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UNIVERSITY OF CALIFORNIA SAN DIEGO

Absolute Extreme Points of Matrix Convex Sets

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

Eric Evert

Committee in charge:

Professor J. William Helton, Chair Professor Jim Agler Professor Jorge Cortés Professor Jiawang Nie Professor Maurício de Oliveira

2018

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Chair

University of California San Diego

2018

DEDICATION

To Bryana.

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Chapter III, in part, is currently being prepared for the submission for publication of the material. The dissertation author was one of the primary authors and investigators of this material. This material is joint work with J. William Helton.

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ABSTRACT OF THE DISSERTATION

Absolute Extreme Points of Matrix Convex Sets

by

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Professor J. William Helton, Chair

Let $\mathbb{S}(\mathcal{H}_n)^g$ denote g-tuples of self-adjoint operators on a Hilbert space \mathcal{H}_n with dim $\mathcal{H}_n = n$. Given tuples $X = (X_1, \ldots, X_g) \in \mathbb{S}(\mathcal{H}_{n_1})^g$ and $Y = (Y_1, \ldots, Y_g) \in \mathbb{S}(\mathcal{H}_{n_2})^g$, a matrix convex combination of X and Y is a sum of the form

$$V_1^* X V_1 + V_2^* Y V_2 \qquad V_1^* V_1 + V_2^* V_2 = I_n$$

where $V_1 : \mathcal{H}_n \to \mathcal{H}_{n_1}$ and $V_2 : \mathcal{H}_n \to \mathcal{H}_{n_2}$ are contractions. Matrix convex sets are sets which are closed under matrix convex combinations. A key feature of matrix convex combinations is that the *g*-tuples X, Y, and $V_1^* X V_1 + V_2^* Y V_2$ do not need to have the same size. As a result, matrix convex sets are a dimension-free analog of convex sets.

While in the classical setting there is only one notion of an extreme point, there are three main notions of extreme points for matrix convex sets: ordinary, matrix, and absolute extreme points. Absolute extreme points are closely related to the classical Arveson boundary. A central goal in the theory of matrix convex sets is to determine if one of these types of extreme points for a matrix convex set minimally recovers the set through matrix convex combinations.

Chapter II shows the existence of a class of closed bounded matrix convex sets which do not have absolute extreme points. The sets we consider are noncommutative sets, K_X , formed by taking matrix convex combinations of a single tuple X. If X is a tuple of compact operators with no nontrivial finite dimensional reducing subspaces and 0 is in the finite interior of K_X , then K_X is a closed bounded matrix convex set with no absolute extreme points.

In Chapter III, we show that every real compact matrix convex set which is defined by a linear matrix inequality is the matrix convex hull of its absolute extreme points, and that the absolute extreme points are a minimal set with this property. Furthermore, we give an algorithm which expresses a tuple as a matrix convex combination of absolute extreme points with optimal bounds. Similar results hold when working over the field of complex numbers rather than the reals.

Chapter I

Introduction

This dissertation will discuss convexity and extreme points for a type of dimensionfree set called noncommutative (nc) sets. NC sets are sets which contain g-tuples of self-adjoint operators acting on an n dimensional Hilbert space where n ranges over all natural numbers; hence comes the term "dimension-free".

I.1 Background

In the classical setting, a convex combination is a sum of the form

$$\lambda x_1 + (1 - \lambda) x_2 \qquad \qquad 0 \le \lambda \le 1$$

where x_1 and x_2 are elements of a subset K of a vector space V. The set of convex combinations of elements of K is called the convex hull of K, and K is said to be convex if K is equal to its convex hull. An element x of a convex set K is an extreme point of Kif, roughly speaking, x cannot be expressed as a nontrivial convex combination of other elements of K. An important result in the theory of convex sets is the following theorem due to Minkowski: Let $K \subset \mathbb{R}^g$ be a compact convex set. Then K is equal to the convex hull of its extreme points. Furthermore, the extreme points of K are the minimal set with this property.

The natural notion of a convex combination for a dimension-free set must also be dimension-free. These dimension-free convex combinations are called matrix convex combinations, and an nc set which is closed under matrix convex combinations is said to be matrix convex.

A central goal in the study of matrix convex sets is to determine if there is a type of extreme point for matrix convex sets which is minimal with respect to spanning the set through matrix convex combinations. That is, we desire a generalization of Minkowski's classical result to the dimension-free setting. In this dimension-free setting, there are three main types of extreme points: Euclidean (classical), matrix, and absolute extreme points.

The study of matrix convex sets is closely related to the study of completely positive maps on an operator system. Indeed, matrix convex sets are in one-to-one correspondence with sets of completely positive maps on an operator system. Under this correspondence, a matrix extreme point becomes a pure completely positive map [F04] while an absolute extreme point becomes an irreducible boundary representation [KLS14] in the sense of Arveson [A69].

Nearly fifty years ago, Arveson conjectured that, in the infinite dimensional setting, the set of completely positive maps on an operator system is spanned by its irreducible boundary representations [A69]. In our language, Arveson conjectured that infinite dimensional "operator" convex sets are spanned by their infinite dimensional absolute extreme points. Little progress was made on Arveson's conjecture until 2005 when Dritschel and McCullough showed that the set of completely positive maps on any operator system is the span of its (not necessarily irreducible) infinite dimensional boundary representations [DM05]. A decade later, Davidson and Kennedy gave a complete and positive answer to Arveson's original question in the infinite dimensional setting [DK15]. The finite dimensional version of the problem has been pursued for some time but until now has remained unsettled.

The Euclidean and matrix extreme points of a matrix convex set are well understood. Both the Euclidean and the matrix extreme points of a compact matrix convex set are known to span the set through matrix convex combinations [WW99]. However these extreme points do not fulfill satisfactory notions of minimality as spanning sets [A69, F00, F04].

Absolute extreme points are the most restricted type of extreme point for matrix convex sets [KLS14, EHKM18]. The additional restrictions placed on absolute extreme points guarantee that if a matrix convex set is spanned by its absolute extreme points, then the absolute extreme points are a minimal spanning set.

The advantage of absolute extreme points over matrix extreme points or Euclidean extreme points in terms of minimality can be significant. In fact, there are examples of matrix convex sets spanned by their absolute extreme points which have finitely many absolute extreme points (up to unitary equivalence) but infinitely many Euclidean and matrix extreme points, see [EHKM18, Theorem 1.2].

Furthermore, absolute extreme points have computational advantages over Euclidean and matrix extreme points. As an example, determining if a g-tuple of self-adjoint operators acting on an n-dimensional Hilbert space is an absolute extreme point of a "noncommutative semialgebraic" matrix convex set, that is, a matrix convex set which is defined by polynomial inequalities in matrix variables, is equivalent to solving a linear system in ng unknowns. In comparison, checking if such a tuple is a Euclidean extreme point is equivalent to solving a linear system in n(n + 1)g/2 unknowns (see Section III.2.2). No algorithm is known to determine if a tuple is a matrix extreme point of an nc semialgebraic matrix convex set. These nc semialgebraic matrix convex sets are often called free spectrahedra.

This dissertation answers the long-standing open question, "is every closed bounded matrix convex set (free spectrahedron) the matrix convex hull of its absolute extreme points?" In Chapter II we show that the question has a negative answer for general matrix convex sets.

Theorem I.1.1. There exists a compact matrix convex set which has no absolute extreme points.

Chapter III provides a positive answer to the question for free spectrahedra.

Theorem I.1.2. Every compact free spectrahedron which is closed under complex conjugation is the matrix convex hull of its absolute extreme points. Furthermore, the set of absolute extreme points is the minimal set which spans the free spectrahedron through matrix convex combinations.

Both proofs are constructive. The proof of Theorem I.1.1 constructs a class of compact matrix convex sets which do not have absolute extreme points. A specific example of such a set is given in Section II.3. The proof of Theorem I.1.2 provides an algorithm which writes any element of a compact free spectrahedron which is closed under complex conjugation as a matrix convex combination of absolute extreme points of the free spectrahedron.

The remainder of this section introduces our basic definitions and notation and gives precise statements of our main results, Theorem I.4.1, Theorem I.6.1 and Theorem I.7.2.

I.2 Notation and definitions

Let \mathcal{H} be a separable Hilbert space over \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and take $(\mathcal{H}_n)_n$ to be a nested sequence of subspaces of \mathcal{H} such that

$$\dim(\mathcal{H}_n) = n \text{ for all } n \in \mathbb{N} \text{ and } \mathcal{H} = \overline{\cup_n \mathcal{H}_n}$$

where the closure is in norm. For any Hilbert space \mathcal{M} over \mathbb{K} , we use the notation $\mathcal{M}^d = \bigoplus_1^d \mathcal{M}$ where $d \in \mathbb{N} \cup \{\infty\}$. We say an operator on \mathcal{M} is **self-adjoint** to mean it is self-adjoint if $\mathbb{K} = \mathbb{C}$ or symmetric if $\mathbb{K} = \mathbb{R}$. We use $B(\mathcal{M})^g, \mathbb{S}(\mathcal{M})^g$, and $\mathfrak{K}(\mathcal{M})^g$ to denote the sets of g-tuples of bounded operators, bounded self-adjoint operators, and compact self-adjoint operators on \mathcal{M} , respectively. Similarly, given Hilbert spaces $\mathcal{M}_1, \mathcal{M}_2$, we let $B(\mathcal{M}_1, \mathcal{M}_2)^g$ be the set of g-tuples of bounded operators mapping $\mathcal{M}_1 \to \mathcal{M}_2$. Say an operator $U \in B(\mathcal{M})$ is a **unitary** if $U^*U = I_{\mathcal{M}}$. Similarly, an operator $V \in B(\mathcal{M}_1, \mathcal{M}_2)$ is an **isometry** if $V^*V = I_{\mathcal{M}_1}$.

Fix a Hilbert space \mathcal{M} over \mathbb{K} and a subspace $\mathfrak{M} \subset \mathcal{M}$. Say \mathfrak{M} is a **reducing** subspace for an operator $Z \in B(\mathcal{M})$ if \mathfrak{M} is an invariant subspace of both Z and Z^* . Say the tuple

$$Y = (Y_1, \ldots, Y_g) \in B(\mathcal{M})^g$$

is **irreducible** (over \mathbb{K}) if the operators Y_1, \ldots, Y_g have no common reducing subspaces.

Given a g-tuple $Y \in \mathbb{S}(\mathcal{M})^g$ and an operator $W \in B(\mathcal{M})$ we define the **conjugation** of Y by W by

$$W^*YW = (W^*Y_1W, \ldots, W^*Y_aW).$$

If W is a unitary (isometry) then we say W^*YW is a **unitary (isometric) conjugation** of Y.

Given tuples $Y, Z \in B(\mathcal{M})^g$ say Y and Z are **unitarily equivalent**, denoted by $Y \sim_u Z$, if there exists a unitary $U : \mathcal{M} \to \mathcal{M}$ such that

$$U^*YU = (U^*Y_1U, \dots, U^*Y_qU) = Z.$$

I.2.1 Free sets

Matrix convexity is a property possessed by some noncommutative (dimension-free) sets. A **noncommutative set** or **nc set** is a set $\Gamma \subset (\mathbb{S}(\mathcal{H}_n)^g)_{n=1}^{\infty}$ which contains g-tuples of self-adjoint operators acting on \mathcal{H}_n for all positive integers n. Given a noncommutative set Γ and positive integer n, we define the set Γ **at level** n, denoted $\Gamma(n)$, by

$$\Gamma(n) = \Gamma \cap \mathbb{S}(\mathcal{H}_n)^g.$$

That is, $\Gamma(n)$ is the set of g-tuples of self-adjoint operators acting on \mathcal{H}_n which are elements of Γ .

Say an nc set Γ is **closed with respect to direct sums** if for any pair of positive integers $n, m \in \mathbb{N}$ and tuples $Y \in \Gamma(n)$ and $Z \in \Gamma(m)$ we have $Y \oplus Z \in \Gamma(n+m)$ where

$$Y \oplus Z = (Y_1 \oplus Z_1, \dots, Y_q \oplus Z_q).$$

We say Γ is closed under unitary conjugation if $X \in \Gamma$ and $Y \sim_u X$, that is, $Y = U^*XU$ for some unitary U, implies $Y \in \Gamma$.

An nc set $\Gamma \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$ is a **free set** if Γ is closed with respect to direct sums and unitary conjugation. A free set Γ is **bounded** if there is a real number C > 0 such that

$$C-\sum_{i=1}^g X_i^2 \geq 0$$

for every tuple $X \in \Gamma$. We say Γ is **closed** if $\Gamma(n)$ is closed for all $n \in N$ and we say Γ is **compact** if Γ is closed and bounded.

I.3 Matrix convex sets

Let $K \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$. A matrix convex combination of elements of K is a finite sum of the form

$$\sum_{i=1}^{k} V_i^* Y^i V_i \qquad \qquad \sum_{i=1}^{k} V_i^* V_i = I_n$$

where $Y^i \in K(n_i)$ and $V_i \in B(\mathcal{H}_n, \mathcal{H}_{n_i})$ for i = 1, ..., k. If additionally $V_i \neq 0$ for each i, then the sum is said to be **weakly proper**. If K is closed under matrix convex combinations then K is **matrix convex**.

Given a set $K \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$, define the **matrix convex hull** of K, denoted

$$\operatorname{co}^{\mathrm{mat}}K,$$

to be the smallest matrix convex set containing K. Equivalently, $co^{mat}K$ is the set of all matrix convex combinations of elements of K. We emphasize that $co^{mat}K$ is not assumed to be closed.

Every matrix convex set is a free set. Let K be a matrix convex set and let $X \in K$. Notice that U^*XU is a matrix convex combination of X for any unitary U, so it follows that K is closed under unitary conjugation. Now let $\{Y^i\}_{i=1}^k$ be a collection of tuples such that $Y^i \in K(n_i)$ for i = 1, ..., k. Setting

$$V_i = \begin{pmatrix} 0_{n_i \times n_1} & \cdots & 0_{n_i \times n_{i-1}} & I_{n_i \times n_i} & 0_{n_i \times n_{i+1}} & \cdots & 0_{n_i \times n_k} \end{pmatrix}$$

gives

$$\oplus_{i=1}^{k} Y^{i} = \sum_{i=1}^{k} V_{i}^{*} Y^{i} V_{i} \qquad \sum_{i=1}^{k} V_{i}^{*} V_{i} = I.$$

Thus K is closed under unitary conjugation and direct sums, so K is a free set.

Matrix convex combinations can equivalently be expressed via isometric conjugation.

As before, let $\{Y^i\}_{i=1}^k \subset K$ be a finite collection of elements of K and let $\{V_i\}_{i=1}^k$ be a collection of mappings from \mathcal{H}_n to \mathcal{H}_{n_i} such that $\sum_{i=1}^k V_i^* V_i = I_n$. Define the *g*-tuple Y and the isometry V by

$$Y = \bigoplus_{i=1}^{k} Y^{i} \qquad V^{*} = \left(V_{1}^{*} \cdots V_{k}^{*} \right).$$

Then

$$V^*YV = \sum_{i=1}^k V_i^*Y^iV_i \qquad V^*V = \sum_{i=1}^k V_i^*V_i = I_n.$$
(I.3.1)

In words, V^*YV is an isometric conjugation which is equal to the matrix convex combination $\sum_{i=1}^{k} V_i^*Y^iV_i$. A matrix convex combination of the form V^*YV is called a **compression** of Y. As an immediate consequence, a free set is matrix convex if and only if it is closed under isometric conjugation.

In the construction of a matrix convex set which has no absolute extreme points we will often consider matrix convex combinations of a single tuple. In other words, we often consider the case where $Y^1 = Y^2 = \cdots = Y^k$. In this case we use the observation

$$\oplus_{i=1}^k Y^i = (I_k \otimes Y^1)$$

to write equation (I.3.1) in the form

$$V^{*}(I_{k} \otimes Y^{1})V = \sum_{i=1}^{k} V_{i}^{*}Y^{i}V_{i} \qquad V^{*}V = \sum_{i=1}^{k} V_{i}^{*}V_{i} = I_{n}$$

where $V^* = \begin{pmatrix} V_1^* & \cdots & V_k^* \end{pmatrix}$ as before.

I.3.1 Extreme points of matrix convex sets

Let K be a matrix convex set. It is easy to show that the set K(n) is convex for each integer n. Indeed, given positive real numbers $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i = 1$ and tuples $Y^1, \ldots, Y^k \in K(n)$, setting $V_i = \sqrt{\lambda_i} I_n$ shows

$$\sum_{i=1}^{k} \lambda_i Y^i = \sum_{i=1}^{k} V_i^* Y^i V_i \qquad \sum_{i=1}^{k} \lambda_i I_n = \sum_{i=1}^{k} V_i^* V_i = I_n.$$

Since K is matrix convex, it follows that K(n) is convex.

Since each K(n) is convex, it is natural to consider the tuples Y which are extreme points of K(n) in the classical sense. We say $Y \in K(n)$ is a **Euclidean extreme point** of K if Y is a classical extreme point of K(n). As an immediate consequence of Minkowski's result for compact convex sets, every compact matrix convex set is the matrix convex hull of its Euclidean extreme points.

Say $Y \in K(n)$ is an **absolute extreme point** of K if whenever Y is written as a weakly proper matrix convex combination $Y = \sum_{i=1}^{k} V_i^* Z^i V_i$, then for all i either $n_i = n$ and $Y \sim_u Z^i$ or $n_i > n$ and there exists a tuple $\tilde{Z}^i \in K$ such that $Y \oplus \tilde{Z}^i \sim_u Z^i$. We let $\partial^{\text{abs}} K$ denote the set of absolute extreme points of K and we call $\partial^{\text{abs}} K$ the **absolute boundary** of K. We comment that an absolute extreme point X has the property that X_1, \ldots, X_g is an irreducible collection of operators. We omit a formal definition of matrix extreme points as we will make little use of this type of extreme point.

I.4 Matrix convex sets without absolute extreme points

Our first main result is Theorem I.4.1 which gives a class of compact matrix convex sets each of which has no absolute extreme points. Our candidate sets are each noncommutative convex hulls, sets we now define. Let $X \in \mathbb{S}(\mathcal{H})^g$ and for each $n \in \mathbb{N}$ define the set $K_X(n) \subset \mathbb{S}(\mathcal{H}_n)^g$ by

$$K_X(n) = \{ Y \in \mathbb{S}(\mathcal{H}_n)^g | Y = V^* (I_{\mathcal{H}} \otimes X) V \text{ for some isometry } V : \mathcal{H}_n \to \bigoplus_{1}^{\infty} \mathcal{H} \}.$$
(I.4.1)

We then define $K_X \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$ by

$$K_X = (K_X(n))_{n=1}^{\infty}.$$
 (I.4.2)

We call K_X the **noncommutative convex hull** of X.

Given a g-tuple X, we say 0 is in the finite interior of K_X if there exists an integer $d \in \mathbb{N}$ and a unit vector $v \in \mathcal{H}^d = \bigoplus_{i=1}^d \mathcal{H}$ such that

$$v^*(I_d \otimes X)v = 0 \in \mathbb{R}^g.$$

Theorem I.4.1. Let $X \in \mathfrak{K}(\mathcal{H})^g$ be a g-tuple of compact self-adjoint operators on \mathcal{H} and let K_X be the noncommutative convex hull of X. Assume that X has no nontrivial finite dimensional reducing subspaces and assume 0 is in the finite interior of K_X . Then K_X is a compact matrix convex set which has no absolute extreme points.

Proof. The proof of Theorem I.4.1 is given in Section II.2.

I.4.1 Noncommutative convex hulls in relation to matrix ranges

We remark that our notion of the noncommutative convex hull of X is closely related to Arveson's notion of the matrix range of X [A72]. Given a tuple $X \in S(\mathcal{H})^g$ the **matrix range** of X, denoted $\mathcal{W}(X)$, is the set

$$\mathcal{W}(X) = (\mathcal{W}_n(X))_{n=1}^{\infty}$$

where

$$\mathcal{W}_n(X) = \{(\phi(X_1), \dots, \phi(X_g)) | \phi : C^*(X) \to B(\mathcal{H}_n) \text{ is a unital completely positive map} \}.$$

Here $C^*(X)$ denotes the unital C^* -algebra generated by X.

[P02, Theorem 7.4] shows that the matrix range of a tuple is always closed. Furthermore, as a consequence of Voiculescu's Weyl-von Neumann Theorem (e.g. [D96, Theorem II.5.3]), we have $\overline{K}_X = \mathcal{W}_X$ for any $X \in \mathbb{S}(\mathcal{H})^g$. However, K_X may fail to be closed.

As an example, set

$$X = \operatorname{diag}(\alpha/n)_n + \beta I \in \mathbb{S}(\mathcal{H})$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$. Then

$$K_X(1) = (\beta, \alpha + \beta] \subset \mathbb{R}.$$

It follows that K_X is not closed.

In addition to [A72], see [DDSS17] and [PSS18] for further discussion of matrix ranges.

I.5 Free spectrahedra

Free spectrahedra are a class of matrix convex sets which are the solution set of a linear matrix inequality. Every closed matrix convex set is an intersection of free spectrahedra [EW97]. Furthermore, all closed noncommutative semialgebraic matrix convex sets, that is, matrix convex sets which are defined by noncommutative polynomial inequalities in matrix variables, are free spectrahedra [HM12]. Let $d \in \mathbb{N}$ be a positive integer and let

$$A = (A_1, \dots, A_g) \in B(\mathcal{H}_d)^g$$

be a g-tuple of self-adjoint operators on \mathcal{H}_d . A homogeneous linear pencil, denoted by Λ_A , is the map $x \mapsto \Lambda_A(x)$ defined by

$$\Lambda_A(x) = A_1 x_1 + \dots + A_g x_g.$$

The monic linear pencil L_A is the map $x \mapsto L_A(x)$ defined by

$$L_A(x) = I_d + A_1 x_1 + \dots + A_g x_g.$$
(I.5.1)

A linear matrix inequality is an inequality of the form

$$L_A(x) \geq 0.$$

Given a positive integer $n \in \mathbb{N}$ and an $X \in \mathbb{S}(\mathcal{H}_n)^g$, the **evaluation** of the monic linear pencil L_A on X is defined by

$$L_A(X) = I_{dn} + \Lambda_A(X) = I_{dn} + A_1 \otimes X_1 + \dots + A_g \otimes X_g.$$

The **free spectrahedron at level** n, denoted $\mathcal{D}^{\mathbb{K}}_{A}(n)$, is the set

$$\mathcal{D}_A^{\mathbb{K}}(n) = \{ X \in \mathbb{S}(\mathcal{H}_n)^g | L_A(X) \ge 0 \}.$$

The corresponding free spectrahedron is the set $(\mathcal{D}_A^{\mathbb{K}}(n))_n \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$. In other words,

$$\mathcal{D}_A^{\mathbb{K}} = \{ X \in (\mathbb{S}(\mathcal{H}_n)^g)_n | L_A(X) \ge 0 \}.$$

For emphasis, the elements of the **real free spectrahedron** $\mathcal{D}_{A}^{\mathbb{R}}$ are *g*-tuples of real symmetric operators where \mathcal{H} is a real Hilbert space, while the elements of the **complex free spectrahedron** $\mathcal{D}_{A}^{\mathbb{C}}$ are *g*-tuples of complex self-adjoint operators and where \mathcal{H} is a complex Hilbert space.

We say a free spectrahedron $\mathcal{D}_A^{\mathbb{K}}$ is **closed under complex conjugation** if $X \in \mathcal{D}_A^{\mathbb{K}}$ implies

$$\overline{X} = (\overline{X}_1, \dots, \overline{X}_g) \in \mathcal{D}_A^{\mathbb{K}}$$

Note that when $\mathbb{K} = \mathbb{R}$ the real free spectrahedron $\mathcal{D}_A^{\mathbb{R}}$ is trivially closed under complex conjugation. See [HKM13], [Z17], and [K+] for further discussion of linear pencils and free spectrahedra.

Remark I.5.1. Given a tuple $Z \in S(\mathcal{H})^g$, [DDSS17, Proposition 3.1] shows that $\mathcal{D}_Z^{\mathbb{K}}$ is the polar dual of K_Z , where $\mathcal{D}_Z^{\mathbb{K}}$ is the matrix convex set defined by the **monic operator pencil**

$$L_Z(x) = I + Z_1 x_1 + \dots + Z_g x_g.$$

We emphasize that $\mathcal{D}_Z^{\mathbb{K}}$ is seldom a free spectrahedron since Z is not a tuple of finite dimensional operators. Also see [HKM17, Theorem 4.6] for the finite dimensional case and extensions.

I.6 Absolute extreme points span

Our second main result shows that every compact free spectrahedron which is closed under complex conjugation is the matrix convex hull of its absolute extreme points. Furthermore, it shows that the absolute boundary is the smallest set of irreducible tuples which is closed under unitary conjugation and spans the free spectrahedron.

Theorem I.6.1. Assume $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and let $\mathcal{D}_A^{\mathbb{K}}$ be a compact free spectrahedron which is closed under complex conjugation. Then $\mathcal{D}_A^{\mathbb{K}}$ is the matrix convex hull of its absolute extreme points. In notation,

$$\mathcal{D}_A^{\mathbb{K}} = \mathrm{co}^{\mathrm{mat}} \partial^{\mathrm{abs}} \mathcal{D}_A^{\mathbb{K}}.$$

Furthermore, if $E \in \mathcal{D}_A^{\mathbb{K}}$ is a set of irreducible tuples which is closed under unitary conjugation and whose matrix convex hull is equal to $\mathcal{D}_A^{\mathbb{K}}$, then E contains the absolute boundary of $\mathcal{D}_A^{\mathbb{K}}$. In other words,

$$\mathcal{D}_A^{\mathbb{K}} = \operatorname{co}^{\mathrm{mat}} E \quad \Rightarrow \quad \partial^{\mathrm{abs}} \mathcal{D}_A^{\mathbb{K}} \subset E.$$

In this sense the absolute extreme points are the minimal spanning set of $\mathcal{D}_A^{\mathbb{K}}$.

Proof. The fact that $\mathcal{D}_A^{\mathbb{K}}$ is the matrix convex hull of its absolute extreme points follows immediately from the forthcoming Theorem I.7.2.

We now prove the second part of the result. Let $E \subset \mathcal{D}_A^{\mathbb{K}}$ be a set of irreducible tuples which is closed under unitary conjugation and satisfies $\operatorname{co}^{\operatorname{mat}} E = \mathcal{D}_A^{\mathbb{K}}$, and let $X \in \partial^{\operatorname{abs}} \mathcal{D}_A^{\mathbb{K}}(n)$. By assumption $X \in \operatorname{co}^{\operatorname{mat}} E$, so there must exist a finite collection of tuples $\{Y^i\} \subset E$ and contractions $V_i : \mathcal{H}_n \to \mathcal{H}_{n_i}$ such that

$$X = \sum_{i=1}^{\text{finite}} V_i^* Y^i V_i.$$

Since X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{K}}$ and each Y^i is irreducible, we conclude that for each i we have $n_i = n$ and there is a unitary $U_i : \mathcal{H}_n \to \mathcal{H}_n$ such that $U_i^* Y^i U_i = X$. By assumption E is closed under unitary conjugation, so it follows that $X \in E$. \Box

I.7 Dilations to Arveson extreme points

Our third main result is a more quantitative version of Theorem I.6.1. This result will view matrix convex combinations and absolute extreme points from a dilation theoretic perspective and will give a Caratheodory-like bound on the number of terms needed to express a tuple as a matrix convex combination of absolute extreme points.

I.7.1 Dilations

Let $K \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$ be a matrix convex set and let $X \in K(n)$. If there exists a positive integer $\ell \in \mathbb{N}$ and g-tuples $\beta \in B(\mathcal{H}_\ell, \mathcal{H}_n)^g$ and $\gamma \in \mathbb{S}(\mathcal{H}_\ell)^g$ such that

$$Y = \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} = \left(\begin{pmatrix} X_1 & \beta_1 \\ \beta_1^* & \gamma_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & \beta_g \\ \beta_g^* & \gamma_g \end{pmatrix} \right) \in K,$$

then we say Y is an ℓ -dilation of X. The tuple Y is said to be a trivial dilation of X if $\beta = 0$. Note that, if $V^* = \begin{pmatrix} I_n & 0 \end{pmatrix}$, then $X = V^*YV$ with $V^*V = I_n$. That is, X is a matrix convex combination of Y in the spirit of equation (I.3.1).

Given tuples $A \in \mathbb{S}(\mathcal{H}_d)^g$ and $X \in \mathbb{S}(\mathcal{H}_n)^g$ the **dilation subspace of** \mathcal{D}_A at X, denoted $\mathfrak{K}_{A,X}^{\mathbb{K}}$, is defined by

$$\mathfrak{K}_{A,X}^{\mathbb{K}} = \{ \beta \in B(\mathcal{H}_1, \mathcal{H}_n)^g | \ker L_A(X) \subset \ker \Lambda_A(\beta^*) \}.$$

In this definition, \mathcal{H} is a Hilbert space over \mathbb{K} and ker $L_A(X)$ and ker $\Lambda_A(\beta^*)$ are subspaces of \mathcal{H}_{dn} . The dilation subspace is examined in greater detail in Section III.1.1.

I.7.2 Arveson extreme points span

The Arveson boundary of a matrix convex set K is a classical dilation theoretic object which is closely related to the absolute boundary of K. We say a tuple $X \in K$ is an **Arveson extreme point** of K if K does not contain a nontrivial dilation of X. In other words, $X \in K$ is an Arveson extreme point of K if and only if, if

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in K$$

for some tuples $\beta \in B(\mathcal{H}_{\ell}, \mathcal{H}_n)^g$ and $\gamma \in \mathbb{S}(\mathcal{H}_{\ell})^g$, then $\beta = 0$. The set of Arveson extreme points of K, denoted by $\partial^{\operatorname{Arv}} K$, is called the **Arveson boundary** of K. If Y is an Arveson extreme point of K and Y is an $(\ell$ -)dilation of $X \in K$ then we will say Y is an **Arveson** $(\ell$ -)dilation of X.

The Arveson and absolute extreme points of a matrix convex set are closely related. Indeed the following theorem shows that a tuple is an absolute extreme point if and only if it is an irreducible Arveson extreme point.

Theorem I.7.1. Let $\mathcal{D}_{A}^{\mathbb{K}}$ be a free spectrahedron which is closed under complex conjugation. Then $X \in \mathcal{D}_{A}^{\mathbb{K}}$ is an absolute extreme point of $\mathcal{D}_{A}^{\mathbb{K}}$ if and only if X is irreducible over \mathbb{K} and X is an Arveson extreme point of $\mathcal{D}_{A}^{\mathbb{K}}$.

Proof. The original statement and proof of this result is given as [EHKM18, Theorem 1.1 (3)] over the field of complex numbers. A proof for the case where $\mathbb{K} = \mathbb{R}$ is given in Section III.6. We comment that the original statement handles more general complex dimension-free sets; however, this version is well suited to our needs.

Our next theorem shows that every element of a compact free spectrahedron $\mathcal{D}_A^{\mathbb{K}}$ which is closed under complex conjugation dilates to the Arveson boundary of $\mathcal{D}_A^{\mathbb{K}}$.

Theorem I.7.2. Let A be a g-tuple of self-adjoint operators on $\mathbb{S}(\mathcal{H}_d)^g$ and let $\mathcal{D}_A^{\mathbb{K}}$ be a compact free spectrahedron which is closed under complex conjugation. Let $X \in \mathcal{D}_A^{\mathbb{K}}(n)$ with

$$\dim \mathfrak{K}_{A,X}^{\mathbb{K}} = \ell.$$

- 1. There exists an integer $k \leq 2\ell + n \leq 2ng + n$ and k-dilation Y of X such that Y is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{C}}$. Thus, X is a matrix convex combination of absolute extreme points of $\mathcal{D}_A^{\mathbb{C}}$ whose sum of sizes is equal to n + k.
- Suppose X is a tuple of real symmetric matrices, then there exists an integer k ≤ l ≤ ng and k-dilation Y of X such that Y is an Arveson extreme point of D^K_A. Thus, X is a matrix convex combination of absolute extreme points of D^K_A whose sum of sizes is equal to n + k.

As an immediate consequence, $\mathcal{D}^{\mathbb{K}}_{A}$ is the matrix convex hull of its absolute extreme points.

Proof. The proof that $X \in \mathcal{D}_A^{\mathbb{R}}$ dilates to an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$ is given in Section III.1.3. We prove that $X \in \mathcal{D}_A^{\mathbb{C}}$ dilates to an Arveson extreme point of $\mathcal{D}_A^{\mathbb{C}}$ in Section III.3.

We now prove that $\mathcal{D}_{A}^{\mathbb{K}}$ is the matrix convex hull of its absolute extreme points. Let $X \in \mathcal{D}_{A}^{\mathbb{K}}$. The first part of Theorem I.7.2 shows that, in the complex setting, there is an Arveson extreme point $Y \in \mathcal{D}_{A}^{\mathbb{K}}(n+k)$ for some $k \leq 2 \dim \mathfrak{K}_{A,X}^{\mathbb{K}} + n$ such that X is a compression of Y.

The g-tuple Y is unitarily equivalent to a direct sum of m irreducible tuples $\{Y^i\}_{i=1}^m$ for some integer m. These too are Arveson, hence absolute, extreme points, see Theorem I.7.1. Since X is a compression of Y, it follows that X is a compression of $\bigoplus_{i=1}^m Y^i$. Equivalently, there is an isometry $V : \mathcal{H}_n \to \mathcal{H}_{n+k}$ such that $X = V^*(\bigoplus_{i=1}^m Y^i)V$.

Decomposing $V^* = \begin{pmatrix} V_1^* & \cdots & V_m^* \end{pmatrix}$ with respect to the block structure of $(\bigoplus_{i=1}^m Y^i)$ gives

$$X = \sum_{i=1}^{m} V_i^* Y^i V_i \qquad \sum_{i=1}^{m} V_i^* V_i = I_n \qquad \text{with } Y^i \in \mathcal{D}_A^{\mathbb{K}}(n_i) \text{ and } \sum_{i=1}^{m} n_i = n + k.$$
(I.7.1)

That is, X is a matrix convex combination of the absolute extreme points Y^1, \ldots, Y^m .

The proof when X is a g-tuple of real symmetric operators on $\mathbb{S}(\mathcal{H}_n)^g$ is identical with n + k replaced by $n + \tilde{k}$ where $\tilde{k} \leq \dim \mathfrak{K}_{A,X}^{\mathbb{K}}$.

We comment that there are examples of a free spectrahedron $\mathcal{D}_A^{\mathbb{K}}$ and an irreducible tuple $X \in \mathcal{D}_A^{\mathbb{K}}$ and an Arveson dilation Y of X that has minimal size such that Y is reducible.

I.8 Reader's guide

Chapter II discusses the absolute extreme points of general matrix convex sets. The main goal of this chapter is to construct a class of compact matrix convex sets which do not have absolute extreme points. In Section II.1 we show that the noncommutative convex hull K_X is a bounded matrix convex set for any tuple $X \in \mathbb{S}(\mathcal{H})^g$. We then show that such a set is closed provided that X is compact and 0 is in the finite interior of K_X . Section II.2 completes the proof of Theorem I.4.1 by showing that every element of K_X has a nontrivial dilation when X has no nontrivial finite dimensional reducing subspaces. Section II.3 gives an explicit example of a tuple X which satisfies the assumptions of Theorem I.4.1. The chapter ends with Section II.4 which gives an alternative proof of Theorem I.4.1. This proof is done in the original language of Arveson and further discusses the correspondence between matrix convex sets and completely positive maps. In this proof we consider an operator system R_{K_X} such that $CS(R_{K_X})$, the set of unital completely positive maps on \mathcal{R}_{K_X} with finite dimensional range, is matrix affine homeomorphic to the noncommutative convex hull K_X . In particular, we show that there are no maximal unital completely positive maps in $CS(\mathcal{R}_{K_X})$.

In Chapter III, we turn to showing that every compact free spectrahedron which is closed under complex conjugation is the matrix convex hull of its absolute extreme points. Section III.1 introduces the notion of a maximal 1-dilation of an element of a free spectrahedron. The main result of this section is Theorem III.1.3 which implies that Arveson dilations of a tuple X in a real free spectrahedron can be constructed by taking a sequence of maximal 1-dilations of X. This result is then used to prove Theorem 1.7.2when $\mathbb{K} = \mathbb{R}$. In Section III.2 we discuss numerical algorithms for real free spectrahedra. Our first algorithm is Proposition III.2.1 which gives a numerical algorithm that can be used to construct Arveson dilations. The second algorithm we discuss is Proposition III.2.2 which describes a linear system that can be solved to determine if a tuple is an Arveson extreme point of a free spectrahedron. Section III.3 completes the proof of Theorem I.7.2 in the general setting. Section III.4 expands on the historical context of our results for free spectrahedra. Section III.4.1 describes a count on the number of parameters needed to express a tuple as a matrix convex combination of absolute extreme points which is given by Theorem I.7.2. Section III.4.2 compares our results to result for general matrix convex sets, and Section III.4.3 discusses the original terminology and viewpoint of [A69], [DM05], and [DK15]. In the second to last section of the chapter, Section III.5, we discuss the NC LDL^{*} calculation that appears in the proof of Theorem III.1.3. We end with Section III.6 which proves the real analogue of Theorem I.7.1.

Chapter II

Matrix convex sets without absolute extreme points

This chapter will give the construction of a class of compact matrix convex sets which have no absolute extreme points. We briefly recall the main definitions that will be used in the section.

Throughout the chapter let \mathcal{H} be a separable Hilbert space over \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and take $(\mathcal{H}_n)_n$ to be a nested sequence of subspaces of \mathcal{H} such that

 $\dim(\mathcal{H}_n) = n \text{ for all } n \in \mathbb{N} \text{ and } \mathcal{H} = \overline{\cup_n \mathcal{H}_n}$

where the closure is in norm. Let $X \in \mathbb{S}(\mathcal{H})^g$. The noncommutative convex hull of X is the set $K_X = (K_X(n))_n$ where

$$K_X(n) = \{ Y \in \mathbb{S}(\mathcal{H}_n)^g | Y = V^*(I_{\mathcal{H}} \otimes X) V \text{ for some isometry } V : \mathcal{H}_n \to \bigoplus_{i=1}^{\infty} \mathcal{H} \}$$

for each positive integer n. We say 0 is in the finite interior of K_X if there exists an integer

 $d \in \mathbb{N}$ and a unit vector $v \in \mathcal{H}^d$ such that

$$v^*(I_d \otimes X)v = 0 \in \mathbb{R}^g.$$

Equivalently, 0 is in the finite interior of K_X if 0 is in the convex hull of the joint numerical range of X.

II.1 Matrix convex sets which are matrix convex combinations of a compact tuple

Our first objective in constructing a compact matrix convex set without absolute extreme points is to show that noncommutative convex hulls are bounded matrix convex sets and that there are reasonable assumptions which can be made on X such that K_X is closed. The main result of this section is Theorem II.1.5 which shows that K_X is a compact matrix convex set when X is a tuple of compact operators and 0 is in the finite interior of K_X .

II.1.1 Convexity and closedness of K_X

We begin by giving results related to the convexity and closedness of K_X . We first state a lemma which shows that K_X is a bounded matrix convex set for any $X \in \mathbb{S}(\mathcal{H})^g$.

Lemma II.1.1. Let $X \in S(\mathcal{H})^g$ be a g-tuple of self-adjoint operators on \mathcal{H} and let K_X be the noncommutative convex hull of X. Then K_X is a bounded matrix convex set.

Proof. To see K_X is bounded, observe that for any n and any isometry $V : \mathcal{H}_n \to \mathcal{H}^\infty$ we have the inequality

$$\|V^*(I_{\mathcal{H}} \otimes X)V\| \le \|(I_{\mathcal{H}} \otimes X)\| = \|X\|.$$

Furthermore, it is straightforward to show that K_X is closed under direct sums. Since K_X is closed under isometric and unitary conjugation by definition, it follows that K_X is matrix convex.

We now aim to prove that K_X is closed when X is compact and 0 is in the finite interior of K_X . Proposition II.1.2 shows that, with these assumptions, for each fixed n the set $K_X(n)$ can be defined as the set of compressions of a tuple of compact operators and is the key result in proving that K_X is closed.

Proposition II.1.2. Let $X \in \mathfrak{K}(\mathcal{H})^g$ be a g-tuple of self-adjoint compact operators on \mathcal{H} .

- 1. For each $n \in \mathbb{N}$ there exists an integer m_n depending only on n and g such that for each $Y \in K_X(n)$ there is a contraction $W : \mathcal{H}_n \to \mathcal{H}^{m_n}$ such that $Y = W^*(I_{m_n} \otimes X)W$.
- 2. Assume 0 is in the finite interior of K_X . Then there exists an integer m_0 depending only on g such that for each $Y \in K_X(n)$ there is an isometry $T : \mathcal{H}_n \to \mathcal{H}^{m_n + nm_0}$ such that $Y = T^*(I_{m_n + nm_0} \otimes X)T$. In particular we have

$$K_X(n) = \{Y \in \mathbb{S}(\mathcal{H}_n)^g | Y = T^*(I_{m_n + nm_0} \otimes X)T \text{ for some isometry } T : \mathcal{H}_n \to \mathcal{H}^{m_n + nm_0}\}.$$
(II.1.1)

Before giving the proof of Proposition II.1.2, we state two lemmas which will be useful in the proofs of Proposition II.1.2 and Theorem II.1.5. The first lemma is a convergence argument, while the second lemma shows that, when 0 is in the finite interior of K_X , contractive conjugation can be replaced with isometric conjugation.

Lemma II.1.3. Let $m, n \in \mathbb{N}$ be positive integers. Let $Z \in \mathfrak{K}(\mathcal{H}^m)^g$ be a g-tuple of compact self-adjoint operators on \mathcal{H}^m and let $\{W^\ell\}_{\ell=1}^{\infty} \subset B(\mathcal{H}_n, \mathcal{H}^m)$ be a sequence of contractions which converges in the weak operator topology on $B(\mathcal{H}_n, \mathcal{H}^m)$ to some operator $W \in B(\mathcal{H}_n, \mathcal{H}^m)$. Then the sequence $\{W_\ell^* Z W_\ell\}_{\ell=1}^{\infty} \subset \mathbb{S}(\mathcal{H}_n)^g$ converges in norm to $W^* Z W \in$ $\mathbb{S}(\mathcal{H}_n)^g$. *Proof.* Since W^*ZW and $W_{\ell}^*ZW_{\ell}$ for all ℓ are all tuples of finite dimensional operators, it is sufficient to show that for each i = 1, ..., g and for all $h \in \mathcal{H}_n$ we have

$$\lim \| \langle (W_{\ell}^* Z_i W_{\ell} - W^* Z_i W) h, h \rangle \| = 0.$$
 (II.1.2)

To this end, fix $h \in \mathcal{H}_n$ and observe that

$$\lim \|\langle (W_{\ell}^{*}Z_{i}W_{\ell} - W^{*}ZW)h, h \rangle\| \leq \lim \|\langle W_{\ell}^{*}Z_{i}(W_{\ell} - W)h, h \rangle\| + \lim \|\langle (W_{\ell}^{*} - W^{*})Z_{i}Wh, h \rangle\|.$$
(II.1.3)

Note that the sequence $\{W_{\ell}^*\}$ converges to W^* in the strong operator topology since these operators map into a finite dimensional space. This implies that

$$\lim \|\langle (W_{\ell}^* - W^*) Z_i W h, h \rangle \| \le \|h\| \lim \| (W_{\ell}^* - W^*) (Z_i W h) \| = 0.$$
(II.1.4)

To handle the remaining term in equation (II.1.3) note that $Z_i W_\ell$ converges strongly to $Z_i W$ since Z_i is compact. Also note that $\sup_{\ell} ||W_{\ell}^*|| \leq 1$ since the W_ℓ are contractions. Using these facts we have

$$\lim \| \langle W_{\ell}^* Z_i (W_{\ell} - W) h, h \rangle \| \le \| h \| \lim \| W_{\ell}^* \| \| (Z_i W_{\ell} - Z_i W) h \| = 0.$$
(II.1.5)

Combining equations (II.1.4) and (II.1.5) shows $\lim W_{\ell}^* Z_i W_{\ell} = W^* Z_i W$ for $i = 1, \ldots, g$. \Box

Lemma II.1.4. Let $X \in \mathfrak{K}(\mathcal{H})^g$ be a g-tuple of compact self-adjoint operators on \mathcal{H} and assume 0 is in the finite interior of K_X . Then there exists an integer m_0 depending only on g such that, given any integers $m, n \in \mathbb{N}$ and any contraction $W : \mathcal{H}_n \to \mathcal{H}^m$, there exists an isometry $T: \mathcal{H}_n \to \mathcal{H}^{m+nm_0}$ such that

$$W^*(I_m \otimes X)W = T^*(I_{m+nm_0} \otimes X)T.$$

Proof. By assumption 0 is in the finite interior of K_X . It follows that there is an integer $m_0 \in \mathbb{N}$ and an isometry $Z_0 : \mathcal{H}_1 \to \mathcal{H}^{m_0}$ such that $Z_0^*(I_{m_0} \otimes X)Z_0 = 0$. This implies that for each *n* there is an isometry $Z_{0_n} : \mathcal{H}_n \to \mathcal{H}^{nm_0}$ such that $0_n = Z_{0_n}^*(I_{nm_0} \otimes X)Z_{0_n}$. Define $T : \mathcal{H}_n \to \mathcal{H}^{m+nm_0}$ by

$$T = \begin{pmatrix} Z_{0_n} (I_n - W^* W)^{\frac{1}{2}} \\ W \end{pmatrix}.$$
 (II.1.6)

Then T is an isometry and $T^*(I_{m+nm_0} \otimes X)T = W^*(I_m \otimes X)W$, and the integer m_0 depends only on g.

We now prove Proposition II.1.2.

Proof. Given $Y = V^*(I_{\mathcal{H}} \otimes X)V \in K_X(n)$ where $V : \mathcal{H}_n \to \mathcal{H}^\infty$ is an isometry, let $P_\ell : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection of \mathcal{H} onto \mathcal{H}_ℓ . Since X is compact, the sequence $P_\ell X P_\ell$ converges to X in norm, and $I_{\mathcal{H}} \otimes P_\ell X P_\ell$ converges to $I_{\mathcal{H}} \otimes X$ in norm. Defining

$$Y^{\ell} = V^* (I_{\mathcal{H}} \otimes P_{\ell} X P_{\ell}) V, \tag{II.1.7}$$

it follows that $\lim Y^{\ell} = Y \in \mathbb{S}(\mathcal{H}_n)^g$.

Observe that

$$Y^{\ell} = ((I_{\mathcal{H}} \otimes \iota_{\ell}^{*})V)^{*}(I_{\mathcal{H}} \otimes (\iota_{\ell}^{*}X\iota_{\ell}))((I_{\mathcal{H}} \otimes \iota_{\ell}^{*})V)$$

for all ℓ where $\iota_{\ell} : \mathcal{H}_{\ell} \to \mathcal{H}$ is the inclusion map. Defining the quantities

$$X^{\ell} = \iota_{\ell}^* X \iota_{\ell} \in B(\mathcal{H}_{\ell})^g \quad \text{and} \quad V_{\ell} = ((I_{\mathcal{H}} \otimes \iota_{\ell}^*)V) \in B(\mathcal{H}_n, \mathcal{H}_{\ell}^{\infty}),$$

we have $V_{\ell}^*(I_{\mathcal{H}} \otimes X^{\ell})V_{\ell} = Y^{\ell}$ and V_{ℓ} is a contraction for all ℓ .

Fix $d, \ell \in \mathbb{N}$ and a g-tuple $A \in \mathbb{S}(\mathcal{H}_d)^g$. It is straightforward to show

$$L_{Y^{\ell}}(A) = (V_{\ell}^* \otimes I_d) (I_{\mathcal{H}} \otimes L_{X^{\ell}}(A)) (V_{\ell} \otimes I_d).$$

It follows that $L_{Y^{\ell}}(A) \geq 0$ if $L_{X^{\ell}}(A) \geq 0$. Therefore $\mathcal{D}_{X^{\ell}} \subseteq \mathcal{D}_{Y^{\ell}}$ for all $\ell \in \mathbb{N}$.

Using [HKM12, Theorem 1.1] we conclude that there is an integer m_n depending only on n and g such that for each $\ell \in \mathbb{N}$ there is a contraction $W'_{\ell} : \mathcal{H}_n \to \mathcal{H}_{\ell}^{m_n}$ such that

$$(W'_{\ell})^* (I_{m_n} \otimes X^{\ell}) W'_{\ell} = Y^{\ell}.$$
 (II.1.8)

For each ℓ define the operator $W_{\ell}: \mathcal{H}_n \to \mathcal{H}^{m_n}$ by

$$W_{\ell} = (I_{m_n} \otimes \iota_{\ell}) W_{\ell}'. \tag{II.1.9}$$

Then each W_{ℓ} is a contraction and

$$Y^{\ell} = W_{\ell}^{\star} (I_{m_n} \otimes X) W_{\ell}$$

for all $\ell \in \mathbb{N}$.

By passing to a subsequence if necessary, we may assume that the W_{ℓ} converge to some contraction $W : \mathcal{H}_n \to \mathcal{H}^{m_n}$ in the weak operator topology on $B(\mathcal{H}_n, \mathcal{H}^{m_n})$. By assumption X is compact, so $I_{m_n} \otimes X$ is compact. Using Lemma II.1.3, it follows that

$$W^*(I_{m_n} \otimes X)W = \lim W^*_{\ell}(I_{m_n} \otimes X)W_{\ell}.$$

It follows that $Y = W^*(I_{m_n} \otimes X)W$, which completes the proof of item (1).

Item (2) is immediate from Lemma II.1.4 and the assumption that 0 is in the finite
interior of K_X .

We now prove K_X is a compact matrix convex set.

Theorem II.1.5. Let $X \in \mathfrak{K}(\mathcal{H})^g$ be a g-tuple of self-adjoint compact operators on \mathcal{H} and let K_X be the noncommutative convex hull of X. Assume that 0 is in the finite interior of K_X . Then K_X is a compact matrix convex set.

Proof. Lemma II.1.1 shows that K_X is bounded and matrix convex. It remains to show that $K_X(n)$ is closed for each n. Let $\{Y^\ell\} \subset K_X(n)$ be a sequence of elements of $K_X(n)$ converging to some g-tuple $Y \in \mathbb{S}(\mathcal{H}_n)^g$. By Proposition II.1.2 there exists a fixed integer m_n depending only on n and g and contractions $W_\ell : \mathcal{H}_n \to \mathcal{H}^{m_n}$ such that $W_\ell^*(I_{m_n} \otimes X)W_\ell = Y^\ell$ for all ℓ . By passing to a subsequence if necessary, we can assume that the W_ℓ converge in the weak operator topology to some contraction $W : \mathcal{H}_n \to \mathcal{H}^{m_n}$. By assumption X and $I_{m_n} \otimes X$ are compact, so Lemma II.1.3 shows that

$$Y = W^* (I_{m_n} \otimes X) W.$$

Furthermore, we assumed 0 is in the finite interior of K_X , so using Lemma II.1.4 there exists an integer m_0 depending only on g and an isometry $T : \mathcal{H}_n \to \mathcal{H}^{m_n + nm_0}$ such that

$$T^*XT = W^*XW = Y.$$

We conclude $Y \in K_X(n)$ and $K_X(n)$ is closed.

Remark II.1.6. Using Theorem II.1.5 with Voiculescu's Weyl-von Neumann Theorem (e.g. [D96, Theorem II.5.3]) it follows that K_X is equal to the matrix range of X when X is a g-tuple of compact operators and 0 is in the finite interior of K_X .

II.2 Absolute extreme points of noncommutative convex hulls

We are almost in position to prove Theorem I.4.1. We first give a lemma that describes the reducing subspaces of a direct sum of a fixed g-tuple with itself.

Lemma II.2.1. Let \mathcal{H} be an infinite dimensional Hilbert space and let $X \in \mathbb{S}(\mathcal{H})^g$ be a g-tuple of self-adjoint operators on \mathcal{H} . Assume that every nontrivial reducing subspace of X is infinite dimensional. Then, for any integer $N \in \mathbb{N}$, every nontrivial reducing subspace of $I_N \otimes X \in \mathbb{S}(\mathcal{H}^N)^g$ is infinite dimensional.

Proof. Fix an integer $N \in \mathbb{N}$ and let $W \subset \mathcal{H}^N$ be any reducing subspace for $I_N \otimes X$. For each $n = 1, \ldots, N$ define $\mathcal{M}_n = \mathcal{H}$. Then $\mathcal{H}^N = \bigoplus_{n=1}^N \mathcal{M}_n$. For each $n = 1, \ldots, N$ let $\iota_n : \mathcal{M}_n \to \mathcal{H}^N$ be the inclusion map of \mathcal{M}_n in \mathcal{H}^N . Given $v \in W$, define $v_n \in \iota_n^* W$ by $v_n = \iota_n^* v$ for all $n = 1, \ldots, N$. Then v can be written $v = \bigoplus_{n=1}^N v_n$. Observe that $(I_N \otimes X_i)v = \bigoplus_{n=1}^N (X_i v_n)$ for all $i = 1, \ldots, g$. Since W is a reducing subspace for $I_N \otimes X$, it follows that $\bigoplus_{n=1}^N (X_i v_n) \in W$ for all $v \in W$ and all $i = 1, \ldots, g$. Fix an $n_0 \in \{1, \ldots, N\}$. Then applying $\iota_{n_0}^*$ to both sides of the equality we find

$$X_i v_{n_0} = \iota_{n_0}^* \left(\bigoplus_{n=1}^N \left(X_i v_n \right) \right) \in \iota_{n_0}^* W$$

for all $v_{n_0} \in \iota_{n_0}^* W$ and all i = 1, ..., g. Since X is a tuple of self-adjoint operators, this shows that $\iota_{n_0}^* W$ is a reducing subspace for X for all $n_0 \in \{1, ..., N\}$. As X was assumed to have no nontrivial finite dimensional reducing subspaces, it follows that either $\iota_n^* W = \{0\}$ or $\iota_n^* W$ is infinite dimensional for each n = 1, ..., N. If $\iota_n^* W = \{0\}$ for all n = 1, ..., N, then $W = \{0\}$. If $\iota_n^* W \neq \{0\}$ for any n, then W is infinite dimensional.

We now complete the proof of Theorem I.4.1.

Proof. Theorem II.1.5 shows that K_X is a compact matrix convex set, so we only need to show $\partial^{abs}K_X = \emptyset$. Let $Y \in K_X(n)$. Using Proposition II.1.2, there is an integer m_n depending on n and g and an isometry $V: \mathcal{H}_n \to \mathcal{H}^{m_n}$ such that

$$Y = V^* (I_{m_n} \otimes X) V.$$

It follows that there are tuples $\alpha \in B(\mathcal{H}^{m_n} \ominus \mathcal{H}_n, \mathcal{H}_n,)^g$ and $\beta \in \mathbb{S}(\mathcal{H}^{m_n} \ominus \mathcal{H}_n)^g$ and a unitary $U \in B(\mathcal{H}^{m_n})$ such that

$$U^*(I_{m_n} \otimes X)U = \begin{pmatrix} Y & \alpha \\ \alpha^* & \beta \end{pmatrix} \in \mathbb{S}(\mathcal{H}^{m_n})^g.$$
(II.2.1)

Furthermore, Lemma II.2.1 shows $I_{m_n} \otimes X$ has no nontrivial finite dimensional invariant subspaces, so $\alpha \neq 0$.

Since $\alpha \neq 0$, there is a unit vector $v \in \mathcal{H}^{m_n} \ominus \mathcal{H}_n$ such that $\alpha v \neq 0$. Let $W : \mathcal{H}_{n+1} \to \mathcal{H}^{m_n}$ be the isometry

$$W = \begin{pmatrix} I_n & 0\\ 0 & v \end{pmatrix}.$$

Then

$$W^*U^*(I_{m_n} \otimes X)UW = \begin{pmatrix} Y & \alpha v \\ v^*\alpha^* & v^*\beta v \end{pmatrix} \in \mathbb{S}(\mathcal{H}_{n+1})^g$$

where $\alpha v \neq 0$. Additionally, $W^*U^*(I_{m_n} \otimes X)UW \in K_X$ since UW is an isometry. Using [EHKM18, Theorem 1.1 (3)], it follows that $Y \notin \partial^{abs}K_X$.

II.3 Examples

The following section gives an explicit example of a tuple $X \in \mathfrak{K}(\mathcal{H})^2$ of compact operators with no nontrivial finite dimensional reducing subspaces so that 0 is in the finite interior of K_X . Throughout the section set $\mathcal{H} = \ell^2(\mathbb{N})$ and $\mathcal{H}_n = \ell^2(1, \ldots, n) \subset \mathcal{H}$ for all $n \in \mathbb{N}$. Given a weight vector $w = (w_1, w_2, ...) \in \mathbb{R}^{\infty}$ define the weighted forward shift $S_w : \mathcal{H} \to \mathcal{H}$ by

$$S_w v = (0, w_1 v_1, w_2 v_2, \dots)$$

for all $v \in \mathcal{H}$. Additionally, for each $n \in \mathbb{N}$, let $\mathfrak{I}_n : \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$\mathfrak{I}_n v = (v_1, \ldots, v_n, 0, 0, \ldots)$$

for all $v \in \mathcal{H}$.

Proposition II.3.1. Let $X_1 = \text{diag}(\lambda_1, \lambda_2, ...)$ where the λ_i nonzero real numbers converging to 0 with distinct norms and let S_w be a weighted shift where $w \in \mathbb{R}^{\infty}$ is chosen so $w_i \neq 0$ for all i and S_w is compact. Set

$$X_2 = S_w + S_w^*.$$

Then there exist real numbers α_1, α_2 so that the tuple

$$\tilde{X} = (X_1 + \alpha_1 \mathfrak{I}_2, X_2 + \alpha_2 \mathfrak{I}_2)$$

is a tuple of compact self-adjoint operators with no finite dimensional reducing subspaces and so that 0 is in the finite interior of $K_{\tilde{X}}$.

Before giving the proof of Proposition II.3.1, we state a lemma which describes the closed invariant subspaces of a diagonal operator.

Lemma II.3.2. Let $X = \operatorname{diag}(\lambda_1, \lambda_2, ...)$ where the λ_i nonzero real numbers converging to 0 with distinct norms, and let W be a closed invariant subspace of X. Then $W = \bigoplus_{j \in J} E_j$ for some index set $J \subset \mathbb{N}$ where E_j denotes jth coordinate subspace.

Proof. If $W = \{0\}$ then the proof is trivial, so assume $W \neq \{0\}$. Define

$$J = \{j \in \mathbb{N} | \text{ there exists a vector } v = (v_1, v_2, \dots) \in W \text{ so that } v_j \neq 0 \}.$$

Since $W \neq \{0\}$ we have $J \neq \emptyset$. We will show $W = \bigoplus_{j \in J} E_j$.

By assumption the λ_i have distinct norms and converge to zero so there is a unique index $j_0 \in J$ such that $|\lambda_{j_0}| = \max_{j \in J} |\lambda_j|$. Choose a vector $v \in J$ so that $v_{j_0} \neq 0$. Then

$$\lim_{\ell \to \infty} \frac{X^{\ell} v}{\lambda_{j_0}^{\ell}} = v_{j_0} e_{j_0}$$

Therefore $e_{j_0} \in W$ since W is closed.

Since E_{j_0} and W are closed invariant subspaces of X with $E_{j_0} \subset W$, it follows that $W = E_{j_0} \oplus W'$ where W' is a closed invariant subspace of X. Proceeding by induction completes the proof.

We now prove Proposition II.3.1.

Proof. We first prove the existence of the real numbers α_1, α_2 so that 0 is in the finite interior of K_X . Let $\iota_2 : \mathcal{H}_2 \to \mathcal{H}$ be the inclusion map of $\mathcal{H}_2 \to \mathcal{H}$. Since $\lambda_1 \neq \lambda_2$, there exists a unit vector $v_0 \in \mathcal{H}_2$ so that, setting

$$\alpha_1 = -\langle \iota_2^* X_1 \iota_2 v_0, v_0 \rangle,$$

the eigenvalues of

$$X_1 + \alpha_1 \mathfrak{I}_2 = \operatorname{diag}(\lambda_1 + \alpha_1, \lambda_2 + \alpha_1, \lambda_3, \lambda_4, \dots)$$

are nonzero real numbers with distinct norms.

 Set

$$\alpha_2 = -\langle \iota_2^* X_2 \iota_2 v_0, v_0 \rangle.$$

Then

$$v_0^*(\iota_2 X_1 \iota_2 + \alpha_1 I_2, \iota_2 X_2 \iota_2 + \alpha_2 I_2)v_0 = 0 \in \mathbb{R}^2.$$

Setting $\tilde{X}_i = X_i + \alpha_i \Im_2$ for i = 1, 2, it follows that

$$v_0^*(\iota_2^* \tilde{X}_1 \iota_2, \iota_2^* \tilde{X}_2 \iota_2) v_0 = 0 \in \mathbb{R}^2.$$
(II.3.1)

Therefore, 0 is in the finite interior of $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$.

It is clear that \tilde{X} is a tuple of compact self-adjoint operators, so it remains to show that \tilde{X} has no finite dimensional reducing subspaces. Assume towards a contradiction that W is a finite dimensional reducing subspace for \tilde{X} . Then W must be a closed invariant subspace of \tilde{X}_1 . Recall that α_1 was chosen so that \tilde{X}_1 is a diagonal operator whose diagonal entries are real numbers that converge to 0 with distinct norms. Using Lemma II.3.2, it follows that $W = \bigoplus_{j \in J} E_j$ for some finite index set $J \subset \mathbb{N}$ where E_j denotes *j*th coordinate subspace.

Let j_0 be the largest integer in J. Since S_w is a weighted forward shift with nonzero weights, it is straightforward to see that

$$\tilde{X}_2 E_{j_0} = (S_w + S_w^* + \alpha_2 \mathfrak{I}_2) E_{j_0} \notin W.$$

This shows that W cannot be a reducing subspace of \tilde{X} . Therefore \tilde{X} has no finite dimensional reducing subspaces.

II.4 Alternate proof of Theorem I.4.1

We now give an alternate proof of Theorem I.4.1. This proof is accomplished by considering an operator system \mathcal{R}_{K_X} such that the set of unital completely positive maps on \mathcal{R}_{K_X} with finite dimensional range, denoted $CS(\mathcal{R}_{K_X})$, is matrix affine homeomorphic to K_X . We show that $CS(\mathcal{R}_{K_X})$ has no maximal completely positive maps. We begin with a brief collection of definitions related to completely positive maps and operator systems.

II.4.1 Completely positive maps

We will assume the reader's familiarity with operator systems and completely positive maps. For a comprehensive discussion of these subjects see [P02, Chapter 3].

Let \mathcal{R} be an operator system and let $\phi : \mathcal{R} \to B(\mathcal{M}_1)$ be a unital completely positive map for some Hilbert space \mathcal{M}_1 . A **dilation** of ϕ is a unital completely positive map of the form $\psi : \mathcal{R} \to B(\mathcal{M}_2)$ such that \mathcal{M}_2 is a Hilbert space containing \mathcal{M}_1 and $\iota^*_{\mathcal{M}_1}\psi(r)\iota_{\mathcal{M}_1} = \phi(r)$ for all $r \in \mathcal{R}$. Here $\iota_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_2$ is the inclusion map of \mathcal{M}_1 into \mathcal{M}_2 . A unital completely positive map ϕ is called **maximal** if any dilation ψ of ϕ has the form $\psi = \phi \oplus \psi'$ for some unital completely positive ψ' .

We use the notation

$$CS_n(\mathcal{R}) = \{ \phi : \mathcal{R} \to B(\mathcal{H}_n) | \phi \text{ is u.c.p.} \}$$

to denote the set of unital completely positive maps sending \mathcal{R} into $B(\mathcal{H}_n)$ and we define

$$CS(\mathcal{R}) = (CS_n(\mathcal{R}))_n$$

to be the set of unital completely positive maps on \mathcal{R} with finite dimensional range.

II.4.2 Matrix affine maps

We now introduce and briefly discuss the notion of matrix affine maps on a matrix convex set. We direct the reader to [WW99, Section 3] for a more detailed discussion of matrix affine maps. Let $K \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$ be a compact matrix convex set. A **matrix affine map** on Kis a sequence $\theta = (\theta_n)_n$ of mappings $\theta_n : K(n) \to M_n(W)$ for some vector space W, such that

$$\theta_n\left(\sum_{\ell=1}^k V_\ell^* Y^\ell V_\ell\right) = \sum_{\ell=1}^k V_\ell^* \theta_{n_\ell}(Y^\ell) V_\ell, \qquad (\text{II.4.1})$$

for all $Y^{\ell} \in K(n_{\ell})$ and $V_{\ell} \in B(\mathcal{H}_{n_{\ell}}, M_n(W))$ for $\ell = 1, ..., k$ satisfying $\sum_{\ell=1}^{k} V_{\ell}^* V_{\ell} = I_n$. If each θ_n is a homeomorphism, then we will say θ is a **matrix affine homeomorphism**. Given a matrix convex set K we will let

$$\mathcal{R}_K = \{ \theta = (\theta_n)_n | \ \theta_n : K(n) \to B(\mathcal{H}_n) \text{ for all } n \in \mathbb{N} \text{ and } \theta \text{ is matrix affine} \}$$

denote the set of matrix affine maps on K sending K(n) into $B(\mathcal{H}_n)$. As an example, if $A \in B(\mathcal{H}_d)^g$ is a g-tuple of operators on \mathcal{H}_d , then the monic linear pencil L_A and the homogeneous linear pencil $\Lambda_A(x)$, i.e. the maps $x \mapsto L_A(x)$ and $x \mapsto \Lambda_A(x)$, are elements of $M_d(\mathcal{R}_K)$.

In [WW99, Proposition 3.5], Webster and Winkler show that the set \mathcal{R}_K is an operator system if $K \subset (\mathbb{S}(\mathcal{H}_n)^g)_n$ is a compact matrix convex set. Given a positive integer d, the positive cone in $M_d(\mathcal{R}_K)$ is defined by $\theta \in M_d(\mathcal{R}_K)^+$ if and only if $\theta(Y) \ge 0$ for all $Y \in K$.

Additionally, [WW99, Proposition 3.5] shows that, with these assumptions, K is matrix affinely homeomorphic to $CS(\mathcal{R}_K)$. In particular the map sending $Y \in K(n)$ to $\phi_Y \in CS_n(\mathcal{R}_K)$ where ϕ_Y is defined by

$$\phi_Y(\theta) = \theta_n(Y) \text{ for all } \theta \in \mathcal{R}_K \tag{II.4.2}$$

defines a matrix affine homeomorphism from K to $CS(\mathcal{R}_K)$. The identification between the matrix convex set K and $CS(\mathcal{R}_K)$ is strengthened by [KLS14, Theorem 4.2] where Kleski shows that Y is an absolute extreme point of K if and only if $\phi_Y \in CS(\mathcal{R}_K)$ is an irreducible maximal completely positive map on \mathcal{R}_K .

II.4.3 Matrix affine maps on K are affine linear pencils

Webster and Winkler [WW99] comment that, if all the elements of K(1) are selfadjoint, as is the case in our setting, then the set of matrix affine maps on K is equivalent to the set of affine maps on K. [WW99] does not give a proof of this fact, as they do not use it, so for the reader's convenience we provide a proof here.

We first introduce the notion of an affine linear pencil. Let $A = (A_0, A_1, \ldots, A_g) \in B(\mathcal{H}_d)^g$ for some positive integer d. The **affine linear pencil** defined by A, denoted $\mathcal{L}_A(x)$ is the map $x \mapsto \mathcal{L}_A(x)$ defined by

$$\mathcal{L}_A(x) = A_0 + A_1 x_1 + \dots + A_g x_g.$$

Similar to the monic case, the evaluation of \mathcal{L}_A on a tuple $X \in \mathbb{S}(\mathcal{H}_n)^g$ is defined by

$$\mathcal{L}_A(X) = A_0 \otimes I_n + A_1 \otimes X_1 + \dots + A_g \otimes X_g.$$

We emphasize that the operators A_0, \ldots, A_g are not required to be self-adjoint in this definition.

Proposition II.4.1. Let K be a compact matrix convex set and assume $0 \in K$. Fix a $d \in \mathbb{N}$ and let $\theta \in M_d(\mathcal{R}_K)$. Then there exists a g + 1 tuple of operators

$$A = (A_0, A_1, \dots, A_q) \in B(\mathcal{H}_d)^{g+1}$$

such that $\theta(Y) = \mathcal{L}_A(Y)$ for all $Y \in K$.

Proof. Note that all the elements of K are self-adjoint, so if $Y \in K$ and $\alpha \in \mathbb{K}$ satisfy $\alpha Y \in K$ then $\alpha \in \mathbb{R}$. Therefore, to show $\theta \in \mathcal{R}_K$ is affine it is sufficient to show θ is affine over the reals. Temporarily assume $\theta \in \mathcal{R}_K$ is self-adjoint. Define the map $\psi = (\psi_n)_n$ by

$$\psi_n(Y) = \theta_n(Y) - \theta_n(0_n)$$
 for all $Y \in K(n)$ and all $n \in \mathbb{N}$.

We will show that ψ is linear over \mathbb{R} for each n.

Fix *n* and a tuple $Y \in K(n)$ and let $\alpha \in [0,1]$. Since $0 \in K$ and matrix convex sets are closed under taking direct sums we have $Y \oplus 0_n \in K(2n)$. Let $V : \mathbb{K}^n \to \mathbb{K}^{2n}$ be the isometry

$$V = \begin{pmatrix} \sqrt{\alpha} I_n \\ \sqrt{1 - \alpha} I_n \end{pmatrix}$$

and set $V_1 = \sqrt{\alpha} I_n$ and $V_2 = \sqrt{1 - \alpha} I_n$. Then we have

$$\alpha \psi_n(Y) = \alpha \theta_n(Y) + (1 - \alpha) \theta_n(0) - \theta_n(0)$$
$$= V_1^* \theta_n(Y) V_1 + V_2^* \theta_n(0) V_2 - \theta_n(0)$$
$$= \theta_n(V^*(Y \oplus 0) V) - \theta_n(0) = \psi_n(\alpha Y)$$

Now let $\alpha > 1$ and assume $\alpha Y \in K(n)$. Then $\frac{1}{\alpha} \in [0,1]$ so $\frac{1}{\alpha} \psi_n(\alpha Y) = \psi_n(Y)$. It follows that

$$\psi_n(\alpha Y) = \alpha \psi_n(Y)$$

for all $\alpha \geq 0$ satisfying $\alpha Y \in K$.

We now show that, given $Y^1, Y^2 \in K(n)$ such that $Y^1 + Y^2 \in K(n)$, we have

$$\psi_n(Y^1 + Y^2) = \psi_n(Y^1 + Y^2).$$

To this end, set

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}I_n \\ \frac{1}{\sqrt{2}}I_n \end{pmatrix}.$$

Since ψ is matrix affine we have

$$\begin{split} \psi_n(\frac{1}{2}Y^1) + \psi_n(\frac{1}{2}Y^2) &= \frac{1}{2}\psi_n(Y^1) + \frac{1}{2}\psi_n(Y^2) \\ &= V_1^*\psi_n(Y^1)V_1 + V_2^*\psi_n(Y^2)V_2 \\ &= \psi_{2n}(V^*(Y^1 \oplus Y^2)V) \\ &= \psi_n(\frac{1}{2}Y^1 + \frac{1}{2}Y^2). \end{split}$$

By assumption $Y^1 + Y^2 \in K$ so we find

$$\psi_n(Y^1) + \psi_n(Y^2) = 2\left(\psi_n(\frac{1}{2}Y^1) + \psi_n(\frac{1}{2}Y^2)\right) = 2\psi_n(\frac{1}{2}Y^1 + \frac{1}{2}Y^2) = \psi_n(Y^1 + Y^2).$$

Additionally, given $Y \in K(n)$ such that $-Y \in K$ we have

$$0_n = \psi(0_n) = \psi_n(Y - Y) = \psi_n(Y) + \psi_n(-Y).$$

Therefore $\psi_n(-Y) = -\psi_n(Y)$. We conclude that ψ_n is linear for each n and θ_n is affine for each n.

Now recall that we are dealing with self-adjoint θ . In particular θ_1 is affine and selfadjoint, so there exists a g + 1-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_g) \in \mathbb{R}^{g+1}$ such that $\theta_1(Y) = \alpha_0 + \sum_{i=1}^g \alpha_i Y_i$ for all $Y \in K(1)$. Since θ is self-adjoint and matrix affine, each θ_n is determined by the equality

$$\langle \theta_n(Y)\zeta,\zeta\rangle = \theta_1(\zeta^*Y\zeta)$$

for all $Y \in K(n)$ and all unit vectors $\zeta \in \mathbb{K}^n$ and all n. It follows that

$$\theta_n(Y) = \alpha_0 I_n + \sum_{i=1}^g \alpha_i Y_i$$

for all $Y \in K(n)$ and all n.

If \mathcal{H} is a real Hilbert space, then every element of \mathcal{R}_K is self-adjoint, so this completes the proof for d = 1 when $\mathbb{K} = \mathbb{R}$. If \mathcal{H} is a complex Hilbert space and $\theta \in \mathcal{R}_K$ is not selfadjoint then θ can be written $\theta = \theta^1 + i\theta^2$ where θ^1 and θ^2 are self-adjoint. It follows from above that there is a g + 1 tuple $(\alpha_0, \alpha_1, \ldots, \alpha_g) \in \mathbb{C}^{g+1}$ such that

$$\theta_n(Y) = \alpha_0 I_n + \sum_{i=1}^g \alpha_i Y_i \tag{II.4.3}$$

for all $Y \in K(n)$ and all n.

It immediately follows that if $\theta \in M_d(\mathcal{R}_K)$, then there exists a tuple

$$A = (A_0, A_1, \dots, A_g) \in M_d(\mathbb{K})^{g+1}$$

such that

$$\theta(Y) = A_0 \otimes I_n + \sum_{i=1}^g A_i \otimes Y_i = \mathcal{L}_A(Y)$$

for all $Y \in K(n)$ and all n. Identifying $M_d(\mathbb{K})^{g+1}$ with $B(\mathcal{H}_d)^{g+1}$ for each d completes the proof.

In light of Proposition II.4.1, if $0 \in K$, then for a fixed $d \in \mathbb{N}$ we have

$$M_d(\mathcal{R}_K) = \{\mathcal{L}_A : K \to M_d(K) \mid A \in B(\mathcal{H}_d)^{g+1}\}$$
(II.4.4)

where \mathcal{L}_A is the map $x \mapsto \mathcal{L}_A(x)$.

Remark II.4.2. As an aside for the reader interested in polar duals, we note that Proposition II.4.1 points towards a strong relationship between positive cone in $\cup_d M_d(\mathcal{R}_K)$ and K° , the polar dual of K. In particular, given a tuple $A = (A_0, A_1, \ldots, A_g) \in B(\mathcal{H}_d)^{g+1}$ it can be shown that $\mathcal{L}_A \in M_d(\mathcal{R}_K)^+$ if and only if there exists a tuple $\tilde{A} \in K^\circ(m)$ for some $m \leq d$ and a positive definite operator $\tilde{A}_0 \in B(\mathcal{H}_m)$ such that A is unitarily equivalent to the tuple

$$(\tilde{A}_0 \oplus 0_{d-m}, \tilde{A}_0^{1/2} \tilde{A}_1 \tilde{A}_0^{1/2} \oplus 0_{d-m}, \dots, \tilde{A}_0^{1/2} \tilde{A}_g \tilde{A}_0^{1/2} \oplus 0_{d-m}).$$

We omit the proof of this fact as we will not make use of the fine structure of the positive cone in $\cup_d M_d(\mathcal{R}_K)$. See [HKM17] for a general discussion of polar duals and [EHKM18] for a discussion of the extreme points of polar duals of free spectrahedra.

II.4.4 Maps on \mathcal{R}_K

Given a Hilbert space \mathcal{M} and a g-tuple of operators $Z \in B(\mathcal{M})^g$ we define the map $\phi_Z : \mathcal{R}_K \to B(\mathcal{M})$ by

$$\phi_Z(\mathcal{L}_A) = \mathcal{L}_A(Z)$$
 for all affine linear pencils $\mathcal{L}_A \in \mathcal{R}_K$. (II.4.5)

We are particularly interested in the case where $K = K_X$ for some g-tuple of selfadjoint compact operators $X \in \mathfrak{K}(\mathcal{H})^g$. The following proposition shows that the map $\phi_X : \mathcal{R}_{K_X} \to B(\mathcal{H})$ is a unital completely positive map on \mathcal{R}_{K_X} when 0 is in the finite interior of K_X .

Proposition II.4.3. Let $X \in \mathfrak{K}(\mathcal{H})^g$ and assume 0 is in the finite interior of K_X . Then $\phi_X : \mathcal{R}_{K_X} \to B(\mathcal{H})$ as defined by equation (II.4.5) is a unital completely positive map on \mathcal{R}_{K_X} .

Proof. By assumption 0 is in the finite interior of K_X . Therefore, Proposition II.4.1 shows

that \mathcal{R}_{K_X} is equal to the set of affine linear pencils on K_X .

For all integers $n \in \mathbb{N}$ let $P_n : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection of \mathcal{H} onto \mathcal{H}_n , and define $X^n \in \mathfrak{K}(\mathcal{H})^g$ to be the tuple $X^n = P_n X P_n$. Observe that $X^n = \iota_n^* X \iota_n \oplus 0_{\mathcal{H}_n^{\perp}}$ where $\iota_n : \mathcal{H}_n \to \mathcal{H}$ is the inclusion map of \mathcal{H}_n into \mathcal{H} .

From the definition of K_X we know that $\iota_n^* X \iota_n \in K_X$. Since $0, \iota_n^* X \iota_n \in K_X$, using [WW99, Proposition 3.5] we find $\phi_0, \phi_{\iota_n^* X \iota_n} \in CS(\mathcal{R}_{K_X})$. As such, the equality

$$\phi_{X^{n}}(\mathcal{L}_{A}) = \mathcal{L}_{A}(\iota_{n}^{*}X\iota_{n} \oplus (I_{\mathcal{H}_{n}^{\perp}} \otimes 0)) = \mathcal{L}_{A}(\iota_{n}^{*}X\iota_{n}) \oplus (I_{\mathcal{H}_{n}^{\perp}} \otimes \mathcal{L}_{A}(0)) = \phi_{\iota_{n}^{*}X\iota_{n}} \oplus (I_{\mathcal{H}_{n}^{\perp}} \otimes \phi_{0}(\mathcal{L}_{A}))$$
(II.4.6)

for all $\mathcal{L}_A \in \mathcal{R}_{K_X}$ shows that ϕ_{X^n} is completely positive for all $n \in \mathbb{N}$.

Now fix $d \in \mathbb{N}$ and $\mathcal{L}_A \in M_d(\mathcal{R}_{K_X})$. Since X is compact we have $\lim X^n = X$ where the convergence is in norm. Furthermore, affine linear pencils are continuous maps, so

$$\lim \phi_{X^n}(\mathcal{L}_A) = \lim \mathcal{L}_A(X^n) = \mathcal{L}_A(X) = \phi_X(\mathcal{L}_A).$$

Since each ϕ_{X^n} is completely positive, it follows that ϕ_X is completely positive on \mathcal{R}_{K_X} as claimed.

To see that ϕ_X is unital let $1_{\mathcal{R}}$ be the identity in \mathcal{R}_{K_X} . Then $1_{\mathcal{R}}$ is the linear pencil

$$1_{\mathcal{R}} = \mathcal{L}_{(1,0,0,...,0)}$$

The evaluation

$$\phi_X(1_{\mathcal{R}}) = \mathcal{L}_{(1,0,0,\dots,0)}(X) = I_{\mathcal{H}}$$

shows ϕ_X is unital.

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II.4.5 Proof of Theorem I.4.1 via completely positive maps

We now give the alternate proof of Theorem I.4.1.

Proof. Theorem II.1.5 shows that K_X is a compact matrix convex set, so we only need to show $\partial^{abs}K_X = \emptyset$. Pick an element $Y \in K_X(n)$ and let R_{K_X} be the operator system of matrix affine maps on K_X . [KLS14, Theorem 4.2] shows that $Y \in \partial^{abs}K_X$ if and only if ϕ_Y is in the Arveson boundary of $CS(\mathcal{R}_K)$ and [A08, Proposition 2.4] shows that if ϕ_Y is a boundary representation, then ϕ_Y is maximal. Therefore, to show $Y \notin \partial^{abs}K_X$ it is sufficient to show that the unital completely positive map $\phi_Y : \mathcal{R}_{K_X} \to \mathcal{H}_n$ is not maximal.

Using Proposition II.1.2 there exists an integer m_n depending only on n and g and an isometry $V : \mathcal{H}_n \to \mathcal{H}^{m_n}$ so that $Y = V^*(I_{m_n} \otimes X)V$. This implies that there is a unitary $U : \mathcal{H}^{m_n} \to \mathcal{H}^{m_n}$ so that $U(I_{m_n} \otimes X)U^*$ is a dilation of Y. It follows that the map

$$\phi_{U(I_{m_n}\otimes X)U^*}:\mathcal{R}_{K_X}\to B(\mathcal{H}^{m_n})$$

is a dilation of the unital completely positive map $\phi_Y : \mathcal{R}_{K_X} \to \mathcal{H}_n$. Furthermore, Proposition II.4.3 shows ϕ_X is a unital completely positive map on \mathcal{R}_{K_X} , so the equality

$$\phi_{U(I_{m_n}\otimes X)U^*}(\mathcal{L}_A) = U(I_{m_n}\otimes \phi_X(\mathcal{L}_A))U^*$$

for all $\mathcal{L}_A \in \mathcal{R}_{K_X}$ shows $\phi_{U(I_{m_n} \otimes X)U^*}$ is a unital completely positive map on \mathcal{R}_{K_X} .

Assume towards a contradiction that ϕ_Y is a maximal unital completely positive map. Since $\phi_{U(I_{m_n}\otimes X)U^*}$ is a unital completely positive dilation of ϕ_Y , there must exist some unital completely positive map $\psi : \mathcal{R}_{K_X} \to \mathcal{H}_n^{\perp}$ so that $\phi_{U(I_{m_n}\otimes X)U^*} = \phi_Y \oplus \psi$. Note that in this definition, \mathcal{H}_n is viewed as a subspace of \mathcal{H}^{m_n} , so \mathcal{H}_n^{\perp} is the orthogonal complement of \mathcal{H}_n in \mathcal{H}^{m_n} . For $i = 1, \ldots, g$ let $\eta_i \in R_{K_X}$ be the evaluation map defined by

$$\phi_S(\eta_i) = S_i \text{ for all } S \in K_X$$

Define $Z \in \mathbb{S}(\mathcal{H}_n^{\perp})^g$ to be the tuple

$$Z = (\psi(\eta_1), \ldots, \psi(\eta_q)).$$

Considering the evaluations

$$\phi_{U(I_{m_n}\otimes X)U^*}(\eta_i) = \phi_Y(\eta_i) \oplus \psi(\eta_i)$$

for i = 1, ..., g shows that $I_{m_n} \otimes X = U^*(Y \oplus Z)U$. In particular, the invariant subspaces of $I_{m_n} \otimes X$ must be equal to the invariant subspaces of $U^*(Y \oplus Z)U$.

Since X is assumed to have no nontrivial finite dimensional reducing subspaces, Lemma II.2.1 shows that any nontrivial reducing subspace of $I_{m_n} \otimes X$ is infinite dimensional. Observe that the subspace $W \subset \mathcal{H}^{m_n}$ defined by

$$W = \{ U^* (v \oplus 0_{\mathcal{H}_n^{\perp}}) | v \in \mathcal{H}_n \}$$

is a nontrivial *n* dimensional reducing subspace for $U^*(Y \oplus Z)U$, and hence for $I_{m_n} \otimes X$, which contradicts Lemma II.2.1. It follows that there is no unital completely positive map ψ so that $\phi_Y \oplus \psi = \phi_{U(I_{m_n} \otimes X)U^*}$. In particular, ϕ_Y is not maximal. This shows that $Y \notin \partial^{abs} K_X$ and $\partial^{abs} K_X = \emptyset$.

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Chapter II, in part, is a reprint of the material as it appears in [E18] E. Evert: Matrix convex sets without absolute extreme points, Linear Algebra Appl. 537 (2018) 287-301. The dissertation author was the primary investigator and author of this paper.

Chapter III

Arveson extreme points span free spectrahedra

In this chapter we focus our attention on the absolute extreme points of free spectrahedra. Throughout the chapter we will not need the overarching Hilbert space \mathcal{H} . For this reason we now fix $\mathcal{H}_n = \mathbb{K}^n$ for each n where $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . In this setting, an nc set is a set which contains g-tuples of $n \times n$ self-adjoint matrices for all positive integers n.

We now let $SM_n(\mathbb{K})^g$ denote the set of g-tuples $X = (X_1, \ldots, X_g)$ of $n \times n$ self-adjoint matrices over \mathbb{K} and let $SM(\mathbb{K})^g$ denote the nc set $SM(\mathbb{K})^g = (SM_n(\mathbb{K})^g)_n$. Similarly, for positive integers n, ℓ and g let $M_{n \times \ell}(\mathbb{K})^g$ denote the set of g-tuples $\beta = (\beta_1, \ldots, \beta_g)$ of $n \times \ell$ matrices with entries in \mathbb{K} .

III.1 Absolute extreme points of free spectrahedra

Recall that a free spectrahedron is a type of matrix convex set which is the set of solutions to a linear matrix equality. That is, given a tuple $A \in SM_d(\mathbb{K})^g$, the free spectrahedron $\mathcal{D}_A^{\mathbb{K}}$ is the set $(\mathcal{D}_A^{\mathbb{K}}(n))_n$ where

$$\mathcal{D}_A^{\mathbb{K}}(n) = \{ X \in SM_n(\mathbb{K})^g | L_A(X) = I_{dn} + A_1 \otimes X_1 + \cdots + A_g \otimes X_g \ge 0 \}$$

for each positive integer n.

III.1.1 The dilation subspace

We begin by more carefully examining the dilation subspace. This subspace will play an important roll in the proof of Theorem III.1.3. Recall that the dilation subspace is the set

$$\mathfrak{K}_{A,X}^{\mathbb{K}} = \{\beta \in M_{n \times 1}(\mathbb{K})^g | \ker L_A(X) \subset \ker \Lambda_A(\beta^*) \}.$$

Here the kernels ker $L_A(X)$ and ker $\Lambda_A(X)$ are subspaces of \mathbb{K}^{dn} .

The subspace $\mathfrak{K}_{A,X}^{\mathbb{K}}$ is called the dilation subspace since, by considering the Schur complement, a tuple $\beta \in M_{n\times 1}(\mathbb{K})^g$ is an element of $\mathfrak{K}_{A,X}^{\mathbb{K}}$ if and only if there is a real number c > 0 and a tuple $\gamma \in \mathbb{R}^g$ such that

$$Y = \begin{pmatrix} X & c\beta \\ c\beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}.$$
 (III.1.1)

The following lemma explains the relationship between the dilation subspace $\mathfrak{K}_{A,X}^{\mathbb{K}}$ and dilations of the tuple $X \in \mathcal{D}_A^{\mathbb{K}}$ in greater detail.

Lemma III.1.1. Let $\mathcal{D}_A^{\mathbb{K}}$ be a free spectrahedron and let $X \in \mathcal{D}_A^{\mathbb{K}}(n)$.

1. If $\beta \in M_{n \times 1}(\mathbb{K})^g$ and

$$Y = \begin{pmatrix} X & \beta \\ & \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}(n+1)$$

is a 1-dilation of X, then $\beta \in \mathfrak{K}_{A,X}^{\mathbb{K}}$.

2. Let $\beta \in M_{n \times 1}(\mathbb{K})^g$. Then $\beta \in \mathfrak{K}_{A,X}^{\mathbb{K}}$ if and only if there is a real number c > 0 such that

$$\begin{pmatrix} X & c\beta \\ c\beta^* & 0 \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}(n+1).$$

3. X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{K}}$ if and only if dim $\mathfrak{K}_{A,X}^{\mathbb{K}} = 0$.

Proof. Items (1) and (2) follow from considering the Schur complement of $L_A(Y)$ for a dilation

$$Y = \begin{pmatrix} X & \beta \\ \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}(n+1)$$

of X. Indeed, multiplying $L_A(X)$ by permutation matrices, sometimes called canonical shuffles, see [P02, Chapter 8], shows

$$L_A(Y) \ge 0$$
 if and only if $\begin{pmatrix} L_A(X) & \Lambda_A(\beta) \\ \Lambda_A(\beta^*) & L_A(\gamma) \end{pmatrix} \ge 0.$ (III.1.2)

Taking the appropriate Schur compliment then implies that

$$L_A(Y) \ge 0$$
 if and only if $L_A(\gamma) \ge 0$ and $L_A(X) - \Lambda_A(\beta)L_A(\gamma)^{\dagger}\Lambda_A(\beta^*) \ge 0$ (III.1.3)

where † denotes the Moore-Penrose pseudoinverse. Item (1) and item (2) are immediate consequences of equation (III.1.3). See [EHKM18, Corollary 2.3] for a related argument.

Item (3) follows from items (1) and (2). \Box

III.1.2 Maximal 1-dilations

An important aspect of the proof of our main result is constructing dilations which satisfy a notion of maximality. Given a matrix convex set K and a tuple $X \in K(n)$, say the dilation

$$Y = \begin{pmatrix} X & c\hat{\beta} \\ c\hat{\beta}^* & \hat{\gamma} \end{pmatrix} \in K(n+1)$$

is a **maximal 1-dilation** of X if Y is a 1-dilation of X and $\hat{\beta}$ is nonzero and the real number c and tuple $\hat{\gamma} \in \mathbb{R}^g$ are solutions to the sequence of maximization problems

$$c \coloneqq \operatorname{Maximizer}_{\alpha \in \mathbb{R}, \gamma \in \mathbb{R}^{g}} \alpha$$

s.t.
$$\begin{pmatrix} X & \alpha \hat{\beta} \\ \alpha \hat{\beta}^{*} & \gamma \end{pmatrix} \in K(n+1)$$

and
$$\hat{\gamma} \coloneqq$$
 A Local Maximizer $\|\gamma\|$
s.t. $\begin{pmatrix} X & c\hat{\beta} \\ c\hat{\beta}^* & \gamma \end{pmatrix} \in K(n+1)$

where $\|\cdot\|$ denotes the usual norm on \mathbb{R}^{g} . We note that maximal 1-dilations can be computed numerically, see Proposition III.2.1. We emphasize that $\hat{\gamma}$ produced by the second optimization need only be any local maximizer, and global maximality is not required.

Remark III.1.2. If K is a compact matrix convex set and $X \in K$ is not an Arveson extreme point of K, then a routine compactness argument shows the existence of nontrivial maximal 1-dilations of X.

Other notions of maximal dilations (in the infinite dimensional setting) are discussed in [DM05], [A08, Section 2] and [DK15, Section 1].

III.1.3 Maximal dilations reduce the dimension of the dilation subspace

Let $A \in SM_d(\mathbb{R})^g$, let $\mathcal{D}_A^{\mathbb{R}}$ be a compact real free spectrahedron, and let $X \in \mathcal{D}_A^{\mathbb{R}}$. The following theorem shows that maximal 1-dilations of X reduce the dimension of the dilation subspace.

Theorem III.1.3. Let $A \in SM_d(\mathbb{R})^g$ be a g-tuple of self-adjoint matrices over \mathbb{R} such that $\mathcal{D}_A^{\mathbb{R}}$ is a compact real free spectrahedron and let $X \in \mathcal{D}_A^{\mathbb{R}}(n)$. Assume X is not an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$. Then there exists a nontrivial maximal 1-dilation $\hat{Y} \in \mathcal{D}_A^{\mathbb{R}}(n+1)$ of X. Furthermore, any such \hat{Y} satisfies

$$\dim \mathfrak{K}^{\mathbb{R}}_{A,\hat{Y}} < \dim \mathfrak{K}^{\mathbb{R}}_{A,X}$$

Proof. Let \hat{Y} be a maximal 1-dilation of X. Equivalently, choose the dilation \hat{Y} (choose $\hat{\beta}$ and $\hat{\gamma}$) such that

$$\hat{Y} = \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \end{pmatrix} \text{ is in } \mathcal{D}^{\mathbb{R}}_A(n+1),$$

and if

$$\tilde{Y}_c = \begin{pmatrix} X & c\hat{\beta} \\ c\hat{\beta}^* & \gamma \end{pmatrix} \text{ is in } \mathcal{D}_A^{\mathbb{R}}(n+1)$$

for a tuple $\gamma \in \mathbb{R}^g$ and a real number $c \in \mathbb{R}$, then $c \leq 1$.¹ Furthermore, if c = 1 and $\tilde{Y} \in \mathcal{D}_A^{\mathbb{R}}(n+1)$, then there exists an $\epsilon > 0$ such that $\|\hat{\gamma} - \gamma\| < \epsilon$ implies $\|\gamma\| \leq \|\hat{\gamma}\|$. As mentioned in Remark III.1.2, the existence of such a \hat{Y} follows from the assumptions that X is not an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$ and that $\mathcal{D}_A^{\mathbb{R}}$ is level-wise compact.

We will show that

$$\dim \mathfrak{K}^{\mathbb{R}}_{A,\hat{Y}} < \dim \mathfrak{K}^{\mathbb{R}}_{A,X}.$$

¹ If \tilde{Y}_c is an element of $\mathcal{D}^{\mathbb{R}}_A(n+1)$ then so is \tilde{Y}_{-c} . For this reason, it is equivalent to require $|c| \leq 1$.

First consider the subspace

$$\mathfrak{E}_{A,\hat{Y}} \coloneqq \{\eta \in M_{n \times 1}(\mathbb{R})^g \mid \text{there exists a } \sigma \in \mathbb{R}^g \text{ so that } \ker L_A(\hat{Y}) \subseteq \ker \Lambda_A \begin{pmatrix} \eta^* & \sigma \end{pmatrix} \}.$$

In other words $\mathfrak{E}_{A,\hat{Y}}$ is the projection ι of $\mathfrak{K}_{A,\hat{Y}}^{\mathbb{R}}$ defined by

$$\mathfrak{E}_{A,\hat{Y}} \coloneqq \iota(\mathfrak{K}_{A,X}^{\mathbb{R}}) \text{ where } \iota\begin{pmatrix} \eta \\ \sigma \end{pmatrix} = \eta$$

for $\eta \in M_{n \times 1}(\mathbb{R})^g$ and $\sigma \in \mathbb{R}^g$. We will show dim $\mathfrak{E}_{A,\hat{Y}} < \dim \mathfrak{K}_{A,X}^{\mathbb{R}}$.

If $\eta \in \mathfrak{E}_{A,\hat{Y}}$, then there exists a tuple $\tilde{\sigma} \in \mathbb{R}^g$ such that

$$\begin{pmatrix} \eta^* & \tilde{\sigma} \end{pmatrix} \in \mathfrak{K}^{\mathbb{R}}_{A,\hat{Y}}.$$

From Lemma III.1.1 (2), it follows that there is a real number c > 0 so that setting $\sigma = c\tilde{\sigma}$ gives

$$\begin{pmatrix} X & \hat{\beta} & c\eta \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\eta^* & \sigma^* & 0 \end{pmatrix} \in \mathcal{D}_A^{\mathbb{R}}.$$

Since $\mathcal{D}^{\mathbb{R}}_A$ is matrix convex it follows that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X & \hat{\beta} & c\eta \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\eta^* & \sigma^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X & c\eta \\ c\eta^* & 0 \end{pmatrix} \in \mathcal{D}_A^{\mathbb{R}},$$

so Lemma III.1.1 (1) shows $\eta \in \mathfrak{K}_{A,X}^{\mathbb{R}}.$ In particular this shows

$$\mathfrak{E}_{A,\hat{Y}} \subset \mathfrak{K}_{A,X}^{\mathbb{R}}.$$
 (III.1.4)

Now, assume towards a contradiction that

$$\dim \mathfrak{E}_{A,\hat{Y}} = \dim \mathfrak{K}_{A,X}^{\mathbb{R}}.$$

Using equation (III.1.4) this implies that

$$\mathfrak{E}_{A,\hat{Y}} = \mathfrak{K}_{A,X}^{\mathbb{R}}.$$

In particular we have $\hat{\beta} \in \mathfrak{E}_{A,\hat{Y}}$. It follows that there is a real number $c \neq 0$ and a tuple $\sigma \in \mathbb{R}^{g}$ so that

$$L_A \begin{pmatrix} X & \hat{\beta} & c\hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\hat{\beta}^* & \sigma & 0 \end{pmatrix} \ge 0.$$
(III.1.5)

Using the NC LDL^{*}-decomposition (up to canonical shuffles) shows that inequality (III.1.5) holds if and only if $L_A(X) \ge 0$ and the Schur complements

$$I_d - c^2 Q \ge 0 \tag{III.1.6}$$

and

$$L_A(\hat{\gamma}) - Q - (\Lambda_A(\sigma) - cQ)^* \left(I_d - c^2 Q \right)^\dagger (\Lambda_A(\sigma) - cQ) \ge 0$$
 (III.1.7)

where

$$Q \coloneqq \Lambda_A(\hat{\beta}^*) L_A(X)^{\dagger} \Lambda_A(\hat{\beta}).$$
(III.1.8)

It follows that

$$L_A(\hat{\gamma}) - Q \ge 0 \tag{III.1.9}$$

and

$$\ker[L_A(\hat{\gamma}) - Q] \subseteq \ker[\Lambda_A(\sigma) - cQ]. \tag{III.1.10}$$

Inequalities (III.1.9) and (III.1.10) imply that there exists a real number $\tilde{\alpha} > 0$ such that $0 < \alpha \leq \tilde{\alpha}$ implies

$$L_A(\hat{\gamma}) - Q \pm \alpha (\Lambda_A(\sigma) - cQ) \geq 0.$$

It follows from this that

$$L_{A}(\hat{\gamma} \pm \alpha \sigma) - (1 \pm c\alpha)Q$$

= $L_{A}(\hat{\gamma} \pm \alpha \sigma) - (\Lambda_{A}(\sqrt{1 \pm c\alpha}\hat{\beta}^{*})L_{A}(X)^{\dagger}\Lambda_{A}(\sqrt{1 \pm c\alpha}\hat{\beta})) \geq 0.$ (III.1.11)

Since $L_A(X) \geq 0$, equation (III.1.11) implies

$$L_A \begin{pmatrix} X & \sqrt{1 \pm c\alpha}\hat{\beta} \\ \sqrt{1 \pm c\alpha}\hat{\beta}^* & \hat{\gamma} \pm \alpha\sigma \end{pmatrix} \ge 0.$$
(III.1.12)

Therefore, from our choice of \hat{Y} , hence of $\hat{\beta}$, we must have

$$\sqrt{1 \pm c\alpha} \le 1$$

It follows that $c\alpha = 0$. However, we have assumed $\alpha > 0$ and $c \neq 0$, so this is a contradiction. We conclude

$$\dim \mathfrak{E}_{A,\hat{Y}} < \dim \mathfrak{K}_{A,X}^{\mathbb{R}}. \tag{III.1.13}$$

Now seeking a contradiction assume that $\dim \mathfrak{K}^{\mathbb{R}}_{A,\hat{Y}} = \dim \mathfrak{K}^{\mathbb{R}}_{A,X}$. Then inequality (III.1.13) implies that there exist tuples $\eta \in M_{n\times 1}(\mathbb{R})^g$ and $\sigma^1, \sigma^2 \in \mathbb{R}^g$ such that $\sigma^1 \neq \sigma^2$ and so

$$\begin{pmatrix} \eta \\ \sigma^1 \end{pmatrix}, \begin{pmatrix} \eta \\ \sigma^2 \end{pmatrix} \in \mathfrak{K}_{A,\hat{Y}}^{\mathbb{R}}.$$

It follows that

$$\begin{pmatrix} 0\\ \sigma^1 - \sigma^2 \end{pmatrix} \in \mathfrak{K}^{\mathbb{R}}_{A,\hat{Y}}.$$
 (III.1.14)

Set $\hat{\sigma} = \sigma^1 - \sigma^2 \neq 0 \in \mathbb{R}^g$. As before, equation (III.1.14) with Lemma III.1.1 (2) implies that there is a real number $c \neq 0 \in \mathbb{R}$ so that

$$L_A \begin{pmatrix} X & \hat{\beta} & 0\\ \hat{\beta}^* & \hat{\gamma} & c\hat{\sigma}\\ 0 & c\hat{\sigma} & 0 \end{pmatrix} \ge 0.$$
(III.1.15)

Considering the NC LDL* decomposition shows that equation (III.1.15) holds if and only if

$$L_A(X) \ge 0$$
 and $L_A(\hat{\gamma}) - Q - c^2 \Lambda_A(\hat{\sigma}) \Lambda_A(\hat{\sigma}) \ge 0,$ (III.1.16)

where $Q = \Lambda_A(\hat{\beta}^*)L_A(X)^{\dagger}\Lambda_A(\hat{\beta})$ as before. It follows from this that

$$\ker[L_A(\hat{\gamma}) - Q] \subseteq \ker \Lambda_A(\hat{\sigma}) \quad \text{and} \quad L_A(\hat{\gamma}) - Q \ge 0.$$
(III.1.17)

This implies that there is a real number $\tilde{\alpha} > 0$ so that, for all $\alpha \in \mathbb{R}$ satisfying $0 < \alpha \leq \tilde{\alpha}$, we have

$$L_A(\hat{\gamma}) - Q \pm \Lambda_A(\alpha \hat{\sigma}) = L_A(\hat{\gamma} \pm \alpha \hat{\sigma}) - Q \ge 0.$$

Since this is the appropriate Schur complement and since $L_A(X) \ge 0$ it follows that

$$L_A \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \pm \alpha \hat{\sigma} \end{pmatrix} \ge 0$$
 (III.1.18)

whenever $0 < \alpha \leq \tilde{\alpha}$. Therefore, the local maximality of $\hat{\gamma}$ implies

$$\|\hat{\gamma} + \alpha\hat{\sigma}\| \leq \|\hat{\gamma}\|$$
 and $\|\hat{\gamma} - \alpha\hat{\sigma}\| \leq \|\hat{\gamma}\|$

for sufficiently small $\alpha \in (0, \tilde{\alpha}]$, a contradiction to the assumptions that $\alpha \neq 0$ and $\hat{\sigma} \neq 0$. We conclude that $\dim \mathfrak{K}_{A,\hat{Y}}^{\mathbb{R}} < \dim \mathfrak{K}_{A,X}^{\mathbb{R}}$ as asserted by Theorem III.1.3.

Proof of Theorem I.7.2 for real free spectrahedra

We are now in position to prove Theorem I.7.2 in the case where $\mathcal{D}_A^{\mathbb{R}}$ is a compact real free spectrahedron.

Proof of Theorem I.7.2 when $\mathbb{K} = \mathbb{R}$. Given a tuple $X \in \mathcal{D}_A^{\mathbb{R}}$ with dim $\mathfrak{K}_{A,X}^{\mathbb{R}} = \ell$, the existence of a k-dilation Y of X such that $Y \in \partial^{\operatorname{Arv}} \mathcal{D}_A^{\mathbb{R}}$ for some $k \leq \ell$ is an immediate consequence of Theorem III.1.3 and Lemma III.1.1 (3).

The fact that $\mathcal{D}_A^{\mathbb{R}}$ is the matrix convex hull of its Arveson extreme points, hence of its absolute extreme points, is proved immediately after the statement of Theorem I.7.2. \Box

III.2 Numerical computation

In this section we provide an algorithm which constructs Arveson dilations of elements of a real free spectrahedron. Additionally we describe computational methods to determine if a tuple X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$.

III.2.1 Computation of maximal 1-dilations

Given a compact real free spectrahedron $\mathcal{D}_A^{\mathbb{R}}$, the following algorithm dilates a tuple $X \in \mathcal{D}_A^{\mathbb{R}}$ to an Arveson extreme point $Y \in \mathcal{D}_A^{\mathbb{R}}$ in dim $\mathfrak{K}_{A,X}^{\mathbb{R}}$ steps or less.

Proposition III.2.1. Let $A \in SM_d(\mathbb{R})^g$ be a *g*-tuple of self-adjoint matrices over \mathbb{R} such that $\mathcal{D}_A^{\mathbb{R}}$ is a compact real free spectrahedron. Given a tuple $X \in \mathcal{D}_A^{\mathbb{R}}(n)$, set $Y^0 = X$. For integers k = 0, 1, 2... and while dim $\mathfrak{K}_{A,Y^k}^{\mathbb{R}} > 0$ define

$$Y^{k+1} := \begin{pmatrix} Y^k & c_k \hat{\beta}^k \\ c_k (\hat{\beta}^k)^* & \hat{\gamma}^k \end{pmatrix}$$

where $\hat{\beta}^k$ is any nonzero element of $\mathfrak{K}^{\mathbb{R}}_{A,Y^k}$ and

$$c_k := \underset{c \in \mathbb{R}, \gamma \in \mathbb{R}^g}{\operatorname{minimizer}} c$$

s.t.
$$L_A \begin{pmatrix} Y^k & c\hat{\beta}^k \\ c(\hat{\beta}^k)^* & \gamma \end{pmatrix} \geq 0,$$

and
$$\hat{\gamma}^k := A \operatorname{Local}_{\gamma \in \mathbb{R}^g} \operatorname{Maximizer} \|\gamma\|$$

s.t. $L_A \begin{pmatrix} Y^k & c_k \hat{\beta}^k \\ c_k (\hat{\beta}^k)^* & \gamma \end{pmatrix} \geq 0.$

Then dim $\mathfrak{K}_{A,Y^{\ell}}^{\mathbb{R}} = 0$ for some integer $\ell \leq \dim \mathfrak{K}_{A,X}^{\mathbb{R}} \leq ng$ and Y^{ℓ} is an Arveson ℓ -dilation of X.

Proof. This follows from the proof of Theorem III.1.3.

The optimization over c in Proposition III.2.1 is a semidefinite program, while the optimization over γ is a local maximization of a convex quadratic over a spectrahedron.

III.2.2 Classification of extreme points using linear systems

Lemma III.1.1 (3) gives a linear system which can be solved to determine if an element of a free spectrahedron is an Arveson extreme point.

Proposition III.2.2. Let $A \in SM_d(\mathbb{K})^g$ and let $\mathcal{D}_A^{\mathbb{K}}$ be a free spectrahedron. Let X be an element of $\mathcal{D}_A^{\mathbb{K}}$ at level n and let $P_{A,X}$: ker $L_A(X) \to \mathbb{K}^{nd}$ be the inclusion map of ker $L_A(X)$ into \mathbb{K}^{nd} . Then X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{K}}$ if and only if the only solution to the homogeneous linear system

$$\Lambda_A(\beta^*)P_{A,X} = 0 \tag{III.2.1}$$

where $\beta \in M_{n \times 1}(\mathbb{K})^g$ is $\beta = 0$.

Proof. If $\Lambda_A(\beta^*)P_{A,X} = 0$ for a tuple $\beta \in M_{n \times 1}(\mathbb{K})^g$, then $\beta \in \mathfrak{K}_{A,X}^{\mathbb{K}}$. Lemma III.1.1 (3) completes the proof.

Set $\ell = \dim \mathfrak{K}_{A,X}^{\mathbb{K}}$. Since β is a *g*-tuple of $M_{n\times 1}$ matrices and A is a *g*-tuple of $n \times n$ self-adjoint matrices, the linear system in equation (III.2.1) is a system of $d\ell$ equation in ng unknowns.

A similar linear system can be solved to determine if a tuple is a Euclidean extreme point of a free spectrahedron. In this case the linear system is a system of $d\ell n$ equation in n(n+1)g/2 unknowns.

III.3 Complex free spectrahedra

This section will prove that every element of a compact complex free spectrahedron which is closed under complex conjugation is the matrix convex hull of its absolute extreme points. We begin with a lemma which shows that the set of real elements in the absolute boundary of a complex free spectrahedron $\mathcal{D}_A^{\mathbb{C}}$ which is closed under complex conjugation is exactly equal to the absolute boundary of $\mathcal{D}_A^{\mathbb{R}}$.

Lemma III.3.1. Let A be a g-tuple of $d \times d$ real symmetric matrices and let $X \in \mathcal{D}_A^{\mathbb{C}}$ be a g-tuple of $n \times n$ real symmetric matrices.

- X is an Arveson extreme point of D^C_A if and only if X is an Arveson extreme point of D^R_A.
- 2. X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{C}}$ if and only if X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{R}}$.

Proof. We first prove item (1). It is straightforward to show that X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$ if X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{C}}$. To prove the converse, assume X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$ and let $\beta \in M_{n \times 1}(\mathbb{C})^g$ be a tuple such that

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{C}}.$$

By assumption A is a tuple of real symmetric matrices so $\mathcal{D}_A^{\mathbb{C}}$ is closed under complex conjugation. It follows that

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} = \begin{pmatrix} X & \overline{\beta} \\ \overline{\beta}^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{C}}$$

Since $\mathcal{D}_A^{\mathbb{C}}$ is convex we conclude that

$$\begin{pmatrix} X & \operatorname{Re}(\beta) \\ \operatorname{Re}(\beta)^* & \gamma \end{pmatrix} = \frac{1}{2} \left(\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} + \begin{pmatrix} X & \overline{\beta} \\ \overline{\beta}^* & \gamma \end{pmatrix} \right) \in \mathcal{D}_A^{\mathbb{C}}.$$

This matrix has real entries so it is an element of $\mathcal{D}_A^{\mathbb{R}}$. However, X was assumed to be an absolute extreme point of $\mathcal{D}_A^{\mathbb{R}}$ so we must have $\operatorname{Re}(\beta) = 0$.

Now, $\mathcal{D}_A^{\mathbb{C}}$ is matrix convex so we know

$$\begin{pmatrix} X & i\beta \\ (i\beta)^* & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in \mathcal{D}_A^{\mathbb{C}}.$$

However, this matrix is in $\mathcal{D}_{A}^{\mathbb{R}}$ since $\operatorname{Re}(\beta) = 0$ from which it follows that $\operatorname{Im}(i\beta) = 0$. We have assumed that X is an Arveson extreme point of $\mathcal{D}_{A}^{\mathbb{R}}$, so $i\beta = 0$, hence $\beta = 0$. We conclude that X is an Arveson extreme point of $\mathcal{D}_{A}^{\mathbb{C}}$, as claimed.

Item (2) immediately follows from item (1) and Theorem I.7.1 together with Lemma III.6.1 which shows that a real symmetric tuple is irreducible over \mathbb{R} if and only if it is irreducible over \mathbb{C} . Note that the issue of irreducibility is independent of the other aspects of the proof, hence its delay until Section III.6.

Our next lemma gives a list of equalities for the dilation subspace which will be used in proving the bound on the dimension of the absolute extreme points appearing in Theorem I.7.2.

Lemma III.3.2. Let $\mathcal{D}_A^{\mathbb{K}}$ be a real or complex free spectrahedron. The following equalities hold for the dilation subspace:

1. Let
$$X \in \mathcal{D}_A^{\mathbb{K}}(n_1)$$
 and $Z \in \mathcal{D}_A^{\mathbb{K}}(n_2)$. Then

$$\mathfrak{K}_{A,X\oplus Z}^{\mathbb{K}} = \left\{ \begin{pmatrix} \beta^* & \sigma^* \end{pmatrix}^* \in M_{(n_1+n_2)\times 1}(\mathbb{K})^g \middle| \beta \in \mathfrak{K}_{A,X}^{\mathbb{K}} \text{ and } \sigma \in \mathfrak{K}_{A,Z}^{\mathbb{K}} \right\}.$$

Additionally,

$$\mathfrak{K}_{A,X\oplus Z}^{\mathbb{K}}=\dim \mathfrak{K}_{A,X}^{\mathbb{K}}+\dim \mathfrak{K}_{A,Z}^{\mathbb{K}}.$$

2. Let $X \in \mathcal{D}^{\mathbb{K}}_{A}(n)$ and let $U \in M_{n}(\mathbb{K})$ be a unitary. Then

$$\mathfrak{K}_{A,X}^{\mathbb{K}} = U^* \mathfrak{K}_{A,U^*XU}^{\mathbb{K}} \quad \text{and} \quad \dim \mathfrak{K}_{A,X}^{\mathbb{K}} = \dim \mathfrak{K}_{A,U^*XU}^{K} \; .$$

3. Assume $\mathcal{D}^{\mathbb{K}}_{A}$ is closed under complex conjugation. Then

$$\mathfrak{K}_{A,X}^{\mathbb{K}} = \overline{\mathfrak{K}_{A,\overline{X}}^{\mathbb{K}}}$$
 and $\dim \mathfrak{K}_{A,X}^{\mathbb{K}} = \dim \mathfrak{K}_{A,\overline{X}}^{\mathbb{K}}$

Proof. The proof of item (1) is immediate from the fact that $\ker L_A(X \oplus Z) \subset \ker \Lambda_A \left(\beta^* \quad \sigma^* \right)$ if and only if $\ker L_A(X) \subset \ker \Lambda_A(\beta^*)$ and $\ker L_A(Z) \subset \ker \Lambda_A(\sigma^*)$.

To prove item (2) let $U \in M_n(\mathbb{K})$ be a unitary and observe that

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}} \quad \Leftrightarrow \quad \begin{pmatrix} U^* X U & U^* \beta \\ \beta^* U & \gamma \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}.$$

To prove item (3): assume $\mathcal{D}_A^{\mathbb{K}}$ is closed under complex conjugation. Then

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}} \quad \Leftrightarrow \quad \begin{pmatrix} \overline{X} & \overline{\beta} \\ \overline{\beta^*} & \overline{\gamma} \end{pmatrix} \in \mathcal{D}_A^{\mathbb{K}}.$$

We now give a classification of free spectrahedra which are closed under complex conjugation.

Lemma III.3.3. Let A be a g-tuple of $d \times d$ complex self-adjoint matrices. Then the complex free spectrahedron $\mathcal{D}_A^{\mathbb{C}}$ is closed under complex conjugation if and only if there is a g-tuple B of real symmetric matrices of size less than or equal to $2d \times 2d$ such that $\mathcal{D}_A^{\mathbb{C}} = \mathcal{D}_B^{\mathbb{C}}$.

Proof. We first prove the forwards direction. Let X be a g-tuple of complex self-adjoint matrices. Since $\mathcal{D}_A^{\mathbb{C}}$ is closed under complex conjugation we know that $X \in \mathcal{D}_A^{\mathbb{C}}$ if and only if

$$L_A(X) \ge 0$$
 and $L_{\overline{A}}(X) \ge 0.$ (III.3.1)

Thus $X \in \mathcal{D}_A^{\mathbb{C}}$ if and only if $L_{A \oplus \overline{A}}(X) \geq 0$.

Write A = S + iT where S is a tuple of $n \times n$ real symmetric matrices and T is a tuple of $n \times n$ real skew symmetric matrices. Then $A \oplus \overline{A}$ is unitarily equivalent to the

g-tuple of real symmetric matrices B defined by

$$B := \begin{pmatrix} S & -T \\ T & S \end{pmatrix} = U^* \begin{pmatrix} S + iT & 0 \\ 0 & S - iT \end{pmatrix} U$$
(III.3.2)

where $U \in M_{2n}(\mathbb{C})$ is the unitary

$$U = \frac{\sqrt{2}}{2} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}.$$

We conclude that $X \in \mathcal{D}_A^{\mathbb{C}}$ if and only if

$$L_B(X) \geq 0.$$

It follows that $\mathcal{D}_A^{\mathbb{C}} = \mathcal{D}_B^{\mathbb{C}}$.

The converse is straightforward.

We are now in position to complete the proof of the Theorem I.7.2.

Proof of Theorem I.7.2. Let $\mathcal{D}_A^{\mathbb{C}}$ be a compact complex free spectrahedron which is closed under complex conjugation and let $X \in \mathcal{D}_A^{\mathbb{C}}(n)$. In light of Lemma III.3.3, we may without loss of generality assume that A is a g-tuple of real symmetric matrices. Set $\ell = \dim \mathfrak{K}_{A,X}^{\mathbb{C}}$. If X is an element of $\mathcal{D}_A^{\mathbb{R}}$, that is, if X is a tuple of real symmetric matrices, then the proof that X dilates to an Arveson extreme point $Y \in \mathcal{D}_A^{\mathbb{C}}(n+k)$ for some integer $k \leq \ell$ is immediate from Theorem III.1.3 with Lemma III.3.1.

To handle the general case where $\operatorname{Im}(X) \neq 0$, write X = S + iT where S is a g-tuple of $n \times n$ real symmetric matrices and T is a g-tuple of $n \times n$ real skew symmetric matrices. By assumption $\mathcal{D}_A^{\mathbb{C}}$ is closed under complex conjugation so we know $S - iT \in \mathcal{D}_A^{\mathbb{C}}$. As shown in equation (III.3.2), the tuple $(S + iT) \oplus (S - iT)$ is unitarily equivalent to the tuple

 $Z \in \mathcal{D}_A^{\mathbb{C}}(2n)$ defined by

$$Z := \begin{pmatrix} S & -T \\ T & S \end{pmatrix}.$$

It follows that X is a compression of Z.

Observe that Z is a tuple of $2n \times 2n$ real symmetric matrices so $Z \in \mathcal{D}_A^{\mathbb{C}}$ implies $Z \in \mathcal{D}_A^{\mathbb{R}}$. Furthermore, an application of Lemma III.3.2 shows that dim $\mathfrak{K}_{A,Z}^{\mathbb{C}} = 2\ell$, hence dim $\mathfrak{K}_{A,Z}^{\mathbb{R}} \leq 2\ell$. Theorem III.1.3 shows that Z dilates to an Arveson extreme point $\tilde{Z} \in \mathcal{D}_A^{\mathbb{R}}(2n+k)$ for some integer $k \leq 2\ell \leq 2ng$ and Lemma III.3.1 implies that \tilde{Z} is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{C}}$. It follows that X is a compression of the Arveson extreme point \tilde{Z} .

As in the real case, the proof that $\mathcal{D}_A^{\mathbb{C}}$ is the matrix convex hull of its absolute extreme points is given immediately after the statement of Theorem I.7.2.

III.4 Remarks

This section contains remarks which expand on the historical context of our results. Section III.4.1 discusses the number of parameters needed to express a tuple as a matrix convex combination of absolute extreme points, while Section III.4.2 explores the relationship between the absolute extreme points of free spectrahedra and of general matrix convex sets. Section III.4.3 discusses infinite dimensional operator convex sets in Arveson's original context.

III.4.1 Parameter counts for (matrix) convex combinations of extreme points

The classical Caratheodory Theorem gives an upper bound on how many terms are required to represent an element of a convex set as a convex combination of its extreme points. Theorem I.7.2 is the analog of this for a free convex set. In addition to giving a bound on the number of absolute extreme points needed to express an arbitrary tuple $X \in \mathcal{D}_A^{\mathbb{K}}(n)$, Theorem I.7.2 gives a bound on the number of parameters needed to express the absolute extreme points appearing in the matrix convex combination for X.

Given a compact free spectrahedron $\mathcal{D}_A^{\mathbb{K}}$, the classical Caratheodory Theorem states that a tuple $X \in \mathcal{D}_A^{\mathbb{K}}(n) \subset SM_n(\mathbb{K})^g$ can be written as a convex combination of dim $SM_n(\mathbb{K})^g + 1$ classical extreme points of $\mathcal{D}_A^{\mathbb{K}}(n)$, each an element of $SM_n(\mathbb{K})^g$. The maximum number of parameters in the extreme points required by this classical representation is

$$(\dim SM_n(\mathbb{K})^g + 1)(\dim SM_n(\mathbb{K})^g) = (n(n+1)g/2 + 1)(n(n+1)g/2) = O(n^4g^2).$$

In contrast, Theorem I.7.2 shows that $X \in \mathcal{D}_A^{\mathbb{K}}(n)$ can be written as a matrix convex combination of a single Arveson extreme point $Y \in \mathcal{D}_A^{\mathbb{K}}(n+k)$ for some integer $k \leq 2ng + n$. The maximum parameter count on the Arveson extreme point required in this dimension-free representation is

$$\dim SM_{2n(g+1)}(\mathbb{K})^g = 2(n+ng)(n+ng+1)g = O(n^2g^3).$$

This suggests that matrix convex combinations are advantageous over classical convex combinations in terms of the number of parameters needed to store the representation of a tuple as a (matrix) convex combination of extreme points when n is large but that they are disadvantageous if g is large.

III.4.2 Absolute extreme points of general matrix convex sets

Let $K \subset SM(\mathbb{K})^g$ be a compact matrix convex set. It is well known that there is a Hilbert space \mathcal{M} and a self-adjoint operator $\mathcal{A} \in B(\mathcal{M})$ such that $K = \mathcal{D}_{\mathcal{A}}^{\mathbb{K}}$, i.e.,

$$K = \{ X \in SM(\mathbb{K})^g \mid L_{\mathcal{A}}(X) \ge 0 \},\$$

where $L_{\mathcal{A}}(X)$ is defined as in the introduction [EW97].

While Theorem I.7.2 shows every compact real free spectrahedron $\mathcal{D}_A^{\mathbb{R}}$ is spanned by its absolute extreme points, Theorem I.4.1 shows the existence of a compact real matrix convex set $\mathcal{D}_A^{\mathbb{R}}$ which has no finite dimensional absolute extreme points.

The critical failure of our proof for a general matrix convex set $\mathcal{D}_{\mathcal{A}}^{\mathbb{R}}$ occurs at equation (III.1.10) in Theorem III.1.3. In Theorem III.1.3 the tuple A is finite dimensional, while \mathcal{A} being discussed here in Section III.4.2 is a tuple of operators acting on \mathcal{M} which may be infinite dimensional. Thus, the kernel containment

$$\ker[L_{\mathcal{A}}(\hat{\gamma}) - Q] \subset \ker[\Lambda_{\mathcal{A}}(\sigma) - cQ]$$

along with

$$L_{\mathcal{A}}(\hat{\gamma}) - Q \ge 0$$

does not imply the existence of a real number $\alpha > 0$ such that

$$L_{\mathcal{A}}(\hat{\gamma}) - Q \pm \alpha (\Lambda_{\mathcal{A}}(\sigma) - cQ) \ge 0.$$

Here $Q = \Lambda_{\mathcal{A}}(\hat{\beta}^*)L_{\mathcal{A}}(X)^{\dagger}\Lambda_{\mathcal{A}}(\hat{\beta})$ similar to before.

A concrete example of this failure follows. Let $\mathcal{M} = \ell^2(\mathbb{N})$, let $M = \operatorname{diag}(1/n^2) \in B(\mathcal{M})$, and let $N = \operatorname{diag}(1/n) \in B(\mathcal{M})$. Then $M \ge 0$ and $\{0\} = \ker M \subseteq \ker N$, however
$M - \alpha N \nleq 0$ for any real number $\alpha > 0$.

III.4.3 Alternative contexts

Much of the literature such as [A69], [DM05], and [DK15] referred to in the introduction takes a different viewpoint than the one here. We now briefly describe the correspondence.

As discussed in Section II.4, operator convex sets are in one-to-one correspondence with the set of completely positive maps on an operator system [WW99], an area which has received great interest over the last several decades. Under this correspondence, an absolute extreme point of an operator convex set becomes a boundary representation of an operator system [KLS14].

Arveson's original question was phrased in the setting of completely positive maps on an operator system. In this language, Arveson conjectured that every operator system has sufficiently many boundary representations to "completely norm it". Additionally, Arveson conjectured that these boundary representations generate the C^* -envelope. Roughly speaking, the C^* -envelope of an operator system is the "smallest" C^* -algebra containing that operator system [P02]. In this language, Theorem I.6.1 shows that every operator system with a finite-dimensional realization (see [FNT17]) is completely normed by its finite dimensional boundary representations. For further material related to operator systems, completely positive maps, boundary representations, and the C^* -envelope we direct the reader to [Ham79], [D96], [MS98], [F00], [F04], [FHL18], and [PSS18].

III.5 The NC LDL^{*} of block 3×3 matrices

This section contains a brief discussion of the NC LDL^{*} decomposition of the evaluation of a linear pencil L_A on a block 3×3 matrix. Consider a general block 3×3 tuple

$$Z \coloneqq \begin{pmatrix} X & \beta & \eta \\ \beta^* & \gamma & \sigma \\ \eta^* & \sigma^* & \psi \end{pmatrix}$$

where $X \in SM_{n_1}(\mathbb{K})^g$ and $\gamma \in SM_{n_2}(\mathbb{K})^g$ and $\psi \in SM_{n_3}(\mathbb{K})^g$ and β, η , and σ are each g-tuples of matrices of appropriate size. We know that

$$L_{A}\begin{pmatrix} X & \beta & \eta \\ \beta^{*} & \gamma & \sigma \\ \eta^{*} & \sigma^{*} & \psi \end{pmatrix} \sim_{\text{c.s.}} \begin{pmatrix} L_{A}(X) & \Lambda_{A}(\beta) & \Lambda_{A}(\eta) \\ \Lambda_{A}(\beta^{*}) & L_{A}(\gamma) & \Lambda_{A}(\sigma) \\ \Lambda_{A}(\eta^{*}) & \Lambda_{A}(\sigma^{*}) & L_{A}(\psi) \end{pmatrix} =: \mathfrak{Z}$$

where $\sim_{c.s.}$ denotes equivalence up to permutations (canonical shuffles). It follows that

$$L_A(Z) \geq 0$$
 if and only if $\mathfrak{Z} \geq 0$.

The NC LDL* of \mathfrak{Z} has as its block diagonal factor D the matrix

$$D = \begin{pmatrix} L_A(X) & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & L_A(\gamma) - \Lambda_A(\beta^*) L_A(X)^{\dagger} \Lambda_A(\beta) - W^* S^{\dagger} W \end{pmatrix}$$

where

$$S = L_A(\psi) - \Lambda_A(\eta^*) L_A(X)^{\dagger} \Lambda_A(\eta)$$
$$W = \Lambda_A(\sigma^*) - \Lambda_A(\eta^*) L_A(X)^{\dagger} \Lambda_A(\beta).$$

It follows that $L_A(Z) \ge 0$ if and only if $L_A(X) \ge 0$ and $S \ge 0$ and

$$L_A(\gamma) - \Lambda_A(\beta^*) L_A(X)^{\dagger} \Lambda_A(\beta) - W^* S^{\dagger} W \ge 0.$$

Considering the case where $\mathbb{K} = \mathbb{R}$ and $\gamma \in \mathbb{R}^g$ and $\psi = 0 \in \mathbb{R}^g$, hence $\sigma = \sigma^* \in \mathbb{R}^g$, and substituting $\eta = c\hat{\beta}$ or $\eta = 0$ gives equations (III.1.7) and (III.1.16), respectively.

III.6 Proof of Theorem I.7.1 over the real numbers

We now give a proof of Theorem I.7.1 over the real numbers. Recall that a tuple $X \in SM_n(\mathbb{K})^g$ is irreducible over \mathbb{K} if the matrices X_1, \ldots, X_g have no common reducing subspaces in \mathbb{K}^n ; a tuple is reducible over \mathbb{K} if it is not irreducible over \mathbb{K} . We begin with a lemma which shows that a tuple of real symmetric matrices is reducible over \mathbb{R} if and only if it is reducible over \mathbb{C} .

Lemma III.6.1. Let X be a g-tuple of real symmetric $n \times n$ matrices. Then X is reducible over \mathbb{R} if and only if X is reducible over \mathbb{C} .

Proof. The forward direction of the proof is straightforward.

Now assume X is reducible over \mathbb{C} . Then the assumption that X is real symmetric implies that there exists a nonzero self-adjoint matrix $W \in M_n(\mathbb{R}) \subset M_n(\mathbb{C})$ such that $W \neq \alpha I_n$ for any $\alpha \in \mathbb{C}$ and WX - XW = 0. Let $E_1, \ldots, E_k \subset \mathbb{C}^n$ denote the real eigenspaces of W corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ of W, respectively. Since X is real and WX - XW = 0, each E_j is a reducing subspace for X. Additionally, we must have $k \geq 2$ since W is not a constant multiple of the identity. Therefore, each E_j is a nontrivial real reducing subspace for X. We conclude that X is reducible over \mathbb{R} .

We now prove our real analogue of [EHKM18, Theorem 1.1 (3)], Theorem I.7.1.

Proof of Theorem I.7.1 when $\mathbb{K} = \mathbb{R}$. First assume that $X \in SM_n(\mathbb{R})^g$ is an irreducible Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$. Lemma III.3.1 (1) shows that X is an Arveson extreme point of $\mathcal{D}_A^{\mathbb{C}}$. Furthermore, Lemma III.6.1 shows that X is irreducible over \mathbb{C} . Therefore, [EHKM18, Theorem 1.1 (3)] shows that X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{C}}$. It immediately follows that X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{R}}$.

We now prove the converse. The proof that X is irreducible over \mathbb{R} when X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{R}}$ is straightforward. The fact that X must be an Arveson extreme point of $\mathcal{D}_A^{\mathbb{R}}$ when X is an absolute extreme point of $\mathcal{D}_A^{\mathbb{R}}$ is immediate from [EHKM18, Lemma 3.13], the proof of which is identical over the reals or complexes. \Box

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