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Local Indecomposability of Hilbert Modular representations and Mumford-Tate conjecture

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Local Indecomposability of Hilbert Modular representations and Mumford-Tate conjecture

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Abstract of the Dissertation

Local Indecomposability of Hilbert Modular representations and Mumford-Tate conjecture

by

Bin Zhao

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Haruzo Hida, Chair

In this thesis, we use the Serre-Tate deformation theory for ordinary abelian varieties to study its associated p-adic Galois representations. As applications, we study two types of questions. The first is to determine the indecomposability of the Galois representations restricted to the p-decomposition group attached to a non CM nearly ordinary weight two Hilbert modular form over a totally real field. Then second is to study the Mumford-Tate conjecture for absolutely simple abelian fourfolds with trivial endomorphism algebras.

The dissertation of Bin Zhao is approved.

Don Blasius

Chandrashekhar Khare

Hongquan Xu

Haruzo Hida, Committee Chair

University of California, Los Angeles

2014

To my parents

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CHAPTER 1

Introduction

Let $A_{/k}$ be an abelian variety over an algebraically closed field k of characteristic $p > 0$. A theorem of Serre and Tate states that the infinitesimal deformation of $A_{/k}$ is equivalent to that of its p-divisible group. When $A_{/k}$ is ordinary, the formal moduli space $\widehat{\mathfrak{M}}_{A/k}$ has a formal group structure. To be more precise, let $\widehat{\mathfrak{M}}_{A/k}$ be the set-valued functor on the category of local artinian rings with residue field k such that:

$$\widehat{\mathfrak{M}}_{A/k}(R) = \{ \textit{ isomorphism classes of liftings of } A_{/k} \textit{ to } R\}.$$

Then there is an isomorphism of functors:

$$\widehat{\mathfrak{M}}_{A/k} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m).$$

This theorem of Serre and Tate has a lot of applications. It can be used to study the local model of Shimura varieties of Hodge type at an ordinary closed point. It can also be used to study the Galois representation attached to an abelian variety over a local field with good ordinary reduction. In this thesis, we use the above ideas to study two types of questions in number theory and arithmetic geometry, which will be sketched below.

1.1 Local Indecomposability of Hilbert Modular Galois Representations

Let F be a totally real field and f be a Hilbert modular form of level $\mathfrak{m}$ over F. Assume that f is a Hecke eigenform and let K_f be its Hecke field. For any prime λ of K_f over a rational prime p, let $K_{f,\lambda}$ be the completion of K_f at λ. It is well known that there is a

Galois represention $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(K_{f,\lambda})$ attached to f. Moreover if the eigenform f is nearly p-ordinary, then up to equivalence the restriction of ρ_f to the decomposition group $D_{\mathfrak{p}}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$ is of the shape (see [52] Theorem 2 for the ordinary case and [19] Proposition 2.3 for the nearly ordinary case):

$$\rho_f|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}.$$

R.Greenberg once asked when the local representation $\rho_f|_{D_{\mathfrak{p}}}$ splits. In the elliptic modular case, it was studied by Ghate and Vatsal and they gave an answer in [14] under some conditions. In a recent paper [2], joint with Balasubramanyam, they generalized their result to the Hilbert modular case under some restrictive conditions.

In this thesis, we are mainly concerned with the case that f is parallel weight two. We also need to put the following technical condition on f when the degree of F over $\mathbb{Q}$ is even: there exists a finite place v of F such that π_v is square integrable (i.e. special or supercuspidal) where $\pi_f = \otimes_v \pi_v$ is the automorphic representation of $GL_2(F_{\mathbb{A}})$ associated to f ($F_{\mathbb{A}}$ is the adele ring of F). Then the first main result of this thesis is:

Theorem 1. *If f does not have complex multiplication, then $\rho_f|_{D_{\mathfrak{p}}}$ is indecomposable.*

We remark here that the above theorem can help us to study a problem raised by Coleman on the existence of companion forms.

1.2 Mumford-Tate Conjecture for Abelian Fourfolds

Let $A_{/F}$ be an abelian variety defined over a number field F of dimension n. Fix an algebraic closure $\bar{F}$ of F and a complex embedding $\bar{F} \to \mathbb{C}$. Let $V = H_1(A_{/\mathbb{C}}, \mathbb{Q})$ be the first singular homology group of $A_{/\mathbb{C}}$ with coefficients in $\mathbb{Q}$. Then we denote by $MT(A)_{/\mathbb{Q}}$ (resp. $Hg(A)_{/\mathbb{Q}}$) the Mumford-Tate group (resp. Hodge group) associated to the natural Hodge structure of V.

On the other hand, for any rational prime l, let $T_lA(\bar{F})$ be the l-adic Tate module of A and set $V_l = T_lA(\bar{F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then we have a Galois representation

$$\rho_l : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathbb{Q}_l}(V_l).$$

We define an algebraic group $G_{l/\mathbb{Q}_l}$ as the Zariski closure of the image of ρ_l inside the algebraic group $\mathrm{Aut}_{\mathbb{Q}_l}(V_l)$, and let $G^\circ_{l/\mathbb{Q}_l}$ be its identity component. By comparison theorem, we have an isomorphism $V \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong V_l$. Under this isomorphism, the Mumford-Tate conjecture states that:

Conjecture 1.1. *For any prime l, we have the equality $G^\circ_{l/\mathbb{Q}_l} = \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_l$.*

In this thesis, we are interested in the case that $A_{/F}$ is an absolutely simple abelian fourfolds (so in particular $n = 4$).

Let $\mathfrak{g}_{/\mathbb{Q}}$ (resp. $\mathfrak{g}_{l/\mathbb{Q}_l}$) be the Lie algebra of the algebraic group $MT(A)_{/\mathbb{Q}}$ (resp. $G_{l/\mathbb{Q}_l}$). Then let $\mathfrak{h}_{/\mathbb{Q}}$ (resp. $\mathfrak{h}_{l/\mathbb{Q}_l}$) be the subalgebra of $\mathfrak{g}_{/\mathbb{Q}}$ (resp. $\mathfrak{g}_{l/\mathbb{Q}_l}$) consisting of elements of trace 0. So we have $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{Q} \cdot \mathrm{Id}$ (resp. $\mathfrak{g}_l = \mathfrak{h}_l \oplus \mathbb{Q}_l \cdot \mathrm{Id}$). In [32], Moonen and Zarhin computed the Lie algebras $\mathfrak{h}_{/\mathbb{Q}}$ and $\mathfrak{h}_{l/\mathbb{Q}_l}$. From their result, the endomorphism $\mathrm{End}^0(A_{/\bar{F}})$ together with its action on the Lie algebra $\mathrm{Lie}(A_{/\bar{F}})$ determines the Lie algebras $\mathfrak{h}_{/\mathbb{Q}}$ and $\mathfrak{h}_{l/\mathbb{Q}_l}$ uniquely except in the case that $\mathrm{End}^0(A_{/\bar{F}}) = \mathbb{Q}$. When $\mathrm{End}^0(A_{/\bar{F}}) = \mathbb{Q}$, we have two possibilities for $\mathfrak{h}$: either $\mathfrak{h} = \mathfrak{sp}_4$ over $\bar{\mathbb{Q}}$ or $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\bar{\mathbb{Q}}$. And similarly we have two possibilities for $\mathfrak{h}_l$: either $\mathfrak{h}_l = \mathfrak{sp}_4$ over $\bar{\mathbb{Q}}_l$ or $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\bar{\mathbb{Q}}_l$. The first case happens when $A_{/F}$ comes from a generic element in the Siegel moduli space while the second happens when $A_{/F}$ comes from an analytic family of abelian varieties constructed by Mumford in [34]. The second main result of this thesis is the following:

Theorem 2. *If $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\bar{\mathbb{Q}}_l$, then $A_{/F}$ comes from a Shimura curve constructed by Mumford in [34].*

Since Deligne proved the inclusion $\mathfrak{h}_l \subseteq \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_l$, combining Theorem 2 with previous work of Moonen and Zarhin, the Mumford-Tate conjecture holds for all absolutely simple abelian fourfolds.

1.3 Organization of the Thesis

In chapter 2, we review the Serre-Tate deformation theory for abelian varieties and the construction of Serre-Tate coordinates of ordinary abelian varieties. We also explain how to get p-th power roots of the Serre-Tate coordinates from a splitting of the connected-étale exact sequence of the p-divisible groups.

In chapter 3, we explain how the Serre-Tate coordinates of an abelian variety over a local field with good ordinary reduction determine the Galois representation associated to its p-adic Tate module.

In chapter 4, we give a brief review of Hilbert modular Shimura varieties and Siegel modular Shimura varieties. We give the integral models of these Shimura varieties we want to work with and list some properties which will be used in later argument.

In chapters 5 and 6, we give the proofs of Theorem 1 and Theorem 2 respectively. As both proofs are quite technical, we postpone to chapters 5 and 6 for summaries of techniques and tools employed in the proofs.

CHAPTER 2

Serre-Tate deformation theory for abelian varieties

In this chapter, we review the deformation theory of abelian varieties. We do not present the results in fully generality and refer the readers to [29] and [20] for more details.

2.1 Cartier duality theorem

In this section we recall the Cartier duality theorem for abelian schemes. For more details, see [33] section 14, 15 when the base is the spectrum of an algebraically closed field and [37] Chapter 1 for the general base.

Theorem 3. *(See [37] Theorem 1.1) Let $\mathbb{A}$ and $\mathbb{B}$ be two abelian schemes over a scheme S, and $f : \mathbb{A} \to \mathbb{B}$ be an S-isogeny. Let $\mathbb{A}^t$ (resp. $\mathbb{B}^t$) be the dual of $\mathbb{A}$ (resp. $\mathbb{B}$) and $f^t : \mathbb{B}^t \to \mathbb{A}^t$ be the dual of f. Then there is a pairing of finite flat group schemes over S:*

$$\langle \cdot, \cdot \rangle_f : ker(f) \times ker(f^t) \to \mathbb{G}_{m/S},$$

which is non-degenerate, bilinear and compatible with arbitrary base change.

Now let S be an arbitrary scheme and $\mathbb{A}_{/S}$ be an abelian scheme. For any integer $N > 0$, the multiplication by N map $[N] : \mathbb{A} \to \mathbb{A}$ is an S-isogeny. In this case we denote the pairing $\langle \cdot, \cdot \rangle_{[N]}$ in Theorem 3 by

$$\mathrm{E}_{\mathbb{A}/S,N} : \mathbb{A}[N] \times \mathbb{A}^t[N] \to \mathbb{G}_{m/S}.$$

2.2 Serre-Tate Theorem on the deformation of abelian varieties

Let p be a prime number. For any ring R, we use $\mathcal{A}(R)$ to denote the category of abelian schemes over R. We also use R_{red} to denote the quotient $R/nil(R)$, where $nil(R)$ is the nilradical of R. Let $I \subseteq R$ be an ideal and set $R_0 = R/I$. We use $\mathrm{Def}(R, R_0)$ to denote the category of triples (A_0, G, i) consisting of an abelian scheme A_0 over R_0, a p-divisible group G over R, and an isomorphism of p-divisible groups $i : G_0 \to A_0[p^\infty]$ over R_0, where G_0 is the p-divisible group over R_0 obtained from G under the base change $R \to R_0$, and $A_0[p^\infty]$ is the p-divisible group of A_0. Later we also write $G_0 = G \otimes_R R_0$ to simplify the notation.

For any object A in $\mathcal{A}(R)$, we have a natural isomorphism of p-divisible groups over R_0:

$$i : (A[p^\infty])_0 = A[p^\infty] \otimes_R R_0 \to A_0[p^\infty],$$

where $A_0 = A \otimes_R R_0$ is the abelian scheme over R_0 obtained by base change. So we have a functor:

$$\begin{aligned} \varphi : \quad \mathcal{A}(R) \quad &\to \mathrm{Def}(R, R_0) \\ A \quad &\mapsto (A_0, A[p^\infty], i). \end{aligned}$$

The Serre-Tate theorem tells us that the above functor φ is an equivalence of categories under certain conditions. More precisely, we have:

Theorem 4. *If p is nilpotent in R and the ideal I is nilpotent in R, then the functor φ is an equivalence of categories.*

The proof of the above theorem is long so we divide it into several lemmas.

By the assumptions, there exist integers n and v, such that $q = p^n$ vanishes in R and $I^{v+1} = (0)$ in R.

Lemma 2.1. *Let G and H be two abelian sheaves on the f.p.p.f. cite of R, which satisfy the following conditions:*

1. *G is q-divisible, i.e. the multiplication by q homomorphism $[q] : G \to G$ is an epimorphism;*

2. *the subsheaf $\hat{H}$ of H defined by $\hat{H}(A) = \ker(H(A) \to H(A_{red}))$ for any R-algebra A, is locally represented by a formal Lie group;*

3. *H is formally smooth, i.e. for any R-algebra A and any nilpotent ideal J of A, the map $H(A) \to H(A/J)$ is surjective.*

Let $G_0 = G \otimes_R R_0$ (resp. $H_0 = H \otimes_R R_0$) be the inverse image of G (resp. H) under the base change $R \to R_0$. Then the following statements hold:

1. *the morphism $\psi : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G_0, H_0)$ obtained by the base change $R \to R_0$ is injective;*

2. *for any morphism $f_0 : G_0 \to H_0$, there exists a unique morphism $F(v, f_0) : G \to H$ which lifts $q^v f_0$;*

3. *a morphism $f_0 : G_0 \to H_0$ can be lifted to R if and only if the morphism $F(v, f_0)$ defined above annihilate the subsheaf $G[q^v] = \ker([q^v] : G \to G)$ of G.*

Proof. We begin by making two remarks. The first remark is that the sheaves $\mathrm{Hom}(G, H)$ and $\mathrm{Hom}(G_0, H_0)$ are q-torsion free. In fact, as G is q-divisible, we have the exact sequence:

$$0 \to G[q] \to G \xrightarrow{[q]} G \to 0.$$

As the functor $\mathrm{Hom}(-, H)$ is left exact, we see that $\mathrm{Hom}(G, H)$ is q-torsion free. Since G is q-divisible, so is G_0. We repeat the above argument to G_0 and it follows that $\mathrm{Hom}(G_0, H_0)$ is also q-torsion free.

The second remark is that the subsheaf H_I of H defined by $H_I(A) = \ker(H(A) \to H(A/IA))$ for any R-algebra A is killed by q^v. In fact, as I is nilpotent in R, H_I is a subsheaf of $\hat{H}$. As the question is local, we can assume that $\hat{H}$ is represented by a formal

Lie group. After choosing suitable coordinates $X_1, \ldots, X_r$ of $\hat{H}$, the morphism $[q] : \hat{H} \to \hat{H}$ can be expressed in terms of the coordinates:

$$([q](X))_i = qX_i + \textit{ higher degree terms of } X\text{'s} .$$

If x is a point of $H_I(A)$ for an R-algebra A, its coordinates lie in IA by definition. As q vanishes in R, the coordinates of $[q](x)$ lie in I^2A and hence $[q](H_I) \subseteq H_{I^2}$. Since the above argument is true for any nilpotent ideal I, we have the inclusion $[q](H_{I^k}) \subseteq H_{I^{2k}} \subseteq H_{I^{k+1}}$ for any integer $k \geq 1$. In particular, $[q^v](H_I) \subseteq H_{I^{v+1}} = 0$ as $I^{v+1} = (0)$.

Notice that $\ker\psi \subseteq \mathrm{Hom}(G, H_I)$. From the first remark above, the sheaf $\mathrm{Hom}(G, H_I)$ is q-torsion free and hence q^v-torsion free. From the second remark, the sheaf H_I is killed by q^v, and so is $\mathrm{Hom}(G, H_I)$. So $\mathrm{Hom}(G, H_I) = 0$ and ψ is injective. This proves the first statement of the lemma.

For the second statement, the uniqueness of $F(v, f_0)$ follows directly from the injectivity of ψ. For the existence, we give the explicit expression of the morphism $F(v, f_0)$. For any R-algebra A, since the sheaf H is formally smooth, the reduction map $H(A) \to H(A/IA)$ is surjective. We define a homomorphism:

$$\begin{aligned} j : \quad H(A/IA) &\to H(A) \\ h \quad &\mapsto q^v\tilde{h}, \end{aligned}$$

where $\tilde{h} \in H(A)$ is any lifting of $h \in H(A/IA)$. Notice that any two liftings of h are different by an element in $H_I(A)$, which is killed by q^v by the second remark above. It follows that the homomorphism j is well defined. Now we define a homomorphism $F(v, f_0)(A) : G(A) \to H(A)$ as the composite:

$$G(A) \xrightarrow{\textit{reduction}} G(A/IA) \xrightarrow{f_0} H_0(A/IA) = H(A/IA) \xrightarrow{j} H(A).$$

It is easy to check that the formation of $F(v, f_0)(A)$ is functorial in A and hence we get a morphism $F(v, f_0)$ which lifts $q^v f_0$.

We remain to prove the last statement. First notice that if $f_0 \in \mathrm{Hom}(G, H)$ is a lifting of $f_0 \in \mathrm{Hom}(G_0, H_0)$, then $q^v f = F(v, f_0)$ due to the injectivity of the map ψ. This proves the 'only if' part. Now we prove the 'if' part.

Applying the left exact functor $\mathrm{Hom}(-, H)$ to exact sequence

$$0 \to G[q] \to G \xrightarrow{[q]} G \to 0,$$

we get another exact sequence

$$0 \to \mathrm{Hom}(G, H) \xrightarrow{[q]} \mathrm{Hom}(G, H) \to \mathrm{Hom}(G[q^v], H).$$

By assumption, the restriction of $F(v, f_0)$ to the subsheaf $G[q^v]$ is zero. So we can find $f \in \mathrm{Hom}(G, H)$ such that $q^v f = F(v, f_0)$. Since $q^v f$ lifts $q^v f_0$, and the sheaf $\mathrm{Hom}(G_0, H_0)$ is q-torsion free, f is a lifting of f_0. $\square$

Now we can prove the full-faithfulness in Theorem 4. We need to prove the following statement: given two abelian schemes A, B over R, a homomorphism $f[p^\infty] : A[p^\infty] \to B[p^\infty]$ of p-divisible groups over R, and a homomorphism $f_0 : A_0 \to B_0$ of abelian schemes over R_0, such that the induced homomorphism $f_0[p^\infty] : A_0[p^\infty] \to B_0[p^\infty]$ coincides with $(f[p^\infty])_0 = f[p^\infty] \otimes_R R_0$, then there exists a unique homorphism $f : A \to B$ which induces $f[p^\infty]$ and lifts f_0.

We remark that if we regard abelian schemes and p-divisible groups over R as abelian sheaves on the f.p.p.f site of R, then they satisfy all the assumptions in Lemma 2.1. So the uniqueness of f follows from the injectivity of the morphism $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A_0, B_0)$ proved in the first part of Lemma 2.1.

We continue to prove the existence of f. By Lemma 2.1, we have a homomorphism $F(v, f_0) : A \to B$ which lifts the homomorphism $q^v f_0$. The induced homomorphism $F(v, f_0)[p^\infty] : A[p^\infty] \to B[p^\infty]$ of p-divisible groups is a lifting of $q^v f_0[p^\infty] = q^v(f[p^\infty])_0 = (q^v f[p^\infty])_0$.

From the injectivity of the morphism

$$\mathrm{Hom}(A[p^\infty], B[p^\infty]) \to \mathrm{Hom}(A_0[p^\infty], B_0[p^\infty]),$$

we have the equality $F(v, f_0) = q^v f[p^\infty]$. It follows that $F(v, f_0)$ annihilates the group $A[q^v]$. From Lemma 2.1, the homomorphism f_0 can be lifted uniquely to a homomorphism $F : A \to B$. Since $F[p^\infty]$ and $f[p^\infty]$ both lift $f_0[p^\infty]$, we have $F[p^\infty] = f[p^\infty]$.

To prove Theorem 4, it remains to prove that given an abelian scheme A_0 over R_0, a p-divisible group G over R and an isomorphism $i : A_0[p^\infty] \to G_0 = G \otimes_R R_0$, there exists an abelian scheme A over R which induces the triple (A_0, G, i).

As R is a nilpotent thickening of R_0, we can always lift the abelian scheme A_0 to an abelian scheme B over R. Let $i_0 : B_0 = B \otimes_R R_0 \to A_0$ be the corresponding isomorphism and $i_0[p^\infty] : B_0[p^\infty] \to A_0[p^\infty]$ be the induced isomorphism of p-divisible groups.

From Lemma 2.1, there exists a morphism $F(v, i_0[p^\infty]) : B[p^\infty] \to G$ (resp. $F(v, (i_0[p^\infty])^{-1}) : G \to B[p^\infty]$) which lifts the morphism $q^v i_0[p^\infty] : B_0[p^\infty] \to A_0[p^\infty]$ (resp. $q^v(i_0[p^\infty])^{-1} : A_0[p^\infty] \to B_0[p^\infty]$). From the injectivity of ψ in Lemma 2.1, the composite of $F(v, i_0[p^\infty])$ and $F(v, (i_0[p^\infty])^{-1})$ (in either order) is the endomorphism $[q^{2v}]$. Hence we have an exact sequence of f.p.p.f. sheaves over R:

$$0 \to K \to B[p^\infty] \xrightarrow{F(v, i_0[p^\infty])} G \to 0,$$

where K is a subgroup scheme of $B[q^{2v}]$.

As the morphism $[q^v](i_0[p^\infty]) : B_0[p^\infty] \to A_0[p^\infty]$ is an isogeny and hence flat, and flatness is an open property, the morphism $F(v, i_0[p^\infty])$ is flat and K is a finite flat subgroup scheme of $B[q^{2v}]$. So we can consider the quotient $A = B/K$, which is also an abelian scheme over R. Since K lifts $B_0[q^v]$, A lifts $B_0/B_0[q^v] \cong B_0 \cong A_0$, and the exact sequence

$$0 \to K \to B[p^\infty] \xrightarrow{F(v, i_0[p^\infty])} G \to 0$$

gives an isomorphism $A[p^\infty] \cong B[p^\infty]/K \cong G$,which is exactly what we wanted to prove.

2.3 Serre-Tate coordinates for ordinary abelian varieties

In this section we fix an algebraically closed field k of characteristic $p > 0$. Let A be an abelian variety over k of dimension $g \geq 1$. We have seen from Theorem 4 that the infinitesimal deformation of the abelian variety $A_{/k}$ is equivalent to that of its p-divisible group $A[p^\infty]_{/k}$. In the rest of this section, we always make the assumption that the abelian variety $A_{/k}$ is ordinary, i.e. its p-adic Tate module $\mathrm{T}_pA(k)$ is a free $\mathbb{Z}_p$-module of rank g. It turns out that under this assumption, the formal moduli space $\widehat{\mathfrak{M}}_{A/k}$ of $A_{/k}$ has a formal group structure. To be more precise, we have the following:

Theorem 5. *Let A be an ordinary abelian variety over k and R be an artinian local ring with residue field k. Then the following statements hold:*

1. *there is a bijection:*

$$\begin{aligned}\{\text{ isomorphism classes of liftings of } A_{/k} \text{ to } R\} &\to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_pA(k)\otimes_{\mathbb{Z}_p}\mathrm{T}_pA^t(k),\widehat{\mathbb{G}}_m)\\ \mathbb{A}_{/R} &\mapsto q(\mathbb{A}_{/R};-,-).\end{aligned}$$

   Moreover, the above bijection is functorial for various R's and give an isomorphism of functors:

$$\widehat{\mathfrak{M}}_{A/k}\to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_pA(k)\otimes_{\mathbb{Z}_p}\mathrm{T}_pA^t(k),\widehat{\mathbb{G}}_m);$$

2. *let $\mathbb{A}_{/R}$ be a lifting of $A_{/k}$ and let $\mathbb{A}^t_{/R}$ be its dual. under the canonical isomorphism $A\cong(A^t)^t$, we have the formula:*

$$q(\mathbb{A}_{/R};\alpha,\alpha_t)=q(\mathbb{A}^t_{/R};\alpha_t,\alpha),$$

   for any $\alpha\in\mathrm{T}_pA(k)$ and $\alpha_t\in\mathrm{T}_pA^t(k)$;

3. *let $B_{/k}$ be another ordinary abelian variety over k and $\mathbb{A}_{/R}$ (resp. $\mathbb{B}_{/R}$) be a lifting of $A_{/k}$ (resp. $B_{/k}$). Let $f:A\to B$ be a homomorphism and $f^t:B^t\to A^t$ be its dual. Then f can be lifted to a homomorphism $\mathbb{f}:\mathbb{A}\to\mathbb{B}$ if and only if*

$$q(\mathbb{A}_{/R};\alpha,f^t(\beta_t))=q(\mathbb{B}_{/R};f(\alpha),\beta_t),$$

   for any $\alpha\in\mathrm{T}_pA(k)$ and $\beta_t\in\mathrm{T}_pB^t(k)$.

Proof. 1. From Theorem 4, to get a lifting $\mathbb{A}_{/R}$ of $A_{/k}$ is equivalent to getting a lifting $\mathbb{A}[p^\infty]_{/R}$ of its p-divisible group $A[p^\infty]_{/k}$. Since $A_{/k}$ is ordinary and k is algebraically closed, the p-divisible group $A[p^\infty]_{/k}$ is a product:

$$A[p^\infty] = \hat{A} \times \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p).$$

For any $n \geq 1$, the paring $\mathrm{E}_{A_{/k},p^n} : A[p^n] \times A^t[p^n] \to \mu_{p^n}$ defines an isomorphism of k-group schemes:

$$\hat{A}[p^n] \to \mathrm{Hom}(A^t[p^n](k), \mu_{p^n}).$$

Taking the inverse limit, we have an isomorphism of formal groups over k:

$$\hat{A} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m),$$

and it induces a pairing:

$$\mathrm{E}_{A_{/k}} : \hat{A} \times \mathrm{T}_p A^t(k) \to \widehat{\mathbb{G}}_m.$$

Since R is local artinian ring with residue field k, the p-divisible group $\mathbb{A}[p^\infty]_{/R}$ of $\mathbb{A}_{/R}$ sits in the connected-étale exact sequence:

$$0 \to \hat{\mathbb{A}} \to \mathbb{A}[p^\infty] \to \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

We remark here that if we regard $\mathbb{A}$ as an f.p.p.f. sheaf on R, then $\hat{\mathbb{A}}$ is defined in the previous section as a subsheaf of $\mathbb{A}$. In fact, $\hat{\mathbb{A}}$ is the unique formal subgroup of $\mathbb{A}[p^\infty]$ which lifts $\hat{A}$. Since $\hat{\mathbb{A}}$ and $\hat{\mathbb{A}}[p^n]$ are multiplicative, and R is local artinian with residue field k, the isomorphisms of k-groups $\hat{A}[p^n] \to \mathrm{Hom}(A^t[p^n](k), \mu_{p^n})$ and $\hat{A} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m)$ extends uniquely to isomorphisms of R-groups $\hat{\mathbb{A}}[p^n] \to \mathrm{Hom}(A^t[p^n](k), \mu_{p^n})$ and $\hat{\mathbb{A}} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m)$, and hence we have the pairing over R:

$$\mathrm{E}_{\mathbb{A}/R,p^n} : \hat{\mathbb{A}}[p^n] \times A^t[p^n](k) \to \mu_{p^n},$$

and

$$\mathrm{E}_{\mathbb{A}/R} : \hat{\mathbb{A}} \times \mathrm{T}_p A^t(k) \to \widehat{\mathbb{G}}_m.$$

We want to construct a homomorphism $\varphi_{\mathbb{A}/R} : \mathrm{T}_pA(k) \to \hat{\mathbb{A}}(R)$ such that the connected-étale exact sequence of $\mathbb{A}[p^\infty]$ is obtained by pushing out the exact sequence

$$0 \to \mathrm{T}_pA(k) \to \mathrm{T}_pA(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{T}_pA(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

along the homomorphism $\varphi_{\mathbb{A}/R}$. Let $\mathfrak{m}$ be the maximal ideal of R. We can choose an integer n such that $\mathfrak{m}^{n+1} = (0)$. Since $p \in \mathfrak{m}$ and $\hat{\mathbb{A}}$ is a formal Lie group over R, the group $\hat{\mathbb{A}}(R)$ is annihilated by p^n. So we can define a group homomorphism:

$$\begin{aligned} \varphi_n : \quad A(k) &\to \mathbb{A}(R) \\ x &\mapsto p^n \tilde{x}, \end{aligned}$$

where $\tilde{x} \in \mathbb{A}(R)$ is any lifting of $x \in A(k)$. (Notice that $\tilde{x}$ always exists as $\mathbb{A}_{/R}$ is smooth). The restriction of φ_n to $A(k)[p^n]$ gives a homomorphism $\varphi_n : A(k)[p^n] \to \hat{\mathbb{A}}(R)$. These φ_n's are compatible in the sense that $\varphi_n \circ [p] = \varphi_{n+1} : A(k)[p^{n+1}] \to \hat{\mathbb{A}}(R)$. Hence we have a homomorphism $\varphi_{\mathbb{A}/R} : \mathrm{T}_pA(k) \to \hat{\mathbb{A}}(R)$ by taking the limit of φ_n's.

Now we define the Serre-Tate coordinates $q(\mathbb{A}_{/R}; -, -) \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_pA(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_pA^t(k), \widehat{\mathbb{G}}_m)$, such that $q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}/R}(\varphi_{\mathbb{A}/R}(\alpha), \alpha_t)$, for any $\alpha \in \mathrm{T}_pA(k)$ and $\alpha_t \in \mathrm{T}_pA^t(k)$.

From the above construction, we have a chain of isomorphisms of functors:

$$\begin{aligned} \{\text{ isomorphism classes of } \mathbb{A}_{/R} \text{ lifting } A_{/k}\} &\cong \{\text{ isomorphism classes of } \mathbb{A}[p^\infty]_{/R} \text{ lifting } A[p^\infty]_{/k}\} \\ &\cong \mathrm{Ext}_R(\mathrm{T}_pA(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \hat{\mathbb{A}}) \\ &\cong \mathrm{Hom}(\mathrm{T}_pA(k), \hat{\mathbb{A}}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_pA(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_pA^t(k), \widehat{\mathbb{G}}_m) \end{aligned}$$

So the construction of $q(\mathbb{A}_{/R}; -, -)$ is functorial and we get (1).

2. Recall that we fix an integer n, such that $\mathfrak{m}^{n+1} = (0)$. Let α_n (resp. $\alpha_{n,t}$) be the image of α (resp. α_t) under the projection $\mathrm{T}_pA(k) \to A(k)[p^n]$ (resp. $\mathrm{T}_pA^t(k) \to A^t(k)[p^n]$). By construction, we have:

$$q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}/R, p^n}(\varphi_n(\alpha_n), \alpha_{t,n}),$$

$$q(\mathbb{A}^t_{/R};\alpha_t,\alpha)=\mathrm{E}_{\mathbb{A}^t/R,p^n}(\varphi_n(\alpha_{t,n}),\alpha_n).$$

Now we need the following:

Lemma 2.2. *Fix $x\in\hat{\mathbb{A}}(R)[p^n]$ and $y\in A^t(k)[p^n]$. There exists an artinian local ring R' which is finite flat over R, and a point $Y\in\mathbb{A}^t(R')[p^n]$ lifting y. For any such R' and Y, we have the equality in $\widehat{\mathbb{G}}_m(R')$:*

$$\mathrm{E}_{\mathbb{A}/R',p^n}(x,y)=\mathrm{E}_{\mathbb{A}/R',p^n}(x,Y).$$

We remark here that in the above equality, the element $\mathrm{E}_{\mathbb{A}/R',p^n}(x,y)$ is the image of $\mathrm{E}_{\mathbb{A}/R,p^n}(x,y)\in\widehat{\mathbb{G}}_m(R)$ under the finite flat extension $R\to R'$.

Proof. Since the abelian scheme $\mathbb{A}^t$ is smooth over R, we can find $Y_1\in\mathbb{A}^t(R)$ which lifts $y\in A^t(k)[p^n]$. Then $Y_2=p^nY_1$ lies in $\hat{\mathbb{A}}^t(R)$. Since $\hat{\mathbb{A}}^t$ is a formal Lie group and hence p-divisible, and R is local artinian, we can find another local artinian ring R' which is finite flat over R and $Y_3\in\hat{\mathbb{A}}^t(R')$ such that $p^nY_3=Y_2$. Then $Y=Y_1-Y_3$ belongs to $\mathbb{A}^t(R')[p^n]$ and lifts $y\in A^t(k)[p^n]$. Notice that $\mathrm{E}_{\mathbb{A}/R',p^n}(-,y),\mathrm{E}_{\mathbb{A}/R',p^n}(-,Y):\hat{\mathbb{A}}[p^n]\to\mu_{p^n}$ both lifts the homomorphism $\mathrm{E}_{A/k,p^n}(-,y):\hat{A}[p^n]\to\mu_{p^n}$. Since both $\hat{A}[p^n]$ and μ_{p^n} are multiplicative, the lifting of $\mathrm{E}_{A/k,p^n}(-,y)$ to R' is unique. So we get the desired equality. □

Now we choose $\mathfrak{A}_n\in\mathbb{A}(R)$ (resp. $\mathfrak{A}_{t,n}\in\mathbb{A}^t(R)$) as a lifting of $\alpha_n\in A(k)[p^n]$ (resp. $\alpha_{t,n}\in A^t(k)[p^n]$). Since the group $\hat{\mathbb{A}}(R)$ and $\hat{\mathbb{A}}^t(R)$ are killed by p^n, we have $\mathfrak{A}_n\in\mathbb{A}(R)[p^{2n}]$ and $\mathfrak{A}_{t,n}\in\mathbb{A}^t(R)[p^{2n}]$.

Now we want to prove the following formula:

$$\frac{q(\mathbb{A}_{/R};\alpha,\alpha_t)}{q(\mathbb{A}^t_{/R};\alpha_t,\alpha)}=\mathrm{E}_{\mathbb{A}/R,p^{2n}}(\mathfrak{A}_n,\mathfrak{A}_{t,n}).$$

By Lemma 2.2, we can find a local artinian ring R' which is finite flat over R, and $B_n\in\mathbb{A}(R')[p^n]$ (resp. $B_{n,t}\in\mathbb{A}^t(R')[p^n]$) lifting $\alpha_n\in A(k)[p^n]$ (resp. $\alpha_{t,n}\in A^t(k)[p^n]$). Define

$$\mathcal{E}_n=\mathfrak{A}_n-B_n\in\hat{\mathbb{A}}(R')[p^{2n}],\mathcal{E}_{t,n}=\mathfrak{A}_{t,n}-B_{t,n}\in\hat{\mathbb{A}}^t(R')[p^{2n}].$$

Again by Lemma 2.2, we have:

$$\begin{aligned} q(\mathbb{A}_{/R};\alpha,\alpha_t) &= \mathrm{E}_{\mathbb{A}/R,p^n}(\varphi_n(\alpha_n),\alpha_{t,n}) \\ &= \mathrm{E}_{\mathbb{A}/R',p^n}(\varphi_n(\alpha_n),B_{t,n}) = \mathrm{E}_{\mathbb{A}/R',p^n}(p^n\mathfrak{A}_n,B_{t,n}) \\ &= \mathrm{E}_{\mathbb{A}/R',p^n}(p^n\mathcal{E}_n,B_{t,n}) = \mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathcal{E}_n,B_{t,n}), \end{aligned}$$

and similarly

$$q(\mathbb{A}^t_{/R};\alpha_t,\alpha) = \mathrm{E}_{\mathbb{A}^t/R',p^{2n}}(\mathcal{E}_{t,n},B_n) = \frac{1}{\mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n,\mathcal{E}_{t,n})}.$$

So to prove the above formula, it is enough to prove:

$$\mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathcal{E}_n,B_{t,n})\cdot\mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n,\mathcal{E}_{t,n}) = \mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathfrak{A}_n,\mathfrak{A}_{t,n}).$$

By direct calculation, we have:

$$\begin{aligned} \mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathfrak{A}_n,\mathfrak{A}_{t,n}) &= \mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n+\mathcal{E}_n,B_{t,n}+\mathcal{E}_{t,n}) \\ &= \mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n,B_{t,n})\cdot\mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathcal{E}_n,\mathcal{E}_{t,n})\cdot\mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n,\mathcal{E}_{t,n})\cdot\mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathcal{E}_n,B_{t,n} \end{aligned}$$

Since B_n is killed by p^n,

$$\mathrm{E}_{\mathbb{A}/R',p^{2n}}(B_n,B_{t,n}) = \mathrm{E}_{\mathbb{A}/R',p^n}(p^nB_n,B_{t,n}) = 1.$$

Since $\mathcal{E}_n\in\hat{\mathbb{A}}(R')[p^{2n}]$, $\mathcal{E}_{t,n}\in\hat{\mathbb{A}}^t(R')[p^{2n}]$, and both $\hat{\mathbb{A}}[p^{2n}]$ and $\hat{\mathbb{A}}^t[p^{2n}]$ are multiplicative, we have $\mathrm{E}_{\mathbb{A}/R',p^{2n}}(\mathcal{E}_n,\mathcal{E}_{t,n})=1$. So we get the desired formula.

Finally we choose liftings $\mathfrak{A}_{2n}\in\mathbb{A}(R)$ (resp. $\mathfrak{A}_{t,2n}\in\mathbb{A}^t(R)$) lifting $\alpha_{2n}\in A(k)[p^{2n}]$ (resp. $\alpha_{t,2n}\in A^t(k)[p^{2n}]$). Then $p^n\mathfrak{A}_{2n}$ (resp. $p^n\mathfrak{A}_{t,2n}$) is a lifting of α_n (resp. $\alpha_{t,n}$). From the above formula, we have:

$$\frac{q(\mathbb{A}_{/R};\alpha,\alpha_t)}{q(\mathbb{A}^t_{/R};\alpha_t,\alpha)} = \mathrm{E}_{\mathbb{A}/R,p^{2n}}(p^n\mathfrak{A}_{2n},p^n\mathfrak{A}_{t,2n}) = (\mathrm{E}_{\mathbb{A}/R,p^{3n}})(\mathfrak{A}_{2n},\mathfrak{A}_{t,2n})^{p^n}\in(1+\mathfrak{m})^{p^n}.$$

We can take n large enough so that $(1+\mathfrak{m})^{p^n}=(1)$. So we have the desired equality

$$q(\mathbb{A}_{/R};\alpha,\alpha_t) = q(\mathbb{A}^t_{/R};\alpha_t,\alpha).$$

3. By Theorem 4, the homomorphism $f : A \to B$ can be lifted to a homomorphism $\mathbb{f} : \mathbb{A} \to \mathbb{B}$ over R if and only if $f[p^\infty]$ lifts to an $\mathbb{f}[p^\infty] : \mathbb{A}[p^\infty] \to \mathbb{B}[p^\infty]$. Such an $\mathbb{f}[p^\infty]$ must be compatible with the exact sequences:

$$0 \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m) \to \mathbb{A}[p^\infty] \to \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m) \to \mathbb{B}[p^\infty] \to \mathrm{T}_p B(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

and the homomorphisms $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m)$ (induced from f^t) and $\mathrm{T}_p A(k) \to \mathrm{T}_p B(k)$ (induced from f).

On the other hand, the first exact sequence gives an element in the group

$$\mathrm{Ext}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m)).$$

By pushing out along $f^t : \mathrm{T}_p B^t(k) \to \mathrm{T}_p A^t(k)$, we get an element in

$$\mathrm{Ext}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m)) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m).$$

Under the above isomorphism, this element corresponds to the pairing

$$(\alpha, \beta_t) \mapsto q(\mathbb{A}_{/R}; \alpha, f^t(\beta_t)).$$

Similarly, the second exact sequence gives an element in

$$\mathrm{Ext}(\mathrm{T}_p B(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m)).$$

When pulling back along $f : \mathrm{T}_p A(k) \to \mathrm{T}_p B(k)$, we get an element in

$$\mathrm{Ext}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m)) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p B^t(k), \widehat{\mathbb{G}}_m).$$

Under the above isomorphism, this element corresponds to the pairing

$$(\alpha, \beta_t) \mapsto q(\mathbb{B}_{/R}; f(\alpha), \beta_t).$$

Hence $\mathbb{f}[p^\infty]$ exists if and only if the two pairings coincide, i.e.

$$q(\mathbb{A}_{/R}; \alpha, f^t(\beta_t)) = q(\mathbb{B}_{/R}; f(\alpha), \beta_t),$$

for any $\alpha \in \mathrm{T}_p A(k)$, $\beta_t \in \mathrm{T}_p B^t(k)$. This finishes the proof of the theorem.

□

Remark 2.3. We extract the following fact from the above proof. This fact will be frequently used in the following calculation. For any $\alpha \in \mathrm{T}_pA(k)$ (resp. $\alpha_t \in \mathrm{T}_pA^t(k)$), let α_n (resp. $\alpha_{t,n}$) be the image of α (resp. α_t) under the projection $\mathrm{T}_pA \to A(k)[p^n]$ (resp. $\mathrm{T}_pA^t \to A^t(k)[p^n]$). Let $\tilde{\alpha}_n \in \mathbb{A}(R)$ be an arbitrary lifting of $\alpha_n \in A(k)[p^n]$. Then by definition $\varphi_{\mathbb{A}/R}(\alpha) = p^n\tilde{\alpha}_n \in \hat{\mathbb{A}}(R)$. Since the group $\hat{\mathbb{A}}(R)$ is killed by p^n, we have $\tilde{\alpha}_n \in \mathbb{A}[p^{2n}](R)$ and $q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}_{/R},p^n}(p^n\tilde{\alpha}_n, \alpha_{t,n})$.

2.4 Section of the connected-étale sequence and p-th power roots of the Serre-Tate coordinate

We keep the notations in the previous section. Let R be an artinian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Assume that $\mathfrak{m}^{n+1} = (0)$. Let $\mathbb{A}_{/R}$ be a lifting of $A_{/k}$. Then for each integer $m > 0$, we have an exact sequence of finite group schemes over R:

$$0 \to \hat{\mathbb{A}}[p^m] \to \mathbb{A}[p^m] \to A(k)[p^m] \to 0. \tag{2.4}$$

By Cartier duality, we get an exact sequence:

$$0 \to \hat{\mathbb{A}}^t[p^m] \to \mathbb{A}^t[p^m] \to A^t(k)[p^m] \to 0 \tag{2.5}$$

over R. The splitting of the above two exact sequences are equivalent.

The sequence (1.3) does not necessarily split over R in general. The splitting of (1.3) is equivalent to the existence of an étale subgroup of $\mathbb{A}^t[p^m]$ which lifts $A^t(k)[p^m]$. Hence the exact sequence (1.3) splits after an fppf extension of R. Now we fix some integer $m \geq n$ in the following discussion. Then we can find an artinian local ring R' finite flat over R such that each $\alpha_{t,m} \in A^t(k)[p^m]$ is lifted to some $\tilde{\alpha}_{t,m} \in \mathbb{A}^t[p^m](R')$. From Lemma 2.2, we have the following equality in $\widehat{\mathbb{G}}_m(R')$:

$$q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}_{/R},p^m}(\varphi_{\mathbb{A}/R}(\alpha), \alpha_{t,m}) = \mathrm{E}_{\mathbb{A}_{/R},p^m}(\varphi_{\mathbb{A}/R'}(\alpha), \tilde{\alpha}_{t,m}) = \mathrm{E}_{\mathbb{A}_{/R'},p^m}(p^m\tilde{\alpha}_m, \tilde{\alpha}_{t,m}),$$

where $\tilde{\alpha}_m \in \mathbb{A}(R)$ is a lifting of $\alpha_m \in A(k)[p^m]$. As $\tilde{\alpha}_m \in \mathbb{A}[p^{2m}](R)$ by Remark 2.3, we have

$$q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}_{/R'},p^m}(p^m\tilde{\alpha}_m, \tilde{\alpha}_{t,m}) = \mathrm{E}_{\mathbb{A}_{/R'},p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,m}).$$

For any $s \geq 0$, we assume further that $\alpha_{t,m+s} \in A^t(k)[p^{m+s}]$ lifts to some $\tilde{\alpha}_{t,m+s} \in \mathbb{A}^t[p^{m+s}](R')$ and $p^s\tilde{\alpha}_{t,m+s} = \tilde{\alpha}_{t,m}$. Then

$$q(\mathbb{A}_{/R}; \alpha, \alpha_t) = \mathrm{E}_{\mathbb{A}_{/R'},p^{2m}}(\tilde{\alpha}_m, p^s\tilde{\alpha}_{t,m+s}) = \mathrm{E}_{\mathbb{A}_{/R'},p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,m+s})^{p^s}.$$

In other words, $\mathrm{E}_{\mathbb{A}_{/R'},p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,m+s})$ is a p^s-th root of the Serre-Tate coordinate $q(\mathbb{A}_{/R}; \alpha, \alpha_t)$. The element $\mathrm{E}_{\mathbb{A}_{/R'},p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,m+s})$ definitely depends on the choice of $\tilde{\alpha}_{t,m+s}$. In the following we want to determine how it depends on the choice of the integer m and the lifting $\tilde{\alpha}_m$.

First let $\tilde{\alpha}'_m \in \mathbb{A}(R)$ be another lifting of $\alpha_m \in A(k)[p^m]$, then $\tilde{\beta}_m = \tilde{\alpha}_m - \tilde{\alpha}'_m \in \hat{\mathbb{A}}(R)$, and hence $\tilde{\beta}_m$ is killed by p^n. When $s + n \leq m$, by Lemma 2.2 we have

$$\begin{aligned}\mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\beta}_m, \tilde{\alpha}_{t,s+m}) &= \mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\beta}_m, \alpha_{t,s+m}) = \mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\beta}_m, p^n\alpha_{t,s+m+n}) \\ &= \mathrm{E}_{\mathbb{A}/R',p^{2m}}(p^n\tilde{\beta}_m, \alpha_{t,s+m+n}) = 1.\end{aligned}$$

Hence when $0 \leq s \leq m-n$, the element $\mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,s+m})$ does not depend on the choice of $\tilde{\alpha}_m$.

Now let $m' \geq m$ be another integer and assume that $0 \leq s \leq m-n$. Let $\tilde{\alpha}'_m$ be a lifting of $\alpha_{m'} \in A(k)[p^{m'}]$. Then we have

$$\mathrm{E}_{\mathbb{A}/R',p^{2m'}}(\tilde{\alpha}_{m'}, \tilde{\alpha}_{t,s+m'}) = \mathrm{E}_{\mathbb{A}/R',p^{2m}}(p^{m'-m}(\tilde{\alpha}_{m'}), p^{m'-m}(\tilde{\alpha}_{t,s+m'})) = \mathrm{E}_{\mathbb{A}/R',p^{2m}}(p^{m'-m}(\tilde{\alpha}_{m'}), \tilde{\alpha}_{t,s+m}).$$

As $p^{m'-m}(\tilde{\alpha}_{m'})$ is a lifting of $p^{m'-m}(\alpha_{m'}) = \alpha \in A(k)[p^m]$, from the previous argument, we see that

$$\mathrm{E}_{\mathbb{A}/R',p^{2m'}}(\tilde{\alpha}_{m'}, \tilde{\alpha}_{t,s+m'}) = \mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,s+m}).$$

In other words, the element $\mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,s+m})$ does not depend on the choice of m.

Since the integer m can be as an arbitrary integer greater than n, from the above discussion we see that for every integer $s \geq 1$, there exists a p^s-th root $\mathrm{E}_{\mathbb{A}/R',p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,s+m})$ of the Serre-Tate coordinate $q(\mathbb{A}_{/R}; \alpha, \alpha_t)$ as long as we choose a compatible lifting $(\tilde{\alpha}_{t,m})_m$ of $(\alpha_{t,m})_m$.

2.5 Extension to more general bases

Let $W = W(k)$ be the ring of Witt vectors with coefficients in k and $CL_{/W}$ be the category of complete local W-algebras with residue field k. Fix an object $\mathcal{R}$ in $CL_{/W}$ with maximal ideal $\mathfrak{m}$. For each $n \geq 0$, set $R_n = \mathcal{R}/\mathfrak{m}^{n+1}$, which is an artinian local ring with residue field k and $\mathcal{R} = \varprojlim R_n$. As before we fix an ordinary abelian variety $A_{/k}$. By passing to the projective limit, we have a bijection:

$$\{\text{ isomorphism classes of liftings of } A_{/k} \text{ to } \mathcal{R}\} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m(\mathcal{R})),$$

$$\mathbb{A}_{/\mathcal{R}} \mapsto q(\mathbb{A}_{/\mathcal{R}}; -, -)$$

such that for any $\alpha \in \mathrm{T}_p(A)(k)$ and $\alpha_t \in \mathrm{T}_p(A^t)(k)$,

$$q(\mathbb{A}_{/\mathcal{R}}; \alpha, \alpha_t) = \varprojlim q(\mathbb{A}_{n/R_n}; \alpha, \alpha_t),$$

where $\mathbb{A}_n = \mathbb{A} \otimes_{\mathcal{R}} R_n$.

For any lifting $\mathbb{A}_{/\mathcal{R}}$ of $A_{/k}$ to $\mathcal{R}$, we have the connect-étale exact sequence of Barsotti-Tate groups over $\mathcal{R}$:

$$0 \to \hat{\mathbb{A}}[p^\infty] \xrightarrow{i} \mathbb{A}[p^\infty] \xrightarrow{\pi} \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Suppose that the above exact sequence splits after a faithfully flat extension of $\mathcal{R}$, i.e. there exist a W-algebra $\mathcal{R}'$ finite and flat over $\mathcal{R}$, and a morphism of Barsotti-Tate groups over $\mathcal{R}'$ $j : \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{A}[p^\infty]$, such that $\pi \circ j = id : \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

For each $n \geq 1$, set $R'_n = \mathcal{R}' \otimes_{\mathcal{R}} R_n = \mathcal{R}'/\mathfrak{m}^{n+1}\mathcal{R}'$. Then we have a split exact sequence of Barsotti-Tate groups over R'_n:

$$0 \to \hat{\mathbb{A}}_n[p^\infty] \to \mathbb{A}_n[p^\infty] \to \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

By the discussion in the previous section, for any $\alpha \in \mathrm{T}_p(A)(k)$, $\alpha_t \in \mathrm{T}_p(A^t)(k)$ and $m \geq 0$, we have a p^m-th root of the Serre-Tate coordinate $q(\mathbb{A}_{n/R_n}; \alpha, \alpha_t)$ in R'_n, which is denoted by $t_{m,n} \in \widehat{\mathbb{G}}_m(R'_n)$. By taking projective limit, we have that $t_m = \varprojlim t_{m,n} \in \widehat{\mathbb{G}}_m(\mathcal{R}')$ is a p^m-th root of the Serre-Tate coordinate $q(\mathbb{A}_{\mathcal{R}}; \alpha, \alpha_t)$ in $\mathcal{R}'$.

2.6 Frobenius action on the Serre-Tate coordinate

In this section we take k to be an algebraic closure of the prime field $\mathbb{F}_p$. Recall that $W = W(k)$ is the ring of Witt vectors with coefficients in k and K^{ur} is the quotient field of W, which can be identified with the p-adic completion of the maximal unramified extension $\mathbb{Q}_p^{ur}$ of $\mathbb{Q}_p$. Let σ (resp. Σ) be the absolute Frobenius automorphism of k (resp. W).

As before we fix an ordinary abelian variety $A_{/k}$. Consider the functor

$$\begin{aligned} \mathrm{Def}_{A_{/k}} : \ CL_{/W} &\to \mathrm{Sets} \\ \mathcal{R} &\mapsto \{ \text{ isomorphism classes of liftings of } A_{/k} \text{ to } \mathcal{R} \}. \end{aligned}$$

The Serre-Tate deformation theorem tells us that the functor $\mathrm{Def}_{A_{/k}}$ is represented by some object $\mathcal{R}^{univ}$ in $CL_{/W}$, and the Serre-Tate coordinate gives us an isomorphism of functors:

$$q(-;-,-) : \mathrm{Def}_{A_{/k}} \to \mathrm{Hom}(\mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p A^t(k), \widehat{\mathbb{G}}_m).$$

Set $\widehat{\mathfrak{M}}_{A/k} = \mathrm{Spf}(\mathcal{R}^{univ})$ which is a formal W-torus. Define formal W-torus $\widehat{\mathfrak{M}}^{(\Sigma)}_{A/k}$ by following the Cartesian diagram:

$$\begin{array}{ccc} \widehat{\mathfrak{M}}^{(\Sigma)}_{A/k} & \longrightarrow & \widehat{\mathfrak{M}}_{A/k} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(W) & \xrightarrow{\mathrm{Spec}(\Sigma)} & \mathrm{Spec}(W), \end{array}$$

and define the abelian variety $A^{(\sigma)}_{/k}$ by the following Cartesian diagram:

$$\begin{array}{ccc} A^{(\sigma)} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(k). \end{array}$$

By [29] Lemma 4.1.1, we have

Lemma 2.6. *There is a canonical isomorphism Σ of formal W-torus: $\widehat{\mathfrak{M}}^{(\sigma)}_{A/k} \to \widehat{\mathfrak{M}}_{A^{(\sigma)}/k}$, under which $\Sigma(q(\alpha, \alpha_t))$ corresponds to $q(\sigma(\alpha), \sigma(\alpha_t))$.*

Since the abelian variety $A_{/k}$ is projective, it is defined over a finite field $\mathbb{F}_q$ inside k, where $q = p^l$ for some integer $l \geq 1$. Let σ_l (resp. Σ_l) be the l times composition of

σ (resp. Σ) with itself. Then $A_{/k}^{(\sigma_l)} \cong A_{/k}$. Then Lemma 2.6 indicates that Σ_l induces an automorphism of the deformation space $\widehat{\mathfrak{m}}_{A/k}$ which sends the Serre-Tate coordinate $q(\mathbb{A}_{/W}; \alpha, \alpha_t)$ to $q(\mathbb{A}_{/W}^{(\Sigma_l)}; \sigma_l(\alpha), \sigma_l(\alpha_t))$.

For later argument, we need the following result:

Lemma 2.7. *Let K/K^{ur} be a finite Galois representation inside $\mathbb{C}_p$. Write $\mathcal{O}_K$ for the valuation ring of K. Let $\mathbb{A}_{/\mathcal{O}_K}$ be a lifting of an ordinary abelian variety $A_{/k}$ to $\mathcal{O}_K$. For any $\sigma \in \mathrm{Gal}(K/K^{ur})$, define an abelian scheme $\mathbb{A}_{/\mathcal{O}_K}^{(\sigma)}$ by the following Cartesian diagram:*

$$\begin{array}{ccc} \mathbb{A}^{(\sigma)} & \longrightarrow & \mathbb{A} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_K) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\mathcal{O}_K). \end{array}$$

Then $\mathbb{A}_{/\mathcal{O}_K}^{(\sigma)}$ is also a lifting of $A_{/k}$ and we have the equality:

$$\sigma(q(\mathbb{A}_{/\mathcal{O}_K}; \alpha, \alpha_t)) = q(\mathbb{A}_{/\mathcal{O}_K}^{(\sigma)}; \alpha, \alpha_t),$$

for any $\alpha \in \mathrm{T}_p A(k)$ and $\alpha_t \in \mathrm{T}_p A^t(k)$.

The above lemma is nothing but the functorial property of Serre-Tate coordinates so we omit the proof here.

In the end of this section, we give a discussion of CM liftings of the abelian variety $A_{/k}$. Recall that we assume that the abelian variety $A_{/k}$ is defined over a finite field $\mathbb{F}_q$. Let $\pi : A \to A$ be the Frobenius endomorphism of $A_{/\mathbb{F}_q}$. We are interested in characterization of liftings of $A_{/\mathbb{F}_q}$ with complex multiplication. When $A_{/\mathbb{F}_q}$ is an arbitrary abelian variety, this question can be quite difficult. However, under the assumption that $A_{/\mathbb{F}_q}$ is ordinary, we have the following:

Proposition 2.8. *([31] Lemma 2.8) Let R be a complete local noetherian $W(\mathbb{F}_q)$-algebra with residue field $\mathbb{F}_q$ and $\mathbb{A}_{/R}$ be a lifting of $A_{/\mathbb{F}_q}$ to R. Let $\bar{R} = R \otimes_{W(\mathbb{F}_q)} W$.*

1. *The following conditions are equivalent:*

(a) $\mathbb{A} \otimes_R \bar{R}$ *is the identity element in* $\widehat{\mathfrak{M}}_{A_{/k}}(\bar{R})$*;*

(b) *the reduction map* $\mathrm{End}(\mathbb{A}_{/R}) \to \mathrm{End}(A_{/\mathbb{F}_q})$ *is an isomorphism;*

(c) *the homomorphism* π^m *lifts to an endomorphism of* $\mathbb{A}_{/R}$ *for some integer* $m \geq 1$.

2. *The following conditions are equivalent:*

(a) $\mathbb{A} \otimes_R \bar{R}$ *is a torsion element in* $\widehat{\mathfrak{M}}_{A_{/k}}(\bar{R})$*;*

(b) *the reduction map* $\mathrm{End}(\mathbb{A}_{/R}) \to \mathrm{End}(A_{/\mathbb{F}_q})$ *is an isomorphism after inverting* p*, i.e. we have an isomorphism* $\mathrm{End}(\mathbb{A}_{/R}) \otimes \mathbb{Z}[\frac{1}{p}] \cong \mathrm{End}(A_{/\mathbb{F}_q}) \otimes \mathbb{Z}[\frac{1}{p}]$*;*

(c) $\mathbb{A} \otimes_R \bar{R}$ *is isogenous to the abelian scheme* $\mathbb{A}_{1/\bar{R}}$ *where* $\mathbb{A}_{1/\bar{R}}$ *is the identity element in* $\widehat{\mathfrak{M}}_{A_{/k}}(\bar{R})$*;*

(d) $\mathbb{A}_{/R}$ *has complex multiplication.*

Definition 2.9. *Under the notations and assumptions of Proposition 2.8, if* $\mathbb{A}_{/R}$ *satisfies the equivalent conditions in part* (1)*, we say that* $\mathbb{A}_{/R}$ *is the canonical lifting of* $A_{/\mathbb{F}_q}$*; if* $\mathbb{A}_{/R}$ *satisfies the equivalent conditions in part* (2)*, we say that* $\mathbb{A}_{/R}$ *is a quasi-canonical lifting of* $A_{/\mathbb{F}_q}$.

2.7 Partial Serre-Tate coordinates

In this section we write R for a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k and an abelian variety $A_{/k}$ which is not necessarily ordinary. Suppose that the Barsotti-Tate group $A[p^\infty]_{/k}$ is not local-local, i.e. the slopes of $A[p^\infty]_{/k}$ contains 0 and 1 (see [3] for the definition of slopes of Barsotti-Tate groups over an arbitrary field) or equivalently, the Tate module $\mathrm{T}_p(A)(k)$ is nontrivial.

We say that a Barsotti-Tate group $G_{/R}$ is multiplicative if its dual $G^t_{/R}$ is ind-étale. We say that $G_{/R}$ is local-local if both $G_{/R}$ and $G^t_{/R}$ are connected. As k is algebraically closed, we can decompose the Barsotti-Tate group $A[p^\infty]_{/k}$ as $A[p^\infty] = A[p^\infty]^{ord} \times A[p^\infty]^{ll}$, where $A[p^\infty]^{ord}$ is a product of a multiplicative Barsotti-Tate group with an ind-étale Barsotti-Tate group, and $A[p^\infty]^{ll}$ is local-local.

Now we assume that R is artinian and $\mathfrak{m}^{n+1} = (0)$ for some $n \geq 1$. Let $\mathbb{A}_{/R}$ be a lifting of $A_{/k}$ to R. We assume further that we have a decomposition of Barsotti-Tate groups over R:

$$\mathbb{A}[p^\infty] = \mathbb{A}[p^\infty]^{ord} \times \mathbb{A}[p^\infty]^{ll},$$

where $\mathbb{A}[p^\infty]^{ord}$ (resp. $\mathbb{A}[p^\infty]^{ll}$) lifts $A[p^\infty]^{ord}$ (resp. $A[p^\infty]^{ll}$). This is equivalent to saying that $\hat{\mathbb{A}}[p^\infty]$ can be decomposed as $\mathbb{A}[p^\infty]^{mult} \times \mathbb{A}[p^\infty]^{ll}$ over R, where $\mathbb{A}[p^\infty]^{mult}$ is multiplicative and $\mathbb{A}[p^\infty]^{ll}$ is local-local. Then we have an exact sequence of Barsotti-Tate groups over R:

$$0 \to \mathbb{A}[p^\infty]^{mult} \to \mathbb{A}[p^\infty]^{ord} \to \mathrm{T}_p(A)(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$

Similar with the ordinary case we define a homomorphism

$$\begin{aligned} p^n: \quad A[p^n](k) &\to \hat{\mathbb{A}}(R) \\ x &\mapsto p^n\tilde{x}, \end{aligned}$$

where $\tilde{x}$ is an arbitrary lifting of x in $\mathbb{A}(R)$. Write $\varphi_{\mathbb{A}/R}$ as the composition

$$\mathrm{T}_p(A)(k) \to A[p^n](k) \xrightarrow{p^n} \hat{\mathbb{A}}(R) \to \mathbb{A}[p^\infty]^{mult}(R).$$

On the other hand we have a perfect pairing of k-group schemes $A^{mult}[p^n] \times A^t[p^n] \to \mu_{p^n}$, which can be lifted uniquely to a perfect pairing of R-group schemes $\mathbb{A}[p^n]^{mult} \times A^t(k)[p^n] \to \mu_{p^n}$. By taking limit, we have a perfect pairing

$$e_{\mathbb{A}/R} : \mathbb{A}[p^\infty]^{mult} \times \mathrm{T}_p A^t(k) \to \widehat{\mathbb{G}}_m$$

over R. Then we can define the partial Serre-Tate coordinate by the formula:

$$\begin{aligned} q(\mathbb{A}_{/R}; -, -): \quad \mathrm{T}_p(A)(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_p(A^t)(k) &\to \widehat{\mathbb{G}}_m(R) \\ \alpha \otimes \alpha_t &\mapsto e_{\mathbb{A}/R}(\varphi_{\mathbb{A}/R}, \alpha_t), \end{aligned}$$

for any $\alpha \in \mathrm{T}_p(A)(k)$ and $\alpha_t \in \mathrm{T}_p(A^t)(k)$.

As in the ordinary case, we assume that the exact sequence

$$0 \to \mathbb{A}[p^\infty]^{mult} \to \mathbb{A}[p^\infty]^{ord} \to \mathrm{T}_p(A)(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

splits after some faithfully flat extension R' of R. Then for any $\alpha_t = (\alpha_{t,n}) \in \mathrm{T}_p A^t(k)$, we choose a compatible lifting $(\tilde{\alpha}_{t,n})$ of $(\alpha_{t,n})$ in $\mathbb{A}[p^\infty]^{ord}(R')$, i.e. $\tilde{\alpha}_{t,n} \in \mathbb{A}[p^n]^{ord}(R')$ and $p(\tilde{\alpha}_{t,n+1}) = \alpha_{t,n}$. Then for any $s > 0$ and $m \geq n + s$, the element $\mathrm{E}_{\mathbb{A}_{R'},p^{2m}}(\pi(\tilde{\alpha}_m), \tilde{\alpha}_{t,m+s})$ is a p^s-th root of the partial Serre-Tate coordinate $q(\mathbb{A}_{/R}; \alpha, \alpha_t)$, where $\pi : \hat{\mathbb{A}} \to \mathbb{A}[p^\infty]^{mult}$ is the natural projection and $\tilde{\alpha}_m \in \mathbb{A}(R)$ is an arbitrary lifting of $\alpha_m \in A[p^m](k)$.

Also similar to section 2.5 by taking projective limit, we can extend the above result to $R \in CL_{/W}$.

CHAPTER 3

Galois representations attached to ordinary abelian varieties

Let $A_{/\mathbb{F}_q}$ be an ordinary abelian variety over a finite field $\mathbb{F}_q$, where q is a power of a prime p. We have seen in the previous chapter that the liftings of $A_{/\mathbb{F}_q}$ to characteristic 0 are determined by the Serre-Tate coordinates. On the other hand, if K is a finite extension of $\mathbb{Q}_q$ and $\mathbb{A}_{/K}$ is a lifting of $A_{/\mathbb{F}_q}$, we can consider the Galois representation ρ of $\mathrm{Gal}(\bar{K}/K)$ associated to the p-adic Tate module of $\mathbb{A}_{/K}$. In this chapter, we explain how the Serre-Tate coordinates of $\mathbb{A}_{/K}$ determine the Galois representation ρ. For later argument, we want the representation ρ valued in the symplectic group so we need to impose a polarization on the abelian scheme $\mathbb{A}_{/K}$. So we divide our discussion into two cases: elliptic curves and polarized abelian varieties.

3.1 The case of elliptic curves

Fix an algebraically closure k of the prime field $\mathbb{F}_p$ for a fixed prime $p > 0$ and an ordinary elliptic curve $E_{/k}$. Let K be a finite extension of $\mathbb{Q}_p^{ur}$ with valuation ring $\mathcal{O}_K$ and recall that K^{ur} is the quotient field of the ring $W = W(k)$. Let Ω be the algebraic closure of K^{ur} inside $\mathbb{C}_p$. Then we have the isomorphism of Galois groups: $\mathrm{Gal}(\Omega/K^{ur}) \cong \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{ur})$. As K and K^{ur} are linearly disjoint over $\mathbb{Q}_p^{ur}$, we take L to be the composite of K and K^{ur} over $\mathbb{Q}_p^{ur}$, which is the p-adic completion of K inside Ω. Let $\mathcal{O}_L$ be the valuation ring of L.

Suppose that $\mathbb{E}_{/\mathcal{O}_K}$ is a lifting of $E_{/k}$ to $\mathcal{O}_K$. Since $E_{/k}$ is an elliptic curve, it is naturally isomorphic to its dual $E^t_{/k}$, and hence we have a Serre-Tate coordinate $q(\mathbb{E}_{/\mathcal{O}_K}; -, -)$:

$\mathrm{T}_pE(k) \otimes_{\mathbb{Z}_p} \mathrm{T}_pE(k) \to \widehat{\mathbb{G}}_m(\mathcal{O}_K)$.

From the exact sequence of Barsotti-Tate groups over $\mathcal{O}_K$:

$$0 \to \hat{\mathbb{E}} \to \mathbb{E}[p^\infty] \to \mathrm{T}_pE(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0,$$

we have an exact sequence of Tate modules:

$$0 \to \mathrm{T}_p\hat{\mathbb{E}}(\bar{\mathbb{Q}}_p) \xrightarrow{i} \mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p) \xrightarrow{\pi} \mathrm{T}_pE(k) \to 0.$$

We identify $\mathrm{T}_p\hat{\mathbb{E}}(\bar{\mathbb{Q}}_p)$ as a submodule of $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$ under i. Then we can choose a $\mathbb{Z}_p$-basis $\{v^\circ = (v_n^\circ), v^{et} = (v_n^{et})\}$ of $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$ (with $v_n^\circ, v_n^{et} \in \mathbb{E}[p^n](\bar{\mathbb{Q}}_p)$) such that v° is a basis of the $\mathbb{Z}_p$-module $\mathrm{T}_p\hat{\mathbb{E}}(\bar{\mathbb{Q}}_p)$ and v^{et} is mapped to a basis $u = (u_n)$ of $\mathrm{T}_pE(k)$ under the map π (with $u_n \in E[p^n](k)$). Then set $t = q(\mathbb{E}_{/\mathcal{O}_K}; u, u)$.

Under the $\mathbb{Z}_p$-basis $\{v^\circ, v^{et}\}$ of $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$, we have a Galois representation attached to $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$:

$$\begin{aligned} \rho: \quad \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{GL}_2(\mathbb{Z}_p) \\ \sigma \qquad &\mapsto \begin{pmatrix} \chi(\sigma) & b(\sigma) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character.

On the other hand, for each integer $n \geq 1$, the element $v_n^{et} \in \mathbb{E}[p^n](\bar{\mathbb{Q}}_p)$ generates an étale subgroup of $\mathbb{E}[p^n]$ which lifts the constant group scheme $E[p^n](k)_{/k}$. Thus we can find a (possibly infinite) extension $\tilde{K}$ of K inside $\bar{\mathbb{Q}}_p$, such that v_n^{et} is defined over $\mathcal{O}_{\tilde{K}}$ for all n ($\mathcal{O}_{\tilde{K}}$ is the valuation ring of $\tilde{K}$). Replacing $\tilde{K}$ by its Galois closure inside $\bar{\mathbb{Q}}_p$, we can assume that $\tilde{K}/K$ is Galois. Let $\tilde{L}$ be the composite of $\tilde{K}$ and K^{ur} over $\mathbb{Q}_p^{ur}$, and $\mathcal{O}_{\tilde{L}}$ be the valuation ring of $\tilde{L}$. Under the above notations, the exact sequence of Barsotti-Tate groups:

$$0 \to \hat{\mathbb{E}} \to \mathbb{E}[p^\infty] \to \mathrm{T}_pE(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0,$$

splits when base change to $\mathcal{O}_{\tilde{L}}$. By the discussion in Chapter 2, for any integer $s \geq 1$, we have a unique p^s-th root $\sqrt[p^s]{t} \in \mathcal{O}_{\tilde{L}}$ of the Serre-Tate coordinate t which depends only on $v^{et} = (v_n^{et})$. Our main result is the following:

Theorem 6. *For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ and integer $s \geq 1$, under the isomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \cong \mathrm{Gal}(\Omega/L)$, we have the equality:*

$$\frac{\sigma(\sqrt[p^s]{t})}{\sqrt[p^s]{t}} = \mathrm{E}_{\mathbb{E}/\bar{\mathbb{Q}}_p,p^s}(v_s^{et}, v_s^{\circ})^{b(\sigma)}.$$

Proof. For any integer $n \geq 1$, let $W_n = \mathcal{O}_K/\mathfrak{m}_K^{n+1}\mathcal{O}_K = \mathcal{O}_L/\mathfrak{m}_L^{n+1}\mathcal{O}_L$ ($\mathfrak{m}_K$ (resp. $\mathfrak{m}_L$) is the maximal ideal of $\mathcal{O}_K$ (resp. $\mathcal{O}_L$)) and $\widetilde{W}_n = W_n \otimes_{\mathcal{O}_K} \mathcal{O}_{\tilde{K}}$, which is an artinian local ring faithfully flat over W_n. Set $\mathbb{E}_{/W_n} = \mathbb{E} \times_{\mathcal{O}_K} W_n$ and $t_n = q(\mathbb{E}_{/W_n}; u, u)$. Then we have $t = \varprojlim t_n \in \widehat{\mathbb{G}}_m(\mathcal{O}_L)$. From the discussion in Section 2.4, for each $m \geq n + s$, we have a unique p^s-th root of t_n, which is given by the formula:

$$\sqrt[p^s]{t_n} = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{et}) \in \mu_{p^{2m}}(\widetilde{W}_n),$$

where $\tilde{\alpha}_{m,n} \in \mathbb{E}(W_n)$ is an arbitrary lift of $u_m \in E[p^m](k)$, and $\bar{v}_{m+s}^{et}$ is the reduction of v_{m+s}^{et} in $\mathbb{E}[p^{m+s}](\widetilde{W}_n)$. From the discussion in Section 2.4, we have $\sqrt[p^s]{t} = \varprojlim \sqrt[p^s]{t_n} \in \widehat{\mathbb{G}}_m(\mathcal{O}_{\tilde{L}})$.

Now for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$, since $\tilde{K}/K$ is Galois, σ induces an automorphism of $\tilde{K}$, and hence induces an automorphism of $\widetilde{W}_n$ for each n. We still denote this automorphism by σ.

Since $\tilde{\alpha}_{m,n} \in \mathbb{E}(W_n)$ and σ fixes K, $\sigma(\tilde{\alpha}_{m,n}) = \tilde{\alpha}_{m,n}$. By our assumption on the expression of the Galois representation ρ, we have $\sigma(v^{et}) = b(\sigma)v^{\circ} + v^{et}$, and hence $\sigma(\bar{v}_{m+s}^{et}) = b(\sigma)\bar{v}_{m+s}^{\circ} + \bar{v}_{m+s}^{et}$.

As the pairing $\mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(-,-)$ is compatible with arbitrary base change, we have

$$\begin{aligned}
\sigma(\sqrt[p^s]{t_n}) &= \sigma(\mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{et})) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\sigma(\tilde{\alpha}_{m,n}), \sigma(\bar{v}_{m+s}^{et})) \\
&= \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, b(\sigma)\bar{v}_{m+s}^{\circ} + \bar{v}_{m+s}^{et}) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{\circ})^{b(\sigma)} \cdot \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{et}) \\
&= \sqrt[p^s]{t_n}\mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{\circ})^{b(\sigma)}.
\end{aligned}$$

Now we analyze the term $\mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{\circ})$. As $\bar{v}_{m+s}^{\circ} \in \hat{\mathbb{E}}(\widetilde{W}_n)$ and $\tilde{\alpha}_{m,n} \in \mathbb{E}[p^{2m}](W_n)$ lifts $u_m \in E[p^m](k) \subseteq E[p^{2m}](k)$, from Lemma 2.2, we have

$$\begin{aligned}
\mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}_{m+s}^{\circ}) &= \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(u_m, \bar{v}_{m+s}^{\circ}) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(\bar{v}_m^{et}, \bar{v}_{m+s}^{\circ}) \\
&= \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m}}(p^s\bar{v}_{m+s}^{et}, \bar{v}_{m+s}^{\circ}) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n,p^{2m+s}}(\bar{v}_{m+s}^{et}, \bar{v}_{m+s}^{\circ}).
\end{aligned}$$

The last term is the projection of $\mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^{2m+s}}(v^{et}_{m+s}, v^{\circ}_{m+s})$ under the base change $\mathcal{O}_{\tilde{K}} \to \widetilde{W}_n$, which is denoted by $\overline{\mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^s}(v^{et}_s, v^{\circ}_s)}$.

By a direct computation, we have

$$\mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^{2m+s}}(v^{et}_{m+s}, v^{\circ}_{m+s}) = \mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^s}(p^m v^{et}_{m+s}, p^m v^{\circ}_{m+s}) = \mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^s}(v^{et}_s, v^{\circ}_s).$$

Hence we have

$$\sigma(\sqrt[p^s]{t_n}) = \sqrt[p^s]{t_n} \cdot (\overline{\mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^s}(v^{et}_s, v^{\circ}_s)})^{b(\sigma)}.$$

By taking projective limit for n, we have the desired equality:

$$\sigma(\sqrt[p^s]{t}) = \sqrt[p^s]{t} \cdot \mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}},p^s}(v^{et}_s, v^{\circ}_s)^{b(\sigma)} = \sqrt[p^s]{t} \cdot \mathrm{E}_{\mathbb{E}/\bar{\mathbb{Q}}_p,p^s}(v^{et}_s, v^{\circ}_s)^{b(\sigma)}.$$

□

3.2 Generalization to higher dimensions

We keep the same notations as in section 3.1. In this section we want to generalize the result in the previous section to higher dimensions. Let $K/\mathbb{Q}_p^{ur}$ be a finite extension inside $\bar{\mathbb{Q}}_p$ with valuation ring $\mathcal{O}_K$. The field L is defined to be the composite of K and K^{ur} over $\mathbb{Q}_p^{ur}$.

Fix an algebraic closure k of the prime field $\mathbb{F}_p$. Let $A_{/k}$ be an abelian variety and $\mathbb{A}_{/\mathcal{O}_K}$ be a lifting of $A_{/k}$ to $\mathcal{O}_K$. Then we have a connected-étale exact sequence of Barsotti-Tate groups over $\mathcal{O}_L$:

$$0 \to \hat{\mathbb{A}} \to \mathbb{A}[p^\infty] \to \mathrm{T}_p A(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$

For every integer $n \geq 1$,we have a perfect pairing:

$$e_{p^n} : \hat{\mathbb{A}}[p^n] \times A^t[p^n](k) \to \widehat{\mathbb{G}}_m$$

over $\mathcal{O}_L$. Taking projective limits, we have a perfect pairing:

$$e_{p^\infty} : \mathrm{T}_p\hat{\mathbb{A}}(\mathbb{C}_p) \times \mathrm{T}_p A^t(k) \to \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)$$

over $\mathcal{O}_L$. For later argument, we fix a basis $\zeta_{p^\infty} = (\zeta_{p^n})_n$ of the $\mathbb{Z}_p$-module $\mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)$.

Now suppose that we are given a polarization $\lambda : \mathbb{A} \to \mathbb{A}^t$ of $\mathbb{A}_{/\mathcal{O}_K}$ whose degree is prime to p. This polarization induces isomorphisms of p-adic Tate modules $\mathrm{T}_p\mathbb{A}(\mathbb{C}_p) \to \mathrm{T}_p\mathbb{A}^t(\mathbb{C}_p)$ and $\mathrm{T}_pA(k) \to \mathrm{T}_pA^t(k)$, which are still denoted by λ. We can take a $\mathbb{Z}_p$-basis $\{v_1^\circ, \ldots, v_n^\circ, v_1^{et}, \ldots, v_n^{et}\}$ of the p-adic Tate module $\mathrm{T}_p\mathbb{A}(\mathbb{C}_p)$ such that:

1. $\{v_1^\circ, \ldots, v_n^\circ\}$ is a $\mathbb{Z}_p$-basis of $\mathrm{T}_p\hat{\mathbb{A}}(\mathbb{C}_p)$;
2. $\{v_1^{et}, \ldots, v_n^{et}\}$ is a lifting of a basis $\{u_1, \ldots, u_n\}$ of the Tate module $\mathrm{T}_pA(k)$;
3. under the pairing e_{p^∞} and the isomorphism $\lambda : \mathrm{T}_pA(k) \to \mathrm{T}_pA^t(k)$, we have $e_{p^\infty}(v_i^\circ, \lambda(u_j)) = 1$ if $i \neq j$ and $e_{p^\infty}(v_i^\circ, \lambda(u_j)) = \zeta_{p^\infty}$ if $i = j$.

Under the basis $\{v_1^\circ, \ldots, v_n^\circ, v_1^{et}, \ldots, v_n^{et}\}$, the Galois representation attached to the Tate module $\mathrm{T}_p\mathbb{A}(\mathbb{C}_p)$ is of the shape:

$$\begin{aligned} \rho : \ \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{Gsp}_{2n}(\mathbb{Z}_p) \\ \sigma &\mapsto \begin{pmatrix} \chi_p(\sigma) \cdot \mathrm{I}_n & B(\sigma) \\ 0 & \mathrm{I}_n \end{pmatrix}, \end{aligned}$$

where $\chi_p : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character, I_n is the $n \times n$ identity matrix and $B = (b_{ij})_{1 \leq i,j \leq n} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to M_{n\times n}(\mathbb{Z}_p)$ is a map valued in the set of $n \times n$ symmetric matrices.

Now we consider the Serre-Tate coordinates $t_{ij} = q(\mathbb{A}_{\mathcal{O}_L}; u_i, \lambda(u_j))$. From the discussion in section 2.4, the lifting v_i^{et} of u_i gives a compatible sequence of p-th power roots $\{\sqrt[p^s]{t_{ij}}\}_{s=1,2,\ldots}$ of the Serre-Tate coordinates t_{ij}, for $1 \leq i, j \leq n$.

Under the above notations, we have:

Theorem 7. *For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ and integer $s \geq 1$, under the isomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \cong \mathrm{Gal}(\Omega/L)$, we have the equality:*

$$\frac{\sigma(\sqrt[p^s]{t_{ij}})}{\sqrt[p^s]{t_{ij}}} = \zeta_{p^s}^{b_{ij}(\sigma)}.$$

3.3 Extension to the decomposition group

In this section we want to extend the previous results to the decomposition group, i.e. we want to use the Serre-Tate coordinates to study the Galois representation of the decomposition group. First we give a cohomological interpetation of Theorem 7.

Remark 3.1. Recall that we have a Galois representation

$$\begin{aligned} \rho: \quad \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{Gsp}_{2n}(\mathbb{Z}_p) \\ \sigma \quad &\mapsto \begin{pmatrix} \chi_p(\sigma)\cdot \mathrm{I}_n & B(\sigma) \\ 0 & \mathrm{I}_n \end{pmatrix}. \end{aligned}$$

By direct calculation, for every pair $1 \le i, j \le n$, the map $b_{ij} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ is a 1-cocycle if we define the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ on $\mathbb{Z}_p$ by the p-adic cyclotomic character χ_p. Hence we have an element in the cohomology group $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathbb{Z}_p(\chi_p))$, which is denoted by $\bar{b}_{ij}$.

On the other hand, under the basis ζ_{p^∞} of $\mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)$, we have an isomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-modules $\mathbb{Z}_p(\chi_p) \to \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)$ which sends 1 to ζ_{p^∞}. So we have an isomorphism of cohomology groups:

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathbb{Z}_p(\chi_p)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)).$$

Now by Kummer theory, we have an isomorphism:

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)) \to \widehat{K^\times},$$

where $\widehat{K^\times}$ is the pro-p-completion of the multiplicative group $K^\times$. As $\widehat{\mathbb{G}}_m(\mathcal{O}_L) = 1 + \mathfrak{m}_L$, we can regard $\widehat{\mathbb{G}}_m(\mathcal{O}_L)$ as a subgroup of $\widehat{K^\times}$. Then Theorem 7 tells us that under the isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathbb{Z}_p(\chi_p)) \to \widehat{K^\times},$$

the element $\bar{b}_{ij}$ coming from the Galois representation ρ corresponds to the Serre-Tate coordinate $t_{ij} \in \widehat{\mathbb{G}}_m(\mathcal{O}_L) \subseteq \widehat{K^\times}$.

We start from the case of elliptic curves. Let $K/\mathbb{Q}_p$ be a finite extension with valuation ring $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ ($q = p^r$ for some integer $r \geq 1$) and let $\mathbb{E}_{/\mathcal{O}_K}$ be an elliptic curve whose special fiber $E_{/\mathbb{F}_q}$ is ordinary. Recall that we fix an algebraic closure $\bar{\mathbb{Q}}_p$ (resp. k) of $\mathbb{Q}_p$ (resp. $\mathbb{F}_q$). Under the ordinary assumption, we have an exact sequence of the p-adic Tate modules:

$$0 \to \mathrm{T}_p\hat{\mathbb{E}}(\bar{\mathbb{Q}}_p) \xrightarrow{i} \mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p) \xrightarrow{\pi} \mathrm{T}_pE(k) \to 0.$$

As in section 3.1, we choose a $\mathbb{Z}_p$-basis $\{v^\circ, v^{et}\}$ of $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$ such that v° is a basis of $\mathrm{T}_p\hat{\mathbb{E}}(\bar{\mathbb{Q}}_p)$ and v^{et} is mapped to a basis u of $\mathrm{T}_pE(k)$ under the reduction map. Under this basis, we have a Galois representation attached to $\mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$:

$$\begin{aligned} \rho: \ \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{GL}_2(\mathbb{Z}_p) \\ \sigma &\mapsto \begin{pmatrix} \chi_p(\sigma)\cdot\eta^{-1}(\sigma) & b(\sigma) \\ 0 & \eta(\sigma) \end{pmatrix}, \end{aligned}$$

where $\chi_p : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $\eta : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p^\times$ is an unramified character. Now we define a map $c : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ by setting $c(\sigma) = \eta^{-1}(\sigma)b(\sigma)$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$. As ρ is a representation, a direct calculation shows that $c : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ is a 1-cocycle valued in $\mathbb{Z}_p(\chi_p\eta^{-2})$. If we choose a different lifting $v^{et} \in \mathrm{T}_p\mathbb{E}(\bar{\mathbb{Q}}_p)$ of $u \in \mathrm{T}_pE(k)$, the 1-cocycle $c : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ is changed by a 1-coboundary valued in $\mathbb{Z}_p(\chi_p\eta^{-2})$. Hence to determine the Galois representation ρ (up to isomorphism), it is enough to determine the corresponding element of $c : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ in the cohomology group $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathbb{Z}_p(\chi_p\eta^{-2}))$. In fact, we have the following relation:

Theorem 8. *For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ and integer $s > 0$, we have the equality:*

$$\frac{\sigma(\sqrt[p^s]{t})^{\eta(\sigma)^{-2}}}{\sqrt[p^s]{t}} = \mathrm{E}_{\mathbb{E}/\bar{\mathbb{Q}}_p,p^s}(v_s^{et}, v_s^\circ)^{c(\sigma)}.$$

Proof. The proof is quite similar with that of Theorem 6, so we do not give all the details here.

We assume that there is a Galois extension $\tilde{K}/L$ such that the exact sequence

$$0 \to \hat{\mathbb{E}} \to \mathbb{E}[p^\infty] \to \mathrm{T}_pE(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

splits over $\mathcal{O}_{\tilde{K}}$. As in the proof of Theorem 6, we can define $W_n, \widetilde{W}_n, t_n$. Then for $m \geq s+n$,

$$\sqrt[p^s]{t_n} = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}^{et}_{m+s}).$$

For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$, we have

$$\sigma(\sqrt[p^s]{t_n}) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\sigma(\tilde{\alpha}_{m,n}), \sigma(\bar{v}^{et}_{m+s})) = \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\sigma(\tilde{\alpha}_{m,n}), b(\sigma)\bar{v}^{\circ}_{m+s} + \eta(\sigma)\bar{v}^{et}_{m+s}).$$

As $\tilde{\alpha}_{m,n} \in \mathbb{E}(W_n)$ is a lifting of $u_m \in E[p^m](k)$, the element $\sigma(\tilde{\alpha}_{m,n}) \in \mathbb{E}(W_n)$ is a lifting of $\sigma(u_m) = \eta(\sigma) \cdot u_m \in E[p^m](k)$. From the argument in section 2.4, the element $\sqrt[p^s]{t_n}$ is independent of the choice of the lifting of u_m. Thus we have

$$\begin{aligned} \sigma(\sqrt[p^s]{t_n}) &= \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\eta(\sigma)\tilde{\alpha}_{m,n}, b(\sigma)\bar{v}^{\circ}_{m+s} + \eta(\sigma)\bar{v}^{et}_{m+s}) \\ &= \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}^{\circ}_{m+s})^{b(\sigma)\eta(\sigma)} \cdot \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}^{et}_{m+s})^{\eta(\sigma)^2} \\ &= (\sqrt[p^s]{t_n})^{\eta(\sigma)^2} \cdot \mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}^{\circ}_{m+s})^{b(\sigma)\eta(\sigma)}. \end{aligned}$$

By the same analysis on the term $\mathrm{E}_{\mathbb{E}/\widetilde{W}_n, p^{2m}}(\tilde{\alpha}_{m,n}, \bar{v}^{\circ}_{m+s})$ as in the proof of Theorem 6, and taking limit for various n, we have the desired equality:

$$\sigma(\sqrt[p^s]{t}) = (\sqrt[p^s]{t})^{\eta(\sigma)^2} \cdot \mathrm{E}_{\mathbb{E}/\mathcal{O}_{\tilde{K}}, p^s}(v^{et}_s, v^{\circ}_s)^{b(\sigma)\eta(\sigma)} = (\sqrt[p^s]{t})^{\eta(\sigma)^2} \cdot \mathrm{E}_{\mathbb{E}/\bar{\mathbb{Q}}_p, p^s}(v^{et}_s, v^{\circ}_s)^{b(\sigma)\eta(\sigma)}.$$

Taking the $\eta^{-2}(\sigma)$-th power on both sides, we get the desired equality. □

Remark 3.2. As in Remark 3.1, we can give a cohomological intepretation of Theorem 8. Let K^{ur} be the maximal unramified extension of K in $\bar{\mathbb{Q}}_p$ and let $I = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K^{ur}) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) = G$ be the inertia group. Then we have the inflation-restriction exact sequence:

$$0 \to H^1(G/I, \mathbb{Z}_p(\chi_p\eta^{-2})^I) \to H^1(G, \mathbb{Z}_p(\chi_p\eta^{-2})) \to H^1(I, \mathbb{Z}_p(\chi_p\eta^{-2}))^{G/I} \to H^2(G/I, \mathbb{Z}_p(\chi_p\eta^{-2})^I).$$

As the character η is unramified, the inertia group I acts on $\mathbb{Z}_p(\chi_p\eta^{-2})$ by the p-adic cyclotomic character. Hence $\mathbb{Z}_p(\chi_p\eta^{-2})^I = 0$. Se the restriction map induces an isomorphism:

$$H^1(G, \mathbb{Z}_p(\chi_p\eta^{-2})) \to H^1(I, \mathbb{Z}_p(\chi_p))^{G/I}.$$

From Remark 3.1, under the isomorphism of I-modules $\mathbb{Z}_p(\chi_p) \to \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)$ which sends 1 to $\lim \mathrm{E}_{\mathbb{E}/\bar{\mathbb{Q}}_p, p^s}(v_s^{et}, v_s^\circ)$ and the isomorphism $H^1(I, \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)) \cong \widehat{(K^{ur})^\times}$, the image of c in $H^1(I, \mathbb{Z}_p(\chi_p))$ corresponds to the Serre-Tate coordinate $t \in \widehat{(K^{ur})^\times}$.

On the other hand, it is easy to check that the map

$$\begin{aligned} \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p) \\ \sigma &\mapsto \varprojlim \frac{\sigma(\sqrt[p^s]{t})^{\eta^{-2}(\sigma)}}{\sqrt[p^s]{t}}, \end{aligned}$$

is a 1-cocycle valued in $H^1(G, \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)(\chi_p\eta^{-2}))$ whose restriction to the inertia group corresponds to the Serre-Tate coordinate t under the above isomorphism. Using this cohomological interpretation, we get another proof of Theorem 8.

Moreover, from the restriction map, we see that the image of c in $H^1(I, \mathbb{Z}_p(\chi_p))$ is invariant under the action of G. Let $f : I \to \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p), \sigma \mapsto \varprojlim \frac{\sigma(\sqrt[p^s]{t})}{\sqrt[p^s]{t}}$ be the 1-cocycle corresponding to the Serre-Tate coordinate t. For any $g \in G$, the action of g on the cocycle f is given by the formula:

$$f^g(\sigma) = g \cdot f(g^{-1}\sigma g).$$

Hence

$$f^g(\sigma) = (\varprojlim \frac{\sigma(\sqrt[p^s]{t})}{\sqrt[p^s]{t}})^{\chi_p(g)\eta^{-2}(g)} = (g(\varprojlim \frac{\sigma(\sqrt[p^s]{t})}{\sqrt[p^s]{t}}))^{\eta^{-2}(g)} = (\varprojlim \frac{\sigma \cdot g(\sqrt[p^s]{t})}{g(\sqrt[p^s]{t})})^{\eta^{-2}(g)}.$$

As $\{\sqrt[p^s]{t}\}$ is a compatible p-th power roots of t, $\{g(\sqrt[p^s]{t})\}$ is a compatible p-th power roots of $g(t)$. Under the isomorphism induced by Kummer theory, the 1-cocycle $\sigma \mapsto (\varprojlim \frac{\sigma \cdot g(\sqrt[p^s]{t})}{g(\sqrt[p^s]{t})})^{\eta^{-2}(g)}$ corresponds to $g(t)^{\eta^{-2}(g)}$. So we have the equality $g(t) = t^{\eta^2(g)}$. The cohomological interpretation gives another proof of Lemma 2.6.

The case of higher dimensions is more complicated. Let $\mathbb{A}_{/\mathcal{O}_K}$ be an abelian scheme of relative dimension n whose special fiber $A_{/\mathbb{F}_q}$ is ordinary. As in section 3.2, we can choose a $\mathbb{Z}_p$-basis of the p-adic Tate module $\mathrm{T}_p\mathbb{A}(\bar{\mathbb{Q}}_p) = \mathrm{T}_p\mathbb{A}(\mathbb{C}_p)$ under which the Galois representation attached to $\mathrm{T}_p\mathbb{A}(\bar{\mathbb{Q}}_p)$ is of the shape:

$$\begin{aligned}\rho: \quad \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) &\to \mathrm{GSp}_{2n}(\mathbb{Z}_p)\\ \sigma \quad &\mapsto \begin{pmatrix} \chi_p(\sigma)\cdot T(\sigma) & B(\sigma) \\ 0 & (T(\sigma)^{-1})^t \end{pmatrix},\end{aligned}$$

here again $\chi_p : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character, $B = (b_{ij})_{1\le i,j\le n}$: $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to M_{n\times n}(\mathbb{Z}_p)$ is a map, and $T(\cdot) : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathrm{GL}_n(\mathbb{Z}_p)$ is the unramified homomorphism which sends (any) Frobenius element in $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ to a matrix $X \in \mathrm{GL}_n(\mathbb{Z}_p)$.

Under the above setting, we define a map $C : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ by requiring that $C(\sigma) = B(\sigma)\cdot T(\sigma)^t$ for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$. A direct calculation shows that $C : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to M_{n\times n}(\mathbb{Z}_p)$ is a 1-cocycle if we define the $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-action on $M_{n\times n}(\mathbb{Z}_p)$ by the formula: $\sigma\cdot M = \chi_p(\sigma)T(\sigma)\cdot M\cdot T(\sigma)^t$. Let $I \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ be the inertia group. Again the inflation-restriction exact sequence tells us that the restriction map induces an isomorphism:

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), M_{n\times n}(\mathbb{Z}_p)) \to H^1(I, M_{n\times n}(\mathbb{Z}_p)(\chi_p))^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)/I}.$$

For $1 \le i,j \le n$, the restriction of the map $b_{ij} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathbb{Z}_p$ to the inertia group I is a 1-cocycle valued in $\mathbb{Z}_p(\chi_p)$. From Theorem 7 and Remark 3.1, under the isomorphism

$$H^1(I, \mathbb{Z}_p(\chi_p)) \cong H^1(I, \mathrm{T}_p\mu_{p^\infty}(\mathbb{C}_p)) \cong \widehat{(K^{ur})^\times},$$

the images of b_{ij}'s correspond to the Serre-Tate coordinates t_{ij}'s. Hence the Serre-Tate coordinates t_{ij}'s determine the images of the 1-cocycle C in the cohomological group $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), M_{n\times n}(\mathbb{Z}_p))$ and hence determine the Galois representation ρ (up to isomorphism).

Since we know little about the matrix A, we cannot expect to get an explicit expression of the 1-cocycle C as in Theorem 8. For later argument, we consider a special case: suppose that there exists a finite extension $L/\mathbb{Q}_p$ with valuation ring $\mathcal{O}_L$ and a matrix $W \in \mathrm{GL}_n(\mathcal{O}_L)$ such that $WXW^{-1} = D = diag\{d_1, \ldots, d_n\}$ is a diagonal matrix in $\mathrm{GL}_n(\mathcal{O}_L)$. Hence

$(W_t)^{-1}X^{t-1}W^t = D^{-1}$. Now we consider a conjugation of the Galois representation ρ:

$$\rho' = \begin{pmatrix} W & 0 \\ 0 & (W^t)^{-1} \end{pmatrix} \rho \begin{pmatrix} W^{-1} & 0 \\ 0 & W^t \end{pmatrix} : \begin{array}{ccc} \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) & \to \mathrm{GSp}_{2n}(\mathcal{O}_L) \\ \sigma & \mapsto \begin{pmatrix} \chi_p(\sigma) \cdot T'(\sigma) & B'(\sigma) \\ 0 & (T'(\sigma)^{-1})^t \end{pmatrix}, \end{array}$$

where $T' : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathrm{GL}_n(\mathcal{O}_L)$ is the unramified homomorphism sending (any) Frobenius element to the matrix $D \in \mathrm{GL}_n(\mathcal{O}_L)$. By direct calculation, $B'(\sigma) = WB(\sigma)W^t$. So for any $1 \leq i, j \leq n$, the map $b'_{ij} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathcal{O}_L$ is an $\mathcal{O}_L$-linear combination of b_{kl}'s. From our previous discussion, the Serre-Tate coordinates t_{kl}'s of $\mathbb{A}_{\mathcal{O}_K}$ determine the images of b_{kl}'s in $H^1(I, \mathbb{Z}_p(\chi_p))$, and hence determine the images of b'_{ij}'s in $H^1(I, \mathcal{O}_L(\chi_p)) = H^1(I, \mathbb{Z}_p(\chi_p)) \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. On the other hand, if we define $\eta_j : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathcal{O}_L^\times$ as the unramified character which sends (any) Frobenius element to $d_j \in \mathcal{O}_L$, then the map $c'_{ij} = \eta_j \cdot b'_{ij} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to \mathcal{O}_L$ is a 1-cocycle valued in $\mathcal{O}_L(\chi_p\eta_i\eta_j)$. Again the restriction map gives us an isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathcal{O}_L(\chi_p\eta_i\eta_j)) \to H^1(I, \mathcal{O}_L(\chi_p))^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)/I}.$$

So in this way, the Serre-Tate coordinates determine the images of c'_{ij} in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/K), \mathcal{O}_L(\chi_p\eta_i\eta_j))$.

CHAPTER 4

Shimura varieties

In this chapter, we give a review on the Hilbert modular Shimura varieties and Siegel modular Shimura varieties. We give the construction of the integral models of these Shimura varieties and study the local model of these Shimura varieties at closed ordinary points.

Throughout this chapter we always fix a totally real field F with integer ring $\mathcal{O}_F$. The degree of F over $\mathbb{Q}$ is denoted by d.

4.1 Abelian varieties with real multiplication

In this section we introduce the notion of abelian varieties with real multiplication (AVRM for short).

Fix an invertible $\mathcal{O}_F$-module L, with a notion of positivity L_+ on it: for each real embedding $\tau : F \to \mathbb{R}$, we give an orientation on the line $L \otimes_{\mathcal{O}_F,\tau} \mathbb{R}$. First we recall the following definition in [7]:

Definition 4.1. *An L-polarized abelian scheme with real multiplication by $\mathcal{O}_F$ is the triple $(A_{/S}, \iota, \varphi)$ consisting of*

1. *$A_{/S}$ is an abelian scheme of relative dimension d;*
2. *$\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ is an algebra homomorphism which gives $A_{/S}$ an $\mathcal{O}_F$-module structure;*
3. *$\varphi : \underline{L} \to \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$ is an $\mathcal{O}_F$-linear morphism of sheaves of $\mathcal{O}_F$-modules on the étale site $(Sch_{/S})_{\acute{e}t}$ of the category of S-schemes, such that φ sends totally positive*

elements of L *into polarizations of* $A_{/S}$, *and the natural morphism* $\alpha : A \otimes_{\mathcal{O}_F} \underline{L} \to A^t$ *is an isomorphism. Here* A^t *is the dual abelian scheme of* A, *and* $\underline{L}$ *is the constant sheaf valued in* L, *and the sheaf* $\mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$ *is defined by :*

$(Sch_{/S})_{\acute{e}t} \ni T \mapsto \mathrm{Hom}^{Sym}_{\mathcal{O}_F,T}(A_{T_{/T}}, A^t_{T_{/T}}) = \{\lambda : A_{T_{/T}} \to A^t_{T_{/T}} | \lambda$ *is* $\mathcal{O}_F$*-linear and symmetric*$\}$

When $L = \mathfrak{c}$ *is a fractional ideal of* $\mathcal{O}_F$ *with the natural notion of positivity, we call the isomorphism* $\alpha : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$ *a* $\mathfrak{c}$*-polarization of* A *(see [28]1.0 for more discussion). We also make the convention that for* $c \in \mathfrak{c}$, *the morphism* $\lambda(c) : A \to A^t$ *is the corresponding symmetric* $\mathcal{O}_F$*-linear homomorphism.*

Remark 4.2. The f.p.p.f. abelian sheaf $A \otimes_{\mathcal{O}_F} \underline{L}$ is the sheafication of the functor

$$(Sch_{/S})_{\text{f.p.p.f.}} \ni T \mapsto A(T) \otimes_{\mathcal{O}_F} L.$$

This sheaf is represented by an abelian scheme over S, which is denoted by $A \otimes_{\mathcal{O}_F} L$. Hence the isomorphism α in (3) can be regarded as an isomorphism of abelian schemes over S.

Definition 4.3. *Let* $A_{/S}$ *be an abelian scheme over a scheme* S *of relative dimension* d, *and* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *be an algebra homomorphism. We say that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (DP) *if the natural morphism* $\alpha : A \otimes_{\mathcal{O}_F} \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S}) \to A^t$ *is an isomorphism. We say that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (RA) *if Zariski locally on* S, $\mathrm{Lie}(A_{/S})$ *is a free* $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{O}_F$*-module of rank* 1.

We remark here that the two conditions (DP) and (RA) in Definition 4.3 can be checked at each geometric point of the base scheme S. When the pair $(A_{/S}, \iota)$ satisfies the condition (RA), we come to the notion of abelian schemes with real multiplication (by $\mathcal{O}_F$) defined in [41]. As explained in [7]2.9, when d_F is invertible on S, condition (DP) in Definition 4.3 implies (RA). For later use, we explain that condition (RA) implies (DP) under some assumption on S and by a suitable choice of the pair (L, L_+), we can make $A_{/S}$ be an L-polarized abelian scheme with real multiplication by $\mathcal{O}_F$. First we need the following:

Proposition 4.4. *([41]1.17,1.18) Let $A_{/S}$ be an abelian scheme of relative dimension d,and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ be an algebra homomorphism. Then the étale sheaf $\mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$ defined above is locally constant with values in a projective $\mathcal{O}_F$-module of rank 1, endowed with a notion of positivity corresponding to polarizations of $A_{/S}$. In particular, when S is normal and connected, this sheaf is constant.*

Here we remark that in [41], the abelian scheme $A_{/S}$ is assumed to satisfy condition (RA). But this condition is not necessary in the proof of the above proposition.

Now assume that S is normal and connected (e.g. S is the spectrum of the integer ring of a number field). Then from Proposition 4.4 we can find a projective $\mathcal{O}_F$-module $\mathcal{M}$ of rank 1 with a notion of positivity $\mathcal{M}_+$ and an $\mathcal{O}_F$-linear isomorphism $\varphi : \underline{\mathcal{M}} \to \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$. To check this φ satisfies condition (3) in Definition 4.1, we still need to check that the morphism $\alpha : A \otimes_{\mathcal{O}_F} \underline{\mathcal{M}} \to A^t$ is an isomorphism.

We can assume that $S = \mathrm{Spec}(k)$, where k is an separably closed field and we want to prove that α is an isomorphism of abelian varieties over k. Then it suffices to show that for any rational prime l, there exists $0 \neq \lambda \in \mathcal{M}$, such that $\deg(\varphi(\lambda))$ is prime to l. In fact, for any $\alpha \in \mathcal{M}$, we have a natural morphism $A \to A \otimes_{\mathcal{O}_F} \mathcal{M}$ whose effect on R-valued points is given by the formula (R is an k-algebra):

$$A(R) \ni a \mapsto a \otimes_{\mathcal{O}_F} \lambda \in A(R) \otimes_{\mathcal{O}_F} \mathcal{M}.$$

The composition of this morphism with α is $\varphi(\lambda)$. Hence $\deg(\alpha) | \deg(\varphi(\lambda))$. In particular, $\deg(\alpha)$ is prime to l. As l is arbitrary, $\deg(\alpha) = 1$ and hence α is an isomorphism.

To prove the existence of λ, we apply an argument in [15] Chapter 3 Section 5: when $char(k) > 0$, by [41]1.13, we can always lift the pair $(A_{/k}, \iota)$ to an abelian scheme with real multiplication $(\tilde{A}_{/W(k)}, \tilde{\iota})$ satisfying (RA). Here $W(k)$ is the ring of Witt vectors of k. Hence we can assume that $char(k) = 0$. By Lefschetz principle, we can assume that k is the complex filed. Then the existence of λ follows from the complex uniformization [15]Chapter 2 Section 2.2.

The following proposition tells us that when S is a scheme of characteristic 0, condition (RA) and hence (DP) is automatically satisfied.

Proposition 4.5. *Let k be a field of characteristic 0, $A_{/k}$ be an abelian variety of dimension d, and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$ be an algebra homomorphism. Then $\mathrm{Lie}(A_{/k})$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-module of rank 1.*

Proof. By Lefschetz principle we can again work over the complex field. Then the result follows from [15] Chapter 2, Corollary 2.6. □

4.2 Hilbert modular Shimura varieties

In this section we introduce the integral model of Hilbert modular Shimura varieties we will work with.

Fix a finite set of primes Ξ. Set

$$\mathbb{Z}_{(\Xi)} = \{\frac{m}{n} \in \mathbb{Q} | m, n \in \mathbb{Z}, (n, p) = 1, \forall p \in \Xi\}.$$

Then define $\mathcal{O}_{(\Xi)} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$, and $\mathcal{O}^{\times}_{(\Xi),+}$ as the set of totally positive units in $\mathcal{O}_{(\Xi)}$. Also we define:

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z},\ \widehat{\mathbb{Z}}^{(\Xi)} = \varprojlim \mathbb{Z}/n\mathbb{Z},\ \mathbb{Z}_{\Xi} = \prod_{l \in \Xi} \mathbb{Z}_l,$$

where in the first inverse limit, n ranges over all positive numbers, and in the second inverse limit, n ranges over all positive integers prime to Ξ. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. Then set

$$\mathbb{A}^{(\Xi\infty)} = \{x \in \mathbb{A} | x_l = x_\infty = 0, \forall l \in \Xi\},$$

and $F_{\mathbb{A}^{(\Xi\infty)}} = F \otimes_{\mathbb{Q}} \mathbb{A}^{(\Xi\infty)}$.

Define the algebraic group $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL(2))$ and let Z be its center. K is an open compact subgroup of $G(\widehat{\mathbb{Z}})$ which is maximal at Ξ, in the sense that $K = G(\mathbb{Z}_{\Xi}) \times K^{(\Xi)}$, where

$$K^{(\Xi)} = \{x \in K | x_p = 1 \text{ for all } p \in \Xi\}.$$

Definition 4.6. *Define the functor* ${\mathcal{E}'}_K^{(\Xi)} : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$*, such that for each* $\mathbb{Z}_{(\Xi)}$*-scheme* S*,* ${\mathcal{E}'}_K^{(\Xi)}(S) = [(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$*. Here* $[(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$ *is the set of isomorphism classes of quadruples* $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ *consisting of:*

1. *an abelian scheme* $A_{/S}$ *of relative dimension* d*;*
2. *an algebra homomorphism* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *such that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (DP) *(see Definition 4.3);*
3. *a subset* $\{\lambda \circ \iota(b) : b \in \mathcal{O}_{(\Xi),+}^{\times}\}$ *of* $\mathrm{Hom}(A_{/S}, A^t_{/S}) \otimes_{\mathbb{Z}} \mathbb{Q}$*, where* $\lambda : A_{/S} \to A^t_{/S}$ *is an* $\mathcal{O}_F$*-linear polarization of* A*, whose degree is prime to* Ξ*;*
4. $\bar{\eta}^{(\Xi)}$ *is a rational* K*-level structure of the abelian scheme* $A_{/S}$ *(see Remark 4.8 below).*

An isomorphism from one quadruple $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ *to another* $(A'_{/S}, \iota', \bar{\lambda}', \bar{\eta}'^{(\Xi)})$ *is an element* $f \in \mathrm{Hom}(A_{/S}, A'_{/S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$ *whose degree is prime to* Ξ *such that:*

1. $f \circ \iota(b) = \iota'(b) \circ f$ *for all* $b \in \mathcal{O}_F$*;*
2. $f^t \circ \bar{\lambda}' \circ f = \bar{\lambda}$ *as subsets of* $\mathrm{Hom}(A_{/S}, A^t_{/S}) \otimes_{\mathbb{Z}} \mathbb{Q}$*;*
3. *we have the equality of level stuctures:* $V^{(\Xi)}(f)(\bar{\eta}^{(\Xi)}) = \bar{\eta}'^{(\Xi)}$.

Now we choose a representative $I = \{\mathfrak{c}\}$ of fractional ideals in the finite class group

$$Cl(K) = (F_{\mathbb{A}^{(\Xi\infty)}})^{\times} / \mathcal{O}_{(\Xi),+}^{\times} \det(K).$$

For each $\mathfrak{c}$, fix an $\mathcal{O}_F$-lattice $L_{\mathfrak{c}} \subseteq V = F^2$ such that $\wedge(L_{\mathfrak{c}} \wedge L_{\mathfrak{c}}) = \mathfrak{c}^*$. Here $\wedge : V \wedge V \to F$ is the alternating form given by $((a_1, a_2), (b_1, b_2)) \mapsto a_1 b_2 - a_2 b_1$.

Definition 4.7. *Define the functor* $\mathcal{E}_{K,\mathfrak{c}}^{(\Xi)} : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$*,such that for each* $\mathbb{Z}_{(\Xi)}$*-scheme* S*,* $\mathcal{E}_{K,\mathfrak{c}}^{(\Xi)}(S) = \{(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})\}_{/\cong}$*, where* $\{(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})\}_{/\cong}$ *is the set of isomorphic classes of quadruples* $(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})$ *consisting of*

1. *an abelian scheme* $A_{/S}$ *of relative dimension* d*;*

2. *an algebra homomorphism* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *such that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (DP) *(see Definition 4.3);*

3. *a* $\mathfrak{c}$*-polarization* $\phi : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$ *of* $A_{/S}$ *(see Definition 4.1);*

4. $\bar{\alpha}^{(\Xi)}$ *is an integral* K*-level structure of the abelian scheme* $A_{/S}$ *(see Remark 4.8 below).*

An isomorphism from one quadruple $(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})$ *to another* $(A'_{/S}, \iota', \phi', \bar{\alpha}'^{(\Xi)})$ *is an isomorphism* $f : A \to A'$ *of abelian schemes over* S *such that*

1. $f \circ \iota(b) = \iota'(b) \circ f$ *for all* $b \in \mathcal{O}_F$*;*

2. $f^t \circ \phi' \circ (f \otimes_{\mathcal{O}_F} Id_{\mathfrak{c}}) = \phi : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$*;*

3. *we have an equality of integral level structures:* $T^{(\Xi)}(f)(\bar{\alpha}^{(\Xi)}) = \bar{\alpha}'^{(\Xi)}$.

Remark 4.8. Here we briefly recall the notion of level structures on an abelian scheme with real multiplication. As in Definition 4.6 and 4.7, we fix an abelian scheme $A_{/S}$ and a homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$. Take a point $s \in S$ and let $\bar{s} : \mathrm{Spec}(k(\bar{s})) \to S$ be a geometric point of S over s, where $k(\bar{s})$ is a separably closed field extension of the residue field $k(s)$ of S at the point s. Consider the prime-to-Ξ Tate module

$$T^{\Xi}(A_{\bar{s}}) = \varprojlim_{N} A[N](k(\bar{s})),$$

where N runs through all positive integers prime to Ξ, and set $V^{\Xi}(A_{\bar{s}}) = T^{\Xi}(A_{\bar{s}}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\Xi}$, which is a free $F_{\mathbb{A}^{(\Xi\infty)}}$-module of rank 2. When N is invertible on S, the finite scheme $A[N]$ is étale over S. The algebraic fundamental group $\pi(S, \bar{s})$ acts on $A[N](k(\bar{s}))$, and hence on $T^{\Xi}(A_{\bar{s}})$ and $V^{\Xi}(A_{\bar{s}})$. This action is compatible with the action of $G(\widehat{\mathbb{Z}}^{(\Xi)})$ (resp. $G(F_{\mathbb{A}^{(\Xi\infty)}})$) on $T^{\Xi}(A_{\bar{s}})$ (resp. $V^{\Xi}(A_{\bar{s}})$).

We define a sheaf of sets $ILV^{(\Xi)} : (Sch_{/S})_{\acute{e}t} \to Set$ on the étale site of the category of S-schemes such that for any connected S-scheme S', we have:

$$ILV^{(\Xi)}(S') = H^0(\pi(S', \bar{s}'),\ \mathrm{Isom}_{\mathcal{O}_F}(L_{\mathfrak{c}} \otimes_{\mathcal{O}_F} \widehat{\mathbb{Z}}^{(\Xi)}, T^{\Xi}(A_{\bar{s}'}))),$$

where $\bar{s}'$ is a geometric point of S' over a point s' of S'. The étale sheaf $ILV^{(\Xi)}$ is independent of the choice of s' (see [20] Section 6.4.1). The group $G(\widehat{\mathbb{Z}}^{(\Xi)})$ acts on the sheaf $ILV^{(\Xi)}$ through its action on the Tate module $T^{\Xi}(A_{\bar{s}'})$, and we denote by $ILV^{(\Xi)}/K$ the quotient sheaf of $ILV^{(\Xi)}$ under the group action of $K^{(\Xi)}$. An integral K-level structure of $A_{/S}$ is a section $\bar{\alpha}^{(\Xi)} \in ILV^{(\Xi)}/K(S)$. Similarly we define another sheaf $RLV^{(\Xi)} : (Sch_{/S})_{ét} \to Set$ such that for any connected S-scheme S', we have:

$$RLV^{(\Xi)}(S') = H^0(\pi(S', \bar{s}'), \mathrm{Isom}_{\mathcal{O}_F}(V \otimes_{\mathbb{Z}} \mathbb{A}^{(\Xi\infty)}, V^{\Xi}(A_{\bar{s}'}))),$$

and define the quotient sheaf $RLV^{(\Xi)}/K$ in the same way. Then a rational K-level structure of $A_{/S}$ is a section $\bar{\eta}^{(\Xi)} \in RLV^{(\Xi)}/K(S)$.

Suppose that we have another abelian scheme $A'_{/S}$ and a homomorphism $\iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/S})$. We can similarly define two étale sheaves $ILV'^{(\Xi)}$ and $RLV'^{(\Xi)}$ replacing $A_{/S}$ by $A'_{/S}$ in the above construction. If $f : A \to A'$ is an $\mathcal{O}_F$-linear isomorphism of abelian schemes, the isomorphism f induces an isomorphism of Tate modules $T^{(\Xi)}(A_{\bar{s}}) \cong T^{(\Xi)}(A'_{\bar{s}})$ for any geometric point $\bar{s}$ of S. Hence f induces an isomorphism of étale sheaves $T^{(\Xi)}(f) : ILV^{(\Xi)} \to ILV'^{(\Xi)}$ which is compatible with the $G(\widehat{\mathbb{Z}}^{(\Xi)})$-action. Thus f also induces an isomorphism $T^{(\Xi)}(f) : ILV^{(\Xi)}/K \to ILV'^{(\Xi)}/K$ for all subgroup K of $G(\widehat{\mathbb{Z}})$. For any integral K-level structure $\bar{\alpha}^{(\Xi)} \in ILV^{(\Xi)}/K(S)$, we use $T^{(\Xi)}(f)(\bar{\alpha}^{(\Xi)})$ to denote its image under the isomorphism $T^{(\Xi)}(f)$. Similarly if $f : A \to A'$ is an $\mathcal{O}_F$-linear prime-to-Ξ isogeny of abelian schemes, then f induces an isomorphism $V^{(\Xi)}(A_{\bar{s}}) \cong V^{(\Xi)}(A'_{\bar{s}})$ and hence isomorphisms of étale sheaves $V^{(\Xi)}(f) : RLV^{(\Xi)} \to RLV'^{(\Xi)}$ and $V^{(\Xi)}(f) : RLV^{(\Xi)}/K \to RLV'^{(\Xi)}/K$. For any rational K-level structure $\bar{\eta}^{(\Xi)} \in RLV^{(\Xi)}/K(S)$, we use $V^{(\Xi)}(f)(\bar{\eta}^{(\Xi)})$ to denote its image under the isomorphism $V^{(\Xi)}(f)$. We refer to [21] Section 4.3.1 for more discussion on this topic.

Theorem 9. *When K is small enough (e.g. $\det(K^{(\Xi)}) \cap \mathcal{O}_+^{\times} \subseteq (K^{(\Xi)} \cap Z(\mathbb{Z}))^2$), then we have a natural isomorphism of functors:*

$$i : \coprod_{\mathfrak{c} \in I} \mathcal{E}^{(\Xi)}_{K,\mathfrak{c}} \to \mathcal{E}'^{(\Xi)}_{K}.$$

The proof is essentially given in [20] Section 4.2.1 so we omit the proof here. The only thing we want to remark here is that for any quadruple $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ considered in Definition 4.6, we can find an abelian scheme $A'_{/S}$ with real multiplication ι',and an $\mathcal{O}_F$-linear prime-to-Ξ isogeny $f : A \to A'$ of abelian schemes over S such that $A'_{/S}$ admits an integral level structure. Since S is a $\mathbb{Z}_{(\Xi)}$-scheme, the isogeny f is étale. From Lemma 5.9, the pair $(A'_{/S}, \iota')$ also satisfies the condition (DP). Then we can follow the argument in [20] Section 4.2.1 to conclude this theorem.

From [7], the functor $\mathcal{E}^{(\Xi)}_{K,\mathfrak{c}}$ is representable. By Theorem 9, when K is small enough, we can assume that the functor $\mathcal{E}'^{(\Xi)}_K$ is represented by a $\mathbb{Z}_{(\Xi)}$-scheme $Sh^{(\Xi)}_K$. From [7] Theorem 2.2, the scheme $Sh^{(\Xi)}_K$ is flat of complete intersection over $\mathbb{Z}_{(\Xi)}$, and smooth over $\mathbb{Z}_{(\Xi)}[\frac{1}{d_F}]$.

Now we take the projective limit of $Sh^{(\Xi)}_K$ for varying K, and get a $\mathbb{Z}_{(\Xi)}$-scheme $Sh^{(\Xi)}$. It is clear that $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ represents the moduli problem $\mathcal{E}'^{(\Xi)} : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$, such that for each $\mathbb{Z}_{(\Xi)}$-scheme S, $\mathcal{E}'^{(\Xi)}_K(S) = [(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})]$. where $[(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$ is the set of isomorphism classes of quadruples $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ considered in Definition 4.6, except that $\eta^{(\Xi)} \in RLA^{(\Xi)}(S)$ is a rational level structure instead of a rational K-level structure for some open compact subgroup K. An isomorphism from one quadruple $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ to another $(A'_{/S}, \iota', \bar{\lambda}', \eta'^{(\Xi)})$ is an element $f \in \mathrm{Hom}(A_{/S}, A'_{/S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$ whose degree is prime to Ξ such that it satisfies the first two conditions in Definition 4.6, and also $V^{(\Xi)}(f)(\eta^{(\Xi)}) = \eta'^{(\Xi)}$ instead of that last condition there.

For any $g \in G(F_{\mathbb{A}^{(\Xi\infty)}})$, the map sending each quadruple $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ to another quadruple
$(A_{/S}, \iota, \bar{\lambda}, g(\eta^{(\Xi)}))$ induces an automorphism of the functor $\mathcal{E}'^{(\Xi)}$, and hence an automorphism of the Shimura variety $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ by universality. We still denote this action by g.

For simplicity we denote the Shimua variety $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ by $X_{/\mathbb{Z}_{(\Xi)}}$ in the following discussion. Pick a closed point $x_p \in X(\bar{\mathbb{F}}_p)$. Let K be a neat subgroup of $G(F_{\mathbb{A}^{(\Xi\infty)}})$. Then the natural morphism $X \to X_K = X/K$ is étale. Let $\mathcal{O}_{X,x_p}$ and $\mathcal{O}_{X_K,x_p}$ be the stalk of X and X_K at x_p, respectively. The completion of $\mathcal{O}_{X,x_p}$ is canonically isomorphic to the completion of $\mathcal{O}_{X_K,x_p}$, and we denote this completion by $\widehat{\mathcal{O}}_{x_p}$. Suppose that x_p is represented by a quadruple $(A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0, \bar{\alpha_0}^{(\Xi)}) \in \mathcal{E}^{(\Xi)}_{K,\mathfrak{c}}(\bar{\mathbb{F}}_p)$.

Let $CL_{/W_p}$ be the category of complete local W_p-algebras with residue field $\bar{\mathbb{F}}_p$. Consider the local deformation functor $\widehat{D}_p : CL_{/W_p} \to Set$, given by

$$\widehat{D}_p(R) = \{(A_{/R}, \iota_R, \phi_R) | (A_{/R}, \iota_R, \phi_R) \times_R \bar{\mathbb{F}}_p \cong (A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0)\}_{/\cong},$$

here the triple $(A_{/R}, \iota_R, \phi_R)$ consists of an abelian A schemes over R, an algebra homomorphism $\iota_R : \mathcal{O}_F \to \mathrm{End}(A_{/R})$ and a $\mathfrak{c}$-polarization ϕ_R of $A_{/R}$. An isomorphism from a triple $(A_{/R}, \iota_R, \phi_R)$ to another $(A'_{/R}, \iota'_R, \phi'_R)$ is an isomorphism $f : A \to A'$ of abelian schemes over R such that

1. for all $a \in \mathcal{O}_F$, we have $f \circ \iota_R(a) = \iota'_R(a) \circ f : A \to A'$;

2. $f^t \circ \phi'_R \circ (f \otimes Id_{\mathfrak{c}}) = \phi_R : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$.

Define a functor $DEF_p : CL_{/W_p} \to Set$ by the formula:

$$DEF_p(R) = \{(D_{/R}, \Lambda_R, \varepsilon_R)\}_{/\cong},$$

where $D_{/R}$ is a Barsotti-Tate $\mathcal{O}_F$-module over R, $\Lambda_R : D \otimes_{\mathcal{O}_F} \mathfrak{c} \to D^t$ is an $\mathcal{O}_F$-linear isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over R (D^t is the Cartier dual of D), and $\varepsilon_R : D_0 = D \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]$ is an isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over the special fiber $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ of $\mathrm{Spec}(R)$.

For any triple $(A_{/R}, \iota_R, \phi_R)$ in $\widehat{D}_p(R)$, let $A[p^\infty]_{/R}$ be its p-divisible Barsotti-Tate $\mathcal{O}_F$-module over R. The $\mathfrak{c}$-polarization ϕ_R of $A_{/R}$ gives an isomorphism $\Lambda_R : A[p^\infty] \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t[p^\infty] \cong (A[p^\infty])^t$. The isomorphism $(A_{/R}, \iota_R, \phi_R) \times_R \bar{\mathbb{F}}_p \cong (A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0)$ gives an isomorphism $\varepsilon_R : A[p^\infty] \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]$. By the Serre-Tate deformation theory, we have:

Proposition 4.9. *The above association $(A_{/R}, \iota_R, \phi_R) \mapsto (A[p^\infty]_{/R}, \Lambda_R, \varepsilon_R)$ induces an equivalence of functors $\widehat{D}_p \to DEF_p$.*

We define two more functors $DEF^?_p : CL_{/W_p} \to Set, ? = ord, ll$, by:

$$DEF^?_p(R) = \{(D^?, \phi^?, \varepsilon^?)\}_{/\cong},$$

here in the triple $(D^?, \phi^?, \varepsilon^?)$, $D^?$ is a Barsotti-Tate $\mathcal{O}_F$-module over R,$\phi^? : D^? \otimes_{\mathcal{O}_F} \mathfrak{c} \to (D^?)^t$ is an isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over R, and $\varepsilon^? : D^? \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]^?$ is an isomorphism over $\bar{\mathbb{F}}_p$.

Since $A_{0/\bar{\mathbb{F}}_p}$ admits an actoin of $\mathcal{O}_F$, we have the decomposition of Barsotti-Tate $\mathcal{O}_F$-modules:

$$A_0[p^\infty] = \bigoplus_{\mathfrak{p}|p} A_0[\mathfrak{p}^\infty].$$

Here $\mathfrak{p}$ ranges over the primes ideals of $\mathcal{O}_F$ over p and for each $\mathfrak{p}$, let

$$A_0[\mathfrak{p}^\infty] = \varinjlim A_0[\mathfrak{p}^n]$$

be the $\mathfrak{p}$-divisible Barsotti-Tate group of A_0. We also define

$$T_{\mathfrak{p}}(A_0) = \varprojlim A_0[\mathfrak{p}^n](\bar{\mathbb{F}}_p)$$

as the $\mathfrak{p}$-divisible Tate module of A_0.

Similar with [24] Proposition 1.2, we have the following facts:

1. the functor DEF_p is represented by the formal scheme $\widehat{S}_{p/W_p}$ associated to $\widehat{\mathcal{O}}_{x_p}$;

2. there is a natural equivalence of functors: $DEF_p \cong DEF_p^{ord} \times DEF_p^{ll}$,and hence the formal $\widehat{S}_{p/W_p}$ is a product of two formal schemes $\widehat{S}^{ord}_{p/W_p}$ and $\widehat{S}^{ll}_{p/W_p}$ such that $DEF_p^?$ is represented by $\widehat{S}^?_{p/W_p}$ for $? = ord, ll$;

3. For each $\mathfrak{p} \in \Sigma_p^{ord}$, fix an isomorphism $\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_0)$. Since $\mathfrak{c}$ is prime to p, by the $\mathfrak{c}$-polarization ϕ_0, we also have an isomorphism $\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_0^t)$. Then $\widehat{S}^{ord}_{p/W_p}$ is a smooth formal scheme over W_p which is isomorphic to

$$\prod_{\mathfrak{p}\in\Sigma^{ord}} \mathrm{Hom}(T_{\mathfrak{p}}(A_0) \otimes_{\mathcal{O}_{\mathfrak{p}}} T_{\mathfrak{p}}(A_0^t), \widehat{\mathbb{G}}_m) \cong \prod_{\mathfrak{p}\in\Sigma^{ord}} \mathrm{Hom}(\mathcal{O}_{\mathfrak{p}}, \widehat{\mathbb{G}}_m) = \prod_{\mathfrak{p}\in\Sigma^{ord}} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*,$$

   here $\mathcal{O}_{\mathfrak{p}}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathfrak{p}}, \mathbb{Z}_p)$.

In fact, for any triple $(A_{/R}, \iota_R, \phi_R)$ in $\widehat{D}_p(R)$, the level structure $\bar{\alpha_0}^{(\Xi)}$ on A_0 can be extended uniquely to a level structure on $A_{/R}$. Then the functor $\widehat{D}_p$, and hence the functor DEF_p by Proposition 4.9, is represented by the the formal scheme $\widehat{S}_{p/W_p} = Spf(\widehat{\mathcal{O}}_{x_p})$.

For a triple $(D_{/R}, \Lambda_R, \varepsilon_R) \in DEF_p(R)$, we have a canonical decomposition the Barsotti-Tate $\mathcal{O}_F$-module $D = D^{ord} \times D^{ll}$, where D^{ord} is the maximal ordinary Barsotti-Tate $\mathcal{O}_F$-submodule of D, and D^{ll} is its local-local complement. From this we have a morphism

$$DEF_p(R) \ni (D_{/R}, \Lambda_R, \varepsilon_R) \mapsto \{(D^{ord}_{/R}, \Lambda_R|_{D^{ord}}, \varepsilon_R|_{D^{ord}}), (D^{ll}_{/R}, \Lambda_R|_{D^{ll}}, \varepsilon_R|_{D^{ll}})\} \in DEF^{ord}_p(R) \times DEF^{ll}_p(R),$$

from which we get a equivalence of functors between DEF_p and $DEF^{ord}_p \times DEF^{ll}_p$. Hence the formal scheme $\widehat{S}_{p/W_p}$ is a product of two formal schemes $\widehat{S}^{ord}_{p/W_p} \times \widehat{S}^{ll}_{p/W_p}$.

In contrast with [24] Proposition 1.2, the formal scheme $\widehat{S}_{p/W_p}$ may not be smooth when p divides the discriminant d_F of F since the Shimura variety $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ we consider here is not smooth. But from the Serre-Tate deformation theory, the formal scheme $\widehat{S}^{ord}_{p/W_p}$ is always smooth, and this is the part we are interested in.

4.3 Siegel modular Shimura varieties

In this section we recall basic results on Siegel modular Shimura varieties. Our main reference is [20].

Fix a positive integer d and a prime p. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at (p). Let $G_{/\mathbb{Q}} = \mathrm{GSp}(2d)_{/\mathbb{Q}}$ be the symplectic similitude group over $\mathbb{Q}$, i.e. for any $\mathbb{Q}$-algebra R, we have

$$G(R) = \{X \in \mathrm{GL}_{2d}(R) | X^t J_d X = \nu(X) J_d, \ \textit{for some } \nu(A) \in R^\times\},$$

where $J_d = \begin{pmatrix} 0 & -1_d \\ 1_d & 0 \end{pmatrix}$. Define the Siegle upper half space

$$\mathcal{H}_d = \{Z = X + iY \in M_{d\times d}(\mathbb{C}) | Z = Z^t, Y > 0\}.$$

Set $\mathfrak{X} = \mathcal{H}_d \sqcup \bar{\mathcal{H}}_d$. For any integer $N > 0$, define

$$\widehat{\Gamma}(N) = \{\alpha \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) | \alpha \equiv 1 \ mod \ N\}.$$

Let $W = \mathbb{Q}^{2d}$ with the alternating form $\psi(x, y) = x^t J_d y$. For each $\mathbb{Q}$-algebra R, $G(R) = \mathrm{GSp}_{2d}(R)$ acts on $W \otimes_{\mathbb{Q}} R$ in the natural way, preserving the alternating form ψ up to scalar multiplication. Set $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p, W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each rational prime p. Let $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_{2d} = (0, 0, \dots, 1)\}$ be the standard $\mathbb{Z}_p$-basis of L_p.

For $N \geq 3$, consider the following moduli problem:

$$\begin{aligned} \mathcal{E}_N : \quad Sch_{/\mathbb{Z}[\frac{1}{N}]} &\to \text{Sets}, \\ S \quad &\mapsto \mathcal{E}_N(S) = \{(A, \lambda, \eta_N)\}_{/\cong}, \end{aligned}$$

such that for each $\mathbb{Z}[\frac{1}{N}]$-scheme S, $\mathcal{E}_N(S)$ is the set of isomorphism classes of the triples (A, λ, η_N) consisting of:

1. an abelian scheme $A_{/S}$ of relative dimension d;
2. a principal polarization $\lambda : A \to A^t$ of A;
3. a level N structure $\eta_N : (\mathbb{Z}/N\mathbb{Z})^{2d} = L/NL \cong A[N](k(s))$, under which the symplectic pairing $< \cdot, \cdot >$ on L/NL is sent to the Weil pairing on $A[N]$ induced by the polarization λ, and $s : \mathrm{Spec}(k(s)) \to S$ is a geometric point of S.

It is well known that the moduli problem $\mathcal{E}_N$ is represented by a scheme $\mathcal{A}_{1,N/\mathbb{Z}[\frac{1}{N}]}$. Moreover, the $\mathbb{C}$-valued point of $\mathcal{A}_{1,N}$ is given by

$$\mathcal{A}_{1,N}(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathfrak{X} \times G(\mathbb{A}_f)) / \widehat{\Gamma}(N).$$

Then we define two pro-schemes:

$$Sh_{/\mathbb{Q}} = \varprojlim_N \mathcal{A}_{1,N/\mathbb{Z}[\frac{1}{N}]}, Sh^{(p)}_{\mathbb{Z}_{(p)}} = \varprojlim_{(p,N)=1} \mathcal{A}_{1,N/\mathbb{Z}[\frac{1}{N}]}$$

Take a closed point $x_p = (A_0, \bar{\lambda}_0, \eta_0^{(p)})_{/\bar{\mathbb{F}}_p} \in Sh^{(p)}(\bar{\mathbb{F}}_p)$ such that the abelian variety $A_{0/\bar{\mathbb{F}}_p}$ is ordinary. Under this assumption, the endomorphism algebra $D = \mathrm{End}^\circ(A_{0/\bar{\mathbb{F}}_p})$ is a matrix algebra over a CM algebra (i.e. a finite product of CM fields) M. The CM algebra M

is generated by the Frobenius endomporphism of $A_{0/\bar{\mathbb{F}}_p}$ over $\mathbb{Q}$. Let R be the order of M generated by the Frobenius map of $A_{0/\bar{\mathbb{F}}_p}$ over $\mathbb{Z}$. Let $R_{(p)} = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Define a torus $T_{/\mathbb{Z}_{(p)}}$ by setting

$$T(\mathbb{Z}_{(p)}) = \{a \in R_{(p)}^{\times} | x \cdot \bar{x} \in \mathbb{Z}_{(p)}^{\times}\}.$$

For each $g \in G(\mathbb{A}^{(p,\infty)})$, it acts on the moduli problem $\mathcal{E}^{(p)}$ by sending a triple $(A, \bar{\lambda}, \eta^{(p)})_{/S} \in \mathcal{E}^{(p)}(S)$ to the triple $(A, \bar{\lambda}, \eta^{(p)} \circ g)_{/S}$. By universal property, g induces an automorphism of the Shimura variety $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$, which is still denoted by g.

Define a homomorphism $\hat{\rho} : T(\mathbb{Z}_{(p)}) \to G(\mathbb{A}^{(p,\infty)})$ by the formula $a \circ \eta_0^{(p)} = \eta_0^{(p)} \circ \hat{\rho}(a)$, for $a \in T(\mathbb{Z}_{(p)})$. The image of $T(\mathbb{Z}_{(p)})$ under $\hat{\rho}$ stabilizes the closed point x_p under the action of $G(\mathbb{A}^{(p,\infty)})$ on $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ explained as above.

Let $\widehat{S}_{p/W_p}$ be the formal completion of the Shimura variety $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ along the closed point x_p, where $W_p = W(\bar{\mathbb{F}}_p)$ is the ring of Witt vectors with coefficients in $\bar{\mathbb{F}}_p$. As the abelian variety $A_{0/\bar{\mathbb{F}}_p}$ is ordinary, by Serre-Tate deformation theory, we have an isomorphism:

$$\widehat{S}_p \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sym}(T_pA_0(\bar{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} T_pA_0(\bar{\mathbb{F}}_p)), \widehat{\mathbb{G}}_m).$$

Each $a \in T(\mathbb{Z}_{(p)})$ gives an automorphism on the Serre-Tate deformation space $\widehat{S}_p$. In terms of the Serre-Tate coordinates, this action is give by the formula:

$$a \circ t = (t \circ (a \otimes a)^{-1})^{a \cdot \bar{a}}, \ \textit{for } t \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sym}(T_pA_0(\bar{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} T_pA_0(\bar{\mathbb{F}}_p)), \widehat{\mathbb{G}}_m).$$

For simplicity, we assume that the abelian variety $A_{0/\bar{\mathbb{F}}_p}$ is simple. But the following results can be generalized to non-simple cases without any difficulty. Under this assumption, M is a CM field and if $A_{0/\bar{\mathbb{F}}_p}$ is defined over a finite field $\mathbb{F}_q$ (q is a power of p), then M is generated by $\mathbb{F}_q$ over $\mathbb{Q}$. Let F be the maximal totally real subfield of M. We make another assumption that the degree of M over $\mathbb{Q}$ is $2d$. We choose embeddings $\varphi_1, \ldots, \varphi_d : M \to \bar{\mathbb{Q}}$, such that all the embeddings of M into $\bar{\mathbb{Q}}$ are given by the set $\{\varphi_1, \ldots, \varphi_d, \bar{\varphi}_1, \ldots, \bar{\varphi}_d\}$, where $\bar{\cdot}$ means a complex conjugation in $\bar{\mathbb{Q}}$, and M acts on the rational Tate module $T_pA_0(\bar{\mathbb{F}}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the character $\Pi_{i=1}^d \bar{\varphi}_i$. Then we have chosen the embeddings $\varphi_1, \ldots, \varphi_d$ so that the deformation space $\widehat{S}_p$ has canonical coordinates $t_{i,j}$ on which the group $T(\mathbb{Z}_{(p)})$ acts through the character $\varphi_i \cdot \bar{\varphi}_j$,$1 \leq i, j \leq d$.

CHAPTER 5

Local Indecomposability of Hilbert Modular Galois Representations

In this chapter, we fix F to be a totally real field with degree d over $\mathbb{Q}$ and use $\mathcal{O}_F$ to denote its integer ring. Let $\mathcal{D} = \mathcal{D}_{F/\mathbb{Q}}$ be the different of $F/\mathbb{Q}$ and $d_F = \mathrm{Norm}_{F/\mathbb{Q}}(\mathcal{D})$ be its discriminant. For any prime $\mathfrak{p}$ of $\mathcal{O}_F$, let $\mathcal{O}_{\mathfrak{p}}$ (resp. $F_{\mathfrak{p}}$) be the completion of $\mathcal{O}_F$ (resp. F) with respect to $\mathfrak{p}$. We use $\mathbb{A}$ to denote the adele ring of $\mathbb{Q}$, and use $F_{\mathbb{A}}$ (resp. $F_{\mathbb{A}_f}$) to denote the adele ring (resp. finite adele ring) of F.

Let f be a parallel weight two Hilbert modularo form of level $\mathfrak{m}$ over F. Assume that f is a Hecke eigenform and let K_f be its Hecke field. For any prime λ of K_f over a rational prime p, let $K_{f,\lambda}$ be the completion of K_f at λ. It is well known that there is a Galois represention $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(K_{f,\lambda})$ attached to f. Moreover if the eigenform f is nearly p-ordinary, then up to equivalence the restriction of ρ_f to the decomposition group $D_{\mathfrak{p}}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$ is of the shape (see [52] Theorem 2 for the ordinary case and [19] Proposition 2.3 for the nearly ordinary case):

$$\rho_f|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}.$$

Recall that we put the following technical condition on f when the degree of F over $\mathbb{Q}$ is even: there exists a finite place v of F such that π_v is square integrable (i.e. special or supercuspidal) where $\pi_f = \otimes_v \pi_v$ is the automorphic representation of $GL_2(F_{\mathbb{A}})$ associated to f ($F_{\mathbb{A}}$ is the adele ring of F). In this chapter, we prove the first main result in this thesis, i.e. the following:

Theorem 10. *If f does not have complex multiplication, then $\rho_f|_{D_{\mathfrak{p}}}$ is indecomposable.*

Before starting the argument, we want to give a sketch of our argument. Under the assumption on f, there exist an abelian variety $A_{f/F}$ and a homomorphism $L \to \mathrm{End}^0(A_{f/F})$ where L/K_f is a finite extension and the degree of L over $\mathbb{Q}$ equals to the dimension of A_f, such that the Galois representation ρ_f comes from the λ-adic Tate module of A_f (at least upto a twist of a character). Hence the theorem is reduced to prove: if the abelian variety $A_{f/F}$ does not have complex multiplication, then its λ-adic Tate module $T_\lambda(A_f)$ is indecomposable as an $I_\mathfrak{p}$-module, where $I_\mathfrak{p}$ is the inertia group of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at a prime $\mathfrak{p}$ of F over p. By an analysis of the endomorphism algebra of an abelian variety of $GL(2)$-type in section 5.1, we can always take L to be a totally real field (see Proposition 5.4). Moreover, we can assume that A_f is absolutely simple and has good reduction at $\mathfrak{p}$. Then the key argument can be divided into two steps:

First, under the assumption that $A_{f/F}$ does not have complex multiplication, we can find two distinct primes $\mathfrak{Q}$ and $\mathfrak{L}$ of F not lying over p with the following property: the abelian variety $A_{f/F}$ has good reduction at $\mathfrak{Q}$ and $\mathfrak{L}$, and if we use $A_\mathfrak{Q}$ (resp. $A_\mathfrak{L}$) to denote the reduction of A_f at $\mathfrak{Q}$ (resp. $\mathfrak{L}$), then $\mathrm{End}^0_L(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ and $\mathrm{End}^0_L(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ are non-isomorphic CM quadratic extension of L (see Lemma 5.13). Here q (resp. l) is the residue characteristic of the prime $\mathfrak{Q}$ (resp. $\mathfrak{L}$). The proof is a slight modification of the argument given in [24] using Faltings's isogeny theorem, a Serre-type open image theorem due to Ribet, and some standard results on the density of primes. As is clear from the argument given in the proof of Lemma 5.13, when the prime p is ramified in the field L, we need to construct an extra auxiliary prime in our argument.

Second, we prove that if the λ-adic representation of $I_\mathfrak{p}$ attached to the Tate module of A_f is decomposable, it is impossible to find the primes $\mathfrak{Q}$ and $\mathfrak{L}$ with the property in the first step. The idea is that by putting polarization and level structure on $A_{f/F}$, the abelian variety $A_{f/F}$ gives rise to a point on the Hilbert modular Shimura variety we defined in section 4.2. In section 5.2 we prove that each L-linear isogeny of $A_{\mathfrak{Q}/\bar{F}_q}$ with degree prime to q induces an automorphism of the Shimura variety, and hence an automorphism of the ordinary deformation space of the mod q reduction of A_f sitting in the special fiber of x

at q. Using the rigid analytic logarithms of the Serre-Tate coordinates on the ordinary deformation space (see section 5.2 below), we can prove that this automorphism must also fixes the special fiber of x at l. Then we can conclude that $\mathrm{End}^0_L(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ and $\mathrm{End}^0_L(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ must be isomorphic as L-algebras.

We prove the main result in Section 5.3, and we give an Λ-adic version of our result by applying an argument in [14]. At the end we explain how our result can be applied to study a problem of Coleman on determining which classical elliptic modular forms lie in the image of the operator defined in [5].

5.1 Abelian Varieties of GL(2)-type

Let E be a number field with degree d over $\mathbb{Q}$, and $A_{/\overline{\mathbb{Q}}}$ be an abelian variety of dimension d. Set $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}}) = \mathrm{End}(A_{/\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a finite dimensional semisimple algebra over $\mathbb{Q}$. Suppose that we have an algebra homomorphism $E \to \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$, which identifies E with a subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$. Recall that the abelian variety $A_{/\overline{\mathbb{Q}}}$ has complex multiplication if $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ contains a commutative semisimple subalgebra of dimension $2d$ over $\mathbb{Q}$. Then from [23] Section 5.3.1, we have the following two results:

Proposition 5.1. *If $A_{/\overline{\mathbb{Q}}}$ does not have complex multiplication, then $A_{/\overline{\mathbb{Q}}}$ is isotypic (i.e. there exists a simple abelian variety $B_{/\overline{\mathbb{Q}}}$ such that $A_{/\overline{\mathbb{Q}}}$ is isogeneous to $(B_{/\overline{\mathbb{Q}}})^e$ for some $e \geq 1$), and $\mathrm{End}^0_E(A_{/\overline{\mathbb{Q}}}) = E$.*

Proposition 5.2. *Under the conditions of Proposition 5.1, if we assume further that $A_{/\overline{\mathbb{Q}}}$ is simple, then one of the following four possibilities holds for $D = \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$:*

1. *E is a quadratic extension of a totally real field Z and D is a totally indefinite division quaternion algebra over Z;*

2. *E is a quadratic extension of a totally real field Z and D is a totally definite division quaternion algebra over Z;*

3. *E is a quadratic extension of a CM field Z and D is a division quaternion algebra over Z;*

4. *$E = D$ and E is totally real.*

Remark 5.3. 1. A quaternion algebra D over a totally real field Z is called totally indefinite if for any real embedding $\tau : Z \to \mathbb{R}$, the $\mathbb{R}$-algebra $D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the matrix algebra $M_2(\mathbb{R})$; the quaternion algebra $D_{/Z}$ is called totally definite if for any real embedding $\tau : Z \to \mathbb{R}$, the $\mathbb{R}$-algebra $D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$.

2. From Proposition 5.1, we see that $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ is always a central simple algebra and E is a maximal commutative subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$;

3. As remarked in [23], case 2 in Proposition 5.2 cannot happen by [47], Theorem 5(a) and Proposition 15.

Proposition 5.4. *Under the notations and assumptions in Proposition 5.1, assume further that there exists a totally real field k such that the abelian variety $A_{/\overline{\mathbb{Q}}}$ is defined over k, and the homomorphism $E \to \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ factors through $\mathrm{End}^0(A_{/k})$. Then we can find a totally real field F with degree d over $\mathbb{Q}$, which can be embedded into $D = \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ as a unital subalgebra of D.*

Proof. By Propositon 5.1, we can find a simple abelian variety $B_{/\overline{\mathbb{Q}}}$ and an integer e such that $A_{/\overline{\mathbb{Q}}}$ is isogeneous to $(B_{/\overline{\mathbb{Q}}})^e$. Hence we have an isomorphism of simple algebras $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}}) \cong M_e(\mathrm{End}^0(B_{/\overline{\mathbb{Q}}}))$,and $d = e \cdot d_1$, where d_1 is the dimension of $B_{/\overline{\mathbb{Q}}}$. Since any maximal commutative subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ has degree d over $\mathbb{Q}$, any maximal commutative subfield of $D_1 = \mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ should have dimension $d/e = d_1$. In other words, we can find number field E_1 of degree d_1 over $\mathbb{Q}$, which can be embedded into $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ as a subalgebra. Since $A_{/\overline{\mathbb{Q}}}$ does not have complex multiplication, neither does $B_{/\overline{\mathbb{Q}}}$. In summary, $B_{/\overline{\mathbb{Q}}}$ satisfies all the assumptions in Proposition 5.2. Assume that $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is of type 3 as in Proposition 5.2, i.e. $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is a division quaternion algebra over a CM field Z and $[E_1 : Z] = 2$.

Since $d_1 = [E_1 : \mathbb{Q}] = 2[Z : \mathbb{Q}]$, the degree s of Z over $\mathbb{Q}$ equals to $\frac{d_1}{2}$. Since Z is a CM field, we can find $s' = \frac{s}{2}$ different embeddings $\tau_i : Z \to \overline{\mathbb{Q}}, i = 1, ..., s'$, such that $\mathrm{Hom}_{\mathbb{Q}}(Z, \overline{\mathbb{Q}}) = \{\tau_1, ..., \tau_{s'}, \bar{\tau}_1, ..., \bar{\tau}_{s'}\}$, where $\bar{\tau}_i$ is the complex conjugation of τ_i for $i = 1, ..., s'$. Then we have an isomorphism

$$\theta : D_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau_i, i=1,...,s'} M_2(\mathbb{C}).$$

Let π_i be the composition

$$D_1 \hookrightarrow D_1 \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\theta} \prod_{\tau_i, i=1,...,s'} M_2(\mathbb{C}) \xrightarrow{\pi_i} M_2(\mathbb{C}),$$

where the map π_i is the i-th projection, for $i = 1, ..., s'$. Let $\bar{\pi}_i$ be the complex conjugation of π_i. Then $\{\pi_1, ..., \pi_{s'}, \bar{\pi}_1, ..., \bar{\pi}_{s'}\}$ are all the absolutely irreducible (complex) representations of D_1 (up to isomorphism).

On the other hand, we have a representation of D_1 by $\rho_1 : D_1 \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(B) \otimes_{\bar{\mathbb{Q}}} \mathbb{C})$. Let r_i (resp. s_i) be the multiplicity of π_i (resp $\bar{\pi}_i$) in ρ_1. Then for any $z \in Z$, the trace of $\rho_1(z)$ is given by the formula:

$$\mathrm{Tr}(\rho_1(z)) = 2\sum_{i=1}^{s'}(r_i\tau_i(z) + s_i\bar{\tau}_i(z)).$$

Since $\mathrm{Lie}(A_{/\bar{\mathbb{Q}}}) \cong (\mathrm{Lie}(B_{/\bar{\mathbb{Q}}}))^e$, we have the representation $\rho : D \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(A) \otimes_{\bar{\mathbb{Q}}} \mathbb{C})$, such that for any $z \in Z$,

$$\mathrm{Tr}(\rho(z)) = e\mathrm{Tr}(\rho_1(z)) = 2e\sum_{i=1}^{s'}(r_i\tau_i(z) + s_i\bar{\tau}_i(z)).$$

Since $Z \subseteq E$ and the homomorphism $E \to \mathrm{End}^0(A_{/\bar{\mathbb{Q}}})$ factors through $\mathrm{End}^0(A_{/k})$, we have $\mathrm{Tr}(\rho(z)) \in k$, for any $z \in Z$. From [47] Section 4, we have $r_i + s_i = 2$, for all $i = 1, ..., s'$. Thus for each i, either $r_i = s_i = 1$ or $r_i \cdot s_i = 0$. If $r_i \cdot s_i = 0$ for at least one i, then $\mathrm{Tr}(\rho(z))$ cannot lie in the totally real field k for all $z \in Z$ as Z is assumed to be a CM field. Hence $r_i = s_i = 1$ for all i. Then by [47] Theorem 5(e) and Proposition 19, this case cannot happen.

Combined with Remark 5.3(3), we see that $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is either a totally real field or a totally indefinite division algebra over a totally real field. Then the existence of F results from:

Lemma 5.5. *Let D be a central simple algebra over a totally real field Z with $[D:Z]=d^2$. If for all real embeddings $\tau: Z \to \mathbb{R}$, the $\mathbb{R}$-algebra $D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the matrix algebra $M_d(\mathbb{R})$, then we can find a field extension F/Z with degree d such that F is totally real and can be embedded into D as an Z-subalgebra.*

Proof of the lemma: We use an argument similar with the proof of Lemma 1.3.8 in [3]. It is enough to find a field extension F/Z with degree d such that F is totally real and splits D(i.e $D \otimes_Z F \cong M_d(F)$).

Let Σ be a non empty set of non-archimedean places of Z containing all the finite places where D does not split, and Σ_∞ be the set of archimedean places of Z. By the weak approximation theorem, the natural map:

$$Z \to \prod_{v \in \Sigma} Z_v \times \prod_{v \in \Sigma_\infty} Z_v$$

has dense image. Hence we can find a monic polynomial $f(X) \in Z[X]$ of degree d, such that it is sufficiently close to a monic irreducible polynomial of degree d over Z_v for all $v \in \Sigma$, and it is sufficiently close to a totally split polynomial of degree d over $\mathbb{R}$ for all $v \in \Sigma_\infty$. Set $F = Z[X]/(f(X))$. Then F/Z is a degree d field extension such that F is totally real and for any $v \in \Sigma$, there is exactly one place w of F lying over v and hence F_w/Z_v is a degree d extension of local fields.

We still need to check that F splits D. Since $D \otimes_Z F$ is a central simple algebra over F and F is a global field, it is enough to prove that for any place w of F (archimedean and non-archimedean), we have an isomorphism $D \otimes_Z F_w \cong M_d(F_w)$. Let v be the place of Z over which w lies.

If w is archimedean, then $Z_v \cong F_w \cong \mathbb{R}$, and hence

$$D \otimes_Z F_w \cong (D \otimes_Z Z_v) \otimes_{Z_v} F_w \cong M_d(\mathbb{R}), \tag{5.6}$$

by our assumption on D.

If w is non-archimedean and v is not in Σ,then $D \otimes_Z Z_v$ is already isomorphic to the matrix algebra over Z_v, so we are safe in this case.

Finally, assume that w is non-archimedean and $v \in \Sigma$. As F_w/Z_v is a degree d extension of local field, the base change from Z_v to F_w induces a homomorphism of Brauer groups $\mathrm{Br}(Z_v) \to \mathrm{Br}(F_w)$, which under the isomorphism $\mathrm{Br}(Z_v) \cong \mathrm{Br}(F_w) \cong \mathbb{Q}/\mathbb{Z}$ by local class field theory, is nothing but multiplication by d. As $[D : Z] = d^2$, the order of $D \otimes_Z Z_v$ in $\mathrm{Br}(Z_v)$ is divisible by d. This implies that $D \otimes_Z F_w$ represents the identity element in $\mathrm{Br}(F_w)$; i.e.$D \otimes_Z F_w \cong M_d(F_w)$. Hence F/Z is the desired extension.

□

Hereafter we always work with the pair $(A_{/\overline{\mathbb{Q}}}, \iota : F \to \mathrm{End}^0(A_{/\overline{\mathbb{Q}}}))$, where F is a totally real field with degree d over $\mathbb{Q}$. Since the abelian variety $A_{/\overline{\mathbb{Q}}}$ is projective, we can find a number field k such that A is defined over k, and $\mathrm{End}(A_{/\overline{\mathbb{Q}}})) = \mathrm{End}(A_{/k})$. Let $\mathcal{O}_k$ be the integer ring of k, and for all prime ideals $\mathfrak{P}$ of $\mathcal{O}_k$ over some rational prime p, let $\mathcal{O}_{(\mathfrak{P})}$ be localization of $\mathcal{O}_k$ at the prime $\mathfrak{P}$ and $\mathbb{F}_{\mathfrak{P}} = \mathcal{O}_k/\mathfrak{P}$ be its residue field. As in [24], we make the following assumption:

(NLL) the abelian variety $A_{/k}$ has good reduction at $\mathfrak{P}$ and the reduction $A_0 = A \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathbb{F}_{\mathfrak{P}}$ has nontrivial p-torsion $\bar{\mathbb{F}}_p$-points.

Change the abelian variety $A_{/k}$ if necessary, we can assume that ι gives a homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$. Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_{\mathfrak{P}}$. $W_p = W(\bar{\mathbb{F}}_p)$ is the ring of Witt vectors of $\bar{\mathbb{F}}_p$. We have the decomposition of Barsotti-Tate $\mathcal{O}_F$-modules:

$$A_0[p^\infty] = \bigoplus_{\mathfrak{p}|p} A_0[\mathfrak{p}^\infty].$$

Here $\mathfrak{p}$ ranges over the primes ideals of $\mathcal{O}_F$ over p and for each $\mathfrak{p}$, let

$$A_0[\mathfrak{p}^\infty] = \varinjlim A_0[\mathfrak{p}^n]$$

be the $\mathfrak{p}$-divisible Barsotti-Tate group of A_0. We also define

$$T_{\mathfrak{p}}(A_0) = \varprojlim A_0[\mathfrak{p}^n](\bar{\mathbb{F}}_p)$$

as the $\mathfrak{p}$-divisible Tate module of A_0.

We say that a prime $\mathfrak{p}$ of $\mathcal{O}_F$ over p is ordinary if $A_0[\mathfrak{p}]$ has nontrivial $\bar{\mathbb{F}}_p$-points, otherwise we say that $\mathfrak{p}$ is local-local. When $\mathfrak{p}$ is ordinary and p is unramified in k, we have an exact sequence of Barsotti-Tate $\mathcal{O}_\mathfrak{p}$-modules over W_p:

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_\mathfrak{p}^* \to A[\mathfrak{p}^\infty]_{/W_p} \to F_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \to 0.$$

Here $\mathcal{O}_\mathfrak{p}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p)$ is the $\mathbb{Z}_p$-dual of $\mathcal{O}_\mathfrak{p}$.

Let Σ_p^{ord} be the set of all ordinary primes of $\mathcal{O}_F$ over p, and Σ_p^{ll} be the set of all local-local primes. Then the condition (NLL) is equivalent to the fact that Σ_p^{ord} is not empty. Also we define:

$$A_0[p^\infty]^{ord} = \bigoplus_{\mathfrak{p} \in \Sigma_p^{ord}} A_0[\mathfrak{p}^\infty], A_0[p^\infty]^{ll} = \bigoplus_{\mathfrak{p} \in \Sigma_p^{ll}} A_0[\mathfrak{p}^\infty].$$

At the end of this section, we explain how the pair $(A_{/k}, \iota : F \to \mathrm{End}^0(A_{/k}))$ can give a point in the Hilbert modular Shimura variety defined in section 4.2. More precisely, we want to prove that there is a $\mathfrak{c}$-polarized abelian variety $A'_{/\mathcal{O}_k}$ with real multiplication by $\mathcal{O}_F$ which is isogenous to $A_{/k}$.

We can find an order $\mathcal{O}$ in F which is mapped into $\mathrm{End}(A)$ under ι. By Serre's Tensor construction ([3]1.7.4.), we can find an isogeny $f : A \to A'$ over k, and the induced isomorphism $\mathrm{End}^0(A_{/k}) \to \mathrm{End}^0(A'_{/k})$ carries $\mathcal{O}_F \subseteq \mathrm{End}^0(A_{/k})$ into $\mathrm{End}(A'_{/k})$. Hence we have an algebra homomorphism $\iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/k})$. By our assumption, $A_{/k}$ has good reduction at the prime $\mathfrak{P}$ of $\mathcal{O}_k$. By the criterion of Néron-Ogg-Shafarevich ([48] Section 1 Corollary 1), $A'_{/k}$ also has good reduction at $\mathfrak{P}$, and hence can be extended to an abelian scheme $A'_{/\mathcal{O}_{(\mathfrak{P})}}$ (recall that $\mathcal{O}_{(\mathfrak{P})}$ is the localization of $\mathcal{O}_k$ at the prime $\mathfrak{P}$). Since $\mathcal{O}_{(\mathfrak{P})}$ is a normal domain, by a lemma of Faltings (see [11] Lemma 1), the restriction to the generic fiber induces a bijection

$$\mathrm{End}(A'_{/\mathcal{O}_{(\mathfrak{P})}}) \to \mathrm{End}(A'_{/k}).$$

So we have an algebra homomorphism $\mathcal{O}_F \to \mathrm{End}(A'_{/\mathcal{O}_{(\mathfrak{P})}})$, which is again denoted by ι'.

From Proposition 4.4, the étale sheaf $\mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A'_{/\mathcal{O}_{(\mathfrak{P})}}, A'^t_{/\mathcal{O}_{(\mathfrak{P})}})$ is a constant sheaf $\underline{\mathfrak{c}}$ for some fractional ideal $\mathfrak{c}$, with the natural notion of positivity $\mathfrak{c}_+$. Thus we have a natural isomorphism $\varphi : \underline{\mathfrak{c}} \to \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A'_{/\mathcal{O}_{(\mathfrak{P})}}, A'^t_{/\mathcal{O}_{(\mathfrak{P})}})$ which sends totally positive elements of $\mathfrak{c}$ to polarizations of $A'_{/\mathcal{O}_{(\mathfrak{P})}}$. We still need to check that the natural morphism $\alpha : A' \otimes_{\mathcal{O}_F} \underline{\mathfrak{c}} \to A'^t$ is an isomorphism over $\mathcal{O}_{(\mathfrak{P})}$. As $char(k) = 0$, by Proposition 4.5, α is an isomorphism at the generic fiber of $\mathcal{O}_{(\mathfrak{P})}$. Hence α is an isomorphism again by Faltings lemma.

In summary, we have:

Proposition 5.7. *Let $A_{/k}$ be an abelian variety of dimension d satisfying the condition* (NLL) *above, and $\iota : F \to \mathrm{End}^0(A_{/k})$ be an algebra homomorphism. Then we can find a fractional ideal $\mathfrak{c}$ and an $\mathfrak{c}$-polarized abelian scheme $(A'_{/\mathcal{O}_{(\mathfrak{P})}}, \iota', \varphi)$ with real multiplication by $\mathcal{O}_F$ such that $A_{/k}$ is k-isogenous to $A'_{/k}$.*

Remark 5.8. Let $A_{/S}$ be an abelian scheme of relative dimension d and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ be an algebra homomorphism. By a similar argument as above, we see that if S is an integral normal scheme and the generic fiber of S is of characteristic 0, then the pair $(A_{/S}, \iota)$ must satisfy the condition (DP).

For later discussion, we need the following:

Lemma 5.9. *Let $A_{/S}, A'_{/S}$ be two abelian schemes of relative dimension d, and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S}), \iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/S})$ be two algebra homomorphisms. Suppose that there exists an $\mathcal{O}_F$-linear étale homomorphism of abelian schemes $f : A \to A'$. If the pair $(A_{/S}, \iota)$ satisfies the condition* (DP)*, so does $(A'_{/S}, \iota')$.*

Proof. Without loss of generality, we can assume that $S = \mathrm{Spec}(k)$ for some separably closed field k. If $char(k) = 0$, then $(A'_{/S}, \iota')$ satisfies (DP) automatically by Proposition 4.5. So we can assume that $char(k) = p > 0$. From the discussion of [15] Page $100 - 101$, the pair $(A_{/k}, \iota)$ can be lifted to characteristic 0; i.e., there exist:

1. a normal local domain W with maximal ideal $\mathfrak{m}$ and residue field k such that the quotient field of W is of characteristic 0;

2. an abelian scheme $\tilde{A}_{/W}$ with an $\mathcal{O}_F$-action $\tilde{\iota} : \mathcal{O}_F \to \mathrm{End}(\tilde{A}_{/W})$ such that $(A_{/k}, \iota)$ is isomorphic the the pull back of $(\tilde{A}_{/W}, \tilde{\iota})$ under the natural morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(W)$.

Replacing W by its $\mathfrak{m}$-adic completion if necessary, we can assume that W is complete.

Since $f : A \to A'$ is étale and $\mathcal{O}_F$-linear, $C = ker(f)$ is a finite étale $\mathcal{O}_F$-submodule of $A_{/k}$. Then we can lift C to an étale $\mathcal{O}_F$-submodule $\tilde{C}_{/W}$ of $\tilde{A}_{/W}$. Let $\tilde{A}'_{/W}$ be the quotient of $\tilde{A}_{/W}$ by $\tilde{C}_{/W}$, with the natural homomorphism $\tilde{\iota}' : \mathcal{O}_F \to \mathrm{End}(\tilde{A}'_{/W})$ induced from $\tilde{A}_{/W}$. By the above construction it is easy to see that $(\tilde{A}'_{/W}, \tilde{\iota}')$ lifts $(A'_{/k}, \iota')$. Then from Remark 5.8, $(A'_{/k}, \iota')$ satisfies (DP). □

5.2 Eigen coordinates

At the beginning of this section we set up some notations. Let $k \subseteq \bar{\mathbb{Q}}$ be a number field and Ξ be a finite set of primes. For each $p \in \Xi$, choose a finite extension $\widetilde{L}_p$ of L_p in $\mathbb{C}_p$ such that:

1. $k \subseteq i_p^{-1}(\widetilde{L}_p)$;
2. $i_p^{-1}(\widetilde{L}_p)$ contains the Galois closure of F in $\bar{\mathbb{Q}}$.

Denote by $\widetilde{W}_p$ the valuation ring of $\widetilde{L}_p$. Then define:

$$\widetilde{\mathcal{W}}_\Xi = \bigcap_{p \in \Xi} i_p^{-1}(\widetilde{W}_p) \subseteq \bar{\mathbb{Q}}, \widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_\Xi \cap k.$$

The ring $\widetilde{\mathcal{W}}_\Xi$ is a semilocal ring, and for each $l \in \Xi$, there is a unique maximal ideal $\mathfrak{m}_l$ with residue characteristic l. Let $\widetilde{\mathcal{L}}_\Xi$ be the quotient field of $\widetilde{\mathcal{W}}_\Xi$.

Given the totally real field F, let $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ be the Hilbert modular Shimura variety constructed in section 4.2. Suppose that the quadruple $(A_{/\widetilde{\mathcal{W}}_\Xi}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ represents a point $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$ such that the image of x lies in $Sh^{(\Xi)}(\widetilde{\mathcal{W}}_k)$. For each $p \in \Xi$, x induces an

$\bar{\mathbb{F}}_p$-valued point $x_p \in Sh^{(\Xi)}(\bar{\mathbb{F}}_p)$. Then the quadruple $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \eta_{\mathfrak{P}}^{(\Xi)})$ obtained by mod p reduction represents the point x_p.

This section consists of the key arguments and ideas used in the proof of the local indecomposability result. We give a sketch of what we want to do in this section before we start the down to earth arguments.

First we construct a torus $R^{\times}_{(\Xi)}$ acting on the Hilbert modular Shimura variety which fixes the closed point x_p. Hence this action induces an automorphism on the formal completion $\widehat{S}_p$ of the Shimura variety $Sh^{(\Xi)}$ at the closed point x_p. From the previous section, we have a decomposition $\widehat{S}_{p/W_p} = \widehat{S}^{ord}_{p/W_p} \times \widehat{S}^{ll}_{p/W_p}$. Then we recall the construction of $\hat{\rho}$-eigen σ-coordinates in [24] and give the explicit expression of the action of $R^{\times}_{(\Xi)}$ on these coordinates. When the ind-étale exact sequence of the Barsotti-Tate $\mathcal{O}_{\mathfrak{p}}$-module $A[\mathfrak{p}^{\infty}]$ splits over $\widetilde{W}_p$, we calculate its Serre-Tate coordinates in Lemma 5.12. It turns out that when p is ramified in the base field (so $W_p \neq \widetilde{W}_p$) this Serre-Tate coordinate is a p-th power root of unity and the abelian variety $A_{/\widetilde{W}_p}$ is isogenous to an abelian variety whose Serre-Tate coordinate at $\mathfrak{p}$ is 1. From the construction of the eigencoordinates, the $\hat{\rho}$-eigen σ-coordinates of these abelian varieties are all 0 for any embedding $\sigma : F \to \bar{\mathbb{Q}}_p$ which induces the prime $\mathfrak{p}$ in F. Since we can change our abelian variety by an isogenous abelian variety, the eigen coordinates should be the right object to study.

The above calculation is local at p. We want to transit the action of $R^{\times}_{(\Xi)}$ on $\widehat{S}_{p/W_p}$ to the deformation space $\widehat{S}_{l/W_l}$ for some other prime l with the property that there exists a prime $\mathfrak{L}$ of k over l and $A_{/\widetilde{\mathcal{W}}_{\Xi}}$ has partially ordinary reduction at $\mathfrak{L}$. Let $\pi : A \to \mathrm{Spec}(\widetilde{\mathcal{W}}_{\Xi})$ be the structure morphism and set $\omega = \pi_*(\Omega_{A/\widetilde{\mathcal{W}}_{\Xi}})$ which is an $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_{\Xi}$-module and define $\omega^{\otimes 2} = \omega \otimes_{\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_{\Xi}} \omega$. This is the global object which allows us to compare the action of $R^{\times}_{(\Xi)}$ at different local deformation space. The sheaf $\omega^{\otimes 2}$ is related with the Serre-Tate coordinates (or the eigen coordinates) through the Kodaira-Spencer map. The Kodaira-Spencer map is not an isomorphism in general if the reduction of $A_{/\widetilde{\mathcal{W}}_{\Xi}}$ at $\mathfrak{P}$ is not ordinary. We want to have decomposition of $\omega^{\otimes 2}$ by its $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_{\Xi}$-module structure as in [29]. Recall $I = \mathrm{Hom}(F, \bar{\mathbb{Q}})$.

The natural homomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi \to \widetilde{\mathcal{W}}_\Xi^I, a \otimes b \mapsto (\sigma(a) \cdot b)_{\sigma \in I}$$

is not an isomorphism when the prime $p \in \Xi$ is ramified in $\mathcal{O}_F$. It becomes an isomorphism when base change to the quotient field $\widetilde{\mathcal{L}}_\Xi$ of $\widetilde{\mathcal{W}}_\Xi$. On the other hand, the formation of the sheaf $\omega_{/\widetilde{\mathcal{W}}_\Xi}$ is compatible with arbitrary base change. So we can decompose the sheaf $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ as a direct sum $\oplus_{\sigma \in I} \tilde{\omega}^{\otimes 2\sigma}$ such that on $\tilde{\omega}^{\otimes 2\sigma}$, the ring $\mathcal{O}_F$ acts by the embedding $\sigma : F \to \bar{\mathbb{Q}}$. Under this decomposition and the Kodaira-Spencer map, we can compare the endomorphism algebras of the reductions of $A_{/\widetilde{\mathcal{W}}_\Xi}$ at different primes and get our main result Theorem 11 at the end of this section.

5.2.1 Construction and properties of eigen coordinates

By [24] Lemma 2.2, we have

Lemma 5.10. *If $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$ is not supersingular (i.e. $\Sigma_p^{ord} \neq \emptyset$), then there exists a CM quadratic extension M of F, and an isomorphism of F-algebras $\theta_{\mathfrak{P}} : M \cong \mathrm{End}_F^0(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$. Set $R = M \cap \theta_{\mathfrak{P}}^{-1}(\mathrm{End}_{\mathcal{O}_F}(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}))$, which is an order in M. If a prime ideal $\mathfrak{p}$ in $\mathcal{O}_F$ belongs to Σ_p^{ord}; i.e. $A_{\mathfrak{P}}[\mathfrak{p}]$ has nontrivial $\bar{\mathbb{F}}_p$-rational points, then $\mathfrak{p}$ splits into two primes $\mathcal{P}\bar{\mathcal{P}}$ in R with $\mathcal{P} \neq \bar{\mathcal{P}}$.*

As in [24], we make the convention that we choose $\mathcal{P}$ such that $A_{\mathfrak{P}}[\mathcal{P}]$ is connected and $A_{\mathfrak{P}}[\bar{\mathcal{P}}]$ is étale.

By the above lemma, we have an isomorphism $M \otimes_F F_{\mathfrak{p}} \cong F_{\mathfrak{p}} \times F_{\mathfrak{p}}$, such that the first factor corresponds to $\mathcal{P}$ and the second factor corresponds to $\bar{\mathcal{P}}$. As M can be naturally embedded into $M \otimes_F F_{\mathfrak{p}}$, we have two embeddings from M to $F_{\mathfrak{p}}$, which correspond to the two factors of $F_{\mathfrak{p}} \times F_{\mathfrak{p}}$. We always regard M as a subfield of $F_{\mathfrak{p}}$ by the first embedding, while the second embedding is denoted by $c : M \hookrightarrow F_{\mathfrak{p}}$.

Let $R_{(\Xi)} = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$. For $\alpha \in R_{(\Xi)}^\times$, $\theta_{\mathfrak{P}}(\alpha)$ is a prime-to-Ξ isogeny of $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$, and hence induces an endomorphism of $V^{(\Xi)}(A_{\mathfrak{P}})$. We still denote this endomorphism by $\theta_{\mathfrak{P}}(\alpha)$. Define a map $\hat{\rho} : R^{\times}_{(\Xi)} \to G(F_{\mathbb{A}^{(\Xi\infty)}})$ such that for each $\alpha \in R^{\times}_{(\Xi)}$, $\hat{\rho}(\alpha)$ is given by the formula: $\eta^{(\Xi)}_{\mathfrak{P}} \circ \hat{\rho}(\alpha) = \theta_{\mathfrak{P}}(\alpha) \circ \eta^{(\Xi)}_{\mathfrak{P}}$.

Fix a prime-to-Ξ polarization $\lambda_{\mathfrak{P}}$ of $A_{\mathfrak{P}}$ as a representative of $\bar{\lambda}_{\mathfrak{P}}$. Under the isomorphism $\theta_{\mathfrak{P}}$, the Rosati involution associated to $\lambda_{\mathfrak{P}}$ on $\mathrm{End}^0_F(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$ induces a positive involution on field M. As M is CM, this involution must be the complex conjugation on M. Hence for any $\alpha \in R^{\times}_{(\Xi)}$, $\lambda^{-1}_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha)^t \circ \lambda_{\mathfrak{P}} = \theta_{\mathfrak{P}}(\bar{\alpha})$. Then $\theta_{\mathfrak{P}}(\alpha)^t \circ \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha) = \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\bar{\alpha}) \circ \theta_{\mathfrak{P}}(\alpha) = \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha\bar{\alpha})$. Since $\alpha\bar{\alpha} \in \mathcal{O}^{\times}_{(\Xi),+}$, we have $\theta_{\mathfrak{P}}(\alpha)^t \circ \bar{\lambda}_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha) = \bar{\lambda}_{\mathfrak{P}}$. So $\theta_{\mathfrak{P}}(\alpha)$ is an isogeny from the quadruple $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \eta^{(\Xi)}_{\mathfrak{P}})$ to $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \theta_{\mathfrak{P}}(\alpha)(\eta^{(\Xi)}_{\mathfrak{P}})) = (A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \hat{\rho}(\alpha)(\eta^{(\Xi)}_{\mathfrak{P}}))$ in the sense of Definition 4.6; in other words, the automorphism $g = \hat{\rho}(\alpha)$ of the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi} = Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}} \times_{\mathbb{Z}_{(\Xi)}} \mathcal{W}_\Xi$ fixes the closed point x_p.

Denote the formal scheme $\widehat{S}_{p/W_p}$ as the completion of the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi}$ along the closed point x_p, and $\nu_p : \widehat{S}_{p/W_p} \to Sh^{(\Xi)}_{/W_p}$ is the natural morphism. As explained in Section 4.2, $\widehat{S}_{p/W_p}$ is the product of two formal schemes $\widehat{S}^{ord}_{p/W_p}$ and $\widehat{S}^{ll}_{p/W_p}$, and if we fix an isomorphism $\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_{\mathfrak{P}})$ for each $\mathfrak{p} \in \Sigma^{ord}$, then $\widehat{S}^{ord}_{p/W_p}$ is isomorphic to $\Pi_{\mathfrak{p}\in\Sigma^{ord}} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$. By deformation theory, we have a Serre-Tate coordinate $t_{\mathfrak{p}} \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Sigma^{ord}$. Then for each object $\mathcal{R}$ in the category $CL_{/W_p}$, and an $\mathcal{R}$-valued point $x \in \widehat{S}_p(\mathcal{R})$, the Serre-Tate coordinate gives us an element $t_{\mathfrak{p}}(x) \in \widehat{\mathbb{G}}_m(\mathcal{R}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} = (1 + \mathfrak{m}_{\mathcal{R}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$, here $\mathfrak{m}_{\mathcal{R}}$ is the maximal ideal of $\mathcal{R}$. In particular, when $\mathcal{R}$ is a subring of $\mathbb{C}_p$, we can consider the p-adic logarithm $log_p : \mathcal{R} \to \mathbb{C}_p$. Consider the following map:

$$log_p \otimes Id : (1 + \mathfrak{m}_{\mathcal{R}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} \to \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} \cong \mathrm{Hom}(\mathcal{O}_{\mathfrak{p}}, \mathbb{C}_p) \cong \prod_{\sigma:F\to\bar{\mathbb{Q}},\sigma\sim\mathfrak{p}} \mathbb{C}_p.$$

Here the notation $\sigma \sim \mathfrak{p}$ means that the composite map $i_p \circ \sigma : F \to \bar{\mathbb{Q}}_p$ induces the prime $\mathfrak{p}$ of F. For such σ, let π_σ be the projection of $\Pi_{\sigma:F\to\bar{\mathbb{Q}},\sigma\sim\mathfrak{p}}\mathbb{C}_p$ to its σ-factor. Then we get an element $\tau_\sigma(x) = \pi_\sigma \circ (log_p \otimes Id)(t_{\mathfrak{p}}(x)) \in \mathbb{C}_p$. The association $x \in \widehat{S}_p(\mathcal{R}) \mapsto \tau_\sigma(x) \in \mathbb{C}_p$ gives p-adic rigid analytic functions on the rigid analytic space $(\widehat{S}^{ord}_p)^{p-an}$ associated to $\widehat{S}^{ord}_p$.

Remark 5.11. From the above construction, we can see that actually the eigen coordinates take values in the valuation ring of the field $\mathbb{C}_p$. But in later argument, we need to invert the prime p when comparing the eigen coordinates and the invariant differential sheaf of

$A_{\widetilde{\mathcal{W}}_\Xi}$ by the Kodaira-Spencer map. Hence we always regard the coordinates τ_σ as $\mathbb{C}_p$-valued functions on the formal scheme $\widehat{S}_p^{ord}$ or $\mathbb{C}_p$-valued rigid analytic functions on $(\widehat{S}_p^{ord})^{p-an}$.

Since the action of $g = \hat{\rho}(\alpha)$ on the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi}$ fixes the closed point x_p, this action also preserves the formal schemes $\widehat{S}_p^{ord}$ and $\widehat{S}_p^{ll}$, and hence $g = \hat{\rho}(\alpha)$ acts on the function τ_σ for each $\sigma \sim \mathfrak{p}$, $\mathfrak{p} \in \Sigma^{ord}$. By [22] Lemma 3.3, the action of $g = \hat{\rho}(\alpha)$ on the Serre-Tate coordinate $t_\mathfrak{p}$ is given by the formula $g(t_\mathfrak{p}) = t_\mathfrak{p}^{\alpha^{1-c}}$. (See the explanation after Lemma 5.10 for the two embeddings of M to $F_\mathfrak{p}$). Then by the construction of τ_σ, we see that the action of $g = \hat{\rho}(\alpha)$ on the function τ_σ is given by the formula: $g(\tau_\sigma) = \tau_\sigma \circ \hat{\rho}(\alpha) = i_p \circ \sigma(\alpha^{1-c}) \cdot \tau_\sigma$. We remark here that $i_p \circ \sigma : F \to \bar{\mathbb{Q}}_p$ naturally extends to an embedding $i_p \circ \sigma : F_\mathfrak{p} \to \bar{\mathbb{Q}}_p$, and hence the expression $i_p \circ \sigma(\alpha^{1-c})$ is well defined. As in [24], the function τ_σ is called a $\hat{\rho}$-eigen σ-coordinate.

Now consider the original point $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$, which is represented by the quadruple $(A_{/\widetilde{\mathcal{W}}_\Xi}, \iota, \bar{\lambda}, \eta^{(\Xi)})$.

Lemma 5.12. *Assume that we have a prime $\mathfrak{p} \in \Sigma^{ord}$, such that the exact sequence of Barsotti-Tate $\mathcal{O}_\mathfrak{p}$-modules :*

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_\mathfrak{p}^* \to A[\mathfrak{p}^\infty] \to F_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \to 0$$

splits over over $\widetilde{W}_p$. In this case, the Serre-Tate coordinate $t_\mathfrak{p}(x)$ for the prime $\mathfrak{p}$ at the point x must be a p-th power root of unity. In particular, for the $\hat{\rho}$-eigen coordinate we have $\tau_\sigma(x) = 1$ for all $\sigma \sim \mathfrak{p}$.

This fact is proved in [1] Section 7 or [25] Section 6.3.4 in the case of elliptic curves. The higher dimensional case is considered in [6] when the abelian variety has ordinary reduction at $\mathfrak{P}$. Since the discussion in the partially ordinary case may not exist in the references, for the sake of completeness we give a proof here.

Proof. First we assume that the ring $R = M \cap \theta_\mathfrak{P}^{-1}(\mathrm{End}_{\mathcal{O}_F}(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}))$ in Lemma 5.10 is the integer ring $\mathcal{O}_M$ of M. From Lemma 5.10 , the prime $\mathfrak{p}$ in $\mathcal{O}_F$ splits into two primes $\mathcal{P}$ and

$\bar{\mathcal{P}}$ in $\mathcal{O}_M$ such that the finite group scheme $A_{\mathfrak{P}}[\mathcal{P}]_{/\bar{\mathbb{F}}_p}$ (resp. $A_{\mathfrak{P}}[\bar{\mathcal{P}}]_{/\bar{\mathbb{F}}_p}$) is connected (resp. étale).

From the splitting of the exact sequence

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^* \to A[\mathfrak{p}^\infty] \to F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}} \to 0$$

over $\widetilde{W}_p$, for each integer n, there exists a finite subgroup scheme $A[\bar{\mathcal{P}}^n]_{/\widetilde{W}_p}$ of $A[\mathfrak{p}^n]_{/\widetilde{W}_p}$ which projects isomorphically to $A_{\mathfrak{P}}[\bar{\mathcal{P}}]_{/\bar{\mathbb{F}}_p}$ under the reduction map. Denote the quotient abelian scheme $(A/A[\bar{\mathcal{P}}^n])_{/\widetilde{W}_p}$ by $A'_{n/\widetilde{W}_p}$ and let $\pi_n : A \to A'_n$ be the natural projection defined over $\widetilde{W}_p$.

As M is a number field, there exists a positive integer N and an element $a \in \mathcal{O}_M$ such that $\bar{\mathcal{P}}^N = (a)$ in $\mathcal{O}_M$. Under the isomorphism $\theta_{\mathfrak{P}} : M \cong \mathrm{End}^0_F(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$, the element $a \in \mathcal{O}_M$ gives an isogeny of $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$ whose kernel is $A_{\mathfrak{P}}[\bar{\mathcal{P}}^N]_{/\bar{\mathbb{F}}_p}$, which is still denoted by a.

From the above construction, the projection $\pi_N : A \to A'_N$ is a lifting of the isogeny $a : A_{\mathfrak{P}} \to A_{\mathfrak{P}}$ to $\widetilde{W}_p$. From [29] Theorem 2.1(4) or [6] Formula 3.7.2, we have the following equation:

$$t_{\mathfrak{p}}(A'_{N/\widetilde{W}_p}; a(\alpha), \alpha') = t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \bar{a}(\alpha')),$$

for $\alpha, \alpha' \in T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$. Here $\bar{a}$ is the complex conjugate of a in M. From our choice of the element $a \in \mathcal{O}_M$, the action of a (resp. $\bar{a}$) on $T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$ is divisible by p (resp. invertible). Hence the above equation tells us that the Serre-Tate coordinate $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha')$ is a p-th power. Now we replace a by arbitrary power of a, and repeat the above argument. It follows that $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha')$ is a p^n-th power for all $n \geq 1$. As $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha') \in \widehat{\mathbb{G}}_m(\widetilde{W}_p)$, we have $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha') = 1$ for all $\alpha, \alpha' \in T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$. So we have $t_{\mathfrak{p}}(x) = 1$.

In the general case, as the ring R is an order in M, we can find a positive integer m such that $ma \in R$. We replace a by ma in the above argument, and it is easy to see that $t_{\mathfrak{p}}(x)^m = 1$ in this setting. As the Serre-Tate coordinate $t_{\mathfrak{p}}(x)$ belongs to $\widehat{\mathbb{G}}_m(\widetilde{W}_p)$, we can take m as a power of p, as desired. □

5.2.2 Comparison of endomorphism algebras at different special fibers

In this section we want to compare the endomorphism algebras of the special fibers of the abelian scheme $A_{/\widetilde{\mathcal{W}}_\Xi}$. The key ingredient is the Kodaira-Spencer map, which we will recall below.

As we can regard $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$ as a $\widetilde{W}_p$-rational point the point x actually sits in the formal scheme $\widehat{S}_{p/W_p}$, in other words, if we regard x as a morphism $\mathrm{Spec}(\widetilde{\mathcal{W}}_\Xi) \to Sh^{(\Xi)}$, then this morphism factors through $\nu_p : \widehat{S}_p \to Sh^{(\Xi)}$.

Let $(A_p^{univ}, \iota_p^{univ}, \phi_p^{univ})$ be the universal object over $\widehat{S}_p$. Let $\pi_p : A_p^{univ} \to \widehat{S}_p$ be the structure morphism and $e_p : \widehat{S}_p \to A_p^{univ}$ be the morphism corresponding to the identity element. Consider the sheaf $\omega_p^{univ} = (\pi_p)_*(\Omega_{A_p^{univ}/\widehat{S}_p}) = e_p^*(\Omega_{A_p^{univ}/\widehat{S}_p})$ over $\widehat{S}_{p/W_p}$, which has a natural $\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module structure, and compatible with arbitrary base change. Set $(\omega_p^{univ})^{\otimes 2} = \omega_p^{univ} \otimes_{(\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F)} \omega_p^{univ}$. Then we have the Kodaira-Spencer map:

$$KS : (\omega_p^{univ})^{\otimes 2} \to \Omega_{\widehat{S}_p/W_p}.$$

We remark here that the Kodaira-Spencer map is $\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F$-linear and compatible with the $g = \hat{\rho}(\alpha)$-action on both sides.

By the isomorphism $\widehat{S}_p \cong \widehat{S}_p^{ord} \times \widehat{S}_p^{ll}$ over W_p, we have the decomposition: $\Omega_{\widehat{S}_p/W_p} = (\pi^{ord})^* \Omega_{\widehat{S}_p^{ord}/W_p} \oplus (\pi^{ll})^* \Omega_{\widehat{S}_p^{ll}/W_p}$, where $\pi^{ord} : \widehat{S}_p \to \widehat{S}_p^{ord}$ and $\pi^{ll} : \widehat{S}_p \to \widehat{S}_p^{ll}$ are the natural projection. Since $\widehat{S}_p^{ord} \cong \Pi_{\mathfrak{p} \in \Sigma^{ord}} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*$, if we set $\widehat{S}_{\mathfrak{p}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*$, then we have $\Omega_{\widehat{S}_p^{ord}/W_p} = \oplus_{\mathfrak{p} \in \Sigma^{ord}} (\pi_{\mathfrak{p}})^* \Omega_{\widehat{S}_{\mathfrak{p}}/W_p}$, where $\pi_{\mathfrak{p}} : \widehat{S}_p^{ord} \to \widehat{S}_{\mathfrak{p}}$ is the natural projection. To express the g-action on $\Omega_{\widehat{S}_p^{ord}/W_p}$ in a simple way, we base change this module to $\widetilde{L}_p$, i.e. we consider $\Omega_{\widehat{S}_p^{ord}/W_p} \otimes_{W_p} \widetilde{L}_p = \Omega_{\widehat{S}_p^{ord}/\widetilde{L}_p} = \oplus_{\mathfrak{p} \in \Sigma^{ord}} \Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p}$, which is free of finite rank over $(\widehat{S}_p^{ord})_{/\widetilde{L}_p}$. Moreover, for each $\mathfrak{p} \in \Sigma^{ord}$, the set $\{d\tau_\sigma | \tau \sim \mathfrak{p}\}$ forms a basis of the module $\Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p}$ over $\widehat{S}_{\mathfrak{p}}$, here τ_σ's are the $\hat{\rho}$-eigen coordinates constructed above.

On the other hand, we consider the cotangent bundle $(\omega_p^{univ})^{\otimes 2} \otimes_{W_p} \widetilde{L}_p = (\widetilde{\omega}_p^{univ})^{\otimes 2}$, which has a natural $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p$-module structure. By our construction of $\widetilde{L}_p$, for any embedding $\sigma : F \to \bar{\mathbb{Q}}$, $\sigma(\mathcal{O}_F)$ is contained in $i_p^{-1}(\widetilde{L}_p)$. Hence we have the isomorphism $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p \cong \Pi_{\sigma : F \to \bar{\mathbb{Q}}} \widetilde{L}_p$. By this isomorphism we can decompose the $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p$-module

$(\widetilde{\omega}_p^{univ})^{\otimes 2}$ as $(\widetilde{\omega}_p^{univ})^{\otimes 2} = \oplus_{\sigma:F\to\bar{\mathbb{Q}}}(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ such that on the bundle $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$, $\mathcal{O}_F$ acts through the embedding σ.

Then by [29] section 1.0, for each $\mathfrak{p} \in \Sigma^{ord}$, the Kodaira-Spencer map induces an isomorphism

$$\bigoplus_{\sigma:F\to\bar{\mathbb{Q}},\sigma\sim\mathfrak{p}} (\widetilde{\omega}_p^{univ})^{\otimes 2\sigma} \to (\pi_{\mathfrak{p}} \circ \pi^{ord})^*\Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p},$$

under which the bundle $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ corresponds to the sub-bundle generated by $d\tau_\sigma$. Hence the action of $g = \hat{\rho}(\alpha)$ preserves each $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ and acts it by multiplying the scalar $i_p \circ \sigma(\alpha^{1-c})$. Moreover, as we assume that $\tau_\sigma(x) = 0$ for all $\sigma \sim \mathfrak{p}$, g also preserves $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}(x)$.

Now we can state the main result in this section:

Theorem 11. *Fix an embedding $\sigma_1 : F \to \bar{\mathbb{Q}}$, such that $i_p \circ \sigma_1$ induces $\mathfrak{p}$. If there exists some prime $l \neq p$ in Ξ, such that the prime $\mathfrak{l}$ induced from $i_l \circ \sigma_1$ belongs to Σ_l^{ord}, then we have an isomorphism of F-algebras: $\mathrm{End}_F^0(A_{\mathfrak{P}/\bar{F}_p}) \cong \mathrm{End}_F^0(A_{\mathfrak{L}/\bar{F}_l})$. Here $A_{\mathfrak{L}/\bar{F}_l}$ sits in the quadruple $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta_{\mathfrak{L}}^{(\Xi)})$ obtained by mod l reduction of the point $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$.*

Proof. Set $\omega = \pi_*(\Omega^1_{A/\widetilde{\mathcal{W}}_\Xi})$, which is naturally an $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi$-module. Again we set $\omega^{\otimes 2} = \omega \otimes_{(\mathcal{O}_F\otimes_{\mathbb{Z}}\widetilde{\mathcal{W}}_\Xi)} \omega$. The base change $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ is an $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{L}}_\Xi$-module. By our construction of $\widetilde{\mathcal{L}}_\Xi$, we have an isomorphism:

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{L}}_\Xi \cong \bigoplus_{\sigma:F\to\bar{\mathbb{Q}}} \widetilde{\mathcal{L}}_\Xi.$$

From this we have the decomposition: $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi = \oplus_{\sigma:F\to\bar{\mathbb{Q}}}\tilde{\omega}^{\otimes 2\sigma}$.

Since the formation of the cotangent sheaf ω_p^{univ} over $\widehat{S}_p$ is compatible with arbitrary base change, by the Cartesian diagram:

$$\begin{array}{ccc} A & \longrightarrow & A_p^{univ} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\widetilde{L}_p) & \xrightarrow{x} & \widehat{S}_p, \end{array}$$

we see that $\tilde{\omega}^{\otimes 2\sigma} \otimes_{\widetilde{\mathcal{L}}_\Xi} \widetilde{L}_p = (\omega_p^{univ})^{\otimes 2\sigma}(x)$. As $g = \hat{\rho}(\alpha)$ acts on the Shimura variety $Sh^{(\Xi)}_{/\widetilde{\mathcal{W}}_\Xi}$, g sends the bundle $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ and hence each factor $\tilde{\omega}^{\otimes 2\sigma}$ to the corresponding bundles over $g(x)$. As g preserves $(\omega_p^{univ})^{\otimes 2\sigma}(x)$ for all $\sigma \sim \mathfrak{p}$, it also preserves $\tilde{\omega}^{\otimes 2\sigma}$. In particular, g preserves $\tilde{\omega}^{\otimes 2\sigma_1}$.

As $\tilde{\omega}^{\otimes 2\sigma_1} \otimes_{\widetilde{\mathcal{L}}_\Xi} \widetilde{L}_l = (\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}(x)$, g also preserves the fiber $(\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}(x)$ of the bundle $(\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}$ at the point x_l and acts on it by multiplication by $i_l \circ \sigma_1(\alpha)$. Hence g must act on the eigen coordinate $\tau_{\sigma_1,l}(x)$ by multiplying $i_l \circ \sigma_1(\alpha)$, and g preserves the sub-bundle of $\Omega_{\widehat{S}_l/W_l}(x)$ generated by $d\tau_{\sigma_1,l}(x)$. If g sends $x_l \in Sh^{(\Xi)}(\bar{\mathbb{F}}_l)$ to another point $x_l' \neq x_l$, the action of g has to move the deformation space $\widehat{S}_l$ over x_l to the deformation space $\widehat{S}_l'$ over x_l', where $\widehat{S}_l'$ is the completion of $Sh^{(\Xi)}_{/\widetilde{\mathcal{W}}_\Xi}$ along the closed point x_l'. Then g induces an isomorphism of cotangent bundles $g : \Omega_{\widehat{S}_l/W_l}(x) \to \Omega_{\widehat{S}_l'/W_l}(g(x))$ and hence g cannot preserve any sub-bundle of $\widehat{S}^{ord}_{l/W_l}(x)$, which is a contradiction. So g fixes the point x_l, i.e. there exists a prime-to-Ξ isogney $\tilde{\theta}_{\mathfrak{L}}(\alpha)$ of $A_{\mathfrak{L}}$,such that $\tilde{\theta}_{\mathfrak{L}}(\alpha) \circ \eta^{(\Xi)}_{\mathfrak{L}} = \eta^{(\Xi)}_{\mathfrak{L}} \circ \hat{\rho}(\alpha)$, and hence establishes an isomorphism from the quadruple $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta^{(\Xi)}_{\mathfrak{L}})$ to the quadruple $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta^{(\Xi)}_{\mathfrak{L}} \circ \hat{\rho}(\alpha))$. The association $\alpha \mapsto \tilde{\theta}_{\mathfrak{L}}(\alpha)$ gives us an embedding $M \hookrightarrow \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{F}_l})$. Since $\mathrm{End}^0_F(A_{\mathfrak{L}/\bar{F}_l})$ is also a CM quadratic extension of F by Lemma 5.10, this embedding must be an isomorphism. Hence we get the desired isomorphism of F-algebras.

□

5.3 Main result on local indecomposability and applications

Let k be a number field. Suppose that we are given an abelian variety $A_{/k}$ and an algebra homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$ (recall that F is a totally real field of degree d over $\mathbb{Q}$ and $\mathcal{O}_F$ is its integer ring). Assume that there is a prime ideal $\mathfrak{P}$ of k over a rational prime p, such that $A_{/k}$ satisfies the condition (NLL) in section 5.1. From Proposition 5.1, the abelian variety $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ is isotypic. Without loss of generality, we can assume that $A_{/k}$ is absolutely simple. Let $I_{\mathfrak{P}}$ be the inertia group of $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ at the prime $\mathfrak{P}$.

Before we give the main result, we need the following:

Lemma 5.13. *If the abelian variety $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ does not have complex multiplication, we can find two primes $\mathfrak{L}$ and $\mathfrak{Q}$ of k lying over l and q respectively (p, l, q are distinct primes), such that $A_{/k}$ has good reduction at $\mathfrak{L}$ and $\mathfrak{Q}$, and F-algebras $\mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ and $\mathrm{End}^0_F(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ are non-isomorphic CM quadratic extension of F , here $A_{\mathfrak{L}/\bar{\mathbb{F}}_l}$ (resp. $A_{\mathfrak{Q}/\bar{\mathbb{F}}_q}$) is the reduction of $A_{/k}$ at $\mathfrak{L}$ (resp. $\mathfrak{Q}$).*

Proof. Fix an embedding $\sigma : F \to \bar{\mathbb{Q}}$ such that the composition $i_l \circ \sigma$ induces the prime $\mathfrak{p}$. From [24] Proposition 7.1, the set

$$\{\mathfrak{L} | \mathfrak{L} \text{ is a prime of } k \text{ over a rational prime } l \neq p \text{ such that } A_{/k} \text{ has good reduction at } \mathfrak{L}, \text{ and } \Sigma_l^{ord} \neq \emptyset\}$$

has Dirichlet density 1. On the other hand, the primes $\mathfrak{l}$ in F which splits completely over $\mathbb{Q}$ also has Dirichlet density 1, we can find a prime $\mathfrak{L}$ of k over a rational prime l such that:

1. l is unramified in F;
2. $A_{/k}$ has good reduction at $\mathfrak{L}$ and Σ_l^{ord} contains the prime $\mathfrak{l}$ induced by $i_l \circ \sigma$ and $\mathfrak{l}$ splits over $\mathbb{Q}$.

Let $A_{\mathfrak{L}/\bar{\mathbb{F}}_l}$ be the reduction of $A_{/k}$ at $\mathfrak{L}$, and set $M_{\mathfrak{L}} = \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$. By Lemma 5.10, $M_{\mathfrak{L}}$ is a quadratic CM extension of the field F.

Now by an argument in [24] Proposition 5.1 we can find a prime $\mathfrak{Q}$ of k over a rational prime $q \neq p, l$, such that

1. $A_{/k}$ has good reduction at $\mathfrak{Q}$;
2. Σ_q^{ord} contains the prime induced by $i_q \circ \sigma$;
3. $M_{\mathfrak{Q}} = \mathrm{End}^0_F(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ is a CM quadratic extension of F which is non-isomorphic to $M_{\mathfrak{L}}$.

For completeness, we give a sketch of the construction of $\mathfrak{Q}$ and refer to [24] Proposition 5.1 for more details. We use D to denote the division algebra $\mathrm{End}^0(A_{/k})$ and let Z be the center of D. From Proposition 5.2, Z is totally real and either $Z = F = D$ or D is a quaternion division algebra over Z and $[F : Z] = 2$.

For any prime $\mathfrak{q}$ of F, we fix an isomorphism $T_{\mathfrak{q}}(A) \cong (\mathcal{O}_{F,\mathfrak{q}})^2$, and denote by $r_{\mathfrak{q}}$: $\mathrm{Gal}(\bar{\mathbb{Q}}/k) \to GL_2(\mathcal{O}_{F,\mathfrak{q}})$ as the induced Galois representation on $T_{\mathfrak{q}}(A)$. Define the algebra $C_{\mathfrak{q}} = Z_{\mathfrak{q}}[r_{\mathfrak{q}}(\mathrm{Gal}(\bar{\mathbb{Q}}/k))]$ as the subalgebra of $\mathrm{End}^0_{Z_{\mathfrak{q}}}(T_{\mathfrak{q}}(A)) = \mathrm{End}_{\mathcal{O}_{Z,\mathfrak{q}}}(T_{\mathfrak{q}}(A)) \otimes_{\mathcal{O}_{Z,\mathfrak{q}}} Z_{\mathfrak{q}}$, of $r_{\mathfrak{q}}$ generated over $Z_{\mathfrak{q}}$ by the image of $r_{\mathfrak{q}}$. Then by Faltings' isogeny theorem, $C_{\mathfrak{q}}$ is either isomorphic to a quaternion division algebra over $Z_{\mathfrak{q}}$ or isomorphic to $M_2(Z_{\mathfrak{q}})$. In the case $\mathfrak{q} = \mathfrak{l}$, $C_{\mathfrak{l}}$ is isomorphic to $M_2(F_{\mathfrak{l}}) = M_2(Z_{\mathfrak{l}})$. Under this assumption,we can apply an argument in [42] Chapter 4 to prove that the image $\mathrm{Im}(r_{\mathfrak{l}})$ contains an open subgroup of $SL_2(\mathbb{Z}_l) \subseteq C_{\mathfrak{l}}^{\times}$.

Choose a quadratic ramified extension $K/\mathbb{Q}_l$. Since $F_{\mathfrak{l}}/\mathbb{Q}_l$ is unramified, K and $F_{\mathfrak{l}}$ are linearly disjoint over $\mathbb{Q}_l$. Let L be the compositum field of K and $F_{\mathfrak{l}}$. Define the torus $T_{/\mathcal{O}_{F,\mathfrak{l}}}$ of $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$ as the norm 1 subgroup of $Res_{\mathcal{O}_L/\mathcal{O}_{F,\mathfrak{l}}}(\mathbb{G}_m)$; i.e.

$$T(\mathcal{O}_{F,\mathfrak{l}}) = \{x \in \mathcal{O}_L^{\times} | Norm_{L/F_{\mathfrak{l}}}(x) = 1\}.$$

Hence $T_{/\mathcal{O}_{F,\mathfrak{l}}}$ is a maximal anisotropic torus of $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$, and $T(\mathcal{O}_{F,\mathfrak{l}}) \cap SL_2(\mathbb{Z}_l)$ is a maximal anisotropic torus of $GL_{2/\mathbb{Z}_l}$.

Choose $\alpha \in T(\mathcal{O}_{F,\mathfrak{l}}) \cap \mathrm{Im}(r) \cap SL_2(\mathbb{Z}_l)$, such that α has two different eigenvalues in $\bar{\mathbb{Q}}_l$. Then $T(\mathcal{O}_{F,\mathfrak{l}})$ is the centralizer T_α of α in $GL_2(\mathcal{O}_{F,\mathfrak{l}})$. Since the isomorphism classes of maximal torus in $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$ is finite, the isomorphism class of the centralizer of α is determined by α mod p^j, for some integer j large enough. In other word, if $\beta \in SL_2(\mathbb{Z}_l)$, such that $\alpha \equiv \beta$ mod p^j, then the centralizer T_β of β is isomorphic to $T_\alpha = T$. By Chebotarev density, we can find a prime $\mathfrak{Q}$ of k over a rational prime $q \neq p, l$, such that $A_{/k}$ has good reduction at $\mathfrak{Q}$ and $r(Frob_{\mathfrak{Q}}) \equiv \alpha$ mod p^j. Hence the commutator $T_{r(Frob_{\mathfrak{Q}})}$ of $r(Frob_{\mathfrak{Q}})$ is isomrphic to T. Let $M_{\mathfrak{Q}}$ be the field generated over F by the eigenvalues of $r(Frob_{\mathfrak{Q}})$. By the above construction, $\mathfrak{l}$ does not split in $M_{\mathfrak{Q}}$, and hence $M_{\mathfrak{Q}}$ is not isomorphic to $M_{\mathfrak{L}}$. Further by [24] Proposition 7.1, we can assume that Σ_q^{ord} contains the prime induce from $i_q \circ \sigma$.

Then it is clear from the above construction that the primes $\mathfrak{Q}$ and $\mathfrak{L}$ satisfy the desired property. ∎

Now we can state and prove the main theorem in this section:

Theorem 12. *Under the above notations and assumptions, suppose further that* $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ *does not have complex multiplication, then for any* $\mathfrak{p} \in \Sigma_p^{ord}$*, the* $\mathfrak{p}$*-adic Tate module* $T_{\mathfrak{p}}(A)$ *of* A *is indecomposable as an* $I_{\mathfrak{P}}$*-module.*

Proof. Let the prime $\mathfrak{Q}$ and $\mathfrak{L}$ be the primes of k in the previous lemma. Define a finite set of primes $\Xi = \{p, q, l\}$. For this set Ξ, we define the semilocal ring $\widetilde{\mathcal{W}_k}$ as in section 5.2. Hence the abelian variety $A_{/k}$ can be extended to an abelian scheme $A_{/\widetilde{\mathcal{W}_k}}$. From Proposition 5.7, replacing $A_{/k}$ by an isogenous abelian variety if necessary, we can assume that the abelian scheme $A_{/\widetilde{\mathcal{W}_k}}$ admits an $\mathcal{O}_F$-action $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/\widetilde{\mathcal{W}_k}})$ and a $\mathfrak{c}$-polarization ϕ for some fractional ideal $\mathfrak{c}$ of F. Then by choosing a integral level structure α^{Ξ} of A, we get a quadruple $(A_{/\widetilde{\mathcal{W}_k}}, \iota, \phi, \alpha^{\Xi})$, which represents a point in the Shimura variety $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}_k})$.

Now assume that the Tate module $T_{\mathfrak{p}}(A)$ is decomposable as an $I_{\mathfrak{P}}$-module. Then the exact sequence of Barsotti-Tate $\mathcal{O}_{\mathfrak{p}}$-modules over $\widetilde{W_p}$:

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}} \to A[\mathfrak{p}^\infty] \to F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}} \to 0$$

splits. Then by Theorem 11, we must have isomorphisms of F-algebras: $M_{\mathfrak{Q}} \cong \mathrm{End}^0_F(A_{\mathfrak{Q}/\bar{\mathbb{F}}_p})$ and $M_{\mathfrak{L}} \cong \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_p})$. But this contradicts with our construction $M_{\mathfrak{Q}} \ncong M_{\mathfrak{L}}$. Hence $T_{\mathfrak{p}}(A)$ must be indecomposable as an $I_{\mathfrak{P}}$-module. ∎

5.3.0.1 Application to Hilbert modular Galois represenations

As the first application of Theorem 12, we study the Galois representation attached to certain Hilbert modular forms. First we recall the notions of Hilbert modular forms and Hecke operators.

Let $\mathrm{I} = \mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}})$, and let $\mathbb{Z}[\mathrm{I}]$ be the set of formal $\mathbb{Z}$-linear combinations of elements in I. Then $\mathbb{Z}[\mathrm{I}]$ can be identified with the character group $X(T)$ of the torus T. Take $k = (k_\sigma)_{\sigma \in \mathrm{I}}$

such that $k_\sigma \geq 2$ for all $\sigma \in \mathrm{I}$ and all the k_σ's have the same parity. Set $t = (1, \dots, 1) \in \mathbb{Z}[\mathrm{I}]$ and $n = k - 2t$. Choose $v = (v_\sigma)_{\sigma \in \mathrm{I}}$ such that $v_\sigma \geq 0$, for all σ, $v_\sigma = 0$ for at least one σ, and there exists $\mu \in \mathbb{Z}$ such that $n + 2v = \mu t \in \mathbb{Z}[\mathrm{I}]$. Then define $w = v + k - t$.

Recall that in Section 4.2 we define the algebraic group $G = Res_{\mathcal{O}_F/\mathbb{Z}}(GL_2)$ and $T = Res_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)$. Denote by $\nu : G \to T$ the reduced norm morphism. Fix an open subgroup U of $G(\widehat{\mathbb{Z}}) = GL_2(\widehat{\mathcal{O}}_F)$ where $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \Pi_{\mathfrak{p}} \mathcal{O}_{F,\mathfrak{p}}$. In the last product, $\mathfrak{p}$ ranges over all the prime ideals of $\mathcal{O}_F$ and $\mathcal{O}_{F,\mathfrak{p}}$ is the completion of $\mathcal{O}_F$ at $\mathfrak{p}$. Let $F_{\mathbb{A}} = F \otimes_{\mathbb{Z}} \mathbb{A}$ be the adele ring of F. We can decompose the group $G(F_{\mathbb{A}})$ as the product $G_\infty \times G_f$, where G_∞ (resp. G_f) is the infinite (resp. finite) part of $G(F_{\mathbb{A}})$, and for each $u \in G(F_{\mathbb{A}})$, we have the corresponding decomposition $u = u_\infty u_f$.

Let $\mathfrak{h}$ be the complex upper half plane and $i = \sqrt{-1} \in \mathfrak{h}$. Let $\mathfrak{h}^{\mathrm{I}}$ be the product of d copies of $\mathfrak{h}$ indexed by elements in I and $z_0 = (i, \dots, i) \in \mathfrak{h}^{\mathrm{I}}$. Define a function $j : G_\infty \times \mathfrak{h}^{\mathrm{I}} \to \mathbb{C}^{\mathrm{I}}$ by the formula:

$$\left(\begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}, z_\tau \right)_{\tau \in \mathrm{I}} \mapsto (c_\tau z_\tau + d_\tau)_{\tau \in \mathrm{I}}.$$

Definition 5.14. *Define the space of Hilbert modular cusp forms $S_{k,w}(U; \mathbb{C})$ as the set of functions $f : G(F_{\mathbb{A}}) \to \mathbb{C}$ satisfying the following conditions:*

1. *$f|_{k,w} u = f$, for all $u \in UC_{\infty+}$ where $C_{\infty+} = (\mathbb{R}^\times \cdot SO_2(\mathbb{R}))^{\mathrm{I}} \subseteq G_\infty$, and*

$$f|_{k,w} u(x) = j(u_\infty, z_0)^{-k} \nu(u_\infty)^w f(xu^{-1});$$

2. *$f(ax) = f(x)$ for all $a \in G(\mathbb{Q}) = GL_2(F)$;*

3. *For any $x \in G_f$, the function $f_x : \mathfrak{h}^{\mathrm{I}} \to \mathbb{C}$ defined by $u_\infty(z_0) \mapsto j(u_\infty, z_0)^k \nu(u_\infty)^{-w} f(xu_\infty)$ for $u_\infty \in G_\infty$ is holomorphic;*

4. *$\int_{F_{\mathbb{A}}/F} f \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x \right) da = 0$ for all $x \in G(F_{\mathbb{A}})$ and additive Haar measure da on $F_{\mathbb{A}}/F$.*

When $F=\mathbb{Q}$, we also add the following condition: the function $|\mathrm{Im}(z)^{k/2} f_x(z)|$ is uniformly bounded on $\mathfrak{h}$ for all $x \in G_f = GL_2(\mathbb{A}_f)$.

Fix an integral ideal $\mathfrak{m}$ of F, we define three open subgroups of $GL_2(\widehat{\mathcal{O}}_F)$:

$$U_0(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

$$U_1(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F, a \equiv 1 \ mod\ \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

$$U(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F, a \equiv d \equiv 1 \ mod\ \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

and set $S_{k,w}(\mathfrak{m}, \mathbb{C}) = S_{k,w}(U_1(\mathfrak{m}), \mathbb{C})$.

Let U, U' be two open compact subgroups of G_f and fix $x \in G_f$. Define a Hecke operator

$$[UxU'] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U';\mathbb{C}), f \mapsto \sum_i f|_{k,w} x_i,$$

where $\{x_i\}$ is a set of representatives of the left cosets $U \backslash UxU'$; i.e., we have $UxU' = \coprod Ux_i$ and when we consider the action $f|_{k,w}x_i$, we regard $x_i \in G_f$ as an element in $G(F_{\mathbb{A}})$ such that its infinite part consists of d copies of identity matrices. For all prime ideal $\mathfrak{q}$ of F, fix a uniformizer $\pi_{\mathfrak{q}}$ of $F_{\mathfrak{q}}$, and define the Hecke operator

$$T(\mathfrak{q}) = \left[U \begin{pmatrix} 1 & 0 \\ 0 & \beta_{\mathfrak{q}} \end{pmatrix} U \right] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}),$$

where $\beta_{\mathfrak{q}} \in F_{\mathbb{A}_f}^{\times}$ is the finite idele whose $\mathfrak{q}$-component is $\pi_{\mathfrak{q}}$ and all the other components are 1. For each fractional ideal $\mathfrak{n}$ of F, set $\alpha = \Pi_{\mathfrak{q}} \pi_{\mathfrak{q}}^{v_{\mathfrak{q}}(\mathfrak{n})} \in F_{\mathbb{A}_f}^{\times}$, and define the Hecke operator

$$\langle \mathfrak{n} \rangle = \left[U \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} U \right] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}).$$

Let $f \in S_{k,w}(\mathfrak{m}, \mathbb{C})$ be a normalized Hilbert modular eigenform in the sense that for any prime ideal $\mathfrak{q}$ of F, there exists $c(\mathfrak{q}, f) \in \bar{\mathbb{Q}}$ and $d(\mathfrak{q}, f) \in \bar{\mathbb{Q}}$ such that $T(\mathfrak{q})(f) = c(\mathfrak{q}, f) \cdot f$

and $\langle \mathfrak{q} \rangle (f) = d(\mathfrak{q}, f) \cdot f$. Let K_f be the field generated over $\mathbb{Q}$ by all the $c(\mathfrak{a}, f)$'s and $d(\mathfrak{a}, f)$'s. Shimura proved that K_f is a number field which is either totally real or CM. Denote by $\mathcal{O}_f$ the integer ring of K_f.

For such an f, let $\pi_f = \otimes \pi_v$ be the automorphic representation of $GL_2(F_{\mathbb{A}})$ on the linear span of all the right translations of f by elements of $GL_2(F_{\mathbb{A}})$, here $F_{\mathbb{A}}$ is the adele ring of F,and π_v is a representation of $GL_2(F_v)$ for each finite place v of F. We assume that one of the following two statements holds:

1. $[F : \mathbb{Q}]$ is odd;

2. there exists some finite place v of F such that π_v is square integrable.

For such an eigenform f, the following result is known (see [21] Theorem 2.43 for details and historical remarks). For each prime λ of $\mathcal{O}_f$ over a rational prime p, there is a continuous representation $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(\mathcal{O}_{f,\lambda})$, which is unramified outside primes dividing $\mathfrak{m}p$ such that for any primes $\mathfrak{q} \nmid \mathfrak{m}p$, we have:

$$\mathrm{trace}(\rho_{f,\lambda}(Frob_{\mathfrak{q}})) = c(\mathfrak{q}, f), \text{ and } \det(\rho_{f,\lambda}(Frob_{\mathfrak{q}})) = d(\mathfrak{q}, f) N\mathfrak{q}.$$

Here $\mathcal{O}_{f,\lambda}$ is the completion of $\mathcal{O}_f$ at λ, $Frob_{\mathfrak{q}}$ is the Frobenius of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{q}$,and for any ideal $\mathfrak{b}$ of $\mathcal{O}_F$, $N\mathfrak{b}$ is the cardinality number of the ring $\mathcal{O}_F/\mathfrak{b}$.

Fix a prime $\mathfrak{p}$ of $\mathcal{O}_F$ over a rational prime p, let $D_{\mathfrak{p}}$(resp. $I_{\mathfrak{p}}$) be the decomposition group (resp. inertia group) of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$. Let λ be a prime of $\mathcal{O}_f$ over p. From [52] Lemma 2.1.5, if $c(\mathfrak{p}, f)$ is a unit mod λ, then the restriction of $\rho_{f,\lambda}$ to $D_{\mathfrak{p}}$ is upper triangular, i.e. there exist two characters ϵ_1, ϵ_2 of $D_{\mathfrak{p}}$, such that

$$\rho_{f,\lambda}|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}.$$

Lemma 5.15. *Suppose that $k = 2t$ and f is nearly $\mathfrak{p}$-ordinary in the sense that $c(\mathfrak{p}, f)$ is a unit mod λ. Then there exists an abelian variety $A_{f/F}$, a finite extension L/K_f and an homomorphism $L \to \mathrm{End}^0(A_{f/F})$ such that degree of L over $\mathbb{Q}$ equals to the dimension of A_f and up to a character the λ-adic representation $\rho_{f,\lambda}$ comes from the Tate module of A_f.*

Proof. As the Hecke operator $T(\mathfrak{p})$ acts nontrivially on f, from [19] Corollary 2.2, the local representation $\pi_{\mathfrak{p}}$ of $GL_2(F_{\mathfrak{p}})$ is either a principal representation $\pi(\xi_{\mathfrak{p}}, \eta_{\mathfrak{p}})$ or a special representation $\sigma(\xi_{\mathfrak{p}}, \eta_{\mathfrak{p}})$. From the argument in [19] Section 2, we can find a finite character $\chi : F_{\mathbb{A}}^{\times}/F^{\times} \to \bar{\mathbb{Q}}^{\times}$ ($F_{\mathbb{A}}$ is the adele ring of F) such that the $\mathfrak{p}$-component of χ satisfies $\chi_{\mathfrak{p}} = \xi_{\mathfrak{p}}$ on $\mathcal{O}_{F,\mathfrak{p}}^{\times}$ and unramified at every infinite place of F. Then the argument in [19] Section 2 implies that the automorphic representation $\chi \otimes \pi$ corresponds to a primitive $\mathfrak{p}$-ordinary newform f_0. If we regard the representations $\rho_{f,\lambda}$ and $\rho_{f_0,\lambda}$ as representations in $GL_2(\bar{\mathbb{Q}}_p)$, then they are related by the formula $\rho_{f,\lambda} \otimes \chi^{-1} = \rho_{f_0,\lambda}$. It is enough to prove the statement for the newform f_0 and henceforth we assume that the Hilbert modular form f is a primitive $\mathfrak{p}$-ordinary newform with character ψ for some idele class character ψ of F with finite order.

From [16] Theorem 4.4 or [51] Theorem 2.1, there exists an abelian variety A_f defined over F , a finite extension L/K_f whose degree equals to the dimension of A_f and an embedding $\theta : L \to \mathrm{End}(A_{f/F})$ such that the λ-adic representation associated to the Tate module of A_f is isomorphic to $\rho_{f,\lambda}$. Moreover the number field L is either totally real or CM. To be more precise, there exists an integer e such that $dim(A_{f/F}) = e[K_f : \mathbb{Q}]$. When $[F : \mathbb{Q}]$ is odd, $e = 1$ and there is nothing to explain in this situation. When $[F : \mathbb{Q}]$ is even, e can be bigger than 1, and a priori the p-adic Tate module of $A_{f/F}$ gives us a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ in $GL_2(L_\lambda)$,where L_λ is a finite extension of $K_{f,\lambda}$. Since this representation is odd, by choosing suitable eigenvectors of a complex conjugation $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/F)$ as basis for $T_p(A_f)$, we can realize this representation in $GL_2(K_{f,\lambda})$. (See [52] Section 2.1 for details.)

□

Remark 5.16. As $c(\mathfrak{p}, \lambda)$ is a unit mod λ, the abelian variety A_f has potentially semistable reduction at $\mathfrak{p}$ by the lemma in [51] Section 2. More precisely, if we denote by F_ψ the number field corresponding to the character ψ by class field theory, then A_f has semistable reduction over F_ψ. In fact, choose a prime λ' of $\mathcal{O}_f$ over a rational prime $l \neq p$ and consider the λ'-adic representation $\rho_{f,\lambda'}$. When $\mathfrak{p}$ does not divide the level $\mathfrak{m}$, the abelian variety A_f has good reduction at $\mathfrak{p}$ because the representation $\rho_{f,\lambda'}$ is unramified at $\mathfrak{p}$. If $\mathfrak{p}$ divides $\mathfrak{m}$, one can consider the complex representation $\sigma_{\mathfrak{p}}$ of the local Weil-Deligne group $W'_{F_{\mathfrak{p}}}$ of F

at $\mathfrak{p}$ associated to $\rho_{f,\lambda'}$ (see [50]). Then by a result of Carayol [4], we have an isomorphism $\pi(\sigma_\mathfrak{p}) \cong \pi_\mathfrak{p}$, where $\pi(\sigma_\mathfrak{p})$ is the representation of $GL_2(F_\mathfrak{p})$ associated to $\sigma_\mathfrak{p}$ under the local Langlands correspondence. In particular, the Euler factor $L(\pi_\mathfrak{p}, s)$ of the L-series at $\mathfrak{p}$ is given by $(1 - c(\mathfrak{p}, f)N\mathfrak{p}^{-s})^{-1}$. As $c(\mathfrak{p}, f) \neq 0$ by assumption, $L(\pi_\mathfrak{p}, s)$ is nontrivial. Hence $\pi_\mathfrak{p}$ is either a special representation $\sigma(\alpha_\mathfrak{p}, \beta_\mathfrak{p})$ or a principal series representation $\pi(\alpha_\mathfrak{p}, \beta_\mathfrak{p})$, where $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ are two quasi-characters of $F_\mathfrak{p}^\times$. In the first case, from [51] Theorem 2.2, the reduction of A_f at $\mathfrak{p}$ is purely multiplicative. From the uniformization result in [36], $\rho_{f,\lambda}|_{I_\mathfrak{p} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/F_\psi)}$ is indecomposable. As $I_\mathfrak{p} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/F_\psi)$ is a subgroup of $I_\mathfrak{p}$ with finite index, and $char(K_f) = 0$, the representation $\rho_{f,\lambda}|_{I_\mathfrak{p}}$ is also indecomposable. In the second case, as the Euler factor $L(\pi_\mathfrak{p}, s) \neq 1$, one of the quasi-characters $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ is unramified. By comparing the determinant of the two representations $\pi_\mathfrak{p}$ and $\sigma_\mathfrak{p}$, we see that the product $\psi_\mathfrak{p}^{-1}\alpha_\mathfrak{p}\beta_\mathfrak{p}$ is unramified, where $\psi_\mathfrak{p}$ is the $\mathfrak{p}$-component of the idele class character ψ. Hence over F_ψ, both quasi-characters $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ are unramified. Then from the criterion of Néron-Ogg-Shafarevich, the abelian variety A_f has good reduction over F_ψ at $\mathfrak{p}$.

Now we would like to prove the following:

Theorem 13. *Under the above notations and assumptions in lemma 5.15, if f does not have complex multiplication, then the representation $\rho_{f,\lambda}|_{I_\mathfrak{p}}$ is indecomposable.*

Proof. From Lemma 5.15 and Remark 5.16 we can assume that A_f has good reduction over F_ψ. From Proposition 5.2, we see that $A_{f/\bar{\mathbb{Q}}}$ is isotypic; i.e. there exists a simple abelian variety $B_{/\bar{\mathbb{Q}}}$ such that there exists an isogeny $\varphi : A_f \to B^e$ for some integer $e \geq 1$. This isogeny induces an isomorphism of simple algebras $i : \mathrm{End}^0(A_{f/\bar{\mathbb{Q}}}) \to \mathrm{End}^0((B_{/\bar{\mathbb{Q}}})^e)$. Hence we have an embedding $\theta_B = i \circ \theta : L \to \mathrm{End}^0((B_{/\bar{\mathbb{Q}}})^e)$.

From Proposition 5.4, we can find a totally real field F_B and a homomorphism $\iota_B : F_B \to \mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$, such that $[F_B : \mathbb{Q}] = dim(B_{/\bar{\mathbb{Q}}})$. Let Z be the center of the division algebra $\mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$. From the proof of Proposition 5.4, if we identify F_B as a subalgebra of $\mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$ by ι_B, then $Z \subseteq F_B$ and $[F_B : Z] \leq 2$.

If $[F_B : Z] = 1$, we have $F_B = Z$ and hence $F_B \subseteq \theta_B(L)$. Since both A_f and B are projective varieties, we can find a finite extension M of F_ψ such that

1. the abelian variety B is defined over M;
2. we have the equalities of endomorphism algebras: $\mathrm{End}(A_{f/\bar{\mathbb{Q}}}) = \mathrm{End}(A_{f/M})$ and $\mathrm{End}(B_{/\bar{\mathbb{Q}}}) = \mathrm{End}(B_{/M})$.
3. the isogeny φ is defined over M.

Under the above notations, the isogeny φ gives an isomorphism of p-adic Tate modules $T_p(B) \otimes_{F_B} L \cong T_p(A)$, which is equivariant under the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

If $[F_B : Z] = 2$, F_B may not contained in the image $\theta_B(L)$. In this case, we can find a quadratic extension K/L such that F_B can be embedded into K. As the homomorphism $\theta : L \to \mathrm{End}^0(A_{f/\bar{\mathbb{Q}}})$ identifies L with a maximal commutative subfield of the simple algebra $\mathrm{End}^0(A_{f/\bar{\mathbb{Q}}})$, we can extend this homomorphism to a homomorphism $\theta' : K \to \mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$, which identifies K with a maximal commutative subfield of $\mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$. Similarly we can extend the homomorphism θ_B to a homomorphism $\theta'_B : K \to \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$. Since $A^2_{f/\bar{\mathbb{Q}}}$ is isogeneous to $B^{2e}_{/\bar{\mathbb{Q}}}$, the simple algebras $\mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$ and $\mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$ are isomorphic. Since all automorphisms of a simple algebra are inner, by choosing a suitable isogeny from $A^2_{f/\bar{\mathbb{Q}}}$ to $B^{2e}_{/\bar{\mathbb{Q}}}$, we have an isomorphism $i' : \mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}}) \cong \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$, such that $i' \circ \theta' = \theta'_B : K \cong \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$. By the same argument as above, we can find a finite extension M/F_ψ such that we have an isomorphism of p-adic Tate modules: $T_p(B) \otimes_{F_B} K \cong T_p(A_f) \otimes_L K$, which is equivariant under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

As $B^e_{/M}$ is isogenous to $A_{f/M}$, $B_{/M}$ has good reduction at a prime $\mathfrak{p}'$ of M over the prime $\mathfrak{p}$ of F. By Theorem 12, for any place λ_B of F_B such that the λ_B-divisible Barsotti-Tate module of $B_{/M}$ is ordinary, the corresponding λ_B-adic Tate module is indecomposable as a $\mathrm{Gal}(\bar{\mathbb{Q}}/M) \cap I_{\mathfrak{p}}$-module. By the above isomorphism of Tate modules, we see that $\rho_{f,\lambda}|_{\mathrm{Gal}(\bar{\mathbb{Q}}/M)\cap I_{\mathfrak{p}}}$ is indecomposable. Since $\mathrm{Gal}(\bar{\mathbb{Q}}/M) \cap I_{\mathfrak{p}}$ is a subgroup of $I_{\mathfrak{p}}$ with finite index, and $char(K_f) = 0$, the representation $\rho_{f,\lambda}|_{I_{\mathfrak{p}}}$ must be also indecomposable. □

From Theorem 13, we can prove a result on local indecomposability of Λ-adic Galois representations. First we briefly recall the definition of ordinary Hecke algebras defined in [17] Section 3.

Let Φ be the Galois closure of F in $\bar{\mathbb{Q}}$. The embedding $i_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ induces a p-adic valuation on Φ and we denote by $\mathcal{O}_\Phi$ the valuation ring. Let K be a finite extension of the p-adic closure of Φ in $\bar{\mathbb{Q}}_p$, and $\mathcal{O}_K$ be the valuation ring of K. Let F_∞/F be the maximal abelian extension of F unramified outside p and ∞, and Z be its Galois group. Let Z_1 be the torsion free part of Z. Let $\Lambda = \mathcal{O}_K[[Z_1]]$ be the continuous group algebra of Z_1 over $\mathcal{O}_K$. Then Λ is (noncanonically) isomorphic to the formal power series ring of $1+\delta$ variables over $\mathcal{O}_K$, where δ is the defect in Leopoldt's conjecture. Let $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathbb{Z}_p^\times$ be the cyclotomic character. The restriction of χ to Z_1 gives a character of Z_1, which is still denoted by χ. For any integer $k \geq 2$ and a finite order character $\epsilon : Z_1 \to \bar{\mathbb{Q}}_p$. The character $\epsilon\chi^{k-1} : Z_1 \to \bar{\mathbb{Q}}_p$ gives a homomorphism $\kappa_{k,\epsilon} : \Lambda \to \bar{\mathbb{Q}}_p$.

For any two open compact subgroups U, U' of G_f and $x \in G_f$, we have the modified Hecke operator defined in [17] Section 3:

$$(UxU') : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U';\mathbb{C}).$$

For each prime ideal $\mathfrak{q}$ of F, set

$$T_0(\mathfrak{q}) = \left(U \begin{pmatrix} 1 & 0 \\ 0 & \beta_\mathfrak{q} \end{pmatrix} U \right) : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}),$$

where $\beta_\mathfrak{q}$ is the same as in the definition of $T(\mathfrak{q})$.

Fix an integral ideal $\mathfrak{n}$ of F which is prime to p, and for each integer $\alpha \geq 1$, set $S_{k,w}(\mathfrak{n}p^\alpha;\mathbb{C}) = S_{k,w}(U_1(\mathfrak{n} \cap U(p^\alpha));\mathbb{C})$. Define the Hecke algebra $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_\Phi)$ as the $\mathcal{O}_\Phi$-subalgebra of $\mathrm{End}_\mathbb{C}(S_{k,w}(\mathfrak{n}p^\alpha;\mathbb{C}))$ generated by all the $T_0(\mathfrak{q})$'s over $\mathcal{O}_\Phi$ and define $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K) = h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} \mathcal{O}_K$. Inside $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K)$ we have the p-adic ordinary projector $e_\alpha = \lim_{n\to\infty} T_0(p)^{n!}$ and we have the ordinary Hecke algebra $h^{ord}_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K) = e_\alpha h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K)$. For $\beta \geq \alpha \geq 0$, we have a natural surjective $\mathcal{O}_K$-algebra homomorphism $h^{ord}_{k,w}(\mathfrak{n}p^\beta;\mathcal{O}_K) \to$

$h^{ord}_{k,w}(\mathfrak{n}p^{\alpha}; \mathcal{O}_K)$, and we define

$$h^{ord}_{k,w}(\mathfrak{n}p^{\infty}; \mathcal{O}_K) = \varprojlim_{\alpha} h^{ord}_{k,w}(\mathfrak{n}p^{\alpha}; \mathcal{O}_K).$$

From [17] Theorem 3.3, the ordinary Hecke algebra $h^{ord}_{k,w}(\mathfrak{n}p^{\infty}; \mathcal{O}_K)$ is a torsion free Λ-module of finite type, and the isomorphism class of $h^{ord}_{k,w}(\mathfrak{n}p^{\infty}; \mathcal{O}_K)$ as an $\mathcal{O}_K$-algebra only depends on the class of v in $\mathbb{Z}[\mathrm{I}]/\mathbb{Z}t$, and hence we denote this algebra by $h^{ord}_{v}(\mathfrak{n}p^{\infty}; \mathcal{O}_K)$.

Now set $\mathbf{h} = h^{ord}_{0}(\mathfrak{n}p^{\infty}; \mathcal{O}_K)$. Fix $\mathrm{Spec}(\Lambda_L) \to \mathrm{Spec}(\mathbf{h})$ a (reduced) irreducible component of $\mathbf{h}$ and let $\mathcal{F} : \mathbf{h} \to \Lambda_L$ be the corresponding homomorphism. Then Λ_L is finite free over Λ, and the quotient field L of Λ_L is a finite extension of the quotient field of Λ. Let P be a $\bar{\mathbb{Q}}_p$-valued point of Λ_L, and let $\varphi_P : \Lambda_L \to \bar{\mathbb{Q}}_p$ be the corresponding homomorphism. The point P is called an arithmetic point if φ_P is an extension of $\kappa_{k,\epsilon}$ for some k and ϵ. If P is an arithmetic point, then the composition $\varphi_P \circ \mathcal{F} : \mathbf{h} \to \bar{\mathbb{Q}}_p$ gives the Hecke eigenvalues of a classical Hilbert modular form f of weight k and tame level $\mathfrak{n}$. We also say that the Hilbert modular form f corresponds to P, and f belongs to the family $\mathcal{F}$. We say that $\mathcal{F}$ has complex multiplication if there exists an arithmetic point P in $\mathcal{F}$, such that the corresponding Hilbert modular form has complex multiplication. Once this is the case, then for all arithmetic point in $\mathcal{F}$, the corresponding Hilbert modular form also has complex multiplication.

It's well known that there is a 2-dimensional Galois representation $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(L)$ attached to $\mathcal{F}$ such that for each prime $\mathfrak{p}$ of F over p, the restriction of $\rho_{\mathcal{F}}$ to the decomposition $D_{\mathfrak{p}}$ is upper triangula;i.e. $\rho_{\mathcal{F}}|_{D_{\mathfrak{p}}}$ is of the shape:

$$\rho_{\mathcal{F}}|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \delta_{\mathfrak{p}} & u_{\mathfrak{p}} \\ 0 & \varepsilon_{\mathfrak{p}} \end{pmatrix},$$

here $\delta_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}} : D_{\mathfrak{p}} \to \Lambda_L$ are two characters of $D_{\mathfrak{p}}$.

Theorem 14. *Suppose that $\mathcal{F}$ does not have complex multiplication, and $\mathcal{F}$ has an arithmetic point P which corresponds to a weight 2 Hilbert modular form satisfying the condition required in Theorem 13. Then there exists a proper closed subscheme S of $\mathrm{Spec}(\Lambda_L)$ such that for all arithmetic points P of $\mathrm{Spec}(\Lambda_L)$ outside S which corresponds to a classical form f, the* *representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable, where ρ_f is the Galois representation attached to f. In particular, when Leopoldt conjecture holds for F and p, then for all but finitely many classical forms f belonging to $\mathcal{F}$, the representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable.*

The proof follows essentially from the argument in [14] Theorem 18. For the sake of completeness, we give a proof here.

Proof. By the assumption and Theorem 13, the representation $\varphi_P \circ \rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable. Hence $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable either. Define $c_\mathfrak{p} = \varepsilon_\mathfrak{p}^{-1} \cdot u_\mathfrak{p} : D_\mathfrak{p} \to \Lambda_L$. Then it's easy to check that $c_{\mathcal{F}}$ satisfies the cocycle condition and $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable if and only if the class $[c_\mathfrak{p}]$ of $c_\mathfrak{p}$ in $H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$ is nontrivial. Since Λ_L is finite over Λ, the residue field of Λ_L is finite and let q be its order. Let E_1 be the compositum of the finitely many tamely ramified abelian extension of $F_\mathfrak{p}$ whose order divides $q-1$, and E_2 be the maximal abelian pro-p-extension of $F_\mathfrak{p}$. Denote by E the compositum field of E_1 and E_2 and set $H = \mathrm{Gal}(\bar{\mathbb{Q}}_p/E) \subseteq D_\mathfrak{p}$. Then the characters $\delta_\mathfrak{p}$ and $\varepsilon_\mathfrak{p}$ are trivial when restricted to H. Hence the restriction of $\rho_{\mathcal{F}}$ to H is of the shape:

$$\rho_{\mathcal{F}}|_H \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

for some (additive) homomorphism $\lambda : H \to \Lambda_L$. From [14] Lemma 19, the restriction

$$H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1})) \to H^1(H, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$$

is injective. Since $[c_\mathfrak{p}]$ is nontrivial in $H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$, the homomorphism $\lambda : H \to \Lambda_L$ is nontrivial. Let I be the ideal of Λ_L generated by $\lambda(H)$. Then I is nonzero and I defines a proper closed subscheme S of $\mathrm{Spec}(\Lambda_L)$. If f is a classical Hilbert modular form in $\mathcal{F}$, then $\rho_f|_H$ is decomposable if and only if f corresponds an arithmetic point in S. Hence for any arithmetic point P of $\mathcal{F}$ outside S, which corresponds to the modular form f, the representation $\rho_f|_H$, and hence $\rho_f|_{D_\mathfrak{p}}$ is indecomposable. $\square$

Now we consider the nearly ordinary case. Let

$$\mathcal{O}_{F,p} = \varprojlim \mathcal{O}_F / p^n \mathcal{O}_F$$

be the p-adic completion of $\mathcal{O}_F$ at p, and U_F be the torsion free part of $\mathcal{O}_{F,p}^{\times}$. Then set $\Gamma = Z_1 \times U_F$ and let $\Lambda' = \mathcal{O}_K[[\Gamma]]$ be the continuous group algebra. For any finite character $\varepsilon : \Gamma \to \bar{\mathbb{Q}}_p^{\times}$, we have another character

$$\Gamma = Z_1 \times U_F \to \bar{\mathbb{Q}}_p^{\times}, (a,d) \mapsto \chi(a)^{\mu} d^{v} \varepsilon((a,d)),$$

which induces a homomorphism $\kappa_{n,v,\varepsilon} : \Lambda' \to \bar{\mathbb{Q}}_p$.

We briefly recall the definition of nearly ordinary Hecke algebras defined in [18] Section 1. For any $\alpha \geq 1$, set $U_\alpha = U_1(\mathfrak{n}) \cap U(p^\alpha)$, and let $\mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_\Phi)$ be the $\mathcal{O}_\Phi$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_{k,w}(\mathfrak{n}p^\alpha; \mathbb{C}))$ generated by all the Hecke operators $(U_\alpha x U_\alpha)$ for $x \in U_0(\mathfrak{n}p^\alpha)$ over $\mathcal{O}_\Phi$. Set $\mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_K) = \mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} \mathcal{O}_K$. Applying the ordinary projector e_α we get the nearly ordinary Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\alpha; \mathcal{O}_K)$, and by taking limit, we have the Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$. From [18] Theorem 2.3, the Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$ are all isomorphic to each other for all pair (k,w) as $\mathcal{O}_K$-algebras and denote this algebra by $\mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$, which is a torsion free Λ'-module of finite type. Let $\mathrm{Spec}(\Lambda'_L)$ be an irreducible component of $\mathrm{Spec}(\mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K))$ and let $\mathcal{F} : \mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K) \to \Lambda'_L$ be the corresponding homomorphism. We know that Λ'_L is free of finite rank over Λ'. A $\bar{\mathbb{Q}}_p$-rational point $P \in \mathrm{Spec}(\Lambda'_L)(\bar{\mathbb{Q}}_p)$ is called an arithmetic point if the corresponding homomorphism φ_P extends $\kappa_{n,v,\varepsilon}$ for some n, v. For such an arithmetic point, the composition $\varphi_P \circ (\mathcal{F})$ gives the eigenvalues of a Hilbert modular form of weight (k,w) and tame level $\mathfrak{m}$.

For such an $\mathcal{F}$, we have a two dimensional Galois representation $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\Lambda'_L)$ such that for any prime $\mathfrak{p}$ of F over p, the restriction $\rho_{\mathcal{F}}|_{D_{\mathfrak{p}}}$ is upper triangular. Similarly to Theorem 14, we have the following result:

Theorem 15. *Suppose that $\mathcal{F}$ does not have complex multiplication, and $\mathcal{F}$ has an arithmetic point P which corresponds to a (parallel) weight 2 Hilbert modular form satisfying the condition required in Theorem 13. Then there exists a proper closed subscheme S of* $\mathrm{Spec}(\Lambda'_L)$

such that for all arithmetic points P of $\mathrm{Spec}(\Lambda'_L)$ *outside S which corresponds to a classical form f, the representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable, where ρ_f is the Galois representation associated to f.*

5.3.0.2 Application to a question of Coleman

In the rest of this chapter, we work with elliptic modular forms. Let $p > 3$ be a prime number and N be a positive number prime to p. For each integer k we use $M_k^\dagger(\Gamma_1(N))$ (resp. $S_k^\dagger(\Gamma_1(N))$) to denote the space of overconvergent p-adic modular forms (resp. cuspforms) of level N over $\mathbb{C}_p$ (see [27] for the definitions). In [5] Proposition 6.3, Coleman proved that there is a linear map $\theta^{k-1} : M_{2-k}^\dagger(\Gamma_1(N)) \to M_k^\dagger(\Gamma_1(N))$ such that the effect of θ^{k-1} on the q-expansions is given by the differential operator $(q\frac{d}{dq})^{k-1}$. Also there is an operator U on $M_k^\dagger(\Gamma_1(N))$ such that if $F(q) = \Sigma_{n\geq 0} a_n q^n$ is an overconvergent modular form, then $U(F)(q) = \Sigma_{n\geq 0} a_{pn} q^n$. Recall that if F is a generalized eigenvector for U with eigenvalue λ in the sense that there exists some $n \geq 1$ such that $(U-\lambda)^n(F) = 0$, then the p-adic valuation of λ is called the slope of F. From [5] Lemma 6.3, if $f \in S_k^\dagger(\Gamma_1(N))$ is a normalized classical eigenform of slope strictly smaller than $k-1$, then f cannot be in the image of θ^{k-1}. On the other hand, a classical eigenform cannot have slope larger than $k-1$. Then it remains to consider the remaining boundary case; i.e. overconvergent modular forms of slope one less than the weight. In [5] Proposition 7.1, Coleman proved that for $k \geq 2$, every classical CM cuspidal eigenform of weight k and slope $k-1$ is in the image of θ^{k-1}. Then he asked whether there is non-CM classical cusp forms in the image of θ^{k-1}. Since the only possible slope for new forms of weight k is $\frac{k}{2}-1$ (see [12] Section 4), it's enough to consider old forms.

Let $g = \Sigma_{n\geq 1} a_n q^n$ be a classical normalized eigenform of level N and weight $k \geq 2$. Denote by $K_g = \mathbb{Q}(a_n | n = 1, 2, \ldots)$ the Hecke field of g, which is known to be a number field. For each prime $\mathfrak{p}$ of K_g over the rational prime p, it induces an embedding $i_\mathfrak{p} : K_g \to \bar{\mathbb{Q}}_p$ and let $v_\mathfrak{p}$ be the corresponding valuation on K_g. Then we can regard g as a modular form over $\bar{\mathbb{Q}}_p$ by $i_\mathfrak{p}$. As explained in [12] Section 4, one can attach to g two oldforms on $\Gamma_1(N) \cap \Gamma_0(p)$ whose slopes add up to $k-1$. When the eigenform g is $\mathfrak{p}$-ordinary; i.e. $v_\mathfrak{p}(a_p) = 1$, one of the associated oldforms has slope 0 and the other has slope $k-1$. We denote the latter oldform by f. What we can prove is the following:

Proposition 5.17. *Let g be a weight two normalized classical cusp eigenform on $\Gamma_1(N)$ with the Hecke field K_g. Suppose that there exists a prime $\mathfrak{p}$ of K_g over the rational prime p such that g is $\mathfrak{p}$-ordinary, and the associated slope one oldform f is in the image of the operator θ. Then g is a CM eigenform.*

Proof. Let $\rho_{g,\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{g,\mathfrak{p}})$ be the $\mathfrak{p}$-adic Galois representation attached to g. As explained in [10] Proposition 1.2 or [13] Proposition 11, when f is in the image of θ, the restriction of $\rho_{g,\mathfrak{p}}$ to an inertia group I_p of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p splits as the direct sum of the trivial character and the character χ_p, where χ_p is the p-adic cyclotomic character. Then from Theorem 13, the eigenform g must have complex multiplication. □

Remark 5.18. In [10] Theorem 1.3, Emerton proved that if the assumption in the above proposition is true for all primes $\mathfrak{p}$ of K_g over p, then g is a CM eigenform. Hence the above proposition can be regarded as an improvement of his theorem. Also in [13] Section 6, Ghate discussed the case when p divides the level N. In this case he explained that one can also attach to the eigenform g a primitive form f with the same weight and level as g. Then he proved that f is in the image of θ if and only if the restriction of $\rho_{g,\mathfrak{p}}$ to the inertia group I_p splits (we need to emphasize here that Ghate's argument works for all weights, but we restrict ourselves to the weight two case where Theorem 13 is applicable). Hence the result in Theorem 13 also applies and the above proposition still holds in this case.

CHAPTER 6

Mumford-Tate conjecture for abelian fourfolds

6.1 Background about the Mumford-Tate conjecture

First we summarize the known results towards the Mumford-Tate conjecture.

Let A be an abelian variety of dimension d over a number field F. Fix an embedding $F \hookrightarrow \mathbb{C}$ and an algebraic closure $\bar{F}$ of F.

The singular homology group $V = H_1(A(\mathbb{C}), \mathbb{Q})$ is a $2d$-dimensional vector space over $\mathbb{Q}$. Then we have the Hodge decomposition $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$, such that $\overline{V^{-1,0}} = V^{0,-1}$. We define a cocharacter $\mu_\infty : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{Aut}_{\mathbb{C}}(V_{\mathbb{C}})$ such that any $z \in \mathbb{C}^\times$ acts on $V^{-1,0}$ by multiplication by z^{-1} and acts trivially on $V^{0,-1}$.

Definition 6.1. *The Mumford-Tate group of the abelian variety $A_{/F}$ is the smallest algebraic subgroup $\mathrm{MT}(A) \subset \mathrm{Aut}_{\mathbb{Q}}(V)$ defined over $\mathbb{Q}$ such that the cocharacter μ_∞ factors through $\mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{C}$.*

For any rational prime l, let $T_lA(\bar{F})$ be the l-adic Tate module of A and set $V_l = T_lA(\bar{F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which is a $2d$-dimensional vector space over $\mathbb{Q}_l$. Then we have a Galois representation:

$$\rho_l : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathbb{Q}_l}(V_l).$$

We define $G_{l/\mathbb{Q}_l}$ as the Zariski closure of the image of ρ_l inside $\mathrm{Aut}_{\mathbb{Q}_l}(V_l)$ and let $G^\circ_{l/\mathbb{Q}_l}$ be its identity component. From Faltings' theorem, the group $G^\circ_{l/\mathbb{Q}_l}$ is reductive.

The Mumford-Tate conjecture predicts that

Conjecture 6.2. *For any prime l, we have the equality $G^\circ_{l/\mathbb{Q}_l} = \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_l$.*

Deligne proved the following:

Theorem 16. *For any prime l, we have the inclusion $G^{\circ}_{l/\mathbb{Q}_l} \subseteq \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_l$.*

The Mumford-Tate conjecture for abelian varieties with trivial endomorphism algebras was first studied by Serre. In [43],[45] and [46], he proved the Mumford-Tate conjecture for such abelian varieties whose dimensions satisfy certain numerical conditions. In this thesis we assume that $d = 4$ and the abelian variety $A_{/F}$ is absolutely simple. Then the absolute endomorphism algebra $\mathrm{End}^{\circ}(A_{/\bar{F}})$ is a division algebra. In [32], Moonen and Zarhin proved that in almost all cases, the endomorphism algebra $\mathrm{End}^{\circ}(A_{/\bar{F}})$ together with its action on the Lie algebra $\mathrm{Lie}(A_{/\bar{F}})$ uniquely determines the Lie algebras of the Mumford-Tate group $\mathrm{MT}(A)_{/\mathbb{Q}}$ and the reductive group $G^{\circ}_{l/\mathbb{Q}_l}$. Then only exception happens when $\mathrm{End}^{\circ}(A_{/\bar{F}}) = \mathbb{Q}$. In this case, there are two possibilities for the Lie algebra of $\mathrm{MT}(A)_{/\mathbb{Q}}$ together with its action on V (resp. the Lie algebra of $G^{\circ}_{l/\mathbb{Q}_l}$ together with its action on V_l):

1. $\mathfrak{c} \oplus \mathfrak{sp}_4$ with the standard representation, where $\mathfrak{c}$ is the 1-dimensional center $\mathfrak{c}$ of the Lie algebra;

2. $\mathfrak{c} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, with the 1-dimensional center $\mathfrak{c}$, and the representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is the tensor product of the standard representation of $\mathfrak{sl}_2$.

Together with Theorem 16, to prove the Mumford-Tate conjecture for simple 4-dimensional abelian varieties, it is enough to prove the following:

Theorem 17. *Let $A_{/F}$ be an abelian variety of dimension 4 over a number field F. Suppose that $\mathrm{End}(A_{/\bar{F}}) = \mathbb{Z}$. If for some prime l, the group $G^{\circ}_{l/\mathbb{Q}_l}$ together with its action on V_l belongs to the second case listed above, then the same is true for the the group $\mathrm{MT}(A)_{/\mathbb{Q}}$ together with its action on V, i.e. the Mumford-Tate conjecture holds for $A_{/F}$.*

Theorem 17 is the second main result in this thesis. We give a sketch of proof before we proceed to the serious proof.

Suppose that the abelian variety $A_{/F}$ satisfies $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\bar{\mathbb{Q}}_l$. By a theorem of Pink, there exists a set V of finite places of F of density 1 at which the abelian variety $A_{/F}$ has good ordinary reduction. For $v \in V$, let k_v be the residue field of F at v with characteristic $p = p_v$ and we use A_{v/k_v} to denote the reduction of $A_{/F}$ at v. If the abelian variety $A_{/F}$ comes from a Shimura curve $Z \hookrightarrow A_{4,1,n}$ where $A_{4,1,n}$ is the Siegel moduli space of principally polarized 4-dimensional abelian varieties with a suitable level structure, then $A_{v/\bar{k}_v}$ gives a closed ordinary point x_v of Z. As Z is a Shimura variety of Hodge type, from a result of Noot ([31]Theorem 4.2), it is formally linear at x_v, i.e. the formal completion of Z at x_v is a formal torus of rank 1. Since the abelian variety $A_{/F}$ gives a non-torsion point on this formal torus, this torus can be determined by the Serre-Tate coordinates of the abelian variety $A_{/F}$.

Conversely, we start with an abelian variety whose Galois representation is of type (2) and we try to prove that it comes from a Shimura curve. Let D_v (resp. I_v) be the decomposition (resp. inertia) group of $\mathrm{Gal}(\bar{F}/F)$ at v. After choosing a suitable symplectic $\mathbb{Z}_p$-basis of the p-adic Tate module $\mathrm{T}_pA(\bar{F})$, the local Galois representation is of the shape:

$$\begin{aligned} \rho_p : \quad I_v \quad &\longrightarrow \mathrm{GSp}_8(\mathbb{Z}_p) \\ \sigma \quad &\longmapsto \begin{pmatrix} \chi_p(\sigma)\cdot \mathrm{I}_n & B(\sigma) \\ 0 & \mathrm{I}_n \end{pmatrix}, \end{aligned}$$

where I_4 is the 4×4 identity matrix, $B(\sigma) = (b_{ij}(\sigma))_{1\leq i,j\leq 4}$ is a symmetric 4×4 matrix depending on σ and $\chi_p : I_v \longrightarrow \mathbb{Z}_p^\times$ is the p-adic cyclotomic character.

In [40], Noot gave a detailed analysis of the isogeny types of the abelian variety $A_{v/\bar{k}_v}$ and the local Galois representation $\rho_p : I_v \longrightarrow \mathrm{GSp}_8(\mathbb{Z}_p)$. He proved that for any Frobenius element $Frob_v \in D_v$, the element $\rho_p(Frob_v) \in G_p(\mathbb{Q}_p)$ generates a maximal torus of $G_{p/\mathbb{Q}_p}$. Also he got a control on the image $\rho_p(I_v)$. This information imposes restrictive conditions on the 1-cocycles b_{ij}'s. Recall that we have studied the relationship of the Serre-Tate coordinates of $A_{/F}$ and the local representation ρ_p. Based the results in chapter 3, we get an explicit description on the Serre-Tate coordinates of $A_{/F}$. In particular, we show that $A_{/F}$ sits in a rank 1 formal subtorus of the local deformation moduli space of the abelian variety of $A_{v/\bar{k}_v}$.

Finally we consider the torsion points on the formal torus we get in the previous step. These points correspond to the quasi-canonical CM liftings of the abelian variety $A_{v/\bar{k}_v}$ in the sense of [31] Definition 2.9. Given the analysis of the Serre-Tate coordinates explained as above, we can use the Mumford-Tate groups of these abelian varieties to generate a candidate of the Mumford-Tate group of $A_{/F}$ and then construct a Shimura curve which contains all these quasi-canonical liftings of $A_{v/\bar{k}_v}$. Then by formal linearity of Shimura varieties of Hodge type, we see that $A_{/F}$ comes from this Shimura curve and then we conclude.

6.2 Reductions of abelian varieties with Galois representations of Mumford's type

Definition 6.3. *Let K be a field of characteristic 0 and fix an algebraic closure $\bar{K}$ of K. Let $G_{/K}$ be an algebraic group and let V be a finite dimensional K-vector space with a faithful representation of G. We say that the pair (G, V) is of Mumford's type if the following three conditions are satisfied:*

1. $\mathrm{Lie}(G)$ *has one dimensional center* $\mathfrak{c}$*;*

2. $\mathrm{Lie}(G)_{\bar{K}} \cong \mathfrak{c}_{\bar{K}} \oplus \mathfrak{sl}^3_{2,\bar{K}}$*;*

3. $\mathrm{Lie}(G)_{\bar{K}}$ *acts on* $V_{\bar{K}}$ *by the tensor product of the standard representations of* $\mathfrak{sl}_{2,\bar{K}}$*.*

For any semisimple group $G_{/K}$, there exist (up to isomorphism) a simply connected group $\tilde{G}$ (resp. adjoint group G^{ad}) such that there exists central isogenies $\tilde{G} \to G$ (resp. $G \to G^{ad}$) over K.

Let F be a number field and $A_{/F}$ be a four dimensional abelian variety. Let $G_F = \mathrm{Gal}(\bar{F}/F)$ be the Galois group of F and we use v to indicate a finite place of F and use p_v to denote its residue characteristic. Let F_v be the completion of F at v and $G_{F_v} \subseteq G_F$ be the decomposition group at v. Let k_v be the residue field whose cardinality is q_v. Fix a Frobenius element Frob_v at v. Let l be any prime number. Recall that we have the

Galois representation $\rho_l : G_F \to \mathrm{Aut}_{\mathbb{Q}_l}(V_l)$ and G_l is the Zariski closure of the image of ρ_l in $\mathrm{Aut}_{\mathbb{Q}_l}(V_l)$. From a result of Serre ([38] Theorem 3.6), replacing F by a finite extension if necessary, we can assume that the algebraic group $G_{l,\mathbb{Q}_l}$ is connected for every prime l and it is a reductive group by a result of Faltings ([38] Corollary 5.8).

From [40] Lemma 1.3, we know that if the pair (G_l, V_l) is of Mumford's type for one prime l, then the same is true for all primes, and we have $\mathrm{End}(A_{/\bar{F}}) = \mathbb{Z}$.

Definition 6.4. *If the abelian variety $A_{/F}$ has the property that the pair (G_l, V_l) is of Mumford's type for some prime l, we say that $A_{/F}$ is an abelian variety with Galois representation of Mumford's type.*

From [40] Corollary 2.2, if $A_{/F}$ is an abelian variety with Galois representation of Mumford's type, it has potentially good reduction at all places of F. Hence replacing F by a finite extension, we can assume that $A_{/F}$ has good reduction everywhere.

For any finite place v of F, we choose a semisimple element $t_v \in \mathrm{GL}_8(\mathbb{Q})$ such that its characteristic polynomial is equal to the characteristic polynomial of the element $\rho_l(\mathrm{Frob}_v)$. By Weil's theorem, the conjugacy class of element t_v in $\mathrm{GL}_8(\mathbb{Q})$ exists and does not depend on l. Let $T_v \subseteq \mathrm{GL}_{8,\mathbb{Q}}$ be the Zariski closure of the subgroup generated by t_v, which is unique up to conjugation in $\mathrm{GL}_{8,\mathbb{Q}}$. From [38] Theorem 3.7, we have:

Theorem 18. *There exists a set V_{max} of finite places of F of Dirichlet density 1, such that for all $v \in V_{max}$, we have:*

1. *the group $T_{v,\mathbb{Q}_l}$ is connected and hence a torus;*
2. *for any $l \neq p_v$, the torus $T_{v,\mathbb{Q}_l}$ is conjugate to a maximal torus of $G_{l/\mathbb{Q}_l}$ under $\mathrm{GL}_8(\mathbb{Q}_l)$.*

As $A_{/F}$ is an abelian variety with Galois representation of Mumford's type, for each prime l, the root system of the simple factors of $G_{l,\bar{\mathbb{Q}}_l}$ has type A_1. In particular, the abelian variety $A_{/F}$ satisfies the hypothesis in [38] Theorem 7.1 and it follows that there exists a subset $V_{good} \subseteq V_{max}$ of finite places of F with Dirichlet density 1 such that $A_{/F}$ has ordinary reduction at v for all $v \in V_{good}$.

Fix a place $v \in V_{good}$. First we want to study the isogeny type of the reduction A_{v/k_v} of $A_{/F}$ at v. From [40] Lemma 1.3, there exist infinitely many primes l's such that the derived group G_l^{der} of G_l is $\mathbb{Q}_l$-simple. We fix such a prime $l \neq p_v$. Then we have:

Proposition 6.5. *Let $\bar{k}_v$ be an algebraic closure of k_v. Then the reduction $A_{v/\bar{k}_v}$ is either simple or isogenous to a product of an elliptic curve and a simple abelian threefold. In particular, the eigenvalues of the Frobenius $Frob_v$ on V_l are all distinct.*

The above proposition is an immediate consequence of [40] Proposition 4.1. But to establish notations used in our later argument, we give a sketchy proof here.

Proof. Let $\rho_{v,l} : G_{F_v} \to G_l(\mathbb{Q}_l)$ be the local l-adic Galois representation attached to V_l. From the proof of [40] Proposition 4.1, replacing F by a finite extension if necessary, we can assume the following conditions:

1. the cardinality q_v of the residue field k_v is a perfect square;
2. for any $\sigma \in G_{F_v}$, we have the congruence $\rho_{v,l}(\sigma) \equiv \mathrm{I}_8 \ (mod\ l^2)$ in $G_l(\mathbb{Z}_l)$;
3. all the simple factors of $A_{v/\bar{k}_v}$ are defined over k_v.

Recall that $\tilde{G}_{l/\mathbb{Q}_l}$ is the simply connected group with a central isogeny $\tilde{G}_l \to G_l$. From the second assumption above, the representation $\rho_{v,l} : G_{F_v} \to G_l(\mathbb{Z}_l)$ lifts uniquely to a representation $\tilde{\rho}_{v,l} : G_{F_v} \to \tilde{G}_l(\mathbb{Z}_l)$.

Now set $\pi = \tilde{\rho}_{v,l}(\mathrm{Frob}_v) \in \tilde{G}(\mathbb{Q}_l)$, and let $\tilde{T}$ be the Zariski closure of the subgroup of $\tilde{G}(\mathbb{Q}_l)$ generated by π, which is a connected torus. We can assume that the residue field k_v has even degree over its prime field and hence its cardinality q_v is a perfect square. Set $\alpha = \frac{\pi}{\sqrt{q_v}} \in \tilde{G}(\mathbb{Q}_l)$ and let $\tilde{T}'$ be the Zariski closure of the subgroup of $\tilde{G}(\mathbb{Q}_l)$ generated by α. Then $\tilde{T} \cong \mathbb{G}_{m,\mathbb{Q}_l} \times \tilde{T}'$ for some torus $\tilde{T}'$ of the derived subgroup $\tilde{G}^{der}$ of $\tilde{G}$. Let $\bar{T}_{\mathbb{Q}_l}$ be a maximal torus of $\tilde{G}^{der}$ containing $\tilde{T}'$.

As the pair (G_l, V_l) is of Mumford's type, from the above construction, the torus $\bar{T}_{/\mathbb{Q}_l}$ has rank 3 and we have an isomorphism $X(\bar{T}) \cong \mathbb{Z}^3$ such that the weights of the representation of

$\bar{T}_{/\mathbb{Q}_l}$ on V_l correspond to $(\pm 1, \pm 1, \pm 1) \in \mathbb{Z}^3$. The evaluation at the element $\alpha \in \bar{T}(\mathbb{Q}_l)$ gives an additive map $ev : X(\bar{T}) \to (\bar{\mathbb{Q}}_l)^*$. As $\tilde{T}'$ is a subtorus of $\bar{T}$, the restriction gives a natural surjection $X(\bar{T}) \to X(\tilde{T}')$, whose kernel is the same as the kernel of the map ev. Hence we have an injective map $ev' : X(\tilde{T}') \to (\bar{\mathbb{Q}}_l)^*$. By construction, the values $ev((\pm 1, \pm 1, \pm 1))$ are exactly the eigenvalues of α on V_l, and hence they are all in $\bar{\mathbb{Q}}$ and have absolute value 1. The injection $X(\tilde{T}') \to (\bar{\mathbb{Q}})^*$ gives an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which extends the $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$-action. It follows that actually the torus $\tilde{T}'$ is defined over $\mathbb{Q}$ and the map $ev' : X(\tilde{T}') \to (\bar{\mathbb{Q}}_l)^*$ takes values in $(\bar{\mathbb{Q}})^*$ and is $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant. The decomposition group $G_{\mathbb{Q}_l}$ acts on the character group $X(\bar{T}) \cong \mathbb{Z}^3$ through the group $\{\pm 1\}^3 \rtimes S_3$ and similarly the Galois group $G_{\mathbb{Q}}$ acts on the character group $X(\tilde{T}')$ in a similar way.

We fix an embedding $i_{p_v} : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_{p_v}$ which induces a p_v-adic valuation v_{p_v} on $\bar{\mathbb{Q}}$, normalized by $v_{p_v}(q_v) = 1$ and define $\varphi_v = v_{p_v} \circ ev' : X(\tilde{T}') \to \mathbb{Q}$, which is $\mathbb{Z}$-linear.

When the reduction A_{v/k_v} is ordinary, from the argument in [40] Proposition 4.1, we see that $\ker(ev)$ is trivial, i.e. $X(\bar{T}) = X(\tilde{T}')$ and hence $\bar{T} = \tilde{T}'$. Under the isomorphism $X(\bar{T}) = X(\tilde{T}') \cong \mathbb{Z}^3$, the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $X(\tilde{T}')$ permutes the set $\{(\pm 1, \pm 1, \pm 1)\}$ and induces a group homomorphism

$$h_v : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \{\pm 1\}^3 \rtimes S_3 = \mathrm{Aut}(X(\tilde{T}')).$$

As the derived group $G^{der}_{l/\mathbb{Q}_l}$ is assumed to be $\mathbb{Q}_l$-simple, the image of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ under h_v contains a cycle in S_3 of length 3. Hence the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the set $\{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\} \subset X(\tilde{T}')$. On the other hand, any complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X(\tilde{T}')$ by multiplication by -1. So we have the following possibilities:

1. the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set $\{ev((\pm 1, \pm 1, \pm 1))\} \subset \bar{\mathbb{Q}}$ is transitive. In this case, the abelian variety $A_{v/\bar{k}_v}$ is simple and $ev((1, 1, 1)) \in \bar{\mathbb{Q}}$ generates a CM field of degree 8 over $\mathbb{Q}$;

2. the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set $\{ev((\pm 1, \pm 1, \pm 1))\} \subset \bar{\mathbb{Q}}$ has two orbits: $\{ev((1, 1, 1), ev((-1, -1, -1))\}$ and

$\{ev((1,1,-1)), ev((1,-1,1)), ev((-1,1,1)), ev((1,-1,-1)), ev((-1,1,-1)), ev((-1,-1,1))\}$.
In this case the abelian variety $A_{v/\bar{k}_v}$ is isogenous to a product of an elliptic curve and a simple abelian threefold. The element $ev((1,1,1)) \in \bar{\mathbb{Q}}$ generates a quadratic field over $\mathbb{Q}$ and $ev((1,1,-1)) \in \bar{\mathbb{Q}}$ generates a CM field of degree 6 over $\mathbb{Q}$.

□

We keep the notation as in the above proof. Since the abelian variety A_{v/k_v} is ordinary, its slopes are 0 and 1, each of which has multiplicity 4. On the other hand, the slopes of A_{v/k_v} are given by the values $\{v_{p_v}(\sqrt{q_v}) \cdot \varphi_v((\pm 1, \pm 1, \pm 1))\}$. Hence the set $\{\varphi_v((\pm 1, \pm 1, \pm 1))\}$ takes values in the set $\{\pm \frac{1}{2}\}$. Then we can choose an isomorphism $X(\bar{T}) \cong \mathbb{Z}^3$ such that $\varphi_v((1,1,1)) = \frac{1}{2}$. As the map $\varphi_v : \bar{T} = X(\tilde{T}')$ is additive, we have

$$\varphi_v((1,1,-1)) + \varphi_v((1,-1,1)) + \varphi_v((-1,1,1)) = \frac{1}{2}.$$

It follows that one of the three numbers $\varphi_v((1,1,-1)), \varphi_v((1,-1,1)), \varphi_v((-1,1,1))$ is $-\frac{1}{2}$ and the other two are $\frac{1}{2}$. Without loss of generality, we can assume that $\varphi_v((1,1,-1)) = -\frac{1}{2}$. Then $\varphi_v((1,0,0)) = \frac{1}{2}$ and $\varphi_v((0,1,0)) = \varphi_v((0,0,1)) = 0$.

Now consider the composition:

$$\bar{h}_v : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{h_v} \{\pm 1\}^3 \rtimes S_3 \to S_3,$$

where the second map the the natural projection. Define a number field $K(v)$ in $\bar{\mathbb{Q}}$ as the fixed field of the group $H_v = \bar{h}_v^{-1}(\{id, (23)\}) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (H_v is the subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which fixes the first component of $X(\bar{T}) \cong \mathbb{Z}^3$). As the image of $\bar{h}_v$ contains a cycle of length 3 in S_3, $K(v)$ is a cubic field. Since the image of any complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ under h_v is $((-1,-1,-1), id) \in \{\pm 1\}^3 \rtimes S_3$, the field $K(v)$ is necessarily totally real.

If we consider another place $v' \in V_{good}$, we can get another totally real cubic field $K(v')$ by the same construction as above. The fields $K(v)$ and $K(v')$ are isomorphic. In fact, from the proof of 6.5, we see that $\tilde{T}_{/\mathbb{Q}_l} \subset \tilde{G}_{l/\mathbb{Q}_l}$ is a maximal torus and we can consider the associated reduced root system Ψ. As the torus $\tilde{T}$ can be defined over $\mathbb{Q}$, we have the continuous group homomorphism $h_v : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\Psi) \cong \{\pm 1\}^3 \rtimes S_3$. If we consider another place $v' \in V_{good}$, we have another maximal torus $\tilde{T}'_{/\mathbb{Q}_l} \subset \tilde{G}_{l/\mathbb{Q}_l}$ which can be defined over $\mathbb{Q}$. As the tori $\tilde{T}$ and $\tilde{T}'$ are conjugate over $\bar{\mathbb{Q}}_l$ inside $\tilde{G}_l(\bar{\mathbb{Q}}_l)$, it induces an isomorphism between the root data associated to these two tori. Such an isomorphism is unique up to conjugation by elements in the Weyl group $W(\Psi)$ of Ψ. Let $\mathrm{Out}(\Psi) = \mathrm{Aut}(\Psi)/W(\Psi)$ be the outer automorphism group of Ψ, which is isomorphic to S_3. Then the composite $\bar{h}_v : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\Psi)$ does not depend on the choice of the maximal torus $\tilde{T}$. Hence $\bar{h}_v$ is independent of v and so is the cubic field $K(v)$. In the following, we just denote this field by K.

Set $H'(v) = h_v^{-1}(\{1, \pm 1, \pm 1\} \rtimes \{id, (23)\}) \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and let $L(v)$ be the fixed field of $H'(v)$ inside $\bar{\mathbb{Q}}$. Then $L(v)/K$ is necessarily a quadratic extension. Moreover, as any complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X(\bar{T}) \cong \mathbb{Z}^3$ by inversion, one can check that $L(v)$ is a CM field by direct calculation.

As the torus $\tilde{T} = \mathbb{G}_m \times \tilde{T}' = \mathbb{G}_m \times \bar{T}$ is generated by $\pi = \tilde{\rho}_{v,l}(Frob_v)$, from the above construction, we see that $\bar{T}_{/\mathbb{Q}_l} = \tilde{T}'_{/\mathbb{Q}_l} = T'_{L(v)/\mathbb{Q}_l}$ and $\tilde{T}_{/\mathbb{Q}_l} = \mathbb{G}_{m,\mathbb{Q}_l} \times T'_{L(v)/\mathbb{Q}_l}$. Here $T'_{L(v)/\mathbb{Q}}$ is a torus defined over $\mathbb{Q}$ such that $T'_{L(v)}(\mathbb{Q}) = \{x \in L^\times | \mathrm{Norm}_{L(v)/K}(x) = 1\}$.

Since the subset V_{good} of finite places has Dirichlet density 1, we can find a place $v \in V_{good}$ over a rational prime $p = p_v$ such that p splits completely in the cubic totally real field K and for simplicity we write $L = L(v)$. So there are three different places $v = v_1, v_2, v_3$ of K lying over p. Since we fix an embedding $i_p = i_{p_v} : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, we have three embeddings $\varphi_1, \varphi_2, \varphi_3 : K \to \bar{\mathbb{Q}}$ such that φ_i induces the place v_i for $i = 1, 2, 3$. As L/K is a totally imaginary quadratic extension, we can denote the embeddings of L to $\bar{\mathbb{Q}}$ by $\psi_i, \bar{\psi}_i : L \to \bar{\mathbb{Q}}, i = 1, 2, 3$ such that $\psi_i, \bar{\psi}_i$ extend the embedding φ_i for $i = 1, 2, 3$.

Now recall that in the proof of Proposition 6.5 we considered the element $\alpha = \frac{\pi}{\sqrt{q_v}} \in \tilde{T}'(\mathbb{Q}) = T'_L(\mathbb{Q}) \subseteq L^\times$, which satisfies:

$$v_p(\psi_1(\alpha)) = \frac{1}{2}, v_p(\bar{\psi}_1(\alpha)) = -\frac{1}{2},$$

and

$$v_p(\psi_i(\alpha)) = v_p(\bar{\psi}_i(\alpha)) = 0$$

for $i = 2, 3$. This implies that the place $v = v_1$ of K splits into two different places $w_1, \bar{w}_1$ of L. Since p splits in K, we see that the w_1-adic (resp. $\bar{w}_1$-adic) completion of L is isomorphic to $\mathbb{Q}_p$. We keep the choice of the place v and the above property will be used in later argument.

6.3 Linear relations of the Serre-Tate coordinates

Fix the place v as in the preceding section and set $p = p_v$. We then have the Galois representation attached to the p-adic Tate module of $A_{/F}$:

$$\rho_p : \mathrm{Gal}(\bar{F}/F) \to G_p \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V_p).$$

In this section we want to study the local Galois representation

$$\rho_{v,p} : D_v = \mathrm{Gal}(\bar{F}_v/F_v) \to G_p \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V_p)$$

and its restriction to the inertia group $I_v \subset D_v$. As the abelian variety $A_{/F}$ has good reduction at v, the representation $\rho_{v,p}$ is crystalline with Hodge-Tate weight 0 and 1.

6.3.0.3 Filtered modules and Newton cocharacters

First we recall the notions of filtered modules and Newton cocharacters.

Let Rep_{D_v} be the tannakian category of all finite dimensional continuous representation of the decomposition group D_v over $\mathbb{Q}_p$ and let $((V_p))$ be the full tannakian subcategory of Rep_{D_v} generated by V_p. Let $\mathrm{Vec}_{\mathbb{Q}_p}$ be the category of finite dimensional $\mathbb{Q}_p$-vector spaces, and we have the forgetful functor $\omega_{V_p} : ((V_p)) \to \mathrm{Vec}_{\mathbb{Q}_p}$, which is a fiber functor of the tannakian categories. The automorphism group $H_{V_p} = \mathrm{Aut}^{\otimes}(\omega_{V_p})$ of the fiber functor ω_{V_p} can be identified with the Zariski closure of the image of the local Galois representation $\rho_{v,p}$.

Let $\sigma : F_v \to F_v$ be the Frobenius automorphism. By p-adic Hodge theory, one can associate a filtered module M_p to the crystalline representation V_p. The filtered module M_p is a finite dimensional F_v-vector space with a σ-linear automorphism $Fr_{M_p} : M_p \to M_p$. Let MF_{F_v} be the tannakian category of weakly admissible filtered modules over F_v, and let $((M_p))$ be the full tannakian subcategory of MF_{F_v} generated by M_p. Then we have the forgetful functor $\omega_{M_p} : ((M_p)) \to \mathrm{Vec}_{F_v}$, which is a fiber functor of the tannakian categories. Let $H_{M_p} = \mathrm{Aut}^{\otimes}(\omega_{M_p}) \subset \mathrm{Aut}_{F_v}(M_v)$ be the automorphism group of the fiber functor ω_{M_v} defined over F_v. Then from [8] Theorem 3.2, the algebraic group H_{M_p} is an inner form of $H_{V_p} \times_{\mathbb{Q}_p} F_v$. Hence we can identify $(H_{M_p})_{\bar{\mathbb{Q}}_p}$ with $(H_{V_p})_{\bar{\mathbb{Q}}_p}$.

Let $m_v = [F_v : \mathbb{Q}_p]$. Then the morphism $Fr_{M_p}^{m_v} : M_p \to M_p$ is F_v-linear, and it gives a $\mathbb{Q}$-grading

$$M_p = \bigoplus_{i \in \mathbb{Q}} M_{p,i},$$

such that the eigenvalues of $Fr_{M_p}^{m_v}$ on $M_{p,i}$ has valuation $m_v i$ (the valuation on $\bar{\mathbb{Q}}_p$ is normalized so that the valuation of p is 1). Then we can define the Newton cocharacter of M_p:

$$\mu_{M_p,F_v} : \mathbb{G}_{m,F_v} \to H_{M_p,F_v},$$

such that $\mathbb{G}_{m,F_v}$ acts on $M_{p,i}$ by $(\cdot)^{m_v i}$.

6.3.0.4 Application to the study of local Galois representations

As mentioned above, the algebraic group H_{M_p,F_v} is an inner form of $H_{V_p} \times_{\mathbb{Q}_p} F_v$, the cocharacter $\mu_{M_p,\bar{\mathbb{Q}}_p} = \mu_{M_p,F_v} \times_{F_v} \bar{\mathbb{Q}}_p$ gives a cocharacter

$$\mu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to H_{V_p,\bar{\mathbb{Q}}_p} \hookrightarrow G_{p,\bar{\mathbb{Q}}_p}.$$

As we have the central isogenies $\tilde{G}_{p,\bar{\mathbb{Q}}_p} \cong \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \times (\mathrm{SL}_{2,\bar{\mathbb{Q}}_p})^3 \to G_{p,\bar{\mathbb{Q}}_p}$, there exists a positive integer k such that $\mu^k : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to G_{p,\bar{\mathbb{Q}}_p}$ can be lifted to a homomorphism:

$$\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to \tilde{G}_{p,\bar{\mathbb{Q}}_p},$$

where $\tilde{\mu}_0 : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to \mathbb{G}_{m,\bar{\mathbb{Q}}_p}$ and $\tilde{\mu}_i : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to \mathrm{SL}_{2,\bar{\mathbb{Q}}_p}$, $i = 1, 2, 3$ are homomorphism of algebraic groups.

Hence there exists $n_0, n_1, n_2, n_3 \in \mathbb{Q}$, such that the slopes of M_p are given by the numbers $n_0 \pm n_1 \pm n_2 \pm n_3$. From the argument in the proof of Proposition 3.2 in [40], we have $n_0 = \frac{1}{2}$. Moreover, when $A_{/F}$ has good ordinary reduction at v, i.e. the Newton polygon of M_p is $4 \times 0, 4 \times 1$, we have that one of the three numbers n_1, n_2, n_3 is $\frac{1}{2}$ and the other two are 0. Without loss of generality, we can assume that $n_1 = \frac{1}{2}$ and $n_2 = n_3 = 0$.

On the other hand, by a theorem of Katz-Messing ([26] 1.3.5) we have the following:

Theorem 19. *The characteristic polynomial of $Fr_{M_p}^{m_p}$ on M_p is equal to the characteristic polynomial of $\rho_l(Frob_v)$ for any $l \neq p$.*

First we assume that all the eigenvalues of $\rho_l(Frob_v)$ are in $\mathbb{Z}_p$. In this case we have an explicit expression for the Serre-Tate coordinates of $A_{/F}$. By this assumption and our choice of the place v together with the above theorem, we see that the 8 eigenvalues of $Fr_{M_p}^{m_p}$ on M_p are all distinct and lie in $\mathbb{Z}_p$. As the reduction of $A_{/F}$ at v is ordinary, we can choose a symplectic basis $\{v_1^\circ, v_2^\circ, v_3^\circ, v_4^\circ, v_1^{et}, v_2^{et}, v_3^{et}, v_4^{et}\}$ of the p-adic Tate module $T_pA(\bar{\mathbb{Q}})$, under which the local Galois represenation is of the shape:

$$\begin{aligned} \rho_{v,p}: \quad D_v = \mathrm{Gal}(\bar{F}_v/F_v) &\to \mathrm{GSp}_8(\mathbb{Z}_p), \\ D_v \ni \quad \sigma &\mapsto \begin{pmatrix} T_1(\sigma) & B(\sigma) \\ 0 & T_2(\sigma) \end{pmatrix}, \end{aligned}$$

where $B(\sigma) = (b_{ij}(\sigma))_{1 \le i,j \le 4}$ is a matrix in $M_{4\times4}(\mathbb{Z}_p)$ depending on σ, and $T_1(\sigma), T_2(\sigma)$ are diagonal matrices of the shapes:

$$T_1(\sigma) = \begin{pmatrix} (\chi_p\psi^{-1}_{(-1,1,1)})(\sigma) & 0 & 0 & 0 \\ 0 & (\chi_p\psi^{-1}_{(-1,1,-1)})(\sigma) & 0 & 0 \\ 0 & 0 & (\chi_p\psi^{-1}_{(-1,-1,1)})(\sigma) & 0 \\ 0 & 0 & 0 & (\chi_p\psi^{-1}_{(-1,-1,-1)})(\sigma) \end{pmatrix},$$

and

$$T_2(\sigma) = \begin{pmatrix} \psi_{(-1,1,1)}(\sigma) & 0 & 0 & 0 \\ 0 & \psi_{(-1,1,-1)}(\sigma) & 0 & 0 \\ 0 & 0 & \psi_{(-1,-1,1)}(\sigma) & 0 \\ 0 & 0 & 0 & \psi_{(-1,-1,-1)}(\sigma) \end{pmatrix}.$$

Here $\chi_p : D_v \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $\psi_{(i,j,k)} : D_v \to \mathbb{Z}_p^\times$ is the unramified character which sends $Frob_v$ to the element $\sqrt{q_v} \cdot ev(i,j,k)$ for $(i,j,k) \in \{\pm 1, \pm 1, \pm 1\}$ defined in section 6.2.

Now we consider the Hodge cocharacter $\mu_{HT} : \mathbb{G}_{m,\mathbb{C}_p} \to G_{p,\mathbb{C}_p}$ associated to the Galois representation V_p. From Sen's theory ([44] Theorem 2), the Zariski closure of the image of μ_{HT} in G_p over $\mathbb{Q}_p$ is equal to the Zariski closure of $\rho_{v,p}(I_v)$ inside $G_{/\mathbb{Q}_p}$, which is denoted by $H'_{V_p/\mathbb{Q}_p}$.

Consider the representation

$$\rho^{ad}_{v,p} : I_v \xrightarrow{\rho_{v,p}} G_p(\mathbb{Q}_p) \to G^{ad}_p(\mathbb{Q}_p).$$

From [40] Proposition 3.5, the representation $\rho^{ad}_{v,p}$ projects I_v nontrivially to exactly one of the $\mathbb{Q}_p$-simple factor of $G^{ad}_{p/\mathbb{Q}_p}$, which is denoted by $G^{ad}_{p,1/\mathbb{Q}_p}$. From [40] Proposition 3.6, when $A_{/F}$ has good ordinary reduction at v, we have an isomorphism $(G^{ad}_{p,1})_{\bar{\mathbb{Q}}_p} \cong \mathrm{PSL}_{2,\bar{\mathbb{Q}}_p}$. Hence the root system of $H'_{V_p/\mathbb{Q}_p}$ is $(\pm 2, 0, 0) \in X(\bar{T})$ under the isomorphism $X(\bar{T}) \cong \mathbb{Z}^3$ defined in the previous section.

Fix a Frobenius element $Frob_v$ in D_v. As we explain above, the eigenvalues of the matrix $\rho_{v,p}(Frob_v)$ are all distinct and lie in $\mathbb{Z}_p^\times$. So we can modify the basis of $T_pA(\bar{\mathbb{Q}})$ if necessary and assume that the matrix $\rho_{v,p}(Frob_v)$ is diagonal. As $\rho_{v,p}(Frob_v)$ generates a maximal torus in $G_{p/\mathbb{Q}_p}$, by the explicit calculation of the conjugation on $\rho_{v,p}(I_v)$ by the matrix $\rho_{v,p}(Frob_v)$, we see that the entries $b_{i,5-i} : I_v \to \mathbb{Z}_p$, $i = 1,2,3,4$ of B give the weight $(2,0,0) \in X(\bar{T})$, and no entry of B gives the weight $(-2,0,0)$. Hence $b_{ij} = 0 : I_v \to \mathbb{Z}_p$ if $i + j \neq 5$,and the set $\{(b_{14}(\sigma), b_{23}(\sigma), b_{32}(\sigma), b_{41}(\sigma)) | \sigma \in I_v\}$ spans a 1-dimensional $\mathbb{Q}_p$ vector space inside $\mathbb{Q}_p^4$.

Now from the discussion in chapter 3, we see that the Serre-Tate coordinates $q(A_{/F_v};-)$: $\mathrm{Sym}(T_pA_v(k)\otimes_{\mathbb{Z}_p}T_pA_v(k))\to\widehat{\mathbb{G}}_m(W(k))$ satisfies the following properties: there exists $(\lambda_1,\lambda_2)\in\mathbb{Z}_p^2\setminus\{(0,0)\}$, such that

$$q(u_1\otimes u_4)^{\lambda_1}=q(u_4\otimes u_1)^{\lambda_1}=q(u_2\otimes u_3)^{\lambda_2}=q(u_3\otimes u_2)^{\lambda_2},$$

and

$$q(u_i\otimes u_j)=0\ for\ i+j\neq 5.$$

Let $\mathfrak{U}_{/W(k)}$ be the formal torus $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sym}^2(T_pA_v(k)),\widehat{\mathbb{G}}_m)$. The $\mathbb{Z}_p$-basis $\{u_1,u_2,u_3,u_4\}$ of $T_pA_v(k)$, we get a $\mathbb{Z}_p$-basis $\{u_i\otimes u_j|1\leq i\leq j\leq 4\}$ of $\mathrm{Sym}^2(T_pA_v(k))$. Under this basis, we have ten coordinates t_{ij}, $1\leq i\leq j\leq 4$ on $\mathcal{U}_{W(k)}$. Set $T_{ij}=t_{ij}-1$, for $1\leq i\leq j\leq 4$, and then we have an isomorphism of formal tori over $W(k)$:

$$\mathfrak{U}\to\mathrm{Spf}(W(k)[[T_{ij}]]_{1\leq i\leq j\leq 4}).$$

Now we define a rank one formal subtorus $\mathfrak{Z}_{/W(k)}$ of $\mathfrak{U}_{/W(k)}$, such that $\mathfrak{Z}$ corresponds to the formal torus $\mathrm{Spf}(W(k))[[T_{ij}]]_{1\leq i\leq j\leq 4}/(T_{11},T_{22},T_{33},T_{44},T_{12},T_{13},T_{24},T_{34},(1+T_{14})^{\lambda_1}-(1+T_{23})^{\lambda_2})$ under the above isomorphism. From the discussion in this section, we see that the abelian variety $A_{/F}$ sits on the subtorus $\mathfrak{Z}_{/W(k)}$ of $\mathfrak{U}_{/W(k)}$.

In general, we do not assume that all the eigenvalues of $\rho_l(Frob_v)$ are in $\mathbb{Z}_p$. Then we can choose a symplectic basis $\{v_1^\circ,v_2^\circ,v_3^\circ,v_4^\circ,v_1^{et},v_2^{et},v_3^{et},v_4^{et}\}$ of the p-adic Tate module $T_pA(\bar{\mathbb{Q}})$ such that the local Galois representation is of the shape:

$$\begin{aligned}\rho_{v,p}:\quad D_v=\mathrm{Gal}(\bar{F}_v/F_v)\ &\to \mathrm{GSp}_8(\mathbb{Z}_p),\\ D_v\ni\qquad \sigma\qquad &\mapsto\begin{pmatrix}T_1(\sigma)&B(\sigma)\\0&T_2(\sigma)\end{pmatrix},\end{aligned}$$

where χ_p is again the p-adic cyclotomic character, $B:D_v\to M_{4\times4}(\mathbb{Z}_p)$ is a map valued in 4×4 symmetric matrices, and $A(\cdot)$ (resp. $A^{-1}(\cdot)$): $D_v\to\mathrm{GL}_4(\mathbb{Z}_p)$ is an unramified homomorphism which send any Frobenius $Frob_v\in D_v$ to a matrix $A\in\mathrm{GL}_4(\mathbb{Z}_p)$ (resp. $A^{-1}\in\mathrm{GL}_4(\mathbb{Z}_p)$). From the discussion in section 6.2, there exists a Galois extension $M/\mathbb{Q}_p$

with degree at most 4, such that all the eigenvalues of $\rho_l(Frob_v)$ are in M. Then we can find a matrix $W \in \mathrm{GL}_4(\mathcal{O}_M)$ such that

$$WAW^{-1} = \begin{pmatrix} \psi_{(-1,1,1)}(Frob_v) & 0 & 0 & 0 \\ 0 & \psi_{(-1,1,-1)}(Frob_v) & 0 & 0 \\ 0 & 0 & \psi_{(-1,-1,1)}(Frob_v) & 0 \\ 0 & 0 & 0 & \psi_{(-1,-1,-1)}(Frob_v) \end{pmatrix}.$$

Then we consider the conjugation of the Galois representation

$$\rho'_{v,p} = \begin{pmatrix} W & 0 \\ 0 & (W^t)^{-1} \end{pmatrix} \rho_{v,p} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^t \end{pmatrix} : \quad D_v \longrightarrow \mathrm{GSp}_8(\mathcal{O}_M),$$

$$D_v \ni \quad \sigma \quad \mapsto \begin{pmatrix} T'_1(\sigma) & B'(\sigma) \\ 0 & T'_2(\sigma) \end{pmatrix},$$

where $T'_2 : D_v \to \mathrm{GL}_4(\mathcal{O}_M)$ is an unramified homomorphism sending (any) Frobenius element to the matrix WAW^{-1}, and $T'_1 = \chi_p \cdot (T'_2)^{-1}$, and $B' = (b'_{ij})_{1\le i,j\le 4} : D_v \to M_{4\times 4}(\mathcal{O}_M)$ is a map.

Take another conjugation if necessary, we can assume that $\rho'_{v,p}(Frob_v)$ is diagonal for some Frobenius element $Frob_v \in D_v$. As $\rho'_{v,p}(Frob_v)$ generates a maximal torus of $G_{p/M}$, we can again apply Noot's results to conclude that $b'_{ij} = 0$ if $i + j \neq 5$ and the set $\{b'_{14}(\sigma), b'_{23}(\sigma), b'_{32}(\sigma), b'_{41}(\sigma) | \sigma \in I_v\}$ spans a 1-dimensional M-vector space inside M^4. For each pair $1 \le i, j \le 4$, the map $b'_{ij} : I_v \to \mathcal{O}_M$ is an $\mathcal{O}_M$-linear combination of the maps $b_{kl} : I_v \to \mathbb{Z}_p$, $1 \le k, l \le 4$. From Theorem 8 and Remark 3.2, the entries b_{kl}'s determines the Serre-Tate coordinates of $A_{/W(k)}$. Hence the above restrictive conditions on the entries b'_{kl}'s can be translated to the restrictive conditions on the Serre-Tate coordinates of $A_{/W(k)}$. It may not be obvious from this observation that we get a rank 1 formal subtorus of $\mathcal{U}_{/W(k)}$, but we will use this observation in next section.

To see that the above restrictive conditions define a rank 1 formal subtorus of $\mathcal{U}_{/W(k)}$, we use our special choice of the finite place v at the end of section 6.2. Replacing the number field by a finite extension if necessary, we can assume that the representation $\rho_{v,p} : D_v \to G_p(\mathbb{Q}_p)$ can be lifted to the semisimple group $\tilde{\rho}_{v,p} : D_v \to \tilde{G}_p(\mathbb{Q}_p)$. Consider the element $\tilde{\rho}_{v,p}(Frob_v) \in \tilde{G}_p(\mathbb{Q}_p) \subseteq \tilde{G}_p(\bar{\mathbb{Q}}_p)$. By our assumption on the algebraic group G_p, we know that over $\bar{\mathbb{Q}}_p$, we have an isomorphism:

$$\tilde{G}_{p/\bar{\mathbb{Q}}_p} \cong \mathrm{G}_{m/\bar{\mathbb{Q}}_p} \times (\mathrm{SL}_{2,\bar{\mathbb{Q}}_p})^3.$$

By our choice of the place v, the projection of $\tilde{\rho}_{v,p}(Frob_v)$ to the first factor of $(\mathrm{SL}_{2,\bar{\mathbb{Q}}_p})^3$ actually sits inside $\mathrm{SL}_2(\mathbb{Q}_p)$ and hence generates a torus over $\mathbb{Q}_p$. On the other hand, from the previous discussion, the conjugation action of the maximal torus $\bar{T}_{/\mathbb{Q}_p}$ generated by $\rho_{v,p}(Frob_v)$ on the group $\rho(I_v)$ can only gives the root $(2,0,0) \in X(\bar{T})$. From the general theory of reduction groups (see [49]), the Lie algebra of G_p on which the maximal torus $\bar{T}$ acts through the root $(2,0,0) \in X(\bar{T})$ has dimension 1 over $\mathbb{Q}_p$. This mean that the set $\{b_{ij}(\sigma) | 1 \le i,j \le 4, \sigma \in I_v\}$ lies in a 1-dimensional $\mathbb{Q}_p$ vector space of $\mathbb{Q}_p^{16}$. Again from Theorem 8 and Remark 3.2, we see that the above conditions define a rank 1 formal subtorus $\mathfrak{Z}$ of $\mathcal{U}_{/W(k)}$.

6.4 Conclusion

In this section, we prove the main result Theorem 17 in this chapter. It is enough to prove the following:

Theorem 20. *Let F be a numbe field. If $A_{/F}$ is an abelian variety with Galois representation of Mumford's type, then $A_{/F}$ come from a Shimura curve constructed by Mumford in [34]. In particular, the Mumford-Tate conjecture holds for $A_{/F}$.*

Replacing F by a finite extension if necessary, we can assume that $A_{/F}$ has a principal polarization $\lambda : A \to A^t$ and the algebraic groups $G_{l/\mathbb{Q}_l}$ are connected for all primes l. For each integer $N \ge 3$, we choose a symplectic level N structure η_N of $A_{/F}$. Then the triple $(A_{/F}, \lambda, \eta_N)$ gives an F-valued point x on the Siegel moduli space $\mathcal{A}_{1,N}$.

Now recall that we choose a finite place v of F in section 6.2 at which the abelian variety

$A_{/F}$ has good ordinary reduction. Let k_v be the residue field of F at v with characteristic $p = p_v$ and fix an algebraic closure k of k_v. The reduction $A_{v/k}$ of $A_{/F}$ at v gives a closed point $x_v \in \mathcal{A}_{1,N}(k)$. As explained in section 4.3, the formal completion $\mathfrak{U}_{/W(k)}$ of $\mathcal{A}_{1,N}$ along the closed point x_v has a formal group structure and is isomorphic to

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sym}(\mathrm{T}_p A_v(k)) \otimes_{\mathbb{Z}_p} \mathrm{T}_p A_v(k), \widehat{\mathbb{G}}_m)_{/W(k)}.$$

In the previous section, we determine a rank 1 formal subtorus $\mathfrak{Z}$ of $\mathfrak{U}_{/W(k)}$ on which the point x lies.

In section 6.3, we fix a basis $\{v_1^\circ, \ldots, v_4^\circ, v_1^{et}, \ldots, v_4^{et}\}$ of $\mathrm{T}_p A(\bar{\mathbb{Q}}_p)$ such that $\{v_1^\circ, \ldots, v_4^\circ\}$ is a basis of $\mathrm{T}_p \widehat{A}(\bar{\mathbb{Q}}_p)$, and $\{v_1^{et}, \ldots, v_4^{et}\}$ corresponds to a basis $\{u_1, \ldots, u_4\}$ of $\mathrm{T}_p A_v(k)$ under the reduction map. Moreover, this basis is symplectic in the sense that under the Weil pairing $E_{p^\infty} : \mathrm{T}_p A(\bar{\mathbb{Q}}_p) \times \mathrm{T}_p A(\bar{\mathbb{Q}}_p) \to \mathrm{T}_p \mu_{p^\infty}(\bar{\mathbb{Q}}_p)$ induced from the polarization λ, we have $E_{p^\infty}(v_i^\circ, v_j^{et}) = \zeta_{p^\infty}$ if $i = j$ and $E_{p^\infty}(v_i^\circ, v_j^{et}) = 1$ if $i \neq j$, where ζ_{p^∞} is a fixed basis of $\mathrm{T}_p \mu_{p^\infty}(\bar{\mathbb{Q}}_p)$.

The basis $\{v_1^\circ, \ldots, v_4^\circ, v_1^{et}, \ldots, v_4^{et}\}$ gives a full level structure at p of the abelian variety $A_{/F}$ $\eta_{p^\infty} : L_p \to \mathrm{T}_p A(\bar{\mathbb{Q}}_p)$ such that $\eta_{p^\infty}(e_i) = v_i^\circ$, for $1 \leq i \leq 4$, and $\eta_{p^\infty}(e_i) = v_i^{et}$, for $5 \leq i \leq 8$. The level structure η_{p^∞} induces an isomorphism $W_p = L_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{T}_p A(\bar{\mathbb{Q}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_p$, which gives an isomorphism of algebraic groups: $\mathrm{Aut}_{\mathbb{Q}_p}(W_p) \to \mathrm{Aut}_{\mathbb{Q}_p}(V_p)$. As $G_{p/\mathbb{Q}_p}$ is an algebraic subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(V_p)$, we can regard $G_{p/\mathbb{Q}_p}$ as a subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(W_p)$ under the above isomorphism.

Now let $(A_{can/W(k)}, \lambda_{can}, \eta_{N,can})$ be the canonical lifting of x_v, which corresponds to the identity element in the group $\mathfrak{U}(W(k))$. From [31] Lemma 2.8, the abelian variety A_{can} has complex multiplication and hence is defined over some number field F_1.

Fix a complex embedding $\iota : F_1 \hookrightarrow \mathbb{C}$ and set $A_{can/\mathbb{C}} = A \times_{F_1,\iota} \mathbb{C}$. Let $H_1(A_{can/\mathbb{C}}, \mathbb{Q}) = V_{can}$ be the first rational homology group of $A_{can/\mathbb{C}}$ and let $\mathrm{MT}(A_{can})_{/\mathbb{Q}} \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}(V_{can})$ be the Mumford-Tate group of A_{can}. On the other hand, fix an algebraic closure $\bar{F}_1$ of F_1. Let $\mathrm{T}_p A_{can}(\bar{F}_1)$ be the p-adic Tate module of A_{can} and set $V_{can,p} = \mathrm{T}_p A_{can}(\bar{F}_1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By comparison theorem, we have an isomorphism $V_{can} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to V_{can,p}$, which induces an isomorphism of algebraic groups: $\mathrm{Aut}_{\mathbb{Q}}(V_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Aut}_{\mathbb{Q}_p}(V_{can,p})$.

Now we give a full level structure at p of A_{can/F_1}. Recall that $A_{can/W(k)}$ is the canonical lifting of the ordinary abelian variety $A_{v/k}$. Then connected-étale exact sequence of Barsotti-Tate groups

$$0 \to \widehat{A}_{can} \to A_{can}[p^\infty] \to \mathrm{T}_p A_v(k) \times_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

splits over $W(k)$. As we have an inclusion from F_1 to the fractional field of $W(k)$, it induces a finite place v_1 of F_1 over p. Let $I \subset \mathrm{Gal}(\bar{F}_1/F_1)$ be the inertia group at v_1. The above splitting exact sequence of Barsotti-Tate groups gives a splitting of the exact sequence of the p-adic Tate modules as I-modules:

$$0 \to \mathrm{T}_p\widehat{A}_{can}(\bar{\mathbb{Q}}_p) \to \mathrm{T}_p A_{can}(\bar{\mathbb{Q}}_p) \to \mathrm{T}_p A_v(k) \to 0.$$

Under the Weil pairing on $\mathrm{T}_p A_{can}(\bar{\mathbb{Q}}_p)$ induced from the polarization λ_{can} of A_{can}, we can choose a symplectic basis $\{v^\circ_{1,can}, \ldots, v^\circ_{4,can}, v^{et}_{1,can}, \ldots, v^{et}_{4,can}\}$ of $\mathrm{T}_p A_{can}(\bar{\mathbb{Q}}_p)$ such that $\{v^\circ_{1,can}, \ldots, v^\circ_{4,can}\}$ is a basis of $\mathrm{T}_p\widehat{A}_{can}(\bar{\mathbb{Q}}_p)$, and $\{v^{et}_{1,can}, \ldots, v^{et}_{4,can}\}$ corresponds to the basis $\{u_1, \ldots, u_4\}$ of $\mathrm{T}_p A_v(k)$ under the splitting of the above exact sequence. This symplectic basis allows us to endow A_{can/F_1} with a full level structure at p $\eta_{can,p^\infty} : L_p \to \mathrm{T}_p A_{can}(\bar{\mathbb{Q}}_p)$ such that $\eta_{can,p^\infty}(e_i) = v^\circ_{i,can}$, for $1 \le i \le 4$, and $\eta_{can,p^\infty}(e_i) = v^{et}_{i,can}$, for $5 \le i \le 8$. By inverting p, the level structure η_{can,p^∞} gives an isomorphism $W_p = L_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{T}_p A_{can}(\bar{\mathbb{Q}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{p,can}$.

Then we define an embedding of algebraic groups over $\mathbb{Q}_p$:

$$i_{can} : \mathrm{MT}(A_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Aut}_{\mathbb{Q}}(V_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Aut}_{\mathbb{Q}_p}(V_{p,can}) \to \mathrm{Aut}_{\mathbb{Q}_p}(W_p).$$

Similarly, for each p-th power root of unity $\zeta \in \bar{\mathbb{Q}}_p$, let $x_\zeta = (A_{\zeta/\mathcal{R}}, \lambda_\zeta, \eta_{N,\zeta}) \in \mathfrak{Z}(\zeta)$ be any nontrivial torsion point on the rank 1 formal torus $\mathfrak{Z}$ where $\mathcal{R}$ is a finite flat $W(k)$-algebra. From [31] Lemma 2.8 and Definition 2.9, the abelian scheme $A_{\zeta/\mathcal{R}}$ is a quasi-canonical lifting of $A_{v/k}$ and has complex multiplication. In particular A_ζ is defined over some number field F_1'.

As before, let $V_\zeta = H_1(A_{\zeta/\mathbb{C}}, \mathbb{Q})$ be the first rational homology group of A_ζ and let $\mathrm{MT}(A_\zeta)_{/\mathbb{Q}} \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}(V_\zeta)$ be its Mumford-Tate group. As $A_{\zeta/\mathcal{R}}$ is a lifting of the ordinary abelian variety $A_{v/k}$, we can choose a symplectic basis $\{v^{\circ}_{1,\zeta}, \dots, v^{\circ}_{4,\zeta}, v^{et}_{1,\zeta}, \dots, v^{et}_{4,\zeta}\}$ with respect to the Weil pairing induced from the polarization λ_ζ such that $\{v^{et}_{1,\zeta}, \dots, v^{et}_{4,\zeta}\}$ corresponds to the basis $\{u_1, \dots, u_4\}$ of $\mathrm{T}_p A_v(k)$ under the reduction map. This basis gives a full level structure at p of A_ζ which induces an isomorphism $W_p \to \mathrm{T}_p A_\zeta(\bar{\mathbb{Q}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{p,\zeta}$. Similarly, we define an embedding of algebraic groups over $\mathbb{Q}_p$:

$$i_\zeta : \mathrm{MT}(A_\zeta) \times_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Aut}_{\mathbb{Q}}(V_\zeta) \times_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Aut}_{\mathbb{Q}_p}(V_{p,\zeta}) \to \mathrm{Aut}_{\mathbb{Q}_p}(W_p).$$

From the above construction we have:

Lemma 6.6. *The embeddings i_{can} and i_ζ's factor through $G_{p/\mathbb{Q}_p}$.*

Proof. As the canonical lifting can also be regarded as a quasi-canonical lifting, we only prove that i_ζ factors through $G_{p/\mathbb{Q}_p}$. Recall that the quasi-canonical lifting $A_{\zeta/\mathcal{R}}$ has complex multiplication and hence is defined over some number field F'. Fix an algebraic closure $\bar{F}'$ of F'. Under the symplectic basis $\{v^{\circ}_{1,\zeta}, \dots, v^{\circ}_{4,\zeta}, v^{et}_{1,\zeta}, \dots, v^{et}_{4,\zeta}\}$ of $\mathrm{T}_p A_\zeta(\bar{F}')$, we can consider the Galois representation $\rho_\zeta : \mathrm{Gal}(\bar{F}'/F') \to \mathrm{GSp}_8(\mathbb{Z}_p)$. Let $G_{\zeta/\mathbb{Q}_p}$ be the Zariski closure of the image of ρ_ζ inside $\mathrm{GSp}_{8/\mathbb{Q}_p}$. As the Mumford-Tate conjecture is known to be true for abelian varieties with complex multiplication, from the construction of the embedding i_ζ, the image of i_ζ is nothing but $G_\zeta(\mathbb{Q}_p)$.

As we have an embedding from F' to the quotient field of $\mathcal{R}$, it induces a p-adic place v' of F'. Let $D_{v'} \subseteq \mathrm{Gal}(\bar{F}'/F')$ (resp. $I_{v'} \subseteq \mathrm{Gal}(\bar{F}'/F')$) be the decomposition group (resp. inertia group) at v'. The local Galois representation $\rho_{\zeta,v'} = \rho_\zeta|_{D_{v'}}$ is of the shape:

$$\begin{aligned} \rho_{\zeta,v'} : \quad D_{v'} &\to \mathrm{GSp}_8(\mathbb{Z}_p), \\ D_{v'} \ni \quad \sigma &\mapsto \begin{pmatrix} T_1(\sigma) & B_\zeta(\sigma) \\ 0 & T_2(\sigma) \end{pmatrix}, \end{aligned}$$

where T_1, T_2 have the same meaning as the previous section, and $B_\zeta(\sigma) \in M_{4\times 4}(\mathbb{Z}_p)$ is a matrix depending on σ. Since the quasi-canonical lifting A_ζ comes from the rank 1 formal subtorus $\mathfrak{Z}_{/W(k)}$, we see that if we consider the conjugation of $\rho_{\zeta,v'}$:

$$\rho'_{\zeta,v'} = \begin{pmatrix} W & 0 \\ 0 & (W^t)^{-1} \end{pmatrix} \rho_{\zeta,v'} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^t \end{pmatrix} : \quad D_v \to \mathrm{GSp}_8(\mathcal{O}_M),$$

$$D_v \ni \sigma \mapsto \begin{pmatrix} T'_1(\sigma) & B'_\zeta(\sigma) \\ 0 & T'_2(\sigma) \end{pmatrix},$$

then we have $b'_{\zeta,ij} = 0$ if $i+j \neq 5$ and the set $\{b'_{\zeta,14}(\sigma), b'_{\zeta,23}(\sigma), b'_{\zeta,32}(\sigma), b'_{\zeta,41}(\sigma) | \sigma \in I_v\}$ lies in the same 1-dimensional vector space in M^4 as in the previous section. Here $B'_\zeta = (b'_{\zeta,ij})_{1\leq i,j\leq 4}$ are the entries of the matrix B'_ζ. In particular,we see that the local Galois representation $\rho_{\zeta,v'}$ factors through $G_p(\mathbb{Q}_p)$.

On the other hand, from the analysis in Section 6.2, the special fiber $A_{v/k}$ is either a product of an elliptic curve and a simple abelian threefold, or s simple abelian fourfold. From the analysis of the isogeny type of $A_{v/k}$ in section 6.2, the Mumford-Tate group of A_ζ is contained in the torus $\mathbb{G}_{m/\mathbb{Q}} \times T'_{L/\mathbb{Q}}$, which is a rank 4 torus (here recall that $T'_{L/\mathbb{Q}}$ is a torus such that $T'_L(\mathbb{Q}) = \{x \in L^\times | \mathrm{Norm}_{L/K}(x) = 1\}$). But from the calculation in [32] Section 7, the Mumford-Tate group of A_ζ is either a rank 4 torus or a rank 5 torus. Hence we must have the equality $\mathrm{MT}(A_\zeta) = \mathbb{G}_{m/\mathbb{Q}} \times T'_{L/\mathbb{Q}}$. On the Galois side, we see that the algebraic group $G_\zeta(\mathbb{Q}_p)$ is the Zariski closure of the image of the local representation $\rho_{\zeta,v'}$ and is generated by $\rho_{\zeta,v'}(Frob_{v'})$ for any Frobenius element in $D_{v'}$.

Combining the above facts together, we see that the embedding i_ζ factors through $G_p(\mathbb{Q}_p)$. □

We fix a compatible sequence $(\zeta_n)_{n\geq 1}$ of p-th power roots of unity in the sense that ζ_n is a primitive p^n-th root of unity and $\zeta_n^p = \zeta_{n-1}$ for each n.

As the above construction is valid for any integer N prime to p, we have $\bar{\mathbb{Q}}$-valued point $x_{can} = \varprojlim_{(p,N)=1}(A_{can}, \lambda_{can}, \eta_{N,can}, \eta_{can,p^\infty}) \in Sh(\bar{\mathbb{Q}})$ (corresponding to the canonical lifting of x_v) and $x_{\zeta_n} = \varprojlim_{(p,N)=1}(A_{\zeta_n}, \lambda_{\zeta_n}, \eta_{N,\zeta_n}, \eta_{\zeta_n,p^\infty}) \in Sh(\bar{\mathbb{Q}})$ (corresponding to the quasi-canonical liftings of x_v). As the abelian variety $A_{can/\bar{\mathbb{Q}}}$ is the canonical lifting of the ordinary abelian variety $A_{v/k}$, it has complex multiplication by a CM-algebra $M = \mathrm{End}^\circ(A_{v/k}) =$

$\mathrm{End}(A_{v/k}) \otimes_{\mathbb{Z}} \mathbb{Q}$. From the reciprocity law at special points ([20] section 7.2.2), we have an embedding of groups: $T_M = \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m(\mathbb{Q}) \to \mathrm{GSp}_8(\mathbb{A}^{(\infty)})$ which acts on the Shimura variety $Sh_{/\bar{\mathbb{Q}}}$ which stabilizes the point x_{can} and acts transitively on the set $\{x_{\zeta_n}|n \geq 1\}$.

As T_M is a $\mathbb{Q}$-torus, the closure of $\{x_{\zeta_n}|n \geq 1\}$ in $Sh(\mathbb{C})$ under the complex topology is contained in a set Ω homeomorphic to $(S^1)^n$ for some $n \geq 1$, where $S^1 = \{z \in \mathbb{C}||z| = 1\}$ is the unit circle in the complex place. As the set $\{x_{\zeta_n}|n \geq 1\}$ is countable, we can find a simply connected open subset $\Omega' \subseteq \Omega$ containing $\{x_{\zeta_n}|n \geq 1\}$.

Now let $f : \mathcal{A} \to Sh_{\mathbb{C}}$ be the universal abelian scheme, the restriction of the local system $\mathrm{R}^1 f_* \mathbb{Q}$ to Ω' is constant. Hence we can identify all the cohomology groups $H^1(A_{\zeta_n}, \mathbb{Q})$'s with the 8-dimensional $\mathbb{Q}$-vector space W.

Definition 6.7. *Define an algebraic group $G_{/\mathbb{Q}}$ to be the smallest algebraic subgroup of $\mathrm{Aut}_{\mathbb{Q}}(W)$ with the property that the embeddings i_{ζ_n}'s factor through $G(\mathbb{Q}_p)$ for all $n \geq 1$.*

From the above definition, the algebraic group $G_{/\mathbb{Q}}$ is an algebraic subgroup of $\mathrm{GSp}(W, \psi)$.

As the algebraic group is defined over $\mathbb{Q}$, for any field automorphism $\tau : \mathbb{Q}_p \to \mathbb{Q}_p$ (which of course fix $\mathbb{Q}$ pointwise), we have the inclusions $\tau(i_{can}(\mathrm{MT}(A_{can})(\mathbb{Q}_p))) \subseteq G_p(\mathbb{Q}_p)$ and $\tau(i_{\zeta}(\mathrm{MT}(A_{\zeta})(\mathbb{Q}_p))) \subseteq G_p(\mathbb{Q}_p)$. From our construction, the algebraic group $\mathrm{MT}(A_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p$ gives a maximal torus of $G_{p/\mathbb{Q}_p}$ under the embedding i_{can}. On the other hand, the group generated by $i_{\zeta}(\mathrm{MT}(A_{\zeta})) \times_{\mathbb{Q}} \mathbb{Q}_p$ and $i_{can}(\mathrm{MT}(A_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p)$ contains a unipotent such that the action of $i_{can}(\mathrm{MT}(A_{can}) \times_{\mathbb{Q}} \mathbb{Q}_p)$ on this unipotent by conjugation corresponds to the root $(2, 0, 0) \in X(T'_L)$ ($X(T'_L)$ is the character group of the torus T'_L). From the analysis in Section 6.2, the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the set $\{(\pm 2, 0, 0), (0, \pm 2, 0), (0, 0, \pm 2)\} \subseteq X(T'_L)$. It follows that the groups $\tau(i_{can}(\mathrm{MT}(A_{can})(\mathbb{Q}_p))) \subseteq G_p(\mathbb{Q}_p)$ and $\tau(i_{\zeta}(\mathrm{MT}(A_{\zeta})(\mathbb{Q}_p))) \subseteq G_p(\mathbb{Q}_p)$ for all τ generate the group $G_p(\mathbb{Q}_p)$ and hence we have:

Lemma 6.8. *We have the equality*

$$G \times_{\mathbb{Q}} \mathbb{Q}_p = G_p$$

over $\mathbb{Q}_p$. In other words, $G_{/\mathbb{Q}}$ is a $\mathbb{Q}$-form of the algebraic group $G_{p/\mathbb{Q}_p}$.

Now we can give a proof of Theorem 20:

Proof. Replacing the algebraic group $G_{/\mathbb{Q}}$ by the semisimple group $\tilde{G}_{/\mathbb{Q}}$ if necessary, we can assume that $G_{/\mathbb{Q}}$ is semisimple. Since the Lie algebra of G_p is isomorphic to $\mathfrak{c} \oplus \mathfrak{sl}_2^3$ (where $\mathfrak{c}$ is the one dimensional center) over an algebraic closure of $\mathbb{Q}_p$, we have an isomorphism:

$$G_{/\mathbb{R}} \cong \mathbb{G}_{m/\mathbb{R}} \times \mathrm{SL}_{2/\mathbb{R}}^i \times \mathrm{SO}_{2/\mathbb{R}}^{3-i}.$$

On the other hand, the morphism $G_{/\mathbb{Q}} \hookrightarrow \mathrm{GSp}_{8/\mathbb{Q}}$ gives faithful symplectic representation of $G_{/\mathbb{Q}}$, hence $i = 1$ or $i = 3$.

Now consider the homomorphism $h_{can} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \to \mathrm{GSp}_8(\mathbb{R})$ (resp. $h_\zeta : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \to \mathrm{GSp}_8(\mathbb{R})$) which defines the complex structure of the abelian variety A_{can} (resp. A_ζ). These homomorphisms factor through $G(\mathbb{R})$ by our construction. Let $\mathfrak{X}$ be the $G(\mathbb{R})$-conjugacy class of h_{can}. Then the pair $(G_{/\mathbb{Q}}, \mathfrak{X})$ is a Shimura datum. From [39] Lemma 3.3, for the fixed prime p,we can find a integer n prime to p, and a Shimura variety Sh_G coming from the Shimura datum $(G_{/\mathbb{Q}}, \mathfrak{X})$ by adding a sufficient deep level structure, such that there is a closed immersion $Sh_G \hookrightarrow \mathcal{A}_{1,n}$. The abelian varieties A_{can} and A_ζ's certainly lie on the Shimura variety Sh_G by our construction. Their special fiber $A_{v/k}$ gives a closed ordinary point x_v of Sh_G. From [39] Theorem 3.7 or [31] Theorem 4.2, the formal completion of Sh_G along the closed point x_v is a union of formal tori. As the canonical lifting A_{can} and the quasi-canonical liftings A_ζ's are dense in the rank 1 formal torus $\mathfrak{Z}_{/W(k)}$, $\mathfrak{Z}_{/W(k)}$ is contained in the formal completion of Sh_G along the point x_v. As the abelian variety $A_{/F}$ sits on the formal torus $\mathfrak{Z}_{/W(k)}$, it is a point of the Shimura variety Sh_G. Since the absolute endomorphism algebra of $A_{/F}$ is $\mathbb{Z}$, we see that $i = 1$ and thus

$$G_{/\mathbb{R}} \cong \mathbb{G}_{m/\mathbb{R}} \times \mathrm{SL}_{2/\mathbb{R}} \times \mathrm{SO}_{2/\mathbb{R}}^2.$$

This shows that $A_{/F}$ arises from a Shimura curve constructed by Mumford in [34], which is exactly what we want to prove. $\square$

Remark 6.9. From the above proof, we see that after constructing the reductive group $G_{/\mathbb{Q}}$, the quasi-canonical lifting A_ζ's give points on the Shimura variety Sh_G after choosing a suitable level structure. In particular, we have an embedding $i_\zeta : \mathrm{MT}(A_\zeta) \hookrightarrow G_{/\mathbb{Q}}$ for each ζ, such that $i_\zeta(\mathrm{MT}(A_\zeta))$ is a maximal torus of $G_{/\mathbb{Q}}$, and $G_{/\mathbb{Q}}$ is the smallest algebraic subgroup of $\mathrm{GSp}_{8/\mathbb{Q}}$ containing all these tori.

However, before we construct the group $G_{/\mathbb{Q}}$, it might be difficult to find an appropriate embeddings $i_\zeta : \mathrm{MT}(A_\zeta) \hookrightarrow \mathrm{GSp}_{8/\mathbb{Q}}$ which factors through $G_{/\mathbb{Q}}$. In fact, if we add a suitable level structure $\eta_{N,\zeta} : L/NL \to A_\zeta[N](\mathbb{C})$ on A_ζ, the triple $(A_{\zeta/\mathbb{C}}, \lambda_\zeta, \eta_{N,\zeta})$ gives a point on the Siegel moduli space $\mathcal{A}_{1,N}(\mathbb{C}) = \mathcal{H}_4/\Gamma(N)$, where $\Gamma(N) = \widehat{\Gamma}(N) \cap \mathrm{GSp}_8(\mathbb{Q})$. In this setting the embedding $\mathrm{MT}(A_\zeta) \hookrightarrow \mathrm{GSp}_{8/\mathbb{Q}}$ is determined up to conjugation in $\Gamma(N)$ as the isomorphism $H_1(A_{\zeta/\mathbb{C}}, \mathbb{Z}) \to L$ is so. Of course not all of these conjugations factor through the group $G_{/\mathbb{Q}}$, but it is difficult to tell which embeddings have this property as we have not constructed the group $G_{/\mathbb{Q}}$ yet. So we consider a base change. After giving a full level structure $\eta_{p^\infty,\zeta}$ at p on A_ζ, it allows us to give an embedding $i_\zeta : \mathrm{MT}(A_\zeta) \times_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow G_p$. To satisfy the last condition, we cannot chosse an arbitrary level structure. In fact, if $\eta_{p^\infty,\zeta} : L_p \to \mathrm{T}_p A_\zeta(\bar{\mathbb{Q}}_p)$ is such a level structure constructed in the proof of 6.6, all the other level structures satisfying the last condition are $\eta \circ g$, where $g \in G_p(\mathbb{Q}_p) \cap \mathrm{GSp}_8(\mathbb{Z}_p)$. As we see in the proof of 6.6, the determination of the Serre-Tate coordinates of $A_{/F}$ and the rank 1 formal torus $\mathfrak{Z}_{/W(k)}$ is crucial to find a desired level structure $\eta_{p^\infty,\zeta}$.

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