that's gonna give us **ten** when we multiply it again, so I, so I, just took the, I just figured it was, the **distance away** was **ten to the two** and then **times the square root of ten**, which is **three point one six two.** So I got **three point one six two times ten to the two**.

At the core of his explanation seems to be the idea that the macro multiplicative pattern should be extended to within the subsections. He said, "I was looking at it where, in the more general sense…**each thing apart on the bigger one** is the same **distance** multiplicatively apart, so we're going to do the same thing here." I interpret his phrase "in the more general sense" to be referencing the macro structure of the line that was established in Math Practice 1, the times ten pattern between tick marks. His use of the phrase "same distance multiplicatively apart" is more ambiguous. He could be saying that the ticks on the number line are the same distance apart and this shows a multiplicative relationship or he could be using the word distance as a general term to mean any sort of comparison of values, as was done when students used it as synonym for difference. In either case, this represents a shift from reasoning linearly with the exponents as he focuses on the multiplicative relationship between the segments rather than the additive pattern in the values of the exponents.

He continued his line of reasoning to explain how continuing this pattern forces the midpoint to be 10^2 times the square root ten. He said, "We have **two sections** that when **multiplied** all together are **ten**. So, each side we'd had better have the square root of ten, because that's the only thing that's gonna give us ten when we multiply it again". In his explanation, he suggested that the macro multiplicative pattern IMPLIES that each subsection should be the square root of ten, because the subsections are multiplied together to get ten, a PROCESS that RESULTS in ten.

Kathy's Explanation. The next instance of multiplicative reasoning came from Kathy. After students had time to talk in small group about Nathan's way of reasoning the teacher asked Farah to present what they had talked about. Farah's way of reasoning was not multiplicative and will not be examined here. However, it led to a discussion about the meaning of square root. During this discussion, Kathy articulated the difference between linear and the exponential reasoning expressed by Nathan, which led to the following exchange.

Kathy: I was going to say it makes sense to me because when we were doing it like half exponential half linear we were **adding** the **two halves**, but now we need to have like the **first half times** the **second half** gives us **ten to the third.** Before we were doing like **five hundred plus five hundred** needs to give us a **thousand**, but that's linear; and we need to do something **this half times this half** needs to give us **ten to the third**.

Teacher: I think what she's saying is when we were doing it linear we…were **adding these chunks**, but what you really want to do is continue this pattern, it's this **times** something is this **times** something is this **times** something is this. So now we have **ten squared times square root of ten** is **ten to the two point five**, **times square root of ten** is **ten to the third.** Is that what you're saying?

After Kathy confirmed the accuracy of the teacher's summary. Nathan also elaborated on this idea.

Nathan: Well, just to get to is you know, I didn't just have **square root of ten** out of the blue, this was actually my process, my process for this was I realized **the whole thing was ten**, I need a number that I can **multiply by itself** to get ten, oh, right, that's **square root of ten.** That's the only thing that will do that… Because I have **two segments** here so I need one **multiplication** of it.

Teacher: I think this is very powerful. And we're going to do an exercise in a minute to try and emphasize this. Is he saying? Some of you, I think, saw patterns down here [sweeps pen back and forth under the number line]. But he's saying another way to approach it is just say wait, **this** [the whole **segment**] is chunked into **two chunks**. What **factor** times itself gives me **ten**? What, what **number times itself** gives me **ten,** because this has to be **times ten**? And that by definition has to be the square root of ten. Kathy.

In these exchanges several semantic relationships are expressed. Kathy talked about the PROCESS of multiplying the two halves, which the teacher called chunks. Nathan reiterated this, but talked about the repeated multiplication in terms of numbers rather than the actual subsections, "I need a number that I can multiply by itself." The teacher restated Nathan's idea and also talked about number that when multiplied by itself yields ten and also referred to this number as a factor. One difference between Kathy's statement and Nathan's explanation is that she talked about getting ten to the third, whereas Nathan said the RESULT of the multiplication should be ten. These two ideas are consistent with each other; Kathy is focusing on the endpoint whereas Nathan is drawing attention to the factor of increase from 10^2 to 10^3.

Seven Subsections. The teacher's reaction to the conversation was to introduce a new task that focused on multiplicative relationships. She asked, "I want you to tell me, how do you think about the relationship between these **two factors** [Draws an **line between 10^3 and 10^2** and another **line between 10^2 and the first tick mark,** (see Figure 6.17)?" This led to the third and last exchange where a student expressed multiplicative reasoning.

After several people raised their hands, the teacher called on Jade, who started by using linear reasoning, but as the teacher pressed, moved to multiplicative reasoning. She said, "Going from **ten squared** to **ten to the third** you have to **multiply ten squared times ten** to get to ten to the third." The teacher then drew an arrow from 10^2 to 10^3 and labeled it "$x10^1$" (see Figure 6.18). Jade continued, “And since you have **seven subsections**, what I did to the **exponent of ten**, since it's ten to the one, I **divided** that **exponent** by **seven**.” The teacher asked her to come to the document camera where she reiterated her thinking, saying, “Since we have **seven subsections**, each **subsection** would be **ten to the one seventh**,” while labeling the first subsection $10^{1/7}$. At this point, Jade is reasoning linearly with the exponent, but the teacher pressed further, saying “Why? Why is that?...Why is it ten to the one seventh?” This led to multiplicative talk. Jade explained, “Because if we multiply **by ten to the seven ten times** it's going to give us **ten to the one**.” Mallory reiterated Jade’s multiplicative thinking during this discussion when she said, “You need to multiply by the same thing to keep it exponential…. Since there's **seven sections** you have to **multiply the same number seven times**, so that's why it's [the exponent’s] **divided by seven**.”

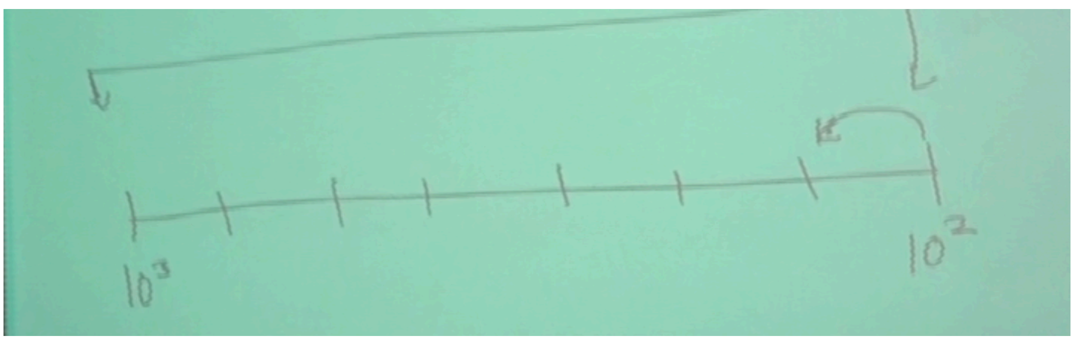
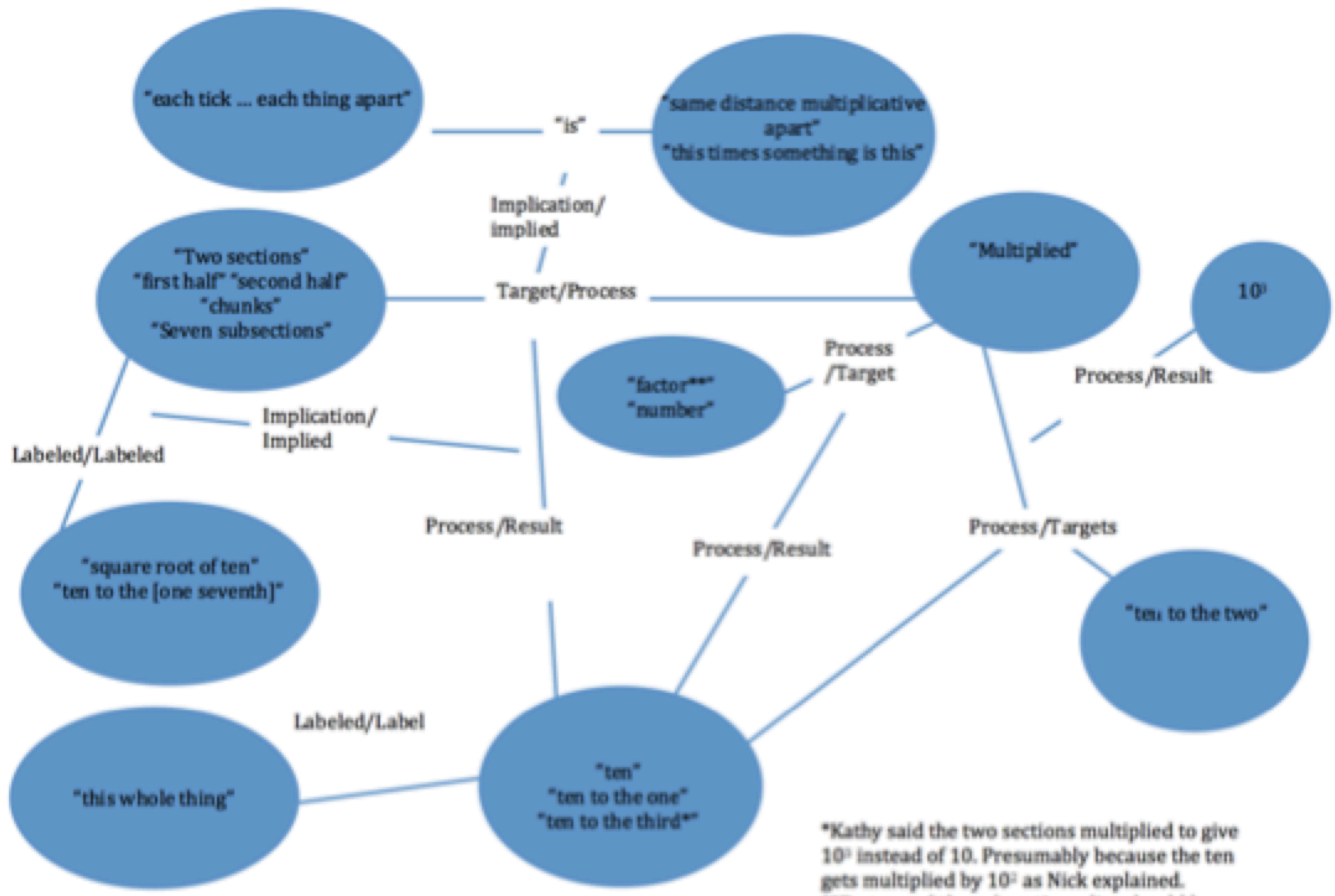


Figure 6.17: Inscription on the Document Camera for the Seven Subsections Task.

In this exchange, Jade LABELS the subsections as ten to the one seventh. Although her original justification focused more on the exponents, with pressing from the teacher, she eventually gave the REASON that repeated multiplication gives ten to the one. Although she originally, inaccurately, said that one would multiply ten times, this is corrected by Mallory who said one would multiply seven times.



*Kathy said the two sections multiplied to give 10^3 instead of 10. Presumably because the ten gets multiplied by 10^2 as Nick explained.
**Factor and the subsections disc should have a labeled/label relationship.

Figure 6.18: The Classroom Map for Multiplicative Reasoning

Comparison of two Classroom Maps for Exponential Reasoning. The maps that represent the semantic relationships expressed while reasoning linearly with the exponents (Figure 6.16) and while reasoning multiplicatively (Figure 6.18) are similar in that they both represent exponential reasoning, which will result in the same number being placed in the same spot. The difference lies in how those placements are justified. In particular, it depends on how students talk about the segments, the subsections, and their operations on those objects. In the map that describes students' talk as they reasoned linearly with the exponents the number line and exponents are divided. This differs from how students talked when they reasoned multiplicatively, where subsections and factors were multiplied. With such a striking difference, it may seem odd that students did not distinguish between the two methods.

One reasonable explanation for why the students did not attend to multiplication as a distinguishing factor is because there was "multiplication talk" in the linear method as well. In particular the students used of the word factor to describe segments. For example, Lacey started her explanation by labeling the segment from 10^2 to 10^3 as a *factor* of 10^1 by pointing to the segment while saying, "**this whole thing** is a **factor** of **ten to the one**." She continued to refer to segments as factors when the teacher asked where she was seeing factors she put braces over the various segments. Consistent with the labeling of segments as factors, she also called each of the subsections "factors of point five." This makes it sound like students are talking multiplicatively, when they are really just reasoning linearly about the exponents. This gives support for *Claim 3: students used a term that suggests multiplicative reasoning, factor, to refer to segments as they described additive patterns in the exponents*. Thus, when true multiplicatively thinking was expressed, students may have interpreted it as the same thing that was said before.

Claim 4: Only Two Ways of Reasoning, Linear and Exponential

In the previous sections, I have analyzed classroom discourse where students used various methods to subdivide segments. However, there were times when students reflected on and talked about these methods themselves. In SFL, this is called condensation (Lemke, 1990), which means a group of semantic relationships are talked about as a single lexical item. In this case, the group of semantic relationships is the network of relationships expressed when subdividing a segment and the condensed version of those relationships is naming the method by which the subdivision took place.

As students talked about these methods, they rarely explicitly established semantic relationships between the methods and other lexical items. Rather, they tended to use equivalence and contrast strategies (see Appendix D in Lemke, 1990) to talk about whether they the saw methods as the same or different. This occurred during four episodes. In two of these episodes they contrasted linear and exponential methods using the strategy of parallel environments. In this strategy speakers place thematic elements, in this case the strategies, so that they have the same function in the grammar of two phrases. For example, students might say something like, in a linear method you do *abc*, but in an exponential method you do *xyz*. In the other two episodes students explicitly said strategies were the same. Analysis of these four episodes will provide evidence for *Claim 4: Students distinguished in their talk between linear and exponential ways of reasoning, but did not distinguish between reasoning linearly with the exponents and multiplicative ways of reasoning. In fact, students seemed to think of both of these methods as the same.* As I analyze these episodes I will bold references to the methods or what the students are comparing the methods to. Students may reference other lexical items, but these will be ignored as this analysis focuses on references to the methods themselves.

Background. Students talked about subdivision methods during both Day 2 and Day 3. The first episode occurred near the end of Day 2, when students were still reasoning linearly to place events. They had already done two tasks, placing the Renaissance and the Ordovician Period, using linear reasoning. Presumably to problematize this way of reasoning, the teacher asked the students to place the Renaissance again, using the same method as before, but using 1 and 1,000 and endpoints instead of 10^2 and 10^3. As she did this, she specifically asked if 500 would end up in the same place. The students worked in small groups for about 15 minutes before they interactively placed the Renaissance as a whole class using these endpoints. This led to the realization that using the different endpoints lead to different placements for the Renaissance. The students then considered the idea that the method they were using to subdivide was problematic. As they talked about their method they contrasted how they subdivided the segments with the macro exponential structure.

Talk about methods continued on Day 3 as well. The day began with Lacey subdividing a segment from 10^2 to 10^3 by reasoning linearly with the exponents. While discussing this task, Nathan introduced a multiplicative way of reasoning. Presumably because the teacher noticed this was a distinct way of reasoning, she asked the students to talk about it in small group. In their small group discussions, the students explicitly talked about Nathan's method as a lexical item. During this second episode of talking about methods, the students said that Nathan's method and Lacey's method were the same.

Later on Day 3, the teacher asked how their method of halving the exponent was different from the linear method they were using before. In this episode, students again contrasted linear and exponential ways of reasoning. Then finally, in the last episode Kathy asked if notation determined the method. In this conversation, students determined that various methods were the same, regardless of how they were notated.

Discussing the Problem of the Renaissance Moving. On Day 2, students were reasoning linearly to place events. At the teacher's request they placed the Renaissance in two ways, both using the same linear method, but with two different sets of endpoints, which resulted in two different placements. This led to students speculating about what the problem was. Some students seemed to think the problem lay with ignoring the 10^1 and 10^2 tick marks rather than with the method itself. For example, Mallory said, "[Using the endpoints of 10^0 and 10^3] only works if you take out the ten to the one and ten to the two." Other students focused more on the linear method itself. These arguments won out eventually and will be what is analyzed here. In these arguments students talked about the method they were using. Brittany started out pointing out that the method could be flawed.

Brittany:	Are you trying to get at **the method** that we're not using, or we're not **using the right method** to plot this point?
Teacher:	What do you think?
Brittany:	Yes

While Brittany introduced the idea that the method might be problematic, she did not offer up an idea to explain why the method might be problematic. This did not come until a bit later when Nathan gave a general rationale. He said the following.

Nathan: Yeah, so ultimately, the issue is it seems like we're trying to apply a **method that's completely linear in nature**, when our **graph is not it's exponential**. That is, that's the problem, so that means that right there, the solution will not work because why would it.

Here is the first instance of parallel environments to contrast the linear method with the exponential graph. Here, Nathan referenced the idea that they were using a “method that’s completely linear in nature” and a “graph… [that’s] exponential.” While Nathan argued that there was mismatch between the nature of the method and the nature of the graph, Danna provided even more detail as to what the problem might be. She argued that a linear placement would not work by using it to show it inaccurately predicts the placement of the known point 10^3.

Danna: I started with first doing basically **what we did up here** [the linear way of reasoning]. So, we looked at the difference between ten to the fourth and ten squared, which was nine thousand nine hundred years and half of that [points to halfway between 10^4 and 10^2] should have been the forty eight or should be, what did I put, four thousand nine hundred fifty, but we know that it's actually ten to the third, so that right there told me **linear** doesn't work and it pushes the halfway mark closer to the ten to the fourth side. So applying the five hundred to this one, I knew it was going to be closer to ten to the three, just `cause, five hundred years it's halfway if it's **linear**, but when it's **exponential**, you know it's not, based on this [points to the number line that she used to argue “linear doesn’t work”]. Then, I realized you can do it this way.

At this point, she put her image of a segment divided into ten sections on the document camera. The subsections were labeled with a noticeable linear pattern in the exponents [Figure 6.19]. The teacher quickly asked her to remove the image from the document camera, which Danna did. The teacher then continued.

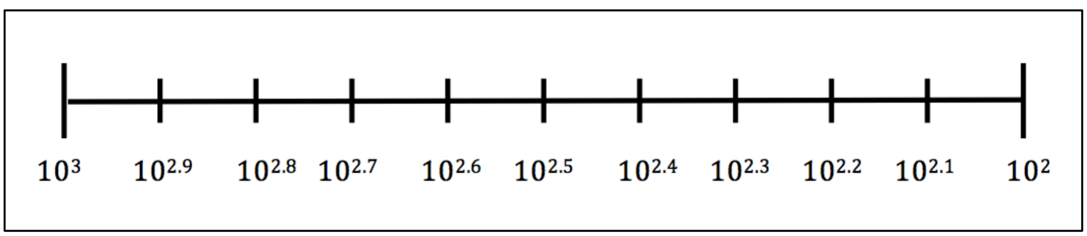


Figure 6.19: Author’s recreation of Danna’s Subdivision

Teacher: ... Instead, in your own words, Danna, what do you think the basic problem is with the methods we were using to place the Renaissance?

Danna: We were trying to look at it **linearly in between each chunk**, but the **entire timeline is exponential**.... So you can't break it up based off of, like on the first one you can't say there's ten hash marks and each mark is one year, it's **an exponential line**.

Teacher: So I'm going to summarize. Thank you, that was really good. Here's what she, I think she's saying is, you guys didn't actually have **a fully exponential time line**. You guys were doing it as **exponential** at this **macro level**, for these **big segments** but in each one you wanted to do **linear**. The problem is, you have to then put a constraint on yourself, that you can't look across different sections to place time lines because that linear inside is going to result in the same time being placed in different parts of your timeline. Which is a problem. There should be one placement for each time period. Right, so, if you have a **linear inside a segment** an **exponential for the segment**, you're going to have problems, you won't get unique placements for any of your times. So our goal is to make **fully exponential timeline**.

After this, the teacher introduced the homework, which the students were to work on over the weekend.

Again, Danna used the strategy of parallel environments to contrast linear and exponential ways of reasoning. She began using the linear method to predict the placement of a year whose placement was known, 10^3. She then extrapolated, “five hundred years, it's halfway if it's **linear**, but when it's **exponential**, you know it's not.” Here Danna used a similar grammatical structure, basically saying if it’s linear, then 500 is halfway, but if it’s exponential, then 500 is not halfway. She did this again when she said, “We were trying to look at it **linearly in between each chunk**, but the **entire timeline is exponential**.” Again, she contrasted linear and exponential ways of reasoning by saying that in each chunk it’s linear, but the structure of the whole timeline is exponential.

Reactions to Nathan’s Ideas. Multiplicative reasoning arose the next day, on Day 3. Nathan originally voiced the idea when the class was talking about Lacey’s placement of $10^{2.5}$ midway between 10^2 and 10^3. He said the following.

Nathan: Well, the way I did this one was I was looking at it where, in the more general sense, each tick was, each,

> each thing apart on the bigger one is the same distance…multiplicatively apart, so we're going to do the same thing here. We have two so, we have two sections that when multiplied all together are ten. So, each side we'd had better have the square root of ten, because that's the only thing that's gonna give us ten when we multiply it again, so I, so I, just took the, I just figured it was, the distance away was ten to the two and then times the square root of ten, which is three point one six two. So I got three point one six two times ten to the two.

After Nathan gave his explantion, the teacher prompted them to think about it in small group. She started by saying, "Nathan said a lot of juicy stuff. He's the first person to bring up square root." She then asked for a student to revoice his idea. When no one volunteered, she asked the students to talk about it in small group—specifically asking, "Where do you see the square root coming up?"

In both small groups, they failed to see the difference between what Nathan said and how they had reasoned about the subdivisions before, reasoning linearly with the exponents. In Group 1, instead of engaging with the ideas of factors and multiplication, Tanya started the discussion of square roots by talking about the exponents. She said, "Well, the exponent one half is the square root right? ... So if it's ten to the two, multiplied by ten to the one half, right? So it's one hundred times the square root of ten." Here we see Tanya following the teacher's prompt to attend to the square root, however Tanya is arriving at the square root in a much different way than Nathan did. Instead of continuing the multiplicative patterns that existed at the macro level, she is arriving at the square root via the previously known rule that $10^{.5}$ is the square root. This allowed her to still preserve her linear ways of reasoning about the exponents, yet explain where the square root is coming from as the teacher asked. This made it so she did not have to distinguish between the two ways of reasoning.

This analysis is consistent with Kathy's comment in small group as well, "[Nathan's way of reasoning is] the **same thing**, because if you're doing ten to the two times ten to the square root that's the same thing as point five." Rachel concurred as she said, "He just thought of it as square root instead of…one half." Kathy summarized by saying, "Yeah, it's the same thing, he just wrote it differently."

In Group 2, the students also continued to focus on exponents. However, instead of engaging with Nathan's idea, they explicitly said they did not understand it and ignored it. Farah said, "Well, I don't understand what he [Nathan] said, but this is how, when she said square root, this is how I thought if it." She then continued with her own idea.

In these small group interactions, the students either explicitly said Nathan's and Lacey's way of reasoning were equivalent, saying "Yeah, it's the same thing," or they ignored Nathan completely saying, "I don't understand what he said." Even though the teacher prompted them to attend to the square root, an idea that was central to Nathan's idea and absent from Lacey's, the students treated this as simply a notational difference. Tanya began by asking "the exponent one half is the square root right?" Rachel echoed this connection when she said, "He just thought of it as square root instead of…one half." This interpretation may have allowed them see

Nathan's idea as simply another expression of Lacey's ideas rather than a new idea worthy of examination.

Discussion About Halving. The next instance where students talked about their method came after Yessica presented the way she placed the introduction of the bow and arrow, which occurred $10^{4.5}$ years ago. She did this by reasoning linearly with the exponents, placing the event halfway between 10^4 and 10^5. After she presented, the teacher asked why it was okay to halve in this method, while it was not okay when they were reasoning linearly.

Teacher:	Now here's my question for you. … I see you halving things. … Why is it working to halve when it did not work when you were using **a halving linear method** for the Renaissance last Thursday? Let me pose my question again. Last Thursday you were using a sort of **halving and a linear approach** of cutting things up to place 500 years ago. Now this has some feeling that feels similar to me. What's similar and what's different?...
Samantha:	This way we're **halving the actual exponential value** and last time we were **solving like ten to the fourth and ten to the fifth and we were halving the, what the answers were to it**. I don't know how to say it…. The ten to the fourth and ten to the fifth you would solve it and you just take half it. Rather than take half of five and four. …
Chris:	… I don't know, but the way I think about it, like exponents and stuff and exponential functions. Like so on the **linear parts** when we were adding together, that's how you add things together in **linear forms**, fashions, but if we want to "add" [airquotes] them together in an **exponential form** you have to multiply them to get that "adding" [airquotes] instead of, 'cause otherwise what we were doing yesterday was like not right.

David:	For the Renaissance … we were looking at a thousand, instead of ten to the three, and so we were taking half a thousand being five hundred and trying to place it on the half line.
Teacher:	And now what are you taking half of?
David:	Now we're just taking half of the, half of the chunk which is ten, ten to the one.
Teacher:	What are you taking half of here? What are you taking half of here?
Students:	The exponent
Teacher:	The exponents, you're not taking half of the thousand or half of ten thousand or one hundred thousand, you're taking half of the exponent.

In this episode, the students again distinguished between linear and exponential ways of reasoning using parallel environments. For example, Samantha said, "This way we're **halving the actual exponential value** and last time we were **solving like ten to the fourth and ten to the fifth and we were halving the, what the answers were to it**." Here, she contrasted the way they were reasoning at that point, "this way," with the way they reasoned previously, "last time." Similarly, Chris compared reasoning linearly and exponentially by comparing how you "add" in each of the situations. He said, "That's how you add things together in **linear forms**, fashions, but if we want to 'add' [airquotes] them together in an **exponential form** you have to multiply them to get that 'adding.'"

Notation. Right after this exchange, Kathy wanted to know if the notation changed the method. In particular, Kathy was asking how to make the whole timeline linear. She wanted to know if simply changing the way the numbers were written, from scientific notation to regular base ten notation, would make the time line linear. This was determined to not be the case.

Kathy: … If we were doing what we were doing on Thursday where, like, if this is a hundred thousand and this is ten thousand [draws a number line from 10,000 to 100,000] and we took one hundred thousand minus ten thousand which is ninety thousand and then split it in half which is forty five thousand, like, is that not an accurate placement? If we change these from, like, scientific notation to number form. … So then if we had to do it, if we did, like, ten thousand to one hundred thousand, those gaps would need to be bigger than a hundred to a thousand? Like what solves the problem? Where we can do it all **linearly**. Lin-e-arly.

Farah: To fix the problem, if we go back to this way, you have to make your graph longer. [Several students say "yeah"] So it, it goes back to the question we had in the homework… If it said, to go from start to finish.

Kathy: You'd need a hundred thousand inches.

Danna: I think part of the confusion also is changing the numbers to this notation does not change the timeline. Just because you're representing ten to the fourth as ten thousand doesn't make them any different. So you can still have that half way point be ten to the four point five in whatever the actual number is and **still have your exponential timeline**. So the numbers don't matter as long as you understand the relationships between them. Which the notation is what tells you the relationships.

Again, the students distinguished between linear and exponential methods, though the contrast is not made as strongly in this episode. The larger point here is that notation does not determine the method. In this way, methods are positioned as independent of notation.

Summary. These four episodes provide evidence for *Claim 4: Students distinguished in their talk between linear and exponential ways of reasoning, but did not distinguish between reasoning linearly with the exponents and multiplicative ways of reasoning. In fact, students seemed to think of both of these methods as the same.* In the first episode, when students discussed the problem of the Renaissance moving, students distinguished between linear and exponential ways of reasoning. They talked about subdividing of the segments as a linear process while the macro structure was exponential in nature. This contrast between linear and exponential came up again, in the third episode I discussed, when students contrasted halving the values on the line with halving the values of the exponents. While this contrast is important, it does not help to disambiguate between the two exponential ways of reasoning. Furthermore, when the students talked about Nathan's multiplicative way of reasoning in small group, they referred to it as the same as Lacey's method, which relied on linear patterns in the exponents. This was again emphasized when Kathy asked about notation and the two exponential methods were positioned as the same. Thus, it is possible that students participating in the class discussion could see the two exponential methods as the same, which leaves little intellectual encouragement for students who can reason successful by focusing on the exponents to adopt multiplicative ways of reasoning.

Discussion

In this chapter I have provided evidence, which together can provide a response to Research Question 2. This research question focused on how the nature of the classroom discourse may have supported the development of the three ways of reasoning exhibited in the post interview by the focus students. In particular, I sought to develop an explanation for how students could participate in a classroom where multiplicative ways of reasoning were developed and accepted by the class community, but not adopt those ways of reasoning as individuals.

Through discourse analysis I discovered that exponential and linear ways of reasoning were contrasted, but reasoning multiplicatively and reasoning linearly with the exponents were not (Claim 4). This may mean that students thought there was primarily two ways of reasoning, a wrong way and a right way—linear reasoning and exponential reasoning. That means that when students heard multiplicative explanations, they may have thought that what they were hearing was no different from reasoning linearly with the exponents, since both were exponential. This is further supported by Claim 3, which suggests that students could hear multiplicative explanations as no different from those that focused on the exponents because in both types of explanations students used multiplicative talk.

Finally, students may not have been well positioned to make the switch to multiplicative reasoning since. Claim 1 suggests that central to this shift is a reinterpretation of the length of segments. While students seemed to implicitly attend to the length of segments as they were reasoning, this attribute of the inscription was not explicitly named or discussed. Rather, students tended to use the terms *distance* and *difference* interchangeably to refer to both quantities that measure attributes of the line and those that measure attributes in the problem situation (Claim 2). Since the students did not disambiguate between these types of quantities and in particular did not disambiguate length and elapsed years, it may have been difficult for students to reassign meaning to the length of segments.

Without reflecting on the meaning of length explicitly, it may have been easier for students to reason linearly with the exponents, as this did not require a reinterpretation of length.

Chapter 7: Discussion

This dissertation study contributed to our understanding of the teaching and learning of exponential and logarithmic relationships on several levels. At a general level, it contributed to researchers' emerging understanding of the relationship between individual ways of reasoning and emergent classroom practices, including an examination of the mathematical meanings constructed through classroom discourse as practices are being established. At a more specific level, it helped create a vision in the research literature for a productive way of teaching students about exponential and logarithmic relationships. In this chapter I talk about these contributions in more detail, as well as acknowledge the limitations of this study and consider directions for future research.

Theoretical Significance

Cobb and Yackel's (1996) emergent perspective gave researchers a framework that outlined the relationship between students' participation in emergent mathematical practices and their evolving conceptions and mathematical activity. They said this relationship was reciprocal. On the one hand mathematical practices arise out of individuals' activity in that individual students' ideas provide the fodder for class discussions, which leads the negotiation and eventual establishment of accepted classroom practices. On the other hand, students' participation in shapes their developing personal conceptions. This articulation contributed greatly to the way researchers conceptualized classroom interactions. However, there are only a limited number of students that have investigated this relationship empirically (Bowers, Cobb, & McClain, 1999; Cobb, 1999; Rasmussen, Wawro, & Zandieh, 2015; Stephan, Cobb, & Gravemeijer, 2003; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014) making the phenomenon not well understood. Both Research Questions in this dissertation contribute to educators' understanding of this process.

In my analysis for Research Question 1 I examined the relationship between the ways of reasoning in *Math Practice 2: Subdividing the Segments*, and individuals' ways of subdividing the segments. I found that in this case, only three of seven students reasoned in a way that was fully consistent with Math Practice 2 on the post interview. Two students relied on reasoning consistent with NWR 2.1: Subdividing Segments by Reasoning Linearly About Exponents. The last two students were still grappling with when to apply linear and exponential reasoning, but had also at some point successfully reasoned linearly with the exponents. Thus, contrary to what previous research suggests (Bowers et al., 1999; Stephan et al., 2003), this study demonstrated that students can continue to reason in ways that are qualitatively different from an established practice, even after instruction has ended. As such, it is important to gain greater understanding of why these different ways of reasoning were allowed to persist.

Part of the answer can be found in the nature of variation in reasoning from the established practice and its relationship to subsequent class activities. In Stephan et al.'s (2003) study, they observed a student who, at times held qualitatively different ways of reasoning from established practices, but eventually reorganized her knowledge. This reorganization seemed to be spurred by continued participation in class in which her reasoning became problematic. In other words, the differences between her way of reasoning and the established practice were significant because her way of reasoning was mathematically problematic. This is in contrast to the difference reported in this study between reasoning linearly with the exponents and Math Practice 2. It is important to note that reasoning linearly with the exponents yields the correct placements of numbers on a number line. Thus, the differences here were significant, not because one way was incorrect, but because students who solely reason linearly with the exponents are missing opportunities to see mathematical connections between the reasoning linearly with the exponents and multiplicative ways of reasoning that were established in Math Practice 2. These mathematical connections are important to understand if the exponential number line is to become a powerful tool to reason about exponential and logarithmic relationships.

Answering Research Question 2, provided further insight into how students could participate in class interactions and not adopt multiplicative ways of reasoning that were established in class. I answered this question by investigating the mathematical meanings constructed in class through detailed analysis of classroom talk. This is analysis is significant because although Cobb and Yackel's (1996) framework offers a broad description of the learning process as it occurs in classrooms, researchers' understanding of the nature of the discourse of classroom interactions that contribute to the emergence of mathematical practices is only beginning to be understood (Rasmussen et al., 2015; Stephan et al., 2003; Tabach et al., 2014).

The analysis for Research Question, in which I explored the semantic relationships expressed in those interactions and thereby examined the constructed mathematical meanings, may help explain why different students reasoned in different ways after participation in classroom mathematical practices. Specifically, that exponential and linear reasoning were more strongly contrasted than the two different ways of reasoning exponentially. This may have resulted in either students not noticing a difference in the two ways of reasoning exponentially or thinking the differences were minor, rather than differentiating the two ways and exploring their relationship. The analysis also revealed that the length of subsections was not explicitly named or discussed. This may be important since students needed to reinterpret the meaning of the length of the subsections to productively engage with NWR 2.2. Thus, it may have been easier for students to reason solely in a way that was consistent with NWR 2.1, which did not require this reinterpretation.

These results are examples of how examining the mathematical content of discursive interactions in which mathematical practices are being established can yield greater insights into the relationship between mathematical practices and individual students' ways of reasoning. In this case, the analysis gave a partial explanatory account of how students could participate the math practice yet end with differing conceptions.

Reflections on the Relationship Between Norms and Mathematical Practices

Gaining greater understanding of the relationship between emergent practices and individual ways of reasoning gave rise to new hypotheses about the relationship between students' engagement in mathematical practices and their engagement in social norms. Often scholars working from the emergent perspective to document the evolution of mathematical practices make note of the norms that were present in the classroom. Specifically, scholars often report students were expected to engage with other students' explanations, including asking questions when explanations do not make sense (e.g. Bowers et al., 1999; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Stephan & Akyuz, 2012). Reporting this norm makes the research more compelling because it implies that the mathematical progress of the classroom was generated through the participation of a variety of students. This makes it more likely that many students advanced their own personal ways of reasoning through participation in the class. This suggests how norms may affect students' engagement with mathematical practices. Namely, that with the proper norms in place, more students are able to intellectually engage with the mathematical practices, which in turn may mean that the established practices are fairly representative of individuals' ways of reasoning. However, this study illustrates the complexity of this relationship in that it provided evidence that students could intellectually engage with class materials, yet end with ways of reasoning that differ from established practices.

Furthermore, this study provided instances that suggest a reciprocal relationship may exist as well. Namely, that students' intellectual engagement with the mathematical practices may affect their participation in social norms. While not the focus of this study, some interactions suggest that students' ways of participating in Math Practice 2 may have affected how they engaged with the social norm of questioning strategies you do not understand. In Chapter 5 I argued that some students could subdivide segments by reasoning linearly with the exponents, but were still grappling with multiplicative reasoning. In other words, their interpretation of *Math Practice 2: Subdividing the Segments* may have been that it was wholly consistent of NWR 2.1. One possible explanation for this is that once they found a way to reason that was sufficient to solve the problems they were given, they did not engage with other ways of reasoning. This was seen most clearly when Farah responded to Nathan's way of reasoning by ignoring it and doing something else (see the small group reaction to Nathan's argument described in Chapter 6). This seems to go against the social norm of asking questions when you do not understand another student's explanation. However, this social norm may not be as straightforward as it seems.

To participate in the social norm of asking questions when you do not understand another student's explanation, students have to monitor their own understanding. Students' personal understanding of the topic likely influences their interpretations of an argument. In the case of Nathan's explanation, Farah was able to articulate that she did not understand what he was saying, but did not feel it necessary to question him. This could be for several reasons. First, Nathan's explanation inspired her thinking of another idea, so it reasonable that she wanted to explore her own idea instead of delving deeply into Nathan's idea. Second, Farah may have, perhaps implicitly, thought of Nathan's idea was essentially the same as NWR 2.1, which she may have already understood. In Chapter 6, I argued students the classroom discourse did not contain strong contrasts between NWR 2.1 and NWR 2.2. This means that as Farah heard Nathan speaking, even though she knew she did not fully understand the details of his argument, she may have interpreted it as consistent with ways of reasoning she already understood. Since, she had another idea it may have seemed more fruitful to explore that idea, rather than take the time to more fully engage with Nathan's idea.

In this way, Farah's understanding of Math Practice 2 may have influenced how she participated in the social norm of asking questions to her peers. This is reasonable since it seems overly onerous for students to make sure they understand every word from every person, especially in an information dense university classroom. Rather, as a student, it seems more important to make sure you understand novel ideas that are presented. These observations are admittedly largely speculatively, but investigating this further in future studies may be fruitful to further advancing the emergent perspective.

Teaching Implications

This dissertation also produced results that have implications for teaching. First, the teaching and learning of exponential and logarithmic relationships has not been thoroughly studied. With the exceptions of the work of Confrey and Smith (Confrey, 1994; Confrey & Smith, 1994, 1995) and Kastberg (2002), research on student thinking about exponential and logarithmic relationships has focused on the mistakes students make (Alagic & Palenz, 2006; Barnes, 2006; Berezovski, 2004; Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013; Davis, 2009; De Bock, van Dooren, Janssens, & Verschaffel, 2002; Hoon, Singh, & Ayop, 2010; Liang & Wood, 2005; Nogueira de Lima & Tall, 2006). As such, the literature is limited in its ability to provide insights into how educators could leverage students' cognitive resources to develop powerful ways of reasoning about exponential and logarithmic relationships. There are a few scholars that have written about this, but the work here largely is hypothetical in that it was not empirically based (Katz, 1986; Van Maanen, 1997; Webb, Kooij, & Geist, 2011; Weber, 2002). The major exception to this was a study reported by Ellis et al. (2015). In this study, the researchers performed a teaching experiment with three students and argued that focusing on the covariation of quantities was helpful in developing powerful ways of reasoning about exponential relationships. This dissertation study also contributes to this area, showing a productive path to develop students' ideas.

This productive path started with the timeline task (Confrey, 1993). Future teachers could use this task and help students notice multiplicative patterns (MP1) and then leverage those patterns to subdivide the segments in an exponential way (MP2 and MP3). As she did so, she should note that students might begin by subdividing segments linearly. Asking students to place the Renaissance may be a way for them to reconsider their initial ways of subdividing. Once students have developed a fully exponential number line, the teacher could then have them look for additive and multiplicative sequences (MP4) and explore the relationships between the sequences. She could then help them find how logarithms are represented on the number line (MP5), which could then serve as a way to reason about and make sense of logarithm rules.

The answer to Research Question 1, that many students' ways of reasoning were qualitatively different from Math Practice 2, also has an implication for teaching. One aspect of Math Practice 2 that made it difficult to encourage students to adopt multiplicative ways of reasoning was that NWR 2.1 provided correct answers. This raises the question of how to encourage students to see the differences between two correct ways of reasoning so that they can distinguish them and then explore their relationships. In Chapter 6, I argued that these two ways of reasoning were not explicitly named or contrasted, which may have contributed to students seeing them as essentially the same strategy. As such, an implication for teachers of this specific unit is to consider asking students to explicitly name and contrast the two strategies. This strategy could also be used in other units as well, but the teacher likely needs to think ahead of time in sufficient detail about what strategies she wants to elicit and what relationships she wants students to talk about.

Thinking at the level of detail necessary to make this move effective may be difficult. In the case of Math Practice 2, the goal would need to go beyond saying she wants students to subdivide exponentially, to thinking about how the multiplicative reasoning relates to the additive reasoning in the exponents. The teacher would then need to think about how to elicit the differences in reasoning. One way this could happen is by pressing students to think about what the lengths of segments represent (multiplication by a factor and addition in the exponents). This may open up a way to differentiate between the methods and talk about their relationship. While this preparation is more obvious in hindsight, it was difficult to see before instruction. This is evidenced by the fact that it was not clear to an experienced teacher working with a team of research associates.

In fact, it is important to note that the teacher in this classroom did many of the things I suggested. For example, I recommended that teachers plan student strategies in detail. The teacher of this course anticipated NWR 2.1 and NWR 2.2 coming up as ways of reasoning. However, she imagined the ideas developing differently, with students first reasoning multiplicatively and then moving to reasoning linearly with the exponents. The fact that things did not proceed as expected complicated the development of ideas in the classroom as the teacher had to try and get students to reason multiplicatively after they already had a way that successfully subdivided the number line exponentially. While she was successful in doing this at the classroom level, interview results suggest that some students did not engage in the multiplicative reasoning. She was more successful in getting students to transition from reasoning linearly to reasoning exponentially. Here she asked students to contrast the two ways of reasoning, which may have helped them see the two ways of reasoning as different. Part of her success in this transition may have been due to the fact that she anticipated this transition would be challenging for her students. She thought ahead of time about how to problematize linear reasoning for her students and developed the Renaissance task. However, the conceptual complexity involved in distinguishing between reasoning linearly with the exponents and reasoning multiplicatively was not as well anticipated. This shows that even though the teacher was thoughtful in her planning, anticipated students strategies at a detailed level, and used discourse moves effectively in the classroom to orchestrate student thinking (such as asking students to name and contrast the linear and exponential methods), there were still some students who did not fully understand the relationship between NWR 2.1 and NWR 2.2. This underscores the point that to some extent, competence in general teaching moves only goes so far in teaching and even highly effective teachers need support garnered through research that illuminates the conceptual difficulties of particular topics and gives insights into how to teach those topics. More of this type of research is needed.

Study Limitations

This study had several limitations. First, the use of clinical interviews to assess students' knowledge is limited. Because of the situated nature of knowledge (Boaler, 1998; Brown, Collins, & Duguid, 1989; Nunes, Schliemann, & Carraher, 1993), students' ways of reasoning may have looked differently if a different problem was posed or if they were interacting in a real world or classroom setting. In particular, students may have had additional knowledge that was not revealed in the interview. In fact this is certainly true as Lacey reasoned linearly about the values on the interview task, but she reasoned exponentially, more specifically linearly with the exponents on a task in class. Had she reasoned this way on the interview task and not considered linear reasoning she would have been placed in a different category. Thus, I am not comfortable claiming that the category the student was placed in based on the interview results represents the full extent of their reasoning.

However, more crucial to the results of this study than categorizing individual students is making sure the categories themselves are meaningful. I believe this is the case. The categories represent the areas of transition that students were still struggling with after instruction. Even though Lacey and Santiago seemed to be able to reason linearly with the exponents, the fact that they reasoned linearly on the interview task suggests that they were still struggling with this transition. Similarly, even if Farah and Brittany were able to reason multiplicatively, they did not see that as essential to an explanation, despite probes for multiplicative reasoning in the interview. Furthermore, it seems very unlikely that Lacey and Santiago were able to reason multiplicatively given how appealing they found linear reasoning. Thus it seems that shifting to multiplicative reasoning was a significant struggle for many students.

Another limitation of this study was the collection of additional data sources that were not used in the analysis of this study, but which may have affected the results. In particular, I conducted a clinical interview with the seven focus students before the unit. This may have altered what students attended to or intellectually engaged with during instruction.

Another limitation of this study is that no one study can fully articulate all possible relationships between emergent mathematical practices and individuals' subsequent ways of reasoning, since this relationship may depend on the nature of the math practice or the setting of the study. It is important to note that I chose to investigate Math Practice 2 because I thought it was conceptually complex and therefore thought meaningful variation was likely to exist. There may be less variation in individuals' ways of reasoning from established practices when the practices are less complex. Similarly, the teacher was special in that she was an experienced teacher and mathematics education researcher. This helps explain her proficiency in anticipating student thinking, engaging students in meaningful tasks, and orchestrating productive mathematical discourse. A less supportive learning environment for students may have affected the relationship between emergent practices and individuals' ways of reasoning, perhaps resulting in more variation in ways of reasoning and potentially a greater prevalence of less productive ways of reasoning. Finally, the university setting may have affected the nature of the relationship. In university courses, there is typically less class time for a given topic than there would be in a high school setting. If students had more time to negotiate ideas as they were developing, this may have resulted in a different relationship between the emergent practices and individuals' ways of reasoning.

Finally, there are likely other explanations for the nature of individual variation besides classroom discourse. Specifically, issues of language and culture likely affect the way students engage in classrooms. Various ways of engaging would likely affect the nature of the relationship between students' individual ways of reasoning and established practices. These issues were not the focus of this study, making the results only a partial explanation for the variation observed.

Future Research

This study contributed to educators understanding of the teaching and learning of logarithms, the nature of the relationship between emergent mathematical practices and individuals' ways of reasoning, and understanding how examining the content do discourse can help provide plausible explanations for the variation in student thinking from established practices. However, much more work is needed in all three of these areas.

First, future studies could investigate further issues of teaching and learning exponential and logarithmic relationships. This study was successful in uncovering difficulties for students and suggesting possible ways to mitigate those difficulties. However, different types of students may have different difficulties. One population of interest could be high school students, since this is where exponential and logarithmic relationships are generally introduced. Rethinking instructional approaches to exponential and logarithmic relationships may be fruitful, given the challenging nature of this topic (Barnes, 2006; Berezovski, 2004; Cangelosi et al., 2013; De Bock et al., 2002; Hoon et al., 2010; Liang & Wood, 2005; Nogueira de Lima

& Tall, 2006). One way researchers could examine the teaching and learning of exponential and logarithmic relationships in high school settings is to use this unit. They could then explore the adaptations needed for a high school setting and whether or not high school students had different difficulties than the college students in this study. Also, ideas might emerge in a different order in another instantiation of this unit, which may affect how students engaged with the ideas. For example, the teacher in this study reported the expectation that multiplicative reasoning would emerge before reasoning linearly with the exponents. In another instantiation of this unit, the ideas might unfold in this way, which may have an affect on students' ability to coordinate the different ways of reasoning.

Second, more research needs to be done to develop educators' understanding of the relationship between emergent mathematical practices and individuals' ways of reasoning. In this study, the students' ways of reasoning were not idiosyncratic interpretations of Math Practice 2. Rather, some students seemed to have found reasoning linearly with the exponents more compelling than multiplicative reasoning and seemed to have difficulty understanding the relationship between these two ways of reasoning. While this is a beginning to researchers' understanding of this relationship, it is not definitive. It may be helpful to examine this relationship under the development different content. The nature of the content may affect what students make of the content and how they interpret emergent practices. It also may be helpful to examine this relationship with different types of students. It is possible that the fact that these students were college math majors may have affected how they participated in the course, which in turn likely affected how their personal ways of reasoning developed.

Third, this study has contributed to our understanding of how discourse can help explain variation in students' thinking from established practices. This could be expanded on in at least two ways. First, this could be broadened in that future studies could examine other aspects of the instructional environment, features of the individual students, or the intersection of those two things to help explain variation. For example, researchers have already argued that discontinuities between home and school cultures can affect learning (Heath, 1982; Labov, 1972; Mejía-Arauz, Rogoff, Dexter, & Najafi, 2007). Future research could make more explicit the connections between the continuity between cultures and the nature of the relationship between emergent practices and individuals' ways of reasoning. Second, future studies could also examine more closely how individuals are interpreting classroom interactions and the content of those interactions. In this study I examined meaning potentials created through discourse, but did not dive into students' personal interpretations of that discourse. To do this type of work, researchers would likely need to work with fewer students and have multiple debriefing sessions with these students as they experienced a unit of instruction. This way the researcher could more easily understand students' experiences in the classroom, how those experiences changes over time, and how those experiences relates to their emerging ways of reasoning.

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