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Toric Stacks

by

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Abstract

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The first purpose of this dissertation is to introduce and develop a theory of toric stacks which encompasses and extends the notions of toric stacks defined in [Laf02, BCS05, FMN09, Iwa09a, Sat09], as well as classical toric varieties. In addition to introducing a broader class of smooth toric stacks, the definition we introduce allows singularities.

The second purpose is to characterize toric stacks in a "bottom up" fashion, similar to the treatment of smooth toric Deligne-Mumford stacks in [FMN09] and the characterization of toric varieties as "abstract toric schemes" which are reduced, separated, and normal.

Contents

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1 Introduction

Recently, a number of theories of toric stacks have been introduced [Laf02, BCS05, FMN09, Iwa09a, Sat09]. There are several reasons one may be interested in developing such a theory. First, these stacks provide a natural place to test conjectures and develop intuition about algebraic stacks, much the same way that toric varieties do for schemes. After developing an adequate theory, they are easy to work with combinatorially, just as toric varieties are. Second, in some situations toric stacks can serve as better-behaved substitutes for toric varieties. A toric variety has a canonical overlying smooth stack. It is sometimes easier to prove results on the smooth stack and "push them down" to the toric variety. Third, with the appropriate machinery, one can show that an "abstract toric stack" (e.g. the closure of a torus within some stack of interest) often arises from a combinatorial "stacky fan." The combinatorial theory of stacky fans then allows one to effectively investigate the stack in question.

There are three distinct kinds of toric stacks in the literature.

Lafforgue's Toric Stacks. In [Laf02], Lafforgue defines a toric stack to be the stack quotient of a toric variety by its torus. These stacks are very "small" in the sense that they have a dense open point. They are rarely smooth.

Smooth Toric Stacks. Borisov, Chen, and Smith defined smooth toric Deligne-Mumford stacks in [BCS05]. These are the stacks studied in [FMN09] and [Iwa09b]. They are smooth and have simplicial toric varieties as their coarse moduli spaces. Satriano generalized this approach in [Sat09] to include certain smooth toric Artin stacks which have toric varieties as their good moduli spaces.

Toric Varieties. Toric varieties are neither "small" nor smooth in general, so the standard theory of toric varieties is not subsumed by the above approaches.

The following definition unifies and extends these approaches. We define a *toric stack* to be the stack quotient of a toric variety X by a subgroup G of its torus T_0. The stack $[X/G]$ has a dense open torus $T = T_0/G$ which acts on $[X/G]$. An integral T-invariant substack of $[X/G]$ is necessarily of the form $[Z/G]$ where $Z \subseteq X$ is an integral T_0-invariant subvariety of X.[1] The subvariety Z is naturally a toric variety whose torus T' is a quotient of T_0. The quotient stack $[Z/G]$ contains a dense open "stacky torus" $[T'/G]$ which acts on $[Z/G]$.

Definition 1.1. In the notation of the above paragraph, a *toric stack* is an Artin stack of the form $[X/G]$, together with the action of the torus $T = T_0/G$. A *generically stacky toric stack* is an Artin stack which is a closed substack of a toric stack, i.e. is of the form $[Z/G]$, together with the action of the stacky torus $[T'/G]$.

[1] Note that Z must be irreducible because G cannot permute the irreducible T_0-invariant subvarieties of X.

Remark 1.2. This definition neatly encompasses and extends the three kinds of toric stacks listed above.

- Taking G to be trivial, we see that any toric variety X is a toric stack.
- Smooth toric Deligne-Mumford stacks in the sense of [BCS05, FMN09, Iwa09a] are smooth generically stacky toric stacks which happen to be separated and Deligne-Mumford. This is explained in the discussion immediately after Definition 2.18.
- Toric stacks in the sense of [Laf02] are toric stacks that have a dense open point with no stabilizer (i.e. toric stacks for which $G = T_0$).
- A toric Artin stacks in the sense of [Sat09] is a smooth generically stacky toric stack with finite generic stabilizer and a toric variety of the same dimension as a good moduli space. See Sections 4 and 6.

We develop the theory of toric stacks in two essentially different ways, which we refer to as the combinatorial approach and the intrinsic approach.

1.1 The Combinatorial Approach

Just as toric varieties can be understood in terms of fans, toric stacks can be understood in terms of combinatorial objects called *stacky fans*. The first part of this dissertation is dedicated to developing the dictionary between the combinatorics of stacky fans and the geometry of toric stacks. We define the basic objects of study, stacky fans, in Section 2.

In Section 3, we prove that any toric morphism of toric stacks is induced by a morphism of stacky fans (Theorem 3.5), a key result in the dictionary between toric stacks and stacky fans.

Sections 2 and 3 provide a sufficient base to generate interesting examples. In Section 4, we highlight a particularly easy to handle class of toric stacks, which we call *fantastacks*. The stacks defined in [BCS05] and [Sat09] which have no generic stabilizer are fantastacks. Generically stacky toric stacks are considerably more general than fantastacks, but it is sometimes easiest to understand a generically stacky toric stack in terms of its relation to some fantastack. For example, the stacks defined in [BCS05] and [Sat09] are closed substacks of fantastacks.

$$\{\text{fantastacks}\} \subset \{\text{toric stacks}\} \subset \{\text{generically stacky toric stacks}\}$$

Section 5 is devoted to the construction of the *canonical stack* over a toric stack. The main result is Proposition 5.7, which justifies the terminology by showing that canonical stacks have a universal property. Canonical stacks are minimal “stacky resolutions” of singularities. Heuristically, the existence of a canonical resolution of singularities is desirable because it is sometimes possible to prove theorems on the smooth resolution and then descend them to the singular base. Indeed, the main result of this dissertation, Theorem 12.1, is proved in this way.

In Section 6, we prove several results that identify toric good moduli space morphisms in the sense of [Alp08]. Good moduli space morphisms generalize the notion of a coarse moduli space and that of a good quotient in the sense of [GIT]. Good moduli space morphisms are of central interest in the theory of moduli, so it is useful to have tools for easily identifying and handling many examples.

In Section 7, we prove a moduli interpretation for smooth toric stacks (Theorem 7.7). That is, we characterize morphisms to smooth toric stacks from an arbitrary source, rather than only toric morphisms from toric stacks. The familiar moduli interpretation of $\mathbb{P}^n$ is that specifying a morphism to $\mathbb{P}^n$ is equivalent to specifying a line bundle, together with $n+1$ sections that generate it. Cox generalized this interpretation to smooth toric varieties in [Cox95], and Perroni further generalized it to smooth toric Deligne-Mumford stacks in [Per08]. Smooth toric stacks are the natural closure of this class of moduli problems. In other words, any moduli problem of the same sort as described by Cox and Perroni is represented by a smooth toric stack (see Remark 7.10).

Much of this first part of the dissertation can be done over an arbitrary base, but we work over an algebraically closed field in order to avoid imposing confusing hypotheses (e.g. every subgroup of a torus we consider is required to be diagonalizable).

1.2 The Intrinsic Approach

As for toric varieties, there is an intrinsic approach to toric stacks. A toric variety can be defined as a reduced finite type scheme with a torus action which contains a dense open copy of the torus (i.e. stabilizer-free orbit). It is a classical result that such an "abstract toric variety" arises from a fan (i.e. is a "combinatorial toric variety") if and only if it is separated and normal.

In [FMN09], Fantechi, Mann, and Nironi develop an analogous approach to smooth toric Deligne-Mumford stacks. They define "abstract" smooth toric Deligne-Mumford stacks roughly as a smooth separated Deligne-Mumford stack with an action of a "stacky torus" and a dense open copy of this torus. They show in [FMN09, Theorem 7.24] that such a stack arises from a stacky fan as defined in [BCS05, §3].

In Sections 8–12, we aim to give a similar intrinsic characterization of toric stacks. Theorem 12.1 achieves this goal. It would be desirable to prove an analogous result for the generically stacky case, but this would require a careful study of actions of stacky tori and some additional hypotheses (see Example 11.10).

Theorem 12.1. *Let $\mathcal{X}$ be an Artin stack over an algebraically closed field k of characteristic 0. Suppose $\mathcal{X}$ has an action of a torus T and a dense open substack which is T-equivariantly isomorphic to T. Then $\mathcal{X}$ is a toric stack if and only if the following conditions hold:*

1. *$\mathcal{X}$ is normal, reduced, and of finite type,*

2. *$\mathcal{X}$ has affine diagonal,*

3. *geometric points of $\mathcal{X}$ have linearly reductive stabilizers, and*

4. *every point of $[\mathcal{X}/T]$ is in the image of an étale representable map from a stack of the form $[U/G]$, where U is quasi-affine and G is an affine group.*

It is possible that condition 4 is always satisfied given the other hypotheses in the theorem. If this is true, then as a special case, we recover a new proof of [FMN09, Theorem 7.24] in the case when $\mathcal{X}$ has trivial generic stabilizer.

It is worth noting that unlike the classical result about toric varieties and the result in [FMN09], we cannot require our stacks to be separated. Indeed, algebraic stacks which are not Deligne-Mumford are hardly ever separated. The condition that the stack have affine diagonal essentially replaces the separatedness condition (see Remark 11.6). In particular, there are toric stacks which are schemes, but which are not toric varieties because they are not separated (see Example 2.15).

This part of the dissertation can be done over a separably closed field k, but we additionally impose the condition that k is of characteristic zero to avoid confusing hypotheses (e.g. that every group we consider is smooth).

Remark 1.3 (The Log Geometric Approach). There is yet another approach to toric geometry, namely that of log geometry. In this dissertation, we do not develop this approach to toric stacks. We refer the interested reader to [Sat09, §§5–6], in which the log geometric approach is taken for fantastacks.

Logical Dependence of Sections

The logical dependence of sections is roughly as follows.

combinatorial approach intrinsic approach

$$\begin{tikzcd}
& 2 \arrow[d] \arrow[dr] & & & 9 \arrow[d] \\
& 3 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & 7 & & 10 \arrow[d] \\
6 & 4 & 5 \arrow[drr] & 8 \arrow[dr] & 11 \arrow[d] \\
& & & & 12
\end{tikzcd}$$

2 Definitions

For a brief introduction to algebraic stacks, we refer the reader to [Ols08, Chapter 1]. For a more detailed treatment, we refer to [Vis05] or [LMB00]. If the reader is unfamiliar with stacks, we encourage her to continue reading, simply treating $[X/G]$ as a formal quotient of a scheme X by an action of a group G. Just as it is possible to learn the theory of toric varieties as a means of learning about varieties in general, we hope the theory of toric stacks can serve as an introduction to the theory of algebraic stacks.

We refer the reader to [Ful93, Chapter 1] or [CLS11, Chapter 3] for the standard correspondence between fans on lattices and toric varieties, and for basic results about toric varieties. We will follow the notation in [CLS11] whenever possible.

2.1 The Toric Stack of a Stacky Fan

Suppose X is a toric variety and $G \subseteq T_0$ is a subgroup of its torus. We may then encode the toric stack $[X/G]$ combinatorially as follows.

Associated to the toric variety X is a fan Σ on the lattice of 1-parameter subgroups of T_0, $L = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{G}_m, T_0)$ (see [Ful93, §1.4] or [CLS11, §3.1]). The surjection of tori $T_0 \to T_0/G$ is encoded by the induced homomorphism of lattices of 1-parameter subgroups, $\beta\colon L \to N = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{G}_m, T_0/G)$. We may therefore recover the toric stack $[X/G]$ from the pair $(\Sigma, \beta\colon L \to N)$. We will refer to such a pair as a "stacky fan." In order to generalize this notion to include generically stacky toric stacks, we first introduce some terminology.

Definition 2.1. Suppose B is a finitely generated abelian group and $A \subseteq B$ is a subgroup. The *saturation of A in B* is the subgroup

$$\mathrm{sat}_B A = \{b \in B | n \cdot b \in A \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

We say A is *saturated in B* if $A = \mathrm{sat}_B A$. We say that a homomorphism $f\colon A \to B$ is *saturated* if $f(A)$ is saturated in B.

Remark 2.2. Saturated morphisms are precisely morphisms whose cokernels are lattices (free abelian groups). In particular, the image of a saturated morphism has a direct complement.

Definition 2.3. A homomorphism of finitely generated abelian groups $f\colon A \to B$ is *close* if $\mathrm{sat}_B f(A) = B$.

Remark 2.4. Equivalently, f is close if the dual homomorphism $f^*\colon B^* = \mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}) = A^*$ is injective. Note that the property of being a close morphism depends only on the quasi-isomorphism class of the mapping cone $C(f) = [A \xrightarrow{f} B]$ in the derived category of abelian groups.

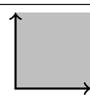
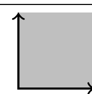
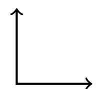
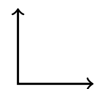
Definition 2.5. A *generically stacky fan* is a pair (Σ, β), where Σ is a fan on a lattice L, and $\beta\colon L \to N$ is a homomorphism to a finitely generated abelian group. If N is a lattice and β is close, we say (Σ, β) is a *stacky fan*.

Definition 2.6. If L is a lattice, we denote by T_L the torus $D(L^*) = \mathrm{Hom}_{\mathrm{gp}}(L^*, \mathbb{G}_m)$ whose lattice of 1-parameter subgroups is naturally isomorphic to L.

Remark 2.7. Here, the functor $D(-)$ is the *Cartier dual* $\mathrm{Hom}_{\mathrm{gp}}(-, \mathbb{G}_m)$. It is an anti-equivalence of categories between finitely generated abelian groups and diagonalizable group schemes. See [SGA3, Exposé VIII].

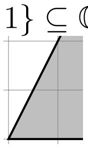
Given a stacky fan (Σ, β), we construct a toric stack as follows. Let X_Σ be the toric variety associated to Σ. The dual of β, $\beta^*\colon N^* \to L^*$, induces a homomorphism of tori $T_\beta\colon T_L \to T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. Since β is close, β^* is injective, so T_β is surjective. Let $G_\beta = \ker(T_\beta)$. Note that T_L is the torus of X_Σ, and $G_\beta \subseteq T_L$ is a subgroup.

Definition 2.8. Using the notation in the above paragraph, if (Σ, β) is a stacky fan, we define the toric stack $\mathcal{X}_{\Sigma,\beta}$ to be $[X_\Sigma/G_\beta]$, with the torus $T_N = T_L/G_\beta$.

Example	2.9	2.10	2.12	2.13	2.15	2.16
Σ	Σ					
L	N	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$
$\downarrow \beta$	$\downarrow \mathrm{id}$	$\downarrow \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$	$\downarrow (1\ 0)$	$\downarrow (1\ {-1})$	$\downarrow (1\ 1)$	$\downarrow 2$
N	N	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Example 2.9 (Toric Varieties). Suppose Σ is a fan on a lattice N. Letting $L = N$ and $\beta = \mathrm{id}_N$, we see that the induced map $T_N \to T_N$ is the identity map, so G_β is trivial. So $\mathcal{X}_{\Sigma,\beta}$ is the toric variety X_Σ. $\diamond$

Example 2.10. Here $X_\Sigma = \mathbb{A}^2$. We have that β^* is given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) : \mathbb{Z}^2 \to \mathbb{Z}^2$, so the induced map on tori is $\mathbb{G}_m^2 \to \mathbb{G}_m^2$ given by $(s, t) \mapsto (st, t^2)$. The kernel is $G_\beta = \mu_2 = \{(\zeta, \zeta) | \zeta^2 = 1\} \subseteq \mathbb{G}_m^2$.



So we see that $\mathcal{X}_{\Sigma,\beta} = [\mathbb{A}^2/\mu_2]$, where the action of μ_2 is given by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. Note that this is a smooth stack. It is distinct from the singular toric variety with the fan shown to the left. $\diamond$

Since quotients of subschemes of $\mathbb{A}^n$ by subgroups of $\mathbb{G}_m^n$ appear frequently, we often include the weights of the action in the notation.

Notation 2.11. Let $G \hookrightarrow \mathbb{G}_m^n$ be the subgroup corresponding to the surjection $\mathbb{Z}^n \to D(G)$. Let g_i be the images of e_i in $D(G)$. Let $X \subseteq \mathbb{A}^n$ be a subscheme. We denote the quotient $[X/G]$ by $[X/_{(g_1\ \cdots\ g_n)} G]$.

In this notation, the stack in Example 2.10 would be denoted $[\mathbb{A}^2/_{(1\ 1)}\mu_1]$.

Example 2.12. Again we have that $X_\Sigma = \mathbb{A}^2$. This time $\beta^* = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : \mathbb{Z} \to \mathbb{Z}^2$, which induces the homomorphism $\mathbb{G}_m^2 \to \mathbb{G}_m$ given by $(s, t) \mapsto s$. Therefore, $G_\beta = \mathbb{G}_m = \{(1, t)\} \subseteq \mathbb{G}_m^2$, so $\mathcal{X}_{\Sigma,\beta} = [\mathbb{A}^2/_{(0\ 1)}\mathbb{G}_m] \cong \mathbb{A}^1 \times [\mathbb{A}^1/\mathbb{G}_m]$. $\diamond$

Example 2.13. This time $X_\Sigma = \mathbb{A}^2 \smallsetminus \{(0,0)\}$. We see that $\beta^* = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) : \mathbb{Z} \to \mathbb{Z}^2$, which induces the morphism $\mathbb{G}_m^2 \to \mathbb{G}_m$ given by $(s,t) \mapsto st^{-1}$. So $G_\beta = \mathbb{G}_m = \{(t,t)\} \subseteq \mathbb{G}_m^2$. We then have that $\mathcal{X}_{\Sigma,\beta} = [(\mathbb{A}^2 \smallsetminus \{(0,0)\})/_{(1\ 1)}\mathbb{G}_m] = \mathbb{P}^1$. $\diamond$

Warning 2.14. Examples 2.9 and 2.13 show that non-isomorphic stacky fans (see Definition 3.2) can give rise to isomorphic toric stacks. The two presentations $[(\mathbb{A}^2 \smallsetminus \{(0,0)\})/_{(1\ 1)}\mathbb{G}_m]$ and $[\mathbb{P}^1/\{e\}]$ of the same toric stack are produced by different stacky fans. $\lrcorner$

Example 2.15 (The non-separated line). Again we have that $X_\Sigma = \mathbb{A}^2 \smallsetminus \{(0,0)\}$. However, this time we see that $\beta^* = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) : \mathbb{Z} \to \mathbb{Z}^2$, which induces the homomorphism $\mathbb{G}_m^2 \to \mathbb{G}_m$ given by $(s,t) \mapsto st$. Therefore, $G_\beta = \mathbb{G}_m = \{(t,t^{-1})\} \subseteq \mathbb{G}_m^2$. So we have that $\mathcal{X}_{\Sigma,\beta} = [(\mathbb{A}^2 \smallsetminus \{(0,0)\})/_{(1\ -1)}\mathbb{G}_m]$ is the affine line $\mathbb{A}^1$ with a doubled origin.

This example shows that there are toric stacks which are schemes, but are not toric varieties because they are non-separated. $\diamond$

Example 2.16. Here $X_\Sigma = \mathbb{A}^1$, and $\beta^* = 2 \colon \mathbb{Z} \to \mathbb{Z}$, which induces the map $\mathbb{G}_m \to \mathbb{G}_m$ given by $t \mapsto t^2$. So $G_\beta = \mu_2 \subseteq \mathbb{G}_m$ and $\mathcal{X}_{\Sigma,\beta} = [\mathbb{A}^1/\mu_2]$. $\diamond$

2.2 The Generically Stacky Case

Before we generalize Definition 2.8 to produce a generically stacky toric stack from a generically stacky fan, it is convenient to define some groups associated to 2-term complexes of finitely generated abelian groups with free kernels.

Definition 2.17. Suppose $f \colon A \to B$ is a homomorphism of finitely generated abelian groups so that $\ker f$ is free. For $i = 0, 1$, let $D(G_f^i)$ be $H^i(C(f)^*)$, where $C(f) = [A \xrightarrow{f} B]$ is the mapping cone of f and $(-)^*$ is the derived functor $R\operatorname{Hom}_{\mathrm{gp}}(-, \mathbb{Z})$. We define G_f^i to be the diagonalizable groups corresponding to $D(G_f^i)$, and we define $G_f = G_f^0 \oplus G_f^1$.

Note that the homomorphism $A^* \to D(G_f^1)$ induces a homomorphism $G_f \to D(A^*)$ (that is trivial on G_f^0). In the case where A and B are free abelian groups, $H^i(C(f)^*)$ are simply the kernel ($i = 0$) and cokernel ($i = 1$) of f^*. If we additionally assume f is close (as was the case in §2.1), then f^* has no kernel, so G_f^0 is trivial. In particular, the notation is consistent with the notation in the paragraph above Definition 2.8, where $f = \beta$.

We now generalize Definition 2.8. If β is assumed to be close, the definition essentially agrees with the ones in [BCS05, §3] and [Sat09, §5]. However, those constructions effectively impose additional conditions on Σ (e.g. that Σ is a subfan of the fan of $\mathbb{A}^n$) since it is required to be induced by a fan on $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.18. If (Σ, β) is a generically stacky fan, we define $\mathcal{X}_{\Sigma,\beta}$ to be $[X_\Sigma/G_\beta]$, where the action of G_β on X_Σ is induced by the homomorphism $G_\beta \to D(L^*) = T_L$.

Now we give a more explicit description of $\mathcal{X}_{\Sigma,\beta}$, which also has the benefit of demonstrating that it is a generically stacky toric stack according to Definition 1.1. See Example 4.15 for an illustration of this approach.

Let $(\Sigma, \beta\colon L \to N)$ be a generically stacky fan. Let

$$\mathbb{Z}^s \xrightarrow{Q} \mathbb{Z}^r \to N \to 0$$

be a presentation of N, and let $B\colon L \to \mathbb{Z}^r$ be a lift of β.

Define the fan Σ' on $L \oplus \mathbb{Z}^s$ as follows. Let τ be the cone generated by $e_1, \dots, e_s \in \mathbb{Z}^s$. For each $\sigma \in \Sigma$, let σ' be the cone spanned by σ and τ in $L \oplus \mathbb{Z}^s$. Let Σ' be the fan generated by all the σ'. Corresponding to the cone τ, we have the closed subvariety $Y \subseteq X_{\Sigma'}$, which isomorphic to X_Σ since Σ is the *star* (sometimes called the *link*) of τ [CLS11, Proposition 3.2.7]. We define

$$\begin{aligned} \beta' = B \oplus Q\colon\ L \oplus \mathbb{Z}^s &\longrightarrow \mathbb{Z}^r \\ (l, \mathbf{a}) &\longmapsto B(l) + Q(\mathbf{a}). \end{aligned}$$

Then (Σ', β') is a stacky fan and we see that $\mathcal{X}_{\Sigma,\beta} \cong [Y/G_{\beta'}]$. Note that $C(\beta')$ is quasi-isomorphic to $C(\beta)$, so $G_{\beta'} \cong G_\beta$.

Remark 2.19. Note that if σ is a smooth cone,[1] then the cone spanned by σ and τ is also a smooth cone. So if $\mathcal{X}_{\Sigma,\beta}$ is a *smooth* generically stacky toric stack, then it is a closed substack of a *smooth* toric stack.

Remark 2.20 (On the condition "β is close"). Since the action of G^0_β on X_σ is trivial, we have that $\mathcal{X}_{\Sigma,\beta} = [X_\Sigma/G^1_\beta] \times BG^0_\beta$. It is often easiest to treat this extra stackiness separately. Let N_1 be the saturation of $\beta(L)$ in N, let N_0 be a direct complement, and let $\beta_1\colon L \to N_1$ be the factorization of β through N_1. Then $G^0_\beta = D(N_0^*) = \mathbb{G}_m^{\mathrm{rk}(N_0)}$ and $[X_\Sigma/G^1_\beta] = \mathcal{X}_{\Sigma,\beta_1}$.

We therefore typically assume β is close (or equivalently that $\mathcal{X}_{\Sigma,\beta}$ has finite generic stabilizer), with the understanding that the non-close case can usually be handled by replacing β by β_1.

Remark 2.21 (On the generically stacky case). In this dissertation, we opt to work primarily with toric stacks, since generically stacky toric stacks can be described as closed substacks.

The primary reason for this focus is that we would like to avoid discussing stacky tori and their actions. We refer the interested reader to [FMN09, §1.7, §2, and Appendix B] for a discussion on stacky tori and their actions. All stacky tori that arise in generically stacky toric stacks are of the form $T \times BG$, where T is a (non-stacky) torus, and G is a diagonalizable group.

However, we do deal with generically stacky fans whenever it is possible to do so without delving too heavily into the theory of stacky tori.

[1] A *smooth cone* is a cone whose corresponding toric variety is smooth. See [CLS11, Definition 1.2.16].

3 Morphisms of Toric Stacks

The main goal of this section is to define morphisms of toric stacks and stacky fans, and to show (in Theorem 3.5) that every morphism of toric stacks is induced by a morphism of stacky fans.

After proving Theorem 3.5, we mitigate the problem introduced in Warning 2.14 by studying presentations of cohomologically affine toric stacks and morphisms of stacky fans which induce isomorphisms of toric stacks.

Definition 3.1. A *toric morphism* or a *morphism of (generically stacky) toric stacks* is a morphism which restricts to a homomorphism of (stacky) tori.

Definition 3.2. A *morphism of generically stacky fans* $(\Sigma, \beta : L \to N) \to (\Sigma', \beta' : L' \to N')$ is a pair of group morphisms $\Phi\colon L \to L'$ and $\phi\colon N \to N'$ so that $\beta' \circ \Phi = \phi \circ \beta$ and so that for every cone $\sigma \in \Sigma$, $\Phi(\sigma)$ is contained in a cone of Σ'.

We typically draw a morphism of generically stacky fans as a commutative diagram.

$$\begin{tikzcd} \Sigma \arrow[r] & \Sigma' \\ L \arrow[r, "\Phi"] \arrow[d, "\beta"'] & L' \arrow[d, "\beta'"] \\ N \arrow[r, "\phi"] & N' \end{tikzcd}$$

A morphism of generically stacky fans $(\Phi, \phi)\colon (\Sigma, \beta) \to (\Sigma', \beta')$ induces a morphism of toric varieties $X_\Sigma \to X_{\Sigma'}$ and a compatible morphism of groups $G_\beta \to G_{\beta'}$, so it induces a toric morphism of (generically stacky) toric stacks $\mathcal{X}_{(\Phi,\phi)}\colon \mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma',\beta'}$.

Proposition 3.3. *Let* $\mathbf{P}$ *be a property of morphisms which is stable under composition and base change. For* $i = 0, 1$*, let* $(\Phi_i, \phi_i)\colon (\Sigma_i, \beta_i\colon L_i \to N_i) \to (\Sigma_i, \beta_i\colon L_i \to N_i)$ *is a morphism of generically stacky fans. Then the product morphism* $(\Phi_0 \times \Phi_1, \phi_0 \times \phi_1)$ *induces an morphism of generically stacky toric stacks which has property* $\mathbf{P}$ *if and only if each* (Φ_i, ϕ_i) *does.*

$$\begin{tikzcd} \Sigma_0 \times \Sigma_1 \arrow[r] & \Sigma'_0 \times \Sigma'_1 \\ L_0 \times L_1 \arrow[r, "\Phi_0\times\Phi_1"] \arrow[d, "\beta_0\times\beta_1"'] & L'_0 \times L'_1 \arrow[d, "\beta'_0\times\beta'_1"] \\ N_0 \times N_1 \arrow[r, "\phi_0\times\phi_1"] & N'_0 \times N'_1 \end{tikzcd}$$

(see [CLS11, Proposition 3.1.14] for basic facts about product fans)

Proof. Products of morphisms with property $\mathbf{P}$ have property $\mathbf{P}$ because $f \times g = (f \times \mathrm{id}) \circ (\mathrm{id} \times g)$, and $f \times \mathrm{id}$ (resp. $\mathrm{id} \times g$) is a base change of f (resp. g).

The constructions of $\mathcal{X}_{\Sigma,\beta}$ from (Σ,β) and of $\mathcal{X}_{(\Phi,\phi)}$ from (Φ,ϕ) commute with products, so $(\Phi_0\times\Phi_1,\phi_0\times\phi_1)$ induces the product morphism $\mathcal{X}_{(\Phi_0\times\Phi_1,\phi_0\times\phi_1)}=\mathcal{X}_{(\Phi_0,\phi_0)}\times\mathcal{X}_{(\Phi_1,\phi_1)}$. It follows that if $\mathcal{X}_{(\Phi_i,\phi_i)}$ have property $\mathbf{P}$, then so does $\mathcal{X}_{(\Phi_0\times\Phi_1,\phi_0\times\phi_1)}$.

Conversely, suppose $\mathcal{X}_{(\Phi_0\times\Phi_1,\phi_0\times\phi_1)}=\mathcal{X}_{(\Phi_0,\phi_0)}\times\mathcal{X}_{(\Phi_1,\phi_1)}$ has property $\mathbf{P}$. We have that $\mathcal{X}_{\Sigma'_0,\beta'_0}$ has a k-point, induced by the identity element of its torus, which induces a morphism $\mathcal{X}_{\Sigma'_1,\beta'_1}\to\mathcal{X}_{\Sigma'_0,\beta'_0}\times\mathcal{X}_{\Sigma'_1,\beta'_1}$. Base changing $\mathcal{X}_{(\Phi_0,\phi_0)}\times\mathcal{X}_{(\Phi_1,\phi_1)}$ by this morphism, we get $\mathcal{X}_{(\Phi_1,\phi_1)}$, so $\mathcal{X}_{(\Phi_1,\phi_1)}$ has property $\mathbf{P}$. Similarly, $\mathcal{X}_{(\Phi_0,\phi_0)}$ has property $\mathbf{P}$. $\square$

3.1 Toric Morphisms are Induced by Morphisms of Stacky Fans

Lemma 3.4. *Let X be a connected scheme, G a group, and $P\to X$ a G-torsor. Suppose $Q\subseteq P$ is a connected component of P.*[1] *Then $Q\to X$ is an H-torsor, where H is the subgroup of G which sends Q to itself.*

Proof. Let ϕ be an automorphism of P. Since Q is a connected component of P, $\phi(Q)$ is either equal to Q or is disjoint from Q. It follows that $(G\times Q)\times_P Q=(H\times Q)\times_P Q\cong H\times Q$, where the map $G\times Q\to P$ is induced by the action of G on P.

The diagonal $Q\to Q\times_X Q\subseteq P\times_X Q$ is a section of the G-torsor $P\times_X Q\to Q$, so it induces a G-equivariant isomorphism $P\times_X Q\cong G\times Q$. We then have the following cartesian diagram.

$$\begin{tikzcd}
H\times Q\cong (G\times Q)\times_P Q \arrow[r]\arrow[d] & Q\times_X Q \arrow[r]\arrow[d] & Q \arrow[d] \\
G\times Q \arrow[r, phantom, "\cong"] & P\times_X Q \arrow[r]\arrow[d] & P \arrow[d] \\
& Q \arrow[r] & X
\end{tikzcd}$$

In particular, the map $H\times Q\to Q\times_X Q$, given by $(h,q)\mapsto(h\cdot q,q)$, is an isomorphism. This shows that Q is an H-torsor. $\square$

Theorem 3.5. *Let $(\Sigma,\beta\colon L\to N)$ and $(\Sigma',\beta'\colon L'\to N')$ be stacky fans, and suppose $f\colon\mathcal{X}_{\Sigma,\beta}\to\mathcal{X}_{\Sigma',\beta'}$ is a toric morphism. Then there exists a stacky fan (Σ_0,β_0) and morphisms $(\Phi,\phi)\colon(\Sigma_0,\beta_0)\to(\Sigma,\beta)$ and $(\Phi',\phi')\colon(\Sigma_0,\beta_0)\to(\Sigma',\beta')$ such that the following triangle commutes and $\mathcal{X}_{(\Phi,\phi)}$ is an isomorphism.*

$$\begin{tikzcd}
\mathcal{X}_{\Sigma_0,\beta_0} \arrow[dr, "\mathcal{X}_{(\Phi',\phi')}"] \arrow[dd, "\mathcal{X}_{(\Phi,\phi)}"', "\wr"] & \\
& \mathcal{X}_{\Sigma',\beta'} \\
\mathcal{X}_{\Sigma,\beta} \arrow[ur, "f"'] &
\end{tikzcd}$$

[1]The hypothesis that X is connected is actually unnecessary, but it makes the condition on Q simpler. It is enough to assume that every connected component of X has exactly one connected component of Q lying over it.

Proof. By assumption, f restricts to a homomorphism of tori $T_N \to T_{N'}$, which induces a homomorphism of lattices of 1-parameter subgroups $\phi\colon N \to N'$.

We define $Y = X_\Sigma \times_{\mathcal{X}_{\Sigma',\beta'}} X_{\Sigma'}$. Since $X_\Sigma \to \mathcal{X}_{\Sigma',\beta'}$ and $X_{\Sigma'} \to \mathcal{X}_{\Sigma',\beta'}$ are toric, $T_L \times_{T_{N'}} T_{L'}$ is a diagonalizable group. The connected component of the identity, $T_0 \subseteq T_L \times_{T_{N'}} T_{L'}$, is a connected diagonalizable group, so it is a torus. Let Y_0 be the connected component of Y which contains T_0, and let G_Φ be the kernel of the homomorphism $T_0 \to T_L$. We then have the following diagram.

$$\begin{tikzcd}
G_\Phi \arrow[rr, hook] \arrow[d, hook] & & G_{\beta'} \arrow[rr, equal] \arrow[d, hook] & & G_{\beta'} \arrow[d, hook] & \\
T_0 \arrow[rr, hook] \arrow[dr, hook] & & T_L \times_{T_{N'}} T_{L'} \arrow[rr] \arrow[dr, hook] \arrow[dd, two heads] & & T_{L'} \arrow[dr, hook] \arrow[dd, two heads] & \\
& Y_0 \arrow[rr, hook] & & Y \arrow[rr] \arrow[dd] & & X_{\Sigma'} \arrow[dd, "G_{\beta'}\text{-torsor}"] \\
& & T_L \arrow[rr] \arrow[dr, hook] & & T_{N'} \arrow[dr, hook] & \\
& & & X_\Sigma \arrow[rr] & & \mathcal{X}_{\Sigma',\beta'}
\end{tikzcd}$$

Since X_Σ is normal and separated, and Y is a $G_{\beta'}$-torsor over X_Σ, we have that Y is normal and separated, so Y_0 is normal, separated, and connected. In particular, Y_0 is irreducible. We have that T_0 is an open subscheme of Y_0, and T_0 acts on Y_0 in a way that extends the multiplication, so Y_0 is a toric variety with torus T_0. Say it corresponds to a fan Σ_0 on the lattice L_0 of 1-parameter subgroups of T_0.

Now $Y_0 \to X_\Sigma$ and $Y_0 \to X_{\Sigma'}$ are morphisms of toric varieties, so they are induced by morphisms of fans $\Sigma_0 \to \Sigma$ and $\Sigma_0 \to \Sigma'$. Defining β_0 to be the composition $L_0 \xrightarrow{\Phi} L \xrightarrow{\beta} N$, we have morphisms of stacky fans

$$\begin{tikzcd}
\Sigma & \Sigma_0 \arrow[l] \arrow[r] & \Sigma' \\
L \arrow[d, "\beta"'] & L_0 \arrow[l, "\Phi"'] \arrow[r] \arrow[d, "\beta_0"'] & L' \arrow[d, "\beta'"] \\
N \arrow[r, equal] & N \arrow[r, "\phi"] & N'
\end{tikzcd}$$

Note that G_Φ is the kernel of the surjection $T_0 \to T_L$, so the notation is consistent with Definition 2.17. By construction, G_Φ is the subgroup of $G_{\beta'}$ which takes T_0 to itself, so it is the subgroup which takes $Y_0 = \overline{T_0}$ to itself. By Lemma 3.4, Y_0 is a G_Φ-torsor over X_Σ.

Since T_0 is a connected component of a group that surjects onto T_L, the induced morphism $T_0 \to T_L$ is surjective, so Φ is close. By Lemma A.2, G_{β_0} is an extension of G_β by G_Φ. The morphism of stacky fans $(\Sigma_0, \beta_0) \to (\Sigma, \beta)$ induces the isomorphism $\mathcal{X}_{\Sigma_0,\beta_0} = [Y/G_{\beta_0}] \to [(Y/G_\Phi)/G_\beta] = \mathcal{X}_{\Sigma,\beta}$.

On the other hand, the morphism $(\Sigma_0, \beta_0) \to (\Sigma', \beta')$ induces the morphism $[Y_0/G_{\beta_0}] \cong [X_\Sigma/G_\beta] \to \mathcal{X}_{\Sigma',\beta'}$. $\square$

3.2 The Cohomologically Affine Case

In this subsection, we study cohomologically affine generically stacky toric stacks in some detail. Roughly, the goal is to mitigate Warning 2.14 and establish a tight connection between cohomologically affine generically stacky toric stacks and their stacky fans. This is important in the proofs of some technical results, specifically Lemma 5.5 and Theorem 8.12.

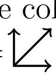
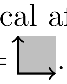
Definition 3.6. A generically stacky toric stack $\mathcal{X}_{\Sigma,\beta}$ is *cohomologically affine* if X_Σ is affine (c.f. Definition 6.1).

A toric variety is affine if and only if its fan is the set of faces of a single cone σ.

Notation 3.7. As a slight abuse of notation, we use the symbol σ to denote a cone as well as the fan consisting of the cone σ and all of its faces.

Lemma 3.8. *Let $(\Phi, \phi)\colon (\sigma, \beta) \to (\Sigma', \beta')$ be a morphism of generically stacky fans, with σ a single cone. Suppose every torus orbit of $\mathcal{X}_{\Sigma',\beta'}$ is in the image of the induced morphism $\mathcal{X}_{(\Phi,\phi)}$ (e.g. if $\mathcal{X}_{(\Phi,\phi)}$ is surjective). Then the map $\sigma \to \Sigma'$ is a surjection. In particular, Σ' is a single cone.*

Proof. The cones of Σ' correspond to torus orbits of $X_{\Sigma'}$, and to those of $\mathcal{X}_{\Sigma',\beta'}$. We see that every torus orbit of $X_{\Sigma'}$ is in the image of X_σ. Thus, the relative interior of every cone of Σ' contains the image of some face of σ. Therefore $\Phi(\sigma)$ intersects the relative interiors of all cones, and in particular all rays, of Σ'. Since $\Phi(\sigma)$ is a convex polyhedral cone, it follows that the $\Phi(\sigma)$ is the cone generated by the rays of Σ'. In particular, $\Sigma' \subseteq \Phi(\sigma)$. $\square$

Remark 3.9. The cohomological affineness condition in Lemma 3.8 cannot be removed. For example, let $\Sigma =$  and $\Sigma' =$ . Then X_Σ is the blowup of $\mathbb{A}^2$ at the origin, minus the two torus-invariant points of the exceptional divisor, and $X_{\Sigma'}$ is $\mathbb{A}^2$. The natural map $X_\Sigma \to X_{\Sigma'}$ is surjective, but the map on fans is not.

Definition 3.10. An affine toric variety X_σ is *pointed* if it contains a torus-invariant point. Alternatively, X_σ is pointed if σ spans the ambient lattice L. If (σ, β) is a generically stacky fan and X_σ is pointed, we say that $\mathcal{X}_{\sigma,\beta}$ is a *pointed generically stacky toric stack*. In this case, we say that (σ, β) is a *pointed generically stacky fan*.

Note that a pointed toric variety has a *unique* torus-invariant point.

Remark 3.11. Note some immediate consequences of the equivalence of categories between toric varieties and fans. For any affine toric variety X_σ, there is a canonical pointed toric subvariety. Explicitly, let $L_\sigma \subseteq L$ be the sublattice spanned by σ. Then X_{σ,L_σ} is pointed, and the inclusion $L_\sigma \to L$ induces a toric closed immersion $X_{\sigma,L_\sigma} \to X_{\sigma,L}$. We have that $X_{\sigma,L}$ is (non-canonically) isomorphic to the product of X_{σ,L_σ} and the torus T_{L/L_σ}.

Remark 3.12. Similarly, if $(\sigma, \beta\colon L \to N)$ is a generically stacky fan, with σ a single cone, there is a canonical morphism from a pointed toric stack. Let $L_\sigma \subseteq L$ be the saturated sublattice generated by σ, let $N_\sigma = \mathrm{sat}_N(\beta(L_\sigma))$, and let $\beta_\sigma\colon L_\sigma \to N_\sigma$ be the morphism induced by β. Then we have a morphism of generically stacky fans $(\sigma, \beta_\sigma) \to (\sigma, \beta)$. We see that $\mathcal{X}_{\sigma,\beta}$ is (non-canonically) isomorphic to the product of the pointed generically stacky toric stack $\mathcal{X}_{\sigma,\beta_\sigma}$ and the stacky torus $\mathcal{X}_{*,L/L_\sigma \to N/N_\sigma}$, where $*$ is the trivial fan.

Remark 3.13. Note that $\mathcal{X}_{\sigma,\beta_\sigma}$, as defined in Remark 3.12, has the following property. Any toric morphism to $\mathcal{X}_{\sigma,\beta}$ from a pointed toric stack factors uniquely through $\mathcal{X}_{\sigma,\beta_\sigma}$. This follows immediately from the fact that any toric morphism from a pointed toric stack to a stacky torus must be trivial.

Lemma 3.14. *Let X_σ be a pointed affine toric variety, and let $Y \to X_\sigma$ be a toric morphism from a toric variety making Y into a G-torsor over X_σ for some group G. Then $Y \to X_\sigma$ is a trivial torsor and G is a torus. In particular, $Y \cong G \times X_\sigma$ as a toric variety, so there is a canonical toric section $X_\sigma \to Y$.*

Proof. We have that G is the kernel of the homomorphism of tori induced by $Y \to X_\sigma$, so it is diagonalizable. We decompose G as a product of a finite group G_0 and a torus $\mathbb{G}_m^r$.

We then have that $Y_0 = Y/\mathbb{G}_m^r$ is a toric variety which is a G_0-torsor over X_σ. The fiber over the torus-invariant point of X_σ is then a torus invariant finite subset of Y_0. A torus has no finite-index subgroups, so any finite torus-invariant subset of a toric variety must consist of *fixed points* of the torus action. On the other hand, Y_0 is affine over X_σ, so it is affine, so it contains at most one torus fixed point. Therefore, G_0 must be trivial, so $G \cong \mathbb{G}_m^r$.

We have that G-torsors over X_σ are parametrized by $H^1(X_\sigma, G) \cong H^1(X_\sigma, \mathbb{G}_m)^r \cong \mathrm{Pic}(X_\sigma)^r$. By [CLS11, Proposition 4.2.2], we have that $\mathrm{Pic}(X_\sigma) = 0$, so all $\mathbb{G}_m^r$-torsors on X_σ are trivial. It follows that $Y = \mathbb{G}_m^r \times X_\sigma$, so there is a canonical toric section. □

Corollary 3.15. *Let $\mathcal{X}_{\sigma,\beta\colon L\to N}$ be a pointed generically stacky toric stack, where σ spans the lattice L. Let $f\colon \mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\Sigma',\beta'}$ be a homomorphism of toric stacks. Then f is induced by a morphism of stacky fans $(\sigma,\beta) \to (\Sigma',\beta')$.*

Proof. Following the proof (and notation) of Theorem 3.5, we see that there is toric variety Y_0 with a toric morphism $Y_0 \to X_\sigma$ making Y_0 into a G_Φ-torsor over X_σ. By Lemma 3.14, there is a canonical toric section s. This toric section induces a section of the morphism of stacky fans $(\Phi, \mathrm{id}_N)\colon (\Sigma_0, \beta_0) \to (\sigma, \beta)$.

$$\begin{tikzcd}
\sigma \arrow[r, bend left, "s"] & \Sigma_0 \arrow[l] \arrow[r] & \Sigma' \\
L \arrow[r, bend right] \arrow[d, "\beta"'] & L_0 \arrow[l, "\Phi"'] \arrow[r] \arrow[d, "\beta_0"] & L' \arrow[d, "\beta'"] \\
N \arrow[r, equal] & N \arrow[r, "\phi"] & N'
\end{tikzcd}$$

The composition $(\sigma,\beta) \to (\Sigma',\beta')$ then induces the morphism f. □

Corollary 3.16. *Let $(\Sigma, \beta\colon L \to N)$ be a generically stacky fan. There is a natural bijection between toric morphisms $\mathbb{A}^1 \to \mathcal{X}_{\Sigma,\beta}$ and elements of $\Sigma \cap L$.*

Proof. This follows immediately from Corollary 3.15 and the usual description of the fan of $\mathbb{A}^1$, namely $(\longrightarrow, \mathrm{id}\colon \mathbb{Z} \to \mathbb{Z})$. □

Lemma 3.17. *Let $(\Phi, \phi)\colon (\sigma, \beta\colon L \to N) \to (\sigma', \beta'\colon L' \to N')$ be a morphism of generically stacky fans, with σ and σ' single cones. Suppose the induced morphism $\mathcal{X}_{(\Phi,\phi)}$ is an isomorphism. Then Φ induces an isomorphism of monoids $\sigma \cap L \to \sigma' \cap L'$.*

Proof. By Corollary 3.16, elements of the monoid $\sigma \cap L$ are in bijection with toric morphisms from $\mathbb{A}^1$ to $\mathcal{X}_{\sigma,\beta}$. The isomorphism $\mathcal{X}_{(\Phi,\phi)}$ induces a bijection of these sets. On the other hand, the induced morphism is a morphism of monoids. □

3.3 Isomorphisms From Morphisms of (Generically) Stacky Fans

As we saw in Warning 2.14, non-isomorphic stacky fans can induce isomorphic toric stacks. In this section, we prove some useful results for identifying morphisms of stacky fans which induce isomorphisms of toric stacks.

Lemma 3.18. *For $i = 0, 1$, let (Φ_i, ϕ_i) be a morphism of generically stacky fans. Then $(\Phi_0 \times \Phi_1, \phi_0 \times \phi_1)$ induces an isomorphism of generically stacky toric stacks if and only if each (Φ_i, ϕ_i) does.*

Proof. This is an immediate corollary of Proposition 3.3. □

Lemma 3.19. *Let Σ be a fan on a lattice L and let $\beta\colon L \to N$ be a close morphism to a finitely generated abelian group. Let L_0 be a lattice, and $\beta_0\colon L_0 \to N$ any homomorphism. Let $\Sigma \times 0$ be the fan Σ regarded as a fan on $L \oplus L_0$, supported entirely on L. Then the morphism of generically stacky fans $(\Sigma, \beta\colon L \to N) \to (\Sigma \times 0, \beta \oplus \beta_0\colon L \oplus L_0 \to N)$ induces an isomorphism $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma\times 0, \beta\oplus\beta'}$.*

Proof. Since $\Sigma \times 0$ is the product of Σ on L and the trivial fan on L_0, we have that $X_{\Sigma\times 0} = X_\Sigma \times T_{L_0}$.

The first diagram has exact rows.

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\beta"'] & L \oplus L_0 \arrow[r] \arrow[d, "\beta\oplus\beta_0"] & L_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, equal] & N \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & G_\beta \arrow[r] \arrow[d] & G_{\beta\oplus\beta'} \arrow[r] \arrow[d] & G_{L_0\to 0} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & T_L \arrow[r] & T_L \oplus T_{L_0} \arrow[r] & T_{L_0} \arrow[r] & 0
\end{tikzcd}
$$

Since β is close, Lemma A.2 implies that the second diagram has exact rows. We see that the induced morphism is then

$$
\begin{aligned}
\mathcal{X}_{\Sigma,\beta} = [X_\Sigma/G_\beta] \xrightarrow{\sim} \big[[X_{\Sigma\times 0}/T_{L_0}]/G_\beta\big] &= \big[[X_{\Sigma\times 0}/G_{L_0\to 0}]/G_\beta\big] \\
&= [X_{\Sigma\times 0}/G_{\beta\oplus\beta_0}] = \mathcal{X}_{\Sigma\times 0,\beta\oplus\beta_0}.
\end{aligned}
$$ □

Remark 3.20. The condition that β is close is necessary in the above argument. As a simple counterexample to the lemma when β is not close, consider the morphism of generically stacky fans $(0, \mathrm{id})\colon (\Sigma, 0 \to \mathbb{Z}) \to (\Sigma, \mathrm{id}\colon \mathbb{Z} \to \mathbb{Z})$, where Σ is the trivial fan. This induces the morphism $B\mathbb{G}_m \to \mathbb{G}_m$.

Lemma 3.21. *Let σ be a cone on L. Suppose $(\Phi, \phi)\colon (\sigma, \beta\colon L \to N) \to (\Phi(\sigma), \beta'\colon L' \to N')$ is a morphism of generically stacky fans so that ϕ is an isomorphism, Φ is close, and so that Φ induces an isomorphism of monoids $(\sigma \cap L) \to (\Phi(\sigma) \cap L')$. Then the induced morphism $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\Phi(\sigma),\beta'}$ is an isomorphism.*

Proof. Let $L_\sigma \subseteq L$ be the sublattice generated by $\sigma \cap L$, and let L_1 be a direct complement. Let X_σ denote the toric variety corresponding to the cone σ, regarded as a fan on L_σ. The fan σ (on L) is the product $\sigma \times 0$ on $L_\sigma \times L_1$, so $\mathcal{X}_{\sigma,\beta} = [(X_\sigma \times T_{L_1})/G_\beta]$.

By assumption, the sublattice of L' generated by $\Phi(\sigma) \cap L'$ is isomorphic to L_σ, and the isomorphism identifies σ with $\Phi(\sigma)$. Let L_1' be a direct complement to L_σ in L'. As above, we have that $\mathcal{X}_{\Phi(\sigma),\beta'} = [(X_\sigma \times T_{L_1'})/G_{\beta'}]$.

Note that $T_L = T_{L_\sigma} \times T_{L_1}$ and $T_{L'} = T_{L_\sigma} \times T_{L_1'}$. Since Φ is close, Lemma A.1 tells us that the following diagram has exact rows.

$$\begin{tikzcd}
0 \arrow[r] & G_\Phi \arrow[r] \arrow[d, equal] & G_\beta \arrow[r] \arrow[d] & G_{\beta'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_\Phi \arrow[r] & T_{L_1} \arrow[r] & T_{L_1'} \arrow[r] & 0 \\[-20pt]
 & \times & \times & \times & \\[-20pt]
0 \arrow[r] & 0 \arrow[r] & T_{L_\sigma} \arrow[r, equal] & T_{L_\sigma} \arrow[r] & 0
\end{tikzcd}$$

So we see that the toric morphism induced by (Φ, ϕ) is $\mathcal{X}_{\sigma,\beta} = [(X_\sigma \times T_{L_1})/G_\beta] = \big[[(X_\sigma \times T_{L_1})/G_\Phi]/G_{\beta'}\big] = [(X_\sigma \times T_{L_1'})/G_{\beta'}] = \mathcal{X}_{\Phi(\sigma),\beta'}$. $\square$

Lemmas 3.19 and 3.21 can be combined and extended.

Proposition 3.22. *Let $(\Phi, \phi)\colon (\sigma, \beta\colon L \to N) \to (\sigma', \beta'\colon L' \to N')$ be a morphism of generically stacky fans, with σ and σ' single cones. Suppose*

1. *ϕ is an isomorphism,*
2. *Φ induces an isomorphism of the monoids $(\sigma \cap L)$ and $(\sigma' \cap L')$, and*
3. *$\phi\big(\mathrm{sat}_N \beta(L)\big) = \mathrm{sat}_{N'} \beta'(L')$.*

Then the induced morphism $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma',\beta'}$ is an isomorphism.

Proof. First we reduce to the case when β is close. Let N_0 be a direct complement to $\mathrm{sat}_N \beta(L)$. Then we see that $(\Sigma, \beta\colon L \to N)$ is the product of $\big(\Sigma, \beta_1\colon L \to \mathrm{sat}_N \beta(L)\big)$ and $(*, \beta_0\colon 0 \to N_0)$, where $*$ is the trivial fan (which contains only the zero cone). Condition 3 implies that $\phi(N_0)$ is a direct complement to $\mathrm{sat}_{N'} \beta'(L')$, so the same argument shows that

$(\Sigma', \beta' \colon L' \to N')$ is the product of $\big(\Sigma', \beta'_1 \colon L' \to \mathrm{sat}_{N'}\beta'(L')\big)$ and $\big(*, \beta'_0 \colon 0 \to \phi(N_0)\big)$. We see that (Φ, ϕ) is a product of $\big(\Phi\colon L \to L', \phi_1 \colon \mathrm{sat}_N(\beta(L) \to \mathrm{sat}_{N'}(\beta'(L'))\big)$ and $(0, \phi|_{N_0})$. The later is an isomorphism by conditions 1 and 3, so by Lemma 3.18, we have reduced to the case where $N = \mathrm{sat}_N\beta(L)$.

Let L_σ be the sublattice of L generated by $\sigma \cap L$. Let N_1 be the free part of $\beta(L)$, and choose a splitting $s\colon N_1 \to L$. Let $L_1 = L_\sigma + s(N_1)$ in L.

Applying Lemmas 3.21 and 3.19 in succession, we see that the morphism induced by the composition $(\sigma, \beta|_{L_1}) \to (\sigma, \beta|_{\mathrm{sat}_L L_1}) \to (\sigma, \beta)$ is an isomorphism.

Note that $\Phi|_{L_1}$ has no kernel, so we may identify $\Phi(L_1)$ with L_1. Then the same argument shows that $(\sigma, \beta|_{L_1}) \to (\sigma', \beta')$ induces an isomorphism.

We then have a factorization $(\sigma, \beta|_{L_1}) \to (\sigma, \beta) \xrightarrow{(\Phi,\phi)} (\sigma', \beta')$. Since the first morphism induces an isomorphism and the composite induces an isomorphism, it follows that (Φ, ϕ) induces an isomorphism. □

Definition 3.23. Suppose $\Phi\colon \Sigma \to \Sigma'$ is a morphism of fans and $\sigma' \in \Sigma'$. The *pre-image* of σ', $\Phi^{-1}(\sigma')$, is the subfan of Σ consisting of cones whose image lie in σ'.

Remark 3.24. Suppose $f\colon X_\Sigma \to X_{\Sigma'}$ is the morphism of toric varieties corresponding to the map of fans $\Sigma \to \Sigma'$. The cone $\sigma' \in \Sigma'$ corresponds to an affine open subscheme $U_{\sigma'} = \mathrm{Spec}(k[\sigma'^\vee \cap N']) \subseteq X_{\Sigma'}$, the complement of the divisors corresponding to rays not on σ'. The key property of $\Phi^{-1}(\sigma')$ is that $X_\Sigma \times_{X_{\Sigma'}} U_{\sigma'}$ is naturally the open subvariety $X_{\Phi^{-1}(\sigma')} \subseteq X_\Sigma$.

Proposition 3.25. *Let $(\Phi, \phi)\colon (\Sigma, \beta\colon L \to N) \to (\Sigma', \beta'\colon L' \to N')$ be a morphism of generically stacky fans. Suppose*

1. *β is close,*
2. *ϕ is an isomorphism,*
3. *for every cone $\sigma' \in \Sigma'$, $\Phi^{-1}(\sigma')$ is a single cone, and*
4. *for every cone $\sigma' \in \Sigma'$, Φ induces a isomorphism of monoids $\Phi^{-1}(\sigma') \cap L \to \sigma' \cap L'$.*

Then the induced morphism $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma',\beta'}$ is an isomorphism.

Proof. We may check whether the map is an isomorphism locally on the base, so we may assume Σ' is a single cone σ'. By Proposition 3.22, we get the result. □

Remark 3.26. Suppose X_Σ is a smooth toric variety and $\Phi\colon (\widetilde{\Sigma}, \mathbb{Z}^n) \to (\Sigma, L)$ is its Cox construction [CLS11, §5.1]. Then the induced morphism $\mathcal{X}_{\widetilde{\Sigma},\Phi} \to \mathcal{X}_{\Sigma, id_L} = X_\Sigma$ is an isomorphism. In particular, any smooth toric variety can be expressed as a quotient of a $\mathbb{G}_m^n$-invariant open subvariety of $\mathbb{A}^n$ by a subgroup of $\mathbb{G}_m^n$. Toric stacks of this form are called *fantastacks* (see §4 and Example 4.7).

4 Fantastacks: Easy-to-Draw Examples

In this section, we introduce a broad class of smooth toric stacks which are easy to handle because N is a lattice and the fan on L is induced by a fan on N.

Definition 4.1. Let Σ be a fan on a lattice N, and $\beta\colon \mathbb{Z}^n \to N$ a close homomorphism so that every ray of Σ contains some $\beta(e_i)$ and every $\beta(e_i)$ lies in the support of Σ. For a cone $\sigma \in \Sigma$, let $\hat{\sigma} = cone(\{e_i | \beta(e_i) \in \sigma\})$. We define the fan $\widehat{\Sigma}$ on $\mathbb{Z}^n$ as the fan generated by all the $\hat{\sigma}$. We define $\mathcal{F}_{\Sigma,\beta} = \mathcal{X}_{\widehat{\Sigma},\beta}$. Any toric stack isomorphic to some $\mathcal{F}_{\Sigma,\beta}$ is called a *fantastack*.

Remark 4.2. The cones of $\widehat{\Sigma}$ are indexed by sets $\{e_{i_1}, \dots, e_{i_k}\}$ such that $\{\beta(e_{i_1}), \dots, \beta(e_{i_k})\}$ is contained in a single cone of Σ. It is therefore easy to identify which open subvariety of $\mathbb{A}^n$ is represented by $\widehat{\Sigma}$. Explicitly, define the ideal

$$J_\Sigma = \Big(\prod_{\beta(e_i)\notin\sigma} x_i \;\Big|\; \sigma \in \Sigma \Big).$$

(Note, as in the Cox construction of a toric variety, that J_Σ is generated by the monomials $\prod_{\beta(e_i)\notin\sigma} x_i$, where σ varies over *maximal* cones of Σ.) Then $X_{\widehat{\Sigma}} = \mathbb{A}^n \smallsetminus V(J_\Sigma)$.

Remark 4.3. Since β is a homomorphism of lattices, $C(\beta)^*$ can be computed by simply dualizing β. Since β is assumed to be close, $G_\beta^0 = 0$, so $G_\beta = D(\operatorname{cok}\beta^*)$.

If $f\colon \mathbb{Z}^n \to \operatorname{cok}\beta^*$ is the cokernel of β^*, with $g_i = f(e_i)$, then we have that $\mathcal{F}_{\Sigma,\beta} = \big[(\mathbb{A}^n \smallsetminus V(J_\Sigma))/_{(\,g_1\ \cdots\ g_n\,)} D(\operatorname{cok}\beta^*)\big]$ using Notation 2.11.

Remark 4.4. Fantastacks are precisely the toric Artin stacks in [Sat09] which have trivial generic stabilizers.

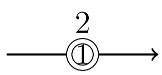
Remark 4.5. The fantastack $\mathcal{F}_{\Sigma,\beta}$ has the toric variety X_Σ as its good moduli space, as we will show in Corollary 6.11 (this is also proved in [Sat09, Theorem 5.5]). In fact, a smooth toric stack $\mathcal{X}$ is a fantastack if and only if it has a toric variety X as a good moduli space and the morphism $\mathcal{X} \to X$ restricts to an isomorphism of tori.

Example 4.6. Let $N = 0$ and Σ the trivial fan on N. Let $\beta\colon \mathbb{Z}^n \to N$ be the zero map. Then $\widehat{\Sigma}$ is the fan of $\mathbb{A}^n$, and $G_\beta = \mathbb{G}_m^n$, so $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^n/\mathbb{G}_m^n]$. ⋄

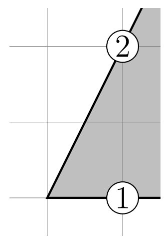
Example 4.7. By Remark 3.26, any smooth toric variety is a fantastack. If X_Σ is a smooth toric variety, then we construct $\beta\colon \mathbb{Z}^n \to \Sigma$ by sending the generators of $\mathbb{Z}^n$ to the first lattice points along the rays of Σ. Then $X_\Sigma \cong \mathcal{F}_{\Sigma,\beta}$.

For a general (non-smooth) fan Σ, one can still construct β as above, but the resulting fantastack $\mathcal{F}_{\Sigma,\beta}$ is not isomorphic to X_Σ. However, it is the canonical stack over X_Σ, a sort of minimal stacky resolution of singularities. See Section 5. ⋄

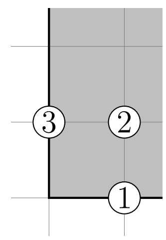
Notation 4.8. When describing fantastacks, we draw the fan Σ and label $\beta(e_i)$ with the number i.



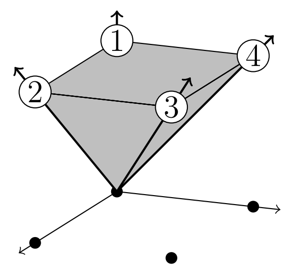
Example 4.9



Example 4.10



Example 4.11



Example 4.12

Example 4.9. Since a single cone contains all the $\beta(e_i)$, we have that $X_{\widehat{\Sigma}} = \mathbb{A}^2$ (see Remark 4.2). We have $\beta = (1\ 1)\colon \mathbb{Z}^2 \to \mathbb{Z}$, so we compute the cokernel

$$\mathbb{Z} \xrightarrow{\beta^* = \binom{1}{1}} \mathbb{Z}^2 \xrightarrow{(1\ \text{-}1)} \mathbb{Z}.$$

We see that $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^2/_{(1\ \text{-}1)}\mathbb{G}_m]$ (see Remark 4.3). ◇

Example 4.10. Since a single cone contains all the $\beta(e_i)$, we have that $X_{\widehat{\Sigma}} = \mathbb{A}^2$ (c.f. Remark 4.2). We compute the cokernel of β^*

$$\mathbb{Z}^2 \xrightarrow{\beta^* = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}/2.$$

Note that β^* can be read off of the picture directly: the rows of β^* are simply the coordinates of the $\beta(e_i)$.

Therefore, $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^2/_{(1\ 1)}\mu_2]$ (c.f. Remark 4.3). This is a "stacky resolution" of the A_1 singularity

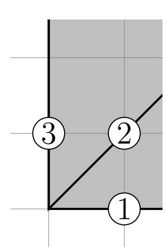
$$\begin{aligned} \mathbb{A}^2/_{(1\ 1)}\mu_2 &= \operatorname{Spec}(k[x_1, x_2]^{\mu_2}) \\ &= \operatorname{Spec} k[x_1^2, x_1x_2, x_2^2] = \operatorname{Spec}\big(k[x,y,z]/(xy - z^2)\big). \end{aligned}$$ ◇

Example 4.11. As in the previous examples, $X_{\widehat{\Sigma}}$ is all of $\mathbb{A}^3$. We compute the cokernel of β^*

$$\mathbb{Z}^2 \xrightarrow{\beta^* = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} \mathbb{Z}^3 \xrightarrow{(1\ -1\ 1)} \mathbb{Z}.$$

So $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^3/_{(1\ \text{-}1\ 1)}\mathbb{G}_m]$.

Note that refining the fan yields an *open* substack. In this example, consider what happens when we refine the fan Σ to the fan Σ' below.



Here G_β is unchanged; indeed, G_β depends only on β, not on Σ. However, we remove the cone $cone(e_1, e_2)$ from $\widehat{\Sigma}'$. The resulting stack is therefore the open substack $\mathcal{F}_{\Sigma',\beta} = [(\mathbb{A}^3 \smallsetminus V(x_1, x_2))/_{(1\ -1\ 1)}\mathbb{G}_m]$, which is the blowup of $\mathbb{A}^2$ at the origin (c.f. Example 4.7).

The birational transformation $Bl_0(\mathbb{A}^2) \to \mathbb{A}^2$ can therefore be realized as the morphism of good moduli spaces induced by the open immersion $\mathcal{F}_{\Sigma',\beta} \to \mathcal{F}_{\Sigma,\beta}$. ⋄

Example 4.12. We have that $X_{\widehat{\Sigma}} = \mathbb{A}^4$. We compute the cokernel

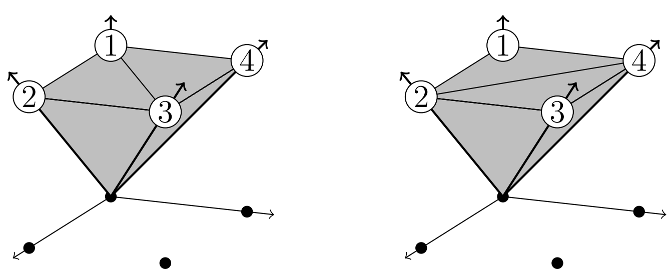
$$\mathbb{Z}^3 \xrightarrow{\beta^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{(1\ \text{-}1\ 1\ \text{-}1)} \mathbb{Z},$$

so $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^4/_{(1\ \text{-}1\ 1\ \text{-}1)}\mathbb{G}_m]$. We will see in Section 5 that $\mathcal{F}_{\Sigma,\beta}$ is the canonical stack over the singular toric variety

$$\begin{aligned} X_\Sigma = \operatorname{Spec} k[x_1, x_2, x_3, x_4]^{\mathbb{G}_m} &= \operatorname{Spec} k[x_1x_2, x_3x_4, x_1x_4, x_2x_3] \\ &= \operatorname{Spec}\big(k[x, y, z, w]/(xy - zw)\big). \end{aligned}$$

It can be regarded as a "stacky resolution" of the singularity.

Note that the two standard toric small resolutions of this singularity are both open substacks of this stacky resolution.



The fan on the left is $[(\mathbb{A}^4 \smallsetminus V(x_2, x_4))/_{(1\ \text{-}1\ 1\ \text{-}1)}\mathbb{G}_m]$ and the fan on the right is $[(\mathbb{A}^4 \smallsetminus V(x_1, x_3))/_{(1\ \text{-}1\ 1\ \text{-}1)}\mathbb{G}_m]$. These are both toric varieties (c.f. Example 4.7). ⋄

4.1 Some Non-fantastack Examples

Example 4.13. Suppose $\{n_1, \ldots, n_k\}$ is a set of positive integers. Let N be $\mathbb{Z}^r \oplus \bigoplus_{i=1}^k (\mathbb{Z}/n_i\mathbb{Z})$, L be 0, Σ be the trivial fan on L, and $\beta\colon L \to N$ the zero map.

To compute G_β, we take a free resolution of $C(\beta)$, namely

$$\mathbb{Z}^k \xrightarrow{diag(n_1,\ldots,n_k)\oplus 0} \mathbb{Z}^k \oplus \mathbb{Z}^r.$$

Then we see that

$$H^0(C(\beta)^*) = D(G^0_\beta) = \mathbb{Z}^r$$
$$H^1(C(\beta)^*) = D(G^1_\beta) = \bigoplus_{i=1}^{k}(\mathbb{Z}/n_i\mathbb{Z})$$

Therefore, $G_\beta = \mathbb{G}_m^r \times \prod \mu_{n_i}$.

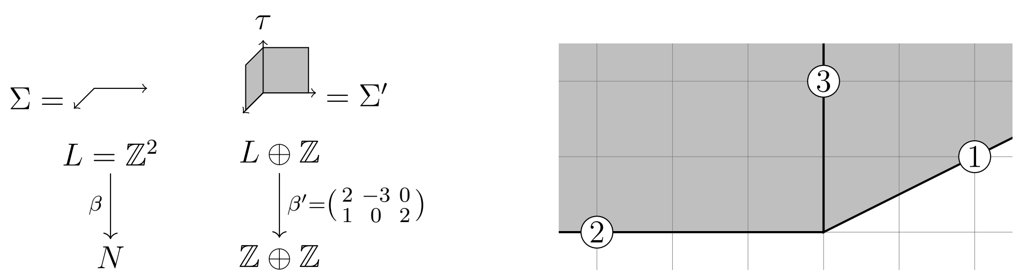
Since $X_\Sigma = \operatorname{Spec} k$, we have that $\mathcal{X}_{\Sigma,\beta} = BG_\beta$. $\diamond$

Example 4.14 (C.f. [BCS05, Examples 2.1 and 3.5]). Consider the stacky fan in which $L = \mathbb{Z}^2$, $\Sigma =$ is the subfan of $\mathbb{A}^2$ corresponding to $\mathbb{A}^2 \smallsetminus \{0\}$, $N = \mathbb{Z} \oplus (\mathbb{Z}/2)$, and $\beta = \left(\begin{smallmatrix} 2 & -3 \\ 1 & 0 \end{smallmatrix}\right) : \mathbb{Z}^2 \to \mathbb{Z} \oplus (\mathbb{Z}/2)$. Then $C(\beta)^*$ is represented by the map $\begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 0 & 2 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^3$. This map is injective, and its cokernel is $(6\ 4\ \text{-}3)\colon \mathbb{Z}^3 \to \mathbb{Z}$. Therefore, $G_\beta = \mathbb{G}_m$, and the induced map to $T_L = \mathbb{G}_m^2$ is given by $t \mapsto (t^6, t^4)$. So $\mathcal{X}_{\Sigma,\beta} = [(\mathbb{A}^2 \smallsetminus \{0\})/_{(6\ 4)}\mathbb{G}_m]$. This is the weighted projective stack $\mathbb{P}(6,4)$, or the moduli stack of elliptic curves $\overline{\mathcal{M}}_{1,1}$. $\diamond$

We repeat the previous example to illustrate that it can be realized as a closed substack of a fantastack. This approach is explained in the discussion following Definition 2.18.

Example 4.15. Consider the stacky fan in which $L = \mathbb{Z}^2$, $\Sigma =$ is the subfan of $\mathbb{A}^2$ corresponding to $\mathbb{A}^2 \smallsetminus \{0\}$, $N = \mathbb{Z} \oplus (\mathbb{Z}/2)$, and $\beta = \left(\begin{smallmatrix} 2 & -3 \\ 1 & 0 \end{smallmatrix}\right) : \mathbb{Z}^2 \to \mathbb{Z} \oplus (\mathbb{Z}/2)$.

We replace β by the quasi-isomorphic map $\beta' = \left(\begin{smallmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right) : \mathbb{Z}^3 \to \mathbb{Z}^2$, and the fan Σ by the fan Σ' obtained by adding the cone τ.



We see that $\mathcal{X}_{\Sigma',\beta'}$ is the fantastack corresponding to the fan on the right. Explicitly, it is the fantastack $[(\mathbb{A}^3 \smallsetminus V(x_1, x_2))/_{(6\ 4\ \text{-}3)}\mathbb{G}_m]$. The closed substack $\mathcal{X}_{\Sigma,\beta}$ is the divisor corresponding to the "extra ray," which is numbered 3 in the picture. That is, it is the divisor $V(x_3) = [(\mathbb{A}^2 \smallsetminus V(x_1, x_2))/_{(6\ 4)}\mathbb{G}_m]$. $\diamond$

5 Canonical Stacks

The Cox construction demonstrates that any toric stack is the good moduli space of a smooth stack. Given a generically stacky toric stack, there is a canonical smooth generically stacky toric stack of which it is a good moduli space. The purpose of this section is to construct and characterize this canonical smooth stack.

Given a fan Σ on a lattice L, the Cox construction [CLS11, §5.1] of the toric variety X_Σ produces an open subscheme U of $\mathbb{A}^n$ and a subgroup $H \subseteq \mathbb{G}_m^n$ so that $X_\Sigma = U/H$. That is, so that $[U/H] \to X_\Sigma$ is a good moduli space in the sense of [Alp08]. We recall and generalize this construction here.

Let (Σ, β) be a generically stacky fan. Let $\Sigma(1)$ be the set of rays of Σ. Let $M \subseteq L$ be the saturated sublattice spanned by Σ, and let $M' \subseteq L$ be a direct complement to M. For each ray $\rho \in \Sigma(1)$, let u_ρ be the first element of M along ρ, and let e_ρ be the generator in $\mathbb{Z}^{\Sigma(1)}$ corresponding to ρ. We then have a morphism $\Phi\colon \mathbb{Z}^{\Sigma(1)} \times M' \to L$ given by $(e_\rho, m) \mapsto u_\rho + m$. We define a fan $\widetilde{\Sigma}$ on $\mathbb{Z}^{\Sigma(1)} \times M'$. For each $\sigma \in \Sigma$, we define $\tilde{\sigma} \in \widetilde{\Sigma}$ as the cone generated by $\{e_\rho | \rho \in \sigma\}$. The morphism of generically stacky fans

$$\begin{tikzcd}
\widetilde{\Sigma} \arrow[r] & \Sigma \\[-20pt]
\mathbb{Z}^{\Sigma(1)} \times M' \arrow[r, "\Phi"] \arrow[d, "\tilde{\beta}"'] & L \arrow[d, "\beta"] \\
N \arrow[r, equal] & N
\end{tikzcd}$$

induces a toric morphism $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$.

Remark 5.1. The usual Cox construction expresses X_Σ as a quotient of $X_{\widetilde{\Sigma}}$ by G_Φ. Applying Lemma A.1 (Φ is close by construction), we see that the morphism constructed above is obtained by quotienting the morphism $[X_{\widetilde{\Sigma}}/G_\Phi] \to X_\Sigma$ by the action of G_β. This shows that the construction above commutes with quotienting $\mathcal{X}_{\Sigma,\beta}$ by its torus (i.e. replacing G_β by $G_{L\to 0}$).

Remark 5.2. Moreover, since $[\mathcal{X}_{\widetilde{\Sigma}}/G_\Phi] \to X_\Sigma$ is a good moduli space morphism, so is $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$ by [Alp08, Proposition 4.6].

Definition 5.3. We call $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}}$ the *canonical stack* over $\mathcal{X}_{\Sigma,\beta}$, and we say that the morphism $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$ is a *canonical stack morphism.*

The remainder of this subsection is dedicated to justifying this terminology by showing that the canonical stack has a universal property (Proposition 5.7).

Remark 5.4. The universal property itself is basically useless. There is a more practical universal property using the language of log geometry, but a useless universal property is good enough for our needs. The only purpose of demonstrating the universal property is to show that the canonical stack is uniquely determined by the stack $\mathcal{X}_{\Sigma,\beta}$ together with the torus action (rather than by the stacky fan (Σ, β)). In particular, if a stack $\mathcal{X}$ (with a dense open torus that acts on it) has an open cover by toric stacks, the canonical stacks over the open substacks are canonically isomorphic on overlaps, so they glue to a canonical stack over $\mathcal{X}$. This will be important in the proof of Theorem 12.1.

Lemma 5.5. *Let $\mathcal{X}_{\Sigma,\beta}$ be a cohomologically affine toric stack with n torus-invariant irreducible divisors. Suppose $f\colon \mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ is a toric surjection from a smooth cohomologically affine toric stack with n torus-invariant divisors, which restricts to an isomorphism on tori. Then $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ factors uniquely through the canonical stack over $\mathcal{X}_{\Sigma,\beta}$.*

Proof. Applying Theorem 3.5, we may assume f is induced by a morphism of stacky fans $(\Phi,\phi)\colon (\Sigma',\beta') \to (\Sigma,\beta)$. Since f restricts to an isomorphism on tori, ϕ must be an isomorphism.

Since we are only considering torus-equivariant morphisms, we may verify the property after quotienting by the action of the torus, so we may assume $\mathcal{X}_{\Sigma,\beta}$ is the quotient of an affine toric variety by its torus, and $\mathcal{X}_{\Sigma',\beta'}$ is the quotient of a smooth affine toric variety with n-divisors by its torus. This identifies $\mathcal{X}_{\Sigma',\beta'}$ as $[\mathbb{A}^n/\mathbb{G}_m^n]$, so we may assume Σ' is the fan of $\mathbb{A}^n$. We identify the first lattice points along the rays of Σ' with the generators $e_i \in \mathbb{Z}^n$.

Since f is surjective, the induced morphism $\Sigma' \to \Sigma$ is surjective by Lemma 3.8. Every ray of Σ is then the image of a unique ray of Σ', since Σ' has only n rays. Suppose $\Phi(e_i) = k_i\rho_i$. Then we see that Φ factors uniquely through the canonical stack via the morphism of fans $\Sigma' \to \widetilde{\Sigma}$ given by sending e_i to $k_i e_{\rho_i}$. $\square$

Remark 5.6 ("Canonical stacks are stable under base change by open immersions"). The preimage of a torus-invariant divisor of $\mathcal{X}_{\Sigma,\beta}$ is a divisor in its canonical stack. So by the above lemma, restricting a canonical stack morphism to the open complement of a torus-invariant divisor yields a canonical stack morphism.

As a corollary, we get the following Proposition.

Proposition 5.7 (Universal property of the canonical stack). *Suppose $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ is a toric morphism from a smooth toric stack, which restricts to an isomorphism of tori, and which restricts to a canonical stack morphism over every torus-invariant cohomologically affine open substack of $\mathcal{X}_{\Sigma,\beta}$. Then $\mathcal{X}_{\Sigma',\beta'}$ is canonically isomorphic to the canonical stack over $\mathcal{X}_{\Sigma,\beta}$.*

Remark 5.8. We will see in Lemma 6.5 that the morphism $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$ is a good moduli space morphism.

Remark 5.9. Note that if $\pi\colon \mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$ is the canonical stack, then the restriction of π to any torus-invariant open substack of $\mathcal{X}_{\Sigma,\beta}$ is also a canonical stack morphism.

By Remark 3.26, we see that the canonical stack morphism over a smooth toric stack is an isomorphism. In particular, this shows that for any generically stacky toric stack $\mathcal{X}_{\Sigma,\beta}$, the canonical stack is isomorphic to $\mathcal{X}_{\Sigma,\beta}$ over its smooth locus. Thus, the canonical stack can be regarded as a (canonical!) "stacky resolution of singularities." (c.f. Examples 4.10 and 4.12)

By Remark 5.9, the definition of a canonical stack morphism can be extended to stacks which are only locally known to be toric stacks.

Definition 5.10. Suppose $\mathcal{X}$ is a stack with an open cover by generically stacky toric stacks with a common torus. A morphism from a smooth stack $\mathcal{Y} \to \mathcal{X}$ is a *canonical stack morphism* if it restricts to canonical stack morphisms on the open toric substacks of $\mathcal{X}$.

6 Toric Good Moduli Space Morphisms

6.1 Good Moduli Space Morphisms

In [Alp08], Alper introduces the notion of a good moduli space morphism, which generalizes the notion of a good quotient in geometric invariant theory and is moreover a common generalization of the notion of a tame Artin stack [AOV08] and of a coarse moduli space [KM97].

Definition 6.1. A quasi-compact and quasi-separated morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is a *good moduli space morphism* if

- (f is Stein) the morphism $\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and
- (f is cohomologically affine) the pushforward functor $f_*\colon \operatorname{QCoh}(\mathcal{O}_{\mathcal{X}}) \to \operatorname{QCoh}(\mathcal{O}_{\mathcal{Y}})$ is exact.

Remark 6.2. An algebraic stack $\mathcal{X}$ is said to be cohomologically affine if the structure morphism $\mathcal{X} \to \operatorname{Spec} k$ is cohomologically affine. This is in agreement with Definition 3.6. If a toric variety X_σ is an affine toric variety and G_β is an affine group, then $[X_\sigma/G_\beta] \to \operatorname{Spec} k$ is cohomologically affine by [Alp08, Proposition 3.13].

Conversely, suppose $\mathcal{X}_{\Sigma,\beta} = [X_\Sigma/G_\beta] \to \operatorname{Spec} k$ is cohomologically affine. Since X_Σ is affine over $\mathcal{X}_{\Sigma,\beta}$ (because it is a G_β-torsor), we have that $X_\Sigma \to \operatorname{Spec} k$ is cohomologically affine, so X_Σ is affine by Serre's criterion [EGA, Corollary 5.2.2]. It follows that Σ is a single cone.

Our main goal in this section is to identify many examples of toric good moduli space morphisms.

Lemma 6.3. *For $i = 0, 1$, let (Φ_i, ϕ_i) be a morphism of generically stacky fans. Then $(\Phi_0 \times \Phi_1, \phi_0 \times \phi_1)$ induces a good moduli space morphism of generically stacky toric stacks if and only if each (Φ_i, ϕ_i) does.*

Proof. Good moduli spaces are stable under composition (this follows quickly from the definition) and base change [Alp08, Proposition 4.6(i)], so the result follows from Proposition 3.3. □

6.2 Local Results

In this subsection, we prove several criteria for a toric morphism to be a good moduli space morphism. See §6.4 for examples.

Throughout this subsection, we use the following setup. We have a morphism of generically stacky fans $(\Phi, \phi)\colon (\sigma, \beta\colon L \to N) \to (\sigma', \beta'\colon L' \to N)$, where σ is a single cone on L, and $\sigma' = \Phi(\sigma)$.

$$\begin{tikzcd} \sigma \arrow[r, two heads] & \sigma' \\ L \arrow[r, "\Phi"] \arrow[d, "\beta"'] & L' \arrow[d, "\beta'"] \\ N \arrow[r, "\phi"] & N' \end{tikzcd}$$

Lemma 6.4 ("Quotient by the kernel of a toric map is a GMS")**.** *Suppose Φ is close. Then the induced morphism $\mathcal{X}_{\sigma,\Phi} \to X_{\sigma'}$ is a good moduli space morphism.*

Proof. Since $X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap L^*]$ and $X_{\sigma'} = \operatorname{Spec} k[\sigma'^\vee \cap L'^*]$ are affine and G_Φ is reductive, it suffices to show that the induced ring homomorphism $k[\sigma'^\vee \cap L'^*] \to k[\sigma^\vee \cap L^*]$ is the inclusion of $k[\sigma^\vee \cap L^*]^{G_\Phi}$. Since Φ is close, we have that $\Phi^*\colon L'^* \to L^*$ is an inclusion, so the ring homomorphism is injective. We begin by showing that its image is the ring of G_Φ-invariants. We have the short exact sequence

$$0 \to L'^* \xrightarrow{\Phi^*} L^* \to D(G_\Phi) \to 0.$$

The subring of G_Φ-invariants in $k[L^*]$ is precisely the subring generated by monomials in L^* whose image in $D(G_\Phi)$ is zero. That is, $k[L^*]^{G_\Phi} = k[L'^*]$.

Since $\Phi(\sigma) = \sigma'$, an element $\lambda \in L'^*$ is non-negative on σ if and only if $\Phi^*(\lambda)$ is non-negative on σ'. That is, $\sigma'^\vee \cap L'^* = (\sigma^\vee \cap L^*) \cap L'^*$ (we are identifying L'^* with its image $\Phi^*(L'^*)$). This shows that $k[\sigma'^\vee \cap L'^*]$ is $k[\sigma^\vee \cap L^*]^{G_\Phi}$. □

Lemma 6.5 ("Isomorphism on tori implies GMS")**.** *Suppose ϕ is an isomorphism and β is close. Then $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma',\beta'}$ is a good moduli space.*

Proof. First we reduce to the case where Φ is close. Let L'' be the saturation of $\Phi(L)$ in L', let β'' be the restriction of β' to L'', and let σ'' be the cone σ', regarded as a fan on L''. By assumption (i.e. the case where Φ is close), the induced morphism $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma'',\beta''}$ is a good moduli space morphism. Note that since β is close, so is β'', so by Lemma 3.19, the induced morphism $\mathcal{X}_{\sigma'',\beta''} \to \mathcal{X}_{\sigma',\beta'}$ is an isomorphism. Therefore the composition $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma',\beta'}$ is a good moduli space morphism.

Now we consider the case when Φ is close. By Lemma A.1, G_β is an extension of $G_{\beta'}$ by G_Φ. The induced map is then $[X_\sigma/G_\beta] = \big[[X_\sigma/G_\Phi]/G_{\beta'}\big] \to [X_{\sigma'}/G_{\beta'}]$. By Lemma 6.4, $[X_\sigma/G_\Phi] \to X_{\sigma'}$ is a good moduli space. Since the property of being a good moduli space can be checked locally in the smooth topology (even in the fpqc topology, [Alp08, Proposition 4.6]), $[X_\sigma/G_\beta] \to [X_{\sigma'}/G_{\beta'}]$ is a good moduli space. □

Definition 6.6. We say that a cone τ of a generically stacky fan $(\Sigma, \beta\colon L \to N)$ is *unstable* if every linear functional $N \to \mathbb{Z}$ which is non-negative on the image of τ vanishes on the image of τ.

Equivalently, τ is unstable if the relative interior of the image of τ in the lattice N/N_{tor} contains 0.

Lemma 6.7 ("Quotient by an unstable cone is a GMS")**.** *Suppose β is surjective. Suppose τ is an unstable face of σ, and that* $\ker\beta$ *is generated by* $\tau \cap \ker\beta$*. Suppose Φ is surjective with* $\ker\Phi$ *generated by* $\tau \cap L$*. Suppose $N' = N/\beta(\ker\Phi)$. Then $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma',\beta'}$ is a good moduli space.*

Proof. We have the following diagram, in which the rows are exact and the first two columns are exact. By the snake lemma, β' is an isomorphism, so $G_{\beta'}$ is trivial.

$$\begin{tikzcd}
& \ker\beta \arrow[r, equal] \arrow[d, hook] & \ker\beta \arrow[d, hook] & & \\
0 \arrow[r] & \ker\Phi \arrow[r] \arrow[d, two heads] & L \arrow[r, "\Phi"] \arrow[d, two heads, "\beta"] & L' \arrow[r] \arrow[d, "\beta'"] & 0 \\
0 \arrow[r] & \beta(\ker\Phi) \arrow[r] & N \arrow[r, "\phi"] & N' \arrow[r] & 0
\end{tikzcd}$$

So we aim to show that $[X_\sigma/G_\beta] \to X_{\sigma'}$ is a good moduli space. As in the proof of Lemma 6.4, it suffices to show that we get an induced isomorphism $k[\sigma^\vee \cap L^*]^{G_\beta} \cong k[\sigma'^\vee \cap L'^*]$. Since Φ and β are surjective, $L'^* \to L^*$ and $N^* \to L^*$ are injective, so we have the exact sequence

$$0 \to N^* \xrightarrow{\beta^*} L^* \to D(G^1_\beta).$$

Since G^0_β acts trivially, the G_β-invariants of $k[L^*]$ are precisely $k[N^*]$, so it suffices to show that $N^* \cap \sigma^\vee = L'^* \cap \sigma'^\vee$. (Here we are identifying N^* and L'^* with their images in L^*.)

First we show that $L'^* \cap \sigma'^\vee \subseteq N^* \cap \sigma$. Any linear functional on L' induces a linear functional on L which vanishes on $\ker\Phi$, and so on $\ker\beta$, so it must be induced by a linear functional on $L/\ker\beta = N$. This shows that $L'^* \subseteq N^*$. Since σ' is the image of σ, any linear functional which is non-negative on σ' must induce a linear functional on L which is non-negative on σ.

Now we show $N^* \cap \sigma \vee \subseteq L'^* \cap \sigma'^\vee$. Since τ is assumed to be unstable, any linear functional ϕ on N which is non-negative on σ must vanish on τ, and therefore on $\ker\Phi$ since $\ker\Phi$ is generated by $\tau \cap L$. Any linear functional that vanishes on $\ker\Phi$ is induced by a linear functional on $L/\ker\Phi = L'$. Since σ' is the image of σ and ϕ is non-negative on σ, we have that the corresponding element of L'^* is non-negative on $\sigma'^\vee$. $\square$

Lemma 6.8 ("Removing trivial generic stackiness is a GMS")**.** *Suppose $N = N' \oplus N_0$ and that β factors through N'. Let $L' = L$. Then $[X_\sigma/G_\beta] \to [X_{\sigma'}/G_{\beta'}]$ is a good moduli space.*

Proof. We have that $\beta = \beta' \oplus 0\colon L \oplus 0 \to N' \oplus N_0$, so $G_\beta = G_{\beta'} \oplus G_0$, where G_0 acts trivially on X_σ. The map $[X_\sigma/G_0] \to X_\sigma$ is then a good moduli space, so $[X_\sigma/G_\beta] = \big[[X_\sigma/G_0]/G_{\beta'}\big] \to [X_\sigma/G_{\beta'}]$ is a good moduli space. □

Lemma 6.9 ("Removing finite generic stackiness is a GMS")**.** *Suppose* $\ker\phi = N_0$ *is finite and* ϕ *is surjective. Let* $L' = L$*. Then* $[X_\sigma/G_\beta] \to [X_{\sigma'}/G_{\beta'}]$ *is a good moduli space.*

Proof. Consider the left-hand diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d, "\beta_0"'] & L \arrow[r, equal] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "\beta'"] & 0 \\ 0 \arrow[r] & N_0 \arrow[r] & N \arrow[r, "\phi"] & N' \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & G_{\beta_0} \arrow[r] \arrow[d] & G_\beta \arrow[r] \arrow[d] & G_{\beta'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & T_L \arrow[r, equal] & T_L \arrow[r] & 0 \end{tikzcd}$$

Since N_0 is finite, β_0 is close. By Lemma A.2, we get the induced right-hand diagram with exact rows. In particular, the action of G_{β_0} on X_σ is trivial, so the map $[X_\sigma/G_{\beta_0}] \to X_\sigma$ is a good moduli space. Thus, the map $[X_\sigma/G_\beta] = \big[[X_\sigma/G_{\beta_0}]/G_{\beta'}\big] \to [X_\sigma/G_{\beta'}]$ is a good moduli space map. □

6.3 Globalizing Local Results

The following proposition makes it possible to check if a morphism of non-cohomologically affine toric stacks is a good moduli space morphism.

Proposition 6.10. *Let* $(\Phi,\phi)\colon (\Sigma, \beta\colon L \to N) \to (\Sigma', \beta'\colon L' \to N')$ *be a morphism of generically stacky fans. The induced morphism* $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma',\beta'}$ *is a good moduli space morphism if and only if*

1. *For every* $\sigma' \in \Sigma'$*, the pre-image* $\Phi^{-1}(\sigma')$ *is a single cone* $\sigma \in \Sigma$ *with* $\Phi(\sigma) = \sigma'$*. In particular, the pre-image of the zero cone is some cone* $\tau \in \Sigma$*, and the pre-image of any other cone has* τ *as a face.*

2. *The restrictions* $\mathcal{X}_{\sigma,\beta} \to \mathcal{X}_{\sigma',\beta'}$ *are good moduli space morphisms.*

Proof. For $\sigma' \in \Sigma'$, let $X_{\sigma'}$ be the open subscheme of $X_{\Sigma'}$ corresponding to σ'. The property of being a good moduli space can be checked Zarsiki locally on the base, so it is equivalent to checking that $[X_{\Phi^{-1}(\sigma')}/G_\beta] \to [X_{\sigma'}/G_{\beta'}]$ is a good moduli space morphism for each $\sigma' \in \Sigma'$. Therefore, the two given conditions imply that $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma',\beta'}$ is a good moduli space morphism.

Conversely, if $[X_{\Phi^{-1}(\sigma')}/G_\beta] \to [X_{\sigma'}/G_{\beta'}]$ is a good moduli space morphism, then we have that $[X_{\Phi^{-1}(\sigma')}/G_\beta]$ is cohomologically affine since it is cohomologically affine over a cohomologically affine stack. So $\Phi^{-1}(\sigma')$ is a single cone $\sigma \in \Sigma$. Good moduli space morphisms are surjective [Alp08, Theorem 4.14(i)], so by Lemma 3.8, $\Phi(\sigma) = \sigma'$. □

The above results allow us to determine when a generically stacky toric stack has a toric variety as a good moduli space. The following corollary generalizes [Sat09, Theorem 5.5].

Corollary 6.11. *Let $(\Sigma, \beta\colon L \to N)$ be a generically stacky fan. Let $N_1 = \mathrm{sat}_N \beta(L)$, $L' = N_1/(N_1)_{\mathrm{tor}}$, and $\Phi\colon L \to L'$ the natural map. Suppose $\Phi^{-1}(\Phi(\sigma))$ is a single cone of Σ for every $\sigma \in \Sigma$. Let Σ' be the fan on L' generated by the images of the maximal cones of Σ. Then $\mathcal{X}_{\Sigma,\beta} \to X_{\Sigma'}$ is a good moduli space.*

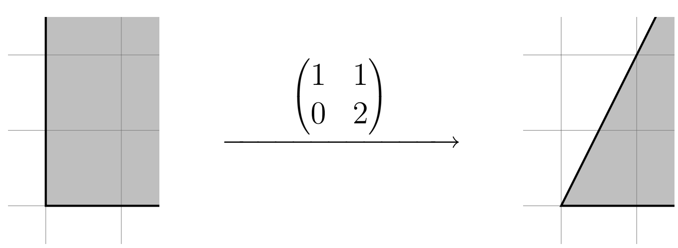
Proof. By Proposition 6.10, we may assume Σ is a single cone. Let N_0 be a complement to N_1, the span of Σ, in N. By Lemma 6.8, $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma,L\to N_1}$ is a good moduli space, and by Lemma 6.9, $\mathcal{X}_{\Sigma,L\to N_1} \to \mathcal{X}_{\Sigma,L\to N_1/(N_1)_{\mathrm{tor}}}$ is a good moduli space. Since $L \to N_1/(N_1)_{\mathrm{tor}}$ is surjective, Lemma 6.4 tells us that $\mathcal{X}_{\Sigma,L\to N_1/(N_1)_{\mathrm{tor}}} \to X_{\Sigma'}$ is a good moduli space. Therefore, the composite map $\mathcal{X}_{\Sigma,\beta} \to X_{\Sigma'}$ is a good moduli space. $\square$

Remark 6.12. We recover [Sat09, Theorem 5.5] as the situation where β is close and N is a lattice. In that case, $L' = N_1 = N$, so the morphism of stacky fans is of the following form.

$$\begin{tikzcd} \Sigma \arrow[r] & \Sigma' \\ L \arrow[r, "\beta"] \arrow[d, "\beta"'] & N \arrow[d, equal] \\ N \arrow[r, equal] & N \end{tikzcd}$$

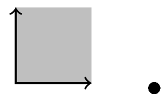
6.4 Examples

Example 6.13. Lemma 6.4 (with Proposition 6.10) shows that the Cox construction of a toric variety is a good moduli space, and more generally that canonical stack morphisms are good moduli space morphisms. For example, consider the morphism of toric varieties given by the following map of fans:



Lemma 6.4 tells us that the toric variety on the right, the A_1 singularity, is the good moduli space of $[\mathbb{A}^2/G_\Phi] = [\mathbb{A}^2/\mu_2]$. $\diamond$

Example 6.14. Lemma 6.7 is illustrated by the following example.



$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r] \arrow[d, "(1\ \text{-}1)"'] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

This morphism of stacky fans represents the good moduli space morphism $[\mathbb{A}^2/_{(1\ 1)}\mathbb{G}_m] \to \operatorname{Spec} k$. Note that the unstable cone (the 2-dimensional cone) corresponds to the origin in $\mathbb{A}^2$. ◇

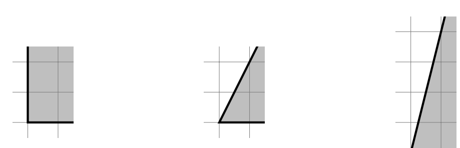
Example 6.15. This example illustrates Lemma 6.9.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "2"] \arrow[d, "2"'] & \mathbb{Z} \arrow[d, "id"] \\
\mathbb{Z} \arrow[r, "id"] & \mathbb{Z}
\end{tikzcd}$$

This morphism of stacky fans represents the good moduli space morphism $[\mathbb{A}^1/\mu_2] \to \mathbb{A}^1/\mu_2 \cong \mathbb{A}^1$. ◇

It is often useful to think about a toric stack as "sandwiched" between its canonical stack and its good moduli space (if it has one). In this way, we often regard a toric stack as a "partial good moduli space" of its canonical stack, or as a "stacky resolution" of its good moduli space.

Example 6.16. Consider the stacky fan (Σ, β) shown in the center below. On the left we have the stacky fan of the corresponding canonical stack (see §5). On the right we have a toric variety. By Lemma 6.5, the two morphism of stacky fans induce good moduli space morphisms of toric stacks.



$$\begin{tikzcd}
\widetilde{\Sigma} & \Sigma & \Sigma' \\
\mathbb{Z}^2 \arrow[r, "\Phi=\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)"] \arrow[d, "\tilde{\beta}=\left(\begin{smallmatrix}1&1\\0&4\end{smallmatrix}\right)"'] & \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)"] \arrow[d, "\beta=\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)"'] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2
\end{tikzcd}$$

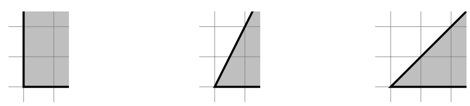
We can easily see that X_Σ is the A_1 singularity $\mathbb{A}^2/_{(1\ 1)}\mu_2$, and that G_β is μ_2, but it is easiest to describe the action of μ_2 on X_Σ in terms of the canonical stack.

The canonical stack is $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} = [\mathbb{A}^2/_{(1\ \text{-}1)}\mu_4]$. We get the induced short exact sequence (c.f. Lemma A.1)

$$\begin{tikzcd} 0 \arrow[r] & G_\Phi \arrow[r] \arrow[d, equal] & G_{\tilde{\beta}} \arrow[r] \arrow[d, equal] & G_\beta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mu_2 \arrow[r, "2"] & \mu_4 \arrow[r] & \mu_2 \arrow[r] & 0 \end{tikzcd}$$

We can therefore express $\mathcal{X}_{\Sigma,\beta}$ as $\big[(\mathbb{A}^2/\mu_2)\ /\ (\mu_4/\mu_2)\big]$. We can view this either as a "partial good moduli space" of $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} = [\mathbb{A}^2/_{(1\ \text{-}1)}\mu_4]$, or as a "partial stacky resolution" of the singular toric variety $\mathcal{X}_{\Sigma'} = \mathbb{A}^2/_{(1\ \text{-}1)}\mu_4$. ◇

Example 6.17. Here is another interesting example of a non-smooth toric stack which is not a scheme. Consider the stacky fan (Σ, β) shown in the center below. On the left we have the stacky fan of the corresponding canonical stack (see §5). On the right we have a toric variety. By Lemma 6.5, the two morphism of stacky fans induce good moduli space morphisms of toric stacks.



$$\begin{tikzcd} \widetilde{\Sigma} & \Sigma & \Sigma' \\[-20pt] \mathbb{Z}^2 \arrow[r, "\Phi = \left(\begin{smallmatrix}1 & 1\\ 0 & 2\end{smallmatrix}\right)"] \arrow[d, "\tilde{\beta} = \left(\begin{smallmatrix}2 & 2\\ 0 & 4\end{smallmatrix}\right)"'] & \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix}2 & 0\\ 0 & 1\end{smallmatrix}\right)"] \arrow[d, "\beta = \left(\begin{smallmatrix}2 & 0\\ 0 & 1\end{smallmatrix}\right)"'] & \mathbb{Z}^2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2 \end{tikzcd}$$

Like the previous example, $\mathcal{X}_{\Sigma,\beta}$ is a quotient of the A_1 singularity $X_\Sigma = \mathbb{A}^2/_{(1\ 1)}\mu_2$ by an action of μ_2. The canonical stack over it is $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}} = \big[\mathbb{A}^2/_{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)}(\mu_2 \times \mu_2)\big]$, and it is the "partial good moduli space" $\big[(\mathbb{A}^2/_{(1\ 1)}\mu_2)/((\mu_2 \times \mu_2)/_{(1\ 1)}\mu_2)\big]$. The toric stack $\mathcal{X}_{\Sigma,\beta}$ has $\mathbb{A}^2$ as its good moduli space.

In Example 4.10, we constructed a stack which "resolves the A_1 singularity by introducing stackiness." In the same informal language, this example *introduces a singularity* at a smooth point of $\mathbb{A}^2$ by introducing stackiness. ◇

7 Moduli Interpretation of Smooth Toric Stacks

A morphism $f \colon Y \to \mathbb{P}^n$ is equivalent to the data of a line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and a choice of $n+1$ sections $\mathcal{O}_Y^{n+1} \to \mathcal{L}$ which generate $\mathcal{L}$. Cox generalized this moduli interpretation to smooth toric varieties [Cox95], and Perroni generalized it further to smooth toric Deligne-Mumford stacks [Per08]. The main goal for this section is to generalize it further to smooth generically stacky toric stacks.

In fact, we will see (Remark 7.10) that smooth generically stacky toric stacks are precisely the moduli stacks parametrizing tuples of Cartier pseudodivisors satisfying any given linear relations and any given intersection relations.

Proposition 7.1 ([SGA3, Exposé VIII, Proposition 4.1])**.** *Let G be a diagonalizable group scheme, and Y a scheme. Suppose $\mathcal{A}$ is a quasi-coherent sheaf of algebras on Y, together with an action of G (i.e. a grading $\mathcal{A} = \bigoplus_{\chi\in D(G)} \mathcal{A}_\chi$). Then $\mathcal{S}pec_Y\, \mathcal{A}$ is a G-torsor if and only if*

- *$\mathcal{A}_\chi$ is a line bundle for each $\chi \in D(G)$, and*
- *the homomorphism induced by multiplication $\mathcal{A}_\chi \otimes_{\mathcal{O}_Y} \mathcal{A}_{\chi'} \to \mathcal{A}_{\chi+\chi'}$ is an isomorphism.*

Since any G-torsor is affine over Y, it is clear that any G-torsor is of this form.

Notation 7.2. Given a collection of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathrm{Pic}(Y)$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, let $\mathcal{L}^{\mathbf{a}} = \mathcal{L}_1^{\otimes a_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes a_n}$.

Let $\beta\colon \mathbb{Z}^n \to N$ be the close morphism of lattices so that $G_\beta = G \subseteq \mathbb{G}_m^n$. Then we have the presentation $N^* \xrightarrow{\beta^*} \mathbb{Z}^n \xrightarrow{\phi} D(G) \to 0$. A quasi-coherent $\mathcal{O}_Y$-algebra as in Proposition 7.1 is therefore equivalent to a collection of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n \in \mathrm{Pic}(Y)$ with isomorphisms $c_\psi\colon \mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}^{\beta^*(\psi)}$ for $\psi \in N^*$, such that $c_\psi \otimes c_{\psi'} = c_{\psi+\psi'}\colon \mathcal{L}^{\beta^*(\psi)+\beta^*(\psi')} \to \mathcal{O}_Y \otimes \mathcal{O}_Y \cong \mathcal{O}_Y$.

Definition 7.3. Suppose Σ is a subfan of the fan of $\mathbb{A}^n$, and $\beta\colon \mathbb{Z}^n \to N$ is a close lattice homomorphism. A (Σ, β)-*collection* on a scheme Y consists of

- an n-tuple of line bundles $(\mathcal{L}_1, \dots, \mathcal{L}_n)$,
- global sections $s_i \in H^0(Y, \mathcal{L}_i)$ so that for each point $y \in Y$, there is a cone $\sigma \in \Sigma$ so that $s_i(y) \neq 0$ for all $e_i \notin \sigma$.
- trivializations $c_\psi\colon \mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}^{\beta^*(\psi)}$ for each $\psi \in N^*$, satisfying the compatibility condition $c_\psi \otimes c_{\psi'} = c_{\psi+\psi'}$.

An *isomorphism* of (Σ, β)-collections is an n-tuple of isomorphisms of line bundles respecting the associated sections and trivializations.

Remark 7.4. Note that since N^* is a free subgroup of $\mathbb{Z}^n$, it suffices to specify c_ψ where ψ varies over a basis of N^*. If specified this way, the isomorphisms do not need to satisfy any compatibility condition. Different choices of these trivializations are related by the action of the torus (see Remark 7.8), so we often suppress the trivializations.

Remark 7.5. A line bundle with section is an *effective Cartier pseudodivisor*. If the section is non-zero, the vanishing locus of the section is an effective Cartier divisor. Therefore, a (Σ, β)-collection can be defined as an n-tuple of effective Cartier pseudodivisors $(D_1, \dots, D_n)$ such that $\sum a_i D_i$ is linearly equivalent to zero whenever $(a_1, \dots, a_n) \in N^*$ and such that $D_{i_1} \cap \dots, D_{i_k} = \varnothing$ whenever $e_{i_1}, \dots, e_{i_k}$ do not all lie on a single cone of Σ. Here we are suppressing the trivializations as in Remark 7.4.

Remark 7.6. Given a morphism $Y' \to Y$ and a (Σ, β)-collection on Y, we may pull back the line bundles, sections, and trivializations to produce a (Σ, β)-collection on Y'. This makes the category of (Σ, β)-collections into a fibered category over the category of schemes.

Theorem 7.7 (Moduli interpretation of smooth $\mathcal{X}_{\Sigma,\beta}$)**.** *Let Σ be a subfan of the fan for $\mathbb{A}^n$, and let $\beta\colon \mathbb{Z}^n \to N$ be close. Then $\mathcal{X}_{\Sigma,\beta}$ represents the fibered category of (Σ, β)-collections.*

Proof. A morphism $f\colon Y \to [\mathbb{A}^n/G_\beta]$ consists of a G_β-torsor $P \to Y$ and a G_β-equivariant morphism $P \to \mathbb{A}^n$. By Proposition 7.1, the data of a G_β-torsor is equivalent to a $D(G_\beta)$-graded quasi-coherent sheaf of algebras $\mathcal{A}$ such that $\mathcal{A}_\chi$ is a line bundle for each $\chi \in D(G_\beta)$. A G_β-equivariant morphism $P \to \mathbb{A}^n$ is then equivalent to a homomorphism of $\mathcal{O}_T$-algebras $\bigoplus_{\mathbf{a}\in\mathbb{N}^n} \mathcal{O}_T \to \mathcal{A}$ which respects the $D(G_\beta)$-grading (the $D(G_\beta)$-grading on the former algebra is induced by the $\mathbb{Z}^n$-grading and the homomorphism $\phi\colon \mathbb{Z}^n \to D(G_\beta)$). This is equivalent to homomorphisms of $\mathcal{O}_T$-modules $s_i\colon \mathcal{O}_T \to \mathcal{A}_{\phi(e_i)}$. Under this correspondence, the vanishing locus of s_i is the pre-image $[\mathbb{A}^{n-1}_i/G_\beta]$, where $\mathbb{A}^{n-1}_i$ is the i-th coordinate hyperplane. In particular, $(\mathcal{A}_{\phi(e_1)}, \dots, \mathcal{A}_{\phi(e_n)}, s_1, \dots, s_n)$ (along with the implicit trivializations) form a (Σ, β)-collection if and only if f factors through the open substack $\mathcal{X}_{\Sigma,\beta}$.

It is straightforward to verify that the above correspondence induces an equivalence. □

Remark 7.8. Carefully following the construction in the proof shows that the action of the torus is as follows. Suppose $N^* \subseteq \mathbb{Z}^n$ is the sublattice of trivialized line bundles. Then the trivializations c_ψ have natural weights of the torus $T = \mathrm{Hom}_{\mathrm{gp}}(N^*, \mathbb{G}_m)$ associated to them. T acts on the trivializations via these weights.

Remark 7.9. This moduli interpretation is stable under base change by open immersions. Suppose a morphism $Y \to \mathcal{X}_{\Sigma,\beta}$ corresponds to the (Σ, β)-collection $(\mathcal{L}_i, s_i, c_\psi)$. Let Σ' be a subfan of Σ. Then the pullback $Y' = Y \times_{\mathcal{X}_{\Sigma,\beta}} \mathcal{X}_{\Sigma',\beta}$ is the open subscheme of Y where a subset of sections may simultaneously vanish only if Σ' contains the cone spanned by the rays corresponding to those sections.

Remark 7.10. As a sort of converse to Theorem 7.7, note that any set of intersection relations among an n-tuple of effective Cartier pseudodivisors (i.e. any specification of which subsets of divisors should have empty intersection) determines a subfan Σ of the fan of $\mathbb{A}^n$. Furthermore, any[1] compatible collection of trivializations determines a subgroup $N^* = \{\mathbf{a} \in \mathbb{Z}^n | \mathcal{L}^{\mathbf{a}}$ is trivialized$\}$. The dual of the inclusion of N^* is a close homomorphism $\beta\colon \mathbb{Z}^n \to N$. Then $\mathcal{X}_{\Sigma,\beta}$ is the moduli stack of n-tuples of effective Cartier pseudodivisors with the given intersection relations and linear relations.

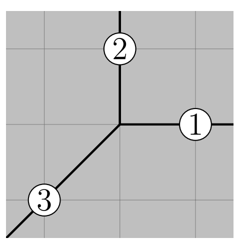
[1]There is *one* required relationship between the intersection relations and the trivializations. Namely, if the intersection of a *single* Cartier divisor is required to be empty (i.e. if the corresponding section of the line bundle is nowhere vanishing), then the line bundle must be trivialized. That is, if the intersection relations explicitly require the section to trivialize the line bundle, then it must be trivialized.

7.1 Examples

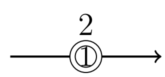
The simplest examples to describe are fantastacks. See Notation 4.8 and Examples 4.9–4.12 for an explanation of the notation used below.

Remark 7.11. Any smooth toric stack contains an open substack which has a toric open immersion into a fantastack. Remark 7.9 therefore allows us to understand the moduli interpretation of non-fantastack smooth toric stacks by appropriately modifying the intersection relations.

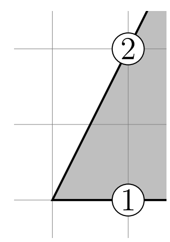
Remark 7.12. We explicitly obtain linear relations by choosing a basis for N^*. For each basis element ψ, we get a trivialization of $L^{\beta^*(\psi)}$. That is, we get trivializations of the divisors whose coefficients appear in the rows of β^*.



Example 7.13



Example 7.14



Example 7.15

We follow the less formal approach to (Σ, β)-collections explained in Remark 7.5.

Example 7.13. A morphism to the leftmost stack is a choice of three effective Cartier pseudo-divisors D_1, D_2, D_3 such that $D_1 \cap D_2 \cap D_3 = \varnothing$ (because no cone contains all three dots), and so that $D_1 - D_3 \sim \varnothing$ and $D_2 - D_3 \sim \varnothing$ (because $\beta^* = \left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{smallmatrix}\right)$; see Remark 7.12). Here, $\varnothing$ denotes the empty divisor.

In other words, it is a choice of a line bundle and three global sections that do not all vanish at any point. This is the usual description of morphisms to $\mathbb{P}^2$. ⋄

Example 7.14. A morphism to the middle stack is a choice of two effective Cartier pseudo-divisors D_1 and D_2 so that $D_1 + D_2 \sim \varnothing$ (because $\beta^* = (1\ 1)$; see Remark 7.12). Notice two particular morphisms from $\mathbb{A}^1$ to this stack; one given by setting $D_1 = 0$ and $D_2 = \varnothing$, and another by setting $D_1 = \varnothing$ and $D_2 = 0$. Indeed, the open substack where we impose the condition $D_1 \cap D_2 = \varnothing$ is the non-separated line. ⋄

Example 7.15. A morphism to the rightmost stack is a choice of two effective Cartier pseudo-divisors D_1 and D_2 so that $D_1 + D_2 \sim \varnothing$ and $2D_2 \sim \varnothing$ (because $\beta^* = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$; see Remark 7.12). Since there is a single cone that contains all the dots, there is no intersection condition on D_1 and D_2. Notice that two such divisors satisfy the relation necessary to specify a map to the stack in Example 7.14, so this stack has a morphism to the stack in that example. ⋄

7.2 Generically Stacky Smooth Toric Stacks

In this subsection we use the moduli interpretation to show that any smooth generically stacky toric stack is an essentially trivial gerbe over a toric stack.

Suppose $Y \to \mathcal{X}_{\Sigma,\beta} = [X_\Sigma/G_\beta]$ is the morphism to a smooth toric stack corresponding to the (Σ, β)-collection $(\mathcal{L}_i, s_i, c_\psi)$. It factors through the closed substack corresponding to the j-th coordinate hyperplane of $\mathbb{A}^n$ if and only if $s_j = 0$. Theorem 7.7 (together with Remark 2.19) therefore gives us the following moduli interpretation of generically stacky smooth toric stacks.

Corollary 7.16 (of Theorem 7.7)**.** *The generically stacky smooth toric substack $\mathcal{Z}$ of $\mathcal{X}_{\Sigma,\beta}$ corresponding to a coordinate subspace H of $\mathbb{A}^n$ has the following moduli interpretation. Morphisms $Y \to \mathcal{Z}$ correspond to (Σ, β)-collections on Y in which we require that $s_j = 0$ if H does not contain the j-th coordinate axis.*

Definition 7.17. Suppose $\mathcal{K}$ is a line bundle on a stack $\mathcal{X}$ and b is a positive integer. The *root stack* $\sqrt[b]{\mathcal{K}/\mathcal{X}}$ is defined as the fiber product in the following diagram, where the map $\mathcal{X} \to B\mathbb{G}_m$ is the one induced by $\mathcal{K}$.

$$\begin{tikzcd}
\sqrt[b]{\mathcal{K}/\mathcal{X}} \arrow[r] \arrow[d] & B\mathbb{G}_m \arrow[d, "\hat{}b"] \\
\mathcal{X} \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

The map $\hat{}b\colon B\mathbb{G}_m \to B\mathbb{G}_m$ is given by sending a line bundle to its b-th tensor power, induced by the group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by $t \mapsto t^b$.

If $\underline{\mathcal{K}} = (\mathcal{K}_1, \dots, \mathcal{K}_r)$ is an r-tuple of line bundles and $\mathbf{b} = (b_1, \dots, b_r)$ is an r-tuple of positive integers, we similarly define $\sqrt[\mathbf{b}]{\underline{\mathcal{K}}/\mathcal{X}}$ as $\mathcal{X} \times_{B\mathbb{G}_m^r} B\mathbb{G}_m^r$, where the map $\mathcal{X} \to B\mathbb{G}_m^r$ is induced by the tuple $(\mathcal{K}_1, \dots, \mathcal{K}_r)$ and the map $\hat{}\mathbf{b}\colon B\mathbb{G}_m^r \to B\mathbb{G}_m^r$ is induced by the homomorphism $\mathbb{G}_m^r \to \mathbb{G}_m^r$ given by $(t_1, \dots, t_r) \mapsto (t_1^{b_1}, \dots, t_r^{b_r})$. It is straightforward to check that $\sqrt[\mathbf{b}]{\underline{\mathcal{K}}/\mathcal{X}}$ is the fiber product of the $\sqrt[b_i]{\mathcal{K}_i/\mathcal{X}}$ over $\mathcal{X}$.

Remark 7.18. Explicitly, a morphism from a scheme (or stack) Y to the root stack $\sqrt[\mathbf{b}]{\underline{\mathcal{K}}/\mathcal{X}}$ is a morphism $f\colon Y \to \mathcal{X}$, an r-tuple of line bundles $(\mathcal{L}_1, \dots, \mathcal{L}_r)$, and isomorphisms $\mathcal{L}_i^{\otimes b_i} \cong \mathcal{K}_i$.

Definition 7.19. We say that $\mathcal{X} \to \mathcal{Y}$ is an *essentially trivial gerbe* if $\mathcal{X}$ is of the form $B(\mathbb{G}_m^s) \times \sqrt[\mathbf{b}]{\underline{\mathcal{K}}/\mathcal{Y}}$.

Proposition 7.20. *Let $\mathcal{Z}$ be a smooth generically stacky toric stack. Then $\mathcal{Z}$ is an essentially trivial gerbe over a toric stack.*

Proof. Suppose $\mathcal{Z}$ is a closed torus-invariant substack of a toric stack $\mathcal{X}_{\Sigma,\beta}$, with Σ a subfan of the fan of $\mathbb{A}^n$ (this is possible by Remark 2.19). Let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be the torus-invariant divisors of $\mathcal{X}_{\Sigma,\beta}$. Without loss of generality, $\mathcal{Z} = \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_l$. By Corollary 7.16, $\mathcal{Z}$ is the stack of (Σ, β)-collections where $s_i = 0$ for $1 \le i \le l$.

Let Σ' be the restriction of Σ to the sublattice $\mathbb{Z}^{n-l} \subseteq \mathbb{Z}^n$ given by the last $n-l$ coordinates, let $N'^* = \mathbb{Z}^{n-l} \cap N^*$, and let $\beta' \colon \mathbb{Z}^{n-l} \to N'$ be the dual to the inclusion. For each i between 1 and l, let b_i be the smallest positive integer (if it exists) so that $(0, \dots, 0, b_i, 0, \dots, 0, a_{i,l+1}, \dots, a_{i,n}) \in N^*$. Without loss of generality, we may assume these integers exist for $1 \le i \le r$. Define $\mathcal{K}_i = \mathcal{L}_{l+1}^{-a_{i,l+1}} \otimes \cdots \otimes \mathcal{L}_n^{-a_{i,n}}$.

Suppose $(\mathcal{L}_i, s_i, c_\psi)$ is a (Σ, β)-collection on a scheme such that $s_i = 0$ for $1 \le i \le l$. Then the last $n-l$ line bundles with sections form a (Σ', β')-collection, the line bundles $\mathcal{L}_i$ for $r < i \le l$ satisfy no relations, and for $1 \le i \le r$, we have isomorphisms $\mathcal{L}_i^{b_i} \cong \mathcal{K}_i$. Therefore, a morphism to $\mathcal{Z}$ is precisely the data of a morphism to $B(\mathbb{G}_m^{l-r}) \times \sqrt[(b_1,\dots,b_r)]{(\mathcal{K}_1, \dots, \mathcal{K}_r)/\mathcal{X}_{\Sigma',\beta'}}$. $\square$

8 Local Construction of Toric Stacks

The main goal of this section is to prove Theorem 8.12.

8.1 Colimits of Toric Monoids

Definition 8.1. A *toric monoid* is any monoid of the form $\sigma \cap L$, where σ is a cone in a lattice L.

Remark 8.2. Toric monoids are precisely the finitely generated, commutative, torsion-free monoids M so that $M \to M^{\mathrm{gp}}$ is injective and saturated.

Remark 8.3. Colimits exist in the category of toric monoids, and have a nice description. A diagram of toric monoids D induces a diagram of free abelian groups D^{gp}. Let L be the colimit of D^{gp} in the category of free abelian groups. Then the colimit of D is the image in L of the direct sum of all the objects of D. In particular, $\operatorname{colim}(D)^{\mathrm{gp}} = \operatorname{colim}(D^{\mathrm{gp}})$.

Definition 8.4. A *face* of a monoid M is a submonoid F so that $a+b \in F$ implies $a, b \in F$.

Remark 8.5. For a toric monoid $\sigma \cap L$, the faces are precisely submonoids of the form $\tau \cap L$, where τ is a face of σ. So the faces of $\sigma \cap L$ are obtained as the vanishing locus of linear functionals on L which are non-negative on σ.

Remark 8.6. If F is a face of a toric monoid M, then $F^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is a saturated inclusion, so it is the inclusion of a direct summand. In particular, any linear functional on F^{gp} can be extended to a linear functional on M^{gp}. Since F is a face of M, there is a linear functional χ on M^{gp} which is non-negative on M and vanishes precisely on F. Given any linear functional on F^{gp} which is non-negative on F, we extend it arbitrarily to a linear functional on M^{gp}. By then adding a large multiple of χ, we can guarantee that the extension is positive away from F.

Definition 8.7. Let D be a diagram in the category of toric monoids (i.e. D is a collection of toric monoids D_i and a collection of morphisms between the monoids). We say D is *tight* if

1. every morphism is an inclusion of a proper face,
2. if D_i appears in D, then all the faces of D_i appear in D,
3. the diagram commutes, and
4. for any two objects D_i and D_j in the diagram, there is a unique maximal common face in the diagram.

Definition 8.8. The subdiagram D^0 of a tight diagram D generated by a set of objects is *join-closed* if it is tight and for every pair of objects D_i and D_j of D^0, if they are both faces of an object D_k of D, then the smallest face of D_k containing D_i and D_j is in D^0.

Lemma 8.9. *Let D^0 be a join-closed subdiagram of a tight diagram D. Suppose χ is a linear functional on $\mathrm{colim}(D^0)^{\mathrm{gp}}$. Then χ can be extended to a linear functional on $\mathrm{colim}(D)^{\mathrm{gp}}$. Moreover, if χ induces non-negative functions on all objects of D^0, then the extension can be chosen to be non-negative on all objects of D, and if $D \neq D^0$, it can be chosen to be positive away from D^0.*

Remark 8.10. By the universal property of a colimit, a linear functional on a colimit of groups is equivalent to a compatible collection of linear functionals on the groups in the diagram.

Proof. We induct on the size of $D \smallsetminus D^0$. Let D_b be a maximal object of D which is not in D^0. Let D^1 be the subdiagram of D consisting of D^0 and all the faces of D_b. Since D_b is maximal, D^1 is join-closed. It suffices to extend the linear functional to $\mathrm{colim}(D^1)^{\mathrm{gp}}$.

Since D^0 is join-closed, there is a maximum object D_m of D^0 which is a face of D_b. We may extend $\chi|_{D_m}$ to a linear functional on D_b as in Remark 8.6. If $\chi|_{D_m}$ is non-negative, we may choose the extension to be positive away from D_m. □

Corollary 8.11. *Let D be a tight diagram of toric monoids with colimit M. Then for every object D_i of D, $D_i \to M$ is an inclusion of a face.*

Proof. To show that $D_i \to M$ is an inclusion, it suffices to show that $D_i^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is an inclusion, for which it suffices to show that the dual map is surjective. The subdiagram consisting of all the faces of D_i is join-closed, so every linear functional on D_i^{gp} can be extended to a linear functional on M^{gp} by Lemma 8.9, so the dual map is surjective.

To show that D_i is a face, it suffices to find a linear functional on M^{gp} which is non-negative on M and vanishes exactly on D_i. Such a linear functional exists by Lemma 8.9. □

8.2 Constructing Toric Stacks Locally

We saw in §5 that every toric stack is a good moduli space of a canonical smooth toric stack. In this subsection, we show that we can construct a toric stack by starting with a smooth toric stack and specifying compatible good moduli space maps from an open cover. In other words, given a canonical stack morphism from a smooth toric stack, the property of being a toric stack can be checked locally. This result will be very important in the proof of Theorem 12.1.

Theorem 8.12. *Let $\mathcal{X}$ be a stack with an action of a torus T and a dense open T-orbit which is T-equivariantly isomorphic to T. Let $\mathcal{Y} \to \mathcal{X}$ be a morphism from a toric stack. Suppose $\mathcal{X}$ has a cover by T-invariant open substacks $\mathcal{X}_i$ which are toric stacks with torus T, and that the maps $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_i \to \mathcal{X}_i$ are canonical stacks. Then $\mathcal{X}$ is a toric stack.*

Proof. Let $N = \mathrm{Hom}_{\mathrm{gp}}(\mathbb{G}_m, T)$. Refining the cover, we may assume each $\mathcal{X}_i$ is of the form $\mathcal{X}_{\sigma_i,\beta_i\colon L_i\to N}$ with σ_i a single cone. Moreover, we may assume that if $\mathcal{X}_{\sigma_i,\beta_i}$ is in the open cover, then the open substacks corresponding to the faces of σ_i are as well. Then $\mathcal{X}$ is the gluing of this diagram of open immersions of toric stacks.

Suppose $\mathcal{Y} = \mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}}$, where $\widetilde{\Sigma}$ is a subfan of the fan of $\mathbb{A}^n$ and $\tilde{\beta}\colon \mathbb{Z}^n \to N$ is a close homomorphism. The canonical stack over each $\mathcal{X}_{\sigma_i,\beta_i}$ is an open substack of $\mathcal{Y}$, so it is the open substack corresponding to a cone $\tilde{\sigma}$ of $\widetilde{\Sigma}$. We may assume we have compatible factorizations of β as $\mathbb{Z}^n \to L_i \xrightarrow{\beta_i} N$, where the first map sends $\tilde{\sigma}_i$ to σ_i. Then the colimit of toric monoids $\sigma_i \cap L_i$ is of the form $\sigma \cap L$, where L is the colimit of the L_i. By Corollary 8.11, the σ_i are faces of σ. By Proposition 3.22, the induced morphisms $\mathcal{X}_{\sigma_i,\beta_i} \to \mathcal{X}_{\sigma,\beta}$ are the open immersions corresponding to the inclusions of the faces $\sigma_i \to \sigma$. The diagram of open immersions of the $\mathcal{X}_{\sigma_i,\beta_i}$ can therefore be realized as the diagram of inclusions of open substacks of $\mathcal{X}_{\sigma,\beta}$. Therefore, $\mathcal{X}$ is the union of these open substacks of $\mathcal{X}_{\sigma,\beta}$. In particular, it is toric. $\square$

Remark 8.13. Note that the proof in fact shows that $\mathcal{X}$ is an open substack of a cohomologically affine toric stack $\mathcal{X}_{\sigma,\beta}$.

Remark 8.14. Let $\mathcal{X}_{\Sigma',\beta'\colon L'\to N}$ be an arbitrary toric stack, and let $\mathcal{X}_{\widetilde{\Sigma},\tilde{\beta}}$ be its canonical stack. Applying Remark 8.13, we see that $\mathcal{X}_{\Sigma',\beta'}$ is an open substack of a cohomologically affine toric stack. In fact, if Σ' spans L' (i.e. $X_{\Sigma'}$ has no torus factors), $\mathcal{X}_{\Sigma',\beta'}$ is an open substack of a *canonical* cohomologically affine toric stack.

9 Preliminary Technical Results

In this section, we gather technical results that will be used in the proofs of Theorems 10.2, 11.2, and 12.1.

9.1 Some Facts About Stacks

Lemma 9.1. *Let $\mathcal{Z}$ be an irreducible Weil divisor (i.e. a reduced irreducible closed substack) of a stack $\mathcal{X}$. Suppose $U \to \mathcal{X}$ is a smooth cover. Then $\mathcal{Z}$ is a Cartier divisor of $\mathcal{X}$ if and only if $\mathcal{Z} \times_{\mathcal{X}} U$ is a Cartier divisor of U. In particular, on any smooth stack, every Weil divisor is Cartier.*

Proof. If $\mathcal{I}$ is the ideal sheaf of the Weil divisor $\mathcal{Z}$, then $\mathcal{Z}$ is Cartier if and only if $\mathcal{I}$ is a line bundle. One may verify that a quasi-coherent sheaf is locally free of a given rank locally in the smooth topology [Mil, Theorem 11.4]. Since smooth morphisms are flat, the pullback to U of ideal sheaf $\mathcal{I}$ is the ideal sheaf of the fiber product $\mathcal{Z} \times_{\mathcal{X}} U$. □

Proposition 9.2. *Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable étale morphism of algebraic stacks. Then f induces finite-index inclusions on stabilizers of geometric points.*

Proof. Since f is representable, it is faithful [LMB00, Proposition 2.4.1.3 with Corollary 8.1.2], so the induced maps on stabilizers are inclusions. Suppose $x\colon \operatorname{Spec} K \to \mathcal{X}$ is a geometric point, and let G be the stabilizer of $f(x)$. The residual gerbe of $\mathcal{Y}$ at $f(x)$ must be trivial since K is separably closed, so we have a stabilizer-preserving morphism $BG \to \mathcal{Y}$ through which $f(x)$ factors. Since stabilizer-preserving morphisms are stable under base change, it suffices to show that the morphism $BG \times_{\mathcal{Y}} \mathcal{X} \to BG$ induces finite-index inclusions on stabilizers. Base changing along $\operatorname{Spec} K \to BG$, we get an étale cover U of $\operatorname{Spec} K$, which must be a finite disjoint union of copies of $\operatorname{Spec} K$.

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & BG \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d, "\text{ét, rep}"] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} K \arrow[r] & BG \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

We have that U is a G-torsor over $BG \times_{\mathcal{Y}} \mathcal{X}$. If $H \subseteq G$ is the stabilizer of a point of U, then the orbit is isomorphic to G/H. Since U is finite, any such G/H must be finite, so H must have finite index inside of G. The stabilizers at points of $BG \times_{\mathcal{Y}} \mathcal{X}$ are precisely such H. □

Lemma 9.3. *Suppose $\mathcal{X}$ is an algebraic stack with affine diagonal. Suppose G is an affine algebraic group with an action on $\mathcal{X}$. Then $[\mathcal{X}/G]$ has affine diagonal.*

Proof. The following diagram is cartesian:

$$\begin{tikzcd}
G \times \mathcal{X} \cong \mathcal{X} \times_{[\mathcal{X}/G]} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{X} \arrow[d] \\
{[\mathcal{X}/G]} \arrow[r, "\Delta_{[\mathcal{X}/G]}"] & {[\mathcal{X}/G] \times [\mathcal{X}/G]}
\end{tikzcd}$$

Since $\mathcal{X} \times \mathcal{X} \to [\mathcal{X}/G] \times [\mathcal{X}/G]$ is a smooth cover, it suffices to verify that the action morphism $G \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is affine.

Composing with the projections gives the projection and action maps $p_2, \alpha\colon G \times \mathcal{X} \to \mathcal{X}$. The projection p_2 is affine because G is affine, and α is isomorphic to p_2, so it is also affine. The top map is then the composition $G \times \mathcal{X} \xrightarrow{\Delta} (G \times \mathcal{X}) \times (G \times \mathcal{X}) \xrightarrow{\alpha \times p_2} \mathcal{X} \times \mathcal{X}$. Since $\alpha \times p_2$ is a product of affine maps, it is affine. Since G is affine, it has affine diagonal. By assumption, $\mathcal{X}$ also has affine diagonal, so $G \times \mathcal{X}$ has affine diagonal. So $G \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a composition of affine morphisms. □

Lemma 9.4. *If $\mathcal{X}$ has affine diagonal and $\mathcal{Y} \to \mathcal{X}$ is a canonical stack morphism, then $\mathcal{Y}$ has affine diagonal.*

Proof. Consider the following diagram, in which the square is cartesian.

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\Delta_{\mathcal{Y}/\mathcal{X}}"'] \arrow[rr, bend left, "\Delta_{\mathcal{Y}}"] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \times \mathcal{Y} \arrow[d] \\
& \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

Since $\Delta_{\mathcal{Y}}$ is a composition of $\Delta_{\mathcal{Y}/\mathcal{X}}$ and a pullback of $\Delta_{\mathcal{X}}$ (which is assumed to be affine), it suffices to show that $\Delta_{\mathcal{Y}/\mathcal{X}}$ is affine.

Affineness can be verified locally on the base in the smooth topology, so we may assume $\mathcal{Y} = [X_{\widetilde{\Sigma}}/G_{\Phi}]$ and $\mathcal{X} = X_{\Sigma}$ (see Remark 5.1). In this case, $\mathcal{Y}$ has affine diagonal by Lemma 9.3, so in the above diagram, $\mathcal{Y}$ and $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ are both affine over $\mathcal{Y} \times \mathcal{Y}$, so $\Delta_{\mathcal{Y}/\mathcal{X}}$ is affine. □

Lemma 9.5. *Let $\mathcal{X}$ be an algebraic stack over a field k, with reductive stabilizers at geometric points, and let G be a diagonalizable group over k which acts on $\mathcal{X}$. Then $[\mathcal{X}/G]$ has reductive stabilizers at geometric points.*

Proof. Let $f\colon \operatorname{Spec} K \to [\mathcal{X}/G]$ be a geometric point (i.e. K be a separably closed extension of the field k). Then f is the image of some geometric point $\tilde{f}\colon \operatorname{Spec} K \to \mathcal{X}$. We have the following diagram, in which the square is cartesian:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \mathcal{X} \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
& {[\mathcal{X}/G]} \arrow[r, "\pi"] & BG
\end{tikzcd}$$

An automorphism ϕ of f in $[X/G]$ induces an automorphism of $\pi \circ f$, which is a K-point of G. By cartesianness of the square, this image in G is the identity if and only if ϕ is induced by an automorphism of $\tilde{f}$, so we get an exact sequence

$$1 \to \operatorname{Aut}_{\mathcal{X}}(\tilde{f}) \to \operatorname{Aut}_{[\mathcal{X}/G]}(f) \to G$$

(exactness on the left follows from the fact that $\mathcal{X} \to [\mathcal{X}/G]$ is representable). So the stabilizer of the point of $[\mathcal{X}/G]$ is an extension of a subgroup of G by the stabilizer of a pre-image in $\mathcal{X}$. Since G is diagonalizable, any subgroup of it is diagonalizable, and so reductive. An extension of reductive groups is reductive. □

Lemma 9.6. *Let $\mathcal{X}$ be a normal noetherian algebraic stack, and let $\mathcal{U} \subseteq \mathcal{X}$ be an open subscheme whose complement is of codimension at least 2. Then the inclusion $\mathcal{U} \hookrightarrow \mathcal{X}$ is Stein.*

Proof. By cohomology and base change [Har77, Proposition 9.3], the property of being Stein is local on the base in the smooth topology, so we may assume $\mathcal{X} = \operatorname{Spec} R$, with R a normal noetherian domain. Then $\mathcal{U}$ is a scheme, and we must show that any regular function on $\mathcal{U}$ arises as an element of R. Any regular function on $\mathcal{U}$ is a rational function on R, so it is of the form f/g, with $f, g \in R$. Since the complement of $\mathcal{U}$ is of codimension at least 2, we see that $f/g \in R_{\mathfrak{p}}$ for any codimension 1 prime $\mathfrak{p}$. A noetherian normal domain is the intersection in its fraction field of its localizations at codimension 1 primes [Eis95, Corollary 11.4], so $f/g \in R$. □

Corollary 9.7. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of normal noetherian algebraic stacks. Suppose there is an open substack $\mathcal{U} \subseteq \mathcal{X}$ so that $f|_{\mathcal{U}}$ is an isomorphism, and so that $\mathcal{U} \subseteq \mathcal{X}$ and $f(\mathcal{U}) \subseteq \mathcal{Y}$ have complements of codimension at least 2. Then f is Stein.*

Proof. Let $i\colon \mathcal{U} \to \mathcal{X}$ be the inclusion. By Lemma 9.6, we have that i and $f \circ i$ are Stein. It follows that $f_*\mathcal{O}_{\mathcal{X}} = f_*i_*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{Y}}$, so f is Stein. □

Proposition 9.8. *Let $D_1, \dots, D_n$ be effective Cartier divisors on a locally finite type scheme X over a field k. Let $x \in X$ be a point at which X is smooth, at which each D_i is smooth, and at which these divisors have simple normal crossings. Then the induced morphism $\phi\colon X \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is smooth at x.*

Remark 9.9. Smoothness of X and the divisors at x can be checked on a smooth cover of X, as can the property of having simple normal crossings. Therefore, this smoothness criterion applies to stacks as well.

However, note that smoothness of the map $\phi\colon \mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ *does not* entail representability of the map. It simply means that for any smooth cover by a scheme $U \to \mathcal{X}$, the composite map $U \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is smooth.

Proof. If some D_i does not pass through x, then there is an open neighborhood of x such that ϕ factors through $[(\mathbb{A}^{n-1} \times \mathbb{G}_m)/\mathbb{G}_m^n] = [\mathbb{A}^{n-1}/\mathbb{G}_m^{n-1}]$. Smoothness can be checked on this neighborhood. We may therefore assume that all the divisors pass through x.

We may verify formal smoothness at x after restricting to the completed local ring $\widehat{\mathcal{O}}_{X,x}$. Since X is locally of finite type, formal smoothness implies smoothness. By the Cohen structure theorem [Eis95, Theorem 7.7], $\widehat{\mathcal{O}}_{X,x} = k[[x_1, \dots, x_r]]$, and since the divisors are smooth with simple normal crossing, we may choose coordinates so that the divisor D_i is the vanishing locus of the coordinate x_i. Then ϕ is a composition of three formally smooth morphisms: the "inclusion" of the complete local ring $\operatorname{Spec}\widehat{\mathcal{O}}_{X,x} \to \mathbb{A}^r$, the coordinate projection $\mathbb{A}^r \to \mathbb{A}^n$, and the quotient morphism $\mathbb{A}^n \to [\mathbb{A}^n/\mathbb{G}_m^n]$. □

9.2 Luna's Slice Argument

Here we prove a weak form of Luna's slice theorem. Our hypotheses are weaker than those in Luna's slice theorem (e.g. we do not assume an action of a reductive group, only that the stabilizers are reductive), as is the conclusion (we do not get *strong* étaleness). Since the hypotheses differ from the standard result significantly, we reproduce the proof here.

Definition 9.10. Let Z be a scheme with an action of a group scheme H, and let $H \subseteq G$ be a subgroup. Then $Z \times^H G$ denotes $[G \times Z/H]$, where the action of H is given by $h \cdot (g, z) = (gh^{-1}, h \cdot z)$.

Lemma 9.11. *Let Z be a scheme over a field k of characteristic 0. Let G be a group scheme over k, and let $H \subseteq G$ be a subgroup. The tangent space to $G \times^H Z$ at the image of (g, z) is $(T_gG \oplus T_zZ)/T_eH$, where the inclusion $T_eH \to T_gG \oplus T_zZ$ is induced by the inclusion $H \to G \times Z$, $h \mapsto (gh^{-1}, h \cdot z)$.*

Moreover, $G \times^H Z$ is smooth at the image of a k-point (g, z) if and only if Z is smooth at z.

Proof. We have a smooth map $G \times Z \to G \times^H Z$ whose fiber over the image of (g, z) is $\{(gh^{-1}, h \cdot z) | h \in H\}$. For any smooth map, the tangent space of an image point is the quotient of the tangent space of the point by the tangent space of the fiber at that point. This proves the first statement.

We have that $G \times Z$ is an H-torsor over $G \times^H Z$ and a G-torsor over Z. Smoothness can be checked locally in the smooth topology. Since G and H are smooth as we are over a field of characteristic zero, we see that Z is smooth at z if and only if $G \times Z$ is smooth at (g, z) if and only if $G \times^H Z$ is smooth at the image of (g, z). □

Lemma 9.12. *Let $f : Y \to X$ be a quasi-compact morphism of schemes and $x \in X$ a point so that f is étale at every point in the pre-image of x. Then there is an open neighborhood $U \subseteq X$ of x so that the restriction $f^{-1}(U) \to U$ is étale.*

Proof. For every point $y \in f^{-1}(x)$, let $V_y \subseteq Y$ be an open neighborhood of y so that $f|_{V_y}$ is étale. Since étale morphisms are open, $f(V_y) \subseteq X$ is open. We have that the fiber $f^{-1}(x)$ is quasi-compact and étale over the point x, so it is finite. Let $U = \bigcap_{y \in f^{-1}(x)} f(V_y)$. Then U is an open neighborhood of x such that $f^{-1}(U) \subseteq \bigcup_{y \in f^{-1}(x)} V_y$, so the induced morphism $f^{-1}(U) \to U$ is étale. □

Proposition 9.13 (Luna slice argument)**.** *Let G be an affine algebraic group acting on a quasi-affine scheme X over an algebraically closed field k of characteristic 0. Suppose $x \in X$ is a k-point whose stabilizer $H \subseteq G$ is linearly reductive. Then there exists a connected locally closed H-invariant subscheme $Z \subseteq X$ such that $x \in Z$ and such that the induced morphism $Z \times^H G \to X$ is étale.*

This roughly says that at a point with linearly reductive stabilizer H, a quotient stack $[X/G]$ is étale locally a quotient by H. Explicitly, we have the étale representable morphism $[Z/H] \cong [(Z \times^H G)/G] \to [X/G]$.

Proof. We first consider the case where X is smooth. Let $A = \mathcal{O}_X(X)$. By [StPrj, Lemma 01P9], the natural map $X \to \operatorname{Spec} A$ is an open immersion, so we identify X with an open subscheme of $\operatorname{Spec} A$. Let $\mathfrak{m}$ be the maximal ideal in A corresponding to $x \in X$. The surjection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \cong (T_xX)^*$ is H-equivariant. Since H is linearly reductive, there is an H-equivariant splitting, which induces an H-equivariant ring homomorphism $\operatorname{Sym}^*(\mathfrak{m}/\mathfrak{m}^2) \to A$ sending the positive degree ideal into $\mathfrak{m}$. This corresponds to an H-equivariant map $\operatorname{Spec} A \to T_xX$ sending x to 0 and inducing an isomorphism on tangent spaces at x. Since X and T_xX are smooth, the map is étale at x [BLR90, §2.2, Corollary 10].

The tangent space T_xX has a natural action of H. The tangent space to the G-orbit through x is an H-invariant subspace of T_xX. Since H is linearly reductive, this subspace has an H-invariant complement V.

$$\begin{tikzcd}
V \arrow[d, hook] & Z' \arrow[l, "H\text{-eq}"'] \arrow[r, hook] \arrow[d, hook] & G \times^H Z' \arrow[d] \\
T_xX & X \arrow[l, "H\text{-eq}"'] \arrow[r, equal] & X
\end{tikzcd}$$

We define Z' as $V \times_{T_xX} X$. This is a closed H-invariant subscheme of X which contains x. The map $Z' \to V$ is H-equivariant and and is étale over V at x. In particular, Z' is smooth at x. The action of G induces a morphism $G \times^H Z' \to X$. By Lemma 9.11, $G \times^H Z'$ is smooth at the image of (e, x), X is smooth at x, and the map induces an isomorphism of tangent spaces since $T_xZ' \cong V$ is complementary to $T_x(G \cdot x)$. By [BLR90, §2.2, Corollary 10], the map $G \times^H Z' \to X$ is étale at the image of (e, x). Since the morphism is G-equivariant and every point in the fiber over x is in a single G-orbit, it is étale at every point in the fiber, and therefore étale over a neighborhood of $x \in X$ by Lemma 9.12. Since this map is G-equivariant, the locus in X where it is étale is a G-invariant open neighborhood U of x. Setting $Z = Z' \cap U$, we get that $Z \times^H G \to X$ is étale. This completes the proof in the case when X is smooth.

Now consider the case where X is not smooth. We may choose a G-equivariant immersion of X into a smooth scheme X_0. Indeed, X_0 can be chosen to be a finite-dimensional representation of G [PV94, Theorem 1.5]. As shown above, there are representations $V \subseteq W$

of H, a G-invariant open neighborhood U_0 of x, and a closed subscheme $Z_0 \subseteq U_0$ such that $Z_0 = V \times_W U_0$ and $Z_0 \times^H G \to U_0$ is étale. Setting $U = U_0 \times_{X_0} X$ and $Z = Z_0 \cap X$, we have the following cartesian diagram.

$$\begin{tikzcd} Z \times^H G \arrow[r] \arrow[d] & Z_0 \times^H G \arrow[d] \\ U \arrow[r] \arrow[d] & U_0 \arrow[d] \\ X \arrow[r] & X_0 \end{tikzcd}$$

Since $Z_0 \times^H G \to U_0$ is étale, so is $Z \times^H G \to U$.

Finally, since x is fixed by H, the connected component of Z which contains x is H-invariant. We may replace Z by this connected component. □

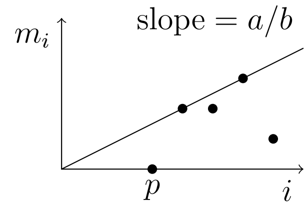
9.3 A Characterization of Pointed Toric Varieties

The proof of the following proposition is due to Vera Serganova (see [Ger]).

Proposition 9.14. *Let V be a representation of a linearly reductive group G over a field k of characteristic 0, and let $Z = \overline{G \cdot v} \subseteq V$ be the closure of an unstable G-orbit (i.e. $0 \in Z$). If Z is not contained in a direct sum of 1-dimensional representations of G, then it contains a positive highest weight vector (with respect to some Borel subgroup of G).*

Proof. We may assume v is not contained in a direct sum of 1-dimensional representations. By the Hilbert-Mumford criterion [GIT, Proposition 2.4], there is a 1-parameter subgroup $\gamma\colon \mathbb{G}_m \to G$ so that $\gamma(t) \cdot v$ contains 0 in its closure. We have the weight space decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where $V_i = \{x \in V | \gamma(t)x = t^i x\}$. Let $v = \sum_{i\geq p} v_i$, where $v_i \in V_i$ and $v_p \neq 0$. We may assume $p > 0$ (replacing γ by its inverse if necessary).

Let T be a maximal torus containing the image of γ, and let $B \subseteq H$ be a Borel subgroup containing T so that γ pairs non-negatively with all positive roots. Since only a finite number of weights appear in V, we may modify γ so that it pairs *positively* with all positive roots. If v is a highest weight vector with respect to B, then we are done. Otherwise, there is some positive root α so that $e_\alpha \cdot v \neq 0$, with $e_\alpha \in \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space corresponding to α in the Lie algebra of G. Let $\exp(te_\alpha) \cdot v = \sum_{i\geq p} f_i(t)$, where $f_i(t) \in V_i \otimes k[t]$. Let $m_i = \deg f_i$. Since α pairs positively with γ (and $e_\alpha \cdot V_i \subseteq V_{i+\langle\gamma,\alpha\rangle}$), we have $e_\alpha \cdot v \in \bigoplus_{i>p} V_i$, so $m_p = 0$. Moreover, since $e_\alpha \cdot v \neq 0$, some m_i is positive.



Let $\frac{a}{b} \in \mathbb{Q}$ be the rational number so that $m_i \leq \frac{a}{b}i$ for all i and $m_j = \frac{a}{b}j$ for some j. Consider the function $g\colon \mathbb{A}^1 \to V$ given by $g(t) = \sum t^{a\cdot i} f_i(t^{-b})$. Note that this is well-defined since $\deg f_i = m_i \leq \frac{a}{b}i$ for all i, so $\deg\big(t^{a\cdot i} f_i(t^{-b})\big) = a \cdot i - b \cdot m_i \geq 0$. Note also that $g(0) \neq 0$ since $m_j = \frac{a}{b}j$ for some j. For $t \neq 0$, we have that $g(t) = \gamma(t^a) \cdot \exp(t^{-b}e_\alpha) \cdot v \in Z$. Since Z is closed, we have that $g(0) \in Z$. Note that the minimal weight (with respect to γ) appearing in $g(0)$ is greater than p, and that $g(0)$ does not lie in a direct sum of 1-dimensional representations since it is in the image of e_α. Since V is finite-dimensional, repeating this procedure a finite number of times produces a positive highest weight vector in Z. $\square$

Proposition 9.15. *Suppose Z is an irreducible affine scheme over an algebraically closed field k of characteristic 0, with an action of a linearly reductive group H. Suppose that $x \in Z$ is an H-invariant k-point, that Z contains a dense open stabilizer-free orbit, and that the stabilizer of each k-point of Z is linearly reductive. Then H is a torus. In particular, if Z is reduced and normal, it is a toric variety.*

Proof. Since H is dense in Z, it is irreducible. Let $Z = \operatorname{Spec} A$, and let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to x. We may choose a finite-dimensional H-invariant subspace $V^* \subseteq \mathfrak{m}$ such that V^* generates A as a k-algebra . Then $\operatorname{Spec} A \to \operatorname{Spec}(\operatorname{Sym}^*(V^*)) = V$ is a closed H-equivariant immersion of Z into a finite-dimensional representation of H, sending x to the origin. Since Z contains a dense open stabilizer-free H-orbit, the subrepresentation spanned by Z is faithful. If Z is contained in a direct sum of 1-dimensional representations, then H is diagonalizable, so it is a torus. Otherwise, Z contains a positive highest weight vector v by Proposition 9.14. Then v is stabilized by the unipotent radical of some Borel subgroup of H. Since v has reductive stabilizer, it must be stabilized by the opposite unipotent group, so by the derived group of H, contradicting the assumption that it is a *positive* weight vector. $\square$

10 The Local Structure Theorem

The main result of this section is Theorem 10.2. Together with Lemma 10.1, this theorem serves as our main tool for showing that a stack is toric.

Lemma 10.1. *Let $\mathcal{X}$ be an algebraic stack over a field k with an action of a torus T and a dense open substack which is T-equivariantly isomorphic to T. Then $\mathcal{X}$ is a toric stack if and only if $[\mathcal{X}/T]$ is a toric stack.*

Proof. If $\mathcal{X} = [X/G]$ is a toric stack, where X is a toric variety and $G \subseteq T_X$ is a subgroup of the torus, then $T = T_X/G$. We see that $[\mathcal{X}/T] \cong [X/T_X]$ is a toric stack.

Now suppose $[\mathcal{X}/T] = [X/G]$ is a toric stack, where X is a toric variety and $G \subseteq T_X$ is a subgroup of the torus. Since $[\mathcal{X}/T]$ has a dense open point, we have $G = T_X$ is the torus of X. Consider the following cartesian diagram.

$$\begin{tikzcd}
T_X \times T \arrow[rr] \arrow[dr, hook] \arrow[dd] & & T \arrow[dr, hook] \arrow[dd] & \\
& X \times_{[\mathcal{X}/T]} \mathcal{X} \arrow[rr] \arrow[dd] & & \mathcal{X} \arrow[dd] \\
T_X \arrow[rr] \arrow[dr, hook] & & {[T_X/T_X] = [T/T] = *} \arrow[dr, hook] & \\
& X \arrow[rr] & & {[X/T_X] \cong [\mathcal{X}/T]}
\end{tikzcd}$$

The stack $Z \times^H G \to X$ has an action of the torus $T_X \times T$, and a dense open substack isomorphic to $T_X \times T$. Since $X \times_{[\mathcal{X}/T]} \mathcal{X}$ is a T-torsor over X, it is a normal separated scheme, so it is a toric variety with torus $T_X \times T$. It is also a T_X-torsor over $\mathcal{X}$, so $\mathcal{X} = [(X \times_{[\mathcal{X}/T]} \mathcal{X})/T_X]$ is a toric stack. $\square$

Theorem 10.2. *Suppose $\mathcal{X}$ is a reduced finite type Artin stack over an algebraically closed field k of characteristic 0, with a dense open (non-stacky) point k-point. Let $\xi\colon \operatorname{Spec} k \to \mathcal{X}$ be a point. Suppose*

1. *$\mathcal{X}$ is normal,*
2. *$\mathcal{X}$ has affine diagonal,*
3. *$\mathcal{X}$ has linearly reductive stabilizers at geometric points, and*
4. *ξ is in the image of an étale representable map $[U/G] \to \mathcal{X}$, where U is quasi-affine and G is an affine group scheme (see Remark 10.3 below),*

Then there is an affine toric variety X with torus T and an open immersion $[X/T] \hookrightarrow \mathcal{X}$ sending the distinguished closed point of $[X/T]$ to ξ.[1]

Remark 10.3. A quasi-compact stack in which every point is in the image of an étale representable map from a quotient of a quasi-affine scheme by an affine group is said to be *locally of global type* [Ryd09]. It is possible that every quasi-compact quasi-separated stack with locally separated diagonal and affine stabilizers is of global type. In particular, it is possible that condition 4 of Theorem 10.2 is unnecessary given the other hypotheses.

Since $\mathcal{X}$ is assumed to be normal, finite type, and to have affine diagonal, any stack étale over it is normal, Noetherian, and has affine stabilizers at closed points. Totaro has shown [Tot04, Theorem 1.1] that such a stack is a quotient of a quasi-affine scheme by an affine

[1] Note that $[X/T]$ has a distinguished closed point, even if X does not. An affine toric variety X can only fail to have a distinguished closed point if it is of the form $X' \times T_0$, where X' has a distinguished closed point and T_0 is a torus. In this case, $[X/T] \cong [X'/(T/T_0)]$.

group if and only if it has the resolution property. Thus, to verify that $\mathcal{X}$ is of global type, it suffices to find a cover by étale representable morphism from stacks with the resolution property.

Finally, any stack of the form $[X/G]$ where X is a normal Noetherian scheme and G is a connected affine group is of global type [Ryd09, Remark 2.3]. In practice, this is probably the most useful way to verify this condition.

Proof. Let $x \in U$ be a k-point mapping to ξ, and let $H \subseteq G$ be the stabilizer of x. Since the morphism $[U/G] \to \mathcal{X}$ is étale representable, Proposition 9.2 implies that the stabilizers of geometric points of $[U/G]$ are finite index subgroups of the stabilizers of geometric points of $\mathcal{X}$. That is, given a point $y\colon \operatorname{Spec} k \to [U/G]$, $\operatorname{Stab}_{\mathcal{X}}(y)/\operatorname{Stab}_{[U/G]}(y)$ is finite, and therefore affine. Since the stabilizers of geometric points of $\mathcal{X}$ are linearly reductive, Matsushima's criterion (see [Alp08, Proposition 11.14(i)]) implies that the stabilizers of geometric points of $[U/G]$ are linearly reductive.

Applying Proposition 9.13, there is a connected locally closed H-invariant subscheme $Z \subseteq U$ so that $x \in Z$ and $Z \times^H G \to U$ is étale. We have that Z is smooth over $\mathcal{X}$. Since $\mathcal{X}$ is normal, and normality is local in the smooth topology, Z is normal. Since Z is also connected, it is irreducible.

The map $[Z/H] \to [U/G] \to \mathcal{X}$ is étale and representable. Base changing $[Z/H] \to \mathcal{X}$ to the dense open k-point of $\mathcal{X}$, we get an irreducible étale cover of $\operatorname{Spec} k$, which must be trivial since k is algebraically closed. In particular, $[Z/H]$ has a dense open k-point, so Z contains a dense open stabilizer-free H-orbit.

Next we show that Z must be affine. Since Z is quasi-affine, it is a dense open subscheme of $\operatorname{Spec} \mathcal{O}_Z(Z)$ [StPrj, Lemma 01P9]. The action of H on Z induces an action of H on $\operatorname{Spec} \mathcal{O}_Z(Z)$. By [GIT, Theorem 1.1], $\operatorname{Spec}\big(\mathcal{O}_Z(Z)\big)/H = \operatorname{Spec}\big(\mathcal{O}_Z(Z)^H\big)$ is a good quotient. Since $\mathcal{O}_Z(Z)$ contains a dense open copy of H, any H-invariant regular function must be constant, so the good quotient is $\operatorname{Spec} k$. It follows that the closures of any two H-orbits intersect. But, $x \in Z$ is a closed H-orbit, and $Z \subseteq \operatorname{Spec} \mathcal{O}_Z(Z)$ is an H-invariant open neighborhood of x, so $Z = \operatorname{Spec} \mathcal{O}_Z(Z)$.

By the same argument we used in the first paragraph of this proof, the stabilizers of $[Z/H]$ are linearly reductive. Since Z is smooth over $\mathcal{X}$, it is normal and reduced. By Proposition 9.15, H is a torus and Z is a toric variety.

Finally, we have an étale representable map $[Z/H] \to \mathcal{X}$, whose image is an open substack. Replacing $\mathcal{X}$ by this open substack, we may assume the map is surjective. Now we have that

$Z \to \mathcal{X}$ is a smooth cover. Consider the following cartesian diagram:

$$\begin{tikzcd}
Z \times_{\mathcal{X}} Z \arrow[r] \arrow[d] & Z \arrow[d, "H\text{-torsor}"] \\
Y \arrow[r] \arrow[d] & {[Z/H]} \arrow[d] \\
Z \arrow[r] & \mathcal{X}
\end{tikzcd}$$

Since Z is affine and $\mathcal{X}$ has affine diagonal, we have that $Z \times_{\mathcal{X}} Z$ is affine. This affine space is the total space of an H-torsor over an algebraic space Y. Since H is linearly reductive, Y is an affine scheme [GIT, Theorem 1.1]. Since Y and Z are both affine, $Y \to Z$ is separated. Separatedness is local on the base in the smooth topology, so $[Z/H] \to \mathcal{X}$ is separated.

Now $[Z/H] \to \mathcal{X}$ is representable separated étale birational and surjective, so it is an isomorphism by Zariski's Main Theorem [LMB00, Theorem 16.5]. □

11 Main Theorem: Smooth Case

Lemma 11.1. *Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a smooth (but not necessarily representable) morphism of Artin stacks. Then f is codimension-preserving: if $\mathcal{Z} \subseteq \mathcal{Y}$ is a closed substack of codimension d, then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \subseteq \mathcal{X}$ is of codimension d.*

Proof. Let $\pi\colon U \to \mathcal{X}$ be a smooth cover by a scheme. Then $g = f \circ \pi\colon U \to \mathcal{Y}$ is smooth and representable, so it is open and codimension-preserving. If $\mathcal{Z} \subseteq \mathcal{Y}$ is a closed substack of codimension d, then $\mathcal{Z} \times_{\mathcal{Y}} U \subseteq U$ is closed of codimension d. On the other hand, $U \to \mathcal{X}$ is codimension-preserving, and $\mathcal{Z} \times_{\mathcal{Y}} U = (\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X}} U$, so $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \subseteq \mathcal{X}$ is of codimension d. □

Theorem 11.2. *Let $\mathcal{X}$ be a smooth Artin stack over an algebraically closed field k of characteristic 0. Suppose $\mathcal{X}$ has an action of a torus T and a dense open substack which is T-equivariantly isomorphic to T. Then $\mathcal{X}$ is a toric stack if and only if the following conditions hold:*

1. *$\mathcal{X}$ is reduced and of finite type,*
2. *$\mathcal{X}$ has affine diagonal,*
3. *geometric points of $\mathcal{X}$ have linearly reductive stabilizers, and*
4. *every point of $[\mathcal{X}/T]$ is in the image of an étale representable map from a stack of the form $[U/G]$, where U is quasi-affine and G is an affine group.*

Remark 11.3. The final condition in the theorem is required to apply Theorem 10.2. It is possible that it is unnecessary given the other hypotheses (see Remark 10.3).

Proof. It is clear that smooth toric stacks satisfy the conditions, so we focus on the converse.

By Lemma 10.1, it suffices to check that $[\mathcal{X}/T]$ is a toric stack. By Lemma 9.3, $[\mathcal{X}/T]$ has affine diagonal. By Lemma 9.5, $[\mathcal{X}/T]$ has linearly reductive stabilizers. Thus, we have reduced to the case where T is trivial and $\mathcal{X}$ has a dense open k-point.

Consider the set of divisors of $\mathcal{X}$. By Lemma 9.1, these divisors are Cartier, so they are induced by line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ with non-zero global sections $s_i \in \Gamma(\mathcal{X}, \mathcal{L}_i)$. These line bundles and sections induce a morphism $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$. We will show that this morphism is an open immersion—and therefore that $\mathcal{X}$ is a toric stack—by induction on n.

The case $n = 0$

If $\mathcal{X}$ has no divisors, then we claim that $\mathcal{X} = \operatorname{Spec} k$. By Theorem 10.2, every point of $\mathcal{X}$ has an open neighborhood of the form $[X/T_X]$, where X is a toric variety and T_X is its torus. Every point of a toric variety lies either in the torus or on a torus-invariant divisor. Since $\mathcal{X}$ has no divisors, X can have no torus-invariant divisors. It follows that X must be a torus, and so $\mathcal{X}$ is covered by its dense open point.

The case $n = 1$

Suppose $\mathcal{D} \subseteq \mathcal{X}$ is the unique divisor. Our aim is to show that the morphism $\mathcal{X} \to [\mathbb{A}^1/\mathbb{G}_m]$ is an isomorphism.

Applying Theorem 10.2 to points of $\mathcal{D}$, we see that $\mathcal{D}$ has a dense (stacky) geometric point, and that any other point must lie on the intersection of two or more distinct divisors (because this is true for torus-invariant divisors on a smooth toric variety). Since $\mathcal{D}$ is the unique divisor of $\mathcal{X}$, it has only one geometric point p. Aside from this point, $\mathcal{X}$ has only one other point: the dense open point t. Applying Theorem 10.2 around p, we get an open neighborhood of the form $[X/T]$, where X is a toric variety and T is its torus. But any open neighborhood p must be all of $\mathcal{X}$, so $\mathcal{X} = [X/T]$ is a toric stack. Moreover, the toric variety X has precisely one torus-invariant divisor, so $[X/T] = [\mathbb{A}^1/\mathbb{G}_m]$.

The general case $n \geq 2$

Suppose $\mathcal{D}_1, \dots, \mathcal{D}_n$ are the divisors cut out by the sections $s_i \in \Gamma(\mathcal{X}, \mathcal{L}_i)$. By induction on n, $\mathcal{X} \smallsetminus \mathcal{D}_i$ is a smooth toric stack, so the morphism $\mathcal{X} \smallsetminus \mathcal{D}_i \to [\mathbb{A}^{n-1}/\mathbb{G}_m^{n-1}]$ is an open immersion. On the other hand, this morphism is part of the following cartesian diagram.

$$\begin{tikzcd}
 & \mathcal{X} \smallsetminus \mathcal{D}_i \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\
{[\mathbb{G}_m/\mathbb{G}_m] \times [\mathbb{A}^{n-1}/\mathbb{G}_m^{n-1}]} \arrow[r, phantom, "="] & {[\mathbb{A}^{n-1}/\mathbb{G}_m^{n-1}]} \arrow[r, hook] & {[\mathbb{A}^n/\mathbb{G}_m^n]}
\end{tikzcd}$$

Therefore, we see that the morphism $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ restricts to an open immersion $\mathcal{U} := \mathcal{X} \smallsetminus (\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n) \to [(\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m^n]$. If $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n = \varnothing$, then we are done, so we may assume $\mathcal{Z} = \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ is non-empty. Then any subset of divisors intersect, but the divisors are distinct, so $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is set-theoretically surjective. In particular, $\mathcal{U} \to [(\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m^n]$ is an isomorphism. By Theorem 10.2, $\mathcal{Z}$ is of codimension $n \geq 2$. So by Lemma 9.6, $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is Stein.

By Theorem 10.2, $\mathcal{X}$ is Zariski locally a quotient of a smooth toric variety. In particular, the divisors are smooth and have simple normal crossings, so by Proposition 9.8, $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is smooth (but may not be representable). So $\mathcal{X} \times_{[\mathbb{A}^n/\mathbb{G}_m^n]} \mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is smooth, and is an isomorphism over the complement of the closed point of $[\mathbb{A}^n/\mathbb{G}_m^n]$. Since smooth maps are codimension preserving (Lemma 11.1), the complement of $\mathcal{U} \cong \mathcal{U} \times_{[\mathbb{A}^n/\mathbb{G}_m^n]} \mathcal{U} \subseteq \mathcal{X} \times_{[\mathbb{A}^n/\mathbb{G}_m^n]} \mathcal{X}$ is of codimension $n \geq 2$. In particular, the diagonal $\Delta_{\mathcal{X}/[\mathbb{A}^n/\mathbb{G}_m^n]}$ is Stein by Lemma 9.6.

Consider the following diagram, in which the square is cartesian.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\Delta_{\mathcal{X}/[\mathbb{A}^n/\mathbb{G}_m^n]}"'] \arrow[rr, bend left, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times_{[\mathbb{A}^n/\mathbb{G}_m^n]} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \times \mathcal{X} \arrow[d] \\
& {[\mathbb{A}^n/\mathbb{G}_m^n]} \arrow[r, "\Delta_{[\mathbb{A}^n/\mathbb{G}_m^n]}"] & {[\mathbb{A}^n/\mathbb{G}_m^n] \times [\mathbb{A}^n/\mathbb{G}_m^n]}
\end{tikzcd}$$

Since $\Delta_{\mathcal{X}}$ and $\Delta_{[\mathbb{A}^n/\mathbb{G}_m^n]}$ are affine, we see that $\Delta_{\mathcal{X}/[\mathbb{A}^n/\mathbb{G}_m^n]}$ is affine.

Now $\Delta_{\mathcal{X}/[\mathbb{A}^n/\mathbb{G}_m^n]}$ is Stein and affine, so it is an isomorphism. Thus, $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is a monomorphism, so it is representable [LMB00, Corollary 8.1.2], separated, and quasi-finite. Since $[\mathbb{A}^n/\mathbb{G}_m^n]$ is normal, Zariski's Main Theorem [LMB00, Theorem 16.5] implies that $\mathcal{X} \to [\mathbb{A}^n/\mathbb{G}_m^n]$ is an open immersion. $\square$

11.1 Counterexamples

There are varieties X that contain a dense open torus T, in which T cannot possibly act on X. For example, blowing up a torus-non-invariant point on a divisor of a toric variety will produce such a variety. When working with algebraic spaces and stacks, the action can fail to extend for more subtle reasons.

Example 11.4 (Torus action does not always extend). Let U be the affine line with a doubled origin over a field not of characteristic 2. Let $\mathbb{Z}/2$ act on U by $x \mapsto -x$ (and switching the two origins). Then $X = [U/(\mathbb{Z}/2)]$ is a smooth algebraic space with a dense open torus $[\mathbb{G}_m/(\mathbb{Z}/2)] \cong \mathbb{G}_m$. This space is a "bug-eyed cover" of $\mathbb{A}^1$ [Kol92]. We claim that the torus cannot act on X.

If it did, the étale cover $\mathbb{A}^1 \to X$ would be toric, inducing the degree 2 map of tori $\mathbb{G}_m \to \mathbb{G}_m/(\mathbb{Z}/2)$. This map induces an isomorphism of $\mathbb{G}_m$-representations between the tangent space to $\mathbb{A}^1$ at 0 and the tangent space to X at "the bug eye." It would follow that the 1-dimensional weight 1 representation of $\mathbb{G}_m$ (i.e. the tangent space to $\mathbb{A}^1$ at 0) factors through the degree 2 map $\mathbb{G}_m \to \mathbb{G}_m/(\mathbb{Z}/2)$, which it clearly does not. $\diamond$

Although we have shown that the previous example is not a toric stack, it is nonetheless interesting to observe that it can be extended in an interesting way.

Example 11.5. Consider the stack $\mathcal{X} = [\mathbb{A}^2/(\mathbb{Z}/2 \ltimes \mathbb{G}_m)]$, with the action given by $(0,t)\cdot(x,y) = (tx, t^{-1}y)$ and $(1,1)\cdot(x,y) = (-y,x)$. This contains the "bug-eyed cover" from the previous example as an open substack (it is the image of $\mathbb{A}^2 \smallsetminus \{0\}$).

What makes $\mathcal{X}$ particularly interesting is that it is a smooth stack with a dense open torus (whose action does not extend, of course) so that the complement of the torus is a single *singular* divisor. $\diamond$

Remark 11.6. A notable difference between toric stacks and toric varieties is that toric varieties are required to be separated. Artin stacks are almost never separated, but the affine diagonal condition seems to play the role of separatedness. Heuristically, toric stacks are entirely controlled by their torus-invariant divisors (this is made precise by Theorem 7.7 and the canonical stack construction in Section 5). The condition that a stack have affine diagonal "forces all non-separatedness to occur in codimension 1" and therefore be controlled by the combinatorics.

Example 11.7 (Non-affine diagonal). The affine plane with a doubled origin is a variety with a torus action satisfying nearly all the conditions of Theorem 11.2, except it does not have affine diagonal.

Note however, that the affine *line* with a double origin does have affine diagonal, and is in fact a toric stack. It is $[(\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by $t\cdot(x,y) = (tx, t^{-1}y)$. $\diamond$

In the world of stacks, non-affine diagonals can occur in stranger ways as well.

*Example 11.8 (Non-*separated *diagonal).* Let G be the affine line with a doubled origin, regarded as a relative group over $\mathbb{A}^1$. The fibers away from the origin are trivial, and the fiber over the origin is given the structure of $\mathbb{Z}/2$. We see that $G \to \mathbb{A}^1$ is an étale relative group scheme. In fact, it is the quotient of the relative group $\mathbb{Z}/2 \times \mathbb{A}^1$ by the open subgroup of Example 11.10.

Let $\mathcal{X} = [\mathbb{A}^1/G]$, where G acts trivially on $\mathbb{A}^1$. Since $\mathcal{X}$ has an étale cover by $\mathbb{A}^1$, it is finite type, normal, and of global type. Moreover, it has linearly reductive stabilizers at geometric points. It contains a dense open torus $T \cong \mathbb{G}_m$ which acts on it. However, $\mathcal{X}$ has *non-separated* diagonal. $\diamond$

In Theorem 11.2, the condition that $\mathcal{X}$ have linearly reductive stabilizers is necessary. It is easy to produce many examples of stacks that satisfy all the other conditions of the theorem, but fail to be toric stacks.

Example 11.9 (Non-reductive stabilizers). If X is any smooth scheme of finite type with an action of a connected affine group G and a dense open copy of G, then $\mathcal{X} = [X/G]$ has a dense open torus (the *trivial* torus $[G/G]$) which acts (trivially). Since G is affine, $\mathcal{X}$ has affine diagonal by Lemma 9.3. By the final paragraph of Remark 10.3, $\mathcal{X}$ is of global type.

For example, consider the stack $\mathcal{X} = [M_{2\times 2}/GL_2]$, where the action of GL_2 on $M_{2\times 2} \cong \mathbb{A}^4$ is given by left multiplication. $\diamond$

We saw in Corollary 7.16 that smooth generically stacky toric stacks arise as essentially trivial gerbes over toric stacks. Theorem 11.2 gives us a good handle on smooth "abstract toric stacks," but the analogous result is false for "abstract generically stacky toric stacks," as the following example demonstrates. In fact, it is not even completely clear what the analogous statement is. An analogue of the theorem for the generically stacky case would have to include some hypothesis that prevents the stabilizers from being "smaller than expected" along a closed substack.

Example 11.10. Let G be the disjoint union of $\mathbb{A}^1$ and $\mathbb{A}^1 \smallsetminus \{0\}$, regarded as a relative group over $\mathbb{A}^1$. This is an open subgroup of the constant group $\mathbb{Z}/2$. Let $\mathcal{X} = [\mathbb{A}^1/G]$, where G acts trivially on $\mathbb{A}^1$.

This is stack containing a dense open stacky torus $\mathcal{T} \cong \mathbb{G}_m \times B(\mathbb{Z}/2)$ whose action on itself extends to an action on $\mathcal{X}$. Since it has an étale cover by $\mathbb{A}^1$, it is normal and of finite type. The stabilizers of geometric points are linearly reductive, and it even has affine diagonal. However, since it contains a point with trivial stabilizer, it is not an essentially trivial gerbe on anything, so it is not a generically stacky toric stack.

Note that we did not verify the final condition, that $[\mathcal{X}/\mathcal{T}]$ is locally of global type. This is because it is not clear how to define the quotient $[\mathcal{X}/\mathcal{T}]$ when $\mathcal{T}$ is a stacky torus. An analogue of Theorem 11.2 would have to address this question. $\diamond$

12 Main Theorem: Non-smooth Case

Theorem 12.1. *Let $\mathcal{X}$ be an Artin stack over an algebraically closed field k of characteristic 0. Suppose $\mathcal{X}$ has an action of a torus T and a dense open substack which is T-equivariantly isomorphic to T. Then $\mathcal{X}$ is a toric stack if and only if the following conditions hold:*

1. *$\mathcal{X}$ is normal, reduced, and of finite type,*
2. *$\mathcal{X}$ has affine diagonal,*
3. *geometric points of $\mathcal{X}$ have linearly reductive stabilizers, and*
4. *every point of $[\mathcal{X}/T]$ is in the image of an étale representable map from a stack of the form $[U/G]$, where U is quasi-affine and G is an affine group.*

Proof. It is clear that any toric stack satisfies the conditions.

As in the proof of Theorem 11.2, we immediately reduce to the case where T is trivial and $\mathcal{X}$ has a dense open point. By Lemma 10.1, it suffices to check that $[\mathcal{X}/T]$ is a toric stack. By Lemma 9.3, $[\mathcal{X}/T]$ has affine diagonal. By Lemma 9.5, $[\mathcal{X}/T]$ has linearly reductive stabilizers. Normality and reducedness are local in the smooth topology, so those hypotheses descend from $\mathcal{X}$ to $[\mathcal{X}/T]$.

Applying Theorem 10.2, we obtain an open cover $\coprod \mathcal{X}_i \to \mathcal{X}$, where each $\mathcal{X}_i$ is of the form $[X_i/T_i]$, with X_i an affine toric variety. Let $\mathcal{Y}_i$ be the canonical smooth toric stack over $\mathcal{X}_i$ (see §5). Since the maps $\mathcal{Y}_i \to \mathcal{X}_i$ have the universal property in Proposition 5.7, they are canonically isomorphic when pulled back to intersections, so they glue together into a smooth stack $\mathcal{Y} \to \mathcal{X}$.

The diagonal of $\mathcal{Y}$ is affine by Lemma 9.4, and it satisfies the other hypotheses of Theorem 11.2 by construction (they are local conditions which all canonical stacks satisfy), so $\mathcal{Y}$ is a smooth toric stack. So by Theorem 8.12, $\mathcal{X}$ is a toric stack. □

Remark 12.2. As mentioned in Remark 10.3, one way to verify that $[\mathcal{X}/T]$ is locally of global type (i.e. that it satisfies condition 4 of the theorem) is to show that it is locally a quotient of a normal noetherian scheme by a connected affine group.

The typical application of this approach is as follows. Let X be a noetherian normal scheme, and let G be an extension of a torus T by a connected affine group H. Suppose G acts on X and that X contains a dense open subscheme isomorphic to G. Then $\mathcal{X} = [X/H]$ inherits an action of T and contains a dense open copy of T. In this situation, $[\mathcal{X}/T] \cong [X/G]$ satisfies condition 4 of the theorem.

Appendix A: Short Exact Sequences of G_βs

In this appendix, we prove two results which allow us to relate the groups of Definition 2.17. The basic advantage of expressing a group G as an extension of a quotient H by a normal subgroup N is that any quotient stack $[X/G]$ can be identified with $\big[[X/N]/H\big]$. This trick is used heavily throughout the dissertation.

We refer the reader to [GM96] for the relevant homological algebra.

Lemma A.1. *Suppose L, L', and N' are finitely generated abelian groups. Suppose $\Phi\colon L \to L'$ is close and $\beta'\colon L' \to N'$ is a homomorphism. Suppose* $\ker\Phi$ *and* $\ker g$ *are free. Then we have the following diagram, in which the rows are exact and the morphisms to $D(L^*)$ and $D(L'^*)$ are the ones described above.*

$$\begin{tikzcd}
0 \arrow[r] & G_\Phi \arrow[r] \arrow[d, equal] & G_{\beta'\circ\Phi} \arrow[r] \arrow[d] & G_{\beta'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_\Phi \arrow[r] & D(L^*) \arrow[r] & D(L'^*) \arrow[r] & 0
\end{tikzcd}$$

Proof. By the octahedron axiom, the commutative triangle $L \to L' \to N'$ induces an exact triangle on cones $C(\Phi) \to C(\beta' \circ \Phi) \to C(\beta') \to C(\Phi)[1]$. This induces an exact triangle of duals, which induces the long exact sequence of homology groups

$$0 \to D(G^0_{\beta'}) \to D(G^0_{\beta'\circ\Phi}) \to \underbrace{D(G^0_\Phi)}_{0} \to D(G^1_{\beta'}) \to D(G^1_{\beta'\circ\Phi}) \to D(G^1_\Phi) \to 0.$$

Since Φ is close, $D(G^0_\Phi) = 0$. We therefore get the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & G^0_{\beta'\circ\Phi} & \longrightarrow & G^0_{\beta'} & \longrightarrow & 0 \\
 & & \oplus & & \oplus & & \oplus & & \\
0 & \longrightarrow & G^1_\Phi & \longrightarrow & G^1_{\beta'\circ\Phi} & \longrightarrow & G^1_{\beta'} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_\Phi & \longrightarrow & D(L^*) & \longrightarrow & D(L'^*) & \longrightarrow & 0
\end{array}$$

□

Lemma A.2. *Suppose we have the following commutative diagram, in which the rows are exact, in which L_0, L, L', N_0, N, and N' are finitely generated abelian groups, and* $\ker\beta_0$*,* $\ker\beta$*, and* $\ker\beta'$ *are free.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_0 & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} & & \\
0 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & N' & \longrightarrow & 0
\end{array}$$

Suppose β_0 is close. Then the top row in the following diagram is exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{\beta_0} & \longrightarrow & G_\beta & \longrightarrow & G_{\beta'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D(L_0^*) & \longrightarrow & D(L^*) & \longrightarrow & D(L'^*) & \longrightarrow & 0
\end{array}$$

Proof. We are given a short exact sequence $0 \to C(\beta_0) \to C(\beta) \to C(\beta') \to 0$, which gives us an exact triangle in the derived category. The dual is then again an exact triangle, so we get a long exact sequence of homology groups

$$0 \to D(G^0_{\beta'}) \to D(G^0_\beta) \to \underbrace{D(G^0_{\beta_0})}_{0} \to D(G^1_{\beta'}) \to D(G^1_\beta) \to D(G^1_{\beta_0}) \to 0.$$

Since β_0 is close, we have that $D(G^0_{\beta_0}) = 0$, so we get the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & G^0_\beta & \longrightarrow & G^0_{\beta'} & \longrightarrow & 0 \\
 & & \oplus & & \oplus & & \oplus & & \\
0 & \longrightarrow & G^1_{\beta_0} & \longrightarrow & G^1_\beta & \longrightarrow & G^1_{\beta'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D(L_0^*) & \longrightarrow & D(L^*) & \longrightarrow & D(L'^*) & \longrightarrow & 0
\end{array}$$

□

Index of Terminology and Notation

canonical stack (morphism)	Definitions 5.3 and 5.10
close morphism	Definition 2.3
cohomologically affine	Definitions 3.6 and 6.1
essentially trivial gerbe	Definition 7.19
fantastack	Definition 4.1
join-closed subdiagram of monoids	Definition 8.8
(locally) of global type	Remark 10.3
good moduli space morphism	Definition 6.1
pointed	Definition 3.10
root stack	Definition 7.17
saturated	Definition 2.1
Stein morphism	Definition 6.1
tight diagram	Definition 8.7
toric monoid	Definition 8.1
unstable cone	Definition 6.6

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