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Similarity and Spacetime: Studies in Intertheoretic Reduction and Physical Significance

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Similarity and Spacetime: Studies in Intertheoretic Reduction and Physical Significance

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Philosophy

by

Samuel Craig Fletcher

Dissertation Committee:
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2014

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ABSTRACT OF THE DISSERTATION

Similarity and Spacetime: Studies in Intertheoretic Reduction and Physical Significance

By

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Doctor of Philosophy in Philosophy

University of California, Irvine, 2014

Professor James O. Weatherall, Chair

This dissertation explores in particular a few ways in which similarity bears on general relativity, our best physical theory of space and time. The first way concerns the role of the concept of "physicality" in theorizing and using spacetime models, which manifests in at least two different ways. One is whether particular models are (or have properties that are) "physically unreasonable"—they are deemed to be pathological or otherwise artifactual, and so must excluded from the theory. Another is whether particular properties of those models are "physically significant"—warranted to be inferred about the properties of a physical system they represent. This distinction, which I draw in more detail in chapter 2, separates these two senses of physicality, the former modal-metaphysical and the latter inferential-epistemic. It is this latter that more intimately involves the notion of similarity, particular notions of which can be encoded through the use of topology. I argue then that there is no canonical topology on the models of general relativity. Rather, the choice of topology must be nontrivially dependent on the context of investigation.

In chapter 3, I bring attention to another way in which similarity, as encoded in topology, can be put to substantive use, namely in better understanding the nature of the reductive relationship between general relativity and its predecessor, the Newtonian theory of space, time, and gravitation. In particular, I show how this relationship can be made both perfectly general—applying to any situation the theories describe—and explanatory of the Newtonian theory's success.

Finally, in chapter 4, I explicate what I think are the deeper structural issues with certain examples due to Robert Geroch that are intended to show a certain topology commonly used in the physics literature in fact has several counterintuitive features. I show that there is in fact no topology that meets a strong version of Geroch's demands, but if those demands are weakened slightly, such a topology may then be constructed, one that corresponds to a natural interpretation for spacetime similarity in terms of certain classes of global observables.

Chapter 1

Introduction: Why Spacetime and Similarity?

Everyone is familiar with and makes use of the notion of similarity: wallabies are similar to kangaroos; navy blue is similar to ultramarine; there is a similar number of students (about 21,000) enrolled at UC Riverside as there is at UC Santa Barbara; and so on. Its meaning is hardly univocal, but it is nevertheless so central in everyday life that its use is typically unremarkable. Perhaps it is then not too surprising that the use of similarity *between* scientific models, despite being so essential to understanding certain parts of science, has largely been ignored by philosophers of science.[1] Indeed, even though its use is often informal, it has played an ineliminable role in some of our most formalized scientific theories.

This dissertation explores in particular a few ways in which similarity bears on general relativity, our best physical theory of space and time. The first way concerns the role of the concept of "physicality" in theorizing and using spacetime models, which manifests in at least two different ways.

[1]By contrast, there has been much work on analogical (i.e., similarity-based) *reasoning* (Bartha 2013) and on scientific representation, i.e., how models are similar to the targets they represent (Frigg and Hartmann 2012). The focus here is on similarity between the models themselves.

One is whether particular models are (or have properties that are) "physically unreasonable"—they are deemed to be pathological or otherwise artifactual, and so must excluded from the theory. Another is whether particular properties of those models are "physically significant"—warranted to be inferred about the properties of a physical system they represent. This distinction, which I draw in more detail in chapter 2, separates these two senses of physicality, the former modal-metaphysical and the latter inferential-epistemic. As I shall explain, it is this latter that more intimately involves the notion of similarity.

The connection between physicality and similarity, though perhaps not immediately apparent, arose from some of the most important developments in relativity theory's golden era (roughly 1960–1975) concerning spacetime singularities. In the early history of the theory, singularities, though not well understood, were strongly suspected to arise only as mathematical artifacts, or at most under extraordinary conditions considered physically unrealistic. In the 1960s, however, a series of results, whose apex is found in theorems by Hawking, Geroch, and Penrose, punctured this conventional wisdom, showing that singularities could arise under much more general circumstances that before thought.[2] The questions that naturally arose from these developments concerned the extent of these singularities' prevalence. Are singular relativistic spacetimes—those whose metrics are solutions to Einstein's equation—exceptional or special (despite being seemingly less exceptional than before), or are they commonplace? Are they distributed evenly among spacetimes, or are they clustered together? These kinds of questions, however ambiguous, were nonetheless seen as important enough for Geroch to ask in print:

> Let us represent the collection of all solutions of Einstein's equations diagrammatically by a piece of paper. . . . Now blacken with a pencil all the points of this figure which represent "singular solutions" in some appropriate sense. The question is: What does the resulting figure look like? Is it mostly black with a few white points, a uniform shade of gray (how dark?), or covered with black and white patches?

[2]See, e.g., Earman (1995, Ch.2) for some of this history.

(Geroch 1970, p. 264)

Just as there are points on the paper more or less close to one another, the solutions (i.e., spacetimes) represented by those points are taken to be more or less "close," or similar, to one another. The *absolute shape* of the figures should be of no consequence, only the "relative proximity" of their points. Mathematically, this suggests using a topological space to represent the space of spacetimes, a space equipped with a structure that determines edges and interiors of collections of spacetimes:

> How would we say that [the collection of universes] contains singular patches and nonsingular patches? One possibility would be in terms of topology: one could interpret a "patch" to mean the existence of a set with an interior. Thus, we might ask whether the singular and nonsingular elements of [the collection of universes] form sets with nonempty interior. That there are no gray regions in [this collection] could be interpreted to mean that every point . . . lies either in one of the patches or on the edge of one of the patches.[3] (Geroch 1970, p. 268)

Intuitively, topology is "rubber sheet" geometry, in that topological properties are just those that are preserved under continuous deformations—stretching and bending, but no tearing or puncturing. Other authors echoed both the questions and the endorsement of topology:

> The question naturally arises as to how prevalent these singular solutions are. Is it true that "nearly all" physically reasonable, spatially closed cosmologies are singular? And if a given space-time *is* singular, does it remain so under small perturbations of the metric tensor (are the singularities stable, in other words)? What do we mean by "perturbing the metric"?

[3]Geroch is restricting attention to "open" universes here, but that distinction is unimportant for the present discussion.

> It is evident that the precise mathematical formulation of these questions proceeds in the following manner: we must topologize the set of [relativistic spacetimes].
>
> (Lerner 1972, p. 1–2)

A perturbation can be understood as a transformation that takes one spacetime to another that is very similar to it. It is here the connection with physicality is made. There seems to be an abiding conviction that stability is necessary for physical significance (the inferential-epistemic sense of physicality):

> It is a general feature of the description of physical systems by mathematics that only conclusions which are stable, in an appropriate sense, are of physical interest. ... To obtain a precise notion of stability in general relativity we must say what "sufficiently small perturbation" means, i.e., we must find a suitable topology on the space of solutions of Einstein's equations. (Geroch 1971, p. 70)

Similarly with Hawking: "the only properties of space-time that are physically significant are those that are stable in some appropriate topology" (Hawking 1971, p. 395).

Hawking alludes to the issue I address explicitly in chapter 2, namely that there are infinitely many different topologies one can place on the collection of all spacetimes, topologies that can well differ in how they provide answers to questions of stability, hence physical significance. It would be convenient if there were some canonical topology with respect to which all questions should be answered. Against some suggestions that this convenience is (or should be) realized, I argue that the choice of topology—that is, the choice of a particular notion of similarity—must be nontrivially dependent on the context of investigation. This contextuality does not undermine the fruitfulness of topology, which formalizes similarity in these theoretical enterprises, but rather reveals that each topology captures something meaningful about how spacetimes are similar, hence should be chosen justifiably, not (say) for mere technical convenience.

The physics literature has, unfortunately, not been sufficiently sensitive to this point. Geroch (1970, 1971) has pointed out that the topology typically used to determine the stability of global spacetime properties has certain troublesome defects, yet there is no obvious candidate to replace it: "We have an intuitive idea of what it means to say that 'two metrics are close', but to make this idea precise turns out to be surprisingly difficult. ... Unfortunately, we do not yet have a fully satisfactory topology (on the space of metrics) with which to treat stability" (Geroch 1971, p. 70–1). In chapter 4, I provide what I think are the deeper structural issues with this topology that underlie Geroch's examples, and show that there is in fact no topology that meets a strong version of his demands. However, I show that if those demands are weakened slightly, such a topology may then be constructed, one that corresponds to a natural interpretation for spacetime similarity in terms of certain classes of global observables. This provides a satisfactory response to Geroch's query which, as far as I am aware, has gone unanswered for over 40 years.

Before that chapter, though, I bring attention to another way in which similarity, as encoded in topology, can be put to substantive use, namely in better understanding the nature of the reductive relationship between general relativity and its predecessor, the Newtonian theory of space, time, and gravitation. Intertheoretic reduction has of course long been within the locus of core issues for philosophers of science. Besides being of intrinsic philosophical and scientific interest, reductive relationships between theories can also inform conceptions of emergence, scientific progress, the (dis)continuity and unity of science, and the realism/antirealism debate. As important as these general issues are, I do not address them directly in any sustained way in this dissertation. Again, this is not because they are not important, but rather because I am concerned that much of the philosophical debate has proceeded without concrete and robust examples of relationships between actual scientific theories.[4] Rather than proceed completely from the top down, as it were, by only theorizing the general form that reduction should take and exploring whether it applies to prospective theory pairs, it seems more productive to work a bit from the bottom up, by reconstructing in detail

[4]I contend that nearly all discussion of traditional examples of reduction, such as that between the ideal gas law and kinetic theory, does not meet sufficient standards of conceptual and technical rigor.

the relationship that some working theories do have.

The case on which I focus is that between general relativity and Newtonian gravitation. Despite often being cited as the paragon of a straightforward reductive relationship (Nickles 1973; Batterman 1995), the results backing this claim are very narrow and often contain mathematically ambiguous statements and inferences. Attempts to construct a more general account of their relationship, by understanding Newtonian gravitation as the limit of general relativity as the speed of light approaches infinity, have foundered on the conceptual difficulties associated with understanding the physical significance of what it means for a constant of nature to vary (and without bound). In chapter 3, I build substantively on a framework originally developed by the physicist Jürgen Ehlers (1981, 1988, 1991) to solve this interpretive problem while maintaining the generality of the limiting approach—that is, without restriction to special classes of relativistic spacetimes. I first present a framework, or metatheory, in which the models of both general relativity and Newtonian gravitation are particular specializations. Placing a topology on this joint collection then determines which sequences in the collection convergence. One can then say that general relativity reduces to Newtonian gravitation just in case for every model of the latter, there is a sequence of models of the former that converge to it. I suggest that the notion of similarity that should be captured by a relevant topology is that of similarity of observable quantities. Thus a sequence of spacetimes converging to another just means that certain classes of observables quantities in the members of the sequence approach arbitrarily closely the values those observables would take in the limit spacetime. Importantly, one can choose sequences of relativistic spacetimes such that the speed of light as measured by any observer is the same in each model.

In line with my conclusions from chapter 2, I lastly question *which* topology could be appropriate for these questions of reduction. In this case, the answer covaries with which class of observable quantities one demands be well-approximated in the limit. Although I eschew trying to provide a definitive and final answer, I do argue that the class so far considered in the literature—observables defined at individual spacetime points—is unduly restrictive. For instance, there are many observables, such as the lengths of worldlines representing the time observers experience as they travel though spacetime, that are defined on extended regions. Demanding that observables defined at points to converge in a limit does not entail that those defined on regions do so as well. Thus a different topology for convergence should be used than has been heretofore used implicitly in the literature. By understanding that there are a variety of notions of similarity that could be used in understanding intertheoretic reduction in this sense, one begins to see new possibilities and potential errors that were not before apparent.

Indeed, there are many further possible routes of investigation regarding similarity and spacetime that this dissertation does not explore, ones pertinent both to conceptual issues within physics and broader issues in the philosophy of science. Chapter 5 discusses some of these directions for future research, some natural extensions of the chapters included and some wholly new.

Chapter 2

Similarity, Topology, and Physical Significance in Relativity Theory

2.1 Introduction

There are many reasons to consider notions of similarity in philosophy and, more specifically, philosophy of science. They underlie Lewis's famous system for counterfactual semantics, modal logic, laws of nature and causation (1973; 1986). More recently, Halvorson (2012) has suggested that similarity is salient for characterizing scientific theories formally: one cannot recover a theory from its models unless one encodes the similarity amongst the models' truth-valuations topologically. And in philosophy of physics, Manchak (2012) has remarked that placing a topology on the models of general relativity can describe how different possible relativistic worlds (i.e., relativistic spacetimes) are "nearby" one another.

This topological approach to similarity has been used by physicists working in mathematical relativity since the 1970s. In this context the notion of *stability* has been crucial. Roughly, a property of an object O of a specified class (like mathematical models, solutions to a differential equation, etc.) is stable when all objects in that class sufficiently similar to O also have that property—the name "stability" comes from the intuitive picture that the property is preserved under arbitrary (but sufficiently small) perturbations. In this paper I am interested in exploring how this literature connects similarity—in particular, stability—with judgments of physical significance. For example, Hawking has asserted that "the only properties of space-time that are physically significant are those that are stable in some appropriate topology"[1] and that "[f]or physical purposes it is sufficient to prove that a theorem holds generically" (1971, p. 395).

One attraction of Hawking's proposal is that, given an "appropriate topology," it would seem to reduce part of a philosophical question to a technical question: in certain cases, instead of puzzling over whether a property is "physically significant," one instead may determine on which sets it is unstable; instead of assessing the import of apparent isolated counterexamples to a theorem, one simply proves that the theorem holds generically. Whether this reduction is successful, however, depends first on justifying why there should be a connection between physical significance and topological stability at all. Then, given a satisfactory justification, one must explicate what an "appropriate topology" is supposed to be. After addressing the former query in §2.2, I focus on the latter in the remainder. Since a choice of topology on spacetimes encodes particular ways in which those spacetimes are similar, an appropriate topology is one that gets this notion of similarity right. While it would greatly simplify matters if there were a canonical such topology, in §2.3–2.4 I consider two classes of topologies often considered in the literature and find them flawed for this purpose. (The propositions I prove to argue this point may be of independent foundational interest.) Against suggestions from some physicists, in §2.5 I suggest instead that there cannot be a canonical topology, and that an "appropriate" topology must covary with the context of inquiry. Without a canonical topology, however, stability itself does not directly settle any conceptual questions about what is physically significant. Rather, whether a property counts as physically significant in a model depends upon the prior choice of topology—how one considers models to be relevantly similar.

[1] See also Geroch (1971, p. 70) and Hawking and Ellis (1973, p. 197).

2.2 Similarity, Topology, and Physical Significance

What does it mean to say that a property is physically significant? There is not likely any straightforward univocal concept that wholly underlies the broad use of this phrase in the physics literature, but it will suffice here to draw out some of its connections with epistemic warrant and stability.[2] Physicists use mathematical models to represent physical phenomena, including past observations and potential predictions. But they must be judicious in using models to make inferences about phenomena, as models are often idealized or only approximate. "Physical significance" expresses an aspect of this partial connection between models and phenomena. One says that a property of a model of physical phenomena is physically significant to the degree to which one has warrant to infer that property about the physical phenomena. For instance, one would say that the past singularity of the standard cosmological model is physically significant to the extent that one can infer about the actual universe that there is a past singularity, the Big Bang.

The connection with stability becomes evident in considering that the observed data used to build such models, based on quantities like length, energy and position, are typically imprecise. Imprecise data usually yield imprecise models, or rather a range of models compatible with the data's imprecision. For convenience, however, scientists typically build a single model and represent the imprecision in other ways. But the inferences warranted through that model must cohere with the inferences one would have made through the range of models—that is, these inferences must be compatible not just with the observed data, but also with the whole set of data values falling within the measurements' range of imprecision. Thus any inferences that crucially depend on perfectly precise data will never be warranted. Requiring the stability of a property as a necessary condition for its physical significance enforces this compatibility between imprecise data and the inferences one draws: the only properties about phenomena one can infer from a model are those that all arbitrarily similar models share.[3]

[2] I take myself to be making some tentative suggestions rather than undertaking any robust project of conceptual analysis.

[3] Stability plays an analogous role in securing inference under idealization. Scientists often use idealized models to

Returning to the example above, the data cosmologists have of the universe is imprecise, with a range of compatible cosmological models, including the standard one and a neighborhood of models similar to it. The standard model has a past singularity, and so one may inquire whether one has warrant to infer about the actual universe that there was a Big Bang. The requirement of stability would demand for such an inference that some neighborhood of cosmological models similar to the standard one also have a past singularity. For, one might argue, the data available is compatible with at least some such neighborhood, and any inferences about the world one draws should not depend on which data-compatible model one chooses. Things seem to turn out fortunately for the conclusions of standard cosmology, as there is in fact a sense in which the past singularity of the standard cosmological model is stable.[4]

It is important to contrast the epistemological character of physical significance as such with the more metaphysical character of being "physically (un)reasonable" (Smeenk and Wüthrich 2011; Manchak 2011; Earman 1995, ch. 3.4) or just plain "(un)physical" (Norton 2008, §3.2). Although physicists do not usually explicitly distinguish these, they tend to say that a model is physically unreasonable when they wish to exclude it as a genuine physical possibility countenanced by a theory. By contrast, physical significance tends to refer to the warrant to infer *properties* of physical phenomena from a model. There are interesting connections to explore between the physically significant and the physically reasonable,[5] but the following discussion shall focus on the former.

In order to apply the stability criterion for physical significance precisely, one should formalize the notion of similarity it depends upon. Topology is a natural choice.[6] A topology on a class of objects determines notions of convergence and continuity, respectively, for sequences and param-

make inference more tractable. In these cases, one would like to infer properties of phenomena the de-idealized model represents from those of the idealized model. Even if the idealization is not too severe, such an inference will not be warranted unless the property in question is stable.

[4]Specifically, the property of the standard cosmological model that each inextendible timelike geodesic is past incomplete is stable in the C^2 open topology (Lerner 1973, theorem 6.1) introduced in §2.3.

[5]See, for example, Fletcher (2012, §3.1) for such connections in the case of excluding the indeterministic trajectories of Norton's dome (2008).

[6]Many other choices, like uniform spaces, are strictly stronger than a topology, so one expects *at least* to reckon with topological structure.

eterized families of such objects, using its system of open neighborhoods to encode a weak sense of similarity. A property P of an object O (or, more generally, of each of a set of objects $\{O_\alpha\}$) is then *stable* just in case there is an open set containing O (resp. $\{O_\alpha\}$), all of whose members also have P.[7] Further, property P holds *generically* on a collection S when it holds on an open subset of S that is dense in S. Density ensures that elements of S that do not have P do so unstably, and openness ensures that elements that do have P do so stably.

However, if the collection of objects is infinite, as is the case with relativistic spacetimes, then there will be infinitely many topologies one can place on the collection, topologies that can differ regarding whether a property is stable or generic on a subcollection. How can one decide which topology is appropriate?[8] Perhaps there is in fact a *canonical* topology: a single choice of topology over the collection of relativistic spacetimes that should apply whenever such a topology is needed. Such a position had at one time been suggested by Geroch, who writes, "It is important, I feel, that one settles on one (or possibly two) topologies in which to work rather than discovering a new topology for each new theorem" (1970, p. 269),[9] and more strongly by Lerner (1972, 1973) and Lerner and Porter, who advocate for a particular choice: "if one regards all Lorentz metrics on M as being on an equal (mathematical [*sic*] footing, it appears that the only acceptable choice for a topology is the Whitney fine C^k topology" (1974, p. 1413). Those familiar with Lewis's (1986) influential (albeit controversial) analysis of comparative similarity for possible worlds based on laws of nature and particular matters of fact might also wonder if a similar general analysis might be given for relativistic spacetimes. In the remainder I investigate the viability of this *canonicalism* about topologies over spacetime, considering Lerner's choice in §2.3 and another initially plausible

[7] There is another related sense of stability associated with the initial value problem in general relativity: one says that the evolution of initial data on a Cauchy surface is stable when its map into the evolved relativistic spacetime is continuous (Hawking and Ellis 1973, p. 253). Because this requires considering only globally hyperbolic spacetimes, here I focus on the "geometric" version of stability.

[8] While the same question arises for topologies on finite collections of objects, in that case one might hold out hope to be able to decide by direct comparison.

[9] I do not attribute to him outright advocacy, since a careful reading reveals an admixture of methodological pragmatism: "I think it is important [. . .] to eventually settle on one or possibly two topologies with which to work. Hardly any economy of thought results if there are hundreds of topologies in use" (1971, p. 73). Moreover, later writings indicate a preference for the methodologically contextualist approach I take in §2.5: "The topology one chooses in practice depends on what one wants the topology to do" (1985, pp. 175–6).

class of candidates in §2.4. In examining the problems each faces, the case against canonicalism will emerge.

2.3 The Open Topologies

Recall that a *relativistic spacetime* is an ordered pair (M, g_{ab}), where M is a four-dimensional smooth manifold[10] and g_{ab} is a smooth Lorentzian metric on M, whose indices are abstract.[11] Then the collection of objects to topologize consists of the Lorentzian metrics on a fixed manifold M, which I denote $L(M)$.[12] The Whitney fine C^k topology, also called the C^k *open* topology, may then be defined as follows. First, let h^{ab} be some smooth Riemannian (inverse) metric on M, and define the "distance" function between the k^{th} partial derivatives of two Lorentz metrics g_{ab} and g'_{ab}, relative to h^{ab} and at each point of M, as the scalar field

$$d(g, g'; h, k) = \begin{cases} [h^{ru}h^{sv}(g_{rs} - g'_{rs})(g_{uv} - g'_{uv})]^{1/2}, & k = 0, \\ [h^{a_1b_1} \cdots h^{a_kb_k}h^{ru}h^{sv} & \\ \quad \otimes \nabla_{a_1} \cdots \nabla_{a_k}(g_{rs} - g'_{rs})\nabla_{b_1} \cdots \nabla_{b_k}(g_{uv} - g'_{uv})]^{1/2}, & k > 0, \end{cases} \tag{2.1}$$

where ∇ is the Levi-Civita derivative operator compatible with h_{ab}. I have omitted the abstract indices in the arguments of d since they needlessly clutter the notation, and I will hereafter continue to drop them when they will never be contracted.

The function $d(g, g'; h, k)$ compares g and g' at each point of M through the Euclidean distance between the components of their k^{th} order partial derivatives as determined by h. Then the sets of

[10]One also requires M to be connected, paracompact, and Hausdorff.

[11]That is, the super- and subscripts of tensor fields like g_{ab} label copies of vector spaces in which the fields reside. See, e.g., Malament (2012, §1.4).

[12]One might of course also wish to compare spacetimes whose underlying manifolds are not diffeomorphic. Although Hawking and Ellis (1973, p. 198) state that this can be done, to my knowledge no one has done so nontrivially for *all* spacetimes.

the form

$$B_k(g, \epsilon; h) = \{g' : \sup_M d(g, g'; h, 0) < \epsilon, \ldots, \sup_M d(g, g'; h, k) < \epsilon\} \tag{2.2}$$

constitute a basis for the C^k open topology, where g ranges over all Lorentz metrics, ϵ ranges over all positive constants,[13] and h ranges over all Riemannian (inverse) metrics. One can view these basis elements as generalizations of the ϵ-balls familiar to metric spaces.

But how does one justify the C^k open topology as canonical? For instance, how should one choose the right value of k? One way is to investigate examples of stability about which one has a strong intuition, ruling out available topologies that do not meet them. For example, in discussing a theorem proving the stability of the strong energy condition[14] in the C^2 open topology, Lerner writes,

> It should be pointed out that [this theorem] is *not* true in any of the weaker topologies frequently used [. . .]. If we agree that any reasonable topology [. . .] should allow perturbations preserving the existence of non-zero rest mass, we may take this as further evidence in favor of the [open] topologies. (Lerner 1973, p. 28)

Indeed, it seems that virtually all of the results regarding stability and genericity of global properties of spacetimes have used one of the open topologies. For example, the encyclopedic monograph *Global Lorentzian Geometry*, which has an entire chapter on "stability of [geodesic] completeness and incompleteness," defines only the open topologies for these purposes (Beem et al. 1996, p. 63 & ch. 7).

However widely accepted, the universal appropriateness of the open topologies has not gone unquestioned. Geroch (1970, 1971) has provided a pair of examples that illustrate some surprising features of the C^0 open topology in particular.[15] His first example is a sequence that seems like it

[13]One can choose a different ϵ for each derivative order, but the resulting basis generates the same topology.

[14]This is the condition that for any timelike vector ξ^a at any point of M, $\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\xi^a\xi^b \geq 0$, where T_{ab} is the stress-energy tensor and T is its trace. See Hawking and Ellis (1973, p. 95) or Malament (2012, p. 166).

[15]In fact, they work just as well for any of the C^k open topologies but I follow Geroch in presenting them in the C^0

should converge to Minkowski spacetime but in fact does not. Explicitly, the sequence of metrics

$$\overset{m}{g}_{ab} = \left(1 + \frac{1}{m^2 + x^2 + y^2 + z^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z) \tag{2.3}$$

on $\mathbb{R}^4$, where t, x, y, z are scalar coordinate fields, does not converge as $m \to \infty$ to the Minkowski metric

$$\eta_{ab} = (d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z), \tag{2.4}$$

even though the "bump," remaining centered at the coordinate origin, decreases in amplitude to zero. This is because $\overset{m}{g} \to \eta$ in the C^0 open topology if and only if for *every* neighborhood of the form $B_0(\eta, \epsilon; h)$, we have $\overset{m}{g} \in B_0(\eta, \epsilon; h)$ for m sufficiently large. But one can always pick an h growing sufficiently rapidly towards infinity that, for all m, $\sup_M d(\eta, \overset{m}{g}; h, 0) = \infty$.

Geroch's second example is the one-parameter family

$$\Lambda = \{\lambda g_{ab} : \lambda > 0\}, \tag{2.5}$$

with a fixed g_{ab} on a non-compact M, which strikingly does not trace out a continuous curve in the C^0 open topology—indeed, it is everywhere discontinuous. To see this, note that the family is continuous in the C^0 open topology if and only if for every $\lambda_0 > 0$ and every neighborhood of the form $B_0(\lambda_0 g, \epsilon; h)$, there is a positive open interval $I \ni \lambda_0$ such that $\{\lambda g_{ab} : \lambda \in I\} \subseteq B_0(\lambda_0 g, \epsilon; h)$. But, as with the first example, one can pick an h growing sufficiently rapidly and without bound—recall M is non-compact—so that for any $\delta \neq 0$, $\sup_M d(\lambda_0 g, (\lambda_0 + \delta)g; h, 0) = \infty$.

This example is particularly surprising because the elements of Λ have the same representational capacities—each can represent precisely the same class of spacetimes, and one can interpret the parameter λ as a mere change of units.[16] In fact, one can prove quite general results regarding

context where the calculations are simplest.

[16]It also demonstrates that the open topologies, like the other topologies that I will consider, "over-represent" the Lorentz metrics on M since in general they represent isometric spacetimes through distinct points. One can compensate for this defect somewhat by ensuring one constructs only invariant topologies (Geroch 1970, pp. 281–2),

the conditions under which a sequence converges or a family is continuous in the open topologies. Specifically, the following is sketched by Golubitsky and Guillemin (1973, pp. 43–4):

Proposition 2.3.1. *Let* $g, \{\overset{n}{g}\}_{n\in\mathbb{N}}$ *be Lorentz metrics on a non-compact manifold* M. *Then* $\overset{n}{g} \to g$ *in the open* C^k *topology on* $L(M)$ *iff there is a compact* $C \subset M$ *such that:*

1. *for sufficiently large n,* $\overset{n}{g}_{|M-C} = g_{|M-C}$*; and*
2. $\overset{n}{g}_{|\,\mathrm{int}(C)} \to g_{|\,\mathrm{int}(C)}$ *in the open* C^k *topology on* $L(\mathrm{int}(C))$.

In other words, a sequence converges in the C^k open topology just in case its elements eventually *equal* the limit point everywhere except at most on (the interior of) a compact set, a criterion of convergence even stronger than uniform! One can then use this proposition to prove (see §2.6.1) a necessary condition for a family of Lorentz metrics to be continuous.

Proposition 2.3.2. *Suppose that* $L(M)$ *is given the* C^k *open topology, with* M *non-compact. If* $f : \mathbb{R} \to L(M)$ *is continuous, then for every* $x_0, x_1 \in \mathbb{R}$*, there is some compact* $C \subset M$ *such that* $f(x_0)_{|M-C} = f(x_1)_{|M-C}$.

Thus any pair from a continuous one-parameter family of Lorentz metrics must always be equal everywhere except at most on a compact set. Intuitively, one might picture the difference between the members of such a pair as a "bump in a rug" that the function f pushes around. Although the bump may be bigger or smaller, wider or narrower, it always has compact support. This is clearly a quite restricted class of continuous families. For example, because any two distinct members from the scale family of metrics given by eq. 2.5 are equal nowhere, proposition 2.3.2 ensures that this family is everywhere discontinuous.

One might object that the discussion of canonicalism was motivated by considering the connection of stability with physical significance, but Geroch's examples and propositions 2.3.1 and 2.3.2 bear

ones for which the pushforward map induced by any element of the diffeomorphism group of M acts on $L(M)$ as a homeomorphism. Indeed, all of the topologies considered in this paper are invariant in this way.

on convergence and continuity. Why should problems with the latter bear on the use of topology for the former? Two responses are on offer. First, if a canonicalist would maintain this objection while affirming that continuity and convergence are worthy of investigation,[17] she would already concede her position in suggesting that different purposes—stability, as opposed to convergence and continuity—may require different topologies. Second, convergence, continuity and stability are all interdependent, each determined by a topology's lattice of open sets. Recall, for instance, that the stability of a property depends on the *existence* of a certain open set. Thus it is in a sense *easier* for a property to be stable in a finer topology, since there are more open sets available that could be witnesses to stability. In particular, if a property is stable on a certain set in a given topology $\mathcal{T}$, it is stable in *every* topology finer than $\mathcal{T}$. Recall as well that the convergence of a sequence depends on certain aspects of *every* open neighborhood of its purported limit point. Thus it is in a sense *easier* for a sequence to converge in a coarser topology, since there are fewer open sets that must fulfill the proper role. In particular, if a sequence converges in a given topology $\mathcal{T}$, it converges in *every* topology coarser than $\mathcal{T}$.

Geroch's examples and propositions 2.3.1 and 2.3.2 therefore suggest that there are more open sets in the open topologies than one might have initially thought, which would perhaps make stability too easily achieved. For example, if a property is stable for g in the C^0 open topology on $L(M)$ for non-compact M, then it obtains on some basic neighborhood $B_0(g, \epsilon; h)$. However, for any $g' \in B_0(g, \epsilon; h)$ distinct from g on a non-compact set, by choosing $h' = \Omega h/d(g, g'; h, 0)$ for some unbounded Ω, there is no ϵ' for which $g' \in B_0(g, \epsilon'; h')$. But one might think that whether g' can be sufficiently close to g, for *some* standard of sufficiency, should not depend on the choice of h, which is simply a device for comparing g and g'. That the open topologies do have this dependence is ultimately what is responsible for propositions 2.3.1 and 2.3.2.[18]

[17] Lerner (1972, 1973) is aware of Geroch's examples and propositions 2.3.1 and 2.3.2, but concludes from them that one must give up talking about continuity and convergence. Considering that topology is typically introduced (in part) in the first place to treat continuity and convergence, a more measured response would be to reject the open topologies as canonical.

[18] Like with Geroch's examples, eqs. 2.3 and 2.5, this dependence applies for all the open topologies. I've only presented the C^0 case to illustrate it economically.

2.4 Continuity in the Geometric Sense and the Compact-Open Topologies

Given the problems that the open topologies face, one might very well abandon Lerner's suggestion and investigate other possible choices for a canonical topology. One idea comes from Geroch (1969), who has proposed a way of interpreting certain limiting relations entirely geometrically through the continuity (smoothness, etc.) of certain fields. Roughly, in the simplest case of a one-parameter family, one constructs a 5-dimensional manifold from the 4-dimensional manifolds of the family "stacked" by their identifying parameter.

More precisely, suppose that one is given a family of metrics $\{\overset{t}{g}_{ab}\}_{t\in\mathbb{R}}$ on a fixed manifold M.[19] Let $\mathcal{M}$ be a manifold diffeomorphic to $M \times \mathbb{R}$ and let $\psi^{(t)} : M \to \mathcal{M}$ be a family of embeddings defined, using this diffeomorphism, by $p \mapsto (p, t)$. Thus the field $\tilde{t} : \mathcal{M} \to \mathbb{R}$ defined by $(p, t) \mapsto t$ is smooth and labels the 4-dimensional hypersurfaces foliating $\mathcal{M}$ by their parameter value t. Then one can define a symmetric field Γ^{ab} on $\mathcal{M}$ with signature $(+, -, -, -, 0)$ by stipulating that $(\Gamma^{ab})_{|(p,t)} = (\psi^{(t)}_p)_*(\overset{t}{g}^{ab})$. In other words, Γ^{ab} is the field that on each $\tilde{t}$-constant hypersurface is just the pushforward of the inverse Lorentz metric $\overset{t}{g}^{ab}$. One can find a fixed derivative operator ∇ that is compatible with Γ^{ab} and $\nabla_a\tilde{t}$, i.e., $\nabla_a\Gamma^{bc} = \mathbf{0}$ and $\nabla_b\nabla_a\tilde{t} = \mathbf{0}$, and that makes these two fields orthogonal, i.e., $\Gamma^{ab}\nabla_a\tilde{t} = \mathbf{0}$.[20]

Now, for each $p \in M$ the points $\psi^{(t)}(p)$ for all t form a smooth curve and the collection of all such curves for all p form a congruence on $\mathcal{M}$ that indicates which points on different $\tilde{t}$-constant hypersurfaces are counterparts. Thus there is a vector field τ^a on $\mathcal{M}$ tangent to the curves of this congruence satisfying $\tau^a\nabla_a\tilde{t} = 1$. This allows one at last to define a unique symmetric field Γ_{ab}

[19]Geroch does not require that the metrics be defined on diffeomorphic manifolds, but I can confine attention to that case here.

[20]For those familiar with the geometrized formulation of Newtonian gravitation, $\nabla_a\tilde{t}$ functions much like the temporal metric and Γ^{ab} like the spatial metric, except that the latter is Lorentzian instead of Riemannian and only assumed to be smooth on each $\tilde{t}$-constant hypersurface. I would like to thank Jim Weatherall for emphasizing this point to me.

such that $\Gamma_{ab}\tau^a = \mathbf{0}$ and $\Gamma_{ab}\Gamma^{bc} = \delta^c_a - \tau^c\nabla_a\tilde{t}$.[21] With this construction in place, we can say that the family $\overset{t}{g}_{ab}$ on M is *continuous in the geometric sense* when the corresponding field Γ_{ab} is continuous everywhere on $\mathcal{M}$. (Analogous definitions would apply to smoothness, etc.) One can similarly define the limit of a sequence of metrics by embedding the sequence in a one-parameter family.

A great appeal of this proposal is that it uses the natural, widely accepted geometrical formulation of a relativistic spacetime to do the work of choosing the canonical topology. On its face it does not seem to involve the kind of arbitrary decisions used in selecting an open topology. It turns out that the topology determined by *all* the families continuous in the geometric sense is well-known:

Proposition 2.4.1. *A family of Lorentz metrics $\{\overset{t}{g}\}_{t\in\mathbb{R}}$ is continuous in the geometric sense iff it is continuous in the C^0 compact-open topology.*[22]

(For a proof, see §2.6.2.) A basis for the C^k compact-open topologies, for any non-negative integer k, may be written as sets of the form

$$B_k(g,\epsilon;h,C) = \{g' : \sup_C d(g,g';h,0) < \epsilon, \ldots, \sup_C d(g,g';h,k) < \epsilon\}, \tag{2.6}$$

where g ranges over all Lorentz metrics, ϵ ranges over all positive constants, h ranges over all Riemannian (inverse) metrics,[23] and C ranges over all compact subsets of M. The essential difference between the open and the compact-open topologies is that the former “control” behavior everywhere on the manifold whereas the latter do so only on compact subsets.

Notably, one can show that, unlike with the open topologies, the sequence defined by eq. 2.3 converges to the Minkowski metric and the family defined by eq. 2.5 is continuous relative to the

[21] This also parallels the construction of the covariant spatial metric in geometrized Newtonian gravitation. Cf. fn. 20 and Malament (2012, p. 254, proposition 4.1.12).

[22] In particular, the C^0 compact-open topologies are the *final* topologies respectively associated with the families continuous in the geometric sense, the finest topologies on $L(M)$ that make those families continuous.

[23] Strictly speaking, letting h range is superfluous, as the same topology is generated through a single choice (Geroch 1970, p. 280 fn.).

C^k compact-open topologies. These topologies are also attractive for having a number of other interesting features. First, they coincide with the topology of C^k compact convergence—that is, a sequence of metrics $\overset{n}{g} \to g$ on M just when it and its partial derivatives to order k (with respect to the Levi-Civita derivative operator compatible with an arbitrary Riemannian metric on M) do so uniformly on each compact $C \subseteq M$ (Munkres 2000, p. 283, theorem 46.2).[24] Second, if a sequence of C^k metrics $\overset{n}{g}$ converges to g, then g is guaranteed to be at least C^k as well (Munkres 2000, p. 284, corollary 46.6). Third, there is a close connection with homotopy. One can show that a family of Lorentz metrics is continuous in the C^k compact-open topology if and only if it traces out a C^k path in $L(M)$. So, in a way, the C^k compact-open topology encodes which Lorentz metrics can be continuously (to order k) deformed into one another.[25]

Like with the open topologies, however, Geroch has criticized the general appropriateness of the compact-open topologies, contending that they rule counterintuitively on the sequence of metrics

$$\overset{m}{g}'_{ab} = \left(1 + \frac{m}{1 + (x - m)^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z) \tag{2.7}$$

on $\mathbb{R}^4$, where t, x, y, z are natural scalar coordinate fields.[26] "The 'bump' in the metrics becomes larger as it recedes to infinity," he writes, but the "sequence *does* approach Minkowski space in the [C^0 compact-open] topology (because the metrics become Minkowskian in every compact set)" (1971, p. 71). In other words, since $d(\eta, \overset{m}{g}'; h, 0)$ is continuous for any choice of (smooth) Riemannian h, its supremum is bounded on any compact set and will become as small as one likes for sufficiently large m. However, "[i]ntuitively, we would not think of this sequence as approaching Minkowski space" (1971, p. 71) (or presumably any spacetime at all). Thus he takes the C^0 compact-open topology to be too coarse.

[24]The compact-open topology coincides with the topology of compact convergence on a function space when the range of the functions is a metrizable space (Munkres 2000, p. 285–6), and the bundle of Lorentz tensors over M, being a finite-dimensional manifold, is metrizable.

[25]Equivalently, the family is continuous in the C^k compact-open topology just when the k-jets of the family belong to the same path component.

[26]The formula for the first term is garbled in Geroch (1971, p. 71), but appears without error in Geroch (1970, p. 280).

This example is less convincing than his examples for the open topology. It is instructive to compare eq. 2.7 with the sequence of Taylor expansions of a real function like $\sin(x)$. For any particular finite-order expansion, one can find a sufficiently large x such that the expansion, evaluated at this x, differs from $\sin(x)$ by as much as one wishes. But if one fixes some compact region of $\mathbb{R}$, then the Taylor series converges uniformly on that region. Similarly, the sequence given by eq. 2.7 converges to Minkowski spacetime because the C^0 compact-open topology corresponds with the topology of compact convergence. Just as the compact convergence of Taylor expansions seems perfectly reasonable, it is not clear why the same cannot be said in the case of sequences of Lorentz metrics.

However, there are other counterintuitive features of the compact-open topologies that bear even more directly on stability. For example, consider Hawking's theorem (Hawking and Ellis 1973, p. 198, proposition 6.4.9):

Theorem 2.4.2 (Hawking)**.** *The existence of a global time function on a relativistic spacetime is equivalent to* stable causality*, an absence of closed causal curves that is stable in the* open C^0 *topology.*

If one of the compact-open topologies were to be canonical, one would want to know whether Hawking's theorem holds with respect to it as well. It turns out that it does not, and spectacularly so. In fact, according to the compact-open topologies, spacetimes generically contain closed timelike curves.

Proposition 2.4.3. *Chronology violating spacetimes are generic in $L(M)$ in any of the C^k compact-open topologies.*[27]

Corollary 2.4.4. *No Lorentz metric is stably causal in any of the C^k compact-open topologies on $L(M)$.*[28]

[27] This slightly improves statements by Hawking (1971, p. 396–7) and Hawking and Ellis (1973, p. 198), who advert without proof to the density of chronology violating spacetimes in $L(M)$ in any of the C^k compact-open topologies.

[28] Cf. proposition 5.1 of Manchak (2012), who shows that each Lorentz metric is homotopic to one that violates

In particular, according to the compact-open topologies, not only does Minkowski spacetime fail to be stably causal despite its global time function, it turns out that *not* having closed timelike curves is not physically significant! In other words, one would never have warrant to infer from a model of relativistic spacetime that the physical situation it represents does not permit a form of time travel.[29] An alternative but equivalent definition of stable causality brings out why: a spacetime (M, g) is stably causal with respect to the C^0 open topology just when there is a metric g' for which there are no closed causal curves and whose light cones everywhere lie outside those of g. By contrast, when stable causality is defined with respect to the C^0 compact-open topology, the light cones of g' need only lie outside of those of g on a compact subset of M, leaving the rest unconstrained and ripe for the sprouting of closed causal curves.

By contrast, if (M, g) already contains a closed timelike curve $\gamma : I \to M$, then one can pick a local basis element $B_k(g, \epsilon; h, C)$ from any compact-open topology so that $\gamma[I] \subseteq C$ and ϵ is small enough so that, for any $g' \in B_k(g, \epsilon; h, C)$, γ is still g'-timelike.

Proposition 2.4.5. *Every spacetime containing a closed timelike curve does so stably in any of the C^k compact-open topologies.*

Thus one always has warrant to infer from spacetimes with closed timelike curves that they represent the possibility for a type of time travel. One ought also, for *some* relativistic spacetime models without closed timelike curves, have warrant to infer that they do not represent this possibility. That this never occurs under the compact-open topologies militates against taking any of them as canonical.

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chronology. As alluded to above, there is a close connection between homotopy and the compact-open topologies: the C^k homotopy classes correspond with the path components of the C^k compact-open topologies.

[29] This result does not state that one can have no inductive evidence for certain global properties, as one might interpret proposition 2 of Manchak (2011, p. 414). Rather, it states that even if one had data for the whole universe, however imprecise, and fit that data to a relativistic spacetime, one could never conclude that there were no closed timelike curves in the universe if one takes any compact-open topology as canonical.

2.5 Methodological Contextualism

Any canonical topology on $L(M)$ should have the ability to properly distinguish which sequences converge, which families are continuous, and which properties are stable or generic. But as the previous two sections laid out, the two main classes of topologies in the literature fall short of these goals. The open topologies, advocated by Lerner, seem too fine to treat convergence and continuity. The compact-open topologies, naturally suggested through continuity in the geometric sense, seem too coarse for stable causality because their neighborhoods control behavior only on compact sets. Of course, that Geroch's examples *do* evince genuine problems for the former can well be challenged, and one may decide, according to one's inclinations, to bite the bullets of propositions 2.3.1 and 2.3.2, or 2.4.3, but this does not completely resolve the issue of how to choose the canonical topology. Proponents of a canonical topology must decide without being *ad hoc* on which counterintuitive results to accept and are obliged to provide an explanation as to why the intuitive features thereby denied do not have the significance they seemed to.

The considerations already raised for the canonicalist can be cast in terms of a no-go result. Given some manifold M, proposition 2.4.1 entails that the C^0 compact-open topology on $L(M)$ is the finest topology in which all the one-parameter families continuous in the geometric sense are continuous. (See fn. 22.) But it follows from corollary 2.4.4 that no Lorentz metric is stably causal in the C^0 compact-open topology or any topology coarser than it. So if there is some Lorentz metric on M that admits of a global time function, Hawking's theorem fails for it in such a topology. (Without the existence of such a metric, Hawking's theorem is vacuous.) This yields the following.

Proposition 2.5.1. *If there is some Lorentz metric on M that admits of a global time function, then there is no topology on $L(M)$ relative to which both*

1. *all one-parameter families continuous in the geometric sense are continuous, and*
2. *Hawking's theorem holds.*

One might object that asking for such compatibility is too much; perhaps there is hope for compatibility with a weakened version of Hawking's theorem, in which one demands the existence of a global time function just in case the spacetime is stably causal when restricted to compact sets (or rather their interiors). But this fails, too.

Corollary 2.5.2. *Suppose $\mathcal{T}_M$ is a topology on $L(M)$, for any M, that makes continuous all the one-parameter families continuous in the geometric sense. Then, for any $g \in L(M)$ there is no compact $C \subseteq M$ such that $g_{|\,\mathrm{int}(C)}$ is stably causal in the topology $\mathcal{T}_{\mathrm{int}(C)}$ on $L(\mathrm{int}(C))$.*

Because there are no subsets of M with the desired property, the analog of Hawking's theorem fails (as there remain many spacetimes with global time functions).[30] This is just one of a possibly large family of "no-go" results that the canonicalist must face.

But reminding oneself of the way these topologies are used suggests that one need not pick any canonical topology at all. Examining the consequences of adopting one topology over another is a part of the process of deciding which topology will be relevant for a given type of problem. Hawking has emphasized as much: "A given property may be stable or generic in some topologies and not in others. Which of these topologies is of physical interest will depend on the nature of the property under consideration" (1971, p. 396). Indeed, Geroch's later writings (see fn. 9) have indicated the same. If different topologies correspond to different ways one can specify how spacetimes are similar, it is not surprising that *different* topologies would be natural choices for *different* kinds of questions if those questions bear upon different kinds of properties. It thus seems best to accept a kind of methodological contextualism, where the best choice of topology is the one that captures, as best as one can manage, at least the properties *relevant* to the type of question at hand, ones that relevantly similar spacetimes should share. Thus, in contrast to the canonicalist, I would demand that particular choices of topology must be justified relative to a context as much

[30] In fact, the stated weakening of stable causality with respect to the C^0 open topology is known to be equivalent to non-total imprisonment, the condition that no future-inextendible causal curve eventually enters but does not leave some compact subset of spacetime (Minguzzi 2009, theorem 1). That non-total imprisonment is much weaker than stable causality reveals that finding a weakening of stable causality that preserves Hawking's theorem is a subtle matter.

as one feasibly can.

Methodological contextualism about topologies—at least in the sense of allowing oneself to pick the most appropriate topology for a given application instead of deciding on one in advance—would make all the above worries associated with picking a canonical topology moot. But the contextualist fortunately still has the resources to choose reasonable topologies—resources not so different from the canonicalists, but without the demand to select a single topology for all purposes. One will expect that similar questions will tend to demand similar topologies, so the process of justification need not be started afresh each time. In particular, one should arrive at a particular choice of topology through reflective equilibrium, balancing the demands of the current understanding of what different topologies capture physically and what notions of similarity one is trying to capture with the implications of new mathematical results, as the many examples and propositions of §2.3–2.4 did for the open and compact-open topologies. These new results may then change one's intuitions, which in turn may suggest further results to investigate. The more one can accumulate these kinds of facts, the more there will be relevant data at hand for a particular type of inquiry so that one can make a sharper, better justified conceptual decision regarding which topology to use. Sometimes this will lead one to reject initially promising and intuitive choices, and sometimes it will reinforce them. One need not postulate that this reflective equilibrium lead to a stable limit; even if one has accumulated many results in favor of using a particular topology for some narrow type of inquiry, one should still be open to new facts and connections that will disturb one's equilibrium.[31]

Nevertheless, more work needs to be done characterizing how particular choices of topology may be appropriate for a given kind of question. Consider again, for example, Hawking's stability criterion for physical significance. Per the discussion at the end of §2.3, one might expect that some topologies fine enough for the stability of some properties—like the open topologies—are too fine for certain sequences to converge, or vice versa—like the compact-open topologies. So

[31] One can find this dynamic and non-teleological conception of reflective equilibrium in the literature on moral theorizing as well (Schroeter 2004).

the topologies that might be natural candidates for inquiries about physical significance through stability may be different from those for inquiries about continuity or convergence.[32] At least in the case of stability, one may be able to characterize classes of properties to which particular topologies are (in)sensitive, or the range of topologies in which interesting properties, like stable causality, behave as one might expect. The usual classification of spacetime properties into local and global (Manchak 2011, p. 413) is too coarse for these purposes.[33] Part of the difficulty in answering this question for stability—indeed, for any inquiry—stems from the small variety of topologies used in the literature.[34] Theorems about the stability and genericity of global properties generally use the open topologies (e.g., see Hawking and Ellis (1973, p. 198), Lerner (1973), and Beem et al. (1996, ch. 7)). Theorems about the stability of Cauchy developments use variants on the coarser compact-open topologies (see Hawking (1971, p. 398–9) and Hawking and Ellis (1973, p. 252–254)). Theorems concerning the convergence of relativistic spacetimes to Newtonian spacetimes (e.g., Malament (1986b)) have (implicitly) used a point-open topology, which is even coarser.[35]

It turns out that there is a simple modification to the C^0 open topology that makes the one-parameter families defined by eq. 2.5 everywhere continuous while, unlike the compact-open topology, still preventing the sequence defined by eq. 2.7 from converging. Take the basis elements of the C^0 open topology (eq. 2.2), but restricted only to *bounded pairs* (g, h), ones for which $\sup_M d(g, \lambda g; h, 0) < \infty$ for any positive λ. This prohibits choosing any h that grows too rapidly, eliminating the open neighborhoods of each Lorentzian g that forced eq. 2.5 to be everywhere

[32] I do not preclude there being different contexts in which the same kind of question ultimately demands a different topology. There may very well be different inquiries into physical significance, for example, that are best served by different choices.

[33] This classification takes a property P of a spacetime (M, g) to be local if and only if all spacetimes locally isometric to (M, g) also have P, and global otherwise. Thus both the topology of M and the existence of a closed timelike curve are global properties, whereas only the latter has any hope of having an analog to proposition 2.4.3.

[34] One pocket of innovation in this regard has been the community working on causal set theory, one proposal for a theory of quantum gravity. (For an introduction, see Henson (2009, 2012).) They have proposed topologies on $L(M)$ that treat the conformal and causal structure of spacetime separately, although some (Bombelli and Meyer 1989) suffer problems similar to those of the open topologies, and others (Noldus 2002) are restricted to globally hyperbolic spacetimes, hence are inappropriate for studying, e.g., the stability of causality conditions. A fuller discussion of these, however, lies outside the scope of this paper.

[35] The point-open topologies are defined similarly to the compact-open topologies (eq. 2.6), but require that the suprema be taken over only finitely many points in each basis element instead of over compact sets.

discontinuous. One can show, moreover, that this topology lies between the open and compact-open topologies in coarseness. However the sequence defined by eq. 2.3 still does not converge to Minkowski spacetime according to this topology, so it still would not rule in the intuitively "right" way according to Geroch.

But if further refinements are found that produce a topology satisfying Geroch's desiderata, might that topology end up being satisfactory for all demands? If I allow for the possibility that the methods available for picking an appropriate topology may eventually single out a unique choice, or perhaps very few, to what extent is methodological contextualism really distinguished from a slightly liberalized canonicalism? The answer is methodological. The two positions are not distinct because of differing ends—whether to use one topology or many—but because of their differing means: what grounds we might have to prefer one topology over another, and how those grounds need to be articulated. A canonicalist holds that because there are definitive reasons always to choose a single topology (or perhaps very few), there is no further reason to say why that choice is appropriate for a given type of inquiry. By contrast, the contextualist takes the relevant reasons to be provided by the type of problem at hand, not in advance, and that they should therefore be articulated and reasonably defended. It bears emphasizing that the latter does *not* deny that there can be principled reasons to pick out a certain topology, only that those reasons can ever be given in enough generality to preclude attention to the details of the type of situation at hand. We indeed be may be lucky for the sake of our economy of thought if a few topologies are always appropriate, but we should not obstruct the development of new ones if they fit particular purposes better.

2.6 Proofs of Propositions

2.6.1 Continuity and the open topologies

Proposition 2.6.1 (2.3.2)**.** *Suppose that $L(M)$ is given the C^k open topology, with M non-compact. If $f : \mathbb{R} \to L(M)$ is continuous, then for every $x_0, x_1 \in \mathbb{R}$, there is some compact $C \subset M$ such that $f(x_0)_{|M-C} = f(x_1)_{|M-C}$.*[36]

Proof. The case where f is a constant function is immediate, so suppose otherwise and pick arbitrary distinct $x_0, x_1 \in \mathbb{R}$, assuming without loss of generality that $x_0 < x_1$. I claim that, given any $r \in [x_0, x_1]$, there is an open (relative to $[x_0, x_1]$) interval $I_r \subseteq [x_0, x_1]$, containing r, such that for any $q \in I_r$, there is some compact $C(r, q) \subset M$ for which $f(r)_{|M-C(r,q)} = f(q)_{|M-C(r,q)}$. For suppose otherwise, and consider any sequence of intervals $I_r^1 \supset I_r^2 \supset \ldots$ such that $I_r^n \subseteq [x_0, x_1]$ for each n and $\bigcap_{n=1}^{\infty} I_r^n = \{r\}$. One can then construct by induction a sequence of metrics that converges to $f(r)$. For the base step, let $I_r^{s_1} = I_r^1$ and note that there is some $q_1 \in I_r^{s_1}$ distinct from r such that $f(r)_{|M-C} \neq f(q_1)_{|M-C}$ for any compact $C \subset M$. (Such a $q_1 \neq r$ exists because f is continuous.) For the inductive step, suppose $I_r^{s_n}$ is given so that there is some $q_n \in I_r^{s_n}$ distinct from r such that $f(r)_{|M-C} \neq f(q_n)_{|M-C}$ for any compact $C \subset M$. Then pick some $I_r^{s_{n+1}}$ such that $s_{n+1} > s_n$ and $q_n \notin I_r^{s_{n+1}}$, noting that there is some $q_{n+1} \in I_r^{s_{n+1}}$ distinct from r such that $f(r)_{|M-C} \neq f(q_{n+1})_{|M-C}$ for any compact $C \subset M$. The induction is complete, so by construction the sequence $q_n \to r$ as $n \to \infty$, and for each n, $f(r)_{|M-C} \neq f(q_n)_{|M-C}$ for any compact $C \subset M$. But because f is continuous, it follows that $f(q_n) \to f(r)$ as $n \to \infty$ (Munkres 2000, p. 130, theorem 21.3), and by proposition 2.3.1, this implies in turn that there is a compact $C \subset M$ for which $f(r)_{|M-C} = f(q_n)_{|M-C}$ for sufficiently large n, which is a contradiction.

Next, note that the $\{I_r : r \in [x_0, x_1]\}$ form an open cover of $[x_0, x_1]$ (relative to $[x_0, x_1]$). The interval is compact, so by definition there is some finite subcover $\{I_{r_i} : i = 1, \ldots, m\}$, each of

[36] This proposition may be generalized to families of Lorentz metrics parameterized by any path-connected space.

whose elements has, for all $q \in I_{r_i}$, an associated compact $C(r_i, q) \subset M$ for which $f(r_i)_{|M-C(r_i,q)} = f(q)_{|M-C(r_i,q)}$. One may assume, without loss of generality, that $r_1 < \ldots < r_m$ and that, because the interval is one-dimensional, no point of $[x_0, x_1]$ is included in more than two of the I_{r_i}. Thus pick any $q_i \in I_{r_i} \cap I_{r_{i+1}}$ for $i = 1, \ldots, m-1$ and put $q_0 = x_0$ and $q_m = x_1$. Let $C = \bigcup_{i=1,\ldots,m} C(r_i, q_{i-1}) \cup C(r_i, q_i)$ and observe that

$$\begin{aligned} f(x_0)_{|M-C} &= f(q_0)_{|M-C} = f(r_1)_{|M-C} = f(q_1)_{|M-C} = \cdots \\ &= f(q_{m-1})_{|M-C} = f(r_m)_{|M-C} = f(q_m)_{|M-C} = f(x_1)_{|M-C}. \end{aligned}$$

Since C is compact and x_0, x_1 were arbitrary, the proof is complete. □

2.6.2 Equivalence of continuity in the geometric sense and compact-open continuity

To prove proposition 2.4.1, it will be helpful to use the bundles of Lorentz tensors over M and degenerate metrics over $M \times \mathbb{R}$, denoted $\hat{L}(M)$ and $\hat{\Gamma}(M, \mathbb{R})$, respectively. The desired equivalence is essentially a corollary of the following (adapted from Munkres (2000, p. 287, Theorem 46.11)).

Lemma 2.6.2. *Let X and Y be topological spaces, and give the set of continuous functions from X to Y, denoted $\mathcal{C}(X, Y)$, the (C^0) compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is the induced function $F : Z \to \mathcal{C}(X, Y)$ defined by the equation $(F(z))(x) = f(x, z)$. The converse holds if X is locally compact*[37] *and Hausdorff.*

Proposition 2.6.3 (2.4.1)**.** *A family of Lorentz metrics $\{\overset{t}{g}\}_{t \in \mathbb{R}}$ on M is continuous in the geometric sense iff it is continuous in the C^0 compact-open topology.*[38]

Proof. Let $\psi_p^{(t)} : M \to M \times \mathbb{R}$ be the embeddings that define the 5-dimensional metric Γ_{ab}, which

[37] A topological space is locally compact when each point has a compact neighborhood.

[38] Using jet bundles (Golubitsky and Guillemin 1973, ch. 2.2–2.3), this proposition may be generalized to any C^k compact-open topology and families of Lorentz metrics parameterized by any smooth manifold.

corresponds to a cross-section $\tilde{\Gamma}$ of a bundle $\hat{\Gamma}(M, \mathbb{R})$ of 5-dimensional metrics. The bundle homomorphism $\phi : \hat{\Gamma}(M, \mathbb{R}) \to \hat{L}(M)$ defined by $\phi(\tilde{\Gamma}'_{|(p,t)}) = (\psi_p^{(t)})^*(\Gamma'_{ab})$, for any local section $\tilde{\Gamma}'$ at $(p, t) \in M \times \mathbb{R}$, is smooth because by definition the $\psi_p^{(t)}$ are jointly smooth in p and t. Thus, if the family $\overset{t}{g}_{ab}$ is continuous in the geometric sense, $f = \phi \circ \tilde{\Gamma} : M \times \mathbb{R} \to \hat{L}(M)$ is continuous. Lemma 2.6.2 then entails that the map $F : \mathbb{R} \to C(M, \hat{L}(M))$ defined by $F : t \mapsto \overset{t}{\hat{g}}$, where $\overset{t}{\hat{g}}$ is the cross-section of $\hat{L}(M)$ corresponding to $\overset{t}{g}_{ab}$, is continuous when its range is given the C^0 compact-open topology.

Conversely, suppose that the family $\overset{t}{g}_{ab}$ is continuous in the C^0 compact-open topology, i.e., that $F : \mathbb{R} \to C(M, \hat{L}(M))$ defined above is continuous. Since M is locally compact and Hausdorff, lemma 2.6.2 entails that $f : M \times \mathbb{R} \to \hat{L}(M)$ is continuous. Thus $(\overset{t}{g}_{ab})_{|p}$ is jointly continuous in t and p. Note that Γ^{ab} is continuous when, for any smooth field α_{ab} on $M \times \mathbb{R}$, $\alpha_{ab}\Gamma^{ab}$ is continuous. Now, for any $(p, t) \in M \times \mathbb{R}$, $(\alpha_{ab}\Gamma^{ab})_{|\psi^{(t)}(p)} = (\psi_p^{(t)})^*(\alpha_{ab})\overset{t}{g}^{ab}$; by assumption $(\psi_p^{(t)})^*(\alpha_{ab})$ is smooth; and $\overset{t}{g}^{ab}$ is continuous because its inverse is. Thus Γ^{ab} is continuous, so Γ_{ab} must be so as well by construction. □

2.6.3 Stable causality and the compact-open topologies

The proof of proposition 2.4.3 uses proposition 2.4.5, but because the latter is a straightforward computation, the order of presentation follows that of the main text. First, a lemma is required allowing one to "interpolate" between any spacetime and a chronology-violating region of Gödel spacetime.[39]

Lemma 2.6.4. *Let (M, g) and $(\mathbb{R}^4, g')$ be two spacetimes. For any open $S \subset M$ and any open $R \subset \mathbb{R}^4$ with compact closure, there is a spacetime (M, g') such that $g'_{|M-S} = g_{|M-S}$ and $g'_{|R}$ is isometric to $g'_{|U}$ for some $U \subset S$.*[40]

[39] For more on the properties of Gödel spacetime, see Malament (2012, ch. 3.1).

[40] Following Manchak (2012), one can use this lemma to answer affirmatively a question posed by Stein (1970, p. 594) about whether it is always possible to continuously deform a spacetime into one containing a closed timelike curve.

Proof. Pick a chart (V, φ) of M such that $V \subseteq S$ and $\varphi[V]$ is an open ball of radius 4, i.e., $\varphi[V] = B_{\mathbb{R}^4}(\vec{0}, 4) = \{\vec{x} \in \mathbb{R}^4 : \|\vec{x}\| < 4\}$, where $\|\cdot\|$ is the Euclidean norm on the coordinates $\vec{x} \in \mathbb{R}^4$. For brevity, define $A_i = \varphi^{-1}[B_{\mathbb{R}^4}(\vec{0}, i)]$ for $i = 1, 2, 3$, and let r be a scalar field on V defined by $r_{|p} = \|\varphi(p)\|$. Finally, let $\psi : \mathbb{R}^4 \to V$ be a diffeomorphism such that $\psi[R] \subseteq A_1$ and define $U = \psi[R]$.

Because all Lorentz metrics on $\mathbb{R}^4$ are homotopic (Finkelstein and Misner 1959), $\psi_*(g')$ is homotopic to $g_{|V}$, considering V as a submanifold. Thus there is some continuous function $f : [0,1] \to L(V)$ such that $f(0) = \psi_*(g')$ and $f(1) = g_{|V}$. One can then define the continuous Lorentz metric

$$\gamma_{|p} = \begin{cases} g_{|p}, & p \in M - A_3, \\ f(r_{|p} - 2)_{|p}, & p \in A_3 - A_2, \\ [\psi_*(g')]_{|p}, & p \in A_2. \end{cases}$$

In order to produce the desired smooth metric g', one can convolve γ with an appropriate positive, symmetric mollifier on the region $V - A_1$. In more detail, define $w : \mathbb{R} \to \mathbb{R}$ to be the smooth function

$$w(x) = \begin{cases} ce^{-1/(1-x^2)}, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where c is a positive constant chosen so that $\int_{\mathbb{R}} w(x)dx = 1$. Further, define $W : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ as the jointly smooth function $W(\vec{x}, \epsilon) = \epsilon^{-n} w(\|\vec{x}\|/\epsilon)$, where $W(\vec{x}, 0) = \lim_{\delta \to 0} W(\vec{x}, \delta)$ is the Dirac delta, the convergence being understood in the distributional sense. Now, one can express γ in terms of its matrix components $\gamma_{\alpha\beta}(\vec{x})$ determined by the chart (V, φ), allowing one to define on $\varphi[V - A_1]$ for some fixed ϵ the new components

$$\tilde{\gamma}_{\alpha\beta}(\vec{x}) = \int_{\varphi[\mathrm{int}(V - A_1)]} W(\vec{x} - \vec{y}, e\epsilon c^{-1} w(\|\vec{x}\| - 5/2))\gamma_{\alpha\beta}(\vec{y})d\vec{y}, \tag{2.8}$$

which are smooth on $\varphi[V - A_1]$.[41] Moreover, for sufficiently small ϵ, the $\tilde{\gamma}_{\alpha\beta}$ approximate the $\gamma_{\alpha\beta}$ arbitrarily well on $V - A_1$.[42] Therefore such $\tilde{\gamma}_{\alpha\beta}$ are the components of a smooth Lorentz metric $\tilde{\gamma}$ on $V - A_1$. Note that, in the integrand of eq. 2.8 the function W becomes the Dirac delta for $\|\vec{x}\| \geq 7/2$ and $\|\vec{x}\| \leq 3/2$, so on the points of V corresponding to these coordinate regions, $\tilde{\gamma}$ is equal to g and $\psi_*(g')$, respectively. We can define at last

$$g'_{|p} = \begin{cases} g_{|p}, & p \in M - V, \\ \tilde{\gamma}_{|p}, & p \in V - A_1, \\ [\psi_*(g')]_{|p}, & p \in A_1. \end{cases}$$

By construction, $g'_{|M-S} = g_{|M-S}$ since $M - S \subseteq M - V$ and $g'_{|U}$ is isometric to $g'_{|R}$ since $g'_{|A_1} = [\psi_*(g')]_{|A_1}$ and $U = \psi[R] \subseteq A_1$. □

Proposition 2.6.5 (2.4.3)**.** *Chronology violating spacetimes are generic in $L(M)$ for every compact-open topology.*

Proof. Every spacetime with a compact M contains timelike curves (Hawking and Ellis 1973, p. 189, proposition 6.4.2), so suppose M is non-compact. Select any neighborhood $N(g)$ of an arbitrary g, which must contain a set of the form $B_k(g, \epsilon; h, C)$. Letting $S = M - C$, by lemma 2.6.4 there is some $g' \in N(g)$ such that $g'_{|U}$ is isometric to a chronology violating region of Gödel spacetime for some $U \subset M - C$. By proposition 2.6.6, there is an open neighborhood of g' consisting only of chronology violating metrics. Let $A_k(g, N(g), B_k)$ be the union of all such open neighborhoods determined by the choices of g, $N(g)$, and $B_k(g, \epsilon; h, C)$, and consider $A = \bigcup_g \bigcup_{N(g)} \bigcup_{B_k} A_k(g, N(g), B_k)$. By construction, every neighborhood $N(g)$ of each g contains an element of A, i.e., A is dense in $L(M)$; A is open, being the union of open sets; and A contains only chronology violating spacetimes. So by definition the chronology violating spacetimes are

[41] This follows essentially from theorem 2.6 of Oden and Reddy (1976, p. 48–49).

[42] Theorem 2.7 of (Oden and Reddy 1976, p. 49) shows that, for the case where the integrand contains $W(\vec{x} - \vec{y}, \delta)$ with a fixed δ, the analog of eq. 2.8 would converge to $\gamma_{\alpha\beta}$ as $\delta \to 0$ in $L^p(\varphi[\mathrm{int}(V - A_1)])$-norm. As before (cf. footnote 41), allowing δ to smoothly vary introduces no new complications.

generic in $L(M)$. $\square$

Proposition 2.6.6 (2.4.5)**.** *If the spacetime (M, g) contains a closed timelike curve, then g is stably chronology violating in every compact-open topology.*

Proof. Fix any Riemannian metric h_{ab} and note that one can write $g_{ab} = h_{am}\mu^m h_{bn}\mu^m - h_{ab}$ for some smooth vector field μ^a (Hawking and Ellis 1973, p. 39). One can thus express that $\gamma : I \to M$ is a closed g-timelike curve with unit tangent vector ξ^a as the condition that $|h_{ab}\xi^a\mu^b|_{|\gamma[I]} > (h_{ab}\xi^a\xi^b)^{1/2}_{|\gamma[I]}$. Pick

$$\epsilon = \inf_{\gamma[I]} \min\left\{1, \left(\frac{|h_{ab}\xi^a\mu^b|}{(h_{ab}\xi^a\xi^b)^{1/2}} - 1\right)^2\right\},$$

and consider any $g' \in B_0(g, \epsilon; h, C)$, where $\gamma[I] \subseteq C$. Writing $g'_{ab} = h_{am}\mu'^m h_{bn}\mu'^m - h_{ab}$, γ is g'-timelike just in case $|h_{ab}\xi^a\mu'^b|_{|\gamma[I]} > (h_{ab}\xi^a\xi^b)^{1/2}_{|\gamma[I]}$. Now, one can calculate that

$$h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) = h_{am}h_{bn}(\mu^a\mu^b - \mu'^a\mu'^b)(\mu^m\mu^n - \mu'^m\mu'^n) = [h_{ab}(\mu^a - \mu'^a)(\mu^b - \mu'^b)]^2,$$

so putting $\eta^a = \mu'^a - \mu^a$ yields that $\sup_C h_{ab}\eta^a\eta^b < \epsilon$. (The remaining calculations involve fields defined on $\gamma[I]$, so the subscript indicating as much will be omitted.) It follows from this inequality and the Cauchy-Schwartz inequality that

$$|h_{ab}\xi^a\eta^b| \leq (h_{ab}\xi^a\xi^b)^{1/2}(h_{ab}\eta^a\eta^b)^{1/2} < (\epsilon h_{ab}\xi^a\xi^b)^{1/2} \leq (h_{ab}\xi^a\xi^b)^{1/2}, \tag{2.9}$$

where the last inequality uses the fact that, by definition, $\epsilon \leq 1$. Then the reverse triangle inequality entails that

$$|h_{ab}\xi^a\mu'^b| = |h_{ab}\xi^a(\mu^b + \eta^b)| \geq \left||h_{ab}\xi^a\mu^b| - |h_{ab}\xi^a\eta^b|\right| = |h_{ab}\xi^a\mu^b| - |h_{ab}\xi^a\eta^b|,$$

where the last equality follows since $|h_{ab}\xi^a\mu^b| > (h_{ab}\xi^a\xi^b)^{1/2} > |h_{ab}\xi^a\eta^b|$ by the hypothesis and equation 2.9. Applying this equation again along with the definition of ϵ yields that

$$|h_{ab}\xi^a\mu'^b| > |h_{ab}\xi^a\mu^b| - (\epsilon h_{ab}\xi^a\xi^b)^{1/2} \geq |h_{ab}\xi^a\mu^b| - (h_{ab}\xi^a\xi^b)^{1/2}\left|\frac{|h_{ab}\xi^a\mu^b|}{(h_{ab}\xi^a\xi^b)^{1/2}} - 1\right| = (h_{ab}\xi^a\xi^b)^{1/2}.$$

Thus γ is g'-timelike, but g' was arbitrary so each element of $B_0(g,\epsilon;h,C)$ contains a closed timelike curve. Since $B_0(g,\epsilon;h,C)$ is open in every C^k compact-open topology, g must be stably chronology violating in each. □

Corollary 2.6.7 (2.4.4)**.** *Suppose $\mathcal{T}_M$ is a topology on $L(M)$, for any M, that makes continuous all the one-parameter families continuous in the geometric sense. Then, for any $g \in L(M)$ there is no compact $C \subseteq M$ such that $g_{|\,\mathrm{int}(C)}$ is stably causal in the topology $\mathcal{T}_{\mathrm{int}(C)}$ on $L(\mathrm{int}(C))$.*

Proof. Consider an arbitrary $g \in L(M)$ and a compact $C \subseteq M$. By corollary 2.4.4, $g_{|\,\mathrm{int}(C)}$ is not stably causal in the C^0 compact-open topology on $L(\mathrm{int}(C))$. But if $\mathcal{T}_M$ makes continuous all the one-parameter families continuous in the geometric sense, it follows from proposition 2.4.1 that it is no finer that the C^0 compact-open topology, hence $g_{|\,\mathrm{int}(C)}$ is not stably causal in the topology $\mathcal{T}_{\mathrm{int}(C)}$ on $L(\mathrm{int}(C))$ either. □

Chapter 3

On the Reduction of General Relativity to Newtonian Gravitation

3.1 Introduction

In physics the concept of reduction is often used to describe how features of one theory can be approximated by those of another under specific circumstances. In such circumstances physicists say the former theory reduces to the latter, and often the reduction will induce a simplification of the features in question. (By contrast, the standard terminology in philosophy is to say that the less encompassing, approximating theory reduces the more encompassing theory being approximated.) Accounts of reductive relationships aspire to generality, as broader accounts provide a more systematic understanding of the relationships between theories and which features are relevant under which circumstances.

Thus reduction is naturally taken to be physically explanatory. Reduction can be more explanatory in other ways as well: sometimes the simpler theory is an older, predecessor of the theory being reduced. These latter "theories emeriti" are retained for their simplicity and other pragmatic virtues despite being acknowledged as incorrect.[1] Under the right circumstances, one incurs sufficiently little error in using the older theory. Reduction explains why these older, admittedly false (!) theories can still be enormously successful and, indeed, explanatory themselves.

Despite the philosophical and scientific importance of reduction, it is astounding that so few reductive relationship in physics are understood with any detail. It is often stated that relativity theory reduces to Newtonian theory, but supposed demonstrations of this fact are almost always extremely specific, not coming close to the level of generality to which an account of reduction aspires, which is to describe the relationship between the collections of all possibilities each theories allows. Indeed, much discussion of the Newtonian limit of relativity theory (e.g., Batterman (1995)) has focused on the "low velocity limit." For example, the relativistic formula for the three-momentum p of a particle of mass m becomes the classical formula in the limit as the magnitude of its velocity v (as measured in some fixed frame) becomes small compared with the speed of light c:

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = mv\left(1 - \frac{1}{2}(v/c)^2 - \frac{1}{8}(v/c)^4 - \frac{1}{16}(v/c)^6 + \cdots\right) \underset{v/c \ll 1}{\approx} mv. \qquad (3.1)$$

Similar formulas may be produced for other point quantities. Another class of narrow demonstrations concerns gravitation sufficiently far from an isolated source. In those cases, the trajectories of massive test particles far from the source will approximately follow trajectories according to a Newtonian gravitational potential (relative to a fixed reference frame).[2]

But it is not often explicitly recognized that even the collection of all point quantity formulas, such as eq. 3.1, along with trajectories of test particles far from an isolated massive body, together constitute only a small fragment of relativity theory. In particular, they say nothing about the nature of gravitation in other circumstances, or exactly how the connection between matter, energy, and spacetime geometry differs between relativistic and classical spacetimes.[3] Insofar as one is

[1] The concept and terminology are adapted from Belot (2005).

[2] These latter cases usually involve invoking mathematically questionable assumptions, and treat Newtonian gravitation as a theory with a preferred set of global inertial coordinates. Wald (1984, Ch. 4.4, p. 74) is one of the few authors who makes these issues the least bit explicit.

[3] Cf. the comments of Roger Jones paraphrased by Batterman (1995, p. 198–199n1).

interested in the general account of the reduction of one theory to another, these particular limit relations and series expansions cannot be understood to be a reduction of relativity theory to classical physics in any strict sense. Even the operationally minded would be interested in an account of how arbitrary relativistic observables can be approximated by their Newtonian counterparts.

One way to do so—to organize in a relatively succinctly the relationships between arbitrary relativistic observables and their classical counterparts—is to provide a sense in which the relativistic spacetimes themselves, and fields defined on them, reduce to classical spacetimes (and their counterpart fields), as all spacetime observables are defined in these terms. This way of understanding the Newtonian limit of relativity theory has been recognized virtually since the former's beginning (Minkowski 1952), where it was observed that as the light cones of spacetime flatten out in Minkowski spacetime, hyperboloids of constant coordinate time become hyperplanes in the limit. This geometric account has since been developed further by Ehlers (1981, 1986, 1988, 1991, 1997, 1998) and others (e.g., Künzle (1976); Malament (1986a,b)) for general, curved spacetimes.

But the nature and interpretation of this limit has sat uneasily with many. The image of the widening light cones seems to suggest an interpretation in which the speed of light $c \to \infty$. In one of the first discussions of this limiting type of reduction relation in the philosophical literature, Nickles (1973) considered this interpretation and pointed out some of its conceptual problems: what is the significance of letting a constant vary, and how is such variation to be interpreted physically? Rohrlich (1989) suggested that "$c \to \infty$" can only be interpreted counterfactually, which is to say that it corresponds literally to a sequence of relativistic worlds whose speeds of light grow without bound. Thus interpreted, the limit only serves to connect the mathematical structure of relativistic spacetime with that of classical spacetime. It cannot explain the success of Newtonian physics, since such an explanation "specifies what quantity is to be neglected relative to what other quantity" (Rohrlich 1989, p. 1165) in determining the relative accuracy of classical formulas as approximations to the outcomes of observations.

The purpose of this paper is to develop and provide an interpretation of the geometrical limiting account that could in fact meet the explanatory demand required of it. It is thus both perfectly general and explanatory.[4] To explicate how it works, I first present a unified framework for classical and relativistic spacetimes in §3.2. The models of each are instantiations of a more general "frame theory" that makes explicit the conceptual and technical continuity between the two. To define the limit of a sequence of spacetimes, however, one needs more structure than just the collection of all spacetime models. As I point out in §3.3, one way to obtain this structure is by putting a topology on this collection. The key interpretative move, as explicated in §3.4, exploits the freedom of each spacetime to represent many different physical situations through the representation of different physical magnitudes. To illustrate this move I describe three classes of examples, involving Minkowski, Schwarzschild, and cosmological (FLRW) spacetimes, that I also show have Newtonian limits in the sense defined here, with respect to a certain topology. (It still remains an open question—though I conjecture it to be true—whether every Newtonian spacetime is an appropriate limit of relativistic spacetimes.)

The choice of topology, though, can make a significant difference to the evaluation of a potential reduction. In particular, whether the Newtonian limit of a particular sequence of spacetimes exists at all can depend on the choice of topology, which is not determined automatically from the spacetime theories themselves. I indicate in §3.5 one way to understand this dependence, which is that a choice of topology corresponds with a choice of the class of observables that one demands must converge. Requiring that more observables converge in the limit leads in general to more stringent convergence criteria, thus a finer topology. In light of this, I argue for a slightly more stringent criterion than has been used by other authors so that observables depending on compact sets of spacetime, such as the proper times along (bounded segments of) timelike curves, also converge. Finally, in §3.6, I summarize what topical and methodological conclusions I think can be drawn about these results for gravitation and philosophy of science, and indicate how they might be applied more generally to other reduction relations of a similar limiting type.[5]

[4]In fact, one can use the methods developed here to describe counterfactual limits—where the speed of light does grow—and even hybrid factual/counterfactual limits of a sort, but that shall not be my focus here.

[5]One might wonder whether the limit falls under the Nagelian theory of reduction (Nagel 1961), according to

3.2 Ehlers's Frame Theory

The conceptual unification of relativistic and classical spacetimes that the frame theory affords requires some preliminaries. In particular, one must recast the empirical content of Newtonian gravitation, a theory of flat spacetime with a gravitational potential, in the language of Newton-Cartan theory, which describes gravitation through a curved connection.[6] Both formulations of Newtonian gravitation share the following structure: a quintuple $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, where M is a four-dimensional smooth manifold of (possible) coincidence events;[7] t_{ab} and s^{ab}, which are called the *temporal* and *spatial metrics*, respectively, are smooth symmetric tensor fields on M with respective signatures $(+, 0, 0, 0)$ and $(0, +, +, +)$; ∇ is a torsion-free derivative operator compatible with t_{ab} and s^{ab}, i.e., such that $\nabla_a t_{bc} = \mathbf{0}$ and $\nabla_a s^{bc} = \mathbf{0}$; and T^{ab}, called the *stress-energy tensor*, is a smooth symmetric field on M representing various matter fields. The latter contracted with the temporal metric defines the *mass density field* $\rho = T^{ab} t_{ab}$. Moreover, one requires t_{ab} and s^{ab} to be orthogonal in the sense that $t_{ab} s^{bc} = \mathbf{0}$.

The temporal and spatial metrics determine, respectively, the (proper) times elapsed along curves representing the possible paths of massive particles and the (proper) lengths of "spatial" paths. More precisely, call some vector $\xi^a \in T_p M$ *timelike* when $t_{ab}\xi^a\xi^b > 0$ and *spacelike* when there exists a covector ξ_a at p satisfying $s^{ab}\xi_a\xi_b > 0$ and $s^{ab}\xi_a = \xi^b$. Most treatments of spacetime geometry do not go into detail about the relations between dimensional quantities and the numerical values produced in calculations, but here it is important to be explicit about them. Given a fixed set of units, there is a freedom in choosing which numerical values for length and times repre-

which a reduction is a derivation of the laws of one theory from another. It is not my intent to engage at length with this theory, but its applicability here turns on whether one takes the existence of a mathematical limit of a theory's models to be a (component of a) derivation. Although Nagel (1970) later considered allowing a broader sense of derivation than just logical deduction, and others (e.g., Rohrlich (1988)) have suggested that limits provide a way of logically deducing one mathematical framework from another, a detailed account of limits as derivations is still not yet forthcoming.

[6]For more details, see Malament (2012, Ch. 4). Ehlers (1998) argues that it should be called the Cartan-Friedrichs theory, since Cartan (1923, 1924) and Friedrichs (1927) bear most responsibility for formulating it, but I will follow the standard terminology here.

[7]Throughout I shall also assume that all manifolds are connected and Hausdorff.

sent one such unit that is correlated with the parameterization of timelike and spacelike curves. The standard convention is to parameterize them so that their tangent vectors are normalized with magnitude 1, but in what follows it will be helpful to relax this. Tangent vectors of curves remain normalized, but their magnitude may not be 1.

Specifically, parameterize every timelike curve—i.e., every curve whose tangent vector ξ^a is always timelike—so that $\alpha = (t_{ab}\xi^a\xi^b)^{1/2}$ at each point of the curve is constant, hence represents a unit of time. Quantities with the dimension of time are then expressed in multiples of this magnitude. For example, suppose the chosen temporal unit is seconds. If $\alpha = 3$, then a timelike curve of length 6 would represent 2 second. Thus, given any timelike vector ζ^a at a point and a parameterization of timelike curves determining α, the vector's *temporal magnitude* is $\|\zeta^a\| = \alpha^{-1}(t_{ab}\zeta^a\zeta^b)^{1/2}$. Thus if ζ^a is the tangent vector to a timelike curve, it always has temporal magnitude 1.

Similarly, one parameterizes every spacelike curve—i.e., every curve whose tangent vector ξ^a is always spacelike—so that the quantity $\beta = (s^{ab}\xi_a\xi_b)^{1/2}$ at each point of the curve is constant, hence represents a unit of distance. (Proposition 4.1.1 of Malament (2012, p. 255) guarantees that the choice of covector ξ_a makes no difference.) Thus, given any spacelike vector ζ^a at a point and a parameterization of spacelike curves determining β, the vector's *spatial magnitude* is $\|\zeta^a\| = \beta^{-1}(s^{ab}\zeta_a\zeta_b)^{1/2}$. Thus if ζ^a is the tangent vector to a spacelike curve, it always has spatial magnitude 1.

These dimensional considerations extend to the computation of proper lengths of timelike and spacelike curves. The *temporal length* of a timelike curve $\gamma : I \to M$ with tangent vector ξ^a is given by $\int_I \|\xi^a\| ds = \alpha^{-1} \int_I (t_{ab}\xi^a\xi^b)^{1/2} ds$. Similarly, the *spatial length* of a spacelike curve γ is $\int_I \|\xi^a\| ds = \beta^{-1} \int_I (s^{ab}\xi_a\xi_b)^{1/2} ds$. Thus while the integrals in the formulas for both temporal and spatial length are invariant under reparameterization of γ, a change in α or β does change their numerical value. Although this extra flexibility in representing dimensional quantities may seem like a gratuitous complication, it will be essential in describing the interpretation of the Newtonian limit in §3.4.

So far the definitions and structures described apply to models of both standard Newtonian gravitation and Newton-Cartan theory. Models of the former require in addition that ∇ be flat and postulate a further smooth scalar field, the gravitational potential ϕ, that satisfies Poisson's equation, $s^{ab}\nabla_a\nabla_b\phi = 4\pi\rho$.[8] The potential determines the gravitational force incurred by a test particle with mass m to be $ms^{ab}\nabla_b\phi$. By contrast, instead of having a gravitational potential, Newton-Cartan theory allows ∇ to be curved, and the trajectories of massive particles in a Newton-Cartan spacetime follow geodesics according to the curvature determined by

$$R_{ab} = 4\pi\rho t_{ab}, \tag{3.2}$$

the "geometrized" Poisson's equation.[9] There is nevertheless a systematic relationship between the former and a subset of models of the latter, captured in part by the following proposition adapted from Malament (2012, Prop. 4.2.1).

Proposition 3.2.1. *Let* $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ *be a classical spacetime with* ∇ *flat, and let* ϕ *be a smooth scalar field satisfying Poisson's equation,* $s^{ab}\nabla_a\nabla_b\phi = 4\pi\rho$, *where* $\rho = T^{ab}t_{ab}$. *Then there is a unique derivative operator* ∇' *such that:*

1. $(M, t_{ab}, s^{ab}, \nabla', T^{ab})$ *is a classical (Newton-Cartan) spacetime;*
2. *for all timelike curves* γ *in* M, $\xi^n\nabla'_n\xi^a = \mathbf{0}$ *iff* $\xi^n\nabla_n\xi^a = s^{ab}\nabla_b\phi$, *where* ξ^a *is the tangent vector field to* γ*; and*
3. *the Ricci curvature* R'_{ab} *associated with* ∇' *satisfies eq. 3.2, the "geometrized" Poisson's equation, i.e.,* $R'_{ab} = 4\pi\rho t_{ab}$.[10]

The biconditional second on the list states that a timelike curve is a geodesic according to ∇'—i.e., its acceleration vanishes—just in case its acceleration according to ∇ is given by the acceleration

[8] I have chosen units for mass so that numerically the gravitational constant $G = 1$.

[9] See the following footnote for the justification of this name.

[10] One can show that $R'_{ab} = (s^{mn}\nabla_m\nabla_n\phi)t_{ab}$, so the "geometrized" Poisson's equation holds iff Poisson's equation holds.

due to gravity. Thus every classical spacetime with a gravitational potential can be "geometrized" to form an empirically equivalent Newton-Cartan spacetime. Under certain conditions the converse holds as well,[11] although this "de-geometrization" is unique only up to a gauge transformation of the potential ϕ (as one might expect). Thus there is a robust sense in which a subclass of models of Newton-Cartan theory is equivalent to the models of Newtonian gravitation (Weatherall 2012), so in what follows it will suffice to consider the reduction of models of general relativity to models of Newton-Cartan theory.

Models of the frame theory are very similar to models of classical spacetimes. They consist too in quintuples $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, where M is a four-dimensional smooth manifold, t_{ab} and s^{ab} are the symmetric temporal and spatial metrics, ∇ is a torsion-free derivative operator compatible with the metrics, and T^{ab} is the symmetric stress-energy tensor.[12] However, one does not impose a signature on the metrics or require that they be orthogonal. Instead, one only requires that $t_{ab}s^{bc} = -\kappa\delta^c_a$ for some real constant κ, called the model's *causality constant*. The suggestive notation makes it clear that a model of Newton-Cartan theory counts as a model of the frame theory with causality constant $\kappa = 0$.[13]

Models of general relativity are models of the frame theory, too. To see why, recall that a relativistic spacetime is a pair (M, g_{ab}), where M is (again) a four-dimensional smooth manifold and g_{ab} is a smooth Lorentzian metric on M, i.e., a symmetric invertible tensor field with signature $(+, -, -, -)$. The temporal metric t_{ab} for a relativistic spacetime is just g_{ab}, while the spatial metric s^{ab} is given by $-\kappa g^{ab}$, and these define temporal and spatial lengths in the same way. The causality constant is typically fixed as $\kappa = c^{-2}$, where c is the speed of light, and ∇ is the Levi-Civita derivative operator

[11] One can always de-geometrize a model of Newton-Cartan theory for which $R^{[a\ \ c]}{}_{(b\ d)} = 0$, but the resulting gravitational force may not be expressible in terms of a gravitational potential. For the latter to be possible, there must be no "global rotation" in a sense that can be made precise. (See Malament (2012, Ch. 4.5).)

[12] Ehlers also included a cosmological constant Λ, which I've assumed to vanish for simplicity since including it does not introduce any new conceptual subtleties.

[13] This formulation is slightly more general than that of Ehlers, who introduces some restrictions to make the models of the general frame theory more easily interpretable in physical terms at the expense of slightly further technical complication.

	Frame theory	Specialization Newton-Cartan theory	General Relativity
manifold	M	M	M
temp. metric	symmetric t_{ab}	t_{ab} has sig. $(+,0,0,0)$	$t_{ab} = g_{ab}$ has sig. $(+,-,-,-)$
spatial metric	symmetric s^{ab}	s^{ab} has sig. $(0,+,+,+)$	$s^{ab} = -\kappa g^{ab}$
derivative	torsion-free ∇	torsion-free ∇	torsion-free ∇
compatibility	$\nabla_a t_{bc} = \mathbf{0}$	$\nabla_a t_{bc} = \mathbf{0}$	$\nabla_a g_{bc} = \mathbf{0}$
	$\nabla_a s^{bc} = \mathbf{0}$	$\nabla_a s^{bc} = \mathbf{0}$	
causality	$t_{ab} s^{bc} = -\kappa \delta_a^c$	$\kappa = 0$	$\kappa = c^{-2} > 0$
stress-energy	T^{ab}	T^{ab}	$T^{ab} = \frac{1}{8\pi\kappa^2}(s^{am}s^{bn} - \frac{1}{2}s^{ab}s^{mn})R_{mn}$
field equation		$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}T t_{ab})$	$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}T t_{ab})$

Table 3.1: The components and conditions thereon of models of the frame theory, and the conditions under which these models specialize to those of Newton-Cartan theory and general relativity.

compatible with g_{ab}. If one then defines the stress-energy tensor as

$$T^{ab} = \frac{1}{8\pi}(g^{am}g^{bn} - \frac{1}{2}g^{ab}g^{mn})R_{mn} = \frac{1}{8\pi\kappa^2}(s^{am}s^{bn} - \frac{1}{2}s^{ab}s^{mn})R_{mn}, \quad (3.3)$$

where R_{mn} is the Ricci curvature associated with ∇, then Einstein's equation, $R_{ab} = 8\pi(T_{ab} - \frac{1}{2}T g_{ab})$, is satisfied, where $T_{ab} = T^{mn}t_{am}t_{bn}$ and $T = T^{mn}t_{mn}$. Although Einstein's equation and eq. 3.2, the "geometrized" Poisson's equation, appear to be different, the latter is actually a special case of Einstein's equation understood in a suitably generalized sense for any model of the frame theory. Note that $\rho = T$, so using the fact that for a model of Newton-Cartan theory $T^{mn}t_{am}t_{bn} = T^{mn}t_{mn}t_{ab}$,[14] one can rewrite eq. 3.2 as

$$\begin{aligned} R_{ab} = 4\pi\rho t_{ab} &= 8\pi(T^{mn}t_{mn}t_{ab} - \frac{1}{2}T^{mn}t_{mn}t_{ab}) \\ &= 8\pi(T^{mn}t_{am}t_{bn} - \frac{1}{2}T^{mn}t_{mn}t_{ab}) = 8\pi(T_{ab} - \frac{1}{2}T t_{ab}). \end{aligned}$$

The last expression is formally identical to Einstein's equation when $\kappa > 0$ and the temporal metric is the Lorentz metric, i.e., when $t_{ab} = g_{ab}$. All these relationships are summarized in table 3.1.

Although the representation in the frame theory of classical and relativistic spacetimes requires

[14]Locally there is always a covector field t_a such that $t_{ab} = t_a t_b$ (Malament 2012, p. 250), so at any point, $t_{am}t_{bn} = t_a t_m t_b t_n = t_{mn}t_{ab}$.

nothing further, introducing a certain redundancy in the latter will prove enormously useful. As described so far, the causality constant for all relativistic spacetimes (M, g_{ab}) is a single fixed number. Now we let this number vary, hence consider the triples (M, g_{ab}, κ), with $\kappa \in (0, \infty)$, where all of the geometrical structure is otherwise the same. The advantage of this factor is that, combined with an appropriate choice of α, β, one is able to interpret the a family of relativistic spacetimes $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ with a Newtonian limit as all agreeing on the speed of light.

To see this, it is illuminating to compare the function of κ with constant conformal factors. Multiplying the Lorentz metric of a relativistic spacetime with such a factor is commonly interpreted as a "change of units." This can be understood in the terminology introduced so far by noting that a parameterization of timelike and spacelike curves with tangent vectors ξ^a determine $\alpha = (\xi^a \xi^b t_{ab})^{1/2}$ and $\beta = (\xi_a \xi_b s^{ab})^{1/2}$, respectively, so (keeping the parameterization of curves the same) the map $g_{ab} \mapsto \Omega^2 g_{ab}$ for some $\Omega > 0$ induces the map $\alpha \mapsto \alpha\Omega$ and $\beta \mapsto \beta/\Omega$. (Recall that $t_{ab} = g_{ab}$ and $s^{ab} = -\kappa g^{ab}$, so $g_{ab} \mapsto \Omega^2 g_{ab}$ induces the map $s^{ab} \mapsto \Omega^{-2} s^{ab}$.) If $\Omega = 1000$, then a timelike curve that previously represented 1s would after the mapping represent .001s or 1ms.

The role of allowing for different values of the causality constant κ is similar to that for constant conformal factors, but with more flexibility. Application of constant conformal factors transforms both α and β, but varying the causality constant κ only transforms β. In particular, the map $\kappa \mapsto \omega^2 \kappa$, with $\omega > 0$, induces $\alpha \mapsto \alpha$ and $\beta \mapsto \omega\beta$. In other words, as the causality constant varies, the values of spatial magnitudes change but those of temporal magnitudes stay the same. For example, if $\omega = 1000$, then the spatial length of a spacelike curve that was originally 1m would become .001m, or 1mm. With both constant conformal factors and varying κ one can thus transform α and β independently, although it will only be necessary to vary κ in considering the Newtonian limit.

3.3 The Newtonian Limit

Before developing how limits can be interpreted in this approach, one must first determine what, mathematically, a limit of a family of spacetimes is supposed to be in the first place. One way to formalize this is to put a topology on the models of the frame theory, or at least the subclass of those models that represents relativistic and classical spacetimes. In particular, if O_0 and O_λ, with $\lambda \in (0,1]$, are models of the frame theory, then $\lim_{\lambda\to 0} O_\lambda = O_0$ if and only if for any open neighborhood N of O_0, there is some λ_0 such that $O_\lambda \in N$ for all $\lambda < \lambda_0$. Because these models consist in quintuples of the form $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$, it is natural to consider product topologies on them, i.e., topologies whose open sets consist of the Cartesian products of the open sets of the topologies on the temporal and spatial metrics, derivative operators, and stress-energy tensors, and then restrict to the subspace of those objects that satisfy the constraints of the frame theory, namely the causality and compatibility conditions and Einstein's equation, and its natural subspace topology.

Although Ehlers was not completely explicit about the choice of topology for the frame theory, preferring to describe the Newtonian limit directly as a kind of pointwise convergence, one can reconstruct in topological terms the particular convergence condition he (and others) used in discussing the Newtonian limit of spacetimes. To do so requires some preliminaries. First, let h_{ab} be some smooth Riemannian metric on M, and define the "distance" function between the k^{th} partial derivatives of two symmetric tensors t_{ab}, t'_{ab}, relative to h_{ab} and at each point of M, as the scalar field

$$d(t,t';h,k) = \begin{cases} [h^{ru}h^{sv}(t_{rs}-t'_{rs})(t_{uv}-t'_{uv})]^{1/2}, & k=0, \\ [h^{a_1b_1}\cdots h^{a_kb_k}h^{ru}h^{sv} & \\ \quad \otimes \bar{\nabla}_{a_1}\cdots\bar{\nabla}_{a_k}(t_{rs}-t'_{rs})\bar{\nabla}_{b_1}\cdots\bar{\nabla}_{b_k}(t_{uv}-t'_{uv})]^{1/2}, & k>0, \end{cases} \tag{3.4}$$

where $\otimes$ is the tensor product and $\bar{\nabla}$ is the Levi-Civita derivative operator compatible with h_{ab}. (I will omit the abstract indices of tensors when they appear as arguments so as not to clutter the notation needlessly.) A particular choice of h_{ab} provides a standard of comparison for the components of the k^{th} order derivatives of t_{ab} and t'_{ab} via $d(t, t'; h, k)$, which is in fact a true distance function (i.e., a metric) at each point of M.

One defines the distance function similarly for symmetric contravariant tensors, replacing h^{ab} with h_{ab}. (One can add requisite copies of h_{ab} and its inverse to extend the definition of d to any tensor fields.) The case of derivative operators requires only a bit more attention. One can relate any derivative operator ∇ on M to $\bar{\nabla}$ through a symmetric connection field C^a_{bc}, denoted $\nabla = (\bar{\nabla}, C^a_{bc})$.[15] Then one can define the k^{th}-order distance between two derivative operators ∇ and ∇' as $d(\nabla, \nabla'; h, k) = d(C, C'; h, k)$, the k^{th}-order distance between their connection fields, using $2k+2$ copies of h^{ab} and $k + 1$ copies of h_{ab}.

Finally, consider the set S of tensor fields on M that one wishes to topologize—the cases of interest will be the temporal metrics, the spatial metrics, the stress-energy tensors, and the derivative operators (or rather their connection fields) associated with classical and relativistic spacetimes. Then sets of the form

$$B_k(t, \epsilon; h, p) = \{t' \in S : d(t, t'; h, 0)_{|p} < \epsilon, \ldots, d(t, t'; h, k)_{|p} < \epsilon\} \tag{3.5}$$

constitute a basis for the C^k *point-open* topology on S, where t ranges over tensor fields in S, ϵ ranges over all positive constants,[16] and p ranges over all points of M. (One can show that any single choice of h_{ab} will do, as ranging over Riemannian metrics adds no new open sets.) One can view these basis elements as generalizations of the ϵ-balls familiar to metric spaces. And given finitely many spaces of tensor fields with their respective point-open topologies, the product point-open topology on the Cartesian product of those spaces is just the topology whose open sets are Cartesian products of the open sets of the respective topologies.

[15] $\nabla = (\bar{\nabla}, C^a_{bc})$ means that for every smooth tensor field $\alpha^{a_1 \ldots a_r}_{b_1 \ldots b_s}$ on M, $(\nabla_m - \bar{\nabla}_m)\alpha^{a_1 \ldots a_r}_{b_1 \ldots b_s} = \alpha^{a_1 \ldots a_r}_{n b_2 \ldots b_s} C^n_{m b_1} + \cdots + \alpha^{a_1 \ldots a_r}_{b_1 \ldots b_{s-1} n} C^n_{m b_s} - \alpha^{n a_2 \ldots a_r}_{b_1 \ldots b_s} C^{a_1}_{mn} - \cdots - \alpha^{a_1 \ldots a_{r-1} n}_{b_1 \ldots b_s} C^{a_r}_{mn}$.

[16] One could choose a different ϵ for each derivative order, but the resulting basis generates the same topology.

This particular description of the point-open topologies with an auxiliary Riemannian metric h_{ab} will make plain their similarities with another class of topologies, the compact-open topologies, which I consider in §3.5. But on its own one might wonder if introducing h_{ab} is really necessary. The following proposition shows in a sense that it is not and exhibits the connection between the point-open topologies and the notion of pointwise convergence used in the literature. (One can find an explicit formulation of this notion in Malament (1986b, p. 192).)

Proposition 3.3.1. *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on* M*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *point-open topology iff for all points* $p \in M$*,* $\lim_{\lambda\to 0}(\overset{0}{\psi}{}^{bc}_a \overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{0}{\psi}{}^{bc}_a \phi^a_{bc})_{|p}$ *for all tensors* $\overset{0}{\psi}{}^{bc}_a$ *at* p *and, for all positive* $i \leq k$*,* $\lim_{\lambda\to 0}(\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a \nabla_{d_1} \cdots \nabla_{d_i} \phi^a_{bc})_{|p}$ *for all tensors* $\overset{i}{\psi}{}^{bcd_1\ldots d_i}_a$ *at* p*.*[17]

(Unless otherwise stated, proofs of all propositions may be found in the appendix.) Analogous results hold for tensor fields of other index structures. The proposition states that a sequence of tensor fields $\overset{\lambda}{\phi}$ on M converges to a tensor field ϕ in the C^k point-open topology just in case the all the sequences of real numbers formed by contracting the tensor field and its derivatives up to order k with any other tensor field ψ at a point converge, for all points in M. In other words, the point-open topology defines a notion of pointwise convergence.

With that definition in place, we are ready to state Ehlers's definition of a Newtonian limit of a family of relativistic spacetimes:

Newtonian Limit (Ehlers) Let $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab})$, with $\lambda \in (0, a)$ for some $a > 0$, be a one-parameter family of models of general relativity that share the same underlying manifold M. Then $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ is said to be a Newtonian limit of the family when it is a model of Newton-Cartan theory and $\lim_{\lambda\to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab}) = (t_{ab}, s^{ab}, \nabla, T^{ab})$ in the C^2 point-open

[17] Topological spaces that are completely determined by their convergent sequences are called *sequential*. I do not know if the point-open topologies are sequential; if they are not, then there are other topologies that would also describe the convergence condition used in the literature. But in any case, the point-open topologies are clearly *sufficient* to do so, allowing one to clarify the significance of the relevant convergence condition.

product topology.[18]

In practice, to prove that a family of relativistic spacetimes has a Newtonian limit, one usually just needs show that $\lim_{\lambda\to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{T}{}^{ab}) = (t_{ab}, s^{ab}, T^{ab})$ since a theorem of Malament (1986b, p. 194) guarantees that, if the family is at least twice differentiable in λ, convergence of the temporal and spatial metrics entails convergence of their associated derivative operators,[19] and convergence of the stress-energy then ensures that the "geometrized" Poisson's equation (3.2) holds in the limit (Malament 1986b, p. 197).[20] The convergence of each of these three fields must be checked, for the convergence of one does not in general imply the convergence of any others.

Now, since there are infinitely many topologies one can place on the collections of tensor fields on a fixed manifold and good reason to believe that there is no canonical topology on these collections (see ch. 2), one may wonder why the C^2 point-open topology is an appropriate choice here. The choice of $k = 2$ reflects the fact that one needs the Riemann curvature tensor of a spacetime, which is defined in terms of twice repeated covariant differentiation, to converge in order to guarantee that the geometrized Poisson's equation will hold in the limit. But one may still question further why the *point-open* topology (of some flavor or other) is appropriate. I return to this important question in §3.5, but for now take the Newtonian limit condition as given. Let us consider the mathematics of the Newtonian limit through the simple example of a family $\overset{\lambda}{\eta}_{ab}$ of Minkowskian spacetimes with causality constant $\overset{\lambda}{\kappa} = \lambda$ that has Galilean spacetime as its Newtonian limit. Using standard

[18]One actually only requires C^1 point-open convergence of derivative operators and C^0 point-open convergence of stress-energy, since the definition of the associated connection fields for the former already involves first-order derivatives and the definition of the stress-energy (eq. 3.3) involves the Ricci tensor, hence second-order derivatives.

[19]The same theorem ensures that $R^{[a\ \ c]}{}_{(b\ d)} = 0$ also holds for the Newtonian limit, which is required for the limit Newton-Cartan model to be equivalent to a model of Newtonian gravitation. (See fn. 11.)

[20]Technically, Malament's results posit that the classical limit spacetime is temporally orientable, but this assumption can be relaxed without consequence.

global coordinate fields t, x, y, z, the temporal and spatial metrics of this family may be written as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{\eta}_{ab} = (d_a t)(d_b t) - \lambda(d_a x)(d_b x) - \lambda(d_a y)(d_b y) - \lambda(d_a z)(d_b z), \tag{3.6}$$

$$\begin{aligned} \overset{\lambda}{s}{}^{ab} &= -\lambda \overset{\lambda}{\eta}{}^{ab} \\ &= -\lambda \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b, \end{aligned} \tag{3.7}$$

while the temporal and spatial metrics of Galilean spacetime may be written as

$$t_{ab} = (d_a t)(d_b t), \tag{3.8}$$

$$s^{ab} = \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b. \tag{3.9}$$

The stress-energy $\overset{\lambda}{T}{}^{ab}$ vanishes for all λ. A straightforward calculation shows that for every $\epsilon > 0$, Riemannian h, $p \in M$ and $k \in \{0, 1, 2\}$, there is a sufficiently small $\lambda > 0$ such that $d(t, \overset{\lambda}{t}; h, k) < \epsilon$, i.e., $\lim_{\lambda \to 0} \overset{\lambda}{t}_{ab} = t_{ab}$ in the C^2 point-open topology; similarly, $\lim_{\lambda \to 0} \overset{\lambda}{s}{}^{ab} = s^{ab}$ in this topology, verifying that Galilean spacetime is in fact the Newtonian limit of the Minkowskian family (as the derivative operators are flat for all λ).

This does not, of course, show that general relativity as a whole reduces to Newtonian gravitation as a whole, which would require attention to whether each Newtonian spacetime is the limit of a sequence or family of relativistic spacetimes. The point here is rather that the definition of the Newtonian Limit provides a precise sense in which the question of reduction can be answered definitively, and that there are particular examples (here and in §3.4.1–3.4.3) where that answer is affirmative.

3.4 Interpretation of the Newtonian Limit

As mathematical objects, the models of the frame theory are completely well-defined, and it is a mathematical matter whether a particular family, parameterized by λ, has a Newtonian limit. According to proposition 3.3.1, the existence of such a limit as $\lambda \to 0$ means that, given some $\epsilon > 0$ and some finite set of spacetime points, the values in any basis of the components of the temporal and spatial metrics, connection, and stress-energy (and their partial derivatives to second order) of members of the family can be approximated at those points within ϵ by those of the Newtonian limit spacetime for sufficiently small λ.

One can thus understand the Newtonian limit through its connection with the observables of the frame theory (or at least of its specialization to GR and Newton-Cartan theory). Now, there is some controversy regarding just what quantities count as observables in spacetime theories. Some of it arises in the context of the constrained Hamiltonian formalism (e.g., Rovelli (1991)) and so is exogenous to my concerns here. There is also the difficulty, by no means unique to these theories, that evenly remotely realistic representations of actual measurements with experimental apparatuses are too complicated to model in detail. Thus any tractable discussion of observables must proceed at some level of abstraction. In particular, I shall assume that one can represent observables with scalar fields that are definable from the temporal and spatial metrics, the connection, and the stress-energy (and their derivatives), along with tetrad fields associated with (the worldlines of) observers. These will be (continuous functions of) the kinds of quantities considered in proposition 3.3.1. Elliptically, one can then speak of tensor fields as observables in the sense that their collections of components, relative to some tetrad field, are observable scalar fields. Note that I only take this to be a necessary condition for observability; some tensor fields so definable, e.g., very high order derivatives of the Riemann curvature, may exceed the boundaries of idealization reasonable for the inquiry at hand. But considering a slightly broad class of tensor fields will nevertheless serve the present needs, for proposition 3.3.1 ensures that any observables falling within such a class and defined at finitely many points will be well-approximated in the limit.

Recall now that the outcome of an observation is typically a *dimensional* quantity, i.e., one involving time, length, etc. Hence in determining the convergence of observables one must consider the parameterizations of curves, insofar as they define α and β, which make the numerical values of observables physically meaningful. The key is to choose them to vary with λ in such a way as to make the speed of light in each model of the family $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ the same. A convenient such choice is $\alpha(\lambda) = 1$ and $\beta(\lambda) = \overset{\lambda}{\kappa}^{-1/2}$.[21] In other words, one retains the same parameterization for timelike curves but linearly reparameterizes spacelike curves so that their tangent vectors ξ^a are mapped as $\xi^a \mapsto \overset{\lambda}{\kappa}\xi^a$. Parameterizing α, β, κ in this way is somewhat analogous to having renormalization group transformations on the relativistic spacetime models, where the time and distance scales (in general) are set only by the arbitrary choice of units, but their ratio is constrained by the speed of light. Indeed, for the above choices turns out that the speed of light in each of the models $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ will be the same, identically 1. Note that the element of a given such family may well be isometric. All the members of the Minkowskian family defined by eqs. 3.6 and 3.7, for example, are flat complete spacetimes on $\mathbb{R}^4$, and under the above choices for $\alpha(\lambda)$ and $\beta(\lambda)$, the speed of light in each of them is the same.

Nevertheless, if $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ has a Newtonian limit as $\lambda \to 0$, then observables definable at points in M that are continuous in λ will converge in the limit as well. In other words, given some such observable and an acceptable margin of error $\epsilon > 0$, that observable will be approximated within that error by its Newtonian counterpart for sufficiently small λ. Instead of corresponding necessarily to different relativistic worlds, the members of $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ for such λ correspond to a range of physical conditions under which the Newtonian approximation is valid. Exactly what these conditions will be in specific cases will depend on the observable considered and the specification of the convergent family. But the general schema for determining them is roughly the same in most cases.[22]

[21] This is not the only choice of units that will yield the factual interpretation—any in which $\alpha(\lambda)/\beta(\lambda) = \overset{\lambda}{\kappa}^{1/2}$ will do, but for simplicity I consider just the choice described.

[22] The description given will suffice when considering an observable definable from just one of the temporal metric, spatial metric, or stress-energy. The complications encountered for others add no new conceptual difficulty, however.

First, pick a tetrad $\{\overset{i}{e}{}^a\}_{i=0}^3$ at a point and an observable definable there that converges in the limit. The former encodes the frame of an observer at that point; if the observer is represented by a timelike worldline that intersects the point, the timelike component will typically be the worldline's tangent vector. It also determines an inverse Riemann metric $h^{ab} = \sum_{i=0}^3 \overset{i}{e}{}^a \overset{i}{e}{}^b$. Second, compute the value of the observable for the spacetime $(M, \overset{\lambda}{g}_{ab}, \overset{\lambda}{\kappa})$ as well as the limit point, using the choices $\alpha(\lambda) = 1$ and $\beta(\lambda) = \overset{\lambda}{\kappa}{}^{-1/2}$ for both. The absolute value of their difference will in general depend on λ. Third, pick some suitable maximum $\epsilon > 0$ for this difference, and compute the bound on λ below which the difference is below ϵ. This bound will in general depend not just on ϵ but on other quantities representing physical magnitudes that appear in the difference. Finally, consider any open neighborhood of the limit point whose intersection with the convergent family consists in exactly those members whose parameter λ is below this bound. It represents precisely those constraints on the aforementioned physical magnitudes under which the formula for the observable under consideration may be approximated, within ϵ, by its Newtonian counterpart.

Conversely, one can also work from a choice of open neighborhood of the limit point to conditions under which certain observables will be well-approximated by their values in the Newtonian limit. If one such neighborhood is a strict subset of another, its members in general correspond to better approximation by the Newtonian limit point, hence typically to more restricted physical circumstances. In many cases, the intersection of these "smaller" open sets with the convergent family will yield members with smaller values of λ.

To illustrate the schema, I will consider a variety of observables drawn from three convergent families of relativistic spacetimes: relative velocity and three-momenta of massive particles and light rays in a Minkowskian family, the acceleration of a static observer in a Schwarzschildian family, and the mass-energy and average radial acceleration in a cosmological (FLRW) family. In most cases I will also calculate them for different choices of α and β to illustrate the role they play.

3.4.1 Relative Velocity and Momentum in a Minkowskian Family

Consider an observer in the Minkowskian spacetimes introduced above (eqs. 3.6 and 3.7) whose worldline has tangent vector

$$\mu^a = \left(\frac{\partial}{\partial t}\right)^a,$$

at a point q and who will measure at that point the three-momentum p of a particle of known mass m with tangent vector

$$\overset{\lambda}{\xi}{}^a = \frac{1}{\sqrt{1-\lambda v^2}}\left[\left(\frac{\partial}{\partial t}\right)^a + v\left(\frac{\partial}{\partial x}\right)^a\right], \tag{3.10}$$

where $0 \leq v < 1$. The coefficients for both tangent vectors have been chosen so that they each have magnitude 1, i.e., so that $\alpha(\lambda) = 1$. Further, parameterize spacelike curves so that $\beta(\lambda) = \lambda^{-1/2}$ so to make the speed of light constant throughout the family. To calculate the speed and three-momentum of the particle relative to the observer, one can decompose $\overset{\lambda}{\xi}{}^a$ into its components collinear and orthogonal to μ^a:

$$\mu^a = \left(\frac{\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c}{\overset{\lambda}{\eta}_{bc}\mu^b\mu^c}\right)\mu^a + \left(\overset{\lambda}{\xi}{}^a - \left(\frac{\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c}{\overset{\lambda}{\eta}_{bc}\mu^b\mu^c}\right)\mu^a\right) = (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a + (\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a).$$

The relative speed is given by the ratio of the magnitude of the spatial component to that of the temporal component, which are given respectively by

$$\begin{aligned}
\|\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a\| &= (\beta(\lambda))^{-1}[\overset{\lambda}{s}{}^{ad}(\overset{\lambda}{\xi}_a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\overset{\lambda}{\mu}_a)(\overset{\lambda}{\xi}_d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\overset{\lambda}{\mu}_d)]^{1/2} \\
&= \lambda^{1/2}[-\lambda^{-1}\overset{\lambda}{\eta}_{ad}(\overset{\lambda}{\xi}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a)(\overset{\lambda}{\xi}{}^d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\mu^d)]^{1/2} \\
&= ((\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)^2 - 1)^{1/2},
\end{aligned} \tag{3.11}$$

$$\|(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a\| = (\alpha(\lambda))^{-1}[\overset{\lambda}{t}_{ad}(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c)\mu^a(\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\xi}{}^f)\mu^d]^{1/2} = |\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\xi}{}^c|. \tag{3.12}$$

Thus the speed of the particle is given by

$$\frac{((\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}^b)^2 - 1)^{1/2}}{|\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}^b|} = \left(1 - \frac{1}{(\eta_{ab}\xi^a\mu^b)^2}\right)^{1/2} = \sqrt{\lambda}v$$

and the magnitude of its three-momentum by

$$p = m(\overset{\lambda}{\eta}_{ab}\mu^a\overset{\lambda}{\xi}^b)^2 - 1)^{1/2} = \frac{m\sqrt{\lambda}v}{\sqrt{1-\lambda v^2}}.$$

As remarked in the introduction, one can approximate this momentum by the classical formula of the product of the mass with the speed, $p = m\sqrt{\lambda}v$, which is indeed the measured momentum in the Newtonian limit, if λv^2—hence just λ for a fixed v—is sufficiently small. Thus choosing some allowed error $\epsilon > 0$ entails there is some $\delta > 0$, depending on v, such that $|p - m\sqrt{\lambda}v| < \epsilon$ when $\lambda < \delta$. This condition will be satisfied for the members of the Minkowskian family that lie in a certain neighborhood of the limiting Galilean spacetime. For simplicity, let

$$h^{ab} = \left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a\left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial z}\right)^b$$

be the (inverse) Riemannian metric constructed from the tetrad compatible with the standard global coordinates t, x, y, z on $(\mathbb{R}^4, \overset{\lambda}{\eta}_{ab}, \lambda)$. (An analogous calculation can be performed for Riemannian metrics constructed from other tetrads.) Then the intersection of the open neighborhood $B_0(s, \delta; h, q)$ of the Galilean spatial metric s^{ab} with the spatial metrics of the Minkowskian family yields $\{\overset{\lambda}{s}^{ab} : 0 < \lambda < \delta\}$, exactly those spatial metrics for which measurements of the particle's three-momentum will be within the allowed error of the classical formula. One can show the analogous neighborhood for the temporal metrics will be $B_0(t, 3\delta; h, q)$—the factor of 3 comes from the dimension of space.

One can also calculate the speed that any observer will measure of a light ray at q. The value will differ depending on the choices of $\alpha(\lambda)$ and $\beta(\lambda)$, which I shall demonstrate below. I shall also perform the calculation of this value for the Minkowskian family, but it generalizes straightforwardly to any spacetime. To begin, consider a family of $\overset{\lambda}{\eta}_{ab}$-null vectors $\overset{\lambda}{\nu}{}^a$ representing the trajectories of light rays at q. These will not in general be collinear because they must lie along the light cone, which is (eventually) widening as $\lambda \to 0$. The speed that the observer with tangent vector μ^a at q will assign to the light ray will be the ratio of the spatial magnitude of the component of $\overset{\lambda}{\nu}{}^a$ orthogonal to μ^a to the temporal magnitude of the component of $\overset{\lambda}{\nu}{}^a$ collinear to μ^a, just like with the speed of the massive particle. Now suppose we set $\alpha(\lambda) = \beta(\lambda) = 1$. So after decomposing $\overset{\lambda}{\nu}{}^a$ into these components via $\overset{\lambda}{\nu}{}^a = (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a + (\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a)$, one can calculate the spatial (eq. 3.11) and temporal (eq. 3.12) magnitudes

$$\begin{aligned}
\|\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a\| &= (\beta(\lambda))^{-1}[\overset{\lambda}{s}{}^{ad}(\overset{\lambda}{\nu}_a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\overset{\lambda}{\mu}_a)(\overset{\lambda}{\nu}_d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\overset{\lambda}{\mu}_d)]^{1/2} \\
&= [-\lambda^{-1}\overset{\lambda}{\eta}_{ad}(\overset{\lambda}{\nu}{}^a - (\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a)(\overset{\lambda}{\nu}{}^d - (\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\mu^d)]^{1/2} \\
&= \lambda^{-1/2}|\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c|^{1/2}, \\
\|(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a\| &= (\alpha(\lambda))^{-1}[\overset{\lambda}{t}_{ad}(\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c)\mu^a(\overset{\lambda}{\eta}_{ef}\mu^e\overset{\lambda}{\nu}{}^f)\mu^d]^{1/2} = |\overset{\lambda}{\eta}_{bc}\mu^b\overset{\lambda}{\nu}{}^c|.
\end{aligned}$$

Thus the observer will determine the speed of a light ray with tangent vector $\overset{\lambda}{\nu}{}^a$ to be $\lambda^{-1/2}$, the ratio of the magnitude of the spatial component to that of the temporal component of $\overset{\lambda}{\nu}{}^a$ (relative to μ^a). Since the choice of this observer was arbitrary, arbitrary observers in the spacetimes $(M, \overset{\lambda}{\eta}_{ab}, \lambda)$ will measure larger and larger speeds of light as $\lambda \to 0$. On the other hand, if one uses instead $\beta(\lambda) = \overset{\lambda}{\kappa}{}^{-1/2} = \lambda^{-1/2}$ while retaining $\alpha(\lambda) = 1$, the above calculation yields a speed of 1 for the speed of light independently of λ. In other words, in this interpretation all observers will always measure the speed of light to be the same fixed value.[23]

To see this significance of these calculations, it may be illuminating to focus on the tangent space at q and consider the relative velocity that a certain observer passing through that point would attribute to particles with various other tangent vectors. (See figure 3.1 for an illustration.) When

[23]Note that, for Newton-Cartan theory, one does not usually countenance particles (massive or otherwise) whose trajectories are not timelike. Nevertheless one can still consider the behavior of the "relative speed" observable in the limit regardless of the choices of α and β. (Cf. Weatherall (2011, p. 430, fn. 16).)

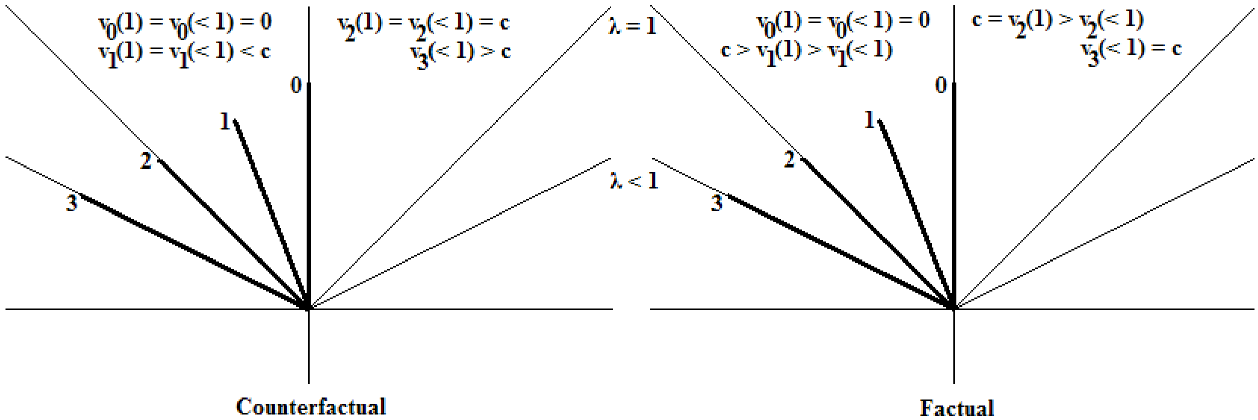

Figure 3.1: Depicted are four vectors (labeled 0, 1, 2, and 3) at some point q, along with the light cones associated with the $\lambda = 1$ member and some $\lambda < 1$ member from a family of relativistic spacetimes with $\kappa = \lambda$. The $v_i(\lambda)$ are the speeds that an observer at q with tangent vector proportional to vector 0 would measure of a particle whose worldline at q has (co-directed) tangent vector proportional to vector i. For the figure labeled "counterfactual," $\alpha(\lambda) = \beta(\lambda) = 1$, and the relative speed corresponding to each vector does not change, but as $\lambda \to 0$ the speed of a light ray increases, and more vectors become timelike. For the figure labeled "factual," on the other hand, $\alpha(\lambda) = 1$ and $\beta(\lambda) = \lambda^{-1/2}$, and as $\lambda \to 0$ the relative speed of each vector decreases, but the speed of a light ray stays the same. More vectors become timelike on for these choices of α and β, too, but as λ becomes sufficiently small their speeds relative to vector 0 also decrease.

$\alpha(\lambda) = \beta(\lambda) = 1$, as the light cone widens, the velocity she would attribute to particles with a given tangent vector (up to normalization) would remain the same, but she would count more and more vectors as timelike. She would still attribute a fixed speed to a particle whose tangent vector was initially null for $\lambda = 1$ but became timelike as the cone widened. By contrast she would judge the speed of a light ray whose tangent vector must lie along the widening cone to be larger than that of a light ray for $\lambda = 1$. By contrast, when $\alpha(\lambda) = 1$ but $\beta(\lambda) = \lambda^{-1/2}$, as the light cone widens, more and more vectors count as timelike, but particles with null tangent vectors are always measured with the same speed. Accordingly, a particle with a fixed tangent vector at q not comoving with the observer will be attributed smaller and smaller relative velocities as the light cone widens. (Only particles comoving with the observer maintain the same (vanishing) measured velocity as the cone widens.) Heuristically, one can interpret v/c becoming small as $\lambda \to 0$ as v being fixed but $c \to \infty$, or as c being fixed but $v \to 0$. Of course, these remarks bear only on relative velocity observables at individual points. Other observables deserve their own treatment, to some of which

I turn presently.

3.4.2 Acceleration in a Schwarzschildian Family

Consider a family of Schwarzschildian spacetimes, whose temporal (hence Lorentz) metrics are given in the standard spherical coordinates t, r, θ, ϕ as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{g}_{ab} = (1 - 2M\lambda/r)(d_a t)(d_b t) - \left(\frac{\lambda}{1 - 2M\lambda/r}\right)(d_a r)(d_b r) \\ - \lambda r^2[(d_a\theta)(d_b\theta) + \sin^2\theta(d_a\phi)(d_b\phi)],$$

where d_a denotes the exterior derivative and M the mass of the black hole. For present purposes I will be concerned with the "external" region of the spacetime, i.e., that for which $r > 2M\lambda$. When restricted to this region, the family $(M, \overset{\lambda}{t}_{ab}, \lambda)$ has as a Newtonian limit the spacetime associated with a point mass (Ehlers 1981, 1991, 1997). (Note that the Schwarzschild radius, $2M\lambda$, vanishes in the limit.) Thus one can evaluate in this family the conditions under which various observables may be approximated by their Newtonian counterparts.

In particular, consider a static observer, i.e., one whose tangent vector at some point p is given by the unit vector

$$\overset{\lambda}{\mu}{}^a = \frac{1}{\sqrt{1 - 2M\lambda/r}}\left(\frac{\partial}{\partial t}\right)^a. \tag{3.13}$$

Such an observer is always accelerating, for she maintains her position (with respect to the static coordinates) despite the gravitational "attraction" of the black hole. When will the acceleration she experiences be approximated by that she would experience if she were in the gravitational field of a Newtonian point mass? Let $\overset{\lambda}{\nabla}$ be the covariant derivative operator associated with the spacetime $(M, \overset{\lambda}{t}_{ab}, \lambda)$, let ∇ be that associated with the Newtonian limit spacetime, and let $\bar{\nabla}$ be that compatible with the (flat) Riemannian metric

$$h_{ab} = (d_a t)(d_b t) + (d_a r)(d_b r) + r^2[(d_a\theta)(d_b\theta) + \sin^2(d_a\phi)(d_b\phi)]$$

arising from the standard coordinates and according to which the observer's worldline is a geodesic. (Again, one can perform analogous calculations with respect to Riemannian metrics constructed from other observers.) In both the Schwarzschildian family and its Newtonian limit spacetime, the observer's acceleration is completely radial, i.e., proportional to $(\partial/\partial r)^a$. Thus the quantity of interest will be the difference in magnitude of these accelerations. The acceleration in the family is given by

$$\overset{\lambda}{\mu}{}^b \overset{\lambda}{\nabla}_b \overset{\lambda}{\mu}{}^a = \overset{\lambda}{\mu}{}^b \bar{\nabla}_b \overset{\lambda}{\mu}{}^a - \overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc} = -\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc},$$

where $\overset{\lambda}{\nabla} = (\bar{\nabla}, \overset{\lambda}{C}{}^a{}_{bc})$. Using the standard result (see, e.g., Malament (2012, p. 78)) that

$$\overset{\lambda}{C}{}^a{}_{bc} = \frac{1}{2} \overset{\lambda}{g}{}^{ad} (\bar{\nabla}_d \overset{\lambda}{t}_{bc} - \bar{\nabla}_b \overset{\lambda}{t}_{dc} - \bar{\nabla}_c \overset{\lambda}{t}_{db}),$$

one can compute that

$$-\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc} = \frac{M}{r^2} \left(\frac{\partial}{\partial r} \right)^a,$$

the spatial magnitude of which is then given by

$$\begin{aligned} \overset{\lambda}{a} = \| -\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc} \| &= (\beta(\lambda))^{-1} [\overset{\lambda}{s}{}^{ad} (-\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}_{abc})(-\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}_{dbc})]^{1/2} \\ &= (\beta(\lambda))^{-1} [-\lambda^{-1} \overset{\lambda}{g}_{ab} (-\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^a{}_{bc})(-\overset{\lambda}{\mu}{}^b \overset{\lambda}{\mu}{}^c \overset{\lambda}{C}{}^d{}_{bc})]^{1/2} \\ &= \frac{M/r^2}{\beta(\lambda)\sqrt{1 - 2M\lambda/r}}. \end{aligned} \tag{3.14}$$

Because these observables are continuous in λ and are functions of tensors that converge in the Newtonian limit, their limiting values must match their corresponding values in the limit. Indeed, a similar calculation for the Newtonian limit spacetime yields

$$a = \| - \mu^b \mu^c C^a_{bc} \| = (\beta(\lambda))^{-1} M/r^2. \tag{3.15}$$

When $\beta(\lambda) = 1$, the magnitude of the acceleration in the Schwarzschildian family approaches that of its Newtonian limit spacetime as $\lambda \to 0$. When $\beta(\lambda) = \lambda^{-1/2}$, eq. 3.15 still approximates eq. 3.14 when $2M\lambda/r$, i.e., the ratio of the Schwarzschild radius to the radial distance of the observer, is sufficiently small. For fixed M, one can interpret this as a sufficiently large r, or for a fixed r, as a sufficiently small M. In other words, given some allowed error $\epsilon > 0$, there is some $\delta > 0$, such that when the ratio of the Schwarzschild radius to the radial distance of the observer is less than it, the magnitude of the acceleration experienced by the observer can be approximated within ϵ by the formula for its Newtonian counterpart. (The calculations showing this are identical to the one for relative velocity in §3.4.1.)

3.4.3 Mass-Energy and Average Radial Acceleration in a Cosmological (FLRW) Family

Consider a family of spatially flat cosmological (FLRW) spacetimes, whose temporal (hence Lorentz) metrics are given in Cartesian coordinates as

$$\overset{\lambda}{t}_{ab} = \overset{\lambda}{g}_{ab} = (d_a t)(d_b t) - \lambda a^2[(d_a x)(d_b x) + (d_a y)(d_b y) + (d_a z)(d_b z)], \tag{3.16}$$

where the cosmological scale factor $a > 0$ depends only on t and is normalized so that $a_{|t=0} = 1$. The stress-energy tensor has the form of that of a perfect fluid with density ρ and pressure p, both of which also only depend on t:[24]

$$\overset{\lambda}{T}{}^{ab} = (\rho + \lambda p)\left(\frac{\partial}{\partial t}\right)^{a}\left(\frac{\partial}{\partial t}\right)^{b} + p\overset{\lambda}{s}{}^{ab}. \tag{3.17}$$

The family $(\mathbb{R}^4, \overset{\lambda}{t}_{ab}, \lambda)$ has, as $\lambda \to 0$, a Newtonian limit representing a homogeneous universe (Ehlers 1988, 1997). Thus observables that are continuous functions in λ will converge to their Newtonian counterparts as well.

Consider, for example, an observer whose tangent vector at some point is given by

$$\overset{\lambda}{\xi}{}^{a} = \frac{1}{\sqrt{1 - \lambda a^2 v^2}}\left[\left(\frac{\partial}{\partial t}\right)^{a} + v\left(\frac{\partial}{\partial x}\right)^{a}\right], \tag{3.18}$$

with $0 \leq v < a^{-1}$. What mass-energy density $\overset{\lambda}{\rho}$ would the observer measure? Under what circumstances can it be approximated by its corresponding Newtonian observable, the mass density ρ? For the relativistic family, one can calculate

$$\overset{\lambda}{\rho} = \overset{\lambda}{t}_{ac}\overset{\lambda}{t}_{bd}\overset{\lambda}{T}{}^{ab}\overset{\lambda}{\xi}{}^{c}\overset{\lambda}{\xi}{}^{d} = \frac{\rho + \lambda p}{1 - \lambda a^2 v^2} - \lambda p = \frac{\rho + \lambda a^2 v^2}{1 - \lambda a^2 v^2}, \tag{3.19}$$

which yields $\lim_{\lambda \to 0} \overset{\lambda}{\rho} = \rho$ as expected. Clearly there will only be a discrepancy from ρ when the observer is not comoving with the fluid, i.e., when $v > 0$, but for *any* $\epsilon > 0$ one can find a sufficiently small λ such that $|\overset{\lambda}{\rho} - \rho| < \epsilon$. The bound for this will depend on $a^2 v^2$, so under the factual interpretation, one can interpret the smallness of λ to be the smallness of a relative to v, or vice versa.

One can also examine the limiting behavior of another observable, sometimes called the *average radial acceleration* (*ARA*), which measures (in a sense) the average tidal forces that a small cluster of massive test particles undergoing geodesic motion would experience. To calculate it, one must

[24]One may also require them to satisfy various energy condition or equations of state, but these play no role in the calculations below.

first invert eq. 3.3 to yield the Ricci tensor

$$\overset{\lambda}{R}_{ab} = 8\pi(\overset{\lambda}{t}_{am}\overset{\lambda}{t}_{bn} - \frac{1}{2}\overset{\lambda}{t}_{ab}\overset{\lambda}{t}_{mn})\overset{\lambda}{T}^{mn} = 8\pi(\rho + \lambda p)(d_a t)(d_b t) - 4\pi(\rho - \lambda p)\overset{\lambda}{t}_{ab}, \tag{3.20}$$

where I have substituted in eq. 3.17. Now, suppose that the observer with tangent vector $\overset{\lambda}{\xi}^a$ at some point is undergoing geodesic motion in a neighborhood of that point, and pick a smooth tetrad field whose timelike component is the observer's tangent vector field and whose spacelike components vanish when Lie derived by that field. The *ARA* is then defined as the average of the magnitudes of the relative acceleration between the observer and "infinitesimally close" observers "connected" by a spacelike component, for each of the three components. It turns out that the *ARA* is independent of the choice of these spacelike components and can be determined from the observer's tangent vector and the Ricci tensor (Malament 2012, p. 165–6):[25]

$$\begin{aligned} ARA = -\frac{1}{3\beta(\lambda)}\overset{\lambda}{R}_{ab}\overset{\lambda}{\xi}^a\overset{\lambda}{\xi}^b &= -\frac{1}{3\beta(\lambda)}\left(\frac{8\pi(\rho + \lambda p)}{1 - \lambda a^2 v^2} - 4\pi(\rho - \lambda p)\right) \\ &= -\frac{4\pi}{3\beta(\lambda)}\left(\frac{\rho + 3\lambda p + \lambda a^2 v^2(\rho - \lambda p)}{1 - \lambda a^2 v^2}\right). \end{aligned} \tag{3.21}$$

Because the *ARA* is composed from tensorial fields that have a Newtonian limit and is continuous in λ, its limiting value as $\lambda \to 0$ must be the value it takes on in the Newtonian limit model. Indeed, when $\beta(\lambda) = 1$, this value is $-4\pi\rho/3$, just as expected (Malament 2012, p. 281). Under the factual interpretation, the conditions under which eq. 3.21 may be approximated by its Newtonian formula are somewhat complicated, but roughly, this will be when $\lambda a^2 v^2$ is much less than 1 and p is much less than ρ, i.e., when the relative velocity of the observer to the integral curves of the cosmic fluid is sufficiently small and the pressure is small compared to the mass density.

[25]The factor of β arises because one is averaging the magnitudes of components of acceleration vectors, which are spacelike.

3.5 Topology and Observables

We can now return to the question I posed after the definition of the Newtonian limit: why use the point-open topology? As I have argued in chapter 2, there is no canonical topology for the spacetime metrics, so this choice must be justified relative to the nature of the investigation. In light of the foregoing discussion of the factual interpretation of the limit, it is clear that a choice of topology corresponds with a set of relevant observables that one requires be well-approximated by the Newtonian limit spacetime. In the case of the C^2 point-open topology, the observables are scalar point quantities at finitely many points arising from contraction of the temporal and spatial metrics, the stress-energy tensor, and their derivatives to second order.

At least this much seems warranted, since we do measure, at least in some idealized sense, relativistic point observables that we can approximate with their classical counterparts.[26] But our experiments and observations are not confined to points: there are many observables corresponding more generally to *extended compact* regions for whose classical approximations one should account as well. These may include continuous measurements of point quantities, such as the momentum flux over time, integrated observables, such as the proper time along a worldline, and analogous quantities over areas and volumes in spacetime.

To see why any point-open topology is insufficient to take these kinds of observables into account, consider the following family of relativistic temporal metrics on $\mathbb{R}^4$ restricted to temporal coordinates in the range $[0,1]$:

$$(\overset{\lambda}{t}_{ab})_{|t^{-1}[0,1]} = (1+\lambda^{-3}t(1-t)^{1/\lambda})(d_a t)(d_b t) - \lambda(d_a x)(d_b x) - \lambda(d_a y)(d_b y) - \lambda(d_a z)(d_b z), \qquad (3.22)$$

where $0 < \lambda \leq 1$. Restricted to the strip $[0,1]\times\mathbb{R}^3$, the relativistic family $(\mathbb{R}^4, \overset{\lambda}{t}_{ab}, \lambda)$ has the "Galilean" spacetime $([0,1]\times\mathbb{R}^3, t_{ab}, s^{ab}, \nabla, \mathbf{0})$ (cf. eqs. 3.8,3.9) as its Newtonian limit in the C^2

[26] A topology's system of open sets must be closed under arbitrary union and finite intersection, so any topology characterized by the sufficient approximation of a class of observables is sensitive as well to finite conjunctions and arbitrary disjunctions of well-approximations of those observables.

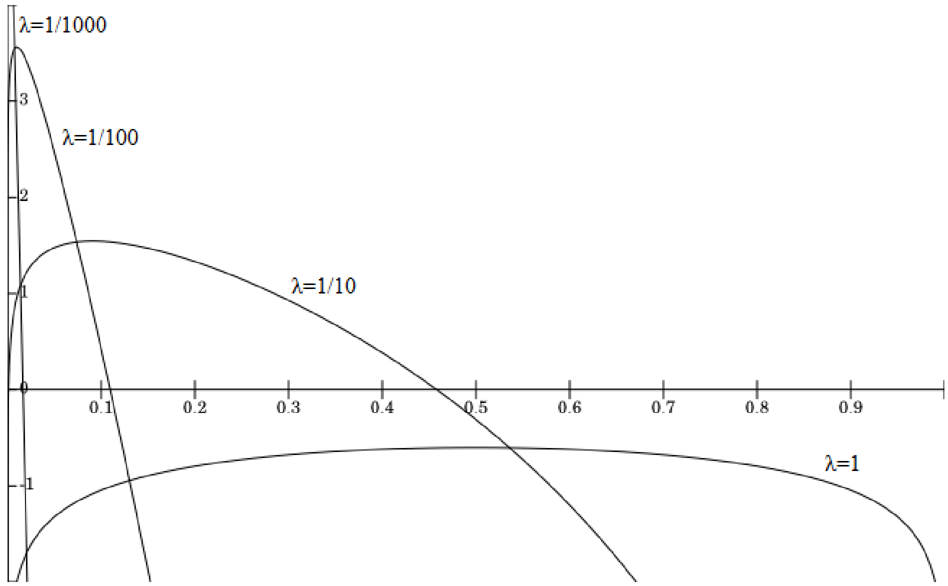

Figure 3.2: Log-plot of the conformal "bump" in eq. 3.22 as a function of t-coordinate. At $\lambda = 1/1000$, it is sharply peaked near $t = 0$ and nearly vanishing elsewhere.

point-open topology. (The members of this family have many smooth extensions to the whole spacetime, but which extension one chooses is immaterial for my purposes here as long as the extensions for the whole family also have a Newtonian limit.) The limit exists because, for every point p with t-coordinate within $[0, 1]$ and any $\epsilon > 0$, one can find a sufficiently small λ such that $d(t, \overset{\lambda}{t}; h, i)_{|p} < \epsilon$ for $i = 0, 1, 2$, and similarly for the spatial metric. (The exponential term $(1 - t)^{1/\lambda}$ dominates as $\lambda \to 0$.) But one cannot find any λ such that the distance function will be *uniformly* bounded by ϵ for all $t \in [0, 1]$. This is because as $\lambda \to 0$, the "bump" in the metric moves toward $t = 0$, becoming more localized but also taller. (See fig. 3.2.) Consequently there will be observables, depending on the values the metric takes on intervals containing a point with $t = 0$, that do not converge in the Newtonian limit.

For example, consider a timelike curve passing through the region with t-coordinate in $[0, 1]$ and with tangent vector everywhere proportional to $(\frac{\partial}{\partial t})^a$. Since the lengths of timelike curves are invariant under reparameterization, the proper time elapsed along the curve between temporal coordinates $t = 0$ and $t = 1$ according to $\overset{\lambda}{t}_{ab}$ is given by

$$\int_0^1 \left[\overset{\lambda}{t}_{ab} \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b\right]^{1/2} dt = 1 + \frac{1}{\lambda(2\lambda^2 + 3\lambda + 1)}, \tag{3.23}$$

whereas the same quantity according to the Galilean temporal metric $(d_a t)(d_b t)$ is just 1. The former diverges as $\lambda \to 0$, meaning that there are *no* experimental contexts in which observations adhere to their classical formulas within some ranges of error. Because these observations include measurable quantities, such as proper time, the point-open topology is evidently too coarse to capture the demand that the Newtonian limit point of a family of relativistic spacetimes can approximate standard observables that depend on extended (but compact) regions.

Requiring the convergence of more kinds of observables corresponds with introducing more open sets into the topology on temporal and spatial metrics, i.e., requiring convergence in a *finer* topology. There is a natural such choice to supplant the point-open topology, one that is defined quite similarly but controls convergence uniformly on compacta instead of pointwise. A basis for the C^k *compact-open* topology on the smooth tensor fields in some set S may be written as sets of the form

$$B_k(t, \epsilon; h, C) = \{t' \in S : \sup_C d(t, t'; h, 0) < \epsilon, \ldots, \sup_C d(t, t'; h, k) < \epsilon\}, \tag{3.24}$$

where, much as with the point-open topologies, t ranges over all tensor fields in S, ϵ ranges over all positive constants, and C ranges over all compact subsets of M. The compact-open topologies have many nice properties (§2.4), the most important of which for this context is that the sequence given by eq. 3.22 does *not* converge—it is sensitive to the fact that the lengths of some timelike curves diverge in the $\lambda \to 0$ limit. Besides what's already been said, one can see this as a consequence of the following proposition that gives convergence conditions for the compact-open topologies analogous to those for the point-open topologies.

Proposition 3.5.1. *A family of tensor fields $\overset{\lambda}{\phi}{}^a_{bc}$ on M, with $\lambda \in (0, a)$ for some $a > 0$, converges to a tensor field ϕ^a_{bc} as $\lambda \to 0$ in the C^k compact-open topology iff for all compacta $C \subseteq M$,*

$\lim_{\lambda\to 0}\sup_C(\overset{0}{\psi}{}_a^{bc}\overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{0}{\psi}{}_a^{bc}\phi^a_{bc})$ *for all tensor fields* $\overset{0}{\psi}{}_a^{bc}$ *on* C *and, for each positive* $i \leq k$, $\lim_{\lambda\to 0}\sup_C(\overset{i}{\psi}{}_a^{bcd_1\ldots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{i}{\psi}{}_a^{bcd_1\ldots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\phi^a_{bc})$ *for all tensor fields* $\overset{i}{\psi}{}_a^{bcd_1\ldots d_i}$ *on* C. *Moreover, the* C^k *compact-open topology is the unique topology with this property.*

Analogous results hold for tensor fields of other index structures. One can interpret the proposition as showing just how the compact-open topologies formalize a notion of uniform convergence of observables defined on compacta.[27]

Although the sequence given by eq. 3.22 does not converge in any compact-open topology, the Minkowskian family given by eqs. 3.6 and 3.7 does still have Galilean spacetime as its Newtonian limit. In light of these considerations, I propose modifying Ehlers's definition of the Newtonian limit to require convergence in the C^2 compact-open topology:[28]

Newtonian Limit (Revised) Let $(M, \overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab})$, with $\lambda \in (0, a)$ for some $a > 0$, be a one-parameter family of models of general relativity that share the same underlying manifold M. Then $(M, t_{ab}, s^{ab}, \nabla, T^{ab})$ is said to be a Newtonian limit of the family when it is a model of Newton-Cartan theory and $\lim_{\lambda\to 0}(\overset{\lambda}{t}_{ab}, \overset{\lambda}{s}{}^{ab}, \overset{\lambda}{\nabla}, \overset{\lambda}{T}{}^{ab}) = (t_{ab}, s^{ab}, \nabla, T^{ab})$ in the C^2 compact-open product topology.

This characterization depends, of course, on considering as relevant all and only observables subsisting on compact regions of spacetime. Thus one need not be too insistent that this is the sole "right" topology to characterize the Newtonian limit, for this class of observables may very well be too expansive or too meager for certain contexts. For example, one might want to consider observables associated not just with compact regions of spacetime, but also with non-compact curves

[27] There is a slightly coarser topology than the compact-open that prevents eq. 3.23 from converging, in which one replaces the distance function (eq. 3.4) with the L^2 norm. The essential point I want to make, which would still stand under this proposal, is that one needs to use a topology that is sensitive (in some appropriate way) to observables defined on extended regions.

[28] This is not a significant departure from Ehlers. Although he effectively stated the definition of the Newtonian limit using the C^2 point-open topology (i.e., "pointwise convergence"), he did on occasion discuss compact observables, such as proper time, in which case he referred to locally uniform convergence. This mode of convergence is equivalent to compact convergence for locally compact spaces. Since every finite-dimensional manifold is locally compact, the corresponding topology is the compact-open topology.

with finite proper length, as arise in singular spacetimes. (Consider starting a stopwatch and then throwing it into a black hole.)

Another case of interest in this regard is whether one should include some global (non-compact) observables, especially in the context of cosmological models. This case is less clear because even in cosmology, it is not obvious that one can have experimental access to observables depending on (data on) non-compact sets, without which it seems one cannot determine the global structure of virtually all spacetimes of interest.[29] In any case, it is harder to find an even plausible topology to encode such global observables. The most common topologies chosen in the literature to control the global behavior of smooth tensor fields in a collection S are the C^k open topologies, a basis for which may be given as sets of the form

$$B_k(t, \epsilon; h) = \{t' \in S : \sup_M d(t, t'; h, 0) < \epsilon, \ldots, \sup_M d(t, t'; h, k) < \epsilon\}, \tag{3.25}$$

where t ranges over all tensor fields in S, ϵ ranges over all positive constants, and (importantly) h ranges over all Riemannian metrics. One can show that, when M is compact, the open topologies are identical to the compact-open topologies. But when M is non-compact, in contrast to the point-open and compact-open topologies, different choices of h in general generate different collections of open sets. One is thus obliged to consider all possible choices of h because, as an arbitrary smoothly varying choice of basis at each point, no particular choice corresponds with anything physically meaningful in a spacetime model. And in this case, the convergence condition for the open topologies is *not* as similar to those for the compact-open and point-open topologies as one might have expected:[30]

Proposition 3.5.2. *Let $g, \{\overset{n}{g}\}_{n\in\mathbb{N}}$ be tensor fields on a non-compact manifold M. Then $\overset{n}{g} \to g$ in the open C^k topology iff there is a compact $C \subset M$ such that:*

[29] In particular, if a spacetime is not causally bizarre, then it is observationally indistinguishable from a spacetime that has holes and is extendible, anisotropic, and not globally hyperbolic. See Manchak (2011) and references cited therein.

[30] For a proof sketch, see Golubitsky and Guillemin (1973, p. 43–44).

1. *for sufficiently large n,* $\overset{n}{g}_{|M-C} = g_{|M-C}$*; and*

2. $\overset{n}{g}_{|C} \to g_{|C}$ *in the compact-open* C^k *topology.*

Thus the mere fact that the temporal and spatial metrics of a relativistic spacetimes differ in signature from those of a Newton-Cartan spacetime is sufficient to entail the following negative result:

Corollary 3.5.3. *No family of relativistic spacetimes on a non-compact manifold has a Newtonian limit in any open topology.*

3.6 Conclusions

One can draw a number of topical and methodological conclusions from the above discussion. In §3.2, I described how one can give a unified description of both general relativity and Newton-Cartan theory under the banner of Ehlers's frame theory. It is only superficially an example of a unification in the traditional sense in philosophy of science, for the theories thereby "unified" do not have different domains but in fact concern the same phenomena: one (general relativity) is a successor to the other (Newtonian gravitation) and improves upon it. Thus the frame theory should perhaps more properly be called a metatheory, since it provides a common terminology for and exhibits the conceptual continuity between Newtonian and relativistic gravitation (Ehlers 1981, 1986, 1988, 1998).

Even considered as a metatheory, it is not without philosophical interest. It reveals that, if only in retrospective rational reconstruction, the transition to relativity theory from Newtonian physics involves much more conceptual continuity than is usually emphasized. This kind of claim is of interest for structural realists, who are keen to find the structural continuity between old and new theories. Metatheories like the frame theory prove just the technical apparatus needed to draw a comparison. Certain critics of structural realism, both to its right (realists) and otherwise, would also find the frame theory of interest, for they can point to the rather minimal structure that relativity and Newtonian theory share, skeptical that something so meager commits one to much at all, ontologically. More generally, the methodology used here, in which the models of two theories of the same (or at least overlapping) subject matter are united in a common framework (that is to be given a topology), does not require in any essential way that the theories under consideration be physical theories—any sufficiently mathematized theories will do. Thus it could also prospectively apply to certain theories of, say, economics, climate science, or population ecology.

Coming back concerns more internal to the conceptual structure of physics, once the frame theory is constructed, one can place a topology on its space of models, thereby defining a convergence condition and making precise what it means to take a limit within that space. In §3.3, I proved how the convergence condition used in the literature can be understood topologically and used this reconstruction to define the Newtonian limit of a family of models of general relativity. The reduction relation (under the factual interpretation) may then be understood as supporting an explanation of why Newtonian gravitation was successful by providing the conditions under which its models successfully approximate those of general relativity. Because the pattern of providing a unifying framework for models of the two theories and then topologizing those models does not appear in principle to depend on anything topically specific to gravitation, it might be applied fruitfully to other reduction relations. It may even serve as a general pattern for how one can explain the success of one (sufficiently mathematized) theory from another. Of course, further case studies must be pursued to test the viability of this hint.

A longstanding obstacle to physically interpreting the geometrical methods used to define the Newtonian limit is the ambiguous conceptual status of letting the limiting parameter be the value of a fixed physical constant, namely the speed of light. I attempted to clarify the nature of the limit in §3.4 by showing through explication and example how it may be interpreted so that the speed of light is in fact the same in all models. This makes the geometric limit, already completely general in its capability to capture properties and observable of interest, appropriate for showing how models of classical spacetime can approximate relativistic spacetimes in specified observational contexts. In some ways, the geometric Newtonian limit subsumes the older, more limited local approximation methods (like the "low velocity limit") more commonly used, although I do not expect this way of approaching approximations to supplant traditional methods. Rather, I see one of the primary upshots of this section to be a general way to understand reductive limiting relationships between theories that are usually described in terms of the elimination of a constant of nature.

In §3.5 I returned to a question broached in §3.3 about the definition of the Newtonian limit: why pick the C^2 point-open topology? Indicating how the choice of topology corresponds with a class of observables one demands be well-approximated in the Newtonian limit, I argued that the point-open topology is not fine enough—it allows too many convergent sequences of spacetimes, including ones in which observables one would think should converge do not, such as the proper time measured along a compact timelike worldline. These observables depend on data on compact sets, so I tentatively suggest using the C^2 compact-open topology instead. This allows one to recover the convergence of observables depending on data on compact sets. An even finer class of topologies seemingly attuned to global features, the open topologies, however allows for no sequences of relativistic spacetimes with a Newtonian limit. This exhibits the fact that whether a particular sequence converges (or is "singular") depends on the choice of topology. How we understand the relationships theories have with each other depends on what features we take to be important in judging the similarities of their models. Choosing a topology implicitly picks out such features, but the significance of these choices in understanding limiting relationships has been unduly neglected.

3.7 Proofs of Propositions

Proposition 3.7.1 (3.3.1). *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on* M*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *point-open topology iff for all points* $p \in M$*,*

$\lim_{\lambda\to 0}(\overset{0}{\psi}{}_a^{bc}\overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{0}{\psi}{}_a^{bc}\phi^a_{bc})_{|p}$ *for all tensors* $\overset{0}{\psi}{}_a^{bc}$ *at p and, for all positive* $i \leq k$*,* $\lim_{\lambda\to 0}(\overset{i}{\psi}{}_a^{bcd_1\dots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\overset{\lambda}{\phi}{}^a_{bc})_{|p} = (\overset{i}{\psi}{}_a^{bcd_1\dots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\phi^a_{bc})_{|p}$ *for all tensors* $\overset{i}{\psi}{}_a^{bcd_1\dots d_i}$ *at p.*

Proof. Consider the case $k = 0$, as the others can be treated similarly, and fix some Riemannian metric h_{ab} on M. First suppose that $\lim_{\lambda\to 0}\overset{\lambda}{\phi}{}^a_{bc} = \phi^a_{bc}$ in the C^0 point-open topology, and consider some tensor ψ_a^{bc} at an arbitrary $p \in M$. In the basis for the tangent and cotangent spaces at p that makes h_{ab} and h^{ab} the identity matrices,

$$|\psi_a^{bc}(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} = \left|\sum_{\alpha,\beta,\gamma=0}^{3}\psi_\alpha^{\beta\gamma}(\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma})\right|$$
$$\leq \left|\sum_{\alpha,\beta,\gamma=0}^{3}(\psi_\alpha^{\beta\gamma})^2\right|^{1/2}\left|\sum_{\alpha,\beta,\gamma=0}^{3}(\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma})^2\right|^{1/2} = d(\psi, 2\psi; h, 0)_{|p}\, d(\phi, \overset{\lambda}{\phi}; h, 0)_{|p},$$

using the Cauchy-Schwarz inequality. Since $d(\psi, 2\psi; h, 0)_{|p}$ is constant with respect to λ, by hypothesis we have that $\lim_{\lambda\to 0}|\psi_a^{bc}(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} = 0$.

For the reverse direction, assume instead that for all $p \in M$ and all tensors of the form ψ_a^{bc} at p, $\lim_{\lambda\to 0}|\psi_a^{bc}(\overset{\lambda}{\phi}{}^a_{bc} - \phi^a_{bc})|_{|p} = 0$. In particular choosing ψ_a^{bc} to vanish in all but one component shows that that each component of $\overset{\lambda}{\phi}{}^a_{bc}$ converges to each component of ϕ^a_{bc}. Since $d(\phi, \overset{\lambda}{\phi}; h, 0)_{|p} = |\sum_{\alpha,\beta,\gamma=0}^{3}(\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma})^2|^{1/2}$ is a continuous function in each $\overset{\lambda}{\phi}{}^\alpha_{\beta\gamma} - \phi^\alpha_{\beta\gamma}$ and $d(\phi, \phi; h, 0)_{|p} = 0$, by definition $\lim_{\lambda\to 0}\overset{\lambda}{\phi}{}^a_{bc} = \phi^a_{bc}$ in the C^0 point-open topology. □

Proposition 3.7.2 (3.5.2)**.** *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on M, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *as* $\lambda \to 0$ *in the* C^k *compact-open topology iff for all compacta* $C \subseteq M$*,* $\lim_{\lambda\to 0}\sup_C(\overset{0}{\psi}{}_a^{bc}\overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{0}{\psi}{}_a^{bc}\phi^a_{bc})$ *for all tensor fields* $\overset{0}{\psi}{}_a^{bc}$ *on C and, for each positive* $i \leq k$*,* $\lim_{\lambda\to 0}\sup_C(\overset{i}{\psi}{}_a^{bcd_1\dots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\overset{\lambda}{\phi}{}^a_{bc}) = \sup_C(\overset{i}{\psi}{}_a^{bcd_1\dots d_i}\nabla_{d_1}\cdots\nabla_{d_i}\phi^a_{bc})$ *for all tensor fields* $\overset{i}{\psi}{}_a^{bcd_1\dots d_i}$ *on C. Moreover, the* C^k *compact-open topology is the unique topology with this property.*

Proof. The proof of the biconditional is analogous to that of proposition 3.3.1. To prove uniqueness, it suffices to prove that the C^k compact-open topology is first-countable, i.e., it has a countable local neighborhood base (Willard 1970, Corollary 10.5, p. 71). Fix a Riemannian h, and let $C_i \subset M$ for $1 \leq i < \infty$ be a sequence of compacta such that $\bigcup_{i=1}^{\infty} C_i = M$. I claim that, for each tensor field α, $B_k(\alpha, 1/n; h, \bigcup_{i=1}^{n} C_i)$ for $1 \leq n < \infty$ is a countable local basis at α if M is non-compact. For consider some open neighborhood of α, which by definition must contain a set of the form $B_k(\alpha, \epsilon; h, C)$ for some $\epsilon > 0$ and compact $C \subset M$. Let $m = \max\{\arg\min_n 1/n < \epsilon, \arg\min_n C \subseteq \bigcup_{i=1}^{n} C_i\}$. Then clearly $B_k(\alpha, 1/m; h, \bigcup_{i=1}^{m} C_i) \subseteq B_k(\alpha, \epsilon; h, C)$. If M is compact, then similarly for any set of the form $B_k(\alpha, \epsilon; h, C)$, $B_k(\alpha, 1/m; h, M) \subseteq B_k(\alpha, \epsilon; h, C)$ with $m = \arg\min_n 1/n < \epsilon$, so $B_k(\alpha, 1/n; h, M)$ for $1 \leq n < \infty$ is a countable local basis at α. □

Chapter 4

Global Spacetime Similarity

4.1 Introduction

A topology on a collection of mathematical objects formalizes a precise notion of similarity amongst them. It determines which sequences of such objects converge, which parameterized families vary continuously, and which properties of subcollections are generic or are stable under sufficiently small perturbations. When these objects are spacetimes, examining which limits of sequences and families of spacetimes converge yields both a way to construct new ones and further understanding of the relationships between them, such as how Newtonian spacetimes can be understood as certain limits of relativistic spacetimes (ch. 3). A topology on the collection of spacetimes also provides a framework for addressing questions of the genericness and stability, hence physical significance, of acausalities and singularities (Geroch 1970, 1971; Hawking 1971; Lerner 1973).

Of course, there are always many topologies from which to choose, each capturing a slightly different type of similarity. Typically, though, one does not begin an investigation with a predetermined topology in hand, but rather with certain intuitions, examples, and other desiderata a desired notion of similarity should respect. One then attempts to formalize that notion with a choice of topology. But when such topologies are introduced, insufficient attention is usually paid to justifying how they respect the relevant desiderata, that is, how they really do capture the relevant notion of similarity. Ignoring this demand can lead to counterintuitive and misleading conclusions, such as it being generic for a spacetime to contain closed timelike curves (prop. 2.4.3).

One notable exception to this omission has been Geroch (1970, 1971), who argues that no instance of either of the two most commonly used topologies, the *compact-open* and *open* topologies,[1] meets the demands on convergence and continuity imposed by three simple examples. In the next section I discuss his examples and interpret his desiderata as a demand for a topology that captures a notion of global similarity analogous to the topology of uniform convergence for real functions of a single variable. In this light the unsuitability of the compact-open topologies is manifest, as they correspond to topologies of uniform convergence on compacta.[2] The failure of the open topologies is more subtle, however, since it straightforwardly *is* a topology of uniform convergence in the mathematical sense. The key to comprehending this failure is the algebraic structure of the Lorentz metrics, understood as a subset of the twice-covariant smooth tensor fields on a manifold, which together form a real vector space of uncountable dimension under pointwise addition. My diagnosis is that the force of Geroch's examples comes from their implicit demand to make vector addition and scalar multiplication in this space continuous, a demand that the open topologies do not in general satisfy. Indeed, the topology of uniform convergence on an infinite-dimensional vector space is not necessarily compatible with the space's linear structure.

Next, I articulate a sense in which these demands cannot be jointly met. That is, I show that there is no topology that both satisfies Geroch's desiderata and is compatible with the linear structure of the Lorentz metrics (as a subset of the twice-covariant smooth tensor fields). The heart of

[1] I follow here the terminology of Hawking (1971); Hawking and Ellis (1973). Geroch (1970, 1971) calls the compact-open and open topologies the *coarse* and *fine* topologies, respectively. (Caution: Hawking (1971) defines a class of fine topologies that are not in general homeomorphic to the open topologies.) The open topologies are sometimes called the *Whitney* topologies by mathematicians (Golubitsky and Guillemin 1973).

[2] I assume the fixed spacetime manifold under consideration is non-compact, as most cases of interest are. Otherwise the compact-open and open topologies coincide.

this incompatibility is the fact that every topology compatible with the linear structure of a space must be translation-invariant: roughly put, the neighborhood structure must look the same at every point. I shall argue that, properly interpreted, one of Geroch's examples requires that whether a "perturbation" (in a sense that I make precise) of a certain Lorentz metric is "small" or "large" must depend on that metric in such a way as to make the same perturbation small for some metrics and large for others. But if the neighborhood structure at every point is the same, then a perturbation that is small for one metric must be small for all others. Thus if the twice-covariant smooth tensor fields on a manifold are given any translation-invariant topology, the subspace topology it induces on the Lorentz metrics cannot satisfy Geroch's desiderata.

Fortunately there *is* a way to satisfy Geroch's desiderata if one weakens the requirement of compatibility slightly. To do so, I construct the *global topology* on the Lorentz metrics, which divides them into an (uncountably) infinite number of uniform components, each corresponding to different "asymptotic behavior" (in a sense that I make precise). This topology does makes scalar multiplication continuous, but vector addition will not be so in general. However, through translation the subspace topology on each uniform component induces a topology on the whole space of twice-covariant smooth tensor fields that *does* make vector addition everywhere continuous, hence *does* make the space a topological vector space. In other words, "within" each uniform component the linear operations are continuous. The construction of the global topologies may be of mathematical interest on its own, for it requires almost no details specific to general relativity. Indeed, it can be used to construct a global topology on the cross-sections of any seminorm bundle, i.e., a smoothly varying seminorm on the fibers of some vector bundle. For instance, each of the temporal and spatial metrics with which Newtonian spacetimes are equipped can be understood as smoothly varying seminorms on the tangent bundle of the underlying manifold (Malament 2012, p. 250–4).[3] In addition to these technical properties, the global topologies also have a natural physical interpretation. They encode similarity amongst spacetimes as similarity of values of globally defined

[3]There is only an indirect sense in which the Newtonian spatial metric, as a symmetric $(2,0)$-tensor field, assigns a length to the spacelike component of a vector at a point, but the length it does assign is unique (Malament 2012, Prop. 4.1.1).

quantities that are measurable, in a precise sense, with bounded precision. This contrasts with the analogous characterization of the open topologies, which also encode similarity of values of globally defined observables, but allow for measurement apparatuses of arbitrary precision.

Before concluding with some open questions concerning invariant and geometrical properties of the global topologies, I discuss their application to our understanding of the stability and genericness of global properties of spacetimes. How do results proved using the open topology fare with the global topologies? I show that for virtually any theorem concerning the stability and genericness of *conformally invariant* properties with respect to an open topology, there is a counterpart with respect to the analogous global topology that also holds. In particular, the counterpart of Hawking's theorem, expressing the equivalance of stable causality and the existence of a global time function, holds with respect to the global topology. But it remains an open question whether and which theorems concerning the stability of other properties, such as geodesic (in)completeness, have counterparts that hold in the global topologies.

4.2 Geroch's Complaint

In what follows, M will always denote some smooth, paracompact, Hausdorff manifold of dimension $n \geq 2$, g_{ab} some smooth Lorentz metric on M, and h^{ab} some (inverse of a) smooth Riemannian metric on M, where the indices are abstract. When these metric fields appear in contexts where their indices will not be contracted, I will often relax the notation by dropping the indices. The collections of all such Lorentzian and Riemannian metrics will be denoted $L(M)$ and $\mathcal{R}(M)$, respectively.

4.2.1 Preliminaries

The basic idea for topologies on the Lorentzian g considered in this paper is that one can divide the task of measuring the similarity of each pair g, g' into two parts: first, that of encoding their relevant differences at each point of M into a real number in some systematic way, and second, evaluating the variability of the resulting scalar field in some way over regions of M. The differences between the topologies considered arise ultimately from different choices of how to implement these two tasks. But one feature they all share is the use of the Riemannian h to define a *norm* in every fiber of each tensor bundle $T^r_s M \to M$, i.e, a real function ν_p of the (r, s)-tensors $K^{a_1\cdots a_r}_{b_1\cdots b_s}$ at $p \in M$ with the following properties:

Homogeneous For any $\alpha \in \mathbb{R}$, $\nu_p(\alpha K^{a_1\cdots a_r}_{b_1\cdots b_s}) = |\alpha|\nu_p(K^{a_1\cdots a_r}_{b_1\cdots b_s})$.

Subadditive For any (r, s)-tensor $\hat{K}^{a_1\cdots a_r}_{b_1\cdots b_s}$ at p, $\nu_p(K^{a_1\cdots a_r}_{b_1\cdots b_s} + \hat{K}^{a_1\cdots a_r}_{b_1\cdots b_s}) \leq \nu_p(K^{a_1\cdots a_r}_{b_1\cdots b_s}) + \nu_p(\hat{K}^{a_1\cdots a_r}_{b_1\cdots b_s})$

Separating If $\nu_p(K^{a_1\cdots a_r}_{b_1\cdots b_s}) = 0$, then $K^{a_1\cdots a_r}_{b_1\cdots b_s}$ is the zero tensor.

It will turn out while the choice of norm will not matter substantively to the results in the sequel, the Frobenius norm offers certain computational advantages.[4]

Definition 4.2.1. For any (r, s)-tensor $K^{a_1\cdots a_r}_{b_1\cdots b_s}$ at $p \in M$, define the *h-fiber norm* of $K^{a_1\cdots a_r}_{b_1\cdots b_s}$ as

$$|K^{a_1\cdots a_r}_{b_1\cdots b_s}|_h = |K^{a_1\cdots a_r}_{b_1\cdots b_s} K^{c_1\cdots c_r}_{d_1\cdots d_s} h_{a_1 c_1} \cdots h_{a_r c_r} h^{b_1 d_1} \cdots h^{b_s d_s}|^{1/2}, \tag{4.1}$$

for $r, s > 0$, and with no copies of h and its inverse when, respectively, $r = 0$ and $s = 0$.

Note that the h-fiber norm of a scalar is just the absolute value of that scalar, hence independent of the choice of h. Now, because h is positive definite, one can choose a (co)basis for the (co)tangent

[4]This choice is also the unique l^p norm on a tensor space that allows one to construct an inner product from the norm according to the polarization identity, but this fact will not play a role in this paper.

space at p in which the matrix representation of $h_{|p}$ is the identity. It then follows immediately from the definition of the h-fiber norm that:

Proposition 4.2.2. *For every $p \in M$ and each non-negative r and s, the h-fiber norm is a norm on the (r, s)-tensors at p, namely the Frobenius norm with respect to the basis in which h is the identity.*

An example that will be central in the sequel will be the h-fiber norm of the difference of two Lorentz metrics g and g':

$$|g - g'|_h = [(g_{ab} - g'_{ab})(g_{cd} - g'_{cd})h^{ac}h^{bd}]^{1/2}$$

at each $p \in M$. This (and indeed any h-fiber norm) defines a scalar field on M whose evaluation is the concern of the second aforementioned task in measuring the similarity between g and g'. All the topologies considered also share a common approach to this task.

Definition 4.2.3. Let $|\cdot|_h$ be some h-fiber norm on M and let $S \subseteq M$ be nonempty. Then for any (r, s)-tensor field $K^{a_1\cdots a_r}_{b_1\cdots b_s}$, define its (h, S)-*uniform norm* as

$$\|K^{a_1\cdots a_r}_{b_1\cdots b_s}\|_{h,S} = \sup_S |K^{a_1\cdots a_r}_{b_1\cdots b_s}|_h. \tag{4.2}$$

More generally, one could consider an (h, S)-L^p norm,

$$\left(\int_S \left(|K^{a_1\cdots a_r}_{b_1\cdots b_s}|_h\right)^p dV\right)^{1/p},$$

where V is the volume measure determined by h,[5] but I will restrict attention to the uniform ($p = \infty$) case here.[6]

[5]The case $p = 2$ is used in applications to the Cauchy problem in general relativity (Hawking and Ellis 1973, Ch. 7.4).

[6]Technically, the $p = \infty$ case uses the essential supremum, but this is equal to the supremum for continuous functions.

It will be absolutely central to the sequel that, in general, the (h, S)-uniform norm for some tensor fields is infinite.

Proposition 4.2.4. *For every nonnegative integer r and s, each* (h, S)*-uniform norm is an* extended norm *on the vector space of continuous* (r, s)*-tensor fields on M, i.e., it is a homogeneous, subadditive, and separating function whose range is* $[0, \infty]$.

Proof. Immediate from proposition 4.2.2 and the subadditivity of the supremum. □

4.2.2 The Compact-Open and Open Topologies

Using the above framework, all topologies under consideration render two tensor fields similar, roughly speaking, when their values and partial derivatives up to kth order are sufficiently similar at points of some $S \subseteq M$. This can be captured for smooth (r, s)-tensor fields K through an analog to the ϵ-ball notion of similarity familiar from metric spaces, with the relevant distance function defined from an (h, S)-norm:

$$B^k(K, \epsilon; h, S) = \{K' \in \Gamma^r_s : \forall j \leq k, \|\nabla^{(k)}K - \nabla^{(k)}K'\|_{h,S} < \epsilon\}, \tag{4.3}$$

where Γ^r_s denotes the collections of smooth (r, s)-tensors on M, $\nabla^{(k)}K$ the field $\nabla_{c_1} \cdots \nabla_{c_k} K^{a_1 \cdots a_r}_{b_1 \cdots b_s}$, and ∇ the Levi-Civita derivative operator compatible with h. The topologies that will be under consideration here are all determined by a suitable collection of such ϵ-balls as a *subbasis*, i.e., a topology's open sets will be generated through finite intersection and arbitrary unions of these ϵ-balls.[7] In particular, a C^k topology on Γ^r_s generated in this way is determined by the appropriate choice of quadruples $(K, \epsilon; h, S)$. Letting $\mathcal{P}(M)$ denote the power set of M and $\mathcal{R}(M)$ the smooth Riemannian metrics on M, these collections may be called *definitional subbases*:

Definition 4.2.5. A collection of quadruples $\Xi \subseteq \Gamma^r_s \times (0, \infty) \times \mathcal{R}(M) \times \mathcal{P}(M)$ is a *definitional*

[7] In some circumstances they form a *basis*, which generates the topology only through finite intersection.

subbasis for a topology on Γ^r_s when $\bigcup \pi_1[\Xi] = \Gamma^r_s$, where $\pi_1 : (K, \epsilon; h, S) \mapsto K$ is the projection onto the first component.

Indeed, the C^k compact-open and open topologies can be defined this way. Letting C denote the collection of all compact subsets of M,

Definition 4.2.6. The definitional subbasis for the C^k *compact-open* topology on Γ^r_s is $\Xi_{CO} = \Gamma^r_s \times (0, \infty) \times \mathcal{R}(M) \times C$.

Definition 4.2.7. The definitional subbasis for the C^k *open* topology on Γ^r_s is $\Xi_O = \Gamma^r_s \times (0, \infty) \times \mathcal{R}(M) \times \{M\}$.[8]

The C^∞ compact-open topology on Γ^r_s is generated from the union of the C^k compact-open topologies for all k, and similarly for the open topologies. Also, if M is compact, the C^k compact-open topology coincides with the C^k open topology. Otherwise, the C^k compact-open topology is strictly coarser than the C^k open topology, i.e, the system of open sets of the former is strictly contained in the system of the latter. For $j < k$, the C^j compact-open topology is also strictly coarser than the C^k compact-open topology, and similarly for the open topologies. Finally, the compact-open and open topologies on the smooth Lorentz metrics on M, denoted $L(M)$, are just the relevant subspace topologies on Γ^0_2.

4.2.3 Geroch's Examples

Geroch (1970, 1971) uses three examples—two sequences and a single one-parameter family of Lorentz metrics—to illustrate his claim that each of the C^k compact-open topologies is too coarse while each of the C^k open topologies is too fine. Of course, a topology can only be too coarse or fine for a particular purpose. There are good reasons to believe that there is no canonical topology

[8]This topology is called *open* since replacing $\{M\}$ with the collection of all open subsets of M generates the same topology.

on the space of smooth Lorentz metrics, so particular choices of topology must be well adapted to the constraints on a notion of similarity relevant for the context at hand (ch. 2). I shall argue that one can thus interpret these examples as illustrating the inadequacy of these topologies for encoding a notion of *global* similarity.

Let t, x, y, z be scalar coordinate fields on $\mathbb{R}^4$. Geroch's two sequences of metrics on this manifold are

$$\overset{m}{g}_{ab} = \left(1 + \frac{m}{1 + (x - m)^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z), \tag{4.4}$$

$$\overset{m}{g}{}'_{ab} = \left(1 + \frac{1}{m^2 + x^2 + y^2 + z^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z), \tag{4.5}$$

where d is the exterior derivative.[9] His one-parameter family is simply

$$\{\lambda g_{ab} : \lambda > 0\}, \tag{4.6}$$

for some arbitrary fixed Lorentz metric g_{ab} on any non-compact M. Regarding eq. 4.4, Geroch writes that "The 'bump' in the metrics becomes larger as it recedes to infinity," but the "sequence *does* approach Minkowski space in the [compact-open] topology (because the metrics become Minkowskian in every compact set)." However, "[i]ntuitively, we would not think of this sequence as approaching Minkowski space" (Geroch 1971, p. 71).

On the other hand, with respect to any open topology the sequence defined by eq. 4.5 does not converge to the Minkowski metric even though "[t]he 'bump' in the metrics remains centered at the origin and decreases in amplitude" to zero (Geroch 1970, p. 280). Perhaps more surprisingly, the family defined in eq. 4.6 does not trace out a continuous curve in the open topology. In fact, it is everywhere discontinuous since the induced subspace topology on this set is discrete (Geroch 1971, p. 71). This is particularly striking since each member of this family can represent precisely

[9]The formula for the first term of eq. 4.4 is garbled in Geroch (1971), but appears without error in Geroch (1970, p. 280).

the same physical spacetimes.[10]

But the compact-open topologies *do* determine that the sequence defined in eq. 4.5 converges to Minkowski spacetime and that the family defined in eq. 4.6 *is* continuous, while according to the open topologies the sequence defined in eq. 4.4 does *not* converge. Thus the compact-open and open topologies are obverses on these examples, each intuitively ruling wrongly on the ones the other rules rightly. The former is too coarse because it permits too many sequences to converge, while the latter is too fine because it permits too few sequences to converge (and too few continuous families).

4.2.4 Diagnosis and Desiderata

But what, more precisely, is undergirding these intuitions? In the first place, as Geroch alludes, the compact-open topology coincides with the topology of compact convergence—that is, a sequence of metrics $\overset{m}{g} \to g$ on M just when its elements converge uniformly on each compact $C \subseteq M$ (Willard 1970, p. 284, Theorem 43.6a). This is similar to the sense in which the k^{th}-order Taylor expansion of a real function such as $\sin(x)$ converges to it as $k \to \infty$. (For any particular finite-order expansion, one can find a sufficiently large x such that the expansion, evaluated at x, differs from $\sin(x)$ by as much as one wishes. But if one fixes some compact neighborhood region of $\mathbb{R}$, then the Taylor series converges uniformly on that neighborhood.) It seems, then, that Geroch is searching for a topology that compares two metrics across the *whole* (non-compact) spacetime, not just on compacta. This would generate the analog to *uniform convergence* for real functions.

Yet the open topologies *are* topologies of uniform convergence in the usual way the latter are defined.[11] In the present context, however, where the space of functions to topologize are smooth

[10] One might thus object there is no problem here, since after taking the quotient of the space of Lorentz metrics by the isometry relation the family becomes a point. Such a strategy does not help with the non-convergence of eq. 4.5, however, since no two elements of that sequence are isometric.

[11] This is for the same reason as the fact that the compact-open topology coincides with the topology of uniform convergence on compacta. (See the previous footnote.)

sections of a vector bundle over a non-compact manifold, Geroch's examples are evidence that the criteria for convergence and continuity that the open topologies require is much too strong. The following propositions (respectively adapted from propositions 2.3.1 and 2.3.2) make this precise.

Proposition 4.2.8. *Let $K, \{\overset{m}{K}\}_{m\in\mathbb{N}}$ be tensor fields on M. Then $\overset{m}{K} \to K$ in the C^k open topology iff there is a compact $C \subseteq M$ such that:*

1. *for sufficiently large m, $\overset{m}{K}_{|M-C} = K_{|M-C}$; and*
2. *$\overset{m}{K}_{|\,\mathrm{int}(C)} \to K_{|\,\mathrm{int}(C)}$ in the C^k open topology.*

Proposition 4.2.9. *A family $\{\overset{\lambda}{K}\}_{\lambda\in I}$ of tensor fields on M is continuous in the C^k open topology only if for every $\lambda_1, \lambda_2 \in I$, there is some compact $C \subseteq M$ such that $\overset{\lambda_1}{K}_{|M-C} = \overset{\lambda_2}{K}_{|M-C}$.*

In light of these propositions, the open topologies would in a sense be appropriate if the sections under consideration had only compact support. But this restriction is clearly inappropriate for Lorentz metrics (as they are everywhere non-degenerate).[12]

One way to capture the essential insight of Geroch's latter two examples is that in addition to requiring a topology that is "sensitive" to differences across all of M—that is, demanding that the last component of each member of its defining subbasis should be M—one should also require that the topology respect the linear structure of the Lorentz metrics. Each class of (r, s)-tensor fields (indeed, any vector bundle) on M forms a real vector space, with vector addition defined pointwise on M and scalar multiplication acting fiberwise.[13] As a subset of the smooth $(0, 2)$-tensor fields, the Lorentz metrics inherit this algebraic structure.[14] Demanding the continuity of the one-

[12] Hawking has expressed doubt that another class of topologies, which he calls the fine topologies and which requires two metrics (and their derivatives to order k) to be equal outside of a compact set to be similar, is as "physical" as the open topology "since to establish that two metrics actually coincide outside some compact set would require an exact measurement which is not physically possible" (Hawking 1971, p. 397). But this is exactly what is required to establish that a sequence of metrics converges in one of the open topologies, so if one accepts this line of reasoning, one must also reject the open topologies on the same grounds.

[13] They also form a module over the ring of continuous real scalar fields on M, but this fact will play no role in what follows.

[14] The Lorentz metrics themselves are not a vector space, as they are not closed under scalar multiplication (consider negative scalars) or vector addition. Nevertheless, what's important is that these operations be continuous on the domain of Lorentz metrics on which they *are* defined.

parameter family exhibited in eq. 4.6 matches exactly the requirement that scalar multiplication be continuous; the sequence defined in eq. 4.5 suggests the requirement that vector addition be continuous as well.[15]

There is no paradox that the open topologies do not make the vector space operations continuous, for the topology of uniform convergence for a vector space need not be compatible with its linear structure, i.e., make it into a *topological* vector space, as illustrated through the following proposition (adapted from (Schechter 1997, Theorem 26.29, p. 705)):

Proposition 4.2.10. *A topology on a vector space V makes it into a (locally convex) topological vector space if and only if the topology*

1. *is translation-invariant, i.e., each translation map $v \mapsto u + v$ for a fixed $u \in V$ is a homeomorphism of V, and*
2. *has a 0-neighborhood base consisting of ϵ-balls generated from some collection of seminorms, i.e., a collection of homogeneous and subadditive (but not necessarily separating) functions.*

Both the open and compact-open topologies are translation-invariant. But the open topologies have 0-neighborhoods bases generated from the collection of the (h, M)-uniform norms, which are *extended* norms (hence extended seminorms)—they assign infinite length to some tensor fields. Thus they are not compatible with the linear structure of Γ^r_s for any $r, s > 0$. The compact-open topologies, meanwhile, have 0-neighborhood bases generated from the (h, C)-uniform norms for a compact C, which are in fact norms (hence seminorms) because the h-fiber norms, as continuous real functions on M, are always bounded on compacta.

[15] At least for "small" vectors. See the discussion at the end of this section motivating the C^k global topologies constructed in the next section.

4.3 The Global Topologies

4.3.1 Motivation: The Problem with Translation-Invariance

Intuitively, a sequence should converge just when it eventually becomes arbitrarily similar to a limit point. The open topologies determine that eq. 4.5 fails that criterion essentially because they allow any choice of Riemannian metric h in their definitional subbasis Ξ_{CO}. When M is non-compact, there will always be some h-fiber norm that "blows up" sufficiently rapidly "at infinity." More precisely, given any two tensors K and K' of the same rank that differ outside of each compact subset of M, there will always be some (h, M)-uniform norm of their difference that will be infinite. This explains why the open topology has the convergence condition that it does: only when K and K' differ on a compact set will each (h, M)-uniform norm assign a finite value to their difference (by the extreme value theorem).

Now, Geroch characterizes the sequence defined by eq. 4.5 as one where the "bump" flattens further and further as the sequence progresses. The open topologies do not capture this since there are some h-fiber norms for which $|\overset{m}{g} - \eta|_h$ does not flatten but is in fact unbounded on $\mathbb{R}^4$. (Here η is the Minkowski metric.) A natural response, which (Geroch 1970, p. 288) has suggested, is to restrict the choices of h to be used in the definitional subbasis. (Such a restriction should also not be *ad hoc*. One could easily fix a *single* choice of h, but this would be completely arbitrary. Moreover, it would not be an *invariant* topology in the sense articulated below.)

But before attempting such a response, it will be helpful to reflect on why Geroch's claim that eq. 4.5 should converge seems plausible in the first place. The difference between the Minkowksi metric and the m^{th} term in the sequence is

$$\overset{m}{g}'_{ab} - \eta_{ab} = (m^2 + x^2 + y^2 + z^2)^{-1}(d_a t)(d_b t). \tag{4.7}$$

This difference will be everywhere small only according to an h-fiber norm that is well adapted to the chosen coordinates, which are in turn well adapted to the limit point, the Minkowskian metric. That is, if

$$h^{ab} = \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b, \tag{4.8}$$

then $|\overset{m}{g}' - \eta|_h = (m^2 + x^2 + y^2 + z^2)^{-1}$ and $\|\overset{m}{g}' - \eta\|_{h,M} = m^{-2}$, which vanishes as $m \to \infty$.

But if the example were modified slightly the result would be different. Consider the sequence

$$\overset{m}{g}''_{ab} = \left(e^t + \frac{1}{m^2 + x^2 + y^2 + z^2}\right)(d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z) \tag{4.9}$$

on $\mathbb{R}^4$ with plausible limit point

$$\eta'_{ab} = e^t (d_a t)(d_b t) - (d_a x)(d_b x) - (d_a y)(d_b y) - (d_a z)(d_b z). \tag{4.10}$$

Their difference is the same tensor as that for the analogous case introduced by Geroch, eq. 4.7. But the (inverse) Riemannian metric well adapted to η' is

$$h'^{ab} = e^{-t}\left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b, \tag{4.11}$$

which yields that $|\overset{m}{g}'' - \eta'|_{h'} = e^{-t}/(m^2 + x^2 + y^2 + z^2)$, hence is unbounded. Thus even though $\overset{m}{g}' - \eta = \overset{m}{g}'' - \eta'$, the former sequence is taken to converge while the latter is not.

This observation poses a dilemma for Geroch's suggestion that one should try to find a topology intermediate between the compact-open and open topologies by means of some definitional subbasis $(\Gamma^0_2, \mathbb{R}_+, \mathcal{R}', \{M\})$, where $\mathcal{R}' \subset \mathcal{R}(M)$, that respects the linear structure of Γ^0_2. By proposition 4.2.10, such a topology must be translation-invariant. But any translation-invariant topology on Γ^0_2 must arrive at the same answer as to whether the sequences $\overset{m}{g}'$ and $\overset{m}{g}''$ converge. This is in conflict with the above conclusion that the $\overset{m}{g}'$ but not the $\overset{m}{g}''$ should converge.

I would like to suggest that one can still satisfy a weakened form of Geroch's desiderata by giving up translation invariance and a topology fully compatible with linear structure. As I adumbrated in the introduction, one can still retain a weakened form of these features, in which the Lorentz metrics are partitioned into components, each of which is itself a subset of a topological vector space. In other words, "within" each of these components, scalar multiplication and vector addition are continuous, so Geroch's desiderata are satisfied for "small" but not "large" perturbations—those that lie in the same component as as the unperturbed metric. This is made precise in the next subsection.

4.3.2 The Definition and Position of the Global Topologies

First, one can extend the notion of a fiber norm in the following way. Given *any* non-degenerate[16] metric f and any (r, s)-tensor field K on M, define the *f-fiber norm of K* to be

$$|K^{a_1\cdots a_r}_{b_1\cdots b_s}|_f = |K^{a_1\cdots a_r}_{b_1\cdots b_s} K^{c_1\cdots c_r}_{d_1\cdots d_s} f_{a_1c_1}\cdots f_{a_rc_r} f^{b_1d_1}\cdots f^{b_sd_s}|^{1/2}, \tag{4.12}$$

and the *(f, S)-uniform norm of K* for any $S \subseteq M$ to be $\|K\|_{f,S} = \sup_S |K|_f$. Note that if K is a scalar field then its fiber norm (hence the uniform norm over any set as well) is independent of the choice of f.

Next, for any $p \in M$, define $\ker_{r,s}(f_{|p}) = \{K \in \Gamma^r_s : (|K|_f)_{|p} = 0\}$ and $\ker_{r,s}(f_{|S}) = \bigcup_{p\in S} \ker_{r,s}(f_{|p})$. These sets, respectively called the kernels of f at p and S, are empty for positive and negative definite metrics but non-empty for metrics of indefinite signature. Finally, let

$$\mathcal{K}^r_s(f, S) = \mathrm{span}(\{K \in \Gamma^r_s - \ker_{r,s}(f_{|S}) : \|K\|_{f,S} < \infty\}). \tag{4.13}$$

This set consists of all the linear combinations of (r, s)-tensor fields on M that are nowhere in the

[16]A metric is non-degenerate when it has an inverse at every point. With a bit more work the present framework can very likely be extended to degenerate metrics as well.

kernel of f at S and are bounded with respect to the (f, S)-uniform norm. Finally:

Definition 4.3.1. Two metrics f, f' are *norm-equivalent on* S, denoted $f_{|S} \asymp f'_{|S}$, when $\mathcal{K}^r_s(f, S) = \mathcal{K}^r_s(f', S)$ for all (r, s).

Metrics that are norm-equivalent on S agree regarding which tensor fields are bounded on S. By definition, norm-equivalence on S is an equivalence relation, hence partitions the metrics on M into equivalence classes $[f_{|S}] = \{f' : f_{|S} \asymp f'_{|S}\}$.[17] (When $S = M$, I shall omit reference to it, writing $[f] = \{f' : f_{|M} \asymp f'_{|M}\}$ and calling them (simply) the metrics norm-equivalent to f.)

Of course, to determine whether two metrics f, f' are norm-equivalent on S, one does not need to check that the collections of tensor fields they determine by eq. 4.13 for each r, s are respectively equal. It is enough to do so on appropriate subcollections:

Proposition 4.3.2. *Let $\mathcal{BK}^r_s(f, S) \subset \mathcal{K}^r_s(f, S)$ denote a collection of tensor fields that spans Γ^r_s. Then two metrics f, f' are norm-equivalent on S if and only if $\mathcal{BK}^r_{s'}(f, S) = \mathcal{BK}^r_{s'}(f', S)$ or $\mathcal{BK}^{r'}_s(f, S) = \mathcal{BK}^{r'}_s(f', S)$ for some $r, s > 0$.*

Thus the values that the (f, S)-uniform norm assigns to vector fields essentially determines which norm-equivalence class it belongs to. In particular, it is their asymptotic behavior that norm-equivalence encodes. One can make this precise by adopting the following definition:

Definition 4.3.3. Let K, K' be two smooth tensor fields and f be a metric. K and K' are *mutually bounding to order k with respect to f on $S \subseteq M$* when there are constants $c, c' > 0$ such that $c|\nabla^{(j)}K|_f \leq |\nabla^{(j)}K'|_f \leq c'|\nabla^{(j)}K|_f$ everywhere on S for each $j \leq k$, where ∇ is the Levi-Civita derivative operator compatible with f.[18]

[17] One may also partition the metrics into classes that are projectively and affinely equivalent, each of which take on particularly simple forms. Further, norm-equivalence is refined by projective equivalence, which in turn is refined by affine equivalence. Nevertheless, I conjecture that replacing norm-equivalence by one of the latter two in the definition of the gloabl topologies (eq. 4.14) yields the same topology, and that no equivalence that strictly refines norm equivalence does so.

[18] Sometimes the notation $K' \in \Theta(K)$ is used for an analogous notion in computer science, especially the analysis of algorithms, although its origins are in analytic number theory.

Note that mutual boundedness is a symmetric relation, for if there are constants $c, c' > 0$ such that $c|\nabla^{(j)}K|_f \leq |\nabla^{(j)}K'|_f \leq c'|\nabla^{(j)}K|_f$ everywhere on some S, then $c'^{-1}|\nabla^{(j)}K'|_f \leq |\nabla^{(j)}K|_f \leq c^{-1}|\nabla^{(j)}K|_f$ everywhere on S. There is a close connection between norm-equivalence and the mutual boundedness of metrics:[19]

Proposition 4.3.4. *Two metrics f, f' are norm-equivalent on some $S \subseteq M$ if and only if they are mutually bounding to order* 0 *on S with respect to every metric f''.*

Proof. Suppose that $f_{|S} \asymp f'_{|S}$ and let f'' be arbitrary. Because f'' and f are metrics, the field $\Omega = 1/|f''|_f$ is well-defined on S. Since then $|\Omega f''|_f = 1$ on S, $\Omega f'' \in \mathcal{K}_2^0(f, S)$, hence by assumption $\Omega f'' \in \mathcal{K}_2^0(f', S)$. By definition $\|\Omega f''\|_{f'} \geq |\Omega f''|_{f'} = \Omega |f''|_{f'}$, so $|f''|_{f'} \leq \|\Omega f''\|_{f'} |f''|_f$. A similar argument returns that $|f''|_f \leq \|\Omega' f''\|_f |f''|_{f'}$ for $\Omega' = 1/|f''|_{f'}$. Combining the two then yields $(\|\Omega' f''\|_f)^{-1}|f''|_f \leq |f''|_{f'} \leq \|\Omega f''\|_{f'} |f''|_f$. Finally, note that $|f''|_f = |f|_{f''}$ and $|f''|_{f'} = |f'|_{f''}$, recalling that f'' was arbitrary.

Conversely, suppose that f, f' are mutually bounding to order 0 with respect to every metric f''. Then by definition there are constants $c, c' > 0$ such that $c|f''|_f \leq |f''|_{f'} \leq c'|f''|_f$. Hence if $f'' \in \mathcal{K}_2^0(f, S)$, then $f'' \in \mathcal{K}_2^0(f', S)$ and vice versa. A similar argument yields the analogous conclusion for the inverses of f. Since these together are tensor fields whose basis components in each index span the corresponding (co-)tangent space at every point of S, proposition 4.3.2 and the definition of norm-equivalence yield that $f_{|S} \asymp f'_{|S}$. □

Two tensor fields that are mutually bounding to order k with respect to some metric need not be mutually bounding to any higher order. (For instance, consider any field with support on $\mathbb{R}^2$, and the conformally related field obtained through the factor $1 + a\sin(x^2)$ for $0 < |a| < 1$.) Thus there is a sense in which the above proposition is tight.[20]

[19]This has been anticipated in the notation for norm-equivalence, which has been adapted from Hardy and Wright (1979, §1.6)

[20]It does suggest, though, investigating equivalence relations defined through analogs of proposition 4.3.4 formed by replacing "0" with some positive integer. Would such a relation have any interesting physical significance?

We are now ready to describe the C^k global topologies, using the definitional subbasis[21]

$$\Xi_G = \{(g, \epsilon; h, S) \in L(M) \times (0, \infty) \times \mathcal{R}(M) \times \{M\} : h \asymp g\}. \tag{4.14}$$

In fact, the neighborhoods $B^k(g, \epsilon; h, M)$ defined by Ξ_G form a (local) basis for the global C^k topology on $L(M)$. Showing this will be crucial in proving many facts about the global topology, but it requires a pair of preliminary propositions.

Proposition 4.3.5. *If $g' \in B^k(g, \epsilon; h, M)$ with $(g, \epsilon; h, M) \in \Xi_G$, then $h \asymp g'$.*

Proof. Let g' be given, and recall from the proof of proposition 4.2.2 that, at any $p \in M$, the h-fiber norm $(|g - g'|_h)_{|p} = \|(g_{\alpha\beta})_{|p} - (g'_{\alpha\beta})_{|p}\|_F$ is just the Frobenius norm with respect to the basis in which the matrix representation of $h_{|p}$ is the identity. Thus

$$\begin{aligned}(|g'|_h)_{|p} &= \|(g'_{\alpha\beta})_{|p}\|_F \\ &\leq \|(g_{\alpha\beta})_{|p}\|_F + \|(g'_{\alpha\beta})_{|p} - (g_{\alpha\beta})_{|p}\|_F = (|g|_h)_{|p} + (|g - g'|_h)_{|p}\end{aligned}$$

by the triangle inequality. Since $\|g\|_h < \infty$ and $\|g - g'\|_h < \epsilon$ by hypothesis, we have that $h \asymp g'$. □

Next we need some information about the geometric structure of the Riemannian metrics in each $[f]$.

Proposition 4.3.6. *For any smooth, non-degenerate metric f, the set $[f] \cap \mathcal{R}(M)$ is a (blunt) convex cone, i.e., if $h_{ab}, h'_{ab} \in [f]$ and $c, c' > 0$, then $ch_{ab} + c'h'_{ab} \in [f]$.*

Proof. It suffices to show that $h_{ab} + h'_{ab} \in [f]$ for any $h_{ab}, h'_{ab} \in [f]$. Recall again from the proof of proposition 4.2.2 that, at any $p \in M$, we can express $(|g|_{h+h'})_{|p} = \|G(h + h', p)\|_F$, where $G(h + h', p)$ is the matrix representing the components of $g_{|p}$ in the basis in which the matrix representation of

[21] Cf. the construction of the Banach space $\mathcal{B}_k$ in Lerner and Porter (1974), which however only yields neighborhoods of the Minkowski metric.

$(h_{ab} + h'_{ab})_{|p}$ is the identity. Note that for any real matrix A of rank r, $||A||_F \leq \text{tr}(\sqrt{A^T A}) \leq \sqrt{r}||A||_F$, where A^T is the transpose of A. So

$$
\begin{aligned}
(|g|_{h+h'})_{|p} &\leq \text{tr}(G(h+h',p)) = ((h^{ab} + h'^{ab})g_{ab})_{|p} \\
&\leq 2\max\{(h^{ab}g_{ab})_{|p}, (h'^{ab}g_{ab})_{|p}\} = 2\max\{\text{tr}(G(h,p)), \text{tr}(G(h',p))\} \\
&\leq 2\sqrt{n}\max\{||G(h,p)||_F, ||G(h',p)||_F\} = 2\sqrt{n}\max\{(|g|_h)_{|p}, (|g|_{h'})_{|p}\}
\end{aligned}
$$

using in the second line the fact that $(\frac{1}{2}(h^{ab} + h'^{ab})\alpha_{ab})_{|p} \leq \max\{(h^{ab}\alpha_{ab})_{|p}, (h'^{ab}\alpha_{ab})_{|p}\}$ for any α_{ab} at p. The rightmost-hand side is bounded by hypothesis, so since p was arbitrary, the conclusion follows. □

Proposition 4.3.7. *For a given g, the sets $B^k(g, \epsilon; h, M)$, ranging over $\epsilon > 0$ and $h \in [g]$, form a local basis at g. Ranging over g as well, they form a basis.*

Proof. To show that the sets $B^k(g, \epsilon; h, M)$, with $(g, \epsilon; h, M) \in \Xi_G$, form a basis, it suffices to show that if $g'' \in B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M)$ for arbitrary $B^k(g, \epsilon; h, M)$ and $B^k(g', \epsilon'; h', M)$, then there is some other $B^k(g'', \epsilon''; h'', M)$ such that $B^k(g'', \epsilon''; h'', M) \subseteq B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M)$. So let some such $B^k(g, \epsilon; h, M)$ and $B^k(g', \epsilon'; h', M)$ be given and pick any g'' in their intersection. Put $h''^{ab} = h^{ab} + h'^{ab}$ and

$$\epsilon'' = \min\{\epsilon - ||g - g''||_{h,M}, \epsilon' - ||g' - g''||_{h',M}\}$$

Note that $g'' \in B^k(g, \epsilon; h, M)$ by hypothesis, so proposition 4.3.5 entails that $h \asymp g''$. Similar reasoning gives that $h' \asymp g''$, so by proposition 4.3.6, $h^{ab} + h'^{ab} \in [g'']$. Thus $B^k(g'', \epsilon''; h'', M)$ is well-defined.

Now consider any $g''' \in B^k(g'', \epsilon''; h'', M)$. By definition, $\|g'' - g'''\|_{h+h'} < \epsilon - \|g - g''\|_h$, hence

$$\begin{aligned}\epsilon &> \|g'' - g'''\|_{h+h',M} + \|g - g''\|_{h,M} > \|g'' - g'''\|_{h,M} + \|g - g''\|_{h,M} \\ &\geq \sup_M \left[|g'' - g'''|_h + |g - g''|_h\right] \geq \|g - g'''\|_{h,M},\end{aligned}$$

using the fact that $\|g'' - g'''\|_{h+h',M} > \|g'' - g'''\|_{h,M}$ in the first line and the triangle inequality for the supremum and then the h-fiber norm in the second. Therefore $g''' \in B^k(g, \epsilon; h, M)$ and a similar argument shows that $g''' \in B^k(g', \epsilon'; h', M)$. Hence $B^k(g'', \epsilon''; h'', M) \subseteq B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M)$ since g''' was arbitrary.

To show that the sets $B^k(g, \epsilon; h, M)$, ranging only over ϵ and $h \in [g]$, form a local basis at g, it suffices to show that, for every basic open neighborhood $B^k(g', \epsilon'; h', M) \ni g$ such that $h' \asymp g'$, there is some $B^k(g, \epsilon; h, M) \subseteq B^k(g', \epsilon'; h, M)$ such that $h \asymp g$. So consider some particular such $B^k(g', \epsilon'; h', M)$ and put $h = h'$ and $\epsilon = \epsilon' - \|g - g'\|_{h',M}$. By proposition 4.3.5, $h \asymp g'$. Now, for an arbitrary $g'' \in B^k(g, \epsilon; h, M)$, $\|g - g''\|_{h,M} < \epsilon' - \|g - g'\|_{h',M}$, so applying similar reasoning as above yields that $\|g - g''\|_{h',M} < \epsilon'$, i.e., $g'' \in B^k(g', \epsilon'; h', M)$, hence $B^k(g, \epsilon; h, M) \subseteq B^k(g', \epsilon'; h', M)$. □

Now, the definitional subbases for the compact-open and open topologies, Ξ_{CO} and Ξ_O, respectively, are plainly closely related—the only difference between them is in their fourth component. Despite initial appearances, there is a definitional subbasis Ξ'_{CO} generating the compact-open topology that is plainly closely related to Ξ_G, also differing only in its fourth component.

Proposition 4.3.8. *The definitional subbasis* $\Xi'_{CO} = \{(g, \epsilon; h, C) : g \in L(M), \epsilon \in (0, \infty), C \in \mathcal{C}, h_{|C} \asymp g_{|C}\}$ *generates the compact-open topology; in fact,* $\Xi'_{CO} = \Xi_{CO}$.

Proof. Let any $g \in L(M)$, $C \in \mathcal{C}$, $K \in \Gamma^r_s$, and $h \in \mathcal{R}(M)$ be given. Since C is compact, the continuous scalar fields $|K|_f$ and $|K|_h$ are both bounded on C (Willard 1970, Theorem 17.13, p. 123), hence $\mathcal{K}^r_s(f, C) = \Gamma^r_s = \mathcal{K}^r_s(h, C)$ and by definition $\mathcal{R}(M) \subseteq [g_{|C}]$. □

Thus the global topologies can be understood as a natural variation on the compact-open topologies that controls similarity across M in just the same way as the open topologies.

4.3.3 Properties of the Global Topologies

Note that the global topologies are indeed topologies for $L(M)$ since $[g] \cap \mathcal{R}(M)$ is non-empty for each g. In particular, for any $h \in \mathcal{R}(M)$ and $g \in L(M)$, $h/|g|_h \asymp g$. Further, they are *diffeomorphism-invariant* (Geroch 1970, p. 281–2), in the sense that for any element ψ of the diffeomorphism group of M, the pushforward ψ_* acts as a homeomorphism on the Lorentz metrics equipped with a global topology. To see this, first note that

$$\begin{aligned}|\psi^* g|_{\psi^* h} &= [\psi^*(h^{am})\psi^*(h^{bn})\psi^*(g_{ab})\psi^*(g_{mn})]^{1/2} \\ &= [\psi^*(h^{am}h^{bn}g_{ab}g_{mn})]^{1/2} = [h^{am}h^{bn}g_{ab}g_{mn}]^{1/2} = |g|_h,\end{aligned}$$

so $h \asymp g$ if and only if $\psi^* h \asymp \psi^* g$. Using similar reasoning, $\psi^*[B^k(g, \epsilon; h, M)] = B^k(\psi^* g, \epsilon; \psi^* h, M)$. Thus the preimage of every basis element is open, meaning ψ_* is continuous, hence acts as a homeomorphism.

The C^0 global topology rules in the "right" way on Geroch's three examples. First, the sequence defined by eq. 4.4 does not converge to the Minkowski metric η. To see why, consider

$$h^{ab} = \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b + \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial x}\right)^b + \left(\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial y}\right)^b + \left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial z}\right)^b,$$

noting that $\|\eta\|_h = 2$, so $h \asymp \eta$. This choice yields that

$$\|\eta - \overset{m}{g}\|_h = \sup_M \frac{m}{1 + (x - m)^2} = m,$$

which cannot be as small as one wishes for sufficiently large m. Hence $\overset{m}{g}$ does not converge to the

Minkowski metric.

Second, the sequence defined by eq. 4.5 *does* converge to the Minkowski metric. To see why, let

$$h^{ab} = \alpha\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^b + \beta_1\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial x}\right)^b + \beta_2\left(\frac{\partial}{\partial y}\right)^a\left(\frac{\partial}{\partial y}\right)^b + \beta_3\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial z}\right)^b + \cdots \tag{4.15}$$

be any Riemannian metric in $[\eta]$ in coordinates determined by η. (Note that h need not be diagonal in these coordinates.) Since $\eta \in \mathcal{K}_0^2(\eta, M)$, by definition $\eta \in \mathcal{K}_0^2(h, M)$, hence

$$\infty > ||\eta||_{h,M} = \sup_M (h^{ab}h^{cd}\eta_{ac}\eta_{bd})^{1/2} = \sup_M(\alpha^2 + \beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2} > \sup_M |\alpha|. \tag{4.16}$$

Consequently

$$||\eta - \overset{m}{g}'||_{h,M} = \sup_M \frac{|\alpha|}{m^2 + x^2 + y^2 + z^2} \leq \frac{\sup_M |\alpha|}{m^2},$$

whose right-hand side is finite and can be made as small as one wishes by choosing a sufficiently large m. Hence $\overset{m}{g}' \to \eta$ as $m \to \infty$. Third, and finally, by similar calculations one can show that the family $\{\lambda g_{ab} : \lambda > 0\}$ is continuous in the C^0 global topology.

We can now use these facts and proposition 4.3.7 to show that each C^k global topology lies strictly between the C^k compact-open and C^k open topologies for non-compact manifolds. (Since the compact-open and open topologies coincide when M is compact, clearly the global topologies do as well.)

Proposition 4.3.9. *For M non-compact, the C^k global topology on $L(M)$ is strictly finer that the C^k compact-open topology and strictly coarser than the C^k open topology.*

Proof. Since $\Xi_G \subset \Xi_O$, each basic neighborhood of the C^k global topology is a basic neighborhood of the C^k open topology, but the former makes every family of the form $\{\lambda g_{ab} : \lambda > 0\}$ is continuous, while the latter does not. Thus the C^k global topology is strictly coarser than the C^k open topology.

Next let an arbitrary $B(g, \epsilon; h, C)$ be given, and pick any $h' \in [g] \cap \mathcal{R}(M)$. Define $\Omega = |g|_{h'}/|g|_h$ and put $h''^{ab} = \Omega h^{ab} / \inf_C \Omega$. Since

$$\|g\|_{h'',M} = \sup_M (h''^{am} h''^{bn} g_{ab} g_{mn})^{1/2} = \frac{\|g\|_{h',M}}{\inf_C \Omega} < \infty,$$

we have that $h'' \asymp g$, so $B^k(g, \epsilon; h'', M)$ is a basic neighborhood of the C^k global topology. Now, for any $g' \in B^k(g, \epsilon; h'', M)$,

$$\epsilon > \|g - g'\|_{h'',M} \geq \|g - g'\|_{h'',C} = \frac{\sup_C (\Omega |g - g'|_h)}{\inf_C \Omega} \geq \left(\frac{\sup_C \Omega}{\inf_C \Omega} \right) \sup_C |g - g'|_h \geq \|g - g'\|_{h,C},$$

thus $g' \in B(g, \epsilon; h, C)$. Since g' was arbitrary, we have $B^k(g, \epsilon; h'', M) \subseteq B(g, \epsilon; h, C)$, so by the Hausdorff criterion (Willard 1970, Theorem 4.8, p. 35), the C^k compact-open topology is coarser than the C^k global topology.

Lastly, for the sake of contradiction, suppose that every basic neighborhood of the C^k global topology $B^k(g, \epsilon; h, M)$ contains a basic neighborhood of the C^k compact-open topology, $B^k(g, \epsilon'; h', C)$. Consider $g' = (1 + \rho / |g|_h) g$, where ρ is a smooth positive scalar field such that

$$\sup_C \rho < \epsilon' \left(\sup_C \frac{|g|_{h'}}{|g|_h} \right)^{-1},$$

but $\sup_M \rho$ does not exist, i.e., is infinite. Note that

$$\|g - g'\|_{h',C} = \sup_C \rho \frac{|g|_{h'}}{|g|_h} \leq (\sup_C \rho) \left(\sup_C \frac{|g|_{h'}}{|g|_h} \right) < \epsilon',$$

so $g' \in B^k(g, \epsilon'; h', C)$. But $|g - g'|_h = \rho$, which is unbounded on M, hence a contradiction. Thus by the Hausdorff criterion the C^k global topology is not coarser than the C^k compact-open topology. □

Because the global topologies are generated from a collection of norms, the corresponding collection of metrics (defined by $d(g, g'; h) = \|g - g'\|_h$) in fact define a uniform structure on $L(M)$.

Instead of digressing into the details of the theory of uniform spaces—that is, sets endowed with uniform structure—the following definition relevant to the present investigation may be abstracted from that theory.

Definition 4.3.10. A pair of points x, x' in a space X whose topology is generated from the ϵ-balls $\mathcal{B}(x, \epsilon; d_\alpha)$ of a collection of metrics $\{d_\alpha\}$ is said to be *uniformly connected* when for every $\epsilon > 0$ and metric d_α, there is a finite sequence $\mathcal{B}(y_1, \epsilon; d_\alpha), \ldots, \mathcal{B}(y_n, \epsilon; d_\alpha)$ such that $x \in \mathcal{B}(y_1, \epsilon; d_\alpha)$, $x' \in \mathcal{B}(y_n, \epsilon; d_\alpha)$, and $\mathcal{B}(y_i, \epsilon; d_\alpha) \cap \mathcal{B}(y_j, \epsilon; d_\alpha) \neq \emptyset$ if $|i - j| = 1$. The space X is uniformly connected when each pair of its points is connected.[22]

Such a space is *locally uniformly connected* when every point has a local neighborhood basis consisting of sets that are uniformly connected (in the subspace topology).

Proposition 4.3.11. *Each $B^k(g, \epsilon; h, M)$ (with $h \asymp g$) is uniformly connected in the C^k global topology.*

Proof. Let some such $B^k(g, \epsilon; h, M)$ be given and consider some arbitrary $h' \asymp g$, $\epsilon' > 0$, and $g' \in B^k(g, \epsilon; h, M)$. Since then $\|g - g'\|_{h'} < \infty$, pick some positive $c < \epsilon'/\|g - g'\|_{h'}$, so that $\|cg - cg'\|_{h'} < \epsilon'$. Further, note that $\|g - cg\|_{h'} = |1 - c| \cdot \|g\|_{h'}$ and $\|g' - cg'\|_{h'} = |1 - c| \cdot \|g'\|_{h'}$ are both finite, so put $N = \lfloor 1 + \|g - cg\|_{h'}/\epsilon' \rfloor$, $N' = \lfloor 1 + \|g' - cg'\|_{h'}/\epsilon' \rfloor$, $C = \|g - cg\|_{h'}/N$, and $C' = \|g' - cg'\|_{h'}/N'$. The families $\Lambda = \{g - \lambda g : \lambda \in [0, c]\}$ and $\Lambda' = \{g' - \lambda g' : \lambda \in [0, c]\}$ can then be covered by neighborhoods of the form $B^k(g - nCg, \epsilon'; h', M)$ and $B^k(g' - n'C'g', \epsilon'; h', M)$, respectively, with $n \in \{0, \ldots, N\}$ and $n' \in \{0, \ldots, N'\}$. This shows that g and g' are uniformly connected, but since g' was arbitrary and uniform connection is an equivalence relation, each pair of elements in $B^k(g, \epsilon; h, M)$ is uniformly connected. □

Combining propositions 4.3.11 and 4.3.9 immediately yields the following.

[22] Sometimes this property is called uniform chain connectedness (Mrówka and Pervin 1964), in analogy with the chain connectedness for general topological spaces, although this latter property is strictly weaker than (topological) connectedness (Willard 1970, Theorem 26.15, p. 195).

Corollary 4.3.12. *Each C^k global topology is locally uniformly connected.*

Despite being locally uniformly connected, the global topologies are not uniformly connected. In fact, they have uncountably many uniform components that can be described using the following lemma.

Proposition 4.3.13. *Given $g, g' \in L(M)$, $g \asymp g'$ if and only if there are constants $\epsilon, \epsilon' > 0$ and $h, h' \in \mathcal{R}(M)$ such that $h \asymp g$, $h' \asymp g'$, and $B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M) \neq \emptyset$.*

Proof. Suppose that $g \asymp g'$. Pick any $h \in [g] \cap \mathcal{R}(M)$, noting that $h \asymp g'$ by transitivity, and then pick any $\epsilon > \|g - g'\|_{h,M}$. Hence $g' \in B^k(g, \epsilon; h, M)$, so arbitrarily setting $\epsilon' = \epsilon$ and $h' = h$ yields that $B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M) \neq \emptyset$.

Conversely, suppose that $h \asymp g$, $h' \asymp g'$, and $g'' \in B^k(g, \epsilon; h, M) \cap B^k(g', \epsilon'; h', M)$. By proposition 4.3.5, $h \asymp g''$ and $h' \asymp g''$, hence by transitivity $g \asymp g'$. □

Proposition 4.3.14. *The uniformly connected components of the C^k global topology on $L(M)$ have the following properties:*

1. *they are identical with the norm-equivalence classes $[f] \cap L(M)$;*
2. *through translation each such component (under the subspace topology) generates a locally convex topological vector space on Γ^0_2 (and similarly on Γ^2_0 for the inverse metrics) compatible with the original topology on the component, in the sense that taking the subspace topology again returns the original topology.*

Proof. 1. Let $\mathcal{G}(g) = \bigcup_{\epsilon>0} B^k(g, \epsilon; h, M)$ for each $g \in L(M)$. Now, by proposition 4.3.13, $g' \not\asymp g$ if and only if $\mathcal{G}(g) \cap \mathcal{G}(g') = \emptyset$. So $\mathcal{G}(g) = [g]$. Further, since any pair of elements of $\mathcal{G}(g)$ is in $B^k(g, \epsilon; h, M)$ for some finite $\epsilon > 0$, proposition 4.3.11 entails that $\mathcal{G}^k(g)$ is uniformly connected. Thus they are the maximal uniformly connected components.

2. Since the Lorentz metrics span Γ^0_2, a topology on the latter is generated from each uniform component as the final topology induced from the translation maps. Moreover, these are generated from the (g, M)-uniform norms, so by proposition 4.2.10 the resulting locally convex topology is compatible with the linear structure of Γ^0_2. The local bases for each g in the same component are generated from the same collections of norms, so the translation maps add no new open neighborhoods to the Lorentz metrics (modulo elements that are not Lorentz metrics).

□

Although the topology that each uniform component generates is locally convex, it bears remarking that the collection $L(M)$ is not itself a convex set, for in general the set $\{\lambda g_{ab} + (1-\lambda)g'_{ab} : \lambda \in [0,1]\} \not\subset L(M)$ for arbitrary $g, g' \in L(M)$.

Lastly, one can characterize the global topologies in terms of similarity of observable quantities, the fields definable on M in terms of g and any collection of frame fields whose (g, M)-uniform norm is bounded.

Proposition 4.3.15. *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on corresponding spacetimes* $(M, \overset{\lambda}{g})$*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *on a spacetime* (M, g) *in the* C^k *global topology iff* $\lim_{\lambda\to 0} \sup_M (\overset{\lambda}{\phi}{}^a_{bc} \overset{0}{\psi}{}^{bc}_a) = \sup_M \phi^a_{bc} \overset{0}{\psi}{}^{bc}_a$ *for every tensor field* $\overset{0}{\psi}{}^{bc}_a \in \mathcal{K}^2_1(g, M)$*, and for each positive* $j \leq k$*,* $\lim_{\lambda\to 0} \sup_M (\overset{j}{\psi}{}_a^{bcd_1\cdots d_j} \nabla_{d_j} \cdots \nabla_{d_1} \overset{\lambda}{\phi}{}^a_{bc}) = \sup_M \overset{j}{\psi}{}_a^{bcd_1\cdots d_j} \nabla_{d_j} \cdots \nabla_{d_1} \phi^a_{bc}$ *for every tensor field* $\overset{j}{\psi}{}_a^{bcd_1\cdots d_j}$*. Moreover, the* C^k *global topology is the unique topology with this property.*

Proof. Analogous to that of proposition 3.5.2. □

Analogous propositions hold for tensor fields ϕ of other ranks. One can interpret any particular ψ field as determining a kind of local system of rods and clocks by which the ϕ fields are measured. The restriction on the former amounts to the requirement that they do not get arbitrarily long and rapid, corresponding to arbitrary precision at infinity, percentage-wise. In a word, then, a field converges in the C^k global topology just in case all observers with bounded precision agree that their measurements converge to those they would make of the limit field. It is instructive to contrast this description with that for the C^k open topologies:

Proposition 4.3.16. *A family of tensor fields* $\overset{\lambda}{\phi}{}^a_{bc}$ *on corresponding spacetimes* $(M, \overset{\lambda}{g})$*, with* $\lambda \in (0, a)$ *for some* $a > 0$*, converges to a tensor field* ϕ^a_{bc} *on a spacetime* (M, g) *in the* C^k *open topology iff there is some compact* $C \subseteq M$ *such that* $(\overset{\lambda}{\phi}{}^a_{bc})_{|M \setminus C} = (\phi^a_{bc})_{|M \setminus C}$ *for sufficiently small* λ*,* $\lim_{\lambda \to 0} \sup_C (\overset{\lambda}{\phi}{}^a_{bc} \overset{0}{\psi}{}^{bc}_a) = \sup_C \phi^a_{bc} \overset{0}{\psi}{}^{bc}_a$ *for every tensor field* $\overset{0}{\psi}{}^{bc}_a \in \mathcal{K}^2_1(g)$*, and for each positive* $j \leq k$*,* $\lim_{\lambda \to 0} \sup_M (\overset{j}{\psi}{}^{bcd_1 \cdots d_j}_a \nabla_{d_j} \cdots \nabla_{d_1} \overset{\lambda}{\phi}{}^a_{bc}) = \sup_C \overset{j}{\psi}{}^{bcd_1 \cdots d_j}_a \nabla_{d_j} \cdots \nabla_{d_1} \phi^a_{bc}$ *for every tensor field* $\overset{j}{\psi}{}^{bcd_1 \cdots d_j}_a$.

Proof. Apply proposition 4.2.8 and the proof of proposition 3.3.1. □

Again, analogous propositions hold for tensor fields ϕ of other ranks. In contrast to proposition 4.3.15, the ψ fields by which the convergence of ϕ is determined are unrestricted, but in balance with that weakening a much stronger condition is placed on the suprema, namely that they equal the limit point except for a bounded region. This makes sense, for if observers are allowed to have arbitrary precision at infinity, the only way a field can converge is if there is no difference between it and its limit point outside of a bounded region.

4.4 Conclusions and Prospects

Geroch has remarked that finding an appropriate topology is surprisingly difficult, and at times has expressed some measure of doubt regarding whether there even *is* such a topology. Instead of reading his demand as one for a canonical topology, one can see him searching for a particular topology with certain properties. The construction of the global topologies meet that goal: it is sensitive to globally defined properties but retains enough continuity for the linear operations defined on $L(M)$ to capture one's intuitive judgments regarding the convergence and continuity of his examples.

A natural query to then pose regards whether, and which, theorems in the literature that use the open topologies have analogs using their global topology counterparts that are more physically relevant in light of the problems with the former. Much work remains to be done for this query, but it turns out that at least certain types of theorems carry over exactly. Recall that a property P is *conformally invariant* when P holds of g if and only if for every scalar field $\Omega > 0$, P holds for Ωg.

Proposition 4.4.1. *A conformally invariant property P of a spacetime g is stable (resp. dense) on some $S \subseteq L(M)$ in the C^k global topology if and only if it is stable (resp. dense) on S in the C^k open topology.*

Proof. For stability, it suffices to show the above equivalence holds when $S = g$, a point. One direction follows immediately from proposition 4.3.9, so suppose that some conformally invariant property of g is stable in the C^k open topology, i.e., there is some $B^k(g, \epsilon; h, M)$ all of whose elements have that property. Let $\Omega = 1/|g|_h$, one can express $B^k(g, \epsilon; h, M) = \{g' : \|\Omega^{-1}g - \Omega^{-1}g'\|_{\Omega h} < \epsilon\}$. Because P is conformally invariant, it must hold on the set $B^k(\Omega g, \epsilon; h, M) = \{g' : \|g - g'\|_{\Omega h} < \epsilon\} = B^k(g, \epsilon; \Omega h, M)$, which is a basic open neighborhood of g in the C^k global topology since $|g|_{\Omega h} = 1$ implies that $g \asymp \Omega h$.

For denseness, as before, one direction follows immediately from proposition 4.3.9, so suppose that some conformally invariant property P of g is dense in some $S \subseteq L(M)$ in the C^k global topology, i.e., for every $g \in S$ there is some $B^k(g, \epsilon; h, M)$ with $h \asymp g$ that contains an element g' with P. Now consider any basic neighborhood $B^k(g, \epsilon'; h', M)$ in the C^k open topology. Using the same reasoning as above, one can conclude that it contains an element with property P if $B^k(\Omega' g, \epsilon'; h', M) = B^k(g, \epsilon'; \Omega' h', M)$ does so as well for some scalar field $\Omega' > 0$. Picking $\Omega' =$

$\max\{\epsilon''/2|g|_{h'}, \epsilon''/2|g'|_{h'}\}$, where $\epsilon'' \in (0, \epsilon')$, yields that

$$|g - g'|_{\Omega' h'} \leq |g|_{\Omega' h'} + |g'|_{\Omega' h'} = \Omega|g|_{h'} + \Omega|g'|_{h'} \leq \epsilon''/2 + \epsilon''/2 < \epsilon'.$$

Hence $\|g - g'\|_{\Omega' h'} < \epsilon'$, meaning $g' \in B^k(g, \epsilon'; \Omega' h', M)$. □

Recall that (Hawking 1969) showed that the existence of a global time function, a smooth scalar field strictly increasing on each future-directed timelike curve, is equivalent to stable causality, i.e., the stability of the property of having no closed causal curves in the C^0 open topology.[23] We thus have the following:

Corollary 4.4.2. *Hawking's theorem holds for the C^0 global topology, i.e., a spacetime admits of a global time function if and only if it is stable in the C^0 global topology.*

There are many properties, such as being singular, not covered by proposition 4.4.1. Geodesic (in)completeness, for instance, is sometimes but not always stable (Beem et al. 1996, Ch. 7.1), and I suspect one can find examples thereof for which the open and global topologies render different judgments. Again, in light of the open topologies' many problems, the global topologies seem a better choice for formulating these kinds of questions regarding global properties.

Of course, I have not shown that the global topologies are the unique topologies meeting Geroch's desiderata. While I have given several characterizations of their structure, proving that they have further invariant characteristics, such as being "maximal" or "minimal" in some relevant way, would further illuminate their status. For instance, one might try to make precise and then investigate a sense in which the global topologies might make linear operations defined on $L(M)$ "as continuous as they could be" (perhaps subject to some other constraints).

[23]See also Hawking and Ellis (1973, Propopsition 6.4.9, p. 198–201). These early results actually only proved that the existence of a *continuous* time function follows from stable causality; the extension to smoothness remained a folk theorem until surprisingly recently. See Minguzzi and Sánchez (2008, §3.8.3) and references therein for part of this story.

Another related direction to pursue concerns the fact that both the compact-open and open topologies have natural formulations in terms of fiber bundle theory. Recall that each Lorentz metric corresponds to a smooth cross-section of a $(0, 2)$-tensor bundle over M, and its derivatives to cross-sections of the appropriate jet bundle. Given an open set of the total space of the bundle, one can define a basis element for the open topology as the set of Lorentz metrics whose corresponding cross-sections' images lie in that open set. One can define a subbasis element for the compact-open topology similarly except one considers the cross-sections' images restricted to compacta of M. I suspect that the global topologies can be given a natural fiber bundle formulation. Like in the present investigation, this would require attention to the algebraic structure of the sections, and so may require something like the principle bundle formalism.

Chapter 5

Conclusions and Beyond

In this dissertation, I have explored just a few of the bearings that notions of similarity have on scientific theorizing and practice. The notion of similarity amongst spacetimes can be formalized through topology, and brought to bear on questions concerning the stability and genericness of spacetime properties, hence concerning their physical significance. But, I argued (ch. 2), because there is no canonical notion of similarity, such a notion must be chosen in consonance with the context of inquiry. Canonicalists about topology—those in opposition to this methodological contextualism—must face the horns of a no-go result, and either reject the significance of Hawking's theorem or of continuity in the geometric sense.[1]

I showed as well in chapter 3 that similarity helps resolve an old issue about the relationship between general relativity and Newtonian gravitation. Building on work by Ehlers, I exhibited a framework for understanding a reduction relation between them that is both perfectly general and explanatory of Newtonian gravitation's success. In particular, a sequence of relativistic spacetimes reduces to a Newtonian spacetime just when it converges to it in an appropriate topology. Once one sees the role that topology plays in defining the convergence (hence reduction) relation, one

[1]Canonicalists attracted by one of the global topologies must reject the latter, that all families of spacetimes continuous in the geometric sense are in fact continuous.

is led, in light of the conclusions of chapter 2, to consider which topology is most appropriate for understanding this relation. It turns out that the choice of topology can be understood to align with a choice regarding which observable quantities must be well-approximated in the limit. Once this is clear, it becomes manifest that one must modify the topology that has been implicitly used in the literature regarding the reduction relation to include observables defined beyond points, like the lengths of curves.

Finally, in chapter 4 I addressed the question of global similarity that Geroch (1971) raises: although the open topologies have been used to capture such a notion, in fact they are not so well adapted to the task. My diagnosis of this failure is that they are not sensitive to the algebraic structure of the Lorentz metrics. Although a topology completely meeting Geroch's demands does not exist, I argue, there is a topology that meets a slight weakening of them. I then prove a series of propositions to justify various aspects of the interpretation of these new global topologies, and venture slightly into examining which propositions using them have analogs with those using the open topologies.

These are only a few applications of similarity. To illustrate how broad this project could continue to be, I have collected together a number of further topics, some of which are extensions of those treated in the above chapters. These are organized roughly by the chapter of which they are largely an extension, although there is some overlap that I note when it occurs. The last section concerns a few applications not immediately related to these.

5.1 Similarity, Topology, and Physical Significance in Relativity Theory

5.1.1 Stronger Results in Support of Methodological Contextualism

One of the arguments I made for methodological contextualism regarding the choice of topology on the models of general relativity relied on certain technical results that I proved. In particular, I showed that there is no topology with respect to which both (1) all the one-parameter families of spacetimes continuous in the geometric sense are continuous, and (2) Hawking's theorem holds. Since the latter is an important result and the former formalized a very intuitive conception of continuity, this result shows that a proponent of the opposing canonicalism must either reject the physical relevance of Hawking's theorem, or of continuity in the geometric sense.

Such a proponent might do so on the grounds that, despite the intuitiveness of continuity in the geometric sense, it does not play a sufficiently significant role in the actual application of relativity theory. A rejoinder to this retrenchment could then consist in finding a different, widely used property to feature in a no-go theorem. One possibility involves the stability of initial-value evolution in general relativity. Without going into technical detail, one says that the evolution of initial data on a Cauchy surface is stable when its map into the evolved relativistic spacetime is continuous (Hawking and Ellis 1973, Ch. 7). Intuitively, this means that small changed in the evolved spacetime result from small changes in the initial data defined on this surface. This, of course, requires a topology on relativistic spacetimes (and on Cauchy surfaces). I conjecture that "continuity in the geometric sense" in my no-go theorem can be replaced with "stability of initial-value evolution."

I conjecture as well that one could perform an analogous substitution into the following proposition (whose proof is analogous to that of proposition 2.5.1):

Proposition 5.1.1. *Any topology on $L(M)$ with respect to which all the one-parameter families* *continuous in the geometric sense are continuous also determines that chronology-violating spacetimes are dense.*

This would be an especially puzzling result for the canonicalist to explain (away), as spacetimes that have initial-value formulations do not have closed timelike curves. It would thus show that the kind of perturbations that small changes in the initial values generate are very special, as any evolved spacetime would be arbitrarily similar to another spacetime without an initial value formulation.

5.1.2 Interpreting Topologies Over Spacetimes

If one accepts that the choice of topology on the collection of spacetimes should depend on the context of investigation, a natural question to ask is exactly *how* that dependence should manifest. When it comes to the stability, density, and genericness of certain properties, there is an intuition that the point-open, compact-open, and global topologies are sensitive only to certain properties. To capture this intuition, one could try to prove analogs of proposition 2.4.3, which asserts the genericness of chronology violation according to the compact-open topologies, for whole classes of spacetime properties and for other topologies. For example, it seems roughly that the point-open topologies would rule as generic any property definable on the complement of any local region, and that the compact-open topologies would rule in the same way properties definable on co-compact sets. If correct, such propositions would require a refinement of the usual partition of spacetime properties into local and global (and indeed a modification of the class of global properties, which are usually taken to be the compliment of the local properties).

Another aspect of interpretation is the use of the (f, S)-uniform norm (where, recall, f is some metric and $S \subseteq M$). This yields that similarity is measured in terms of maximal dissimilarity. One could instead use the (f, S)-L^p norm to measure similarity (arguably) in terms of *average* dissimilarity, i.e., for any tensor field K on S,

$$||K||_{f,S,p} = \left(\int_S |K|_f^p dV_f \right)^{1/p} \tag{5.1}$$

where (as before) $|K|_f$ is the f-fiber norm of K and integration is with respect to the volume element defined by f. When the S consist of compact sets, this norm is often used in the context of the initial value problem in general relativity (Hawking and Ellis 1973, Ch. 7.4). The case where $S = M$ for a non-compact M would, of course, require more work along the lines developed for the global topologies.

A third aspect of interpretation is the choice of order of the topology, viz. how many derivatives of the metric should be relevant for measuring similarity. Accordingly, Beem et al. (1996, Remark 3.14) have suggested the following interpretations for first three orders of the open topology:

1. Closeness in the zeroth order open topology means similarity of light cones.

2. Closeness in the first order open topology means similarity of geodesic systems.

3. Closeness in the second order open topology means similarity of curvature tensors.

The suggestion for the second order seems right to me, but the suggestion for the first order is ambiguous and that for the zeroth order incorrect. In the first place, what is a "geodesic system" and what would it mean for two of them to be similar? Given a pair of spacetimes, this could mean that the geodesics in one are curves of sufficiently bounded acceleration in the other. Or it could refer to similarity of connection fields, or of geodesic sprays. Regarding the suggestion for the interpretation of the zeroth order topology, two conformally equivalent metrics will not be "close" if the relevant conformal factor is not unity outside of a compact set (on account of propositions 2.3.1 and 2.3.2 consideration). Since conformally equivalent metrics have the same light cone structure, it is clear that the suggested interpretation neglects the volumetric structure of the spacetime metric. It remains to clarify these interpretations and see how (if at all), once corrected, they may be extended to other classes of topologies besides the open ones.

Finally, a fourth aspect concerns alternative characterizations of the topologies that reveal new angles on their interpretation. Already in the second and third main chapters I characterized the point-open and compact-open topologies, and the global and open topologies, respectively, in terms of similarity of observable quantities. The point-open, compact-open, and open topologies have entirely geometric formulations using the formalism of fiber bundles, and it remains to be seen whether the global topologies can be given such a formulation as well. Yet another reformulation that eschews the uniform norm instead uses effectively continuously varying ϵ-balls. In a bit more detail, one considers a topology generated from sets of the form $\{g' \in L(M) : (|g - g'|_h)_{|S} < \delta_{|S}\}$, with g ranging over all of $L(M)$, h and S over some collection of Riemannian metrics and subsets of M, respectively, and $\delta : M \to (0, \infty)$ ranging over some collection of continuous fields. Again, appropriate choices of these collections yield equivalent formulations of the point-open, compact-open, and open topologies, and it remains to be seen whether this is so for the global topologies also.

5.1.3 Stability and Physicality

A much broader, multifaceted project stemming from the work in chapter 2 concerns the stability principle suggested independently by Hawking (1971, p. 395), Geroch (1971, p. 70), and Hawking and Ellis (1973, p. 197). The brief form in which they present it elicits several questions. First, *what is its history?* Here I can make only a few schematic remarks. One finds an expression of the principle in Duhem (1954, p. 143), who stresses that

> a mathematical deduction is of no use to the physicist so long as it is limited to asserting that a given *rigorously* true proposition has for its consequence the *rigorous* accuracy of some such other proposition. To be useful to the physicist, it must still be

> proved that the second proposition remains *approximately* exact when the first is only *approximately* true. . . . Such are the rigorous conditions that we are bound to impose on mathematical deduction if we wish this absolutely precise language to be able to translate without betraying the physicist's idiom, for the terms of this latter idiom are and always will be vague and inexact like the perceptions which they are to express.

A thorough discussion of this point occupies the whole of Part II, Ch. 3 of *The Aim and Structure of Physical Theory*, where Duhem explicitly cites Hadamard for a mathematical basis for his interpretation of the role of approximation. Hadamard, as early as 1898, had defined a system of differential equations to be *well-posed* just in case, given some appropriate set of initial data, there is a *unique evolution* that varies *continuously* with the initial data.[2] Otherwise, the problem is called *ill-posed*. There was a sense that only well-posed systems could represent real physical processes,[3] a sense which perhaps has precedent even in Leibniz's Law of Continuity and beyond. Tracing this history, besides being of independent interest, could inform the philosophical understanding of the stability principle and its justification.

Second, *how does the stability principle compare with other notions in the literature?* For instance, what is the relationship of this notion of stability with the so-called stability dogma[4] in dynamical systems theory, and with notions of robustness in modeling? Such a comparison will show more clearly that the stability principle has connections to ideas already in the philosophy of science literature, but that it is also novel and perhaps in some senses more general than certain of them. A natural follow-up question would be to ask *what are the implications of accepting the principle for the epistemology of modeling?* I can see already a number of such implications that could sustain further discussion:

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[2]See, e.g., Hadamard (1923).

[3]The astute reader may notice here that the import has shifted to the modal-metaphysical (what's possible) from the inferential-epistemological (what we have warrant to infer). Was it thus Duhem's phenomenological influence that drove the shift to epistemology?

[4]For a discussion, see, e.g., Schmidt (2011).

- What we have warrant to infer is interdependent on what we think is possible, and how we think those possibilia are similar.

- In general there will be properties of the world that models describe that, in principle, one will never have warrant to infer.

- Accounts of scientific theories must thus distinguish between the modal possibilities for which a theory allows and the possibilities about the world one can infer.[5]

Do these implications hold in general for scientific modeling, or are they restricted to cases in which the models are sufficiently mathematized?

Third, *what is the status of the principle and its assumptions?* Are there arguments for it, or other assumptions that imply it? Or is a contingent a priori principle, one that must be presupposed for or is mutually constitutive of certain kinds of scientific inquiry?[6] At issue, essentially, is whether there are plausible examples in conflict with the stability principle, or whether the principle follows from some very basic epistemological assumptions or observations. One plausible example of the former are so-called inverse problems, like inferring the structure of a medium from measurements of waves on its boundary. (They are inverses to the perhaps more traditional problem of knowing the structure of the medium and then predicting how the waves will propagate.) Many inverse problems are ill-conditioned in the sense that the inferences made (e.g., the structure of the medium) does not depend continuously on the data. Yet practitioners often employ "regularization" methods to transform or alter the problem at hand into one in which this dependence *is* continuous. Much work needs to be done to sort out the status of this field of applied mathematics vis-à-vis the stability principle.

[5] Does this pose a problem for inferentialist conceptions of scientific theories, according to which a theory is just a certain collection of inferences that are (conditionally) warranted?

[6] For example, one might think that a kind of discreteness must be presupposed in order to enter into the activity of counting. For more examples and explication, see Chang (2008).

On the other hand, one might want to show that the stability principle follows from certain assumptions about:

1. the fundamental imprecision (relative to some justified topology) of data, or more generally of a model's representation of the world;
2. the continuity (?) of the map from data (or representation) into the space of models under consideration; and
3. the kinds of claims about the world that can be warranted from models thereof.

Does such an argument succeed? What, in turn, is the status of *these* assumptions?

5.2 On the Reduction of General Relativity to Newtonian Gravitation

5.2.1 The Completeness of Reduction

The framework I developed in chapter 3 provides a way to understand the relationship between classical and relativistic gravitational theory that is both perfectly general—not confined to special cases of the theory—and explanatory of the classical theory's success. This framework collected together the models of general relativity and Newtonian gravitation, and took particular reductions to consist in sequences of models of general relativity that converge to a model of Newtonian gravitation, under an appropriately chosen topology. I also exhibited particular simple examples of this reduction, such as a sequence of Minkowskian spacetimes that reduce to Galilean spacetime. But it still remains to be demonstrated that this reduction is complete—that for every model of Newtonian gravitation, there is some sequence of models of general relativity that converges to it.

I conjecture that this can in fact be proved. Such a proof would be, to my knowledge, the first perfectly general rigorous demonstration of this reduction relation for gravitational theories.

Another important question is to determine which topologies are relevant for formulating this reduction. Since these topologies track certain classes of observable quantities, this question reduces to that of deciding which observables need be well-approximated in the limit. I suggested one emendation for the assumed answer that these are point-observables, pointing out that the lengths of curves, defined on compact regions, should be included as well. But should one include other, say, globally defined observables as well?

5.2.2 Models, Laws, Reduction, Emergence

The framework I developed for reduction in the case of gravitation is essentially model-based: it is a relationship between models of one theory and sequences of models of another, all collected together in a common space endowed with a topology. Yet traditional discussions of intertheoretic reduction have concerned the relationship between the laws of the pair of theories of interest, which has led to a voluminous literature analyzing, e.g., the minutiae of how the (mostly theoretic) terms of the two theories are to be related. By contrast, the approach I develop avoids these kinds of problems. Nevertheless, insofar as one is concerned with the relationship of the laws of the theory, one might wonder if reduction of models bears on reduction of laws (and vice versa). I conjecture that there is in fact a close connection between the two, the exploration of which might also illuminate the recently rekindled debate on the nature (semantic or syntactic?) of scientific theories.

I see question not just as a way to connect more clearly with the existing literature on intertheoretic reduction, but also as a way to understand in precise terms the many lawlike approximations employed in relativity theory. For example, applications to much astrophysical modeling and to gravitational waves require a so-called linear approximation of the Einstein equation interrelating the presence of matter with spacetime curvature. But what does it even mean for one *equation* to approximate another in the first place? Is this linearized gravity itself really an intermediate theory lying between Newtonian and relativistic gravitation, with its own class of models? I believe the techniques developed in this dissertation could be applied to better understand the nature of such linear approximations and their broader relationship to spacetime theory.

More generally, because these techniques do not in principle depend on the models in questions being models of spacetimes or even of physical phenomena, they could be extended to other sufficiently mathematized theories. In particular, in the mathematical sciences they could provide a more formal account of reduction and emergence advocated by Butterfield (2011).[7] One can understand reduction to be a kind of approximation, spelled out as above with limits of sequences of models of one theory converging to models of another. One can then understand a property of a model in the limit theory to be emergent just when it is definable in the limit theory but not in the limiting theory. For example, having a well-defined notion of absolute simultaneity would be emergent in Newtonian gravitation with respect to general relativity. There is good reason to believe this account would capture much of the usage of the terms "reduction" and "emergence" in the physics literature. Because of its dependence on the choice of topology, it also highlights a crucial observation that is often overlooked, namely that whether one theory reduces to another depends essentially on the features of the theories we deem relevant to compare.

[7] I demur from Butterfield in classifying this notion of reduction as Nagelian, since I do not see any substantial sense in which the models of the limit theory are *derived* from those of the limiting theory: one must postulate them both in order to formulate the limit at all.

5.3 Global Spacetime Similarity

5.3.1 Exploring and Expanding the Global Topologies

I argued in chapter 4 that the global topologies that I constructed were a suitable replacement for the open topologies: they are "sensitive" to differences between spacetimes across the whole underlying manifold, while avoiding the problems of the latter. Thus, although the conditions that I interpreted Geroch (1971) to demand cannot be all satisfied, a weakened version of them can. Yet virtually all of the propositions proved in the literature concerning stability and genericness of spacetime properties have used one of the open topologies, so it remains to check which have analogs with respect to the global topologies. I was able to prove that results concerning the stability and density of conformal properties do carry over, but the status of many other significant results remain. (Are there general results one can prove for other classes besides conformal properties?) Of particular interest are the results of Williams (1984) and others concerning stability of geodesic incompleteness.[8] (Note that if a property is *unstable* with respect to the C^k open topology, it must be unstable with respect to the C^k global topology.) Are standard cosmological models still stably singularity-free according to the global topologies?

Other interesting topics include extending and exploring further properties of the global topologies. I developed these topologies under the assumption that the basic objects were smooth manifolds with a non-degenerate metric, but I conjecture that my construction may be naturally extended to certain other spacetime theories using degenerate metrics, such as Newton-Cartan theory, the "geometrized" analog of Newtonian gravitation. This would allow one to have a notion of global spacetime similarity for classical spacetimes, and also to consider whether general relativity reduces to Newtonian gravitation with respect to the (say, C^2) global topology. (Notably, this reduction always fails for the open topologies.) As an example of an exploration of further properties, one may observe that the point-open, compact-open, and open topologies all have a canonical

[8]See Beem et al. (1996, Ch. 7) for a review.

uniform structure associated with them, which one can interpret as providing a way to order the "sizes" of neighborhoods of different points, without necessarily providing a numerical value for this size.[9] It remains to be seen whether there is a canonical such uniform structure for the global topologies, or whether there is no canonical choice but a range of choices that, perhaps, must also be decided according to methodological contextualism.

5.3.2 Approximate Global Symmetries

In physical cosmology one typically considers models of the universe that homogeneous and isotropic—at any given time, the distribution of matter is the same everywhere and appears the same in every direction. Homogeneity and isotropy are introduced to give the spacetime a high degree of symmetry, thus simplifying the analysis of the spacetime dramatically. Indeed, the conclusion that there was a Big Bang—a past singularity for all observers—rests (at least in part) on this analysis. But, needless to say, as one looks around, one can see that these assumptions are not strictly true (and thankfully so). Yet there is a sense that, on "large enough scales," these assumptions are *approximately* true. But what is this supposed to mean? And if the symmetries these assumptions introduce are broken, if only "slightly", can one still draw similar conclusions about the universe? There are many questions here regarding idealization, symmetry, and modeling. But answers to nearly all them will presuppose some notion of approximate symmetry yet to be elucidated.

In relativity theory, a spacetime symmetry is given by a Killing vector field, local flows along which generate a one-parameter family of isometries. Given a *prospective* symmetry, then, one can consider the one-parameter family of mappings induced by its local flow and the corresponding one-parameter family of spacetimes thus generated. One can then characterize the "degree" to which the symmetry holds by the collection of basic open neighborhoods of the original spacetime

[9] In the cases at hand, there is in fact no such numerical value; the corresponding topologies are *non-metrizable*. For more on uniform structures, see, e.g., Willard (1970, Ch. 9).

that contain this family. If one is considering the symmetry as a *global* symmetry, one can use the global topology; for local symmetries, one might use instead the compact-open topology. (If the global topology admits of a canonical uniform structure, as discussed above, then one can give a bit more structure to the notion of the degree of approximation.) One consequence of this account of approximate symmetry is that, in some sense, the degree of approximation will ultimately be observer dependent: given some fixed standard, some will consider a certain (non-exact) approximate symmetry good and others bad. The benefit of introducing the machinery of topology is that one can make absolutely precise within a unified treatment the degree to which any given observer will judge the symmetry to be approximate. In the case of cosmology, where the observers of interest are nearly all (approximately!) co-moving with the matter content of the universe, nearly all observers of interest (with about the same standards) will agree on which symmetries holds approximately.

5.4 Further Topics

5.4.1 Topologies on *All* Spacetimes

All of the topologies considered are on spacetimes with the same underlying manifold, but there are many situations in which one might want to compare the similarity of spacetimes with different underlying manifolds (though of the same dimension). Hawking and Ellis (1973, p. 198) state that this can be done so that the resulting space is connected, but I have not yet seen a demonstration of this claim. Indeed, although some work has been done to produce a tractable such topology (Bombelli 2000; Noldus 2004), and although the topic remains of mathematical interest (Sormani 2012), no approach has been clearly successful. Since a topology on a space can be characterized by the collection of continuous maps into that space,[10] one might adapt the technique of Geroch

[10]In particular, the final topology on a space X induced from a collection of maps $\psi_i : Y_i \to X$, where the Y_i are topological spaces, is the finest topology on X making each of the ψ_i continuous.

(1969) discussed in the second chapter and consider one-parameter families of spacetimes "pasted together," without assuming they share the same underlying manifold. The topology one then choses for the resulting 5-manifold may then induce a topology on the spacetimes, just as choosing the (5-)manifold topology when the underlying spacetimes shared the same manifold yielded the compact-open topology. Of course, a host of subtle technical issues remain to be solved before this intuition can be implemented with full rigor.

5.4.2 Semantics for Scientific Counterfactuals

Lewis (1973) famously gave a semantics for counterfactual statements in terms of similarity of possible worlds. One of the main points of criticism of his account, however, was that it said little regarding what similarity was supposed to amount to. Later, admitting that similarity was ultimately contextual, he made a proposal in terms of minimizing the scope and scale of "miracles"—worlds in which the laws of nature are violated. This might (arguably) be a plausible proposal for everyday discourse, where there is only at most a vague restriction on what is possible or not. But it cannot suffice for scientific counterfactuals, where the possibilities, even those not actually realized, are described by models of some theory. Incredibly, the uniform structures canonically associated with many of the topologies considered in this dissertation satisfy Lewis's axioms for a similarity relation among possible worlds. In other words, as uniform spaces, the models of gravitational theory form a collection of worlds with a three-place similarity relation (taken as "x is more similar to y than it is to z") that ground the semantics for counterfactual statements for spacetimes.[11] Developing a modal logic internal to a scientific theory has a certain appeal, for it eliminates much of the need for extravagant modal metaphysics to enumerate and distinguish possibilities. In addition, it might be applied to answer concrete scientific questions in cosmology

[11] One point of dispute between Lewis and Stalnaker regarding possible worlds was the co-called Limit Assumption: for every world and every antecedent of a counterfactual conditional entertainable in that world, there is effectively some (not necessarily unique) possible world most similar to it. Stalnaker upheld it always, while Lewis said it could fail in many cases of interest. The similarity relations under consideration defined by uniform structures generically violate the Limit Assumption.

involving fine-tuning, e.g., “If these constants of nature or initial conditions were only a little bit different, then the universe would be radically different.” As far I understand, this would be a novel approach to questions of fine-tuning.

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