Title
Combinatorial Topology and Applications to Quantum Field Theory

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Publication Date
2018

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Fall 2018

Abstract

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Topology has become increasingly important in the study of many-body quantum mechanics, in both high energy and condensed matter applications. While the importance of smooth topology has long been appreciated in this context, especially with the rise of index theory, torsion phenomena and discrete group symmetries are relatively new directions. In this thesis, I collect some mathematical results and conjectures that I have encountered in the exploration of these new topics. I also give an introduction to some quantum field theory topics I hope will be accessible to topologists.

To my loving parents, kind friends, and patient teachers.

Contents

Part I

Discrete Topology Toolbox

Introduction

In this part we will develop some mathematical machinery for performing topological computations using simplicial and cellular cochains. Our focus is mainly on things that are useful for the physics applications discussed in part II, but there is some novel mathematics here as well. For these purposes, we will need formulas for manipulating cohomology operations and characteristic classes on the cochain level. The approach is pitched at a mixed audience of physicists and mathematicians. The mathematicians may find it a bit pedantic, but hopefully they learn something new by the end.

Usually in topology we manipulate cohomology classes directly. In this case they form a graded ring $H^*(X, R)$, where X is a space and R is a coefficient ring. Then, cohomology operations like Steenrod squares and Massey products are added after the fact, and their origin can seem mysterious. By working on the cochain level however, $C^*(X, R)$ becomes an ∞-categorical version of a graded ring, where the algebraic origin of these cohomology operations is clear and they are computable, even manipulable with formulas. (Because of slippery foundations I won't state what kind of "∞-ring" $C^*(X, R)$ is.)

Among the mathematical novelties in this part is a discrete Morse flow (first developed by Robin Forman [1]) on the barycentric subdivision of a CW complex X whose unstable cells are the original cells of X. This discrete Morse flow gives a geometric picture of the cap product as the infinite time flow from the dual complex which I don't believe has appeared anywhere. The usual geometric picture of the cup product as an intersection form also follows.

This Morse flow also gives us a conjectural geometric formula for the $\cup_i$ products [2, 3], whose geometry has remained confusing because of their complicated combinatorial formula and that they don't descend to a product on cohomology. One can think of these formulas and their construction as a discrete version of the work of Fukaya, Oh, Ohto, and Ono [4], who used smooth Morse theory to describe the A_∞ structure of the usual Morse complex. Being finite, our approach is free of analytical difficulties and our formula for $\cup_i$ hides all of the combinatorics in the definition of the Morse flow. The important properties of the $\cup_i$ products are obvious in this formulation.

We also use this Morse flow to give cocycles representing the Stiefel-Whitney classes of vector bundles. These cocycles are priveleged among their cohomology classes because they satisfy a cochain-level version of René Thom's theorem [5] (English translation in [6]) on Steenrod squares and Stiefel-Whitney classes. While at the time Thom's theorem elucidated the geometry of the Steenrod squares and required the use of the Thom space, for us it is completely apparent from our description of the $\cup_i$ products, of which the Steenrod squares are a special case. I show that it is impossible to give us a cocycle refinement of the Wu formula.

It seems likely that an analogous construction using discrete Morse flow on the Grassmannian bundle $Gr_2(TX)$ will give rise to a cocycle representing the Pontryagin classes with nice properties but we don't attempt it here. It seems likely a natural triangulation with branching structure on $Gr_2(TX)$ would reproduce Israel Gelfand and Robert MacPherson's cocycles [7].

Our Stiefel-Whitney cocycles can also be used to give simplicial and cellular descriptions of tangent structures associated with the Whitehead tower of BO. These descriptions have such a pleasant form that we hope it will inspire mathematicians to make simpler and more functorial obstruction theories. Indeed, for us a spin structure for the tangent bundle TX is simply a 1-cochain η with $d\eta = w_2(TX)$. For surfaces, we show how the extra structure on X which defines the cocycle $w_2(TX)$ gives a functorial correspondence between such η and Kastelyn orientations, which have been the go-to choice for describing discrete spin structures in two dimensions. Our approach works in all dimensions, however, and for all Stiefel-Whitney classes, most of whose associated geometric structure is yet to be explored. Using it we describe spin structures in all dimensions using cellular data. One can think of our approach as a discrete version of Mike Hopkins and Isadore Singer's construction of integral Wu structures on spin manifolds [8]. I hope that once the Pontryagin cocycles are worked out, there is an analogous construction of discrete differential Pontryagin structure. This would be very valuable for physicists who wish to construct discrete versions of gravitational Chern-Simons terms [9].

Chapter 1

Basics

Introduction

This chapter contains some standard definitions in combinatorial and algebraic topology. A textbook reference is [10]. We also describe some aspects of Forman's discrete Morse theory [1]. We are especially concerned with simplicial and cellular cochain-level aspects of duality, which in our discussion we separate into Poincaré duality, Hodge duality, and a duality map derived using discrete Morse theory. The main theorem in this chapter is the construction of this duality map in terms of a particular discrete Morse flow on the barycentric subdivision of a CW complex equipped with a branching structure.

1.1 Discrete Spaces

In this section we introduce our basic combinatorial notion of space:

Definition 1. A *CW complex* ("closure-finite, weak topology") or sometimes cell complex is a Hausdorff space X together with a decomposition of X into open *cells*, such that each cell

- is homeomorphic to an open ball B^k for some k, in which case we call it a k-cell;
- has boundary contained in the union of all j-cells for $j < k$. This union is called the $k-1$-*skeleton* and is denoted X_{k-1}.

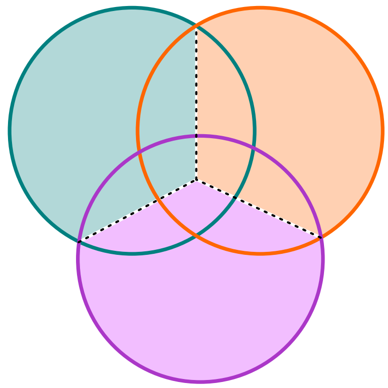

Figure 1.1: The relationship between a fine atlas or Cech covering and a CW complex. Points are colored by the closest center of a patch. For a generic choice of local metric, this CW complex is dual to a triangulation.

It is useful to construct CW complexes by specifying the collection of k-cells for each k along with attaching maps $f : \partial \bar{B}_k \to X_{k-1}$, where B_k is a standard open k-ball, $\bar{B}_k$ is the standard closed k-ball, and $\partial \bar{B}_k$ is its boundary. For example, the complex projective plane $\mathbb{CP}^2$ may be constructed by taking a single 0-cell (so $X_0 = \star$); no 1-cells (so $X_1 = X_0 = \star$); a 2-cell attached to it by the constant map $\partial \bar{B}_2 = S^2 \to X_1 = \star$ (so $X_2 = S^2$); no 3-cells (so $X_3 = X_2 = S^2$); and finally a single 4-cell attached by the *Hopf map* $\partial \bar{B}_4 = S^3 \to X_3 = S^2$.

Often we will restrict our CW complexes to be combinatorial, meaning that the attaching maps are injective and their image is itself a union of cells. Note that any n-manifold may be given the structure of a combinatorial CW complex by choosing an appropriately fine atlas. We can then choose a metric and reduce each patch to its Voronoi cell. See Fig 1.1. Alternatively we can take the nerve of this covering to construct a dual cell complex (the Delauney cell complex). Note that the above CW complex for $\mathbb{CP}^2$ is *not* combinatorial. The smallest known combinatorial CW complex for $\mathbb{CP}^2$ is considerably more complicated. See [11] for example.

To manipulate expressions involving cells and their boundaries, we will need to introduce some extra structure:

Definition 2. A k-cell together with an orientation of its interior is called an oriented k-cell. A local orientation of a CW complex is a choice of orientation for each of its k-cells. If an oriented k-cell agrees with the local orientation it is called positive while if it disagrees with the local orientation it is called negative.

1.1.1 Cellular Maps and Cellular Approximation

A map $f : X \to Y$ between CW complexes is called weakly cellular if it sends the k-skeleton of X into the k-skeleton of Y for every k. This means that f sends 0-cells to 0-cells, but note this is not true for $k > 0$. Instead, f sends 1-cells of X to paths between 0-cells in the 1-skeleton of Y, which may consist of several 1-cells of Y. If the image of each k-cell of X is a union of closures of k-cells of Y, then the map is called cellular.

A refinement or refinement X' of a CW complex X is a cellular homeomorphism $X \to X'$. Thus we may rephrase the above to say that a map $f : X \to Y$ is cellular if there is a refinement of X such that $f' : X' \to Y$ cells k-cells to k-cells for all k. By definition, a PL homemorphism between CW complexes X and Y is a common refinement of both:

$$X \to Z \leftarrow Y.$$

An important theorem for us is the following, which can be found in many standard references, for example [12, 10].

Theorem 1. *Cellular Approximation Theorem* If $f : X \to Y$ is a continuous map of CW complexes, then f is homotopic to a cellular map.

Such a homotopy can be constructed inductively, starting by moving the images of the 0-cells of X to some nearby 0-cells of Y and then proceeding to move the image of X_1 into Y_1 cell-by-cell.

1.1.2 Triangulations and Barycentric Subdivision

Combinatorial CW complexes capture the most common notions of combinatorial topological spaces, such as the hypercubic or other crystalline lattices, but for writing algebraic expressions of cocycles, we will need something whose combinatorics is based on the n-simplex:

Definition 3. The standard geometric n-simplex Δ^n is convex hull of the points $e_1, \ldots, e_{n+1}$ where $e_1, \ldots, e_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$. For every $k+1$-subset $\{i_0, \ldots, i_k\}$, we obtain a k-simplex denoted $(i_0 \cdots i_k)$ given by the convex hull of $e_{i_0}, \ldots, e_{i_k}$. These simplices give Δ^n the structure of a combinatorial CW complex.

Definition 4. For a combinatorial CW complex of X, we define its face poset to be the set of all closed cells of X with the partial ordering $V < W$ if $V \subset \partial W$.

We can now phrase our stronger notion of combinatorial space, that will allow us to manipulate algebraic expressions:

Definition 5. A triangulation of a space X is a special combinatorial CW structure on X in whose face poset the set of cells lying below any k-cell is equivalent to the face poset of Δ^k. To emphasize this we refer to the k-cells as k-simplices. When X comes with a triangulation we call it a triangulated space. Any combinatorial CW complex may be refined to a triangulation without adding 0-cells just like any polytope may be triangulated.

Definition 6. A branching structure on a CW complex is a choice of partial ordering of the 0-cells, such that on the boundary of any k-cell, the 0-cells are totally ordered.

The branching structure gives us a cellular homeomorphism between each k-cell and the standard k-simplex Δ^k which glues appropriately across neighboring k-simplices. In this way, a branching structure behaves much like an atlas of local coordinate charts.

A branching structure is easily constructed on any triangulation by simply choosing a total ordering of all the vertices. Note that a branching structure determines a local orientation. Later we will see that a branching structure determines a framing of the tangent bundle with singularities.

We include some other useful notions for us.

- The (open) star of a k-simplex σ is the union of all simplices τ with $\sigma \subset \bar{\tau}$.
- The link of a k-simplex is the boundary of the star.
- A k-chain is a sequence of simplices σ_j such that

$$\bar{\sigma}_0 \subset \bar{\sigma}_2 \subset \cdots \subset \bar{\sigma}_k.$$

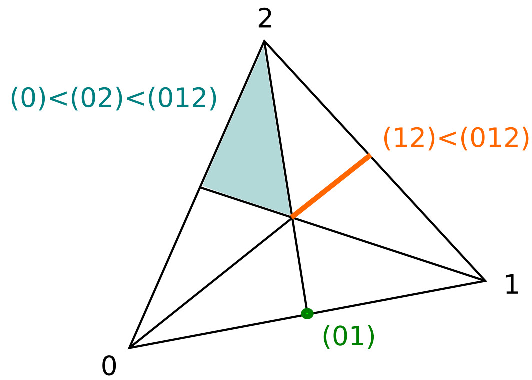

Figure 1.2: The barycentric subdivision of the triangle (012) with some simplices labeled by their chains.

Finally, we describe the barycentric subdivision [10, 13]. Roughly, this subdivision is constructed beginning with a 0-cell (a barycenter) for each simplex and proceeding by joining them according to the face poset of X. We denote this X^b. A picture is shown in Fig 1.2.

The face poset of the barycentric subdivision is most easily described in terms of chains. The k-simplices of the barycentric subdivision are the k-chains of X. The boundary of a k-chain $(\sigma_0, \ldots, \sigma_k)$ is

$$(\sigma_1, \ldots, \sigma_k), (\sigma_0, \sigma_2, \ldots, \sigma_k), \ldots, (\sigma_0, \ldots, \sigma_{k-1}).$$

In particular the 1-simplices of the barycentric subdivision are inclusions

$$\sigma_0 \subset \bar{\sigma}_1,$$

so the subdivision admits two natural branching structures, one where σ_0 points to σ_1, which we will call the ascending branching structure and its opposite the descending branching structure. The fact that the face poset of X has no ordered loops implies that these are indeed branching structures.

1.1.3 PL-Manifolds and Combinatorial Duality

Let X be a combinatorial CW complex. We wish to construct a CW complex $X^\vee$ whose face poset is the opposite of the face poset of X. This is difficult unless X is a manifold:

Definition 7. A triangulated PL n-manifold is a triangulated space such that the link of every k-simplex is PL homeomorphic to either an $n-k-1$-simplex or the boundary of an $n-k$-simplex. [12, 14]

For a triangulated PL n-manifold X, we may define the dual CW complex $X^\vee$ whose 0-cells are the n-simplices of X, whose 1-cells are junctions between two n-simplices, whose 2-cells are junctions between several $n-1$-simplices, and so on. See [10, 13].

Theorem 2. *Combinatorial Duality* For a triangulated PL n-manifold X, X and $X^\vee$ are PL-homeomorphic.

To prove this theorem, we need to exhibit a common refinement of both X and $X^\vee$:

$$X \to Z \leftarrow X^\vee.$$

It is clear that the barycentric subdivision is a common subdivision of both X and $X^\vee$, proving the theorem.

If $\sigma \in X_k$, then $\sigma^\vee \in X_{n-k}$ admits a decomposition into $n-k$-simplices in X^b_{n-k} given by $n-k$-chains

$$\sigma = \sigma_0 < \cdots < \sigma_{n-k} \in X_n.$$

1.1.4 Discrete Morse Flows

In this section we describe some aspects of Robin Forman's discrete Morse theory [1]. Our main application of the theory will be a Morse flow that lets us return from the barycentric subdivision X^b of a cell complex X back to X. I don't believe this Morse flow has been constructed anywhere.

We define a discrete flow V to be a collection of pairs $\sigma_k \to \tau_{k+1}$ where σ_k is a k-cell on the boundary of the $k+1$-cell τ_{k+1} such that each cell appears in *at most* one pair. A V-path is a sequence of pairs

$$\sigma_k^0 \to \tau_{k+1}^0 > \sigma_k^1 \to \tau_k^1 > \cdots > \tau_k^m \to \sigma_k^m,$$

such that $\sigma_k^j \neq \sigma_k^{j+1}$ (no backtracking). V is called a discrete Morse flow if it has no cyclic V-paths, ie. $\sigma_k^0 \neq \sigma_k^m$ for all V-paths. In this case, one can actually define V as a discrete gradient of a function. We will not pursue this here, however.

Given a discrete Morse flow V, we define a critical cell to be any cell which does not occur in one of the pairs of V. For each critical cell σ_k^* of the original CW complex, we call the union of V-paths beginning on the boundary of σ_k^* the unstable manifold of σ_k^*, while the union of all V-paths ending on the boundary of σ_k^* we call the stable manifold of σ_k^*. X is both a union of all unstable manifolds of critical cells and a union of all stable manifolds of critical cells. That V has no cyclic paths implies that these cells are all polyhedra. Thus, either of these unions define a CW coarsening of X.

Theorem 3. *Branching Morse Flow* A branching structure on a combinatorial CW complex defines a Morse flow on its barycentric subdivision whose unstable cells are the cells of the original triangulation.

Proof. For convenience we first describe the critical simplices, for simplicity focusing on a triangulated space X. They correspond to the k-simplices of X $\sigma_k = (i_0 \cdots i_k)$ of X by

$$(i_0 \cdots i_k) \mapsto ((i_0) < (i_0 i_1) < \cdots < (i_0 \cdots i_k)) \tag{1.1}$$

where we use the branching structure to order $i_0 < \cdots < i_k$. We denote a subsequence of this kind, namely of the form

$$(i_0 \cdots i_m) < (i_0 \cdots i_{m+1}) < \cdots < (i_0 \cdots i_{m+l})$$

such that $i_0 < \cdots < i_{m+l}$ a *frozen* sequence.

We extend a total ordering of vertices of X to a total ordering on all the simplices of X^b by extending each k-simplex $(i_0 \cdots i_k)$ to a list of length $n+1$: $(i_0 \cdots i_k) \mapsto (i_k, \ldots, i_0, \infty, \ldots, \infty)$ and then using lexicographical ordering. We denote this ordering $\lhd$ and call it the simplex ordering. In this ordering, the largest simplex is the vertex of highest degree, followed by other simplices containing this vertex. Then it goes on to the vertex of next highest degree, followed by all the simplices containing this one but not the highest one, and so on.

Given a k-simplex

$$(\sigma_{i_0} < \sigma_{i_1} < \cdots < \sigma_{i_k}),$$

we let j be the least j such that

$$\sigma_{i_{j+1}} < \cdots < \sigma_{i_k}$$

is frozen. We then look for simplices ρ which may be inserted in the initial "unfrozen" subsequence

$$\sigma_{i_0} < \cdots < \sigma_{i_j}$$

such that

$$\sigma_{i_m} \lhd \rho$$

for all $m \leq j$. We call these *admissible insertions.* We look for the largest possible ρ with an admissible insertion and insert it to form a $k+1$-simplex

$$(\sigma_{i_0} < \cdots \sigma_{i_l} < \rho < \sigma_{i_{l+1}} \cdots < \sigma_{i_k}).$$

Note that the final frozen subsequence of this $k+1$-simplex is the same as our original k-simplex. Indeed, this is clear if $l \neq j$. However, if $l = j$, then we have a situation like

$$\rho < (i_0 \cdots i_m) < \cdots ,$$

where $i_0 < \cdots < i_m$ and if ρ were to be added to the frozen subsequence then we would necessarily have

$$\rho = (i_0 \cdots i_{m-1}),$$

but there are larger insertions. Thus, the final frozen subsequence stays the same, and since there is no bigger admissible insertion than ρ in the intial unfrozen subsequence, our $k+1$-simplex so constructed admits no admissible insertions of its own. Therefore, the set of pairs

$$(\sigma_{i_0} < \sigma_{i_1} < \cdots < \sigma_{i_k}) \to (\sigma_{i_0} < \cdots \sigma_{i_l} < \rho < \sigma_{i_{l+1}} \cdots < \sigma_{i_k})$$

defines a discrete Morse flow on X^b.

Note that the last element of any k-simplex forms a frozen subsequence of length 1, so ρ is always inserted to the left of σ_k. It follows that this Morse flow pairs simplices of $(X_m)^b$ with other simplices of $(X_m)^b$ for all m. That is, it preserves the skeleton of X, and can be considered as glued together from this same Morse flow constructed n-simplex by n-simplex.

It remains to show that the k-simplices of (1.1) are precisely the critical simplices of this Morse flow. These sequences are frozen and have no admissible insertions so they are critical. We need only show that they are the only critical simplices. Suppose a sequence

$$\sigma_{i_0} < \cdots < \sigma_{i_k}$$

admits no admissible insertions. If this sequence is completely frozen, it must be one of the k-simplices of (1.1), otherwise, σ_{i_0} is not a vertex and it admits an insertion at the left.

Otherwise, we let

$$\sigma_{i_0} < \cdots < \sigma_{i_m}$$

be the initial unfrozen subsequence. Let σ_{i_l} be the largest in the simplex ordering among these. Suppose if we drop σ_{i_l} from the sequence, that it admits an admissible insertion of a ρ which is bigger than σ_{i_l}. The only way this can be is if $\rho < \sigma_{i_l}$, in which case ρ is an admissible insertion in the original sequence. Otherwise, σ_{i_l} is the largest admissible insertion of the sequence obtained by dropping σ_{i_l}, so again the sequence is not critical.

For more general combinatorial CW complexes, a construction like the above is also possible, except now the dimensions of the cells of X are not fixed to their number of vertices (although they can still be described by their set of vertices) and this makes the notation especially cumbersome. In this more general case, the construction of the flow is identical except for the definition of a frozen sequence. A frozen sequence is a sequence of cells

$$V_1 < \cdots < V_k$$

such that $\dim V_{j+1} = \dim V_j + 1$ and V_j is the least cell of its dimension in the cell ordering of V_{j+1}.

$\square$

It's also possible to derive our Morse flow from a Morse function on the standard n-simplex Δ^n, formed as the convex hull of orthonormal basis vectors. Using that coordinate system we consider the linear function

$$f(x_0, \ldots, x_n) = x_0 + ax_1 + a^2x_2 + \cdots + a^nx_n$$

for a a positive real number. We assign values to simplices of Δ^n by the value of f at their centroids. It is easy to check that for $a \gg 1$ the Morse flow of this function as defined by Forman agrees with our combinatorial description above.

Note that the barycentric subdivision X^b has a natural branching structure which defines a Morse flow on the second barycentric subdivision X^{bb}. So X^{bb} has a natural Morse flow. This explains why some combinatorial topology constructions use X^{bb}. Throughout however we will use the above Morse flow to pass from X^b, which has many nice properties, back to X.

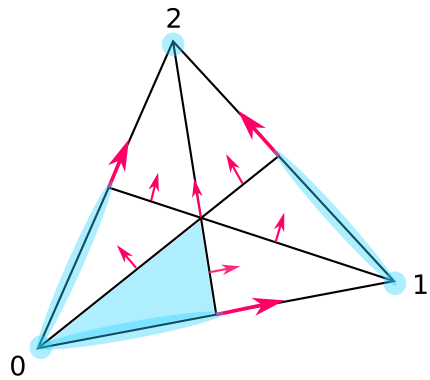

Figure 1.3: A picture of the branching Morse flow on a triangle (012), with critical simplices highlighted in blue.

1.2 Chains, Cycles, Cochains, Cocycles

1.2.1 Chains, Cycles, and Homology

Let X be a CW complex, A be an abelian group. We will construct an abelian group $C_k(X, A)$ spanned by symbols $[V]$ for each *oriented* k-cell V, such that if V^{op} is its orientation-reversed counterpart, then $[V^{op}] = -[V]$. With a choice of local orientation, its oriented k-cells form a linear basis.

The elements of this group are called (cellular) A-valued k-chains in X. A general element is an expression

$$\sum_i a_i[V_{k,i}],$$

where $a_i \in A$ and $[V_{k,i}]$ is the symbol associated to the ith oriented k-cell in the sum. For computational purposes we will often choose an ordering of all the k-cells so that $V_{k,i}$ refers to a specific oriented k-cell of X outside of the context of a particular sum. The union of closures of the $[V_{i,k}]$ with $a_i \neq 0$ is called the support of the k-chain.

These is an apparent group structure on k-chains by collecting terms:

$$\sum_i a_i[V_{k,i}] + \sum_i b_i[V_{k,i}] = \sum_i (a_i + b_i)[V_{k,i}].$$

If X is a combinatorial CW complex, then the boundary of any k-cell V is the union of closures of a set of $k-1$-cells. Further, an orientation of this

k-cell will induce orientations of these boundary $k-1$-cells. Thus we define $\partial[V]$ to be the sum of the symbols associated to these oriented boundary $k-1$-cells. Note that to put it into the positive basis defined by a local orientation of X, we may need to flip some signs according to $[W^{op}] = -[W]$.

We extend ∂ to the boundary map

$$\partial : C_k(X, A) \to C_{k-1}(X, A)$$

by linearity, meaning for a general k-chain,

$$\partial \sum_i a_i[V_i] = \sum_i a_i \partial[V_i].$$

If X is moreover a triangulation with branching structure, then the closure of any positive k-cell V is combinatorially a k-simplex with its vertices identified with the numbers $0, 1, \ldots, k$. Each $k-1$-cell on its boundary is identified with a k-subset of this set, of which there are exactly $k+1$, each defined by their missing vertex. If we denote the symbol of the oriented $k-1$-cell associated to the ordered k-subset $0, \ldots, i-1, i+1, \ldots, k$ as $[\hat{i}]$ (this is the $k-1$-simplex opposite vertex i), then we have

$$\partial[V] = \sum_{0 \leq i \leq k} (-1)^i [\hat{i}].$$

For instance, for a 3-simplex (tetrahedron) (0123), we have

$$\partial[0123] = [123] - [023] + [013] - [012].$$

A chain Γ satisfying $\partial\Gamma = 0$ is called a cycle. group of cycles is denoted $Z_k(X, A)$ while the group of boundaries is denoted $B_k(X, A)$.

The very important property of the boundary map ∂ is that two neighboring oriented k-cells whose orientations agree induce opposite orientations on their share boundary $k-1$-cells. Thus,

$$\partial^2 = 0.$$

In other words, boundary of chains are cycles, $B_k(X, A) \subset Z_k(X, A)$. However, not all cycles are boundaries. The group that captures the obstructions for a cycle to be a boundary is by definition the kth homology of X with coefficients in A, defined as the quotient of the cycles by the boundaries:

$$H_k(X, A) = Z_k(X, A)/B_k(X, A).$$

It is an amazing fact, proved in any textbook on algebraic topology, that these abelian groups $H_k(X, A)$ are independent of the CW complex structure on X. In particular, they are unchanged by refining or coarsening any given CW complex. They are topological invariants of X.

As an example, we write out the groups of chains and boundary maps (the "chain complex") $C_*(X, A)$ for the simple CW complex described above for $X = \mathbb{CP}^2$ with $A = \mathbb{Z}$ coefficients

$$C_5 = 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} = C_0,$$

where all of the boundary maps are zero because the boundary of each k-cell contains no $k-1$-cells. Thus,

$$H_4(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$$
$$H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$$
$$H_0(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z},$$

while the rest are zero. Often when the coefficients $A = \mathbb{Z}$ we will suppress them in the notation.

1.2.2 Pushforward of Chains

Suppose $f : X \to Y$ is a cellular map of CW complexes. Rephrasing our earlier definition, this means that the image of a k-cell V_k of X is the support of a k-chain of Y. This allows us to define a pushforward of CW chains by defining

$$f_*[V_k] = \sum_{U_k \in f(V_k)} [U_k],$$

where we take care to note how the local orientation of V_k maps, taking the U_k to have the orientation induced from f. We extend this to

$$f_* : C_*(X, A) \to C_*(Y, A)$$

by linearity. It is clear that this definition satisfies

$$f_* \partial = \partial f_*.$$

Note that $f(V_k)$ may be a union of $j < k$-dimensional cells, in which case we say that f collapses V_k and by our definition

$$f_*[V_k] = 0.$$

We may have $f_*[V_k] = 0$ even if f doesn't collapse V_k. Indeed, f may fold V_k in half, and it would assign opposite local orientations to the two halves, which would cancel in the above sum.

1.2.3 Cochains, Cocycles, and Cohomology

Let X be a locally oriented combinatorial CW complex, A an abelian group.

If cellular chains are combinatorial analogues of integration cycles, cellular cochains are combinatorial analogues of differential forms. To turn a differential form into a finite piece of data, we keep only its integrals over our cellular chains. Because integration is linear in the integration domain, we thus define an A-valued k-cocycle as a linear map

$$\alpha : C_k(X, \mathbb{Z}) \to A$$

where its value on a k-chain Γ is written suggestively

$$\int_\Gamma \alpha.$$

If $\Gamma = \sum_i n_i[V_{k,i}]$, $n_i \in \mathbb{Z}$, then by linearity

$$\int_\Gamma \alpha = \sum_i n_i \int_{V_{k,i}} \alpha,$$

so α is determined by its values on the positive k-cells of X. For this reason, it is often said that an A-valued k-cochain is simply an assignment of elements of A to each (positive) k-cell of X.

Like the chains, the set of cochains also forms a group, denoted $C^k(X, A)$, defined to mimic linearity in the integrand:

$$\int_\Gamma \alpha + \beta = \int_\Gamma \alpha + \int_\Gamma \beta.$$

Another important property of integration is the Stokes theorem. We *define* the exterior derivative or coboundary map or just differential

$$d : C^k(X, A) \to C^{k+1}(X, A)$$

to satisfy the Stokes theorem:

$$\int_\Gamma d\alpha := \int_{\partial\Gamma} \alpha,$$

for any chain Γ. On a k-simplex of a triangulated CW complex with branching structure, we have

$$\int_{0\dots k} d\alpha = \int_{1\dots k} \alpha - \int_{02\dots k} \alpha + \cdots + (-)^k \int_{0\dots k-1} \alpha.$$

If we denote the value of α on a $k-1$-simplex opposite the ith vertex as $\alpha(\hat{i})$, this can also be written

$$d\alpha(0\dots k) = \sum_i (-1)^i \alpha(\hat{i}).$$

A cochain α with $d\alpha = 0$ is called a cocycle and is said to be *closed*, while $d\beta$ for a $k-1$ cochain β is called a coboundary and is said to be *exact*. The group of k-cocycles is denoted $Z^k(X, A)$ while the group of k-coboundaries is denoted $B^k(X, A)$. One can check, either directly or using the Stokes theorem and $\partial^2 = 0$, that $d^2 = 0$. Thus, $B^k(X, A) \subset Z^k(X, A)$. However, as with the cycles, not every cocycle is a coboundary, and we may define the obstruction group, the kth cohomology of X with coefficients in A as the quotient

$$H^k(X, A) = Z^k(X, A)/B^k(X, A).$$

As with homology, these groups are topological invariants, independent of the specific CW complex we use to describe X.

If we have a map of CW complexes $f : X \to Y$, we may dualize the pushforward of chains $f_* : C_k(X, \mathbb{Z}) \to C_k(Y, \mathbb{Z})$ to obtain the pullback map

$$f^* : C^k(Y, A) \to C^k(X, A)$$

given by

$$\int_\Gamma f^*\alpha = \int_{f_*\Gamma} \alpha.$$

Because d satisfies the Stokes theorem and f_* intertwines ∂, we have

$$df^* = f^*d.$$

Thus, the pullback also gives us a map on cohomology, denoted the same way.

Note that if $\partial U = \Gamma - \Gamma'$ for k-cycles Γ, Γ', then for any k-cochain α,

$$\int_\Gamma \alpha - \int_{\Gamma'} \alpha = \int_U d\alpha,$$

so if α is closed, its value on any k-cycles depends only on the homology class of that k-cycle. It therefore defines a map

$$\int_{-} \alpha : H_k(X, \mathbb{Z}) \to A.$$

1.2.4 Universal Coefficient Theorem

A very important observation about cohomology is that, while the group of cochains is dual to the group of chains, the group of cocycles is *not dual* to the group of cycles. In other words, the map above does not determine α.

There are "phantom k-cocycles", which integrate to zero on every k-cycle and yet are nonzero on some k-chains. One might assume that these values are thus determined on the boundaries of these k-chains, and therefore that the k-cycle is exact. However, there is the possibility for $k-1$-chains Γ which are not boundaries but which for some integer n, $n\Gamma$ is a boundary. For these, the k-cycle might assign to a k-chain whose boundary is $n\Gamma$ a value $a \in A$ such that $a/n \notin A$. In this case, there is no possible value in A that a $k-1$-chain could assign to Γ, so the k-cycle cannot be exact.

This intuition may be turned into a proof of the following:

Theorem 4. *Universal Coefficient Sequence* There is an exact sequence

$$0 \to \mathrm{Ext}(H_{k-1}(X), A) \to H^k(X, A) \to \mathrm{Hom}(H_k(X), A) \to 0,$$

where all the homology coefficients are $\mathbb{Z}$ (the "universal coefficients"). On the right hand side we have the cocycles evaluated just against the chains. On the left hand side we have the group of abelian extensions of $H_{k-1}(X)$ by A, which precisely captures the divisibility obstruction we explained may arise.

If $H_{k-1}(X)$ is torsion-free, meaning it is isomorphic to a free abelian group $\mathbb{Z}^r$ for some r, then $H^k(X, A) \simeq \mathrm{Hom}(H_k(X), A)$. In general, we may split the above sequence,

$$H^k(X, A) \sim \mathrm{Hom}(H_k(X), A) \oplus \mathrm{Ext}(H_{k-1}(X), A),$$

but there is no canonical choice of isomorphism, meaning that there is no way to choose a splitting for all spaces such that the pullback f^* factorizes into block diagonal form for all maps f.

1.2.5 Twisted Cohomology

There is a generalization of $H^1(X,G)$ for G a nonabelian group called nonabelian cohomology (later we discuss a very broad generalization of this). The 1-cochains are assignments of elements $a(e) \in G$ to edges $e \in X_1$. The 1-cocycle equation for $a \in Z^1(X,G)$ is that for every 2-cell with boundary $e_1, \ldots, e_n$,

$$a(e_1) \cdots a(e_n) = 1.$$

A 0-cochain is an assignment $f(x) \in G$ to vertices $x \in X_0$. They act on 1-cochains by

$$a(01) \mapsto a^f(01) = f(0)^{-1} a(01) f(1).$$

The quotient of $Z^1(X,G)$ by this action is $H^1(X,G)$. Warning: when G is nonabelian, $H^1(X,G)$ does not have a natural group structure.

Now suppose G acts on an abelian group M and $a \in Z^1(X,G)$. We can define the a-twisted differential:

$$D^a : C^k(X,M) \to C^{k+1}(X,M)$$

$$\begin{aligned}(D^a\omega)(0\cdots k{+}1) &= a(01)\cdot\omega(1\cdots k{+}1) - \omega(02\cdots k{+}1) + \omega(013\cdots k{+}1) - \cdots \\ &= (d\omega)(0\cdots k+1) + (a(01) - \epsilon(01))\cdot\omega(1\cdots k+1),\end{aligned}$$

where $\epsilon(01) \in Z^1(X,G)$ assigns the identity element of G to every edge. One checks that the cocycle condition on a is equivalent to

$$(D^a)^2 = 0.$$

The cohomology of the twisted differential is denoted $H^k(X, M^a)$ (this is a group). Under the action of a 0-cochain $f \in C^0(X,G)$, there is a corresponding natural transformation

$$H^k(X, M^a) \to H^k(X, M^{a^f})$$

given by

$$\omega(0\cdots k) \mapsto f(0)^{-1}\omega(0\cdots k).$$

1.3 Dualities

1.3.1 Intersection Number and Poincaré Duality

Much of the material in this section is standard and a textbook reference is [10].

Let X be an oriented, closed PL n-manifold with branching structure. We will describe an R-bilinear pairing

$$\#(- \cap -) : C_k(X, R) \otimes C_{n-k}(X^\vee, R) \to R$$

For simple chains $\sigma, \tau^\vee$, where σ and τ are k-simplices in X with σ oriented and τ co-oriented (ie. $\tau^\vee$ oriented), σ and $\tau^\vee$ either don't intersect or they meet transversely in a single point x lying in the interior of both. In the first case $\#(\sigma \cap \tau^\vee) = 0$ and in the second $\#(\sigma \cap \tau^\vee) = \pm 1$. The sign is determined by comparing the orientation of $T_x\sigma \oplus T_x\tau^\vee = T_xX$ induced by the orientations of σ and $\tau^\vee$ with the ambient orientation of X. We define opposite intersection number $\#(\tau^\vee \cap \sigma)$ likewise. It satisfies

$$\#(\tau^\vee \cap \sigma) = (-1)^{k(n-k)} \#(\sigma \cap \tau^\vee).$$

Clearly this pairing is nondegenerate.

We use the intersection number to construct an isomorphism

$$W : C^k(X, A) \to C_{n-k}(X^\vee, A).$$

The notation comes from the concept of a domain wall of a gauge field, which we will discuss in the second half of the thesis. We will define this map on "indicator cochains". Written I_σ, for $\sigma \in X_k$, I_σ is defined to be the k-cochain which is 1 on σ (with its branching structure orientation) and 0 on other k-simplices. $C^k(X, \mathbb{Z})$ is spanned by the indicator cochains. The $n-k$-cells of $X^\vee$ are in bijection with the k-simplices of X. W is defined so that

$$W(I_\sigma) = \sigma^\vee,$$

where $\sigma^\vee$ receives an orientation given an orientation of σ so that

$$\#(\sigma \cap W(I_\sigma)) = 1.$$

Thus, for an arbitrary $\alpha \in C^k(X, A)$, we have

$$\#(\sigma \cap W(\alpha)) = \alpha(\sigma)$$

and
$$W(\alpha) = \sum_{\#(\sigma \cap \sigma^\vee)=1} \alpha(\sigma)\sigma^\vee.$$

Using Stokes theorem, for $\sigma \in X_{k+1}$, we have
$$\#(\sigma \cap W(d\alpha)) = (d\alpha)(\sigma) = \alpha(\partial\sigma) = \#(\partial\sigma \cap W(\alpha)).$$
Then, since
$$\#(\partial A \cap B) = (-1)^{|A|}\#(A \cap \partial B),$$
it follows
$$\#(\sigma \cap \partial W(\alpha)) = (-1)^{k+1}\#(\sigma \cap W(d\alpha)).$$
Since this holds for all σ and the intersection pairing is nondegenerate, we have
$$W(d\alpha) = (-1)^{k+1}\partial W(\alpha).$$
The sign is annoying but unavoidable.

We also define an inverse map
$$\delta_- : C_{n-k}(X^\vee, A) \to C^k(X, A)$$
such that
$$W(\delta_\Sigma) = \Sigma.$$
For this we must have
$$\delta_\Sigma(\tau) = \#(\tau \cap W(\delta_\Sigma)) = \#(\tau \cap \Sigma).$$
By a similar argument as above, we find
$$d\delta_\Sigma = (-1)^{k+1}\delta_{\partial\Sigma}.$$

The notation δ comes from the Dirac delta distributions. Indeed, given a point $x \in X^\vee$ and a region R represented as a chain in $C_n(X, \mathbb{Z})$ whose coefficients on simplices are one or zero,
$$\int_R \delta_x = \begin{cases} 1 & x \in R \\ 0 & x \notin R \end{cases}.$$

These maps together give isomorphisms
$$H^k(X, A) \simeq H_{n-k}(X^\vee, A)$$

for all k. These isomorphisms usual go by the name Poincaré duality [15], especially when combined with the barycentric subdivision, which further gives $H_{n-k}(X^\vee, A) \simeq H_{n-k}(X, A)$.

Sometimes Poincaré duality is phrased as a nondegenerate pairing between cochains $\alpha \in C^k(X, A)$ and $\beta \in C^{n-k}(X^\vee, A)$, which we define by

$$(\alpha, \beta) = \int_{W(\alpha)} \beta = \sum_{\#(\sigma \cap \sigma^\vee)=1} \alpha(\sigma)\beta(\sigma^\vee).$$

1.3.2 Hodge Duality and the Laplacian

We describe an isomorphism called the Hodge star[16]:

$$\star : C^k(X, A) \to C^{n-k}(X^\vee, A).$$

This map is very similar to the Poincaré duality map, except it maps cochains to cochains. For $\alpha \in C^k(X, A)$ and $\sigma \in X_k$ a k-simplex, we define

$$(\star\alpha)(\sigma^\vee) = \alpha(\sigma), \qquad \#(\sigma \cap \sigma^\vee) = 1.$$

This is related to the indicator cochains:

$$\star I_\sigma = \delta_\sigma.$$

Unlike the Poincaré duality maps, $\star$ does not give an isomorphism of chain complexes. Indeed, it doesn't commute with the differential d. In fact, the degrees go the wrong way, and let us define a codifferential

$$d^\dagger := \star d \star : C^k(X, A) \to C^{k-1}(X, A),$$

satisfying

$$(d^\dagger)^2 = 0.$$

We also define the Laplacian

$$\Delta := \frac{1}{2}(dd^\dagger + d^\dagger d) : C^k(X, A) \to C^k(X, A).$$

One can check that this definition agrees with the usual definitions of the discrete Laplacian by difference operators, for example with $k = 0$ it captures the Kirchoff Laplacian [17, 18]. We say that a cochain α is harmonic if

$$\Delta\alpha = 0.$$

By manipulating d and $d^\dagger$, one may easily prove the following:

Theorem 5. *Hodge-Helmholtz Decomposition* Every cochain ω may be decomposed uniquely as

$$\omega = \lambda + d\alpha + d^\dagger \beta$$

where λ is harmonic. In particular, every cohomology class has a unique harmonic representative.

1.3.3 Morse Flow of Chains and Duality

In this section we describe a map induced on chains by a Morse flow which was described by Forman in [19]. Using the branching Morse flow we prove a new duality theorem.

Suppose X is a CW complex equipped with a discrete Morse flow f. We denote X^f the critical cells of X. We define a map

$$\tilde{f}_\infty : C_k(X, A) \to C_k(X^f, A) \hookrightarrow C_k(X, A).$$

We define this map when f consists of a single pair so that for different pairs the $\tilde{f}_\infty$'s commute.

For a k-cell V_k there are 3 possibilities:

- V_k is critical and so $\tilde{f}_\infty[V_k]$ is the symbol of the unstable k-cell of V_k.
- The Morse flow pairs $V_k \to U_{k+1}$, from which we define $\tilde{f}_\infty[V_k]$ to be the sum of symbols of the unstable k-cells of the boundary k-cells of U_{k+1} other than V_k, which are all critical (with signs induced by the boundary orientations of U_{k+1}).
- The Morse flow pairs $W_{k-1} \to V_k$. In this case $\tilde{f}_\infty[V_k] = 0$.

One can check that indeed this assembles into a map for general Morse flows. Indeed, if our Morse flow consists of the single pair

$$V_k \to U_{k+1},$$

then we have isomorphisms

$$C_k(X, A)/\partial[U_{k+1}] \simeq C_k(X^f, A)$$

$$C_{k+1}(X, A)/[U_{k+1}] \simeq C_{k+1}(X^f, A)$$

$$C_j(X, A) \simeq C_j(X^f, A) \qquad j \neq k, k+1,$$

where in the first we eliminate $[V_k]$ in favor of $[V_k] \pm \partial[U_{k+1}]$ depending on the orientation of V_k induced by the orientation of U_{k+1}. Meanwhile in the second, the space occupied by U_{k+1} is now accounted for by the unstable cell of the $k+1$ cell W_{k+1} that flows into it, while U_{k+1} collapses to a k-dimensional cell, and may be set to zero. It is clear that these form a chain isomorphism and that the quotient maps commute.

We may also define the trace of the flow f applied to Σ as the sum of all U_{k+1} encountered in the tree recursion which computes $\tilde{f}_\infty\Sigma$, with appropriate sign. We denote this $k+1$-chain $f_+\Sigma$. By construction,

$$\partial f_+\Sigma = \tilde{f}_\infty\Sigma - \Sigma + f_+\partial\tilde{f}_\infty\Sigma - f_+\partial\Sigma. \tag{1.2}$$

This motivates the definition of the flow map:

$$f_\infty = \tilde{f}_\infty + f_+\partial\tilde{f}_\infty,$$

which takes k-chains in X to k-chains whose coefficients are constant on the unstable cells of the Morse flow. It's equivalent to first performing $\tilde{f}_\infty$, which lands on critical cells only, and then enlarging the critical cells to their unstable cells. From (1.2) we now have

Lemma 1.

$$\partial f_+\Sigma = f_\infty\Sigma - \Sigma - f_+\partial\Sigma. \tag{1.3}$$

Note that this means that f_+ is a chain homotopy (see [20]) from the identity map to f_∞. Applying this relation twice to $\partial\Sigma$ we find

$$\partial f_\infty = f_\infty\partial.$$

Indeed, $f_\infty\partial$ is the Morse complex differential of Forman [1], and

$$f_\infty\partial f_\infty\partial = f_\infty^2\partial^2 = 0.$$

The flow map f_∞ is a projector,

$$f_\infty^2 = f_\infty,$$

while the trace of the flow is nilpotent,

$$f_+^2 = 0.$$

In fact, the flow map fixes any chain whose coefficients are constant on the unstable cells. This is witnessed by the useful relations

$$f_+ f_\infty = f_\infty f_+ = 0.$$

We will often use the flow map of the branching Morse flow, since it gives us a map

$$C_k(X^\vee, A) \hookrightarrow C_k(X^b, A) \xrightarrow{f_\infty} C_k(X, A).$$

This allows us to improve somewhat on Poincaré duality.

Theorem 6. *Morse flow and Duality* Let X be a PL manifold with branching structure. By restricting f_∞ to $C_k(X^\vee, A)$ we obtain a map

$$f_\infty : C_k(X^\vee, A) \to C_k(X, A)$$

which yields an isomorphism

$$H_k(X^\vee, A) \simeq H_k(X, A).$$

Proof. Both the inclusion map $C_k(X^\vee, A) \hookrightarrow C_k(X^b, A)$ and f_∞ commute with ∂ and both induce isomorphisms on homology, the first because it is a refinement [10], the second because of the homotopy f_+ [19]. Therefore, the composition does as well. □

Note that for a typical branching structure, f_∞ may be neither surjective nor injective. For example, the boundary $\partial\Delta^3$ of a 3-simplex is a triangulation of S^3 and for any branching structure (they are all related by S_4 symmetry) the map

$$f_\infty : C_0(\partial\Delta^\vee, \mathbb{Z}) \to C_0(\partial\Delta, \mathbb{Z})$$

has kernel and cokernel $\mathbb{Z}^2$. This is related to the difficulty of defining an inverse flow map.

Suppose we have another branching structure, and we denote g_∞ the branching Morse flow obtained from this branching structure. Since the two Morse flows have the same unstable cells,

$$f_\infty g_\infty = g_\infty$$

and vice versa. Likewise

$$f_+ g_\infty = g_+ f_\infty = 0.$$

The interesting nonzero combinations are

$$f_\infty g_+.$$

1.3.4 Halperin-Toledo Vector Fields

In this section, we describe a framing with singularities that was constructed on a barycentric subdivision of a PL n-manifold by Whitney [21] and Halperin and Toledo [22], and explain how to extend it to an arbitrary PL n-manifold with branching structure. This makes precise our intuition that a branching structure plays the role of a local coordinate system in the world of combinatorial manifolds.

Let X be a PL n-manifold with branching structure. We choose a PL embedding of X into some $\mathbb{R}^N$, meaning that all k-simplices of X are embedded as simplices inside an affine k-subspace of $\mathbb{R}^N$. If $x \in \sigma = (v_0 \dots v_k) \in X_k$, ordered using the branching structure, we can write

$$\vec{x} = \sum_{0 \leq j \leq k} \lambda_{v_j}(x)\ \vec{v}_j,$$

where we use vector notation to emphasize that we are using the PL embedding. For each vertex $v \in X_0$ function λ_v can be extended to a continuous function on X by $\lambda_v(x) = 0$ whenever x is not in the star of v.

Further, for every vertex v we can define a vector field on the star of v called the radial vector field which points radially into v, vanishing only at v. We denote this vector field R_v.

We define the *Halperin-Toledo vector fields* by

$$F_k(x) = \sum_{(v_0 \dots v_k) \in X_k} \lambda_{v_0}(x) \cdots \lambda_{v_k}(x)\ R_{v_k}(x).$$

When X is a barycentric subdivision with the ascending branching structure, this reduces to the definition of the "fundamental vector fields" of Halperin-Toledo, but they are more general.

These have some important properties (see [22]):

Lemma 2. The Halperin-Toledo vector fields F_k satisfy the following properties:

- F_k are continuous vector fields on X, and smooth on each simplex.
- $F_k(x) = 0$ for all x in the $k-1$-skeleton.
- For x in the interior of a k-simplex, $F_1(x), \dots, F_k(x)$ is a basis for $T_x X$.

Observe that F_1 is our branching Morse flow. For instance, consider the 1-simplex in $\mathbb{R}^N$ with vertices $\vec{v}_0$ and $\vec{v}_1$. We can coordinatize this 1-simplex along the branching structure using $\vec{x}(t) = (1-t)\vec{v}_0 + t\vec{v}_1$. In these coordinates, $\lambda_{v_0}(t) = 1-t$ and $\lambda_{v_1}(t) = t$. Further, we see that the radial vector fields are

$$R_{v_0}(t) = t(\vec{v}_0 - \vec{v}_1)$$

$$R_{v_1}(t) = (1-t)\vec{v}_1 - \vec{v}_0.$$

We check indeed,

$$(1-t)R_{v_0}(t) + tR_{v_1}(t) = 0 \qquad \forall t.$$

Now we see

$$F_1(t) = t(1-t)(\vec{v}_1 - \vec{v}_0)$$

describes a monotonic flow from v_0 to v_1 which fixes these points.

For a 2-simplex spanned by $\vec{v}_0, \vec{v}_1, \vec{v}_2$, we choose right triangle coordinates

$$\vec{x}(s,t) = (1-s-t)\vec{v}_0 + s\vec{v}_1 + t\vec{v}_2,$$

so that

$$\lambda_{v_0}(s,t) = 1-s-t$$

$$\lambda_{v_1}(s,t) = s$$

$$\lambda_{v_2}(s,t) = t.$$

For any coordinates, the radial vector fields are

$$R_{v_0}(s,t) = \lambda_{v_1}(s,t)(\vec{v}_0 - \vec{v}_1) + \lambda_{v_2}(s,t)(\vec{v}_0 - \vec{v}_2)$$

$$R_{v_1}(s,t) = \lambda_{v_0}(s,t)(\vec{v}_1 - \vec{v}_0) + \lambda_{v_2}(s,t)(\vec{v}_1 - \vec{v}_2)$$

$$R_{v_2}(s,t) = \lambda_{v_0}(s,t)(\vec{v}_2 - \vec{v}_0) + \lambda_{v_1}(s,t)(\vec{v}_2 - \vec{v}_1),$$

and therefore

$$F_1 = \lambda_{v_0}\lambda_{v_1}R_{v_1} + \lambda_{v_1}\lambda_{v_2}R_{v_2} + \lambda_{v_0}\lambda_{v_1}R_{v_2}.$$

For $N = 2$, $v_0 = (0,0)$, $v_1 = (1,0)$, $v_2 = (0,1)$, we have

$$R_{v_0} = (-x, -y)$$

$$R_{v_1} = (1-x, -y)$$

$$R_{v_2} = (-x, 1-y)$$

and

$$F_1 = ((x+y-1)(x-1)x + x(y-1)y, (x+y-1)xy + y(y-1)^2),$$

$$F_2 = xy(x+y-1) \cdot (x, y-1).$$

Observe how F_2 vanishes on the lines $x+y=1$, $x=0$, and $y=0$ which bound the 2-simplex. This vector field has appeared in the physics literature, eg. [23] in the study of discrete spin structures. We will make the relation precise in Chapter 3.

It is a corollary of the lemma that the Halperin-Toledo vector fields define a trivialization of the tangent bundle away from the $n-1$-skeleton. Along the $n-1$-skeleton, F_n vanishes and the rest are linearly indepedent away from the $n-2$-skeleton, and so on. In this way, a branching structure nicely defines a "framing with singularities" of X, which justifies its ubiquity in the theory.

Now we define what we call the *discrete Halperin-Toledo Morse flows*, f^k. Let X be a PL n-manifold with branching structure. f^1 is defined to be the branching Morse flow. To define f^2, we will take the subflow $f' \subset f^1$ which is trivial on the 1-skeleton $X_1 \subset X^b$. Then, in the interior of every j-simplex, we will apply the permutation to f' which exchanges the 1st and 2nd vertex. This defines a discrete Morse flow on X^b. To construct f^k, we take the subflow of f^{k-1} which is trivial on the $k-1$-skeleton $X_{k-1} \subset X^b$, and in the interior of all higher simplices we apply the permutation exchanging the $k-1$st and kth vertices. We will phrase a precise conjecture that we expect these vector fields to satisfy in Chapter 4.

Chapter 2

Products and Intersections

In this chapter we will develop a product structure on cochains which is related to the geometric intersection of chains by Poincaré duality. The basic results relating the cup product to intersections were the original motivation for the cup product and can be found in most older references, eg. [14]. We will use our branching Morse flow to also give a geometric interpretation of the cap product (a discrete Morse flow formula for the cup product appeared in [19]) and the trace of the Morse flow to give geometric interpretation (and pleasant formulas) for the $\cup_i$ products of Steenrod [2].

2.1 Basic Products

2.1.1 Cup Product

Let X be a combinatorial CW complex and let us study cohomlogy where the coefficient group $A = R$ is a ring.

Given a triangulation with branching structure of X it is possible to define a product on the cochain complex $C^*(X, R) = \bigoplus_k C^k(X, R)$ which imitates the wedge product of differential forms. Given a j-cochain α and a k-cochain β we construct a $j+k$ cochain $\alpha \cup \beta$ by assigning its value on a $j+k$-simplex:

$$(\alpha \cup \beta)(0 \cdots j+k) = \alpha(0 \cdots j)\beta(j \cdots j+k),$$

where on the right hand side we use the product in R. There are generalizations of this cup product when R has different sorts of products. For instance, an important case for us with when the coefficients form a Lie algebra $A = \mathfrak{g}$.

In this case one may define a cup product or "cup bracket"

$$(\alpha \cup \beta)(0 \cdots j + k) = [\alpha(0 \cdots j), \beta(j \cdots j + k)].$$

Something at first disturbing but later rather awe-inspiring is that the cup product defined above is not graded-commutative like its cousin, the wedge product. We will appreciate this phenomenon as part of the dependence of $\cup$ on the branching structure of the triangulation. Indeed, reversing the entire branching structure exchanges $\alpha(0 \cdots j)\beta(j \cdots j+k)$ with $\beta(0 \cdots k)\alpha(k \cdots j+k)$.

2.1.2 Cap Product and Morse Flow

We have discussed above an isomorphism for a PL n-manifold X between $C^k(X, A)$ and $C_{n-k}(X^\vee, A)$, where $X^\vee$ denotes the dual PL structure. In this section we will describe a map, which depends on a choice of branching structure,

$$- \cap X : C^k(X, A) \to C_{n-k}(X, A)$$

which depends on a fundamental cycle $X \in Z_n(X, \mathbb{Z})$ which depends on an orientation. We will define the map in terms of the (left) cap "product",

$$- \cap - : C^k(X, R) \otimes C_{k+j}(X, {}_R M) \to C_j(X, {}_R M),$$

which should really be thought of as not so much a product as a left action of C^* on C_*. To emphasize this, we write it in terms of a commutative ring R and a left R-module ${}_R M$. Indeed, the cap product will be defined to associate with the cup product, meaning

$$(\alpha \cup \beta) \cap \Sigma = \alpha \cap (\beta \cap \Sigma).$$

Further, if $j = 0$, then the cap product $\alpha \cap \Sigma \in C_0(X, {}_R M)$ is related to integration. If we write

$$\Sigma = \sum_{(0 \cdots k) \in X_k} \Sigma_{(0 \cdots k)} [0 \cdots k]$$

then

$$\alpha \cap \Sigma = \sum_{(0 \cdots k)} \left(\int_{(0 \cdots k) \in X_k} \alpha \right) \cdot \Sigma_{(0 \cdots k)} [0].$$

For general j, we define the map on a $k+j$-simplex by

$$\alpha \cap (0 \cdots k+j) = \left(\int_{(j \cdots k+j)} \alpha \right) [0 \cdots j]$$

and we extend to the whole domain by linearity. Note that this is supported on the set of initial j-simplices.

We also define a right cap product

$$(0 \cdots k+j) \cap \alpha = [j \cdots j+k] \left(\int_{0 \cdots j} \alpha \right)$$

which is supported on the set of final j-simplices. Likewise, the right cap product defines a second map

$$X \cap - : C^k(X, A) \to C_{n-k}(X, A)$$

which we will show later is homotopic to the first map. The left and right cap product commute with eachother:

$$\alpha \cap (\Sigma \cap \beta) = (\alpha \cap \Sigma) \cap \beta.$$

Indeed, if the coefficients of Σ are 1 or 0, then

$$\alpha \cap (\Sigma \cap \beta) = \alpha \cap (\sum_{(0 \cdots n) \in \Sigma} \beta(0 \cdots j)[j \cdots n])$$

$$= \sum_{(0 \cdots n) \in \Sigma} \beta(0 \cdots j)[j \cdots n-k] \alpha(k \cdots n)$$

$$= (\sum_{(0 \cdots n) \in \Sigma} \alpha(n-k \cdots n)[0 \cdots n-k]) \cap \beta$$

$$= (\alpha \cap \Sigma) \cap \beta.$$

The cap products also play well with the boundary map [10], for $\alpha \in C^k(X, R)$, we have

$$(-1)^k \partial(\Sigma \cap \alpha) = (\partial \Sigma) \cap \alpha - \Sigma \cap d\alpha.$$

We can give a new interpretation of the cap product in terms of the branching Morse flow:

Theorem 7. *Cap Product and Morse Flows* Let $\alpha \in C^k(X, A)$ and denote its Poincaré dual $W(\alpha) \in C_{n-k}(X^\vee, A)$. $X^\vee$ is a coarsening of the barycentric subdivision X^b and so we may pushforward the Poincaré dual chain to $W(\alpha) \in C_{n-k}(X^b, A)$. Then, denoting f the branching Morse flow of X^b, defined by the branching structure of X in theorem 3, we have

$$f_\infty W(\alpha) = X \cap \alpha.$$

$$f_{-\infty} W(\alpha) = \alpha \cap X.$$

Proof. Indeed, let us look at how $X^\vee$ is included in X^b, in particular focusing on an n-simplex $\Delta^n \in X_n$ and a k-simplex $\sigma_k \in \partial \Delta^n$. Its dual $n-k$-cell $\sigma_k^\vee$ is divided into several $n-k$-simplices in $X^\vee$. Those contained in Δ^n are labeled by $n-k$-sequences of simplicies of Δ^n starting with σ_k and ending with Δ^n:

$$\sigma_k < \sigma_{k+1} < \cdots < \sigma_n = \Delta^n.$$

Of these, only one of them is in the stable cell of a critical $n-k$-simplex of Δ^{nb}, namely

$$(0 \cdots k) < (0 \cdots k+1) < \cdots < (0 \cdots n).$$

For this $n-k$-simplex the relevant part of the flow goes

$$\begin{gathered}((0 \cdots k) < \cdots < (0 \cdots n)) \\ \to ((k) < (0 \cdots k) < \cdots < (0 \cdots n)) \\ > ((k) < (0 \cdots k+1) < \cdots < (0 \cdots n)) \\ \to ((k) < (k\ k+1) < (0 \cdots k+1) < \cdots < (0 \cdots n) \\ > \cdots > ((k) < (k\ k+1) < \cdots < (k \cdots n)),\end{gathered}$$

landing on the critical $n-k$-simplex corresponding to the complementary $n-k$-simplex $(k \cdots n)$. Thus, we see that restricting f to those pairs which lie in $\Delta^{nb} \subset X^{nb}$, we have

$$f_\infty W(\alpha) = \alpha(0 \cdots k)[k \cdots n],$$

which proves the first equality. The second equality then follows by reversing the branching structure. $\square$

Corollary 1. If $\Sigma \in C_{j+k}(X, R)$, $\alpha \in C^k(X, R)$ then because the branching Morse flow on Δ^n restricts to the branching Morse flow on its facets,

$$\Sigma \cap \alpha = f_\infty(\Sigma \cap W(\alpha))$$

$$\alpha \cap \Sigma = f_{-\infty}(\Sigma \cap W(\alpha)).$$

2.1.3 Intersections of Chains

In this section, we would like to give some more geometric context for the previous theorem and also explain the geometry behind the combinatorial definition of the cup product.

let $\Sigma \in C_k(X, R)$, $\Xi \in C_{n-k}(X^\vee, R)$ be chains of complementary dimension in a PL n-manifold X. Because each k-cell of X meets exactly one $n-k$-cell of $X^\vee$ in exactly one point, Σ and Ξ are guaranteed to meet transversely. This allowes us to define the intersection number

$$\#(\Sigma \cap \Xi) \in R.$$

Note that these intersection points occur at the vertices of X^b at the barycenters of the k-simplices of X (but for $k \neq 0, n$ these intersection points lie in neither X nor $X^\vee$). We can refine the intersection number to a geometric intersection pairing

$$C_k(X, R) \otimes C_{n-j}(X^\vee, R) \to C_{k-j}(X^b, R).$$

In the barycentric subdivision, a k-simplex $\sigma \in X_k$ is refined to a collection of p-simplices labelled by descending p-chains

$$\rho_0 < \cdots < \rho_p = \sigma,$$

while $\tau^\vee$ for $\tau \in X_j$ is refined to q-simplices labelled by ascending q-chains

$$\tau = \rho_0 < \cdots < \rho_q.$$

The geometric intersection between σ and $\tau^\vee$ is thus given by the collection of chains

$$\tau = \rho_0 < \cdots < \rho_q = \sigma,$$

of which the top dimensional ones have $q = k - j$. Thus we will define

$$\sigma \cap \tau^\vee = \sum \pm(\tau = \rho_0 < \cdots < \rho_q = \sigma) \in C_{k-j}(X^b, R)$$

where the sign must be determined. To understand it, note that using the ambient orientation, a tangent orientation of a submanifold is the same as a normal orientation. Thus, we obtain a normal orientation of $N(\sigma \cap \tau^\vee) = N\sigma \oplus N\tau^\vee$ and hence of $T(\sigma \cap \tau^\vee)$. We choose the signs above to give $\sigma \cap \tau^\vee$ this orientation.

One way to present these signs is to write $\tau = (a_0 \cdots a_j)$, $a_0 < \cdots < a_j$, $\sigma = (b_0 \cdots b_k)$, $b_0 < \cdots < b_k$, $\{c_0, \cdots, c_{n-k}\} = \{0, \cdots, n\} - \{b_0 \cdots b_k\}$, $c_0 < \cdots < c_{n-k}$ and extend each simplex in the sum to an n-simplex:

$$(a_0) < (a_0 a_1) < \cdots < (a_0 \cdots a_j) = \tau = \rho_0 < \cdots$$

$$< \rho_{k-j} = \sigma = (b_0 \cdots b_k) < (b_0 \cdots b_k c_0) < \cdots < (0 \cdots n).$$

This n-simplex receives an orientation from the orientation of X as well as from the ascending branching structure of the barycentric subdivision. The coefficient of the corresponding term in the sum is $+1$ if these agree, -1 otherwise.

Note that when $j = k$, we have

$$\Sigma \cap \Xi^\vee \in C_0(X, R),$$

and the sum of the coefficients is the intersection number $\#(\Sigma \cap \Xi^\vee)$.

There is also an intersection pairing that can be defined for chains on the same CW complex. However, intersections of such chains are never transverse if they are non-empty. In order to define the intersection numbers, we will need to perturb the chains slightly so that the intersections are transverse.

A convenient way of doing this is to choose a vector field on X and let one of the chains flow for a small time ϵ along the vector field. For a generic vector field, the result will be transverse.

In fact, given a branching structure on a triangulated n-manifold X, we can choose a very useful vector field, which we already considered in the proof of the branching Morse flow theorem 3. On the standard n-simplex embedded in $\mathbb{R}^n$ with coordinates $x_0, \ldots, x_n$ it is the gradient of

$$f(x_0, \ldots, x_n) = x_0 + a x_1 + a^2 x_2 + \cdots + a^n x_n,$$

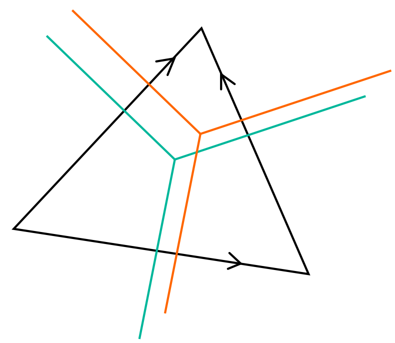

Figure 2.1: A 2-simplex of the triangulation X is drawn in black, with branching structure indicated by the directed edges. Part of the dual CW complex $X^\vee$ is drawn in teal with its pushoff drawn in orange. We see that the intersection between $X^\vee$ and its pushoff occurs between the teal edge dual to (12) and the orange edge dual to (01). We recommend the reader to draw the version for a 3-simplex.

projected onto the n-simplex

$$\Delta^n = \{\vec{x} \quad | \quad x_0 + \cdots + x_n = 1\}.$$

The branching structure gives us an identification of each n-simplex of X with the standard n-simplex, but they may glue along $n-1$-simplices corresponding to different $n-1$-subsets of $\{0, \ldots, n\}$. We are guaranteed that the labels from each side form a monotonically increasing bijection between these subsets though, which means there are j, k such that

$$f_{\text{left}}(ax_0, \ldots, ax_j, x_{j+1}, \ldots x_{n-1}) = f_{\text{right}}(ax_0, \ldots, ax_k, x_{k+1}, \ldots x_{n-1}),$$

after using the branching structure to identify the $n-1$-simplex with the standard one. j, k are the "missing indices" from the n-simplices on either side. This allows us to interpolate the vector fields by a diagonal transformation. This interpolation happens near the $n-1$-skeleton and won't be important for computing intersection numbers of chains in $X^\vee$.

Let $\Sigma \in C_k(X^\vee, R), \Xi \in C_{n-k}(X^\vee, R)$. We can define the pushoffs $f_\epsilon\Sigma$ and $f_\epsilon\Xi$ to be the small-time flows of Σ and Ξ respectively along this vector field. The results are Whitney chains transverse to X^b. In particular, the intersections $\Sigma \cap \Xi_{+\epsilon}$ and $\Sigma_{+\epsilon} \cap \Xi$ are both transverse.

Theorem 8. *Intersection Theorem* For $\Sigma \in C_k(X^\vee, R), \Xi \in C_{n-k}(X^\vee, R)$, f the branching Morse flow on X,

$$\#(f_\epsilon \Xi \cap \Sigma) = \#(f_\infty \Xi \cap \Sigma) = \int_{f_\infty \Xi} \delta_\Sigma = \int_X \delta_\Xi \cup \delta_\Sigma,$$

further,

$$f_\infty(f_\infty \Xi \cap \Sigma) = X \cap (\delta_\Xi \cup \delta_\Sigma).$$

Equivalently, for $\alpha \in C^k(X, R)$, $\beta \in C^j(X, R)$,

$$f_\infty(f_\infty W(\alpha) \cap W(\beta)) = X \cap (\alpha \cup \beta).$$

Proof. The first equality holds because

$$f_\infty(f_\epsilon \Xi \cap \Sigma) = f_\infty((f_\infty f_\epsilon \Xi) \cap \Sigma) = f_\infty((f_\infty \Xi) \cap \Sigma) \in C_0(X, R)$$

and f_∞ does not affect the point count. The second equality follows from our theorem on the cap product and f_∞ of theorem 3, the third and the last follow from the properties relating $\cup$ and $\cap$. The equivalent statements follow from Poincaré duality. □

The equality between the fourth and first expression is illustrated in Fig 2.1.

Corollary 2. By exchanging the branching structure with its reverse, we obtain

$$\#(f_{-\infty}\Sigma \cap \Xi) = (-1)^{k(n-k)} \int_X \delta_\Xi \cup \delta_\Sigma$$

as well as, for arbitrary $\alpha \in C^j(X, R)$, $\beta \in C^k(X, R)$,

$$f_\infty(f_\infty W(\alpha) \cap W(\beta)) - f_{-\infty}(f_{-\infty} W(\beta) \cap W(\alpha)) = (f_\infty - f_{-\infty}) W(\alpha \cup \beta).$$

Indeed, while $\alpha \cup \beta$ is invariant under simultaneous exchange of α, β and the branching structure with its reverse,

$$X \cap (\alpha \cup \beta) \mapsto (\alpha \cup \beta) \cap X.$$

There is an interesting symmetrical combination,

$$\alpha \cap X \cap \beta,$$

which places the intersection point between complementary dual simplices $(0 \cdots j)^\vee$ and $(j \cdots n)^\vee$ at vertex j. It can be written

$$f_{-\infty}(W(\alpha) \cap f_\infty W(\beta)) = f_\infty(f_{-\infty} W(\alpha) \cap W(\beta)), \tag{2.1}$$

where equality comes from the commutativity between left and right cap product.

2.2 (Non)-Commutativity

2.2.1 Cup-1 Product

The intersection theorem highlights the geometric difficulties in making a commutative cup product on cochains, namely fixing the location of the intersection points (which in either case geometrically lie disjoint from the 0-cells of either X or $X^\vee$), and the boundaries of the dual chains. The former ambiguity does not contribute to the count of the intersection points, but the latter does. Incredibly, both of these ambiguities have a common origin, and can be quantified together algebraically by the definition of a new product $\cup_1$ which satisfies

$$\alpha \cup \beta - (-)^{jk} \beta \cup \alpha = (-)^{j+k+1} d(\alpha \cup_1 \beta) + (-)^{j+k} (d\alpha) \cup_1 \beta + (-)^k \alpha \cup_1 (d\beta) \quad (2.2)$$

for $\alpha \in C^j(X, R)$, $\beta \in C^k(X, R)$,

$$\alpha \cup_1 \beta \in C^{j+k-1}(X, R).$$

This was first realized by Norman Steenrod [2]. The first term above encodes the first ambiguity mentioned above and the second two terms encode the second. This formula implies that the cup product is graded-commutative on cohomology, but actually we will see it has deep geometric content as well.

To facilitate the definition of the cup-1 product and further products, we define an alternating l-spine of an n-simplex to be a sequence of l consecutive subsets $A_1, B_1, A_2, B_2, \ldots$ of vertices of the n-simplex $(0 \cdots n)$ such that consecutive subsets share first and last elements. For instance, we have already seen alternating 2-spines A_1, B_1 in the definition of the cup product, which involves evaluating the first cochain on the simplex spanned by A_1 and the second cochain on the simplex spanned by B_1. Note that in [2], the pair $(A_1 \cdots), (B_1 \cdots)$ is called $l-2$-regular.

Observe that under a reversal of the branching structure, we get a bijection on the set of alternating l-spines which for even l exchanges the A's and B's and which for odd l preserves them. This will later give us a method for constructing the $\cup_{2i+1}$ products in terms of the $\cup_{2i}$ products, and in particular for $i = 0$ will give us a geometric picture of the $\cup_1$ product.

The $\cup_1$ product of j and k cochains is defined on a $j+k-1$-simplex by a sum over alternating 3-spines of that simplex A_1, B_1, A_2 such that $|B_1| = k+1$

by

$$(\alpha \cup_1 \beta)(0 \cdots j+k-1) = (-1)^{(j+1)(k+1)} \sum_{(A_1,B_1,A_2)} (-1)^{|A_1||B_1|} \alpha(A_1 \cup A_2)\beta(B_1). \tag{2.3}$$

As a baby example, for 1-cochains α, β, $\alpha \cup_1 \beta$ is also a 1-cochain, and one can check

$$(\alpha \cup_1 \beta)(01) = \alpha(01)\beta(01).$$

As a second, slightly more complex example, for a 1-cochain α and a 2-cochain γ, $\alpha \cup_1 \gamma$ is a 2-cochain, and one can check

$$(\alpha \cup_1 \gamma)(012) = \alpha(02)\gamma(012)$$

$$(\gamma \cup_1 \alpha)(012) = \gamma(012)(\alpha(01) + \alpha(12)).$$

With these in hand, we can understand the commutativity relation for the cup product of 1-chains α and β by following $\#(\Sigma_t \cap \Xi)$ through a homotopy singularity-by-singularity from $t = \epsilon$ to $t = -\epsilon$, from which we will show

$$\begin{aligned} \alpha \cup \beta + \beta \cup \alpha &= -d(\alpha \cup_1 \beta) + (d\alpha) \cup_1 \beta - \alpha \cup_1 (d\beta) \\ &= -\Big(\alpha(01)\beta(01) + \alpha(12)\beta(12) - \alpha(02)\beta(02)\Big) \\ &\quad + d\alpha(012)\Big(\beta(01) + \beta(12)\Big) - \alpha(02)d\beta(012). \end{aligned}$$

Shown in the figures, each of these terms comes from a singularity encountered during this homotopy.

This will lead us to a geometric picture of the $\cup_1$ product in general dimensions. Again as a warm-up, let us focus for the moment on $(\alpha \cup_1 \beta)(01) = \alpha(01)\beta(01)$. We can think of α and β as Poincaré dual to labelled points $W(\alpha), W(\beta)$ each lying at the midpoint of the 1-simplex (01). Accordingly, the intersection between these two points is not transverse and we must use the branching structure on the edge to separate them. Whether we use the positive flow or the negative flow along the vector field we get zero every time:

$$\#(W(\alpha)_{+\epsilon} \cap W(\beta)) = \#(W(\alpha)_{-\epsilon} \cap W(\beta) = 0.$$

However, any homotopy $W(\alpha)_t$ from $W(\alpha)_{-\epsilon}$ to $W(\alpha)_{+\epsilon}$ will have an unavoidable intersection with $W(\beta)$, and we see

$$\#(W(\alpha)_t \cap W(\beta)) = (\alpha \cup_1 \beta)(01) = (\beta \cup_1 \alpha)(01).$$

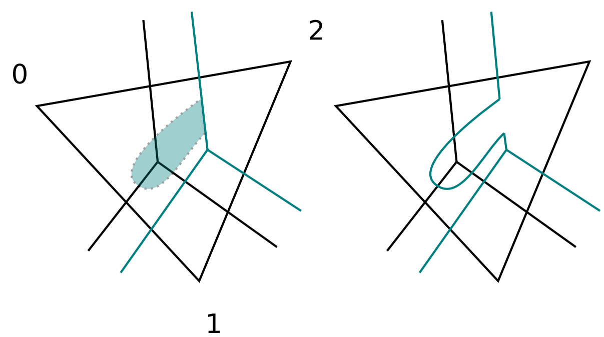

Figure 2.2: In this figure the Poincaré dual of α is drawn in blue and that of β is drawn in black. In the first step of the homotopy, the intersection number changes by an amount equal to the integral of $d\beta$ over the blue disc, times $\alpha(02)$, for a total contribution $\alpha(02)d\beta(012) = \alpha \cup_1 d\beta(012)$.

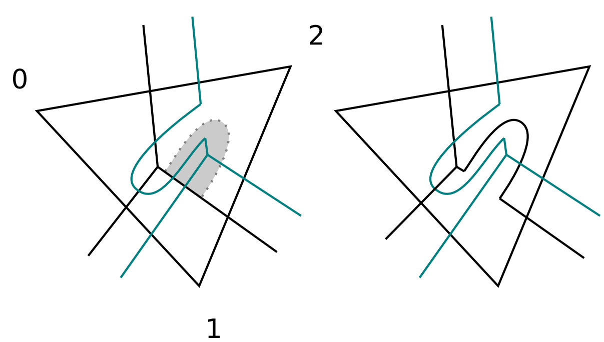

Figure 2.3: Here the intersection changes by $-d\alpha(012)\beta(12)$.

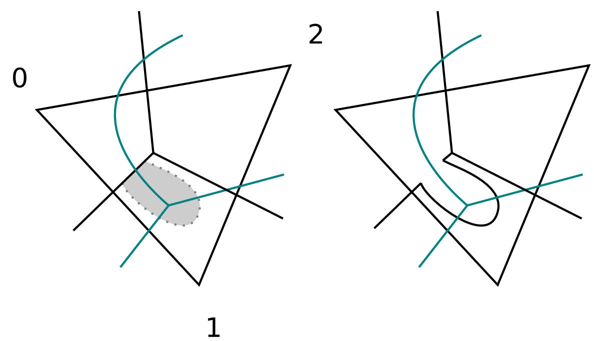

Figure 2.4: Here the intersection number changes by $-d\alpha(012)\beta(01)$.

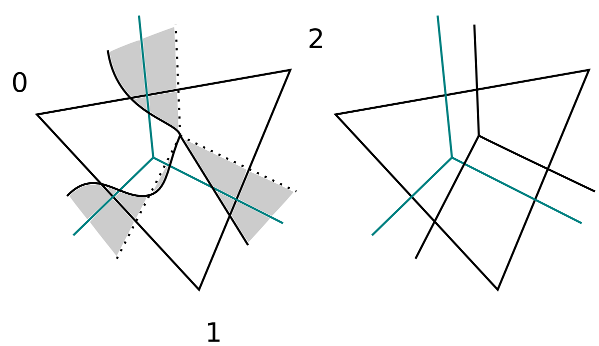

Figure 2.5: In the last step, intersection points are pushed to the boundary of the 2-simplex, and we see a variation $\alpha(01)\beta(01)+\alpha(12)\beta(12)-\alpha(02)\beta(02) = d(\alpha \cup_1 \beta)(012)$. The remaining intersection is $-\beta(01)\alpha(12) = -(\beta \cup \alpha)(012)$, finishing the proof of the commutativity relation.

As a second exercise, consider the intersection of a 1-chain $\gamma \in C_1(X^\vee, R)$ and a 0-chain $x \in C_0(X^\vee, R)$ inside a triangle of X. This intersection is not transverse. The novelty in this situation is that there are *two* choices of homotopies from $x_{-\epsilon}$ to $x_{+\epsilon}$. From the figure, we see that these intersection of γ with these two homotopies compute $\delta_x \cup_1 \delta_\gamma$ and $\delta_\gamma \cup_1 \delta_x$. The reader versed in the yoga of homotopy theory will guess that we will shortly understand the difference between these two as a homotopy of homotopies, and this will lead us to a geometric definition of a $\cup_2$ product. For now, let us note that the branching structure allows us to choose a preferred homotopy, namely one which passes above the 0-cell of $X^\vee$ wrt to the branching flow. We denote this homotopy x_{t+} and we see

$$\#(x_{t+} \cap \gamma) = \delta_x \cup_1 \delta_\gamma$$

while for the other homotopy x_{t-} we have

$$\#(x_{t-} \cap \gamma) = -\delta_\gamma \cup_1 \delta_x.$$

Let us also note that the alternating 3-spines of the 2-simplex (012), namely

$$(0)(012)(2), \quad (0)(01)(12), \quad (01)(12)(2)$$

are in correspondence with the edges of 2-simplex by which piece δ_γ gets evaluated on. These edges coincide by duality with the 1-cells where the homotopy crosses $X^\vee$.

2.2.2 Trace of the Morse Flow

On a manifold with branching structure, we can work instead with the infinite forward and backwards flows $f_{\pm\infty}\Sigma$ instead of the infinitesimal flows. Recall$f_{-\infty}$ is the Morse flow of the reversed branching structure. Likewise we let f_- be the trace of this 'reverse' Morse flow.

Consider for $\Sigma \in C_k(X^b, A)$,

$$(f_+ - f_-)\Sigma.$$

Using (1.3) have

$$\partial(f_+ - f_-)\Sigma = (f_\infty - f_{-\infty})\Sigma - (f_+ - f_-)\partial\Sigma. \tag{2.4}$$

We define

$$h_1(\Sigma) = f_\infty(f_+ - f_-)\Sigma = -f_\infty f_- \in C_{k+1}(X, A).$$

For $\Sigma \in C_k(X^\vee, A)$, which is especially when we'll use it, this is a "bordism with corners" from $f_{-\infty}\Sigma$ to $f_\infty\Sigma$ which has been pushed off from $X^\vee$ to X to be transverse to Σ. One can think of it as a generic 1-parameter family of pushoffs of Σ.

Rephrasing (2.4) with h_1, we have

Theorem 9. For $\Sigma \in C_k(X^b, A)$, we have

$$\partial h_1(\Sigma) + h_1(\partial\Sigma) = f_\infty\Sigma - f_{-\infty}\Sigma = X \cap \Sigma - \Sigma \cap X. \tag{2.5}$$

In other words, h_1 gives a chain homotopy between the left and right cap products.

Corollary 3. Let $\alpha \in C^j(X, R)$, $\beta \in C^k(X, R)$ and consider

$$h_1(W(\beta)) \cap \alpha \in C_{n-j-k+1}(X, R).$$

$$f_\infty(W(\alpha) \cap h_1(W(\beta))) \in C_{n-j-k+1}(X, R).$$

We have

$$\begin{gathered}\partial f_\infty(W(\alpha) \cap h_1(W(\beta)) \\ = X \cap (\alpha \cup \beta - (-1)^{jk}\beta \cup \alpha) + f_\infty(W(d\alpha) \cap W(\beta) + W(\alpha) \cap W(d\beta)).\end{gathered}$$

Proof. Using the theorem,

$$\partial(W(\alpha) \cap h_1(W(\beta)))$$

$$\begin{aligned}&= (\partial W(\alpha)) \cap h_1(W(\beta)) + (-1)^j W(\alpha) \cap \partial h_1(W(\beta)) \\ &= W(d\alpha) \cap h_1(W(\beta)) - (-1)^j W(\alpha) \cap h_1(W(d\beta)) \\ &\quad + (-1)^j W(\alpha) \cap (f_\infty W(\beta) - f_{-\infty} W(\beta)).\end{aligned}$$

□

Corollary 4. Let $\Sigma \in C_j(X^\vee, R)$ $\Xi \in C_k(X^\vee, R)$. $h_1(\Sigma)$ and Ξ are transverse so we can compute their intersection $n-j-k$-chain in X^b

$$\partial(h_1(\Sigma) \cap \Xi) = \partial h_1(\Sigma) \cap \Xi - (-1)^j h_1(\Sigma) \cap \partial\Xi \tag{2.6}$$

$$= (f_\infty \Sigma - f_{-\infty}\Sigma) \cap \Xi - h_1(\partial\Sigma) \cap \Xi - (-1)^j h_1(\Sigma) \cap \Xi$$

$$= f_\infty \Sigma \cap \Xi - (-1)^{jk} f_\infty \Xi \cap \Sigma - h_1(\partial\Sigma) \cap \Xi - (-1)^j h_1(\Sigma) \cap \Xi.$$

Recalling from theorem 8 that

$$\int_X \delta_\Sigma \cup \delta_\Xi = f_\infty \Sigma \cap \Xi$$

$$\int_X \delta_\Xi \cup \delta_\Sigma = f_\infty \Xi \cap \Sigma,$$

we see that $(-1)^{j+k+1} h_1(\Sigma) \cap \Xi$ satisfies a property completely analogous to $\delta_\Sigma \cup_1 \delta_\Xi$ in (2.2).

In fact, with a bit of combinatorics, one can prove the following theorem, which gives a geometric interpretation of the $\cup_1$ product:

Theorem 10. $\cup_1$ *theorem* For $\Sigma \in C_k(X^\vee, A), \Xi \in C_j(X^\vee, A)$ in a PL n-manifold X with branching structure,

$$(-1)^{j+k+1} f_\infty(\Sigma \cap f_\infty f_- \Xi) = X \cap (\delta_\Sigma \cup_1 \delta_\Xi).$$

Equivalently, for $\sigma \in X_j$, $\tau \in X_k$,

$$(-1)^{j+k+1} f_\infty(\sigma^\vee \cap f_\infty f_- \tau^\vee) = X \cap (I_\sigma \cup_1 I_\tau).$$

Proof. The theorem follows from comparing with the "join formulas" of Steenrod [2], which provide an inductive definition of $\cup_1$.

Given a vertex v in the link of a k-simplex σ, we may define the join (σv) to be the $k+1$-simplex spanned by v and the vertices of σ, with orientation induced by the branching structure of X. For a fixed vertex v, this defines a pair of maps

$$-v : C_k(X, \mathbb{Z}) \to C_{k+1}(X, \mathbb{Z}),$$

which on k-simplices σ is

$$\sigma v = \begin{cases} (\sigma v) & v \in \mathrm{Link}(\sigma) \\ 0 & \text{otherwise} \end{cases},$$

as well as

$$J_v : C^k(X, \mathbb{Z}) \to C^{k+1}(X, \mathbb{Z})$$

which on indicator k-cochains is

$$J_v(\sigma) = \begin{cases} I_{\sigma v} & v \in \mathrm{Link}(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

From the rest of the proof we take $X = \Delta^n$.

One can easily verify from the definition of the Morse flow that if v is the top vertex of an n-simplex and σ is a k-simplex in the "bottom facet", that is the $n-1$-simplex opposite v, then

$$h_1(\sigma)^\vee = h_1(\sigma)v.$$

Further, if τ is a j-simplex in the bottom facet, it is clear that if $\tau^\vee$ and σ intersect at the barycenter (ρ) that $(\tau v)^\vee$ and σv intersect at the barycenter (ρv). If τ is a j-simplex in the bottom facet, it follows when $n = j + k - 1$ that

$$f_\infty((\tau v)^\vee \cap h_1(\sigma^\vee)) = (-1)^k f_\infty(\tau^\vee \cap h_1(\sigma^\vee))v, \tag{2.7}$$

where the sign comes from the orientation of the join as we move the join to the outside of the expression.

The second property is

$$h_1(\sigma v)^\vee = (f_{\mp\infty}\sigma)v.$$

The proof of this property is much like the proof of the cap product formula since there is only one nonzero path through the Morse flow $f_\infty f_-$. This path occurs for $\sigma v = (n-k-1 \cdots n)$. In the Poincaré dual, there is the $n-k$-simplex

$$\rho_0 = (n-k \cdots n) < (n-k-1 \cdots n) < \cdots < (0 \cdots n).$$

f_- is computed as follows:

$$\rho_0 \to (n-k) < (n-k \cdots n) < \cdots < (0 \cdots n)$$

$$\to (n-k) < (n-k-1, n-k) < (n-k-1 \cdots n) < \cdots < (0 \cdots n)$$
$$\to (n-k-1) < (n-k-1, n-k) < (n-k-1 \cdots n)$$
$$\to \cdots \to (0) < (01) < \cdots < (0 \cdots k-1) < (0 \cdots n).$$

Then f_∞ is computed in a single step

$$(0) < (01) < \cdots < (0 \cdots k-1) < (0 \cdots n)$$

$$\to (0) < (01) < \cdots < (0 \cdots k-1) < (0 \cdots k-1, n),$$

which is the critical simplex in the join of $f_{-\infty}(k \cdots n-1)^\vee$.

From this it follows

$$\tau^\vee \cap h_1((\sigma v)^\vee) = 0 \tag{2.8}$$

$$(\tau v)^\vee \cap h_1((\sigma v)^\vee) = (-1)^k \tau^\vee \cap f_{-\infty}(\sigma^\vee). \tag{2.9}$$

The three "join formulas" (2.7), (2.8), (2.9) coincide with three inductive properties of [2] which characterize $\cup_1$ in terms of $\cup$. Matching the properties, and using the intersection theorem as a base case, we derive the theorem. □

2.2.3 $\cup_i$ Products and i-Parameter Families

One sees an obvious asymmetry in the definition of the $\cup_1$ product (2.3). This leads one to a whole tower of products

$$- \cup_i - : C^j(X, R) \times C^k(X, R) \to C^{j+k-i}(X, R)$$

which satisfy

$$d(\alpha \cup_i \beta) = (d\alpha) \cup_i \beta + (-1)^{i+j} \alpha \cup_i (d\beta) + \alpha \cup_{i-1} \beta + (-)^{jk+i} \beta \cup_{i-1} \alpha. \tag{2.10}$$

One can define the $\cup_i$ product as a sum over alternating $i+2$-spines of the $j+k-i$-simplex as:

$$\alpha \cup_i \beta = \sum_{(A_1 B_2 \cdots B_{i+2}) = (0 \cdots j+k-i)} \pm \alpha(A_1 A_2 \cdots) \beta(B_1 B_2 \cdots) \qquad i = 0 \mod 2,$$

$$\alpha \cup_i \beta = \sum_{(A_1 B_2 \cdots A_{i+2}) = (0 \cdots j+k-i)} \pm \alpha(A_1 A_2 \cdots) \beta(B_1 B_2 \cdots) \qquad i = 1 \mod 2.$$

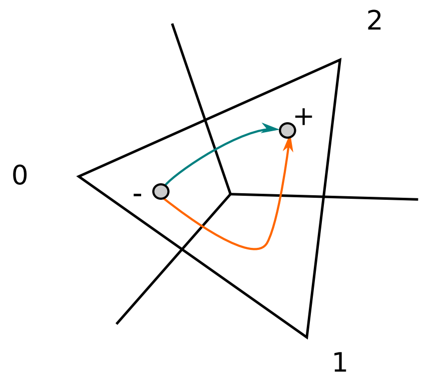

Figure 2.6: Two different homotopies from $x_{-\epsilon}$ to $x_{+\epsilon}$ inside a 2-simplex of X highlight the noncommutativity of the $\cup_1$ product. Each time the homotopy crosses a 1-cell of $X^\vee$, a term in the definition of the $\cup_1$ product is generated. Indeed, such crossings are in bijection with the alternating 3-spines of this 2-simplex.

See Steenrod [2], where such a pair $(A_1A_2\cdots),(B_1B_2\cdots)$ is described as an i-regular pair and the sign is described as the sign of a certain permutation.

Our goal is to understand this in a more geometric way. The simplest case of this noncommutativity is encountered in computing the $\cup_1$ product of a 2-cochain α and a 1-cochain β on a 2-simplex (012). We find, according to (2.3)

$$(\alpha \cup_1 \beta)(012) = \alpha(012)(\beta(01) + \beta(02))$$

$$(\beta \cup_1 \alpha)(012) = \beta(02)\alpha(012).$$

These correspond to the intersection numbers of the orange and blue homotopies of $\alpha^\vee$ with $\beta^\vee$, respectively, depicted in Fig 2.6. The blue homotopy is $h_1(012)^\vee = f_\infty f_-(012)^\vee = f_{-\infty}f_+(012)^\vee$. Note that reversing the branching structure fixes the two products rather than exchanging them, and equivalently h_1 is the same whether we compute it with the chosen branching structure or with its reverse.

To describe the orange homotopy, we need to use a different discrete Morse flow. We invoke the discrete Halperin-Toledo Morse flows we constructed at the end of Chapter 1. We find $f^2_\infty(f^1_+ - f^1_-)(012)^\vee = (01)+(02)$. Summarizing, we have

$$\#(W(\beta) \cap f^2_\infty(f^1_+ - f^1_-)(012)^\vee) = (\delta_{012} \cup_1 \beta)(012),$$

$$\#(W(\beta) \cap f^1_\infty(f^1_+ - f^1_-)(012)^\vee) = (\beta \cup_1 \delta_{012})(012).$$

We find as well

$$\#(W(d\beta) \cap f^1_\infty(f^2_+ - f^1_+)(f^1_+ - f^1_-)(012)^\vee) = (d\beta \cup_2 \delta_{012})(012) = d\beta(012).$$

This leads us to the following conjecture:

Conjecture 1. $\cup_i$ *Conjecture* Given a PL n-manifold X with branching structure, there exists a series of discrete Morse flows $f^1, \dots, f^n$ on the barycentric subdivision X^b (the discrete Halperin-Toledo Morse flows we have constructed), we have (schematically, neglecting signs)

$$X \cap (\alpha \cup_i \beta) = f_\infty(W(\alpha) \cap f^1_\infty(f^i_+ - f^{i-1}_+) \cdots (f^2_+ - f^1_+)(f^1_+ - f^1_-)W(\beta)).$$

The geometric intuition behind this conjecture is that

$$h_i W(\beta) := f_\infty(f^i_+ - f^{i-1}_+) \cdots (f^2_+ - f^1_+)(f^1_+ - f^1_-)W(\beta)$$

is a generic i-parameter family of push-offs of $W(\beta) \in X^\vee$, expressing a homotopy between homotopies implementing the $\cup_i$ property (2.10). Indeed, if V is a k-cycle in $X^\vee$, then V is transverse to X and so does not intersect the $n-k-1$-skeleton. Thus, the Halperin-Toledo vector fields $F_1, \dots, F_{n-k}$ are linearly independent on V. This allows us to define a generic $n-k$-parameter family of push-offs of V by flowing along the Halperin-Toledo vector fields. We may make the analogous $\cup_i$ conjecture in this setting by saying that the $\cup_i$ product is Poincaré dual to intersection with the trace of these push-offs.

2.2.4 Linking Pairing and Steenrod Squares

The geometric takeaway from the previous discussion is that for j and k chains Σ, Ξ in a PL $j+k+1$-manifold $X^\vee$, we have a quantity

$$\int_X \delta_\Sigma \cup_1 \delta_\Xi = \#(h_1(\Sigma) \cap \Xi)$$

which can be thought of as a kind of linking number, because of the dimensions of the chains involved.

However, while the intersection number, which is a point count, can be computed by an integral of Poincaré duals, there is no such integral formula for the global linking number, since there are no points that are being counted. Indeed, usually one has to fill in one of the chains, say $\Sigma = \partial M$, and then one can compute the global linking number as an intersection number:

$$\#(M \cap \Xi).$$

We should think of $h_1(\Sigma)$ as a 1-dimensional thickening of Σ defined by the branching structure. It is thus like a piece of a filling M, so when we compute

$$h_1(\Sigma) \cap \Xi \subset M \cap \Xi$$

it's like we're counting the linking number only of pieces of Σ and Ξ which are very close, in fact abutting the same n-simplex.

This all works out especially nicely when $\Sigma = \Xi$, since in this case, the two 1-cycles are as nearby to eachother as it gets. Indeed in this case

$$\int_X \delta_\Sigma \cup_1 \delta_\Sigma = \#(h_1(\Sigma) \cap \Sigma)$$

is a proper count of the self-linking number of Σ. Indeed, let $\alpha \in Z^k(X, R)$. Under $\alpha \mapsto \alpha' = \alpha + df$ we have

$$\alpha' \cup_1 \alpha' = \alpha \cup_1 \alpha + (1 + (-1)^k)(f \cup df + \alpha \cup_1 df) + d(f \cup_1 df + f \cup f + \alpha \cup_2 df).$$

Thus, in a closed manifold, the self-linking number is a well-defined element of the coefficient ring R when k is odd, otherwise it is a well-defined element mod $2R$. In the case $k = 1$ of curves in a 3-manifold, this reflects the well-known fact that the framed bordism group in one dimension is $\mathbb{Z}_2$ [24]:

$$\Omega_1^{fr} = \mathbb{Z}_2.$$

Likewise, for $\Sigma \in C^k(X, R)$ in a $2k + i$-manifold, we may define a "higher self-linking number"

$$\#(h_i(\Sigma) \cap \Sigma) = \int_X \delta_\Sigma \cup_i \delta_\Sigma,$$

which according to the conjecture is computing the intersection number of Σ with a transverse i-parameter families of pushoffs $h_i(\Sigma)$. These operators in cohomology are called the Steenrod squares and are defined on cochains by

$$Sq^l : C^k(X, A) \to C^{k+l}(X, A),$$

$$Sq^{k-i}\alpha = \alpha \cup_i \alpha,$$

$$\int_X Sq^{k-i}\alpha = \#(h_i(W(\alpha) \cap W(\alpha)) \qquad conjectural.$$

Under $\alpha \mapsto \alpha + df$, $Sq^l\alpha$ changes by an exact quantity mod 2. The Steenrod squares thus descend to cohomology operations

$$Sq^l : H^k(-, \mathbb{Z}_2) \to H^{k+l}(-, \mathbb{Z}_2).$$

The algebra they generate, called the Steenrod algebra, is the algebra of all stable $\mathbb{Z}_2$ cohomology operations [25]. This means that any natural transformation

$$H^k(-, \mathbb{Z}_2) \to H^{k+l}(-, \mathbb{Z}_2)$$

which is a group homomorphism is a sum of products of Steenrod squares of total degree l. They are central to all things in topology modulo 2 and we will even see them make some cameos in physics.

Chapter 3

∞-Groups

Introduction

In this chapter, we explore some of the higher categorical aspects of simplicial cohomology, with a view towards geometric tangent structures and higher symmetries of quantum field theories.

At the center of the discussion is the concept of an ∞-group. A wonderful introduction to this subject is given by Baez and Shulman in [26]. In its most elegant but perhaps most opaque definition, an ∞-group is an ∞-groupoid with a single object, which is an ∞-category all of whose k-morphisms are invertible. Intuitively, this single object is a thing, the 1-morphisms are symmetry transformations of it, the 2-morphisms are transformations between transformations (or symmetries of symmetries), and so on. When the k-morphisms above some n are all the identity, we call it an n-group. A 1-group is the same thing as an ordinary group, realized as the morphisms of a category with one object.

A relevant example for us, the group of cochains $C^n(X, A)$ may be given the structure of an n-category. It's objects are the n-cochains themselves, while morphisms between n-cochains $\alpha_1 \to \alpha_2$ are given by $n-1$-cochains β such that

$$d\beta = \alpha_2 - \alpha_1.$$

2-morphisms $\beta_1 \to \beta_2$ are given by $n-2$-cochains γ with

$$d\gamma = \beta_2 - \beta_1,$$

and so on, until we reach the n-morphisms, which are 0-cochains, and we can go no further. The identity k-morphisms are given by the zero $n-k$-cochains, so every k-morphism β has an inverse $-\beta$, making $C^n(X, A)$ an n-groupoid. Restricting our attention to any particular object yields an n-group. We will see that these n-groups act as symmetries of certain quantum field theories later. Note that the additive structure of $C^n(X, A)$ is on the level of objects, and is additional data on top of the n-groupoid structure[1].

One can construct a triangulated space associated to an ∞-groupoid $\mathbb{G}$ called its classifying space $B\mathbb{G}$. The k-simplices of this space are diagrams in the category in the shape of a k-simplex, meaning that the vertices are labelled with objects, the edges with 1-morphisms, the faces with 2-morphisms, and so on, such that the labels create a sensible diagram (all morphisms are composable).

The classifying space is extremely useful for translating algebraic data about $\mathbb{G}$ into topological data about $B\mathbb{G}$. For instance, a $\mathbb{G}$ action on an object M in an ∞-category $\mathcal{C}$ is a functor from $\mathbb{G}$ to $\mathcal{C}$ which sends the unique object of $\mathbb{G}$ to M. To such an action we can form an "∞-bundle" over $B\mathbb{G}$ which can be visualized intuitively in the "dual CW complex" of $B\mathbb{G}$, which has a single top cell σ_0 (typically an infinite-dimensional polytope) associated to the single vertex of $B\mathbb{G}$, over which we have the trivial bundle $\sigma_0 \times M$. Then, facets of σ_0 meet each other perpendicular to 1-simplices, which are labelled with 1-morphisms in $\mathbb{G}$, and whose action on M specifies how the bundle glues. Where facets meet there are 2-morphisms specifying the gluing of gluing functions, and so on. In fact, all of this data is *equivalent* to the action of $\mathbb{G}$ on M, so bundles over $B\mathbb{G}$ with fiber M are the same as $\mathbb{G}$ actions on M. See [27].

For ordinary 1-groups, this is a familiar situation. Representations of G are equivalent to vector bundles over BG. The characteristic classes of these vector bundles are cohomology classes in $H^*(BG, A)$, which encode algebraic information about the representation. Our approach will be to study everything about the ∞-group using the classfying space.

3.1 The Postnikov Tower

For our purposes, we need a more compact way to describe an ∞-group rather than just its collection of objects, k-morphisms, and composition rules. First, note that we can define a sequence of groups Π_k, where Π_k is the group of k-morphisms modulo $k+1$-morphisms over the unique object when $k = 1$ and over the identity $k-1$-morphism when $k > 1$. Π_1 can be any group, but by the Eckman-Hilton argument [13] $\Pi_{>1}$ are all abelian. One can also define an action α_k of Π_1 on Π_k, which generalizes the concept of conjugation (inner

[1] Later we will see it means that $C^n(X, A)$ is the "loop category" of an $n+1$-group $BC^n(X, A)$.

automorphism) of a 1-group, as well as a number of "associators", which are encoded in the Postnikov classes

$$\beta_k \in H^{k+1}(B\mathbb{G}_{<k}, \Pi_k) \qquad k \geq 2,$$

where $\mathbb{G}_{<k}$ is a $k-1$-group obtained from $\mathbb{G}$ by discarding all non-identity $\geq k$-morphisms. The data of the Π_i, α_j, and β_k specify $\mathbb{G}$ up to isomorphism, so we will often refer to this data as the ∞-group itself. Some references for this are [28, 29, 13, 30, 27, 31].

From this data, we may obtain the classifying space $B\mathbb{G}$ as an iterated fibration of Eilenberg-Maclane spaces $B^k\Pi_k = K(\Pi_k, k)$ called the Postnikov tower:

$$\begin{tikzcd}
 & B\mathbb{G} \arrow[d] \\
 & \vdots \\
B^3\Pi_3 \arrow[r] & B\mathbb{G}_{\leq 3} \arrow[d] \\
B^2\Pi_2 \arrow[r] & B\mathbb{G}_{\leq 2} \arrow[d] \\
 & B\Pi_1.
\end{tikzcd}$$

Any connected CW complex X gives rise to such data where $\Pi_k = \pi_k X$. This ∞-group is called the homotopy type of X, which is justified by the following theorem (see [29]):

Theorem 11. *Geometric Realization of Homotopy Types* Any locally-finite connected CW complex X is homotopy equivalent to the classifying space of its homotopy type.

In [32], I used this theorem to draw an analogy between quantum field theories which depend on a parameter space X and systems with a symmetry given by the homotopy type of X. As far as adiabatic homotopy invariants are concerned, these concepts are exactly equivalent.

3.2 Nonabelian Cohomology

There is a notion of cohomology with coefficients in an ∞-group $\mathbb{G}$ called nonabelian cohomology. The nonabelian cohomology $H^1(X, \mathbb{G})$ is the ∞-category of functors from the homotopy type of X to $\mathbb{G}$. For example, when $\mathbb{G}$ is a 1-group $\Pi_1 = G$ (which may be nonabelian), this is the category of homomorphisms

$$\pi_1 X \to G$$

and morphisms (natural transformations of functors) are labelled by G and act on this map by conjugation. Thus, equivalence classes of objects in $H^1(X, G)$ classify principal G-bundles.

This leads to another definition of nonabelian cohomology: $H^1(X, \mathbb{G})$ is the homotopy type of the mapping space from X to $B\mathbb{G}$. If X is a CW complex, then using cellular approximation we may thus obtain a very concrete cochain picture of the nonabelian cohomology. A map $X \to B\mathbb{G}$ is equivalent to a collection of cochains

$$a_j \in C^j(X, \Pi_j)$$

satisfying the nonabelian cocycle equations:

$$\begin{aligned} da_1 &= 1 \\ D_a a_2 &= \beta_2(a_1) \\ D_a a_3 &= \beta_3(a_1, a_2) \\ D_a a_4 &= \cdots , \end{aligned} \tag{3.1}$$

where the twisted differential D_a (see section 1.2.5) is defined by the action of Π_1 on the Π_j. Such data in physics is called a $\mathbb{G}$ gauge field [33, 34]. We will use the shorthand $a \in Z^1(X, \mathbb{G})$ to discuss this data.

There also is a notion of cellular homotopy of such maps which is important to discuss. In physics these are called $\mathbb{G}$ gauge transformations. They are parametrized by $f_j \in C^{j-1}(X, \Pi_j)$ and act by

$$a_k \mapsto a_k \qquad k < j$$

$$a_j \mapsto a_j + D_a f_j$$

$$a_k \mapsto a_k + \beta_{k,1}(f_j; a_1, \ldots, a_{k-1}) \qquad k > j,$$

where $\beta_{k,1}$ is a first descendant of β_k (which we discuss below). We will often use the shorthand $f \in C^0(X, \mathbb{G})$ to discuss the collection of f_j. These homotopies form the 1-morphisms of $H^1(X, \mathbb{G})$, and one can imagine now how the higher morphisms are defined, analogous to the n-groupoid structure of $C^n(X, A)$.

When $\mathbb{G}$ is stable (see below), there is also an associative, invertible, and functorial multiplication on the elements of $H^1(X, \mathbb{G})$ which makes it the loop category of an ∞-group we abusively denote $BH^1(X, \mathbb{G})$, meaning that the homotopy type of the loop space of $BH^1(X, \mathbb{G})$ (typically disconnected) is $H^1(X, \mathbb{G})$. This multiplication is component-wise in the a_j, corrected by certain descendants of the Postnikov classes β_k. When $\mathbb{G}$ is not stable however, $H^1(X, \mathbb{G})$ has no natural group structure, just like nonabelian principal G bundles have no group structure.

3.2.1 Higher Nonabelian Cohomology

Related to lack of group structure on $H^1(X, \mathbb{G})$, when Π_1 is nonabelian, there is no way to define $H^{\geq 2}(X, \mathbb{G})$. Even when Π_1 is abelian, we also need that the actions α_k are trivial and that the Postnikov classes define stable cohomology operations, meaning that they induce group homomorphisms on nonabelian cohomology:

$$\beta_k : H^1(X, \mathbb{G}_{<k}) \to H^{k+1}(X, \Pi_k).$$

In this case, we can define a delooping of $\mathbb{G}$, abusively denoted $B\mathbb{G}$. This is an ∞-group with a single 1-morphism whose automorphism ∞-group is $\mathbb{G}$. It has

$$\Pi_1(B\mathbb{G}) = 0$$

$$\Pi_{k+1}(B\mathbb{G}) = \Pi_k(\mathbb{G}).$$

Because the Postnikov classes are stable operations, they may also be delooped, beginning with

$$\beta_3(B\mathbb{G}) = B\beta_2(\mathbb{G}) \in H^4(B^2\Pi_1, \Pi_2),$$

which has the property that under the loop map

$$H^4(B^2\Pi_1, \Pi_2) \to H^3(B\Pi_1, \Pi_2)$$

it maps to $\beta_2(\mathbb{G})$. Then we may define

$$H^k(X, \mathbb{G}) = H^1(X, B^{k-1}\mathbb{G}).$$

Such a $\mathbb{G}$ is called a stable ∞-groups or a connective Ω spectrum and defines a generalized cohomology theory. For more, see [35, 36, 37].

3.2.2 Twisted Nonabelian Cohomology

Given a bundle

$$\begin{array}{ccc} B\mathbb{G} & \longrightarrow & E \\ & & \downarrow \\ & & X \end{array}$$

We may define the twisted nonabelian cohomology $H^1(X, E)$ to be sections of this bundle. Note that the data of this bundle is the same as the data of an action of the homotopy type of X on $B\mathbb{G}$ [27].

When $\mathbb{G}$ is stable, we can make this relatively concrete. The action of the homotopy type of X on $B\mathbb{G}$ is given by an action ρ of $\pi_1 X$ on all the Π_k as well as a sequence

$$c_{j+1} \in C^{j+1}(X, \Pi_j)$$

satisfying the twisted nonabelian cocycle conditions for $c \in Z^2(X, \mathbb{G}^\rho)$:

$$D_\rho c_2 = 0$$

$$D_\rho c_3 = B\beta_3(c_2)$$
$$D_\rho c_4 = B\beta_4(c_2, c_3)$$
$$D_\rho c_5 = \cdots,$$

where $B\beta_k$ are the Postnikov classes of the delooping $B\mathbb{G}$.

For example, when $X = BH$ and $\mathbb{G} = G$ is an abelian 1-group, then such actions of H on $B\mathbb{G}$ are equivalent to group extensions, and are given by an action ρ of H on G and an extension class $c \in H^2(BH, G^\rho)$.

With this in hand, we can define a c-twisted nonabelian 1-cocyle $a \in Z^1(X, \mathbb{G}^c)$ as a sequence

$$a_j \in C^j(X, \Pi_j)$$

satisfying

$$da_1 = c_2$$

$$da_2 = B\beta_{3,1}(c_2, a_1) + c_3$$
$$da_3 = B\beta_{4,1}(c_3, c_2, a_2, a_1) + c_4$$
$$\cdots .$$

Gauge transformations act by higher descendants (see section 3.4).

3.3 Maps Between ∞-Groups

If $\mathbb{G}$ and $\mathbb{H}$ are ∞-groups, homomorphisms $\mathbb{G} \to \mathbb{H}$ form an ∞-group, in fact $\mathrm{Hom}(\mathbb{G}, \mathbb{H})$ may be taken to be the homotopy type of the space of continuous maps
$$B\mathbb{G} \to B\mathbb{H}.$$
But this is exactly what is captured by nonabelian cohomology, so we may write
$$\mathrm{Hom}(\mathbb{G}, \mathbb{H}) = H^1(B\mathbb{G}, \mathbb{H}).$$
Any element in this nonabelian cohomology defines a natural transformation of cohomology functors:
$$H^1(-, \mathbb{G}) \to H^1(-, \mathbb{H}).$$
When we express the elements of $\mathrm{Hom}(\mathbb{G}, \mathbb{H})$ using $Z^1(B\mathbb{G}, \mathbb{H})$, then we get a natural transformation on the level of cocycles:
$$Z^1(-, \mathbb{G}) \to Z^1(-, \mathbb{H}).$$
Concretely, describing the elements of the left hand side using the $a_j \in C^k(-, \Pi_k(\mathbb{G}))$, this is equivalently a sequence of cochain operations
$$\phi_k(a_1, \ldots, a_k) \in \Pi_k(\mathbb{H})$$
such that
$$d\phi_k(a_{\leq k}) = \beta_k(\phi_1(a_1), \phi_2(a_1, a_2), \ldots, \phi_{k-1}(a_{<k})),$$
where β_k are the Postnikov classes of $\mathbb{H}$. Those of $\mathbb{G}$ appear when expanding the left hand side of this relation.

In our physics applications of this theory, we are often interested in computing the cohomology operations which take as input a "gauge field"

$A \in Z^1(X, \mathbb{G})$ and produce a "Lagrangian" $\omega(A) \in Z^n(X, \mathbb{R}/\mathbb{Z})$. Thus we are interested in computing

$$H^1(B\mathbb{G}, B^{n-1}\mathbb{R}/\mathbb{Z}) = H^n(B\mathbb{G}, \mathbb{R}/\mathbb{Z}).$$

There is actually a nice technique for computing

$$H^n(B\mathbb{G}, A)$$

for any abelian group A. This is interesting even beyond our physics applications: if $\mathbb{G}$ is the homotopy type of a locally-finite CW complex X, then since cohomology is a homotopy invariant,

$$H^n(X, A) = H^n(B\mathbb{G}, A),$$

so if we can do this, we can compute the cohomology of almost any space whose homotopy type we already know.

The strategy is to consider first the cohomology of the product

$$\prod_k B^k \Pi_k$$

and then "turn on" the Postnikov classes one by one. It is simple to see that if we give $\prod_k B^k \Pi_k$ the product CW structure, then

$$C^*(\prod_k B^k \Pi_k, A) = \prod_k C^*(B^k \Pi_k, A).$$

The differential is computed in terms of the individual differentials by the Leibniz rule:

$$d = \sum_k d_k,$$

where d_k is the differential of $C^*(B^k \Pi_k, A)$. The group of cochains on $B\mathbb{G}$ is the same but now the differentials are the twisted differentials:

$$d'_k = D_k^{a_1} - \beta_k(a_1, \ldots, a_{k-1}),$$

$$d' = \sum_k d'_k.$$

The cohomology of d' is the cohomology of $B\mathbb{G}$. The degrees allows one to solve the cocycle equations $d'\omega = 0$ inductively. We will not spell out the whole procedure here. It is equivalent to iterating the Serre spectral sequence [38] for the Postnikov tower. Later we will see it used in an example.

3.4 Descendants

In this section we delve deeper into the functorial properties of cocycles $\omega \in Z^n(B\mathbb{G}, A)$. There is a very interesting general phenomenon, which is that if we have a homomorphism

$$\omega : Z^1(X, \mathbb{G}) \to C^n(X, A)$$

whose value on an n-cell $\omega(A)(\sigma_n)$ is determined by the retriction of $A \in Z^1(X, \mathbb{G})$ to σ_n, then the simple equation

$$d\omega(A) = 0 \qquad \forall A$$

means that ω gives an element in $H^n(B\mathbb{G}, A)$, which is a natural transformation of ∞-functors

$$H^1(-, \mathbb{G}) \to H^1(-, B^{n-1}A),$$

with all of the attendant coherence relations. These various coherence relations are called the descendants of ω, and we wish to compute them.

To begin, we will show that the cocycle equation $d\omega = 0$ is equivalent to the existence of a 1st descendant, ie. a natural transformation

$$\omega_1 : Z^1(X, \mathbb{G}) \times C^0(X, \mathbb{G}) \to C^{n-1}(X, A)$$

satisfying

$$d\omega_1(a, f) = \omega(a^f) - \omega(a), \tag{3.2}$$

where a^f denotes the action of $C^0(X, \mathbb{G})$ on $Z^1(X, \mathbb{G})$.

To see that this implies the cocycle condition, let V be an $n+1$-cell. For any $a \in Z^1(V, \mathbb{G})$, there is an $f \in C^0(V, \mathbb{G})$ such that $a = 1^f$. Then the descendant equation (3.2) implies

$$\int_V d\omega(a) = \int_{\partial V} \omega(a) = \int_{\partial V} d\omega_1(1, f) = 0.$$

Conversely, to construct the 1st descendant, we study the prism $\Delta^n \times [0, 1]$. The prism has a cell complex divided into "horizontal cells", which are the cells of the n-simplices $\Delta^n \times 1$, $\Delta^n \times 0$ at the "top" and "bottom", respectively; and "vertical cells", which are $k+1$-cells of $\partial\Delta^n \times [0, 1]$ formed by taking the prism between a bottom k-simplex and a top k-simplex. There is one vertical $k+1$-cell for each k-simplex of Δ^n.

We can define a $\mathbb{G}$ cocycle $\tilde{a}$ on $\Delta^n \times [0,1]$ which restricts to a^g on $\Delta^n \times 1$ and a on $\Delta^n \times 0$ (the "horizontal facets") and whose values on the vertical $k+1$-facets, associated to k-facets of Δ^n, are f_k. We now have

$$0 = \int_{\Delta^n \times [0,1]} d\omega(\tilde{a}) = \int_{\Delta^n \times 1} \omega(a^g) - \int_{\Delta^n \times 0} \omega(a) - \int_{\partial\Delta^n \times [0,1]} \omega(\tilde{a}).$$

Since the values of $\tilde{a}$ on the vertical cells $\partial\Delta^n \times [0,1]$ are determined by a and f, we may define

$$\omega_1(a,f)(V) = \omega(\tilde{a})(V \times [0,1])$$

for any $n-1$-cell V. Then we will have

$$\int_{\partial\Delta^n \times [0,1]} \omega(\tilde{a}) = \int_{\partial\Delta^n} \omega_1(a,f),$$

and the 1st descendant equation (3.2) follows.

The 1st descendant defined this way is functorial in ω. Meaning that if we transform $\omega \mapsto \omega + d\lambda$, ω_1 transforms by

$$\omega_1(A,f) \mapsto \omega_1(A,f) + \lambda(A^f) - \lambda(A).$$

We consider stacking two prisms, with horizontal simplices labeled by a,${}^{f_1}a$, and ${}^{f_2 f_1}a$. We have

$$\begin{aligned} 0 = \int_{\Delta^n \times [0,2]} d\omega(\tilde{a}) &= \int_{\Delta^n \times 2} \omega(a^{f_1 f_2}) - \int_{\Delta^n \times 0} \omega(a) \\ &\quad - \int_{\partial\Delta^n} \omega_1(a,f_1) + \omega_1(a^{f_1}, f_2) \\ &= \int_{\partial\Delta^n} \omega_1(a, f_1 f_2) - \omega_1(a,f_1) - \omega_1(a^{f_1}, f_2) \end{aligned}$$

This indicates the existence of a 2nd descendant $\omega_2(a, f_1, f_2)$, satisfying

$$\omega_1(a,f_1) + \omega_1(a^{f_1}, f_2) = \omega_1(a, f_2 f_1) + d\omega_2(a, f_1, f_2), \tag{3.3}$$

which may be constructed by studying a biprism $\Delta^n \times [0,1]^2$ which interpolates between two prisms. Still higher descendants may be computed by studying higher prisms. In practice, our $\omega(A)$ are polynomials in the a_j made from the $\cup_j$ products, so we can compute descendants algebraically, without resorting to thinking about prisms.

We may even compute the cocycle in terms of its 1st descendant. Consider

$$\alpha : Z^1(X, \mathbb{G}) \times C^0(X, \mathbb{G}) \to C^{n-1}(X, A),$$

for which there exists a descendant α_2 satisfying (3.3). Let $A \in Z^1(\Delta^n, \mathbb{G})$ for $\Delta^n = (0 \cdots n)$. We can write

$$A = df$$

for some $f \in C^0(\Delta^n, \mathbb{G})$ and define

$$\omega(A)(\Delta^n) = \alpha(0, f)(\partial \Delta^n).$$

One checks that this is well-defined provided (3.3) holds. Thus, like the existence of the 1st descendant is equivalent to the cocycle condition, the existence of the 2nd descendant is equivalent to such a natural transformation being a 1st descendant of a cocycle.

Chapter 4

Obstruction Theory

In the previous chapters we have already begun to see how the combinatorics of the cup product and related pairings are geometrically encoded in the behavior of certain families of vector fields. In this section we will use these constructions to give special cocycle representatives of some simple characteristic classes of the tangent bundle and other vector bundles. We especially focus on the Stiefel-Whitney classes of the tangent bundle, and provide an extension of Halperin-Toledo's construction of cocycle representatives for them on a barycentric subdivision to a general PL n-manifold with branching structure. We also discuss the case with boundary.

The Stiefel-Whitney classes are closely related to the Steenrod squares through Wu's formula and Thom's theorem. We discuss a conjectural construction of cocycle representatives for the Stiefel-Whitney classes of an arbitrary bundle and describe a conjectural extension of Thom's theorem to a cochain-level formula. On the other hand, we show that there is no cochain-level refinement of the Wu formula, and the obstruction to this formula is an interesting cochain operation.

We discuss the Whitehead tower of $BO(n)$ and how the Stiefel-Whitney classes act as obstructions to lifting the classifying map up this tower. This is standard material in homotopy theory, but what we have developed in the previous chapters allows us to do it on the level of simplicial cochains and cocycles. The advantage is that we can describe things like a (tangent) spin structure on a PL n-manifold X (with branching structure) as a 1-cochain η with $d\eta = w_2(TX)$, where $w_2(TX)$ is our natural cocycle representative of the 2nd Stiefel-Whitney class. This is useful because it works in all dimensions, while previously PL spin structures have only been adequately described in

2d. In 2d, we show that this definition is equivalent to a Kastelyn orientation, which is the traditional description of a spin structure on a PL surface.

While we only discuss tangent structure from the 2-torsion part of the Whitehead tower, our generalized Halperin-Toledo vector fields appear to give a natural construction of Pontryagin cocycles on a PL n-manifold which thus define a PL version of "string" and higher structures. We leave the investigation of this to further work.

4.1 Stiefel-Whitney Classes

4.1.1 General Smooth Version

Let X be a closed n-manifold with a real vector bundle $\pi : E \to X$ of rank k and let there be j sections

$$s_i : X \to E, \qquad \pi s_i = \mathrm{id}$$

which are generic, meaning that their convex hull:

$$B(s_1, \dots, s_j) = \{\sum_i t_i s_i(x) \ \mid \ x \in X,\ t_i \in [0,1],\ \sum_i t_i = 1\} \subset E$$

is transverse to X embedded in E as the zero section. In this case,

$$B(s_1, \dots, s_j) \cap X$$

is a $\mathbb{Z}_2$ Whitney $n+k-j-1$-cycle [39] in X (coefficients must be $\mathbb{Z}_2$ because orientability cannot be guaranteed). It turns out that for different choices of generic sections, these chains are all homologous, so the intersection defines a homology class in $H_{n-k-j+1}(X, \mathbb{Z}_2)$ which is Poincaré dual to the Stiefel-Whitney class [40]

$$[w_{k+j-1}(E)] \in H^{k+j-1}(X, \mathbb{Z}_2).$$

In terms of the sections, this intersection is precisely the locus where the sections are linearly independent. For instance, with $j = 1$, the intersection is the vanishing locus of s_1, which has codimension k, representing the top Stiefel-Whitney class $[w_k]$, which is thus equal to the Euler class mod 2 when E is orientable.

4.1.2 PL Version for the Tangent Bundle

Now we focus on the Stiefel-Whitney classes of the tangent bundle TX specifically, where X in an n-dimensional closed PL manifold (we treat the case with boundary below). Stiefel conjectured and Halperin-Toledo proved [22] (see also Whitney [21] and Cheeger [41]) that the sum of all $n-k$ simplices of the first barycentric subdivision X^b of X, ie.

$$W_k(TX^b) := \sum_{\sigma \in X^b_{n-k}} \sigma \in C_{n-k}(X^b, \mathbb{Z})$$

is a Poincaré dual representative for $[w_k(TX)] \in H^k(X, \mathbb{Z}_2)$. We will later give an obstruction-theoretic argument in support of this formula. For now, let us discuss how we can use it to represent the Stiefel-Whitney classes of the tangent bundle on an arbitrary triangulation.

Let us choose a branching structure on X. In Thm 3, we showed this defines a discrete Morse flow f on X^b whose unstable cells are the simplices of X. We therefore have a flow map

$$f_\infty : C_{n-k}(X^b, \mathbb{Z}_2) \to C_{n-k}(X, \mathbb{Z}_2)$$

and we may define the Stiefel-Whitney cycles on X by

$$W_k(TX) := f_\infty W_k(TX^b) \in Z_{n-k}(X, \mathbb{Z}_2).$$

Because f_∞ induces the identity map on homology and cohomology, we obtain

Theorem 12. *Representation of Stiefel-Whitney Classes* $f_\infty W_k(TX^b) \in Z_k(X, \mathbb{Z}_2)$ *are Poincaré dual representatives of the Stiefel-Whitney classes of* TX.

We note that it is often useful to apply this construction to $X^\vee$ when X is a triangulated PL manifold, since in this case we obtain a cocycle in $Z^*(X, \mathbb{Z}_2)$, which can be manipulated using the cup product, etc. In this case we refer to the cycles and cocycles so constructed as $W_k(TX^\vee)$, $w_k(TX^\vee)$, respectively.

It is worthwhile to discuss W_1, which is the obstruction to orienting X. The natural branching structure on X^b defines a local orientation which flips across every $n-1$-simplex. Thus, if there is a $\Sigma \in C_2(X^b, \mathbb{Z}_2)$ with $\partial\Sigma = W_1$, then we can flip the orientation of every n-simplex in Σ to obtain a consistent orientation of X^b. This way, W_1 represents a "natural obstruction" to an orientation of X^b. We will discuss this concept in full generality later. For now, let us argue $f_\infty W_1$ does the same for X.

The idea is to let the local orientations of X^b "flow" along f. That is, we assign a local orientation of a critical cell to its unstable cell. The branching Morse flow is that combined with the natural orientation of X^b, this induces the same local orientations on X as the branching structure does. This follows from our characterization of the critical cells in the proof of the branching Morse flow theorem.

As the local orientations flow along f, so do the $n-1$-cells of X^b across which the orientations are incompatible. f_∞ precisely counts how these $n-1$-cells pile up onto critical $n-1$-cells of X. Thus, $f_\infty W_1$ is also a natural obstruction to an orientation of X.

4.1.3 Conjectural Version and Thom's Theorem

Now let us make some speculation about a PL formula for the Stiefel-Whitney classes of an arbitary bundle and Thom's theorem.

Recall in the previous chapter we used the branching Morse flow and its generalization to the discrete Halperin-Toledo Morse flows to define a j-parameter family of pushoffs $h_j(\Sigma) \in C_{k+j}(X, A)$ which is transverse to $\Sigma \in C_k(X^\vee, A)$. When Σ is a simple cycle, that is, its coefficients are either zero or one, or equivalently it is the pushforward of the fundamental class of an immersed PL sub-k-manifold:

$$i : Y \to X^\vee$$

$$\Sigma = i_*[Y],$$

then we conjecture that the pushoff $h_j(i_*[Y])$ is a $k+j$-chain which is

$$i_*[B(s_1, \dots, s_{j+1})]$$

for some $j+1$ piecewise-linear sections s_i of the normal bundle of Y in X. Thus, we have:

Conjecture 2. *Representation of Stiefel-Whitney Classes* By forming the pullback of the Poincaré dual of the j-parameter pushoff,

$$w_{n-j-k}(NY) := i^* \delta(h_j(i_*[Y])) \in C^{n-j-k}(Y, \mathbb{Z})$$

$$w_l(NY) = i^*\delta(h_{n-k-l}(i_*[Y])) \in C^l(Y, \mathbb{Z})$$

represents $[w_{n-j-k}(NY)]$ when we reduce coefficients mod 2. Here $\delta(-)$ is the Poincaré duality map elsewhere denoted δ_-.

If X is a vector bundle E over Y and Y is embedded as the zero section, $NY = E$ so the above expression gives a cocycle representative of $[w_{n-j-k}(E)]$ for any vector bundle, once we choose a triangulation of E such that the zero section lives in the dual cell complex.

Combining this with the $\cup_i$ conjecture, we immediately obtain a chain level refinement of Thom's famous theorem [5]:

Conjecture 3. *Thom's Theorem, Chain Version* We use the shorthand $i_*[Y] = Y \in C_k(X, \mathbb{Z})$.

$$i_! w_l(NY) := \delta(i_*(Y \cap w_k(NY))) = \delta(i_*(h_{n-k-l}(Y) \cap Y))$$

$$= \delta_Y \cup_{n-k-l} \delta_Y = Sq^l \delta_Y.$$

The tangent bundle is equivalent to the normal bundle of the diagonal embedding

$$\Delta : X \to X \times X.$$

It is reasonable to expect that the branching structure can be used to define a cellular approximation to this of the form

$$\Delta : X \to (X \times X)^\vee$$

which allows us to apply our construction to define

$$w_{n-j-l}(TX)' = \Delta^*\delta(h_j(\Delta_* X)).$$

I expect it's possible to construct Δ so that the conjectural representation of general Stiefel-Whitney classes coincides with the Halperin-Toledo representation for the tangent bundle of the previous section:

$$w_k(TX)' = w_k(TX^\vee) \in C^k(X, \mathbb{Z}_2).$$

4.1.4 Relative Stiefel-Whitney Cocycles for the Tangent Bundle

If X is a PL n-manifold with boundary, then the barycentric subdivision of X restricts to the barycentric subdivision of ∂X. If we consider the barycentric subdivision of the n-simplex Δ^n, the link of each boundary k-simplex

$$(\sigma_1 < \cdots < \sigma_k) \in \partial\Delta_k^{nb}$$

contains exactly one interior $k+1$ simplex, namely

$$(\sigma_1 < \cdots < \sigma_k < \Delta^n) \in \partial\Delta_{k+1}^{nb}.$$

Thus,

$$\partial W_k(TX^b) = W_{k-1}(T\partial X^b) \mod 2.$$

Then, since $f_\infty \partial = \partial f_\infty$, we have the useful property

Theorem 13. *Relative Stiefel-Whitney Cycles* If X is a PL n-manifold with boundary, then the Stiefel-Whitney chains defined by

$$W_k(TX) = f_\infty\left(\sum_{\sigma \in X_{n-k}^b} \sigma\right) = \sum_{\sigma \in X_{n-k}^b} f_\infty(\sigma)$$

$$W_{k-1}(T\partial X) = f_\infty\left(\sum_{\sigma \in \partial X_{n-k-1}^b} \sigma\right) = \sum_{\sigma \in \partial X_{n-k-1}^b} f_\infty(\sigma)$$

satisfy the relation

$$\partial W_k(TX) = W_{k-1}(T\partial X).$$

Thus, the pair $W_k(TX), W_k(T\partial X)$ define a relative homology class in

$$H_k(X, \partial X, \mathbb{Z}_2)$$

which is Poincaré-Lefschetz dual in X to a Stiefel-Whitney cocycle $w_{n-k}(TX)$. The restriction of this cocycle to the boundary is Poincaré dual in ∂X to $W_{k-1}(T\partial X)$, and hence equals $w_{n-k}(T\partial X)$, as the classes do. This is one of the ways in which our cocycles are natural representatives of the Stiefel-Whitney classes.

4.1.5 Wu Classes

Let X be a closed PL n-manifold with branching structure. The Steenrod squares give rise to homomorphisms

$$Sq^k : H^{n-k}(X, \mathbb{Z}_2) \to H^n(X, \mathbb{Z}_2) \xrightarrow{\int_X -} \mathbb{Z}_2.$$

Since $\mathbb{Z}_2$ is a field, these are represented in Poincaré duality by Wu classes $[u_k]$:

$$Sq^k[\alpha] = [u_k] \cup [\alpha].$$

We wish to have a cocycle refinement of this, that is to represent the $u_k \in Z^k(X, \mathbb{Z}_2)$ such that for every $\alpha \in Z^{n-k}(X, \mathbb{Z}_2)$,

$$Sq^k \alpha = u_k \cup \alpha \mod 2.$$

Such a cocycle may be called a strong Wu cocycle.

It is known that these classes are polynomials in the Stiefel-Whitney classes:

$$[u_k] = P_k([w_1(TX)], \ldots, [w_k(TX)]). \tag{4.1}$$

See [42, 43]. For instance, $[u_1] = [w_1(TX)]$. It follows that

$$u_k := P_k(w_1(TX^\vee), \ldots, w_k(TX^\vee))$$

represents the Wu class, but there may be a correction to the Wu formula

$$u_k \cup \alpha - Sq^k \alpha = dU_k(\alpha) + 2V_k(\alpha)$$

In fact *no matter which representative of* $[u_k]$ *we choose*, there is no way for this correction to be identically zero. Indeed, for $\alpha, \beta \in Z^{n-k}(X, \mathbb{Z}_2)$

$$Sq^k(\alpha + \beta) - Sq^k \alpha - Sq^k \beta$$

$$= (1 + (-1)^{n-k})\ \alpha \cup_{n-2k} \beta \pm d\alpha \cup_{n-2k+1} \beta \pm \alpha \cup_{n-2k+1} d\beta + d(\alpha \cup_{n-2k+1} \beta)$$
$$= d(\alpha \cup_{n-2k+1} \beta) \mod 2.$$

while $u_k \cup -$ is linear on the cocycle level. This means there are no strong Wu cocycles! The best we have is the Wu cocycle u_k above.

The Wu descendants U_k and V_j (see section 3.4) are very interesting functions. For instance, we can rephrase the above formula as

$$d(U_k(\alpha + \beta) - U_k(\alpha) - U_k(\beta)) = d(\alpha \cup_{2j-n+1} \beta) \mod 2. \tag{4.2}$$

In this way, U_k is much like a quadratic form, at least up to closed pieces. We define a Sq^k-Wu structure on an n-manifold to be a function

$$Q : Z^{n-k}(X, \mathbb{Z}_2) \to \mathbb{Z}_2$$

such that

$$Q(\alpha + \beta) - Q(\alpha) - Q(\beta) = \int_X \alpha \cup_{n-k} \beta \mod 2.$$

Compare [8]. Thus, we cannot in general remove the d's from the Wu descendant formula (4.2). It is expected that these Wu structures are closely related to spin structures [23], but we do not discuss it here.

4.2 Obstruction Theory and Tangent Structures

4.2.1 The Whitehead Tower of $BO(n)$

The tangent bundle of an n-manifold (or any rank n vector bundle) is described up to homotopy by a map $X \to BO(n)$, where $BO(n)$ is a classifying space defined as the space of linear n-dimensional subspaces of $\mathbb{R}^\infty$. (There is a construction, suitable for topological ∞-groups, which presents $BO(n)$ analogously to the classifying space $B\mathbb{G}$ we described in the previous chapter, for instance see [44], and by which $O(n)$ bundles are classified by a version of continuous nonabelian cohomology $H^1(X, O(n))$.)

The classifying space $BO(n)$, unlike BG for a discrete group G, has nonzero homotopy groups in arbitrarily high degrees. To understand its homotopy theory, a useful object is the Whitehead tower. This is a sequence of spaces W_k such that $\pi_{<k} W_k = 0$ and $\pi_{\geq k} W_k = BO(k)$. These spaces sit

in a tower of fibrations dual to the Postnikov tower:

$$\begin{array}{ccc} & & \vdots \\ & & \\ B\mathbb{Z}_2 & \longrightarrow & W_2 = BSpin(n) \\ & & \downarrow \\ \mathbb{Z}_2 & \longrightarrow & W_1 = BSO(n) \\ & & \downarrow \\ & & W_0 = BO(n). \end{array}$$

Note the first three Whitehead spaces are classifying spaces of other Lie groups, but the pattern does not continue. $W_3 = W_2$ and see [44] for a discussion of W_4 which has to do with a certain Lie 2-group called the "string 2-group".

If we consider the tangent bundle as a map $X \to BO(n)$, we can ask for a lift of this map to $W_1 = BSO(n)$. Such a lift is equivalent to an orientation. Given an oriented tangent bundle as a map $X \to BSO(n)$ then, a spin structure is equivalent to a lift of this map to $W_2 = BSpin(n)$. This way, the Whitehead tower organizes an infinite sequence of interesting tangent structures.

Of course, there are more tangent structures than these, for example there is also a pair of $B\mathbb{Z}_2$ fibration over $BO(n)$ given by the two pin groups $BPin^{\pm}(n)$ [45]. A $Pin^{\pm}(n)$ structure on the tangent bundle is a lift of the classifying map to these spaces. Further, a certain $B^2\mathbb{Z}_2$ fibration over $BSO(n)$ appeared in my work on bosonization in 3+1D [46].

Essentially, by abstract nonsense [27], any cohomology class on $BO(n)$ defines a type of tangent structure that may be useful in some application. In the following sections we will give a description of the corresponding PL tangent structure for 2-torsion cohomology classes.

4.2.2 Orientations

Suppose we want to endow our PL n-manifold X with an orientation. That is, we wish to lift the classifying map $X \to BO(n)$ to the double cover $BSO(n)$. We can pull back this "universal" double cover to obtain a double cover X. A lift of the classifying map is homotopically the same as a section of this principal $\mathbb{Z}_2$ bundle.

We can speak about this in a more concrete way. Indeed, at any point x, we can define two different orientations of the tangent space $T_x X = \mathbb{R}^n$. These local orientations collect into a double cover of local orientations which is equivalent to the one pulled back from the universal double cover of $BO(n)$. It is dubbed the orientation double cover of X. Now, clearly a section of this $\mathbb{Z}_2$ bundle is exactly equivalent to an orientation, ie. it is a continuous choice of local orientation at every point of X.

The basic idea of obstruction theory is that principal bundles like these admit sections iff they are homotopically trivial. The homotopical classification of these principal bundles is controlled by the (nonabelian) cohomology of X. Thus, for any tangent structure defined by a lift of the classifying map, there is a cohomology class which is zero iff such a structure exists.

For example, the orientation double cover is classified by an element $[w_1(TX)] \in H^1(X, \mathbb{Z}_2)$, which happens to coincide with the 1st Stiefel-Whitney class. A manifold admits an orientation iff $[w_1(TX)] = 0$.

We wish to improve on this basic obstruction theory in the following way. Given a branching structure on X, we have defined in previous sections a *cocycle* representative $w_1(TX) \in Z^1(X^\vee, \mathbb{Z}_2)$. According to our definition, $w_1(TX)$ is nonzero on precisely those 1-cells which are dual to $n-1$-simplices separating two n-simplices which receive opposite local orientations from the branching structure. Thus, if we have a 0-cochain ω with $d\omega = w_1(TX)$, or equivalent an n-chain O with ∂O the union of these $n-1$-simplices, then we can reverse the local branching orientation on any n-simplex in the support of O or equivalently dual to a 0-cell where ω is nonzero.

Further, the only choice in ω is a shift by a closed 0-cocycle ϵ, which by its closedness must be locally constant. On components where $\epsilon = 1$, $\omega + \epsilon$ is oppositely oriented to ω. Thus, the set of *trivializations* ω of the *obstruction cocycle* $w_1(TX)$ is in $H^0(X, \mathbb{Z}_2)$-equivariant bijection with the set of orientations. We will refer to such a pleasant situation as a *simplicial cocycle obstruction theory*, although let us refrain from attempting to state a precise definition.

Most importantly, we would like to say that there is a functorial correspondence between tangent structures and the trivialization of the corresponding obstruction cocycle. In practice this means that constructions made with respect to an orientation can be adapted to depend on ω and the branching structure. For example, to construct the fundamental cycle of X, we consider the action of $C^0(X^\vee, \mathbb{Z}_2)$ on $C_n(X, \mathbb{Z})$ for which (recall section 1.2.5)

$$(f \cdot \sigma_n) = (-1)^{f(\sigma_n^\vee)} \sigma_n.$$

If we let X_n be the sum of all n-simplices of X, given their local orientations from the branching structure, then the fundamental cycle is

$$X = (\omega \cdot X_n).$$

Topologically, what the branching structure allowed us to do is construct a generic "section with singularities" of the orientation bundle by defining a local orientation away from a codimension 1 set (which is the whole $n-1$-skeleton for a barycentric subdivision and its ascending or descending branching structure). This singular locus is Poincaré dual to the obstruction cocycle. A trivialization of the cocycle amounts to a null bordism of the singular locus, and we have a cocycle obstruction theory when we can follow any such null bordism by a deformation of the section with singularities to an honest section, hence a structure.

The reason we like branching structures in general is that by our construction in Chapter 1, they give rise to a (singular) framing of the tangent bundle by Halperin-Toledo vector fields with whose codimension k singularities occur only in the $n-k$-skeleton. Thus it is reasonable to expect that a cocycle obstruction theory is available for all types of tangent structures. We only consider structure associated to Stiefel-Whitney classes in this thesis, but it would be interesting to construct such a cocycle obstruction theory for the so-called String structures, having to do with the Pontryagin classes.

Finally, there is an important generalization which we don't discuss here of these concepts to differential cohomology. For example in [8], the authors constructed a differential cocycle obstruction theory which classifies "differential integral Wu structures". It would be very interesting to find a theory which combines their work with the results on spin structures we now discuss.

4.2.3 2d Spin Structures and Kastelyn Orientations

Orientations are easy to understand, but now we turn our attention to spin structures (see [47] for instance), using obstruction theory based around our cocycle $w_2(TX)$, which depends on a branching structure of a PL n-manifold X. By the yoga we outlined in the previous section, we simply define a discrete spin structure as a 1-cochain η with $d\eta = w_2(TX)$. In this section, we describe a correspondence between such trivializations and an existing notion of a spin structure on an oriented PL surface called a Kastelyn orientation.

A Kastelyn orientation [48, 49, 50, 51] is an orientation of edges of $X^\vee$ such that around every face of $X^\vee$ there are an odd number of edges oriented against the boundary orientation of that face (given by the global orientation of TX). This definition is designed to imitate the fundamental property of spin structures that for the two spin structures on a circle, periodic (P) and anti-periodic (AP), it is the anti-periodic spin structure which extends to the disc. In the string theory literature these are called the Ramond (R) and Neveu-Schwarz (NS) spin structures, respectively.

For a circle γ embedded in a surface X and which bounds a disc, we can think of a spin structure as a framing of $TX|_\gamma$. In this case the framing has a winding number, counted like so:

1. Choose a generic vector in TX at some point of γ.
2. Use the framing of $TX|_\gamma$ to extend this vector to a vector field over γ.
3. Form the pushoff $\hat{\gamma}$.
4. The winding number of the framing is $\#(\hat{\gamma} \cap \gamma)/2 \bmod 2$.

The AP spin structure has odd winding number while the P spin structure has even winding number.

In terms of the Kastelyn orientation, with $\gamma \in X^\vee$, the winding number is the number of arrows encountered along γ, traversed according to the orientation of X, which point against γ. Indeed, the Kastelyn orientation may be extended to a framing of $TX|_\gamma$ using the right hand rule and gluing the framing across vertices by rotating it counterclockwise [50] (both moves defined by the orientation of X).

Now let us suppose X has a branching structure and an orientation and we will attempt to construct a Kastelyn orientation from it. The edges of $X^\vee$ meet those of X at right angles, so the branching structure gives a normal vector $\hat{n}$ to each edge of $X^\vee$. We choose that edge to be oriented with a tangent vector $\hat{t}$ so that $\det(\hat{n}\hat{t}) > 0$ with respect to the orientation of X. One can check that the winding number of this framing of γ is simply $\frac{1}{2}\int_X \delta_\gamma \cup \delta_\gamma$.

Consider this on a barycentric subdivision X^b. The faces of $X^{b\vee}$ correspond to the vertices of X^b, which have degree $0, 1$, or 2 depending on what dimension of simplex they arise from in X. In the canonical branching structure of X^b, all the edges incident at a degree 0 (resp degree 2) vertex are outgoing (resp incoming), while each degree 1 vertex has two incoming and two outgoing edges. We compute the winding number of the boundary of each dual face around these vertices and find that they are all even. That is, the Kastelyn orientation condition fails at every cell of $X^{b\vee}$!

This is actually exactly what we want, and coincides with what we saw for the local orientations of X^b. Indeed, the Poincaré dual Stiefel-Whitney cycle $W_2(TX^b)$ is the sum of all the vertices. If we have an $E \in C_1(X^b, \mathbb{Z}_2)$ with $\partial E = W_2(TX^b)$, then we can produce a Kastelyn orientation of $X^{b\vee}$ by flipping the orientation $\hat{t}$ of each edge $e \in X_1^{b\vee}$ which crosses an edge of E. Note that a complete dimerization of X^b gives rise to such an E, and these have already been shown to have a special relationship to 2d spin structures [50].

We can do the same thing on X given a branching structure by studying the Morse flow of the Stiefel-Whitney cycle $W_2(TX) = f_\infty W_2(TX^b)$. Explicitly,

$$W_2(TX) = \sum_{(0)\in X_0} (0) + \sum_{(01)\in X_1} (1) + \sum_{(012)\in X_2} (2).$$

In terms of a vertex x, this is

$$1 + \#(\text{incoming arrows}) + \#(\text{incoming adjacent pairs of arrows}).$$

One sees that this equivalently counts 1+ the number of intervals of incoming arrows, which is 1+ the number of times the arrows change from incoming to outgoing, which is 1+ half of the self-intersection of the boundary of $x^\vee$, or exactly where our orientation of $(X^\vee)_1$ fails to be Kastelyn! Thus again given an E with $\partial E = W_2(TX)$, we may flip the edge orientations across E to obtain a Kastelyn orientation of $X^\vee$.

Furthermore, this correspondence is equivariant with respect to the action of $Z^1(X, \mathbb{Z}_2)$, which acts by flipping the orientations of edges in $(X^\vee)_1$. Thus we have

Theorem 14. *spin structure obstruction theory in 2d* For a PL surface X with branching structure, there is an equivalence of categories between trivializations of $w_2(TX) \in Z^2(X^\vee, \mathbb{Z}_2)$ and Kastelyn orientations on $X^\vee$, which is equivariant with respect to the action of $Z^1(X^\vee, \mathbb{Z}_2)$.

4.2.4 Discrete Spin Structures in Dimensions ≥ 3

When $n > 2$, $\pi_1 SO(n) = \mathbb{Z}_2$, and a spin structure of an n-manifold is equivalent to an assignment of ± 1 to framed embedded circles which flips sign when the framing is rotated by the nontrivial element of $\pi_1 SO(n)$, which is homotopy equivalent to a 2π rotation in any 2-plane. This is equivalent to a choice of framing of $TX|_\gamma$ for any embedded circle γ, such that if γ extends to a disc, then the framing extends as well.

Let X be a trianguled PL n-manifold with branching structure. The branching structure gives a normal framing to every embedded circle in $X^\vee$, given on an edge e by vectors parallel to the edges $(i, i+1)$ of the $n-1$-simplex $e^\vee$. It also defines a $w_2(TX) \in Z^2(X^\vee, \mathbb{Z}_2)$. One can show $w_2(TX)$ is nonzero on exactly those dual 2-cells whose boundary receives a non-bounding framing from the branching structure. This can be easily checked on a barycentric subdivision either geometrically or algebraically. For the algebraic approach using the mod 2 winding number computed using the Steenrod squares, the key formula is

$$\frac{1}{2} Sq^1 \delta_{\partial D} + 1 = \int_D w_2(TX) \mod 2,$$

which is analogous to how we measured the violation of the Kastelyn condition on triangles of $(X^\vee)_2$ where $w_2(TX) = 1$ by computing $\frac{1}{2}\delta_\gamma \cup \delta_\gamma$. Then, given an $n-2$-chain $E \in C_{n-1}(X, \mathbb{Z}_2)$ with $\partial E = W_2(TX)$, we can rotate by 2π the framings of any curve passing through a facet of E. This will give us a consistent framing of all embedded circles and hence a spin structure.

Geometrically this works because the Halperin-Toledo vector fields define a framing on the barycentric subdivision with an odd singularity on every simplex. If we have an orientation we have an n-chain Ω whose boundary is the $n-1$-skeleton. Recall that F_n vanishes on the $n-1$-skeleton. Thus we can flip the sign of F_n inside the support of Ω to obtain a framing $F_1, \ldots, F_{n-1}, F_n'$ with singularities which defines a consistent orientation on all of X. Further, this allows us to homotopy F_n' to F_n'' which vanishes only on the $n-2$-skeleton.

Thus we obtain a framing $F_1, \ldots, F_{n-1}, F_n''$ away from the $n-2$-skeleton. By Halperin-Toledo, this framing has an odd singularity around every circle linking an $n-2$-simplex. Thus, if we have an $n-1$-chain E with ∂E the $n-2$-skeleton, then we can twist the framing by 2π in the normal coordinates wherever it passes through E. Thus we obtain a framing with only even singularities on $n-2$-skeleton. This is precisely a spin structure. Applying the Morse flow we obtain the result for an arbitrary triangulation.

This is encoded in the following theorem:

Theorem 15. *spin structure obstruction theory* For a PL n-manifold X with branching structure, there is an equivalence of categories between trivializations $\partial E = W_2(TX) \in Z_{n-2}(X, \mathbb{Z}_2)$ and spin structures for TX given by framing each embedded circle C (transverse to the cell structure) according to the branching structure, but applying a 2π rotation when C crosses E.

Part II

Physics Applications

Introduction

In this part, we will discuss some physics applications of the mathematical devices we've developed so far. These applications are the main thrust of my research, but to keep the exposition palatable to a mathematical audience I have chosen just a couple simple, but interesting topics, ones which appear very close to admitting a purely mathematical formulation. I will focus on a path integral approach to these topics, which as a formal manipulation of symbols is totally rigorous. Unfortunately to really obtain the meat of the physics we will have to make some reference to an underlying Hilbert space associated to these formal path integrals, whose construction has been acheived only in the simplest situations. I will talk about what is expected of this Hilbert space and show that in two constructible situations, namely those of finite gauge theories and conformal field theories, that these expectations hold true.

To facilitate the discussion, let me outline a rough definition of a quantum field theory (QFT). To first approximation, one can think of a QFT as a kind of distribution which provides expectation values $\langle \mathcal{O} \rangle$ of some specified set of observables $\mathcal{O}$, which form an algebra.

For us, a QFT always comes along with a spacetime X and observables have a notion of support in X. The central principle of QFT is called cluster decomposition, which says that observables with largely separated support are uncorrelated. For point operators $\mathcal{O}(x)$, whose support is a single point x in spacetime this means that for distant points x and y,

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle \approx \langle \mathcal{O}(x) \rangle \langle \mathcal{O}(y) \rangle.$$

Our notion of support here is defined by the above condition, which is a condition only on large distances. To emphasize this fuzziness, we will say $\mathcal{O}(x)$ is supported "near" x.

The main way we know of constructing QFTs which satisfy cluster decomposition is by the action principle of Feynman and Dirac [52]. It is very hard to state what this sort of thing is in general but I will do my best. To define a QFT by the action principle we first specify a set of (dynamical) fields ϕ on X, which are typically either functions on X or sections of some kind of bundle over X. These are dynamical fields because we're going to integrate over them. Then one defines a Lagrangian density $\mathcal{L}(x)$ which is a function of the fields and their derivatives at x, and background fields like a metric $g_{\mu\nu}(x)$. These are background fields because we're not going to integrate over them (nobody knows how to integrate over a metric). Likewise we take as our observables any local functionals of this sort. Then the expectation values of such observables are defined by

$$\langle \mathcal{O}(\phi(x), \ldots) \rangle = \frac{1}{Z} \int D\phi \; \mathcal{O}(\phi(x), \ldots) \exp i \int_X \mathcal{L},$$

where $D\phi$ is some measure on the space of dynamical fields and Z is a normalization constant called the partition function defined so that $\langle 1 \rangle = 1$. It is more or less clear that if the bare measure $D\phi$ satisfies cluster decomposition, so will these expectional values. Unfortunately, nothing comes for free, and much sweat and many tears have been shed over the proper definition of $D\phi$ [53]. We will get around this issue either by discretization (see eg. [54]) or by modifying established QFTs whose action principle has stood the test of time if not yet yielded to pressure of proof. For mathematicians interested in attempting to understand the construction of $D\phi$ in more complicated cases but unwilling to read physics textbooks I recommend the book [55] and the MIT course by Pavel Etingof.

Among the QFTs we will study are those whose fields include a principal G-bundle with connection, for some group G. Such fields are called gauge fields, theories that have them are called G gauge theories, and G is called the gauge group. $\mathbb{Z}_n$ gauge theory will be the first theory we discuss. It can be presented straightforwardly by an action principle, but even this theory has some nontrivial physical applications.

We are interested in these theories because they have nontrivial operators which are supported along submanifolds of positive dimension. We call a QFT with such operators an extended QFT, following Baez-Dolan [56]. We will see that naive cluster decomposition fails for many extended QFTs presented by action principles. This is nicely demonstrated in the $\mathbb{Z}_n$ gauge theory and seems to be a key feature of (irreducible) topological QFT (TQFT), QFTs with only the identity point operator but nontrivial extended operators whose correlation functions are all isotopy invariant.

We stress that this definition of TQFT differs from the Atiyah-Segal TQFTs usually discussed in the mathematics literature. Such theories also have a topological invariant partition function, which appears to be a stronger condition than isotopy invariance of the correlation functions. However, we will discuss how in three dimensions one can use knot surgery to define a 3-manifold invariant (the Reshitikhin-Turaev invariant) from the correlation functions and how it relates a partition function that may or may not exist.

The main issue is that the path integral has a fundamental ambiguity. Indeed, we can shift the Lagrangian density $\mathcal{L}$ above by anything independent of the dynamical or background fields. We can phrase the locality condition for $\mathcal{L}$ by saying that $\int_X \mathcal{L}$ glues along discs decorated by the germs of the fields on their boundaries.

We will see this ambiguity can complicate matters of symmetry, leading to so-called anomalies. Anomalous symmetry is a very important topic in QFT since it tells us qualitative features of path integral QFTs whose correlation functions we don't know how to compute. For instance, it seriously constrains the form of the standard model of particle physics [53, 57]. Recently, anomalous symmetry has found important application in condensed matter physics, especially in the classification of topological phases of matter [58, 59] and phase transitions [60, 61]. Most of my work on anomalies has been exploring this direction. The study of anomalies turns out to be equivalent to the study of boundary conditions of topological gauge theories through a mechanism called anomaly in-flow [62].

We will also discuss how background structure modifies the construction of topological gauge theories and their anomalies. We are especially interested in spin structures, since they arise in systems of fermions [53]. The electron is a fermion, so almost everything we really care about in the end depends on a spin structure and it's important to find out how.

A tool that has been very effective for 1+1 dimensional systems is called bosonization/fermionization [63], which allows one to trade spin structure dependence for an extra gauge. Recently, it has become apparent that this correspondence also works in higher dimensions, although one must consider gauge ∞-groups and their anomalies [23, 46, 64].

In this thesis, we will consider for the first time how this works in the presence of boundary conditions. We will show how to use fermionization to construct spin Dijkgraaf-Witten theory in any dimension and describe the anomalies associated with it. As a capstone, we will compute the action of bosonization on the chiral anomaly of the 1+1D Dirac fermion.

Chapter 5

Topological Gauge Theory

5.1 $\mathbb{Z}_n$ Gauge Theory

The first gauge theory we will discuss is the topological $\mathbb{Z}_n$ gauge theory. We keep the spacetime dimension $D = d+1$ arbitrary (the notation is customary for d space dimensions and 1 time dimension, although for us all manifolds have Euclidean signature) and denote by X an oriented PL D-manifold, which is our spacetime. Our approach will be overly pedantic for the benefit of nonphysicists and to fix the terminology. In particular, we will try to define the theory on the fly, so the reader can see where certain choices must be made in the formulation of the path integral. We will try to imitate as much as possible the approach one would take in analyzing a more complicated QFT.

5.1.1 path integral of the quantum double

We begin by describing the $\mathbb{Z}_n$ gauge theory in the "quantum double" formalism, so called because it involves two (dynamical) fields, $a \in C^1(X, \mathbb{Z}), b \in C^{d-1}(X^\vee, \mathbb{Z})$. The path integral formulation we present here is equivalent to Turaev-Viro's state-sum [65] applied to the Drinfeld double of $\mathbb{Z}_n$ (see eg. [66]), while Hamiltonian version of the formalism first appeared in [67].

The action functional is defined for a PL D-manifold X with finitely many cells by

$$S(a, b) = \int_X \frac{1}{n}\ (a, db) \in \mathbb{R}/\mathbb{Z}, \tag{5.1}$$

where we have used the Poincaré duality pairing.

Important observables whose expectation values and correlation functions we are interested in calculating are the Wilson lines

$$W(q, \Gamma) = \exp\left(2\pi i q \int_\Gamma a\right)$$

where $\Gamma \in C_1(X, \mathbb{Z})$ and $q \in \mathbb{R}/\mathbb{Z}$ is called the electric charge; as well as the 't Hooft operators

$$H(m, \Sigma) = \exp\left(2\pi i m \int_\Sigma b\right),$$

where $\Sigma \in C_{d-1}(X^\vee, \mathbb{Z})$ and $m \in \mathbb{R}/\mathbb{Z}$ is called the magnetic charge. We will find that this theory violates a naive form of cluster decomposition when Σ and Γ link each other.

To do this computation, we will need to define and evaluate

$$\langle W(q, \Gamma) H(m, \Sigma) \rangle_X = \frac{N(X)}{Z(X)} \sum_{a,b}{}' W(q, \Gamma, a) H(m, \Sigma, b) \exp\Big(2\pi i S(a, b)\Big),$$

where

$$Z(X) = N(X) \sum_{a,b}{}' \exp\Big(2\pi i S(a, b)\Big)$$

is the partition function, and $N(X)$ is a normalization factor we have included for later convenience, on which the expectation values do not depend. The prime above the sum indicates that these expressions are not yet defined, since if we sum over all a, b, there are infinitely many of them.

To define the sums, first note that the bare partition sum in $Z(X)$ has a huge redundancy, since we may transform

$$a \mapsto a + df + nw \qquad (f, w) \in C^0(X, \mathbb{Z}) \oplus C^1(X, \mathbb{Z})$$

$$b \mapsto b + dg + nh \qquad (g, h) \in C^{d-2}(X, \mathbb{Z}) \oplus C^{d-1}(X, \mathbb{Z})$$

under which $S(a, b)$ is invariant modulo fractional boundary terms and integer bulk terms. The first set (f, w) are called the (small and large, resp.) electric (local) transformations, while the second set (g, h) are called the . Sometimes we will specify the degree of the parameter by calling them .

Under the large transformations, a, b have finitely many orbits, labelled by their reduced values mod n:

$$(\bar{a}, \bar{b}) \in C^1(X, \mathbb{Z}_n) \oplus C^{d-1}(X, \mathbb{Z}_n).$$

Thus, one solution to defining the partition sum is to take

$$\sum_{a,b}' = \sum_{(\bar{a},\bar{b})\in C^1(X,\mathbb{Z}_n)\oplus C^{d-1}(X,\mathbb{Z}_n)} . \tag{5.2}$$

If we do this in all of our sums, including the ones in the correlation functions, we are de facto considering all a, b related by the large transformations as physically equivalent. Indeed, it will only make sense to consider correlation functions of observables which are invariant under these transformations, otherwise we will have to make an explicit choice of representatives (a, b) for each mod n equivalence class $(\bar{a}, \bar{b})$. In this case the transformations are called gauge transformations.

Now we study the restrictions on the algebra of observables $W(q, \Gamma)$ and $H(m, \Sigma)$ imposed by gauge invariance, so that we may make the same replacement of a finite sum in the correlation functions. First note that $W(q, \Gamma)$ (resp. $H(m, \Sigma)$) is automatically invariant under the magnetic (resp. electric) local transformations for arbitrary Γ and q (resp. Σ and m). For the other transformations, we summarize below, along with the physical interpretation, to be explained in more detail later.

- *electric charge quantization:* $W(q, \Gamma)$ is invariant under the large electric local transformations parametrized by $w \in C^1(X, \mathbb{Z})$ iff $q \in \frac{1}{n}\mathbb{Z}$.
- *electric charge conservation:* $W(q, \Gamma)$ is invariant under the small electric local transformations parametrized by $f \in C^0(X, \mathbb{Z})$ iff $\partial\Gamma = 0 \mod n$.
- *magnetic charge quantization:* $H(m, \Sigma)$ is invariant under the large magnetic local transformations parametrized by $h \in C^{d-1}(X, \mathbb{Z})$ iff $m \in \frac{1}{n}\mathbb{Z}$.
- *magnetic charge conservation:* $H(m, \Sigma)$ is invariant under the small magnetic local transformations parametrized by $g \in C^{d-2}(X, \mathbb{Z})$ iff $\partial\Sigma = 0 \mod n$.

The relationship between conservation laws and quantization conditions, small and large transformations, is a recurring theme in this work.

Our regulated sum $\sum'$ defined above only requires charge quantization, but we will also require both conservation laws, meaning we will consider also the small local transformations as gauge transformations.

Summarizing:

Definition 8. The topological $\mathbb{Z}_n$ gauge theory on a closed PL D-manifold X is the QFT whose observables are defined by functions $\mathcal{O}(a,b)$, forming a $\mathbb{C}$-algebra generated by Wilson lines

$$W(q,\Gamma)$$

with $q \in \frac{1}{n}\mathbb{Z}$, $\Gamma \in Z_1(X,\mathbb{Z}_n)$, and 't Hooft operators

$$H(m,\Sigma)$$

with $m \in \frac{1}{n}\mathbb{Z}$, $\Sigma \in Z_{d-1}(X,\mathbb{Z}_n)$, and whose correlation functions are given by the path integral

$$\langle \mathcal{O} \rangle = \frac{N(X)}{Z(X)} \sum_{a\in C^1(X,\mathbb{Z}_n)} \sum_{b\in C^{d-1}(X^\vee,\mathbb{Z}_n)} \mathcal{O}(a,b) e^{2\pi i S(a,b)}, \tag{5.3}$$

where $N(X)$ is an arbitrary normalization.

5.1.2 normalization of the partition function

Let us now perform a calculation of the partition function $Z(X)$. We do the sum by parts, first summing over b to obtain an effective action for a:

$$e^{iS_{eff}(a)} := \sum_{b\in C^{d-1}(X,\mathbb{Z}_n)} \exp\left(\frac{2\pi i}{n}\int (da,b)\right), \tag{5.4}$$

where we have integrated $S(a,b)$ by parts to rewrite the weight in this simple form. We can write the exponentiated integral as a product over all triangles (012), since (da,b) contributes exactly one term per triangle, namely

$$da(012)b(012)^\vee,$$

where $(012)^\vee$ is the dual $d-1$-cell in $X^\vee$. Further, every $d-1$-cell of $X^\vee$ is the dual of exactly one triangle of X, so we can split the sum over b into a sum over each $b(012)^\vee$'s. We obtain

$$e^{iS_{eff}(a)} = \prod_{(012)\in X_2} \sum_{b(012)^\vee\in\mathbb{Z}_n} \exp\left(\frac{2\pi i}{n} da(012)b(012)^\vee\right) = \prod_{(012)\in X_2} n\delta(da(012))$$

$$= n^{|X_2|}\delta(da \mod n),$$

where the first δ is an indicator function which is 1 if the argument is 0 mod n and 0 otherwise and the second δ is short-hand for indicating 0 in $C^2(X,\mathbb{Z}_n)$. Since we use this trick so often we will state it in generality:

Lemma 3. For X a closed oriented PL D-manifold, $a \in C^k(X,\mathbb{Z})$, $F \in C^{k+1}(X,\mathbb{Z})$:

$$\sum_{b\in C^{D-k-1}(X^\vee,\mathbb{Z}_n)} \exp\left(\frac{2\pi i}{n}\int_X (da+F,b)\right) = n^{|X_{k+1}|}\delta(da+F \mod n).$$

Proof. Exchange sum and product:

$$\sum_{b\in C^{D-k-1}(X^\vee,\mathbb{Z}_n)} \exp\left(\frac{2\pi i}{n}\int_X (da+F,b)\right)$$

$$= \sum_b \prod_{\sigma\in X_{k+1}} \exp\left(\frac{2\pi i}{n}(da(\sigma)+F(\sigma))b(\sigma)^\vee\right)$$

$$= \prod_{\sigma\in X_{k+1}} n\delta(da(\sigma)+F(\sigma) \mod n) = n^{|X_{k+1}|}\delta(da+F \mod n).$$

□

Thus we see that b acts as a Lagrange multiplier to impose the flatness constraint $a \in Z^1(X,\mathbb{Z}_n)$. Some sources take this as the starting point for describing the topological $\mathbb{Z}_n$ gauge theory by declaring the fundamental field to be a $\mathbb{Z}_n$ 1-cocycle with trivial action principle $S_{eff} = 0$. For example, see [68].

Concluding the calculation, we have

$$Z(X) = N(X)\sum_a e^{iS_{eff}(a)} = N(X)n^{|X_2|}|Z^1(X,\mathbb{Z}_n)|.$$

Recall we are free to choose the normalization function $N(X)$ anyway we like, so long as it is a local function of the lattice, meaning that $N(X)$ receives a multiplicative contribution from each cell, so that all j-cells contribute the same amount. For instance, we will choose

$$N(X) = n^{-|X_2|-|X_0|}$$

so that

$$Z(X) = |H^1(X,\mathbb{Z}_n)|/|H^0(X,\mathbb{Z}_n)|$$

is a topological invariant (!). We will say a theory whose correlation functions are topological invariant and admits a choice of normalization function so that its partition functions are topological a strongly topological QFT.

Note that there is no way to choose $N(X)$ locally so that $Z(X) = 1$. In fact, our choice is the *unique* choice of normalization so that $Z(S^D) = 1$ and $Z(X)$ is a topological invariant. We will discuss later that while all correlation functions are independent of $N(X)$, there are interesting physical interpretations of $Z(X)$ when we can make a choice of normalization like this, having to do with entanglement. For now, we just point it out as something to ponder and move on to computing correlation functions.

Finally, note that we could've done the sum over a instead, and obtained (using lemma 3 applied to $X^\vee$)

$$e^{iS_{eff}(b)} = n^{|X_0|}\delta(db \mod n),$$

which enforces the flatness constraint $b \in Z^{d-1}(X, \mathbb{Z}_n)$. Some sources describe something called $\mathbb{Z}_n$ $d-1$-form gauge theory, whose fundamental field is a $\mathbb{Z}_n$ $d-1$-cocycle and has trivial action principle $S_{eff} = 0$. What we have just derived is that a single parent action principle (the "quantum double" (5.1)) gives rise to an equivalence between the partition functions of these theories. This is a very simple manifestation of electric-magnetic duality, which is incarnated in many forms.

5.1.3 some correlation functions and duality

First let us consider the expectation value of the unit Wilson line

$$W(\Gamma) = \exp\left(\frac{2\pi i}{n}\int_\Gamma a\right)$$

for $\Gamma \in Z_1(X, \mathbb{Z})$. Because $W(\Gamma)$ is independent of b, we can do the sum over b as we did for the partition function:

$$\langle W(\Gamma)\rangle_X = \frac{N(X)}{Z(X)}\sum_a \exp\left(\frac{2\pi i}{n}\int_\Gamma a\right)e^{iS_{eff}(a)}$$

$$= \frac{N(X)}{Z(X)}n^{|X_2|}\sum_{a\in Z^1(X,\mathbb{Z}_n)} \exp\left(\frac{2\pi i}{n}\int_\Gamma a\right).$$

We see that $\langle W(\Gamma)\rangle_X = 0$ if Γ is nontrivial in $\mathbb{Z}_n$ homology, and 1 otherwise. If we think about Γ as the worldline of a particle described by the Wilson line, when $\langle W(\Gamma)\rangle \neq 0$ for arbitrarily large (but nullhomologous) Γ, then we say the particle is deconfined. Likewise, we find $\langle H(\Sigma)\rangle_X = 0$ if Σ is nontrivial in $\mathbb{Z}_n$ homology, and 1 otherwise, so the object whose worldvolume is Σ is also deconfined.

Now we consider the correlation function

$$\langle W(\Gamma)H(\Sigma)\rangle_X,$$

which cannot be straightforwardly computed by simply summing over one of the two gauge fields since the integrand depends on both a and b. However, we can exploit a clever trick which illustrates the quantum nature of these observables. Let $\delta_\Sigma \in Z^2(X,\mathbb{Z})$ denote the Poincaré dual of $\Sigma \in Z_{d-1}(X^\vee,\mathbb{Z})$. We can write

$$H(\Sigma) = \exp\left(\frac{2\pi i}{n}\int_\Sigma b\right) = \exp\left(\frac{2\pi i}{n}\int_X (\delta_\Sigma, b)\right).$$

Thus we can consolidate $H(\Sigma)$ with the path integral weight and compute the partial sum

$$e^{iS_{eff}(a)} = \sum_{b\in C^{d-1}(X,\mathbb{Z}_n)} \exp\left(\frac{2\pi i}{n}\int_X (da + \delta_\Sigma, b)\right).$$

We see that summing over b in $\langle W(\Gamma)H(\Sigma)\rangle$ now yields the modified constraint (see lemma 3)

$$da = -\delta_\Sigma \mod n.$$

In particular we see that if $[\delta_\Sigma] \neq 0 \in H^2(X,\mathbb{Z}_n)$ or equivalently $[\Sigma] \neq 0 \in H_{d-1}(X,\mathbb{Z}_n)$, then $\langle W(\Gamma)H(\Sigma)\rangle$ vanishes. Further, if $[\Gamma] \neq 0 \in H_1(X,\mathbb{Z}_n)$ the correlation function also vanishes. If both Γ and Σ are trivial in homology, then there are two cases remaining. If they are unlinked, it is easy to see the modified constraint doesn't affect the remaining average

$$\sum_a W(\Gamma)e^{iS_{eff}(a)}.$$

On the other hand, if Γ and Σ are linked with linking number k, then the modified constraint contributes

$$W(\Gamma) = \exp\left(\frac{2\pi i}{n}\int_\Gamma a\right) = \exp\left(\frac{2\pi i}{n}\int_D da\right)$$

$$= \exp\left(-\frac{2\pi i}{n} \int_D \delta_\Sigma \right) = e^{-2\pi i k/n},$$

where D is a $\mathbb{Z}_n$ 2-chain bounding Γ and we have used the definition of the linking number

$$\langle \Gamma, \Sigma \rangle = \int_D \delta_\Sigma = \#(D \cap \Sigma).$$

Summarizing,

$$\langle W(\Gamma) H(\Sigma) \rangle = \begin{cases} 0 & [\Sigma] \neq 0 \in H_{d-1}(X, \mathbb{Z}_n) \\ 0 & [\Gamma] \neq 0 \in H_1(X, \mathbb{Z}_n) \\ e^{-2\pi i k/n} & \langle \Gamma, \Sigma \rangle = k \end{cases} .$$

This illustrates that operators in a topological QFT can have a long range effect on each other, in the sense that we have a minor violation of the cluster decomposition principle. Indeed, even for well-separated operators $W(\Gamma)$ and $H(\Sigma)$ (assume third case above), if they are linked, $k \neq 0 \bmod n$, then

$$\langle W(\Gamma) H(\Sigma) \rangle_X \neq \langle W(\Gamma) \rangle_X \langle H(\Sigma) \rangle_X = 1.$$

When this is the case, it is often said that the two operators are not mutually local.

Observe that the 't Hooft operator $H(\Sigma)$, once we integrated out b, acted to modify the flatness constraint of the $\mathbb{Z}_n$ gauge field a. In formulations of the $\mathbb{Z}_n$ gauge theory without b, where we declare our basic field to already satisfy the constraint $a \in Z^1(X, \mathbb{Z}_n)$, the 't Hooft operator must be defined as a singularity along Σ for a. This can be discussed precisely by removing a tubular neighborhood of Σ and specifying boundary conditions for a there. To obtain the modified flatness constraint above, the appropriate boundary condition has $\int_{S^1} a = -1/n$ around the small circular coordinate of the boundary of the tubular neighborhood of Σ. One will often see such disorder operators discussed this way in the literature, and one should imagine that there is secretly some other field, in this case b, which has been integrated out, and that if we include it in our analysis we can treat without discussing singularities and removing pieces of spacetime. As always, which operators should be included in the discussion should be specified at the beginning in the definition of the QFT.

Note that when we integrate out a instead, the presence of $W(\Gamma)$ modifies the flatness constraint of b. In this way, electric-magnetic duality (see section 5.1.2) exchanges ordinary operators with disorder operators.

5.2 Twisted $\mathbb{Z}_n$ Gauge Theory in 2+1D and Knot Invariants

5.2.1 twisted quantum double

In this section we discuss a "twisting" of the topological $\mathbb{Z}_n$ gauge theory by a Dijkgraaf-Witten term [68]. These terms come from the group cohomology of the gauge group, in this case $\mathbb{Z}_n$. The relevant one for us is

$$H^3(B\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n.$$

Such classes define cohomology operations

$$H^1(X, \mathbb{Z}_n) \to H^3(X, \mathbb{R}/\mathbb{Z})$$
$$[a] \mapsto [\omega(a)].$$

In other words, such classes give gauge invariant functions $\int_X \omega(a) \in \mathbb{R}/\mathbb{Z}$ for a (flat) $\mathbb{Z}_n$ gauge field $a \in Z^1(X, \mathbb{Z}_n)$. Thus, the Dijkgraaf-Witten term

$$\int_X \omega(a)$$

is a possible term in the effective action (5.4).

We wish to modify the quantum double action (5.1) to obtain such a term after integrating out the dual field b. To do so, we will need to express our cohomology operation ω as a function on cochains

$$\omega : C^1(X, \mathbb{Z}) \to C^3(X, \mathbb{R}/\mathbb{Z})$$

which reduces to ω on those $a \in C^1(X, \mathbb{Z}_n)$ with $da = 0 \bmod n$.

The cohomology of $B\mathbb{Z}_n$ is understood [69] and a complete basis of

$$H^3(B\mathbb{Z}_n, \mathbb{R}/\mathbb{Z})$$

can be expressed as

$$a \mapsto \frac{k}{n^2} a \cup da,$$

where $k \in \mathbb{Z}$. Note that this term is not a total derivative. For example, on a $\mathbb{Z}_n$ lens space $L(n, 1)$ ($\mathbb{RP}^3$ for $n = 2$) [10], with a representing a generator of $H^1(L(n, 1), \mathbb{Z}_n)$,

$$\exp\left(\frac{2\pi i}{n^2} \int_{L(n,1)} a \cup da\right)$$

is a primitive nth root of unity.

This expression makes sense for cochains, not just cocycles, so we may use it to define the "twisted quantum double" model:

$$S(a,b) = \int_X \frac{1}{n}(a, db) + \frac{k}{n^2} a \cup da, \tag{5.5}$$

where we choose a branching structure on X to define the cup product. The QFT described by this action (with the operators we define below) is referred to as $\mathbb{Z}_n$ Dijkgraaf-Witten theory at level k.

This new term is *not* invariant under 1-form gauge transformations of a:

$$a \mapsto a + nw \qquad w \in C^1(X, \mathbb{Z}) \tag{5.6}$$

instead it has a variation

$$S(a+nw, b) = \int_X \frac{1}{n}(a+nw, db) + \frac{k}{n^2}(a+nw) \cup d(a+nw) \tag{5.7}$$

$$= \int_X \frac{1}{n}(a, db) + (w, db) + \frac{k}{n^2} a \cup da + \frac{k}{n} w \cup da + \frac{k}{n} a \cup dw + kw \cup dw$$

$$= S(a,b) + \frac{k}{n} \int_X w \cup da + a \cup dw,$$

after discarding integer terms (recall the action only appears as $e^{2\pi i S}$). However, the action is invariant under the restricted 1-form gauge transformations

$$a \mapsto a + n^2 w,$$

(as well as other gauge transformations) and this group has a finite orbit space $C^1(X, \mathbb{Z}_{n^2})$. We therefore define the path integral for the twisted gauge theory using

$$\langle \mathcal{O} \rangle = \frac{1}{Z(X)} n^{-|X_2| - 2|X_1|} \sum_{\substack{a \in C^1(X, \mathbb{Z}_{n^2}) \\ b \in C^1(X^\vee, \mathbb{Z}_n)}} \mathcal{O}(a,b) \exp\Big(2\pi i S\Big), \tag{5.8}$$

where $\mathcal{O}(a,b)$ is a function of a and b such that

$$\mathcal{O}(a + dg, b + df) = \mathcal{O}(a,b) \qquad g, f \in C^0(X, \mathbb{Z}).$$

In the path integral for the untwisted $\mathbb{Z}_n$ gauge theory (5.3), the sum was over $a \in C^1(X, \mathbb{Z}_n)$, so $\mathcal{O}(a,b)$ was automatically invariant under the 1-form transformations (5.6). To define the twisted $\mathbb{Z}_n$ gauge theory, we will have to impose a constraint on $\mathcal{O}$ by hand, which is that after performing the sum over b, we obtain a gauge invariant function of a:

$$\sum_{b \in C^1(X^\vee, \mathbb{Z}_n)} \mathcal{O}(a+nw, b) e^{2\pi i S(a+nw,b)} = \sum_{b \in C^1(X^\vee, \mathbb{Z}_n)} \mathcal{O}(a,b) e^{2\pi i S(a,b)} \tag{5.9}$$

for all a.

For example, consider the "fractional" Wilson line

$$W(\Gamma, q) = \exp \frac{2\pi i q}{n^2} \int_\Gamma a,$$

for $q \in \mathbb{Z}$, $\Gamma \in Z_1(X, \mathbb{Z})$. Though this function makes sense in the path integral above for any q, once we perform the partial sum above, we obtain

$$\sum_{b \in C^1(X^\vee, \mathbb{Z}_n)} e^{2\pi i q \int_\Gamma a/n^2} e^{2\pi i S(a,b)} = n^{|X_2|} e^{2\pi i q \int_\Gamma a/n^2} \delta(da \mod n),$$

which is invariant under the 1-form transformations only when $q \in n\mathbb{Z}$. Thus, with this choice of 1-form gauge invariance constraint, we obtain the same quantization rule for Wilson lines as we had in the untwisted $\mathbb{Z}_n$ gauge theory. Note that even if we don't impose this constraint, the fractional Wilson lines are confined, in the sense that

$$\langle W(\Gamma, q) \rangle = 0$$

for all Γ (even contractible) and $q \notin n\mathbb{Z}$, since da/n is unconstrained by integrating over b.

5.2.2 decorated 't Hooft lines

We consider the "bare" 't Hooft operator

$$H(\Gamma) = \exp \frac{2\pi i}{n} \int_\Gamma b$$

where $\Gamma \in Z_1(X^\vee, \mathbb{Z})$. When we perform the sum over b, we obtain (see lemma 3)

$$\sum_{b \in C^1(X^\vee, \mathbb{Z}_n)} H(\Gamma) e^{2\pi i S(a,b)} = n^{|X_2|} \exp\left(\frac{2\pi i k}{n^2} \int_X a \cup da\right) \delta(da + \delta_\Gamma \mod n). \tag{5.10}$$

Let us denote this expression as $H(\Gamma, a)$. Using (5.7) and $da = -\delta_\Gamma \bmod n$ we obtain the variation

$$H(\Gamma, a + nw) = H(\Gamma, a) \exp\left(\frac{2\pi i k}{n} \int_X -w \cup \delta_\Gamma - \delta_\Gamma \cup w\right),$$

so the bare 't Hooft operator is *not* gauge invariant (cf. (5.9)). However, we may add fractional Wilson lines to it, defining the physical 't Hooft line as the decorated 't Hooft line

$$\tilde{H}(\Gamma) = \exp\left(\frac{2\pi i}{n} \int_\Gamma b + \frac{2\pi i k}{n^2} \int_{f_{-\infty}\Gamma} a + \frac{2\pi i k}{n^2} \int_{f_{+\infty}\Gamma} a\right),$$

where we use the branching Morse flow of theorem 3 to define the pushoffs $f_{\pm\infty}\Gamma \in Z_1(X, \mathbb{Z})$. These satisfy

$$\int_X \delta_\Gamma \cup a = \int_{f_{-\infty}\Gamma} a$$

$$\int_X a \cup \delta_\Gamma = \int_{f_{+\infty}\Gamma} a,$$

from which one sees that $\tilde{H}(\Gamma)$ is gauge invariant.

This leads us to the definition of the twisted $\mathbb{Z}_n$ gauge theory:

Definition 9. The 2+1D $\mathbb{Z}_n$ Dijkgraaf-Witten theory at level k on a closed PL 3-manifold X is the QFT whose observables are defined by functions $\mathcal{O}(a, b)$, generated as a $\mathbb{C}$-algebra by the Wilson lines

$$W(\Gamma) = \exp \frac{2\pi i}{n} \int_\Gamma a, \qquad \Gamma \in Z_1(X, \mathbb{Z}_n)$$

and the decorated 't Hooft lines

$$\tilde{H}(\Gamma) = \exp\left(\frac{2\pi i}{n} \int_\Gamma b + \frac{2\pi i k}{n^2} \int_{f_{-\infty}\Gamma} a + \frac{2\pi i k}{n^2} \int_{f_{+\infty}\Gamma} a\right), \qquad \Gamma \in Z_1(X^\vee, \mathbb{Z}_n)$$

and whose correlation functions are given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z(X)} n^{-|X_2|-2|X_1|} \sum_{\substack{a \in C^1(X,\mathbb{Z}_{n^2}) \\ b \in C^1(X^\vee,\mathbb{Z}_n)}} \mathcal{O}(a,b) \exp\left(2\pi i S\right),$$

where we have normalized the sum so that the partition function is a topological invariant.

5.2.3 Framing Dependence

Let us compute the expectation value

$$\langle \tilde{H}(\Gamma) \rangle.$$

Because of the constraint $da = -\delta_\Gamma \mod n$ in the partial sum (5.10), the expectation value vanishes unless $\Gamma = \partial D$ for some 2-chain D. In Poincaré duals we have $d\delta_D = \delta_\Gamma$. We can use δ_D to define a new field variable

$$a = c - \delta_D$$

for which the constraint $da + \delta_\Gamma = 0 \mod n$ reads

$$dc = 0 \mod n.$$

In terms of c, the sum simplifies considerably:

$$\langle \tilde{H}(\Gamma) \rangle = \frac{1}{Z(X)} n^{|X_2|} \sum_{\substack{c \in C^1(X,\mathbb{Z}_{n^2}) \\ dc=0 \mod n}} \exp\left(2\pi i \int_X \frac{k}{n} c \cup \frac{dc}{n} - \frac{k}{n^2} \delta_D \cup \delta_\Gamma\right)$$

$$= \exp\left(-\frac{2\pi i k}{n^2} \int_X \delta_\Gamma \cup \delta_D\right),$$

where the sum over c and $n^{|X_2|}$ exactly cancel the partition function $Z(X)^{-1}$.

We recognize this integral as that which computes the self-linking of Γ with respect to the branching structure on X. This is remarkable for two reasons. The first is that this TQFT computes a (simple) knot invariant of Γ, one which has no formula as a local integral involving just δ_Γ, as we discussed in section 2.2.4.

The second comment about this expectation value is that it depends on the branching structure, as

$$\int_X \delta_\Gamma \cup \delta_D = \#(\Gamma_\infty \cap D).$$

This is interesting because the effective action for a in the absence of any 't Hooft operators is independent of the branching structure on a closed 3-manifold X. In a continuum quantum field theory describing this system, such as a Chern-Simons theory [70], the bare 't Hooft line is a disorder operator, where A has a singularity and is undefined. Therefore, one must define any decoration of it by Wilson lines using a normal framing, since we cannot evaluate the Wilson line on the singularity [71].

5.2.4 fusion rules and modular tensor category

By forgetting that the fractional Wilson lines decorating the physical 't Hooft are slightly displaced from each other, and from the "core" where a is singular, we can say that the unit 't Hooft line carries electric charge $2k/n$ mod n as well as its magnetic charge 1 mod n. This may be expressed symbolically as "fusion rules":

$$H^n = W^{2k}$$

$$W^n = 1,$$

where H represents any 't Hooft line near a fixed 1-cycle, and W represents any Wilson line near that fixed 1-cycle.

In 2+1D, the fusion rules and correlation functions of the line operators can be encoded in a structure called a modular tensor category [72], a braided monoidal category whose simple objects are the unit line operators, whose monoidal structure is the fusion product, and whose braiding comes from the linking phases. A complete description of this formalism would take us too far afield, since we wish to focus on path integrals, but it is important to mention because many mathematicians and even physicists think about 3D TQFT in these terms. Let it suffice to claim that it is simple to construct the modular tensor category from the path integral with slightly more work along the same lines as above.

5.3 Higher Dijkgraaf-Witten Theory

Now we consider one of the most general sorts of topological gauge theories: a theory whose gauge group is an n-group $\mathbb{G}$, which we describe in terms of the Postnikov tower of its classifying space, whose homotopy groups we denote $\Pi_1, \ldots, \Pi_n$ and Postnikov classes $\beta_2, \ldots, \beta_n$. The case of a 1-group was first discussed in [68], while examples implicitly using 2-groups appeared in [73]. The general theory was outlined in [33], while a description like the one I present here first appeared in my paper with Anton Kapustin [34] for 2-groups.

A $\mathbb{G}$ gauge field on a CW complex is a nonabelian 1-cocycle $a \in Z^1(X, \mathbb{G})$ (see section 3.2 of part I) meaning a collection of cochains

$$a_j \in C^j(X, \Pi_j)$$

satisfying the cocycle condition for nonabelian cohomology $Z^1(X, \mathbb{G})$:

$$da_1 = 1$$
$$D_a a_2 = \beta_2(a_1)$$
$$D_a a_3 = \beta_3(a_1, a_2)$$
$$D_a a_4 = \cdots ,$$

where the covariant derivative for twisted cochains is defined by the action of Π_1 on the Π_j (see section 1.2.5). Note that we only consider the case $n \leq D$, since the $a_{>D}$ don't appear anywhere in the CW complex of a D-manifold.

Gauge transformations are parametrized by $f_j \in C^{j-1}(X, \Pi_j)$ and act by

$$a_k \mapsto a_k \qquad k < j$$
$$a_j \mapsto a_j + D_a f_j$$
$$a_k \mapsto a_k + \beta_{k,1}(f_j; a_1, \ldots, a_{k-1}) \qquad k > j,$$

where $\beta_{k,1}$ is a first descendant of β_k (see section 3.4).

Higher Dijkgraaf-Witten theory is defined by $\mathbb{G}$ as well as a cocycle

$$\omega \in Z^D(B\mathbb{G}, \mathbb{R}/\mathbb{Z}).$$

So long as all the Π_j are finite (so $Z^1(X, \mathbb{G})$ is finite), this defines a TQFT via the path integral

$$N(X) \sum_{a \in Z^1(X,\mathbb{G})} \exp 2\pi i \int_X \omega(a),$$

completely analogously to the original paper [68].

5.3.1 Electric and Magnetic Operators

The effect of the Postnikov classes is to modify the spectrum of Wilson operators. Besides the usual Wilson line for a_1, which remains gauge invariant, there are also higher Wilson operators

$$W_k(\Sigma_k, \chi_k) = \exp 2\pi i \int_{\Sigma_k} \chi_k(a_k),$$

where $\Sigma_k \in C_k(X, \mathbb{Z})$ and $\chi_k : \Pi_k \to \mathbb{R}/\mathbb{Z}$ is the logarithm of a 1D representation of Π_k which is the charge of the Wilson operator. However, the Postnikov classes spoil the gauge invariance of these bare Wilson operators.

This can be resolved when β_k is in the kernel of the coefficient map $\chi_k : H^{k+1}(B\tau_{<k}\mathbb{G}, \Pi_k) \to H^{k+1}(B\tau_{<k}\mathbb{G}, U(1))$, meaning there is a

$$\gamma_k(a_1, \ldots, a_{k-1}) \in U(1)$$

such that

$$d\gamma_k(a_1, \ldots, a_{k-1}) = \chi_k(\beta_k(a_1, \ldots, a_{k-1})).$$

For these restricted charges, we may decorate the Wilson operator by

$$\tilde{W}_k(\Sigma_k, \chi_k, \gamma_k) = \exp 2\pi i \int_{\Sigma_k} \chi_k(a_k) - \gamma_k(a_1, \ldots, a_{k-1})$$

and this operator is gauge invariant. This works for any Σ_k.

We can phrase this all very neatly in terms of extended k-dimensional Wilson operators, of which those of Dijkgraaf-Witten type are classified by $H^k(B\mathbb{G}, U(1)$. Indeed, χ_k defines a class in $H^k(B\Pi_k, U(1))$ and we wish to extend it to a class in $H^k(B\mathbb{G}, U(1))$. $B\mathbb{G}$ is an iterated fibration and there is no problem with extending χ_k to those pieces fibered above $B\Pi_k$. However, the pieces below $B\Pi_k$ may form a problem. Thus we study the Serre spectral sequence for $B\Pi_k \to B\tau_{\leq k}\mathbb{G} \to B\tau_{<k}\mathbb{G}$. The only differential comes from the Postnikov class β_k, and this yields our description above.

One can also define codimension k 't Hooft operators which create singularities in the a_k. This requires extending the Postnikov classes to be functions of cochains, not just cocycles. However, the choices of extensions don't matter for correlation functions, and the Postnikov classes don't create any strange selection rules for the 't Hooft operators as they do for the Wilson operators.

5.3.2 Quantum Double and Duality

In a precise sense, the physics of the Postnikov classes is dual to the physics of mixed Dijkgraaf-Witten terms. For instance, consider the $n = 2$ case with Postnikov class $\beta \in H^3(B\Pi_1, \Pi_2)$. We can define the path integral by introducing a Lagrange multiplier λ and writing

$$\sum_{(a_1,a_2)\in Z^1(X,\mathbb{G})}$$

$$\sim \sum_{a_1\in Z^1(X,\Pi_1)} \sum_{a_2\in C^2(X,\Pi_2)} \sum_{\lambda\in C^{D-3}(X^\vee,\Pi_2^\vee)} \exp\left(2\pi i \int_X (\lambda, D_{a_1}a_2 - \beta(a_1))\right).$$

To dualize, we now sum over a_2 instead of λ to obtain

$$\sum_{(a_1,a_2)\in Z^1(X,\mathbb{G})} \sim \sum_{a_1\in Z^1(X,\Pi_1)} \sum_{\lambda\in Z^{D-3}(X^\vee,\Pi_2^{\vee,a_1})} \exp\left(2\pi i \int_X -(\lambda, \beta(a_1))\right)$$

$$\sim \sum_{a_1\in Z^1(X,\Pi_1)} \sum_{\lambda^\vee\in Z^{D-3}(X,\Pi_2^{\vee,a_1})} \exp\left(2\pi i \int_X -\lambda^\vee \cup \beta(a_1)\right),$$

where in the last step we have used the cocycle conditions for λ, β, and a_1 to relate the intersection pairing to the cup product. Here $\sim$ means up to a normalization.

Under this duality, the Wilson surface operator for a_2 transforms into the 't Hooft operator for λ. To see this, note that the bare Wilson surface may be written

$$\exp\left(2\pi i \int_X (\delta_\Sigma, \chi(a_2))\right),$$

which when we insert in the sum above and integrate out a_2 we find modified constraints for the Lagrange multiplier and dual field

$$d\lambda = \chi\delta_\Sigma,$$

$$d\lambda^\vee = \chi\delta_{\Sigma_{-\infty}},$$

where we consider $\chi \in \Pi_2^\vee$ an element of the group of 1D characters, which is the coefficient group of $\lambda, \lambda^\vee$.

There are two cases to consider for this operator. The first case is that β is in the kernel of the coefficient map defined by χ, and we have chosen a decoration $\gamma(a_1) \in C^2(B\Pi_1, U(1))$ as above. The decoration is independent of a_2 and so it passes through the duality to become a decoration of the λ 't Hooft operator, whose necessity may be derived by studying gauge invariance of the topological term $\lambda^\vee \cup \beta(a_1)$ in the presence of this operator, as we did for the $\mathbb{Z}_n$ gauge theory. The second case is that β is not in the kernel of the coefficient map defined by χ. In thise case, there is no way to choose a decoration which makes the operator gauge invariant.

In general we can produce a quantum double formulation of higher DW theory as follows. Let $\mathbb{G}$ be a stable n-group. We can invent a family of Lagrange multiplier fields

$$b_j \in C^{D-j}(X^\vee, \Pi_j^\vee)$$

and an action

$$S(a,b) = \sum_j \int_X (b_j, da_j + \beta_j(a_{<j})).$$

Clearly integrating out b_j yields the cocycle conditions of (3.1). Integrating out the a_j however is difficult because of the β_j's. In general only the last one, a_n, may be integrated out, since it is guaranteed not to occur in any β_j. In that case we get a cocycle condition $db_n = 0$ and a Dijkgraaf-Witten term $(b_n, \beta_n(a_1, \ldots, a_{n-1}))$. The duality properties of these theories were studied by Anton Kapustin and myself in [74].

5.4 Coupling to Background Gravity

In this section we study how finite gauge theories may be coupled explicitly to the tangent bundle of spacetime.

5.4.1 Stiefel-Whitney Terms

Recall (see section 4.1) with a choice of branching structure on a triangulated PL D-manifold X we have a family of Stiefel-Whitney cocycles $w_j(TX) \in Z^j(X^\vee, \mathbb{Z}_2)$. These may be coupled to our gauge field $a \in Z^1(X, \mathbb{G})$ by a choice of cocycle $\Omega_{D-j} \in Z^{D-j}(B\mathbb{G}, \mathbb{Z}_2)$, which defines a term in the action:

$$\frac{1}{2}\int_X (w_j, \Omega_{D-j}(a)),$$

or more generally we can take a degree j polynomial $P(w_1, \dots, w_j)$ in the Stiefel-Whitney classes and consider

$$\frac{1}{2}\int_X (P(w_1, \dots, w_j), \Omega_{D-j}(a)).$$

These are the most general cobordism invariants in

$$\Omega^D_O(B\mathbb{G}, U(1)) = H^D(B\mathbb{G}, \Omega_O),$$

since Ω_O has vanishing Postnikov classes [24].

Because of the Wu formula (4.1), some of these terms are expressible in the Steenrod squares of $\Omega_{D-j}(a)$. These are the classes in the image of the map

$$H^D(B\mathbb{G}, U(1)) \to \Omega^D_O(B\mathbb{G}, U(1)),$$

but there are classes not in the image of this map which are beyond the usual Dijkgraaf-Witten theory. Like the Dijkgraaf-Witten terms they generalize, these Stiefel-Whitney terms have interesting effects on the extended 't Hooft operators of a.

w_2 and emergent spinors

For instance, we consider the topological $\mathbb{Z}_2$ gauge theory in the quantum double formalism with $a \in C^1(X^\vee, \mathbb{Z}_2)$, $b \in C^{D-2}(X, \mathbb{Z}_2)$ with an extra w_2 term:

$$S(a, b) = \frac{1}{2}\int_X (b, da + w_2(TX)).$$

When we integrate out a, we get the topological $D-2$-form gauge theory with (b, w_2) topological term. However, when we integrate out b, we see that w_2 sources 't Hooft operators for a. This means that Wilson lines for a will detect the tangent bundle of X:

$$\langle W(\Gamma)\rangle = \begin{cases} \text{undefined} & [w_2(TX)] \neq 0 \in H^2(X, \mathbb{Z}_2) \\ 0 & [\Gamma] \neq 0 \in H_1(X, \mathbb{Z}_2), \quad \text{X spin} \\ (-1)^{\int_D w_2(TX)} & \partial D = \Gamma, \quad \text{X spin} \end{cases}.$$

The first case is undefined because the partition function vanishes when X is not a spin manifold, so our path integral yields $0/0$.

This means that our path integral yields a QFT only defined on spin manifolds. This is surprising because the action principle did not have any explicit dependence on a spin structure. This is a simple example of a "gravitational anomaly" which we will discuss in great detail later.

For now we note that this theory is equivalent to one whose (dynamical) fields themselves are spin structures. Indeed, once we integrate out b, we have $da = w_2 \bmod 2$, and using our obstruction theory we can identify a with a spin structure η. This identification depends on the choice of branching structure.

To define the Wilson line in terms of η, we use the fact that a spin structure assigns a mod 2 invariant $Q_\eta(\Gamma)$ to a curve Γ carrying a framing of its normal bundle. The obstruction theory is such that this mod 2 invariant is precisely $\int_\Gamma a$ when the curve is $\Gamma \in Z_1(X^\vee, \mathbb{Z})$, given the normal framing induced by the branching structure. See section 4.2.4.

Thus, as we are now familiar, the proper definition of the Wilson line requires us to keep track of this normal framing. And above we were implicitly using the branching structure to define such a framing for $\Gamma \in Z_1(X^\vee, \mathbb{Z})$. Indeed, the expectation value $\langle W(\Gamma)\rangle$ depends on the branching structure through the cocycle w_2. This expectation value (and $Q_\eta(\Gamma)$) has the property that it changes sign when the normal framing is rotated by 2π. In this sense, we say that the Wilson line describes a fermionic particle or more precisely a spinor particle. Because the degrees of freedom that went into the theory were purely "bosonic", meaning there was no explicit spin structure dependence, we sometimes say that these are "emergent" spinors [75]. The existence of the above theories, being topological bosonic theories in any dimensions with a deconfined emergent fermion, has important physical consequences we will see later in chapter 7.

A related version of this theory is

$$S(a,b) = \frac{1}{2}\int_X (b,a) + b \cup w_2(TX^\vee),$$

which is almost the same except that integration over b enforces the constraint

$$\partial W(a) = f_\infty^\vee W_2(TX^\vee).$$

This is also only possible if X is a spin manifold, as

$$\delta f_\infty^\vee W_2(TX^\vee) \in Z^2(X^\vee, \mathbb{Z}_2)$$

is another representative of $[w_2(TX)]$, defined using a branching structure on $X^\vee$ rather than X.

Chapter 6

Higher Symmetries and Anomalies

Introduction

Let $\mathbb{G}$ be a finite ∞-group, ie. one whose Π_k are finite for all k. We say that a path integral QFT in D spacetime dimensions has a global $\mathbb{G}$ symmetry if it has an action of $H^0(X, \mathbb{G})$ on the field space, written

$$f : \text{fields}(X) \to \text{fields}(X)$$

$$f : \phi \mapsto \phi^f$$

such that

$$S(X, \phi) = S(X, \phi^f).$$

We say the symmetry can be extended to act locally if there is a way of defining a field space which includes twisted sectors fields$(X, [A])$, depending on a gauge-equivalence class of background $\mathbb{G}$ gauge field, ie. an element $[A] \in H^1(X, \mathbb{G})$, as well as an extension of the action $S(X, [A], \phi)$ to include these twisted sectors. We require the action of $H^0(X, \mathbb{G})$ to also extend to the twisted sectors. In this case the symmetry is called anomaly free. When $\mathbb{G}$ is not finite but still compact, then the same can be done using nonabelian differential cohomology (see [76] and references therein).

Often in defining the twisted sectors, it is not clear how to define the extended field space and action in a way that explicitly only depends on the cohomology class $[A] \in H^1(X, \mathbb{G})$. Indeed, we have to make sure $S(X, [A], \phi)$ is consistent with locality, which is most easy to do using a local representative A of the cohomology class. What happens in practice then is we end up choosing a cochain theory $Z^1(X, \mathbb{G})$ where A lives and constructing an action $S(X, A, \phi)$ which is local, and then checking how the action transforms under gauge transformations:

$$\delta_f S = S(X, A^f, \phi^f) - S(X, A, \phi).$$

If we cannot choose such an action to be gauge invariant, then we say the symmetry has an anomaly. Intuitively, an anomaly is an unavoidable tension between symmetry and locality.

To be as precise as possible, we will discuss anomalies which satisfy anomaly in-flow. Actually all known anomalies satisfy some form of anomaly in-flow, but there is no proof. See for instance the discussion in [77] of a possible counterexample which turned out to satisfy anomaly in-flow only after extending the symmetry group to a 2-group.

Anomaly in-flow is a situation where a D-dimensional QFT with a $\mathbb{G}$ symmetry is defined on the boundary of an $D+1$-dimensional invertible TQFT with a $\mathbb{G}$ symmetry (the "bulk"). In this case, the bulk action may include a topological term whose boundary variation under a gauge transformation cancels $\delta_f S$ from the boundary action. When the bulk theory may be taken to be trivial, this is equivalent to being anomaly free. Thus, we may think of the $D+1$-dimensional TQFT with $\mathbb{G}$ symmetry as actually labelling some equivalence class of the anomaly. For this reason we will refer to it as the anomaly theory associated with the symmetry. Meanwhile the theory with the anomaly is the anomalous theory.

Invertible TQFTs describe short-range entangled (SRE) phases of matter [78]. Invertible TQFTs for manifolds equipped with a G bundle describe symmetry protected topological (SPT) phases of matter [59]. The study of anomalies in the anomaly in-flow situation is equivalent to the study of these systems and their boundary conditions [79].

The first sort of anomalies we will discuss are those whose anomaly theory is a (higher) Dijkgraaf-Witten theory (where the gauge field is kept as a background field). Because of their relationship to group cohomology, these are called group cohomology anomalies. Indeed, there is a nice correspondence: the projective representations classified by the group cohomology classes actually appear in the Hilbert space of the anomalous theory via the descendants. We will construct some examples of these in 2+1D, which is important because until Anton Kapustin and I constructed an example, it was thought that theories in odd spacetime dimensions were always free of anomalies. This is because most familiar anomalies appear because of chiral fermions [80] or chiral gauge theories [81, 8]. In our example, the chirality is due to the presence of a Dijkgraaf-Witten term, which creates an asymmetry between electric and magnetic operators, as we saw in section 5.2.

Then we will discuss some more exotic anomalies, sometimes called gravitational anomalies [81] whose anomaly theories depend on the topology of the bulk even when the $\mathbb{G}$ background is trivial. While in the case of anomalous $\mathbb{G}$-symmetry we can point to the gauge variance $\delta_f S$ of the boundary action as a symptom of the anomaly, gravitational anomalies tend to occur when the boundary theory depends on a choice of coordinate chart. In our discrete gauge theories we will see this as a dependence on a choice of branching structure. This is closely related to the framing anomaly of [71] but isn't quite the same.

6.1 Boundary Partition Functions and States

Suppose we have a path integral quantum field theory for $D+1$-manifolds X with boundary, such that the fields come in two families: bulk fields supported everywhere on X and boundary fields supported only near ∂X, such that the total space of fields is a fibration over the bulk fields with fiber the boundary fields. We can use the path integral to define a boundary partition function, which takes as argument a bulk field ϕ and returns the integrated path integral weight over all boundary fields with ϕ fixed:

$$Z(X,\phi) := \int_{\text{fields}(\partial X,\phi)} D\psi \ e^{iS(X,\phi,\psi)}.$$

If the action S is local, then we can rewrite this as a usual partition function of a D-dimensional theory, where the ϕ restricted to the boundary act as background fields:

$$Z(X,\phi) = Z(\partial X,\phi) = \int_{\text{fields}(\partial X,\phi)} D\psi \ e^{iS'(\partial X,\phi|_{\partial X},\psi)}.$$

Intuitively, this has the form of a wavefunction in the $D+1$-dimensional theory, and so we obtain a *state* in some Hilbert space, which is (very) schematically

$$|Z(\partial X)\rangle = \int_{\text{fields}(X)} D\phi \ \int_{\text{fields}(\partial X,\phi)} e^{iS'(\phi|_{\partial X},\phi)}|\phi\rangle. \tag{6.1}$$

In Dan Freed and Constantin Teleman's relative QFT [79] (see also Anton Kapustin's ICM lecture [82]), this is phrased by saying that the theory on the boundary defines a morphism from the trivial $D+1$-QFT to the bulk QFT. In particular, on the level of Hilbert spaces we have a map

$$\mathbb{C} \to \mathcal{H}_Y,$$

where $\mathcal{H}_Y$ is the Hilbert space of bulk states on some D-manifold Y. The image of $1 \in \mathbb{C}$ under this map is supposed to be $|Z(Y)\rangle$. Note that states in the Hilbert space and partition functions have similar normalization ambiguities.

We will see this is a powerful tool for studying D-dimensional QFTs coupled to background gauge fields. In this case we will attempt to derive which $D+1$-dimensional gauge theory contains the state

$$|Z\rangle = \sum_{A \in Z^1(Y,G)} Z(Y, A)|A\rangle$$

in its Hilbert space. To do so we will need an understanding of the Hilbert space of some of the gauge theories

The presentation of these Hilbert spaces will depend on a choice of triangulation on Y and the background gauge field A will be defined as a 1-cocycle on this triangulation. Intuitively, we can couple a path integral QFT to such an object as follow. This triangulation gives rise to a decomposition of Y into dual polyhedral D-cells of $Y^\vee$. One can consider the path integral on each D-cell where the boundary fields are fixed. This gives a sum of partition functions, one for each D-cell, which is a function of all of the boundary values. We use the G action on the fields and the 1-cocycle $A \in Z^1(Y, G)$ to form a half dimensional subspace of these boundary values. To do so, we look along the boundarys $(01)^\vee$ where D-cells $(0)^\vee$ and $(1)^\vee$ meet and fix a diagonal subspace of boundary values by

$$A(01) \cdot \phi^0|_{(01)^\vee} = \phi^1|_{(01)^\vee},$$

using the action of $A(01) \in G$ on the fields ϕ, where ϕ^0 are those fields in $(0)^\vee$ and ϕ^1 are those in $(1)^\vee$. The cocycle condition on A ensures that this is consistent across triple and higher junctions. Integrating over this half dimension subspace defines a state $|Z\rangle$.

6.2 Hilbert Spaces of Topological Gauge Theories

6.2.1 Higher Dijkgraaf-Witten Hilbert Space

Fix a finite (∞-)group $\mathbb{G}$, a cocycle $\omega \in Z^D(B\mathbb{G}, U(1))$, and a first descendant ω_1. To a $d = D - 1$ manifold Y we may associate a Hilbert space $\mathcal{H}_Y$, spanned by an orthonormal basis of "classical states" $|A\rangle$, where $A \in Z^1(Y, \mathbb{G})$. Within this Hilbert space is a subspace of "physical states" $\mathcal{H}_Y^{phys}$, which are linear combinations of the $|A\rangle$'s invariant under the local gauge transformations

$$U_g|A\rangle \mapsto e^{-i\int_Y \omega_1(A,g)}|A^g\rangle$$

for all $g \in C^0(Y, \mathbb{G})$. Orbits of the classical states under these transformations,

$$|[A]\rangle = \frac{|H^0(Y,\mathbb{G})|}{|C^0(Y,\mathbb{G})|} \sum_{g \in C^0(Y,\mathbb{G})} e^{-i\int_Y \omega_1(A,g)}|A^g\rangle$$

with this normalization form an orthonormal basis for the physical Hilbert space and are labelled by the (nonabelian) cohomology classes $[A] \in H^1(Y, \mathbb{G})$.

It turns out that one can give an Atiyah-Segal description of higher Dijkgraaf-Witten theory which associates this Hilbert space $\mathcal{H}_Y^{phys}$ to Y [83, 84]. To a D-manifold X with boundary we associate the state $|X\rangle \in \mathcal{H}_{\partial X}^{phys}$ defined in the basis above by

$$\langle [A] | X \rangle = \sum_{\substack{[\hat{A}] \in H^1(X,\mathbb{G}) \\ [\hat{A}]|_{\partial X} = [A]}} \exp i \int_X \omega(\hat{A}).$$

For this reason we refer to $\mathcal{H}_Y^{phys}$ as the (higher) Dijkgraaf-Witten Hilbert space.

Abelian Examples

When G is an abelian 1-group, for instance $G = \mathbb{Z}_n$, then it is useful to write the background gauge field as in integer cochain $A \in C^1(X, \mathbb{Z})$. In this case we have two kinds of gauge transformations, 0-form, $f \in C^0(X, \mathbb{Z})$, and 1-form $w \in C^1(X, \mathbb{Z})$ which act by

$$A \mapsto A + df + nw.$$

In this case the 1st descendant will be a function of both parameters:

$$\delta_{f,w}\omega(A) = d\omega_1(A, f, w).$$

Similar resolutions can be made for stable ∞-groups, although we won't use them here.

For the generator of $H^3(B\mathbb{Z}_n, U(1))$ given by

$$\omega(A) = \frac{1}{n^2} A \cup dA,$$

we can compute the descendant by inspection:

$$\omega(A + df + nw) - \omega(A) = \frac{1}{n^2} df \cup (dA + ndw) + \frac{1}{n}(w \cup dA + A \cup dw)$$
$$= d\left(\frac{1}{n^2} f \cup (dA + ndw) - \frac{1}{n} A \cup w\right).$$
$$\omega_1(A, f, w) = \frac{1}{n^2} f \cup (dA + ndw) - \frac{1}{n} A \cup w.$$

Thus a 1+1D theory that lives on the boundary of the 2+1D $\mathbb{Z}_n$ Dijkgraaf-Witten theory with twist ω will have a partition function which transforms this way.

To extract some physical content from this expression, consider the first term, which comes about from the 0-form variation when $w = 0$:

$$Z(Y, A + df) = \exp\left(\frac{2\pi i}{n^2} \int_Y f \cup dA\right) Z(Y, A),$$

where $Z(Y, A)$ is the partition function of the anomalous boundary theory on a closed surface Y coupled to background gauge field $A \in Z^1(Y, A)$. Let us suppose we insert a 2D 't Hooft operator through a triangle (012) of Y, modifying the cocycle condition for A at that face to $dA(012) = 1$. We see that the transformation law above is now

$$Z(Y, A + df) = \exp\left(\frac{2\pi i}{n^2} \int_Y f \cup \delta_{012}\right) Z(Y, A) = e^{\frac{2\pi i}{n^2} f(0)} Z(Y, A).$$

This is the same transformation rule we would have if there was an A Wilson line with fractional charge $1/n$ is ending at vertex 0. One says that the boundary A flux carries fractional A charge, and this fraction characterizes the anomaly.

In QFTs whose quantum mechanical structure is understood, this fractional charge shows up in the twisted sectors of the Hilbert space. That is, the 1+1D theory has a Hilbert space on a circle where the boundary conditions of the fields are periodic up to an application of a fixed element $m \in \mathbb{Z}_n$.

These are the boundary conditions for fields restricted to a circle enclosing a region R where $\int_R dA = m$, so we can relate it to the above. What happens is that when $m = 0$, the charges of states in the Hilbert space form the lattice of integer charges, while when $m = 1$, because of the anomaly, this charge lattice is shifted by a fractional amount $1/n$, in order to be consistent with the above transformation rule. This allows us to characterize the anomaly precisely. We will see 1+1D examples of this when we discuss some conformal field theories in section 7.3.

6.2.2 Hilbert Space of a Simple Topological Gravity Theory

As we have discussed, it's reasonable to use Stiefel-Whitney and other characteristic classes of the tangent bundle in the construction of action principles, and this leads to some interesting geometric effects, like the appearance of spin structure. Now we investigate the meaning of topological terms made entirely from the Stiefel-Whitney classes, in fact where such a term is the only contribution to the action, and there may as well be no other fields.

For instance, there is a 3+1D unoriented TQFT with the partition function

$$Z(X) = (-1)^{\int_X w_2(TX)^2}.$$

If X is a closed PL 4-manifold with branching structure, then we can give a cocycle defining $w_2(TX) \in Z^2(X^\vee, \mathbb{Z}_2)$. If $X^\vee$ is a triangulation then given it a branching structure as well we can define a Lagrangian density

$$\frac{1}{2} w_2(TX) \cup w_2(TX).$$

When we change the branching structure, $w_2(TX) \mapsto w_2(TX) + d\alpha + 2\beta$ for some $\alpha \in C^1(X, \mathbb{Z})$, $\beta \in C^2(X, \mathbb{Z})$. If X has boundary (and still $X^\vee$ is a triangulation), then the action has a boundary variation

$$\int_{\partial X} \frac{1}{2}\alpha \cup w_2(TX) + \frac{1}{2} w_2(TX) \cup \alpha + \frac{1}{2}\alpha \cup d\alpha. \tag{6.2}$$

Write $\partial X = Y$. We choose to consider only branching structures for which the edges with exactly one vertex in ∂X are oriented towards ∂X. With this convention the branching structure near ∂X depends only on the branching structure on Y. Further, $w_2(TX)|_Y$ is determined by this branching structure. We denote it $w_2(TY \oplus NY)$, since it represents this Stiefel-Whitney class.

Thus we can define a Hilbert space associated to Y which has an orthonormal basis labelled by branching structures of Y. An efficient way of encoding this is to give the vertices of Y a total ordering. The physical states are sums of these which are invariant under a combined change of the branching structure (a re-ordering of the vertices) and multiplication by

$$(-1)^{\int_Y \alpha \cup w_2(TY \oplus NY) + w_2(TY \oplus NY) \cup \alpha + \alpha \cup \alpha},$$

where α is the 1-form transformation of $w_2(TY \oplus NY)$ in caused by this change in branching structure.

Let us note that $w_2(TX)^2$ integrates to the signature of X modulot 2. It follows from Gauss sum formulas (see eg. [34]) that this theory is equivalent to a Crane-Yetter-Walker-Wang model [73, 85] with four *dynamical* 2-form gauge fields $B_i \in Z^2(X, \mathbb{Z}_2)$ and an action which is the sum of their Pontryagin squares:

$$S = \sum_i \frac{1}{4} B_i \cup B_i + \frac{1}{4} B_i \cup_1 dB_i.$$

6.3 Two Anomalous Topological Gauge Theories

6.3.1 Higher Symmetries of Topological Gauge Theories

The global symmetries of Dijkgraaf-Witten theory are symmetry transformations of BG which preserve the Dijkgraaf-Witten class $[\omega] \in H^D(BG, U(1))$. As we discussed around the classifying space, H-group actions on BG are the same as BG-bundles over BH. In turn, this is equivalent to group extensions of H by G. When G is abelian (or more generally a stable ∞-group), then this is the same as an action ρ of H on G, together with a (ρ-twisted) 2-cocycle

$$c \in H^2(BH, G^\rho)$$

which controls all possible anomalies of these symmetries.

Note that even when ρ is trivial, that is, when H doesn't action G, there still may be nontrival extension cocycles c. A puzzle in this case is that the global H symmetries (gauge transformations in $Z^0(X, H)$) may not act on the G gauge field at all. However, in a local formulation of the G Dijkgraaf-Witten theory with no constraints, such as the quantum double, in this case we will see a nontrival action on the dual variables. Physically, gauge theories always come with charged matter, and in the action of the global H symmetry on this charged matter, we will see the cocycle c. We don't discuss this here, but it is discussed in [86, 77].

6.3.2 2+1D 0-form Gauge Anomaly

The example in this section was first discussed by Anton Kapustin and myself in [86, 87] and is the first known example of an anomalous global internal symmetry in an odd spacetime dimension.

Derivation of the Anomaly Theory

We consider $\mathbb{Z}_3 \times \mathbb{Z}_3$ global symmetries of topological $\mathbb{Z}_3$ gauge theory where the global symmetry group is extended by the gauge group. In particular we consider the subgroup $H(3,3)$ of $GL(3,\mathbb{C})$ generated by

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This group is sometimes called the discrete Heisenberg group. These generators satisfy the group algebra $X^3 = Y^3 = 1$,

$$XYXY^{-1} = e^{2\pi i/3} = Z.$$

In terms of $\mathbb{Z}_3$ gauge fields A, B, c governing the X, Y, Z twists, respectively, the relations above imply

$$\begin{aligned} dA = dB = 0 \quad \text{mod } 3, \\ dc = A \cup B \quad \text{mod } 3. \end{aligned} \tag{6.3}$$

This last expression means that under A, B gauge transformations, c must transform. One choice of 0-form transformation rule is

$$A \mapsto A + df \tag{6.4}$$

$$B \mapsto B + dg$$
$$c \mapsto c + f \cup B + A \cup g + f \cup dg.$$

Consider now a 2+1D Dijkgraaf-Witten term for c, which are all proportional to

$$S_0(c) = \int_Y \frac{1}{9} c \cup dc.$$

We wish to modify this action by "counterterms" $\int_Y \Lambda(A,B,c)$, $\Lambda(A,B,c) \in C^3(X, \mathbb{R}/\mathbb{Z})$ which vanish when $A = B = 0$, ie. $\Lambda(0,0,c) = 0$, such that

$$S_0(c) + \int_Y \Lambda(A,B,c)$$

is gauge invariant. From the theory of descendants, this is equivalent to the condition

$$d(\frac{1}{9} c \cup dc + \Lambda(A,B,c)) = 0.$$

In fact we will find that this is an impossible request, although there is an $\omega(A,B)$ for which

$$d(\frac{1}{9} c \cup dc + \Lambda(A,B,c)) = \omega(A,B).$$

This $\omega \in Z^4(B(\mathbb{Z}_3 \times \mathbb{Z}_3), U(1))$ will define the 3+1D anomaly theory for us.

The situation is familiar to mathematicians. Indeed, the action of $B\mathbb{Z}_3 \times B\mathbb{Z}_3$ on $B\mathbb{Z}_3$ defines a fibration

$$\begin{tikzcd} B\mathbb{Z}_3 \arrow[r] & BH(3,3) \arrow[d] \\ & B\mathbb{Z}_3 \times B\mathbb{Z}_3. \end{tikzcd}$$

We begin with a cohomology class on the fiber, namely the Dijkgraaf-Witten Lagrangian, and we wish to extend it to the total space $BH(3,3)$. There are a series of obstructions encoded in the Serre spectral sequence. $\Lambda(A,B,c)$ is built up "page by page" (equivalently order by order in A, B) by trivializing the differentials. In our case, all but the last differential is trivializable, and it is $\omega(A,B)$.

We take

$$\Lambda_0(A,B,c) = \frac{1}{9}(A \cup B \cup c - c \cup A \cup B) - \frac{1}{9} c \cup_1 d(A \cup B)$$

and compute

$$d(\frac{1}{9}c \cup dc + \Lambda_0(A, B, c)) = \frac{2}{3}d(A \cup B) \cup c + \cdots ,$$

where $\cdots$ are terms independent of c. This $\frac{2}{3}d(A \cup B)$ is the first differential we encounter in the Serre spectral sequence of the above fibration. In order to proceed, we need to find a $\lambda_1 \in C^2(B(\mathbb{Z}_3)^2, \mathbb{Z}_3)$ such that

$$d\lambda_1(A, B) = \frac{2}{3}d(A \cup B).$$

In fact, $\frac{2}{3}d(A \cup B)$ represents a trivial class in $H^2(B(\mathbb{Z}_3)^2, \mathbb{Z}_3)$, so we may indeed find such a λ_1. This lets us construct the next counterterm

$$\Lambda_1(A, B, c) = \frac{1}{3}\lambda_1(A, B) \cup c.$$

Now we compute

$$d(\frac{1}{9}c \cup dc + \Lambda_0(A, B, c) + \Lambda_1(A, B, c))$$

$$= -\frac{1}{9}A \cup B \cup A \cup B + \frac{1}{9}A \cup B \cup d(A \cup B) + \frac{1}{3}\lambda_1(A, B) \cup A \cup B,$$

which represents a nontrivial class $[\omega(A, B)] \in H^4(B(\mathbb{Z}_3)^2, U(1))$ called the Pontryagin square of $A \cup B$ [88].

This class defines a 3+1D Dijkgraaf-Witten theory with gauge group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Our 2+1D theory (with counterterms) forms an extension of this TQFT to 4 manifolds whose boundary admits a c satisfying (6.3) (cf. open-closed TQFT [89, 90]) by

$$-\int_X \omega(A, B) + \int_{\partial X} \frac{1}{9}c \cup dc + \Lambda_0(A, B, c) + \Lambda_1(A, B, c).$$

Note that Stokes' theorem does not imply this is zero unless c can be extended into X. However, our calculation does imply that this combination is invariant under all gauge transformations.

This construction of the theory depends nontrivially on the filling (X, A, B). We can demonstrate this by exhibiting a single nontrivial $\int_X \omega(A, B)$ on a closed 4-manifold X. We can take a torus $X = T^4$ with one vertex and four 1-cycles $\Gamma_1, \ldots, \Gamma_4$, $\int_{1,2} A = 1$ and $\int_{3,4} B = 1$, zero otherwise. These are closed as integer cochains, not just modulo 3, so $d(A \cup B) = 0$ and the second and third terms of $\omega(A, B)$ vanish. The first term yields $\int_{T^4} \omega(A, B) = -1/9$. This implies a nontrivial dependence on the filling and therefore that the symmetry is indeed anomalous.

A result of combined bulk-boundary gauge invariance is that when we compute the 1st descendant of $\omega(A, B)$ under the gauge transformations (6.4), we have

$$\int_Y \omega_1(A, B, f, g) = \delta_{f,g} \int_Y \frac{1}{9} c \cup dc + \Lambda_0(A, B, c) + \Lambda_1(A, B, c).$$

Therefore, the boundary partition function (cf. (6.1))

$$|Z\rangle = \sum_{A,B \in Z^1(Y,\mathbb{Z}_3)} \sum_{dc=AB} \exp 2\pi i \left(\int_Y \frac{1}{9} c \cup dc + \Lambda_0(A, B, c) + \Lambda_1(A, B, c) \right) |A, B\rangle$$

is a state in the Hilbert space of the $\omega(A, B)$ Dijkgraaf-Witten theory.

6.3.3 2+1D Gravity Anomaly

A version of this theory was discussed in [91].

Let Y be a triangulated PL 3-manifold with branching structure, possibly unorientable. Recall we defined $w_2(TY \oplus NY)$ in section 6.2.2. Briefly, we compute $w_2(T(Y \times [0, 1]))$ where the prism $Y \times [0, 1]$ is given a branching structure extending that of Y so that all the internal edges point from $Y \times 0$ to $Y \times 1$. Then we restrict $w_2(T(Y \times [0, 1]))$ to $Y \times 1$. We use this to write an action functional for a pair of dynamical fields $a \in C^1(Y^\vee, \mathbb{Z}_2), b \in C^1(Y, \mathbb{Z}_2)$:

$$S(a, b) = \frac{1}{2} \int_Y (a, db - w_2(TY \oplus NY)) + b \cup w_2(TY \oplus NY).$$

Integration over a enforces the constraint

$$db = w_2(TY \oplus NY) \mod 2.$$

This constraint depends on the branching structure of Y, as $w_2(TY \oplus NY)$ does. When we change the branching structure, we may have

$$w_2(TY \oplus NY) \mapsto w_2(TY \oplus NY) + d\alpha$$

for some $\alpha \in C^1(Y, \mathbb{Z}_2)$. In order for the above constraint to be invariant, we need to impose a transformation rule

$$b \mapsto b + \alpha.$$

Like we discussed in section 5.4.1 and 4.2.4, this means b is a spin structure and its Wilson lines are spinors.

Under this transformation, the remaining term

$$\frac{1}{2} b \cup w_2(TY \oplus NY)$$

transforms exactly like the boundary variation (6.2). Thus, we can define an action on 4-manifolds X with boundary $\partial X = Y$

$$\int_X \frac{1}{2} w_2(TX) \cup w_2(TX) + \frac{1}{2} \int_Y (a, db - w_2(TY \oplus NY)) + b \cup w_2(TY \oplus NY).$$

This bulk-boundary action is independent of the branching structure but it depends nontrivially on the filling X. For instance, $\int_{\mathbb{CP}^2} w_2(T\mathbb{CP}^2)^2 = 1$. This means it is impossible to modify $S(a, b)$ so that it is a, b-gauge invariant and independent of a choice of branching structure without adding more background structure. In this sense there is an anomaly.

This is very similar to the framing anomaly of 2+1D Chern-Simons theory [71]. Indeed, this anomaly has been related to the chiral central charge mod 4 [92] and the theory is equivalent to a Walker-Wang theory, which is known to carry chiral Chern-Simons theories as a boundary condition [85]. However, it is slightly different because this anomaly requires us to consider unorientable 3-manifolds, meaning physically that it is only relevant when there is time-reversal, reflection, or some other orientation-reversing spacetime symmetries [93] (Dominic Else and I discussed symmetries which act on spacetime in [94]). Indeed, if we only consider oriented manifolds, then since

$$\Omega_3^{SO} = 0,$$

we can always choose the filling X to be oriented. For oriented closed 4-manifolds it's still possible to have a nontrivial partition function of the anomaly theory (eg. $\mathbb{CP}^2$), but this 3+1D TQFT is continuously connected to the trivial one through a family whose partition functions are

$$Z(X) = e^{i\theta \int_X w_2(TX)^2}, \qquad \theta \in \mathbb{R}.$$

We can only use the anomaly in-flow argument when the bulk theory is disconnected from the trivial theory in the space of TQFTs [93, 78]. Intuitively, we could otherwise define our theory on a slab $Y \times [0, 1]$ with our theory on $Y \times 0$, the trivial theory on $Y \times 1$, and θ varying continuously from π to 0 along the interior direction, and this would be invariant under all transformation.

As we discussed in section 5.4.1, a consequence of the action $S(a, b)$ is that the b-charge, b-flux, and b-charge/flux bound state are all spinors. For this reason the theory is called the all fermion $\mathbb{Z}_2$ gauge theory. These braiding relations define a modular tensor category with an anomalous anti-unitary symmetry [92]. In 3+1D, one can study electrodynamics with a fermionic electron, fermionic monopole, and fermionic dyon, and I showed in [95] (see also [96]) that this theory has an anomaly whose anomaly theory has partition function

$$Z(X^5) = (-1)^{\int_X w_2(TX) \cup w_3(TX)},$$

by reduction to a theory of the form

$$\frac{1}{2} \int_Y (a, db - w_2(TY \oplus NY)) + b \cup w_3(TY \oplus NY),$$

where now $a \in C^2(X^\vee, \mathbb{Z}_2)$ and $b \in C^1(X, \mathbb{Z}_2)$ is descended from the electrodynamic gauge field upon condensing cooper pairs [97]. Interestingly, this anomaly theory is not connected to the trivial TQFT for oriented 5-manifolds. Indeed,

$$\Omega_5^{SO} = \mathbb{Z}_2,$$

characterized by the de Rham invariant of the linking form [98], which computes $w_2 w_3$.

Chapter 7

Bosonization/Fermionization

Introduction

In this chapter we will descibe a number of correspondences between D-dimensional QFTs of oriented manifolds (ie. bosonic QFTs) with an action of a special stable ∞-group E_D, related to the spin cobordism spectrum, with a "canonical" anomaly, and D-dimensional QFTs of spin manifolds (ie. fermionic QFTs). These correspondences are called bosonization going from the latter to the former and fermionization going from the former to the latter.

Bosonization/fermionization is an old subject, going at least as far back as Lars Onsager's computation of the critical Ising model correlation functions [99], and forming the foundation of much of the theory of integrable systems. For example, see [51]. Recently, pushed forward by new understanding of topological aspects of field theories and spin structures [100, 23, 46, 101], we are beginning to understand how to push these techniques into higher dimensions. For instance, see [102, 64].

The most simple form of bosonization can be performed on any fermionic QFT with expectation value

$$\mathcal{O} \mapsto \langle \mathcal{O} \rangle^F_{X,\eta}$$

on a closed D-manifold with spin structure η by simply summing over the spin structures, which form a finite set:

$$\langle \mathcal{O} \rangle^B_X := \sum_\eta \langle \mathcal{O} \rangle^F_{X,\eta}.$$

Because spin structures are locally indistinguishable, the resulting expectation value satisfies cluster decomposition and defines a bosonic QFT. This can be proven rigorously if one has a path integral description of the expectation values and is expected to hold in general. The CPT transformations are more subtle and we will not discuss them here. However, see [103].

We would like to improve this bosonization map to be invertible. We use a trick similar to constructing the Fourier transform, introducing probe charges which see the spin structure. The simplest sort of probe charge is a fermionic particle, which is a sort of Wilson loop for the spin structure. We write it (the bare Wilson loop)

$$(-1)^{\int_\gamma \eta},$$

where $\eta \in C^1(X^\vee, \mathbb{Z}_2)$ is the spin structure and $\gamma \in Z_1(X^\vee, \mathbb{Z}_2)$ is the worldline of the particle. We can use Poincaré duality to realize this probe particle instead as a $\mathbb{Z}_2$ $D-1$-form gauge field $B \in Z^{D-1}(X, \mathbb{Z}_2)$. Thus, in the presence of the probe, we have

$$\langle \mathcal{O} \rangle^b_{X,B} := \sum_\eta \langle \mathcal{O} \rangle^f_{X,\eta} (-1)^{\int_X (B,\eta)}. \tag{7.1}$$

This transformation has an inverse transformation where we use a spin structure η to probe the B field:

$$\langle \mathcal{O} \rangle^f_{X,\eta} = \sum_B \langle \mathcal{O} \rangle^b_{X,B} (-1)^{\int_X (B,\eta)}. \tag{7.2}$$

This works because

$$\sum_{B \in Z^{D-1}(X,\mathbb{Z}_2)} (-1)^{\int_X (B,\eta-\eta')} = \delta(\eta - \eta').$$

We would like to think about B as a gauge field associated to a $D-2$-form $\mathbb{Z}_2$ symmetry of the bosonic theory, and not have to work with simplicial cocycles, especially if the QFTs in question are defined in the continuum. There is a problem, since under a gauge transformation

$$B \mapsto B + d\lambda,$$

we have

$$(-1)^{\int_X (B+d\lambda,\eta)} = (-1)^{\int_X (B,\eta) + (\lambda, w_2)},$$

the second term of which pulls out of the sum over η and so

$$\langle \mathcal{O} \rangle^b_{X,B+d\lambda} = \langle \mathcal{O} \rangle^b_{X,B} (-1)^{\int_X (\lambda, w_2)}. \tag{7.3}$$

This means that the $B^{D-1}\mathbb{Z}_2$ symmetry of the theory $\langle - \rangle^b_{X,B}$ is anomalous. Only after the introduction of the spin factor in (7.2) can the sum be made over $[B] \in H^{D-1}(X, \mathbb{Z}_2)$.

7.1 Spin Dijkgraaf-Witten Theory

As an example of the bosonization procedure we discuss the construction of a topological G gauge theory in 2+1D which depends on a spin structure.

We begin with a bosonic theory whose fields are $a \in Z^1(X, G)$, $b \in C^1(X, \mathbb{Z})$, $c \in C^1(X^\vee, \mathbb{Z})$ and action is

$$S_0(a, b, c) = \int_X \frac{1}{2} b \cup db + \frac{1}{2}(c, db - \nu_2(a)) + \nu_3(a), \tag{7.4}$$

where $\nu_2(a) \in C^2(X, \mathbb{Z}_2)$ and $\nu_3(a) \in C^3(X, \mathbb{R}/\mathbb{Z})$ are functions of a.

One can think of $\nu_3(a)$ as a Dijkgraaf-Witten type topological term for a, although we will soon see it satisfies an interesting conservation law which is unlike the usual $d\nu_3 = 0$. On the other hand we will find $d\nu_2 = 0$. Indeed upon integrating out c the second term imposes a constraint that says that b and a together combine into a single gauge field for a central extension

$$\Omega^1_{spin} = \mathbb{Z}_2 \to \hat{G} \to G,$$

where for later convenience we identify the gauge group of b with the spin cobordism group of 1-manifolds. Equivalently ν_2 describes how the G gauge field sources b flux. Finally, roughly speaking the first term indicates that these (bare) b-fluxes carry unit b-charge, and are thus fermionic particles. The presence of these fermionic particles is a clue that we may be able to fermionize this theory.

7.1.1 Gauge Invariance and the Gu-Wen Equation

The action is invariant under gauge transformations of b after integrating by parts. It is invariant under gauge transformations $c \mapsto c + df$ up to boundary terms iff

$$d\nu_2(a) = 0 \mod 2.$$

Meanwhile, under a gauge transformation of

$$a \mapsto a^f,$$

in order for the constraint imposed by integrating out c:

$$db = \nu_2(a) \mod 2$$

to be gauge invariant, we must have a transformation rule for b:

$$b \mapsto b + \nu_{2,1}(a, f) \mod 2,$$

where $\nu_{2,1}(a, f)$ is a first descendant of ν_2. This creates a variation in the first term:

$$\int_X \frac{1}{2}\nu_{2,1}(a, f) \cup (\nu_2(a^f) - \nu_2(a)).$$

We wish this variation to cancel the variation of ν_3, ie. we need to choose ν_3 so that

$$\int_X \frac{1}{2}\nu_{2,1}(a, f) \cup (\nu_2(a^f) - \nu_2(a)) + \nu_3(a^f) - \nu_3(a) = 0 \mod 1$$

up to boundary terms. To do this, we will assume a relation called the Gu-Wen equation:

$$\frac{1}{2}\nu_2 \cup \nu_2 = d\nu_3 \mod 1.$$

This implies that both

$$\nu_{2,1}(a, f) \cup (\nu_2(a^f) - \nu_2(a)) \quad \text{and} \quad \nu_3(a^f) - \nu_3(a)$$

are first descendants of ν_2^2. Since the first descendants are defined up to exact terms, it follows given the Gu-Wen equation that the action S_0 is invariant under gauge transformations of a, b, and c.

7.1.2 1-form Anomaly

We couple this to a background gauge field $B \in Z^2(X, \mathbb{Z}_2)$, for which a small gauge transformation $\lambda \in C^1(X, \mathbb{Z})$ acts by

$$B \mapsto B + d\lambda$$

$$b \mapsto b + \lambda.$$

And a large gauge transformation $\beta \in C^2(X, \mathbb{Z})$ by

$$B \mapsto B + 2\beta.$$

We add to the action some extra terms

$$S_1 = \int_X \frac{1}{2}(B, c) + \frac{1}{2}B \cup_1 db.$$

There is no β variation, and the total λ variation is

$$\delta_\lambda(S_0+S_1) = \int_X \frac{1}{2}\lambda db + \frac{1}{2}bd\lambda + \frac{1}{2}d\lambda \cup_1 db + \frac{1}{2}B \cup_1 d\lambda + \frac{1}{2}\lambda d\lambda$$
$$= \int_X \frac{1}{2}B \cup_1 d\lambda + \frac{1}{2}\lambda d\lambda + \frac{1}{2}d(\cdots).$$

If $X = \partial Z$, then this is a boundary variation of

$$\int_Z \frac{1}{2}B \cup B,$$

which forms a nontrivial 4D TQFT for oriented manifolds equipped with an equivalence class of $B\mathbb{Z}_2$ gauge field $[B] \in H^2(Z, \mathbb{Z}_2)$. Furthermore, by the Wu formula

$$[w_2(TZ)] \cup [B] = [B] \cup [B] \in H^4(Z, \mathbb{Z}_2),$$

this theory has the same partition function on every closed, orientable 4-manifold as the theory with action

$$\int_Z \frac{1}{2}(w_2(TZ), B)$$

we discussed in the introduction. It is thus reasonable to assume there is a correspondence between their boundary conditions. In particular, we hope to cancel the anomaly (7.3) given a spin structure $\eta \in C^1(X, \mathbb{Z}_2)$ on X.

7.1.3 A Special Quadratic Form and Fermionization

To do so, we will construct a function

$$Q_\eta : Z^2(X, \mathbb{Z}_2) \to \mathbb{Z}_2$$

which satisfies

$$Q_\eta(B + d\lambda) = Q_\eta(B) + \int_X \frac{1}{2}\lambda d\lambda + \frac{1}{2}B \cup_1 d\lambda.$$

Then the combined action

$$S_0 + S_1 + Q_\eta$$

will be completely gauge invariant, and define the spin Dijkgraaf-Witten theory associated to the Gu-Wen data (ν_2, ν_3).

We will construct Q_η first in terms of a filling $\partial Z = X$ which carries an extension of B (as a cocycle). This is always possible since

$$\Omega_3^{SO}(B^2\mathbb{Z}_2) = 0.$$

Recall from the relative Stiefel-Whitney cycle theorem that

$$\partial W_2(TZ) = W_2(TX).$$

If we have a spin structure $\eta \in C^1(X^\vee, \mathbb{Z}_2)$ on the PL 3-manifold X, we can Poincaré dualize it to $E \in C_2(X, \mathbb{Z}_2)$ with $\partial E = W_2(TX)$. In particular,

$$\partial(W_2(TZ) + E) = 0$$

so we define the relative Stiefel-Whitney cycle as

$$W_2(TZ, \eta) := W_2(TZ) + E \in Z_2(Z, \mathbb{Z}_2).$$

This cycle represents the obstruction to extending E to a spin structure on Z. We write

$$Q_\eta(B, Z) = \int_Z B \cup B + (w_2(TZ, \eta), B).$$

If we have a second filling Z', we can glue them together and find

$$Q_\eta(B,Z) - Q_\eta(B,Z') = \int_{Z\cup Z'} B \cup B + \int_Z (w_2(TZ,\eta),B) + \int_{Z'} (w_2(TZ',\eta),B)$$

$$= \int_{Z\cup Z'} B \cup B + (w_2(T(Z\cup Z')), B) = 0 \mod 2,$$

using the Wu relation. Thus, $Q_\eta(B) = Q_\eta(B, Z)$ is *independent* of the filling Z. This argument shows it is a cobordism invariant of PL 3-manifolds equipped with a $B \in Z^2(X, \mathbb{Z}_2)$.

However, it is not invariant under gauge transformations $B \mapsto B + d\lambda$. Instead,

$$Q_\eta(B + d\lambda) = \int_Z d\lambda \cup B + B \cup d\lambda + d\lambda \cup d\lambda + (w_2(TZ, \eta), d\lambda).$$

Using

$$\partial(W_2(TZ, \eta) \cap \lambda) = W_2(TZ, \eta) \cap d\lambda,$$

this becomes the required variation:

$$Q_\eta(B+d\lambda) = \int_X \lambda d\lambda + B \cup_1 \lambda.$$

Therefore

$$S_f = S_0(a,b,c) + S_1(b,c,B) + Q_\eta(B)$$

is invariant under all gauge transformations.

7.1.4 Generalization To Higher Dimensions

There is a simple generalization to higher dimensions, where we take $a \in Z^1(X,G), b \in C^{D-2}(X,\mathbb{Z}_2), c \in C^1(X,\mathbb{Z}_2), \nu_{D-1} \in Z^{D-1}(BG,\Omega^1), \nu_D \in C^D(BG,\mathbb{R}/\mathbb{Z})$ and consider

$$S(a,b,c) = \int_X \frac{1}{2} b \cup_{D-3} db + \frac{1}{2}(c, db - \nu_{D-1}(a)) + \nu_D(a),$$

satisfying the higher Gu-Wen equation

$$d\nu_D = Sq^2 \nu_{D-1} = \nu_{D-1} \cup_{D-3} \nu_{D-1}$$

and where b transforms under a gauge transformations by the 1st descendant of ν_{D-1}. One may check that under the $B^{D-2}\mathbb{Z}_2$ symmetry transformation

$$b \mapsto b + \lambda,$$

this theory has an anomaly

$$Sq^2 B.$$

Then using the Wu formula

$$\int_Z Sq^2 B + (w_2(TZ), B) = 0 \mod 2$$

which holds on closed, orientable $D+1$-manifolds, we may define a

$$Q_\eta(X,B) = \int_Z B^2 + (w_2(TZ,\eta), B)$$

as before, relative to a filling, and this is independent of the filling. The variation of Q_η cancels the anomalous variation of the action, given suitable counterterms as above.

A caveat is that $\Omega_D^{SO}(B^{D-1}\mathbb{Z}_2)$ may be nonzero, and in this case our definition of Q_η becomes difficult to apply since we cannot find a filling for (X, B). We can improve this definition by relaxing the orientability condition on the filling. In this case, the Wu formula is

$$Sq^2[B] = ([w_2] + Sq^1[w_1]) \cup [B] \in H^{D+1}(Z, \mathbb{Z}_2)$$

for closed $D+1$-manifolds Z, not necessarily orientable. We can give a cocycle representative of $Sq^1[w_1]$ using the realization of Sq^1 as the Bockstein operation:

$$Sq^1 w_1 = dw_1/2.$$

Recall our Poincaré dual cycle W_1 is defined first on the barycentric subdivision as the sum of all $D-1$-simplices mod 2. We can endow these $D-1$-simplices with local orientations induced from the canonical branching structure on the barycentric subdivision to define an integer-valued chain which by abuse of notation we also call $W_1(TX^b)$ and which reduces to the usual chain mod 2. Then using the Morse flow we obtain $W_1(TX) = f_\infty W_1(TX^b) \in C_{D-1}(X, \mathbb{Z})$ and

$$\partial W_1/2 \in Z_{D-2}(X, \mathbb{Z})$$

is a Poincaré dual representative of $Sq^1[w_1]$. Then we may define a pin$^-$ structure as a chain $E \in C_{D-1}(X, \mathbb{Z}_2)$ with

$$\partial E = W_2(TX) + \partial W_1(TX)/2.$$

These chains behave well with respect to boundaries, in particular, for a (nonorientable) filling Z and a pin$^-$ structure on its boundary, X, we have

$$\partial(W_2(TZ) + \partial W_1(TZ)/2 + E) = 0.$$

This allows us to define a quadratic form

$$Q_E(B, Z) = \int_Z B \cup B + (W_2(TZ) + \partial W_1(TZ)/2 + E) \cap B$$

which can be used to cancel the anomalous variation (7.3) and which one may check only depends on the boundary X and not the filling Z.

This allows us to use more general nonorientable fillings, but still the relevant bordism group $\Omega_D^O(B^{D-1}\mathbb{Z}_2)$ may be nonzero. The advantage is that this bordism group is 2-torsion. Indeed, the nonorientable bordism spectrum is a product of $B^k\mathbb{Z}_2$'s with no Postnikov invariants [24]. Instead, we can use a trick. If (X, B) does not admit a filling, then two copies $(X, B) \sqcup (X, B)$ admits a filling Z and we define

$$Q_E(B, X) = \frac{1}{2} Q_E(B, Z).$$

This finishes the construction of the general spin Dijkgraaf-Witten theory on arbitrary spin manifolds.

7.1.5 Comments on the Partition Function and More General G-SPT Phases

We can construct the partition function of our theory, where a and η are kept as backgrounds but all other fields are summed over:

$$Z(X, a, \eta) = N(X) \sum_{\substack{b \in C^1(X,\mathbb{Z}_2) \\ c \in C^1(X^\vee,\mathbb{Z}_2)}} \sum_{\substack{B \in C^2(X,\mathbb{Z}_2) \\ B^\vee \in C^0(X^\vee,\mathbb{Z}_2)}} e^{2\pi i S_f(a,b,c,B,\eta) + \pi i \int_X (B^\vee, dB)},$$

where we have introduced a Lagrange multiplier for B which sets the constraint $dB = 0$. With proper normalization

$$N(X) = 2^{-2|X_1| - 2|X_2|}$$

this is a *cobordism invariant* of $(X, [a], [\eta])$,

$$Z \in \mathrm{Hom}(\Omega^3_{spin}(BG), U(1)).$$

This means that it defines a unitary invertible Atiyah-Segal TQFT for spin 3-manifolds over BG, ie. the TQFT associated to a fermionic G-SPT phase [104].

However, it is not the most general cobordism invariant nor the most general such TQFT. The most general sort also admits a description by bosonization, as showed by Gaiotto-Kapustin [23], but the bosonization is a *nonabelian* TQFT carrying Majorana quasiparticles. The description of these more general theories requires describing how the spin structure responds to extended fermionic probes. Beyond simple fermionic particles, the first extended fermionic probe is a string carrying a Kitaev wire [46], a fermionic SRE phase described by

$$\Omega^2_{spin} = \mathbb{Z}_2.$$

The partition function of this SRE phase on a surface with spin structure is the Arf invariant [104]. The Arf invariant is not an integral of a local quantity over the surface, since it is not multiplicative over covering spaces [45]. For this reason one needs to consider these extended objects to carry nontrivial degrees of freedom of their own, which does not fit into any tranditional gauge theory or cocycle framework. To describe these in such a way appears possible, but one must consider "quantum" 1-cycles labeled by elements of a monoidal category. We leave this for future work.

For now we consider that just as ν_2 described how fermionic particles are sourced by the G-gauge field a, these extended objects may also be sourced by the G-gauge field, and this should be described by a

$$\nu_1(a) \in Z^1(BG, \Omega^2_{spin}).$$

The total data describing the most general fermionic G-SPT in 2+1D [23] is

$$\nu_1 \in Z^1(BG, \Omega^2), \nu_2 \in Z^2(BG, \Omega^1), \nu_3 \in C^3(BG, \mathbb{R}/\mathbb{Z})$$

satisfying the Gu-Wen equation

$$d\nu_3 = \nu_2 \cup \nu_2.$$

Presumably in higher dimensions the G gauge field can source even higher dimensional extended fermionic objects, for example the topological superconductors in 2+1D, which are classified by an integer invariant [104], meaning we will have to include a cocycle

$$\nu_{D-3} \in H^{D-3}(BG, \mathbb{Z}).$$

In the case $D = 3$ we've been discussing, this is

$$\nu_0 \in H^0(BG, \mathbb{Z}) = \mathbb{Z},$$

which amounts to adding a number of decoupled topological superconductor to the theory and should change the fermionization procedure accordingly.

In work of Anton Kapustin, and later in collaboration with myself and Alex Turzillo and Zitao Wang, we proposed that the group of fermionic SRE phases in each dimension assemble into a version of the spin cobordism spectrum called the (1-shifted) Anderson dual spin bordism spectrum we denote $\Omega_n^{spin,\vee} = \Omega^n_{spin}(-, U(1))$, which in torsion parts computes spin cobordism invariants valued in $U(1)$ and in nontorision parts computes "gravitational Chern-Simons terms" so that there is a sort of universal coefficient sequence:

$$\Omega_n^{spin,\vee} = \mathrm{Hom}(\Omega_n^{spin,tors}, U(1)) \oplus \Omega_{n+1}^{spin,free}$$

or equivalently

$$\mathrm{Ext}(\Omega_n^{spin}, \mathbb{Z}) \to \Omega_n^{spin,\vee} \to \mathrm{Hom}(\Omega_{n+1}^{spin}, \mathbb{Z}),$$

which splits but not canonically. Dan Freed and Mike Hopkins later proved [78] (using the Baez-Dolan-Lurie cobordism hypothesis and an assumption about the Madsen-Tillman spectrum) that the Anderson dual of the sphere spectrum forms a good target category for unitary invertible TQFTs and that these TQFTs are determined by their partition functions, and are thus classified by $\Omega_n^{spin,\vee}$, verifying our physically-motivated conjecture.

This spectrum $\Omega_n^{spin,\vee}$ defines a cohomology theory and the physical intuition about gauge fields sourcing fermionic SRE phases is precisely captured by the Atiyah-Hirzebruch spectral sequence which computes

$$\Omega^n_{spin}(BG, U(1)).$$

Indeed the ν_j are the cocycles which appear on the E^2 page

$$[\nu_j] \in H^j(BG, \Omega^k_{spin}(\star, U(1))), \qquad j + k = n.$$

The differentials in this spectral sequence are precisely the Gu-Wen equations are their higher-dimensional analogues. In later work of Anton Kapustin's and my own, we argued that these differentials have topological significance and ensure that the singular cycles of the gauge field, that is, those Poincaré dual to the ν_j, have precisely the topological conditions one needs for these singular cycles to be homologous to immersed submanifolds and for the spin structure on X to restrict to tangent structure on these submanifolds. A remaining mystery is to understand why sometimes this tangent structure must be more general than a spin structure. For example, on $\mathbb{RP}^3$, the unique nontrivial homology class is represented by $\mathbb{RP}^2$, which inherits a pin$^-$ structure from a spin structure on $\mathbb{RP}^3$ but there is no way to represent this cycle with an orientable manifold since Sq^1 of the Poincaré dual class is nonzero (cf. Thom's theorem in section 4.1). Indeed, Sq^1 does not appear as a differential in the spectral sequence, so nonorientable cycles are bound to appear. This seems intricately related to the extension problem in the spectral sequence and I hope that eventually a concrete understanding of it will be possible.

As far as bosonization may be generalized to include extended probes, we will need to include gauge fields

$$b_j \in C^j(X, \Omega^{D-j}_{spin}(\star, U(1)))$$

which assemble into a map $b : X \to B\mathcal{F}$ for some ∞-group $\mathcal{F}$ associated with the spin cobordism spectrum such that the ν_j assemble into a map

$$\nu : G \to \mathcal{F}$$

of ∞-groups. One can think of $\mathcal{F}$ as the group of extended invertible fermionic charges.

For this to work we will need to treat $\nu_D \in C^D(X, U(1))$ slightly differently. The coefficient of ν_D comes from

$$\Omega^0_{spin}(\star, U(1)) = U(1)$$

and if we included this piece in $\mathcal{F}$ we would have a $b_D \in C^D(X, U(1))$, which is problematic because the form degree is the same as the spacetime dimension, and physically it does not really correspond to any fermionic probe. Instead we consider this "top" $U(1)$ where the anomaly of the $\mathcal{F}$ symmetry of the bosonic system lives. Indeed, in the Atiyah-Hirzebruch spectral sequence, there is a differential which lands there and that differential is precisely the anomaly of the $\mathcal{F}$ symmetry of the bosonized theory (which is the same among all bosonizations).

When there is also a global G symmetry of the fermionic theory, the G symmetry of the bosonization is embedded diagonally in $G \times \mathcal{F}$ using $\nu : G \to \mathcal{F}$ such that the anomaly restricted to this G is cancelled by a local counterterm ν_D. Meanwhile the anomaly of the $\mathcal{F}$ symmetry is cancelled by a spin factor generalizing Q_η. To form the fermionization we couple to a dynamical $\mathcal{F}$ gauge field. We hope in future work to construct an explicit formula for this generalized spin factor.

7.2 Fermionic Anomalies

7.2.1 Hilbert Space of Spin Dijkgraaf-Witten Theory

To understand the Hilbert space spin Dijkgraaf-Witten theory associates to a spin $D-1$-manifold, we study its bosonization (7.4). We will focus on the case $D = 3$. The other cases are a straightforward generalization.

Gauge Transformations and Modified Gauss Law

To proceed, we must study the gauge transformations of the action on a manifold X with boundary $\partial X = Y$. First we have

$$c \mapsto c + dh$$

$$\delta S = \int_Y (h, db - \nu_2(a)),$$

which indicates that the physical Hilbert space of Y lies in the constrained Hilbert space spanned by classical states

$$|a, b, c\rangle$$

satisfying

$$db = \nu_2(a). \tag{7.5}$$

Indeed, this is the constraint equivalently obtained by integrating out c.

Next we study

$$b \mapsto b + df$$

$$\partial S = \int_Y \frac{1}{2} f \cup db.$$

This indicates a modification of the usual Gauss law which forbids the termination of a Wilson line:

$$W(\Gamma) = (-1)^{\int_\Gamma b}, \qquad \Gamma \in C_1(Y, \mathbb{Z}_2).$$

That is, instead of gauge invariance of this operator requiring $\partial\Gamma = 0$ (cf. section 5.2), it requires that

$$\int_{\partial\Gamma} f = \int_Y f \cup db \qquad \forall f \in C^0(X, \mathbb{Z}_2),$$

ie.

$$f_{-\infty}W(db) = \partial\Gamma. \tag{7.6}$$

This means that an isolated triangle (012) where $db(012) = 1 \bmod 2$ must have a Wilson line ending at the vertex (0). Warning: it's not guaranteed that all vertices can be end points of Wilson lines because the image of $f_{-\infty}$ might not be surjective. This sensitive framing dependence is a feature of framing anomaly. The proper way to say it is that bare b-fluxes are decorated with b-charges, much like in our analysis of the $\mathbb{Z}_n$ Dijkgraaf-Witten theory in section 5.2. These are both bosons with a mutual braiding phase -1, so the composite object is a statistical fermion, which we distinguish from a spin fermion because there is no spin structure dependence yet.

Finally we consider

$$a \mapsto a^g$$

$$b \mapsto b + \nu_{2,1}(a,g).$$

The variation of the action comes from two pieces,

$$\delta(\frac{1}{2}\int_X bdb) = \frac{1}{2}\int_X \nu_{2,1}(a,g)db + bd\nu_{2,1}(a,g) + \nu_{2,1}(a,g)d\nu_{2,1}(a,g)$$

$$= \int_X \frac{1}{2}\nu_2(a) \cup \nu_{2,1}(a,g) + \frac{1}{2}\nu_{2,1}(a,g) \cup \nu_2(a^g) + \int_Y \frac{1}{2} b \cup \nu_{2,1}(a,g),$$

$$\delta(\int_X \nu_3(a)) = \int_X \nu_3(a^g) - \nu_3(a).$$

Setting aside the boundary term involving b, these are both 1st descendants of $\nu_2(a) \cup \nu_2(a)$, and accordingly there is a 2-cochain satisfying

$$d\chi(a,g) = \frac{1}{2}\nu_2(a) \cup \nu_{2,1}(a,g) + \frac{1}{2}\nu_{2,1}(a,g) \cup \nu_2(a^g) + \nu_3(a^g) - \nu_3(a).$$

Thus, the total variation of the action is

$$\int_Y \chi(a,g) + \frac{1}{2} b \cup \nu_{2,1}(a,g).$$

Therefore, the Hilbert space associated to Y shall be considered to be spanned by "classical states" $|a,b,c\rangle$ with $a \in C^1(Y,G)$, $b \in C^1(Y,\mathbb{Z}_2)$, $c \in C^1(Y^\vee,\mathbb{Z}_2)$ which are invariant under the gauge transformations

$$|a,b,c\rangle \mapsto$$

$$\exp\left(2\pi i \int_Y -\chi(a,f) + \frac{1}{2} b \cup \nu_{2,1}(a,f) + \frac{1}{2} f \cup db + \frac{1}{2}(h, db - \nu_2(a))\right)$$
$$\times |a^g, b + df, c + dh\rangle.$$

1-form Symmetry Action

There is also the action of the global 1-form symmetry to consider

$$b \mapsto b + \lambda, \qquad \lambda \in Z^1(X, \mathbb{Z}_2).$$

This produces a boundary variation

$$\frac{1}{2} \int_Y \lambda \cup b.$$

Eigenvectors of this symmetry are labelled by gauge-invariant functions

$$q : Z^1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$$

satisfying

$$q(\alpha + \beta) = q(\alpha) + \int_Y \alpha \cup \beta + q(\beta)$$

by

$$|q, [b]\rangle = \sum_{\lambda \in Z^1(X,\mathbb{Z}_2)} (-1)^{q(\lambda)} |b + \lambda\rangle.$$

Such functions are in 1-to-1 correspondence with spin structures [45], where given a spin structure η we may define

$$\eta \mapsto q_\eta(b)$$

for each $\mathbb{Z}_2$ cohomology class b by choosing a Poincaré dual representative of b by disjoint circles, and counting -1 to the number of periodic spin structures in the restriction of η to these circles.

Examples

The simplest example is $G = \mathbb{Z}_2$. We consider $a \in C^1(X, \mathbb{Z})$ with

$$da = 0 \mod 2$$

and gauge transformations

$$a \mapsto a + df \qquad f \in C^0(X, \mathbb{Z})$$

$$a \mapsto a + 2w \qquad w \in C^1(X, \mathbb{Z}).$$

We choose the Atiyah-Hirzebruch data

$$\nu_2(a) = \frac{1}{2} da,$$

$$\nu_3(a) = \pm \frac{1}{8} a \cup da,$$

and one easily checks the Gu-Wen equations

$$d\nu_2(a) = 0,$$

$$d\nu_3(a) = \pm \frac{1}{8} da \cup da = \frac{1}{2} \nu_2(a) \cup \nu_2(a).$$

We find the descendant

$$d\nu_{2,1}(a, f, w) = \nu_2(a + 2w + df) - \nu_2(a) = dw$$

$$\nu_{2,1}(a, f, w) = w.$$

This gives us the transformation rule

$$b \mapsto b + w$$

under the G gauge transformation. Because of the form of $\nu_3(a)$, this theory is sometimes referred to as level $\pm 1/2$ $\mathbb{Z}_2$ Dijkgraaf-Witten theory. It is closely related to the spin Chern-Simons theory of [70]. Indeed, these theories are obtained from $U(1)_{\pm 1}$ by a Higgs mechanism where the gauge group is reduced to $\mathbb{Z}_2 \subset U(1)$.

The resulting theory is actually equivalent to $\mathbb{Z}_4$ gauge theory with a Dijkgraaf-Witten level ± 2. Indeed, we can rewrite the constraint $db = da/2$ as,

$$d(a + 2b) = 0 \mod 4,$$

so $\hat{a} = a + 2b$ is a $\mathbb{Z}_4$ cocycle. Further, the level 2 Dijkgraaf-Witten term is

$$\pm \frac{1}{2} \hat{a} \cup \frac{d\hat{a}}{4} = \pm \frac{1}{8} a \cup da + \frac{1}{2} b \cup db,$$

up to boundary tersm, which accounts for both topological terms in the action (7.4). Thus, the Hilbert space of this theory can be understood also according to section 6.2.1. The boundary variation of the level 2 Dijkgraaf-Witten term under

$$a \mapsto a + 2w + dg$$

$$b \mapsto b + w + df$$

is

$$\delta S = \int_{\partial X} \frac{1}{2} a \cup w \pm \frac{1}{8}(g + 2f) \cup d(a + 2b).$$

The term

$$\pm \frac{1}{8} g \cup da \in \delta S$$

tells us that the boundary G-flux must carry G charge $\mp 1/4$ to balance this variation. This is analogous to how the bulk 't Hooft line is decorated with a fractional Wilson line (see section 5.2). When the bulk 't Hooft line meets the boundary, its magnetic charge is automatically conserved by the flatness condition $d\hat{a} = 0$, but for the electric charge to be conserved, it must be met by a charge density on the boundary.

7.2.2 Anomaly In-Flow and Bosonization

Let us consider a situation where a spin QFT sits at the boundary of an invertible spin TQFT and both are coupled to a background G gauge field. As before (cf. 6.1), when the bulk TQFT has nontrivial response to this G gauge field, then we say the boundary theory has anomalous G symmetry. The novelty now is that the bulk spin structure restricts to a spin structure on the boundary, and their mutual dependence on this structure leads to more intricate anomalies. For instance, we can now have in-flow of not just gauge charges but also fermionic charges or extended fermionic objects.

In terms of partition functions, the situation is much the same. The bulk theory associates a 1-dimensional Hilbert space to a D-manifold X with spin structure η and gauge background A and the partition function of the boundary theory $|Z(X, \eta, A)\rangle$ defines a vector in this Hilbert space.

We wish to bosonize this setup, coupling the boundary theory to a background $B^d\Omega^1_{spin} = B^d\mathbb{Z}_2$ gauge field B. Assuming that the fermionic nature of the bulk theory is captured by probe particles, its bosonization is described by a class $\nu_{d+1} \in H^{d+1}(BG, \mathbb{Z}_2)$ which describes a central extension of ∞-groups

$$B^d\mathbb{Z}_2 \to \hat{G} \to G.$$

For instance when $D = d+1 = 2$, we assume the bulk theory is of spin Dijkgraaf-Witten type and then $\nu_2 \in H^2(BG, \mathbb{Z}_2)$ describes a group extension

$$\mathbb{Z}_2 \to \hat{G} \to G.$$

As we've seen, the background fields A and B combine into a $\hat{G}$ gauge field $\hat{A}$ when we bosonize.

7.3 1+1D CFT and the Chiral Anomaly

We will use the following bosonization/fermionization relations:

$$2^{-(\chi+1)} \sum_\alpha (-1)^{q_\eta(\alpha)+q_{\eta'}(\alpha)} = \delta(\eta - \eta') \tag{7.7}$$

$$2^{-(\chi+1)} \sum_\eta (-1)^{q_\eta(\alpha)} = \delta(\alpha).$$

These imply the following inverse transformations for arbitrary partition functions

$$Z_f(\eta) = 2^{-(\chi+1)/2} \sum_\alpha (-1)^{q_\eta(\alpha)} Z_b(\alpha)$$

$$Z_b(\alpha) = 2^{-(\chi+1)/2} \sum_\eta (-1)^{q_\eta(\alpha)} Z_f(\eta).$$

Note that the normalization of the partition functions is arbitrary up to exponential factors of the Euler characteristic. We have simply chosen the normalization symmetrically between the bosonic and fermionic transformation. Further, in this form the Ising ferromagnet $Z_b(\alpha) = \delta(\alpha)$ is dual to the trivial fermionic SRE phase while the Ising paramagnet $Z_b(\alpha) = 1$ is dual to the Kitaev fermionic SRE phase, by the relation

$$2^{-(\chi+1)} \sum_\alpha (-1)^{q_\eta(\alpha)} = \mathrm{Arf}(\eta).$$

Furthermore, one can show that these transformations are equivariant with respect to the mapping class group action of the surface on its spin structures and $H^1(\Sigma, \mathbb{Z}_2)$.

We will apply these transformations to some well known partition functions of 1+1D conformal field theories (CFTs). To describe these theories from first principles is beyond the scope of this work. As a stop-gap, one can think of a CFT in terms of Segal's axioms [105]. In particular, a 2D CFT has a partition function associated to a closed surface with conformal structure. The space of conformal surfaces forms a complex manifold, and one of the hallmarks of 2D CFT is that these partition functions admit a factorization

$$Z(\Sigma) = \sum_{\mu\nu} \bar{\chi}_\mu(\tau) M_{\mu\nu} \chi_\nu(\tau), \tag{7.8}$$

where $\chi_\mu(\tau)$ are certain special holomorphic functions of the coordinates τ on the moduli space of conformal structures, called the conformal blocks and M is some (constant!) complex matrix. For instance, when Σ is a torus, τ is the shape parameter, valued in the complex upper half-plane, and the χ are called Virasoro characters.

Indeed, the form of the torus partition function reflects the structure of the CFT Hilbert space, which is a direct sum of representations of two tensored copies of the Virasoro algebra (one acting on holomorphic degrees of freedom, the other on antiholomorphic degrees of freedom), a certain infinite dimensional algebra, basically a central extension of the Lie algebra of $\mathrm{Diff}(S^1)$. The matrix $M_{\mu\nu}$ tells us which representations $V_\mu \otimes \bar{V}_\nu$ appear in the Hilbert space.

There is an action of the (pointed) mapping class group on the vector space of conformal blocks. In particular for Σ a torus we have a representation of $SL(2,\mathbb{Z})$, generated by the usual transformations S, T, under which $Z(\Sigma)$ is invariant, meaning $S^\dagger M S = M$ and $T^\dagger M T = M$. For the torus this is called modular invariance This places serious constraints on the form of the partition functions, and can often be used to compute them exactly. As usual, there is an overall normalization that may or may not be meaningful depending on the theory and is chosen to preserve conformal invariance of $Z(\Sigma)$.

We will also allow for CFTs with extra background structure. For instance, a 2D G-CFT for us will be a 2D Segal CFT for conformal surfaces equipped with a background G gauge field. Similarly, a 2D spin-CFT will be such for conformal surfaces equipped with a spin structure. Thus, we get a partition function for every closed conformal surface with a choice of background structure. The mapping class group acts on the background, and the partition functions must transform equivariantly under this action. For the torus we call this modular equivariance. Thus we will use the bosonization/fermionization relations to form a correspondence between partition functions of spin-CFTs and those of $\mathbb{Z}_2$-CFTs.

The twisted torus partition functions of these more general theories may also be decomposed as a bilinear form of Virasoro characters. The matrix M then tells us also about the G charges and fermion parity of the representations which appear in the Hilbert space, via the identities

$$Z(1/g) = \mathrm{Tr} g q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}$$

$$Z(AP/P) = \mathrm{Tr}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24},$$

where $q = e^{2\pi i \tau}$, $L_0, \bar{L}_0$ are the Hamiltonians in the holomorphic and antiholomorphic sectors, respectively, c is a number characterizing which central extension of $\mathrm{Diff}(S^1)$ appears in the construction of the Virasoro algebra, and $1/g$ indicates no twist around the spatial cycle, but a g twist around the time cycle, and likewise AP/P indicates anti-periodic (periodic) spin structure around space (time). Furthermore, partition functions with the opposite twists, trivial around time but nontrivial around space, reveal the structure of the Hilbert space in twisted sectors, in the presence of G flux or with periodic spin structure or both.

As far as I know, a complete definition at the level of detail of [106] has not yet appeared in the literature for either spin-CFT or G-CFT. However, there are definitions of G-crossed modular tensor category [107] and spin modular tensor category [108] which are sufficient to describe the space of (G- or spin-)conformal blocks and the action of the mapping class groups on them for many CFTs of interest (namely the "rational" ones). For more about the correspondence between conformal blocks and modular tensor categories/3D TQFT, see [109, 72].

7.3.1 Ising/Majorana Correspondence

Partition Functions

This section follows the conventions of [110].

We study the partition functions of the Ising $\mathbb{Z}_2$-CFT (the $\mathbb{Z}_2$ acts by the usual spin-flipping operation) and the Majorana spin-CFT on a torus. The Majorana has a partition function for each of the four spin structures on the torus, labeled antiperiodic (AP) or periodic (P). Writing the cycles as space/time ([110] uses the opposite convention) wrt the usual definition for the Virasoro characters (indexed by their position in the Kac table), by a standard calculation, we have

$$Z_f(AP/AP) = |\chi_{1,1}(q) + \chi_{2,1}(q)|^2$$

$$Z_f(P/AP) = |\chi_{1,1}(q) - \chi_{2,1}(q)|^2$$
$$Z_f(AP/P) = 2|\chi_{1,2}(q)|^2$$
$$Z_f(P/P) = 0.$$

The last partition vanishes because of a gravitational anomaly.

The pointed mapping class group $SL(2,\mathbb{Z})$ acts on these via its generators

$$S : AP/AP \mapsto AP/AP$$

$$P/AP \mapsto AP/P$$
$$AP/P \mapsto P/AP$$
$$P/P \mapsto P/P,$$
$$T : AP/AP \mapsto AP/P$$
$$P/AP \mapsto P/AP$$
$$AP/P \mapsto AP/AP$$
$$P/P \mapsto P/P.$$

Note that the only $SL(2,\mathbb{Z})$-invariant partition function among them is $Z_f(P/P) = 0$.

Now we compute the Ising partition functions in the four symmetry twists, labeled $\pm/\pm$ for the space/time. The untwisted partition function is a "diagonal" modular invariant $M = 1$ (compare (7.8)):

$$Z_b(+/+) = |\chi_{1,1}(q)|^2 + |\chi_{2,1}(q)|^2 + |\chi_{1,2}(q)|^2.$$

These correspond to the superselection sectors $1, \psi, \sigma$, respectively of the Ising TQFT [109]. Only the third is charged under the $\mathbb{Z}_2$ symmetry, so we find

$$Z_b(+/-) = |\chi_{1,1}(q)|^2 + |\chi_{2,1}(q)|^2 - |\chi_{1,2}(q)|^2.$$

This is summarized by taking M (of Eqn. (7.8)) to be the charge matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We obtain the partition function $Z(-/+)$ from $Z(+/-)$ by applying the S transformation

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{bmatrix},$$

which defines the flux matrix

$$F = S^\dagger C S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$Z_b(-/+) = \bar{\chi}_{1,1}(\bar{q})\chi_{2,1}(q) + \bar{\chi}_{2,1}(\bar{q})\chi_{1,1}(q) + |\chi_{1,2}(q)|^2.$$

We can also derive the remaining twisted partition function $Z(-/-)$ from $Z(-/+)$ by applying the T transformation

$$T = \begin{bmatrix} \exp(-2\pi i/48) & 0 & 0 \\ 0 & -\exp(-2\pi i/48) & 0 \\ 0 & 0 & \exp(2\pi i/24) \end{bmatrix},$$

from which we obtain the charge-flux matrix

$$T^\dagger F T = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -CF,$$

and so

$$Z_b(-/-) = -\bar{\chi}_{1,1}(\bar{q})\chi_{2,1}(q) - \bar{\chi}_{2,1}(\bar{q})\chi_{1,1}(q) + |\chi_{1,2}(q)|^2.$$

One checks that these partition functions are reproduced by the bosonization/fermionization formulas:

$$Z_b(+/+) = \frac{1}{2}(Z_f(AP/AP) + Z_f(P/AP) + Z_f(AP/P) + Z_f(P/P))$$

$$Z_b(-/+) = \frac{1}{2}(Z_f(AP/AP) - Z_f(P/AP) + Z_f(AP/P) - Z_f(P/P))$$

$$Z_b(+/-) = \frac{1}{2}(Z_f(AP/AP) + Z_f(P/AP) - Z_f(AP/P) - Z_f(P/P))$$

$$Z_b(-/-) = \frac{1}{2}(-Z_f(AP/AP) + Z_f(P/AP) + Z_f(AP/P) - Z_f(P/P))$$

Or in summary:

$$\vec{Z}_b = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \vec{Z}_f.$$

In the inverse form we have

$$Z_f(AP/AP) = \frac{1}{2}(Z_b(+/+) + Z_b(-/+) + Z_b(+/-) - Z_b(-/-))$$

$$Z_f(P/AP) = \frac{1}{2}(Z_b(+/+) - Z_b(-/+) + Z_b(+/-) + Z_b(-/-))$$

$$Z_f(AP/P) = \frac{1}{2}(Z_b(+/+) + Z_b(-/+) - Z_b(+/-) + Z_b(-/-))$$

$$Z_f(P/P) = \frac{1}{2}(Z_b(+/+) - Z_b(-/+) - Z_b(+/-) - Z_b(-/-))$$

or

$$\vec{Z}_f = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \vec{Z}_b.$$

Note that these two matrices are transpose, and the column (row) sums are the Arf invariant. We refer to these as the bosonization and fermionization matrices for the torus, respectively. They are simply the matrix elements of the kernel $2^{-(\chi+1)/2}(-1)^{q_\eta(\alpha)}$.

Chiral Anomaly of the Majorana CFT

The Majorana CFT has a chiral anomaly [111, 80, 110], which can be seen by studying the chiral fermion parity, a $\mathbb{Z}_2$ symmetry under which only the left moving fermion, ie. the holomorphic sector, is charged. A precise description of this anomaly requires the use of the (nonabelian) Ising TQFT [104, 23, 46] and is beyond the scope of our exposition here. However for completeness we give an intuitive description of what happens.

One can show that at the $\mathbb{Z}_2$ domain wall there is a Majorana zero mode, that is an edge mode of the fermionic SRE phase generating

$$\Omega^2_{spin}(\star, U(1)) = \mathbb{Z}_2.$$

This means that the theory sits at the boundary of a fermionic $\mathbb{Z}_2$ gauge theory with the generating

$$\nu_1 \in H^1(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2.$$

As we have briefly discussed, such $\mathbb{Z}_2$ gauge theories are generators of the group of invertible $\mathbb{Z}_2$ fermionic SPT phases:

$$\Omega^3_{spin}(B\mathbb{Z}_2, U(1)) = \mathbb{Z}_8.$$

Another meaning of ν_1 is that an application of the global $\mathbb{Z}_2$ symmetry is like tensoring with a Kitaev phase. The presence of this symmetry is the reason why $Z_f(P/P) = 0$. Meanwhile one can also check ν_2 and ν_3 are both zero.

When we bosonize the Majorana CFT, we obtain the Ising CFT, and this $\mathbb{Z}_2$ symmetry acts not as the chiral spin-flip symmetry (which has an order 2 anomaly described by bosonic $\mathbb{Z}_2$ gauge theory at Dijkgraaf-Witten level 1) but as Kramers-Wannier duality [51], which maps the Ising CFT to the Ising CFT with spin-flip symmetry gauged. This is not really a $\mathbb{Z}_2$ symmetry of the Ising CFT since it squares to a projection operator, a fact which is apparent from the fusion rules

$$\sigma \otimes \sigma = 1 \oplus \psi.$$

This projection operator preserves all spin-flip-symmetric states, so Kramers-Wannier duality becomes a symmetry when we couple to background $\mathbb{Z}_2$ gauge field, but only then. This is apparently a way that a global symmetry may be extended by a gauge symmetry without having an actual extension of symmetry groups.

7.3.2 Compact Boson/Dirac Fermion Correspondence

Partition Functions

This section follows the conventions of [110].

Now we investigate the correspondence between the Dirac fermion (massless Thirring model) and the compact boson.

The compact boson at radius R has a torus partition function which is a bilinear combination of infinitely many Virasoro characters:

$$Z = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(n/R+mR/2)^2} \bar{q}^{\frac{1}{2}(n/R-mR/2)^2}, \tag{7.9}$$

where $\eta(\tau)$ is the Dedekind η function. It is known that at the radius $R = 2$ the theory is equivalent to a free Dirac fermion, itself equivalent to two decoupled Majorana fermions. At this special radius, the (untwisted) partition function is

$$Z_b(+/+) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(n/2+m)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

Meanwhile, the partition functions of the Dirac fermion are made from the Jacobi theta functions:

$$Z_f(AP/AP) = |\chi_{1,1}(q) + \chi_{2,1}(q)|^4 = \frac{1}{|\eta(\tau)|^2} \sum_{a,b\in\mathbb{Z}} q^{\frac{1}{2}a^2} \bar{q}^{\frac{1}{2}b^2} = \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2$$

$$Z_f(AP/P) = 4|\chi_{1,2}(q)|^4 = \frac{1}{|\eta(\tau)|^2} \sum_{a,b\in\mathbb{Z}} (-1)^{a+b} q^{\frac{1}{2}a^2} \bar{q}^{\frac{1}{2}b^2} = \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2$$

$$Z_f(P/AP) = |\chi_{1,1}(q) - \chi_{2,1}(q)|^4 = \frac{1}{|\eta(\tau)|^2} \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}r^2} \bar{q}^{\frac{1}{2}s^2} = \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2$$

$$Z_f(P/P) = \frac{1}{|\eta(\tau)|^2} \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-s} q^{\frac{1}{2}r^2} \bar{q}^{\frac{1}{2}s^2} = 0.$$

As an example of the bosonization procedure, let us verify the untwisted relation

$$Z_b(+/+) = \frac{1}{2}\Big(Z_f(AP/AP) + Z_f(AP/P) + Z_f(P/AP) + Z_f(P/P)\Big).$$

We can write

$$Z_f(AP/AP) + Z_f(AP/P) = \frac{2}{|\eta(\tau)|^2} \sum_{\substack{a,b\in\mathbb{Z} \\ a=b \bmod 2}} q^{\frac{1}{2}a^2} \bar{q}^{\frac{1}{2}b^2},$$

which we recognize as the subset $n = 0 \mod 2$ in the sum $Z_b(+/+)$. Likewise

$$Z(P/AP) + Z(P/P) = \frac{2}{|\eta(\tau)|^2} \sum_{\substack{r,s \in \mathbb{Z}+\frac{1}{2} \\ r=s \mod 2}} q^{\frac{1}{2}r^2} \bar{q}^{\frac{1}{2}s^2}$$

which we recognize as the subset $n = 1 \mod 2$ in the sum $Z_b(+/+)$.

We wish to identify the Ω^1_{spin} symmetry Q of the compact boson which allows us to invert the bosonization transformation by (7.7). To do so, we can study the partition function

$$\mathrm{Tr}(-1)^Q q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} \sim Z_b(+/-) \tag{7.10}$$

$$= \frac{1}{2}\Big(Z_f(AP/AP) + Z_f(AP/P) - Z_f(P/AP) - Z_f(P/P) \Big)$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n/2+m)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

We are also interested in the twisted partition functions

$$Z_b(-/+) = \frac{1}{2}\Big(Z_f(AP/AP) - Z_f(AP/P) + Z_f(P/AP) - Z_f(P/P) \Big)$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{\substack{a,b \in \mathbb{Z} \\ a-b=1 \mod 2}} q^{\frac{1}{2}a^2} \bar{q}^{\frac{1}{2}b^2} + \frac{1}{|\eta(\tau)|^2} \sum_{\substack{r,s \in \mathbb{Z}+\frac{1}{2} \\ r-s=1 \mod 2}} q^{\frac{1}{2}r^2} \bar{q}^{\frac{1}{2}s^2}$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(n/2+m+1)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2},$$

$$Z_b(-/-) = \frac{1}{2}\Big(- Z_f(AP/AP) + Z_f(AP/P) + Z_f(P/AP) - Z_f(P/P) \Big)$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} (-1)^{n+1} q^{\frac{1}{2}(n/2+m+1)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

We can summarize, writing $B \in Z^1(X, \mathbb{Z}_2)$ as the background Ω^1_{spin} gauge field, with x, t as the space and time cycles:

$$Z_b(B) = \frac{1}{|\eta(\tau)|^2} \sum_{n,m \in \mathbb{Z}} \exp\left[\pi i \Big(\int_t B \Big) \Big(n + \int_x B \Big) \right] q^{\frac{1}{2}(n/2+m+\int_x B)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

We note that under a shift $\int_x B \mapsto \int_x B + 2$, we can compensate by a redefinition of the dummy variables $n \mapsto n - 2$, $m \mapsto m - 1$. These are also clearly invariant under $B \mapsto B + df$, so each of these $Z_b(B)$ defines a state in the untwisted 2+1D $\mathbb{Z}_2$ gauge theory.

Hilbert Space and Chiral Anomaly of the Compact Boson

The structure of the Hilbert space $\mathcal{H}$ this CFT assigns to S^1 (see [105]) is encoded in the torus partition function through the formula

$$\mathrm{Tr}_{\mathcal{H}} q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} = Z_b(+/+),$$

where L_0 and $\bar{L}_0$ are special elements of the holomorphic (ie. left moving by the identification $z = x + it$) and antiholomorphic (right moving) Virasoro algebras, and the normalization factor $q^{-1/24}\bar{q}^{-1/24}$ is chosen to preserve conformal invariance, analogous to how we chose $N(X)$ to have topological invariance for the $\mathbb{Z}_n$ gauge theory of section 5.2.

Characters of the Virasoro algebra for $c = 1$ have the form [112]

$$\mathrm{Tr} q^{L_0} = \frac{q^h}{\eta(\tau)},$$

and are irreducible so long as h is not of the form $l^2/4$ for $l \in \mathbb{Z}$. The ones that appear in the torus partition function are

$$h = \frac{1}{2}(n/2 + m)^2$$

and

$$\bar{h} = \frac{1}{2}(n/2 - m)^2$$

for the antiholomorphic sector. Denoting these modules as $V(h), \bar{V}(\bar{h})$, respectively, we find

$$\mathcal{H} = \bigoplus_{m,n \in \mathbb{Z}} V(\frac{1}{2}(n/2 + m)^2) \otimes \bar{V}(\frac{1}{2}(n/2 - m)^2).$$

We denote the highest weight states of these modules as

$$|h, \bar{h}\rangle,$$

and define

$$|m,n\rangle := |\frac{1}{2}(n/2+m)^2, \frac{1}{2}(n/2-m)^2\rangle.$$

The action of the special $\mathbb{Z}_2 = \Omega^1_{spin}$ symmetry Q commutes with the Virasoro actions and can be read off from (7.10):

$$(-1)^Q|m,n\rangle = (-1)^n|m,n\rangle.$$

It can be factored (nonuniquely) into a tensor product of operators acting on the holomorphic and anti-holomorphic sectors, by

$$Q = Q_L + Q_R.$$

For instance, we may take

$$e^{\pi i Q_L}|m,n\rangle = e^{\pi i(n/2+m)}|m,n\rangle = e^{\pi i\sqrt{2h}}|m,n\rangle$$

$$e^{\pi i Q_R}|m,n\rangle = e^{\pi i(n/2-m)}|m,n\rangle = e^{\pi i\sqrt{2\bar{h}}}|m,n\rangle.$$

We will show Q_L, Q_R are anomalous $\mathbb{Z}_2$ symmetries whose anomaly theory is level 1 $\mathbb{Z}_2$ Dijgraaf-Witten theory. To this end, we first consider

Chiral Anomaly of the Dirac Fermion by Bosonization

Like the Majorana fermion, the Dirac fermion has an anomalous symmetry $(-1)^{F_L}$ which counts the fermion parity of the left-moving (holomorphic) states and acts trivial on the right-moving (antiholomorphic) states. For example, the twisted partition function

$$\begin{aligned}\mathrm{Tr}_{AP}(-1)^{F_L}q^{L_0+1/24}\bar{q}^{\bar{L}_0+1/24} &\sim Z_f(AP/AP, 1/(-1)^{F_L})\\ &= \frac{1}{|\eta(\tau)|^2}\sum_{a,b\in\mathbb{Z}}(-1)^a q^{\frac{1}{2}a^2}\bar{q}^{\frac{1}{2}b^2},\end{aligned}$$

where in the second argument of Z_f we indicate the space/time twists by the chiral symmetry. In the $P/\star$ sectors we choose $(-1)^{F_L} = e^{\pi i r}$. There is another choice $e^{-\pi i r} = -e^{\pi i r}$ which works just as well, and we will see they derive the same anomaly.

In order to describe the anomaly by bosonization, we need to find a corresponding symmetry U of the compact boson such that the bosonization/fermionization relations (7.7) map U-twisted partition functions on the bosonic side to $(-1)^{F_L}$-twisted partition functions on the fermionic side. The first partition function to investigate is:

$$Z_b(+/+,1/U) = \frac{1}{2}\Bigg(Z_f(AP/AP,1/i) + Z_f(P/AP,1/i)$$

$$+Z_f(AP/P,1/i) + Z_f(P/P,1/i)\Bigg)$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} e^{\pi i(n/2+m)} q^{\frac{1}{2}(n/2+m)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

Then using Poisson resummation we can obtain from this its S transformed partition function

$$Z_b(+/+,U/1) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(n/2+m+1/2)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

Then using T:

$$Z_b(+/+,U/U) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} e^{\pi i(n/2+m+1/4)} q^{\frac{1}{2}(n/2+m+1/2)^2} \bar{q}^{\frac{1}{2}(n/2-m)^2}.$$

Observe from $Z_b(+/+,1/U)$ that the U charge of the $|n,m\rangle$ state is half that of its Q charge. Thus, at least in the untwisted sectors,

$$U^2 = Q,$$

which indicates that the bosonic anomalous $\mathbb{Z}_2$ symmetry U is extended by the $\Omega^1_{spin} = \mathbb{Z}_2$ symmetry! This indicates that the anomaly theory has a nontrivial $[\nu_2] \in H^2(B\mathbb{Z}_2, \Omega^1_{spin})$, in the notation of (7.4). If A is the $\mathbb{Z}_2$ background and B is the Ω^1_{spin} background, then we have

$$dB = \nu_2(A) = dA/2.$$

There are two choices of ν_3 which satisfy the Gu-Wen equations, namely

$$\nu_3(A) = \pm\frac{1}{8}AdA.$$

To fully characterize the anomaly we need to determine which of these arises from $(-1)^{F_L}$.

Unfortunately this can't be seen so easily from the torus partition functions without thinking about the Hilbert space of the compact boson in the presence of A fluxes. However, from the prefactor

$$\exp \pi i(n/2 + m + 1/4)$$

in $Z_b(+/+, U/U)$, compared with

$$\exp \pi i(n/2 + m)$$

in $Z_b(+/+, 1/U)$, we see that the A flux carries a $1/4$ $\mathbb{Z}_2$ charge, meaning (see section 6.2.1) that the theory sits at the boundary of spin Dijkgraaf-Witten theory with $\nu_3 = -AdA/8$. In the classification, this anomaly is referred to as the $\nu = 2 \bmod 8$ anomaly.

Note that had we used $\exp -\pi i$ instead of $\exp \pi i$, we would have obtained the same result, since this amounts to a difference of $(-1)^n$ in the definition of U. This is an automorphism of the spin $\mathbb{Z}_2$ Dijkgraaf-Witten theory $B \mapsto B + A$, which does not change the level.

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