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Quasiregularly Elliptic Manifolds

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ABSTRACT OF THE DISSERTATION

Quasiregularly Elliptic Manifolds

by

Eden Prywes

Doctor of Philosophy in Mathematics

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Professor Mario Bonk, Chair

The work in this dissertation is centered around the study of quasiregularly elliptic manifolds. These are manifolds that admit quasiregular maps from Euclidean space. The research of quasiregular maps is motivated by the pursuit of extending theorems from complex analysis and conformal geometry to higher dimensional settings.

We first provide a new proof for the Rickman-Picard theorem, which states that a non-constant quasiregular map from Euclidean space to a sphere may omit a bounded number of points depending on the dilatation of the map.

We next show that a closed, connected and orientable Riemannian manifold that is quasiregularly elliptic must have bounded dimension of the cohomology independent of the distortion of the map. The bound for the dimension is sharp and proves the Bonk-Heinonen conjecture. A corollary of this theorem answers an open problem posed by Gromov in 1981. He asked whether there exists a simply connected manifold that does not admit a quasiregular mapping from Euclidean space. The result shown gives an affirmative answer to this question.

Lastly, we study the behavior of branched covers whose image of their branch set is contained in a simplicial complex. The image of the branch set of a piecewise linear branched cover between piecewise linear manifolds is a simplicial complex. We demonstrate that the reverse implication also holds. A branched cover from a sphere to a sphere with the image of the branch set contained in a codimension two simplicial complex is equivalent up to

homeomorphism to a PL mapping. This extends a result by Martio and Srebro in the three dimensional setting.

The dissertation of Eden Prywes is approved.

Monica Visan

Peter Petersen

John B. Garnett

Mario Bonk, Committee Chair

University of California, Los Angeles

2019

To my mother, Ilana Sasson.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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PUBLICATIONS

Characterization of Branched Covers With Simplicial Branch Sets, (joint work with Rami Luisto), arXiv:1809.10874, submitted for publication.

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CHAPTER 1

Introduction

A classical area of study in complex analysis is the theory of value distribution of holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$, also known as entire functions. The goal of this discipline is to study the possible behaviors of entire functions. For example, according to Liouville's theorem a bounded entire function must be constant. An extension of this theorem is Picard's theorem that states an entire function omitting more than two values in its image must be constant. Out of these results grew an entire area of research pioneered by the Finnish school of mathematics, often called Value Distribution theory or Nevanlinna theory, see [Nev70].

This dissertation focuses on how such results can be generalized to higher dimensions. Since on $\mathbb{R}^d$ there is no longer a natural complex structure, there is no notion of holomorphicity. Instead, the geometric properties of holomorphic maps can be generalized to higher dimensions. Injective holomorphic maps are conformal and non-injective holomorphic maps are conformal away from their critical points. In light of this, conformal maps provide a generalization to d dimensions. Unfortunately, this is a small family compared to conformal maps on $\mathbb{C}$. Conformal maps on $\mathbb{R}^d$ form a finite-dimensional Lie group if $d \geq 3$ (this theorem is also called Liouville's theorem). In order to have a richer class of maps, angle preservation can be weakened to bounded distortion of angles. This is the class of quasiconformal maps. They have been studied in both dimension 2 and in higher dimensions.

A K-*quasiconformal map*, for $K > 1$, is a homeomorphism $f\colon \mathbb{R}^d \to \mathbb{R}^d$ that is in the Sobolev space $W^{1,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies

$$\|Df(x)\|^d \leq K J_f(x), \tag{1.0.1}$$

for almost every $x \in \mathbb{R}^d$, where Df is the differential of f, $J_f = \det(Df)$ and $\|Df\|$ denotes the operator norm. If f is not a homeomorphism but satisfies (1.0.1), then f is called K-*quasiregular*. Inequality (1.0.1) describes the infinitesimal angle distortion properties of f. The constant K, known as the dilatation of f, describes to what extent f maps ellipsoids to spheres of eccentricity dependent on K. If $K = 1$, then f is conformal.

Quasiconformal maps were first introduced by Grötzsch [Gro28] in dimension 2. The study of quasiconformal and quasiregular maps in dimension n was developed by the Finnish school during the previous fifty years. The standard reference for n-dimensional quasiconformal maps is [V71] and for quasiregular maps is [Ric93].

A key difference between quasiregular maps and holomorphic maps is that quasiregular maps are not necessarily smooth. Due to this, different tools have been developed to study quasiregular maps that originate from real analysis and differential geometry. Even though different methods are used, many similar results from the holomorphic theory still apply. Perhaps the most famous of these generalizations is the Rickman-Picard theorem.

The Rickman-Picard theorem was originally proved by Rickman in [Ric80]. The theorem states that if $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a nonconstant K-quasiregular map, then f omits at most C values, where C is a constant that depends on d and K. A difference between this result and the classical case is that the constant depends on the dilatation K. This is unavoidable since Rickman in [Ric85] constructed quasiregular maps from $\mathbb{R}^3 \to \mathbb{R}^3$ that omit arbitrarily many points. Pankka and Drasin in [DP15] generalized this to $\mathbb{R}^d$ showing that the number of points omitted must depend on K in every dimension.

This extension of Picard's theorem has been reproven in several subsequent publications. In [EL91] and in [Lew94], Eremenko and Lewis, and Lewis proved the theorem using tools from potential theory and nonlinear elliptic partial differential equations. In [BP19], Bonk and Poggi-Corradini also used these tools, but provided a new approach in proving the theorem. In chapter 2, I give a new proof that is loosely based on the approach in [EL91]. My approach uses the language of differential forms and hence is closely connected to the material in the subsequent chapters.

The study of the value distribution of a quasiregular map leads to the question: Which

Riemannian manifolds admit nonconstant quasiregular maps from $\mathbb{R}^d$? Such manifolds are generally called *quasiregularly elliptic* and this question is the major focus of the thesis. For $d = 2$, this question simplifies to the holomorphic case. If $f\colon \mathbb{C} \to M$ is quasiregular, then $f = g \circ \phi$, where $\phi\colon \mathbb{C} \to \mathbb{C}$ is quasiconformal and $g\colon \mathbb{C} \to M$ is holomorphic (this result is sometimes known as Stoïlow's theorem, see [Sto28], [LP17]).

Picard essentially showed that if M admits a holomorphic map, then M must be homeomorphic to $\mathbb{C}$, S^2, $S^1 \times \mathbb{R}$ or $S^1 \times S^1$. The proof of this is as follows. Suppose that M does admit a holomorphic map g. By the uniformization theorem, the universal cover of M is either $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{D}$. In the first two cases, M must be one of the manifolds mentioned. If the universal cover is the unit disk $\mathbb{D}$, then g lifts to a holomorphic function $\widetilde{g}\colon \mathbb{C} \to \mathbb{D}$. By Liouville's theorem, $\widetilde{g}$ is constant and therefore g is constant as well. This would mean that M is not quasiregularly elliptic.

There is also a characterization of quasiregularly elliptic compact manifolds in dimension 3. Jormakka in [Jor88] showed, given Thurston's geometrization conjecture, that M must be homeomorphic to a quotient of S^3, $S^2 \times S^1$ or $S^1 \times S^1 \times S^1$. For dimensions d greater than 3 such characterizations do not exist. However, there are several conditions that M must satisfy. Varopolous showed that the fundamental group of M must have polynomial growth order less than d (see [VSC92]). This result crucially does not depend on the dilatation K of the mapping. Relating to this result, Gromov in [Gro81] asked whether there exists any simply-connected manifolds that are not quasiregularly elliptic.

In attempting to answer this question, Bonk and Heinonen in [BH01] studied the dimension of the de Rham cohomology of M. They showed that the dimension of the de Rham cohomology of a compact quasiregular map is bounded above by a constant depending on d and K. However, they conjectured that such a constant should be independent of K. They predicted that the d-dimensional torus is extremal. The dimension of the degree l de Rham cohomology of the torus is $\binom{d}{l}$. So they conjectured that for any quasiregularly elliptic M,

$$\dim H^l(M) \leq \binom{d}{l},$$

where $H^l(M)$ is the degree l de Rham cohomology of M. Note that the bound is independent of the dilatation of the map. Kangasniemi in [Kan17] proved a weaker version of this conjecture. He showed that if M admits a self-map $f\colon M \to M$, such that the iterates of f are all K-quasiregular, then the above bound is satisfied. Such manifolds are called *uniformly quasiregularly elliptic*. Uniform quasiregular ellipticity implies quasiregular ellipticity. In Chapter 3, I show the proof of the conjecture due to Bonk and Heinonen.

This conjecture also answers Gromov's question. Let M be the connected sum of k copies of $S^2 \times S^2$. Then M is simply-connected, but $\dim H^2(M) = 2k$. So for $k \geq 4$, M is not quasiregularly elliptic.

The proof of the Bonk-Heinonen conjecture includes an independently interesting result. I show that the Jacobian of a quasiregular map $f\colon \mathbb{R}^d \to M$ satisfies a reverse Hölder inequality as long as M has a nontrivial cohomology group of degree l, where $1 \leq l \leq d-1$. This was shown for quasiconformal maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ by Gehring in [Geh73]. Bojarski and Iwaniec showed this for quasiregular maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ in [BI83]. It is interesting to note that such an inequality fails when $f\colon \mathbb{R}^d \to S^d$.

A related question can be asked, which noncompact manifolds, M, are quasiregularly elliptic? In dimensions 2, if M is quasiregularly elliptic, then M is homeomorphic to $\mathbb{C}$ or $S^1 \times \mathbb{R}$. In higher dimensions very little is known. A cohomology dimension bound as in the compact case is not possible due to the sharpness of the Rickman-Picard theorem. The manifold $S^d \setminus \{p_1, \dots, p_k\}$ is quasiregularly elliptic for all $k \in \mathbb{N}$. However, $\dim H^{d-1}(S^d \setminus \{p_1, \dots, p_k\}) = k-1$. Additionally, the quasiregular ellipticity of M will depend on the Riemannian metric, which is not the case in the compact case. For example, in [PR11], Pankka and Rajala showed that there exists a Riemannian metric on $M = S^d \setminus L$, where L is the unknot or a Hopf link, such that M is quasiregularly elliptic. If M was given the spherical metric, then it would not admit a nonconstant quasiregular map.

The main result of Chapter 3 gives a condition that restricts the existence of quasiregular maps. In Chapter 4 the main goal is to study the behavior of certain types of quasiregular maps and give an indication of how to construct quasiregular maps. By a theorem due to Reshetnyak (see [Res89]), a quasiregular map is always open and discrete. A map is called *open* if it maps open sets to open sets and *discrete* if the preimage of a point is a discrete set. In this direction, I study a more general class of maps called branched covers.

A *branched cover* is a continuous map that is open and discrete. In dimension 2, Stoïlow's theorem (see [LP17]) states that every branched cover $f\colon \mathbb{C} \to \mathbb{C}$ is equivalent up to a homeomorphism to a holomorphic map. In this sense, this class is a topological generalization of holomorphic maps. A branched cover f has a branch set B_f that is defined to be the set of points where f is not locally injective. The restriction $f\colon \mathbb{R}^d \setminus B_f \to \mathbb{R}^d \setminus f(B_f)$ is a covering map. The study of the branch set of the map can often give information of the map itself.

The fundamental example of a branched cover is the k-winding map. In dimension 2, this map can be written in polar coordinates as $(r, \theta) \mapsto (r, k\theta)$. Topologically this is equivalent to the map $z \mapsto z^k$. If these maps are extended to $\mathbb{R}^d$ by the identity on $\mathbb{R}^{d-2}$, then they are still branched covers (the former will be quasiregular). The branch set will be $\{(0, 0) \times \mathbb{R}^{d-2}$. In dimension 2, locally every branched cover behaves as a winding map. In higher dimensions this is not always true, though there are corresponding results in special cases.

A theorem due to Černavskiĭ and Väisälä [V66] states that the topological dimension of B_f and $f(B_f)$ is bounded by $d-2$. If the image of the branch set is contained in a codimension 2 subspace, then Church and Hemmingsen [CH60] showed that the map is topologically equivalent to a winding map. This is a very specialized case since the assumption is generally not satisfied. Martio and Srebro in [MS79] generalized this to include cases where $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ and the image of the branch set is contained in a collection of rays emanating from a point.

Martio and Srebro showed that if $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a branched cover and near a point x_0, $f(B_f)$ can be embedded into a collection of rays originating at $f(x_0)$, then f is equivalent to a cone of a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. More specifically, near x_0, f is equivalent to a map $\widetilde{g}$ from the ball B^3 to itself, where $\widetilde{g}(r, z) = rg(z)$. Here, r is the radius from 0 and z is a point on $\widehat{\mathbb{C}}$. A consequence of this theorem is that if the image of the branch set of f is contained in a piecewise linear (PL) structure, then f is topologically equivalent to a piecewise linear map. In chapter 4, I show, in a collaboration with Rami Luisto, that this result can be extended to higher dimensions.

We showed that if $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a branched cover and the image of the branch set can be embedded into a simplicial complex, then f is topologically equivalent to a piecewise linear map. As long as the branch set of f satisfies the geometrical hypothesis of the theorem, the map's behavior will be entirely understood.

A corollary of this result pertains to the theory of quasiregular maps. In general, a branched cover will not satisfy the analytic properties required in the definition for quasiregularity. However, a piecewise linear map on a compact set that is also a branched cover will always be quasiregular. The theorem proven in Chapter 4 gives a way to generate piecewise linear maps, and hence quasiregular maps, from the more easily constructed family of branched coverings. An example of this is given to show that $\mathbb{CP}^d$ is quasiregular elliptic for all $d \in \mathbb{N}$.

The thesis is organized in the following way. Chapter 2 contains a new proof for the Rickman-Picard theorem. In Chapter 3, Section 3.1 has a construction for a quasiregular map $f\colon \mathbb{R}^4 \to \mathbb{CP}^2$, showing that $\mathbb{CP}^2$ is quasiregularly elliptic. In Section 3.2, I provide the proof for the theorem that states the dimension of the cohomology of a quasiregularly elliptic manifold is bounded. In Chapter 4, I present the theorem that characterizes the behavior of branched coverings with the images of their branch sets contained in simplicial structures.

CHAPTER 2

Rickman-Picard theorem

2.1 Introduction

The Rickman-Picard theorem is a generalization of the Picard theorem for entire functions.

Theorem 2.1.1 (Picard Theorem). *Let* $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ *be a nonconstant holomorphic function. Then the image of* f *omits at most 2 points.*

This theorem is sharp since e^z omits 0 and ∞. A proof of the theorem can be found in most standard Complex Analysis texts. This theorem also extends to quasiregular mappings since every quasiregular mapping f can be written as $g \circ \phi$, where g is holomorphic and $\phi\colon \mathbb{C} \to \mathbb{C}$ is quasiconformal.

This result is one of the fundamental value distribution results for entire functions. For this reason it is interesting to ask if an analogous result holds for quasiregular mappings from $\mathbb{R}^n$ to S^n.

In [Ric80], Rickman showed that a similar theorem was true for mappings from $\mathbb{R}^n$ to S^n. He proved the following theorem:

Theorem 2.1.2 (Rickman-Picard Theorem). *Let* $f : \mathbb{R}^n \to S^n$ *be a nonconstant* K*-Quasiregular mapping. Then the image of* f *omits at most* $C(n,K)$ *points, where* $C(n,K)$ *only depends on* n *and the dilatation* K *of* f*.*

This theorem is also sharp by results from Rickman in [Ric85] and Drasin and Pankka in [DP15]. Note that in this case there is a dependence on the dilatation constant of f, while in the 2-dimensional case the number of omitted values is independent of K.

There have been several proofs in the literature of Theorem 2.1.2, see [Ric80], [EL91], [Lew94], and [BP19]. The proofs in all these cases used functions similar to $\log|f|$ in order to show the theorem. The function $\log|f|$ notably is $\mathcal{A}$-harmonic, a degenerate elliptic PDE that has often been used in the context of quasiregular mappings.

In this section we will provide a new proof of Theorem 2.1.2 that uses $\mathcal{A}$-harmonic differential forms instead of $\mathcal{A}$-harmonic functions. The use of these forms is inspired by the proofs in [EL91] and [Lew94].

The notation used in this section and the subsequent chapters is as follows:

Euclidean space of dimension d will be denoted by $\mathbb{R}^n$. The d-dimensional sphere will be written as S^n. The symbol $\lesssim$ will denote "less than a constant multiple of". The symbol $\sim$ denotes $\lesssim$ and $\gtrsim$. Often we will also write $C(\cdot,\ldots,\cdot)$ for a constant. This will mean the constant depends on the terms mentioned. For $x \in \mathbb{R}^n$ and $r > 0$, the set $B(x,r) \subset \mathbb{R}^n$ denotes the ball of radius r, centered at x.

Let $\bigwedge^l(\mathbb{R}^n)$ denote the space of degree l exterior powers of the cotangent bundle of $\mathbb{R}^n$, for $1 \leq l \leq n-1$. Let $D \subset \mathbb{R}^n$ be an open domain. By $C_c^\infty(D)$, we denote the space of smooth functions with compact support in D. We say a differential form α is in $L^p(D)$, whenever the component functions of α are in the usual L^p-space. Similarly, α is in the Sobolev space $W^{1,p}(D)$ whenever the component functions are in the standard Sobolev space, i.e., $\alpha_i \in L^p(D)$ of functions and α_i has weak derivatives in $L^p(D)$ of functions. Sometimes these will be written as $L^p(\wedge^l D)$ and $W^{1,p}(\wedge^l D)$ when the degree of the form needs to be emphasized. The norm of a differential form α will be denoted by $|\alpha|$ and will refer to the pointwise ℓ^2-norm on the component functions of α.

The space M will always be a closed, connected and orientable Riemannian manifold of dimension d. By $\Omega^l(M)$, we mean the space of smooth differential forms on M of degree l. On $\Omega^l(M)$, there exists an inner product induced by the Riemannian metric on M. For $\omega \in \Omega^l(M)$, we denote by $\|\omega\|_\infty$ the L^∞-norm given by this inner product. The de Rham cohomology group of M will be denoted by $H^l(M)$. If $\alpha \in \Omega^l(M)$, then $[\alpha] \in H^l(M)$ will denote its equivalence class in the de Rham cohomology group of M.

If M is a Riemannian manifold, then the Hodge star operator, denoted by $*$, is the unique operator $*\colon L^2(\wedge^l M) \to L^2(\wedge^{n-l} M)$ such that for all $\alpha, \beta \in L^2(\wedge^l M)$,

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV,$$

for almost every point on M. Here, $\langle \cdot, \cdot \rangle$ is the inner product induced by the Riemannian metric and dV is the volume form on M.

2.2 Calculus of measurable differential forms and integral inequalities

Before proceeding to the proof of the Rickman-Picard theorem, several analysis lemmas are needed in order to apply the tools of calculus to differential forms that are in L^p-spaces.

The first lemma gives a version of the Poincaré-Sobolev inequality. Let $D \subset \mathbb{R}^n$ be an open domain and let $\alpha \in L^p(\wedge^l D)$. The form α is closed in distribution if for all smooth $(n-l-1)$-forms ϕ, with compact support,

$$\int_D \alpha \wedge d\phi = 0.$$

Lemma 2.2.1. *If D is simply connected, $l \geq 1$ and $p > 1$, then there exists $u \in W^{1,p}(\wedge^{l-1} D)$ such that $du = \alpha$ in distribution and*

$$\|u\|_{1,p} \lesssim \|\alpha\|_p.$$

Additionally, by Sobolev embedding,

$$\|u\|_{p^*} \lesssim \|\alpha\|_p,$$

where $p^* = np/(n-p)$. The constants are independent of u and α. If D is a ball, the constants are independent of D as well.

The content of this lemma is a summary of the results shown in [IL93, Section 4]. The main ingredient of their proof is the use of a bounded homotopy operator $T\colon L^p(\wedge^l D) \to$

$W^{1,p}(\wedge^{l-1}D)$ where $D \subset \mathbb{R}^n$ is a convex domain. They showed that $\alpha = Td\alpha + dT\alpha$, which in this case means $\alpha = dT\alpha = du$.

The next lemma gives a criterion for integration by parts to hold.

Lemma 2.2.2. *If $\beta \in W^{1,q}(\wedge^{n-l-1}D)$ for $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_D \alpha \wedge d\beta = 0.$$

The proof follows from Lemma 2.2.1, [GT01, Theorem 7.4] and the discussion following the theorem in the reference.

Finally, we record here a lemma regarding the pullback of differential forms by quasiregular maps. Let M and N be Riemannian manifolds and let $f\colon M \to N$ be a quasiregular map. Additionally, let $\omega \in \Omega^l(N)$ for an integer $l \in \{0, \dots, n\}$. Then the pullback of $f^*\omega$ depends on the product of l derivatives of f. Since f lies in the Sobolev space $W^{1,n}_{\mathrm{loc}}(M)$, the form $f^*\omega$ is in $L^{n/l}_{\mathrm{loc}}(M)$.

Lemma 2.2.3. *If $\omega \in \Omega^l(N)$, then*

$$d(f^*\omega) = f^*(d\omega).$$

For a proof of this lemma see [DS89, Lemma 2.22] or [IM93, Lemma 3.6].

2.3 Proof of the Rickman-Picard theorem

Suppose $f\colon \mathbb{R}^n \to S^n$ is K-quasiregular and omits the points $\{p_1, \dots, p_m, q_1, \dots, q_m\}$. To prove the theorem, it suffices to show that m is bounded by a constant that depends only on K and n.

Without loss of generality, f is uniformly Hölder continuous. Indeed, if there exists a quasiregular map $f\colon \mathbb{R}^n \to S^n$, then f can be rescaled in a way that gives a uniformly Hölder continuous map from $\mathbb{R}^n$ to S^n. Additionally, the Hölder exponent will only depend on the dilatation K of f. This is result is due to Miniowitz [Min79]. The rescaling technique is similar to the proof of Zalcman's lemma or the Bloch-Brody principle in Complex Analysis.

For a discussion of this result, see [BH01]. The proof below relies on some results relating to this rescaling. They will be stated but used without proof. For their proofs consult [BH01].

In order to prove the theorem, p-harmonic differential forms related to potentials on S^n will be constructed.

Definition 2.3.1. *A differential form ω on a Riemannian manifold M is p-harmonic if $\omega \in L^p(M)$ and*

$$d\omega = 0 \quad \textit{and} \quad d(*|\omega|^{p-2}\omega) = 0$$

in the weak sense. Here, $*$ is the Hodge star operator.

Note that if ω is a degree k-form and if $p = n/k$, then the p-harmonic equation for ω is conformally invariant. That is, if $\psi\colon M \to M$ is conformal, then $\psi^*\omega$ is still p-harmonic.

Define the following $(n-1)$ form on $S^n \setminus \{0, \infty\}$,

$$\omega(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x_i}{|x|^n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^n$$

where $|x|$ denotes the Euclidean norm in the stereographic projection of $S^n \setminus \{\infty\}$ to $\mathbb{R}^n$. The form $\omega(x) = *d(\log x)$. The function $\log x$ is n-harmonic in the sense that

$$d(*|d\log x|^{n-2}\log x) = 0.$$

This equation and the fact that $d^2 = 0$ give that

$$d\omega = 0 \quad \text{and} \quad d(*|\omega|^{n/(n-1)-2}\omega) = 0,$$

and so ω is $n/(n-1)$-harmonic.

Given $p, q \in S^n$, there exists a rotation of S^n so that q is mapped to the north pole. By a stereographic projection, p can be mapped to $\mathbb{R}^n$. Additionally, there exists a translation that maps p to 0. This gives a Möbius transformation $\psi : S^n \to S^n$ that maps p, q to $0, \infty$, respectively. Define

$$\omega_i = \psi_i^*\omega. \tag{2.3.1}$$

The map ψ_i is a Möbius transformation and therefore conformal. So ω_i is still $n/(n-1)$-harmonic. Let $\tau_i = f^*\omega_i$. To study the pullbacks of p-harmonic differential forms, it is useful to consider a new class of forms.

Definition 2.3.2. *A differential form ω on a Riemannian manifold M is $\mathcal{A}$-harmonic if for some $p \geq 1$, $\omega \in L^p(M)$ and ω satisfies*

$$d\omega = 0 \quad \textit{and} \quad d(*\mathcal{A}(\omega)) = 0$$

in the weak sense, where $\mathcal{A}$ is a measurable map from l-forms to l-forms that satisfies the following properties:

(i) For $t \in \mathbb{R}$, $\mathcal{A}(t\omega) = |t|^{p-2}t\mathcal{A}(\omega)$,

(ii) $|\mathcal{A}(\omega)| \leq c_1|\omega|^{p-1}$,

(iii) $\langle \mathcal{A}(\omega), \omega\rangle \geq c_2|\omega|^p$.

The constants $c_1 > 0$ and $c_2 > 0$ are the structure constants associated to $\mathcal{A}$.

The pullback of a p-harmonic form by a K-quasiregular map is $\mathcal{A}$-harmonic, where

$$\mathcal{A}(\omega) = \langle G^{-1}\omega, \omega\rangle^{(p-2)/2} G^{-1}\omega.$$

Here, G is defined by

$$(Df)^t Df = J_f^{2/n} G$$

and

$$\langle Gv, v\rangle \sim \|v\|^2,$$

with constants depending only on n and K. This last condition can be used to show that the properties in the definition for $\mathcal{A}$ are satisfied with structure constants depending only on n and K. So τ_i is an $\mathcal{A}$-harmonic $(n-1)$ form. That is,

$$d(*\langle G^{-1}\tau_i, \tau_i\rangle^{\frac{1}{2}(\frac{n}{n-1}-2)} G^{-1}\tau_i) = 0 \quad \text{and} \quad d\tau_i = 0,$$

The following lemma is a Caccioppoli inequality for $\mathcal{A}$-harmonic exact forms on $\mathbb{R}^n$.

Lemma 2.3.3. *If $d\alpha$ is an $\mathcal{A}$-harmonic l-form on $\mathbb{R}^n$ and $\eta \in C_c^\infty(\mathbb{R}^n)$ non-negative, then*

$$\int_{\mathbb{R}^n} |d\alpha|^{n/l}\eta^{n/l} \lesssim \int_{\mathbb{R}^n} |\alpha|^{n/l}|d\eta|^{n/l},$$

where the constant depends on l, n and K.

Proof. The weak version of the $\mathcal{A}$-harmonic equation for $d\alpha$ gives that

$$0 = \int_{\mathbb{R}^n} *\mathcal{A}(d\alpha) \wedge d(\eta^{n/l}\alpha).$$

Note that $d(\eta^{n/l}\alpha)$ is not necessarily smooth. However it is an element of $L^{n/l}(\mathbb{R}^n)$ so the weak formulation of the $\mathcal{A}$-harmonic equation is still valid. By integration by parts,

$$0 = \int_{\mathbb{R}^n} (*\mathcal{A}(d\alpha) \wedge d\alpha)\eta^{n/l} + \frac{n}{l}\int_{\mathbb{R}^n} *\mathcal{A}(d\alpha) \wedge d\eta \wedge \alpha\eta^{n/l-1}.$$

By Properties (ii) and (iii) of $\mathcal{A}$, the integrand of the first term is comparable with constants depending on n and K to $|d\alpha|^{n/l}(\eta)^{n/l}$. By Property (ii), the integrand of the second term is bounded above by $C(n,K)|d\alpha|^{n/l-1}\eta^{n/l-1}|\alpha||d\eta|$. So if the terms are rearranged the following inequality emerges:

$$\begin{aligned}\int_{\mathbb{R}^n} |d\alpha|^{n/l}\eta^{n/l}dx &\lesssim \int_{\mathbb{R}^n} |d\alpha|^{n/l-1}\eta^{n/l-1}|\alpha||d\eta|dx \\ &\leq \left(\int_{\mathbb{R}^n} |d\alpha|^{n/l}\eta^{n/l}dx\right)^{(n-l)/n}\left(\int_{\mathbb{R}^n} |d\eta|^{n/l}|\alpha|^{n/l}dx\right)^{l/n},\end{aligned}$$

by Hölder's inequality. Combining the like terms gives the lemma. ☐

A remark is that such an inequality also works for an $\mathcal{A}$-harmonic $(n-1)$-form α satisfying $d\alpha = \sigma$, with $*\sigma > 0$. The proof is the same as showing that $\mathcal{A}$-subharmonic functions satisfy a Caccioppoli inequality, see [BP19, Lemma 4.1].

we will also need to use a Poincaré-Sobolev type inequality for differential forms from Lemma 2.2.1. As mentioned in the discussion after the lemma, there exists a bounded operator $T\colon L^p(\wedge^l B) \to W^{1,p}(\wedge^{l-1}B)$. Additionally, $\tau = Td\tau + dT\tau$, which means $\tau = dT\tau$ in this setting. So

$$\left(\frac{1}{|B(0,R)|}\int_{B(0,R)} |T\tau|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leq CR\left(\frac{1}{|B(0,R)|}\int_{B(0,R)} |\tau|^p\right)^{\frac{1}{p}}, \tag{2.3.2}$$

where C depends on n. So if τ is an $(n-1)$-form that is $\mathcal{A}$-harmonic, by Lemma 2.3.3,

$$\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|\tau|^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \lesssim \frac{1}{R}\left(\frac{1}{|B(0,2R)|}\int_{B(0,2R)}|T\tau|^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}},$$

where the implicit test function used in Lemma 2.3.3 is 1 on $B(0,R)$ and 0 outside $B(0,2R)$. Crucially, the gradient of the test function is bounded above by $1/R$. By (2.3.2),

$$\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|\tau|^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \lesssim \left(\frac{1}{|B(0,2R)|}\int_{B(0,2R)}|\tau|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}. \tag{2.3.3}$$

We now proceed with the proof of the Rickman-Picard Theorem.

Proof of Theorem 2.1.2. Let $(\omega_i)_{i=1}^m$ be the differential forms described in (2.3.1) and let $\tau_i = f^*\omega_i$. By the above discussion, τ_i satisfies the reverse Hölder inequality (2.3.3) on $B(0,R)$. Let d be half the minimum (spherical) distance between the omitted points so that the sets $B(p_i,d)$, $B(q_i,d)$ are disjoint for all indices i. Define

$$A_{i,R} = f^{-1}(B(p_i,d)\cup B(q_i,d))\cap B(0,R)$$

and

$$C_{i,R} = B(0,R)\setminus A_{i,R}.$$

Let $p = n/(n-1)$ and $q = (n-1)/(n-2)$. Then by the reverse Hölder inequality (2.3.3) and Hölder's inequality,

$$\begin{aligned}\left(\sum_{i=1}^m \frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q &= \left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\sum_{i=1}^m|\tau_i|^p\mathbb{1}_{A_{i,R}}\right)^q \\ &\le \frac{|B(0,R)|^{q-1}}{|B(0,R)|^q}\int_{B(0,R)}\left(\sum_{i=1}^m|\tau_i|^p\mathbb{1}_{A_{i,R}}\right)^q.\end{aligned}$$

The sets $A_{i,R}$ are disjoint so

$$\begin{aligned}\left(\sum_{i=1}^m \frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q &\le \frac{1}{|B(0,R)|}\sum_{i=1}^m\int_{A_{i,R}}|\tau_i|^{pq} \\ &\le \frac{1}{|B(0,R)|}\sum_{i=1}^m\int_{B(0,R)}|\tau_i|^{\frac{n}{n-2}}.\end{aligned}$$

By (2.3.3),

$$\left(\sum_{i=1}^{m}\frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q \lesssim \sum_{i=1}^{m}\left(\frac{1}{|B(0,2R)|}\int_{B(0,2R)}|\tau_i|^p\right)^q$$

Note that the inequalities above only depend on the $\mathcal{A}$-harmonicity and the dimension. So the constants only depend on n and K and not on m.

Recall that f is uniformly Hölder continuous. By [BH01, Corollary 2.2], this property implies that

$$\int_{B(0,R)} J_f \lesssim R^n,$$

where the constant only depends on n and K. So

$$\begin{aligned}\int_{B(0,R)}|\tau_i|^p &= \int_{B(0,R)}|f^*\omega_i|^p(x)dx\\ &\leq K\int_{B(0,R)}\frac{1}{\min(d_{S^n}(f(x),p_i),d_{S^n}(f(x),q_i))}J_f(x)dx\\ &\lesssim \int_{B(0,R)}|x|^{\alpha n}J_f(x)dx \lesssim R^{(\alpha+1)n},\end{aligned}$$

where α is the Hölder exponent of f. So the growth of the integral is bounded polynomially and therefore is doubling outside a set of radii with finite logarithmic measure by [BH01, Lemma 4.14]. A set $E \subset [0,\infty)$ has finite logarithmic measure if

$$\int_{E\cap[1,\infty)}\frac{1}{t}dt < \infty.$$

Choosing a radius outside this finite logarithmic measure set,

$$\int_{B(0,2R)}|\tau_i|^p \lesssim \int_{B(0,R)}|\tau_i|^p,$$

and by the calculations above,

$$\left(\sum_{i=1}^{m}\frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q \lesssim \sum_{i=1}^{m}\left(\frac{1}{|B(0,2R)|}\int_{B(0,R)}|\tau_i|^p\right)^q.$$

The right hand side splits to get an integral over $A_{i,R}$ and $C_{i,R}$,

$$\begin{aligned}\left(\sum_{i=1}^{m}\frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q &\lesssim \sum_{i=1}^{m}\left(\frac{1}{|B(0,R)|}\int_{A_{i,R}}|\tau_i|^p\right)^q\\ &+\sum_{i=1}^{m}\left(\frac{1}{|B(0,R)|}\int_{C_{i,R}}|\tau_i|^p\right)^q.\end{aligned} \tag{2.3.4}$$

Again, note that the constants depend only on n and K.

The goal now is to normalize by the integrals over $A_{i,R}$. Multiplying τ_i by a constant does not change any of the inequalities proven above. So for a fixed $R > 0$, the above inequality becomes

$$m^q \lesssim m + \sum_{i=1}^{m} \left(\frac{1}{|B(0,R)|} \int_{C_{i,R}} |\tau_i|^p \right)^q \left(\frac{1}{|B(0,R)|} \int_{A_{i,R}} |\tau_i|^p \right)^{-q}. \tag{2.3.5}$$

Showing that the second term is bounded uniformly of m will imply the theorem. To prove this, the following Equidistribution Theorem, due to Pankka [Pan10, Theorem 5], is needed.

Theorem 2.3.4. *Let M be a closed, connected and oriented Riemannian manifold and let $f : \mathbb{R}^n \to M$ be K-quasiregular. Then for every $\alpha \in L^s(\wedge^n M)$ for $s > n$ and $\epsilon > 0$, there exists a set $E \subset [1, \infty)$ of finite logarithmic measure so that*

$$\left(\int_M \alpha - \epsilon \right) \int_{B(0,r)} J_f < \int_{B(0,r)} f^*\alpha < \left(\int_M \alpha + \epsilon \right) \int_{B(0,r)} J_f$$

for $r \in [1, \infty) \setminus E$.

This equidistribution theorem originates from similar results for measures due to Mattila and Rickman that can be found in [MR79]. Since the union of finitely many sets with finite logarithmic measure still has finite logarithmic measure, the above bounds apply to a finite number of differential forms simultaneously.

Since f is K-quasiregular,

$$f^*(|\omega_i|^p dV) \sim |f^*\omega_i|^p,$$

for $p = n/(n-1)$, where the constant depends only on n and K. Let ω_i' be the form ω_i restricted to $S^n \setminus (B(p_i, \rho) \cup B(q_i, \rho))$, where $\rho > 0$ is a small number to be determined. By applying Theorem 2.3.4 to $|\omega_i'|^p dV$, for all $\epsilon > 0$ and for a large radius R satisfying the conditions in Theorem 2.3.4,

$$\begin{aligned} \frac{1}{A_f(R)} \int_{A_{i,R}} |\tau_i|^p &\geq \frac{1}{A_f(R)} \int_{B(0,R) \cap f^{-1}(B(p_i,\rho) \cup B(q_i,\rho))} |\tau_i|^p \\ &\gtrsim \frac{1}{A_f(R)} \int_{B(0,R) \cap f^{-1}(B(p_i,\rho) \cup B(q_i,\rho))} f^*(|\omega_i'|^p dV), \end{aligned}$$

where

$$A_f(R) = \int_{B(0,R)} J_f.$$

The setting is now the same as in Theorem 2.3.4 and therefore

$$\frac{1}{A_f(R)} \int_{A_{i,R}} |\tau_i|^p \gtrsim \int_{(B(p_i,d)\cup B(q_i,d))\setminus(B(p_i,\rho)\cup B(q_i,\rho))} |\omega_i|^p - \epsilon.$$

Computing the last term explicitly gives that

$$\frac{1}{A_f(R)} \int_{A_{i,R}} |\tau_i|^p \gtrsim \log \frac{d}{\rho} - \epsilon.$$

On the other hand,

$$\begin{aligned} \frac{1}{A_f(R)} \int_{C_{i,R}} |\tau_i|^p &= \frac{1}{A_f(R)} \int_{A_{i,R}} |f^*\omega_i'|^p \\ &\lesssim \frac{1}{A_f(R)} \int_{A_{i,R}} f^*(|\omega_i'|^p dV). \end{aligned}$$

Applying Theorem 2.3.4 again gives that

$$\frac{1}{A_f(R)} \int_{C_{i,R}} |\tau_i|^p \lesssim \int_{S^n\setminus(B(q_i,d/2)\cup B(p_i,d/2))} |\omega_i|^p + \epsilon \sim \log \frac{1}{d} + \epsilon.$$

Recall that d was half the distance between the points $\{p_1, \ldots, p_m, q_1, \ldots, q_m\}$. So d is fixed and there exists ρ small enough satisfying

$$\frac{\log \frac{1}{d} - \epsilon}{\log \frac{d}{\rho} + \epsilon} \lesssim m^{-1/q}.$$

Therefore there exists a radius $R > 0$ so that

$$\left(\frac{1}{|B(0,R)|} \int_{C_{i,R}} |\tau_i|^p \right) \left(\frac{1}{|B(0,R)|} \int_{A_{i,R}} |\tau_i|^p \right)^{-1} \lesssim m^{-1/q}.$$

Hence, equation (2.3.5) becomes,

$$m^q \lesssim m + 1.$$

This is only possible if m is bounded by a constant depending only on n and K. $\square$

CHAPTER 3

Quasiregularly elliptic manifolds

3.1 Examples of quasiregularly elliptic manifolds

In this section $\mathbb{CP}^2$ is shown to be quasiregularly elliptic. Recall that complex projective space, $\mathbb{CP}^n$ is defined as $\mathbb{C}^{n+1}/\sim$, where $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{C}$. Below I will denote elements in $\mathbb{CP}^n$ using projective coordinates,

$$z \in \mathbb{CP}^n, z = [z_0 : z_1 : \cdots : z_n].$$

The proof that $\mathbb{CP}^2$ is quasiregularly elliptic involves constructing a quasiregular map $f\colon \mathbb{R}^4 \to \mathbb{CP}^2$. Later, in Chapter 4, it is shown that $\mathbb{CP}^n$ is quasiregularly elliptic for all $n \in \mathbb{N}$. However, it is instructive to see an example of how one constructs such maps. Note that in general a Riemannian metric must be specified when considering quasiregular ellipticity. However, when a manifold is compact, if it is quasiregularly elliptic under one Riemannian metric, then it will be quasiregularly elliptic under all Riemannian metrics.

Theorem 3.1.1. *The manifold $\mathbb{CP}^2$ is quasiregularly elliptic.*

There exists a quasiregular map $G : \mathbb{R}^4 \to S^2 \times S^2$ (see [Ric88]). The map G is constructed by first taking the covering map $p\colon \mathbb{R}^4 \to S^1 \times S^1 \times S^1 \times S^1$, which is a local isometry. There exists a 2-to-1 branched covering map from $S^1 \times S^1 \to S^2$ that has *bounded length distortion* (BLD). The class of BLD maps was first introduced by Martio and Väisälä in [MV88].

Definition 3.1.2. *A map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is L-BLD if f is continuous, open, discrete and there exists a constant $L \geq 1$ such that for any path $\gamma\colon [0,1] \to \mathbb{R}^d$,*

$$\frac{1}{L}\ell(\gamma) \leq \ell(f(\gamma)) \leq L\ell(\gamma),$$

where $\ell(\cdot)$ denotes the length of a path.

Martio and Väisälä in [MV88] showed that orientation-preserving L-BLD maps are special cases of K-quasiregular maps. Given an equivalent definition of BLD maps, see [MV88, Section 2.1], it is not difficult to show that the Cartesian product of BLD maps is BLD and hence quasiregular.

So there exists a two-to-one quasiregular map from $\mathbb{R}^4 \to S^2 \times S^2$. To show that $\mathbb{CP}^2$ is quasiregularly elliptic it therefore suffices to find a map whose domain is $S^2 \times S^2 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

I will construct a candidate for such a map in steps. The general idea behind the proof is first to construct a branched covering from $S^2 \times S^2 \to \mathbb{CP}^2$. The branched covering will not be quasiregular near its branch set. Near the branch set the map will locally behave like a power mapping, i.e., $(z, w) \mapsto (z^2, w)$, where $S^2 \times S^2 \simeq \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. Such a map is not quasiregular, but the topologically equivalent winding map,

$$(z, w) \mapsto (\frac{z^2}{|z|}, w)$$

is quasiregular. So the goal will be to locally alter the map near the branch set so the behavior is that of a winding map while ensuring that the alterations can be glued together in a well-defined manner. This gluing method is delicate since the branch set will twisted in the space.

I now present the full details of the proof. Define $f_1 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ as

$$([z_0 : z_1], [w_0 : w_1]) \mapsto [z_0 w_0 : z_0 w_1 : z_1 w_0 : z_1 w_1].$$

This map is a diffeomorphism onto the quadric $X_0 X_3 = X_1 X_2$. Projecting from a point not on the quadric to the plane $[Y_0 : Y_1 : Y_2 : 0]$ gives a map from Q to $\mathbb{CP}^2$, where

$$Q = \{[X_0 : X_1 : X_2 : X_3] \in \mathbb{CP}^3 : X_0 X_3 = X_1 X_2\}.$$

One version of this map is

$$[X_0 : X_1 : X_2 : X_3] \mapsto [X_0 : -X_1 - X_2 : X_3].$$

Composing the projection with f_1 we define f_2 as

$$([z_0 : z_1], [w_0 : w_1]) \mapsto [z_0 w_0 : -z_0 w_1 - w_0 z_1 : z_1 w_1].$$

Note that entries in the image of f_2 correspond to the coefficients of the polynomial

$$p(u, v) = (z_0 u - z_1 v)(w_0 u - w_1 v).$$

That is, $z_0 w_0$ is the coefficient of u^2, $-z_0 w_1 - w_0 z_1$ is the coefficient of uv and $z_1 w_1$ is the coefficient of v^2. From this representation of f_2, it follows that f_2 is two-to-one with a branch set equal to the diagonal $\Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$.

Outside a small open neighborhood $N(\Delta)$ of the branch set, $f_2 : \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus (N(\Delta)) \to \mathbb{CP}^2 \setminus f(N(\Delta))$ is a covering map. In fact, f_2 restricted as above is quasiregular. However, near the branch set the quasiregularity fails and so f_2 needs to be modified.

The set $N(\Delta)$ is diffeomorphic to a neighborhood of Δ in the normal bundle $\mathcal{N}\Delta$. Denote this neighborhood $U_\epsilon(\Delta) \subset \mathcal{N}\Delta$. The normal bundle of the diagonal is diffeomorphic to the tangent bundle of $\mathbb{CP}^1$ (see [BT82, Lemma 11.23]). Let $X = f(\Delta)$ and consider the diffeomorphism from a neighborhood of X in the normal bundle $\mathcal{N}X$ to a neighborhood of X in $\mathbb{CP}^2$. Let $U_\epsilon(X)$ be the neighborhood of X in $\mathcal{N}X$. This gives the following diagram:

$$\begin{tikzcd} f_2^{-1}(N(X)) \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \arrow[r, "f_2"] \arrow[d, "\phi_1"] & N(X) \subset \mathbb{CP}^2 \arrow[d, "\phi_2"] \\ U_\epsilon(\Delta) \subset \mathcal{N}\Delta \arrow[r, "\widetilde{f_2}"] & U_\epsilon(X) \subset \mathcal{N}X \end{tikzcd}$$

It is not immediately clear that ϕ_1 exists. However, f is injective on Δ and a covering map outside Δ. Additionally, by direct computation, $f_2^{-1}(N(X))$ must be contained in $N(\Delta) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ for a small enough neighborhood of X. Therefore, it can be embedded by a submersion, ϕ_1 into $\mathcal{N}\Delta$ and will be diffeomorphic to $U_\epsilon(\Delta)$.

A rank 2 vector bundle over S^2 is completely characterized up to bundle diffeomorphism by its Euler class (see [Mor01, Theorem 6.22]). In addition, the S^1-bundle generated by considering the unit circle in the vector bundle is always diffeomorphic to a lens space $L_{1/q}$, where $q \in \mathbb{Z}$ is the Euler class. If $S^3 \subset \mathbb{C}^4$, then the $1/q$-lens space can be defined as $L_{1/q} =$

$S^3/\sim$, where the equivalence relation is given by $(z_1, z_2) \sim (w_1, w_2)$ when $z_i = e^{2\pi i/q} w_i$, for $i = 1, 2$. This definition shows that $\pi_1(L_{1/q}) \cong \mathbb{Z}/q\mathbb{Z}$.

The S^1-bundle that comes from the tangent space of S^2 is diffemorphic to $L_{1/2} \simeq \mathbb{RP}^3$. So considering $\widetilde{f}_2$ on the boundary of $U_\epsilon(\Delta)$, $\widetilde{f}_2$ will be a covering map from $\mathbb{RP}^3$ to $L_{1/q}$. Since $\widetilde{f}_2$ is a two-to-one map, $\pi_1(L_{1/q})$ must contain $\mathbb{Z}/2\mathbb{Z}$ as an index 2 subgroup. However, $\pi_1(L_{1/q}) \cong \mathbb{Z}/q\mathbb{Z}$ and hence $q = 4$. In fact, up to a homeomorphism, $\widetilde{f}_2$ is the quotient map when $\mathbb{RP}^3$ and $L_{1/4}$ are considered as quotients of S^3, as described above.

I will now define two new spaces in order to more easily analyze the behavior of $\widetilde{f}_2$. It is convenient to consider S^2 at this point as the Riemann sphere $\widehat{\mathbb{C}}$. Let $E_1(S^2) = \{(u, v) \in \mathcal{N}\Delta : |u|, |v| \leq 1\}$, where u and v are the coordinates of the trivialization of $\mathcal{N}\Delta$ around $(0, 0) \in \Delta$. If (w, v) is the trivialization around $(\infty, \infty) \in \Delta$, then

$$(u, v) \sim (w, u^2 v).$$

Additionally, define $E_2(S^2) = \{(u, v) \in \mathcal{N}X : |u|, |v| \leq 1\}$, i.e., the equivalent construction but for X instead of Δ. Note that, X is diffeomorphic to Δ, but the transition maps are now,

$$(u, v) \sim (w, u^4 v).$$

As mentioned above, $\widetilde{f}_2$ is a covering map from $U_\epsilon(\Delta) \setminus \Delta$ to $U_\epsilon(X) \setminus X$. The behavior of $\widetilde{f}_2$ on the boundary of $U_\epsilon(\Delta)$ is topologically the same as $(u, v) \mapsto (u, v^2)$. Hence there exist diffeomorphisms, ψ_1, of $\overline{U_\epsilon(\Delta)}$ to the unit disk bundle over S^2 and ψ_2 of $\overline{U_\epsilon(X)}$ to the unit disk bundle over S^2 so that $f_3 = \psi_2 \circ \widetilde{f}_2 \circ \psi_1^{-1}(u, v) = (u, v^2)$ on the boundary of $E_1(S^2)$.

$$\begin{tikzcd} U_\epsilon(\Delta) \arrow[r, "\widetilde{f}_2"] \arrow[d, "\psi_1"] & U_\epsilon(X) \arrow[d, "\psi_2"] \\ E_1(S^2) \arrow[r, "f_3"] & E_2(S^2) \end{tikzcd}$$

Initially, ψ_1 exists on $U_\epsilon(\Delta) \setminus \Delta$ and ψ_2 exists on $U_\epsilon(X) \setminus X$. Since $\widetilde{f}_2$ and f_3 extend to Δ and S^2 respectively as diffeomorphisms, ψ_1 and ψ_2 extend to Δ and X respectively so that they are still both diffeomorphisms (in fact, ψ_2 can be chosen to be any diffeomorphism from $U_\epsilon(X) \to E_2(S^2)$).

Next I show that f_3 is well-defined. Consider the two charts for the normal bundle are (u, v) and (w, v) where $(w, v) \sim (u, u^2 v)$ in $E_1(S^2)$ and $(w, v) \sim (u, u^4 v)$ in $E_2(S^2)$. So

$$f_3(u, v) = (u, v^2) \sim (u, u^4 v^2) = f_3(u, u^2 v) \text{ and } (u, v) \sim (u, u^2 v).$$

(This also shows that f_3 is not quasiregular near $v = 0$.)

The map f_3 can be modified to

$$\widetilde{f_3}(u, v) = \begin{cases} \left(u, \frac{v^2}{|v|}\right) & \text{if } |u| \leq 1, \\ \\ \left(w, \frac{v^2}{|v|}\right) & \text{if } |w| \geq 1. \end{cases}$$

The two definitions are in the different trivializations of the bundle. To see what $\widetilde{f_3}$ does in the u trivialization when $|u| > 1$:

$$\widetilde{f_3}(u, v) \sim \widetilde{f_3}(u, u^2 v) = \left(u, \frac{u^4 v^2}{|u|^2 |v|}\right) \sim \left(u, \frac{v^2}{|u|^2 |v|}\right)$$

This shows that if $|u| = |w| = 1$, then the two maps agree. If $|v| = 1$, then the maps also agree since the norms on v agree when $|u| = |w| = 1$.

$\widetilde{f_3}$ is a BLD-mapping since it is a branched cover that satisfies the BLD condition. Define a map $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$. If $([z_0 : z_1], [w_0, w_1]) \notin N(\Delta)$, then

$$([z_0 : z_1], [w_0, w_1]) \mapsto [-z_0 w_1 - z_1 w_0 : z_0 w_0 : z_1 w_1]$$

Otherwise take the map

$$N(\Delta) \xrightarrow{\phi_1} U_\epsilon(\Delta) \subset \mathcal{N}\Delta \xrightarrow{\psi_1} E_1(S^2) \xrightarrow{\widetilde{f_3}} E_2(S^2) \xrightarrow{\psi_2^{-1}} U_\epsilon(X) \subset \mathcal{N}X \xrightarrow{\phi_2^{-1}} N(X). \qquad (3.1.1)$$

Each map in (3.1.1) is a diffeomorphism of compact sets except $\widetilde{f_3}$. So this is a composition of BLD mappings and therefore BLD. Hence, F is a quasiregular mapping. This concludes the proof that $\mathbb{CP}^2$ is quasiregularly elliptic.

3.2 Cohomology of quasiregularly elliptic manifolds

3.2.1 Introduction

In Section 3.1 we constructed an explicit example of a quasiregularly elliptic manifold. In this section we exhibit a condition on M which guarantees that M is not quasiregularly elliptic. The main result of this section is as follows:

Theorem 3.2.1. *Let M be a closed, connected and orientable Riemannian manifold of dimension d. If M admits a nonconstant quasiregular mapping from $\mathbb{R}^d$, then*

$$\dim H^l(M) \leq \binom{d}{l},$$

for $0 \leq l \leq d$, where $H^l(M)$ is the de Rham cohomology group of M of degree l.

Theorem 3.2.1 is the first result that gives a restriction, independent of the fundamental group of M and the distortion K of the mapping, on quasiregular ellipticity of manifolds. A K-dependent version of Theorem 3.2.1 was proved by Bonk and Heinonen [BH01]. They showed that $\dim H^l(M) \leq C(d, l, K)$ and conjectured that the constant is independent of K. Theorem 3.2.1 answers this with a sharp bound. The d-dimensional torus, $T^d = S^1 \times \cdots \times S^1$, is quasiregularly elliptic and $\dim H^l(T^d) = \binom{d}{l}$.

This theorem also leads to an answer of a longstanding open problem first posed by Gromov in 1981 [Gro81, p. 200]. He asked whether their exists a d-dimensional, simply connected manifold that does not admit a nonconstant quasiregular mapping from $\mathbb{R}^d$. Theorem 3.2.1 implies the following corollary.

Corollary 3.2.2. *The simply connected manifold $M = \#^n(S^2 \times S^2)$, the connected sum of n copies of $S^2 \times S^2$, is not quasiregularly elliptic for $n \geq 4$.*

Proof. Firstly, the 2-sphere S^2, and hence $S^2 \times S^2$, is simply connected. Furthermore, since the dimension of M is larger than 2, the connected sum of simply connected manifolds is simply connected. So M is simply connected.

The sphere S^2 satisfies $\dim H^2(S^2) = 1$. By the Künneth formula [BT82, p. 47], $\dim H^2(S^2 \times S^2) = 2$. By the Mayer-Vietoris Theorem [BT82, p. 22], $H^2(\#^n(S^2 \times S^2)) \cong \oplus^n H^2(S^2 \times S^2)$. Therefore $\dim H^2(M) = 2n > \binom{4}{2}$ for $n \geq 4$. So by Theorem 3.2.1, M is not quasiregularly elliptic. $\square$

We next outline the proof for Theorem 3.2.1. We argue by contradiction. Suppose there is a quasiregular map $f\colon \mathbb{R}^d \to M$. Fix an index $l \in \mathbb{N}$ so that $1 \leq l \leq d$. Let $k > \binom{d}{l}$ and let $\alpha_1, \ldots, \alpha_k$ be representatives of cohomology classes that form a basis in $H^l(M)$. Using Poincaré duality we can choose closed $(d-l)$-forms $\beta_1, \ldots, \beta_k$ on M such that

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij},$$

for $1 \leq i, j \leq k$ and where δ_{ij} is the Kronecker delta. In previous papers on quasiregular ellipticity, p-harmonic forms were used instead of smooth forms arising from Poincaré duality. Our approach allows us to avoid the use of this machinery.

The pullbacks, $\eta_i = f^*\alpha_i$ and $\theta_i = f^*(\beta_i)$ are closed forms on $\mathbb{R}^d$ and they satisfy local $L^{d/l}$-bounds depending on the Jacobian of f. Note that the exponent $p = d/l$ is the conformally invariant exponent for l-forms. For this reason, it is appropriate to consider the pullbacks of differential forms by a quasiregular map in this L^p-space. The uniform bounds allow us to use a rescaling procedure to obtain forms on the unit ball in $\mathbb{R}^d$ such that the wedge product of the rescaled forms is equal to 0 almost everywhere.

In the papers by Eremenko and Lewis, [EL91] and [Lew94], the authors applied a similar rescaling to $\mathcal{A}$-harmonic functions in order to prove the Rickman-Picard theorem for quasiregular mappings. Instead of rescaling functions, we consider rescalings of differential forms. We also note that Kangasniemi [Kan17] rescaled differential forms in the uniformly quasiregular case. The differential forms in his case rescale so that they are orthogonal at every point to each other. The main connection between the techniques used in this section and the above two results is that in the limit the rescaled objects obey pointwise results. This is the crucial ingredient of the proof.

The rescaling captures how $f\colon \mathbb{R}^d \to M$ behaves on average. Since quasiregular mappings have equidistribution properties similar to holomorphic mappings, f will map a large set evenly over M. This can be measured by the size of the Jacobian of f on a set. We choose a sequence of balls, B_n, so that the integral of the Jacobian of f on B_n limits to infinity. The differential forms, η_i and θ_i, rescaled from B_n to $B(0,1)$, will converge to averages of themselves on M. The limit, $\widetilde{\eta_i}$ and $\widetilde{\theta_j}$, in this rescaling will be both non-zero and pair to 0 pointwise. On M, we have that

$$\int_M \alpha_i \wedge \beta_j = 0$$

for $i \neq j$. However, the limits of the rescaled forms will satisfy

$$\widetilde{\eta_i} \wedge \widetilde{\theta_j} = 0,$$

for almost every $x \in B(0,1)$.

Once the differential forms on the unit ball are constructed and we know that they pair pointwise to 0, we see that at most $\binom{d}{l} = \dim(\bigwedge^l \mathbb{R}^d)$ of the forms can be non-zero. This will imply that the sets where at least one of the forms is 0 covers the entire ball, apart from a set of measure 0. However, the size of the rescaled forms is governed by the size of the Jacobian of f. In order to prove this we need to first show that the Jacobian of f satisfies a reverse Hölder inequality. In general, the Jacobian of a quasiregular mapping is in $L^1_{\text{loc}}(\mathbb{R}^d)$. Bojarski and Iwaniec [BI83], using a method similar to Gehring's lemma [Geh73], showed that if $f\colon \mathbb{R}^d \to \mathbb{R}^d$, then the Jacobian of f is in $L^{1+\epsilon}_{\text{loc}}(\mathbb{R}^d)$ for a sufficiently small $\epsilon > 0$. In addition, they show that f satisfies a reverse Hölder inequality, i.e.,

$$\left(\frac{1}{|B(x,\frac{r}{2})|}\int_{B(x,\frac{r}{2})} J_f^{(1+\epsilon)}\right)^{1/(1+\epsilon)} \leq C(d,\epsilon,K)\frac{1}{|B(x,r)|}\int_{B(x,r)} J_f, \tag{3.2.1}$$

where $x \in \mathbb{R}^d$ and $r > 0$. If $f\colon \mathbb{R}^d \to M$, then the Jacobian of f will be in $L^{1+\epsilon}_{\text{loc}}(\mathbb{R}^d)$, but it will not necessarily satisfy a reverse Hölder inequality. The reverse Hölder inequality only holds when $H^l(M) \neq 0$ for some l, where $1 \leq l \leq d-1$.

An example of a map that does not satisfy a reverse Hölder inequality is $f(z) = e^z\colon \mathbb{C} \to \widehat{\mathbb{C}}$. The Jacobian of f at $z \in \mathbb{C}$

$$J_f(z) = e^{2x}/(1+e^{2x})^2,$$

where $x = \mathrm{Re}(z)$. If we consider balls of the form $B(0, r)$, then the term on the left hand side of (3.2.1) will be comparable to $r^{-1/(1+\epsilon)}$ while the term on the right hand side be comparable to r^{-1}. This is not possible and hence such an inequality cannot be satisfied. Crucially, $H^1(\widehat{\mathbb{C}}) = 0$ and so (3.2.1) is not expected to hold.

Once we know that the Jacobian of f satisfies a reverse Hölder inequality, we prove that the size of the Jacobian governs the size of the rescaled forms, η_i and θ_i, on B_n. In turn, this shows that the integral of the Jacobian of f on B_n will be arbitrarily small as $n \to \infty$. At this point we arrive at a contradiction since the balls were exactly chosen so that the integral of the Jacobian of f is bounded away from 0. Hence the number of forms is bounded by $\binom{d}{l}$. These forms correspond to the dimension of the degree l de Rham cohomology on M, proving Theorem 3.2.1.

The structure of the rest of the chapter is as follows. Section 3.2.2 gives a brief introduction to differential forms on manifolds and pullbacks of differential forms by quasiregular mappings. We also show the reverse Hölder inequality for the Jacobian of f. For the relationship between quasiregular mappings and differential forms, see [BH01, Section 3] and [IM93]. The use of differential forms in this setting is inspired by the work of Bonk and Heinonen [BH01], Donaldson and Sullivan [DS89] and Iwaniec and Martin [IM93].

In Section 3.2.3 we discuss equidistribution properties for f. In Section 3.2.4 we define the rescalings of the differential forms and prove certain required convergence results. Section 3.2.5 gives the proof of Theorem 3.2.1. Some of the methods in the proof are influenced by techniques developed by Pankka [Pan10]. For a reference on the facts used for quasiregular mappings, see [BH01], [DS89] and [Ric93].

3.2.2 Exterior algebra and differential forms

This section gives an introduction to the tools needed to prove Theorem 3.2.1.

Recall that M is a closed, connected and orientable Riemannian manifold of dimension d. The map $f : \mathbb{R}^d \to M$ is a K-quasiregular map. Let l be an integer corresponding to the degree of a de Rham cohomology group of M. In the following it suffices to consider l such that $1 \leq l \leq d-1$. This is because $H^d(M) \cong H^0(M) \cong \mathbb{R}$ for the manifolds considered in Theorem 3.2.1.

In order to select suitable differential forms from the cohomology classes on M, we use Poincaré duality (see [BT82, p. 44]).

Theorem 3.2.3. *Let $k = \dim H^l(M)$. Then there exists closed forms $\alpha_1, \dots, \alpha_k \in \Omega^l(M)$ and $\beta_1, \dots, \beta_k \in \Omega^{d-l}(M)$ such that $\{[\alpha_i]\}_{i=1}^k$ forms a basis for $H^l(M)$ and*

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij} \tag{3.2.2}$$

for $1 \leq i, j \leq k$.

In estimating integrals of differential forms, the following inequality will be useful later on. If $\alpha \in \bigwedge^{l_1}(\mathbb{R}^d)$ and $\beta \in \bigwedge^{l_2}(\mathbb{R}^d)$, then

$$|\alpha \wedge \beta| \leq C(d)|\alpha||\beta|, \tag{3.2.3}$$

where $C(d)$ only depends on the dimension. To prove this note that the above norms are translation invariant so it suffices to consider the forms evaluated at 0. The bilinear operator $(\alpha(0), \beta(0)) \mapsto \alpha(0) \wedge \beta(0)$ is defined on two finite-dimensional vector spaces. Therefore it is bounded and and we arrive at (3.2.3).

A key tool we use is the pullback of a differential form by a quasiregular map. If $f \colon \mathbb{R}^d \to M$ is quasiregular and $\omega \in \Omega^l(M)$, then $f^*\omega$ is a well-defined measurable form in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$d(f^*\omega) = f^*(d\omega). \tag{3.2.4}$$

Here, $d(f^*\omega)$ is interpreted in the weak sense (see Lemma 2.2.3).

We also have the following well-known inequality for pullbacks of differential forms by f:

$$|f^*\omega(x)| \leq C(d)\|\omega\|_\infty \|Df(x)\|^l \tag{3.2.5}$$

for almost every $x \in \mathbb{R}^d$, where $\|Df\|$ is the operator norm for Df and $C(d) > 0$ is a constant that depends only on d. If f is K-quasiregular, then this becomes

$$|f^*\omega(x)| \leq C(d)K\|\omega\|_\infty J_f(x)^{l/n},$$

for almost every $x \in \mathbb{R}^d$.

The inequality is a pointwise estimate. So to prove it, without loss of generality, we may assume that $\omega \in \Omega^l(B(0,1))$. For almost every $x \in \mathbb{R}^d$,

$$f^*\omega(x) = \sum_I (\omega_I \circ f(x)) df^I(x)$$

where $I = \{i_1, \ldots, i_l\}$ is a multi-index of length l. That is,

$$df^I = df_{i_1} \wedge \cdots \wedge df_{i_l},$$

where f_i is i-th component function of f and we sum over all multi-indices $I = \{i_1, \ldots, i_l\}$ such that $1 \le i_1 < \cdots < i_l \le d$. One can deduce from Hadamard's inequality that

$$|df_{i_1} \wedge \cdots \wedge df_{i_l}| \le |df_{i_1}| \cdots |df_{i_l}| \le \|Df\|^l.$$

Thus,

$$|f^*\omega(x)| \le C(d)\|\omega\|_\infty \|Df(x)\|^l.$$

Bojarski and Iwaniec [BI83, Theorem 5.1] showed that a quasiregular mapping $f\colon \mathbb{R}^d \to \mathbb{R}^d$ has a Jacobian that satisfies a reverse Hölder inequality. That is, there exists $b > 1$ so that if $F, \Omega \subset \mathbb{R}^d$ are sets such that F is compact, Ω is open and $F \subset \Omega$, then

$$\left(\int_F J_f^b\right)^{1/b} \le C(d,b,K)\frac{1}{\operatorname{dist}(F, \partial\Omega)^{d/a}} \int_\Omega J_f \tag{3.2.6}$$

where $\frac{1}{a} + \frac{1}{b} = 1$. Crucially, b and $C(d,b,K)$ are independent of f, F and Ω. They prove this by showing a weaker reverse Hölder inequality, where the exponents are 1 and $1/2$. They then use Gehring's lemma to upgrade to the above inequality. We would like to have such a statement for $f\colon \mathbb{R}^d \to M$. If $H^l(M) = 0$ for $1 \le l \le d-1$, then the Jacobian of f does not necessarily satisfy a reverse Hölder inequality. An example of such a map is $f(z) = e^z$ as a map from $\mathbb{C} \to \widehat{\mathbb{C}}$, as mentioned in the introduction. In the setting of Theorem 3.2.1, we may assume there exists an l such that $H^l(M) \neq 0$, otherwise the theorem is trivially true.

Proposition 3.2.4. *Let M be a closed Riemannian manifold and let $f\colon \mathbb{R}^d \to M$ be K-quasiregular. If there exists an integer l with $1 \le l \le d-1$ such that $H^l(M) \neq 0$,* *then the Jacobian of f satisfies the weak reverse Hölder inequality,*

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \lesssim \left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d},$$

where $B \subset \mathbb{R}^d$ is an arbitrary ball and the constant depends only on K, d and M.

In order to prove the proposition we will need two lemmas. In general, a top-dimensional product that integrates to the volume of M can be expressed as $\alpha \wedge \beta = V + d\tau$, where $\tau \in \Omega^{d-1}(M)$. In order to prove the revere Hölder inequality of Proposition 3.2.4, we would like to write V as solely the sum of products of differential forms. The following lemma describes how to absorb the $d\tau$ term into the product term.

Lemma 3.2.5. *Let l be an integer such that $1 \leq l \leq d-1$. If there exists a pair of differential forms, $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{d-l}(M)$ that are closed and satisfy*

$$\int_M \alpha \wedge \beta = \int_M V,$$

where V is the volume form on M, then V can be expressed as

$$V = \sum_{\nu=1}^{m} \alpha_\nu \wedge \beta_\nu,$$

where $\alpha_\nu \in \Omega^l(M)$ and $\beta_\nu \in \Omega^{d-l}(M)$.

Proof. Without loss of generality, the volume of M is 1. Since $\alpha \wedge \beta$ is a top-dimensional form, we have that

$$\alpha \wedge \beta = gV,$$

where V is the volume form on M and $g \in C^\infty(M)$. There exists a point $a \in M$ so that $g(a) > 0$.

Let $x \in M$. By the isotopy lemma [GP74, p. 142], there exists an orientation-preserving diffeomorphism $\Phi_x \colon M \to M$ such that $\Phi_x(x) = a$. Let U be an open set containing a such that g is positive on U. Then $\{\Phi_x^{-1}(U)\}_{x\in M}$ is an open cover of M and there exists a finite collection of points, $x_1, \ldots, x_m$ such that $\{U_\nu\}_{\nu=1}^m$ cover M, where $U_\nu = \Phi_{x_\nu}^{-1}(U)$. By the construction, $g \circ \Phi_x$ is positive on U_x. Let Φ_ν be the diffeomorphism corresponding to U_ν and let $\{\lambda_\nu\}_{\nu=1}^m$ be a partition of unity subordinate to $\{U_\nu\}_{\nu=1}^m$. Define

$$\omega := \sum_{\nu=1}^{m} \lambda_\nu \Phi_\nu^*(\alpha \wedge \beta).$$

From this definition we get that for any $x \in M$,

$$\begin{aligned}\omega(x) &= \sum_{\nu=1}^{m} \lambda_\nu(x)(g \circ \Phi_\nu(x))\Phi_\nu^*(V)(x)\\ &= \sum_{\nu=1}^{m} \lambda_\nu(x)(g \circ \Phi_\nu(x))J_{\Phi_\nu}(x)V(x),\end{aligned}$$

where J_{Φ_ν} is the Jacobian of Φ_ν. The diffeomorphism Φ_ν is orientation-preserving, so $J_{\Phi_\nu}(x) > 0$. So the term in the sum is non-negative everywhere and for every $x \in M$, at least one term is positive. So ω is a positive top-dimensional form and thus $V = c\omega$, where $c\colon M \to (0,\infty)$ is a positive, smooth function on M. $\square$

The following lemma is well-known to experts.

Lemma 3.2.6. *Let $f\colon \mathbb{R}^d \to M$ be a K-quasiregular mapping and let $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{d-l}(M)$ be closed forms. If B is a ball in $\mathbb{R}^d$ such that $f^*(\alpha \wedge \beta) = g dx^1 \wedge \cdots \wedge dx^d$ for a non-negative function $g\colon B \to \mathbb{R}$, then*

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} f^*(\alpha \wedge \beta) \leq C(d,K)\|\alpha\|_\infty\|\beta\|_\infty\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d},$$

where $C(d,K)$ depends only on d and K.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function that is 1 on $\frac{1}{2}B$ with compact support in B. Note that we can choose ψ so that $|d\psi| \leq 2r^{-1}$, where r is the radius of B. Since $f^*(\alpha \wedge \beta)$ is a non-negative multiple of the volume form,

$$\int_{\frac{1}{2}B} f^*(\alpha \wedge \beta) \leq \int_B \psi f^*(\alpha \wedge \beta).$$

On M, α is closed. By (3.2.4), $f^*\alpha = du$ on B. We can choose u so that u satisfies a Poincaré-Sobolev inequality, see. The precise formulation of this is given in Lemma 2.2.1. Integration by parts (see Lemma 2.2.2) gives that

$$\left|\int_B \psi f^*\alpha \wedge f^*\beta\right| = \left|\int_B d\psi \wedge u \wedge f^*\beta\right|.$$

By (3.2.3), Hölder's inequality and because $|d\psi| \leq 2r^{-1}$,

$$\left|\int_B d\psi \wedge u \wedge f^*\beta\right| \leq \frac{C(d)}{r}\|u\|_{d^2/(l(d+1)-d)}\|f^*\beta\|_{d^2/((d+1)(d-l))}.$$

Since $du = f^*\alpha$ and by the Poincaré-Sobolev inequality,

$$\frac{C(d)}{r}\|u\|_{d^2/(l(d+1)-d)}\|f^*\beta\|_{d^2/((d+1)(d-l))} \leq \frac{C(d)}{r}\|f^*\alpha\|_{d^2/(l(d+1))}\|f^*\beta\|_{d^2/((d+1)(d-l))}.$$

We remark that the Poincaré-Sobolev inequality is only valid here because $1 \leq l \leq d-1$. The forms α and β are smooth on M and therefore are bounded independently of f. So by (3.2.5),

$$\begin{aligned}\frac{C(d)}{r}&\|f^*\alpha\|_{d^2/(l(d+1))}\|f^*\beta\|_{d^2/((d+1)(d-l))}\\ &\leq \frac{C(d,K)}{r}\|\alpha\|_\infty\|\beta\|_\infty\|J_f\|_{d/(d+1)}^{l/d}\|J_f\|_{d/(d+1)}^{(d-l)/d}\\ &= \frac{C(d,K)}{r}\|\alpha\|_\infty\|\beta\|_\infty\left(\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}.\end{aligned}$$

By taking averages we arrive at the lemma. □

We can now proceed to showing the proposition.

Proof of Proposition 3.2.4. Since $H^l(M) \neq 0$ there exists a Poincaré pair, $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{d-l}(M)$, given in Theorem 3.2.3, with

$$\int_M \alpha \wedge \beta = 1.$$

Fix a ball $B \subset \mathbb{R}^d$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function that is 1 on $\frac{1}{2}B$ and 0 with compact support in B. The Jacobian of f satisfies

$$J_f dx^1 \wedge \cdots \wedge dx^d = f^*V.$$

So, by Lemma 3.2.5,

$$\begin{aligned}\int_{\frac{1}{2}B} J_f &\leq \int_B \psi J_f\\ &= \int_B \psi f^*V\\ &= \sum_{\nu=1}^m \int_B f^*(\alpha_\nu \wedge \beta_\nu).\end{aligned}$$

We also know that c and λ_ν are positive and bounded above by constants depending only on M. So by Lemma 3.2.6,

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \leq C(d,K)\sum_{\nu=1}^{m}\|\alpha_\nu\|_\infty\|\beta_\nu\|_\infty\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}.$$

The number m and the L^∞-norms of α_ν and β_ν depend only on M and can be absorbed into the constant. Therefore

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \leq C(d,K,M)\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}.$$

□

Proposition 3.2.4 and [BI83, Theorem 4.2] together imply the following statement:

Proposition 3.2.7. *There exists $b>1$ such that for any ball $B\subset\mathbb{R}^d$*

$$\left(\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f^b\right)^{1/b} \leq C(d,M,K,b)\frac{1}{|B|}\int_B J_f.$$

3.2.3 Equidistribution

In this section we provide equidistribution results for a quasiregular mapping $f\colon\mathbb{R}^d\to M$. These results will help show that the limits of our rescaled forms, which will be constructed in Section 3.2.4, are non-zero. Define

$$A(E) := \int_E J_f, \tag{3.2.7}$$

for a Borel set $E\subset\mathbb{R}^d$, to be the averaged counting function for f (see [Ric93, Chapter IV]). The following theorem [BH01, Theorem 1.11] shows that $A(B(0,r))$ is unbounded.

Theorem 3.2.8. *Let $f\colon\mathbb{R}^d\to M$ be a quasiregular mapping. If $H^l(M)\neq\{0\}$ for some $l\in\{1,\dots,d-1\}$, then there exists a constant $\alpha>0$ such that*

$$\liminf_{r\to\infty}\frac{A(B(0,r))}{r^\alpha}>0.$$

In particular, $A(\mathbb{R}^d)=\infty$.

We also record a lemma due to Rickman; for the proof see [Ric80, Lemma 5.1].

Lemma 3.2.9 (Rickman's Hunting Lemma). *Let μ be a Borel measure on $\mathbb{R}^d$ that is absolutely continuous with respect to Lebesgue measure. If $\mu(\mathbb{R}^d) = \infty$, then, for all $M > 0$, there exists a point $a \in \mathbb{R}^d$ and a radius $r > 0$ such that*

$$\mu(B(a,r)) \geq M \quad \textit{and} \quad \mu(B(a,r)) \leq D(d)\mu(B(a,r/2)),$$

where $D(d)$ is a constant that depends only on the dimension.

We remark that variants of this lemma have been used in most proofs of the Rickman-Picard theorem.

The next proposition is the key equidistribution result for the quasiregular mapping f. Equidistribution results for quasiregular mappings were first shown by Mattila and Rickman in [MR79]. The following result also bears some similarity to an equidistribution result due to Pankka [Pan10, Theorem 4]. The proof also uses some methods developed there.

Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of balls satisfying

$$\lim_{n\to\infty} A(B_n) = \infty.$$

Let $T_n(x) = a_n + r_n x$, where a_n is the center of B_n and r_n is the radius of B_n.

Proposition 3.2.10. *Suppose $\psi \in C_c^\infty(B(0,1))$ is a non-negative function that satisfies*

$$A(B_n) \leq C(d,K) \int_{B_n} (\psi \circ T_n^{-1})^d J_f. \tag{3.2.8}$$

If $\omega \in \Omega^d(M)$ satisfies

$$\int_M \omega = \int_M V,$$

where V is the volume form on M, then

$$\lim_{n\to\infty} \left| \frac{1}{\int_{B_n} (\psi \circ T_n^{-1})^d J_f} \int_{B_n} (\psi \circ T_n^{-1})^d f^*\omega - 1 \right| = 0.$$

We can interpret this result in the following way. The condition that $A(B_n) \to \infty$ corresponds to the blow up of the degree of f on the balls B_n. The proposition then shows that as f wraps B_n around a manifold M an increasing amount of times, f must distribute evenly its values on M. If this were not so, we could choose an ω whose support is where f distributes its values least often and the limit in the proposition would not follow. We now give the proof of the proposition.

Proof. Let $\psi_n(x) = \psi \circ T_n^{-1}$. Since $\int_M \omega = \int_M V$, the d-form $\omega - V$ integrates to 0 on M. By de Rham's theorem, it is exact and $\omega - V = d\tau$, where $\tau \in \Omega^{d-1}(M)$. We apply integration by parts,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| = \left|\int_{B_n} \psi_n^d d(f^*\tau)\right| = \left|d\int_{B_n} \psi_n^{d-1} d\psi_n \wedge f^*\tau\right|.$$

By (3.2.3) and Hölder's inequality,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| \leq C(d)\|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d |f^*\tau|^{d/(d-1)}\right)^{(d-1)/d}.$$

By (3.2.5) and quasiregularity of f,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| \leq C(d)K^{(d-1)/d}\|\tau\|_\infty \|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d J_f\right)^{(d-1)/d}.$$

Thus,

$$\left|\frac{1}{(\int_{B_n} \psi_n^d J_f)}\int_{B_n} \psi_n^d f^*\omega - 1\right| \leq C(K, d, M)\|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d J_f\right)^{-1/d}$$

Note that

$$\|d\psi_n\|_{d,B_n} = \|d\psi\|_{d,B(0,1)},$$

by the conformal invariance of the d-energy. In other words, the term with ψ_n is independent of n. This and (3.2.8) give that

$$\left|\frac{1}{(\int_{B_n} \psi_n^d J_f)}\int_{B_n} \psi_n^d f^*\omega - 1\right| \leq C(K, d, M)\|d\psi\|_{d,B(0,1)} A(B_n)^{-1/d} \to 0$$

as $n \to \infty$. □

3.2.4 Rescaling principle

In this section we construct rescaled forms on $B(0, 1)$. By Theorem 3.2.3, there exist closed differential forms $\alpha_1, \ldots, \alpha_k \in \Omega^l(M)$ and $\beta_1, \ldots, \beta_k \in \Omega^{d-l}(M)$ such that the cohomology classes $[\alpha_1], \dots, [\alpha_k]$ form a basis for $H^l(M)$. In addition, they satisfy the orthogonality relation

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij},$$

for $1 \le i, j \le k$.

By Theorem 3.2.8 and Lemma 3.2.9, there exist balls $B_n \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} A(B_n) = \infty$ and

$$A(B_n) \le D(d) A(\tfrac{1}{2} B_n). \tag{3.2.9}$$

In the following $\{B_n\}_{n\in\mathbb{N}}$ will always refer to a sequence of balls satisfying these conditions.

We can now rescale the pullbacks, $\eta_i = f^*\alpha_i$ and $\theta_i = f^*\beta_i$ on the sequence of balls, $\{B_n\}_{n\in\mathbb{N}}$. Let $T_n \colon B(0,1) \to B_n = B(a_n, r_n)$ be the map $x \mapsto a_n + r_n x$. The rescaled forms are defined as

$$\eta_i^n := \frac{1}{A(B_n)^{1/p}} T_n^* \eta_i \tag{3.2.10}$$

and

$$\theta_i^n := \frac{1}{A(B_n)^{1/q}} T_n^* \theta_i. \tag{3.2.11}$$

Note that by (3.2.4), η_i^n and θ_i^n are closed. By quasiregularity of f, we have that $f \in W^{1,d}_{\mathrm{loc}}(\mathbb{R}^d, M)$. By (3.2.5), $\eta_i^n \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $\theta_i^n \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$, where $p = d/l$ and $q = d/(d-l)$, $1 \le l \le d-1$.

The following lemma provides a convergence result for the sequences, $\{\eta_i^n\}_{n\in\mathbb{N}}$ and $\{\theta_i^n\}_{n\in\mathbb{N}}$.

Lemma 3.2.11. *For $n \in \mathbb{N}$, there exists a $(d-l-1)$-form $u_i^n \in W^{1,q}(B(0,1))$, where $q = d/(d-l)$, such that*

$$du_i^n = \theta_i^n.$$

Furthermore, we can pass to a subsequence so that the following convergence results hold.

(i) *There exists an l-form $\widetilde{\eta}_i \in L^p(B(0,1))$ and a $(d-l)$-form $\widetilde{\theta}_i \in L^q(B(0,1))$ such that*

$$\lim_{n\to\infty} \eta_i^n = \widetilde{\eta}_i \quad \textit{and} \quad \lim_{n\to\infty} \theta_i^n = \widetilde{\theta}_i$$

where the convergence of η_i^n is in the weak topology on $L^p(B(0,1))$ and the convergence of θ_i^n is in the weak topology on $L^q(B(0,1))$.

(ii) *There exists a $(d-l-1)$-form, $\widetilde{u}_i \in W^{1,q}(B(0,1))$ such that*

$$\lim_{n\to\infty} u_i^n = \widetilde{u}_i$$

in the norm topology of $L^q(B(0,1))$.

(iii) *On $B(0,1)$, we have that $d\widetilde{u}_i = \widetilde{\theta}_i$ in the weak sense.*

Proof. In the following proof we will construct several subsequences of the sequences mentioned in the lemma. It is understood that the subsequences should be taken simultaneously for all the forms used.

For the proof of (i), we compute the L^p-norm of η_i^n. Indeed, by (3.2.10),

$$\int_{B(0,1)} |\eta_i^n|^p = \frac{1}{A(B_n)} \int_{B_n} |\eta_i|^p,$$

by the conformal invariance of the integral. By quasiregularity of f and (3.2.5),

$$\frac{1}{A(B_n)} \int_{B_n} |\eta_i|^p \leq KC(d) \frac{\|\alpha_i\|_\infty^p}{A(B_n)} \int_{B_n} J_f \leq KC(d)\|\alpha_i\|_\infty^p.$$

Hence, the L^p-norm of η_i^n is uniformly bounded. By the Banach-Alaoglu theorem, we can pass to a subsequence so that

$$\lim_{n\to\infty} \eta_i^n = \widetilde{\eta}_i$$

in the weak-* topology of $L^p(B(0,1))$. Since the dual of $L^p(B(0,1))$ is $L^q(B(0,1))$ for $\frac{1}{p}+\frac{1}{q} = 1$, the weak-* topology on $L^p(B(0,1))$ coincides with the weak topology. So η_i^n converges to $\widetilde{\eta}_i$ weakly.

The proof for θ_i^n is very similar. By (3.2.11),

$$\begin{aligned}\int_{B(0,1)} |\theta_i^n|^q &= \frac{r_n^d}{A(B_n)} \int_{B(0,1)} |\theta_i(a_n + r_n x)|^q \\ &= \frac{1}{A(B_n)} \int_{B_n} |\theta_i|^q \\ &\leq KC(d) \frac{\|\beta_i\|_\infty^q}{A(B_n)} \int_{B_n} J_f \\ &\leq KC(d) \|\beta_i\|_\infty^q.\end{aligned}$$

Again, by the Banach-Alaoglu theorem, we can pass to a subsequence so that

$$\lim_{n\to\infty} \theta_i^n = \widetilde{\theta}_i$$

weakly in $L^q(B(0,1))$.

We next prove (ii). By part (i), the L^q-norm of θ_i^n is uniformly bounded. The forms θ_i^n are closed by (3.2.4). By the Sobolev embedding theorem, there exists $(d-l-1)$-forms, $u_i^n \in W^{1,q}(B(0,1))$ such that $du_i^n = \theta_i^n$ and $\|u_i^n\|_{d/(d-l-1)} \leq C\|\theta_i^n\|_q$, where C does not depend on n, u_i^n or θ_i^n (see Lemma 2.2.1). Thus there exists a subsequence of u_i^n that converges to $\widetilde{u}_i$ strongly in $L^q(B(0,1))$. We will also denote this subsequence as u_i^n.

Finally, we show (iii). We demonstrate that $d\widetilde{u}_i = \widetilde{\theta}_i$ in the weak sense. By duality, we can consider test forms $\phi \in \Omega^{l+1}(B(0,1))$ with compact support. We pair $\widetilde{u}_i$ with $d\phi$,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{u}_i \wedge d\phi &= \lim_{n\to\infty} \int_{\mathbb{R}^d} u_i^n \wedge d\phi \\ &= \lim_{n\to\infty} (-1)^{d-l} \int_{\mathbb{R}^d} \theta_i^n \wedge \phi \\ &= (-1)^{d-l} \int_{\mathbb{R}^d} \widetilde{\theta}_i \wedge \phi.\end{aligned}$$

This proves the claims in the lemma. $\square$

The following convergence result is a key tool in proving the main result.

Lemma 3.2.12. *Let* $\psi \in C_c^\infty(B(0,1))$. *Then*

$$\lim_{n\to\infty} \int_{B(0,1)} \psi \eta_i^n \wedge \theta_j^n = \int_{B(0,1)} \psi \widetilde{\eta}_i \wedge \widetilde{\theta}_j,$$

for $1 \leq i, j \leq k$.

Proof. Consider the difference,

$$\begin{aligned}\left|\int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n - \int_{B(0,1)} \psi\widetilde{\eta}_i \wedge \widetilde{\theta}_j\right| &\leq \left|\int_{B(0,1)} \psi\eta_i^n \wedge (\theta_j^n - \widetilde{\theta}_j)\right| \\ &\quad + \left|\int_{B(0,1)} \psi(\eta_i^n - \widetilde{\eta}_i) \wedge \widetilde{\theta}_j\right| \\ &= I + II.\end{aligned}$$

Lemma 3.2.11 gives that

$$I = \left|\int_{B(0,1)} \psi\eta_i^n \wedge (du_j^n - d\widetilde{u}_j)\right|.$$

Since ψ has compact support, integrating by parts (see Lemma 2.2.2) yields that

$$\begin{aligned}\int_{B(0,1)} \psi\eta_i^n \wedge d(u_j^n - \widetilde{u}_j) &= (-1)^{l+1} \int_{B(0,1)} d(\psi\eta_i^n) \wedge (u_j^n - \widetilde{u}_j) \\ &= (-1)^{l+1} \int_{B(0,1)} d\psi \wedge \eta_i^n \wedge (u_j^n - \widetilde{u}_j)\end{aligned}$$

because η_i^n is weakly closed and $\psi(u_j^n - \widetilde{u}_j) \in W^{1,q}(\mathbb{R}^d)$, where $q = d/(d-l)$. By (3.2.3),

$$|d\psi \wedge \eta_i^n \wedge (u_j^n - \widetilde{u}_j)| \leq C(d)|d\psi \wedge \eta_i^n||u_j^n - \widetilde{u}_j|,$$

where $C(d)$ only depends on d. By Hölder's inequality,

$$I \leq C(d)\|d\psi \wedge \eta_i^n\|_p\|u_j^n - \widetilde{u}_j\|_q.$$

A computation in the proof of Lemma 3.2.11 implies that $\|d\psi \wedge \eta_i^n\|_p$ is bounded independently of n. Additionally, $u_i^n \to \widetilde{u}_i$ in $L^q(B(0,1))$. So $\lim_{n\to\infty} |I| = 0$.

For the term II, by Lemma 3.2.11, $\eta_i^n \to \widetilde{\eta}_i$ in $L^p(B(0,1))$ in the weak sense. Since $\psi\widetilde{\theta}_j \in L^q(B(0,1))$, it follows that

$$\lim_{n\to\infty} II = \lim_{n\to\infty}\left|\int_{B(0,1)} (\eta_i^n - \widetilde{\eta}_i) \wedge (\psi\widetilde{\theta}_j)\right| = 0.$$

□

We show that, as a result of the rescaling, condition (3.2.2) transfers to a pointwise property of the forms $\widetilde{\eta}_i$ and $\widetilde{\theta}_j$.

Lemma 3.2.13. *For almost every $x \in B(0,1)$,*

$$\widetilde{\eta}_i \wedge \widetilde{\theta}_j(x) = 0 \tag{3.2.12}$$

when $i \neq j$.

Proof. When $i \neq j$,

$$\int_M \alpha_i \wedge \beta_j = 0,$$

by (3.2.2). By de Rham's theorem [BT82, Corollary 5.8], there exists $\tau \in \Omega^{d-1}(M)$ such that $d\tau = \alpha_i \wedge \beta_j$. Let $\psi \in C_c^\infty(B(0,1))$. Define $\psi_n(x) = \psi \circ T_n^{-1}(x)$ for $x \in B_n$, where T_n is the affine transformations that maps the unit ball to B_n. Then

$$\begin{aligned}\int_{B(0,1)} \psi \eta_i^n \wedge \theta_j^n &= \frac{1}{A(B_n)} \int_{B_n} \psi_n d(f^*\tau) \\ &= \frac{-1}{A(B_n)} \int_{B_n} d\psi_n \wedge f^*\tau,\end{aligned}$$

due to integration by parts and the compactness of the support of ψ. By (3.2.3) and Hölder's inequality,

$$\left| \int_{B(0,1)} \psi \eta_i^n \wedge \theta_j^n \right| \leq \frac{C(d)}{A(B_n)} \|d\psi_n\|_{d,B_n} \left(\int_{B_n} |f^*\tau|^{d/(d-1)} \right)^{(d-1)/d}.$$

The two integrals can be estimated separately. By the conformal invariance of the d-energy,

$$\|d\psi_n\|_{d,B_n} = \|d\psi\|_{d,B(0,1)}.$$

By (3.2.5) and quasiregularity of f,

$$\int_{B_n} |f^*\tau|^{d/(d-1)} \leq K \|\tau\|_\infty^{d/(d-1)} \int_{B_n} J_f.$$

Combining these we find that

$$\left| \int_{B(0,1)} \psi \eta_i^n \wedge \theta_j^n \right| \leq C(d) K^{(d-1)/d} \frac{\|d\psi\|_{d,B(0,1)} \|\tau\|_\infty}{A(B_n)} \left(\int_{B_n} J_f \right)^{(d-1)/d} \to 0$$

as $n \to \infty$. By Lemma 3.2.12,

$$\int_{B(0,1)} \psi \widetilde{\eta}_i \wedge \widetilde{\theta}_j = 0.$$

Since ψ was an arbitrary test function, $\widetilde{\eta}_i \wedge \widetilde{\theta}_j(x) = 0$ for almost every $x \in B(0,1)$. □

We finish this section with a corollary of the lemma.

Corollary 3.2.14. *Suppose the differential forms $\widetilde{\eta}_i$ and $\widetilde{\theta}_i$ are as provided by Lemma 3.2.12 for each $i \in \{1, \ldots, k\}$. Under our assumption that $k > \binom{d}{l}$, for almost every $x \in B(0,1)$ there exists an $i \in \{1, \ldots, k\}$ such that*

$$\widetilde{\eta}_i \wedge \widetilde{\theta}_i(x) = 0.$$

Proof. Fix $x \in B(0,1)$ such that (3.2.12) holds for all pairs. Let $\{\widetilde{\eta}_{j_1}(x), \ldots, \widetilde{\eta}_{j_m}(x)\}$ be a basis for $\operatorname{span}(\{\widetilde{\eta}_j(x)\}_{j=1}^k) \subset \bigwedge^l \mathbb{R}^d$. Since the dimension of $\bigwedge^l \mathbb{R}^d$ is $\binom{d}{l}$, we have that $m \leq \binom{d}{l}$. By our assumption $k > \binom{d}{l}$, so there exists an index $j \notin \{j_1, \ldots, j_m\}$. It follows that $\widetilde{\eta}_j(x)$ is a linear combination of the other forms. So

$$\widetilde{\eta}_j \wedge \widetilde{\theta}_j(x) = \sum_{a=1}^{m} \lambda_{i_a} \widetilde{\eta}_{i_a} \wedge \widetilde{\theta}_j(x) = 0$$

by (3.2.12). $\square$

3.2.5 Proof of Theorem 3.2.1

In this section we complete the proof of the main result. Recall that $\widetilde{\eta}_i$ and $\widetilde{\theta}_i$ are the forms that were constructed in Lemma 3.2.11. For each $i = 1, \ldots, k$, let $D_i = \{x \in B(0,1) : \widetilde{\eta}_i \wedge \widetilde{\theta}_i(x) = 0\}$ and define $D_i^n = a_n + r_n D_i$.

We will prove Theorem 3.2.1 by contradiction; assume $k > \binom{d}{l}$. It follows from Corollary 3.2.14 that $|B_n| = |\bigcup_{i=1}^k D_i^n|$ and

$$A(\tfrac{1}{2}B_n) \leq \sum_{i=1}^{k} \int_{D_i^n \cap \frac{1}{2}B_n} J_f.$$

So for each $n \in \mathbb{N}$ there exists an index $1 \leq i_n \leq k$ so that

$$\int_{D_{i_n}^n \cap \frac{1}{2}B_n} J_f \geq \frac{1}{k} A(\tfrac{1}{2}B_n) \geq \frac{A(B_n)}{kD(d)}. \tag{3.2.13}$$

by (3.2.9). We may assume, by taking a subsequence that the index $i_n = i$ is always the same.

Lemma 3.2.15. *For all $\epsilon > 0$, there exists a compact set $C_i \subset D_i \cap B(0, \frac{1}{2})$ and an open set $U_i \subset \overline{U_i} \subset B(0,1)$ containing $D_i \cap B(0, \frac{1}{2})$ such that*

$$\int_{C_i^n} J_f \geq \frac{A(B_n)}{2kD(d)}, \tag{3.2.14}$$

where $C_i^n = a_n + r_n C_i$, and

$$\int_{U_i} |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| < \epsilon. \tag{3.2.15}$$

Proof. Fix $\epsilon > 0$. Since $\int_{D_i} |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| = 0$, there exists an open set U_i containing $D_i \cap B(0, \frac{1}{2})$ such that (3.2.15) is satisfied. To simplify notation, denote $\frac{1}{2}D_i := D_i \cap B(0, \frac{1}{2})$ and $\frac{1}{2}D_i^n := a_n + r_n \frac{1}{2} D_i$, where $D_i^n = a_n + r_n D_i$.

To construct C_i, first note that for each $\delta > 0$, there exist compact sets $C_i(\delta) \subset D_i \cap B(0, \frac{1}{2})$ satisfying

$$|(D_i \cap B(0, \tfrac{1}{2})) \setminus C_i(\delta)| < \delta.$$

Let $C_i^n(\delta) = a_n + r_n C_i(\delta)$. Then, by Hölder's inequality,

$$\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f \leq |\tfrac{1}{2}D_i^n \setminus C_i^n(\delta)|^{1/a} \left(\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f^b \right)^{1/b}$$

where $\frac{1}{a} + \frac{1}{b} = 1$ and $b > 1$ is chosen from Proposition 3.2.7. Continuing the calculation, we get

$$\begin{aligned} \int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f &\leq r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f^b \right)^{1/b} \\ &\leq r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}B_n} J_f^b \right)^{1/b}. \end{aligned}$$

We now use the higher integrability for Jacobians of quasiregular mappings given in Proposition 3.2.7,

$$\begin{aligned} r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}B_n} J_f^b \right)^{1/b} &\leq C(K,M,d,b) |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} r_n^{d/a} r_n^{-d/a} \int_{B_n} J_f \\ &= C(K,M,d,b) |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} A(B_n). \end{aligned}$$

We choose now $\delta > 0$ to be so small that $|\frac{1}{2}D_i \setminus C_i(\delta)|^{1/a} < \frac{1}{2C(K,M,d,b)kD(d)}$. Then

$$\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f \leq \frac{A(B_n)}{2kD(d)}.$$

If we combine this with (3.2.13), then the lemma follows. □

We now have all of the ingredients to finish the proof of the main theorem.

Proof of Theorem 3.2.1. Recall that we proceed by contradiction and assume that $k > \binom{d}{l}$. We may assume that $\mathrm{vol}(M) = 1$. Let C_i and U_i be the sets given in Lemma 3.2.15. Define C_i^n and U_i^n as in Lemma 3.2.15. Choose $\widetilde{\psi} \in C_c^\infty(B(0,1))$ so that $0 \leq \widetilde{\psi} \leq 1$, $\widetilde{\psi} \equiv 1$ on C_i and $\widetilde{\psi} \equiv 0$ on the complement of U_i. Next we define $\psi_n(x) = \widetilde{\psi} \circ T_n^{-1}$. By Lemma 3.2.15,

$$A(B_n) \leq 2kD(d) \int_{C_i^n} J_f \leq 2kD(d) \int_{B_n} \psi_n J_f.$$

By Proposition 3.2.10 and Lemma 3.2.14,

$$\lim_{n\to\infty} \left| \frac{1}{(\int_{B_n} \psi_n^d J_f)} \int_{B_n} \psi_n^d \eta_i \wedge \theta_i - 1 \right| = 0 \tag{3.2.16}$$

By Lemma 3.2.12,

$$\lim_{n\to\infty} \left| \frac{1}{A(B_n)} \int_{B_n} \psi_n^d \eta_i^n \wedge \theta_i^n \right| = \left| \int_{B(0,1)} \widetilde{\psi}^d \widetilde{\eta}_i \wedge \widetilde{\theta}_i \right| \leq \int_{B(0,1)} \widetilde{\psi}^d |\widetilde{\eta}_i \wedge \widetilde{\theta}_i|$$

Since the support of $\widetilde{\psi}$ is contained in U_i,

$$\int_{B(0,1)} \widetilde{\psi}^d |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| \leq \int_{U_i} |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| < \epsilon,$$

by (3.2.15). So, for n sufficiently large, we have that

$$\left| \frac{1}{A(B_n)} \int_{B_n} \psi_n^d \eta_i \wedge \theta_i \right| \leq 2\epsilon. \tag{3.2.17}$$

Therefore, by (3.2.14) and (3.2.17),

$$\frac{1}{(\int_{B_n} \psi_n^d J_f)} \left| \int_{B_n} \psi_n^d \eta_i \wedge \theta_i \right| = \frac{A(B_n)}{(\int_{B_n} \psi_n^d J_f)} \left| \frac{1}{A(B_n)} \int_{B_n} \psi_n^d \eta_i \wedge \theta_i \right| \leq 4kD(d)\epsilon.$$

This bound is independent of n and contradicts (3.2.16) for small ϵ and large n. Therefore $|\bigcup D_i| \neq |B(0,1)|$ and $k \leq \binom{d}{l}$. This proves Theorem 3.2.1. □

CHAPTER 4

A classification of the branch set of branched coverings

4.1 Introduction

In this chapter we discuss the behavior of branched covers whose image of its branch set is contained in a simplicial complex. This work was done jointly with Rami Luisto.

A mapping between topological spaces is said to be *open* if the image of every open set is open and *discrete* if the preimages of points are discrete sets in the domain. A continuous, discrete and open mapping is called a *branched cover*. The canonical example is the winding map in the plane $w_p(z) = \frac{z^p}{|z|^{p-1}}$, $p \in \mathbb{Z}$, and the higher dimensional analogues, $w_p \times \mathrm{id}_{\mathbb{R}^k} \colon \mathbb{R}^{k+2} \to \mathbb{R}^{k+2}$. An important subclass of branched covers is that of quasiregular mappings.

By the Reshetnyak theorem quasiregular mappings are branched covers ([Res89] or [Ric93, Section IV.5, p. 145]) and so branched coverings can be seen as generalizations of quasiregular mappings, see e.g. [LP17] for some further discussion.

We denote by B_f the *branch set* of f. This is the set of points where f fails to be a local homeomorphism. In dimension two the branch set of branched covers is well understood. By the classical Stoïlow theorem (see e.g. [Sto28] or [LP17]) the branch set of a branched cover between planar domain is a discrete set. In higher dimensions the Černavskii-Väisälä theorem [V66] states that the branch set of a branched cover between two n-manifolds has topological dimension of at most $n-2$. Note that the winding map $w_p \colon \mathbb{R}^n \to \mathbb{R}^n$ gives an extremal example as the branch set of w_p is the $(n-2)$-dimensional subspace

$$\{(0, 0, x_3, \ldots, x_n) \in \mathbb{R}^n : (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}\}.$$

On the other hand the Černavskii-Väisälä result is not strict in all dimensions. In Section 4.2.3 we describe an example by Church and Timourian of a branched cover $S^5 \to S^5$ with $\dim_{\mathcal{T}}(B_f) = 1$. It is not currently known if such examples exist in lower dimensions. Church and Hemmingsen asked if there exists a branched cover in three dimensions with a branch set homeomorphic to a Cantor set (see [CH60] and [AP17]). In general the structure of the branch set of a branched cover, or even a quasiregular mapping, is not well understood but the topic garners great interest. In Heinonen's ICM address, [Hei02, Section 3], he asked the following:

Can we describe the geometry and the topology of the allowable branch sets of quasiregular mappings between metric n-manifolds?

In the setting of piecewise linear (PL) branched covers between PL manifolds the result due to Černavskii and Väisälä is exact in the sense that the branch set is $(n-2)$-dimensional. Furthermore, it is a simplicial subcomplex of the underlying PL structure and the branched cover is locally a composition of winding maps. Even without an underlying PL structure of the mapping, we can in some situations identify that a branched cover between Euclidean domains is a winding map. Indeed, by the classical results of Church and Hemmingsen [CH60] and Martio, Rickman and Väisälä [MRV71], if the image of the branch set of a branched cover $f\colon \Omega \to \mathbb{R}^n$ is contained in an $(n-2)$-dimensional affine subset, then the mapping is locally topologically equivalent to a winding map. Winding maps, in turn, admit locally a canonical PL structure.

These notions were improved upon by Martio and Srebro [MS79] in dimension three in the form of the following theorem. For the definition of a cone see Section 4.2.

Theorem (Martio-Srebro). *Let $f\colon D \to \mathbb{R}^3$ be a continuous, open and discrete (or quasiregular) mapping, and let $x_0 \in B_f$. Suppose there exists a neighborhood V of $f(x_0)$ such that $V \cap fB_f$ is a finite union of half-open line segments originating from $f(x_0)$. Then f is topologically (or quasiconformally) equivalent to a cone of a rational function locally at x_0.*

Our following main theorem extends this result to all dimensions. For terminology on simplicial complexes and cones we again refer to Section 4.2. We formulate and prove our results in the topological setting, but a quasiregular version of the theorem in the spirit of the Martio-Srebro result can be acquired using similar methods (see Section 4.5).

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^n$ be a branched cover. Suppose that $f(B_f)$ is contained in a topological simplicial $(n-2)$-complex. Then f is locally topologically equivalent to a piecewise linear map which is a cone of a lower-dimensional PL mapping $g\colon S^{n-1} \to S^{n-1}$.*

Theorem 4.1.1 also yields the following corollary.

Corollary 4.1.2. *Let $f\colon S^n \to S^n$ be a branched cover such that $f(B_f)$ is contained in a topological simplicial $(n-2)$-complex. Then f is topologically equivalent to a PL mapping.*

The previous two statements assume that $f(B_f)$ is contained in a simplicial $(n-2)$-complex. Since the results are stated up to topological equivalence, Theorem 4.1.1 can be proven with the added assumption that $f(B_f)$ is contained in a Euclidean $(n-2)$-simplicial complex.

Note that whenever $f(B_f)$ is contained in a codimension two simplicial complex, the topological dimension of $f(B_f)$ must be exactly $(n-2)$ The removal of $f(B_f)$ must locally generate elements in the fundamental group by the classical result of Church and Hemmingsen [CH60, Corollary 5.3].

However, there are many branched covers for which the image of the branch set is complicated. Indeed, Heinonen and Rickman construct a quasiregular branched cover $f\colon S^3 \to S^3$ containing a wild Cantor set in the branch set. The set $S^3 \setminus f(B_f)$ is not simply connected and so $f(B_f)$ cannot be contained in a codimension 2 simplicial complex (see [HR02] and [HR98]). Here a wild Cantor set refers to any Cantor set C in $\mathbb{R}^n$ such that there is no homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ for which $h(C) \subset \mathbb{R} \times \{0\}^{n-1}$. We also note that the hypothesis of the PL structure must be made on the image of the branch set and not on the branch set itself (again see Section 4.2.3 for an example due to Church and Timourian [CT78]).

A crucial step in the proof of Theorem 4.1.1 is showing that the boundaries of so-called *normal domains* of the mapping f are $(n-1)$-manifolds when $f(B_f)$ is piecewise linear.

This method is also a major step in the proof by Martio and Srebro of the three dimensional case. In higher dimensions the situation is more complicated. In dimensions above three we need to study not only the boundary of a normal domain U, but also the boundaries of the $(n-1)$-dimensional normal domains of the restriction $f|_{\partial U} \colon \partial U \to f\partial U$, and so forth continuing these restrictions to boundaries of normal domains all the way down to dimension 1.

Finally, as an application of our results, we construct examples of quasiregular mappings in Section 4.5 in the form of the following proposition.

Proposition 4.1.3. *For each $n \in \mathbb{N}$ there exists a non-constant quasiregular mapping $f \colon \mathbb{R}^{2n} \to \mathbb{CP}^n$.*

As mentioned above, a large motivation for the contemporary study of branched covers comes from their subclass of quasiregular mappings. Often quasiregular mappings in dimensions larger than 2 are difficult to construct, but it can oftentimes be easier to construct branched covers. Thus Proposition 4.1.3 demonstrates that Theorem 4.1.1 can be applied in some cases to enhance a branched cover into a quasiregular mapping.

4.2 Preliminaries

We follow the conventions of [Ric93] and say that $U \subset X$ is a *normal domain* for $f \colon X \to Y$ if U is a precompact domain such that

$$\partial f(U) = f(\partial U).$$

A normal domain U is *a normal neighborhood* of $x \in U$ if

$$\overline{U} \cap f^{-1}(\{f(x)\}) = \{x\}.$$

By $U(x, f, r)$, we denote the component of the open set $f^{-1}(B_Y(f(x), r))$ containing x. The existence of arbitrarily small normal neighborhoods is essential for the theory of branched covers. The following lemma guarantees the existence of normal domains, the proof can be found in [Ric93, Lemma I.4.9, p. 19] (see also [V66, Lemma 5.1.]).

Lemma 4.2.1. *Let X and Y be locally compact complete path-metric spaces and $f\colon X \to Y$ a branched cover. Then for every point $x \in X$ there exists a radius $r_0 > 0$ such that $U(x, f, r)$ is a normal neighborhood of x for any $r \in (0, r_0)$. Furthermore,*

$$\lim_{r\to 0} \operatorname{diam} U(x, f, r) = 0.$$

The following Černavskii-Väisälä theorem (see [V66]) is of prime importance in the study of branched covers.

Theorem 4.2.2. *Let X and Y be n-dimensional manifolds. If $f\colon X \to Y$ is a branched cover, then the topological dimension of B_f, $f(B_f)$ and $f^{-1}(f(B_f))$ is bounded above by $n-2$. In particular, B_f, $f(B_f)$ and $f^{-1}(f(B_f))$ have no interior points and do not locally separate the spaces X nor Y.*

Another concept that we will use below is that of a *cone*.

Definition 4.2.3. *Let X be a topological space.*

1. *The* cone *of X is the set $(X \times [0,1])/(X \times \{0\}) =: \operatorname{cone}(X)$.*
2. *The* suspension *of X, denoted $S(X)$, is the disjoint union of two copies of $\operatorname{cone}(X)$ glued together by the identity at $X \times \{1\}$.*
3. *If Y is another topological space, a* cone map *$f\colon \operatorname{cone}(X) \to \operatorname{cone}(Y)$ is a continuous map such that $f(x,t) = (h(x), t)$ for some $h\colon X \to Y$ and for all $t \in [0,1]$. Note that a mapping $g\colon X \to Y$ induces a canonical cone map $\operatorname{cone}(X) \to \operatorname{cone}(Y)$, $(x,t) \mapsto (g(x), t)$ which we will denote by $\operatorname{cone}(g)$.*

   The suspension map *of f, denoted $S(f)\colon S(X) \to S(Y)$, is defined in an identical manner.*

Note that $\operatorname{cone}(S^k)$ is homeomorphic to the closed $(k+1)$-ball, and $S(S^k)$ is homeomorphic to S^{k+1}.

Definition 4.2.4. *A mapping $f\colon X \to Y$ is topologically equivalent to $g\colon X' \to Y'$ if there exists homeomorphisms ϕ and ψ such that*

$$f = \psi^{-1} \circ g \circ \phi.$$

In other words the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow{g} & Y' \end{array} .$$

4.2.1 Simplicial complexes and PL structures

We largely follow [RS72] in our notation and terminology. We list some of the basic definitions and concepts in this section for the sake of completeness.

Definition 4.2.5. *Let $\{v_0, \ldots, v_k\} \subset \mathbb{R}^n$ be a finite set of points not contained in any $(k-1)$-dimensional affine subset. The Euclidean k-*simplex *D is defined as*

$$D = \left\{ \sum_{i=1}^{k} \lambda_i v_i : \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

We say D is spanned *by $\{v_1, \ldots, v_k\}$.*

Definition 4.2.6. *A* topological k-simplex *is a set $D \subset \mathbb{R}^n$ for which there exists a homeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$, that maps D to a Euclidean k-simplex.*

A *face* of a simplex D is a simplex spanned by a subset of the vertices that span D.

Definition 4.2.7. *A* Euclidean simplicial complex *X is a finite collection of simplices such that*

1. *if $D_1 \in X$ and D_2 is a face of D_1, then $D_2 \in X$, and*
2. *if $D_1, D_2 \in X$, then $D_1 \cap D_2$ is a face of both D_1 and D_2.*

*The simplicial complex X is k-*dimensional *if the highest degree simplex in X is a k-simplex.*

Definition 4.2.8. *A* topological simplicial complex *X is a collection of subsets of X that are homeomorphic to simplices and a homeomorphism $\phi\colon X \to Y \subset \mathbb{R}^n$ such that ϕ maps the collection of subsets to a simplicial complex Y.*

We will often consider X as a subset of $\mathbb{R}^n$. In this case we tacitly identify X with the union of the simplices contained in X.

Definition 4.2.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain. A mapping $f\colon \Omega \to \mathbb{R}^n$ is piecewise linear if there exists a simplicial complex $X = \Omega$ such that f is linear on each n-simplex in X.*

4.2.2 Algebraic topology

We refer to [Hat02] for basic definitions and theory of homotopy and homology. We denote the homotopy groups and the singular homology groups of a space X by $\pi_k(X)$ and $H_k(X)$, respectively, for $k \in \mathbb{N}$. A closed n-manifold M is said to be a *homology sphere* if $H_0(M) = H_n(M) = \mathbb{Z}$ and $H_k(M) = 0$ for all $k \neq 0, n$.

A homology sphere need not be a sphere. The canonical example of a nontrivial homology sphere is the so-called *Poincaré homology sphere*, defined by gluing the opposing edges of a solid dodecahedron together with a twist (see e.g. [Can78] and [KS79]). We will denote the Poincaré homology sphere by P and note that even though the suspension $S(P)$ of P is not a manifold, the double suspension $S^2(P)$ of P is homeomorphic to S^5 (see again e.g. [Can78] and [KS79]).

An important result for us is the following theorem that is an immediate corollary of the Hurewicz isomorphism theorem [Hat02, Theorem 4.32] combined with the generalized Poincaré conjecture.

Proposition 4.2.10. *If M is a simply connected homology sphere, then M is homeomorphic to the n-dimensional sphere S^n.*

4.2.3 The double suspension of the cover $\mathbb{S}^3 \to P$.

To contrast our results and underline the necessity of the more technical arguments we recall in this section a classical branched cover $S^5 \to S^5$ constructed by Church and Timourian [CT78] with complicated branch behavior. This example shares many of the properties of branched covers with $f(B_f)$ contained in an $(n-2)$-simplicial complex, but it is not a PL mapping. For further discussion on this map see e.g. [AP17].

We note first that the Poincaré homology sphere can be equivalently defined as a quotient of S^3 under a group action of order 120 (see [KS79]). The mapping $f\colon S^3 \to P$ induced by the group action is a covering map, and since S^3 is simply connected we see that S^3 is the universal cover of the Poincaré homology sphere P. As a covering map f has an empty branch set but the suspension of f, $S(f)\colon S(S^3) \to S(P)$, has a branch set equal to the two suspension points. By definition of the cone of a map, the preimage of either suspension point $P \times \{0\}$ or $P \times \{1\}$ is a point and the preimage of any other point is a discrete set of 120 points. Thus the double suspension of f,

$$S^2(f)\colon S^2(S^3) \simeq S^5 \to S^2(P) \simeq S^5$$

is a branched cover between 5-spheres and has a branch set equal to the suspension of the two branch points of $S(f)$. Thus the branch set $B_{S^2(f)}$ is PL equivalent to S^1 and so we see that $S^2(f)$ is a branched cover between two spheres with a branch set of codimension four.

The image of the branch set $B_{S^2(f)}$ is complicated since its complement has a fundamental group of 120 elements. Furthermore even though the branch set is PL equivalent to S^1, the image of the branch set is not PL equivalent to a simplicial complex even though it is a Jordan curve in S^5. Thus the map $S^2(f)$ does not satisfy the hypothesis of our main theorem.

We also remark for future comparison that for $S^2(f)$ the boundaries of normal neighborhoods $U(x_0, f, r)$, where x_0 is one of the two suspension points of the second suspension, are homeomorphic to $S(P)$. This means that the suspension of the Poincaré homology sphere foliates a punctured neighborhood of a point in $\mathbb{R}^5$, but the simply connected space $S(P)$ with homology groups of a sphere is not a manifold.

4.3 Boundary of a normal domain

In this section we show that for a branched cover $f\colon \Omega \to \mathbb{R}^n$ with $f(B_f)$ contained in a simplicial $(n-2)$-complex, the boundaries of sufficiently small normal domains are homeomorphic to a sphere. The main step of the proof takes the form of an inductive argument where in the inductive step we restrict a branched cover to the boundary of a small normal domain and study the new branched cover between the lower dimensional spaces. Since we do not a priori know that the boundary of a normal domain is a manifold, many of the results in this section are proved in a more general setting where the domain of the mapping is not assumed to be a manifold.

We begin with a few preliminary results on the behavior of f on the boundary of a normal domain. The following Lemma 4.3.1 is known to the experts in the field (see e.g. [MS79]) but we give a short proof for the convenience of the reader.

Lemma 4.3.1. *Let X be a locally compact and complete metric space and $f\colon X \to \mathbb{R}^n$ a branched cover. Fix $x_0 \in X$ and let $r_0 > 0$ be such that $U_r := U(x_0, f, r)$ is a normal neighborhood of x_0 for all $r \le r_0$. Then the restriction*

$$f|_{\partial U_r}\colon \partial U_r \to \partial B(f(x_0), r)$$

is a branched cover for all $r < r_0$.

Proof. The restriction is clearly continuous and discrete, so it suffices to show that it is an open map. Let $V \subset \partial U_r$ be a relatively open set and suppose $y = f(x_1) \in f(V)$, where $x_1 \in V$. Additionally, suppose that $\{x_1, \ldots, x_k\} = f^{-1}(y)$. For $\delta > 0$ let $N_\delta(y) = B(y, \delta) \cap \partial B(x_0, r)$ and for $\epsilon > 0$ let $N_\epsilon(x_i) = B(x_i, \epsilon) \cap \partial U_r$.

Fix $\epsilon > 0$ so that $N_\epsilon(x_i) \cap N_\epsilon(x_j) = \emptyset$ for $i \neq j$ and $N_\epsilon(x_i) \subset V$ for $1 \le i \le k$. By [BM17, Lemma 5.15], there exists a $\delta > 0$ such that

$$f^{-1}(B(y, \delta)) \subset \cup_{i=1}^k B(x_i, \epsilon).$$

Let $y' \in N_\delta(y)$. There exists a path γ connecting y to y' in $N_\delta(y)$. By the path-lifting properties of branched covers, (see e.g. [Ric93, Chapter II.3]), γ can be lifted to paths

$\gamma_1, \ldots, \gamma_k$ each contained in $N_\epsilon(x_i)$. The end point of each lift x_i' maps to y'. So $N_\delta(y) \subset f(V)$, which means that $f(V)$ is open.

$\square$

We will repeatedly choose suitably small normal neighborhoods for points in the domain. For clarity we formulate this selection as the following lemma.

Lemma 4.3.2. *Let X be a locally connected, locally compact and complete metric space and $f\colon X \to \mathbb{R}^n$ a branched cover. Then for every $x \in X$ there exists a radius $r(x,f) > 0$ such that for all $r < r(x,f)$, $U(x,f,r)$ is a normal neighborhood of x.*

Furthermore if $f(B_f)$ is contained in an Euclidean $(n-2)$-simplicial complex we may assume that $f(B_f) \cap f(\partial U(x,f,r)) = f(B_f) \cap \partial B(f(x), r)$ is contained in an Euclidean $(n-3)$-simplicial complex (up to a global homeomorphism) for all $r < r(x,f)$.

4.3.1 Radial properties of the mapping f

In the following arguments we need a consistent way of describing boundaries of normal domains of mappings which are themselves restrictions of ambient mappings to boundaries of normal domains. To this end we define nested collections of lower dimensional normal domains.

Definition 4.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f\colon \Omega \to \mathbb{R}^n$ a branched cover. Denote by $\mathcal{U}_{n-1}$ the collection of boundaries of normal domains $U(x,f,r) \subset \Omega$ with $r < r(x,f)$ as in Lemma 4.3.2. For $k = n-1, \ldots, 2$ we similarly define $\mathcal{U}_{k-1}$ to be the collection of boundaries of normal domains $U(x, f|_V, r) \subset V$, $V \in \mathcal{U}_k$, with $r < r(x, f|_V)$ as in Lemma 4.3.2. We call these collections as* lower dimensional normal domains.

By Lemma 4.3.2, in the case where $f(B_f)$ is contained in an $(n-2)$-simplicial complex we may assume that for given $1 \le k \le n-1$ and $V \in \mathcal{U}_k$ that the set $f(B_f) \cap f(\partial U(x, f|_V, r))$ is contained, up to a homeomorphism, in an $(n-3)$-simplicial for $r < r(x, f|_V)$.

Lemma 4.3.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f\colon \Omega \to \mathbb{R}^n$ a branched cover with $f(B_f)$ contained in an $(n-2)$-simplicial complex. Then for any $k = n-1, \ldots, 1$ and $V \in \mathcal{U}_k$, fV*

is homeomorphic to a sphere.

Proof. By using an inductive argument we see that it suffices to study the case where $f(V) \subset f(U)$ with $U \in \mathcal{U}_{k+1}$ and $f(U)$ is a $(k+1)$-sphere. The proof in this setting identical to the proof of [Ric93, Lemma I.4.9]. $\square$

The following proposition is of prime importance to the proof of Theorem 4.1.1. It captures the fact that for branched covers with $f(B_f)$ contained in an Euclidean $(n-2)$-simplicial complex, the branching should occur 'tangentially', i.e., inside the boundaries of normal domains. Some of the steps of the proof are described in Figure 4.1.

Proposition 4.3.5. *Let $f\colon \Omega \to \mathbb{R}^n$ be a branched cover such that $f(B_f)$ is contained in an Euclidean $(n-2)$-simplicial complex. Then for any $x_0 \in \Omega$, there exists a sufficiently small $r < r(x_0, f)$ so that for $v \in \mathbb{S}^{n-1}$, the path*

$$\beta\colon [0,r] \to \overline{B}(f(x_0), r), \quad \beta(t) = (r-t)v + f(x_0)$$

has a unique lift starting from any point $z_0 \in \overline{U}(x_0, f, r) \cap f^{-1}\{\beta(0)\}$.

Proof. Choose r small enough so that $f(B_f) \cap B(f(x_0, r))$ is contained in a codimension-2 radial set. That is, there exists an $(n-2)$-simplicial complex D such that $D \cap B(x_0, r_1) = \frac{r_1}{r_2}(D \cap B(x_0, r_2))$.

Suppose towards contradiction that the claim is false. Then there exists two different lifts of β, say $\alpha_1, \alpha_2\colon [0,r] \to \overline{U}(x_0, f, r)$ satisfying,

$$\alpha_1(0) = \alpha_2(0) = z_0 \quad \text{and} \quad \alpha_1(s_0) \neq \alpha_2(s_0),$$

for some $s_0 \in (0, r)$. Set

$$t_0 = \inf\{t \in [0,r] \mid \alpha_1(t) \neq \alpha_2(t)\}.$$

So $\alpha_1(t) = \alpha(t)$ for all $t \in [0, t_0]$, but for $s \in (t_0, t_0 + \epsilon)$ for small ϵ, $\alpha_1(s) \neq \alpha_2(s)$. Without loss of generality we may assume that $t_0 = 0$ and that

$$\alpha_1(t_0) = \alpha_2(t_0) = z_0.$$

(see top part of Figure 4.1).

Fix a radius $R < r(z_0, f)$ such that $\overline{B}(f(z_0), R) \subset B(f(x_0), r(x_0, f))$ (see middle part of Figure 4.1). Let $s_0 \in (t_0, t_0 + \epsilon)$, we may assume that s_0 is sufficiently small so that $\beta(s_0) \in B(f(x_0), R)$. We now let $U(\alpha_1(s_0))$ and $U(\alpha_2(s_0))$ be normal neighborhoods of $\alpha_1(s_0)$ and $\alpha_2(s_0)$ respectively. Let ζ be a line segment that has one endpoint at $\beta(s_0)$ and intersects $f(B_f)$ only at $\beta(s_0)$. Additionally, suppose that ζ is small so that

$$\zeta \subset fU(\alpha_1(s_0)) \cap fU(\alpha_2(s_0)).$$

Since everything is contained in the image of normal neighborhoods we can lift ζ to $\gamma_1 \subset U(\alpha_1(s_0))$ and $\gamma_2 \subset U(\alpha_2(s_0))$ from the points $\alpha_1(s_0)$ and $\alpha_2(s_0)$, respectively – note though that these lifts might not be unique. Let γ_3 be a path connecting γ_1 and γ_2 that lies outside of $f^{-1}(f(B_f))$. The path $f(\gamma_1 \cup \gamma_2 \cup \gamma_3)$ will be a loop based at $\beta(s_0)$ that consists of a line segment and a loop. The loop will lie outside of $f(B_f)$ (see bottom part of Figure 4.1).

The image of the branch set is contained in a simplicial complex so if the normal neighborhood around x_0 is chosen to be sufficiently small the image of the branch set will be *radial in the normal neighborhood*. By this we mean that it is contained in the union of $(n-2)$-dimensional planes whose intersection contains $f(x_0)$. We may choose the normal neighborhood V of z_0 to be so small that the image of the branch set is also radial with respect to z_0 in this normal neighborhood V. The point $\beta(s_0)$ lies on a path between $f(z_0)$ and $f(x_0)$ so the branch set will be radial at $\beta(s_0)$ with respect to small enough normal domains as well; indeed, for any $w \in B(f(z_0), R) \cap f(B_f)$ the line segment $[w, f(z_0)]$ belongs to the branch, and so will the line $[w, f(x_0)]$. Additionally, for each $w' \in [w, f(z_0)]$, the line $[w', f(x_0)]$ will be in $f(B_f)$ and so we conclude that $f(B_f)$ contains the segment $[w, \beta(s_0)]$.

Define a homotopy that consists of the straight line from each point in $f(\gamma_1 \cup \gamma_2 \cup \gamma_3)$ to $\beta(s_0)$. Due to the local radial structure of $f(B_f)$ at $\beta(s_0)$ we see that the homotopy will take $f(\gamma_1 \cup \gamma_2 \cup \gamma_3)$ to an arbitrarily small neighborhood of $\beta(s_0)$ without intersecting $f(B_f)$. Additionally, the end loop will be contained in the image of the normal neighborhoods, $U(\alpha_1(s_0))$ and $U(\alpha_2(s_0))$.

The homotopy will always preserve a small straight line in ζ and so we can lift the homotopy uniquely. The end loop of the homotopy will be lifted separately to the normal neighborhood of $\alpha_1(s_0)$ and $\alpha_2(s_0)$. This gives a homotopy from a connected curve to two disconnected loops, which is a contradiction.

□

The previous proposition allows us to uniquely lift radial paths in normal neighborhoods. We would also like to be able to lift radial paths uniquely in lower dimensional normal neighborhoods $U \subset V$ with $V \in \mathcal{U}_k$ for any k.

Let $f\colon \Omega \to \mathbb{R}^n$ be a branched cover such that $f(B_f)$ is contained in an $(n-2)$-simplicial complex. Let $x_0 \in V \in \mathcal{U}_k$. By Lemma 4.3.4, $f(V) \simeq S^k$. Up to homeomorphism we can assume that $f(V)$ minus a point maps to a k-dimensional plane. In this case $f|_V$ will have a branch set contained in a $(k-2)$-simplicial complex.

Proposition 4.3.6. *If $r < r(x_0, f|_V)$ and $v \in S^{k-1}$, the path*

$$\beta\colon [0,r] \to \overline{B}(f(x_0), r), \quad \beta(t) = (r-t)v$$

has a unique lift starting from any point $z_0 \in \overline{U}(x_0, f|_V, r) \cap f|_V^{-1}\{\beta(0)\}$.

Proof. By Proposition 4.3.5 we know that β has a unique lift in Ω starting from any preimage of $\beta(0)$. Thus we only need to show that such a lift is contained V. But this is clear since $f(V)$ maps surjectively onto a k-dimensional plane containing β and thus there will be a preimage of $\beta(0)$ in V and the lift of β under $f|_V$ starting from this preimage is contained in V. □

Proposition 4.3.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f\colon \Omega \to \mathbb{R}^n$ a branched cover with $f(B_f)$ contained in an $(n-2)$-simplicial complex. Suppose $k = n-1, \ldots, 2$ and $W \in \mathcal{U}_k$. Then for any $x_0 \in W$ and all normal domains $U(x_0, f|_W, r)$ with $r < r(x, f) =: r_0$ (as in Lemma 4.3.2) there exists a parameterized collection of homeomorphisms*

$$h_t\colon \partial U(x_0, f|_W, r_0) \to \partial U(x_0, f|_W, t),$$

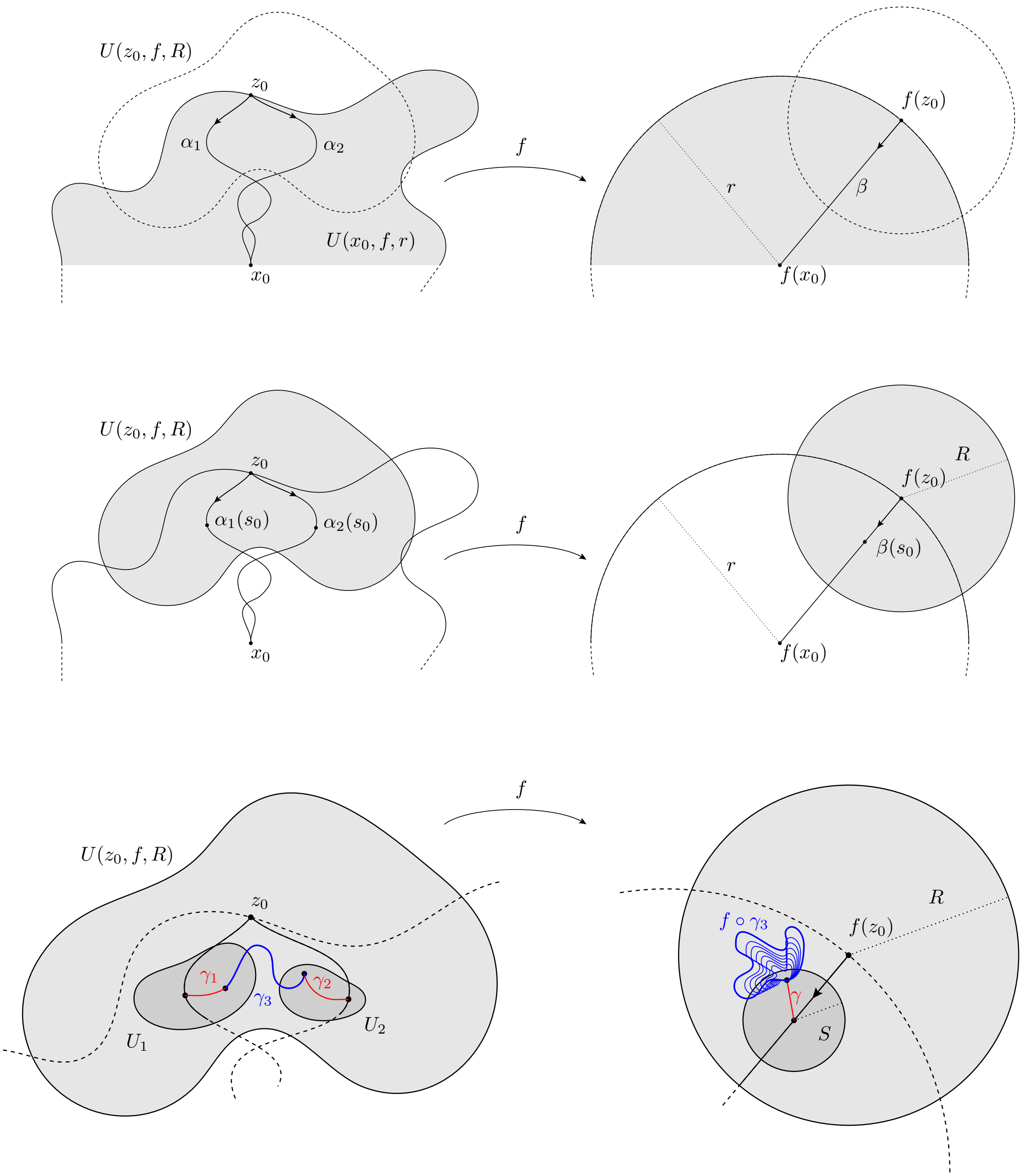

Figure 4.1: Showing that radial lifts are unique.

$t \in (0, r_0)$ such that the mapping

$$H\colon (0, r_0) \times \partial U(x_0, f|_W, r_0) \to U(x_0, f|_W, r_0) \setminus \{x_0\},$$

$$H(t, x) = h_t(x)$$

is also a homeomorphism and $U(x_0, f|_W, r_0) \simeq \operatorname{cone}(\partial U(x_0, f|_W, r_0))$.

Proof. For $t \in (0, r_0)$ and any given point $x \in \partial U(x_0, f|_W, t)$ we define the homeomorphism h_t to map x to the endpoint of the unique lift, guaranteed by Proposition 4.3.6, of the straight line connecting $f(x)$ and $f(x_0)$. Since these lifts are unique, there exists a canonical inverse map for h_t. Since these two maps are defined symmetrically it suffices to show that h_t is continuous to prove the claim.

Suppose that there exists a sequence $\{a_j\}_{j\in\mathbb{N}}$ such that $a_j \in U(x_0, f|_W, r_0)$ and

$$\lim_{j\to\infty} a_j = a \in U(x_0, f|_W, r_0).$$

This would imply that there is a radial line segment I together with a sequence (I_j) of line segments converging to I. We must show that the unique lifts α_j of I_j converge to the unique lift α of I. By compactness of the Hausdorff metric (see e.g. [BH99, pp. 70–77]) $\{\alpha_j\}_{j\in\mathbb{N}}$ must have a converging subsequence. So by taking a subsequence suppose that $\lim_{j\to\infty} \alpha_j = \beta$. Additionally, β will be connected since for each $j \in \mathbb{N}$, α_j is connected.

We can parametrize the I_j by a time parameter t in the obvious way. Similarly, we can parametrize α_j so that $f \circ \alpha_j(t) = I_j(t)$. By [BH99, Lemma 5.32], for every $x \in \beta$, there exists a sequence $\{\alpha_j(t)\}_{j\in\mathbb{N}}$ so that $\lim_{j\to\infty} \alpha_j(t_j) = x$. Since f is continuous and $f(\alpha_j(t_j)) \in I_j$, we have that $f(x) \in I$. So $\beta \subset \alpha$. Note that β cannot be contained in a different preimage of I by f since $\lim_{j\to\infty} a_j = a \in \alpha$ and β is connected.

If $x \in \alpha$, then there exists a sequence of points $y_j \in I_j$ such that $\lim_{j\to\infty} y_j = f(x)$. The point y_j has a unique preimage $x_j \in \alpha_j$ for all $j \in \mathbb{N}$. By [BH99, Lemma 5.32] there exists a subsequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{j_k} = x' \in \beta \subset \alpha$. So $f(x') \in I$ and by uniqueness of lifts we have that $x = x'$. This gives that $\beta = \alpha$. The argument shows that every subsequence of $\{\alpha_j\}_{j\in\mathbb{N}}$ must limit to α and so $\lim_{j\to\infty} h_t(a_j) = h_t(a)$, which gives that h_t is continuous.

Finally, it is straightforward to check that H is also a homeomorphism, which implies that $U(x_0, f|_W, r_0) = \mathrm{cone}(\partial U(x_0, f|_W, r_0))$. □

The previous Proposition 4.3.7 shows that we can foliate the small punctured lower dimensional normal domains with their boundaries. Note that this does not a priori imply that the boundaries are spheres, see again the example in Section 4.2.3.

4.3.2 Boundaries of normal domains are homeomorphic to spheres

We wish to show that the boundary of a normal domain is homeomorphic to a sphere for a branched cover f with $f(B_f)$ contained in an $(n-2)$-simplicial complex. The proof is based on an inductive argument on the dimension of the lower dimensional normal domains. Most of the complications in the statements and proofs of the following proposition arise from the fact that we need to study the restriction of f to the boundary of a normal domain before showing that the boundary is a manifold.

We first compute the homology groups of the boundary of a lower dimensional normal neighborhood.

Lemma 4.3.8. *Fix $k \in \{2, \dots, n-2\}$. Let U be a normal neighborhood in $\mathcal{U}_{k+1}$ centered at a point $x \in \mathbb{R}^n$. Let also $\partial U = V \in \mathcal{U}_k$. If U is sufficiently small, then*

$$H_l(V) = H_l(\mathbb{S}^k)$$

for $0 \le l \le k$, where H_l is the simplicial homology group.

Proof. By Proposition 4.3.7, $U \simeq \mathrm{cone}(V)$ and therefore $U \setminus \{x\} \simeq V \times (0,1)$.

Since $V \in \mathcal{U}_k$ we know that U is a normal neighborhood contained in some $W \in \mathcal{U}_{k+1}$. By Proposition 4.3.7 there exists an open set containing U in W that is homeomorphic to $U \times (0,1)$. Removing the point $x \in U$ thus gives rise to a neighborhood of $U \setminus \{x\}$ homeomorphic to $U \setminus \{x\} \times (0,1) \simeq V \times (0,1)^2$.

We can continue inductively to find an open set containing U in the top level normal neighborhood (which is a domain in $\mathbb{R}^n$) that is homeomorphic to $U \times (0,1)^{n-k-1}$. Furthermore, $U \setminus \{x\}$ is contained in an open set that is homeomorphic to $V \times (0,1)^{n-k}$. These are now open sets in $\mathbb{R}^n$ and are therefore manifolds. Recall that $U \simeq \mathrm{cone}(V)$ and therefore U is contractible. So $U \times (0,1)^{n-k-1}$ is also contractible.

By extending U to an open domain in $\mathbb{R}^n$ the point $x \in U$ is extended radially. Therefore $x \times (0,1)^{n-k-1} \subset U \times (0,1)^{n-k-1}$ is an $(n-k-1)$-submanifold. Consider now a map

$$\gamma \colon \mathbb{S}^l \to (U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1}).$$

Since $U \times (0,1)^{n-k-1}$ is contractible, there is a homotopy H that takes γ to a point $x' \neq x$. The dimension of $\mathbb{S}^l \times (0,1)$ is $l+1$ and, since

$$(l+1) + (n-k-1) < n,$$

we claim that the image of H can be guaranteed to avoid $\{x\} \times (0,1)^{n-k-1}$. To prove this claim note that H can be assumed to be smooth since $U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1})$ is an open set and hence a smooth manifold. By the compactness of the image of H, there exists an $\epsilon > 0$ so that the ϵ-neighborhood of H lies in $U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1})$. Smooth functions are dense in the uniform topology. Therefore there exists a smooth function $\widetilde{H} \colon \mathbb{S}^l \times [0,1] \to U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1})$ such that $\|H - \widetilde{H}\| < \epsilon$. A straight-line homotopy takes H to $\widetilde{H}$ and hence the claim is shown.

We can also assume that the image of H is transverse to the submanifold $\{x\} \times (0,1)^{n-k-1}$ (see [GP74, Chapter 2]). Since the dimensions add up to less than n, the transversality condition implies that they are actually disjoint. The entire homotopy is disjoint from the removed set and thus

$$\pi_l((U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1})) = 0$$

for $1 \leq l < k$. By the above argument, $\pi_l(V) = 0$ for $1 \leq l < k$. The lemma now follows by the Hurewicz theorem [Hat02, p. 366] for this index range.

It remains to show that $H_k(V) = H_k(S^k)$. We show this case by the use of the Mayer-Vietoris theorem. Let $M = U \times (0,1)^{n-k-1}$ and let $L = \{x\} \times \mathbb{R}^{n-k-1}$. Note that

$$M \setminus L = (U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1}).$$

The Mayer-Vietoris theorem implies that

$$\begin{aligned}\cdots \to H_{k+1}(M \cup (\mathbb{R}^n \setminus L)) \to H_k(M \cap (\mathbb{R}^n \setminus L)) \to H_k(M) \oplus H_k(\mathbb{R}^n \setminus L)\\ \to H_k(M \cup (\mathbb{R}^n \setminus L)) \to \cdots\end{aligned}$$

is an exact sequence. We have that M and

$$M \cup (\mathbb{R}^n \setminus L) = \mathbb{R}^n \setminus (\{x\} \times (0,1)^{n-k-1})$$

are contractible. Additionally,

$$M \cap (\mathbb{R}^n \setminus L) = M \setminus (\{x\} \times (0,1)^{n-k-1}).$$

So

$$0 \to H_k(M \setminus (\{x\} \times (0,1)^{n-k-1})) \to H_k(\mathbb{R}^n \setminus L) \to 0.$$

This implies that

$$\begin{aligned}H_k(V) &= H_k((U \times (0,1)^{n-k-1}) \setminus (\{x\} \times (0,1)^{n-k-1})\\ &\cong H_k(\mathbb{R}^n \setminus L) \cong H_k(S^k).\end{aligned}$$

□

We next show that the boundary of normal domains are homeomorphic to spheres.

Proposition 4.3.9. *Let $k \in \{2, \ldots, n-1\}$. If $V \in \mathcal{U}_k$, then $V \simeq S^k$.*

Proof. We begin by noting that by the proof in Lemma 4.3.8,

$$V \times (0,1)^{n-k} \simeq U \times (0,1)^{n-k-1} \setminus (\{x\} \times (0,1)^{n-k-1}),$$

where U is a normal neighborhood on the $(k+1)$-level and $\partial U = V$. Normal neighborhoods are connected and removing a set of dimension $n-k-1$ does not disconnect the set for $k \geq 1$. So $V \times (0,1)^{n-k}$ is connected and therefore V is connected.

We now continue to prove the main claim in the proposition. Suppose first that $k = 1$ and fix $V \in \mathcal{U}_1$. We denote the restriction $f|_V \colon V \to fV$ by g. By Lemma 4.3.4, gV is homeomorphic to a circle. The definition of $\mathcal{U}_k$ gives that

$$g(V) \cap f(B_f) = \emptyset.$$

This implies that $V \cap B_f = \emptyset$ and

$$g \colon V \to g(V) \simeq S^1$$

is a covering map. Since the branched cover f is finite-to-one in any normal domain, we see that g is a finite-to-one cover of S^1. This implies that V is homeomorphic to S^1.

Suppose next that the claim holds true for some $k < n-1$ and $V \in \mathcal{U}_{k+1}$. Fix a point $x \in V$ and take a normal neighborhood W of x such that $\partial W \in \mathcal{U}_k$. By the inductive assumption ∂W is homeomorphic to S^k. By Proposition 4.3.7,

$$W \simeq \operatorname{cone} \partial W \simeq \operatorname{cone} S^k \simeq B^n.$$

The point x has a neighborhood in V homeomorphic to a ball and therefore V is a closed k-manifold. By Lemma 4.3.8,

$$H_l(V) \cong H_l(S^k)$$

for $0 \leq l \leq k$. Combining these we see that V is a simply connected homology k-sphere and so $V \simeq S^k$ by Proposition 4.2.10. □

4.4 PL cone mappings

In this section we prove Theorem 4.1.1. We divide the proof into a local and global part. A branched cover $f \colon S^n \to S^n$ is called *locally PL with respect to a simplicial decomposition*

A of S^n if, for all $x \in S^n$, there exists an open set $U \subset \overline{U} \subset S^n$ containing x and a homeomorphism

$$\phi\colon \overline{U'} \to \overline{U} \subset S^n$$

such that $f \circ \phi$ is a PL mapping. Additionally, if $\overline{U}$ is given a simplicial decomposition defined by $f \circ \phi$, the k-simplices in $\overline{U}$ are mapped to k-simplices in a subdivision of A.

Lemma 4.4.1. *Let* $g\colon S^n \to S^n$ *be a branched cover whose branch set is contained in an* $(n-2)$*-simplicial complex. Let* A *be a simplicial decomposition of* S^n *that contains* $g(B_g)$ *in its* $(n-2)$*-skeleton. Additionally, suppose that* g *is locally PL with respect to* A*. Under these conditions, there exists a homeomorphism*

$$\Phi\colon S^n \to S^n$$

such that $g \circ \Phi$ *is a PL mapping and the* k*-simplices defined by* $g \circ \Phi$ *are mapped to* k*-simplices in* A*.*

We remark that the proof here uses ideas from the proof in [BM17, Lemma 5.12].

Proof. The strategy of the proof will be to pull back the simplicial structure A by g. The set S^n can be covered by finitely many open sets U that satisfy the conditions in the definition of the local PL property of g. We refine A so that g maps the simplices in $\overline{U'}$ to simplices in A, where $\phi\colon \overline{U'} \to \overline{U}$ is a homeomorphism as described above.

In the spirit of pulling back A by g, let B be the set of simplices σ such that $g(\sigma) \in A$ and $g|_\sigma$ is a homeomorphism onto its image. To show that this is a simplicial structure for S^n it suffices to show that every point lies in the interior of a unique simplex and that the intersection of two simplices is a face of those simplices.

We first show that every point lies in the interior of a unique simplex. Let $x \in S^n$ and $y = g(x) \in \Delta_k^o$, where Δ_k is a k-simplex in B and Δ_k^o is the interior of Δ_k. Let U be an open set containing x such that there exists a homeomorphism $\phi\colon U' \to U$ satisfying that $g \circ \phi$ is a PL mapping. By our assumption, $x' = \phi^{-1}(x)$ is contained in a simplex D that is mapped by $g \circ \phi$ onto a simplex in A. Since $g \circ \phi(x') = y \in \Delta_k^o$, the simplex D must be a degree k simplex and $x' \in D^o$. Additionally, $D^o = (g \circ \phi)^{-1}(\Delta_k^o) \cap U'$.

The map $g \circ \phi$ is a PL branched cover. Therefore it is locally injective on D^o. So g defines a covering map from the component τ of $g^{-1}(\Delta_k^o)$ containing x to Δ_k^o. Since Δ_k is simply connected, g is actually a homeomorphism from τ to Δ_k^o.

We claim that g extends to a homeomorphism from $\sigma = \overline{\tau}$ to Δ_k. It suffices to show that $g^{-1}\colon \Delta_k^o \to \sigma$ extends continuously to the boundary. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of points such that $y_n \to y \in \partial\Delta_n$. Then there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ such that $g(x_n) = y_n$. Let a and b be accumulation points of $\{x_n\}_{n\in\mathbb{N}}$. Let a_n be a subsequence that converges to a and b_n a subsequence that converges to b.

By [BM17, Lemma 5.15], for all $\epsilon > 0$, there exists $\delta > 0$ so that

$$g^{-1}(B(y,\delta)) \subset \cup_{z\in g^{-1}(y)} B(z,\epsilon).$$

By choosing ϵ sufficiently small, the sets $B(z,\epsilon)$ will be pairwise disjoint for $z \in g^{-1}(y)$. However, for large n, $g(a_n)$ and $g(b_n)$ will be in $B(y,\delta)$. If a_n and b_n are connected by a path γ, then $g^{-1}(\gamma)$ must be a path connecting $a_n \in B(a,\epsilon)$ and $b_n \in B(b,\epsilon)$. So $g^{-1}(\gamma)$ lies outside $\cup_{z\in g^{-1}(y)} B(z,\epsilon)$, which gives a contradiction if $a \neq b$. Thus g^{-1} extends continuously to $\partial\Delta_k$ and g defines a homeomorphism from σ to Δ_k. This shows that σ defines a k-simplex in B and that $x \in \sigma$. This shows that every x is in a simplex defined by B.

Let σ_1 and σ_2 be simplices in B and suppose $\sigma_1 \cap \sigma_2 \neq \emptyset$. If $\sigma_1^o \cap \sigma_2^o \neq \emptyset$, then they must both be k-simplices and by construction must be mapped homeomorphically onto the same k-simplex $\Delta_k \in A$. This is not possible since Δ_k is simply connected.

If $\sigma_1^o \cap \sigma_2^o = \emptyset$, then suppose τ is a simplex such that $\tau^o \cap \sigma_1 \cap \sigma_2 \neq \emptyset$. It follows that $g(\tau) \subset g(\sigma_1) \cap g(\sigma_2)$. The map g defines an inverse on $g(\tau)^o$, which must agree with the inverses that it defines on $g(\sigma_1)^o$ and $g(\sigma_2)^o$. So the entirety of τ must be contained in $\sigma_1 \cap \sigma_2$. This implies that $\sigma_1 \cap \sigma_2$ is comprised of the union of finitely many simplices.

Finally, we claim that A and B can be refined so that the intersection of two simplices is a face. Let σ_1 and σ_2 be k-simplices. Suppose that $\sigma_1 \cap \sigma_2 \neq \emptyset$ and that there are two $(k-1)$-simplices whose interiors are in $\sigma_1 \cap \sigma_2$. We apply a barycentric subdivision A. Then

g pulls back this decomposition to a refinement of B and the new simplices in σ_1 and σ_2 cannot share more than one $(k-1)$-simplex. We may now proceed by repeated barycentric subdivision to rule out the cases when $\sigma_1 \cap \sigma_2$ contain more than one interior of lower degree shared simplices. At the end of this process, the refined B must be a simplicial decomposition of S^n.

The construction implies that g is a simplicial map from S^n_B to S^n_A. Thus there exists a PL map from S^n_B to S^n_A that is topologically equivalent to g with respect to A. □

Lemma 4.4.2. *Let $f\colon S^n \to S^n$ be a branched cover with $f(B_f)$ contained in a simplicial $(n-2)$-complex. Let A be a simplicial decomposition of S^n that contains $f(B_f)$ in its $(n-2)$-skeleton. Then f is locally PL with respect to A.*

Proof. We proceed by induction on n. The base case, $n = 2$, follows from Stoïlow's theorem (see [Sto28] or [LP17]) as f is topologically equivalent to a rational map $S^2 \to S^2$ and rational maps are topologically equivalent to PL mappings.

We now suppose that $f\colon S^n \to S^n$ is defined as in the statement of the lemma. Then there exists a Euclidean simplicial decomposition A (when S^n is viewed as $\mathbb{R}^n \cup \{\infty\}$) such that $f(B_f)$ is contained in the $(n-2)$-skeleton of A.

Fix $x \in S^n$. For a small radius r_0, there exists a ball $B(f(x), r_0)$ that is radially symmetric with respect to the simplicial decomposition A. More precisely, for any simplex $\Delta \in A$,

$$\Delta \cap \partial B(f(x), r) = \frac{r}{s}(\Delta \cap \partial B(f(x), s))$$

for $0 < r, s \leq r_0$, where r/s is the dilation mapping the s-sphere at $f(x_0)$ to the r-sphere.

By Proposition 4.3.7 and Proposition 4.3.9, for sufficiently small r_0, the normal neighborhood $U(x, f, r_0) \simeq \mathrm{cone}(V)$, where $V = \partial U(x, f, r_0)$, is homeomorphic to S^{n-1}. Let $g = f|_V$. By the construction of the homeomorphism in Proposition 4.3.7, f is topologically equivalent to $\mathrm{cone}(g)\colon \mathrm{cone}(V) \to B(f(x), r_0)$. By the choice of $B(f(x), r_0)$, the map $g\colon V \to \partial B(f(x), r_0)$ sends its branch set into the $(n-3)$-skeleton of $B(f(x), r_0)$ induced by A. The induction hypothesis gives that g is locally a PL mapping which respects the simplicial decomposition A. By Lemma 4.4.1 it is globally a PL mapping, which respects the simplicial decomposition A.

The set $B(f(x), r_0)$ was chosen to be radially symmetric. Therefore, the map cone(g) also respects the simplicial decomposition A on S^n. Thus f satisfies the conclusion of the lemma. □

Theorem 4.1.1 follows immediately from Lemma 4.4.2. The combination of Lemmas 4.4.1 and 4.4.2 proves Corollary 4.1.2.

4.5 Construction of a quasiregular mapping

Our main results, Theorem 4.1.1 and Corollary 4.1.2, can be used to produce examples of quasiregular mappings between manifolds. We give one such construction in this section.

Proof of Theorem 4.1.3. We first note that the manifold $\mathbb{CP}^1$ is homeomorphic to $\widehat{\mathbb{C}}$ and $(\widehat{\mathbb{C}})^n$ is quasiregularly elliptic via e.g. the Alexander mapping, see [Ric93]. Additionally, the composition of quasiregular mappings is still quasiregular. Thus in order to prove quasiregular ellpiticity of $\mathbb{CP}^n$, it suffices to construct a quasiregular mapping $(\mathbb{CP}^1)^n \to \mathbb{CP}^n$.

We first construct a branched covering $f\colon (\mathbb{CP}^1)^n \to \mathbb{CP}^n$. Consider the polynomial

$$p(u, v) = (z_1 u + w_1 v) \dots (z_n u + w_n v).$$

The coefficients of each term are homogeneous polynomials in $([z_i : w_i])_{i=1}^n$, so in particular the coefficients define a continuous map $f\colon (\mathbb{CP}^1)^n \to \mathbb{CP}^n$. By the definition of the mapping, f is locally injective outside the set

$$B_f = \{([z_1 : w_1], \dots, [z_n : w_n]) : [z_i : w_i] = [z_j : w_j] \text{ for } i \neq j\}$$

and at each point $x \in B_f$, f is k-to-1 for some $k = k(x) < \infty$. Thus f is discrete. To see that f is open, we note that away from B_f the mapping is open by local injectivity and on the branch set B_f, f is locally equivalent to a polynomial, and is thus an open map. Thus we conclude that f is a branched cover.

Again by the definition of f, it is clear that B_f has locally a simplicial structure. Since f is locally a polynomial, we see that $f(B_f)$ is also locally topologically equivalent to an $(n-2)$-simplicial complex in $\mathbb{CP}^n$. Thus by Theorem 4.1.1 f is locally equivalent to a PL mapping and hence topologically equivalent to a quasiregular mapping. A similar argument as in Lemma 4.4.1 implies that there exists PL structures on $(\mathbb{CP}^1)^n$ and $\mathbb{CP}^n$ so that f is equivalent to a PL map. That is, there exists a map, $\widetilde{f}\colon X \to Y$ such that X and Y are PL manifolds and the following diagram commutes:

$$\begin{tikzcd} (\mathbb{CP}^1)^n \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{CP}^n \arrow[d, "\psi"] \\ X \arrow[r, "\widetilde{f}"] & Y \end{tikzcd}$$

where the mappings ϕ and ψ are homeomorphisms. The spaces X and Y have a PL structure and so they also have a quasiconformal structure. When the dimension is not 4, that is, $n \neq 2$, by [Sul79] there is in fact a unique quasiconformal structure. Thus we can identify X and Y with $\times_{i=1}^{n}\mathbb{CP}^1$ and $\mathbb{CP}^n$, respectively. In the case $n = 4$, a direct computation of the maps shows the same result. Thus we conclude that there exists a quasiregular mapping

$$\widetilde{f}\colon (\mathbb{CP}^1)^n \to \mathbb{CP}^n$$

and we conclude this implies that $\mathbb{CP}^n$ is quasiregularly elliptic for all $n \geq 2$. $\square$

Remark 4.5.1. In [HR98] Heinonen and Rickman ask the following: *Let $f\colon S^3 \to S^3$ be a branched cover. Does there exist homeomorphisms $h_1, h_2\colon S^3 \to S^3$ such that $h_1 \circ f \circ h_2$ is a quasiregular mapping?* The methods in this section offer an advance in the understanding of the problem; indeed, the techniques here can be used to show that for $n \geq 4$ any branched cover $f\colon S^n \to S^n$ with fB_f contained in a $(n-2)$-simplicial complex is, up to a conjugation by homeomorphisms, a quasiregular mapping.

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