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Combinatorial Invariants on Smooth Projective Spherical Varieties



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Combinatorial Invariants on Smooth Projective Spherical Varieties

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Abstract

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The Knop Conjecture, which was proven by Losev in [Los09a], states that smooth affine spherical varieties are classified up to equivariant isomorphism by their weight monoids. This is in contrast with the standard classification of spherical varieties, which involves combinatorial invariants related to divisors and valuations. In this thesis, we prove that some of these combinatorial invariants are also determined by weight monoids in the smooth projective case. This results in certain partial analogs of the Knop Conjecture for smooth projective spherical varieties. We provide counterexamples to demonstrate that these partial analogs are relatively optimal.

Our results indicate that weight monoids of smooth projective spherical varieties are closely related to the data of certain divisors on these varieties. In analogy with the total coordinate ring discussed in [Bri07], we develop methods for comparing the data of weight monoids with the data of divisors, even without smoothness hypotheses. We then show that, under mild hypotheses, the data provided by weight monoids is equivalent to the data provided by divisors on projective spherical varieties.

To the Lord our God, and to his son, Jesus Christ:

> For by him all things were created, in heaven and on earth, visible and invisible, whether thrones or dominions or rulers or authorities—all things were created through him and for him. And he is before all things, and in him all things hold together. And he is the head of the body, the church. He is the beginning, the firstborn from the dead, that in everything he might be preeminent.
> (Colossians 1:16-18)

He is the reason I'm on this earth, the reason I'm still alive, and the reason I didn't drop out of graduate school. This thesis, like all other things, is for him.

# Contents

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# Chapter 1

# Introduction

## 1.1 Overview of the Subject Matter

Let $k$ be an algebraically closed field of characteristic 0, and let $G$ be a connected reductive group over $k$. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. A *spherical $G$-variety* is a normal $G$-variety $X$ containing a dense $B$-orbit. Spherical varieties are a very nice type of $G$-variety with a very rich theory. Examples include toric varieties (which are precisely spherical varieties in the case where $G = B = T$ is a torus) and flag varieties. These varieties provide useful examples in a number of different fields of study. For instance, all spherical varieties are Mori dream spaces, and their intersection theory is also well-understood.

In addition to enjoying such nice geometric properties, spherical varieties admit a purely combinatorial classification. This classification began with the theory of Luna and Vust [LV83], which classifies the open embedding $G/H \hookrightarrow X$ of the dense $G$-orbit $G/H$ of a spherical variety $X$. The classification of this embedding is given in terms of combinatorial objects called *colored fans* and generalizes the classification of toric varieties in terms of fans. With Luna–Vust theory, the classification of spherical varieties reduces to the case of a homogeneous spherical variety, i.e. to the case where $X = G/H$ is a homogeneous space containing a dense $B$-orbit. This case is rather difficult. Luna first proposed a classification of homogeneous spherical varieties in [Lun01] and proved this classification for certain reductive groups $G$ (specifically, the so-called reductive groups of type A, see Definition 2.2.17). His methods were extended to various other types for $G$ in work by Bravi, Pezzini, and Luna (for a breakdown of which cases were proven, along with precise references to the original papers by these authors, see [Tim11, p. 198]). Meanwhile, Losev proved the uniqueness portion of Luna's classification [Los09c] for all $G$, and an argument for the existence portion was given by Cupit-Foutou [Cup14] using a geometric construction (this paper was uploaded to the Arxiv as a preprint in 2009 and revised in subsequent years, but it seems to have never been published). Later, Bravi and Pezzini [BP16] provided a proof of the existence portion of the classification by extending the methods originally used by Luna in the type A case.

The combinatorial objects used to classify homogeneous spherical varieties pertain to the behavior of valuations and divisors on these varieties. This behavior is greatly restricted by reducing to the case of certain very nice spherical varieties (the *wonderful varieties of rank* $\leq 2$) and appealing to an explicit classification of these varieties, which was given by Wasserman in [Was96] (Wasserman's paper is the original proof of the rank-2 case and gives an overview of the rank-1 case, which was originally proven by Akhiezer [Akh83] in an analytic setting and by Brion [Bri89b] in this algebraic context). This approach, pioneered by Luna, is fundamentally geometric and gives clear restrictions on the combinatorial data that can arise, albeit partly by appealing to a list which can be found in Wasserman's paper. Meanwhile, another sort of combinatorial object, more representation-theoretic in nature and less clearly constrained, has also been studied for spherical varieties. These objects come from observing the $B$-eigenvectors of $G$-actions on global sections of line bundles, which yields certain monoids and polytopes in the $\mathbb{Q}$–vector space spanned by the lattice of weights of $T$. This sort of combinatorial data is appealing in light of the case of toric varieties, where monoids (in the affine case) and polytopes (in the projective case) of $T$-weights are known to classify the variety up to $T$-isomorphism.

Some basic properties of these monoids and polytopes have been studied for spherical varieties, e.g. by Brion in [Bri97], and in [Bri07], Brion studied similar types of data in the context of Cox rings of spherical varieties. The relationship between Cox rings and Luna's combinatorial data for homogeneous spherical varieties has been further studied, e.g. in [Gag14] and [Gag19]. Moreover, F. Knop conjectured that a smooth affine spherical variety $X$ is classifed up to $G$-equivariant isomorphism by the monoid of weights of $B$-eigenvectors in $\Gamma(X, \mathcal{O}_X)$. This conjecture was proven by Losev in [Los09a] using Losev's work on the uniqueness portion of the classification of homogeneous spherical varieties. Later, Pezzini and Van Steirteghem [PV19] were able to completely classify which weight monoids occur as those of a smooth affine spherical variety.

Aside from Losev's proof of the Knop conjecture, not much seems to be known about when these monoids of $T$-weights classify spherical varieties. Our goal is to investigate other situations in which this type of combinatorial data classifies spherical varieties, or more generally, determines some of the combinatorial data used in Luna's method of classification. To be more precise, we will denote such a monoid by $\Lambda^+(X, L)$, where $X$ is a spherical variety and $L$ is a so-called $G$-*linearized* line bundle on $X$, i.e. a line bundle equipped with a piece of data called a $G$-linearization (which is necessary in order to construct the monoid $\Lambda^+(X, L)$). There are three main pieces of data that arise in the classification of spherical varieties:

1. The set of of $B$-stable prime Weil divisors of $X$, which we denote by $\mathcal{D}_{G,X}$. This set comes equipped with certain other data related to these divisors (for instance, some data related to their valuations).

2. The so-called *colored fan* of $X$, denoted $\mathscr{F}_X$. The colored fan is a set whose elements are *colored cones*, which encode certain data related to the $G$-orbits of $X$.

3. The so-called *spherical roots* of $X$. The set of spherical roots is denoted $\Psi_{G,X}$.

The classification of spherical varieties implies that two spherical varieties $X_1$ and $X_2$ are $G$-equivariently isomorphic if and only if we can "equate" (in a sense made precise in Section 4.1) the data of the $\mathcal{D}_{G,X_i}$, the $\mathscr{F}_{X_i}$, and the $\Psi_{G,X_i}$. In light of this fact, our goal is to answer the following question.

**Question 1.1.1.** *Let $X_1$ and $X_2$ be spherical $G$-varieties, and suppose given some $G$-linearized line bundles $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that the monoids $\Lambda^+(X_1, L_1)$ and $\Lambda^+(X_2, L_2)$ are equal. Can we obtain equalities on the $\mathcal{D}_{G,X_i}$, the $\Psi_{G,X_i}$, or the $\mathscr{F}_{X_i}$?*

The main new results of the thesis are motivated by this question. These results are contained in Chapters 4 and 5.

## 1.2 Results in Chapter 4

In Chapter 4, we use the Knop conjecture and the so-called local structure theorem for spherical varieties to show that when $X_1$ and $X_2$ are smooth and projective and $L_1$ and $L_2$ are ample, then the assumption $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ implies that $X_1$ and $X_2$ are locally $B$-equivariantly isomorphic (see Theorem 4.4.6 for a precise statement). On its own, this is not much more than an adaptation of Losev's arguments to prove the Knop conjecture (see [Los09a, Section 5]) from the affine setting to our setup with ample line bundles. (It is perhaps worth noting one key difference here: Losev had to appeal to an inductive argument to get his local isomorphism, but since we will obtain affine spherical varieties in the local setting, we can simply appeal to the Knop conjecture now that Losev has already proven it.)

Our plan in Chapter 4 is to use the local isomorphism given in Theorem 4.4.6 to equate the various combinatorial data of $X_1$ and $X_2$. This is a somewhat subtle task for two reasons: first, because not all combinatorial data is captured locally in this way; and second, because the local isomorphism we get is generally only $B$-equivariant, not $G$-equivariant. We do our best to address this issue for each of the three pieces of combinatorial data mentioned above that classify spherical varieties.

**Remark 1.2.1.** It is generally necessary to assume that $X_1$ and $X_2$ are both smooth and projective for almost all of our arguments using Theorem 4.4.6. On the other hand, we will see in Examples 4.2.7 and Example 4.2.8 that $\Lambda^+(X, L)$ generally does not capture the pieces of combinatorial data we are interested in if $X$ is not smooth and projective (or affine, but this case is already handled by the Knop conjecture). So, while the majority of our results do require hypotheses of smoothness and projectivity, these hypotheses are likely necessary for most statements we are trying to prove.

**Equating $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$**

In Section 4.5, we consider the task of equating $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$. We formalize this notion of "equality" in Section 4.1 by introducing a gadget called a $\mathcal{D}$-*equivalence*, which is a bijection

$\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ that preserves all the relevant combinatorial properties of the $\mathcal{D}_{G,X_i}$. We say that $X_1$ and $X_2$ are *$\mathcal{D}$-equivalent* if such a $\mathcal{D}$-equivalence exists. It turns out that the isomorphism of Theorem 4.4.6 is not enough to give us a $\mathcal{D}$-equivalence, even when $X_1$ and $X_2$ are smooth and projective: indeed, we will provide an explicit counterexample in Examples 4.9.1 and 4.9.2 where $X_1$ and $X_2$ are actually very well-behaved spherical varieties. To fix this, we can keep track of another piece of data in addition to the weight monoids $\Lambda^+(X_i, L_i)$: namely, the *set of type-b roots* of $X_i$, denoted $\Pi^b_{X_i}$. If we additionally assume that $\Pi^b_{X_1} = \Pi^b_{X_2}$ (which is not the case in the aforementioned counterexample), then we obtain the following result.

**Theorem 1.2.2** (see Theorem 4.5.5)**.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively). If $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\Pi^b_{X_1} = \Pi^b_{X_2}$, then $X_1$ and $X_2$ are $\mathcal{D}$-equivalent.*

### Equating $\Psi_{G,X_1}$ and $\Psi_{G,X_2}$

In Section 4.6, we consider equating the sets of spherical roots $\Psi_{G,X_1}$ and $\Psi_{G,X_2}$. Unlike with the $\mathcal{D}_{G,X_i}$, where we need to clearly define what "equality" should mean, in this case we actually mean $\Psi_{G,X_1} = \Psi_{G,X_2}$ as sets.

The standard theory of spherical varieties gives many constraints on the combinatorial behavior of spherical roots. "Most" of the combinatorial behaviors that can possibly arise are such that we can detect the spherical root locally and so use our local isomorphism from Theorem 4.4.6 to identify spherical roots between $X_1$ and $X_2$. More precisely, this gives us the following theorem.

**Theorem 1.2.3** (see Theorem 4.6.8)**.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties, let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively), and suppose that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. For any spherical root $\gamma \in \Psi_{G,X_1}$ be a spherical root, we have $\gamma \in \Psi_{G,X_2}$, exept possibly when $\gamma \in \Pi^b_{X_1}$ or when $\gamma$ lies in a certain list of 4 "exceptional types" spherical roots.*

We remark that the set of type $b$ roots $\Pi^b_{X_i}$ is always a subset of $\Psi_{G,X_i}$, and whether a spherical root $\gamma \in \Psi_{G,X_i}$ lie in $\Pi^b_{X_i}$ is entirely a question of the combinatorial properties of $\gamma$. In particular, if $\gamma \in \Pi^b_{X_1}$ and $\gamma \in \Psi_{G,X_2}$, then $\gamma \in \Pi^b_{X_2}$. So, the exceptional case $\gamma \in \Pi^b_{X_1}$ in Theorem 1.2.3 again comes back to the issue that we may have $\Pi^b_{X_1} \neq \Pi^b_{X_2}$, as we did when equating the $\mathcal{D}_{G,X_i}$. In particular, the same counterexample for the $\mathcal{D}$-equivalences (Examples 4.9.1 and 4.9.2) will give a choice of $X_1$ and $X_2$ as in the above theorem and some $\gamma \in \Pi^b_{X_1} \cap \Psi_{G,X_1}$ such that $\gamma \notin \Psi_{G,X_2}$. Thus, the statement of Theorem 1.2.3 is essentially optimal when it comes to roots of type $b$.

The precise description of the 4 "exceptional types" of spherical roots is somewhat technical; we give a brief overview of the description here and defer a more precise description to Theorem 4.6.8 and Proposition 4.6.5c. For every spherical root $\gamma$ of a spherical $G$-variety $X$, there exists some reductive subgroup $G_0 \subset G$ and some spherical $G_0$-variety $X_0$ which is "nice" (more precisely, $X_0$ is a so-called *wonderful variety of rank 1*) such that the unique spherical root of $X_0$ is $\gamma$. Since the wonderful varieties of rank 1 can be classified, and all of them have been explicitly written down, this fact completely determines all possible spherical roots of any spherical variety for $G$, and in fact, one can write down a full list of spherical roots for each possible choice of $G$ (see e.g. [Tim11, Table 30.2], which gives this list for $G$ a semisimple simply connected group; the list for any other $G$ can be computed from this list using standard combinatorial facts about spherical roots). Also, in the classification of all rank-1 wonderful varieties, one naturally finds certain families $(G_n, X_n)_{n\geq n_0}$ consisting of a rank-1 wonderful $G_n$-variety $X_n$ for each $n \geq n_0$, where $n_0 \in \mathbb{N}$. For instance, one such family is $(\mathrm{SL}_n, \mathbb{P}^n)$ for $n \geq 2$.

Returning to Theorem 1.2.3, there are certain combinatorial conditions on the spherical root $\gamma$ in the theorem that our proof will *not* work for. These combinatorial conditions occur only for all the spherical roots arising from 3 families of rank 1 wonderful varieties $(G_{1,n}, X_{1,n})_{n\geq 2}$, $(G_{2,n}, X_{2,n})_{n\geq 2}$, and $(G_{3,n}, X_{3,n})_{n\geq 2}$, and for the spherical root of one other rank-1 wonderful variety $X_4$ under the action of some reductive group $G = G_4$. So, the spherical roots of the $X_{i,n}$ for $1 \leq i \leq 3$ constitute 3 "exceptional types" of spherical roots, and the unique spherical root of $X_4$ is a 4th "exceptional type" (or more precisely, a single exceptional spherical root).

Now, for every single spherical root $\gamma$ that is one of these 4 exceptional types (i.e. every $\gamma$ that is a spherical root of $X_{i,n}$ for some $i$ and $n$, and for $\gamma$ equal to the spherical root of $X_4$), we will see in Example 4.9.4 that there exist $X_1$ and $X_2$ smooth and projective and $L_1$ and $L_2$ ample such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, but $\gamma$ is a spherical root of $X_1$ and not a spherical root of $X_2$. Thus, Theorem 1.2.3 is essentially optimal, in that there exists a counterexample for every single spherical root that the theorem does not apply to. We remark, however, that unlike the counterexample for the first exceptional type in Examples 4.9.1 and 4.9.1, our counterexample in Example 4.9.4 is not a geometric construction; instead, it is given by simply writing down some valid combinatorial data and noting that a spherical variety with this data exists by the classification of spherical varieties.

### Equating $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$

The colored fans $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$ are sets, but to equate them, we do not mean to use a literal equality of sets. Equating two colored fans actually only makes sense when $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, so to capture what it means for these colored fans to be "equal," we will introduce a notion of a $\mathcal{D}$-equivalence *preserving colored fans*. For the purposes of this introduction, this technicality is not particularly relevant, so we will just speak of these colored fans being "equal" for the time being (by which we really mean that there exists a $\mathcal{D}$-equivalence between $X_1$ and $X_2$ that preserves colored fans).

Our techniques using Theorem 4.4.6 seem to give us very little control over the colored fans $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$. The primary issue here is that the elements of a colored fan are generally not detectable locally, except when $X_1$ and $X_2$ are so-called *toroidal varieties* (not to be confused with toric varieties), which are a particularly nice type of spherical variety. In

Proposition 4.7.1, we will give one very limited result in this vein: roughly speaking, it says that an element $C \in \mathscr{F}_{X_1}$ also lies in $\mathscr{F}_{X_2}$ provided that $C$ has the same properties as an element of a colored fan of a toroidal variety. Being toroidal is a property that can be checked on colored fans, so in fact, Proposition 4.7.1 will only imply that $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$ are equal when $X_1$ and $X_2$ are both toroidal. In the toroidal case, however, Proposition 4.7.1 is actually superceded by a stronger statement (namely, Theorem 1.2.6, which we will discuss below).

On the other hand, In Examples 4.9.3 and 4.9.4, we will obtain counterexamples when the assumptions of Proposition 4.7.1 do not hold. So, while it is rather limited in scope, Proposition 4.7.1 seems to be a relatively optimal result when trying to equate the colored fans of $X_1$ and $X_2$.

**From Valuation Cones to Everything Else**

Remarkably, it turns out that if we are willing to assume $\Psi_{G,X_1} = \Psi_{G,X_2}$ then we actually obtain an "equality" on both the $\mathcal{D}_{G,X_i}$ and the $\mathscr{F}_{X_i}$. Even more remarkably, we don't even need a smoothness assumption for this to be the case. Here is the precise result:

**Theorem 1.2.4** (see Corollary 4.3.5)**.** *Let $X_1$ and $X_2$ be projective spherical $G$-varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ respectively. If $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\Psi_{G,X_1} = \Psi_{G,X_2}$, then there exists a $G$-equivariant isomorphism $i : X_1 \xrightarrow{\sim} X_2$ such that $i^*L_2 \cong L_1$ (as $G$-linearized invertible sheaves).*

Theorem 1.2.4 is essentially a generalization of an analogous result for affine spherical varieties, which Losev proved as a stepping stone to proving the Knop conjecture (see [Los09a, Theorem 1.2]). Indeed, the proof of Theorem 4.4.6 under the assumptions of the above theorem is essentially an application of Theorem 1.2 in [Los09a], and Theorem 1.2.4 follows quickly from Theorem 4.4.6 and our results on equating the $\mathcal{D}_{G,X_i}$. For this method of proving Theorem 1.2.4, see the end of Section 4.5. We will also give a different proof of the theorem in Section 4.3, which hinges on "lifting" certain valuations on a projective variety to valuations on its affine cone.

Write $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ for the subset of spherical roots which are either in $\Pi^b_{X_i}$ or are one of the 4 "exceptional types" in Theorem 1.2.3. By combining Theorem 1.2.4 and Theorem 1.2.3, we obtain a new criterion for when two smooth projective spherical $G$-varieties are $G$-equivariantly isomorphic.

**Corollary 1.2.5** (see Corollary 4.8.1)**.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties. The following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$, and there exist $G$-linearized invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $L_1$ and $L_2$ are both ample and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

We discussed above how the "exceptional types" of spherical roots in Theorem 1.2.3 are obtained by a technical analysis of all possible spherical roots for a spherical $G$-variety. This sort of analysis also shows that "many" spherical roots are not exceptional. Although we have not rigorously quantified what "many" means here, this leads us to make 2 intuively logical claims.

1. There are likely many interesting smooth projective spherical varieties $X$ such that $\Psi_{G,X}^{exc} = \varnothing$. By Corollary 1.2.5, such a variety $X$ is completely determined up to $G$-equivariant isomorphism by a weight monoid $\Lambda^+(X, L)$ for any $G$-linearized ample invertible sheaf $L$ on $X$.

2. For a general smooth projective spherical $G$-variety $X$, the size of the (finite) set $\Psi_{G,X}^{exc}$ is often relatively small compared to the size of $\Psi_{G,X}$. In other words, the data of $\Psi_{G,X}^{exc}$ is less than that of $\Psi_{G,X}$, so the statement of Corollary 1.2.5 classifies $X$ up to $G$-equivariant isomorphism with relatively little data compared to that of Theorem 1.2.4. Moreover, Theorem 1.2.4 itself seems to use less data than the general classification of spherical varieties, because Theorem 1.2.4 does not make any assumptions about colored fans. Thus, both Theorem 1.2.4 and Corollary 1.2.5 seem to be significant improvements on the amount of data needed to classify a (smooth) projective spherical variety up to $G$-equivariant isomorphism.

In spite of these statements, the appearance of the set $\Psi_{G,X}^{exc}$ in Corollary 1.2.5 is rather less nice than the statement of the Knop conjeture, in which one assumes only an equality on weight monoids. However, as mentioned in our discussion of Theorem 1.2.3 above, Examples 4.9.1, 4.9.2, and 4.9.4 will give us for every single possible $\gamma \in \Psi_{G,X_1}^{exc}$ an example in which $\gamma \in \Psi_{G,X_1}$ but $\gamma \notin \Psi_{G,X_2}$. Thus, the assumption that $\Psi_{G,X_1}^{exc} = \Psi_{G,X_2}^{exc}$ seems to be necessary in Corollary 1.2.5.

These examples notwithstanding, there are certain specific "nice" classes of spherical varieties for which we can improve upon Corollary 1.2.5. We can provide partial results for so-called *rank-1* spherical varieties, although these require some other niceness assumptions (see Theorem 4.8.5 and Corollary 4.8.6 for details). Moreover, the so-called *horospherical varieties* are a type of spherical variety characterized by the fact that they have no spherical roots. So, $\Psi_{G,X_1} = \Psi_{G,X_2} = \varnothing$ when $X_1$ and $X_2$ are horospherical, which allows us to remove the assumption that the sets $\Psi_{G,X_i}$ are equal in Theorem 1.2.4. Finally, we can consider the case of toroidal varieties, which we have already mentioned above. Below is our main theorem on horospherical and toroidal varieties.

**Theorem 1.2.6** (see Theorem 4.8.2)**.** *Let $X_1$ and $X_2$ be smooth projective $G$-varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively).*

*(a) Suppose that $X_1$ and $X_2$ are horospherical. If $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, then $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(b) Suppose that $X_1$ and $X_2$ are toroidal. If $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\Pi^b_{X_1} = \Pi^b_{X_2}$, then $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

We remark that horospherical varieties are a very typical nice type of spherical variety in the literature. For instance, all flag varieties over an algebraically closed field are horospherical varieties. Toric varieties are also examples of horospherical varieties; in fact, our result on horospherical varieties in Theorem 1.2.6 may be viewed as a generalization of the fact that a projective toric variety is characterized up to isomorphism by a conex polytope (see e.g. [Oda88, Theorem 2.22]). Toroidal varieties are less commonly studied in their own right, perhaps because their behavior is already very well understood (and in fact is closely related to that of toric varieties). However, there do exist many examples of toroidal varieties: for instance, every spherical variety $X$ is $G$-equivariantly birational to a smooth toroidal variety $X'$, and if $X$ is projective, then $X'$ may be taken to be projective as well (see Proposition 3.5.8 and Theorem 3.5.10 for details).

## 1.3 Results in Chapter 5

Our results in Chapter 4 seem to indicate that, of all the combinatorial invariants that classify spherical varieties, the set of $B$-stable divisors $\mathcal{D}_{G,X}$ is the one that is most directly related to the weight monoid $\Lambda^+(X, L)$. Indeed, we saw in Theorem 1.2.2 that an equality $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ gives us an "equality" between $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$ (i.e. a $\mathcal{D}$-equivalence between $X_1$ and $X_2$), provided that we also have $\Pi^b_{X_1} = \Pi^b_{X_2}$. We are thus led to consider whether weight monoids can also be used to encode the information of the set $\Pi^b_X$. If they can, then an equality on weight monoids might be sufficient on its own to give a $\mathcal{D}$-equivalence.

In Chapter 5, we will consider 2 methods of encoding the information of $\Pi^b_X$ in a weight monoid, and we will studying the ensuing relationship between divisors and weight monoids for both methods. The first method is to simply choose a certain "nice" $G$-linearized ample line bundle $L$ on $X$ such that the weight monoid $\Lambda^+(X, L)$ determines the set $\Pi^b_X$. The second method is to consider the weight monoid $\Lambda^+(X, L)$ for every $G$-linearized invertible sheaf $L$ on $X$, rather than just using the weight monoid for a single invertible sheaf.

### Using "Nice" Ample Line Bundles

Our method of proving Theorem 1.2.2 is close to being able to prove that $\Pi^b_{X_1} = \Pi^b_{X_2}$, but it cannot generally prove this because of one technical issue: a type $b$ root $\alpha \in \Pi^b_X$ corresponds in a nice way to two $B$-stable prime Weil divisors $D_1$ and $D_2$ on $X$, and we would need to be able to find a certain global section of $L$ that does not vanish on either $D_1$ or $D_2$. To fix this issue, we are led to define a notion of a *level line bundle*, which, roughly speaking, is a choice of $L$ such that this necessary global section does exist. We discuss the notion of level line bundles in Section 5.1. In particular, we show that the method of proof in Theorem 1.2.2 does also take care of the sets $\Pi^b_{X_1}$ and $\Pi^b_{X_2}$ in the setting of level line bundles. More precisely, we have the following improvement on Theorem 1.2.2 in the level setting.

**Theorem 1.3.1** (see Theorem 5.1.10 and Corollary 5.1.11)**.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively). If $L_1$ and $L_2$ are level and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, then $\Pi^b_{X_1} = \Pi^b_{X_2}$, and $X_1$ and $X_2$ are $\mathcal{D}$-equivalent.*

The main advantage to using level line bundles is that we no longer run into any issues with roots of type $b$, either in Theorem 1.2.2 or in our consideration of spherical roots in Theorem 1.2.3. In particular, in the statement of Theorem 1.2.3, we only need to except 4 "exceptional types" of spherical roots: the spherical roots of the 5th exceptional type are the elements of $\Pi^b_{X_i}$, and we know from Theorem 1.3.1 that $\Pi^b_{X_1} = \Pi^b_{X_2}$ when $L_1$ and $L_2$ are level. We can thus exclude $\Pi^b_{X_i}$ from the sets $\Psi^{exc}_{G,X_i}$ in Corollary 1.2.5, which leads to the following refinement of that corollary in the level case.

**Corollary 1.3.2** (see Corollary 5.1.12)**.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be smooth polarized spherical $G$-varieties, and suppose that $L_1$ and $L_2$ are level. The following are equivalent.*

*(i) There exists a $G$-equivariant isomorphism $i : X_1 \to X_2$ such that $i^*L_2 \cong L_1$ as $G$-linearized invertible sheaves.*

*(ii) $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$ and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

The main downside to using level line bundles is that there exist spherical varieties $X$ such that no $G$-linearized ample invertible sheaf on $X$ is level. However, we will prove in Lemma 5.1.3 and Proposition 5.1.4 that level line bundles do exist for many spherical varieties.

### Considering All Weight Monoids At Once

Rather than trying to pick a single "nice" ample line bundle to work with, we might simply try to use every ample line bundle at once. In fact, in formulating the theory, it is beneficial to work not just with ample line bundles or even with line bundles at all, but with a more general object called a *divisorial sheaf*. This allows us to associate a sheaf to every Weil divisor on a normal variety, which will help in our goal of understanding the relationship between divisors and weight monoids (which arise from sheaves). The theory of divisorial sheaves works quite generally, for any normal variety; we briefly review this theory in Appendix B.

One way to work with the weight monoids of every divisorial sheaf at once would be to consider an action of $G$ on the Cox ring of $X$. This approach has been considered in work by Brion and Gagliardi, see [Bri07], [Gag14], and [Gag19]. For our purposes, we are most interested in what it means for two spherical varieties $X_1$ and $X_2$ to have an "equality" on every weight monoid at once. So, rather than working with Cox rings, we introduce the notion of a $\Lambda^+$-*equivalence* between $X_1$ and $X_2$, which which is essentially the condition of having $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ for every possible choice of $L_1$ and $L_2$. Our main result on $\Lambda^+$-equivalences is the following theorem, which shows that these "equalities" are very closely connected to $\mathcal{D}$-equivalences in nice cases.

**Theorem 1.3.3** (see Corollary 5.4.2)**.** *Let $X_1$ and $X_2$ be spherical $G$-varieties, and suppose that*

1. $\mathrm{Pic}(G) = 0$,
2. *the function fields $K(X_1)$ and $K(X_2)$ are isomorphic as $G$-modules, and*
3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*(For instance, these 3 assumptions hold if $\mathrm{Pic}(G) = 0$ and $X_1$ and $X_2$ are projective, cf. Corollary 5.3.7.) Then, $X_1$ and $X_2$ are $\mathcal{D}$-equivalent if and only if they are $\Lambda^+$-equivalent.*

**Remark 1.3.4.** One benefit of using divisorial sheaves instead of just invertible sheaves when working with $\Lambda^+$-equivalences is that the above theorem (as well as Corollaries 1.3.6 and 1.3.7 below) does not require $X_1$ and $X_2$ to be smooth.

**Remark 1.3.5.** For technical reasons, we will often require the assumptions $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$ and $\mathrm{Pic}(G) = 0$ when working with $\Lambda^+$-equivalences. The former assumption is a standard constraint for certain technical statements about Cox rings, and it does little harm for our purposes, since we are most interested in the case of projective spherical varieties. The assumption that $\mathrm{Pic}(G) = 0$ is also a common constraint when working with line bundles and weight monoids, and it is not terribly restrictive. Indeed, we can always force $\mathrm{Pic}(G) = 0$ after replacing $G$ by $\tilde{G}$ for some finite surjective homomorphism $\tilde{G} \to G$, and such a replacement is perfectly suitable for most applications. For instance, a replacement of $G$ like this occurs in the work of Brion [Bri07] and Gagliardi [Gag19] on Cox rings and weight monoids.

As for the assumption that $K(X_1) \cong K(X_2)$ as $G$-modules in the above theorem, this is simply necessary for the concept of a $\mathcal{D}$-equivalence to make sense. Part of the data of the set $\mathcal{D}_{G,X_i}$ that we wish to consider is the behavior of valuations of divisors, and this behavior occurs in the context of the canonical $G$-module structure on the function field. For more details, see Sections 2.4.a, 3.1.b, and Definition 4.1.1.

It is not hard to show that if $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, then $\Pi^b_{X_1} = \Pi^b_{X_2}$ (see Lemma 4.1.2). So, one particular consequence of Theorem 1.3.3b is that, if $\mathrm{Pic}(G) = 0$ and $X_1$ and $X_2$ are projective and $\Lambda^+$-equivalent, then $\Pi^b_{X_1} = \Pi^b_{X_2}$. Conceptually, this means that the data of $\Lambda^+(X, L)$ for every possible choice of $L$ does encode the information of $\Pi^b_X$, as well as all the information about every $B$-stable Weil divisor on $X$.

Theorem 1.3.3 also allows us to replace the data of "equality" on $\mathcal{D}_{G,X}$ (i.e. a $\mathcal{D}$-equivalence) with the information of a $\Lambda^+$-equivalence in the classification of spherical varieties. More precisely, just as "equality on colored fans" really means a $\mathcal{D}$-equivalence that preserves colored fans, we define a notion of a $\Lambda^+$-equivalence that *preserves colored fans*. The following corollary is then almost immediate from Theorem 1.3.3 and the classification of spherical varieties.

**Corollary 1.3.6** (see Corollary 5.3.11)**.** *Let $X_1$ and $X_2$ be spherical $G$-varieties. Suppose that*

1. $\mathrm{Pic}(G) = 0$,
2. *the function fields $K(X_1)$ and $K(X_2)$ are isomorphic as $G$-modules, and*
3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*(For instance, these 3 assumptions hold if $\mathrm{Pic}(G) = 0$ and $X_1$ and $X_2$ are projective, cf. Corollary 5.3.12.) Then, the following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) $\Psi_{G,X_1} = \Psi_{G,X_2}$, and there exists a $\Lambda^+$-equivalence between $X_1$ and $X_2$ that preserves colored fans.*

### Compatibility with Ample Line Bundles

As dicsussed above, our main classification statement in Corollary 1.2.5 has a natural improvement in the context of level line bundles. We also wish to obtain an analog of this corollary in the context of $\Lambda^+$-equivalences. However, the main methods of proof of the corollary hinge on having ample line bundles $L_1$ and $L_2$ on $X_1$ and $X_2$ such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. A $\Lambda^+$-equivalence gives us many such line bundles, but it does not guarantee that both $L_1$ and $L_2$ will be ample.

To resolve this, we define a notion of a *strong* $\Lambda^+$*-equivalence*, which is essentially a $\Lambda^+$-equivalence that identifies some $G$-linearized ample line bundle on $X_1$ with a $G$-linearized ample line bundle on $X_2$. This guarantees that we do have the necessary condition on some choice of ample line bundle. Analogously, we define a notion of a *strong* $\mathcal{D}$*-equivalence*, which is a $\mathcal{D}$-equivalence that "maps" some ample divisor on $X_1$ to an ample divisor on $X_2$ in some appropriate sense. The statement of Theorem 1.3.3 leads very easily to an analogous statement for strong equivalences:

**Corollary 1.3.7** (see Corollary 5.5.2)**.** *Let $X_1$ and $X_2$ be quasi-projective spherical $G$-varieties. Suppose that $\mathrm{Pic}(G) = 0$ and that $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$. Then, $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent if and only if they are strongly $\Lambda^+$-equivalent.*

In the case of smooth projective spherical varieties, we can use the techniques of Chapter 4 to show that the data of a strong $\mathcal{D}$-equivalence is equivalent to the data of $\Pi_X^b$ and the weight monoid of an ample line bundle. More precisely:

**Theorem 1.3.8** (see Corollary 5.5.11)**.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties. The following are equivalent:*

*(i) $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent.*

*(ii)* $\Pi^b_{X_1} = \Pi^b_{X_2}$*, and there exist* $G$*-linearized ample invertible sheaves* $L_1$ *and* $L_2$ *on* $X_1$ *and* $X_2$ *(respectively) such that* $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$*.*

Combining Theorem 1.3.8 and Corollary 1.3.7, we see that a strong $\mathcal{D}$-equivalence (hence also a strong $\Lambda^+$-equivalence when $\mathrm{Pic}(G) = 0$) gives no more information than an equality on the sets $\Pi^b_{X_i}$ and on the weight monoids $\Lambda^+(X_i, L_i)$ for some $G$-linearized ample line bundles $L_i$. Thus, we have the following analog of Corollary 1.2.5.

**Corollary 1.3.9** (see Corollaries 5.5.3 and 5.5.4)**.** *Let* $X_1$ *and* $X_2$ *be smooth projective spherical* $G$*-varieties. The following are equivalent.*

*(i)* $X_1$ *and* $X_2$ *are* $G$*-equivariantly isomorphic.*

*(ii)* $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$*, and* $X_1$ *and* $X_2$ *are strongly* $\mathcal{D}$*-equivalent.*

*(iii)* $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$*, and* $X_1$ *and* $X_2$ *are strongly* $\Lambda^+$*-equivalent.*

*If* $\mathrm{Pic}(G) = 0$*, these are also equivalent to the following condition.*

*(iv)* $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$*, and* $X_1$ *and* $X_2$ *are strongly* $\Lambda^+$*-equivalent.*

We noted above that Example 4.9.4 will give for every possible exceptional spherical root $\gamma$ an example of some $X_1$ and $X_2$ smooth and projective and some $G$-linearized ample invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\gamma \in \Psi^{exc}_{G,X_1}$ but $\gamma \notin \Psi^{exc}_{G,X_2}$. This shows that the assumption $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$ in Corollary 1.2.5 is necessary. Likewise, our examples in Example 4.9.4 will actually be strongly $\mathcal{D}$-equivalent, and $\mathrm{Pic}(G) = 0$ in those examples, so they are strongly $\Lambda^+$-equivalent as well (by Corollary 1.3.7 above). Thus, these examples indicate that the equalities on sets of exceptional spherical roots in Corollary 1.3.9 are also necessary.

## 1.4 Applications to Hamiltonian Manifolds

Some of our main results (namely, Corollaries 1.2.5 and 1.3.9) are in some sense partial analogs of the Knop conjecture in the projective setting. They are thus related to certain problems on Hamiltonian manifolds, in a similar way to the Knop conjecture. We briefly discuss this connection here, drawing primarily on Losev's discussion of Hamiltonian manifolds and the Knop conjecture in [Los09a].

Let $K$ be a connected compact Lie group with Lie algebra $\mathfrak{k}$, and fix a Cartain subalgebra $\mathfrak{t} \subset \mathfrak{k}$ and a positive Weyl chamber $\mathfrak{t}^+ \subset \mathfrak{t}$. An action of $K$ on a symplectic manifold $(M, \omega)$ is said to be *Hamiltonian* if there exists a *moment map*, i.e. a $K$-equivariant map $\mu : M \to \mathfrak{k}^*$ such that

$$\omega(\xi_* x, v) = \langle d_x \mu(v), \xi \rangle \ \forall \ x \in M, \ \xi \in \mathfrak{k}, \ v \in T_x M.$$

We call the triple $(M, \omega, \mu)$ a *Hamiltonian manifold*. Choosing a $K$-invariant inner product on $\mathfrak{k}$, we may identify $\mathfrak{k}$ with $\mathfrak{k}^*$. Under this identifiaction, a theorem of Kirwan tells us that

$$\Delta(M) = \mu(M) \cap \mathfrak{t}_+$$

is a convex polytope, called the *moment polytope* of $M$.

We say that a Hamiltonian $K$-manifold $(M, \omega, \mu)$ is *multiplicity-free* if a general $K$-orbit in $M$ is a coisotropic submanifold of $M$. For a compact multiplicity-free Hamiltonian $K$-manifold, it turns out that every fiber of the map $\psi(x) = K\mu(x) \cap \mathfrak{t}_+$ is a single $K$-orbit. Since $\mathfrak{t}_+$ can be identified with the orbit space $\mathfrak{k}^*/K$, we may view $\psi$ as an invariant moment map, and the condition that $M$ is multiplicity-free tells us that $\psi$ induces a homemorphism between the orbit space $M/K$ and the moment polytope $\Delta(M) = \psi(M)$.

An important problem in symplectic geometry is to classify all compact multiplicity-free Hamiltonian manifolds. One important piece of data is the moment polytope $\Delta(M)$. Another is the so-called *principal isotropy subgroup* of $M$, which is defined to be $K_x$ for any $x \in \mu^{-1}(\eta)$, where $\eta$ is a general element of $\Delta(M)$. It turns out that $K_x$ does not depend on the choice of $x$ or $\eta$ and so constitutes an invariant of $M$. Delzant conjectured that the pair $(\Delta(M), K_x)$ classifies a compact multiplicity-free Hamiltonian manifold $M$ up to $K$-equivariant symplectic diffeomorphism.

The Delzant conjecture was proven by Knop [Kno11]. To do this, Knop used the fact that Multiplicity-free Hamiltonian manifolds are in some sense a symplectic analog of spherical varieties. Indeed, there is a very precise way in which one can locally model a multiplicity-free Hamiltonian manifold by an affine spherical variety. This allows one to reduce a local version of the Delzant conjecture to the statement of the Knop conjecture; Knop then used this local version to prove the Delzant conjecture globally.

The Delzant conjecture suggests that we would not expect to obtain as nice a statement in the projective case as we do in the affine case. Indeed, the weight monoid $\Lambda^+(X, L)$ can be used to define a polytope (see e.g. [Bri97, Section 5.3]), which is the algebro-geometric analog of the moment polytope for compact multiplicity-free Hamiltonian manifolds. However, in the analytic setting, Delzant's conjecture requires not just the moment polytope but also the principal isotropy subgroup to classify the manifolds under consideration. On the other hand, our Theorem 1.2.4 already suggests that the algebraic setting of projective spherical varieties is nicer than the analytic setting of compact multiplicity-free Hamiltonian manifolds. Indeed, Theorem 1.2.4 indicates that a projective spherical variety $X$ is determined up to $G$-equivariant isomorphism by a weight monoid $\Lambda^+(X, L)$ and the set of spherical roots $\Psi_{G,X}$, and the data of the spherical roots $\Psi_{G,X}$ seems intuitively "weaker" than the data of the entire principal isotropy subgroup $K_x$. For comparison, the set of spherical roots $\Psi_{G,X}$ alone does *not* uniquely determine the isotropy subgroup $G_x$ for a general point $x$ in a spherical variety $X$: instead, the classification of spherical varieties indicates that $G_x$ is determined up to conjugation by $\Psi_{G,X}$ along with (some of) the data of $\mathcal{D}_{G,X}$. Corollary 1.2.5 further improves the situation in Theorem 1.2.4, because it asserts that when $X$ is smooth, we need not even remember the entire set $\Psi_{G,X}$ but only the set $\Psi_{G,X}^{exc}$ of exceptional spherical roots.

We can thus regard our main results in Chapter 4 as evidence that there is a somewhat nicer structure to the combinatorial data of $\Lambda^+(X, L)$ in the algebro-geometric case than there is for the analogous data in the setting of symplectic geometry.

## 1.5 Outline of the Thesis

This thesis is organized as follows. In Chapters 2 and 3, we discuss in detail the classification of spherical varieties. These chapters are purely expositional. Chapter 2 contains basic results on reductive groups and $G$-varieties. This is meant to serve as a brief introduction to the theory of reductive groups, which is sufficient to understand the rest of the thesis but is also accessible to the reader with only a little exposure to the theory of algebraic groups. Chapter 3 then builds up the whole theory of spherical varieties, from the definition of a spherical variety to the full combinatorial classification of these varieties. We give a very thorough and technically precise treatment of the theory in this chapter. For the reader who is not interested in studying the theory of spherical varieties in this depth, we provide a suggestion at the beginning of Chapter 3 for how to read this chapter in order to gain just enough understanding to be able to read the chapters that follow it.

After giving a thorough treatment of the theory, we give our original work in Chapters 4 and 5. In Chapter 4, we develop our results on weight monoids of ample line bundles, which we discussed in Section 1.2 above. In Chapter 5, we develop our results on level line bundles and $\Lambda^+$-equivalences, which we discussed in Section 1.3 above.

## 1.6 Conventions and Notation

We end this introduction with a few conventions. Throughout, $k$ will denote a field and $G$ will denote an algebraic group over $k$. Except where we explicitly state otherwise, we assume that $k$ is an algebraically closed field of characteristic 0 and that $G$ is a (connected) reductive group. We will also typically assume given some choice of maximal torus $T \subset G$ and Borel subgroup $B \subset G$ containing $T$. For information on what these terms mean and why it is okay to fix some choice of $T$ and $B$, see Sections 2.2 and 2.3.

Everything we say about schemes is assumed to take place in the category of finite-type $k$-schemes (or some appropriate subcategory), i.e. all schemes will be assumed to be of finite type over $k$, morphisms are assumed to be morphisms of $k$-schemes, the product $X \times Y$ denotes the fiber product $X \times_{\mathrm{Spec}(k)} Y$, etc. When referring to points of $G$ (or of schemes with an action of $G$), we mean closed points (unless context clearly indicates otherwise). For instance, $g \in G$ denotes a closed point $g \in G(k)$.

Given an invertible sheaf $L$ on a scheme $X$, we write $\Gamma_*(X, L)$ for the graded ring $\oplus_{n \geq 0} H^0(X, L^{\otimes n})$. When working with this graded ring in the context of affine cones, we are often interested in the group $\tilde{G} = G \times \mathbb{G}_m$ (see Section 2.5 and Appendix A for details). In other places, however, the group $\tilde{G}$ often denotes a finite cover of $G$, i.e. an algebraic group equipped with an isogeny $\tilde{G} \to G$. Because of these two different uses of the notation $\tilde{G}$, we will always explicitly state what $\tilde{G}$ means wherever we use it.

**Definition 1.6.1.** A *variety* is an integral, separated, finite-type scheme over $k$. A $G$-*variety* is a variety equipped with an action of $G$. A $G$-scheme is a (finite-type) scheme over $k$ equipped with an action of $G$.

When working with a lattice (i.e. a free finitely generated abelian group) $X$, we will write $X_{\mathbb{Q}}$ and $X_{\mathbb{R}}$ for the vector spaces $X \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X \otimes_{\mathbb{Z}} \mathbb{R}$ (respectively) and $X^\vee$ for the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$.

For any algebraic group $G$, we denote the derived subgroup of $G$ by $[G, G]$. For any subgroup $H \subset G$, we write $C_G(H)$ and $N_G(H)$ for the centralizer and normalizer (respectively) of $H$ in $G$, and we write $Z(G)$ for the center of $G$. Also, we denote all Lie alebras with lowercase letters in the Fraktur font, e.g. $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{h}$ is the Lie algebra of $H$. Moreover, for any $H$-scheme $Z$, we denote by $G \times^H Z$ the homogeneous fiber bundle over $G/H$ whose fibers are $Z$. More precisely, $G \times^H Z$ is the quotient $(G \times Z)/H$, where $H$ acts on $G \times Z$ by $h(g, z) = (gh^{-1}, hz)$. This quotient can be defined rigorously as an algebraic space in a natural way, or as a ringed space (as in [Tim11, Section 2.1]); however, it turns out to be a $k$-scheme under relatively weak conditions on $Z$ (see e.g. [Tim11, Theorem 2.2]). In particular, the homogeneous fiber bundle $G \times^H Z$ will be a $k$-scheme in every circumstance in which we use it. For more information about $G \times^H Z$, we refer the reader to [Tim11, Section 2.1].

# Chapter 2

# Reductive Groups and $G$-varieties

In this chapter, we review backround material concerning reductive groups and varieties with an action of a reductive group. We assume that the reader is familiar with basic definitions of algebraic groups and their representations but is not necessarily familiar with the theory of algebraic groups or the theory of reductive groups. A fantastic reference for all of these topics (and much more) is [Mil17]. The reader with less background than we assume here may wish to start with Brion [Bri10], which assumes no knowledge of algebraic groups and is geared specifically towards spherical varieties. Other standard and useful references include [Bor91] and SGA 3 (particularly Exposé 26 [Dem66], which gives a much more in-depth treatment of parabolic subgroups than most sources).

This chapter is included mainly for the reader's convenience and to define notation. In particular, many of the propositions are primarily included so that they can be referred to for the occasional technical argument. The reader familiar with reductive groups, group actions, and $G$-linearizations may wish to skim through the definitions and results in Sections 2.5 and 2.6, as these will be the most useful sections later on, and then refer back to the rest of the chapter as needed. For those particularly versed in the standard results on reductive group actions and $G$-linearizations, it may even be sufficient just to consult the Index of Notation.

## 2.1 Algebraic Groups

We begin by collecting some facts about algebraic groups. These are all fairly standard, but they will be useful to refer back to for certain arguments later. We have tried to state these facts in as much generality as possible, simply for precision. However, we will almost always be working in the case where $k$ is of characteristic 0 and algebraically closed and $G$ is affine, hence also smooth ([Mil17, Theorem 3.23]). Thus, most of the assumptions we impose in the results below will always be satisfied for our purposes.

First, we have a standard fact about dimensions of quotients of algebraic groups.

**Lemma 2.1.1.** *Let $G$ be an algebraic group, and let $H \subset G$ be an algebraic subgroup. Then,*

$$\dim(G/H) = \dim(G) - \dim(H).$$

*Proof.* The quotient morphism $\pi : G \to G/H$ is given on $k$-points by $g \mapsto gH$. For any coset $g_0H \in G/H$, we have $\pi(g) = g_0H$ if and only if $g_0h = g$ for some $h \in H$. Thus, the fiber $\pi^{-1}(g_0H)$ is the subvariety $g_0H \subset G$, which has dimension $\dim(H)$. Since every fiber of $\pi$ has dimension $\dim(H)$ and $\pi$ is surjective ([Mil17, Theorem B.37]), it is a general (scheme-theoretic) fact (see e.g. [TY05, Corollary 15.5.5]) that

$$\dim(G) = \dim(G/H) + \dim(H),$$

and the statement now follows. □

Next, we consider some important facts about orbits of algebraic group actions.

**Proposition 2.1.2** ([Mil17, Propositions 7.4, 7.17], [Bor91, Proposition 1.8])**.** *Let $k$ be a field, let $X$ be a separated scheme of finite type over $k$, and let $G$ be a smooth algebraic group acting on $X$. Let $x \in X(k)$ be a $k$-point.*

*(a) The orbit $Gx$ of $x$ is a smooth, $G$-stable locally closed subscheme of $X$.*

*(b) We have $Gx \cong G/G_x$, where $G_x$ is the stablizer of $x$. In particular, if $G$ is irreducible, then $Gx$ is geometrically irreducible (hence so is $\overline{Gx}$).*

*(c) The boundary $\overline{Gx} \setminus Gx$ of the orbit closure is a union of $G$-orbits of dimension strictly smaller than* $\dim(Gx)$*.*

*(d) Any $G$-orbit of $X$ of minimum dimension is closed, and every $G$-orbit contains a closed orbit in its closure.*

*Proof.* See [Mil17, Proposition 7.4] for (a) and [Mil17, Proposition 7.17] for (b). (The subtleties here are primarily that the quotient $G/G_x$ exists and that the canonical map $G/G_x \to X$ is an immersion. These certainly hold when $G$ is smooth, but much can be said when $G$ is not smooth as well; see [Mil17, Chapter 7 and Appendix B] for details. Also, if $G$ is irreducible, then the image of $G$ under the quotient morphism $G \to G/G_x$ is irreducible. But the quotient morphism is surjective, so $G/G_x \cong Gx$ is irreducible.)

For (c), we note that since $Gx$ is $G$-stable by (a), the closure $\overline{Gx}$ is $G$-stable as well. (Proof: any closed subset $Y \subset X$ contains $Gx$ if and only if $gY$ is a closed subset of $X$ containing $gGx = Gx$ for all $g \in G$. Thus, every element of $G$ permutes the closed subsets containing $Gx$ and hence fixes their intersection, which is $\overline{Gx}$.) Hence, the boundary $B = \overline{Gx} \setminus Gx$ is $G$-stable, so it is a union of $G$-orbits. Since $\overline{Gx}$ is irreducible and $Gx$ is open in $\overline{Gx}$ by (a), $B$ is of strictly smaller dimension than $\overline{Gx}$ by definition of the Krull dimension (any chain of irreducible closed subsets of $B$ is a chain of irreducible closed subsets of $\overline{Gx}$, and we can add $\overline{Gx}$ to this chain to get a strictly larger chain). The fact that $Gx$ is open in its closure also gives us $\dim(Gx) = \dim(\overline{Gx})$, so this proves (c).

As for (d), note that if $Gx$ is an orbit of minimal dimension, then (c) gives us $\overline{Gx} \backslash Gx = \varnothing$ and hence $Gx = \overline{Gx}$. The proof that $\overline{Gx}$ always contains a closed orbit is by induction on $\dim(Gx)$. In the base case, we have just seen that any orbit over minimum dimension is closed. For the inductive step, the boundary $\overline{Gx} \setminus Gx$ is a union of orbits of dimension strictly less than that of $Gx$ (by statement c). By the induction hypothesis, these orbits each contain a closed orbit in their closure, so $Gx$ does as well. □

The following lemma collects a few topological facts about algebraic group actions. We could not find a reference that proves statements (a) and (b) in the lemma, so we have supplied proofs here.

**Lemma 2.1.3** ([Tim11, Proposition 2.7])**.** *Let $X$ be a scheme of finite type over a field $k$, and let $G$ be a smooth connected algebraic group over a field $k$.*

*(a) Every irreducible component of $X$ is $G$-stable.*

*(b) Let $U \subset G$ be an open subset of $G$. For any $x \in X(k)$, the image of $U \times \{x\}$ under the action morphism $G \times X \to X$ is an open subscheme of the orbit $Gx$.*

*(c) Let $P \subset G$ be a subgroup such that $G/P$ is complete (i.e. a parabolic subgroup, see Definition 2.2.5 below), and let $Z \subset X$ be a $P$-orbit. If $Z$ is closed, then $GZ \subset X$ is a closed $G$-orbit.*

*Proof.* Let $Z$ be an irreducible component of $X$, and let $z \in Z$ be a $k$-point. Define the closed subscheme $G_{Z,z} \subset G$ to be the fiber product fitting into the following Cartesian diagram:

$$\begin{tikzcd} G_{Z,z} \arrow[r] \arrow[d] & Z \arrow[d, hook] \\ G \arrow[r, "\mu_z"] & X \end{tikzcd}$$

Here, $\mu_z$ denotes the orbit map, i.e. the composition

$$G \cong G \times \{z\} \hookrightarrow G \times X \overset{(g,x)\mapsto gx}{\to} X.$$

Then, we may consider the closed subscheme

$$G_Z = \bigcap_{z \in Z(k)} G_{Z,z} \subset G.$$

On functors of points, we have

$$G_Z(T) = \{g \in G(T) \mid g \cdot z \in Z \ \forall z \in Z\}.$$

To complete the proof of (a), it suffices to show that $G_Z = G$. For this, note that any $g \in G(k)$ defines an automorphism $X \to X$ (with inverse $g^{-1}$), which must permute the irreducible components of $X$. So, the cosets $gG_Z$ are all of the form $G_{Z'}$ for some irreducible component $Z' \subset X$ (specifically, if $g \cdot Z = Z'$, then $gG_Z = G_{Z'}$). Since $X$ has finitely many irreducible components, we see that $G/G_Z$ is a finite group and hence that $\dim(G_Z) = \dim(G)$. $G$ is irreducible (because it is connected, see [Mil17, Corollary 1.35]), and the connected component of the identity $G_Z^0 \subset G_Z$ is likewise irreducible (note that our above expression for $G_Z(T)$ shows that $G_Z$ is in fact an algebraic subgroup of $G$, so [Mil17, Corollary 1.35] applies to $G_Z$ as well). So, $G_Z^0 \subset G$ are two irreducible closed subsets of $G$ with

$$\dim(G_Z^0) = \dim(G_Z) = \dim(G).$$

(For the first equality here, we note that the connected components of $G_Z$ are its irreducible components by [Mil17, Corollary 1.35]. These irreducible components are permuted by the action of $G_Z$ on itself by left multiplication, so they all have the same dimension, and one of them must have the dimension of $G_Z$ by definition.) We conclude that $G_Z^0 = G$, as desired.

For statement (b), let $\rho : G \times X \to X$ be the action morphism. We have the following commutative diagram:

$$\begin{tikzcd}
G \times X \arrow[r, "\rho"] \arrow[d, "(g,x)\mapsto(g,gx)"'] & X \arrow[d, equal, "\mathrm{id}_X"] \\
G \times X \arrow[r, "\mathrm{pr}_X"] & X
\end{tikzcd}$$

The projection map $\mathrm{pr}_X$ is the base change of the structure morphism $G \to \mathrm{Spec}(k)$ and hence is faithfully flat and finitely presented. Since the vertical arrows in the diagram are isomorphisms, we conclude that $\rho$ is faithfully flat and finitely presented as well. In particular, $\rho$ is an open map, hence so is its restriction to the subset $G \times \{x\} \subset G \times X$. Since $U \times \{x\}$ is open in $G \times \{x\}$, the image $\rho(U \times \{x\})$ is open in $\rho(G \times \{x\})$. But $\rho(G \times \{x\})$ is the orbit $Gx$ by definition, so we are done.

Statement (c) is [Tim11, Proposition 2.7]. We repeat the proof given there (but with some details made more explicit). Consider the action of $P$ on $G \times X$ given by $p \cdot (g, x) = (gp^{-1}, px)$. Then, $G \times Z$ is a $P$-stable closed subscheme of $G \times X$. The quotient $(G \times X)/P$ is the homogeneous fiber bundle $G \times^P X$; since $X$ has an action of $G$, we have $G \times^P X \cong G/P \times X$, with the isomorphism given by $(g, x) \mapsto (gP, gx)$. In particular, this quotient is a scheme (see e.g. [Tim11, Section 2.1]). Since $P$ acts freely on $G \times Z$, the quotient $G \times^P Z = (G \times Z)/P$ is an algebraic space, which (by the universal property of the quotient) admits a map $i : (G \times Z)/P \to G/P \times X$ such that the following diagram commutes:

$$\begin{tikzcd}
G \times Z \arrow[r] \arrow[d, "(g,z)\mapsto(g,z)"'] & (G \times Z)/P \arrow[d, "i"] \\
G \times X \arrow[r, "(g,x)\mapsto(gP,gx)"] & G/P \times X
\end{tikzcd}$$

In fact, one can check on functors of points that this diagram is Cartesian. Since The map $G \times Z \to (G \times Z)/P$ is a cover in the fppf topology and the map $G \times Z \hookrightarrow G \times X$ is a closed immersion, we see that $i$ is a closed immersion of an algebraic space into a scheme. This implies that $(G \times Z)/P$ is in fact a scheme, and $i$ is a closed immersion of schemes.

Now, we consider the composition

$$G \times Z \to (G \times Z)/P \overset{i}{\hookrightarrow} G/P \times X \overset{\pi_2}{\to} X.$$

By commutativity of the above diagram, the image of this composition is the $G$-orbit $GZ$. This image is also the image of $\pi_2 \circ i$, because the map $G \times Z \to (G \times Z)/P$ is surjective. But $\pi_2$ is proper (because $G/P$ is proper over $k$) and $i$ is also proper, so $GZ = \mathrm{Im}(\pi_2 \circ i)$ is closed in $X$. □

The following statement follows from the so-called Borel fixed point theorem, a standard theorem in the theory of reductive groups.

**Proposition 2.1.4** (cf. [Mil17, Corollary 17.3])**.** *Let $X$ be a scheme of finite type over a field $k$, and let $G$ be an algebraic group over $k$. Suppose that $k$ is algebraically closed and that $G$ is affine, and let $B \subset G$ be a Borel subgroup (see Definition 2.2.5 for a definition). Let $Y \subset X$ be a (nonempty) $G$-orbit that is proper over $k$. Then, $Y$ contains a unique $B$-fixed point.*

*Proof.* The Borel fixed point theorem (see e.g. [Mil17, Corollary 17.3]) implies the existence of a $B$-fixed point $y$ in $Y$. For uniqueness, let $y$ and $y'$ be any two fixed points of $B$ in $Y$. In particular, there exists some $g \in G$ such that $gy = y'$. We then have $g^{-1}G_{y'}g = G_y$. On the other hand, both $G_y$ and $G_{y'}$ contain $B$ and hence are parabolic, and every parabolic subgroup of $G$ is conjugate to a unique parabolic subgroup containing $B$ (see [Bor91, Corollary 11.17]). So, we have $G_{y'} = G_y$ and hence $g^{-1}G_y g = G_y$. Since a parabolic subgroup is equal to its own normalizer (see [Mil17, Corollary 17.49] or [Bor91, Theorem 11.16]), we have $g \in G_y$ and hence $y = gy = y'$. □

We now recall a standard theorem due to Cartier, which is significant for our purposes because it will tell us that all algebraic groups we intend to work with are smooth.

**Theorem 2.1.5** ([Mil17, Theorem 3.23])**.** *Let $G$ be an algebraic group over a field $k$. If $\mathrm{char}(k) = 0$ and $G$ is affine, then $G$ is smooth.*

We end this section with a couple technical lemmas about dense open subsets. We record them here mainly for later use. We remark that Lemma 2.1.7 is sometimes implicitly used in the literature (see e.g. [Mil17, proof of Proposition 22.17]), but we could not find a proof in the literature, so we have supplied a proof here.

**Lemma 2.1.6.** *Let $G$ be a reduced algebraic group over a field $k$, and let $X$ be a reduced and separated $G$-scheme. If $G$ acts trivially on a dense open subset of $X$, then $G$ acts trivially on $X$.*

*Proof.* Let $X^\circ \subset X$ be the dense open subset on which $G$ acts trivially The action morphism $G \times X \to X$ agrees with the projection morphism on the dense open subset $G \times X^\circ \subset G \times X$. Since $G \times X$ is reduced and $X$ is separated, it follows that the action morphism and the projection morphism are equal (see e.g. [Har77, Exercise II.4.2]). $\square$

**Lemma 2.1.7.** *Let $G$ be an algebraic group over a field $k$. Let $V$ be a $G$-module, and suppose that $f \in V$ is an element which generates $V$ as a $G$-module. For any dense open subset $U \subset G$, the vector space $V$ is spanned by the elements $u \cdot f$ for $u \in U(k)$.*

*Proof.* Let $W \subset V$ be the subspace spanned by the elements $u \cdot f$ for $u \in U(k)$. We show that $W = V$. By definition, giving a $G$-module structure on $V$ is the same as giving an action of $G$ on the scheme $\mathbb{A}(V)$ whose functor of points is $\mathrm{Spec}(R) \mapsto V \otimes_k R$ (c.f. [Mil17, Section 4.a]). Explicitly, the scheme $\mathbb{A}(V)$ is given by $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}^\bullet(V^*))$, so the inclusion map $W \subset V$ gives rise to a surjection $V^* \to W^*$ and hence a closed immersion $\mathbb{A}(W) \hookrightarrow \mathbb{A}(V)$. On functors of points, this closed immersion is just the natural inclusion $W \otimes_k R \hookrightarrow V \otimes_k R$.

Now, we have a morphism of schemes $\psi_f : G \to \mathbb{A}(V)$ which is given on functors of points by sending any $g \in G(\mathrm{Spec}(R))$ to the element $g \cdot (f \otimes 1) \in V \otimes_k R$. The preimage $\psi_f^{-1}(\mathbb{A}(W))$ is a closed subset of $G$ which contains $U$ (by definition of $W$). Since $U$ is dense in $G$, it follows that $\psi_f^{-1}(\mathbb{A}(W)) = G$. By definition of $\psi_f$, this implies that $g \cdot f \in W$ for all $g \in G(k)$. Since $V$ is spanned by the $g \cdot f$ for $g \in G(k)$, this proves that $W = V$. $\square$

## 2.2 Reductive Groups

In this section, we summarize the theory of reductive groups. For simplicity, we only treat the case where $k$ is algebraically closed and $\mathrm{char}(k) = 0$. The assumption that $k$ is algebraically closed can be dropped by proving that every part of the theory works for the base change of a reductive group to some suitable field extension $K \supset k$. The assumption that $\mathrm{char}(k) = 0$ is actually not very relevant at all, except for one detail about isogenies (see Remark 2.2.10 below). We refer the reader to [Mil17] for a treatment of the material in this section over an arbitrary base field $k$.

We also assume (in this section and in every section after it) that every algebraic group $G$ is affine (hence so is every algebraic subgroup $H \subset G$, since subgroups are closed subschemes). Taking $G$ to be affine is a standard assumption in the theory of algebraic groups. This is in part because the behavior of all algebraic groups can largely be reduced from the behavior of affine algebraic groups and abelian varieties (see [Mil17, Chapter 8]). One significant consequence of being in the affine setting is that every algebraic subgroup of $G$ is smooth if $\mathrm{char}(k) = 0$, see Theorem 2.1.5.

### 2.2.a The Classification of Reductive Groups

We first recall several definitions from the theory of algebraic groups.

**Definition 2.2.1.** Let $G$ be an algebraic group.

1. We say that $G$ is a *torus* if $G$ is isomorphic to $\mathbb{G}_m^n$ for some $n$ (i.e. is isomorphic to a product of $n$ copies of $\mathbb{G}_m$ for some $n$).
2. We say that $G$ is *unipotent* if every irreducible representation of $G$ is a one-dimensional space on which $G$ acts trivially (i.e. is isomorphic to the trivial representation).
3. We say that $G$ is *solvable* if there exists a series of subgroups
$$G = G_0 \supset G_1 \supset \cdots \supset G_r = 1$$
such that $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is commutative for all $i$.
4. If $G$ is connected, we define the *unipotent radical* of $G$, denoted $R_u(G)$, to be the largest connected normal unipotent subgroup of $G$ (this group exists when $G$ is connected and contains every other connected normal unipotent subgroup of $G$).
5. We say that $G$ is *reductive* if $G$ is connected and $R_u(G) = 1$, i.e. if $G$ contains no nontrivial connected normal unipotent subgroups.

**Remark 2.2.2.** Some authors do not require reductive groups to be connected by definition. However, connectedness is a useful assumption for certain technical reasons in the theory, and the theory for non-connected reductive groups essentially reduces to the theory for connected reductive groups by taking connected components. As such, little is lost by imposing connectedness in the definition of a reductive group.

Tori and unipotent algebraic groups are relatively well-behaved and straightforward to study; see e.g. [Mil17, Chapters 12 and 14]. In the cases of interest to us, the structure of solvable groups is also quite nice.

**Theorem 2.2.3** ([Mil17, Propositions 16.52, 16.53 and Theorems 16.6, 16.26, 16.33])**.** *Let $G$ be a smooth connected affine algebraic group over an algebraically closed field $k$ (of arbitrary characteristic). The following are equivalent:*

*(i) $G$ is solvable.*

*(ii) $G$ is isomorphic to a subgroup of the upper triangular $n \times n$ matrices for some $n$.*

*(iii) The quotient $G/R_u(G)$ is a torus, and the exact sequence*
$$1 \to R_u(G) \to G \to G/R_u(G) \to 1$$
*is split.*

*(iv) For some (equivalently, any) torus $T \subset G$ which is maximal (under the partial order given by containment), we have*
$$G = R_u(G) \rtimes T \cong R_u(G) \times T.$$

*Proof.* For the equivalence of (i) and (ii), see [Mil17, Proposition 16.52, Corollary 16.53], and for the equivalence of (ii) and (iii), see [Mil17, Section 12.e, Theorems 16.6, 16.26]. (To translate these results from Milne to the form stated here, we have used the assumption that $k$ is algebraically closed, the definition of the unipotent radical $R_u(G)$, and the fact that $G/R_u(G)$ is connected when $G$ is connected, because the image of the quotient map $\pi : G \to G/R_u(G)$ is connected and $\pi$ is surjective.) The equivalence of (iii) and (iv) is completely formal, using the fact that $R_u(G) \cap T = \{1\}$ (see [Mil17, Corollary 14.17]) and the fact that every choice of maximal torus $T$ is conjugate (see [Mil17, Proposition 16.33d]). $\square$

It turns out that unipotent and solvable groups (hence also tori, which are solvable) have particularly nice orbits. We go ahead and record the relevant result, which we will make use of later.

**Theorem 2.2.4** ([Tim11, Lemma 3.4 and Theorem 3.5])**.**

*(a) The orbits of a unipotent group on an affine variety are all closed (and in particular affine).*

*(b) If $G$ is solvable, then $G/H$ is affine for any subgroup $H \subset G$. In particular, the orbits of a smooth solvable group on any scheme are affine.*

To study the structure of reductive groups, one looks at certain nice subgroups of $G$, which we can largely understand using our understanding of tori, unipotent groups, and solvable groups.

**Definition 2.2.5.** Let $G$ be a connected affine algebraic group over $k$.

1. The *character group* of $G$ is the group $\mathcal{X}(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ (here Hom denotes morphisms of algebraic groups). Equivalently, $\mathcal{X}(G)$ is the group of one-dimensional representations of $G$. Elements of $\mathcal{X}(G)$ are called *characters*.

2. A *maximal torus* of $G$ is a torus $T \subset G$ which is maximal (with respect to the partial order induced by inclusion of algebraic subgroups).

3. A *Borel subgroup* of $G$ is a connected solvable algebraic subgroup $B \subset G$ which is maximal among connected solvable algebraic subgroups.

4. A *parabolic subgroup* of $G$ is a subgroup $P \subset G$ which contains a Borel subgroup of $G$ (this is equivalent to saying that $P$ is a subgroup such that $G/P$ is a complete variety, see [Mil17, Theorem 17.16]).

Our plan is to pick a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ such that $T \subset B$, and then obtain certain nice combinatorial data using $T$ and $B$. We first make a few notes on such choices of $T$ and $B$. Note that the existence of Borel subgroups and maximal tori is immediate from the definitions (though these groups could conceivably be trivial or all of $G$). Moreover, since tori are connected solvable groups, every maximal torus is contained in a Borel subgroup. On the other hand, it turns out that every Borel subgroup is conjugate ([Mil17, Theorem 17.9]). Since there exists some Borel subgroup containing a maximal torus, it follows that every subgroup contains a maximal torus. Moreover, any two maximal tori are also conjugate and in fact, for any two choices $T \subset B \subset G$ and $T' \subset B' \subset G$ of a maximal tori $T$ and $T'$ and Borel subgroups $B$ and $B'$, there exists an element $g \in G(k)$ such that we have both $gBg^{-1} = B'$ and $gTg^{-1} = T'$ ([Mil17, Theorem 17.13]). In summary, the pairs $(T, B)$ consisting of a maximal torus $T$ and a Borel subgroup $B$ such that $T \subset B \subset G$ are all conjugate, and every maximal torus and Borel subgroup is in some such pair. In particular, we may always choose $T \subset B \subset G$, and any two such choices are essentially equivalent (in the sense that there exists an inner automorphism of $G$ sending any choice $T \subset B \subset G$ to any other choice $T' \subset B' \subset G$).

For the moment, we will focus on combinatorial data arising from the maximal torus $T$; we will return to the data of Borel subgroups (and, more generally, parabolic subgroups) later on. Let $G$ be a reductive group, and let $T \subset G$ be a maximal torus. Write $\mathfrak{g}$ for the Lie algebra of $G$, and let

$$\mathrm{Ad} : G \to \mathrm{GL}_{\mathfrak{g}}$$

be the adjoint representation. By restricting Ad to $T$, we may view $\mathfrak{g}$ as a representation of $T$. Since any representation of a torus decomposes into a direct sum of one-dimensional representations ([Mil17, Theorem 12.12]), we obtain a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathcal{X}(T) \text{ nontrivial}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the subrepresentation of $\mathfrak{g}$ on which $T$ acts by the character $\alpha$, and $\mathfrak{g}_0$ is the subrepresentation on which $T$ acts trivially. Since $\mathfrak{g}$ is finite-dimensional, only finitely many of the $\mathfrak{g}_\alpha$ are nonzero. We write

$$\Phi = \Phi(G, T) = \{\alpha \in \mathcal{X}(T) \text{ nontrivial} \mid \mathfrak{g}_\alpha \neq 0\},$$

and we call elements of $\Phi(G, T)$ *roots* of the pair $(G, T)$.

The set of roots $\Phi(G, T)$ has a very nice combinatorial structure. To explain, we first define the *Weyl group* of the pair $(G, T)$ to be

$$W(G, T) = N_G(T)/C_G(T),$$

where $N_G(T)$ and $C_G(T)$ denote the normalizer and centralizer (respectively) of $T$ in $G$. It turns out that $W(G, T)$ is a finite constant group, i.e. it is the algebraic group given by a functor which sends every finite-type $k$-scheme to some fixed finite group $W$ (see [Mil17, Proposition 21.1]). Note that any $g \in N_G(T)$ induces an automorphism of $T$ (by conjugation), which in turn induces an automorphism of $\mathcal{X}(T)$ (explicitly, this automorphism maps a character $\lambda : T \to \mathbb{G}_m$ to the character given by $t \mapsto \lambda(gtg^{-1})$). This defines an action of $N_G(T)$ on $\mathcal{X}(T)$, and $C_G(T) \subset N_G(T)$ is precisely the subgroup that acts trivially on $\mathcal{X}(T)$.

Thus, this action descends to a faithful action of $W(G,T)$ on $\mathcal{X}(T)$. Note also that $\mathcal{X}(T)$ is a free finitely generated abelian group (explicitly, we have $\mathcal{X}(\mathbb{G}_m^n) \cong \mathbb{Z}^n$, see [Mil17, Sections 12.b and 12.e]). Hence, we may consider the dual group $\mathcal{X}(T)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}(T), \mathbb{Z})$, which is also a free finitely generated abelian group on which $W(G,T)$ acts faithfully.

Here is the main theorem on the structure of the roots $\Phi(G,T)$.

**Theorem 2.2.6** ([Mil17, Theorems 21.2, 21.11], see also [Mil17, Corollaries 17.59, 21.12])**.** *Let $G$ be a connected reductive group, let $T \subset G$ be a maximal torus, and let $\alpha \in \Phi(G,T)$ be a root. Let $T_\alpha = \ker(\alpha)^0$ (the connected component of the subscheme $\ker(\alpha) \subset T$), and let $G_\alpha = C_G(T_\alpha)$ be the centralizer of $T_\alpha$.*

*(a) $G_\alpha$ is a connected reductive group, and $T$ is a maximal torus of $G_\alpha$.*

*(b) The only rational multiples of $\alpha$ in $\Phi(G,T)$ are $\pm\alpha$.*

*(c) The Weyl group $W(G_\alpha, T)$ (viewed as a finite group) contains a unique nontrivial element $s_\alpha$.*

*(d) There is a unique element $\alpha^\vee \in \mathcal{X}(T)^\vee$ such that $s_\alpha$ acts on $\mathcal{X}(T)$ by the rule*

$$s_\alpha \cdot \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

*for all $\lambda \in \mathcal{X}(T)$. Moreover, $\langle \alpha, \alpha^\vee \rangle = 2$.*

*(e) The action of $s_\alpha$ on $\mathcal{X}(T)$ maps $\Phi(G,T)$ into itself.*

**Definition 2.2.7.** we call $\alpha^\vee$ in the above theorem the *coroot* of $(G,T)$ corresponding to $\alpha$.

**Remark 2.2.8.** Note that, although one often works in the dual vector spaces $\mathcal{X}(T) \otimes F$ and $\mathcal{X}(T)^\vee \otimes F$, where $F$ is either $\mathbb{Q}$ or $\mathbb{R}$, when working with roots and coroots, the element $\alpha^\vee \in \mathcal{X}(T)^\vee \otimes F$ is *not* the "dual" of $\alpha \in \mathcal{X}(T) \otimes F$ in the conventional sense of dual elements in linear algebra. That is, $\alpha^\vee$ is not the map given by projection onto the line $F\alpha$: indeed, we have $\alpha^\vee(\alpha) = 2$, whereas this projection map sends $\alpha$ to 1.

It turns out that the existence of coroots $\alpha^\vee$ satisfying the conditions of the above theorem is actually a very strong condition on the finite set of roots $\Phi(G,T) \subset \mathcal{X}(T)$. Indeed, from this theorem, it follows that the tuple $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$ is a combinatorial object called a *root datum* (and it is even a *reduced* root datum, which is a particularly nice type of root datum).

In summary, we have now attached a very nice combinatorial object (namely, a root datum) to the pair $(G,T)$, and we have completely classified what this combinatorial object. It turns out that these objects can be used to completely classify the reductive group $G$ (up to an isomorphism of algebraic groups). First, we note that any other choice of a maximal torus $T'$ is conjugate to $T$, and conjugation will induce an isomorphism $\mathcal{X}(T) \cong \mathcal{X}(T')$ that restricts to a bijection $\Phi(G,T) \cong \Phi(G,T')$ on roots and a bijection on coroots as well. Thus, the isomorphism class of a root datum of $(G, T)$ does not depend on the choice of $T$, so we may say that we have defined "the" root datum of a reductive group $G$ (by which we mean, we have defined the root datum up to isomorphism). With this, there are a few primary classification results. To state them, we first need a definition.

**Definition 2.2.9.** Let $\varphi : G \to G'$ be a homomorphism of algebraic groups over an algebraically closed field $k$. We say that $\varphi$ is an *isogeny* if $\varphi$ is surjective and $\ker(\varphi)$ is a finite algebraic group. We say that $\varphi$ is a *central isogeny* if $\varphi$ is an isogeny and $\ker(\varphi) \subset Z(G)$.

**Remark 2.2.10.** The only reason we have taken $\mathrm{char}(k) = 0$ in this section is that, in this case, the kernel of any isogeny is smooth ([Mil17, Corollary 11.31]) and finite, hence étale, which implies that the isogeny is central, see [Mil17, Remark 12.39]. In other words, every isogeny is central in characteristic 0. There is a version of the classification of reductive groups that applies to non-central isogenies (see [Mil17, Definition 23.1 and Theorem 23.25]), but it is slightly more technical; since we will not need these technicalities, we avoid them here by sticking to the case where $\mathrm{char}(k) = 0$.

**Definition 2.2.11.** Let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ and $(X', \Phi, \alpha' \mapsto \alpha'^\vee)$ be two root data. A homomorphism $\varphi : X' \hookrightarrow X$ is said to be a *central isogeny* from $(X', \Phi, \alpha' \mapsto \alpha'^\vee)$ to $(X, \Phi, \alpha \mapsto \alpha^\vee)$ if $\varphi$ is injective, $\mathrm{coker}(\varphi)$ is finite, and $\varphi$ and the dual map $\varphi^\vee : X^\vee \to X'^\vee$ restrict to bijections $\Phi' \xrightarrow{\sim} \Phi$ and $\Phi^\vee \xrightarrow{\sim} \Phi'^\vee$ on roots and coroots. Note that a central isogeny $\varphi$ is an isomorphism of root data if and only if $\varphi$ is surjective (i.e. $\mathrm{coker}(\varphi) = 0$).

**Theorem 2.2.12** ([Mil17, Theorem 23.25])**.** *Let $G$ and $G'$ be a homomorphism of algebraic groups over an algebraically closed field $k$, and let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ and $(X', \Phi, \alpha' \mapsto \alpha'^\vee)$ be the root data of $G$ and $G'$ (respectively).*

*(a) (The Isogeny Theorem) There exists a central isogeny $G \to G'$ if and only if there exists a central isogeny from $(X', \Phi, \alpha' \mapsto \alpha'^\vee)$ to $(X, \Phi, \alpha \mapsto \alpha^\vee)$.*

*(b) (The Isomorphism Theorem) The groups $G$ and $G'$ are isomorphic if and only if the root data $(X, \Phi, \alpha \mapsto \alpha^\vee)$ and $(X', \Phi, \alpha' \mapsto \alpha'^\vee)$ are isomorphic.*

**Theorem 2.2.13** (The Existence Theorem; [Mil17, Theorem 23.55])**.** *Every reduced root datum is isomorphic to the root datum of some connected reductive group.*

**Corollary 2.2.14.** *The map which sends a reductive group to its root datum is a bijection between isomorphism classes of reductive groups and isomorphism classes of reduced root data.*

### 2.2.b The Classification of Root Data

We have shown that a reductive group $G$ is completely classified (up to isomorphism) by its corresponding root datum $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$. A root datum is a purely combinatorial object whose main data consists of the (finite) set of roots $\Phi(G,T)$. These roots are subject to strong combinatorial conditions; it turns out that these conditions (plus some linear algebra) allows one to completely classify root data. We briefly sketch this classification here, because some of the details are important for working with root data. We refer the reader to [Mil17, Appendix C] for a much more thorough treatment.

We explain the classification of root data in 4 steps.

Step 1: Reduce to classifying *root systems*, which are slightly simpler combinatorial objects than root data. Given a root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$, define $V$ to be the $\mathbb{Q}$-subspace of $X_\mathbb{Q}$ spanned by $\Phi$. It turns out (see [Mil17, Proposition C.33]) that the coroots $\alpha^\vee$ and the map $\alpha \mapsto \alpha^\vee$ are determined by $V$ and $\Phi$ (due to the combinatorial properties of a root datum). So, the pair $(V, \Phi)$ is a simpler combinatorial object called a *root system*. Every root system comes from some root datum in this way, and the root data corresponding to a specific root system $(V, \Phi)$ are classified by a choice of a lattice $X$ and an inclusion $V \hookrightarrow X_\mathbb{Q}$ that maps $\Phi$ into the lattice $X \subset X_\mathbb{Q}$. Thus, it will suffice to classify root systems up to isomorphism. One further technicality: we are only interested in so-called *reduced* root data and root systems, as these are the ones that arise in the classification of reductive groups. So throughout, we will assume without further mention that our root data and root systems are reduced.

Step 2: Reduce to the case of so-called *indecomposable* root systems, i.e. those which are not made by "adding together" two separate (nontrivial) root systems. way. More precisely, given a family of root systems $(V_i, \Phi_i)_{i\in I}$, we can form their *direct sum*, which we define to be

$$\bigoplus_{i\in I}(V_i, \Phi_i) = \left(\bigoplus_{i\in I} V_i, \coprod_{i\in I} \Phi_i\right).$$

We say that a root system is *indecomposable* if it cannot be written as a direct sum of root sytems $(V_i, \Phi_i)_{I\in I}$ with $|I| \geq 2$ and $\Phi_i \neq \varnothing$ for all $i$. Since any root system $(V, \Phi)$ has $\dim(V) < \infty$, it immediately follows that every root system can be written as a direct sum of finitely many indecomposable root systems, and that this decomposition is unique (up to swapping the order of the indecomposable root systems). Thus, it will suffice to classify the indecomposable root systems.

Step 3: Put a little extra structure on our root systems (and root data) in order to be able to work with their roots more concretely. This extra structure is essentially a notion of a set of roots that "generate" all other roots in an appropriate sense. More precisely, we make the following definition.

**Definition 2.2.15.** Let $(V, \Phi)$ be a root system.

1. A *base* of the root system $(V, \Phi)$ is a subset $\Pi \subset \Phi$ such that every root $\beta \in \Phi$ can be written as a sum
$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha,$$
where the $m_\alpha$ are integers which all have the same sign (i.e. either $m_\alpha \geq 0$ for all $\alpha$ or $m_\alpha \leq 0$ for all $\alpha$).

2. Given a base $\Pi \subset \Phi$, the elements of $\Pi$ are called the *simple roots* (for the base $\Pi$), and the roots of the form $\sum_{\alpha \in \Pi} m_\alpha \alpha$ with $m_\alpha \geq 0$ (resp. $m_\alpha \leq 0$) are called the *positive roots* (resp. *negative roots*).

**Remark 2.2.16.** The behavior of roots in a root datum is the same as that in its corresponding root system; the only difference between the two objects is that a root datum has an ambient lattice $X$ and corresponding vector space $X_{\mathbb{Q}}$, whereas a root system essentially uses the lattice $X = \oplus_{\alpha \in \mathbb{Z}} \mathbb{Z}\alpha$ (which also determines the vector space $V = X_{\mathbb{Q}}$). In particular, we can define a base (hence also simple roots and positive roots) for a root datum in exactly the same way as we did for root systems. As such, we will sometimes use all of these notions in the context of root data rather than root systems.

All of the combinatorial properties in the definition of a root system (and of a root datum) are entirely determined by the combinatorial behavior of a base. Moreover, every choice of base for a root system $(V, \Phi)$ is essentially equivalent, in the sense that for any two bases $\Pi_1, \Pi_2 \subset \Phi$, there exists an automorphism of $(V, \Phi)$ which maps $\Pi_1$ bijectively onto $\Pi_2$ ([Mil17, Propositions C.9 and C.10]; see also the proof of [Mil17, Proposition C.52], and our discussion of Weyl chambers at the beginning of Section 2.2.d). So, if we are trying to classify root systems up to isomorphism, it will suffice to classify triples $(V, \Phi, \Pi)$, where $(V, \Phi)$ is a root system and $\Pi \subset \Phi$ is a given base. In other words, there is no harm in working with a fixed base for each root system in the classification.

Step 4: We are left with determining what possibilities exist for a choice of indecomposable (reduced) root system $(V, \Phi)$ and a base $\Pi \subset \Phi$. Using some linear algebra arguments, one can systematically constrain the possible behavior of the roots $\Phi$ and the base $\Pi$. After enough work, one proves that, up to isomorphism, there are only 4 families of indecomposable root systems (which are typically called: $A_n$ for $n \geq 1$; $B_n$ for $n \geq 2$; $C_n$ for $n \geq 3$; and $D_n$ for $n \geq 4$) and 5 exceptional indecomposable root systems (which are typically called: $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$).

In summary: reductive groups are classified by root data; root data are classified by root systems (plus a little extra information); root systems are all direct sums of indecomposable root systems; and up to isomorphism, we can explicitly write down all indecomposable root systems (they are the 4 infinite families $A_n$, $B_n$, $C_n$, and $D_n$, and the 5 exceptional root systems $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$. Since most of the combinatorial properties of a root datum are captured by its corresponding root system, it is useful to have some terminology to refer to reductive groups with certain corresponding root systems. This prompts the following definition.

**Definition 2.2.17.** Let $G$ be a reductive group.

1. By *the root system corresponding to* $G$ we mean the root system corresponding to the root datum of $G$ (under the construction of root systems from root data in Step 1 above).

2. We say that $G$ is *of type A* if the root system corresponding to $G$ is a direct sum of indecomposable root systems which are all isomorphic to $A_{n_i}$ for some $n_i \geq 1$. We say that $G$ is *of type B, C, D, E, F, G* if an analogous condition holds with the indecomposable root systems $B_{n_i}$, $C_{n_i}$, $D_{n_i}$, $E_{n_i}$, $F_4$, or $G_2$.

Typically, when classifying indecomposable root systems, one actually reformulates these root systems graphically, as so-called *Dynkin diagrams*. The construction is as follows. Let $(V, \Phi)$ be a root system, and let $\Pi \subset \Phi$. We define an undirected graph $C$ by letting $C$ have one vertex for each simple root $\alpha \in \Pi$, and for any two simple roots $\alpha, \beta \in \Pi$, setting the number of edges between the vertex corresponding to $\alpha$ and the vertex corresponding to $\beta$ to be

$$m_{\alpha,\beta} = \langle \alpha^\vee, \beta \rangle \cdot \langle \beta^\vee, \alpha \rangle.$$

Due to the combinatorial properties of root systems, it turns out that for all $\alpha$ and $\beta$, we have $\langle \alpha^\vee, \beta \rangle \leq 0$, and $m_{\alpha,\beta}$ is an integer satisfying $0 \leq m_{\alpha,\beta} \leq 3$ for all $\alpha, \beta \in \Pi$. The graph $C$ defined in this way is called the *Coxeter graph* of $C$. However, the Coxeter graph does not quite encode all of information of the triple $(V, \Phi, \Pi)$: in particular, when $m_{\alpha,\beta}$ is 2 (resp. 3), then one of $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$ is $-1$, and the other is $-2$ (resp. $-3$), but the Coxeter graph does not tell us which one is which. To remedy this, we define the *Dynkin diagram* of the triple $(V, \Phi, \Pi)$ to be a diagram (i.e. a planar embedding) of the Coxeter graph $C$, but where we put either a $<$ or a $>$ over the edges between $\alpha$ and $\beta$ whenever $m_{\alpha,b\eta} \geq 2$ in such a way that the $<$ or $>$ points towards $\alpha$ if $\langle \alpha^\vee, \beta \rangle < -1$, and it points towards $\beta$ if $\langle \beta^\vee, \alpha \rangle < -1$.

The Dynkin diagram completely captures the information of the triple $(V, \Phi, \Pi)$. More precisely, two triples $(V_1, \Phi_1, \Pi_1)$ and $(V_2, \Phi_2, \Pi_2)$ have isomorphic Dynkin diagrams (i.e. there is an isomorphism on their Coxeter graphs which "preserves" the markings $<$ and $>$ in a suitable sense) if and only if the triples themselves are isomorphic (i.e. there is an isomorphism of root systems $(V_1, \Phi_1) \cong (V_2, \Phi_2)$ which identifies $\Pi_1$ and $\Pi_2$). Since we are really only interested in thinking of root systems up to isomorphism, this means that we may think of root systems and Dynkin diagrams interchangeably. Under this identification, we will often identify a simple root $\alpha \in \Pi$ with a vertex of the Dynkin diagram. In particular, this prompts the following definition.

**Definition 2.2.18.** Let $(V, \Phi)$ be a root system, and let $\Pi \subset \Phi$ be a base. We say that two simple roots $\alpha, \beta \in \Pi$ are *adjacent* if $\langle \alpha^\vee, \beta \rangle \neq 0$ (which holds if and only if $\langle \beta^\vee, \alpha \rangle \neq 0$). Equivalently, $\alpha$ and $\beta$ are adjacent if their corresponding vertices in the Dynkin diagram of $(V, \Phi, \Pi)$ have an edge between them.

Note from the above construction that any two simple roots $\alpha, \beta \in \Pi$ have an edge between their corresponding vertices if and only if $\langle \alpha^\vee, \beta \rangle, \langle \beta^\vee, \alpha \rangle \neq 0$. This implies that taking a direct sum of two root systems is the same as taking the disjoint union of their Dynkin diagrams. It follows that a root system is indecomposable if and only if its Dynkin diagram is connected, and decomposing any root system as a direct sum of indecomposable ones corresponds to taking all the connected components of the Dynkin diagram. Thus, the classification of the indecomposable root systems is the same as the classification of connected Dynkin diagrams. Figure 2.1 contains a complete list of all connected Dynkin diagrams (up to isomorphism).

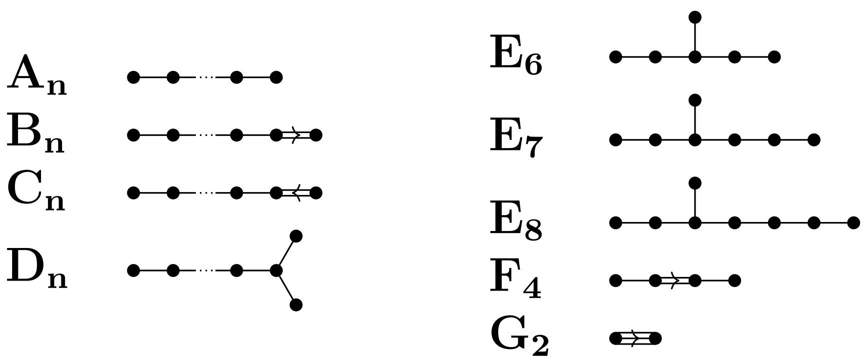



Figure 2.1: List of all connected Dynkin diagrams (up to isomorphism). The diagrams for $A_n$, $B_n$, $C_n$, and $D_n$ all have exactly $n$ vertices.

**Remark 2.2.19.** While the notation $A_n$, $B_n$, etc. was defined above to refer to indecomposable root systems, we often identify indecomposable root systems with their corresponding Dynkin diagrams and so use $A_n$, $B_n$, etc. to refer to a connected Dynkin diagram.

The depictions of the connected Dynkin diagrams in Figure 2.1 are the conventional ways to draw these diagrams. These depictions also give us a conventional way for writing down a base of the corresponding root system. For $A_n$, $B_n$, $C_n$, $D_n$, $F_4$ and $G_2$, we write the base as $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1$ corresponds to the leftmost vertex in the above drawing of the Dynkin diagram, $\alpha_2$ corresponds to the vertex directly to the right of $\alpha_1$, and so on. For $D_n$, this means that the two rightmost vertices (both of which are adjacent to $\alpha_{n-2}$) are $\alpha_{n-1}$ and $\alpha_n$ (it doesn't matter which is which, since swapping these two vertices is an automorphism of the diagram). As for $E_6$, $E_7$, and $E_8$, the convention (as we have seen it in some of the literature, at least) is that $\alpha_2$ is the "topmost" vertex, and then the long row of vertices is $\alpha_1, \alpha_3, \alpha_4$, etc. from left to right.

These choices of bases $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ for each Dynkin diagram are a common but often unmentioned convention in the literature. We will also use them later on, e.g. in Section 4.6. (We remark, however, that the numbering of the vertices in $E_6$, $E_7$, and $E_8$ will not come up in any of our work.)

### 2.2.c Semisimple Groups

Semisimple groups are a special type of reductive group with a particularly nice structure and behavior. The theory of semisimple groups is an important part of the theory of reductive groups; as such, we briefly review that theory here.

**Definition 2.2.20.** Let $G$ be a connected affine algebraic group.

1. The *radical* of $G$, denoted $R(G)$, is the largest connected normal solvable algebraic subgroup of $G$. (Such a largest group exists and contains all other connected normal solvable algebraic subgroups of $G$, see [Mil17, Proposition 6.42 and 6.44].)

2. We say that $G$ is *semisimple* if $R(G) = 1$, or equivalently, if $G$ contains no nontrivial connected normal solvable subgroups.

Note that $R_u(G) \subset R(G)$ by definition, since unipotent groups are solvable. Thus, any semisimple group is reductive. The following statement tells us which reductive groups are semisimple, in terms of both their algebraic behavior and their combinatorial behavior.

**Proposition 2.2.21** ([Mil17, Proposition 19.10] and [Mil17, Proposition 21.48, see also Definition C.34])**.** *Let $G$ be a reductive group over $k$, and let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ be the root datum of $G$. The following are equivalent.*

*(i) $G$ is semisimple.*

*(ii) The center $Z(G)$ is finite.*

*(iii) The elements of $\Phi = \Phi(G, T)$ span the lattice $X = \mathcal{X}(T)$.*

The following proposition gives a couple other useful facts about semisimple groups, which we will need later.

**Proposition 2.2.22.** *Let $G$ be a semisimple group over $k$. Then, $G = [G, G]$, and $\mathcal{X}(G) = 0$.*

*Proof.* The fact that $G = [G, G]$ is [Mil17, Proposition 19.21]. As for $\mathcal{X}(G) = 1$, let $\mu : G \to \mathbb{G}_m$ be a character. The quotient $Q = G/\ker(\mu)$ is isomorphic to a subgroup of $\mathbb{G}_m$ and in particular is commutative. It follows that $[G, G] \subset \ker(\mu)$, so $G = [G, G]$ implies that $\ker(\mu) = G$, i.e. that $\mu$ is trivial. □

In the previous section, we saw that root data are classified by their corresponding root system $(V, \Phi)$, a choice of lattice $X$, and an inclusion $V \hookrightarrow X \otimes_{\mathbb{Z}} \mathbb{Q}$. For semisimple groups, however, one can show ([Mil17, Proposition C.35, see also C.27]) that the root datum is determined only the root system $(V, \Phi)$ and by a choice of lattice $X$, and moreover, that $X$ must be one of finitely many suitable lattices contained in $V$. These suitable lattices can be described in terms of the combinatorics of the root system. More specifically, there is a minimal lattice $Q(\Phi)$ and a maximal lattice $P(\Phi)$, both of which span $V$, and the root data whose corresponding root system is $(V, \Phi)$ are in bijection with the lattices $X$ such that

$$Q(\Phi) \subset X \subset P(\Phi).$$

Choosing such a lattice means choosing a subgroup of the finite group $P(\Phi)/Q(\Phi)$, which is why there are only finitely many choices.

In light of these facts, the classification of reductive groups is even simpler when applied to semisimple groups. Indeed, the Isomorphism Theorem (see Theorem 2.2.12b) along with our above discussion implies that semisimple groups are determined up to isomorphism by a root system $(V, \Phi)$ and a lattice $X$ such that $Q(\Phi) \subset X \subset P(\Phi)$. More generally, let $G$ and $G'$ be semisimple groups, let $(V, \Phi)$ and $(V', \Phi')$ be their corresponding root systems, and consider their corresponding lattices $Q(\Phi) \subset X \subset P(\Phi)$ and $Q(\Phi') \subset X' \subset P(\Phi')$ as well. It follows from the Isogeny theorem (Theorem 2.2.12a) that there exists a central isogeny $G \to G'$ if and only if $(V, \Phi) \cong (V', \Phi')$ and, under this isomorphism, we have $X' \subset X$.

One particularly interesting construction follows somewhat easily from these results. Let $G$ be a semisimple group with root system $(V, \Phi)$, and consider the semisimple group $\tilde{G}$ corresponding to the root system $(V, \Phi)$ taken with the maximal possible lattice $\tilde{X} = P(\Phi)$. Our above discussion immediately implies that there exists a central isogeny $\tilde{G} \to G$ and. Moreover $\tilde{G}$ is *simply connected*, i.e. every isogeny $G' \to \tilde{G}$ is an isomorphism. (Proof: $G'$ must have root system $(V, \Phi)$, and the lattice $X' \subset P(\Phi)$ of $G'$ satisfies $X' \supset \tilde{X} = P(\Phi)$. So, we have $X' = \tilde{X} = P(\Phi)$, and the isogeny $G' \to \tilde{G}$ induces an isomorphism on the data $(V, \Phi, \tilde{X}) \cong (V, \Phi, X')$, which means it is an isomorphism. We remark that this is one of the few places where what we have used isogenies that are not assumed to be central; however, since we are assuming that $\mathrm{char}(k) = 0$, every isogeny is central, see Remark 2.2.10.) The group $\tilde{G}$ is called the *universal cover* of $G$. It follows from our above discussion that the universal cover $\tilde{G}$ is the unique (up to isomorphism) simply connected semisimple group $\tilde{G}$ such that there exists a (central) isogeny $\tilde{G} \to G$. (A construction of the universal cover of a semisimple group can also be given without appealing to the classification of semisimple groups; see e.g. [Mil17, Section 18.d] for a sketch of such a proof.)

This discussion indicates that root systems are the primary piece of data in the classification of semisimple groups. Since root systems are direct sums of indecomposable root systems, which are completely classified, we are led to ask: which semisimple groups $G$ have indecomposable root systems? Conveniently, the 4 families of indecomposable root systems arise from the the so-called "classical semisimple groups," which are familiar families of matrix groups (see [Mil17, Section 21.j]):

- The root system $A_n$ (for $n \geq 1$) is the root system of $\mathrm{SL}_{n+1}$. The group $\mathrm{SL}_{n+1}$ is simply connected, so it is the universal cover of any semisimple group whose corresponding root system is $A_n$.
- The root system of $B_n$ (for $n \geq 2$) is the root system of $\mathrm{SO}_{2n+1}$. The universal cover of $\mathrm{SO}_{2n+1}$ is the spin group $\mathrm{Spin}_{2n+1}$.

- The root system of $C_n$ (for $n \geq 3$) is the root system of $\mathrm{Sp}_{2n}$. The group $\mathrm{Sp}_{2n}$ is simply connected, so it is the universal cover of any semisimple group whose corresponding root system is $C_n$.
- The root system of $D_n$ (for $n \geq 4$) is the root system of $\mathrm{SO}_{2n}$. The universal cover of $\mathrm{SO}_{2n}$ is the spin group $\mathrm{Spin}_{2n}$.

We have seen on a combinatorial level that the information of a root datum can be broken up, first to that of a root system $(V, \Phi)$ (along with a little extra information, namely, a lattice $X$ and an inclusion $V \hookrightarrow X_{\mathbb{Q}}$), and then to a direct sum of indecomposable root systems. It turns out that these ways to "repackage" combinatorial information have interesting algebraic analogs. To explain these analogs, we need a couple of algebraic results.

**Proposition 2.2.23** ([Mil17, Proposition 21.60])**.** *Let $G$ be a reductive group. There exists a central isogeny $G^{ss} \times C \to G$, where $G^{ss}$ is a semisimple simply connected group and $C$ is a torus.*

*Proof.* By [Mil17, Proposition 21.60], there exists an isogeny $G^{ss} \times C \to G$, where $C$ is a torus and $G^{ss}$ is semisimple. After replacing $G^{ss}$ by its universal cover, we may take $G^{ss}$ to be simply connected. Also, our isogeny is central because $\mathrm{char}(k) = 0$, see Remark 2.2.10 above.

□

**Theorem 2.2.24** ([Mil17, Theorems 21.51 and 23.62])**.** *Let $G$ be a semisimple group over $k$. There exists a central isogeny of the form*

$$G_1 \times \cdots \times G_m \to G,$$

*where the $G_i$ are semisimple groups whose root systems are indecomposable.*

*Proof.* The existence of such an isogeny is [Mil17, Theorem 21.51 and 23.62], and the isogeny is central because $\mathrm{char}(k) = 0$, see Remark 2.2.10 above. □

Let $G$ be a reductive group, and consider the central isogeny $G^{ss} \times C \to G$ of Proposition 2.2.23. Note that the root system $(V, \Phi)$ of $G$ is the same as the root system of $G^{ss} \times C$ (because central isogenies of root data induce isomorphisms on the root systems), which in turn is the root system of $G^{ss}$ (this can be computed directly: intuitively, a torus has no roots in its root datum, so taking the direct product with a torus should not affect the roots in a root datum, hence it won't affect the underlying root system). As for root data, let $(X', \Phi', \alpha \mapsto \alpha^\vee)$ be the root datum corresponding to $G^{ss} \times C$. Using the fact that $G^{ss}$ is semisimple and $C$ is a torus, one can show that

$$X' = \mathcal{X}(C) \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} \cdot \alpha \quad \text{and} \quad \Phi' = 0 \oplus \Phi.$$

Thus, replacing $G$ by $G^{ss} \times C$ (which only changes the root datum by a central isogeny) has the effect of "separating out" the information of the root datum nicely into the information of the root system $(V, \Phi)$ and the lattice $\mathcal{X}(C)$. This replacement of $G$ by $G^{ss} \times C$ is essentially an algebraic analog of the combinatorial fact that the root datum of $G$ is determined by its corresponding root system $(V, \Phi)$ along with some extra information (namely, the lattice $\mathcal{X}(T)$, and an inclusion $V \hookrightarrow \mathcal{X}(T)_{\mathbb{Q}}$, which in our above description are both captured by the lattice $\mathcal{X}(C)$).

Similarly, with $G^{ss}$ still a semisimple simply connected group, consider the central isogeny $\rho : G_1 \times \cdots \times G_r \to G^{ss}$ of Theorem 2.2.24. Since $G^{ss}$ is simply connected, $\rho$ is an isomorphism, and $G^{ss}$ is determined up to isomorphism by its root system $(V, \Phi)$ (we saw above that $G^{ss}$ is determined by $(V, \Phi)$ and a lattice $X$, but since $G^{ss}$ is simply connected, this lattice must be the maximal possible lattice, namely $P(\Phi)$). On the other hand, one can show that the root system of $G_1 \times \cdots \times G_r$ is the direct sum $\oplus_{i=1}^{r}(V_i, \Phi_i)$, where $(V_i, \Phi)$ is the root system corresponding to $G_i$. The $(V_i, \Phi_i)$ are all indecomposable by construction, and since $\rho$ is an isomorphism, we conclude that

$$(V, \Phi) \cong \bigoplus_{i=1}^{r}(V_i, \Phi_i).$$

Thus, the existence of the isomorphism $\rho$ is the algebraic analog of the fact that any root system decomposes into a direct sum of indecomposable root systems.

There is one technical consequence of these facts that will be very useful to us later on. To state it, a little background is necessary. Let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ be a root datum, let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a base for the root datum, and let $X_0 \subset X$ be the sublattice generated by elements of $\Phi$. The $\alpha_i$ form a basis for the lattice $X_0$ (this follows almost immediately from the definition of a base of a root datum). There is another basis for $X_0$ that is sometimes useful: namely, the basis dual to $\{\alpha_1^\vee, \alpha_n^\vee\}$. More precisely, we let $\omega_1, \ldots, \omega_n \in X_0$ be the unique elements such that $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$ for all $i$ and $j$. Then, the $\omega_i$ form a basis of $X_0$. We call the $\omega_i$ the *fundamental weights* of the root datum $(X\Phi, \alpha \mapsto \alpha^\vee)$ with respect to the base $\Pi$. (We use the term "weight" because when the root datum comes from a reductive group $G$ with maximal torus $T$, the lattice $X = \mathcal{X}(T)$ is sometimes called the set of "weights" of $G$, see Definition 2.3.1 below.) For more details on the definition of fundamental weights [Mil17, Section 22.a].

**Lemma 2.2.25.** *Let $G$ be a reductive group over $k$, and let $T \subset G$ be a maximal torus. Let $\Pi_G$ be some base for the root datum of $G$, and let $X_\omega \subset \mathcal{X}(T)$ be the sublattice generated by the fundamental weights with respect to $\Pi_G$ (or equivalently, the sublattice generated by $\Phi(G, T)$).*

*(a) The map $\mathcal{X}(G) \to \mathcal{X}(T)$ given by restricting characters from $G$ to $T$ is injective.*

*(b) Viewing $\mathcal{X}(G)$ as a subgroup of $\mathcal{X}(T)$ using (a), we have*

$$\mathcal{X}(T) = X_\omega \oplus \mathcal{X}(G).$$

*In particular, the set of fundamental weights is linearly independent from the characters of $G$.*

*Proof.* For (a), suppose that $\mu : G \to \mathbb{G}_m$ is a character such that $T \subset \ker(\mu)$; we need to show that $\mu$ is trivial. Pick some Borel subgroup $B$ of $G$ containing $T$. Then, we have $R_u(B) \subset \ker(\mu)$ (a homomorphism from a unipotent group to a torus is always trivial, see [Mil17, Corollary 14.18]), and $B = R_u(B) \cdot T$ (see Theorem 2.2.3), so $B \subset \ker(\mu)$. It follows that $\mu$ factors as a map $\mu' : G/B \to \mathbb{G}_m$. But $G/B$ is a complete variety (see e.g. [Mil17, Theorem 17.9]), so the image of $\mu'$ is complete, affine, and connected, hence is a single point. This image must contain the identity $0 \in \mathbb{G}_m$ (since $\mu$ is a homomorphism), so we conclude that $\mathrm{Im}(\mu) = \{0\}$, i.e. that $\mu$ is trivial.

As for (b), by Proposition 2.2.23, there exists a central isogeny $G^{ss} \times C \to G$, where $G^{ss}$ is semisimple and simply connected and $V$ is a torus. Replacing $G$ by $\tilde{G}$ changes nothing about the assumptions or conclusion of the proposition (this replacement changes the root datum of $G$ by a central isogeny, but that does not affect any of the combinatorial properties of the roots in the root datum). So, we may assume that $G = G^{ss} \times C$. In this case, $T_0 = T \cap (G^{ss} \times \{0\})$ is maximal torus (respectively) of $G^{ss}$, and The equality $T = T_0 \times C$ gives us

$$\mathcal{X}(T) = \mathcal{X}(T_0) \oplus \mathcal{X}(C).$$

We will show that $\mathcal{X}(T_0) = X_\omega$ and that $\mathcal{X}(C) = \mathcal{X}(G)$.

To show that $\mathcal{X}(T_0) = X_\omega$, we note that the Lie algebra of $G$ is $\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{c}$, and the Lie algebra of $T$ is $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{c}$. Since $T$ is commutative, it acts trivially on its own Lie algebra in the adjoint representation. In other words, under the decomposition

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha$$

that we used to define the set of roots $\Phi(G,T)$, we have $\mathfrak{t} \subset \mathfrak{g}^T$. So, for each $\alpha \in \Phi(G,T)$, every element of $\mathfrak{g}_\alpha$ must have the same component in $\mathfrak{c}$ (since $T$ acts by the character $\alpha$ on $\mathfrak{g}_\alpha$ and fixes the $\mathfrak{c}$-component of every element of $\mathfrak{g}$). In other words, we have

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}^{ss}) \oplus c$$

for some $c \in \mathfrak{c}$. It follows that the inclusion $\iota : \mathcal{X}(T_0) \hookrightarrow \mathcal{X}(T)$ induces a bijection $\Phi(G^{ss}, T_0) \xrightarrow{\sim} \Phi(G,T)$. In particular, the lattice $X_\omega$ is the sublattice of $\mathcal{X}(T_0)$ generated by $\Phi(G^{ss}, T_0)$. Since $G$ is semisimple, this lattice is $\mathcal{X}(T_0)$, see Proposition 2.2.21.

It remains to show that $\mathcal{X}(C) = \mathcal{X}(G)$ as subgroups of $\mathcal{X}(T)$. The inclusion $\mathcal{X}(C) \hookrightarrow \mathcal{X}(T)$ is given by sending any character $\mu : C \to \mathbb{G}_m$ to the character $\mu_T : T_0 \times C \to \mathbb{G}_m$ given by $(t_0, c) \mapsto \mu(c)$. For any character $\mu : C \to \mathbb{G}_m$, we can define a character $\mu_G : G \to \mathbb{G}_m$ by $(g^{ss}, c) \mapsto \mu(c)$. It follows immediately from the definition that $\mu_G|_T$ is the character $\mu_T$ mentioned above. We conclude that $\mathcal{X}(C) \subset \mathcal{X}(G)$ as subgroups of $\mathcal{X}(T)$. For the reverse inclusion, note that since $G^{ss}$ is semisimple, we have $\mathcal{X}(G^{ss}) = 0$ (see Proposition 2.2.22).

So, for any character $\mu : G \to \mathbb{G}_m$, we have $\mu|_{G^{ss}} = 0$ and in particular $\mu|_{T_0} = 0$. Since $\mu|_T$ is a homomorphism, this gives us

$$\mu|_T(t_0, c) = \mu|_T(t_0, 0) + \mu|_T(0, c) = 0 + \mu|_C(c) = \mu|_C(c)$$

for all $(t_0, c) \in T_0 \times C$. In other words, $\mu|_T$ is the image of $\mu|_C$ under the inclusion $\mathcal{X}(C) \hookrightarrow \mathcal{X}(T)$, which proves that $\mathcal{X}(G) \subset \mathcal{X}(C)$. $\square$

### 2.2.d Borel Subgroups and Parabolic Subgroups

So far, everything we have said about root data and the classification of reductive groups depends only on a choice of a maximal torus $T \subset G$. However, a choice of Borel subgroup $B \subset G$ such that $T \subset B$ happens to put some extra combinatorial structure on the root datum corresponding to $G$. More precisely, choosing a Borel subgroup of $G$ containing $T$ is equivalent to choosing a base for the root datum of $G$.

To explain this equivalence, we first need to characterize bases of root data in a different way. Let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ be a root datum. Given a base $\Pi \subset \Phi$, it follows immediately from the definitions that the simple roots are precisely the positive roots which are not the sum of two other positive roots (note that $0 \notin \Phi$ by definition of a root datum). Thus, choosing a base is essentially the same as choosing a set of positive roots in an appropriate way. We can make this intuition more precise as follows. For any root $\alpha \in \Phi$, consider the hyperplane

$$H_\alpha = \alpha^\perp = \{f \in \mathcal{X}(T)^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle f, \alpha \rangle = 0\}.$$

We define the *Weyl chambers* of the root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$ to be the connected components of the complement

$$X_{\mathbb{Q}}^\vee \setminus \bigcup_{\alpha \in \Phi} H_\alpha.$$

Since $\Phi$ is finite and $\mathbb{Q}$ is an infinite field, there are finitely many Weyl chambers, and each of them is a full-dimensional subset (and even a convex polyhedral cone) in the vector space $X_{\mathbb{Q}}^\vee$.

For any Weyl chamber $W$, let $f \in W$. By definition of a Weyl chamber, we have $\langle f, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$. We define

$$\Phi_W^+ = \{\alpha \in \Phi \mid \langle f, \alpha \rangle > 0\},$$

and we define $\Pi_W \subset \Phi_W^+$ to be the subset of elements which cannot be written as a sum of two elements of $\Phi_W^+$. One can use a little bit of linear algebra (see [Mil17, Proposition C.9]) to show that the following hold.

1. $\Pi_W$ is a base for the root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$ and $\Phi_W^+$ is the set of positive roots for this base.

2. The sets $\Phi_W^+$ and $\Pi_W$ depend only on the Weyl chamber $W$, not on the choice of $f \in W$.

3. Every base for $(X, \Phi, \alpha \mapsto \alpha^\vee)$ has the form $\Pi_W$ for a unique Weyl chamber $W$.

In summary, given a root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$, the map $W \mapsto \Pi_W$ is a bijection between Weyl chambers of the root datum and bases of the root datum, and this bijection also behaves as nicely as one might hope for when it comes to positive roots.

On the other hand, we can relate Weyl chambers of root data to Borel subgroups of reductive groups in the following way. Let $G$ be a reductive group, and let $T \subset G$ be a maximal torus. For any Borel subgroup $B$ of $G$ containing $T$, we define $\Phi^+(B,T)$ to be the set of roots $\Phi(B,T)$ (here viewing $B$ as a reductive group with maximal torus $T$). Equivalently, $\Phi^+(B,T)$ is the set of $\alpha \in \Phi(G,T)$ such that $\mathfrak{g}_\alpha$ is contained in the Lie algebra $\mathfrak{b}$ of $B$. We also define

$$W(B,T) = \{\lambda \in \mathcal{X}(T)^\vee_\mathbb{Q} \mid \langle \lambda, \alpha \rangle > 0 \ \forall \ \alpha \in \Phi^+(B,T)\}.$$

One can show that the following hold.

1. $W(B,T)$ is a Weyl chamber of the root datum $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$ corresponding to $(G,T)$, and $\Phi^+(B,T)$ is the set of positive roots for the base corresponding to the Weyl chamber $W(B,T)$ (see [Mil17, Proposition 21.29]).

2. The map $B \mapsto W(B,T)$ is a bijection between the set of Borel subgroups of $G$ containing $T$ and the set of Weyl chambers of the root datum $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$ (see [Mil17, Theorem 21.32]).

Combining these facts with the bijection between Weyl chambers and bases of a root datum, we see that choosing a base of the root datum $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$ is equivalent to choosing a Borel subgroup of $G$ containing $T$.

When working with root data of reductive groups, we will almost always want to choose a base. As such, we will almost always begin by fixing a choice of Borel subgroup. The following definition will help us refer to both the Weyl chamber and base of the root datum that correspond to a given choice of Borel subgroup.

**Definition 2.2.26.** Let $G$ be a reductive group, let $T \subset G$ be a maximal torus, and let $B \subset G$ be a Borel subgroup containing $T$

1. We call the Weyl chamber $W(B,T)$ the *dominant Weyl chamber* for $B$.

2. We define $\Pi_G(B,T) \subset \Phi^+(B,T)$ to be the base of the root datum $(\mathcal{X}(T), \Phi(G,T), \alpha \mapsto \alpha^\vee)$ corresponding to the Weyl chamber $W(B,T)$. When $B$ and $T$ are clear from context, we will often write $\Pi_G$ (or simply $\Pi$) for $\Pi_G(B,T)$.

The bijection $B \mapsto W(B,T)$ between Borel subgroups and Weyl chambers actually has a very nice inverse, which gives us some further insight to the structure of Borel subgroups (and, more generally, parabolic subgroups) of the reductive group $G$. To understand it, we first recall that there is a canonical isomorphism $\mathcal{X}(\mathbb{G}_m) \cong \mathbb{Z}$. (Explicitly, any homomorphism

$\mathbb{G}_m \to \mathbb{G}_m$ must be given by a ring homomorphism $k[t] \to k[t]$ that sends $t \mapsto t^a$ for some $a \in \mathbb{Z}$, and our canonical isomorphism sending such a character of $\mathbb{G}_m$ to $a$.) This gives rise to a perfect pairing

$$\mathrm{Hom}(\mathbb{G}_m, T) \times \mathcal{X}(T) \to \mathbb{Z}$$

given by sending any pair $(\lambda, \mu)$ to the integer corresponding to the character $\mu \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$. As such, we have a canonical isomorphism of abelian groups $\mathrm{Hom}(\mathbb{G}_m, T) \cong \mathcal{X}(T)^\vee$, so we may view an element $\lambda \in \mathcal{X}(T)^\vee$ as a homomorphism $\lambda : \mathbb{G}_m \to T$. We call such a homomorphism $\lambda$ a *one-parameter subgroup* of $T$.

Now, let $\lambda : \mathbb{G}_m \to T$ be a one-parameter subgroup, and let $X$ be a scheme on which $G$ acts. For any $k$-point $x \in X(k)$, we obtain a map $\varphi : \mathbb{G}_m \to X$ which is given on functors of points by $\varphi(t) = \lambda(t) \cdot x$. On the other hand, there is a canonical inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k$ (given on functors of points by the inclusion $\Gamma(S, \mathcal{O}_S^\times) \subset \Gamma(S, \mathcal{O}_S)$), and $\mathbb{G}_a \setminus \mathbb{G}_m$ consists of a unique $k$-point $0 \in \mathbb{G}_a$. If the map $\varphi$ extends to a map $\varphi' : \mathbb{A}^1_k \to X$, then we say that the limit $\lim_{t\to 0} \lambda(t)x$ exists, and we define

$$\lim_{t\to 0} \lambda(t)x = \varphi'(0).$$

For the moment, we are primarily interested in the case where $X = G$ and $G$ acts on itself by conjugation. In this case, for $g \in G(k)$, the limit defined above is the limit $\lim_{t\to 0} \lambda(t)x\lambda^{-1}(t)$. (Note that $\lambda^{-1}$ is the homomorphism $\mathbb{G}_m \to T$ given by $t \mapsto \lambda(t)^{-1}$, or equivalently, the element $-\lambda$ in the group $\mathrm{Hom}(\mathbb{G}_m, T)$.) This allows us to define the following subgroups of $G$.

$$P_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda^{-1}(t) \text{ exists}\}$$
$$U_\lambda = \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda^{-1}(t) = 1\}$$
$$M_\lambda = \{g \in G \mid \lambda(t) g = g\lambda(t)\ \forall t \in \mathbb{G}_m\}$$

Consider the case where $\lambda$ is contained in some Weyl chamber $W$ (when viewed as an element of $\mathcal{X}(T)^\vee$), or equivalently, when $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Phi(G, T)$.x One can show that $P_\lambda$ is the unique Borel subgroup of $G$ containing $T$ such that $W(P_\lambda, T) = W$, and that $P_\lambda = P_{\lambda'}$ for any two one-parameter subgroups $\lambda, \lambda' \in W$ (see [Mil17, Proposition 21.29]). In other words, the map sending a Weyl chamber $W$ to the group $P_\lambda$ for some $\lambda \in W$ is the inverse to the bijection $B \mapsto W(B, T)$ described above.

What is particularly interesting about this construction is that, when $\lambda$ is not contained in a Weyl chamber, we actually obtain all the parabolic subgroups of $G$ which are not Borel subgroups. More precisely, let $B \subset G$ be any Borel subgroup containing $T$, and let $\lambda : \mathbb{G}_m \to T$ be a one-parameter subgroups. The following statements hold.

1. The group $P_\lambda$ is parabolic, and every parabolic subgroup of $G$ has the form $P_\lambda$ for some $\lambda$ ([Mil17, Theorem 25.1]).

2. The group $P_\lambda$ contains $B$ if and only if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+(B,T)$ ([Mil17, Corollary 21.92]). This is equivalent to the condition that $\lambda$ (viewed as an element of $\mathcal{X}(T)^\vee$) lies in the closure of the dominant Weyl chamber $W(B,T)$. In this case, we call $\lambda$ a *dominant* one-parameter subgroup.

3. If $B \subset P_\lambda$, then the group $P_\lambda$ is completely determined by the set

$$I = \{\alpha \in \Pi_G(B,T) \mid \langle \lambda, \alpha \rangle = 0\}$$

([Mil17, Corollary 21.92, see also Theorem 21.91]).

There is also another useful way to construct parabolic subgroups containing a given Borel subgroup $B$. Namely, given any set of simple roots $I \subset \Pi_G(B,T)$, one can construct a parabolic subgroup $P_I$ of $G$ containing $B$. Moreover, the following statements hold (see [Mil17, Proposition 21.90 and Theorem 21.91]).

1. The map $I \mapsto P_I$ is a bijection between subsets of the base $\Pi_G(B,T)$ and parabolic subgroups of $G$ containing $B$.

2. There exists a canonical subgroup $M_I \subset P_I$ containing $T$ such that $P_I = R_u(P_I) \cdot M_I$.

3. The group $M_I$ is reductive, and its root datum (with respect to the maximal torus $T$) is $(\mathcal{X}(T), \Phi_I, \alpha \mapsto \alpha^\vee)$, where $\Phi_I = \mathbb{Z}I \cap \Phi(G,T)$ is the set of roots which are linear combinations of elements of $I$ (and the map $\alpha \mapsto \alpha^\vee$ is the same as in the root data for $(G,T)$, just restricted to $\Phi_I$).

4. The intersection $B \cap M_I$ is a Borel subgroup of $M_I$ containing $T$, and the base $\Pi_{M_I}(B \cap M_I, T)$ is precisely the set $I$.

5. It follows from the construction of $P_I$ (see the proof of the aforementioned results in Milne) that $P_I \subset P_J$ if and only if $I \subset J$.

In general, a *Levi subgroup* of an algebraic group $G$ is a subgroup $M \subset G$ such that $G = R_u(G) \cdot M$ (or equivalently, $G$ is isomorphic to the semi-direct product of $M$ and $R_u(G)$). Thus, the subgroup $M_I$ is a Levi subgroup of $P_I$ whose construction depends only on $I$. We call $M_I$ the *standard Levi subgroup* of $P_I$. Apart from the explicit construction of $M_I$ in [Mil17, Section 21.i], the standard Levi subgroup $M_I$ can also be characterized as the unique Levi subgroup of $P_I$ containing $T$ ([Tim11, p. 9]).

In general, we call the groups $P_I$ (i.e. the parabolic subgroups containing $B$) the *standard parabolic subgroups* of $G$. As noted above, the standard parabolic subgroups are precisely the parabolic subgroups that contain $B$.

In summary, we have now constructed all parabolic subgroups of $G$ in two different ways. The first is as the group $P_\lambda$ for some one-parameter subgroup $\lambda : \mathbb{G}_m \to T$; and the second is as $P_I$ for some $I \subset \Pi_G(B,T)$ and some $B$. (Note that since every parabolice subgroup contains some Borel subgroup by definition, every parabolic subgroup has the form $P_I$ for some choice of $B$ and $I \subset \Pi_G(B,T)$.) These two constructions are related in the following way. Let $B \subset G$ be a Borel subgroup, and let $\lambda : \mathbb{G}_m \to T$ be such that $B \subset P_\lambda$ (equivalently, such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+(B,T)$). Define

$$I = \{\alpha \in \Phi_G(B,T) \mid \langle \lambda, \alpha \rangle = 0.\}$$

Then, we have $P_\lambda = P_I$ ([Mil17, Corollary 21.92]). Moreover, it follows from the construction of $M_I$ (see [Mil17, Notation 21.89]) that $M_\lambda = M_I$. If $G$ is smooth, then we have $P_\lambda = U_\lambda \cdot M_\lambda$ (see [Mil17, Theorem 13.33]), and it follows formally that $U_\lambda = R_u(P_I)$.

In short, we can use the description of a parabolic subgroups as $P_\lambda$ or as $P_I$ interchangeably. We will mainly be interested in thinking of parabolic subgroups as $P_I$ for some $I \subset \Pi_G(B,T)$, since this allows us to read off the root data of $M_I$ in terms of $I$. The description of a parabolic subgroup as $P_\lambda$ will primarily be useful when we are interested in working with one-parameter subgroups, for instance in Section 3.6.a.

There are a couple trivial examples of the constructions of the $P_\lambda$ and $P_I$, which may serve to illustrate more concretely how they work. First, consider the case where $P = B$ is a Borel subgroup. We have already seen that this occurs when $\lambda$ lies in a Weyl chamber, in which case our above statement gives us $P_\lambda = P_I$ for $I = \varnothing$. In fact, we noted above that $P_I \subset B = P_\varnothing \subset P_I$ if and only if $I \subset \varnothing$. So, we have $B = P_I$ if and only if $I = \varnothing$ and $B = P_\lambda$ if and only if $\lambda$ lies in a Weyl chamber. In this case, $R_u(P) = R_u(B)$, and $T$ itself is a Levi subgroup (see Theorem 2.2.3) containing $T$, so $T = M_I$.

The other trivial case to consider is the case where $P = G$. This choice of $P$ strictly contains all other choices of $G$, so we must have $P = P_I$ for $I$ strictly containing every other possible choice of $I$, i.e. $I = \Pi_G(B,T)$. In terms of one-parameter subgroups, this corresponds to the case where $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Pi_G(B,T)$, hence $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Phi(G,T)$. In other words, $\lambda$ lies in the hyperplane $H_\alpha$ for all $\alpha \in \Phi(G,T)$. So, we have $G = P_I$ if and only if $I = \Pi_G(B,T)$ and $G = P_\lambda$ if and only if $\lambda \in \cap_{alpha \in \Phi(G,T)} H_\alpha$. Since $G$ is reductive, we have $R_u(P_I) = R_u(G) = \{1\}$, so $M_I = G$.

There is one other construction with parabolic subgroups that will sometimes be useful to us. Given a parabolic subgroup $P \subset G$, we say that another parabolic subgroup $Q \subset G$ is *opposite to* $P$ if $P \cap Q$ is a Levi subgroup of both $P$ and $Q$. For any Levi subgroup $M \subset P$, there exists a unique parabolic subgroup $Q$ containing $M$ and opposite to $P$, and $P \cap Q = M$ (see [Bor91, Proposition 14.21] and its proof). For our purposes, we will always be interested in the standard Levi subgroup (i.e. the unique Levi subgroup containing $T$), so we will simply refer to *the* opposite parabolic subgroup to $P$ as the parabolic subgroup which is opposite to $P$ and contains the standard Levi subgroup. We denote this parabolic subgroup by $P^-$. In particular, when $P = B$ is a Borel subgroup, then the *opposite Borel subgroup* $B^-$ is the unique Borel subgroup of $G$ such that $B \cap B^- = T$.

There are a couple other facts about opposite parabolic and opposite Borel subgroups that will sometimes be useful to us.

1. If $P$ and $P^-$ are opposite parabolic subgroups, then $R_u(P^-) \cdot P$ is an open subset of $G$ equal to $P^- \cdot P$.

2. On the level of root data, swapping to the opposite Borel subgroup amounts to negating all positive roots. More precisely, we have $\Phi^+(B^-, T) = -\Phi^+(B, T)$ and hence $W(B^-, T) = -W(B, T)$ (see [Mil17, Summary 21.86]). Since $W(B, T)$ is the dominant Weyl chamber for $B$, we call $-W(B, T)$ the *antidominant Weyl chamber* for $B$.

3. The opposite parabolic subgroup can easily be constructed using one-paremeter subgroups. More precisely, if $P = P_\lambda$ for some dominant one-parameter subgroup $\lambda : \mathbb{G}_m \to T$, then $P^- = P_{\lambda^{-1}}$.

## 2.3 Representation Theory of Reductive Groups

In this section, we assume that $k$ is an algebraically closed field of arbitrary characteristic. As before, we assume throughout that all algebraic groups are affine.

**Definition 2.3.1.** Let $G$ be a connected reductive group, and let $H \subset G$ be any subgroup. Let $T \subset G$ be a maximal torus, and let $B \subset G$ be a Borel subgroup containing $T$. Let $V$ be a $G$-module.

1. We say that $V$ is *simple* (or *irreducible*) if $V$ contains no nontrivial proper $G$-submodules. We say that $V$ is *semisimple* if it is a direct sum of simple representations.

2. We call elements of $\mathcal{X}(T)$ the *weights* of $G$, and we denote the set $\mathcal{X}(T)$ by $\Lambda_G(T)$ (or simply by $\Lambda_G$ when the choice of maximal torus $T$ is clear from context).

3. We say that a weight $\mu \in \Lambda_G$ is *dominant* with respect to $B$ if $\langle \alpha^\vee, \mu \rangle > 0$ for all positive roots $\alpha \in \Phi^+(B, T)$. We denote the set of all dominant weights by $\Lambda_G^+(B, T)$ (or simply $\Lambda_G^+$ when $B$ and $T$ are clear from context).

4. We say that an element $v \in V$ is *$H$-invariant* if for all $h \in H$, we have $h \cdot v = v$. We denote the set of all $H$-invariants by $V^H$. Equivalently, $V^H$ is the largest submodule of $V$ on which $H$ acts trivially.

5. We say that an element $v \in V$ is an *$H$-eigenvector* if the line $k \cdot V$ is an $H$-submodule of $V$, or equivalently, if there exists some character $\lambda : H \to \mathbb{G}_m$ such that $h \cdot v = \lambda(h)v$ for all $h \in H$. If $H \supset T$, then any $H$-eigenvector has a corresponding weight $\mu$ (namely, $\mu = \lambda|_T$); in this case, we say that $v$ is an $H$-eigenvector *of weight* $\mu$. We denote the set of all $H$-eigenvectors in $V$ by $V^{(H)}$. Note that $V^{(H)}$ is an $H$-submodule of $G$ and that $V^H \subset V^{(H)}$.

**Remark 2.3.2.** We note that the notion of dominant weights $\Lambda_G^+(B, T)$ is in some sense "dual to" the notion of the dominant Weyl chamber $W(B, T)$: the former involves the function $\langle \alpha^\vee, \cdot \rangle$ being positive for all positive roots $\alpha$, while the latter involves the function $\langle \cdot, \alpha \rangle$ being positive for all positive roots $\alpha$.

**Remark 2.3.3.** Note that $0 \in V$ is always an $H$-eigenvector (of weight 0 when $H \supset T$). This is convenient inasmuch as it makes the set of $H$-eigenvectors into a vector space, but we are almost never interested in this (somewhat trivial) $H$-eigenvector. As such, whenever we say something like "let $v \in V^{(H)}$ be an $H$-eigenvector," we always mean a *nonzero* $H$-eigenvector (unless we explicitly say otherwise).

All of the above definitions are dependent on the choice of a maximal torus $T$, and the definition of a dominant weight also depends on the choice of a Borel subgroup $B$. However, over an algebraically closed field $k$, given any other choice of maximal torus $T'$ and Borel subgroup $B' \supset T$, there exists some $g \in G(k)$ such that $gBg^{-1} = B'$ and $gTg^{-1} = T'$ ([Mil17, Theorem 17.13]). It follows that, after composing everything with the inner automorphism of $G$ given by conjugation by $g$, all the notions in the above definition are true for $B$ and $T$ if and only if they are true for $B'$ and $T$'. More precisely:

(1) We replace the action of $G$ on $V$ by the action given by $h \cdot w = ghg^{-1}w$. This representation is simple if and only if the original representation was.

(2) We have an isomorphism $\mathcal{X}(T) \cong \mathcal{X}(T')$ induced by conjugation by $g$. Thus, we can identify the weights of $T$ and $T'$.

(3) The isomorphism on weights above induces a bijection on roots $\Phi(G,T) \cong \Phi(G,T')$. Since this bijection is induced by conjugation by $g$, which sends $B$ to $B'$, we see that the bijection also identifies the set of positive roots $\Phi^+(B,T) \cong \Phi^+(B',T')$. Hence, a weight $\mu \in \mathcal{X}(T)$ is dominant with respect to $B$ if and only if the corresponding element of $\mathcal{X}(T')$ is dominant with respect to $B'$.

(4) For any subgroup $H \subset G$, we have $H \supset T$ if and only if $H' = gHg^{-1} \supset T'$, and any $v \in V$ is an $H$-eigenvector (resp. is $H$-invariant) if and only if $v$ is an $H'$-eigenvector (resp. is $H'$-invariant) under the new action of $G$ on $V$ given by (1) above. Moreover, if $H \supset T$ and $v$ is an $H$-eigenvector, then its weight (as an $H$-eigenvector) is identified with its weight as an $H'$-eigenvector unde the isomorphism $\mathcal{X}(T) \cong \mathcal{X}(T')$.

In short, we will always need to fix a choice of $B$ and $T$ in order to talk about (dominant) weights, but every such choice will give us essentially the same representation-theoretic behavior.

As with the algebraic structure of reductive groups in the previous section, we begin our discussion of representations of redutive groups by understanding tori and solvable groups first (cf. Theorem 2.2.3). The representations of these groups turn out to be relatively straightforward to characterize.

**Theorem 2.3.4.**

1. *Let $T$ be a torus, and let $V$ be a $T$-module. Every element of $V$ is a sum of (finitely many) $T$-eigenvectors. In other words, $V$ is a direct sum of one-dimensional $T$-modules.*

2. *Let $G$ be a smooth connected solvable group, and let $T \subset G$ be a maximal torus. The map $\mathcal{X}(G) \to \mathcal{X}(T)$ given by restriction of characters is an isomorphism of abelian groups.*

3. *Let $G$ be a smooth connected solvable group, and let $V$ be a $G$-module. The $G$-eigenvectors of $V$ are precisely the elements that are fixed by $R_u(G)$. In other words, $V^{(G)} = V^{R_u(G)}$.*

*Proof.* For statement (a), see [Mil17, Theorem 12.12]. For statement (b), recall from Theorem 2.2.3 that $G \cong R_u(G) \times T$. So, giving a character $G \to \mathbb{G}_m$ is the same as giving two characters $T \to \mathbb{G}_m$ and $R_u(G) \to \mathbb{G}_m$. But $R_u(G)$ is unipotent, so every character of $R_u(G)$ is trivial (by definition of a unipotent group). Statement (b) follows directly from this.

Statement (c) is essentially just a combination of (a) and (b). More precisely, pick a maximal torus $T \subset G$. For any nonzero element $v \in V$, the isomorphism $G \cong R_u(G) \times T$ implies that $v$ is a $G$-eigenvector if and only if $v$ is an $R_u(G)$-eigenvector and a $T$-eigenvector. But (a) says that $v$ is always a $T$-eigenvector, and since every character of $R_u(G)$ is trivial, the $R_u(G)$-eigenvectors are precisely the elements of $V$ fixed by $R_u(G)$. □

One more theorem about representations of solvable groups is often very useful.

**Theorem 2.3.5** (Lie–Kolchin; [Mil17, Theorem 16.30])**.** *Let $G$ be a smooth connected solvable group. Any $G$-module $V$ contains a nonzero $G$-eigenvector.*

In the previous section, we saw that the classification of reductive groups is largely controlled by the behavior of maximal tori and Borel subgroups. It turns out that simple $G$-modules are also classified by the behavior of maximal tori and Borel subgroups on them. To state this more precisely, we first need to explain the setup of this structure. First, note that $G$ acts on itself via the multiplication map $G \times G \to G$ given by $(g, h) \mapsto gh$. This action induces a natural $G$-module structure on the global sections $\Gamma(G, \mathcal{O}_G)$; we call $\Gamma(G, \mathcal{O}_G)$ with this $G$-module structure the *regular representation* of $G$. (See Section 2.4.a for details on this $G$-module structure.) Moreover, recall from Section 2.2.d that a choice of Borel subgroup $B \subset G$ containing a maximal torus $T$ determines a base $\Pi_G(B, T)$ for the root datum of $G$. We can use this base to define a partial order on the set of weights $\Lambda_G(T) = \mathcal{X}(T)$ as follows: we define $\lambda \geq \mu$ if and only if we can write

$$\lambda - \mu = \sum_{\alpha \in \Pi_G(B,T)} m_\alpha \alpha$$

for some $m_\alpha \geq 0$. Since $\Phi^+(B, T)$ is the set of positive roots for the base $\Pi_G(B, T)$, it follows immediately from the definitions that $\Phi^+(B, T) = \{\alpha \in \Phi(G, T) \mid \alpha > 0\}$. We remark that this partial order is not generally a total order.

**Theorem 2.3.6** ([Mil17, Theorems 22.18, 22.19 and Proposition 22.27, see also Theorem 22.2, Definition 22.21])**.** *Let $G$ be a reductive group, let $T$ be a maximal torus of $G$, and let $B \subset G$ be a Borel subgroup containing $T$.*

(a) *For any simple $G$-module $V$, there exists a unique line of $B$-eigenvectors $\ell \subset V$. Moreover, if $\mu$ is the weight of a $B$-eigenvector $v$ on this line, then $\mu$ is dominant and is the* highest weight *of $V$, i.e. $\mu$ is strictly greater than every other weight of an element of $V$ (under the partial order defined above).*

(b) *Two simple $G$-modules are isomorphic if and only if their highest weights are equal (as elements of $\mathcal{X}(T)$).*

(c) *For every dominant weight $\mu \in \Lambda_G^+(B,T)$, there exists a canonical simple submodule $V(\mu)$ of the regular representation $\Gamma(G, \mathcal{O}_G)$ whose highest weight is $\mu$. In particular, the assignment $\mu \mapsto V(\mu)$ defines a bijection between the set $\Lambda_G^+(B,T)$ and the set of isomorphism classes of simple $G$-modules.*

Now that we understand simple $G$-modules, we also understand semisimple $G$-modules: they are just direct sums of the simple $G$-modules. The following proposition (which is really a general representation-theoretic fact) will help us understand these direct sum decompositions of semisimple modules more concretely. To state it, we need one small piece of notation: given two $G$-modules $V$ and $W$, we denote by $\mathrm{Hom}_G(V,W)$ the $k$–vector space of $G$-equivariant linear maps $f : V \to W$.

**Proposition 2.3.7** (cf. [Bri10, Lemma 2.2]; see also [Mil17, Lemma 4.20])**.** *Let $G$ be an algebraic group, and let $V$ be a $G$-module.*

(a) *(Schur's lemma) Suppose $V$ is simple, and let $W$ be any other simple $G$-module. If $V \cong W$, then*
$$\mathrm{Hom}_G(V,W) \cong \mathrm{Hom}_G(V,V) = k \cdot \mathrm{id}_V .$$
*(Here the isomorphism is induced by $V \cong W$, and $k\,\mathrm{id}_V$ denotes the set of multiples of the identity map, i.e. the set of maps $V \to V$ given by $v \mapsto cv$ for some $c \in k$.) If $V \not\cong W$, then $\mathrm{Hom}_G(V,W) = 0$.*

(b) *If $V$ is semisimple, then the map*
$$\bigoplus_{W \text{ simple}} \mathrm{Hom}_G(W,V) \otimes W \to V$$
*given by $f \otimes w \mapsto f(w)$ is an isomorphism of $G$-modules. (Here we take $\mathrm{Hom}_G(W,V) \otimes W$ to have the $G$-module structure given by $g \cdot f \otimes w = f \otimes gw$, i.e. $G$ acts trivially on $\mathrm{Hom}_G(W,V)$ and by its given action on $W$. Moreover, the direct sum is over each isomorphism class of simple $G$-modules, and we pick some representative $W$ of each class.)*

(c) *Suppose that $V$ is semisimple, and write $V \cong \oplus_i V_i$ with each $V_i$ a simple $G$-module. For any simple $G$-module $W$, the number $\dim_k(\mathrm{Hom}_G(W,V))$ is the number of indices $i$ such that $V_i \cong W$ (considering both numbers as elements of $\mathbb{N} \cup \{+\infty\}$).*

*Proof.* For (a), let $f : V \to W$ be a $G$-equivariant map. Then, $\ker(f)$ is a $G$-submodule of the simple module $V$, and $\mathrm{Im}(f)$ is a submodule of the simple module $W$. Thus, exactly one of the following options occurs:

1. Either $\ker(f) = V$ or $\mathrm{Im}(f) = 0$, in which case $f$ is the 0 map, i.e. we have both $\ker(f) = V$ and $\mathrm{Im}(f) = 0$.

2. We have $\ker(f) = 0$ and $\mathrm{Im}(f) = V$, in which case $f$ is an isomorphism.

Thus, if $V$ and $W$ are not isomorphic, $f$ must be the 0 map, so $\mathrm{Hom}_G(V, W) = 0$. To complete the proof of (a), it remains to show that $\mathrm{Hom}_G(V, V) = k\,\mathrm{id}_V$. For this, let $f : V \to V$ be any element of $\mathrm{Hom}_G(V, V)$. Then, $f$ is a linear map of vector spaces, so it has some nonzero eigenvector $v$ with eigenvalue $\lambda$ (here we are using the fact that $k$ is algebraically closed). So, $f_0 = f - \lambda\,\mathrm{id}_V$ is another element of $\mathrm{Hom}_G(V, V)$, and $f_0(v) = 0$. In particular, $\ker(f_0) \neq 0$, so using our above arguments (with $V$ in place of $W$), we see that option 1 above holds, i.e. that $f_0 = 0$ and hence that $f = \lambda\,\mathrm{id}_V$. Thus $\mathrm{Hom}_G(V, V) \subset k\,\mathrm{id}_V$, and the opposite containment follows immediately from definitions.

As for (b), one can check directly from the definitions that the given map is a well-defined $G$-equivariant map and that formation of this map commutes with taking direct sums of different choices of $V$. Since $V$ is a direct sum of simple modules by assumption, it will suffice to consider the case where $V$ is simple. In this case, (a) tells us that there is only direct summand in the domain of the map that is nonzero, namely $\mathrm{Hom}_G(V, V) \otimes V$. By part (a) again, we have

$$\mathrm{Hom}_G(V, V) \otimes V \cong V$$

as $G$-modules, with the isomorphism given by $c\,\mathrm{id}_V \otimes w \mapsto cw$. This isomorphism identifies the map in (b) with the identity map $V \to V$, which is of course an isomorphism. Finally, (c) follows from (b) using the fact that $\mathrm{Hom}_G(W, V) \otimes W$ is isomorphic as a $G$-module to a direct sum $\dim_k(\mathrm{Hom}_G(W, V))$ copies of $W$ (since $G$ acts trivially on $\mathrm{Hom}_G(W, V)$). ☐

**Remark 2.3.8.** Part (b) of the above proposition is not a canonical decomposition of $V$ into simple modules (indeed, $\mathrm{Hom}_G(W, V) \otimes W$ is generally not simple!) However, from parts (a) and (b), it follows that any such decomposition is unique up to (1) picking a different representative for any isomorphism class of a simple $G$-modules appearing in $V$, and (2) swapping the order of the simple modules in the decomposition. When $G$ is reductive, we have canonical representatives of each isomorphism class of simple $G$-modules (namely, the $V(\mu)$) by Theorem 2.3.6, so any semisimple representation of a reductive group is isomorphic to $\bigoplus_{\mu \in \Lambda_G^+(B,T)} V(\mu)^{m_\mu}$ for a unique choice of $m_\mu \in \mathbb{N}$.

**Definition 2.3.9.** Let $G$ be an algebraic group, and let $V$ be a semisimple $G$-module.

1. For any simple $G$-module $W$, the *multiplicity* of $V$ in $W$ is the number of times that simple modules isomorphic to $W$ appear in a direct sum decomposition of $V$. By the above proposition, the multiplicity of $W$ in $V$ is $\dim_k(\mathrm{Hom}_G(W, V))$ (in particular, it does not depend on the direct sum decomposition and so is well defined).

2. We say that $V$ is *multiplicity-free* (resp. *multiplicity-finite*) if the multiplicity of every simple $G$-module in $V$ is at most one (resp. is finite).

The above proposition is a completely general way to define multiplicities for semisimple $G$-modules. When $G$ is reductive, the notion of multiplicity is slightly nicer, thanks to the classification of simple modules given above. Namely, any semisimple $G$-module $V$ can be written as

$$V \cong \bigoplus_{\mu \in \Lambda_G^+(B,T)} V(\mu)^{m_\mu}$$

for a unique choice of $m_\mu \in \mathbb{N}$. The multiplicity of any simple $G$-module $V(\mu)$ in $V$ is precisely $m_\mu$. On the other hand, since the only $B$-eigenvectors in $V(\mu')$ are those of weight $\mu'$ for any dominant weight $\mu'$, the $\mu$-eigenspace $V_\mu$ of $V$ (i.e. the $B$-submodule $V_\mu \subset V$ generated by every $B$-eigenvector of weight $\mu$) is precisely the subspace of $V$ generated (as a vector space) by the $B$-eigenvectors in each copy of $V(\mu)$ appearing in $V$. It follows that the multiplicity of $V(\mu)$ in $V$ is $\dim_k(V_\mu)$. (See [Tim11, Proposition 2.21] for another proof of this fact.)

One of the reasons that we are mainly interested in the characteristic 0 case is that every representation is semisimple in that setting. This means that the classification of simple representations in fact classifies all representations.

**Theorem 2.3.10** ([Mil17, Theorem 22.42])**.** *Let $k$ be an algebraically closed field of characteristic 0, and let $G$ be a connected algebraic group over $k$. The following are equivalent:*

*(i) $G$ is reductive.*

*(ii) Every finite-dimensional representation of $G$ is semisimple.*

*(iii) Every representation of $G$ is semisimple.*

*Proof.* That (i) and (ii) are equivalent is [Mil17, Theorem 22.42]. Certainly (iii) implies (ii); conversely, suppose that (ii) holds, and let $V$ be any representation of $G$. The key fact is that any representation is the union of all its finite-dimensional subrepresentations ([Mil17, Corollary 4.8]).

Let $W_1 \subset V$ be any subrepresentation. We claim that there exists some subrepresentation $W_2 \subset V$ such that $V = W_1 \oplus W_2$. Consider the set

$$S = \{W \subset V \mid W_1 \cap W = 0\}.$$

For any totally ordered subset $\{W_i\}$ of $S$, the sum $\sum_i W_i$ is also in $S$. (Proof: if $\sum_i w_i \in W_1 \cap \sum_i W_i$, then there is some $W_j$ containing all the nonzero $w_i$; so, $\sum_i w_i \in W_1 \cap W_j$, contradicting $W_j \in S$.) Thus, Zorn's lemma implies that there exists a maximal element $W_2$ of $S$. Since $W_1 \cap W_2 = 0$, to show $V = W_1 \oplus W_2$, it will suffice to show that every element of $V$ lies in $W_1 + W_2$. Given any $v \in V$, there exists a finite-dimensional subrepresentation $W \subset V$ containing $v$. Then, $W_1 \cap W$ is a subrepresentation of $W$, from which it follows that

$$W = (W_1 \cap W) \bigoplus W'$$

for some $W' \subset W$ ([Mil17, Proposition 4.17]). Since $v \in W$, we have $v = w_1 + w'$ for some $w_1 \in W_1 \cap W$ and $w' \in W'$. On the other hand, $W' \cap W_1 = 0$, so $W_2 + W'$ is an element of $S$ containing $W_2$. By maximality of $W_2$, we have $W_2 + W' \subset W_2$, so $v = w_1 + w'$ implies that $v \in W_1 + W' \subset W_1 + W_2$. Thus, $V = W_1 + W_2$, which proves the claim.

The fact that $V$ is semisimple given the above claim is a standard one; for completeness, we give a proof. Let $V_0 = \sum_{W \subset V \text{ simple}} W$. Since the intersection of any two distinct simple subrepresentations is trivial (it is a proper subrepresentation of a simple representation), we have

$$V_0 = \bigoplus_{W \subset V \text{ simple}} W.$$

We just need to show that $V_0 = V$. By our claim above, there exists some $W \subset V$ such that $V_0 \oplus W = V$. If $W \neq 0$, then $W$ contains a nonzero simple representation. (Proof: $W$ contains a nonzero finite-dimensional representation $W'$, and Zorn's lemma implies the existence of a maximal proper subrepresentation $W_1 \subset W'$. By the above claim, we have $W' = W_1 \oplus W_2$ for some $W_2 \subset W'$; but now $W_2 \cong W'/W_1$ is simple by maximality of $W_1$ and is nonzero because $W_1$ is proper.) But every simple subrepresentation of $V$ is contained in $V_0$, and $V_0 \cap W = 0$. Thus, we must have $W = 0$, so that $V_0 = V$. □

One nice application of semisimplicity in characteristic 0 is that we can obtain a canonical description of the regular representation of $G$. This is best stated by considering the regular representation not as a $G$-module but rather as a $(G \times G)$-module, which we can do using the action of $G \times G$ on $G$ given by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$ (see Remark 2.4.6 below for details). On the other hand, we recall that any action of $G$ on a $G$-module $V$ induces a $G$-module structure on the dual vector space $V^*$ in a natural way. Thus, given a $G$-module $V$, we can obtain a $(G \times G)$-module $V^* \otimes V$ by setting $(g, h) \cdot (\phi \otimes v) = \phi h \otimes gv$. In other words, the first copy of $G$ in $G \times G$ acts on $V$, and the second copy of $G$ acts on $V^*$.

**Proposition 2.3.11** ([Bri10, Lemma 2.2], [Tim11, Theorem 2.15])**.** *Let $k$ be an algebraically closed field of characteristic 0. Let $G$ be a reductive group over $k$, let $T \subset G$ be a maximal torus, and let $B \subset G$ be a Borel subgroup containing $T$. There exists a canonical isomorphism of $(G \times G)$-modules*

$$\Gamma(G, \mathcal{O}_G) \cong \bigoplus_{\lambda \in \Lambda_G^+(B,T)} V(\lambda)^* \otimes V(\lambda),$$

*where $V(\lambda)^*$ denotes the dual vector space to $V(\lambda)$ and $G \times G$ acts on $V(\lambda)^* \otimes V(\lambda)$ by $(g, h) \cdot (\phi \otimes v) = h\phi \otimes gv$.*

*sketch of proof.* The statement follows from Proposition 2.3.7 after one shows that

$$\mathrm{Hom}_G(V(\lambda), \Gamma(G, \mathcal{O}_G)) \cong V(\lambda)^*.$$

See [Bri10, Lemma 2.2] for details. □

## 2.4 $G$-modules and Schemes

In order to apply the representation theory of reductive groups to schemes, we first need to be able to obtain a representation from some geometric data on the scheme. In this section, we discuss a few important ways to do this. In this section, $k$ is an arbitrary field (not necessarily algebraically closed and of arbitrary characteristic).

### 2.4.a Global Sections and Function Fields

Let $G$ be an algebraic group (not necessarily reductive), and let $X$ be a $G$-scheme. For any affine $k$-scheme $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, consider the isomorphism $\rho_{g,R} : X \times S \to X \times S$ given on functors of points by mapping any $(x, s) \in (X \times_k S)(S')$ to $(g_s x, s)$, where $g_s \in G(S')$ is the composition $S' \xrightarrow{s} S \xrightarrow{g} G$. Put another way, the morphism $\rho_{g,R}$ is the composition

$$X \times S \xrightarrow{(g \circ \mathrm{pr}_2, \mathrm{id}_{X \times S})} G \times X \times S \xrightarrow{(\rho, \mathrm{id}_S)} X \times S,$$

where $\rho : G \times X \to X$ is the action morphism. Since $S$ is flat over $k$, taking global sections commutes with base changing by $S$, i.e. we have a canonical isomorphism

$$\Gamma(X \times S, \mathcal{O}_{X \times S}) \cong \Gamma(X, \mathcal{O}_X) \otimes_k R$$

which is natural in $S$. Thus, the map $\rho_{g,R}$ on global sections gives us a ring isomorphism

$$\varphi_{g,R} : \Gamma(X, \mathcal{O}_X) \otimes_k R \to \Gamma(X, \mathcal{O}_X) \otimes_k R.$$

Note that $\rho_{g,R}$ is a map of $S$-schemes. It follows that $\varphi_{g,R}$ is $R$-linear, i.e. that $\varphi_{g,R}(1 \otimes r) = 1 \otimes r$ for all $r \in R$.

The $\varphi_{g,R}$ are isomorphisms of $k$–vector spaces, and for all $g, h \in G(S)$, one can check that $\rho_{gh,R} = \rho_g \circ \rho_h$, which gives us

$$\varphi_{gh,R} = \varphi_{h,R} \circ \varphi_{g,R}.$$

It follows that $g \mapsto \varphi_{g,R}$ defines a homomorphism $G^{\mathrm{op}} \to \mathrm{GL}_V$, where $V = \Gamma(X, \mathcal{O}_X)$. Unfortunately, this is not a representation of $G$ (which by definition is a homomorphism $G \to \mathrm{GL}_V$). The issue is essentially that the $\varphi_{g,R}$ define a right action of $G(S)$ on $V \otimes_k R$, whereas representations of algebraic groups require left actions (by definition). Conceptually, this makes sense: actions of $G$ on schemes are also left actions by definition, and passing to ring maps is contravariant, so we end up with a right action on rings instead of a left action.

We could fix this by using right actions of $G$ on schemes instead, but it is conventional to use left actions of algebraic groups on schemes. Instead (and this seems to be the convention throughout the literature), we simply note that given a right action of $G$ on $V$, there is a canonical way to form a left action of $G$ on $V$: we just let elements of $G$ act by their inverse on $G$, i.e. we set $g \cdot v = g^{-1} * v$, where $*$ denotes the right action and $\cdot$ denotes the left action that we are defining. In terms of homomorphisms: given a homomorphism, $r : G^{\mathrm{op}} \to \mathrm{GL}_V$ we can define a homomorphism $\ell : G \to \mathrm{GL}_V$ by $\ell(g) = r(g^{-1})$. Putting this all together, the homomorphism $\ell$ defines a $G$-module structure on $V = \Gamma(X, \mathcal{O}_X)$, which explicitly is given by letting $g \in G(S)$ act by $\varphi_{g,R}^{-1}$.

When $X$ is irreducible, we can define a $G$-module structure on $K(X)$ in a similar way. More precisely, the generic fiber of the action morphism $\rho$ is a morphism $G \times \mathrm{Spec}(K(X)) \to \mathrm{Spec}(K(X))$ which defines an action of $G$ on $\mathrm{Spec}(K(X))$. Following the above construction with $\mathrm{Spec}(K(X))$ in place of $X$, we obtain a $G$-module structure on $K(X)$. From now on, we will use these $G$-module structures on $\Gamma(X, \mathcal{O}_X)$ and $K(X)$ without further mention.

**Remark 2.4.1.** When $X$ is affine, we have $K(X) = \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))$, and the generic fiber of $\rho$ is the localization of a ring map at the prime ideal $(0)$. It follows that the action of any $g \in G(S)$ on $K(X) \otimes_k R$ defined above is simply the localization of the map $\varphi_{g,R}$ at the prime ideal $(0)$. However, because taking invariants or eigenvectors does not commute with localization, we have $K(X)^G \neq \mathrm{Frac}(\Gamma(X, \mathcal{O}_X)^G)$ and $K(X)^{(G)} \neq \mathrm{Frac}(\Gamma(X, \mathcal{O}_X)^{(G)})$ in general.

**Remark 2.4.2.** It's worth noting what differences the convention of left action over right action actually makes for the action of $G$ on $\Gamma(X, \mathcal{O}_X)$ and $K(X)$. Our primary interest will be in the set of $B$-eigenvectors in these $G$-modules. Since the $B$-eigenvectors are the $T$-eigenvectors which are fixed by $R_u(B)$ (Theorem 2.3.4), the set of $B$-eigenvectors is the same whether we use a left or right action of $G$. However, if the character corresponding to an eigenvector $f \in \Gamma(X, \mathcal{O}_X)^{(B)}$ under the left action is $\mu$, then the character corresponding to $f$ under the right action is $-\mu$.

In short, the use of a left action instead of a right action only changes all the weights of $B$-eigenvectors by a minus sign. Keeping track of this sign is important for explicit examples, but there are certain properties that remain unchanged by this convention. For instance, the set of weights of $B$-eigenvectors in $K(X)$ is the same whether we use the left or right action (because if $f \in K(X)^{(B)}$ has weight $\mu$, then $f^{-1}$ has weight $-\mu$).

The following lemma gives us a nice functoriality property for our $G$-module structures on $\Gamma(X, \mathcal{O}_X)$ and $K(X)$.

**Lemma 2.4.3.** *Let $G$ be an algebraic group, and let $f : X \to Y$ be a $G$-equivariant morphism of $G$-schemes.*

*(a) The map on global sections $f^{\#} : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ is a $G$-equivariant map of $G$-modules.*

*(b) If $X$ and $Y$ are irreducible and $f$ is dominant, then the map on function fields $K(Y) \to K(X)$ induced by $f$ is a $G$-equivariant map of $G$-modules.*

*Proof.* Let $\rho : G \times X \to X$ and $\pi : G \times Y \to Y$ be the action morphisms. The statement that $f$ is $G$-equivariant is equivalent to the equation

$$\pi \circ (\mathrm{id}_G, f) = f \circ \rho.$$

So, for any $k$-scheme $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, we have the following commutative diagram:

$$\begin{tikzcd} X \times S \arrow[r, "(g\circ \mathrm{pr}_2, \mathrm{id}_{X\times S})"] \arrow[d, "(f, \mathrm{id}_S)"'] & G \times X \times S \arrow[r, "(\rho, \mathrm{id}_S)"] \arrow[d, "(\mathrm{id}_G, f, \mathrm{id}_S)"] & X \times S \arrow[d, "(f, \mathrm{id}_S)"] \\ Y \times S \arrow[r, "(g\circ \mathrm{pr}_2, \mathrm{id}_{Y\times S})"] & G \times Y \times S \arrow[r, "(\pi, \mathrm{id}_S)"] & Y \times S \end{tikzcd}$$

The top and bottom row here are the maps $\rho_{g,R}$ and $\pi_{g,R}$ (respectively) constructed above. Passing to global sections, we see that the map on global sections $f^\# : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ commutes with the action of $g$ for all $g \in G(S)$, i.e. that $f^\#$ is $G$-equivariant. Similarly, if $X$ and $Y$ are irreducible and $f$ is $G$-equivariant and dominant, then one can check that the composition of the inclusion of $\mathrm{Spec}(K(X))$ with the generic fiber of $f$ gives a $G$-equivariant morphism $\mathrm{Spec}(K(X)) \to \mathrm{Spec}(K(Y))$, which induces a $G$-equivariant map $K(Y) \to K(X)$ by the above arguments. □

When $X$ is affine, giving a $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ is more or less the same as giving an action of $G$ on $X$. The following lemma makes this statement precise.

**Lemma 2.4.4.** *Let $G$ be an algebraic group over $k$, and let $X = \mathrm{Spec}(A)$ be an affine scheme of finite type over $k$. Suppose given a $G$-module structure of $A$ in which every point in $G$ acts by ring homomorphisms. Then, there exists a unique action of $G$ on $X$ such that the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ constructed above is the given $G$-module structure on $A$.*

*Proof.* To construct the action of $G$ on $X$, we essentially reverse the construction of the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$. Giving a $G$-module structure on $A$ as in the lemma statement is equivalent to giving, for every affine $k$-scheme $S = \mathrm{Spec}(R)$ and every point $g \in G(S)$, a ring isomorphism

$$\varphi_{g,R} : A \otimes_k R \xrightarrow{\sim} A \otimes_k R$$

in such a way that $\varphi_{gh,R} = \varphi_{g,R} \circ \varphi_{h,R}$ for all $g, h \in G(S)$. Given such a choice of $\varphi_{g,R}$, we define the isomorphism

$$\rho_{g,R} : X \times S \xrightarrow{\sim} X \times S$$

to be the morphism of affine shemes given on global sections by $\varphi_{g^{-1},R}$. Then, one can check that $\rho_{gh,R} = \rho_{g,R} \circ \rho_{h,R}$ for all $g, h \in G(S)$. We then define a morphism $\rho : G \times X \to X$

which is given on functors of points by $(g, x) \mapsto \mathrm{pr}_X \circ \rho_{g,R}(x, \mathrm{id}_S)$. In other words, for any $(g, x) \in G(S) \times X(S)$, the image $\rho(g, x) \in X(S)$ is the composition

$$S \overset{(x,\mathrm{id}_S)}{\to} X \times S \overset{\rho_{g,R}}{\to} X \times S \overset{\mathrm{pr}}{\to} X.$$

One then checks that $\rho$ defines an action of $G$ on $X$ and that, under the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ constructed above, any $g \in G(S)$ acts on $\Gamma(X, \mathcal{O}_X) \otimes_k R$ by the given ring isomorphism $\varphi_{g,R}$. Moreover, the maps $\rho_{g,R}$ defined above are precisely the maps $\rho_{g,R}$ defined in the construction of the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$. Since $\rho$ is determined by the $\rho_{g,R}$, which in turn are determined by the $\varphi_{g,R}$, it follows that $\rho$ is the unique action of $G$ on $X$ satisfying the condition in the lemma statement. □

One important example of the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ arises when we take $X = G$. In this case, there are two interesting actions of $G$ on itself that we might consider: the action given by left multiplication, i.e. by the morphism $m : G \times G \to G$ given on functors of points by $(g, h) \mapsto gh$; or, the action given by right multiplication, which on functors of points will be $(g, h) \mapsto hg^{-1}$ (remember that an action of algebraic group on a scheme is by definition a left action, hence the use of $g^{-1}$ here). Thus, $\Gamma(G, \mathcal{O}_G)$ has two different natural structures of a $G$-module: the *left regular representation* (which comes from the action given by left multiplication) and the *right regular representation* (which comes from the action given by right multiplication). It turns out that these two representations are canonically isomorphic. Indeed, if $i : G \to G$ is the inversion map (given on funtors of points of $g \mapsto g^{-1}$) and $m' : G \times G \to G$ is the right action $(g, h) \mapsto hg^{-1}$, then we have

$$i \circ m = m' \circ i.$$

(On functors of points, this is just the statement that $(gh)^{-1} = h^{-1}g^{-1}$.) In other words, $i$ is a $G$-equivariant morphism from $G$ (with the left multiplication action) to $G$ (with the right multiplication action). By our discussion of functoriality above, $i$ induces a map from the right regular representation to the left regular representation, and since $i$ is an automorphism of $G$, this map is an isomorphism of representations. For this reason, one typically does not distinguish between the left and right regular representation and just calls the left regular representation *the regular representation* of $G$

**Remark 2.4.5.** The regular representation is faithful when $G$ is affine, so every affine algebraic group is isomorphic to a subgroup of $\mathrm{GL}_n$ for some $n$, see [Mil17, Theorem 4.9 and Corollary 4.10]. Subgroups of $\mathrm{GL}_n$ are sometimes called *linear algebraic groups*, so what we have just said is that every affine algebraic group is linear. This is one part of the reason why so much of the theory of algebraic groups is built on the affine case.

**Remark 2.4.6.** One can also combine the left and right regular representations in an interesting way. Indeed, we have an action of $G \times G$ on $G$ by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$. This action induces a $(G \times G)$-module structure on $\Gamma(G, \mathcal{O}_G)$, which is in some sense the left regular representation in one factor of $G \times G$ and the right regular representation in the other factor.

For our purposes, we will generally be interested in the case where $G$ is reductive, in which case we want to understand the $B$-eigenvectors (and their associated weights) of $\Gamma(X, \mathcal{O}_X)$ and $K(X)$. We now give an explicit example to show how these can be worked out.

**Example 2.4.7.** Let $k$ be a field, and let $V$ be a finite-dimensional $k$–vector space. We consider the $k$-scheme

$$X = \mathbb{A}(V) = \operatorname{Spec}(\operatorname{Sym}_k^\bullet(V^*)).$$

For any $k$-algebra $R$, we have isomorphisms

$$X(\operatorname{Spec}(R)) \cong \operatorname{Hom}_k(\operatorname{Sym}_k^\bullet(V^*), R) \cong \operatorname{Hom}_k(V^*, R) \cong V \otimes_k R$$

which are natural in $R$. (The second isomorphism here follows from the universal property of $\operatorname{Sym}_k^\bullet$, and the third isomorphism is given by sending any $\sum_j v_j \otimes r_j \in V \otimes_k R$ to the map $V^* \to R$ given by $\varphi \mapsto \sum_j \varphi(v_j) r_j$).) In other words, the functor of points of $X$, viewed as a functor $\mathbf{Alg}_k \to \mathbf{Set}$, is (naturally isomorphic to) the functor $R \mapsto V \otimes_k R$.

Suppose now that $V$ is a $G$-module for some algebraic group $G$. By definition, this means that for all $k$-algebras $R$, we have an action of $G(\operatorname{Spec}(R))$ on $X(\operatorname{Spec}(R)) = V \otimes_k R$ that is natural in $R$. In other words, the $G$-module structure on $V$ immediately gives us an action of $G$ on $X$. This action induce a $G$-module structure on $\Gamma(X, \mathcal{O}_X) = \operatorname{Sym}_k^\bullet(V^*)$, which we wish to describe explicitly. To do this, we first construct a $G$-module structure on $\operatorname{Sym}_k^\bullet(V^*)$ and then show that the corresponding action of $G$ on $X$ from Lemma 2.4.4 is the same as the action we just described.

Note that the $G$-module structure of $V$ induces a $G$-module structure on the dual space $V^*$. Explicitly, this structure is given as follows. Let $S = \operatorname{Spec}(R)$ and $g \in G(S)$. We have a canonical isomorphism $V^* \otimes_k R \cong \operatorname{Hom}_k(V, R)$ which is natural in $R$. (Explicitly, the isomorphism is given by $\sum_j \varphi_j \otimes r_j \mapsto (v \mapsto \sum_j \varphi_j(v) r_j)$.) Thus, we may describe the action of $g$ on $V^* \otimes_k R$ by describing the action of $g$ on $\operatorname{Hom}_k(V, R)$. This action is the map

$$\psi_{g,R} : \operatorname{Hom}_k(V, R) \to \operatorname{Hom}_k(V, R)$$

that sends any $k$-linear map $\varphi : V \to R$ to the composition

$$V \overset{v \mapsto v \otimes 1}{\hookrightarrow} V \otimes_k R \overset{g^{-1}\cdot}{\to} V \otimes_k R \overset{v \otimes r \mapsto \varphi(v) r}{\to} R.$$

The map $\psi_{g,R}$ is an isomorphism, so it gives rise to an isomorphism $V^* \otimes_k R \xrightarrow{\sim} V^* \otimes_k R$ and hence to an isomorphism

$$\tilde{\psi}_{g,R} : \operatorname{Sym}_k^\bullet(V^*) \otimes_k R \to \operatorname{Sym}_k^\bullet(V^*) \otimes_k R.$$

One can check that the $\tilde{\psi}_{g,R}$ define a left action of $G(R)$ on $\operatorname{Sym}_k^\bullet(V^*) \otimes_k R$. (Note that the use of $g^{-1}$ in the defintion of $\psi_{g,R}(\varphi)$ above ensures we get a left action instead of a right action.) So, we have defined a $G$-module structure on $\operatorname{Sym}_k^\bullet(V^*)$. We remark that in the case where $R = k$, the action of $g$ on $V$ is a linear map, and $\psi_{g,k}$ is nothing more than the dual linear map $V^* \to V^*$.

Now, consider the action $\rho : G \times X \to X$ induced (via Lemma 2.4.4) by the above $G$-module structure on $\mathrm{Sym}_k^\bullet(V^*)$. Let $S = \mathrm{Spec}(R)$ and $g \in G(S)$. By the proof of Lemma 2.4.4, the map $\rho_{g,R} : X \times S \to X \times S$ is given on global sections by $\tilde{\psi}_{g^{-1},R}$. On the other hand, the proof of the lemma also shows us that, for any $x \in X(S)$, the image $\rho(g,x) \in X(S)$ is the composition

$$S \overset{(x,\mathrm{id}_S)}{\to} X \times S \overset{\rho_{g,R}}{\to} X \times S \overset{\mathrm{pr}}{\to} X.$$

To show that $\rho$ agrees with our original action of $G$ on $X$, we need to show that the above composition, when viewed as an element of $X(S) \cong \mathrm{Hom}_k(V^*, R) \cong V \otimes_k R$, is precisely $g \cdot x$, where $g$ acts on $V \otimes_k R$ via the $G$-module structure of $V$. Viewing $x$ as an element of $V \otimes_k R$, we may write $x = \sum_j v_j \otimes r_j$. Let $i : k \hookrightarrow R$ be the inclusion map. Passing to maps on global sections and then to maps on $\mathrm{Hom}_k(V, R)$, the above composition becomes

$$V^* = \mathrm{Hom}_k(V,k) \overset{\varphi \mapsto i \circ \varphi}{\to} \mathrm{Hom}_k(V,R) \overset{\psi_{g^{-1},R}}{\to} \mathrm{Hom}_k(V,R) \overset{\varphi \mapsto \sum_j \varphi(v_j) r_j}{\longrightarrow} R.$$

Using the definition of $\psi_{g^{-1},R}$, one can check that this composition sends any $\varphi \in V^*$ to the element $\varphi(g \cdot (\sum_j v_j \otimes r_j)) \in R$. Under the isomorphism $\mathrm{Hom}_k(V^*, R) \cong V \otimes_k R$, this map $V^* \to R$ becomes precisely $g \cdot (\sum_j v_j \otimes r_j) = g \cdot x$. This proves the claim.

**Example 2.4.8.** We write out all the representation-theoretic data of interest to us in the above example for a particularly nice case. Suppose $k$ is algebraically closed of characteristic 0. Write $V = \oplus_{i=1}^n k e_i$ for some $n \geq 2$, so that $X = \mathbb{A}(V) = \mathbb{A}_k^n$. Let $G = \mathrm{SL}_n$, let $B$ be the subgroup of upper triangular $n \times n$ matrices (which is a Borel subgroup), and let $T$ be the subgroup of diagonal $n \times n$ matrices (which is a maximal torus of $G$ contained in $B$). By definition of $G$, we have a $G$-module structure on $V$. Explicitly: for any $S = \mathrm{Spec}(R)$, a point $g \in G(S)$ is an $n \times n$ matrix with coefficients in $R$, and the action of $g$ on $V \otimes_k R \cong R^n$ is just given by sending any $(r_1, \ldots, r_n) \in R$ to $g \cdot (r_1, \ldots, r_n)^T$. By Example 2.4.7 above, this induces an action of $G$ on $X$ in such a way that the corresponding $G$-module structure of $\Gamma(X, \mathcal{O}_X) \cong k[x_1, \ldots, x_n]$ is given by letting $g \in G(\mathrm{Spec}(R))$ act by the map $\tilde{\psi}_{g,R}$ in that example. To understand the representation theory of $\Gamma(X, \mathcal{O}_X)$, it will suffice to look at the action of $k$-points $g \in G(\mathrm{Spec}(k))$. By the construction of $\tilde{\psi}_{g,R}$, we see that any $k$-point $g$ acts by the isomorphism

$$k[x_1, \ldots, x_n] \overset{\sim}{\to} k[x_1, \ldots, x_n]$$

which, on degree-1 parts, is the automorphism of $V^* \cong \oplus_{i=1}^n k x_i$ given by the dual map $(g^{-1})^* \in \mathrm{SL}(V^*)$. The $x_i$ are the dual basis for the given basis $e_1, \ldots, e_n$ of $V$, and in the dual basis, the dual map $(g^{-1})^*$ is just the transpose of the matrix $g^{-1}$. Since $g$ sends $e_i$ to the $i$th column of $g$ (i.e. we have $g \cdot e_i = (g_{1,i}, \ldots, g_{n,i})$), we conclude that $g$ acts on $k[x_1, \ldots, x_n]$ by sending $x_i$ to the $i$th column of $(g^{-1})^T$, which is the $i$th row of $g^{-1}$. In equations: $g \cdot x_i = \sum_j (g^{-1})_{ij} x_j$.

Now, the degree-$d$ elements of $k[x_1, \ldots, x_n]$ form a $G$-submodule for all $d \geq 0$. We describe certain $B$-eigenvectors in each of these $G$-submodules. Any $b \in B$ is an upper-triangular matrix, and $b^{-1}$ is upper-triangular as well. It follows that for any $d \geq 0$, we

have
$$b \cdot x_n^d = \left((b^{-1})_{nn}\right)^d x_1^d,$$
and $\det(b) = 1$ implies that $b_{nn} \in k^\times$. Thus, the elements of the form $cx_1^d$ are $B$-eigenvectors for any $c \in k^\times$ and any $d \geq 0$. In general, the degree-$d$ elements of $k[x_1, \ldots, x_n]$ will not form a simple $G$-module, so there are typically other $B$-eigenvectors as well (except in degree 1, as $\oplus_{i=1}^n kx_i$ is actually simple, see [Mil17, Example 22.34]).

Note that if $\epsilon$ is the weight of $x_n$, then the weight of $x_n^d$ is $d\epsilon \in \mathcal{X}(T)$. Also, we have an isomorphism
$$T \xrightarrow{\sim} \mathbb{G}_m^{n-1}, \quad \operatorname{diag}(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{n-1}) \in \mathbb{G}_m^{n-1}.$$
(Note that this is an isomorphism because $\det(\operatorname{diag}(t_1, \ldots, t_n)) = 1$ implies that $t_n = t_1^{-1} \cdots t_{n-1}^{-1}$.) This isomorphism induces an isomorphism $\mathcal{X}(T) \cong \mathbb{Z}^{n-1}$, which identifies $\epsilon$ with the element $(1, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$. One can show that this weight $\epsilon$ is a fundamental weight of $G$ (see e.g. [Mil17, Example 22.34]).

The $x_i$ for $i < n$ are not $B$-eigenvectors, but they are $T$-eigenvectors, so we may consider their weights as well. Under the above isomorphism $\mathcal{X}(T) \cong \mathbb{Z}^{n-1}$, the weight of $x_i$ is identified with is $(0, \ldots, 0, -1, 0, \ldots, 0)$ (with the $-1$ in the $i$th position). We remark that the $-1$ here is coming from the fact that any $t \in T$ acts on the ring $k[x_1, \ldots, x_n]$ via $(t^{-1})^T$ rather than via $t$. Since these $x_i$ are not $B$-eigenvectors, their weights are unrelated to the representation theory of $G$; however, the fact that we get $-1$ instead of 1 for these $T$-eigenvectors will be relevant to our discussion of toric varieties in Section 3.3.a.

### 2.4.b $G$-Linearizations of Line Bundles

Sometimes, the $G$-module structure of $\Gamma(X, \mathcal{O}_X)$ carries a lot of interesting information. For instance, when $X$ is affine, the global sections of $X$ determine all the geometry of $X$, so the $G$-module structure of $\Gamma(X, \mathcal{O}_X)$ can be expected to tell us a lot about the action of $G$ on $X$. When $X$ is projective, however, we have $\Gamma(X, \mathcal{O}_X) = k$, and the $G$-module structure will just be the trivial one (because the map $X \to \operatorname{Spec}(k)$ is $G$-equivariant and so induces a $G$-equivariant map on global sections, see Lemma 2.4.3).

To understand geometry in the projective setting, one typically studies global sections of (especially ample) line bundles, not of $\mathcal{O}_X$. So, we would like to generalize our $G$-module construction for $\Gamma(X, \mathcal{O}_X)$ to get a $G$-action on $H^0(X, L)$ for an invertible sheaf $L$ on $X$. For this, we first need a gadget that plays the role of an "action" of $G$ on $L$.

**Definition 2.4.9.** Let $G$ be an algebraic group, let $X$ be a finite-type $k$ scheme equipped with an action of $G$, and let $L$ be an invertible sheaf on $X$. Let $\rho : G \times X \to X$ and $m : G \times G \to X$ be the action morphism and multiplication morphism (respectively). A *$G$-linearization* of $L$ is an isomorphism
$$\phi : \rho^* L \xrightarrow{\sim} \operatorname{pr}_X^* L$$

of invertible sheaves on $G \times X$ such that the following diagram of invertible sheaves on $G \times G \times X$ commutes:

$$\begin{tikzcd}
(\rho \circ (\mathrm{id}_G, \rho))^* L \arrow[r, "(\mathrm{id}_G, \rho)"] \arrow[dd, equal] & (\mathrm{pr}_X \circ (\mathrm{id}_G, \rho))^* L \arrow[r, equal] & (\rho \circ \mathrm{pr}_{G \times X})^* L \arrow[d, "\mathrm{pr}^*_{G\times X} \phi"] \\
& & (\mathrm{pr}_X \circ \mathrm{pr}_{G \times X})^* L \arrow[d, equal] \\
(\rho \circ (m, \mathrm{id}_X))^* L \arrow[rr, "(\mu, \mathrm{id}_X)^* \phi"] & & (\mathrm{pr}_X \circ (m, \mathrm{id}_X))^* L
\end{tikzcd}$$

(Here, $\mathrm{pr}_X : G \times X \to X$ and $\mathrm{pr}_{G \times X} : G \times G \times X \to G \times X$ are the projection maps, and all equals signs follow from pulling back by two compositions of maps that are equal.) We call commutativity of this diagram the *cocyle condition* of a $G$-linearization.

**Definition 2.4.10.** Let $G$ be an algebraic group, let $X$ be a finite-type $k$ scheme equipped with an action of $G$.

1. A *$G$-linearized invertible sheaf* on $X$ is an invertible sheaf $L$ equipped with a $G$-linearization.

2. We say that an invertible sheaf $L$ on $X$ is *$G$-linearizable* if there exists a $G$-linearization of $L$.

Let $X$ be a $G$-scheme, and let $\rho : G \times X \to X$ be the action morphism. For any affine $k$-scheme $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, we made use in the previous section of the isomorphism $\rho_{g,R} : X \times S \xrightarrow{\sim} X \times S$ given on functors of points by $(x, s) \mapsto (g_s x, s)$, where $g_s = g \circ s$ and $g_s x = \rho(g_s, x)$. Intuitively, one thinks of the map $\rho_{g,R}$ as the morphism by which $g$ acts on $X \times S$. In fact, we can recover the action morphism $\rho$ from the maps $\rho_{g,R}$: indeed, taking $S = G$ and $g_{univ} = \mathrm{id}_G$, we see that $\rho_{g_{univ},R}$ is the morphism $X \times G \xrightarrow{\sim} X \times G$ given by $(x, g) \mapsto (gx, g)$, so we have $\rho = \mathrm{pr}_X \circ \rho_{g_{univ},R}$ in this case (cf. the proof of Lemma 2.4.4). Note that, under this correspondence between $\rho$ and the maps $\rho_{g,R}$, the condition that the morphism $\rho$ is a $G$-action is equivalent to the condition that $\rho_{g,R} \circ \rho_{h,R} = \rho_{gh,R}$ for all $S = \mathrm{Spec}(R)$ and all $g, h \in G(S)$.

Just as the data of $\rho$ can be thought of as the data of the morphisms $\rho_{g,R}$, we would like to think of a $G$-linearization $\phi : \rho^* L \to \mathrm{pr}_X^* L$ as the data of some sheaf morphisms $\phi_{g,R}$ for all $S = \mathrm{Spec}(R)$ and all $g \in G(S)$. Since a $G$-linearization in some sense plays the role of an "action" of $G$ on $L$, the $\phi_{g,R}$ would then play the role of the map by which the point $g$ acts.

To construct the $\phi_{g,R}$, we define $i_{g,S} = (g \circ \mathrm{pr}_S, \mathrm{id}_{X\times S})$. We then have the following commutative diagram.

$$\begin{tikzcd}
X \times S \arrow[r, "i_{g,S}"] \arrow[rr, bend left, "\rho_{g,R}"] & G \times X \times S \arrow[r, "(\rho, \mathrm{id}_S)"] \arrow[d, "\mathrm{pr}_{G\times X}"] & X \times S \arrow[d, "\mathrm{pr}_X"] \\
& G \times X \arrow[r, "\rho"] & X
\end{tikzcd}$$

Set $L_S = \mathrm{pr}_X^* L$ and $\phi_S = \mathrm{pr}_{G\times X}^* \phi$. Then, $\phi_S$ is an isomorphism

$$\phi_S : (\rho, \mathrm{id}_S)^* L_S \xrightarrow{\sim} \mathrm{pr}_{X\times S}^* L_S,$$

so $i_{g,S}^* \phi_S$ is an isomorphism

$$i_{g,S}^* \phi_S : \rho_{g,R}^* L_S \to L_S.$$

The map $i_{g,S}^* \phi_S$ is the map $\phi_{g,R}$ that we were looking for. These maps also come with a sort of "associativity condition," which is the analog of the cocycle condition on the $G$-linearization. Indeed, one can pull back the cocyle condition on $\phi$ under the composition

$$X \times S \xrightarrow{(g\circ \mathrm{pr}_2, h\circ \mathrm{pr}_2, \mathrm{id}_{X\times S})} G \times G \times X \times S \xrightarrow{\mathrm{pr}} G \times G \times X$$

to obtain the following commutative diagram:

$$\begin{tikzcd}
\rho_{h,R}^*(\rho_{g,R}^* L_S) \arrow[r, equal] \arrow[d, "\rho_{h,R}^*(\phi_{g,R})"'] & \rho_{gh,R}^* L_S \arrow[r, "\phi_{gh,R}"] & L_S \\
\rho_{h,R}^* L_S \arrow[urr, "\phi_{h,R}"'] & &
\end{tikzcd}$$

As with $\rho$ and the maps $\rho_{g,R}$, the $G$-linearization $\phi$ can be recovered from the maps $\phi_{g,R}$. This gives us an alternative (and sometimes more straightforward) way of constructing and working with $G$-linearizations.

**Lemma 2.4.11.** *Let $X$ be a $G$-scheme, let $L$ be an invertible sheaf on $X$, and let $\rho : G\times X \to X$ be the action morphism. Suppose that for each affine $k$-scheme $S = \mathrm{Spec}(R)$ and each $g \in G(S)$, we are given a morphism*

$$\phi_{g,R} : \rho_{g,R}^* L_S \to L_S$$

*of sheaves of $\mathcal{O}_{X\times S}$-modules (here $L_S = \mathrm{pr}_X^* L$ and $\rho_{g,R}$ are as above). Suppose moreover that the following hold.*

1. *The $\phi_{g,R}$ are functorial in $R$, in the following sense: for any two affine $k$-schemes $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$ and any morphism $f : S' \to S$, the pullback of $\phi_{g,R}$ under the map $\mathrm{id}_X \times f : X \times S' \to X \times S$ is $\phi_{g',R'}$, where $g' = g \circ f \in G(S')$.*

2. *For every $S = \operatorname{Spec}(R)$ and every $g, h \in G(S)$, we have the following commutative diagram of sheaves of $\mathcal{O}_{X\times S}$-modules:*

$$\begin{tikzcd}
\rho^*_{h,R}(\rho^*_{g,R}L_S) \arrow[r, equal] \arrow[d, "\rho^*_{h,R}(\phi_{g,R})"'] & \rho^*_{gh,R}L_S \arrow[r, "\phi_{gh,R}"] & L_S \\
\rho^*_{h,R}L_S \arrow[urr, "\phi_{h,R}"'] & &
\end{tikzcd}$$

*Then, there exists a unique $G$-linearization $\phi : \rho^* L \to \operatorname{pr}_X^* L$ of $L$ such that for all $S = \operatorname{Spec}(R)$ and all $g \in G(S)$, the map $\phi_{g,R}$ is the map $i^*_{g,S}\phi_S$ defined from $\phi$ above.*

*Proof.* As with recovering the map $\rho$ from the $\rho_{g,R}$, we define $\phi$ by taking $S = G$ and $g_{univ} = \mathrm{id}_G$. Then, we have $L_S = \operatorname{pr}_X^* L$ and

$$\operatorname{pr}_X \circ \rho_{g_{univ},R} = \rho \circ \operatorname{pr}_{G\times X} \circ i_{g_{univ},S} = \rho$$

(the last equality here follows from the fact that $g_{univ} = \mathrm{id}_G$). This implies that $\rho^*_{g_{univ},R}L_S = \rho^* L$, so $\phi_{g_{univ},R}$ is a morphism $\phi : \rho^* L \to \operatorname{pr}_X^* L$.

For any $S = \operatorname{Spec}(R)$ and any $g \in G(S)$, the composition $\operatorname{pr}_{G\times X} \circ i_{g,S}$ is the map $\mathrm{id}_X \times g : X \times S \to X \times G$. Thus, Assumption 1 (applied with $f = g$) tells us that the pullback $i^*_{g,S}\phi_S = (\operatorname{pr}_{G\times X} \circ i_{g,S})^*\phi$ is the map $\phi_{g',R}$, where $g' = g_{univ} \circ g = g$. In other words, we have $\phi_{g,R} = i^*_{g,S}\phi_S$. Moreover, the map $\phi$ can be recovered from the $i^*_{g,S}\phi_S$: taking $S = G$ and $g = g_{univ}$ gives us $i^*_{g_{univ},S}\phi_G = \phi$. It follows that $\phi$ is the unique map $\rho^* L \to \operatorname{pr}_X^* L$ such that $i^*_{g,S}\phi_S = \phi_{g,R}$ for all $S = \operatorname{Spec}(R)$ and all $g \in G(S)$.

It remains to check that $\phi$ satisfies the cocyle condition (and so is actually a $G$-linearization). For this, we note that the cocycle condition is a diagram which commutes if and only if its pullback under every map $S \to G \times G \times X$ commutes. Any such map is given by a pair $g, h \in G(S)$ and a point $x \in X(S)$, and we have the following commutative diagram:

$$\begin{tikzcd}
S \arrow[d, "{(x, \mathrm{id}_S)}"] \arrow[rrd, "{(g,h,x)}"] & & \\
X \times S \arrow[r, "{(g\circ \operatorname{pr}_S, h\circ \operatorname{pr}_S, \mathrm{id}_{X\times S})}"'] & G \times G \times X \times S \arrow[r, "\operatorname{pr}"'] & G \times G \times X
\end{tikzcd}$$

As noted above, the pullback of the cocyle condition under the bottom row of this diagram is the diagram of Assumption 2. Since we have assumed that this diagram comutes, so does its pullback to $S$. □

With $G$, $X$, $\rho$, and $L$, let $\phi : \rho^* L \to \operatorname{pr}_X^* L$ be a $G$-linearization. We can use $\phi$ to construct a $G$-module structure on $H^0(X, L)$ as follows. For any affine $k$-scheme $S = \operatorname{Spec}(R)$ and any $g \in G(S)$, we define $\rho_{g,R}$ as above and set $L_S = \operatorname{pr}_X^* L$ and $\phi_S = \operatorname{pr}_{G\times X}^* \phi$ as above. We already know that $\phi_{g,R} = i^*_{g,S}\phi_S$ is an isomorphism

$$\phi_{g,R} : \rho^*_{g,R}L_S \to L_S.$$

On the other hand, $\rho_{g,R}$ induces a $k$-linear map $H^0(X \times S, L_S) \to H^0(X \times S, \rho_{g,R}^* L_S)$ by pulling back global sections. We consider the composition

$$\psi_{g,R,L} : H^0(X \times S, L_S) \xrightarrow{\rho_{g,R}^*} H^0(X \times S, \rho_{g,R}^* L_S) \xrightarrow{\phi_{g,R}} H^0(X \times S, L_S).$$

Note that $\psi_{g,R,L}$ is $k$-linear (because $\rho_{g,R}$ is a morphism of $k$-schemes, the morphism $i_{g,S}^* \phi_S$ is $\mathcal{O}_X$-linear, and $k \subset \Gamma(X, \mathcal{O}_X)$) and is an isomorphism (because both $\rho_{g,R}^*$ and $i_{g,S}^* \phi_S$ are isomorphisms). Moreover, since $S$ is flat over $k$, taking global sections of $L$ commutes with base change to $S$, i.e. we have a canonical isomorphism

$$H^0(X \times S, L_S) \cong H^0(X, L) \otimes_k R$$

which is functorial in $R$. We claim that the $\psi_{g,R,L}$ define a right action of $G(S)$ on $H^0(X, L) \otimes_k R$. For all $g, h \in G(S)$, we know that

$$\rho_{gh,R} = \rho_{g,R} \circ \rho_{h,R}.$$

Using this fact and the commutative diagram on the $\phi_{g,R}$ given above (see Assumption 2 of Lemma 2.4.11 above), we have

$$\begin{aligned} \psi_{gh,R,L} = \phi_{g,R} \circ \rho_{gh,R}^* &= (\phi_{h,R} \circ \rho_{h,R}^*(\phi_{g,R})) \circ (\rho_{h,R}^* \circ \rho_{g,R}^*) \\ &= \phi_{h,R} \circ (\rho_{h,R}^* \circ \phi_{g,R}) \circ \rho_{g,R}^* \\ &= \psi_{h,R,L} \circ \psi_{g,R,L}. \end{aligned}$$

(The third equality here follows from definition of the pullback map $\rho_{h,R}^*$, which the rest are using the commutativity statements given above.) This proves that the $\psi_{g,R,L}$ define a right action of $G(S)$ on $H^0(X, L) \otimes_k R$. So, we obtain a left action by letting $g \in G(S)$ act by $\psi_{g,R,L}^{-1}$, and this left action defines a $G$-module structure on $H^0(X, L)$.

For our purposes, the main use of $G$-linearizations will be to work with this $G$-module structure on $H^0(X, L)$. Note, however, that whereas the $G$-module structures on $\Gamma(X, \mathcal{O}_X)$ and $K(X)$ can be defined just using the action of $G$ on $X$, the $G$-module structure on $H^0(X, L)$ requires the added data of a $G$-linearization on $L$. $G$-linearizations need not always exist, and they also need not be unique when they do exist. We will discuss these questions of existence and uniqueness in more detail in Section 2.6. For now, we give a few basic properties of $G$-linearizations that help us work with them.

First, we check that the $G$-module structure on $H^0(X, L)$ is functorial in some suitable sense (cf. the analogous statement for $\Gamma(X, \mathcal{O}_X)$ and $K(X)$ in Lemma 2.4.3).

**Lemma 2.4.12.** *Let $f : Y \to X$ be a $G$-equivariant morphism of $G$-schemes, and let $L$ be a $G$-linearized invertible sheaf on $X$. There exists a canonical $G$-linearization on $f^*$ : such that the pullback map*

$$f^* : H^0(X, L) \to H^0(Y, f^* L)$$

*is a $G$-equivariant map of $G$-modules.*

*Proof.* Since $f$ is $G$-equivariant, the following diagram commutes:

$$\begin{array}{ccc} G \times Y & \xrightarrow{(\mathrm{id}_G, f)} & G \times X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ Y & \xrightarrow{f} & X \end{array}$$

The same diagram also commutes with $\mathrm{pr}_Y$ and $\mathrm{pr}_X$ in place of $\pi$ and $\rho$ (by definition of the projection maps). We can thus consider the pullback

$$\psi = (\mathrm{id}_G, f)^*\phi : \pi^*(f^*L) \xrightarrow{\sim} \mathrm{pr}_Y^*(f^*L).$$

One can check that $\psi$ satisfies the cocyle condition and hence is a $G$-linearization of $f^*L$. Arguing as with $\Gamma(X, \mathcal{O}_X)$ in Lemma 2.4.3, we see that for any $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, we have

$$\pi_{g,S}^* \circ (f, \mathrm{id}_S)^* = (f, \mathrm{id}_S)^* \circ \rho_{g,S}^*.$$

Write $j_{g,S} = (g \circ \mathrm{pr}_S, \mathrm{id}_{Y\times S})$ and $\psi_S = \mathrm{pr}_S^* \psi$. Since $g$ acts on $H^0(X, L) \otimes R$ (resp. $H^0(Y, f^*L)$) by $i_{g,S}^*\phi_S \circ \rho_{g,S}^*$ (resp. $j_{g,S}^*\psi_S \circ \pi_{g,S}^*$) and the map $H^0(X, L) \otimes R \to H^0(Y, f^*L)$ induced by $f$ is $(f, \mathrm{id}_S)^*$, it remains to prove that

$$j_{g,S}^*\psi_S \circ (f, \mathrm{id}_S)^* = (f, \mathrm{id}_S)^* \circ i_{g,S}^*\phi_S.$$

For this, we note that the following diagram commutes:

$$\begin{array}{ccc} Y \times S & \xrightarrow{j_{g,S}} & G \times Y \times S \\ \downarrow{\scriptstyle (f, \mathrm{id}_S)} & & \downarrow{\scriptstyle (\mathrm{id}_G, f, \mathrm{id}_S)} \\ X \times S & \xrightarrow{i_{g,S}} & G \times X \times S \end{array}$$

Thus, we have

$$\begin{aligned} j_{g,S}^*\psi_S = [j_{g,S} \circ \mathrm{pr}_S \circ (\mathrm{id}_G, f)]^*\phi &= [j_{g,S} \circ (\mathrm{id}_G, f, \mathrm{id}_S) \circ \mathrm{pr}_S]^*\phi \\ &= [(f, \mathrm{id}_S) \circ i_{g,S}]^*(\mathrm{pr}_S^* \phi) \\ &= (f, \mathrm{id}_S)^*(i_{g,S}^*\phi_S). \end{aligned}$$

It now follows from a general property of the pullback map $(f, \mathrm{id}_S)^*$ that

$$j_{g,S}^*\psi_S \circ (f, \mathrm{id}_S)^* = (f, \mathrm{id}_S)^*(i_{g,S}^*\phi_S) \circ (f, \mathrm{id}_S)^* = (f, \mathrm{id}_S)^* \circ i_{g,S}^*\phi_S.$$

This equality is precisely the statement that the pullback map $f^*$ is $G$-equivariant. $\square$

Next, we show that $G$-linearizations behave nicely with respect to tensor products and inverses of invertible sheaves.

**Lemma 2.4.13.** *Let $X$ be a finite-type $k$-scheme, let $G$ be an algebraic group, and let $\rho : G \times X \to X$ be a group action. Let $L$ and $M$ be invertible sheaves on $X$, and let*

$$\phi : \rho^* L \xrightarrow{\sim} \mathrm{pr}_X^* L, \quad \psi : \rho^* M \xrightarrow{\sim} \mathrm{pr}_X^* M$$

*be $G$-linearizations.*

(a) *There exists a canonical $G$-linearization on $\mathcal{O}_X$ such that the induced $G$-module structure on $H^0(X, \mathcal{O}_X)$ is the usual $G$-module structure on $\Gamma(X, \mathcal{O}_X)$.*

(b) *The map $\phi \otimes \psi$ is a $G$-linearization of $L \otimes M$, and using the actions on global sections resulting from $\phi$, $\psi$, and $\phi \otimes \psi$, the canonical map*

$$H^0(X, L) \otimes_k H^0(X, M) \to H^0(X, L \otimes M)$$

*is $G$-equivariant.*

(c) *There exists a canonical $G$-linearization on $L^{-1}$ such that the $G$-linearization on the tensor product $L \otimes L^{-1} \cong \mathcal{O}_X$ of part (b) is the $G$-linearization on $\mathcal{O}_X$ of part (a).*

*Proof.* We have a canonical $G$-linearization on $\mathcal{O}_X$ arising from the canonical isomorphisms

$$\rho^* \mathcal{O}_X \cong \mathcal{O}_{G \times X} \cong \mathrm{pr}_X^* \mathcal{O}_X.$$

These canonical isomorphisms are such that pulling back by any morphism gives us the same canonical isomorphism (just on a different scheme). Thus, if $\alpha$ is the above composition, then one checks quite quickly that $\alpha$ satisfies the cocyle condition (every map in the cocycle condition is a canonical isomorphism of the structure sheaf), so $\alpha$ is a $G$-linearization on $\mathcal{O}_X$. To see what the corresponding action on $H^0(X, \mathcal{O}_X)$ is, let $S = \mathrm{Spec}(R)$ and $g \in G(S)$. The pullback $i_{g,S}^* \alpha_S$ is the canonical isomorphism

$$\rho_{g,R}^* \mathcal{O}_{X \times S} \cong \mathcal{O}_{X \times S} \cong \mathrm{pr}_X^* \mathcal{O}_X \cong \mathcal{O}_{X \times S}.$$

The second two isomorphisms here are inverses, so $i_{g,S}^* \phi_S$ is just the canonical isomorphism $\rho_{g,R}^* \mathcal{O}_{X \times S} \cong \mathcal{O}_{X \times S}$. The action of $g$ on $H^0(X, \mathcal{O}_X)$ is thus the composition

$$\Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{g,R}^*} \Gamma(X, \rho_{g,R}^* \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X).$$

It follows by definition of the pullback map $\rho_{g,R}^*$ that this is precisely the map on global sections $\rho_{g,R}^{\#}$. Thus, the action of $G$ on $\Gamma(X, \mathcal{O}_X)$ coming from the $G$-linearization $\alpha$ agrees with the $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ constructed above.

For (b), the cocyle condition on $\phi \otimes \psi$ follows formally from the cocyle conditions on $\phi$ and $\psi$ (and the fact that tensor products commute with pullbacks). For any $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, pullback by $\rho_{g,R}^*$ commutes with the canonical maps

$$H^0(X, L) \otimes_k H^0(X, M) \to H^0(X, L \otimes M)$$

and

$$H^0(X, \rho_{g,R}^* L) \otimes_k H^0(X, \rho_{g,R}^*) \to H^0(X, \rho_{g,R}^* L \otimes \rho_{g,R}^* M).$$

(This commutativity statement follows by definition of these maps and of the pullback of sections). The pullback $i_{g,S}^*(\phi \otimes \psi) = i_{g,S}^*\phi_S \otimes i_{g,S}^*\psi_S$ commutes with these maps as well (this is again a general scheme-theoretic fact), so the action of $g$ (which is the composition $i_{g,S}^*(\phi \otimes \psi) \circ \rho_{g,R}^*$) commutes with them, too.

For (c), we wish to construct a $G$-linearization $\phi_- : \rho^* L^{-1} \to \operatorname{pr}_X^* L^{-1}$. Using the identifications $(\rho^* L)^{-1} \cong \rho^*(L^{-1})$ and $(\operatorname{pr}_X^* L)^{-1} \cong \operatorname{pr}_X^*(L^{-1})$, we see that giving the morphism $\phi_-$ is equivalent to giving a morphism

$$\phi'_- : (\rho^* L)^{-1} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\rho^* L, \mathcal{O}_{G\times X}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\operatorname{pr}_X^* L, \mathcal{O}_{G\times X}) = (\operatorname{pr}_X^* L)^{-1}.$$

We can define such a morphism $\phi'_-$ by $f \mapsto f \circ \phi^{-1}$, and this gives us the morphism $\phi_-$ that we wanted. Using the cocyle condition on $\phi$, one can check that $\phi'_-$ (hence also $\phi_-$) satisfies the cocyle condition.

Now, the $G$-linearization on $L \otimes L^{-1}$ from (b) is the tensor product $\phi \otimes \phi_-$. Using the identifications $(\rho^* L)^{-1} \cong \rho^*(L^{-1})$ and $(\operatorname{pr}_X^* L)^{-1} \cong \operatorname{pr}_X^*(L^{-1})$ again, so that $\phi_-$ is identified with $\phi'_-$, this becomes the morphism

$$\phi \otimes \phi'_- : \rho^* L \otimes \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\rho^* L, \mathcal{O}_X) \to \operatorname{pr}_X^* L \otimes \mathcal{H}\mathrm{om}(\operatorname{pr}_X^* L, \mathcal{O}_X)$$

given by $s \otimes f \mapsto \phi(s) \otimes (f \circ \phi^{-1})$. Both the domain and target of $\phi \otimes \phi'_-$ are isomorphic to $\mathcal{O}_{G\times X}$, and under these isomorphisms (which are just the evaluation maps), the morphism $\phi \otimes \phi'_-$ becomes the identity map on $\mathcal{O}_{G\times X}$. On the other hand, we saw in the proof of (a) above that the canonical $G$-linearization on $\mathcal{O}_X$ is also the identity map on $\mathcal{O}_{G\times X}$ (after using the identifications $\rho^*\mathcal{O}_X \cong \mathcal{O}_{G\times X}$ and $\operatorname{pr}_X^* \mathcal{O}_X \cong \mathcal{O}_{G\times X}$). In other words, the $G$-linearization $\phi \otimes \phi_-$ on $\mathcal{O}_X$ is the same as the one from (a). $\square$

The above lemma allows us to define a "$G$-equivariant" version of the Picard group.

**Definition 2.4.14.** Let $X$ be a $G$-scheme, and let $\rho : G \times X \to X$ be the action morphism. Let $L$ and $M$ be invertible sheaves on $X$, and let

$$\alpha : \rho^* L \to \operatorname{pr}_X^* L, \quad \beta : \rho^* M \to \operatorname{pr}_X^* M$$

be $G$-linearizations.

1. A morphism $\varphi : L \to M$ is said to be *$G$-equivariant* (or *a morphism of $G$-linearized invertible sheaves*) if the following diagram commutes:

$$\begin{tikzcd} \rho^* L \arrow[r, "\alpha"] \arrow[d, "\rho^*\varphi"] & \operatorname{pr}_X^* L \arrow[d, "\operatorname{pr}_X^* \varphi"] \\ \rho^* M \arrow[r, "\beta"] & \operatorname{pr}_X^* M \end{tikzcd}$$

   (One can check that this is equivalent to saying that the homomrphism of global sections of $\varphi$ is a $G$-equivariant $H^0(X, L) \to H^0(X, M)$.)

2. We define the *$G$-equivariant Picard group*, denoted $\mathrm{Pic}_G(X)$, to be the set of all $G$-equivariant isomorphism classes of $G$-linearized invertible sheaves. In other words, elements of $\mathrm{Pic}_G(X)$ are invertible sheaves $L$ equipped with a $G$-linearization, modulo the equivalence relation imposed by $G$-equivariant isomorphisms.

**Remark 2.4.15.** Note that Lemma 2.4.13 implies that $\mathrm{Pic}_G(X)$ is an abelian group, with multiplication given by part (b) of the lemma and identity element $\mathcal{O}_X$ with the canonical $G$-linearization given in part (a) of the lemma.

The following lemma gives us another useful fact about the $G$-module structure on $H^0(X, L)$.

**Lemma 2.4.16.** *Let $X$ be a $G$-scheme, let $L$ be a $G$-linearized invertible sheaf on $X$, and let $f \in H^0(X, L)$. Let $S = \mathrm{Spec}(R)$ be an affine $k$-scheme, and write $X_S = X \times S$, $L_S = \mathrm{pr}_X^* L$, and $f_S = \mathrm{pr}_X^* f \in H^0(X_S, L_S)$, where $\mathrm{pr}_X^* : X \times S \to X$ is the projection morphism. For any $g \in G(S)$, we have*

$$g \cdot (X_S)_{f_S} = (X_S)_{g \cdot f_S}.$$

*Proof.* Let $\rho : G \times X \to X$ be the action morphism, and let $\phi : \rho^* L \to \mathrm{pr}_X^* L$ be the $G$-linearization of $L$. By definition, the action of $g \in G(S)$ on $H^0(X_S, L_S)$ is given by the composition $i_{g,R}^* \phi_S \circ \rho_{g,R}^*$. The morphsim $\rho_{g,R} : X \to X$ is the morphism by which $g$ acts on $X_S$, so it follows from the (purely scheme-theoretic) definitions that

$$g \cdot (X_S)_{f_S} = \rho_{g,R}((X_S)_{f_S}) = (X_S)_{\rho_{g,R}^* f_S}.$$

On the other hand, the map $i_{g,R}^* \phi_S$ is an isomorphism $\rho_{g,R}^* L_S \cong L_S$ which identifies $\rho_{g,R}^* f_S$ with $g \cdot f_S \in H^0(X_S, L_S)$. So, we have $(X_S)_{\rho_{g,R}^* f_S} = (X_S)_{g \cdot f_S}$ (because an isomorphism of invertible sheaves does not change the vanishing locus of a section). $\square$

One of the most important uses of $G$-linearizations for our purposes will be $G$-linearizations of an ample line bundle $L$ on a projective scheme $X$. In this setting, there is a canonical isomorphism $X \cong \mathrm{Proj}(\Gamma_*(X, L))$. It turns out that this isomorphism "plays nicely with $G$-actions" in a way made precise by the following proposition.

**Proposition 2.4.17.** *Let $k$ be a field, and let $G$ be an algebraic group over $k$.*

*(a) Let $X$ be a finite-type $k$-scheme with an action of $G$, and let $L$ be a globally generated $G$-linearized invertible sheaf $\Gamma_*(X, L)$ is finitely generated over $\Gamma(X, \mathcal{O}_X)$ (for instance, this holds if $X$ is proper and $L$ is ample, see Proposition A.4). The $G$-linearization on $L$ induces a $G$-module structure on $\Gamma_*(X, L)$ in which $G$ acts by graded ring homomorphisms, and this $G$-module structure induces a $G$-action on $\mathrm{Proj}(\Gamma_*(X, L))$ such that the canonical morphism*

$$X \to \mathrm{Proj}(\Gamma_*(X, L))$$

*is $G$-equivariant.*

(b) *Conversely, let $A = \oplus_{n\geq 0} A_n$ be a graded ring finitely generated in degree 1 over $k \subset A_0$, and suppose that $A$ has the structure of a $G$-module in which $G$ acts by graded ring homomorphisms on $A$. Then, there exists a canonical $G$-action on $Y = \mathrm{Proj}(A)$ and a canonical $G$-linearization of $\mathcal{O}_Y(n)$ for all $n$ such that the canonical morphism*

$$A_n \to H^0(Y, \mathcal{O}_Y(n))$$

*is $G$-equivariant.*

(c) *The constructions of (a) and (b) are inverses, in the following sense. If $f : X \to Y = \mathrm{Proj}(\Gamma_*(X, L))$ is the canonical morphism from (a), then we have $f^*\mathcal{O}_Y(1) \cong L$. In the situation of (a), $f$ is $G$-equivariant, so the $G$-linearization on $\mathcal{O}_Y(1)$ from (b) induces a $G$-linearization on $f^*\mathcal{O}_Y(1)$, hence also on $L$. The $G$-linearization on $L$ induced in this way is the same as the $G$-linearization on $L$ given in (a).*

**Remark 2.4.18.** We fully expect (though we have not checked it) that the above proposition holds without any of the finitely generated assumptions. The only difficulty is that the functor of points of $\mathrm{Proj}(A)$ is somewhat more unwieldy without assuming that the ring $A$ is finitely generated in degree 1 (or at least finitely generated).

As a concrete example of the above proposition, we write out all the actions and representation-theoretic details of the $G$-module $\Gamma_*(X, L)$ in a particularly nice case.

**Example 2.4.19.** Let $k$ be an algebraically closed field of characteristic 0. Let $G = \mathrm{SL}_2$, let $B$ be the subgroup of upper triangular matrices, and let $T$ be the subgroup of diagonal matrices. We define a $G$-module structure on $k[x, y]$ as follows. For any $S = \mathrm{Spec}(R)$, a point $g \in G(S)$ is a $2 \times 2$ matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in $R$. We let $g$ act on the ring $R[x, y] = k[x, y] \otimes R$ via the $R$-linear graded isomorphism $R[x, y] \to R[x, y]$ determined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = dx - by, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y = -cx + ay.$$

Now, Proposition 2.4.17b gives us an action of $G$ on $\mathbb{P}^1 = \mathrm{Proj}(k[x, y])$ and a $G$-linearization of $\mathcal{O}_{\mathbb{P}^1}(1)$ such that the canonical map

$$k[x, y] \to \Gamma_*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

is $G$-equivariant. This map is an isomorphism for $\mathbb{P}^1$, so we may identify $\Gamma_*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with the $G$-module $k[x, y]$. One can check that the above action on $k[x, y]$ agrees with the action of $G$ on $k[x, y]$ described in Example 2.4.8. Indeed, after unwinding definitions, this boils down to the fact that $d$ and $-b$ are the entries in the first column of $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\right)^T$, and $-c$ and $a$ are the entries in the second column of that matrix. It follows that, if we view $\mathbb{A}^2$ as the affine cone over $\mathbb{P}^1$ (see Example A.2), then the resulting action of $G$ on $\mathbb{A}^2$ is the action of

Example 2.4.8. We saw in that example that this action on $\mathbb{A}^2$ is given on functors of points by matrix multiplication, i.e by the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cd + dy).$$

Since the morphism $\pi : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ is $G$-equivariant (see Appendix A), the above equation implies that, in projective coordinates, the action of $G$ on $\mathbb{P}^1$ is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

We are interested in the representation theory of the $G$-module $k[x, y]$. Since $\mathrm{char}(k) = 0$, it is a standard fact from the representation theory of $\mathrm{SL}_2$ (see e.g. [Spr77, Chapter 3]) that for all $d \geq 0$, the $G$-submodule $k[x, y]_d$ of degree-$d$ elements is the unique simple representation of $G$ of dimension $d + 1$. we saw in Example 2.4.8 that $y^d$ is a $B$-eigenvector of $k[x, y]_d$ for all $d$ and that, under the isomorphism $\mathbb{Z} \cong \mathcal{X}(T)$ induced by the isomorphism

$$T \xrightarrow{\sim} \mathbb{G}_m, \quad \mathrm{diag}(t, t^{-1}) \mapsto t,$$

the weight of $y^d$ is $d \in \mathbb{Z}$. (Explicitly, the weight $d \in \mathbb{Z}$ corresponds to the character $T \to \mathbb{G}_m$ given by $\mathrm{diag}(t, t^{-1}) \mapsto t^n$.) Since $k[x, y]_d$ is simple, we conclude that the only $B$-eigenvectors of $k[x, y]$ are the monomials $cy^d$ for any $d \geq 0$ and $c \in k$.

Finally, we note that the $B$-eigenvector $y^d$ vanishes only at the $B$-fixed point $[1 : 0] \in \mathbb{P}^1$. This is an example of the more general fact that divisors cut out by $B$-eigenvectors are $B$-stable, which we will prove in Corollary 2.5.5.

## 2.5 Representation Theory and $G$-Schemes

In the previous section, we defined some very useful $G$-modules arising from actions of $G$ on schemes. In this section, we discuss some general results on the representation theory of these $G$-modules. We assume throughout this section that $k$ is an algebraically closed field of arbitrary characteristic, that $G$ is a reductive group over $k$, and that $T$ and $B$ are a maximal torus and a Borel subgroup (respectively) of $G$ such that $T \subset B \subset G$.

We have seen in Section 2.3 that the representation theory of reductive groups is largely controlled by $B$-eigenvectors and their weights. We now define some notation for certain important sets of weights.

**Definition 2.5.1.** Let $X$ be a $G$-scheme, let $L$ be a $G$-linearized invertible sheaf on $X$, and let $V$ be a $G$-module.

1. We define $\Lambda^+(V)$ to be the set of weights of nonzero $B$-eigenvectors of $V$. Note that $\Lambda^+(V) \subset \Lambda^+_G$ is a set of dominant weights.

2. We define $\Lambda^+(X) = \Lambda^+(\Gamma(X, \mathcal{O}_X))$ (using the usual $G$-module structure on $\Gamma(X, \mathcal{O}_X)$).

3. If $X$ is irreducible, we define $\Lambda(X) = \Lambda^+(K(X))$.

4. We define
$$\Lambda^+(X, L) = \bigcup_{d \geq 0} \left(\Lambda^+(H^0(X, L^{\otimes d})) \times \{d\}\right) \subset \Lambda_G^+ \times \mathbb{N}.$$
(Here, we use the $G$-module structure on $H^0(X, L^{\otimes d})$ coming from the $G$-linearization of $L^{\otimes n}$ from Lemma 2.4.13b.)

Note that our definition of $\Lambda^+(X, L)$ keeps track of both the weights and the degrees of $B$-eigenvectors in the graded ring $\Gamma_*(X, L) = \oplus_{d \geq 0} H^0(X, L^{\otimes d})$. We can relate the degrees of these $B$-eigenvectors to representation theory in the following way. Define

$$\tilde{G} = G \times \mathbb{G}_m, \quad \tilde{B} = B \times \mathbb{G}_m, \quad \tilde{T} = T \times \mathbb{G}_m.$$

Then, $\tilde{G}$ is a reductive group, and $\tilde{T}$ and $\tilde{B}$ are a maximal torus and a Borel subgroup of $\tilde{G}$ (respectively). Moreover, the usual isomorphism $\mathcal{X}(\mathbb{G}_m) \cong \mathbb{Z}$ induces an isomorphism

$$\mathcal{X}(\tilde{T}) \cong \mathcal{X}(T) \times \mathbb{Z}$$

which restricts to a bijection $\Lambda^+_{\tilde{G}}(\tilde{B}, \tilde{T}) \cong \Lambda^+_G(B, T) \times \mathbb{N}$ on dominant weights.

Now, for any $G$-linearized invertible sheaf $L$ on $X$, we can define a $\tilde{G}$-module structure on $\Gamma_*(X, L)$ by letting $G$ act via the $G$-linearization on $L$ and letting $\mathbb{G}_m$ act on $H^0(X, L^{\otimes d})$ by the character $d \in \mathbb{Z} \cong \mathcal{X}(T)$ for all $d$. It follows from the definitions that the $\tilde{B}$-eigenvectors of the $\tilde{G}$-module $\Gamma_*(X, L)$ are precisely the $B$-eigenvectors of $H^0(X, L^{\otimes d})$ for any $d \geq 0$, and the weight of such an eigenvector is $(\mu, d) \in \mathcal{X}(T) \times \mathbb{Z}$, where $\mu$ is the weight of the eigenvector viewed as a $B$-eigenvector. In other words, we have

$$\Lambda^+(X, L) = \Lambda^+(\Gamma_*(X, L)),$$

where we use the $\tilde{G}$-module structure to compute the set on the righthand side. If $X$ is projective, $L$ is ample, and $\tilde{X} = \mathrm{Spec}(\Gamma_*(X, L))$ is the affine cone over $X$, then we could also write the above equation as $\Lambda^+(X, L) = \Lambda^+(\tilde{X})$. Thus, our definition of $\Lambda^+(X, L)$ as "keeping track of both weights and degrees" has a natural representation-theoretic interpretation (involving the $\tilde{G}$-module structure of $\Gamma_*(X, L)$) and a natural geometric interpretation (involving the affine cone $\tilde{X}$).

In the remainder of this section, we will collect several standard results about the $B$-eigenvectors of the $G$-modules $\Gamma(X, \mathcal{O}_X)$ and $\Gamma_*(X, L)$. We begin with a proposition that allows us to understand eigenvectors in function fields using global sections of an ample line bundle.

**Proposition 2.5.2** (cf. [Bri10, Proposition 2.8]). *Let $G$ be a reductive group over an algebraically closed field $k$, let $B \subset G$ be a Borel subgroup, and let $U = R_u(B)$. Let $X$ be a quasi-projective $G$-variety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$.*

(a) $\Gamma_*(X,L)$ *is an integral domain, and we have a canonical* $G$*-equivariant isomorphism of rings*

$$K(X) \cong (\mathrm{Frac}(\Gamma_*(X,L)))_0,$$

*where the* $G$*-module structure on the right-hand side is the one induced by the* $G$*-module structure on* $\Gamma_*(X,L)$ *(see Proposition 2.4.17a).*

(b) *The isomorphism of (a) restricts to an isomorphism* $K(X)^U = \mathrm{Frac}(\Gamma_*(X,L)^U)_0$.

(c) *Under the isomorphism of (a), every* $B$*-eigenvector* $q \in K(X)^{(B)}$ *can be written as* $q = f/g$*, where* $f,g \in \Gamma(X, L^{\otimes d})^{(B)}$ *for some* $d \geq 0$.

*Proof.* For a proof that $\Gamma_*(X,L)$ is an integral domain, see Proposition A.4 (that proposition assumes that $X$ is projective, but projectivity is not needed for the proof that $\Gamma_*(X,L)$ is an integral domain). Moreover, because $L$ is ample, we have a canonical $G$-equivariant dominant open immersion $i : X \hookrightarrow \mathrm{Proj}(\Gamma_*(X,L))$ such that $i^*\mathcal{O}(1) \cong L$ (see Proposition 2.4.17a). The morphism $i$ induces a $G$-equivariant isomorphism on function fields, which is the isomorphism in the proposition statement. One can check that the $G$-module structure of the function field of $\mathrm{Proj}(\Gamma_*(X,L))$ is actually the structure on $(\mathrm{Frac}(\Gamma_*(X,L)))_0$ induced by the $G$-module structure of $\Gamma_*(X,L)$, so this completes the proof of (a).

For statement (c), we may replace $X$ by $\mathrm{Proj}(\Gamma_*(X,L))$. Write $A = \Gamma_*(X,L)$, and for any $q \in K(X)^{(B)}$, consider the "set of denominators" of $q$:

$$D = \{d \in A \mid qd \in A\}.$$

Notice that $D$ is an ideal of $A$, and in particular is a $k$–vector space. We claim that $D$ is a $B$-submodule of $A$. It suffices to check this on $k$-points, i.e. to check that for all $b \in B(k)$ and any $d \in D$, we have $bd \in D$ (see e.g. [Mil17, Corollary 4.5]). For this, let $b \in B(k)$ and $d \in D$, and let $\chi : B \to \mathbb{G}_m$ be the character by which $B$ acts on $q$. Since $b$ acts on $A$ by a ring homomorphism, we have

$$q \cdot (bd) = \chi(b)^{-1}(b \cdot q)(b \cdot d) = \chi(b)^{-1}(b \cdot (qd)),$$

and the righthand side is in $A$ because $qd \in A$. (Here we are also implicitly using that the $G$-module structure on $K(X) = \mathrm{Frac}(A)_0$ is induced by the $G$-module structure on $A$.) Thus, we have $bd \in D$ by definition, which proves that $D$ is $B$-stable.

Now, $D$ is a $B$-module in its own right. Since $B$ is solvable, $D$ must contain some $B$-eigenvector $g \in D^{(B)}$ (Theorem 2.3.5). Write $f = qg$, and let $\lambda : B \to \mathbb{G}_m$ be the character associated to $g$. Then, $f, g \in A$, and for any $b \in B(k)$, we have

$$f/g = q = \chi^{-1}(b)(b \cdot q) = \chi^{-1}(b \cdot f)/(b \cdot g) = \chi^{-1}(b)\lambda(b)^{-1}(b \cdot f)/g,$$

so that $\chi(b)\lambda(b)f = b \cdot f$. Thus, $f$ is also a $B$-eigenvector with associated character $\chi + \lambda$. Finally, we note that $f/g = q \in K(X) = \mathrm{Frac}(A)_0$, so $f$ and $g$ must have the same degree in $A$. This implies that $f, g \in \Gamma(X, L^{\otimes d})$ for some $d \geq 0$.

Statement (b) can be proven by an analogous argument to that for (c) above. Alternately, since the $B$-eigenvectors of $K(X)$ are exactly the $U$-invariant elements (Theorem 2.3.4), statement (b) follows immediately from (c). $\square$

Next, we give a few important facts about divisors cut out by sections of line bundles. For this, we recall that $f \in K(X)$ determines a (principal) Cartier divisor $\operatorname{div}(f)$ on $X$, and for any line bundle $L$ on $X$, a nonzero global section $s \in H^0(X, L)$ cuts out an effective Cartier divisor $\operatorname{div}(s)$ on $X$.

**Lemma 2.5.3.** *Let $X$ be a normal $G$-variety, and let $L$ be a $G$-linearized invertible sheaf on $X$. Let $s_0 \in H^0(X, L)$ be a nonzero section, and let $D_0 = \operatorname{div}(s_0)$.*

*(a) For any other nonzero section $s \in H^0(X, L)$, we have $\operatorname{div}(s) = D_0 + \operatorname{div}(s/s_0)$ (here viewing $s/s_0$ as an element of $K(X)$).*

*(b) We have a $G$-equivariant isomorphism*

$$H^0(X, L) \to \{f \in K(X) \mid D_0 + \operatorname{div}(f) \textit{ is effective}\} \quad s \mapsto s/s_0.$$

*In particular, if $s_0$ is a $B$-eigenvector of weight $\mu_0$ and $f \in K(X)^{(B)}$ is an eigenvector of weight $\mu$ such that $D_0 + \operatorname{div}(f)$ is effective, then the element of $H^0(X, L)$ corresponding to $f$ is a $B$-eigenvector of weight $\mu_0 + \mu$.*

*Proof.* Statement (a) follows directly from the definitions of the divisors involved. More precisely, $\operatorname{div}(s)$ is defined by taking an open cover $\{U_i\}_i$ of $X$ such that $L|_{U_i} \cong \mathcal{O}_{U_i}$ for all $i$ and then letting $\operatorname{div}(s)$ be the divisor which on $U_i$ is cut out by the section $s|_{U_i} \in \Gamma(U_i, \mathcal{O}_{U_i})$. In other words, $\operatorname{div}(s)$ is represented by the data $\{(U_i, s|_{U_i})\}_i$, see [GW10, Section 11.9 and Proposition 11.32]. The divisor $D_0 = \operatorname{div}(s_0)$ is defined analogously, and since we view $s/s_0$ as an element of $K(X)$ via a local isomorphism $L|_{U_i} \cong \mathcal{O}_{U_i}$, the divisor $\operatorname{div}(s/s_0)$ is defined by the data $\{(U_i, s|_{U_i} s_0|_{U_i}^{-1}\}_i$. Thus, statement (a) boils down to the fact that

$$s_0|_{U_i} \cdot s|_{U_i} s_0|_{U_i}^{-1} = s|_{U_i}$$

for all $i$.

Statement (b) is also closely related to standard facts and definitions involving Cartier divisors. In fact, we have an isomorphism $L \cong \mathcal{O}_X(D_0)$ which sends $s_0$ to the canonical section of $\mathcal{O}_X(D_0)$, and the global sections of $\mathcal{O}_X(D_0)$ are precisely the target of the map in (b), see [GW10, Section 11.12 and Proposition 11.32], so this immediately gives us an isomorphism as in (b). However, it is not entirely clear whether this isomorphism is $G$-equivariant. As such, we re-derive this isomorphism in another way to ensure that the map we get is $G$-equivariant.

Let $U \subset X$ be the union of the sets $X_s$ for any global section $s \in H^0(X, L^{\otimes n})$ and any $n \geq 1$. Since the sets $X_s$ cover $U$, there exists a canonical morphism $f : U \to \operatorname{Proj}(\Gamma_*(U, L))$.

The morphism $f$ is $G$-equivariant by Proposition 2.4.17 and dominant by [Sta20, Tag 01Q0], so $f$ induces a $G$-equivariant injection

$$\iota : (\mathrm{Frac}(\Gamma_*(U, L)))_0 \hookrightarrow K(X).$$

On the other hand, $U$ is $G$-stable because $g \cdot X_s = X_{gs}$ for any $g \in G$, so the inclusion map $U \hookrightarrow X$ is $G$-equivariant. This implies that the restriction map $r : H^0(X, L) \to H^0(U, L)$ is $G$-equivariant, and $r$ is injective because $X$ is integral. So, we define a map

$$\alpha : H^0(X, L) \to K(X)$$

by setting $\alpha(s) = \iota(r(s))/\iota(r(s_0))$. The map $\alpha$ is $G$-equivariant and injective because $r$ and $\iota$ are. By tracing through the definitions, one can check that for any $s \in H^0(X, L)$, we have $\alpha(s) = s/s_0$, where we identify $s$ and $s_0$ with elements of $K(X)$ by picking a local isomorphism $L|_V \cong \mathcal{O}_V$ for some open subset $V \subset X$. (Note that the element $s/s_0 \in K(X)$ does not depend on $V$, because we are restricting from $V$ to the function field, and it also does not depend on the choice of isomorphism $L|_V \cong \mathcal{O}_V$, because a different choice of isomorphism would amount to multiplying both $s$ and $s_0$ by the same unit.)

It remains to check that the elements $s/s_0 \in K(X)$ for $s \in H^0(X, L)$ are precisely the rational functions $f \in K(X)$ such that $D_0 + \mathrm{div}(f) \geq 0$. For any $s \in H^0(X, L)$, part (a) gives us $D_0 + \mathrm{div}(s/s_0) = \mathrm{div}(s_0)$, and $\mathrm{div}(s_0)$ is effective by definition. Conversely, let $f \in K(X)$ be such that $D_0 + \mathrm{div}(f) \geq 0$, and as above, let the data $\{(U_i, s_0|_{U_i})\}_i$ represent the Cartier divisor $D_0 = \mathrm{div}(s_0)$. The divisor $D_0 + \mathrm{div}(f)$ is represented by $\{(U_i, s_0|_{U_i} f)\}_i$, and the fact that this divisor is effective means we can take the $U_i$ to be such that $s_0|_{U_i} f \in \Gamma(U_i, \mathcal{O}_X)$. Let $s_i \in \Gamma(U_i, L)$ be the image of $s_0|_{U_i} f$ under the isomorphism $\mathcal{O}_{U_i} \cong L|_{U_i}$. Since $s_0$ and $f$ are defined globally, the $s_i$ agree on intersections $U_i \cap U_j$ and so glue to a global section $s \in H^0(X, L)$. Since $f = s|_{U_i} s_0|_{U_i}^{-1}$ for all $i$, we see that $\mathrm{div}(s/s_0) = \mathrm{div}(f)$. $\square$

**Proposition 2.5.4.** *Let $X$ be a normal $G$-variety, let $L$ be a $G$-linearized invertible sheaf on $X$, and let $f \in H^0(X, L)$. Define two subgroups $H, H' \subset G$ by*

$$H = \{g \in G \mid g \cdot X_f = X_f\}, \quad H' = \{g \in G \mid g \cdot f = cf,\ c \in k^\times\}.$$

*We have $H' \subset H$. If $H$ is connected, then $H = H'$.*

*Proof.* We remark that $H$ and $H'$ are well-defined, because subgroups are determined by their $k$-points ([Mil17, Theorem 1.45]), and that any containments between them can be checked on $k$-points as well (cf. [Mil17, Corollary 1.44]). For any $g \in G(k)$, we have $g \cdot X_f = X_{g \cdot f}$ (Lemma 2.4.16). Since $X_f = X_{cf}$ for any $c \in k^\times$, this immediately implies that $H' \subset H$.

It remains to prove the reverse containment when $H$ is connected. Write $D = \mathrm{div}(f)$, and let $\rho : H \times X \to X$ be the action map. Since $X \setminus D = X_f$, we have $H \cdot D = D$, i.e. $\rho^{-1}(D) = H \times D$. If $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf corresponding to $D$ (as a reduced closed subscheme of $X$), then $H \times D$ has ideal sheaf $\mathrm{pr}_X^* \mathcal{I}$, and $\rho^{-1}(D)$ has ideal sheaf $\rho^* \mathcal{I}$ (both as reduced closed subschemes of $H \times X$). The fact that $\rho^{-1}(D) = H \times D$ as sets implies that $\rho^{-1}(D)$

and $H \times D$ are equal as reduced closed subschemes, so $\rho^*\mathcal{I} = \mathrm{pr}_X^*\mathcal{I}$ (as sheaves of ideals in $\mathcal{O}_{H\times X}$). It follows that the canonical $H$-linearization of $\mathcal{O}_X$ restricts to an $H$-linearization $\rho^*\mathcal{I} \to \mathrm{pr}_X^*\mathcal{I}$ of $\mathcal{I}$. (Here we are implicitly using the fact that the canonical $H$-linearization of $\mathcal{O}_X$ is the composition of the canonical isomorphisms $\rho^*\mathcal{O}_X \cong \mathcal{O}_{H\times X} \cong \mathrm{pr}_X^*\mathcal{O}_X$, which are the isomorphisms that we use to identify $\rho^*\mathcal{I}$ and $\mathrm{pr}_X^*\mathcal{I}$ with sheaves of ideals in $\mathcal{O}_{H\times X}$.) Similarly, the given $G$-linearization on $L$ induces an $H$-linearization on $L$ (by pulling back the $G$-linearization by the inclusion map $H \times X \hookrightarrow G \times X$).

Now, we have a canonical isomorphism

$$\iota : \mathcal{I} \otimes L \xrightarrow{\sim} \mathcal{O}_X.$$

The $H$-linearizations on $\mathcal{I}$ and $L$ give us an $H$-linearization on $\mathcal{I} \otimes L$ (see Lemma 2.4.13), and identifying $\mathcal{I} \otimes L$ with $\mathcal{O}_X$ via $\iota$ then gives us an $H$-linearization on $\mathcal{O}_X$ such that $\iota$ is an isomorphism of $G$-linearized sheaves (hence is a $G$-equivariant isomorphism on global sections). Since $\iota(1 \otimes f) = 1$ and $H$ acts trivially on $1 \in H^0(X, \mathcal{I})$, we can read off the action of $H$ on $f$ from the action of $H$ on $1 \in H^0(X, \mathcal{O}_X)$. (Here we are implicitly using that the "tensor product map" on global sections in Lemma 2.4.13b is $H$-equivariant.) By Theorem 2.6.10 below (see also Corollary 2.6.8 below and its proof), our $H$-linearization $\phi$ of $\mathcal{O}_X$ is given by some character $\mu \in \mathcal{X}(H)$, and with this $H$-linearization, the action of $H$ on $H^0(X, \mathcal{O}_X)$ is the canonical $H$-module structure on $\Gamma(X, \mathcal{O}_X)$ "multiplied by" $\mu$. (Note we need connectedness of $H$ in order to apply Theorem 2.6.10.) In particular, since $H$ acts by the identity on $1 \in \Gamma(X, \mathcal{O}_X)$ under the canonical $H$-module structure, we see that $H$ acts on 1 by $\mu$ under the $H$-module structure induced by $\phi$. So, $H$ acts by $\mu$ on $f$, which implies that $f$ is an $H$-eigenvector and hence that $H \subset H'$. □

**Corollary 2.5.5.** *Let $X$ be a normal $G$-variety, let $L$ be a $G$-linearized invertible sheaf on $X$, and let $f \in H^0(X, L)$. Then, $f$ is a $B$-eigenvector if and only if the divisor* $\mathrm{div}(f)$ *is $B$-stable. Moreover, if this is the case, then for any $g \in G$, we have $g \cdot X_f = X_f$ if and only if $g \cdot f = cf$ for some $c \in k^\times$.*

*Proof.* Let $H$ and $H'$ be as in Proposition 2.5.4. Note that $f$ is a $B$-eigenvector if and only if $k \cdot f$ is a $B$-submodule of $H^0(X, L)$, and this can be checked on $k$-points ([Mil17, Corollary 4.5]). It follows that $f$ is a $B$-eigenvector if and only if $B \subset H'$. Moreover, since $X \setminus \mathrm{div}(f) = X_f$, we see that $f$ is $B$-stable if and only if $B \subset H$. In particular, we have $B \subset H$ in either case, so $H$ is parabolic, hence connected ([Mil17, Corollary 17.49]). The proposition thus tells us that $H' = H$, which in paricular means that $B \subset H'$ if and only if $B \subset H$. □

The following statement says that we can "lift" eigenvectors from $G$-stable closed subschemes.

**Proposition 2.5.6** ([Kno91, Theorem 1.1]; [Tim11, Corollary D.2, Lemma 5.8])**.** *Let $G$ be a connected reductive group over an algebraically closed field $k$, let $p$ be the characteristic exponent of $k$ (i.e. $p = 1$ if* $\mathrm{char}(k) = 0$ *and $p =$* $\mathrm{char}(k)$ *otherwise). Let $X$ be a $G$-scheme, and let $Y \subseteq X$ be a $G$-stable closed subscheme.*

(a) *If $X$ is affine, then for any $B$-eigenvector $f \in \Gamma(Y, \mathcal{O}_Y)^{(B)}$, there exists some $N \in \mathbb{N}$ and some $f' \in \Gamma(X, \mathcal{O}_X)^{(B)}$ such that $f'|_Y = f^{p^N}$.*

(b) *Suppose $X$ and $Y$ are irreducible. For any $B$-eigenvector $f \in K(Y)^{(B)}$, there exists some $N \in \mathbb{N}$ and some $f' \in K(X)^{(B)}$ such that $f'|_Y = f^{p^N}$. In fact, $f' \in \mathcal{O}_{X,\eta}$, where $\eta \in Y$ is the generic point.*

*sketch of proof.* We sketch the proof of (a) in the case that $\operatorname{char}(k) = 0$. The general case is somewhat more delicate; see [Tim11, Corollary D.2] for details. Suppose that $\operatorname{char}(k) = 0$ and that $X = \operatorname{Spec}(A)$. Then, $G$ acts on $A$, and since $Y$ is $G$-stable, we have $Y = \operatorname{Spec}(A/I)$ for some $G$-stable ideal $I \subset A$. Since $\operatorname{char}(k) = 0$, the $G$-modules $A$ and $I$ are both semisimple, so $A \cong \bigoplus_{\mu \in \Lambda_G^+} V(\mu)^{m_\mu}$ and so $I \cong \bigoplus_{\mu \in \Lambda_G^+} V(\mu)^{n_\mu}$ for some $m_\mu$ and $n_\mu$. Since $I \subset A$, we have $n_\mu \leq m_\mu$ for all $\mu$. It follows that

$$A/I \cong \bigoplus_{\mu \in \Lambda_G^+} V(\mu)^{m_\mu - n_\mu}$$

as $G$-modules. In particular, the quotient map $A \to A/I$ admits a section (as a map of $G$-modules). Statement (a) follows formally from the existence of such a section.

For (b), the idea is to reduce to the case where $X$ is affine; in that case, (b) can be deduced somewhat formally from (a). See [Tim11, Lemma 5.8] for details. We briefly sketch the reduction to the affine case, since this is somewhat subtle and is not fully explained in [Tim11]. First, we can replace $X$ by its normalization (see Lemma 2.6.15 below) and so assume that $X$ is normal. Since $X$ is normal, we can replace $X$ by some $G$-stable quasi-projective open subset and then by its closure in $\mathbb{P}(V)$ for some $G$-module $V$ (see Theorem 2.6.12 below) and so assume that $X$ is projective. Finally, we replace $X$ by the affine cone $\tilde{X}$.

□

We know that the representation theory of $G$ is largely controlled by $B$-eigenvectors, which themselves are controlled by $T$-eigenvectors. The following proposition provides a geometric way to pass between the representation theory of $G$ and that of $T$.

**Proposition 2.5.7** (cf. [Tim11, Theorem D.5])**.** *Let $G$ be a reductive group over an algebraically closed field $k$, let $T \subset G$ be a maximal torus, and let $B \subset G$ be a Borel subgroup containing $T$. Let $X$ be an affine $G$-variety, and write $U = R_u(B)$.*

(a) *The GIT quotient $X//U = \operatorname{Spec}(A^U)$ is a $T$-variety. In particular, the $k$-algebra $A^U$ is finitely generated.*

(b) *If $\operatorname{char}(k) = 0$, then $X$ is normal (resp. has rational singularities) if and only if $X//U$ is (resp. does).*

(c) *We have equalities*

$$K(X//U) = K(X)^U, \qquad \Lambda(X) = \Lambda(X//U), \qquad \Lambda^+(X) = \Lambda^+(X//U).$$

*(Here we think of $\Lambda(X//U)$ and $\Lambda^+(X//U)$ as coming from the structure of $X//U$ as a $T$-variety, which makes sense because $T$ is a reductive group.)*

*Proof.* Since $B = U \cdot T$ and $U$ is a normal subgroup of $B$, we see that the action of $T$ on $A$ fixes $A^U$. Thus, $A^U$ is a $T$-submodule of $A$, and the action of $T$ on $A^U$ gives us an action of $T$ on $X//U$. The scheme $X//U$ is separated (because it is affine), and since $A$ is an integral domain, so is $A^U$, i.e. $X//U$ is integral. To show that $X//U$ is a variety, then, we just need to show that $A^U$ is a finitely generated $k$-algebra. This is a rather subtle fact; for a proof, see [Bri10, Theorem 2.7].

For statement (c), the equality $K(X//U) = K(X)^U$ is immediate from Proposition 2.5.2 (applied to the ample line bundle $L = \mathcal{O}_X$). The equality $\Lambda(X) = \Lambda(X//U)$ follows immediately from the fact that the $B$-eigenvectors of $A$ are precisely the $T$-eigenvetors which are fixed by $U$ (Theorem 2.3.4). This same fact (applied to $A$ instead of to $K(X)$) gives us $\Lambda^+(X) = \Lambda^+(X//U)$ as well.

As for (b), if $X$ is normal, then $A$ is integrally closed, so we immediately see that $A^U$ is integrally closed in $K(X)^U = \mathrm{Frac}(A^U)$ (any element integral over $A^U$ must be $U$-invariant), which proves that $X//U$ is normal. For a proof of the converse, see [Tim11, Theorem D.5] (or [Bri10, Proposition 2.8.] for a more elementary version of the argument). For a proof of the rational singularities statement, see [Tim11, Theorem D.5]. $\square$

GIT quotients are also useful in proving the following lemma. This lemma is rarely found in the literature (perhaps because its hypotheses are relatively restrictive), but it will be quite useful to us in a few places.

**Lemma 2.5.8.** *Let $G$ be a reductive group over an algebraically closed field $k$, and let $X$ be an affine integral $G$-scheme. If $X$ has a dense $G$-orbit, then $X$ has a unique closed $G$-orbit.*

*Proof.* Wrtie $X = \mathrm{Spec}(A)$, and let $\pi : X \to X//G = \mathrm{Spec}(A^G)$ be the GIT quotient map (i.e. $\pi$ is the morphism corresponding to the inclusion map $A^G \hookrightarrow A$). It is a standard fact about the GIT quotient (see e.g. [MF82, Corollaries 2.1.2 and Appendix to Chapter 1, Corollary A.1.3]) that every fiber of $\pi$ contains a unique closed $G$-orbit. We claim that $A^G$ is a field: then, $X//G$ is a point and $X$ is itself the unique fiber of $\pi$, so $X$ contains a unique closed $G$-orbit.

That $A^G$ is a field is an immediate consequence of some general standard results from invariant theory (specifically, it follows directly from Fact 1 in our discussion on complexities just before Theorem 3.1.4). We give here a more fundamental and direct proof that $A^G$ is a field. Let $X_G^\circ \subset X$ be the dense $G$-orbit. Note that for any $f \in A^G$ nonzero, the set $X_f$ is a nonempty $G$-stable open subset by Corollary 2.5.5 (applied to $\mathcal{O}_X$ with the canonical $G$-linearization), so we have $X_G^\circ \cap \mathrm{D}(f) \neq \varnothing$ and hence $X_G^\circ \subset X_f$. It follows that if $f$ is a non-unit, it must be contained in one of the finitely many prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ corresponding to the generic points of the irreducible components of $X \setminus X_G^\circ$. Write $\mathfrak{p}_i = \mathfrak{q}_i \cap A^G$ for all $i$. Then, the $\mathfrak{p}_i$ are a set of finitely many prime ideals containing every non-unit in $A^G$. Hence the union $\cup_i \mathfrak{p}_i$ contains every prime ideal of $A^G$, and it follows from the prime avoidance lemma that the $\mathfrak{p}_i$ are the maximal ideals of $A^G$. On the other hand, since $A^G$ is finitely generated over $k$ (see e.g. [MF82, Theorem 1.1] or [Bri10, Theorem 1.24]), it is a Jacobson ring, and $A^G$ is a domain because $A$ is. So, we see that

$$\mathfrak{p}_1 \cdots \mathfrak{p}_r \subset \bigcap_i \mathfrak{p}_i = \mathrm{Nilrad}(A^G) = (0),$$

which is only possible if $\mathfrak{p}_i = (0)$ for all $i$. So $(0)$ is the unique maximal ideal of $A^G$, i.e. $A^G$ is a field, as desired. ☐

We end this section by showing that the monoid $\Lambda^+(X)$ and the abelian group $\Lambda(X)$ are always finitely generated.

**Proposition 2.5.9.** *Let $X$ be a $G$-scheme.*

*(a) If $X$ is affine, then $\Lambda^+(X)$ is a finitely generated commutative submonoid of $\Lambda_G^+$. Moreover, we have*

$$\Lambda(X) = \mathrm{Span}_{\mathbb{Z}}(\Lambda^+(X)) = \Lambda^+(X)^{gp}$$

*as subgroups of $\Lambda_G$.*

*(b) If $X$ is irreducible, then $\Lambda(X)$ is a finitely generated free abelian group.*

*Proof.* $\Lambda(X)$ is by definition a subgroup of $\Lambda_G$, and $\Lambda_G = (T) \cong \mathbb{Z}^n$. Since a subgroup of a finitely generated free abelian group is again a free finitely generated abelian group, we immediately obtain (b). As for (a), we know that $\Lambda^+(X)$ is a submonoid of $\Lambda_G^+$ by definition and hence is commutative (because $\Lambda_G^+ \subset \Lambda_G$ is). The equality $\Lambda(X) = \mathrm{Span}_{\mathbb{Z}}(\Lambda^+(X))$ follows immediately from Proposition 2.5.2 (applied to the ample line bundle $\mathcal{O}_X$ with the canonical $G$-linearization). and the equality $\mathrm{Span}_{\mathbb{Z}}(\Lambda^+(X)) = \Lambda^+(X)^{gp}$ is immediate from the definitions (and the fact that $\Lambda^+(X)$ is commutative).

It remains to show that $\Lambda^+(X)$ is finitely generated. For this, write $X = \operatorname{Spec} A$ and $U = R_u(B)$. Proposition 2.5.7 implies that $A^U$ is a finitely generated $k$-algebra equipped with an action of $T$, and that $\Lambda^+(X) = \Lambda^+(A^U)$. So, we need to show that $\Lambda^+(A^U)$ (i.e. the monoid of $T$-eigenvectors in $A^U$) is finitely generated. Since every $T$-module decomposes as a direct sum of characters, we may find $T$-eigenvectors $f_1, \dots, f_n \in A^U$ that generate $A^U$ as a $k$-algebra (just pick any set of generators and write each of them as a $k$-linear combination of eigenvectors; the set of all the eigenvectors appearing in these linear combinations then generates $A^U$).

Let $\mu_i$ be the character corresponding to $f_i$. We claim that the $\mu_i$ generate $\Lambda^+(A^U)$. Let $a \in (A^U)^{(T)}$ be a $T$-eigenvector; we show that the character $\mu_a$ corresponding to $a$ is in the monoid generated by the $\mu_i$. Write

$$a = \sum_{i=1}^r a_i m_i,$$

where $m_i = \prod_{j=1}^n f_j^{n_{ij}}$ for some $n_{ij} \in \mathbb{N}$ and $a_i \in k^\times$ for all $i$. After replacing some $m_i$ by a linear combination of the other $m_i$ if necessary, we may assume that the $m_i$ are linearly independent (over $k$). Write $\mu_{m_i} = \sum_j n_{ij}\mu_j$ for the character corresponding to $m_i$. For any $t \in T(k)$, we have

$$\sum_{i=1}^r a_i \mu_{m_i}(t) m_i = \sum_{i=1}^r a_i (t \cdot m_i) = t \cdot a = \mu_a(t) a = \sum_{i=1}^r a_i \mu_a(t) m_i,$$

so that

$$\sum_{i=1}^r a_i(\mu_{m_i}(t) - \mu_a(t)) m_i = 0.$$

Since the $m_i$ are linearly independent and $a_i \neq 0$ for all $i$, we conclude that $\mu_{m_i}(t) = \mu_a(t)$ for all $i$ and all $t \in T(k)$, so that $\mu_{m_i} = \mu_a$ for all $i$. But the $\mu_{m_i}$ are in the monoid generated by the $\mu_i$, hence so is $\mu_a$. □

The above proposition justifies the following definition.

**Definition 2.5.10.** Let $X$ be an irreducible $G$-scheme. We define the *rank* of $X$, denoted $r(X)$, to be the rank of $\Lambda(X)$ (as a free finitely generated abelian group).

## 2.6 More on $G$-Linearizations

Given a $G$-linearized invertible sheaf $L$ on a $G$-variety $X$, our results in the previous section indicate that the $G$-module $\Gamma_*(X, L)$ relates to the geometry of $X$ in some interesting ways. As such, we will often be interested in equipping invertible sheaves with $G$-linearizations, so that we can study the $G$-module $\Gamma_*(X, L)$. However, $G$-linearizations are somewhat technical to work with directly. In this section, we collect several key results which together give us a nice foundation for handling many technical details related to $G$-linearizations. Throughout, we assume that $k$ is an algebraically closed field. Some of what we say will have more general analogs, but the algebraically closed case is slightly cleaner (and is all we will need for our purposes). Also, we assume that $G$ is an arbitrary (i.e. not necessarily reductive) algebraic group.

### 2.6.a Uniqueness of $G$-Linearizations

Suppose we are given a $G$-variety $X$ and a $G$-linearized invertible sheaf $L$ on $X$. It is natural to ask: is the given $G$-linarization on $L$ unique? In general, the answer is no. In this section, we determine the extent to which a $G$-linearization of $L$ is non-unique, and we determine how a different choice of $G$-linearization affects the resulting $G$-module structure on $H^0(X, L)$.

We begin by giving an exact sequence which completely describes the extent to which $G$-linearizations are non-unique. To state it, recall that $\mathrm{Pic}_G(X)$ denotes the abelian group of isomorphism classes of $G$-linearized invertible sheaves on a $G$-scheme $X$. By construction of the group operation on $\mathrm{Pic}_G(X)$ (see Lemma 2.4.13), the map $\varphi : \mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$ that "forgets the $G$-linearization" is a homomorphism. We also note that the set of $G$-linearizations on $\mathcal{O}_X$ can be viewed as a group whose operation is given by taking the tensor product of $G$-linearizations (which is again a $G$-linearization on $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$, see Lemma 2.4.13b). The identity element of this group is the canonical $G$-linearization of $\mathcal{O}_X$ given in Lemma 2.4.13a. Moreover, one can check that this group operation is commutative (this essentially boils down to the fact that the isomorphism $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$ identifies a map $\phi \otimes \phi'$ with the product $\phi \cdot \phi'$, and multiplication of sections in $\mathcal{O}_X$ is commutative).

**Proposition 2.6.1.** *Let $X$ be a $G$-scheme. There is an exact sequence of abelian groups*

$$0 \to \Gamma(X, \mathcal{O}_X^\times)^G \to \Gamma(X, \mathcal{O}_X^\times) \to \{G\textit{-linearizations on } \mathcal{O}_X\} \xrightarrow{\iota} \mathrm{Pic}_G(X) \xrightarrow{\varphi} \mathrm{Pic}(X).$$

*Here, $\iota$ is the map which sends a $G$-linearization $\phi$ of $\mathcal{O}_X$ to the isomorphism class of $\mathcal{O}_X$ equipped with $\phi$, and $\varphi$ is the map which "forgets" the $G$-linearization.*

*Proof.* Exactness at $\mathrm{Pic}_G(X)$ is immediate from the definitions of $\varphi$ and $\iota$: indeed, $\ker(\varphi)$ is by definition the set of isomorphism classes of $G$-linearizations of $\mathcal{O}_X$, which is by definition the image of $\iota$. Exactness at $\Gamma(X, \mathcal{O}_X^\times)^G$ is also immediate (since the map to $\Gamma(X, \mathcal{O}_X^\times)$ is just the inclusion map). So, we just need to define the map

$$\mu : \Gamma(X, \mathcal{O}_X^\times) \to \{G\text{-linearizations on } \mathcal{O}_X\}$$

and prove exactness at the domain and target of this map.

To define the map $\mu$, recall that there exists a canonical $G$-linearization $\phi_0$ of $\mathcal{O}_X$ (Lemma 2.4.13) and that giving a $G$-linearization $\phi$ of $\mathcal{O}_X$ is equivalent to giving maps $\phi_{g,R}$ for each $S = \mathrm{Spec}(R)$ and $g \in G(S)$ satisfying certain conditions (see Lemma 2.4.11). For any unit $u \in \Gamma(X, \mathcal{O}_X^\times)$ and any $S$ and $g \in G(S)$, we define

$$\phi_{g,R} = (\rho_{g,R}^* u) \cdot u^{-1} \cdot (\phi_0)_{g,R}.$$

(Here, $\cdot$ denotes multiplication of the map of $\mathcal{O}_{X \times S}$-modules $(\phi_0)_{g,R}$ by an element of $\Gamma(X \times S, \mathcal{O}_{X\times S}^\times)$ in the natural way, and we identify $\Gamma(X, \mathcal{O}_X^\times)$ with its image in $\Gamma(X \times S, \mathcal{O}_{X\times S}^\times)$ under the pullback by the projection map $X \times S \to X$.) One can check that this definition of the $\phi_{g,R}$ does satisfy the conditions in Lemma 2.4.11, so the lemma implies that the $\phi_{g,R}$ correspond to a $G$-linearization $\phi$ of $\mathcal{O}_X$. We define the map $\mu$ by setting $\mu(u) = \phi$. One can check that $\mu$ is a group homomorphism.

Next, we prove exactness at the domain of $\mu$. As noted above, the identity element of the group of $G$-linearizations on $\mathcal{O}_X$ is the canonical $G$-linearization $\phi_0$ from Lemma 2.4.13a. So, the kernel of $\mu$ is the set of all $u \in \Gamma(X, \mathcal{O}_X^\times)$ such that the $G$-linearization $\phi$ defined above is equal to $\phi_0$, or equivalently, such that $\phi_{g,R} = (\phi_0)_{g,R}$ for all $g$ and $S$. On the other hand, the map $(\phi_0)_{g,R}$ is by construction the canonical isomorphism $\rho_{g,R}^* \mathcal{O}_X \cong \mathcal{O}_{X\times S}$ (cf. the proof of Lemma 2.4.13), and when we write $(\rho_{g,R}^* u) \cdot (\phi_0)_{g,R}$ in the definition of $\rho_{g,R}$, we are implicitly identifying $\rho_{g,R}^* u$ with its image under this canonical isomorphism (i.e. we actually mean to multipliy $(\phi_0)_{g,R}$ by $((\phi_0)_{g,R} \circ \rho^*_{g,R})u)$. But $((\phi_0)_{g,R} \circ \rho^*_{g,R})u = g \cdot u$ is by definition the action of $g$ on $u$ under the $G$-module structure on $H^0(X, \mathcal{O}_X)$ induced by $\phi_0$ (which is the same as the usual $G$-module structure on $\Gamma(X, \mathcal{O}_X$, see Lemma 2.4.13a). So, we have

$$\phi_{g,R} = (g \cdot u) \cdot (u^{-1}) \cdot (\phi_0)_{g,R},$$

and it follows that $\phi_{g,R} = (\phi_0)_{g,R}$ if and only if $g \cdot u = u$. Thus, $\mu(u) = \phi_0$ if and only if $g \cdot u = u$ for all $g$ and $S$, i.e. if and only if $u \in \Gamma(X, \mathcal{O}_X)^G$.

As for exactness at the target of $\mu$, we note that the identity in $\mathrm{Pic}_G(X)$ is by definition the isomorphism class of $\mathcal{O}_X$ with the $G$-linearization $\phi_0$. Any $G$-linearization $\phi$ of $\mathcal{O}_X$ is in this class if and only if there exists an automorphism $\alpha : \mathcal{O}_X \to \mathcal{O}_X$ which identifies $\phi$ with $\phi_0$. Equivalently, $\alpha$ must identify $\phi_{g,R}$ with $(\phi_0)_{g,R}$ for all $g$ and $S$. This identification is the statement that the following diagram commutes:

$$\begin{tikzcd}
\rho^*_{g,R}\mathcal{O}_X \arrow[r, "\phi_{g,R}"] \arrow[d, "\rho^*_{g,R}\alpha"'] & \mathcal{O}_X \arrow[d, "\alpha"] \\
\rho^*_{g,R}\mathcal{O}_X \arrow[r, "(\phi_0)_{g,R}"] & \mathcal{O}_X
\end{tikzcd}$$

But any automorphism $\alpha$ of $\mathcal{O}_X$ is given by multiplication by some unit $u \in \Gamma(X, \mathcal{O}_X^\times)$, in which case $\alpha^{-1}$ is multiplication by $u^{-1}$ and $\rho^*_{g,R}\alpha$ is multiplication by $\rho^*_{g,R}u$. Thus, commutativity of the above diagram is equivalent to the statement that

$$\phi_{g,R} = \alpha^{-1} \circ (\phi_0)_{g,R} \circ \rho^*_{g,R}\alpha = (\rho^*_{g,R}u) \cdot u^{-1} \cdot (\phi_0)_{g,R}.$$

By definition of the map $\mu$, this holds for all $g$ and $R$ if and only if $\phi = \mu(u)$. $\square$

Exactness at $\mathrm{Pic}_G(X)$ in the above proposition tells us that the set of $G$-linearizations on any invertible sheaf $L$ is controlled by the set of $G$-linearizations on $\mathcal{O}_X$. In and of itself, this is not saying much. To make use of this fact, we seek a more useful description of the set of $G$-linearizations on $\mathcal{O}_X$. For this, recall that the functor $S \mapsto \mathrm{Hom}(G \times S, \mathbb{G}_m \times S)$ (here Hom denotes homomorphisms of group schemes over $S$) defines a sheaf on the étale site of $\mathrm{Spec}(k)$. We denote this sheaf by $\widehat{G}$. (For the reader unfamiliar with sheaves on sites, all that matters for what follows is the definition of this functor.)

**Proposition 2.6.2.** *Let $X$ be a $G$-scheme. We have an isomorphism of abelian groups*

$$\widehat{G}(X) = \mathrm{Hom}_X(G \times X, \mathbb{G}_m \times X) \cong \{G\text{-}linearizations\ on\ \mathcal{O}_X\}.$$

*Proof.* Lemma 2.4.11 (and the discussion preceding it) tells us that giving a $G$-linearization $\phi : \rho^*\mathcal{O}_X \to \mathrm{pr}_X^* \mathcal{O}_X$ is equivalent to giving maps $\phi_{g,R} : \rho^*_{g,R}\mathcal{O}_{X\times S} \to \mathcal{O}_{X\times S}$ for all $S =$

Spec$(R)$ and $g \in G(S)$ such that (1) the $\phi_{g,R}$ are functorial in $R$, and (2) for all $S = \mathrm{Spec}(R)$ and all $g, h \in G(S)$, the following diagram commutes:

$$\begin{tikzcd}
\rho_{h,R}^*(\rho_{g,R}^*\mathcal{O}_{X\times S}) \arrow[r, equal] \arrow[d, "\rho_{h,R}^*(\phi_{g,R})"'] & \rho_{gh,R}^*\mathcal{O}_{X\times S} \arrow[r, "\phi_{gh,R}"] & \mathcal{O}_{X\times S} \\
\rho_{h,R}^*\mathcal{O}_{X\times S} \arrow[urr, "\phi_{h,R}"'] & &
\end{tikzcd}$$

Using the canonical isomorphisms $\rho^*\mathcal{O}_X \cong \mathcal{O}_{X\times S} \cong \mathrm{pr}_X^*\,\mathcal{O}_X$, we may view the $\phi_{g,R}$ as automorphisms of $\mathcal{O}_{X\times S}$, which are given by multiplication by a unit $u_{g,R} \in \Gamma(X\times S, \mathcal{O}_{X\times S}^\times)$. Under this identification, one can check that the above diagram commutes if and only if $u_{gh,R} = u_{g,R} \cdot u_{h,R}$. If $\phi'$ is any other $G$-linearization of $\mathcal{O}_X$ with corresponding sections $v_{g,R} \in \Gamma(X \times S, \mathcal{O}_{X\times S}^\times)$ for each choice of $S$ and $g$, one can check that the tensor product $\phi \otimes \phi'$ (as a $G$-linearization on $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$, see Lemma 2.4.13b) corresponds to the sections $u_{g,R}u'_{g,R} \in \Gamma(X \times S, \mathcal{O}_{X\times S}^\times)$.

Now, a choice of unit $u_{g,R} \in \Gamma(X \times S, \mathcal{O}_{X\times S}^\times)$ is equivalent to a morphism of $k$-schemes $f_{g,R} : X \times S \to \mathbb{G}_m$, and the product $u_{g,R} \cdot u_{h,R}$ corresponds to the morphism $f_{g,R} \cdot f_{h,R}$ given by $(x, s) \mapsto f_{g,R}(x, s) \cdot f_{h,R}(x, s)$ (here multiplication is given by the algebraic group structure on $\mathbb{G}_m$), and the condition $u_{gh,R} = u_{g,R} \cdot u_{h,R}$ is equivalent to $f_{gh,R} = f_{g,R} \cdot f_{h,R}$. Finally, given such a choice of the maps $f_{g,R}$ for all $g$ and $R$, we can define a morphism $f : G \times X \to \mathbb{G}_m \times X$ of schemes over $X$ on functors of points by setting $(g, x) \mapsto (f_{g,R}(x, \mathrm{id}_S), x)$. The condition that $f_{gh,R} = f_{g,R} \cdot f_{h,R}$ implies that $f$ is a homomorphism of group schemes over $X$.

In summary, identifying $\phi$ first with the maps $\phi_{g,R}$, then with the units $u_{g,R}$, then with the morphisms $f_{g,R}$, and finally with the morphism $f$ gives us a map

$$\varphi : \{G\text{-linearizations on } \mathcal{O}_X\} \to \widehat{G}(X).$$

Given any other $G$-linearization $\phi'$ corresponding to units $u'_{g,R}$, corresponding morphisms $f'_{g,R}$, and corresponding morphism $f' : G \times X \to \mathbb{G}_m \times X$, the $G$-linearization $\phi \otimes \phi'$ has corresponding units $u_{g,R} \cdot u'_{g,R}$, hence corresponding morphisms $f_{g,R} \cdot f'_{g,R}$, and hence maps to $f \cdot f'$ under $\varphi$. In other words, $\varphi$ is a homomorphism. Moreover, it is almost immediate that every correspondence we made above to define $\varphi$ is a bijection except possibly the correspondence between the $f_{g,R}$ and the morphism $f$. To check that this is a bijection (which implies that $\varphi$ is also a bijection), we give an inverse construction. Suppose given a homomorphism $f : G \times X \to \mathbb{G}_m \times X$ of group schemes over $X$. For any $S = \mathrm{Spec}(R)$ and any $g \in G(S)$, we define the map $f_{g,R} : X \times S \to \mathbb{G}_m$ to be the composition

$$X \times S \overset{(g\circ \mathrm{pr}_S, \mathrm{id}_{X\times S})}{\to} G \times X \times S \overset{f\times \mathrm{id}_S}{\to} \mathbb{G}_m \times X \times S \overset{\mathrm{pr}}{\to} \mathbb{G}_m.$$

The fact that $f$ is a homomorphism implies that $f_{gh,R} = f_{g,R} \cdot f_{h,R}$ for all $g, h \in G(S)$, and one can check that this construction is the (two-sided) inverse to the construction of $f$ from the $f_{g,R}$ given above. $\square$

We are primarily interested in the case where $X$ is integral. In this case, we can further improve the above proposition by identifying $\widehat{G}(X)$ with $\mathcal{X}(G)$.

**Proposition 2.6.3.** *Let $G$ be a connected algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-scheme. Consider the group homomorphism*

$$\beta : \mathcal{X}(G) = \widehat{G}(k) \to \widehat{G}(X)$$

*which sends a character $\lambda : G \to \mathbb{G}_m$ to its base change $\lambda \times \mathrm{id}_X : G \times X \to \mathbb{G}_m \times X$. The homomorphism $\beta$ is injective. If $X$ is integral, then $\beta$ is an isomorphism.*

**Remark 2.6.4.** The statement of the above proposition is actually true whenever $X$ is connected. The proof is almost exactly the same as the proof given below, except that Step 1 in the proof (the case where $G = T$ is a torus) is significantly more subtle. Since we will only need the case where $X$ is integral, we stick to that case here. For the interested reader, we remark that Step 1 of the proof for $X$ connected is a consequence of [Gro64, Corollary 1.3].

We also note that since every $k$-scheme of finite type is Noetherian and hence locally connected, the statement of the proposition for all $X$ connected is equivalent to the statement that $\widehat{G}$ is a constant sheaf on the étale site of $\mathrm{Spec}(k)$. A slightly more general version of this fact holds even when $k$ is not algebraically closed, see [Bri15, Lemma 2.3].

*Proof.* For injectivity, let $\lambda : G \to \mathbb{G}_m$ be a character, and let $x \in X(k)$ be any $k$-point. Then, the fiber of $\lambda \times \mathrm{id}_X$ over $x$ is simply $\lambda$. So, base changing to the fiber over $x$ is a left inverse to $\beta$, which proves that $\beta$ is injective.

It remains to prove that $\beta$ is surjective when $X$ is integral. This is much more subtle, primarily because the definition of $\widehat{G}(X)$ involves group schemes over $X$. The details of this proof are irrelevant to everything that follows and can safely be skipped; we include them primarily for completeness. For a version of the proof that includes all the key ideas but skips most of the technical details, we refer the reader to the proof of [Bri15, Lemma 2.3].

We will first sketch the proof of a series of 4 claims about group schemes over $X$. We will then use these to prove the proposition in 4 steps. For convenience, we will write $G_X = G \times X$ (and $H_X = H \times X$ for any subgroup $H \subset G$).

<u>Claim 1</u>: For any subgroup $H \subset G$, the quotient $G_X/H_X$ of group schemes over $X$ is isomorphic to $G/H \times X$, and this isomorphism identifies the quotient map $G_X \to G_X/H_X$ with the base change of the quotient map $G \to G/H$. To see this, recall that the universal property of the quotient map $G_X \to G_X/H_X$ is that it is the coequalizer of the two maps $G_X \times_X H_X \to G_X$ given by $(g, h) \mapsto g$ and $(g, h) \mapsto gh$. Because the multiplication map $G_X \times_X G_X \to G_X$ is the base change of the multiplication map $G \times G \to G$, it is formal to check that base change of the quotient map $G \to G/H$ satisfies the universal property of the quotient $G_X/H_X$.

<u>Claim 2</u>: If $X$ is affine, then every homomorphism of $X$–group schemes $\lambda : \mathbb{G}_a \times X \to \mathbb{G}_m \times X$ is trivial. Write $X = \mathrm{Spec}(A)$. Giving such a morphism $\lambda$ is equivalent to giving

a map of $A$-algebras $\varphi : A[t^{\pm}] \to A[t]$. But $\varphi(t)$ must be a unit, so $\varphi(t) \in A^{\times}$. Since $\lambda$ is a homomorphism, it must map the identity $1 \in (\mathbb{G}_a \times X)(X)$ to the identity in $(\mathbb{G}_m \times X)(X)$; unraveling the definitions, this implies that $\varphi(t) = 1$ and hence that $\lambda$ factors as the composition

$$\mathbb{G}_a \times X \xrightarrow{\mathrm{pr}} X \xrightarrow{1} \mathbb{G}_m \times X$$

of the projection map and the identity. This is precisely the statement that $\lambda(g) = 1$ for every $X$-scheme $Y$ and every $g \in (\mathbb{G}_a \times X)(Y)$, i.e. that $\lambda$ is trivial.

Claim 3: If $X$ is affine, then for any unipotent algebraic group $U$, every homomorphism of $X$–group schemes $\lambda : U \times X \to \mathbb{G}_m \times X$ is trivial. It is a general fact about unipotent algebraic groups (see e.g. [Mil17, Proposition 14.21]) that there exists a central normal series for $U$, i.e. a series of subgroups

$$U = U_n \supset U_{n-1} \supset U_{n-2} \supset \cdots \supset U_0 = \{1\}$$

such that $U_i$ is normal (even central) in $U_{i+1}$ and $U_{i+1}/U_i \cong \mathbb{G}_a$ for all $i$. We use induction on $i$ to show that $\lambda(U_i \times X) = 0$ for all $i$. The base case $i = 0$ is trivial (since $U_0 \times X$ is the trivial $X$–group scheme in this case). For the inductive step, we note that $(U_{i+1} \times X)/(U_i \times X) \cong \mathbb{G}_a \times X$ by Claim 1, and $\lambda(U_i \times X) = 0$ by the inductive hypothesis, so $\lambda|_{U_{i+1} \times X}$ factors as a homomorphism $\lambda' : \mathbb{G}_a \times X \to \mathbb{G}_m \times X$. But $\lambda'$ is trivial by Claim 2, so it follows that $\lambda|_{U_{i+1} \times X}$ is trivial as well.

Claim 4: Suppose $X$ is affine and $G$ is reductive, and let $T \subset G$ be a maximal torus. Restriction of characters to $T \times X \subset G \times X$ gives a homomorphism

$$r : \mathrm{Hom}(G \times X, \mathbb{G}_m \times X) \to \mathrm{Hom}(T \times X, \mathbb{G}_m \times X),$$

and $r$ is injective. The argument here is essentially the same as the proof of Lemma 2.2.25a, except over $X$ instead of over $k$. Let $\lambda : G \times X \to \mathbb{G}_m \times X$ be a homomorphism of $X$–group schemes, and suppose that $\lambda(T_X) = 0$. We show that $\lambda$ is trivial. Pick a Borel subgroup $B \subset G$ containing $T$, and let $U = R_u(B)$. Then, $B = U \cdot T$ implies that $B_X = U_X \cdot T_X$, and Claim 3 implies that $\lambda(U_X) = 0$, so in fact, $\lambda(B_X) = 0$. The map $\lambda$ thus factors as a map $\lambda' : G_X/B_X \to \mathbb{G}_m \times X$.

By Claim 1, the quotient $G_X/B_X$ is isomorphic to $G/B \times X$ and in particular is proper over $X$ (because $G/B$ is complete). Moreover, since $X$ and $G/B$ are both integral, so is $G_X/B_X$. It follows that $p_*\mathcal{O}_{G_X/B_X} = \mathcal{O}_X$, where $p : G_X/B_X \to X$ is the structure morphism (this is a completely scheme-theoretic fact about proper morphisms of geometrically integral schemes). On the other hand, since the morphism $q : \mathbb{G}_m \times X \to X$ is affine, giving a morphism $\lambda' : G_X/B_X \to \mathbb{G}_m \times X$ is equivalent (by the universal property of $\mathbb{G}_m \times X \cong \underline{\mathrm{Spec}}(q_*\mathcal{O}_{\mathbb{G}_a \times X})$) to giving a morphism

$$q_*\mathcal{O}_{\mathbb{G}_a \times X} \to p_*\mathcal{O}_{G_X/B_X},$$

and since the target of this morphism is $\mathcal{O}_X$, this is equivalent (by the universal property of relative Spec again) to giving a section $x : X \to \mathbb{G}_m \times X$ of $q$. On the other hand, the map $\lambda'$ is a factorization of $\lambda$, and since $\lambda$ sends the identity $1 \in G_X(X)$ to the identity $1 \in (\mathbb{G}_m \times X)(X)$, one can check that the section $x$ must be the identity in $(\mathbb{G}_m \times X)(X)$. This implies that $\lambda'$ is the composition

$$G_X/B_X \xrightarrow{p} X \xrightarrow{1} \mathbb{G}_m \times X,$$

and it follows that $\lambda$ is trivial, i.e. that $\lambda(G \times X) = 1$.

With the above claims established, we can now prove the proposition in 4 steps. Step 1: The proposition is true if $X$ is affine and $G = T$ is a torus. In this case, $G \cong \mathbb{G}_m^n$ for some $n$ and $X = \mathrm{Spec}(A)$, so everything can be checked very explicitly on rings. Indeed, any homomorphism $\lambda : \mathbb{G}_m^n \times X \to \mathbb{G}_m \times X$ is given by a map of $A$-algebras

$$\varphi : A[t^{\pm}] \to C = A[t_1^{\pm}, \ldots, t_n^{\pm}],$$

and $\varphi(t)$ must be a unit.

We claim that the only units in $C$ are those of the form $ut_1^{m_1} \cdots t_n^{m_n}$ for some $u \in A^\times$ and some $m_i \in \mathbb{Z}$. Since $C = (A[t_1^{\pm}, \ldots, t_{n-1}^{\pm}])[t_n^{\pm}]$, the claim follows immediately from induction on $n$ provided we can prove the base case $n = 1$. In this case, after "clearing denominators," any element of $C$ can be written as $pt_1^m$ for some $p \in A[t_1]$ such that $p(0) \neq 0$. Suppose that $pt_1^m$ has an inverse $qt_1^{m'}$. Then, we have

$$pqt_1^{m+m'} = 1.$$

If $m+m' > 0$, then this is an equation of elemenets in $A[t_1]$, so plugging in $t_1 = 0$ (i.e. taking the image of the above equation under the $A$-linear homomorphism $A[t_1] \to A$ sending $t_1 \mapsto 0$) gives $0 = 1$ in $A$, contradicting the fact that $A$ is an integral domain. If $m+m' < 0$, then multiplying the above equation by $t_1^{-(m+m')}$ gives us $pq = t_1^{-(m+m')}$ as element in $A[t_1]$, and plugging in $t_1 = 0$ again gives a contradiction (since $p(0), q(0) \neq 0$). It follows that $m + m' = 0$, so the above equation is $pq = 1$. Since $p, q \in A[t_1]$ and $A[t_1]$ is graded, we conclude that $p, q \in A$ and hence $p \in A^\times$. This proves the claim.

Because of this claim, we have $\varphi(t) = ut_1^{m_1} \cdots t_n^{m_n}$ for some $u \in A^\times$ and some $m_i \in \mathbb{Z}$. On the other hand, using the fact that the homomorphism $\lambda$ must send 1 to 1 (cf. the argument in Claim 2 above), one can check that we must have $u = 1$. Define $\varphi_0 : k[t^{\pm}] \to k[t_1^{\pm}, \ldots, t_n^{\pm}]$ to be the map of $k$-algebras given by $t \mapsto t_1^{m_1} \cdots t_n^{m_n}$. One can check that $\varphi_0$ defines a homomorphism $\lambda_0 : \mathbb{G}_m^n \to \mathbb{G}_m$ and that $\lambda = \lambda_0 \times \mathrm{id}_X = \beta(\lambda_0)$. This proves that $\beta$ is surjective.

Step 2: The proposition is true if $X$ is affine and $G$ is reductive. Pick a maximal torus $T \subset G$, let $\beta_T$ be the homomorphism $\beta$ defined for $T$ instead of $G$, and let $x \in X(k)$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(G, \mathbb{G}_m) \arrow[r, hook, "r"] & \mathrm{Hom}(T, \mathbb{G}_m) \\
\mathrm{Hom}(G \times X, \mathbb{G}_m \times X) \arrow[u, "x^*"] \arrow[r, hook, "r"] & \mathrm{Hom}(T \times X, \mathbb{G}_m \times X) \arrow[u, "x^*"]
\end{tikzcd}$$

Here, the maps labelled "r" are given by restriction of characters (and are injective by Claim 4), and the maps labelled "$x^*$" are given by pulling back to the fiber over $x$ (and are left inverses to $\beta$ and $\beta_T$ by our arguments at the beginning of the proof). Since we know that $\beta_T$ is an isomorphism by Step 1, so is the map $x^*$ for $T$. Commutativity of the above diagram then implies that $x^*$ is injective for $G$. But $x^*$ is also surjective (because $\beta$ is its right inverse), so $x^*$ is an isomorphism. Its inverse must be $\beta$, so $\beta$ is also an isomorphism.

Step 3: The proposition is true if $X$ is affine. Let $\lambda : G \times X \to \mathbb{G}_m \times X$ be a homomorphism. By Claim 3, we have $\lambda(R_u(G) \times X) = 0$, so $\lambda$ factors as a homomorphism $\lambda' : G_X/R_u(G)_X \to \mathbb{G}_m \times X$. On the other hand, Claim 1 gives us $G_X/R_u(G)_X \cong G/R_u(G) \times X$. Since $G/R_u(G)$ is reductive, Step 2 implies that $\lambda' = \lambda_0' \times \mathrm{id}_X$ for some $\lambda_0' : G/R_u(G) \to \mathbb{G}_m$. Let $\lambda_0 : G \to \mathbb{G}_m$ be the composition

$$G \to G/R_u(G) \xrightarrow{\lambda_0'} \mathbb{G}_m.$$

Then, the base change $\lambda_0 \times \mathrm{id}_X$ is equal to the composition of $\lambda'$ with the quotient map $G_X \to G_X/R_u(G)_X$, which is $\lambda$ by definition. So, $\lambda = \beta(\lambda_0)$, which proves that $\beta$ is surjective. (Note that we have implicitly used connectedness of $G$ in this step in order to use the unipotent radical $R_u(G)$ and to consider the quotient $G/R_u(G)$ as a reductive group, since reductive groups in our terminology are connected by definition.)

Step 4: The proposition is true for any $X$ integral. This is essentially a gluing argument and so boils down to the fact that $\widehat{G}$ (as a sheaf on the étale site) is determined by its values on affine schemes. More precisely, let $\lambda : G \times X \to \mathbb{G}_m \times X$ be a homomorphism. For every affine open subset $U \subset X$, the base change of $\lambda$ to $U$ is a homomorphism of $U$–group schemes $\lambda_U : G \times U \to \mathbb{G}_m \times U$. Since $X$ is integral, so is $U$, so Step 3 implies that $\lambda_U = \lambda_{0,U} \times \mathrm{id}_U$ for some character $\lambda_{0,U} : G \to \mathbb{G}_m$. Given any other affine open subset $V \subset X$, the pull backs of $\lambda_U$ and $\lambda_V$ to $G \times (U \cap V)$ are equal, hence so are the pullbacks of $\lambda_U$ and $\lambda_V$ to the fiber over any $k$-point $x \in U \cap V$. But the pullbacks to these fibers are precisely $\lambda_{0,U}$ and $\lambda_{0,V}$ by our above arguments, so $\lambda_{0,U} = \lambda_{0,V}$. Setting $\lambda_0 = \lambda_{0,U}$ for any $U$, it follows that $\lambda$ and $\lambda_0 \times \mathrm{id}_X$ agree on $G \times U$ for every affine open subset $U \subset X$. It follows that $\lambda = \lambda_0 \times \mathrm{id}_X$. □

Our above results culminate in the following improvement on Proposition 2.6.1 when $X$ is a $G$-variety. This is the main result that we wanted on the uniqueness of $G$-linearizations.

**Theorem 2.6.5.** *Let $G$ be a connected algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-variety. There is an exact sequence of abelian groups*

$$0 \to \Gamma(X, \mathcal{O}_X^\times)^G \to \Gamma(X, \mathcal{O}_X^\times) \to \mathcal{X}(G) \xrightarrow{\iota} \mathrm{Pic}_G(X) \xrightarrow{\varphi} \mathrm{Pic}(X).$$

*Here, $\varphi$ is the map which "forgets" the $G$-linearization, and for any character $\lambda \in \mathcal{X}(G)$, the isomorphism class $\iota(\lambda)$ is represented by $\mathcal{O}_X$ equipped with a $G$-linearization $\phi$ such that the resulting $G$-module structure on $H^0(X, \mathcal{O}_X)$ is as follows: for any $S = \mathrm{Spec}(R)$, $g \in G(S)$, and $f \in H^0(X, \mathcal{O}_X) \otimes R$, we have*

$$g \cdot f = \lambda(g)(g * f),$$

*where $g \cdot f$ denotes the action induced by $\phi$, $g * f$ denotes the canonical $G$-module structure on $\Gamma(X, \mathcal{O}_X)$, and we identify $\lambda(g) \in \mathbb{G}_m(S) \cong \Gamma(S, \mathcal{O}_S^\times)$ with its pullback under the projection $X \times S \to S$.*

*Proof.* The exact sequence in the statement is precisely the exact sequence of Proposition 2.6.1, just using the identifications

$$\mathcal{X}(G) \cong \widehat{G}(X) \cong \{G\text{-linearizations of } \mathcal{O}_X\}$$

of Propositions 2.6.2 and 2.6.3 to replace the set of $G$-linearizations of $\mathcal{O}_X$ with $\mathcal{X}(G)$ in the sequence. To obtain the statement about $\iota(\lambda)$, we just need to trace through the identifications made in Propositions 2.6.2 and 2.6.3 in order to understand the map $\iota$.

Given a character $\lambda : G \to \mathbb{G}_m$, Proposition 2.6.3 identifies $\lambda$ with the homomorphism of $X$–group schemes $f = \lambda \times \mathrm{id}_X$. The proof of Proposition 2.6.2 then identifies $f$ with maps $f_{g,R} : X \times S \to \mathbb{G}_m$ for each $S = \mathrm{Spec}(R)$ and $g \in G(S)$. Explicitly, the $f_{g,R}$ are given by the composition

$$X \times S \overset{(g \circ \mathrm{pr}_S, \mathrm{id}_{X \times S})}{\to} G \times X \times S \overset{f \times \mathrm{id}_S}{\to} \mathbb{G}_m \times X \times S \overset{\mathrm{pr}}{\to} \mathbb{G}_m.$$

Since $f = \lambda \times \mathrm{id}_X$, this composition is the same as the composition

$$X \times S \overset{\mathrm{pr}_S}{\to} S \overset{g}{\to} G \overset{\lambda}{\to} \mathbb{G}_m.$$

The proof of Proposition 2.6.2 then identifies the $f_{g,R}$ with units

$$u_{g,R} \in \mathbb{G}_m(X \times S) \cong \Gamma(X \times S, \mathcal{O}_{X \times S}^\times).$$

using the functor of points of $\mathbb{G}_m$. By definition of the functor of points of $\mathbb{G}_m$, the fact that $f_{g,R} = \lambda \circ g \circ \mathrm{pr}_S$ implies that

$$u_{g,R} = \mathrm{pr}_S^*(\lambda(g)),$$

where we view $\lambda(g) = \lambda \circ g$ as an element of $\mathbb{G}_m(S) \cong \Gamma(S, \mathcal{O}_S^\times)$. Finally, Proposition 2.6.2 identifies the $u_{g,R}$ with the $G$-linearization $\phi$ of $\mathcal{O}_X$ such that $\phi_{g,R}$ is the map given by multiplication by $u_{g,R}$ under the canonical isomorphisms $\rho^*\mathcal{O}_X \cong \mathcal{O}_{X \times S} \cong \mathrm{pr}_X^* \mathcal{O}_X$, and the map $\iota'$ sends $\phi$ to the class in $\mathrm{Pic}_G(X)$ of $\mathcal{O}_X$ equipped with $\phi$.

We need to understand the $G$-module structure of $H^0(X, \mathcal{O}_X)$ induced by this $G$-linearization $\phi$ in terms of the character $\lambda$. For this, we note that by construction, the canonical $G$-linearization $\phi_0$ on $\mathcal{O}_X$ is given by the canonical isomorphisms $\rho^*\mathcal{O}_X \cong \mathcal{O}_{X \times S} \cong \mathrm{pr}_X^* \mathcal{O}_X$ (see Lemma 2.4.13a), so we have

$$\phi_{g,R} = u_{g,R} \cdot (\phi_0)_{g,R}$$

for all $S = \mathrm{Spec}(R)$ and $g \in G(S)$. The $G$-module structure on $H^0(X, \mathcal{O}_X)$ induced by $\phi$ (resp. $\phi_0$) is given by letting $g$ act by the composition $\phi_{g,R} \circ \rho_{g,R}^*$ (resp. $(\phi_0)_{g,R} \circ \rho_{g,R}^*$). So, if $g \cdot f$ and $g * f$ denote the $G$-actions induced by $\phi$ and $\phi_0$ (respectively), then the above equation implies that

$$g \cdot f = u_{g,R}(g * f)$$

for all $f \in H^0(X, \mathcal{O}_X) \otimes R$. But the $G$-action induced by $\phi_0$ is the canonical $G$-module structure on $\Gamma(X, \mathcal{O}_X)$ (Lemma 2.4.13a), and $u_{g,R} = \mathrm{pr}_S^*(\lambda(g))$, so this is exactly the statement of the corollary. □

**Definition 2.6.6.** Let $X$ be a $G$-variety, and let $\lambda \in \mathcal{X}(G)$ be a character. We denote by $\mathcal{O}_X(\lambda)$ the sheaf $\mathcal{O}_X$ equipped with the $G$-linearization $\phi$ given in Theorem 2.6.5.

**Remark 2.6.7.** Since the map $\mathcal{X}(G) \to \mathrm{Pic}_G(X)$ is a group homomorphism, we immediately see that for any $n \geq 1$, the $G$-linearization on $\mathcal{O}_X(\lambda)^{\otimes n}$ given by Lemma 2.4.13b is isomorphic to that of $\mathcal{O}_X(n\lambda)$. Similarly, when $\lambda = 0$ is the trivial character, then $\mathcal{O}_X(\lambda)$ is $\mathcal{O}_X$ with the canonical $G$-linearization.

Theorem 2.6.5 allows us to completely describe each $G$-linearization on an arbitrary invertible sheaf $L$ as well as the resulting $G$-module structure on $H^0(X, L)$.

**Corollary 2.6.8.** *Let $G$ be a connected algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-variety. Let $L$ be an invertible sheaf on $X$, and suppose given a $G$-linearization on $L$.*

*(a) The isomorphism classes of $G$-linearizations of $L$ in $\mathrm{Pic}_G(X)$ are precisely the isomorphism classes of $L \otimes \mathcal{O}_X(\lambda)$ for any $\lambda \in \mathcal{X}(G)$ (with the $G$-linearization on $L \otimes \mathcal{O}_X(\lambda)$ induced by the given $G$-linearization on $L$ and the one on $\mathcal{O}_X(\lambda)$, see Lemma 2.4.13b.)*

*(b) For any character $\lambda \in \mathcal{X}(G)$, we have*

$$\Lambda^+(X, L \otimes \mathcal{O}_X(\lambda)) = \{(\mu + d\lambda, d) \mid (\mu, d) \in \Lambda^+(X, L)\}.$$

*(Here, we view $\lambda$ as a weight of $G$ by identifying $\lambda$ with $\lambda|_T$ for a maximal torus $T \subset G$. Cf. Lemma 2.2.25.)*

*Proof.* Notice that exactness at $\mathrm{Pic}_G(X)$ in Theorem 2.6.5 implies that the isomorphism classes of $G$-linearizations on $\mathcal{O}_X$ are precisely the classes of the $\mathcal{O}_X(\lambda)$ for $\lambda \in \mathcal{X}(G)$. Statement (a) follows formally from this fact (by making use of the group structure on $\mathrm{Pic}_G(X)$).

As for statement (b), let $\phi$ be the given $G$-linearization on $L$. Since $\mathcal{O}_X(\lambda) = \mathcal{O}_X$ as sheaves of $\mathcal{O}_X$-modules, we have a canonical (but generally *not* $G$-equivariant) isomorphism

$$L \otimes \mathcal{O}_X(\lambda) \cong L.$$

Under this isomorphism, the $G$-linearization on $L \otimes \mathcal{O}_X(\lambda)$ (from Lemma 2.4.13b) induces a different $G$-linearization $\phi'$ on $L$, and the weight monoid $\Lambda^+(X, L \otimes \mathcal{O}_X(\lambda))$ is the same as the weight monoid $\Lambda^+(X, L)$ when we use the $G$-linearization $\phi'$ on $L$. Thus, our goal is to compute the weight monoid $\Lambda^+(X, L)$ for $\phi'$ in terms of the weight monoid $\Lambda^+(X, L)$ for $\phi$.

For this, note that for any $d \geq 1$, our above isomorphism $L \otimes \mathcal{O}_X(\lambda) \cong L$ induces an isomorphism

$$L^{\otimes d} \otimes \mathcal{O}_X(d\lambda) \cong L^{\otimes d}$$

which is $G$-equivariant using the $G$-linearization on $L^{\otimes d}$ induced by $\phi'$. Using this fact and the description of the $G$-module structure of $H^0(X, \mathcal{O}_X(d\lambda))$ given in Theorem 2.6.5, one can check that the $G$-module structure on $H^0(X, L)$ induced by $\phi'$ is as follows: for any $S = \operatorname{Spec}(R)$, any $g \in G(S)$, and any $f \in H^0(X, L^{\otimes d}) \otimes R$, we have

$$g \cdot f = (d\lambda(g)) \cdot (g * f),$$

where $g*a$ denotes the action of $g$ on $f$ induced by $\phi$. Thus, the $B$-eigenvectors of $H^0(X, L^{\otimes d})$ are the same whether we use the action induced by $\phi$ or $\phi$', but if $\mu$ is the weight of an eigenvector using the action induced by $\phi$, then $\mu + d\lambda$ is its weight under the action induced by $\phi'$. $\square$

One important implication of the above corollary is: for any $G$-linearized invertible sheaf $L$, choosing another $G$-linearization amounts to shifting all the weights of $B$-eigenvectors of $H^0(X, L)$ by a character of $G$ (or equivalently, tensoring the $G$-module $H^0(X, L)$ by a character of $G$). In particular, the representation theory on $H^0(X, L)$ does not differ in any significant way based on which $G$-linearization we pick.

One other nice implication of Theorem 2.6.5 and Corollary 2.6.8 is that we characterize precisely when $G$-linearizations of an invertible sheaf are unique.

**Corollary 2.6.9.** *Let $G$ be a connected algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-variety.*

*(a) If $\mathcal{X}(G) = 0$ (this holds, for instance, when $G$ is semisimple, see Proposition 2.2.22), then for any invertible sheaf $L$ on $X$, there exists at most one $G$-linearization on $L$ (up to $G$-equivariant isomorphism of $L$).*

*(b) Conversely, if any invertible sheaf $L$ on $X$ has a unique $G$-linearization (up to $G$-equivariant isomorphism of $L$), then $\mathcal{X}(G) = 0$.*

*Proof.* For (a), suppose $\phi$ and $\psi$ are two $G$-linearizations $\phi$ on an invertible sheaf $L$. We can use $\psi$ to obtain a $G$-linearization on $L^{-1}$, and then using $\phi$ on $L$ induces a $G$-linearization on the tensor product

$$L \otimes L^{-1} \cong \mathcal{O}_X$$

(see Lemma 2.4.13). Since $\mathcal{X}(G) = 0$, exactness at $\operatorname{Pic}_G(X)$ in Theorem 2.6.5 implies that this $G$-linearization on $\mathcal{O}_X$ is the canonical one, so tensoring the above equation by $L$ equipped with $\psi$ gives us a $G$-equivariant isomorphism $L \cong L$, where one copy of $L$ is equipped with $\phi$ and the other is equipped with $\psi$.

As for (b), we note that for any character $\lambda : G \to \mathbb{G}_m$, the unique $G$-linearization $\phi$ on $L$ induces a $G$-linearization on $L \otimes \mathcal{O}_X(\lambda)$ by Lemma 2.4.13b, which in turn induces a $G$-linearization $\psi$ on $L$ via the isomorphism of $\mathcal{O}_X$-modules $L \otimes \mathcal{O}_X(\lambda) \cong L$. If $\lambda \neq 0$, then Corollary 2.6.8 implies that the weight monoid $\Lambda^+(X, L)$ is different if we use $\psi$ instead of $\phi$, so $\phi$ and $\psi$ must be non-isomorphic $G$-linearizations of $L$. This contradicts our assumption that $L$ has a unique $G$-linearization □

### 2.6.b A Few Big Theorems

We have now covered the uniqueness of $G$-linearizations at length. As for existence of $G$-linearizations, it turns out (though we will not prove it here) that the exact sequence of Theorem 2.6.5 can be extended by one more term, so that it also characterizes when $G$-linearizations exist.

**Theorem 2.6.10** ([Bri18, Theorem 4.2.2 and Proposition 4.2.3]; cf. [Bri15, Proposition 2.10])**.** *Let $G$ be a connected algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-variety. There is an exact sequence*

$$0 \to \Gamma(X, \mathcal{O}_X^*)^G \to \Gamma(X, \mathcal{O}_X^*) \to \mathcal{X}(G) \to \operatorname{Pic}_G(X) \xrightarrow{\varphi} \operatorname{Pic}(X) \xrightarrow{\psi} \operatorname{Pic}(G \times X)/\operatorname{pr}_X^* \operatorname{Pic}(X).$$

*Here, $\psi$ is the map that sends the class of $L$ to the class of $\rho^* L$ (modulo the subgroup $\operatorname{pr}_X^* \operatorname{Pic}(X) \subset \operatorname{Pic}(G \times X)$), and $\varphi$ is the map that "forgets" the $G$-linearization.*

*Proof.* Exactness everywhere except at $\operatorname{Pic}(X)$ is Theorem 2.6.5. For a complete proof of the theorem, we refer the reader to Brion's presentations in [Bri18, Section 4] and [Bri15, Section 2]. (The former presentation proves the theorem as stated, and the latter is a generalization which applies when $X$ is a reduced $G$-scheme.) □

Exactness at $\operatorname{Pic}(X)$ in Theorem 2.6.10 tells us that an invertible sheaf $L$ on $X$ admits a $G$-linearization if and only if $\rho^* L \cong \operatorname{pr}_X^* L$. In other words: if an isomorphism $\rho^* L \cong \operatorname{pr}_X^* L$ exists, then there exists such an isomorphism satisfying the cocyle condition. This reduces the question of existence of a $G$-linearization to a calculation on Picard groups. Using some standard facts about Picard groups of algebraic groups and normal varieties, this yields the following theorem on the existence of $G$-linearizations.

**Theorem 2.6.11** ([Sum75, Theorem 1.6], [Kno+89, Propositions 2.4, 4.6]; see also [Bri18, Theorem 2.14], [Bri18, Theorem 5.2.1])**.** *Let $G$ be a connected linear algebraic group over an algebraically closed field $k$, and let $X$ be a normal $G$-variety.*

(a) *Let $n$ be the order of $\operatorname{Pic}(G)$ (which is a finite group). For any invertible sheaf $L$ on $X$, the sheaf $L^{\otimes n}$ is $G$-linearizable. In particular, if $\operatorname{Pic}(G) = 0$ (equivalently, if $G$ is locally factorial), then every invertible sheaf on $X$ is $G$-linearizable.*

(b) *There exists an isogeny of algebraic groups $\pi : \tilde{G} \to G$ such that* $\mathrm{Pic}(\tilde{G}) = 0$. *In particular, if we let $\tilde{G}$ act on $X$ via the action of $G$, then every invertible sheaf on $X$ is $\tilde{G}$-linearizable.*

*Sketch of proof.* Statement (a), as far as we know, was first proven by Sumihiro (for $S$–group scheme actions over a sufficiently nice base scheme $S$) in [Sum75, Theorem 1.6]. There are now many other presentations of this fact, but they all essentially follow the same method of proof. One shows that for every $M \in \mathrm{Pic}(G \times X)$, we have $M^{\otimes n} \in \mathrm{pr}_X^* \mathrm{Pic}(X)$; it follows that $L^{\otimes n}$ lies in the kernel of the map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times X)/\mathrm{pr}_X^* \mathrm{Pic}(X)$ of Theorem 2.6.10, so exactness at $\mathrm{Pic}(G)$ in the theorem implies that $L^{\otimes n}$ is $G$-linearizable.

For this argument to go through, there are two key steps. The first is to show that $\mathrm{Pic}(G)$ is finite in the first place. This fact fundamentally relies on the theory of algebraic groups; for a proof, see [Mil17, Corollary 18.23], [Bri15, Lemma 2.3], or [Kno+89, Proposition 4.5] (all of which use somewhat different methods of proof). The second step is to show that every invertible sheaf on $G \times X$ has the form $\mathrm{pr}_X^* L_X \otimes \mathrm{pr}_G^* L_G$ for some invertible sheaves $L_X$ on $X$ and $L_G$ on $G$. This is an essentially scheme-theoretic fact, and it is the step that relies on $X$ being normal. For a proof, see [Bri15, Lemma 2.12] or [Kno+89, Lemma 4.2]. Once these two steps are done, we need only note that

$$(\mathrm{pr}_X^* L_X \otimes \mathrm{pr}_G^* L_G)^{\otimes n} \cong \mathrm{pr}_X^*(L_X^{\otimes n}) \otimes \mathrm{pr}_X^*(L_G^{\otimes n}) \cong \mathrm{pr}_X^*(L_X^{\otimes n}),$$

so that $n\,\mathrm{Pic}(G \times X) \subset \mathrm{pr}_X^* \mathrm{Pic}(X)$. The rest of the proof now follows from Theorem 2.6.10, as described above.

Statement (b) is another fact from the theory of algebraic groups. In the case where $G$ is reductive, one can use the isogeny $G^{ss} \times C \to G$ of Proposition 2.2.23 and then show that $\mathrm{Pic}(G^{ss} \times C) = 0$ (which follows, for instance, from the above scheme-theoretic fact about $\mathrm{Pic}(G \times X)$ and $\mathrm{pr}_X^* \mathrm{Pic}(X)$ with $G = G^{ss}$ and $X = C$, since $\mathrm{Pic}(G^{ss}) = 0$ by [Mil17, Corollary 18.24] and $\mathrm{Pic}(C) \cong \mathrm{Pic}(\mathbb{G}_m^r) = 0$). For a proof for general $G$, see [Mil17, Proposition 18.22] or [Kno+89, Proposition 4.6]. Finally, since $\mathrm{Pic}(\tilde{G}) = 0$, statement (a) implies that every invertible sheaf on $X$ is $\tilde{G}$-linearizable. □

Sumihiro's main interest in Theorem 2.6.11a was actually to study the notion of quasi-projectivity as it relates to $G$-linearizations. More precisely, $X$ is quasi-projective if and only if there exists a finite-dimensional $k$–vector space $V$ and an immersion $X \hookrightarrow \mathbb{P}(V)$. If $X$ is a $G$-scheme, we say that $X$ is *$G$–quasi-projective* if $X$ is quasi-projective and we can take $V$ to be a $G$-module and the immersion $X \hookrightarrow \mathbb{P}(V)$ to be $G$-equivariant. Recall that if $X$ quasi-projective, the immersion $X \hookrightarrow \mathbb{P}(V)$ can be obtained by letting $V$ be some finite-dimensional subspace of $H^0(X, L)$, where $L$ is a very ample invertible sheaf on $X$. If we may pick $L$ to be $G$-linearized and $V$ to be a certain finite-dimensional $G$-submodule of $H^0(X, L)$, then arguing as in Proposition 2.4.17 will give us a natural $G$-action on $\mathbb{P}(V)$ such that $X \hookrightarrow \mathbb{P}(V)$ is $G$-equivariant, so that $X$ is $G$–quasi-projective. When $X$ is normal, we can always take $L$ to be $G$-linearized by Theorem 2.6.11. So, these arguments (once they are made more rigorous) imply that any normal quasi-projective $G$-variety is $G$–quasi-projective.

In fact, what Sumihiro proved is somewhat more than this. The following theorem summarizes his results, which were first proven in [Sum74]. These results were then generalized to group schemes over a sufficiently nice base scheme $S$ in [Sum75]. We also refer the interested reader to [Kno+89], which provides another nice presentation of Sumihiro's results.

**Theorem 2.6.12** ([Sum74, Lemma 8, and Corollary 2] and [Sum75, proof of Theorem 2.5])**.** *Let $G$ be a connected linear algebraic group over an algebraically closed field $k$, and let $X$ be a normal $G$-variety.*

*(a) For any $G$-orbit $Y \subset X$, there exists a $G$-stable, quasi-projective open subset $U \subset X$ such that $Y \subset U$. In particular, there exists an open cover of $X$ by $G$-stable, quasi-projective open subsets.*

*(b) If $G = T$ is a torus, then the open subsets in (a) can be taken to be affine, not just quasi-projective.*

*(c) If $L$ is a $G$-linearized very ample invertible sheaf on $X$, then there exists a finite-dimensional $G$-module $V$ and a $G$-equivariant (locally closed) immersion $i : X \hookrightarrow \mathbb{P}(V)$ such that $i^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong L$. In particular, if $X$ is quasi-projective, then $X$ is $G$–quasi-projective.*

**Corollary 2.6.13.** *Let $G$ be a connected linear algebraic group over an algebraically closed field $k$, and let $X$ be a normal $G$-variety. If $X$ has a unique closed $G$-orbit $Y$, then $X$ is quasi-projective.*

*Proof.* By the above theorem of Sumihiro, there exists a quasi-projective, $G$-stable open subset $U \subset X$ such that $Y \subset U$. If $Z = X \setminus U$ is nonempty then $Z$ contains a closed $G$-orbit (any orbit of minimal dimension is closed), which is a closed $G$-orbit of $X$ as well (because $Z$ is closed in $X$). Since $Y$ is the unique closed orbit of $X$, we conclude that $Z = \varnothing$ and hence that $X = U$ is quasi-projective. □

Theorem 2.6.12 is a key starting point for working with normal $G$-varieties. It tells us that locally, we may work with quasi-projective normal $G$-varieties, which are locally closed $G$-subvarieties of $\mathbb{P}(V)$ for some $G$-module $V$. For some purposes, this even allows one to take the closure in $\mathbb{P}(V)$ and so reduce to the projective case.

Theorems 2.6.11 and 2.6.12 are two very important and useful theorems, and we will use them frequently in what follows. It is worth noting, however, that both theorems require the $G$-variety $X$ to be normal. We will almost always be working with normal varieties, so this assumption is generally fine for our purposes. For certain technical applications, however, one may be intersted in dealing with the case of non-normal varieties. To this end, we give a couple results that sometimes allow one to either reduce to the normal case (and then to the quasi-projective case using Theorem 2.6.12 if desired) or to reduce directly to the quasi-projective case (without needing normality to apply Theorem 2.6.12).

**Theorem 2.6.14** ("Equivariant Chow lemma"; [Sum74, Theorem 2]). *Let $G$ be a connected linear algebraic group over an algebraically closed field $k$, and let $X$ be a $G$-variety. There exists a quasi-projective $G$-variety $\widetilde{X}$ and a $G$-equivariant morphism $f : \widetilde{X} \to X$ such that*

*(a) $f$ is birational, projective, and surjective, and*

*(b) there exists a nonempty $G$-stable open subset $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an isomorphism of $G$-varieties.*

**Lemma 2.6.15** (cf. [Bri10, proof of Proposition 2.8], [Tim11, Proposition 15.15]). *Let $G$ be a connected algebraic group over an algebraically closed field $k$ which is normal (as a variety). Let $X$ be a $G$-variety, and let $\nu : \tilde{X} \to X$ be the normalization morphism.*

*(a) The morphism $\nu$ is birational, surjective, and finite.*

*(b) There exists a unique $G$-action on $\tilde{X}$ such that $\nu$ is a $G$-equivariant morphism.*

*(c) Suppose that $G$ is reductive and that $X$ can be covered by $G$-stable $G$–quasi-projective open subsets, and let $\tilde{X}$ have the $G$-action given in (b). Then, $\nu$ induces a bijection on the sets of $G$-orbits of $\tilde{X}$ and $X$.*

*Proof.* Statement (a) is a completely general fact about normalizations of varieties; see e.g. [Sta20, Tag 035E]. For statement (b), define a morphism

$$\nu' : G \times \tilde{X} \to G \times X$$

on functors of points by $(g, x) \mapsto (g, \nu(x))$, and let $\rho : G \times X \to X$ be the action morphism. Then, $\nu'$ is surjective (because $\nu$ is), and $\rho$ is surjective, so the composition $\nu' \circ \rho$ is a dominant morphism to $X$ from a normal variety $G \times X$. By the universal property of normalization, then, there exists a unique morphism $\rho' : G \times \tilde{X} \to \tilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times \tilde{X} & \xrightarrow{\rho'} & \tilde{X} \\ \big\downarrow{\scriptstyle \nu'} & & \big\downarrow{\scriptstyle \nu} \\ G \times X & \xrightarrow{\rho} & X \end{array}$$

A $G$-action on $\tilde{X}$ such that $\nu$ is $G$-equivariant is precisely an action morphism $\rho'$ fitting into this commutative diagram. Uniqueness of $\rho'$ tells us there is at most one such action, so to complete the proof, we just need to prove that $\rho'$ is in fact a group action.

For this, the key point is that $\nu$ is an isomorphism on a dense open subset $U \subset \tilde{X}$, which implies that $\nu$ induces an injection on functors of points: indeed, for any morphisms $f, g : T \to \tilde{X}$ such that $\nu \circ f = \nu \circ g$, we have $f|_U = g|_U$ and hence $f = g$ (because $T$ is reduced and $\tilde{X}$ is separated). So, given any $g, h \in G(T)$ for some variety $T$ and any

$x \in \tilde{X}(T)$, commutativity of the above diagram (plus the fact that $\rho$ induces a group action $G(T) \times X(T) \to X(T)$) gives us

$$\begin{aligned} \nu(\rho'(gh, x)) = \rho(gh, \nu(x)) = gh \cdot \nu(x) = g \cdot (h\nu(x)) &= \rho(g, \rho(h, \nu(x))) \\ &= \rho(g, \nu(\rho'(h, x)) \\ &= \nu(\rho'(g, \rho'(h, x))). \end{aligned}$$

Since $\nu$ is injective on points, we conclude that $\rho'(gh, x) = \rho'(g, \rho'(h, x))$. Likewise, we have

$$\nu(\rho'(1, x)) = \rho(1, \nu(x)) = \nu(x),$$

so that $\rho'(1, x) = x$.

Finally, the proof of (c) is a somewhat more technical argument. Our assumption on $X$ allows one to reduce to the case where $X$ is affine, and then some representation-theoretic arguments on $G$-stable ideals of $\Gamma(X, \mathcal{O}_X)$ can be used to reduce to the case where $G = T$ is a torus. In this case, one can describe $T$-stable ideals of $\Gamma(X, \mathcal{O}_X)$ combinatorially, in terms of $\Lambda^+(X)$, and then check that $\tilde{X}$ admits the same description of $T$-stable ideals. See [Tim11, Proposition 15.15] for details. □

# Chapter 3

# The Classification of Spherical Varieties

In this chapter, we discuss the theory of spherical varieties, building up to their full classification by combinatorial invariants. We begin with definitions and general results on spherical varieties in Section 3.1. In Section 3.2, we give a treatment of the so-called *local structure theorem*, which is a key technical result that arises throughout the theory of spherical varieties.

After this background is established, we will begin to discuss the classification of spherical varieties. This classification has two main components: the first is known as *Luna-Vust theory*, which we treat in Section 3.3, and the second is a classification statement by combinatorial objects called *homogeneous spherical data*, which we treat in Section 3.6. The classification by homogeneous spherical data is by far the more complicated part of the theory. To fully explain it, we will need to develop a theory of so-called *$G$-invariant valuations* in Section 3.4 and a theory of certain nice types of spherical varieties (called *toroidal varieties* and *wonderful varieties*) in Section 3.5. Using techniques discussed in Section 3.6, many questions about the behavior of homogeneous spherical data can be reduced to the case of certain toroidal or wonderul varieties which are nice enough to be completely classified. This produces some strong combinatorial conditions on homogeneous spherical data, which is a key feature of the theory.

We remark that there is one section in this chapter which is not relevant to the classification of spherical varieties: namely, Section 3.7, in which we discuss a combinatorial description of Weil and Cartier divisors on spherical varieties. This section is included primarily because it will be needed for arguments in later chapters.

The reader who is not interested in having a deep understanding of the theory of spherical varieties does not need to wade through all of the technicalities in this chapter. For our purposes, the most important parts will be an understanding of the local structure theorem and a familiarity with the combinatorial data used to classify spherical varieties. In particular, the technical details of all of the proofs in this chapter are not so important for what follows (except possibly certain proofs that make use of the local structure theorem). So, for a "crash course" in the theory of spherical varieties, we recommend reading this chapter in the following way.

1. Briefly read through Sections 3.1 and 3.2 to get familiar with spherical varieties and the local structure theorem. There is no need to focus on any proofs in these sections. The main goal is to get familiar with the basic gadgets and terminology introduced in these sections (e.g. $B$-divisors and $G$-invariant valuations) and to understand the main statements of the local structure theorem and related results (namely, Theorems 3.2.2 and 3.2.7 and Proposition 3.2.3).

2. Briefly read through Sections 3.3, 3.4, and 3.5 (in that order) to get familiar with the terminology and combinatorial objects used in the classification of spherical varieties. Focus on any big theorems about the structure of spherical varieties (for instance, Theorem 3.1.4, 3.3.26, 3.4.1, or 3.5.21).

3. In Section 3.6, blackbox the notion of "localization at simple roots" and just read through Subsections 3.6.b and 3.6.c. This will introduce a few more important combinatorial objects and give some insight into their properties. The details of these properties will come up repeatedly in what follows, but one can always refer back to the relevant results in these sections after a brief first read.

4. Go back and read Section 3.2 more carefully to get a solid understanding of the local structure theorem. One does not need to understand the technical aspects of proofs such as Theorem 3.2.2 or Theorem 3.2.7. The goal is to be comfortable applying the statements of these theorems, as we do for instance in the proofs of Lemma 3.2.9 and 3.2.10. The curious reader may also be interested in looking at some of the details of the local structure theorem as it applies to toroidal and wonderful varieties, for instance in the proofs of Theorems 3.5.6, 3.5.9, and 3.5.22.

5. We have now covered all of the necessary material in this chapter except for Subsection 3.6.a and Section 3.7. For our purposes, the former is only used in a couple of technical proofs, and the latter is mainly used to check whether certain line bundles are ample in the counterexamples of Section 4.9 (as well as for a few other technical ideas in Chapters 4 and 5). Depending on the reader's interests, it is possible to skip both Subsection 3.6.a and Section 3.7 and still understand most of our work later on. Alternately, the reader may wish to briefly read Section 3.7 and then refer back to it as needed.

Starting with this chapter, we will assume that $k$ is an algebraically closed field of characteristic 0. Much of the theory can be developed in positive characteristic, and we will give references to characteristic-independent treatments wherever possible. However, the full classification of spherical varieties has only been completed in characteristic 0, so we will not consider the positive-characteristic setting here. We also assume without further mention that $G$ is a reductive group, $T \subset G$ is a maximal torus, and $B \subset G$ is a Borel subgroup containing $T$.

## 3.1 Spherical Varieties

In this section, we define spherical varieties and discuss some of their basic geometric properties.

**Definition 3.1.1.**

1. A *spherical variety* is a normal $G$-variety containing a dense $B$-orbit (which is necessarily open, because orbits are locally closed).

2. A *spherical subgroup* of $G$ is a (closed) subgroup $H \subset G$ such that the quotient $G/H$ is a spherical variety. Spherical varieties of the form $G/H$ are called *homogeneous spherical varieties*.

**Remark 3.1.2.** Note that whether a normal variety is spherical does not depend on our choice of $B$. Indeed, the Borel subgroups are all conjugate, and if $B' = gBg^{-1}$ for any $g \in G$, then $Bx \subset X$ is a dense $B$-orbit if and only if $B'(gx) = g \cdot Bx$ is a dense $B'$ orbit. Alternately, Theorem 3.1.4iv below gives an explicit, geometric characterization of spherical varieties that does not depend on $B$.

**Example 3.1.3.** Recall that a *toric variety* for a torus $T'$ is a normal $T'$-variety $X$ such that there exists a $T'$-equivariant open immersion $T' \hookrightarrow X$ (equivalently, such that there exists an open $T'$-orbit on which $T'$ acts freely). Toric varieties are often cited as the first example of spherical varieties. Indeed, suppose that $G$ is a torus, so that $G = B = T$. Then, any toric $T$-variety $X$ is normal and has open $T$-orbit, so $X$ is a spherical $T$-variety by definition.

There is actually a little more we can say about the relationship between spherical $T$-varieties and toric varieties. With $G = B = T$ as above, let $X$ be a spherical $T$-variety. Then, the open $T$-orbit has the form $T/K$ for some $K \subset T$. Since $T$ is commutative, $K$ is normal and hence is the stabilizer of every point in the open $T$-orbit of $X$. It follows that $K$ acts trivially on all of $X$ (see Lemma 2.1.6). Thus, the action of $T$ on $X$ induces an action of the torus $T' = T/K$ on $X$, and $T'$ acts freely on the open $T$-orbit of $X$. In other words, $X$ is a toric $T'$-variety for the quotient $T' = T/K$ of $T$.

It turns out that, for any normal $G$-variety $X$, there are many conditions which are equivalent to $X$ being spherical. These conditions are very standard, but their proofs require some theory about *complexities* of varieties that we will not need later. We will sketch the relevant details of this theory briefly, in order to present the equivalent conditions to being spherical in Theorem 3.1.4. The reader willing to take this theorem on faith can safely skip this discussion of complexity.

Given any algebraic group $G$ and any $G$-variety $X$, we define the *complexity* of $X$, denoted $c_G(X)$, to be the minimal codimension of a $G$-orbit in $X$. When $G$ is reductive, it immediately from the definition that a normal variety $X$ is spherical if and only if $c_B(X) = 0$. Much can be said about the complexity of a variety; for a thorough treatment, see [Tim11, Chapter 2]. For our purposes, we will need only two facts about complexity:

1. For any algebraic group $G$ and any $G$-variety $X$, we have $c_G(X) = \mathrm{trdeg}_k(K(X)^G)$. This follows from a theorem of Rosenlicht about $G$-invariants in function fields. See [Per18, Theorem 1.1.8 and Proposition 1.2.4] for a statement of Rosenlicht's theorem and a proof of this complexity fact, and see [VP89, Section 2.3] for a proof of Rosenlicht's theorem.

2. For a reductive group $G$, any $G$-variety $X$, and any $B$-stable subvariety $Y$, we have $c_B(Y) \leq c_B(X)$. When $Y$ is $G$-stable, this actually follows from fact (1) above along with the fact that we can lift $G$-invariant elements of the function field from $Y$ to $X$ (see e.g. [Tim11, Lemma 5.8]). One then proves that using a careful geometric argument that for any $B$-stable subvariety $Y$, we have $c_B(Y) \leq c_B(GY)$. For details, see [Per18, Proposition 1.2.10].

Equipped with these facts about complexity, we can now prove the promised equivalent characterizations of spherical varieties.

**Theorem 3.1.4** ([Per14, Theorem 2.1.2], cf. [Bri97, Theorem 2.1], [Tim11, Section 25.1])**.** *Let $X$ be a normal $G$-variety. The following are equivalent.*

*(i) $X$ is spherical.*

*(ii) $K(X)^B = k$.*

*(iii) $X$ has finitely many $B$-orbits.*

*(iv) Every $G$-equivariant birational model of $X$ has finitely many $G$-orbits.*

*If $X$ is also quasi-projective, then these conditions are also equivalent to the following condition:*

*(v) For any $G$-linearized invertible sheaf $L$ on $X$, the $G$-module $H^0(X, L)$ is multiplicity-free.*

*Proof.*

(i) $\Leftrightarrow$ (ii): $X$ is spherical if and only if $c_B(X) = 0$. On the other hand, Fact 1 above gives us $c_B(X) = \mathrm{trdeg}_k(K(X)^B)$, and since $k$ is algebraically closed, $\mathrm{trdeg}_k(K(X)^B) = 0$ if and only if $K(X)^B = k$.

(i) $\Leftrightarrow$ (iii): If $X$ has finitely many $B$-orbits $O_1, \dots, O_r$, then we have

$$X = \bigcup_i \overline{O_i}.$$

Since $X$ is irreducible, this implies that $X = \overline{O_i}$ for some $i$. For the converse, we note that by Fact 2 above, every $B$-stable subvariety $Y \subset X$ has $c_B(Y) = c_B(X) = 0$, i.e. $Y$ has a dense $B$-orbit. We prove by induction on $\dim(X)$ that any (not necessarily normal) $B$-variety $X$ with this property has finitely many $B$-orbits. The base case $\dim(X) = 0$ is trivial (as $X = \operatorname{Spec}(k)$ in this case). So, suppose the claim holds whenever $\dim(X) < d$, and let $Y \subset X$ be the complement of the open $B$-orbit of $X$. Then, $Y$ satisfies the same condition as $X$ (every $B$-stable subvariety of $Y$ is a $B$-stable subvariety of $X$), and $\dim(Y) < \dim(X)$. So, $Y$ has finitely many orbits by the induction hypothesis, and $X$ has only one more $B$-orbit than $Y$ does by construction.

(i) $\Rightarrow$ (iv): Suppose $X$ is spherical. For any $G$-equivariant birational model $\tilde{X}$, let $\varphi : X \dashrightarrow \tilde{X}$ be a $G$-equivariant birational map. The map $\varphi$ is defined on some point in the open $B$-orbit of $X$, and since $\varphi$ is $G$-equivariant, it extends over the entire open $B$-orbit. Thus, $\tilde{X}$ is a $G$-variety with a dense $B$-orbit. By the proof of (i) $\Rightarrow$ (iii) above, we conclude that $\widetilde{X}$ has finitely many $B$-orbits, hence finitely many $G$-orbits.

(iv) $\Rightarrow$ (v): The general idea of this direction is as follows. Suppose that (iv) holds but (v) does not. One then construcs a $G$-equivariant birational model of $X$ with infinitely many $G$-orbits, which contradicts (iv). We omit the details of this construction here and refer the interested reader to the proof given in [Bri97, Theorem 2.1].

(v) $\Rightarrow$ (ii): Let $a \in K(X)^B$, and pick any $G$-linearized ample invertible sheaf $L$ on $X$. By Proposition 2.5.2, we may write $a = f/g$, where $f, g \in \Gamma(X, L^{\otimes d})^{(B)}$ for some $d \geq 0$. Since $B$ acts by some character on both $f$ and $g$ and fixes $a = f/g$, we see that $B$ must act by the same character on both $f$ and $g$. Since $L^{\otimes d}$ is again a $G$-linearized ample invertible sheaf, we know that $\Gamma(X, L^{\otimes d})$ is multiplicity-free by assumption, so we must have $f = cg$ for some $c \in k$. It follows that $a = f/g = c \in k$. (Note that the containment $k \subset K(X)^{(B)}$ follows immediately from the fact that the structure morphism $X \to \operatorname{Spec}(k)$ is $G$-equivariant, so it induces a $G$-equivariant map on function fields by Lemma 2.4.3.)

(iv) $\Rightarrow$ (i): By Theorem 2.6.12, there exists a nonempty $G$-stable open subset $U \subset X$ that is quasi-projective. Note that (iv) also holds for $U$ (since $U$ is itself a $G$-equivariant birational model of $X$). Moreover, $U$ is dense in $X$, so $U$ has a dense $B$-orbit if and only if $X$ does. It thus suffices to prove that (iv) $\Rightarrow$ (i) for $U$. Since $U$ is quasi-projective, this follows from the implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii) $\Rightarrow$ (i). □

**Remark 3.1.5.** Note that (v) in the above theorem requires quasi-projectivity, because we need to consider (v) for an ample line bundle in order to prove that (v) implies (ii). However, Conditions (i)-(iv) in the theorem always imply (v), even when $X$ is not quasi-projective. Indeed, suppose that $X$ is spherical, and let $L$ be a $G$-linearized invertible sheaf on $X$. Restricting global sections to some $G$-stable quasi-projective open subset $U \subset X$ (which exists by Theorem 2.6.12) allows us to view $H^0(X, L)$ as a $G$-submodule of $H^0(U, L|_U)$. Applying Theorem 3.1.4 to $U$ (which is spherical because $X$ is) tells us that $H^0(U, L|_U)$ is multiplicity-free, hence so is the $G$-submodule $H^0(X, L)$.

### 3.1.a The Geometry of Spherical Varieties

Spherical varieties enjoy several nice geometric properties. For instance, every spherical variety is rationa.

**Proposition 3.1.6** ([Per14, Corollary 2.1.3])**.** *Spherical varieties are rational.*

*sketch of proof.* Any spherical variety has a dense $B$-orbit, which is of the form $B/K$. It thus suffices to show that whenever $B$ is connected and solvable and $K \subset B$ is a subgroup, the quotient $B/K$ is rational. For these, one proceeds by inducting first on $\dim(B)$ and then on $\dim(B/K)$. The base cases are trivial. For the induction step, one considers the quotient $B/K \to B/KZ$ for some appropriately chosen 1-dimensional normal subgroup $Z \subset B$. The induction hypothesis on $B/Z$ and $B/KZ$ implies that $B/KZ$ is rational, and by choosing $Z$ correctly and using some facts about solvable groups, the quotient map $B/K \to B/KZ$ turns out to be a locally trivial fibration. It follows that $B/K$ is rational as well. □

We saw in Example 3.1.3 that toric varieties are examples of spherical varieties. Conversely, we can relate any affine spherical variety to a toric variety in the following way. Let $X = \mathrm{Spec}(A)$ be an affine $G$-variety, and consider the GIT quotient $X//U = \mathrm{Spec}(A^U)$. Since $B = T \times U$, we may view $X//U$ as a $T$-variety, and the quotient map $X \to X//U$ induces a bijection between $B$-orbits of $X$ and $T$-orbits of $X//U$. Since Proposition 2.5.7 tells us that $X$ is normal if and only if $X//U$ is, we arrive at the following result.

**Proposition 3.1.7** ([Per18, Corollary 2.3.4])**.** *Let $X$ be an irreducible affine $G$-variety. $X$ is a spherical $G$-variety if and only if $X//U$ is a toric $T$-variety.*

One very useful property of spherical varieties is: any $G$-stable (closed) subvariety is again spherical. The key to this statement is a very nice result on the singularities of $G$-stable subvarieties of spherical varieties.

**Theorem 3.1.8** ([Per14, Corollary 2.3.4 and Theorem 3.1.19], [Tim11, Theorem 15.20])**.** *Let $X$ be a spherical $G$-variety.*

*(a) Any $G$-stable subvariety $Y \subseteq X$ has rational singularities. In particular, $Y$ is normal and in fact Cohen–Macaulay.*

*(b) Any $G$-stable subvariety of $X$ is spherical.*

*sketch of proof.* Note that (b) follows immediately from (a) and Theorem 3.1.4: any $G$-stable subvariety of $X$ is normal by (a) and has finitely many $B$-orbits because $X$ does. As for (a), the idea is to reduce from the case of spherical varieties to that of toric varieties, in which case Statement (a) follows from some standard facts about toric varieties. More precisely: if $X$ is a toric $T'$-variety for some torus $T'$, then $X$ (hence also $Y$) has finitely many $T'$-orbits (by Theorem 3.1.4 applied with $G = T'$, or by standard facts about orbits of toric varieties, see e.g. [Ful93, Proposition 3.1]). It follows that $Y$ is an orbit closure, and it is a standard fact (see e.g. [Ful93, Section 3.1]) that every orbit closure of a toric variety is toric. Finally, one can show that every toric variety has rational singularities ([Ful93, Section 3.5]). So, $Y$ has rational singularities, as desired.

There are a few possible ways to reduce to the toric case. First, one has to reduce to the affine case. This can be done by reducing to the case where $X$ is a locally closed $G$-stable subvariety of $\mathbb{P}(V)$ (using Theorem 2.6.12), then replacing $X$ by the affine cone over the closure of $X$ in $\mathbb{P}(V)$. A simpler way is to apply the local structure theorem from Section 3.2 to the open subset $X_{B,Y'}$ of Theorem 3.2.7, where $Y'$ is any $G$-orbit contained in $Y$. (Note that the sets $Y \cap GX_{B,Y'}$ form an open cover of $Y$, so it suffices to prove that $Y \cap X_{B,Y'}$ has rational singularities for each choice of $Y'$.)

Once we are in the case where $X$ is affine, one possibility is to to construct a flat deformation of $X$ to a toric variety and then use stability of rational singularities under flat deformations. For details on this approach, see [Per14, Corollary 4.3.15]. Alternately, since $X$ is affine and spherical, the GIT quotient $X//U$ is an affine toric variety under some quotient $T'$ of $T$ (Propositions 3.1.7), and $Y//U$ is a $T'$-stable closed subvariety of $X//U$ (this follows from the fact that, given an action of $T$ on a ring $A$ and a $T$-stable ideal $I \subset A$, we have $(A/I)^U = A^U/I^U$, where $U = R_u(B)$; see e.g. [MF82, proof of Theorem 1.1] for a proof). By our above arguments in the toric case, $Y//U$ has rational singularities; Proposition 2.5.7 then implies that $Y$ has rational singularities as well. $\square$

**Remark 3.1.9.** The above result is not true in positive characteristic. For a counterexample, see [Per18, Example 2.3.8].

## 3.1.b Valuations and Divisors

The combinatorial invariants involved in the classification of spherical varieties all have to do with two types of geometric objects: valuations and divisors. In this section, we introduce some notation and give a few basic results regarding valuations and divisors on spherical varieties. We begin with valuations.

**Definition 3.1.10.** Let $G$ be an algebraic group over a field $k$, and let $X$ be a $G$-variety.

1. For any totally ordered abelian group $\Gamma$, a *valuation on $X$ with values in $\Gamma$* is a group homomorphism
$$v : K(X)^\times \to \Gamma$$
satisfying
   a) $v(k^\times) = 0$, and
   b) for any $f, g \in K(X)^\times$ with $f + g \in K(X)^\times$, we have
$$v(f + g) \geq \min\{v(f), v(g)\}.$$

We say that $v$ is a *discrete valuation* if we may take $\Gamma = \mathbb{Q}$ and if the image of $v$ is $a\mathbb{Z}$ for some $a \in \mathbb{Q}$. (Note that the map $K(X)^\times \to \Gamma$ sending every element of $K(X)^\times$ to 0 is a discrete valuation by definition.)

2. Given a valuation $v$ on $X$, we denote by

$$\mathcal{O}_v = \{0\} \cup \{f \in K(X)^\times \mid v(f) \geq 0\}$$

the associated valuation ring.

3. We say that a valuation $v$ on $X$ is *$G$-invariant* if for all $f \in K(X)^\times$ and all $g \in G(k)$, we have

$$v(g \cdot f) = v(f).$$

(Here $g \cdot f$ denotes the action of $g$ on $f$ using the usual $G$-module structure on $K(X)$, see Section 2.4.a.) We denote by $\mathcal{V}(X)$ the set of all $G$-invariant valuations on $X$.

4. Given a valuation $v$ on $X$ we say that a closed subvariety $Z = \overline{\{\zeta\}} \subset X$ is the *center on $X$* of $v$ and write $Z = Z_v$ if the valuation ring $\mathcal{O}_v$ dominates the stalk $\mathcal{O}_{X,\zeta}$, i.e. if $\mathcal{O}_{X,\zeta} \subset \mathcal{O}_v$ (as subrings of $K(X)$) and the inclusion map $\mathcal{O}_{X,\zeta} \hookrightarrow \mathcal{O}_v$ is a local ring homomorphism.

5. Given a prime Weil divisor $D$ on $X$, we write $v_D : K(X)^\times \to \mathbb{Q}$ for the (discrete) valuation corresponding to the DVR $\mathcal{O}_{X,\delta}$, where $\delta \in D \subset X$ is the generic point of $D$.

**Remark 3.1.11.** Throughout this thesis, we will only be interested in valuations with values in $\mathbb{Q}$ (whether discrete or not). The word "valuation" will thus always mean a valuation with values in $\mathbb{Q}$. Note that this does not lose much in the setting we're in: indeed, any Noetherian valuation ring is automatically a DVR, and we're primarily interested in valuations that arise from stalks of Noetherian schemes, so these will automatically be discrete.

The following lemma gives a few standard properties of $G$-valuations and their centers.

**Lemma 3.1.12.** *Let $G$ be an algebraic group over a field $k$, and let $X$ be a $G$-variety.*

*(a) The center of any valuation on $X$ is unique if it exists. Moreover, $X$ is proper if and only if every valuation on $X$ has a (unique) center on $X$.*

*(b) If $X$ is affine, then $v$ has a center on $X$ if and only if $v$ is non-negative on $\Gamma(X, \mathcal{O}_X)$. In this case, the center $Z_v$ of $v$ on $X$ is cut out by the ideal $\mathfrak{m}_v \cap \Gamma(X, \mathcal{O}_X)$, where $\mathfrak{m}_v \subset \mathcal{O}_v$ is the maximal ideal and the intersection is taken in $K(X)$.*

*(c) The $G$-stable subvarieties of $X$ are precisely the centers of $G$-invariant valuations on $X$.*

*Proof.* Statement (a) follows from the valuative criteria of separatedness and properness; see [Har77, Exercise II.4.5]. As for (b), write $A = \Gamma(X, \mathcal{O}_X)$. If $v$ has a center on $X$, then $\mathcal{O}_v$ dominates the stalk of some point $x \in X$, so we have $A \subset \mathcal{O}_{X,x} \subset \mathcal{O}_v$. In particular, $v$ is non-negative on $A$ by definition. Conversely, if $v$ is non-negative on $A$, then $A \subset \mathcal{O}_v$ by definition. Let $\mathfrak{p} = \mathfrak{m}_v \cap A$. Then, $\mathfrak{p}$ is a prime ideal of $A$ (it is the pullback of $\mathfrak{m}_v$ under the inclusion $A \hookrightarrow \mathcal{O}_v$), and $A_\mathfrak{p}$ is dominated by $\mathcal{O}_v$, so $\mathrm{V}(\mathfrak{p}) \subset \mathrm{Spec}(A) \cong X$ is the center of $v$ on $X$.

For statement (c), if $Z = Z_v$ is the center of a $G$-invariant valuation $v$ on $X$, then the fact that $v$ is $G$-invariant implies that the action of $G$ on $K(X)$ fixes the valuation ring $\mathcal{O}_v$. The (unique) morphism $f : \mathrm{Spec}(\mathcal{O}_v) \to X$ is therefore a $G$-equivariant map sending the closed point of $\mathrm{Spec}(\mathcal{O}_v)$ to the generic point of $Z$. Since the action of $G$ fixes the closed point of $\mathrm{Spec}(\mathcal{O}_v)$ and $f$ maps this closed point to the generic point of $Z$, we see that $G$ fixes the generis point of $Z$ as well. It follows from continuity of the action morphism $G \times X \to X$ that we have $G \cdot Z = Z$, i.e. $Z$ is $G$-stable.

Conversely, if $Z$ is $G$-stable, then the blowup $\mathrm{Bl}_Z(X)$ inherits a natural $G$-action such that the blowup morphism $\pi : \mathrm{Bl}_Z(X) \to X$ is $G$-equivariant. It follows that the exceptional divisor $\pi^{-1}(Z)$ is $G$-stable, hence so is any irreducible component $D \subset \pi^{-1}(Z)$. Since the generic point of $D$ is fixed by $G$, we see that the valuation $v_D$ is $G$-invariant. Moreover, since $\pi$ maps $D$ dominantly onto $Z$, we see that $\pi$ induces a local ring homomorphism $\mathcal{O}_{X,\zeta} \hookrightarrow \mathcal{O}_{v_D}$, where $\zeta \in Z$ is the generic point. We conclude that $Z$ is the center of $v_D$ on $X$. □

The main technical ingredient for working with valuations on spherical varieties is the following theorem, which essentially allows us to only consider the values of $G$-invariant valuations on the $B$-eigenvectors of $K(X)^\times$. The proof of the theorem involves several technical arguments about $G$-invariant valuations; we refer the reader to [Kno91, Section 1] for details.

**Theorem 3.1.13** ([Kno91, Corollary 1.7])**.** *Let $G$ be a connected reductive group, let $X$ be a spherical $G$-variety, and let $X_B^\circ \subset X$ be the open $B$-orbit of $X$. For any $G$-invariant valuation $v_0 \in \mathcal{V}(X)$ and any nonzero element $f \in \Gamma(X_B^\circ, \mathcal{O}_X)$, there exists some $f' \in K(X)^{(B)}$ such that*

$v_0(f') = v_0(f)$,

$v(f') \geq v(f)$ *for all* $v \in \mathcal{V}(X)$, *and*

$v_D(f') \geq v_D(f)$ *for every $B$-stable prime divisor $D$ on $X$.*

Inspired by the above theorem, we want to distill the information of a $G$-invariant valuation into a simpler combinatorial object by considering only its values on $B$-eigenvectors. We can do this in the following way. Let $X$ be a spherical variety. Note that the set of nonzero

$B$-eigenvectors $(K(X)^{\times})^{(B)}$ is abelian group (under multiplication), and we can define a homomorphism

$$(K(X)^{\times})^{(B)} \to \Lambda(X)$$

by $f \mapsto \chi_f$, where $\chi_f$ is the character through which $B$ acts on $f$. This map is surjective by definition, and its kernel is precisely the nonzero elements of $K(X)^B$. Since $X$ is spherical, we have $K(X)^B = k$ (Theorem 3.1.4), so we obtain an exact sequence

$$1 \to k^{\times} \to (K(X)^{\times})^{(B)} \to \Lambda(X) \to 0.$$

Now, any valuation $v : K(X)^{\times} \to \mathbb{Q}$ restricts to a group homomorphism $v' : (K(X)^{\times})^{(B)} \to \mathbb{Q}$. Since $v'$ vanishes on $k^{\times}$, it induces a group homomorphism $\varphi(v) : \Lambda(X) \to \mathbb{Q}$. We have thus defined a map

$$\varphi : \mathcal{V}(X) \to N(X) = \operatorname{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}).$$

One of the key implications of Theorem 3.1.13 is that the map $\varphi$ is injective.

**Corollary 3.1.14** ([Kno91, Corollary 1.8])**.** *Any $G$-invariant valuation $v \in \mathcal{V}(X)$ is uniquely determined by its values on $K(X)^{(B)}$. In other words, the map $\varphi$ defined above is injective.*

*Proof.* Let $v' \in \mathcal{V}(X)$ be any $G$-invariant valuation with $v' \neq v$, and let $X_B^{\circ} \subset X$ be the open $B$-orbit. Note that $X_B^{\circ}$ is an orbit of a smooth solvable group and hence is affine (see Theorem 2.2.4), so we have $\operatorname{Frac}(\Gamma(X_B^{\circ}, \mathcal{O}_X)) = K(X)$. Thus, there exists some $f \in \Gamma(X_B^{\circ}, \mathcal{O}_X)$ such that $v(f) \neq v'(f)$. Without loss of generality, suppose that $v(f) < v'(f)$. By applying Theorem 3.1.13 to $v$ and $f$ we, then obtain some $f' \in K(X)^{(B)}$ and some $q$ such that

$$v(f') = v(f^q) < v'(f^q) \leq v'(f').$$

So $v(f') \neq v'(f')$, which shows that $v$ and $v'$ do not have the same values on $K(X)^{(B)}$. $\square$

In light of this corollary, we typically identify $\mathcal{V}(X)$ with its image under $\varphi$ in the $\mathbb{Q}$–vector space $N(X)$. As we will see later (Section 3.4), the subset $\mathcal{V}(X) \subset N(X)$ is actually a polyhedral cone with several nice properties related to the geometry of $X$.

We now turn to divisors on spherical varieties.

**Definition 3.1.15.** Let $X$ be a $G$-variety.

1. By a *$B$-divisor* (resp. *$G$-divisor*) on $X$ we mean a $B$-stable (resp. $G$-stable) prime Weil divisor on $X$.

2. We denote by $\mathcal{D}_{G,X}$ (resp. $\mathcal{D}_{G,X}^G$) the set of all $B$-divisors (resp. $G$-divisors) on $X$.

3. A *color* on $X$ is a $B$-divisor which is not $G$-stable. We define $\Delta(X) = \mathcal{D}(X) \setminus \mathcal{D}_G(X)$ to be the set of all colors of $X$.

Let $X$ be a spherical variety. Note that for a $G$-divisor $D$ of $X$, the associated valuation $v_D$ is $G$-invariant (Lemma 3.1.12). If $D$ is instead a color, the associated valuation $v_D$ will not be $G$-invariant (though it will be $B$-invariant). In either case, however, $v_D$ induces a map $\Lambda(X) \to \mathbb{Q}$ as above. We can thus extend the map $\varphi$ defined above to a map

$$\varphi : \mathcal{V}(X) \cup \mathcal{D}_{G,X} \to N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$$

by sending any $D \in \Delta(X)$ to the map $\varphi_D : \Lambda(X) \to \mathbb{Q}$ determined by the valuation $v_D$. Note that $\varphi$ is generally not injective on $\Delta(X)$ (i.e. a $B$-invariant valuation need not be determined by its value on $B$-eigenvectors), as the following example shows.

**Example 3.1.16** ([Pez10, Examples 2.2.1, 2.3.1, 2.5.1])**.** Let $G = \mathrm{SL}_2$ acting on $X = \mathbb{P}^1 \times \mathbb{P}^1$ diagonally via the action on $\mathbb{P}^1$ given in Example 2.4.19. In projective coordinates, this action is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot ([x : y], [w : z]) = ([ax + by : cx + dy], [aw + bz : cw + dz]).$$

We take $B \subset G$ to be the Borel subgroup of upper triangular matrices and $T \subset B$ to be the subgroup of diagonal matrices. Note that if $[x : y] \neq [w : z]$, then the vectors $(x, y)^T$ and $(w, z)^T$ form a basis for $\mathbb{C}^2$. So for any other such pair $[x' : y'] \neq [w' : z']$, there is an invertible matrix mapping $(x, y)^T \mapsto (x', y')^T$ and $(w, z)^T \mapsto (w', z')^T$. After scaling this matrix, we obtain some matrix $g \in \mathrm{SL}_2$ such that $g \cdot [x : y] = [x' : y']$ and $g \cdot [w : z] = [w' : z']$. If $\Delta \subset X$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, this proves that $X \setminus \Delta$ is a $G$-orbit of $X$, and one can check that $\Delta$ is the only other $G$-orbit of $X$.

We can characterize the $B$-orbits of $X$ in a similar way. Given any point $([x : y], [w : z]) \in \mathbb{P}^1$, one can check that there is a (unique) upper triangular matrix of determinant 1 mapping $[x : y] \mapsto [x' : y]$ and $[w : z] \mapsto [w' : z]$ for any choice of $x'$ and $w'$. On the other hand, the point $([1 : 0], [1 : 0])$ is fixed by every upper triangular matrix and hence is a $B$-fixed point. It follows that the diagonal $\Delta$ is a $B$-stable divisor of $X$, and so are the subvarieties

$$D_1 = \mathbb{P}^1 \times \{[1 : 0]\}, \quad D_2 = \{[1 : 0]\} \times \mathbb{P}^1.$$

Our above statements also imply that the set

$$\{([x : 1], [y : 1]) \mid x \neq y\} = X \setminus (\Delta \cup D_1 \cup D_2)$$

is an open $B$-orbit of $X$. This proves that $X$ is spherical and that the $B$-divisors of $X$ are $\Delta$, $D_1$, and $D_2$. Since $\Delta$ is a $G$-orbit, it is $G$-stable; on the other hand, both $D_1$ and $D_2$ intersect the open $G$-orbit $X \setminus \Delta$ and hence are colors of $X$.

Write $X \cong \mathrm{Proj}(k[x, y]) \times \mathrm{Proj}(k[w, z])$, and consider the open subset

$$X_{B,\Delta} = X \setminus (D_1 \cup D_2) = \{([x : 1], [y : 1]) \mid x, y \in k\} \subset X.$$

We have $X_{B,\Delta} = \mathrm{D}_+(y) \times \mathrm{D}_+(z)$ and hence

$$X_{B,\Delta} \cong \operatorname{Spec}\left(k\left[\frac{x}{y}, \frac{w}{z}\right]\right) \cong \operatorname{Spec}(k[x', w']).$$

Using the action of $G$ on $k[x,y]$ and $k[w,z]$ given in Example 2.4.19, we see that $B$ acts in these coordinates by

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot x' = t^{-2}x' - ut^{-1}, \qquad \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot w' = t^{-2}w' - ut^{-1}.$$

So, the element $(x' - w')^{-1} \in K(X_{B,\Delta}) = K(X)$ is a $B$-eigenvector with character equal to the unique simple root $\alpha_1$ of $G$. (Explicitly, $\alpha_1$ is given by $\mathrm{diag}(t, t^{-1}) \mapsto t^2$). It follows that $\alpha_1 \in \Lambda(X)$.

Now, the isomorphism $X_{B,\Delta} \cong \operatorname{Spec}(k[x', w'])$ identifies $x'$ (resp. $w'$) with the rational function $x/y \in K(X)$ (resp. $w/z \in K(X)$). Thus, the rational function $x' - w' = \frac{x}{y} - \frac{w}{z}$ has zeros of order 1 on the open subset $U \cap \Delta$ of $\Delta$ and has poles of order 1 on the open subsets $D_1 \setminus \{([1:0],[1:0])\}$ and $D_2 \setminus \{([1:0],[1:0])\}$ of $D_1$ and $D_2$. Since $x' - w'$ is a $B$-eigenvector with character $-\alpha_1$, it follows that $\varphi_{D_1}(\alpha_1) = \varphi_{D_2}(\alpha_1) = 1$ and that $\varphi_\Delta(\alpha_1) = -1$. Moreover, we know that

$$\mathbb{Z} \cdot \alpha_1 \subset \Lambda(X) \subset \Lambda_G = \mathbb{Z} \cdot \frac{\alpha_1}{2}.$$

If there were any nonzero $B$-eigenvector $f \in K(X)^{(B)}$ with weight $\alpha_1/2$, then we would have $v_{D_1}(f) = \varphi(\alpha_1)/2 = 1/2$, contradicting the fact that $v_{D_1}$ takes integer values on $K(X)^\times$. It follows that $\Lambda(X) = \mathbb{Z} \cdot \alpha_1$.

Since $\varphi_{D_1}(\alpha_1) = \varphi_{D_2}(\alpha_1) = 1$, we see that $\varphi_{D_1} = \varphi_{D_2}$ as functions $\Lambda(X) \to \mathbb{Q}$. On the other hand, for each $i$, the valuation $v_{D_i}$ has center $D_i$ on $X$; since $X$ is separated, this implies that $v_{D_1} \neq v_{D_2}$. Thus, this example shows that $\varphi$ need not be injective on colors, or equivalently, that valuations which are not $G$-invariant need not be determined by their values on $B$-eigenvectors.

The following lemma gives us a few nice properties about divisors and valuations on spherical varieties.

**Lemma 3.1.17.** *Let $X$ be a spherical $G$-variety.*

*(a) For any $B$-stable affine open subset $U \subset X$, the irreducible components of $X \setminus U$ are all $B$-divisors.*

*(b) The $B$-divisors of $X$ are precisely the irreducible components of the complement of the open $B$-orbit.*

*(c) Either $X$ has no $G$-divisors (i.e. every $G$-stable subset of $X$ is either dense or of codimension $> 1$), or the $G$-divisors of $X$ are precisely the irreducible components of the complement of the open $G$-orbit.*

*(d) If $X$ is affine, we have*

$$\Lambda^+(X) = \{\chi \in \Lambda(X) \mid \varphi_D(\chi) \geq 0 \ \forall \ D \in \mathcal{D}_{G,X}\}.$$

*Proof.* For (a), we note that $X \setminus U$ is $B$-stable, hence so is every irreducible component $X \setminus U$ (see Lemma 2.1.3). Moreover, because $U$ is affine, it is a general scheme-theoretic fact that every irreducible component of $X \setminus U$ has codimension 1 (see e.g. [Sta20, Tag 0BCV]). So, the irreducible components of $X \setminus U$ are $B$-stable irreducible codimension 1 subschemes of $X$, i.e. $B$-divisors.

As for (b), the open $B$-orbit $X_B^\circ$ is affine (Theorem 2.2.4), so (a) implies that every irreducible component of $X \setminus X_B^\circ$ is a $B$-divisor. Conversely, any $B$-divisor $D$ is $B$-stable and does not contain the open $B$-orbit $X_B^\circ$, so $D$ cannot intersect $X_B^\circ$. In other words, $D$ lies in the complement $V$ of $X_B^\circ$, so $D$ must be an irreducible component of $V$ (because $D$ is irreducible and has codimension 1 in $X$).

For (c), the same reasoning as in (b) shows that any $G$-divisor is an irreducible component of the complement $V_G$ of the open $G$-orbit of $X$. So, we just need to show that if a $G$-divisor $D$ exists, then every irreducible component of $V_G$ is a $G$-divisor. For this, note that $V_G$ is a $G$-stable closed subvariety of $X$. So, Theorem 3.1.8 implies that $V_G$ is Cohen-Macaulay and in particular equidimensional. Since $D$ is a component of $V_G$, we conclude that every component of $V_G$ has codimension 1 in $X$ and hence is a prime Weil divisor of $X$. Finally, every component of $V_G$ is $G$-stable by Lemma 2.1.3.

As for (d), write $A = \Gamma(X, \mathcal{O}_X)$ and $A^\circ = \Gamma(X_B^\circ, \mathcal{O}_X)$. The $B$-eigenvectors in $A$ are precisely the $B$-eigenvectors of $A^\circ$ that extend over $X \setminus X_B^\circ$. Combining this fact with Statement (a) gives us

$$A^{(B)} = \{f \in (A^\circ)^{(B)} \mid v_D(f) \geq 0 \ \forall \ D \in \mathcal{D}_{G,X}\}.$$

On the other hand, for any $B$-divisor $D \in \mathcal{D}_{G,X}$ and any eigenvector $f \in (A^\circ)^{(B)}$ of weight $\chi$, we have $v_D(f) = \varphi_D(\chi)$ by definition. So, the above equation will be precisely the equation in (c), provided we can show that $\Lambda^+(A^\circ) = \Lambda(X)$. We already know that $\Lambda(X) = \Lambda^+(X)^{gp}$ (see Proposition 2.5.9). For any $f \in A^{(B)}$, the nonvanishing locus $X_f$ is a $B$-stable open subset, so it intersects $X_B^\circ$ and hence contains $X_B^\circ$. It follows that $f, f^{-1} \in A^\circ$ and hence that

$$\Lambda(X) = \Lambda^+(X)^{gp} \subset \Lambda^+(X_B^\circ).$$

The opposite containment follows immediately from the fact that $A^\circ \subset K(X)$ (and the fact that the $G$-module structure on $A^\circ$ is compatible with that of $K(X)$, see Remark 2.4.1). $\square$

**Remark 3.1.18.** One consequence of parts (a) and (b) of the above lemma is that the sets $\mathcal{D}_{G,X}$, $\mathcal{D}_{G,X}^G$, and $\Delta(X)$ are all finite for any spherical variety $X$.

We have already seen (Example 3.1.3) that toric varieties are spherical varieties when $G = B = T$ is a torus. Notice that toic varieties will never have colors, because $G = B$. Conversely, it turns out that any spherical variety with no colors is a toric variety.

**Proposition 3.1.19** (cf. [Bri97, proof of Theorem 2.3])**.** *Let $X$ be a spherical $G$-variety, and suppose that $X$ has no colors (i.e. that every $B$-divisor of $X$ is $G$-stable). Then, $[G,G]$ acts trivially on $X$, and $X$ is a toric variety for a quotient of the torus $G/[G,G]$.*

*Proof.* Let $X^\circ$ be the open $B$-orbit of $X$. The complement of $X^\circ$ is the union of the $B$-divisors of $X$ (Lemma 3.1.17). All of these divisors are $G$-stable by assumption, so $X^\circ$ is $G$-stable and hence is the open $G$-orbit of $X$. Let $H$ be the stabilizer of a point $x \in X^\circ$. We claim that $G = BH$. Indeed, for any $g \in G$, we have $gx \in X^\circ$. Since $X^\circ$ is a $B$-orbit, there exists some $b \in B$ such that $gx = bx$. It follows that $b^{-1}g \in H$, so that $g = b(b^{-1}g) \in BH$.

Since $G = BH$, a lemma using the theory of semisimple groups (see [Bri97, Lemma 2.3] or [Pez10, Lemma 3.1.1]) implies that $[G,G] \subset H$. Thus, $H$ is a normal subgroup of $G$ and hence is the stabilizer of every point in the open $G$-orbit $X^\circ$. It follows that $H$ acts trivially on all of $X$ (see Lemma 2.1.6). Thus, the action of $G$ on $X$ induces an action by the quotient group $G/H$, and $X^\circ$ is a $G/H$-orbit on which $G/H$ acts freely. On the other hand, $G/H$ is reductive (see [Bor91, Corollary 14.11]) and commutative (because $[G,G] \subset H$), so $G/H$ is a torus (see [Mil17, Proposition 19.13]). It follows that $X$ is a toric varic under $G/H$ (cf. Example 3.1.3). $\square$

The following is a technical (but sometimes very useful) statement about $B$-stable divisors.

**Proposition 3.1.20** (cf. [Bri97, proof of Proposition 2.2])**.** *Let $X$ be a $G$-variety with finitely many $G$-orbits, and let $D \subset X$ be a $B$-stable effective Weil divisor of $X$ such that no $G$-orbit is contained in the support of $D$. Then, $D$ is an effective Cartier divisor, and the invertible sheaf $\mathcal{O}_X(D)$ is generated by global sections.*

*Proof.* Let $\mathcal{O}_X(D)$ be the divisorial sheaf corresponding to $D$ (see Appendix B), and let $I = \mathcal{O}_X(-D)$. Let $\tilde{G} \to G$ be an isogeny of algebraic groups such that $I$ is $\tilde{G}$-linearizable (such an isogeny exists, see Theorem 2.6.11). There exists a Borel subgroup $\tilde{B} \subset \tilde{G}$ mapping to $B \subset G$ (see e.g. [Mil17, Proposition 17.20]). Since $\tilde{G}$ acts on $X$ via its image in $G$, we see that $D$ is $\tilde{B}$-stable. So, replacing $G$ by $\tilde{G}$ affects neither our assumptions nor the statement of the proposition. After this replacement, then, we may assume that $I$ is $G$-linearizable. We fix a $G$-linearization $\phi : \rho^* I \xrightarrow{\sim} \mathrm{pr}_X^* I$ (here $\rho : G \times X \to X$ is the action morphism).

Now, we define

$$Y = \{x \in X \mid I_x \not\cong \mathcal{O}_{X,x}\}.$$

Note that $Y$ is a closed subscheme of $X$ (since the locus where $I$ is invertible is open). We claim that $Y$ is $G$-stable. This can be checked on $k$-points. (Proof: because $X$ is a Jacobson scheme over $k$, the $k$-points of $Y$ are dense in $Y$, so continuity of the action morphism $\rho : G \times X \to X$ implies that if $G$ maps the $k$-points of $Y$ into $Y$, then it maps all of $Y$ into $Y$.) So, let $g \in G(k)$. The point $g$ acts on $X$ by a morphism $\rho_{g,k} : X \to X$, and the $G$-linearization $\phi$ induces an isomorphism of sheaves of $\mathcal{O}_X$-modules $i^*_{g,\mathrm{Spec}(k)}\phi : \rho^*_{g,k} I \to I$

(see the discussion preceding Lemma 2.4.11, applied to $S = \mathrm{Spec}(k)$). For any $x \in X(k)$, we have $\rho_{g,k}(x) = gx$ and hence

$$I_{gx} \cong (\rho_{g,k}^* I)_x \cong I_x.$$

(the first isomorphism here is a general fact about pullbacks of $\mathcal{O}_X$-modules, and the second isomorphism is induced by $i_{g,\mathrm{Spec}(k)}^* \phi$.) It follows that $x \in Y$ if and only if $gx \in Y$, which proves the claim.

On the other hand, Lemma B.11 implies that $I$ is the ideal sheaf corresponding to some closed subscheme structure on $\mathrm{Supp}(D)$. It follows that $Y \subset \mathrm{Supp}(D)$. In particular, $Y$ contains no $G$-orbit by our assumptions on $D$. But $Y$ is $G$-stable, so this is only possible if $Y = \varnothing$, i.e. if $I$ is invertible. This implies that $\mathcal{O}_X(D) \cong I^\vee$ is also invertible and hence that $D$ is a Cartier divisor (see Corollary B.12).

To prove that $\mathcal{O}_X(D)$ is globally generated, let $\sigma \in \Gamma(X, \mathcal{O}_X(D)$ be the canonical section of $\mathcal{O}_X(D)$. We have $X \setminus X_\sigma = D$, so $X_\sigma$ intersects every $G$-orbit. It follows that the sets $g \cdot X_\sigma = X_{g\sigma}$ for $g \in G$ cover $X$, so $\mathcal{O}_X(D)$ is globally generated by the sections $g\sigma$ for $g \in G$. □

The geometry of divisors on spherical varieties is very rich. For instance, one can give a combinatorial description of both Weil and Cartier divisors on a spherical variety, and for complete spherical varieties, the combinatorial invariants of spherical varieties give rise to many interesting intersection-theoretic results regarding Chow groups, cones of effective curves, etc. We will summarize this description of Weil and Cartier divisors later, in Section 3.7. For a discussion of intersection theory on complete spherical varieties, we refer the reader to [Per18, Section 4].

## 3.2 The Local Structure Theorem

In this section, we discuss the local structure theorem, first as a general result on normal $G$-varieties, and then as it applies specifically to spherical varieties. As far as we are aware, the first proof of the local structure theorem was given by Brion, Luna, and Vust ([BLV86, Theorem 1.4]). Since then, however, several different variants of the theorem have been proven by several different authors. We mention here a few significant variations on the local structure theorem that appear in the literature.

1. Knop has proven a weaker variant of the theorem holds in positive characteristic (see [Tim11, Section 4] for a statement and references).

2. Knop has also proven a slight refinement of the result in [BLV86] in characteristic 0. See [Kno94, Theorem 2.3] for the original proof (or [Tim11, Theorem 4.10] for a brief summary of Knop's result).

3. One can carefully keep track of data such as valuations and divisors in the local structure theorem, as Losev does in [Los09a, Section 5]. Since we will want to be able to

keep track of this data in the same way that Losev does, we give statements and proofs analogous to those of Losev's formulation in Propositions 3.2.3 and 4.4.1.

4. The local structure takes on particularly nice forms for certain nice types of spherical varieties. We will prove such variants of the local structure theorem later, in Theorems 3.5.6, 3.5.9, and 3.5.22.

Our presentation of the local structure theorem is similar to many others in the literature (c.f. [BLV86, Theorem 1.4], [Tim11, Section 4.2], and [Bri97, Sections 1.4, 2.2–2.4]). However, our presentation is more nuanced than most. This is due to the following technical details, which are often not discussed in the literature:

(1) An explicit description of the parabolic subgroup $P$ in Theorem 3.2.2. This will be important to us because, along with Corollary 2.5.5, it implies that this subgroup $P$ does not depends on the $G$-variety $X$ but only on the representation theory of a $G$-module of the form $H^0(X, L^{\otimes n})$ (see Lemma 4.4.4 for details).

(2) Certain relationships between the local structure theorem and $B^-$-fixed points in Theorem 3.2.2b and Lemma 3.2.9. These are important for certain applications of the local structure theorem to so-called *wonderful varieties*, which are a certain nice type of spherical variety. More specifically, we use these technicalities on $B^-$-fixed points in the proof of Theorem 3.5.21, which is a very important result on wonderful varieties, as well as in the proof of Theorem 3.5.22.

(3) The fact that certain "nice" open subsets of spherical varieties (namely, the sets $X_{B,Y}$ of Theorem 3.2.7) have the form $X_f$ for some $f \in H^0(X, L^{\otimes n})^{(B)}$ when $X$ is quasi-projective. (See Theorem 3.2.7a for a precise statement of this fact.) For most applications of the local structure theorem to spherical varieties, this fact is entirely unnecessary. However, it will be essential for our purposes, because we will need to apply the local structure theorem to a subset of the form $X_f$ in order to keep track of certain key combinatorial data locally (see Lemma 4.4.4, which is inspired by Losev's approach in [Los09a, Corollary 5.6]).

We are not aware of any presentation of the local structure theorem in the literature that explicitly proves all three of these technical points, or indeed any presentation which explicitly proves either (2) or (3) above. However, (1) does explicitly appear in the literature (see e.g. [Bri97, Theorem 2.3]); the key details of (2) follow from the proofs given in [BLV86, Section 1]; and (3) can be proven by adapting Knop's arguments in [Kno91, Theorem 2.1]. As such, it seems likely that these details are known to experts.

### 3.2.a The General Theorem

The following lemma gives the key construction for the local structure theorem. Some variation on this lemma appears in every proof of the local structure theorem that we are aware of in the literature. The proof of the lemma is essentially a technical algebraic argument involving Lie algebras; we omit the details here.

**Lemma 3.2.1** ([BLV86, Lemma 1.1, Proposition 1.2]; cf. [Bri97, Section 1.4],[Tim11, Lemma 4.4, Corollary 4.5])**.** *Let $N$ be a finite-dimensional $G$-module, and let $v \in N^{(B)}$ and $\omega \in (N^*)^{(B^-)}$ be eigenvectors such that $\langle \omega, v\rangle = 1$. Let $P$ (resp. $P^-$) be the stabilizer $G_{[v]}$ (resp. $G_{[\omega]}$) of the point $[v] \in \mathbb{P}(N^*)$ (resp. $[\omega] \in \mathbb{P}(N)$) corresponding to the line $kv \subset N$ (resp. $k\omega \subset N^*$).*

*(a) $M = P \cap P^-$ is a Levi subgroup of both $P$ and $P^-$, and $T \subseteq M$.*

*(b) Let $\mathfrak{g}$ be the Lie algebra of $G$. The subspace*

$$(\mathfrak{g}\omega)^\perp = \{n \in N \mid \langle g\omega, n\rangle = 0 \ \forall\ g \in \mathfrak{g}\}$$

*is a $P$-stable subspace of $N$, and $N = \mathfrak{g}v \bigoplus (\mathfrak{g}\omega)^\perp$.*

*(c) Write $N = E \bigoplus kv \bigoplus (\mathfrak{g}\omega)^\perp$ for some subspace $E \subset N$, and let*

$$Z = \mathbb{P}(N)_v \cap \mathrm{V}(\mathrm{Sym}^\bullet(E)) = U \cap \mathbb{P}(kv \bigoplus (\mathfrak{g}\omega)^\perp)) \subset \mathbb{P}(N).$$

*Then, $Z$ is an $M$-stable closed subvariety of $\mathbb{P}(N)_v$, and the morphism*

$$R_u(P) \times Z \to \mathbb{P}(N)_v$$

*given by $(p, z) \mapsto pz$ is a $P$-equivariant isomorphism. Here, the action of $P = R_u(P)M$ on $R_u(P) \times Z$ is as follows: we let $R_u(P)$ act on itself by left multiplication and act trivially on $Z$, and we let $M$ act by conjugation on $R_u(P)$ and act on $Z$ via the action of $G$ on $X$.*

**Theorem 3.2.2** (The local structure theorem; cf. [BLV86, Theorem 1.4], [Tim11, Theorem 4.6])**.** *Let $X$ be a $G$-variety, let $L$ be a $G$-linearized globally generated invertible sheaf on $X$, and let $f \in H^0(X, L^{\otimes n})^{(B)}$ for some $n \geq 1$. Set*

$$P = \{g \in G \mid gX_f = X_f\},$$

*and let $M$ be the standard Levi subgroup of $P$ containing $T$.*

*(a) There exists an $M$-stable closed subvariety $Z \subset X_f$ such that the morphism*

$$R_u(P) \times Z \to X_f$$

*given by $(u, z) \mapsto uz$ is a $P$-equivariant isomorphism. Here, the action of $P = R_u(P)M$ on $R_u(P) \times Z$ is as follows: we let $R_u(P)$ act on itself by left multiplication and act trivially on $Z$, and we let $M$ act by conjugation on $R_u(P)$ and act on $Z$ via the action of $G$ on $X$.*

(b) *Suppose that $y \in X_f$ is a point stabilized by $B^-$. Then, the stabilizer $G_y$ is the opposite parabolic subgroup to $P$ containing $T$. Moreover, we may choose the subvariety $Z$ in (a) such that $y \in Z$.*

*Proof.* After replacing $L$ by a tensor power, we may assume that $f \in H^0(X, L)$. Let $N \subset H^0(X, L)$ be a finite-dimensional $G$-submodule such that $f \in N$ and $L$ is generated by global sections in $N$. Then, the surjection $\mathcal{O}_X \otimes N \to L$ gives us a $G$-equivariant morphism $\varphi : X \to \mathbb{P}(N)$ such that $\varphi^*\mathcal{O}_{\mathbb{P}(N)}(1) \cong L$. Since $f \in N$, we may view $f$ as a section of $\mathcal{O}_{\mathbb{P}(N)}(1)$ pulling back to $f \in H^0(X, L)$. In particular, this gives us $\varphi^{-1}(\mathbb{P}(N)_f) = X_f$.

We wish to apply Lemma 3.2.1 to $\mathbb{P}(N)$ with $v = f$ and $\omega \in N^*$ some $B^-$-eigenvector such that $\langle \omega, f \rangle = 1$. To do this, we first need to check that such a choice of $\omega$ exists. Let $V$ be the $G$-submodule of $N$ generated by $f$, and let $\omega \in (V^*)^{(B^-)}$ be a nonzero $B^-$-eigenvector (which exists by Theorem 2.3.6). Suppose that $\langle \omega, f \rangle = 0$. Then, $\omega : V \to k$ is 0 on all of $B^-B \cdot f$ (since $\omega$ and $f$ are eigenvectors of $B^-$ and $B$, respectively). But since $B^-B$ is an open subset of $G$, the $G$-module $V$ is spanned (as a $k$–vector space) by $B^-B \cdot f$ (see Lemma 2.1.7). So in fact, $\omega$ is 0 on all of $V$, contradicting the fact that $\omega$ is nonzero. This proves that $\langle \omega, f \rangle \neq 0$, and after rescalling $\omega$ if necessary, we may assume that $\langle \omega, f \rangle = 1$.

We can now apply Lemma 3.2.1 to $f$ and $\omega$ to get a closed subvariety $Z_0 \subset \mathbb{P}(N)_f$. Pulling back $Z_0$ by $\varphi$ gives us a subscheme $Z \subset X_f$ which satisfies (a), but with $G_{[v]} = G_{[f]}$ in place of $P$. (Here we are implicitly using the fact that $\varphi$ is $G$-equivariant, cf. Proposition 2.4.17.) However, one can check from the definition of the action of $G$ on $\mathbb{P}(N)$ that

$$G_{[f]} = \{g \in G \mid g \cdot f = cf \text{ for some } c \in k^\times\}.$$

Proposition 2.5.4 thus implies that $P = G_{[f]}$. Finally, since the isomorphism $R_u(P) \times Z \cong X_f$ is $R_u(P)$-equivariant, the scheme $Z$ is isomorphic to the geometric quotient $X_f/R_u(P)$ and hence is integral by [MF82, Section 0.2] (see the proof of Proposition 3.2.3b below for details). This proves (a).

For (b), suppose that in our above application of Lemma 3.2.1, we take $\omega$ to be some nonzero element in the line in $N^*$ corresponding to $\varphi(y) \in \mathbb{P}(N)$. In this case, $\varphi(y) \in \mathbb{P}(N)_f$ implies that $\langle \omega, f \rangle \neq 0$, so after rescaling $\omega$ if necessary, we may take $\langle \omega, f \rangle = 1$, and the construction goes through as before. Note that $G_y = G_{\varphi(y)} = G_{[\omega]}$ is now the opposite parabolic subgroup to $P$ containing $T$ (by Lemma 3.2.1a). Moreover, the morphism $\varphi(y) \hookrightarrow \mathbb{P}(N)$ corresponds (via the functor of points of $\mathbb{P}(N)$) to the surjection $\rho : N \to k$ which is the dual of the inclusion $k\omega \hookrightarrow N^*$. Writing $N = E \bigoplus kf \bigoplus (\mathfrak{g}\omega)^\perp$ as in Lemma 3.2.1, we see that $\rho(f) = \langle \omega, f \rangle = 1$ and that

$$\rho\left((E \bigoplus (\mathfrak{g}\omega)^\perp\right) = 0.$$

In particular, $\rho$ factors through the quotient map $N \to kv \bigoplus (\mathfrak{g}\omega)^\perp$, so we have $\varphi(y) \in \mathrm{V}(\mathrm{Sym}^\bullet(E))$ and hence $\varphi(y) \in Z$. $\square$

The following proposition allows us to transfer a lot of interesting data from the variety $X$ to the subvariety $Z$ in the situation of the local structure theorem.

**Proposition 3.2.3** (cf. [Los09a, Propositions 5.3 and 5.4])**.** *Let $X$ be a $G$-variety, and let $L$ be a $G$-linearized invertible sheaf on $X$ generated by global sections. Let $f \in H^0(X, L^{\otimes n})^{(B)}$ for some $n \geq 1$, let $\mathcal{D} \subset \mathcal{D}_{G,X}$ be the set of irreducible components of $X \setminus X_f$, and let $P$, $M$, and $Z$ be as in Theorem 3.2.2.*

*(a) If either (1) $X$ is projective and $L$ is ample or (2) $X$ is affine, then $Z$ is affine.*

*(b) $R_u(P)$ acts freely on $X_f$, and $Z$ is $M$-equivariantly isomorphic to the geometric quotient $X_f/R_u(P)$. Moreover, if $X$ is normal (resp. smooth), then $Z$ is normal (resp. smooth).*

*(c) For any $B$-orbit $\mathcal{O} \subset X_f$, the intersection $\mathcal{O} \cap Z$ is a $(B \cap M)$-orbit of $Z$ (and in particular is nonempty).*

*(d) $T$ is a maximal torus of $M$, and $B \cap M$ is a Borel subgroup of $M$ containing $T$. Moreover, with $T$ and $B \cap M$ as our choices of maximal torus and Borel subgroup, we have $\Lambda(X) = \Lambda(Z)$ as subsets of $\Lambda_G = \mathcal{X}(T)$, hence also $N(X) = N(Z)$.*

*(e) If $X$ is a spherical $G$-variety, then $Z$ is a spherical $M$-variety.*

*(f) The map*

$$\iota : \mathcal{D}_{M,Z} \to \mathcal{D}_{G,X} \setminus \mathcal{D}$$

*given by $D \mapsto \overline{R_u(P) \times Z}$ is a bijection with inverse given by $D \mapsto D \cap Z$. Moreover, for any $D \in \mathcal{D}_{M,Z}$, we have $\varphi_D = \varphi_{\iota(D)}$ as elements of $N(X) = N(Z)$.*

**Remark 3.2.4.** Nothing in the statement or proof of this proposition depends on the fact that the open subset in the local structure theorem has the form $X_f$ for some $f \in H^0(X, L^{\otimes n})^{(B)}$. In other words, the proposition holds equally well for any open subset $U \subset X$ and any $M$-stable closed subvariety $Z \subset U$ such that $R_u(P) \times Z \cong U$ as $P$-varieties (just replace $X_f$ by $U$ everywhere in the proposition). This variation on the proposition is useful for some other formulations of the local structure theorem (see e.g. [Los09a, Section 5]); for our purposes, however, the above proposition as stated will always be sufficient.

*Proof.* For convenience, we write $P_u = R_u(P)$. If $X = \mathrm{Spec}(A)$ is affine, then $X_f \cong \mathrm{Spec}(A_f)$ is affine; likewise, if $L$ is ample and $X$ is projective, then $X \cong \mathrm{Proj}(\Gamma_*(X, L))$, so $X_f \cong \mathrm{Spec}((\Gamma_*(X, L)_f)_0)$ is affine. In either case, $Z$ is affine because it is a closed subscheme of $X_f$. This proves (a).

For (b), we note that $X_f \cong P_u \times Z$ as $P$-varieties. Since $P_u$ acts on $P_u \times Z$ by left multiplication in $P_u$-coordinate, we immediately see that the action of $P_u$ on $X_f$ is free (since left multiplication is a free action). Moreover, $P_u$ acts trivially on the $Z$-coordinate of $P_u \times Z$, it follows formally from definitions that $Z$ is the geometric quotient of $P_u \times Z$ and that the quotient map is just projection map $P_u \times Z \to Z$. Finally, if $X$ is normal (resp. smooth), then $X_f \cong P_u \times Z$ is as well. The projection map $P_u \times Z \to Z$ is faithfully flat and finitely presented, so it follows from standard results on fppf descent (see e.g. [GW10, Proposition 14.57]) that $Z$ is normal (resp. smooth) if $X$ is.

Before we continue with the proof, we note a couple of general facts about unipotent radicals and parabolic subgroups containing $B$.

1. We have $R_u(B) = P_u \cdot (M \cap R_u(B))$. In particular, $P_u \subset R_u(B)$. This can be proven using the structure of standard parabolic groups (see e.g. [Mil17, Theorem 21.91]).

2. We have $B = P_u(M \cap B)$. This follows formally from Fact 1 and the fact that $B = R_u(B)T$.

To prove (c), we make use of Fact 2 above. For any $B$-orbit $\mathcal{O} \subset X_f$, pick any point $(p, z) \in P_u \times Z$ such that $(p, z) \in \mathcal{O}$. Then, $p^{-1} \cdot (p, z) = (1, z) \in \mathcal{O} \cap Z$ because $P_u \subset B$ (and because the isomorphism $X_f \cong P_u \times Z$ identifies $Z$ with $\{1\} \times Z$). So, $\mathcal{O} \cap Z$ is a nonempty $(B \cap M)$-stable subset of $Z$. On the other hand, let $p_1, p_2 \in \mathcal{O} \cap Z$ be any two points. Since $\mathcal{O}$ is a $B$-orbit, there exists some $b \in B$ such that $bp_1 = p_2$. Write $p_i = (1, z_i)$ for some $z_i \in Z$. By Fact 2 above, we can write $b = um$ with $u \in P_u$ and $m \in B \cap M$. We then see that

$$(1, z_2) = p_2 = bp_1 = um(1, z_1) = u(1, mz_1) = (u, mz_1).$$

It follows that $u = 1$, so that $b = m \in B \cap M$ and $mp_1 = p_2$. Thus, $\mathcal{O} \cap Z$ is a single $(B \cap M)$-orbit, as desired.

For (d), note that $T \subset M$ by definition, so $T$ is a maximal torus of $M$. That $B \cap M$ contains $T$ is immediate, and that $B \cap M$ is a Borel subgroup of $M$ is a general fact about Levi subgroups of parabolic subgroups containing $B$ (see e.g. [Mil17, Proposition 21.90, Theorem 21.91]). To show that $\Lambda(Z) = \Lambda(X)$, note that since $Z \cong X_f/P_u$ is a geometric quotient and $P_u$ acts freely (and in particular properly) on $X_f$, we have

$$K(Z) \cong K(X_f/P_u) \cong K(X_f)^{P_u} = K(X)^{P_u}.$$

(This follows from the definition of a geometric quotient plus the fact that the quotient morphism $X_f \to X_f/P_u$ is affine; see [MF82, Definition 0.6(iv) and Proposition 0.7].) Now, Theorem 2.3.4 states that an element of $K(X)$ is a $B$-eigenvector if and only if it is a $T$-eigenvector which is fixed by $R_u(B)$. We noted above that $R_u(B) = P_u \cdot (M \cap R_u(B))$, so being fixed by $R_u(B)$ is equivalent to being fixed by both $P_u$ and $R_u(B \cap M) = M \cap R_u(B)$. Thus, the $B$-eigenvectors of $K(X)$ are precisely the $T$-eigenvectors of $K(X)^{P_u}$ which are fixed by $R_u(B \cap M)$, and these are precisely the $(B \cap M)$-eigenvectors of $K(X)^{P_u}$ (by Theorem 2.3.4 again). Putting this all together, we have

$$K(Z)^{(B \cap M)} \cong (K(X)^{P_u})^{B \cap M} = K(X)^{(B)},$$

which implies that $\Lambda(Z) = \Lambda(X)$.

Statement (e) now follows almost immediately from (b), (c), and (d). If $X$ is spherical, then $Z$ is normal by (b). Moreover, the open $B$-orbit $X_B^\circ$ intersects $X_f$ (because $X$ is irreducible), so $X_B^\circ \subset X_f$ (because $X_f$ is $B$-stable). Statement (c) now says that $X_B^\circ \cap Z$ is an open $(B \cap M)$-orbit of $Z$, so $Z$ is spherical.

Finally, we prove (f). Let $D \subset Z$ be a $(B \cap M)$-stable prime Weil divisor. Then, $P_u \times D \subset P_u \times Z$ is stable under $P_u(B \cap M) = B$. Also, $P_u \times D$ is irreducible (see e.g. [Sta20, Tag 038F]), and is codimension-1 in $P_u \times Z \cong X_f$, hence also in $X$. So the closure $\overline{P_u \times D}$ is a $B$-stable prime divisor of $X$. Conversely, let $D' \subset X$ be any $B$-divisor not contained in $X \setminus X_f$. Then, $D_0 = D' \cap X_f \neq \varnothing$, and $D'$ is the closure of $D_0$ in $X$. Notice that for any $(u, z) \in D_0 \subset P_u \times Z$, we have $u^{-1} \cdot (u, z) = (1, z) \in D_0 \cap Z$. It follows that $D_0 = P_u \times D$, where $D = D_0 \cap Z$. Since $D_0$ is $B$-stable and $Z$ is $M$-stable, $D$ is $(B \cap M)$-stable, and $D$ has codimension 1 in $Z$ because $D_0$ has codimension 1 in $P_u \times Z$. Moreover, $D$ is irreducible because it is (isomorphic to) the image of the irreducible set $D_0$ under the projection morphism $P_u \times Z \to Z$. Thus, both the map $\iota$ and the map $D' \mapsto D' \cap Z$ are well-defined, and one can check that these maps are indeed inverses.

It remains to check that $\varphi_D = \varphi_{\iota(D)}$ for all $D \in \mathcal{D}_{M,Z}$. Fix such a divisor $D$, and set $D' = \iota(D) = \overline{P_u \times Z}$. The proof of (c) above shows that the quotient map $q : X_f \to X_f/P_u \cong Z$ induces an inclusion on function fields

$$K(Z)^{(B\cap M)} \cong K(X)^{(B)} \subset K(X).$$

On the other hand, The isomorphism $Z \cong X_f/P_u$ identifies the inclusion $i : Z \hookrightarrow X_f$ with a section of $q$. It follows that restricting a $B$-eigenvector $a \in K(X)^{(B)}$ to $Z$ (i.e. taking the image of $a$ under the map $K(X) \to K(X)$ induced by $i$) is the same as viewing $a$ as an element of $K(Z)^{(B\cap M)} \cong K(X)^{(B)}$. In particular, the restriction of $a$ to $Z$ is a $(B \cap M)$-eigenvector whose weight $\mu$ is the same as the weight of $a$ (as a $B$-eigenvector in $K(X)$). Since $D = D' \cap Z$ and restricting $a$ to $Z$ does not change the order of $a$ along $D'$, we have $v_D(a) = v_{D'}(a|_Z)$ and hence $\varphi_D(\mu) = \varphi_{D'}(\mu)$. □

### 3.2.b The Local Structure Theorem on Spherical Varieties

For a spherical variety $X$, there exist canonical affine open subsets of $X$ that we can apply the local structure theorem to, and these subsets are nicely related to the $G$-orbits of $X$. This statement, which is made precise in Theorem 3.2.7, will give us our main use of the local structure theorem on spherical varieties.

In order to prove Theorem 3.2.7, we will need a few technical lemmas. The first can be viewed as a sort of generalization of Theorem 2.6.12b. That theorem says that, when $G = T$ is a torus, we can cover a normal $G$-variety by $G$-stable affine open subsets. When $G$ is a general reductive group, our affine open subsets are not $G$-stable but only $B$-stable. However, we still get some useful compatibility conditions.

**Lemma 3.2.5** ([Kno91, Theorem 1.3], [Bri97, Proposition 1.1])**.** *Let $X$ be a quasi-projective $G$-variety, let $Y \subseteq X$ be a $G$-stable (locally closed) subvariety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. There exists some $n \geq 1$ and some $f \in H^0(X, L^{\otimes n})^{(B)}$ such that*

*(a) $X_f$ is affine,*

*(b) $f$ does not vanish on $Y$ (i.e. $X_f \cap Y \neq \varnothing$), and*

*(c) the restriction map on $B$-eigenvectors*

$$\Gamma(X_f, \mathcal{O}_X)^{(B)} \to \Gamma(X_f \cap Y, \mathcal{O}_Y)^{(B)}$$

*is surjective.*

*Proof.* We may replace $L$ by a tensor power and so assume that $L$ is very ample. Then, Theorem 2.6.12 gives us a $G$-equivariant immersion $i : X \hookrightarrow \mathbb{P}(V)$ such that $i^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong L$ for some finite-dimensional $G$-module $V$. Let $\overline{X}$ and $\overline{Y}$ denote the closures of $X$ and $Y$ (respectively) in $\mathbb{P}(V)$, and let $X'$ (resp. $Y'$, $Z'$) denote the affine cone over $\overline{X}$ (resp. $\overline{Y}$, $\overline{X} \setminus X$). Since formation of affine cones commutes with immersions, we have containments

$$Y', Z' \subset X' \subset \mathbb{A}(V).$$

Now, $Y'$ is closed in $X'$ by definition. Since $X$ is locally closed in $\mathbb{P}(V)$, the boundary $\overline{X} \setminus X$ is closed in $\mathbb{P}(V)$, so $Z'$ is closed in $\mathbb{A}(V)$ (hence also in $X'$). Also, we know that $X'$ is equipped with an action of $\tilde{G} = G \times \mathbb{G}_m$. Because $\overline{Y}$ is $G$-stable in $\overline{X}$, the affine cone $Y'$ is $\tilde{G}$-stable in $X'$. Likewise, $Z'$ is $\tilde{G}$-stable because $\overline{X} \setminus X$ is $G$-stable.

We claim that there exists a nonzero $B$-eigenvector $f_1 \in A = \Gamma(X', \mathcal{O}_{X'})$ which vanishes on $Z'$ but not on $Y'$. such that $Y_0' \cup Z' \subset \mathrm{V}(f_1)$. Let $I \subset A' = \Gamma(Y' \cup Z', \mathcal{O}_{Y' \cup Z'})$ be the radical ideal cutting out the closed subscheme $Z'$. Since $Z'$ is $\tilde{G}$-stable, $I$ is a $\tilde{G}$-submodule of $A'$, so there exists some nonzero $\tilde{B}$-eigenvector $f_1' \in I^{(\tilde{B})}$ (see Theorem 2.3.6). Then, $f_1'$ vanishes on $Z'$, so $f_1'$ must not vanish on $Y'$ (otherwise, $f_1$ would be 0 on $Y' \cup Z'$). Since $Y' \cup Z'$ is a $\tilde{G}$-stable subscheme of $X'$, we can lift $f_1'$ to a $\tilde{B}$-eigenvector $f_1 \in A^{(\tilde{B})}$ (Proposition 2.5.6). Since $f_1|_{Y' \cup Z'} = f_1'$, we see that $f_1$ vanishes on $Z'$ but not on $Y'$.

Now, by definition of the affine cone $X'$, we have

$$A = \bigoplus_{n \geq 0} H^0(\overline{X}, \overline{L}^{\otimes n}),$$

where $\overline{L} = \mathcal{O}_{\mathbb{P}(V)}(1)|_{\overline{X}}$. Since $f_1$ is a $\tilde{B}$-eigenvector and $\mathbb{G}_m$ acts on $A$ according to the grading, one can check that $f_1$ must be a homogeneous element of $A$. Moreover, $H^0(\overline{X}, \mathcal{O}_{\overline{X}}) = k$ has no sections that vanish anywhere, so $f_1 \in H^0(\overline{X}, \overline{L}^{\otimes n})$ for some $n \geq 1$. (If $Z = \varnothing$ and $X = \overline{X}$, it is possible that $f_1$ does not vanish anywhere. In this case, we may simply pick $f_1 \in A$ to be any $\tilde{B}$-eigenvector which does not vanish on $Y'$ and also does not lie in $A_0 = k$.) Since $\overline{L}|_X = L$, the restriction $f = f_1|_L$ is now an element of $H^0(X, L^{\otimes n})^{(B)}$. By our choice of $f_1$, the section $f$ vanishes on $\overline{X} \setminus X$ but not on $Y$. It follows that

$$X_f = \overline{X}_{f_1} \subset \overline{X} \cong \mathrm{Proj}(A).$$

In particular, $X_f$ is affine (because $\overline{X}_{f_1}$ is).

This prove that our choice of $f$ satisfies statements (a) and (b). To check statement (c), let $a \in \Gamma(X_f \cap Y, \mathcal{O}_Y)^{(B)}$ be any $B$-eigenvector. Then, the fact that $X_f = \overline{X}_{f_1}$ tells us that

$X_f \cap Y = \overline{Y}_{f_1|_Y}$. So, if $J \subset A$ is the homogeneous ideal cutting out $\overline{Y} \subset \overline{X}$ and $\overline{f_1}$ is the image of $f_1$ in $A/J$, then we have

$$\Gamma(X_f \cap Y, \mathcal{O}_Y) \cong ((A/J)_{\overline{f_1}})_0.$$

There exists some $m \geq 0$ such that $a\overline{f_1}^m \in A/J \cong \Gamma(\overline{Y}, \mathcal{O}_{\overline{Y}})$. Since $\overline{f_1}$ is a $B$-eigenvector of degree $n$, the product $a\overline{f_1}^m$ is a $B$-eigenvector of degree $mn$. By Proposition 2.5.6, we can lift $a\overline{f_1}^m$ to some $a' \in A^{(B)}$. Since the quotient map $A \to A/J$ is homogeneous, $a'$ is homogeneous of degree $mn$. It follows that

$$a' f_1^{-m} \in \Gamma(X_f, \mathcal{O}_X)^{(B)} = \Gamma(\overline{X}_{f_1}, \mathcal{O}_{\overline{X}})^{(B)} \cong (A_{f_1})_0$$

is a $B$-eigenvector whose restriction to $X_f \cap Y$ is $a$. □

In the setting of spherical varieties, we can use our main theorem on $G$-invariant valuations (Theorem 3.1.13) to strengthen the above lemma.

**Lemma 3.2.6** (cf. [Kno91, proof of Theorem 2.1]). *In the scenario of Lemma 3.2.5, let $D$ be a $B$-stable effective Weil divisor of $X$ and $Z \subset X$ be a $G$-stable closed subvariety such that $Y \not\subset D$ and $Y \not\subset Z$. If $X$ is spherical, then we can take the $B$-eigenvetor $f$ of Lemma 3.2.5 to vanish on $D$ and $Z$.*

*Proof.* The proof is almost identical to that of Lemma 3.2.5; the main difference is that we pick the $B$-eigenvector $f_1$ in that proof slightly more carefully. As in the proof of Lemma 3.2.5, we obtain a $G$-equivariant immersion $i : X \hookrightarrow \mathbb{P}(V)$ such that $i^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong L$, and we consider the closure $\overline{X}$ in $\mathbb{P}(V)$. This time, however, we take the affine cone over the normalization of $\overline{X}$. More precisely, let $\nu : \overline{X}^{norm} \to \overline{X}$ be the normalization morphism, and let $\overline{L} = \mathcal{O}_{\mathbb{P}(V)}(1)|_{\overline{X}}$. Since $\nu$ is finite (see Lemma 2.6.15) and $\overline{L}$ is ample on $\overline{X}$, the pullback $\nu^*\overline{L}$ is ample on $\overline{X}^{norm}$. Moreover, $\overline{X}^{norm}$ is projective (because $\overline{X}$ is), so we have $\overline{X}^{norm} \cong \mathrm{Proj}(A)$, where

$$A = \bigoplus_{n \geq 0} H^0(\overline{X}^{norm}, (\nu^*\overline{L})^{\otimes n}).$$

We let $X' = \mathrm{Spec}(A)$ be the affine cone over $\overline{X}^{norm}$ with respect to $\nu^*\overline{L}$. The reason we use $\overline{X}^{norm}$ instead of $\overline{X}$ here is that $\overline{X}^{norm}$ is normal and hence spherical (if $\mathcal{O}$ is the open $B$-orbit of $X$, then $\nu^{-1}(\mathcal{O})$ is an open $B$-orbit of $\overline{X}^{norm}$). So, Corollary A.6 implies that $X'$ is a spherical variety under the action of $\tilde{G} = G \times \mathbb{G}_m$.

Similarly, let $Y'$ (resp. $D'$) be the affine cone over the normalization of $\overline{Y}$ (resp. $\overline{D}$) and let $Z'$ be the affine cone over $\overline{Z}^{norm} \cup (\overline{X}^{norm} \setminus X)$, where $\overline{Z}^{norm}$ denotes the normalization of $\overline{Z}$. As in the proof of Lemma 3.2.5, one can check that $Y'$ is a (locally closed) subvariety of $X'$, that $Z'$ is a closed subvariety of $X'$, and that $D'$ is a union of prime Weil divisors of $X'$. Moreover, the subvarieties $Y'$ and $Z'$ are $\tilde{G}$-stable, and $D'$ is a union of $\tilde{B}$-divisors (here $\tilde{B} = B \times \mathbb{G}_m$). Also, we have $Y' \not\subset Z'$ and $Y' \not\subset D'$ (because $Y \not\subset Z$ and $Y \not\subset D$).

Now, let $I$ and $J$ be the ideals of $A$ cutting out $Y'$ and $D' \cup Z'$ (respectively). The fact that $Y' \not\subset D'$ and $Y' \not\subset Z'$ implies that $Y' \not\subset D' \cup Z'$ (because $Y'$ is irreducible), so $I \not\supset J$. Thus, there exists some $f_0 \in J$ such that $f_0 \notin I$, i.e. $f_0$ vanishes on $D' \cup Z'$ but not on $Y'$. Our plan is to "adjust" $f_0$ to a $\tilde{B}$-eigenvector $f_1 \in A^{(\tilde{B})}$ which still vanishes on $D' \cup Z'$ but not on $Y'$. Suppose for the moment that we have such a choice of $f_1$. Note that $X$ is normal by assumption, so $\nu$ identifies $X$ with a $G$-stable open subvariety of $\overline{X}^{norm}$ in such a way that $L \cong (\nu^*\overline{L})|_X$. Thus, all of our arguments in the proof of Lemma 3.2.5 will go through exactly as they did before (just with $\overline{X}^{norm}$ in place of $\overline{X}$). More specifically, by copying the proof of that lemma, we obtain the following statements:

1. $f_1$ is a homogeneous element of $A$ of positive degree, i.e. we have $f_1 \in H^0(\overline{X}^{norm}, (\nu^*\overline{L})^{\otimes n})^{(B)}$ for some $n \geq 1$;

2. the restriction $f = f_1|_X$ vanishes on $D$ and $Z$ but not on $Y$ (because $f_1$ vanishes on $D' \cup Z'$ but not on $Y'$);

3. we have $X_f = (\overline{X}^{norm})_{f_1}$ (because $f_1$ vanishes on $\overline{X}^{norm} \setminus X$), so $X_f$ is affine; and

4. the restriction map on $B$-eigenvectors
$$\Gamma(X_f, \mathcal{O}_X)^{(B)} \to \Gamma(X_f \cap Y, \mathcal{O}_Y)^{(B)}$$
is surjective.

Thus, the restriction $f = f_1|_X$ is the desired element of $H^0(X, L^{\otimes n})^{(B)}$.

It remains to check that we can in fact find find some $f_1 \in A^{(\tilde{B})}$ which vanishes on $D' \cup Z'$ but not on $Y'$. For this, the key ingredient is Theorem 3.1.13. Let $v_0$ be a $\tilde{G}$-invariant valuation on $X'$ whose center is $Y'$ (such a valuation exists by Lemma 3.1.12), and let $v_1, \ldots, v_m$ be the valuations corresponding to the prime Weil divisors in the support of $D'$. For each $\tilde{G}$-orbit closure $Z_0$ contained in $Z$, pick some $\tilde{G}$-invariant valuation centered on $Z_0$; call these valuations $v_{m+1}, \ldots, v_n$ (there are finitely many because $X'$ is spherical, so it has finitely many $\tilde{G}$-orbits). Our choice of $f_0$ gives us $v_0(f_0) = 0$ but $v_i(f_0) > 0$ for all $i > 0$. Applying Theorem 3.1.13 to the restriction of $f_0$ to the open $\tilde{B}$-orbit of $X'$, we obtain some $\tilde{B}$-eigenvector $f_1 \in K(X)^{(\tilde{B})}$ such that

(1) $v_0(f_1) = v_0(f_0) = 0$,

(2) $v_E(f_1) \geq v_E(f_0)$ for all $\tilde{B}$-divisors $E$ of $X'$, and

(3) $v(f_1) \geq v(f_0)$ for all $\tilde{G}$-invariant valuations on $K(X)$.

In particular, since $f_0 \in A$, we have $v_E(f_0) \geq 0$ for all $\tilde{B}$-divisors $E$ on $X'$, so Statement (2) above gives us $v_E(f_1) \geq 0$ for all $E \in \mathcal{D}_{\tilde{G},X}$. Lemma 3.1.17d then implies that $f_1 \in A$. Moreover, Statement (1) implies that $f_1$ does not vanish on $Y$. Statement (2) implies that for all $1 \leq i \leq m$, we have
$$v_i(f_1) \geq v_i(f_0) > 0$$

(here are implicitly using the fact that $f_0$ vanishes on every prime Weil divisor in the support of $D'$). It follows that $f_1$ vanishes on $D'$. Likewise, Statement (3) implies that $v_i(f_1) > 0$ for $m+1 \leq i \leq n$, and it follows that $f_1$ vanishes on $Z'$ as well. □

We can now prove our main theorem regarding the local structure of spherical varieties.

**Theorem 3.2.7** (cf. [Kno91, Theorem 2.1], [Bri97, Proposition 2.2.2])**.** *Let $X$ be a spherical $G$-variety, and let $Y \subseteq X$ be a $G$-orbit. Define*

$$X_{B,Y} = X \setminus \bigcup_{\substack{D \in \mathcal{D}_{G,X} \\ Y \not\subset D}} D,$$

*where the union is over all $B$-divisors $D \subset X$ not containing $Y$.*

*(a) For any $G$-stable quasi-projective open subset $X_0 \subset X$ containing $Y$ and any $G$-linearized ample invertible sheaf $L$ on $X_0$, there exists some $B$-eigenvector $f \in H^0(X_0, L^{\otimes n})^{(B)}$ for some $n \geq 1$ such that $X_{B,Y} = (X_0)_f$. In particular, the local structure theorem (Theorem 3.2.2) applies to $X_{B,Y}$.*

*(b) $X_{B,Y}$ is a $B$-stable affine open subset of $X$ intersecting $Y$, and $X_{B,Y}$ is the minimal such subset (i.e. every other such subset of $X$ contains $X_{B,Y}$.)*

*(c) $X_{B,Y} \cap Y$ is the unique closed $B$-orbit of $X_{B,Y}$, and*

$$X_{B,Y} = \{x \in X \mid X_{B,Y} \cap Y \subset \overline{B \cdot x}\}.$$

*(d) $Y$ is the unique closed $G$-orbit of $G \cdot X_{B,Y}$, and*

$$G \cdot X_{B,Y} = \{x \in X \mid Y \subset \overline{G \cdot x}\}.$$

*Proof.* Let $X_0 \subset X$ be a $G$-stable quasi-projective open subset containing $Y$, and let $L$ be a $G$-linearized ample invertible sheaf on $X_0$ (note that such a choice of $X_0$ and $L$ exists by Theorems 2.6.12 and 2.6.11). Applying Lemma 3.2.6 to $X_0$ and $L$, we obtain some $f \in H^0(X_0, L^{\otimes n})^{(B)}$ for some $n \geq 1$ such that

(1) $(X_0)_f$ is affine,

(2) $f$ vanishes on $X_0 \setminus X_{B,Y}$ and on the union $Z$ of every $G$-orbit closure not containing $Y$, and

(3) $f$ doesn't vanish on $Y$.

We claim that $X_{B,Y} = (X_0)_f$. For this, let $V = X \setminus (X_0)_f$. Since $(X_0)_f$ is $B$-stable and affine, the irreducible components of $V$ are all $B$-divisors of $X$ (Lemma 3.1.17). Moreover, since $f$ vanishes on $X_0 \setminus X_{B,Y}$, we know that every $B$-divisor $D \in \mathcal{D}_{G,X}$ not containing $Y$ is a component of $V$. On the other hand, if $Y \subset D$, then we have $D \cap X_0 \neq \varnothing$, and because $f$ does not vanish on $Y$, we have $D \cap (X_0)_f \neq \varnothing$ and hence $D \not\subset V$. Thus, the irreducible components of $V$ are precisely the $D \in \mathcal{D}_{G,X}$ not containing $Y$, which means that $X_{B,Y} = (X_0)_f$ by definition. This proves (a), and it also implies that $X_{B,Y}$ is affine. Note also that $X_{B,Y}$ is $B$-stable (because its complement is) and open (because the set $\mathcal{D}_{G,X}$ is finite), and we have $Y \cap X_{B,y} \neq \varnothing$ by definition. Moreover, for any $B$-stable affine open subset $U \subset X$ intersecting $Y$, the complement of $U$ is a union of $B$-divisors of $X$ (cf. our arguments with $V$ above), and none of these $B$-divisors contains $Y$. So, we have

$$X \setminus U \subset X \setminus X_{B,Y},$$

which implies that $X_{B,Y} \subset U$. This proves (b).

Next, we prove (d). Note that any $G$-orbit of $G \cdot X_{B,Y}$ must intersect $X_{B,Y} = (X_0)_f$. On the other hand, in our application of Lemma 3.2.6 above, we took $f$ to vanish on every $G$-orbit whose closure does not contain $Y$. It follows that $Y$ lies in the closure of every $G$-orbit of $G \cdot X_{B,Y}$. In particular, no $G$-orbit of $G \cdot X_{B,Y}$ is closed except possibly $Y$. But $G \cdot X_{B,Y}$ must have some closed $G$-orbit (e.g. any orbit of minimal dimension), so $Y$ must be the unique closed $G$-orbit of $G \cdot X_{B,Y}$. As for the equality on $G \cdot X_{B,Y}$ in (d), our above arguments give us one direction of containment. For the other direction, let $\mathcal{O}$ be a $G$-orbit of $X$ such that $Y \subset \overline{\mathcal{O}}$; we show that $\mathcal{O} \subset G \cdot X_{B,Y}$. The orbit $\mathcal{O}$ must intersect $X_0$ (because $Y \subset X_0$), and $f$ cannot vanish on $\mathcal{O}$ (otherwise $f$ would vanish on $Y$). It follows that $\mathcal{O} \cap (X_0)_f \neq \varnothing$ and hence that

$$\mathcal{O} \subset G \cdot (X_0)_f = G \cdot X_{B,Y},$$

as desired.

It remains to prove (c). By (d), we see that $G \cdot X_{B,Y}$ is a spherical variety which contains both $X_{B,Y}$ and every $B$-orbit whose closure contains $X_{B,Y} \cap Y$, and by (b), we see that $(G \cdot X_{B,Y})_{B,Y} = X_{B,Y}$. So for the rest of the proof, we may replace $X$ by $G \cdot X_{B,Y}$ and so assume that $Y$ is the unique closed $G$-orbit of $X$. In particular, $X$ is quasi-projective (Corollary 2.6.13), so we may take $X_0 = X$ and $X_{B,Y} = X_f$.

We claim that $X_{B,Y} \cap Y$ is a $B$-orbit. Note that $Y$ is a spherical $G$-variety (Theorem 3.1.8), so we may apply Lemma 3.2.6 to $Y$ to obtain a $B$-eigenvector $f' \in H^0(Y, L|_Y^{\otimes m})^{(B)}$ for some $m \geq 0$ such that $f'$ vanishes on every $B$-divisor of $Y$ but not on all of $Y$. In other words, $Y_{f'} = Y_0$ is the open $B$-orbit of $Y$ (see Lemma 3.1.17). Since $L$ is trivial on $X_f \cap Y$, we may view the restriction of $f'$ to $X_f \cap Y$ as an element of $\Gamma(X_f \cap Y, \mathcal{O}_Y)^{(B)}$. By Lemma 3.2.5c (which applies to $X_{f'}$, see Lemma 3.2.6), we can lift $f'|_{X_f \cap Y}$ to a $B$-eigenvector $f'' \in \Gamma(X_f, \mathcal{O}_X)^{(B)}$. Then, $(X_f)_{f''}$ is a $B$-stable affine open subset of $X$ such that $(X_f)_{f''} \cap Y = Y_0$. In particular, $(X_f)_{f''}$ intersects $Y$, so by the minimality statement in (b), we have

$$X_{B,Y} \cap Y \subset (X_f)_{f''} \cap Y = Y_0.$$

On the other hand, since $X_{B,Y} \cap Y$ is an open subset of $Y$, it intersects the dense $B$-orbit $Y_0$ of $Y$, and since $X_{B,Y} \cap Y$ is $B$-stable, we have $Y_0 \subset X_{B,Y} \cap Y$. So, $X_{B,Y} \cap Y = Y_0$ is a $B$-orbit, as desired. Note also that since $Y$ is closed in $X$, the intersection $X_{B,Y} \cap Y$ is closed in $X_{B,Y}$. Since $X_{B,Y} = X_f$, the rest of statement (c) follows from Lemma 3.2.8 below. □

**Lemma 3.2.8.** *Let $X$ be a spherical variety, and let $Y \subset X$ be a $G$-orbit. Let $L$ be a $G$-linearized globally generated sheaf on $X$, and let $f \in H^0(X, L^{\otimes n})^{(B)}$ for some $n \geq 1$. Suppose that $X_f$ is affine and that $X_f \cap Y$ is a closed $B$-orbit of $X_f$. Then, $X_f \cap Y$ is the unique closed $B$-orbit of $X_f$, and*

$$X_f = \{x \in X \mid X_f \cap Y \subset \overline{B \cdot x}\}.$$

*Proof.* Let $Z$, $M$, and $P$ be as in the local structure theorem (Theorem 3.2.2) applied to $X_f$, so that $X_f \cong R_u(P) \times Z$. Since $X_{B,Y}$ is affine, so is $Z$, and $Z$ is also a spherical $M$-variety (see Proposition 3.2.3a,e). So, Lemma 2.5.8 implies that $Z$ has a unique closed $M$-orbit. We claim that this $M$-orbit is $Z \cap Y$. Since $X_f \cap Y$ is a $B$-orbit, the intersection $Z \cap Y$ is a $(B \cap M)$-orbit of $Z$ (Proposition 3.2.3c). On the other hand, $Z \cap Y$ is $M$-stable (because $Z$ and $Y$ are), so $Z \cap Y$ is in fact an $M$-orbit of $Z$. Finally, $Z \cap Y$ is closed because $X_f \cap Y$ is closed in $X_f$.

We claim that $X_f \cap Y$ is the unique closed $B$-orbit of $X_f$. For any closed $B$-orbit $\mathcal{O} \subset X_f$, the intersection $\mathcal{O} \cap Z$ is a closed $(B \cap M)$-orbit of $Z$ (Proposition 3.2.3), so $M(\mathcal{O} \cap Z)$ is a closed $M$-orbit of $Z$ (Lemma2.1.3). But $Z \cap Y$ is the unique closed $M$-orbit of $Z$. So, we must have $M(Z \cap \mathcal{O}) = Z \cap Y$ and hence $Z \cap \mathcal{O} \subset Z \cap Y$. This gives us

$$\mathcal{O} = P_u(Z \cap \mathcal{O}) \subset P_u(Z \cap Y) = X_f \cap Y.$$

But $X_f \cap Y$ and $\mathcal{O}$ are both closed $B$-orbits of $X_f$, so we must have $\mathcal{O} = X_f \cap Y$. This proves the claim.

Now, every $B$-orbit of $X_f$ contains a closed $B$-orbit in its closure (see Proposition 2.1.2). Since the only closed $B$-orbit of $X_f$ is $X_f \cap Y$, we immediately obtain

$$X_f \subset \{x \in X \mid X_f \cap Y \subset \overline{B \cdot x}\}.$$

For the reverse containment, let $\mathcal{O} \subset X$ be a $B$-orbit such that $Y \subset \overline{O}$. Then, we have $\mathcal{O} \cap X_f \neq \varnothing$. Since $X_f$ is $B$-stable, this implies that $\mathcal{O} \subset X_f$. □

As in our general statement of the local structure theorem (see Theorem 3.2.2), there is something we can say about $B^-$-fixed points in the spherical setting. Somewhat surprisingly, the statement is: when $Y \subset X$ is a $G$-orbit containing a $B^-$-fixed point, the *only* $B$-stable affine open subset of the form $X_f$ which intersects $Y$ is the set $X_{B,Y}$. Since $X_{B,Y}$ is in general the minimal such open subset, this means that sometimes the only subset intersecting $Y$ that is of the form $X_f$ is the "smallest possible" one.

**Lemma 3.2.9.** *Let $X$ be a spherical $G$-variety, and let $Y \subset X$ be a $G$-orbit. Let $L$ be any $G$-linearized globally generated invertible sheaf on $X$, let $f \in H^0(X, L)^{(B)}$, and let $P$, $M$, and $Z$ be as in the local structure theorem (Theorem 3.2.2) applied to $X_f$. Suppose that $Y$ is complete, that $X_f$ is affine, and that $X_f \cap Y \neq \varnothing$,*

*(a) $X_f$ contains the unique $B^-$-fixed point $y$ of $Y$, and we may choose $Z$ such that $Y \cap Z = \{y\}$.*

*(b) We have $X_f = X_{B,Y}$.*

*Proof.* First of all, there exists a unique $B^-$-fixed point $y \in Y$ by Proposition 2.1.4. Moreover, Lemma 2.1.3 implies that $BB^-y = By$ is an open subset of the orbit $Gy = Y$. Since $X_f \cap Y \neq \varnothing$, we have $X_f \cap By \neq \varnothing$ (because $Y$ is irreducible). Because $X_f$ is $B$-stable, we conclude that $y \in By \subset X_f$. Theorem 3.2.2 then tells us that $G_y$ is the opposite parabolic subgroup $P^-$ to $P$ containing $M$ and that we may choose $Z$ such that $y \in Y \cap Z$.

We now imitate several arguments from the proof of Lemma 3.2.8. (Unfortunately, we cannot use that lemma directly here, because we don't know that $X_f \cap Y$ is a closed $B$-orbit.) Because $X$ is spherical and $X_f$ is affine, we know that $Z$ is spherial and affine (see Proposition 3.2.3a,e). So, $Z$ contains a unique closed $M$-orbit (Lemma 2.5.8). Since $M \subset P^- = G_y$, that orbit is $\{y\}$.

We claim that $By$ is the unique closed $B$-orbit of $X_f$. Let $\mathcal{O} \subset X_f$ be any closed $B$-orbit. Then, $\mathcal{O} \cap Z$ is a closed $(B \cap M)$-orbit of $Z$ (Proposition 3.2.3c), so $M(\mathcal{O} \cap Z$ is a closed $M$-orbit of $Z$ $Z$ (Lemma2.1.3). But the unique closed $M$-orbit of $Z$ is $\{y\}$, so we have $M(\mathcal{O} \cap Z) = \{y\}$ and hence $\mathcal{O} \cap Z = \{y\}$. This implies that $\mathcal{O} = By$. (Note that since $X_f$ must have some closed $B$-orbit, this argument implies that $By$ is a closed $B$-orbit of $X_f$, and that it is the unique such orbit.)

Since every $B$-orbit of $X_f$ contains a closed $B$-orbit in its closure and $By$ is dense in $Y$, the above claim immediately gives us

$$X_f \subset \{x \in X \mid By \subset \overline{Bx}\} = \{x \in X \mid Y \subset \overline{Bx}\}.$$

On the other hand, for any $B$-orbit $\mathcal{O}$ such that $Y \subset \overline{\mathcal{O}}$, we have $\mathcal{O} \cap X_f \neq \varnothing$ (because $X_f$ is an open set containing points in $Y$) and hence $\mathcal{O} \subset X_f$ (because $X_f$ is $B$-stable). Thus, we have

$$X_f = \{x \in X \mid Y \subset \overline{Bx}\}.$$

The righthand side of this equation is equal to $X_{B,Y}$ by Theorem 3.2.7, so this proves (b). The same theorem thus tells us that $X_f \cap Y = X_{B,Y} \cap Y$ is a $B$-orbit, so $Z \cap Y$ is a $(B \cap M)$-orbit by Proposition 3.2.3c. But $\{y\} \subset Z \cap Y$ is itself a $(B \cap M)$-orbit (since $P^- \supset M \supset B \cap M$), so in fact, we must have $Z \cap Y = \{y\}$. $\square$

One nice application of the local structure theorem is the following formula for the dimension of a spherical variety.

**Lemma 3.2.10** ([Los09c, Lemma 3.5.8]). *Let $X$ be a spherical variety, let $X_B^\circ$ be the open $B$-orbit, and define*

$$P_X = \{g \in G \mid gX_B^\circ = X_B^\circ\}.$$

*Then, we have*

$$\dim(X) = r(X) + \dim(G) - \dim(P_X).$$

**Remark 3.2.11.** The group $P_X$ is often mentioned in the literature on spherical varieties. Note that $P_X$ is parabolic, since $B \subset P_X$ by definition. Moreover, since $X \setminus X_B^\circ$ is the union of all $B$-divisors of $X$ (Lemma 3.1.17), the group $P_X$ is equivalently the subgroup of all $g \in G$ such that $gD = D$ for every $B$-divisor $D$ of $X$ (by Lemma 2.1.3). This characterization is often the more interesting one in light of certain considerations in the theory regarding simple roots and colors (which we will discuss in Section 3.6.b).

*Proof.* Let $Y$ be the open $G$-orbit of $X$. Then, we have $X_{B,Y} = X_B^\circ$ by definition, so Theorem 3.2.7 allows us to apply the local structure theorem (Theorem 3.2.2) to $X_B^\circ$. Writing $M$ for the standard Levi subgroup of $P_X$, this gives us an $M$-stable closed subvariety $Z \subset X_B^\circ$ and a $P_X$-equivariant isomorphism

$$R_u(P_X) \times Z \xrightarrow{\sim} X_B^\circ.$$

Since $X_B^\circ$ intersects no $B$-divisor of $X$, Proposition 3.2.3e implies that $Z$ has no $(B \cap M)$-divisors. Because $Z$ has no colors, $Z$ is a toric variety for some quotient of $M/[M, M]$ (Proposition 3.1.19), and since $Z$ has no $M$-divisors, $Z$ is homogeneous (see Lemma 3.1.17). In other words, $Z$ is isomorphic to some quotient of $M/[M, M]$. In particular, $Z$ is a torus, so $\dim(Z) = r(Z)$, and $r(Z) = r(X)$ by Proposition 3.2.3c. So, we have

$$\dim(X) = \dim(X_B^\circ) = \dim(R_u(P_X)) + \dim(Z) = \dim(R_u(P_X)) + r(X). \tag{3.2.1}$$

On the other hand, consider the opposite parabolic subgroup $P_X^-$ to $P_X$ containing $M$. The product $R_u(P_X) \cdot P_X^-$ is an open subset of $G$, and since $P_X \cap P_X^- = M$, we have $R_u(P_X) \cap P_X^- = \{1\}$. Moreover, it follows from the structure of standard parabolic subgroups (see e.g. [Mil17, Theorem 21.91]) that $\dim(P_X) = \dim(P_X^-)$. Putting all this together, we have

$$\dim(G) = \dim(P_X P_X^-) = \dim(R_u(P_X)P_X^-) = \dim(R_u(P_X)) + \dim(P_X),$$

so that $\dim(R_u(P_X)) = \dim(G) - \dim(P_X)$. Combining this equation with (3.2.1) gives the result. $\square$

## 3.3 Luna–Vust Theory

Our first step in classifying spherical varieties is to reduce to the case of homogeneous spherical varieties. This reduction was first proven by Luna and Vust [LV83] and became known as Luna–Vust theory. Luna and Vust presented the theory in a high degree of generality, as a theory about $G$-equivariant open immersions $G/H \hookrightarrow X$ of arbitrary normal $G$-varieties $X$. While the theory does produce a classification of such open immersions in this generality, it is only when $X$ contains a $B$-orbit of codimension $\leq 1$ (i.e. the complexity of $X$ is $\leq 1$, see the discussion before Theorem 3.1.4) that the classification is truly combinatorial in nature.

In the case of spherical varieties, Luna–Vust theory was later reformulated in a simpler and characteristic-independent way by Knop [Kno91]. Our presentation largely follows that of Knop's paper. A nearly identical presentation appears in [Bri97, Section 3], which we have also referred to in some places. The general Luna–Vust theory has also been reformulated by Timashev in what the author calls a "natural exposition" ([Tim11, p. 57]); we refer the reader to [Tim11, Chapter 3] for this presentation.

Let $X$ be a spherical variety. We know that $X$ has an open $B$-orbit $\mathcal{O}$, hence also an open $G$-orbit $G \cdot \mathcal{O}$. Since $G$ acts transitively on $G \cdot \mathcal{O}$, we have $G \cdot \mathcal{O} \cong G/H$, where $H \subset G$ is the stabilizer of any $k$-point in the orbit $\mathcal{O}$. Note that $G/H$ is a homogeneous spherical variety, since $\mathcal{O} \subset G \cdot \mathcal{O}$ is an open $B$-orbit. Moreover, the inclusion $G \cdot \mathcal{O} \subset X$ gives rise to a $G$-equivariant open immersion $G/H \hookrightarrow X$.

In this way, we obtain for any spherical variety $X$ a $G$-equivariant open immersion $G/H \hookrightarrow X$ with $G/H$ a homogeneous spherical variety. To reduce to classifying homogeneous spherical varieties, then, we need to classify $G$-equivariant open immersions of the form $G/H \hookrightarrow X$ for some fixed spherical subgroup $H \subset G$. This classification is precisely the content of Luna–Vust theory. For the sake of brevity, we give such open immersions a name.

**Definition 3.3.1.** Let $G/H$ be a homogeneous spherical variety. By an *embedding* of $G/H$ we mean a $G$-equivariant open immersion $G/H \hookrightarrow X$ for some normal $G$-variety $X$. (Note that $X$ is necessarily spherical, because $G/H$ has an open $B$-orbit.)

Throughout this section, we take $H \subset G$ to be a spherical subgroup and $G/H \hookrightarrow X$ to be an embedding of $G/H$. We will continue to assume that $k$ is algebraically closed of characteristic 0. However, we note that the characteristic of $k$ affects nothing about the core results of Luna–Vust theory; indeed the presentation in Knop's paper [Kno91] makes no assumptions on the characteristic.

## 3.3.a A First Example: Classifying Toric Varieties

Before we discuss the general theory, we first consider the special case of toric varieties. Recall that every toric variety is a spherical variety, where the reductive group in question is a torus (see Example 3.1.3). The toric case is thus less technical than the general spherical case, mainly because we have $G = B = T$. Moreover, since a toric $T$-variety $X$ is a normal $T$-variety equipped with a $T$-equivariant open immersion $T \hookrightarrow X$, the classification of these open immersions $T \hookrightarrow X$ is nothing more than the classification of toric varieties. Thus, Luna–Vust theory in the toric case boils down to the standard classification of toric varieties in terms of combinatorial objects called *fans*. Our goal in this section is to prove that classification in a way that mimics the key ideas of Luna–Vust theory for spherical varieties in general. We have found the toric case helpful for gaining intuition for the main proofs and statements of Luna–Vust theory. However, nothing in this discussion of toric varieties is essential for what follows (except some standard terminology about cones in Definition 3.3.2 below), so reader who wishes to skip this subsection can safely do so.

To keep this section both self-contained and centered around the main idea of Luna–Vust theory, we will aim to avoid using any of the general machinery that we've built up for spherical varieties so far, opting instead to cite results from the theory of toric varieties wherever possible. There is just one result we will require from our discussion of spherical varieties in general: namely, Lemma 3.1.17, which gives some basic topological and algebraic properties of $G$-divisors. We will not prove this lemma again for toric varieties, since the proof is largely scheme-theoretic and hence is more or less identical in the toric case (except that no distinction is made between $B$- and $G$-divisors in the toric case, since $G = B = T$).

The main combinatorial objects in the classification of toric varieties are cones. We now review some standard definitions pertaining to these objects.

**Definition 3.3.2.** Let $V$ be a $\mathbb{Q}$–vector space, and let $C \subset V$ be a subset. We say that $C$ is a *cone* if $C$ is closed under addition and under multiplication by elements of $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0\}$. If $C$ is a cone, then we have the following definitions.

1. We say that $C$ is *polyhedral* if there exist finitely many elements $v_1, \ldots, v_r \in V$ such that $C = \mathbb{Q}^+ v_1 + \cdots + \mathbb{Q}^+ v_r$.

2. The *dimension* of $C$ is the dimension of the linear subspace $\mathbb{Q}C \subset V$ spanned by $C$.

3. The *dual cone* of $C$ is given by

$$C^\vee = \{\alpha \in V^\vee \mid \alpha(c) \geq 0 \ \forall \ c \in C\}.$$

   As implied by the terminology, $C^\vee$ is a cone in the dual vector space $V^\vee$.

4. A *face* of $C$ is a subset $F \subset C$ of the form

$$F = C \cap \{v \in V \mid \alpha(v) = 0\}$$

   for some fixed $\alpha \in C^\vee$. A face is itself a cone, so in particular, it also has a dimension as defined above. A face of dimension one is called an *extremal ray.*

5. The *relative interior* of $C$ is

$$C^\circ = C \setminus \bigcup_{F \subsetneq C} F,$$

   where the union is over all proper faces of $C$.

6. We say a cone $C \subset V$ is *strictly convex* if $C$ contains no nonzero linear subspaces of $V$, or equivalently, if there exists some $\alpha \in C^\vee$ such that
$$C \cap \alpha^\perp = \{v \in C \mid \alpha(v) = 0\} = \{0\}$$
(in other words, if $\{0\}$ is a face of $C$.)

There is a classical construction of affine toric varieties from cones (see [Ful93, Section 1.3]), which goes as follows. Fix a lattice $N$ and an isomorphism $N \cong \mathbb{Z}^n$, and pick a strictly convex polyhedral cone $\sigma \subset N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$. Write $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$. Then, Gordon's lemma ([Ful93, Section 1.2, Proposition 1]) tells us that $S_\sigma = \sigma^\vee \cap M$ is a finitely generated commutative monoid. So, we can define a finitely generated $k$-algebra $k[S_\sigma]$ which, as a $k$–vector space, has a basis given by the elements of $S_\sigma$, with multiplication in $k[S_\sigma]$ given by the operation in $S_\sigma$. We set $U_\sigma = \mathrm{Spec}(k[S_\sigma])$. Note that the inclusion of monoids $S_\sigma \subset M$ gives rise to an injective homomorphism $k[S_\sigma] \hookrightarrow k[M]$. Since $\sigma$ is strictly convex, one can show ([Ful93, Section 1.2, Proposition 2], applied to the face $\{0\}$ of $\sigma$) that there exists some $u \in S_\sigma$ such that $M = S_\sigma + \mathbb{Z} \cdot u$. It follows that $k[M] = k[S_\sigma]_u$, so the inclusion $k[S_\sigma] \hookrightarrow k[M]$ is just the map $k[S_\sigma] \to k[S_\sigma]_u$ given by $a \mapsto a/1$, and the corresponding morphism
$$i : T_N = \mathrm{Spec}(k[M]) \hookrightarrow \mathrm{Spec}(k[S_\sigma]) = U_\sigma$$
is an open immersion. Since the given isomorphism $N \cong \mathbb{Z}^n$ identifies $M$ with $\mathbb{Z}^n$ and hence $k[M]$ with $k[x_1^\pm, \ldots, x_n^\pm]$, we have an isomorphism $T_N \cong \mathbb{G}_m^n$. Moreover, we can define an action of $T_N$ on $U_\sigma$ by taking the morphism corresponding to the ring homomorphism
$$k[S_\sigma] \to k[S_\sigma] \otimes k[M], \quad s_\mu \mapsto s_\mu \otimes s_\mu$$
(here $s_\mu$ is the element of $k[S_\sigma]$ corresponding to some element $\mu$ of $S_\sigma$). One can check that this extends the action of $T_N$ on itself by left multiplication, i.e. that $i$ is a $T_N$-equivariant open immersion. Since $U_\sigma$ turns out to be a normal variety ([Ful93, Section 2.1, Proposition 2]), this proves that $U_\sigma$ is an affine toric variety.

**Remark 3.3.3.** In the theory of toric varieties, one often works in vector spaces over $\mathbb{R}$ instead of over $\mathbb{Q}$. This is useful for proving some of the technical details involving cones that we cited above, but it is not necessary to work over $\mathbb{R}$ for any of our argumnets here. For us, the only difference it would make to work over $\mathbb{R}$ is that we would have to assume that every cone we use is *rational*, i.e. is generated by elements of the lattice (either $N$ or $M$, depending on which vector space we are in). When we work over $\mathbb{Q}$, this condition holds automatically.

To classify affine toric varieties, we will provide an inverse to the above construction, having to do with the weights of $T_N$-eigenvectors in $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma})$. Since $T_N = \mathrm{Spec}(k[M])$, one can check that there is a natural identification $\mathcal{X}(T_N) \cong M$, so weights of $T_N$-eigenvectors can in fact be viewed as elements of $M$. However, because of our general conventions about the $G$-module structure of $\Gamma(X, \mathcal{O}_X)$ for a $G$-variety $X$ (see Section 2.4.a), these $T_N$-weights are actually the negatives of the weights that one might expect. The following example illustrates this phenomenon.

**Example 3.3.4.** Let $\sigma$ be the positive orthant of $N_{\mathbb{Q}} \cong \mathbb{Q}^n$, i.e. the cone generated by the vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with the 1 in the $i$th coordinate) for $1 \leq i \leq n$. One can check that $U_\sigma = \mathbb{A}^n$, with the action of $T_N \cong \mathbb{G}_m^n$ given on functors of points by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1x_1, \cdots, t_nx_n$. Writing $U_\sigma \cong \mathrm{Spec}(k[x_1, \ldots, x_n])$, an analogous argument to that of Example 2.4.8 (but with $G = \mathbb{G}_m^n$ instead of $G = \mathrm{SL}_n$) implies that $x_i$ is a $T_N$-eigenvector of weight $-e_i^* \in M$ (not $e_i^*$, as one might expect). Thus, if $C \in M$ is the cone over the weights of $T_N$-eigenvectors in $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma})$, then we have $\sigma = -C^\vee$.

More generally, it will turn out for the $T_N$-action on $U_\sigma$ defined above, given any $\mu \in \sigma^\vee$, the element $s_\mu \in k[S_\sigma]$ is a $T_N$-eigenvector of weight $-\mu$. Thus, we need to keep track of this minus sign when defining an inverse to the above construction. In order to align everything with the conventions of Luna–Vust theory for general spherical varieties (cf. Theorem 3.3.20), we prefer to put this minus sign into the construction of $U_\sigma$ by working with $U_{-\sigma}$ instead.

**Proposition 3.3.5.** *Let $T$ be a torus, and let $X$ be an affine toric $T$-variety. Write $M = \mathcal{X}(T) = \Lambda(X)$, and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.*

*(a) The $T$-divisors of $X$ are precisely the irreducible components of the complement $X \setminus T$.*

*(b) For each $T$-divisor $D \subset X$, the valuation $v_D$ induces a $\mathbb{Z}$-linear map $\varphi_D : M \to \mathbb{Z}$ in a natural way, and the cone $\sigma(X)$ generated by the $\varphi_D$ is a strictly convex polyhedral cone.*

*(c) The map $X \mapsto \sigma(X)$ is a bijection between isomorphism classes of affine toric $T$-varieties and strictly convex polyhedral cones in $N_{\mathbb{Q}}$, whose inverse is the map $\sigma \mapsto U_{-\sigma}$.*

**Remark 3.3.6.** Two details about the relationship between this proposition and spherical varieties bear mentioning here.

1. The maps $\varphi_D$ in the above proposition are the same as those defined in Section 3.1.b. We re-define them here for toric varieties for completeness, but the construction is essentially the same (albeit slightly easier in the toric case).

2. Our use of $N$ to denote a lattice here is slightly different than our usual notation of $N(X)$ for spherical varieties. Recall that for any spherical variety $X$, we have defined $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$. Thus, $N(X)$ is actually a vector space, not a lattice, and the lattice $N$ in the above proposition satisfies $N_{\mathbb{Q}} = N(X)$. This difference is notation is due to the fact that lattices play a key role in describing toric varieties explicitly; for the general spherical case, however, the vector space $N(X)$ is typically more important than any lattice inside of it.

*Proof.* Statement (a) is Lemma 3.1.17a. As for (b), we note that $K(X)$ is a $T$-module, and $K(X) \cong K(T)$ implies that $\Lambda(X) = \mathcal{X}(T)$. One can check that for every $\lambda \in \mathcal{X}(T)$, the $T$-module $K(T) \cong K(X)$ has a 1-dimensional $\lambda$-eigenspace $K(X)_\lambda$ (this is an explicit calculation on $K(T) \cong K(\mathbb{G}_m^n)$). Let $f_\lambda \in K(X)_\lambda$ be any nonzero element. Then, $v_D(f_\lambda)$ does not depend on the choice of $f_\lambda$ (because $v_D(k^\times) = 0$), so the map $\varphi_D : M \to \mathbb{Z}$ given by $\lambda \mapsto v_D(f_\lambda)$ is well-defined. Note that $\varphi_D$ is $\mathbb{Z}$-linear because

$$v_D(f_{\mu+\lambda}) = v_D(f_\mu \cdot f_\lambda) = v_D(f_\mu) + v_D(f_\lambda),$$

so $\varphi_D$ is an element of $N$.

As in the proposition, let $\sigma(X)$ be the cone generated by the $\varphi_D$ for all $T$-divisors $D \subset X$. By (a), there are finitely many $T$-divisors of $X$, so $\sigma(X)$ is polyhedral. To prove that $\sigma(X)$ is strictly convex, suppose that there exist some $a_D, b_D \in \mathbb{Q}_+$ for each $T$-divisor $D$ such that

$$\sum_D a_D \varphi_D = -\sum_D b_D \varphi_D.$$

We prove that $a_D = 0$ for all $D$ (which implies that $\sigma(X) \cap (-\sigma(X)) = \{0\}$). Since each $\varphi_D$ is nonnegative on $\Lambda^+(X)$, the above equation implies that if $a_D \neq 0$ for some $D$, then $\varphi_D$ must be 0 on $\Lambda^+(X)$. Suppose this is true for some $D$. Then, we have $\Lambda(X) = \Lambda^+(X)^{gp}$ (see Proposition 2.5.9), so $\varphi_D$ is in fact 0 on all of $\Lambda(X)$. On the other hand, if $\delta \in D$ is the generic point, then $\delta$ is fixed by the action of $T$, which implies that $\mathcal{O}_{X,\delta}$ is a $T$-module and that the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,\delta}$ is a $T$-submodule. So, there exists a nonzero $T$-eigenvector in $\mathfrak{m}$, and the weight $\mu$ of this eigenvector satisfies $\varphi_D(\mu) > 0$. This contradicts the fact that $\varphi_D = 0$. So, we must have $a_D = 0$ for all $D$, as desired.

For (c), write $X = \operatorname{Spec} A$. The $T$-equivariant open immersion $T \hookrightarrow X$ corresponds to an injection of $k$-algebras $A \hookrightarrow \Gamma(T, \mathcal{O}_T) = k[M]$. Thus, $A$ is the $k$-subalgebra of $k[M]$ generated by the $T$-eigenvectors appearing in the $A$. In other words, we have $A \cong k[-\Lambda^+(X)]$. (Note that the minus sign appears here due to details about the $T_N$-module structure on $A = \Gamma(X, \mathcal{O}_X)$, as we discussed above.) On the other hand, Lemma 3.1.17 gives us

$$\Lambda^+(X) = \{\lambda \in M \mid \varphi_D(\lambda) \geq 0 \ \forall\ T\text{-divisors } D \subset X\}.$$

It follows that the cone in $M$ generated by $\Lambda^+(X)$ is precisely the dual cone $\sigma(X)^\vee$, and $\Lambda^+(X) = \sigma(X)^\vee \cap M = S_\sigma$. So, we have

$$A \cong k[-\Lambda^+(X)] = k[S_{-\sigma(X)}]$$

and hence $X = \operatorname{Spec}(A) \cong U_{-\sigma(X)}$.

This proves that the map $\sigma \mapsto U_{-\sigma}$ is surjective, with right inverse given by $X \mapsto \sigma(X)$. To complete the proof, it suffices to show that $\sigma \mapsto U_{-\sigma}$ is injective. Let $\tau_1 \neq \tau_2$ be two distinct cones in $N_\mathbb{Q}$. The cones $-\tau_1^\vee$ and $-\tau_2^\vee$ are not equal, and since the $-\tau_i^\vee$ are are generated by lattice elements over $\mathbb{Q}$, we get $(-\tau_1^\vee) \cap M \neq (-\tau_2^\vee) \cap M$. But it follows from the definitions that

$$\Lambda^+(U_{-\tau_i}) = (\tau_i^\vee) \cap M$$

for $i \in \{1, 2\}$ (see the discussion before this proposition). So, the global sections of the $U_{-\tau_i}$ have different $T$-weights, which means the $U_{-\tau_i}$ cannot be $T$-equivariantly isomorphic. This proves that $\sigma \mapsto U_{-\sigma}$ is injective, as desired. □

The following lemma gives us a few useful properties of the combinatorial data introduced by Proposition 3.3.5.

**Lemma 3.3.7** ([Bri97, Section 3.1.1, Example 1]). *Let $X$ be an affine toric $T$-variety with corresponding cone $\sigma = \sigma(X)$. Write $M = \mathcal{X}(T) = \Lambda(X)$, and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.*

*(a) The extremal rays of $\sigma$ are precisely the subcones $\mathbb{Q}_{\geq 0}\varphi_D$ for any $T$-divisor $D \subset X$.*

*(b) For any $\varphi \in N_{\mathbb{Q}}$, there exists a $T$-invariant valuation $v : K(X)^\times \to \mathbb{Q}$ such that, for any $T$-eigenvector $f \in K(X)^{(T)}$ of weight $\mu$, we have $\varphi(\mu) = v(f)$. In other words, every element of $N_{\mathbb{Q}}$ is induced by some $T$-invariant valuation as in the proof of Proposition 3.3.5b.*

*(c) Let $Y \subset X$ be the unique closed $T$-orbit of $X$ (such an orbit always exists, see Lemma 2.5.8), and for any $\varphi \in \sigma$, let $v$ be the valuation given by (b). If $\varphi \in \sigma^\circ$, then $Y$ is the center of $v$ on $X$.*

**Remark 3.3.8.** In the language of Section 3.1.b, statement (b) of the above lemma says that the map $\varphi : \mathcal{V}(X) \to N(X)$ is surjective. It is an important technical result for spherical varieties that the map $\varphi$ is injective (see Corollary 3.1.14); it follows that the valuation $v$ in statement (b) of the above lemma is unique. However, we will not need this fact in this section.

*Proof.* Let $D_1, \ldots, D_r$ be the $T$-divisors of $X$, and let $\varphi_i = \varphi_{D_i}$. Statement (a) is equivalent to the statement that the $\varphi_i$ are a minimal set of generators for $\sigma$, i.e. that no $\varphi_i$ is a nonnegative linear combination of the others. To get a contradiction, suppose we have $\varphi_1 = \sum_{i=2}^r a_i\varphi_i$ for some $a_i \geq 0$. Then, we see that $\varphi_1(\mu) \geq 0$ whenever $\varphi_i(\mu) \geq 0$ for all $i > 1$. By the prime avoidance lemma, there exists some $f \in \Gamma(X, \mathcal{O}_X)$ such that $f$ vanishes on $D_1$ but not on $D_i$ for any $i > 1$. In fact, since any $T$-module is generated (as a vector space) by $T$-eigenvectors (Theorem 2.3.4), we can take $f$ to be a $T$-eigenvector. (Proof: let $\mathfrak{p}_j \subset \Gamma(X, \mathcal{O}_X)$ be the ideal cutting out $D_j$ for each $j$. If every $T$-eigenvector in $\mathfrak{p}_1$ is contained in one of the $\mathfrak{p}_j$ for $j > 1$, then we have $\mathfrak{p}_1 \subset \cup_j \mathfrak{p}_j$ and hence $\mathfrak{p}_1 \subset \mathfrak{p}_j$ for some $j$. So $D_j \subset D_1$, contradicting the fact that $D_j$ and $D_1$ are both irreducible codimension-1 subschemes of $X$.) Since $f$ is a $T$-eigenvector, the set $X_f$ is $T$-stable and hence is a toric $T$-variety. Moreover, the $T$-divisors of $X_f$ are the intersections $X_f \cap D_i$ for $i > 1$, so Lemma 3.1.17 gives us

$$\Lambda^+(X_f) = \{\lambda \in \Lambda(X) \mid \varphi_i(\lambda) \geq 0 \ \forall\ i > 1\}.$$

But $\varphi_1(f) \geq 0$ whenever $\varphi_1(f) \geq 0$ for all $i > 1$, so in fact, we have

$$\Lambda^+(X_f) = \{\lambda \in \Lambda(X) \mid \varphi_i(\lambda) \geq 0 \ \forall\ i\} = \Lambda^+(X)$$

(here applying Lemma 3.1.17 again to compute $\Lambda^+(X)$). It follows that the $T$-modules $\Gamma(X, \mathcal{O}_X)$ and $\Gamma(\mathrm{D}(f), \mathcal{O}_X)$ are isomorphic via the restriction map, which means the open immersion $X_f \hookrightarrow X$ is an isomorphism of affine schemes. This contradicts the fact that $D_1 \cap X_f = \varnothing$.

For statement (b), we follow [Bri97, Section 3.1.1, Example 1]. Let $f \in K(X) \cong K(T)$. Since the $T$-module $K(X)$ is a direct sum of characters of $T$ (Theorem 2.3.4), we can write $f = \sum_{\mu \in M} a_\mu f_\mu$ for some $a_\mu \in k$ and some eigenvectors $f_\mu$ of weight $\mu$, with all but finitely many $a_\mu$ equal to 0. We define

$$v(f) = \min_{a_\mu \neq 0} \varphi(\mu).$$

(Note that this is well-defined because the $a_\mu$ are uniquely determined by $f$.) One can check that this defines a valuation of $K(X)/k$, and it is immediate from the construction that $v(f) = \varphi(\mu)$ if $f$ is a $T$-eigenvector of weight $\mu$. So, $v$ is the desired valuation.

As for (c), by definition of $\sigma$, we have $\varphi = \sum_{i=1}^n a_i \varphi_i$ for some $a_i \geq 0$. We proved above that the $\varphi_i$ are a minimal set of generators for $\sigma$, so $\varphi \in \sigma^\circ$ gives us $a_i > 0$ for all $i$. Let $v : K(X)^\times \to \mathbb{Q}$ be the valuation given by applying (b) to $\varphi$, and let $\eta \in Y$ be the generic point. To prove that $Y$ is the center of $v$ on $X$, it suffices to prove that the valuation ring $\mathcal{O}_v$ dominates $\mathcal{O}_{X,\eta}$, i.e. that $\mathcal{O}_{X,\eta} \subset \mathcal{O}_v$ and that for any $f \in \mathcal{O}_{X,\eta}$, $f$ is a unit in $\mathcal{O}_{X,\eta}$ if and only if $f$ is a unit in $\mathcal{O}_v$.

Let $f \in \mathcal{O}_{X,\eta}$. Since $Y$ is $T$-stable, $\mathcal{O}_{X,\eta}$ is a $T$-submodule of $K(X)$, so we may write $f = \sum_j c_j f_j$ for some $c_j \in k$ and some $T$-eigenvectors $f_j \in \mathcal{O}_{X,\eta}^{(B)}$ (see Theorem 2.3.4). Let $\mu_j$ be the weight of $f_j$ for all $j$. For all $i$, the divisor $D_i$ is $T$-stable and closed, so it must contain a closed $T$-orbit (Proposition 2.1.2). It follows that $Y \subset D_i$ for all $i$ and hence that $\varphi_i(\mu_j) = v_{D_i}(f_j) \geq 0$ for all $i$ and $j$. So, we have

$$\varphi(\mu_j) = \sum_i a_i \varphi_i(\mu_j) \geq 0$$

for all $j$ and hence

$$v(f) = v\left(\sum_j c_j f_j\right) \geq \min_j\{v(c_j f_j)\} = \min_j\{v(f_j)\} = \min_j\{\varphi(\mu_j)\} \geq 0.$$

By definition, this means that $f \in \mathcal{O}_v$, so we have $\mathcal{O}_{X,\eta} \subset \mathcal{O}_v$.

It remains to show that $f$ is a unit in $\mathcal{O}_{X,\eta}$ if and only if $f$ is a unit in $\mathcal{O}_v$. Since every element of $\mathcal{O}_{X,\eta}$ has the form $f_1/f_2$ for some $f_1, f_2 \in \Gamma(X, \mathcal{O}_X)$, it will suffice to consider the case where $f \in \Gamma(X, \mathcal{O}_X)$. Note that if $f$ is a unit in $\mathcal{O}_{X,\eta}$ (i.e. if $f^{-1} \in \mathcal{O}_{X,\eta}$), then $f$ is a unit in $\mathcal{O}_v$ because $\mathcal{O}_{X,\eta} \subset \mathcal{O}_v$. On the other hand, suppose that $f$ is not a unit in $\mathcal{O}_{X,\eta}$. Then, $f$ lies in the prime ideal $\mathfrak{p} \subset \Gamma(X, \mathcal{O}_X)$ corresponding to the point $\eta$ in the affine scheme $X$, and $\mathfrak{p}$ is a $T$-submodule of $\Gamma(X, \mathcal{O}_X)$ (because $Y$ is $T$-stable, so $\eta$ is fixed by the action of $T$). It follows that we may write $f = \sum_j c_j f_j$ for some $c_j \in k$ and some $T$-eigenvectors $f_j \in \mathfrak{p}$ (Theorem 2.3.4). For any $j$, let $\mu_j$ be the weight of $f_j$. There exists some $i$ such that $X_{f_j} \cap D_i = \varnothing$ (otherwise, the complement $X \setminus X_{f_j}$ contains none of the $D_i$ and hence must be empty by Lemma 3.1.17a; so, we have $X_{f_j} = X$ and $X_{f_j} \cap Y \neq \varnothing$, which contradicts the fact that $f_j|_Y = 0$). It follows that $v_{D_i}(f_j) = \varphi_i(\mu_j) > 0$ for some $i$ and hence that

$$v(f_j) = \varphi(\mu_j) = \sum_i a_i \varphi_i(\mu_j) > 0.$$

(Here we are using the fact that $a_i > 0$ and $\varphi_i(\mu_j) \geq 0$ for all $i$). So, $v(f_j) > 0$ for all $j$, which implies that

$$v(f) = v\left(\sum_j c_j f_j\right) \geq \min_j \{v(f_j)\} > 0.$$

This proves that $f$ is not a unit in $\mathcal{O}_v$. $\square$

We now turn to classifying toric varieties which aren't necessarily affine. The classical construction of such varieties begines with a *fan* $\mathcal{F}$, i.e. a nonempty finite set of strictly convex polyhedral cones in $N$ such that

1. for every cone $\sigma \in \mathcal{F}$ and every face $\tau \subset \sigma$, we have $\tau \in \mathcal{F}$, and
2. for every $\sigma, \tau \in \mathcal{F}$, the intersection $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$ (and in particular is in $\mathcal{F}$).

One then shows ([Ful93, Section 1.3, Lemma]) that for any face $\tau \subset \sigma$, we have an open immersion $U_\tau \hookrightarrow U_\sigma$. Each cone $\sigma \in \mathcal{F}$ determines an affine toric variety $U_\sigma$, and we can glue these varieties along the open subvarieties $U_{\sigma\cap\tau}$ for any two cones $\sigma, \tau \in \mathcal{F}$ (since $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$). The resulting scheme, denoted $X_{\mathcal{F}}$, is immediately seen to be reduced and irreducible (because the $U_\sigma$ are), and $X_{\mathcal{F}}$ turns out to be separated and normal as well ([Ful93, Section 1.4, Lemma and Section 2.1, Proposition]). One then glues the action of $T_N$ on each $U_\sigma$ to obtain an action of $T_N$ on $X_{\mathcal{F}}$. For any $\sigma$, we immediately see that the composition

$$T_N \hookrightarrow U_\sigma \hookrightarrow X_{\mathcal{F}}$$

is a $T_N$-equivariant open immersion, so $X_{\mathcal{F}}$ is a toric $T_N$-variety.

As in the affine case, it turns out that every toric variety can be obtained by the classical construction. To prove this, we will want to "reverse" the above construction by finding a $T$-stable affine open cover of $X$ and then using the classification of affine toric varieties by cones. The main issue is to find for any toric $T$-variety $X$ a canonical choice of $T$-stable affine open subsets that cover $X$.

To this end, the following lemma classifies all the $T$-stable affine open subsets of a toric variety $X$. In particular, there are finitely many of them, and they cover $X$, so we can simply take all of them in our open cover of $X$. The key ingredient of the proof is a rather remarkable theorem of Sumihiro, Theorem 2.6.12, which in the toric case gives us affine open subsets rather than just quasi-projective ones.

**Lemma 3.3.9.** *Let $T$ be a torus, and let $X$ be a toric $T$-variety.*

*(a) $X$ is affine if and only if $X$ has a unique closed $T$-orbit.*

*(b) For any $T$-orbit $Y \subset X$, define*

$$X_Y = \{x \in X \mid Y \subset \overline{Tx}\}.$$

*Then, $X_Y$ is the unique $T$-stable affine open subset of $X$ whose unique closed $T$-orbit is $Y$.*

*(c) For any two $T$-orbits $Y, Z \subset X$, we have $X_Z \subset X_Y$ if and only if $Y \subset \overline{Z}$.*

*(d) Every $T$-stable affine open subset of $X$ is equal to $X_Y$ for some $T$-orbit $Y$ of $X$.*

*Proof.* If $X$ is affine, then the fact that $X$ contains a unique orbit is Lemma 2.5.8 (alternately, it follows from an explicit construction of the $T$-orbits in $X \cong U_\sigma$, see [Ful93, Section 3.1, Exercise]). For the converse, we argue as in the proof of Corollary 2.6.13. If $Y \subset X$ is the unique closed $T$-orbit, then a theorem of Sumihiro (Theorem 2.6.12) tells us that there exists some $T$-stable affine open subset $U \subset X$ such that $Y \subset U$. The complement $X \setminus U$ is now a $T$-stable closed subset, so if $X \setminus U$ is nonempty, it must contain a closed $T$-orbit (e.g. any orbit of minimal dimension). But any closed $T$-orbit in $X \setminus U$ would be a closed $T$-orbit of $X$; since the only such orbit is $Y \subset U$, we conclude that $X \setminus U$ is empty. It follows that $X = U$, so $X$ is affine.

For (b), we will need to use the fact that $X$ has finitely many $T$-orbits. This is a general fact about spherical varieties (see Theorem 3.1.4); alternately, one can avoid using the theory of spherical varieties as follows. By Theorem 2.6.12, there exists an open cover of $X$ by $T$-stable affine open subsets. Since $X$ is quasi-compact, we may take this cover to be finite, so it will suffice to check that any $T$-stable affine open subset $U$ of $X$ has finitely many orbits. Any such $U$ is an affine toric variety, and if $\sigma$ is the corresponding cone, then one use the construction of $U$ from $\sigma$ to show that the $T$-orbits of $U$ are in bijection with faces of $\sigma$ (see e.g. [Oda88, Proposition 1.6]). In particular, since $\sigma$ is polyhedral, it has finitely many faces, so $U$ has finitely many $T$-orbits.

Now, the set $X_Y$ is $T$-stable by definition and is open because

$$X_Y = X \setminus \bigcup_{\substack{Z \text{ a } T\text{-orbit} \\ Y \not\subset \overline{Z}}} \overline{Z}.$$

(Note that since $X$ has finitely many $T$-orbits, the union in the above equation is finite.) Moreover, we have $Y \subset X_Y$ by definition, and $X_Y$ must have some closed $T$-orbit (e.g. any $T$-orbit of minimal dimension). But no $T$-orbit can be closed in $X_Y$ except possibly $Y$, so $Y$ must be the unique closed $T$-orbit of $X_Y$. Since $X_Y$ is a $T$-stable open subset of $X$, it is a toric $T$-variety, so part (a) implies that $X_Y$ is affine. On the other hand, suppose $U \subset X$ is any $T$-stable affine open subset such that $Y \subset U$ is the unique closed $T$-orbit of $U$. For any

$T$-orbit $Z$ such that $Y \subset \overline{Z}$, we have $Z \cap U \neq \varnothing$ and hence $Z \subset U$ (because $U$ is $T$-stable). It follows that $X_Y \subset U$. Conversely, for any $T$-orbit $Z \subset U$, the closure of $Z$ in $U$ must contain some closed $T$-orbit, and the only such orbit is $Y$. It follows that $Y$ is contained in the closure of $Z$ in $X$, so that $Z \subset X_Y$ by definition. This gives us $U \subset X_Y$ and hence $U = X_Y$.

Statement (c) follows almost immediately from the definition of $X_Y$. Explicitly: if $Y \subset \overline{Z}$, then any orbit closure containing $Z$ also contains $Y$, so $X_Z \subset X_Y$ by definition; conversely, if $X_Z \subset X_Y$, then in particular, $Z \subset X_Z$ implies that $Y \subset \overline{Z}$. Finally, any $T$-stable affine open subset $U \subset X$ has a unique closed $T$-orbit $Y$ by (a). Statement (b) then gives us $U = X_Y$, which proves (d). □

**Remark 3.3.10.** Note that part (b) of the above lemma tells us that $X_Y$ is precisely the set $X_{T,Y}$ of Theorem 3.2.7 (remember that $X$ is a spherical variety with $G = B = T$). We can thus view the above lemma as an analog of all the statements in Theorem 3.2.7 (except for the part of Theorem 3.2.7 involving the local structure theorem).

Using the nice $T$-stable affine open cover provided by the sets $X_Y$, we can now classify all toric varieties.

**Theorem 3.3.11.** *Let $T$ be a torus, and let $X$ be a toric $T$-variety. Write $M = \mathcal{X}(T) = \Lambda(X)$, and let $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. For each $T$-orbit $Y \subset X$, let $\sigma_Y$ be the cone corresponding (via Proposition 3.3.5) to the affine toric variety $X_Y$ given in Lemma 3.3.9.*

*(a) For any $T$-orbit $Y \subset X$, the map $Z \mapsto \sigma_Z$ is a bijection between $T$-orbits of $X$ whose closures contain $Y$ and faces of the cone $\sigma_Y$.*

*(b) The set*
$$\mathcal{F}(X) = \{\sigma_Y \mid Y \subset X \text{ a } T\text{-orbit}\}$$
*is a fan of strictly convex polyhedral cones in $N_{\mathbb{Q}}$.*

*(c) The map $X \mapsto \mathcal{F}(X)$ is a bijection between isomorphism classes of toric $T$-varieties and fans of strictly convex polyhedral cones in $N_{\mathbb{Q}}$. Its inverse is given by $\mathcal{F} \mapsto X_{\mathcal{F}}$.*

**Remark 3.3.12.** Statement (a) of the above theorem is a standard result for the classical construction of a toric variety $X_{\mathcal{F}}$ from a fan $\mathcal{F}$ and is often proven explicitly from this construction (see e.g. [Ful93, Section 3.1, Proposition]). If we already knew the classification in statement (c) of the theorem, then this result using the classical construction would be enough to prove (a). However, since we will need (a) to prove (c), we provide an alternative proof of (a).

*Proof.* The proof relies in several places on the following fact about strictly convex cones: given any set of minimal generators $v_1, \ldots, v_r$ of a cone $\sigma$, the faces of $\sigma$ are precisely the cones generated by any subset of the $v_i$. (Proof: faces are sets of the form $\{v \in \sigma \mid \langle v, u\rangle = 0\}$ for some $u \in \sigma^\vee$. Such a subset is generated as a cone by the set of $v_i$ such that $\langle v_i, u\rangle = 0$.

On the other hand, since the $v_i$ are minimal generators in a strictly convex cone, they are linearly independent, so one can pick some $u \in \sigma^\vee$ that vanishes precisely on any given subset of the $v_i$ and is positive on all the others.)

Fix some $T$-orbit $Y$ of $X$. For the map in (a) to make sense, we first need to check that for any $T$-orbit $Z$ such that $Y \subset \overline{Z}$, the cone $\sigma_Z$ is a face of $\sigma_Y$. For this, note that the $T$-divisors of $X_Y$ (resp. $X_Z$) are precisely the nonempty intersections $D \cap X_Y$ (resp. $D \cap X_Z$) for $T$-divisors $D$ of $X$. Since $X_Z \subset X_Y$ (see Lemma 3.3.9), it follows that for any $T$-divisor $D_Z \subset X_Z$, the map $\varphi_{D_Z}$ is equal to $\varphi_{D_Y}$ for some $T$-divisor $D_Y \subset X_Y$. Since the $\varphi_{D_Y}$ (resp. $\varphi_{D_Z}$) are a set of minimal generators for $\sigma_Y$ (resp. $\sigma_Z$) by Lemma 3.3.7a, we can use the above fact on strictly convex cones to conclude that $\sigma_Z$ is a face of $\sigma_Y$.

Next, we check that the map in (a) is injective. For any $T$-orbits $Z_1, Z_2$ such that $Y \subset \overline{Z_i}$ and $\sigma_{Z_1} = \sigma_{Z_2}$, let $\varphi \in \sigma_{Z_1}^\circ$, and let $v : K(X)^\times \to \mathbb{Q}$ be a valuation given by Lemma 3.3.7b. By part (c) of that same lemma (applied to the affine toric varieties $U_{Z_1}$ and $U_{Z_2}$), the valuation $v$ has center $Z_i$ on $U_{Z_i}$ for $i \in \{1, 2\}$. Thus, both $Z_1$ and $Z_2$ are the center of $v$ on $X$. But $X$ is separated, so $v$ has a unique center on $X$. It follows that $Z_1 = Z_2$

We now check that the map in (a) is surjective. Let $\sigma \subset \sigma_Y$ be any face of $\sigma_Y$. Then, $\sigma$ is generated by a subset of the valuations of the $T$-divisors in $X_Y$, hence by a subset of the valuations of the $T$-divisors $D_1, \ldots, D_r \subset X$ such that $Y \subset D_i$. Say $\sigma$ is generated by the valuations corresponding to $D_1, \ldots, D_m$ for some $1 \leq m \leq r$. Now, $Y \subset D_1 \cap \cdots \cap D_m$ implies that $Y$ is contained in some irreducible component $W$ of $D_1 \cap \cdots \cap D_m$. Recall that $X$ has finitely many $T$-orbits (this follows from Theorem 3.1.4 for spherical varieties in general, but see the proof of Lemma 3.3.9 for an argument that only involves the theory of toric varieties). In particular, $W$ contains only finitely many $T$-orbits; since $W$ is irreducible, one of these $T$-orbits, say $Z \subset W$, must be dense in $W$. Then, $Z$ is a $T$-orbit of $X$, and we have $Y \subset \overline{Z} = W$ and $Z \subset D_i$ for $i \in \{1, \ldots, m\}$. Since any $T$-stable divisor containing $Z$ must contain $\overline{Z}$ and hence $Y$, we see that $D_1, \ldots, D_m$ are precisely the $T$-stable divisors of $X$ containing $Z$. It follows that $\sigma_Z$ is generated by $\varphi_{D_1}, \ldots, \varphi_{D_m}$ and hence that $\sigma_Z = \sigma$. This completes the proof of (a).

As for (b), given any $\sigma_Y$, statement (a) tells us that every face of $\sigma_Y$ is $\sigma_Z$ for some $T$-orbit $Z$. To prove (b), then, we just need to show that given any two orbits $Y, Z \subset X$, the intersection $\sigma_Y \cap \sigma_Z$ is a face of both $\sigma_Y$ and $\sigma_Z$. If $\sigma_Y \cap \sigma_Z$ is contained in either a proper face of $\sigma_Y$ or a proper face of $\sigma_Z$, then we may replace $\sigma_Y$ or $\sigma_Z$ by a proper face (by statement a, this amounts to replacing $Y$ or $Z$ by some orbit whose closure contains it, and this replacement is harmless because being a face is a transitive relation, see e.g. [Ful93, Section 1.2, Property (4)]). Making this replacement repeatedly, we may assume that $\sigma_Y \cap \sigma_Z$ is not contained in any proper face of $\sigma_Y$ or $\sigma_Z$, i.e. that $\sigma_Y^\circ \cap \sigma_Z^\circ \neq \varnothing$. Let $\varphi \in \sigma_Y^\circ \cap \sigma_Z^\circ$. Applying Lemma 3.3.7b to both $\varphi$ gives us a $T$-invariant valuation $v : K(X)^\times \to \mathbb{Q}$ corresponding to $\varphi$, and by part (c) of the same lemma (applied to both $X_Y$ and $X_Z$), the valuation $v$ has center $\overline{Y}$ (resp. $\overline{Z}$) on $X_Y$ (resp. $X_Z$). Since $X$ is separated, we conclude that $\overline{Y} = \overline{Z}$ and hence that $Y = Z$. We thus have $\sigma_Y = \sigma_Z$, so $\sigma_Y \cap \sigma_Z$ is certainly a face of both $\sigma_Y$ and $\sigma_Z$.

To prove (c), we note that for any two $T$-orbits $Y$ and $Z$ of $X$, the cone $\sigma_Y \cap \sigma_Z$ is a face of $\sigma_Y$ and $\sigma_Z$ by (b) and so is generated by the minimal generators of $\sigma_Y$ that are also minimal generators of $\sigma_Z$. The minimal generators of $\sigma_Y$ (resp. $\sigma_Z$) are determined by the valuations of $T$-divisors $D$ such that $D \cap X_Y \neq \varnothing$ (resp. such that $D \cap X_Z \neq \varnothing$), so the minimal generators that $\sigma_Y$ and $\sigma_Z$ have in common are given by the $T$-divisors $D$ such that $D \cap X_Y \neq \varnothing$ and $D \cap X_Z \neq \varnothing$, or equivalently, such that $D \cap X_Y \cap X_Z \neq \varnothing$. (The equivalence here follows from the fact that if $D$ intersects both $X_Y$ and $X_Z$, then the dense $T$-orbit of $D$ must lie in both $X_Y$ and $X_Z$, since $X_Y$ and $X_Z$ are $T$-stable.) It follows that $\sigma_Y \cap \sigma_Z$ is precisely the cone corresponding to the toric variety $X_Y \cap X_Z$ (which is affine because $X$ is separated). So, $X_{\mathcal{F}(X)}$ is by definition the varity obtained by gluing the $X_Y$ for any $T$-orbit $Y \subset X$ along the affine open subsets $X_Y \cap X_Z$. But $X$ can be obtained by precisely the same gluing procedure, so $X_{\mathcal{F}(X)} \cong X$.

It remains to prove that if we start with a fan $\mathcal{G}$ and let $X = X_{\mathcal{G}}$, then $\mathcal{F}(X) = \mathcal{G}$. By construction, the scheme $X$ is obtained by gluing the $U_\sigma$ for $\sigma \in \mathcal{G}$ along the open subsets $U_{\sigma \cap \sigma'} = U_\sigma \cap U_{\sigma'}$. Each $U_\sigma$ is a $T$-stable affine open subset of $X$ and so has the form $X_Y$ for some $T$-orbit $Y$ by Lemma 3.3.9d. By definition of $\sigma_Y$, this implies that $\sigma = \sigma_Y$, so $\sigma \in \mathcal{F}(X)$. Conversely, any cone in $\mathcal{F}(X)$ has the form $\sigma_Y$ for some $T$-orbit $Y \subset X$. Since the $U_\sigma$ are $T$-stable and cover $X$, We have $Y \subset U_\sigma$ for some $\sigma \in \mathcal{G}$. Then, the closed orbit $Z \subset U_\sigma$ satisfies $X_Z = U_\sigma$ by Lemma 3.3.9b, so $Y \subset U_\sigma$ gives us $Z \subset \overline{Y}$. Applying statement (a) to $U_\sigma$, we see that $\sigma_Y$ is a face of $\sigma_Z = \sigma$; in particular, $\sigma_Y \in \mathcal{G}$ because $\mathcal{G}$ is a fan. This proves that $\mathcal{F}(X) = \mathcal{G}$, as desired. □

As we discussed at the beginning of this section, this classification of toric varieties is essentially the classification of embeddings $G/H \hookrightarrow X$ in the special case where $G = T$ is a torus. For the more general case, the proof of the Luna–Vust classification will follow exactly the same approach as the above classification for toric varieties. The steps to this approach are as follows.

1. Classify embeddings $G/H \hookrightarrow X$ when $X$ is a certain "nice" type of spherical variety. For toric varieties, this was the classification of affine toric varieties; for the general case, it will be the classification of spherical varieties which have a unique closed $G$-orbit (in the toric case, this is equivalent to being affine by Lemma 3.3.9 above). We will again classify such an $X$ by a cone, but this time there will be some extra data stemming from the colors of $X$. (Recall that colors are $B$-divisors which are not $G$-stable; such a divisor never exists in the toric case, where $G = B = T$.) We will thus obtain a gadget called a *colored cone*. For toric varieties, this step was the classification of affine toric varieties in Proposition 3.3.5; for spherical varieties, it will be Proposition 3.3.15.

2. For a general embedding $G/H \hookrightarrow X$, define a cover of $X$ by $G$-stable open subsets which are "nice" in the sense of Step 1. For toric varieties, this cover was the open subsets $X_Y$ of Lemma 3.3.9; for spherical varieties, it will be the open subsets $G \cdot X_{B,Y}$, where $X_{B,Y} \subset X$ is the open subset from Theorem 3.2.7. Thus, we have already completed this step for spherical varieties (though we remark that Theorem 3.2.7 was considerably more difficult to prove than Lemma 3.3.9, mainly because of added sub-

tleties when $G \neq B$). Notice that our open cover contains one element for each $G$-orbit of $X$; this is important context for what follows.

3. As a key technical ingredient, we need to translate the statement $Y \subset \overline{Z}$ for two $G$-orbits $Y, Z \subset X$ into a combinatorial statement involving the combinatorial data (i.e. colored cones) from Step 1. This will itself require a technical statement about centers of valuations. In the toric case, the valuation statement is Lemma 3.3.7c, which we used in the proof of Theorem 3.3.11a; in the spherical case, our valuation statement will be Lemma 3.3.17, which we will use to prove Proposition 3.3.24.

4. Complete the classification of embeddings $G/H \hookrightarrow X$. By using the cover from Step 2 and the classification of Step 1, we obtain a combinatorial gadget called a (strictly convex) *colored fan* from $X$, which is a finite set of colored cones that obey nice combinatorial properties. Conversely, given a (strictly convex) colored fan, one constructs an embedding $G/H \hookrightarrow X$ by gluing the embeddings defined by the colored cones in the fan. To define this gluing and check that it is spherical, Step 3 will be crucial. One then shows that these two constructions are inverse two each other, which is mostly a formal check (though Step 3 is again needed for a minor technical point). For toric varieties, this was Theorem 3.3.11b,c; for spherical varieties, it will be Theorem 3.3.26.

As noted above, Step 2 is already done for us. Indeed, for any embedding $G/H \hookrightarrow X$, note that $X$ contains a dense $B$-orbit (because $G/H$ does) and hence is spherical. So, for any $G$-orbit $Y \subset X$, Theorem 3.2.7 gives us a $G$-stable open subset $GX_{B,Y} \subset X$ such that $Y$ is the only closed orbit of $GX_{B,Y}$. Since $GX_{B,Y}$ is open in $X$, it intersects $G/H$, and since $GX_{B,Y}$ is $G$-stable, we have $G/H \subset GX_{B,Y}$, so $GX_{B,Y}$ is again spherical. So, our first step will be to classify embeddings that "look like" $GX_{B,Y}$, i.e. those which have a unique closed $G$-orbit.

### 3.3.b Colored Cones and Classifications

We now return to the question of classifying embeddings $G/H \hookrightarrow X$ for an arbitrary homogeneous spherical variety $G/H$. We will omit several technical proofs in our presentation of the general theory. The reader interested in these proofs can find the main ideas in our proofs of the toric case in Section 3.3.a and can find complete proofs of the general spherical case in either [Kno91] or [Bri97, Section 3].

In light of our motivation from the toric case, we begin by considering a certain "nice" type of embedding $G/H \hookrightarrow X$.

**Definition 3.3.13.** We say that a $G$-variety $X$ is *simple* if $X$ has a unique closed $G$-orbit. We say that an embedding $G/H \hookrightarrow X$ is *simple* if $X$ is simple.

Let $G/H \hookrightarrow X$ be a simple embedding. Following the intuition gained from the toric case, we might hope to classify $X$ by using the cone in $N(X)$ generated by the $\varphi_D$, where $D \in \mathcal{D}_{G,X}$ is a $B$-divisor of $X$. This does turn out to be the right idea, and it makes sense because we have $K(X) = K(G/H)$ as $G$-modules, hence also $N(X) = N(G/H)$. The $\varphi_D$ can thus be expressed entirely in terms of $G/H$ (which we have fixed) rather than in terms of $X$ (which is the object we're trying to classify). However, there is one piece of geometric data that these valuations will not tell us: namely, whether a $B$-divisor $D$ contains the unique closed $G$-orbit $Y$. If $D$ is $G$-stable, then it must contain $Y$ ($D$ contains some $G$-orbit, hence also a $G$-orbit closure, and this closure contains a closed $G$-orbit). In particular, in the case of toric varieties, where $G = B$, every such divisor would contain $Y$. In general, however, it is possible that some $B$-divisors of $X$ which are not $G$-stable (i.e. colors of $X$) might not contain $Y$.

We will thus have to keep track of which colors contain $Y$. We cannot keep track of this information in terms of valuations; we will have to remember the divisors themselves. Thankfully, this is acceptable in the case of colors. Indeed, any color $D$ of $X$ cannot lie in $X \setminus (G/H)$, as it would otherwise be a component of $X \setminus (G/H)$ and hence be $G$-stable (Lemma 2.1.3). Thus, we have $D \cap (G/H) \neq \varnothing$. Since $G/H$ is an open subset of $X$, we can recover $D$ from $D \cap (G/H)$ by the rule

$$D = \overline{D \cap G/H}.$$

Thus, if we keep track of $D \cap (G/H)$ for all the colors $D$ containing $Y$, we will still have data depending only on $G/H$, and this data will determine the geometric information about $X$ that we are interetsed in.

With this motivation, we make the following definitions.

**Definition 3.3.14.** Let $G/H \hookrightarrow X$ be an embedding, and let $Y \subset X$ be a $G$-orbit. We define

$$\mathcal{B}_Y = \{v_D \in \mathcal{V}(G/H) \mid D \in \mathcal{D}_{G,X}^G,\ Y \subset D\}$$

and

$$\Delta_Y = \{D \cap G/H \in \mathcal{D}(G/H) \mid D \in \mathcal{D}_{G,X} \setminus \mathcal{D}_{G,X}^G,\ Y \subset D\}.$$

In words, $\mathcal{B}_Y$ is the set of valuations of $G$-divisors containing $Y$, and $\Delta_Y$ is the set of intersections with $G/H$ of every color containing $Y$.

As our above discussion suggests, when $X$ is simple with unique closed $G$-orbit $Y$, the data of $\mathcal{B}_Y$ and $\Delta_Y$ is indeed enough to classify the embeddings $G/H \hookrightarrow X$.

**Proposition 3.3.15** ([Bri97, Proposition 3.2.1]; cf. [Kno91, Theorem 2.3])**.** *Let $G/H \hookrightarrow X$ be a simple embedding, and let $Y \subset X$ be the unique closed $G$-orbit of $X$. The simple embedding $G/H \hookrightarrow X$ is uniquely determined up to $G$-equivariant isomorphism by the pair $(\mathcal{B}_Y(X), \Delta_Y(X))$.*

*Proof.* Let $G/H \hookrightarrow X'$ be another simple embedding and let $Y'$ be the unique closed $G$-orbit of $X$'. Suppose that $(\mathcal{B}_Y, \Delta_Y) = (\mathcal{B}_{Y'}, \Delta_{Y'})$. Let $X_{B,Y} \subset X$ and and $X'_{B,Y'} \subset X'$ be the affine open subsets of Theorem 3.2.7. By definition, the valuations of $B$-divisors of $X_{B,Y}$ are precisely the valuations of the $B$-divisors of $X$ containing $Y$, which are the valuations in $\mathcal{B}_Y$

along with the valuations $v_D$ for $D \in \Delta_Y$. Since $X_{B,Y}$ is normal, the global sections of $X_{B,Y}$ are precisely the sections on the open $B$-orbit $Bx \subset G/H$ which extend over each $B$-stable prime divisor of $X_{B,Y}$ (cf. Lemma 3.1.17). This gives us

$$\Gamma(X_{B,Y}, \mathcal{O}_X) = \{f \in \Gamma(Bx, \mathcal{O}_{G/H}) \mid v(f) \geq 0 \ \forall v \in \mathcal{B}_Y \text{ and } v_D(f) \geq 0 \ \forall \ D \in \Delta_Y\}.$$

Applying the same argument for $X'_{B,Y'}$ instead of $X_{B,Y}$ (and using the fact that $(\mathcal{B}_Y, \Delta_Y) = (\mathcal{B}_{Y'}, \Delta_{Y'})$), we see that $\Gamma(X_{B,Y}, \mathcal{O}_X) = \Gamma(X'_{B,Y'}, \mathcal{O}_{X'})$ as subrings of $\Gamma(Bx, \mathcal{O}_{G/H})$, hence as subrings of $K(X) = K(G/H) = K(X')$. So, the birational morphism $X \dashrightarrow X'$ coming from this equality on function fields induces an isomorphism $\varphi : X_{B,Y} \xrightarrow{\sim} X_{B,Y'}$. In fact, we have $\Gamma(X_{B,Y}, \mathcal{O}_X) = \Gamma(X'_{B,Y'}, \mathcal{O}_{X'})$ as $G$-submodules of $K(G/H)$, which implies that $\varphi$ extends to a $G$-equivariant isomorphism $\tilde{\varphi} : GX_{B,Y} \xrightarrow{\sim} GX'_{B,Y'}$. Since $X$ and $X'$ are simple, we have $GX_{B,Y} = X$ and $GX'_{B,Y'} = X'$ (see Theorem 3.2.7d and Proposition 2.1.2d), so $\tilde{\varphi}$ is an isomorphism of $G$-varieties $X \cong X'$. In particular, $\tilde{\varphi}$ commutes with the open immersions $G/H \hookrightarrow X$ and $G/H \hookrightarrow X'$ because it is $G$-equivariant. □

The key pieces of data in the above proposition are a set of valuations (namely, $\mathcal{B}_Y$) and a set of divisors (namely, $\Delta_Y$). We wish to translate this data into something more combinatorial. For this, we use the map

$$\varphi : \mathcal{V}(X) \cup \Delta D(X) \to N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$$

defined in Section 3.1.b to map our data into the vector space $N(X) = N(G/H)$. Since $\varphi$ is injective on $\mathcal{V}(X)$ (hence also on $\mathcal{B}_Y \subset \mathcal{V}(X)$) by Corollary 3.1.14, we identify $\mathcal{V}(X)$ with its image in $N(X)$ in what follows. This leads us to make the following definition.

**Definition 3.3.16.** Let $G/H \hookrightarrow X$ be an embedding, and let $Y \subset X$ be a $G$-orbit. We define $\mathcal{C}_Y \subset N(G/H)$ to be the cone generated by $\varphi(\mathcal{B}_Y)$ and $\varphi(\Delta_Y)$.

Since $\varphi$ is injective on valuations but not generally on divisors, we can expect the cone $\mathcal{C}_Y$ to capture all the information of the valuations $\mathcal{B}_Y$ but not necessarily the information of $\Delta_Y$. The following lemma makes this rigorous.

**Lemma 3.3.17** ([Kno91, Lemma 2.4])**.** *Suppose that $G/H \hookrightarrow X$ is simple, and let $Y$ be the unique closed $G$-orbit of $X$. The sets of the form $\mathbb{Q}^+ v$ for any $v \in \mathcal{B}_Y$ are precisely the extremal rays of $\mathcal{C}_Y$ which do not contain any element of $\varphi(\Delta_Y)$.*

Now, we claim that any valuation $v \in \mathcal{B}_Y(X)$ is determined uniquely by the set $\mathbb{Q}^+ v \subset N_{G/H}$. Picking any $v' \in \mathbb{Q}^+ v$, we know that $v$ is a discrete valuation, so $v$ takes values in $\mathbb{Z}$, and its image contains 1. Thus, there exists some $n \in \mathbb{Z}$ which "clears denominators" of $v'$, so that $n \cdot v'$ takes values in $\mathbb{Z}$, and the minimal such $n$ is characterized by $1 \in \mathrm{Im}(n \cdot v')$ and hence $n \cdot v' = v$. Combining this with the above lemma and Proposition 3.3.15, we see that when $X$ is simple with closed $G$-orbit $Y$, the embedding $G/H \hookrightarrow X$ is determined up to $G$-isomorphism by the pair $(\mathcal{C}_Y, \Delta_Y)$. We now define some terminology for certain conditions on such a pair $(\mathcal{C}_Y, \Delta_Y)$.

**Definition 3.3.18.**

1. A *colored cone* is a pair $(\mathcal{C}, \Delta)$ with $\mathcal{C} \subset N(G/H)$ a cone and $\Delta \subset \mathcal{D}(G/H)$ a subset of colors such that

   a) $\mathcal{C}$ is generated by the (finite) set $\varphi(\Delta)$ along with finitely many elements of $\mathcal{V}(G/H)$, and

   b) $\mathcal{C}^\circ \cap \mathcal{V}(G/H) \neq \varnothing$.

2. We say that a colored cone $(\mathcal{C}, \Delta)$ is *strictly convex* if the cone $\mathcal{C}$ is strictly convex and $0 \notin \varphi(\Delta)$.

3. A *face* of a colored cone $(\mathcal{C}, \Delta)$ is a pair $(\mathcal{C}_0, \Delta_0)$, where

   a) $\mathcal{C}_0$ is a face of the cone $\mathcal{C}$,

   b) $\Delta_0 = \Delta \cap \varphi^{-1}(\mathcal{C}_0)$, and

   c) the pair $(\mathcal{C}_0, \Delta_0)$ is itself a colored cone (equivalently, $\mathcal{C}_0^\circ \cap \mathcal{V}(G/H) \neq \varnothing$)).

**Remark 3.3.19.** Since we are now accumulating quite a few sets of combinatorial data, we remark briefly on the mnemonics used. $\mathcal{D}$ stands for "divisors" (e.g. $\mathcal{D}_{G,X}$ is the set of $B$-divisors of $X$), while $\Delta$ (the Greek equivalent of a "d") denotes colors, i.e. $B$-divisors that are not $G$-stable (e.g. $\Delta_Y$ is the set of $D \cap G/H$ for $D$ a color containing $Y$). The set $\mathcal{B}_Y$ will not be used anymore from here on out; in view of Lemma 3.3.17, it will suffice to use $\mathcal{C}_Y$ instead, and the $\mathcal{C}$ stands for "cone."

In order to classify simple embeddings $G/H \hookrightarrow X$, it remains to understand which pairs of the form $(\mathcal{C}, \Delta)$ actually arise from these simple embeddings. It turns out that these pairs are precisely the strictly convex colored cones. When $G/H \hookrightarrow X$ is simple with closed $G$-orbit $Y$, checking that the pair $(\mathcal{C}_Y, \Delta_Y)$ is a strictly convex colored cone is not too difficult (though it does require a certain technical result on valuations, namely Proposition 3.3.21 below). The main difficulty is to construct a simple embedding $G/H \hookrightarrow X$ from a strictly convex colored cone $(\mathcal{C}, \Delta)$. This is more subtle than the toric case, because we need to ensure that the set $\Delta$ corresponds to the colors that contain $Y$. We refer the reader to the proof of [Kno91, Theorem 3.1] (or that of [Bri97, Theorem 3.3.]) for the necessary construction.

**Theorem 3.3.20** ([Kno91, Theorem 3.1])**.** *Consider the map*

$$(G/H \hookrightarrow X) \mapsto (\mathcal{C}_Y(X), \Delta_Y(X))$$

*that sends any simple embedding $G/H \hookrightarrow X$ with closed $G$-orbit $Y$ to the pair $(\mathcal{C}_Y(X), \Delta_Y(X))$. This map defines a bijection between the set of $G$-isomorphism classes of simple embeddings of $G/H$ and the set of strictly convex colored cones.*

As mentioned above, proving that $(\mathcal{C}_Y, \Delta_Y)$ is a strictly convex colored cone requires a certain technical result. More precisely, we need to be able to relate the centers of certain valuations to combinatorial properties of the cone $\mathcal{C}_Y$. The following proposition allows us to do just that. This proposition is a key technical tool not just for the proof of Theorem 3.3.20 above but also for the classification statements throughout the rest of this section. The reader who has read our discussion of the toric case in Section 3.3.a may recall that a similar statement (namely, Lemma 3.3.7c) was instrumental in the toric case as well.

**Proposition 3.3.21** ([Bri97, Proposition 3.1.3]; cf. [Kno91, Theorem 2.5])**.** *Let $G/H \hookrightarrow X$ be a simple embedding, let $Y$ be the unique closed $G$-orbit of $X$, and let $v \in \mathcal{V}(G/H)$.*

*(a) We have*
$$\Gamma(X_{B,Y}, \mathcal{O}_X)^{(B)} = \{f \in K(G/H)^{(B)} \mid \chi_f \in \mathcal{C}_Y^\vee\}.$$

*(b) The center of $v$ on $X$ exists if and only if $v \in \mathcal{C}_Y$.*

*(c) The center of $v$ is $Y$ if and only if $v \in \mathcal{C}_Y^\circ$.*

We now turn to classifying embeddings $G/H \hookrightarrow X$ for $X$ not necessarily simple. To do this, note that the sets $G \cdot X_{B,Y}$ for $Y \subset X$ any $G$-orbit are $G$-stable open subsets that cover $X$, and every $G \cdot X_{B,Y}$ is simple with unique closed $G$-orbit $Y$. (see Theorem 3.2.7). We know how to classify simple embeddings, so we just need to understand how to "glue together" that classification over the open cover given by the $G \cdot X_{B,Y}$. To model this"gluing" on a combinatorial level, we use a combinatorial gadget that combines many colored cones; by analogy with the theory of toric varieties, we call such a gadget a *colored fan*.

**Definition 3.3.22.**

1. A *colored fan* is a nonempty, finite set of colored cones $\mathscr{F} = \{(\mathcal{C}_i, \Delta_i)\}_i$ such that

   a) for all $i$, every face of $(\mathcal{C}_i, \Delta_i)$ lies in $\mathscr{F}$, and

   b) for all $v \in \mathcal{V}(G/H)$, there exists at most one $i$ such that $v \in \mathcal{C}_i^\circ$.

2. A colored fan $\mathscr{F}$ is *strictly convex* if $(0, \varnothing) \in \mathscr{F}$. This is equivalent to the condition that every colored cone in $\mathscr{F}$ is strictly convex. (Proof: if any colored cone $(\mathcal{C}, \Delta)$ in $\mathscr{F}$ is strictly convex, then $(0, \varnothing)$ is a face of $(\mathcal{C}, \Delta)$ and hence lies in $\mathscr{F}$. Conversely, if $(0, \varnothing) \in \mathscr{F}$ but some colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}$ is not strictly convex, then $0 \in \mathcal{V}(X)$ and $0 \in \mathcal{C}^\circ \cap 0^\circ$, contradicting the definition of a colored fan.)

3. Given any embedding $G/H \hookrightarrow X$, we define
$$\mathscr{F}_X = \{(\mathcal{C}_Y(X), \Delta_Y(X)) \mid Y \subseteq X \text{ a } G\text{-orbit}\}.$$

**Remark 3.3.23.** Colored cones and colored fans do not have all the combinatorial properties that cones and fans do in toric geometry. More specifically, there are two important pathologies that are impossible with regular (i.e. non-colored) cones and fans.

1. Given a colored cone $(\mathcal{C}, \Delta)$ and any face $\sigma$ of $\mathcal{C}$, it may be the case that there is no face of the colored cone whose cone is $\sigma$. By definition, such a face would have to be $(\sigma, \Delta')$, where $\Delta' = \sigma \cap \varphi^{-1}(\Delta)$. But $(\sigma, \Delta')$ might not be a colored cone, because it might be the case that $\sigma^\circ \cap \mathcal{V}(G/H) = \varnothing$. This will happen, for instance, if $\sigma$ is an extremal ray of $\mathcal{C}$ generated by some element of $\varphi(\Delta)$. Such rays are not always extremal, but they can be; see e.g. [Pez10, Example 2.5.3] for an example.

2. Given two colored cones $(\mathcal{C}_1, \Delta_1)$ and $(\mathcal{C}_2, \Delta_2)$ in the same colored fan $\mathscr{F}$, it is not necessarily the case that the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ is a face of both $\mathcal{C}_1$ and $\mathcal{C}_2$ (and even if it is, it may not be the case that that $\mathcal{C}_1 \cap \mathcal{C}_2$ defines a face of the colored cones $(\mathcal{C}_1, \Delta_1)$ and $(\mathcal{C}_2, \Delta_2)$, as explained above). For an example, see [Pez10, Example 2.5.5] (which gives a picture of the image of $\varphi$ for some choice of $G/H$, and a suitable colored fan can be constructed from this picture).

Note that these two pathologies can never occur in the toric case. This is due to the fact that $\mathcal{V}(T)$ is all of $N(T)$ when $G/H = T$ (see Lemma 3.3.7b). In particular, this implies that every face of a cone contains an element of $\mathcal{V}(T)$ in its relative interior (which implies that pathology 1 cannot happen) and for any $(C_1, \Delta_1)$ and $(\mathcal{C}_2, \Delta_2)$ in $\mathscr{F}$, we must have $\mathcal{C}_1^\circ \cap \mathcal{C}_2^\circ = \varnothing$ (which implies that pathology 2 cannot happen).

There is another important case in which colored cones and colored fans behave like cones and fans from toric geometry: namely, the case where $\Delta = \varnothing$ for every $(\mathcal{C}, \Delta) \in \mathscr{F}$. Such colored fans correspond to a special class of spherical varieties called *toroidal* varieties. As we will see in Theorem 3.5.9, toroidal varieties are related to toric varieties in a very interesting and useful way.

The combinatorial condition of being a face of another colored cone has a useful scheme-theoretic interpretation. An analogous fact is also essential in the toric case (see Theorem 3.3.11a).

**Proposition 3.3.24** ([Kno91, Lemma 3.2])**.** *Let $G/H \hookrightarrow X$ be an embedding, and let $Y \subseteq X$ be a $G$-orbit. The map $Z \mapsto (\mathcal{C}_Z, \Delta_Z)$ is a bijection between $G$-orbits of $X$ whose closure contains $Y$ and faces of the colored cone $(\mathcal{C}_Y, \Delta_Y)$.*

The above proposition is the main ingredient in "gluing" our classification in the case where $X$ is simple. The other main ingredients are Proposition 3.3.21, which is used to prove that the scheme we get from gluing is separated, and the following fact (which is an analog of Lemma 3.3.9d from the toric case).

**Lemma 3.3.25** (cf. Lemma 3.3.9)**.** *Let $X$ be a spherical $G$-variety, and let $Y \subset X$ be a $G$-orbit. Then, $GX_{B,Y}$ is the unique $G$-stable open subset of $X$ which is simple with unique closed orbit $Y$.*

*Proof.* That $GX_{B,Y}$ is such an open subset follows immediately from Theorem 3.2.7. Conversely, if $U \subset X$ is such a subset, then for any $G$-orbit $Z$ with $Y \subset \overline{Z}$, we must have

$Z \cap U \neq \varnothing$ (by definition of the closure) and hence $Z \subset U$ because $U$ is $G$-stable. On the other hand, for any $G$-orbit $Z \subset U$, since $Y$ is the unique closed $G$-orbit of $U$, we have $Y \subset \overline{Z}$. We conclude that $U$ is the union of all the $G$-orbits whose closures contain $Y$, which is precisely $GX_{B,Y}$ by Theorem 3.2.7. □

Bringing together all of our above tools (plus a few more scheme-theoretic technicalities), one arrives at the following classification of all embeddings of homogeneous spherical varieties.

**Theorem 3.3.26** ([Kno91, Theorem 3.3])**.** *Let $G/H$ be a spherical variety. The map $(G/H \hookrightarrow X) \mapsto \mathscr{F}_X$ is a bijection between $G$-isomorphism classes of embeddings of $G/H$ and strictly convex colored fans for $G/H$.*

*sketch of proof.* Let $G/H \hookrightarrow X$ be an embedding. We first prove that $\mathscr{F}_X$ is a strictly convex colored fan. Proposition 3.3.24 implies that every face of a colored cone in $\mathscr{F}_X$ is again a colored cone in $\mathscr{F}_X$. For any $v \in \mathcal{V}(G/H)$, suppose that $v \in \mathcal{C}_{Y_1}(X)^\circ \cap \mathcal{C}_{Y_2}(X)^\circ$ for some $G$-orbits $Y_1, Y_2 \subset X$. Proposition 3.3.21 (applied to $G \cdot X_{B,Y_1}$ and $G \cdot X_{B,Y_2}$) implies that $\overline{Y_1}$ and $\overline{Y_2}$ are both centers for $v$ on $X$. Because $X$ is separated, we must have $\overline{Y_1} = \overline{Y_2}$ and hence $Y_1 = Y_2$. This proves that $\mathscr{F}_X$ is a colored fan, and each colored cone in it is strictly convex by Theorem 3.3.20.

To construct an inverse to the map $X \mapsto \mathscr{F}_X$, let $\mathscr{F} = \{(\mathcal{C}_i, \Delta_i)\}_i$ be a strictly convex colored fan. Every colored cone $(\mathcal{C}_i, \Delta_i)$ corresponds (via Theorem 3.3.20) to a simple embedding $G/H \hookrightarrow X_i$, and for any $i$ and $j$, Proposition 3.3.24 tells us that the faces common to both $(\mathcal{C}_i, \Delta_i)$ and $(\mathcal{C}_j, \Delta_j)$ correspond to $G$-orbits $Z_1 \subset X_1$ and $Z_2 \subset X_2$ such that $G \cdot X_{B,Z_1} \cong G \cdot X_{B,Z_2}$ (because both $G \cdot X_{B,Z_1}$ and $G \cdot X_{B,Z_2}$ are simple and have the same colored cone, namely, a face of both $(\mathcal{C}_i, \Delta_i)$ and $(\mathcal{C}_j, \Delta_j)$). We glue $X_i$ and $X_j$ along the open subsets $(X_1)_{B,Z_1}$ and $(X_2)_{B,Z_2}$ for every face common to $(\mathcal{C}_i, \Delta_i)$ and $(\mathcal{C}_j, \Delta_j)$. After applying this gluing to all the $X_i$ and $X_j$, we obtain a scheme $X_{\mathscr{F}}$ equipped with a $G$-action.

Note that each $X_i$ and $X_j$ will be glued along their open $G$-orbits (which correspond to the common face $(0, \varnothing)$). In particular, $X_{\mathscr{F}}$ is connected, and we have an embedding $G/H \hookrightarrow X_{\mathscr{F}}$. Moreover, $X_{\mathscr{F}}$ is integral, normal, and has finitely many $B$-orbits (because there are finitely many $X_i$, and they all have these properties). To check that $X_{\mathscr{F}}$ is spherical, then, we just need to check that this scheme is separated. Since this is a local question, it suffices to consider the case where $X$ is a gluing of two simple spherical varieties $X_1$ and $X_2$. For the rest of the proof of separatedness, which ultimately boils down to applying Proposition 3.3.21, see [Kno91, Theorem 3.3].

It remains to check that the map $\mathscr{F} \mapsto X_{\mathscr{F}}$ is an inverse to the map $(G/H \hookrightarrow X) \mapsto \mathscr{F}_X$. First, start with an embedding $G/H \hookrightarrow X$, take its fan $\mathscr{F}_X$, and then glue to get $X_{\mathscr{F}_X}$. The $X_i$ in our construction of $X_{\mathscr{F}_X}$ are the simple spherical $G$-varieties $G \cdot X_{B,Y}$ for any $G$-orbit $Y \subset X$. For any two $G$-orbits $Y_1, Y_2 \subset X$, the intersection $G \cdot X_{B,Y_1} \cap G \cdot X_{B,Y_2}$ in $X$ is a union of subsets of the form $G \cdot X_{B,Z}$ (namely, the union over all $Z$ such that $Y_1, Y_2 \subset \overline{Z}$). These are exactly the subsets that $X_1 = G \cdot X_{B,Y_1}$ and $X_2 = G \cdot X_{B,Y_2}$ are glued on in our construction of $X_{\mathscr{F}_X}$. So, $X$ can be obtained by gluing the $G \cdot X_{B,Y}$ in in exactly the same way as the construction of $X_{\mathscr{F}_X}$, which implies that $X \cong X_{\mathscr{F}_X}$.

Now, take any colored fan $\mathscr{G} = \{(\mathcal{C}_i, \Delta_i)\}_i$, and let $X = X_{\mathscr{G}}$. If $G/H \hookrightarrow X_i$ is the simple embedding corresponding to $(\mathcal{C}_i, \Delta_i)$ for each $i$, then by construction, $X$ is covered by the $X_i$. If $Y_i$ is the unique closed $G$-orbit of $X_i$, then $G \cdot X_{B,Y_i} = X_i$ by Lemma 3.3.25. It follows that $(\mathcal{C}_i, \Delta_i) = (\mathcal{C}_{Y_i}, \Delta_{Y_i})$. On the other hand, for any $G$-orbit $Y \subset X$, we have $Y \subset X_i$ for some $i$ and hence $Y_i \subset \overline{Y}$. Proposition 3.3.24 implies that $(\mathcal{C}_Y, \Delta_Y)$ is a face of $(\mathcal{C}_{Y_i}, \Delta_{Y_i}) \in \mathscr{G}$, so we have $(\mathcal{C}_Y, \Delta_Y) \in \mathscr{G}$. In summary, we have

$$\mathscr{G} = \{(\mathcal{C}_i, \Delta_i)\}_i = \{(\mathcal{C}_Y, \Delta_Y) \mid Y \subset X \text{ a } G\text{-orbit}\} = \mathscr{F}_X,$$

which is what we needed to prove. □

### 3.3.c Classifying Morphisms

We have now classified embeddings $G/H \hookrightarrow X$ for any fixed homogeneous spherical variety $G/H$ in terms of certain combinatorial invariants (namely, colored fans). In this section, we describe dominant morphisms embeddings in terms of the same combinatorial invariants. By a "dominant morphism of embeddings," we mean a commutative diagrams of the form

$$\begin{tikzcd} G/H \arrow[r, hook] \arrow[d, "\pi"] & X \arrow[d, "f"] \\ G/H' \arrow[r, hook] & X' \end{tikzcd}$$

where all maps are $G$-equivariant and $\pi$ (hence also $f$) is dominant. Note that giving an embedding $G/H \hookrightarrow X$ is equivalent to giving a spherical variety $X$ whose open $G$-orbit is $G/H$. Under this interpretation, a "dominant morphism of embeddings" is nothing more than a $G$-equivariant dominant morphism $X \to X'$ for any two spherical $G$-varieties $X$ and $X'$.

Let $H, H' \subset G$ be two spherical subgroups, and let $\pi : G/H \to G/H'$ be a dominant $G$-equivariant morphism. Then, $\pi$ maps the generic point of $G/H$ to the generic point of $G/H'$ and so induces an inclusion $\iota : K(G/H') \hookrightarrow K(G/H)$. Because $\varphi$ is $G$-equivariant, the inclusion $\iota$ is a $G$-equivariant map of $G$-modules, so $\iota$ induces an injection

$$\pi^* : \Lambda(G/H') \hookrightarrow \Lambda(G/H).$$

Applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q})$ to this map gives us a surjection

$$\pi_* : N(G/H) \to N(G/H').$$

Now, for any $v \in \mathcal{V}(G/H)$, it follows from the definition of $\pi_*$ that $\pi_*(v)$ is an element of $\mathcal{V}(G/H')$. On the other hand, any element $v' \in \mathcal{V}(G/H')$ lifts to an element of $\mathcal{V}(G)$

([Kno91, Corollary 1.5]), which in turn restricts to a valuation $v \in \mathcal{V}(G/H)$ (via the inclusion $K(G/H) \hookrightarrow K(G)$ induced by the quotient map $G \to G/H$), and one can check that $\pi_*(v) = v'$. So, we see that

$$\pi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H').$$

As for divisors, note that for any $D \in \mathcal{D}(G/H)$, the image $\pi_*(\varphi_D)$ takes values in $\mathbb{Z}$ (because $v_D$ does). So, one of two possibilities must occur: either $\pi_*(\varphi_D)$ is the 0 valuation, in which case the generic point of $D$ maps to the generic point of $G/H'$, i.e. the composition $D \hookrightarrow G/H \xrightarrow{\pi} G/H'$ is dominant; or $\pi_*(\varphi_D)$ is a nonzero discrete valuation, in which case $\pi$ maps $D$ onto a prime Weil divisor $D' \in \mathcal{D}(G/H')$. So, writing

$$\Delta_\pi = \{D \in \mathcal{D}(G/H) \mid \pi|_D : D \to G/H' \text{ is dominant}\},$$

we obtain a map

$$\pi_* : \mathcal{D}(G/H) \setminus \Delta_\pi \to \mathcal{D}(G/H').$$

Our goal is to combine all these induced maps to understand how a morphism from $G/H \hookrightarrow X$ to $G/H' \hookrightarrow X'$ should behave on combinatorial data. We now state the relevant definitions and results.

**Definition 3.3.27.** Let $\pi : G/H \to G/H'$ be a dominant $G$-equivariant morphism.

1. Let $(\mathcal{C}, \Delta)$ (respectively, $(\mathcal{C}', \Delta')$) be a colored cone for $G/H$ (respectively, $G/H'$). We say that $\pi$ *maps* $(\mathcal{C}, \Delta)$ *to* $(\mathcal{C}', \Delta')$ if
   a) the map $\pi_* : N(G/H) \to N(G/H')$ satisfies $\pi_*(\mathcal{C}) \subseteq \mathcal{C}'$, and
   b) the map $\pi_* : \mathcal{D}(G/H) \setminus \Delta_\pi \to \mathcal{D}(G/H')$ satisfies $\pi_*(\Delta \setminus \Delta_\pi) \subseteq \Delta'$.
2. Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a colored fan for $G/H$ (resp. $G/H'$). We say that $\varphi$ *maps* $\mathscr{F}$ *to* $\mathscr{F}'$ if every $\varphi$ maps every colored cone in $\mathscr{F}$ to some colored cone in $\mathscr{F}'$.
3. Let $\mathscr{F}$ be a colored fan for $G/H$. We define the *support of* $\mathscr{F}$ to be

$$\operatorname{Supp}(\mathscr{F}) = \mathcal{V}(G/H) \cap \left( \bigcup_{(\mathcal{C},\Delta)\in\mathscr{F}} \mathcal{C} \right).$$

**Theorem 3.3.28** ([Kno91, Theorems 4.1 and 4.2], [Bri97, Section 3.4, Theorem 2])**.** *Let $G/H$ and $G/H$ be homogeneous spherical varieties, let $G/H \hookrightarrow X$ and $G/H' \hookrightarrow X'$ be embeddings, and let $\pi : G/H \to G/H'$ be a dominant $G$-equivariant morphism.*

*(a) $\pi$ extends to a $G$-equivariant morphism $f : X \to X'$ if and only if $\pi$ maps $\mathscr{F}_X$ to $\mathscr{F}(X')$.*

*(b) Suppose that $f : X \to X'$ is a dominant $G$-equivariant morphism extending $\pi$. Then, $f$ is proper if and only if*

$$\operatorname{Supp}(\mathscr{F}_X) = \pi_*^{-1}(\operatorname{Supp}(\mathscr{F}(X'))).$$

*In particular, any spherical variety $X$ is complete if and only if* $\operatorname{Supp}(\mathscr{F}_X) = \mathcal{V}(G/H)$.

## 3.4 The Valuation Cone

For any homogeneous spherical variety $G/H$, the set $\mathcal{V}(G/H)$ of $G$-invariant valuations plays a large role in the Luna–Vust theory of $G/H$. It is thus natural to ask: what is the structure of $\mathcal{V}(G/H)$? Since the set $\mathcal{V}(G/H)$ is a $G$-equivariant birational invariant of $G/H$ and the open $G$-orbit of any spherical variety has the form $G/H$, this is the same as asking: what is the structure of $\mathcal{V}(X)$ for any spherical variety $X$? In this section, we discuss a few important results in the literature that answer this question.

To start with, it turns out that $\mathcal{V}(G/H)$ is a cone in the vector space $N(G/H)$. There are two main ways to prove this. The first is very geometric: given any two $G$-invariant valuations $v_1, v_2 : K(X)^\times \to \mathbb{Q}$ and any $a_1, a_2 \in \mathbb{Q}_{\geq 0}$, one can use Luna–Vust theory to explicitly construct an embedding $G/H \hookrightarrow X$ such that $a_1 v_1 + a_2 v_2$ has center on a closed $G$-orbit of $X$ (and hence must be a $G$-invariant valuation, since its center is $G$-stable). For a proof using this approach, see [Bri97, Theorem 4.1].

Alternately, one can show that $\mathcal{V}(G/H)$ is a cone in a more representation-theoretic way, by obtaining an explicit description of its dual cone. For our purposes, this description of the dual cone will not be needed; we are only interested in the properties of $\mathcal{V}(G/H)$ that arise from this description, which are stated in Theorem 3.4.1 below. However, this description of the dual cone is common in certain parts of the literature, so we explain it briefly here.

Consider the left regular representation $\Gamma(G, \mathcal{O}_G)$, i.e. the $G$-module structure on $\Gamma(G, \mathcal{O}_G)$ coming from the action of $G$ on itself by left multiplication. Recall that, by the classification of simple $G$-modules (Theorem 2.3.6), any simple $G$-module is isomorphic to $V(\mu)$ for some dominant weight $\mu \in \Lambda_G^+$. So, let $V(\mu_1)$, $V(\mu_2)$, and $V(\lambda)$ be three simple $G$-submodules of the left regular representation $\Gamma(G, \mathcal{O}_G)$. We say that the weight $\mu_1 + \mu_2 - \lambda \in \Lambda_G$ is a *tail* of $G/H$ if

1. every element of $V(\mu_1)$ and $V(\mu_2)$ is an eigenvector for $H$ under the action of $H$ on $\Gamma(G, \mathcal{O}_G)$ given by right multiplication, and
2. $V(\lambda) \subset V(\mu_1)V(\mu_2)$, where $V(\mu_1)V(\mu_2)$ is the $G$-submodule of $\Gamma(G, \mathcal{O}_G)$ generated by elements of the form $f_1 f_2$ for any $f_1 \in V(\mu_1)$ and $f_2 \in V(\mu_2)$.

We claim that any tail $\mu_1 + \mu_2 - \lambda$ is an element of $\Lambda(G/H)$. For this, let $f_1 \in V(\mu_1)$, $f_2 \in V(\mu_2)$, and $f \in V(\lambda)$ be $B$-eigenvectors of weights $\mu_1$, $\mu_2$, and $\mu$ (respectively). Since $K(G/H) = K(G)^H$ (see e.g. [Bor91, Proposition 6.5, Theorem 6.8]), it will suffice to show that the element $f_1 f_2 f^{-1} \in K(G)$ is fixed by $H$. We have $f \in V(\mu_1)V(\mu_2)$, so we can write $f = \sum_m (g_{m,1} f_1)(g_{m,2} f_2)$ for some $g_{m,i} \in G$, and we may take the summands $(g_{m,1} f_1)(g_{m,2} f_2)$ to be linearly independent over $k$. By assumption, $f_1$, $f_2$, and $f$ are all eigenvectors of $H$; let $\eta_1$, $\eta_2$, and $\eta$ (respectively) be the corresponding characters of $H$. For any $h \in H$, we have

$$0 = h \cdot f - \eta(h) f = (\eta_1(h)\eta_2(h) - \eta(h)) \sum_m (g_{m,1} f_1)(g_{m,2} f_2).$$

(Here we have used the fact that $G$ acts by left multiplication while $H$ acts by right multiplication, so the two actions commute.) Since the $(g_{m,1} f_1)(g_{m,2} f_2)$ are linearly independent, we conclude that $\eta_1(h)\eta_2(h) = \eta(h)$ for all $h$. Since $H$ acts on $f_1 f_2$ by $\eta_1\eta_2$ and on $f$ by $\eta$, it follows that $H$ fixes $f_1 f_2 f^{-1}$, as desired.

Now, let $\tau(G/H)$ be the set of all tails of $G/H$. By the above claim, we have $\tau(G/H) \subset \Lambda(G/H)$, so we can consider the cone in $N(G/H)^\vee$ generated by $\tau(G/H)$. It turns out that we can describe $\mathcal{V}(G/H)$ in terms of this cone.

**Theorem 3.4.1** ([Bri97, Proposition 4.2 and Corollary]; cf. [Kno91, Lemma 5.1 and Corollary 5.3])**.** *Let $G/H$ be a homogeneous spherical variety.*

*(a) We have*

$$\mathcal{V}(G/H) = \{v \in N(G/H) \mid v(t) \leq 0 \ \forall \ t \in \tau(G/H)\}.$$

*In particular, $\mathcal{V}(G/H)$ is a convex cone whose dual is the cone generated by $-\tau(G/H)$, and the linear part of $\mathcal{V}(G/H)$ is*

$$\mathcal{V}(G/H) \cap (-\mathcal{V}(G/H)) = \tau(G/H)^\perp.$$

*(b) Let*

$$\mathcal{W}_{G/H} = \{v \in N(G/H) \mid v(\alpha) \leq 0 \ \textit{ for all positive roots } \alpha\}$$

*be the image of the antidominant Weyl chamber of $G$ under the projection map $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_G, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda(G/H), \mathbb{Q}) = N(G/H)$. The valuation cone $\mathcal{V}(G/H)$ contains $\mathcal{W}_{G/H}$. In particular, $\mathcal{V}(G/H)$ is a full-dimensional cone in $N(G/H)$.*

*sketch of proof.* The containment $\mathcal{V}(G/H) \subset (-\tau(G/H))^\vee$ is purely algebraic: for any $G$-invariant valuation $v : K(G/H)^\times \to \mathbb{Q}$ and any tail $\mu_1 - \mu_2 + \lambda$, one can take $f_1$, $f_2$, and $f = \sum_m (g_{m,1} f_1)(g_{m,2} f_2)$ as above and note that

$$v(f f_1^{-1} f_2^{-1}) \geq \min_m \left\{ v\left( \frac{(g_{m,1} f_1)(g_{m,2} f_2)}{f_1 f_2} \right) \right\} = \min_m \{v(1)\} = 0.$$

It follows that $v(f_1 f_2 f^{-1}) \leq 0$, which implies that $v(\mu_1 - \mu_2 + \lambda) \leq 0$ (here viewing $v$ as an element of $N(G/H)$ in the usual way, using the injection $\varphi : \mathcal{V}(G/H) \to N(G/H)$).

The containment $(-\tau(G/H))^\vee \subset \mathcal{V}(G/H)$ is more involved. Given an element $v \in (-\tau(G/H))^\vee$, one constructs a simple spherical variety $X$ with unique closed $G$-orbit $Y \subset X$ and shows that $v = cv'$, where $v'$ is a valuation whose center on $X$ is $Y$. The valuation $v'$ is $G$-invariant because $Y$ is $G$-stable, so this implies that $v \in \mathcal{V}(G/H)$. The construction of $X$ here is essentially the construction in the proof of Theorem 3.3.20 in a special case.

Finally, statement (b) follows directly from (a) using some general facts about simple roots in the root datum of $G$. Note that by "the image of the antiominant Weyl chamber," we mean the following. By definition, the antidominant Weyl chamber $-W(B,T)$ is a subset of $(\Lambda_G)^\vee_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Q}}(\Lambda_G, \mathbb{Z})$. The inclusion $\Lambda(G/H) \hookrightarrow \Lambda_G$ induces a surjection $\rho : (\Lambda_G)^\vee_{\mathbb{Q}} \to \Lambda(G/H)^\vee)_{\mathbb{Q}} = N(G/H)$. The set $\mathcal{W}_{G/H}$ defined in the theorem is precisely the image of $-W(B,T)$ under the map $\rho$. □

Now that we know $\mathcal{V}(G/H)$ is a cone, we can use a construction of a "nice" embedding $G/H \hookrightarrow X$ along with Luna–Vust theory to show that this cone is in fact polyhedral. We defer the construction of this "nice" embedding until Section 3.5, since it relates more closely to the material in that section.

**Proposition 3.4.2.** *The cone $\mathcal{V}(G/H)$ is polyhedral.*

*Proof.* By Proposition 3.5.8 below, there exists a complete embedding of spherical $G$-varieties $G/H \hookrightarrow X$ such that for any colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}_X$, we have $\Delta = \varnothing$. By definition of a colored cone, this implies that $\mathcal{C} \subset \mathcal{V}(X) = \mathcal{V}(G/H)$ for every $(\mathcal{C}, \Delta) \in \mathscr{F}_X$. On the other hand, since $X$ is complete, Theorem 3.3.28 gives us

$$\mathcal{V}(G/H) = \bigcup_{(\mathcal{C},\Delta)\in\mathscr{F}_X} \mathcal{C}.$$

Each of the cones $\mathcal{C}$ is polyhedral (by definition of a colored cone), and there are finitely many of them (by definition of a colored fan), so the above equation implies that $\mathcal{V}(G/H)$ is polyhedral as well. $\square$

It turns out that the cone $\mathcal{V}(G/H)$ is actually a Weyl chamber of a certain root system, which yields further interesting properties of $\mathcal{V}(G/H)$. This was first proven in the characteristic 0 case by Brion [Bri90]. Later, Knop gave a more geometric construction in [Kno94]. Knop has also generalized the statement to the case where $\mathrm{char}(k) \neq 2$ (see [Kno14b]). For our purposes, we will not need to consider this root system; instead, we just need the equation for $\mathcal{V}(G/H)$ in the following theorem, which follows from the description of $\mathcal{V}(G/H)$ as a Weyl chamber.

**Theorem 3.4.3** ([Bri90, Theorem 3.5]; cf. [Kno94, Theorem 1.3, 7.4])**.** *There exists a root system $(V, R)$ with $V = \Gamma(G/H)_{\mathbb{Q}}$ and a base $\Pi$ of $(V, R)$ such that, if $\gamma_1, \ldots, \gamma_r$ are the simple roots of $(V, R)$, then we have*

$$\mathcal{V}(G/H) = \{v \in N(G/H) \mid v(\gamma_i) \leq 0 \ \forall \ i\}.$$

*In other words, $\mathcal{V}(G/H)$ is the antidominant Weyl chamber with respect to $\Pi$.*

**Remark 3.4.4.** In the literature, the root system in the above theorem is often described in terms of its Weyl group $W_{G/H}$, which is called the *little Weyl group* of $G/H$. Also, since $\mathcal{V}(G/H)$ is a $G$-equivariant birational invariant of $G/H$, so is the root system $(V, R)$ in the above theorem and the simple roots $\gamma_1, \ldots, \gamma_r$.

**Remark 3.4.5.** The version of Theorem 3.4.3 proven by Knop in [Kno94] actually applies to non-spherical varieties as well. In fact, many of the results about the valuation cone in this section can be generalized to the non-spherical case; see [Tim11, Section 20-22] for details. For certain of these general results, one has to restrict the discussion to special types of valuations, called *geometric valuations* and *central valuations*. However, it turns out that in the spherical case, every valuation is both geometric and central (see e.g. [Kno94, Theorem 7.2]), so these distinctions do not appear in the theory of spherical varieties.

The characterization of the valuation cone in Theorem 3.4.3 will be instrumental to the classification of spherical varieties. As such, we introduce some terminology to refer to this characterization.

**Definition 3.4.6.** The simple roots $\gamma_1, \ldots, \gamma_r$ in Theorem 3.4.3 are called the *spherical roots* of $G/H$. More generally, we define the spherical roots of any spherical variety $X$ to be the spherical roots of the open $G$-orbit of $X$. We denote the set of spherical roots of $X$ by $\Psi_{G,X}$. (or simply by $\Psi_X$ when the group $G$ is clear from context).

**Remark 3.4.7.** The set of spherical roots $\Psi_{G,X}$ can also be defined without referring to open $G$-orbits or to root systems. Indeed, it follows from Theorem 3.4.3 that $\Psi_{G,X}$ is the unique set of minimal generators of $-\mathcal{V}(X)^\vee$ which are all indivisible elements of the lattice $\Lambda(X)$. Such a set of minimal generators must exist for any polyhedral cone, so this is not a particularly exciting definition on its own. However, since the spherical roots are simple roots of a root system, the set $\Psi_{G,X}$ is linearly independent in the vector space $\Lambda(G/H)_\mathbb{Q}$. In other words, the cone $-\mathcal{V}(X)^\vee$ is a so-called *simplicial* cone. The spherical roots also have many nice combinatorial properties, as we will see in Section 3.6.

**Remark 3.4.8.** Theorem 3.4.1 states that $\mathcal{V}(G/H)$ contains the image $\mathcal{W}_{G/H}$ of the antidominant Weyl chamber in $N(G/H)$. Passing to dual cones, we see that $-\mathcal{V}(G/H)^\vee \subset -\mathcal{W}_{G/H}^\vee$. By definition, the cone $-\mathcal{W}_{G/H}^\vee$ is the intersection with $\Lambda(G/H)_\mathbb{Q}$ of the cone in $\Lambda_G \otimes_\mathbb{Z} \mathbb{Q}$ generated by the simple roots of $G$. It follows that every spherical root of $G/H$ is a sum of simple roots with nonnegative (integer) coefficients.

Since Luna–Vust theory deals exclusively with strictly convex polyhedral cones, it is natural to ask: when is the polyhedral cone $\mathcal{V}(G/H)$ strictly convex? It turns out that the linear part of $\mathcal{V}(G/H)$, i.e. the intersection $\mathcal{V}(G/H) \cap (-\mathcal{V}(G/H)$, is related to the normalizer subgroup $N_G(H) \subset G$ in a precise way. To give a litle more context for the precise statement, we note that Theorem 3.4.1 above gives us $\mathcal{V}(G/H) \cap (-\mathcal{V}(G/H)) = \tau(G/H)^\perp$.

**Theorem 3.4.9** ([Bri97, Theorem 4.3])**.** *Let $G/H$ be a homogeneous spherical $G$-variety.*

*(a) Let $\langle\tau\rangle$ be the subgroup of $\Lambda(G/H)$ generated by the set of tails $\tau(G/H)$. Then, there exists a canonical isomorphism of algebraic groups*

$$N_G(H)/H \cong \mathrm{Hom}(\Lambda(G/H)/\langle\tau\rangle, \mathbb{G}_m).$$

*In particular, $N_G(H)/H$ is diagonalizable, and* $\dim(N_G(H)/H)$ *is the dimension of the linear part $\tau(G/H)^\perp$ of $\mathcal{V}(G/H)$.*

*(b) The isomorphism of part (a) induces an exact sequence*

$$0 \to \tau(G/H)^\perp \to N(G/H) \to N(G/N_G(H)) \to 0,$$

*which in turn induces an exact sequence*

$$0 \to \tau(G/H)^\perp \to \mathcal{V}(G/H) \to \mathcal{V}(G/N_G(H)) \to 0.$$

*(c) We have $N_G(H) = N_G(H^\circ)$, where $H^\circ \subset H$ is the connected component of the identity.*

**Corollary 3.4.10** ([Pez10, Corollary 3.2.1]; [Bri97, Corollary 4.3])**.** *Let $G/H$ be a homogeneous spherical $G$-variety.*

*(a) The cone $\mathcal{V}(G/H)$ is strictly convex if and only if $N_G(H)/H$ is a finite group.*

*(b) The cone $\mathcal{V}(G/H)$ is a vector space (hence is equal to $N(G/H)$, because $\mathcal{V}(G/H)$ is full-dimensional) if and only if $H$ contains a maximal unipotent subgroup of $G$.*

The type of spherical variety described in part (b) of the above corollary has a particularly nice geometry. We now give such spherical varieties a name.

**Definition 3.4.11.** We say that a spherical homogeneous variety $G/H$ is *horospherical* if $H$ contains a maximal unipotent subgroup. We say that a spherical variety $X$ is horospherical if its open $G$-orbit is (or equivalently, if $\mathcal{V}(X) = N(X)$, see Corollary 3.4.10 above).

**Remark 3.4.12.** By some general facts about algebraic groups, it turns out that $X$ is horospherical if and only if the stabilizer of every point in $X$ contains a maximal unipotent subgroup (see [Tim11, Remark 7.2]).

**Example 3.4.13.** Consider the case of toric varieties, where $G = T$. Let $X$ be a toric $T$-variety. The only unipotent subgroup of $T$ is $\{0\}$, which is certainly contained in the stabilizer of every point in $X$, so $X$ is horospherical. Thus, Corollary 3.4.10 implies that every element of $N(T) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda_T, \mathbb{Q})$ comes from a $T$-invariant valuation. This confirms what we have already proven explicitly for toric varieties in Lemma 3.3.7b.

The theory of horospherical varieties is quite interesting in its own right; for a brief overview of the theory, see [Tim11, Sections 7 and 28]. For our purposes, we will think of them mainly as a nice class of spherical varieties corresponding to one extreme case for $\mathcal{V}(X)$: namely, the case where $\mathcal{V}(G/H)$ contains every line in $N(G/H)$. The other extreme, where $\mathcal{V}(G/H)$ is strictly convex and so contains no line in $N(G/H)$, gives rise to another nice type of spherical variety, which we will study in Section 3.5.b.

## 3.5 Special Types of Spherical Varieties

In this section, we discuss a few important "nice" types of spherical varieties. Using the theory we've built up so far, we will be able to say a lot about the geometry of these spherical varieties. These "nice" types of spherical varieties play a crucial role in the classification of spherical varieties, because many proofs about the behavior of combinatorial data on spherical varieties can be reduced to the case where the spherical varieties in question are one of the "nice" types discussed here. We will see this sort of reduction to the "nice" case repeatedly in Section 3.6.

### 3.5.a Toroidal Varieties

**Definition 3.5.1.** Let $X$ be a spherical $G$-variety. We say that $X$ is *toroidal* if no color of $X$ contains a $G$-orbit (or equivalently, if every $B$-divisor of $X$ that contains a $G$-orbit is $G$-stable).

**Remark 3.5.2.** If $G/H \hookrightarrow X$ is an embedding with corresponding colored fan $\mathscr{F}_X$, it follows immediately from the definition of $\mathscr{F}_X$ that $X$ is toroidal if and only if for every colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}$, we have $\Delta = \varnothing$.

**Example 3.5.3.** For the toric case, where $G = B = T$, we note that any toric variety is automatically toroidal, as every $B$-divisor is $G$-stable.

**Example 3.5.4.** Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ for some $e > 0$, and consider the ruled surface $Y = \mathbb{P}(\mathcal{E})$. Let $\pi : Y \to \mathbb{P}^1$ be the structure morphism, and let $G = \mathrm{SL}_2$ act on $\mathbb{P}^1$ by the action of Example 2.4.19, which is given in coordinates by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

We take $B \subset G$ to be the Borel subgroup of upper triangular matrices and $T \subset B$ to be the subgroup of diagonal matrices. We have a canonical $G$-linearization of $\mathcal{O}_{\mathbb{P}^1}$, and Example 2.4.19 gives us a $G$-linearization of $\mathcal{O}_{\mathbb{P}^1}(1)$, which induces a $G$-linearization on $\mathcal{O}_{\mathbb{P}^1}(-e)$. One can show that these $G$-linearizations on $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-e)$ induce a $G$-action on $Y$ such that the structure morphism $\pi : Y \to \mathbb{P}^1$ is $G$-equivariant (cf. Proposition 2.4.17).

We intend to show that $Y$ is a toroidal variety. To do this, we need to compute all the $B$-divisors of $Y$ as well as its $G$-orbits. First, the fiber $C = \pi^{-1}([1 : 0]) \cong \mathbb{P}^1$ is $B$-stable but not $G$-stable (because $[1 : 0]$ is fixed by $B$ but not by $G$), so $C$ is a color of $Y$. Every other $B$-divisor lies in the preimage under $\pi$ of $\mathbb{A}^1 = \mathbb{P}^1 \setminus [1 : 0]$. Since $\mathcal{E}|_{\mathbb{A}^1}$ is trivial, we get $\pi^{-1}(\mathbb{A}^1) \cong \mathbb{A}^1 \times \mathbb{P}^1$. More specifically, $\mathcal{O}_{\mathbb{P}^1}|_{\mathbb{A}^1}$ is generated over $\mathcal{O}_{\mathbb{A}^1}$ by the section 1, which is fixed by $G$, and $\mathcal{O}_{\mathbb{P}^1}(-e)|_{\mathbb{A}^1}$ is generated over $\mathcal{O}_{\mathbb{A}^1}$ by a $B$-eigenvector of weight $-e$. It follows that under the isomorphism $\pi^{-1}(\mathbb{A}^1) \cong \mathbb{A}^1 \times \mathbb{P}^1$, the action of $B$ on $\mathbb{P}^1$ is given by

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot [w : z] = [w : t^{-e} z].$$

There are thus two other $B$-divisors of $Y$, namely

$$E_1 = \overline{\mathbb{A}^1 \times [1 : 0]}, \quad E_2 = \overline{\mathbb{A}^1 \times [0 : 1]}.$$

Note that $\mathbb{A}^1 \times [1 : 0]$ and $\mathbb{A}^1 \times [0 : 1]$ are the images of sections $s_1, s_2 : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ of $\pi|_{\pi^{-1}(\mathbb{A}^1)}$. By the valuative criterion of properness, $s_1$ and $s_2$ extend to sections $\mathbb{P}^1 \to Y$ of $\pi$, and the images of these extensions are necessarily $E_1$ and $E_2$ (respectively). It follows that $E_1$ and $E_2$ are isomorphic to $\mathbb{P}^1$ as $G$-varieties (because $\pi$ is $G$-equivariant). In particular,

$E_1$ and $E_2$ are $G$-divisors of $Y$, and $E_1$ and $E_2$ are $G$-orbits of $Y$ (because $\mathbb{P}^1$ is a single $G$-orbit).

Finally, one can check that $Y_B^\circ = \mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{[1:0],[0:1]\})$ is a single $B$-orbit and is open in $Y$. So, $Y$ is spherical, and since $Y = C \cup E_1 \cup E_2 \cup Y_B^\circ$, we have found all the $B$-divisors of $Y$. Moreover, the $G$-orbits of $Y$ are the open orbit, which is certainly not contained in a color, and the two $G$-divisors $E_1$ and $E_2$, which are not contained in the unique color $C$. So, $Y$ is a toroidal variety by definition.

As the name suggests, toroidal varieties are related to toric varieties. We will make this relationship precise in Theorem 3.5.6 below, which is essentially an application of the local structure theorem for toroidal varieties. To prove it, we first require an auxilliary lemma, which intuitively says that $G$-orbit of toroidal varieties behave like those of toric varieties. (Indeed, the lemma can also be proven in the toric case by essentially the same argument, thanks to Theorem 3.3.11a).

**Lemma 3.5.5** (cf. [Tim11, Proof of Theorem 29.1])**.** *Let $X$ be a toroidal $G$-variety. Then, the $G$-orbit closures of $X$ are precisely the intersections of $G$-divisors of $X$.*

*Proof.* The intersection $Z$ of any set of $G$-divisors of $X$ is $G$-stable and has finitely many $G$-orbits (because $X$ does), so $Z$ contains a dense $G$-orbit. But $Z$ is closed, so it is the closure of its dense $G$-orbit.

Conversely, let $Y$ be any $G$-orbit of $X$, and let $D_1, \ldots, D_m$ be the $G$-divisors of $X$ that $Y$. Note that since $X$ is toroidal, the $D_i$ are the only $B$-divisors of $X$ containing $Y$. Thus, the cone $\mathcal{C}_Y$ in the colored cone corresponding to $Y$ is precisely the cone generated by the $\varphi_{D_i}$. On the other hand, arguing as with $Z$ above, we see that $\bigcap_i D_i$ has a dense $G$-orbit $Y'$. We have $Y \subset \overline{Y'}$, so any $B$-divisor of $X$ containing $Y'$ also contains $Y$ and hence is one of the $D_i$. We conclude that the cone $\mathcal{C}_{Y'}$ is also the cone generated by the $\varphi_{D_i}$, so that $\mathcal{C}_{Y'} = \mathcal{C}_Y$ and $Y = Y'$ (by Proposition 3.3.24). So $Y$ is dense in $\bigcap_i D_i$, which implies that $\overline{Y} = \bigcap_i D_i$. □

**Theorem 3.5.6** ([Bri97, Proposition 2.4.1]; cf. [Tim11, Theorem 29.1])**.** *Let $X$ be a toroidal $G$-variety, let $\Delta \subset X$ be the union of all the colors of $X$, and let $P \subset G$ be the parabolic subgroup given by*

$$P = \{g \in G \mid g \cdot \Delta = \Delta\}.$$

*(Equivalently, $P$ is the subgroup $P_X$ of Lemma 3.2.10.) Let $P_u = R_u(P)$, and let $X_G^\circ$ be the open $G$-orbit of $X$.*

*(a) There exists a Levi subgroup $M \subset P$ depending only on $X_G^\circ$ and an $M$-stable closed subvariety $Z \subset X \setminus \Delta$ such that the map*

$$P_u \times Z \to X \setminus \Delta$$

*given by $(p, z) \mapsto p \cdot z$ is a $P$-equivariant isomorphism.*

(b) *The commutator $[M, M]$ acts trivially on $Z$, and $Z$ is a toric variety for some quotient of the torus $M/[M, M]$.*

(c) *Every $G$-orbit of $X$ intersects $Z$ in a single $M$-orbit. In other words, the map $\mathcal{O} \mapsto \mathcal{O} \cap Z$ is a bijection between $G$-orbits of $X$ and $M$-orbits of $Z$.*

(d) *The cones in the colored fan $\mathscr{F}_X$ are the cones in the fan corresponding to the toric variety $Z$. More precisely: if $\mathscr{F}_X = \{(\mathcal{C}_1, \varnothing), \ldots, (\mathcal{C}_n, \varnothing)\}$ is the colored fan corresponding to $X$, then the fan corresponding to $Z$ is $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.*

*Proof.* Since $X$ is toroidal, applying Proposition 3.1.20 to $\Delta$ shows that $\Delta$ is an effective Cartier divisor and that $\mathcal{O}_X(\Delta)$ is generated by global sections. Let $\sigma \in \Gamma(X, \mathcal{O}_X(\Delta))$ be the canonical section, so that $X_\sigma = X \setminus \Delta$, and then apply the local structure theorem (Theorem 3.2.2) to $X_\sigma$ to get statement (a). Note that $M$ is the standard Levi subgroup of $P$, and we have

$$P = P_X = \{g \in G \mid g \cdot X_B^\circ = X_B^\circ\},$$

where $X_B^\circ$ is the open $B$-orbit of $X$ (see Remark 3.2.11). In particular, since $X_B^\circ \subset X_G^\circ$, both $P$ and $M$ depend only on $X_G^\circ$.

By Proposition 3.2.3, $Z$ is a spherical $M$-variety, and the map $D \mapsto D \cap Z$ is a bijection between the $B$-divisors of $X$ that intersect $X \setminus \Delta$ and the $(B \cap M)$-divisors of $Z$. By definition of $\Delta$, the $B$-divisors intersecting $X \setminus \Delta$ are precisely the $G$-divisors of $X$, so the intersections $D \cap Z$ are all $M$-stable (because $D$ and $Z$ are $M$-stable). Thus, every $(B \cap M)$-divisor of $Z$ is $M$-stable, whence (b) follows immediately from Proposition 3.1.19 (applied to the spherical $M$-variety $Z$). These arguments also show that the map $D \mapsto D \cap Z$ is in this case a bijection between $G$-divisors of $X$ and $M$-divisors of $Z$. This fact will be useful below.

For (c), we first claim that intersecting with $Z$ induces a bijection on orbit closures. This follows directly from Lemma 3.5.5 and the fact that the map $D \mapsto D \cap M$ is a bijection $\mathcal{D}_{G,X}^G \to \mathcal{D}_{M,Z}^M$, as noted above. Indeed, this bijection implies that the intersections of $M$-divisors in $Z$ are the sets of the form $(\bigcap_i D_i) \cap Z$, where the $D_i$ are $G$-divisors of $X$. But Lemma 3.5.5 (applied to $X$) implies that the $G$-orbit closures are the intersections of $G$-divisors, while the same lemma (applied to $Z$) implies that the $M$-orbit closures in $X$ are the intersectsions of $M$-divisors. Putting this all together yields the claim.

It remains to prove that intersecting with $Z$ induces a bijection on orbits, not just orbit closures. The main difficulty is to prove that the intersection of any $G$-orbit with $Z$ is an $M$-orbit. If this is true, then one can use this fact plus the bijection on orbit closures above to prove that the map in (c) is a bijection on orbits. So, let $Y$ be a $G$-orbit of $X$. We prove that $Y \cap Z$ is an $M$-orbit of $Z$ by induction on $\dim(Y)$. Write $Y_0 = Y \cap (X \setminus \Delta)$. Then, we have $\dim(Y) = \dim(Y_0)$ and

$$Y_0 = P_u \cdot (Y_0 \cap Z) \cong P_u \times (Y \cap Z),$$

from which it follows that the codimension of $Y \cap Z$ in $Z$ is the same as the codimension of $Y$ in $X$ (here also using the fact that $X \setminus \Delta \cong P_u \times Z$ implies that $\dim(X) = \dim(P_u) + \dim(Z)$).

This gives us the base case of our induction: if $Y$ is an orbit of minimal dimension, then $Y$ is closed and hence is an orbit closure; so, the intersection $Y \cap Z$ is an orbit closure of minimal dimension in $Z$, which means $Y \cap Z$ is a closed orbit. For the inductive step, given any $G$-orbit $Y$, the intersection $\overline{Y} \cap Z$ is an orbit closure of $Z$ and hence is a union of one $M$-orbit $Y'$ of dimension $\dim(Y \cap Z)$ and other $M$-orbits of strictly smaller dimension (Proposition 2.1.2). For any of these $M$-orbits $\mathcal{O}'$ of dimension $< \dim(Y \cap Z)$, the orbit closure $\overline{\mathcal{O}}'$ has the form $\overline{\mathcal{O}} \cap Z$ for some $G$-orbit $\mathcal{O}$ of $X$ by the above claim. Then, $\mathcal{O} \cap Z$ is an $M$-orbit (by induction hypothesis) that is dense in $\overline{\mathcal{O}'}$, so we have $\mathcal{O} \cap Z = \mathcal{O}'$. Note also that $\mathcal{O} \cap Z \subset \overline{Y} \cap Z$ implies that

$$\mathcal{O} \subset \overline{\mathcal{O}} = \overline{P_u(\mathcal{O} \cap Z)} \subset \overline{P_u(\overline{Y} \cap Z)} = \overline{Y}.$$

In summary, $\overline{Y} \cap Z$ is a union of one dense $M$-orbit $Y'$ and other $M$-orbits which are all of the form $\mathcal{O} \cap Z$ for some $G$-orbit $\mathcal{O} \subset \overline{Y}$. Since $Y$ does not intersect any other $G$-orbit of $X$, it follows that $Y' \supset Y \cap Z$. On the other hand, $Y \cap Z$ is $M$-stable and intersects the $M$-orbit $Y'$, so we must have $Y' \subset Y \cap Z$ and hence $Y' = Y \cap Z$.

As for (d), by the bijection in statement (c), it will suffice to show that for any $G$-orbit $Y \subset X$, the cone $\mathcal{C}_Y$ in the colored cone $(\mathcal{C}_Y, \Delta_Y) \in \mathscr{F}_X$ is equal to the cone $\mathcal{C}$ corresponding to the $M$-orbit $Y \cap Z$ (in the fan for the toric variety $Z$). Because $X$ is toroidal, we have $\Delta_Y = \varnothing$, so the extremal rays of $\mathcal{C}_Y$ are generated by the $\varphi_D$, where $D$ is a $G$-divisor containing $Y$ (Lemma 3.3.17). Similarly, the extremal rays of $\mathcal{C}$ are generated by the $\varphi_{D'}$, where $D'$ is an $M$-divisor containing $Y \cap Z$ (Lemma 3.3.7). Using our above arguments (and the inverse of the map $D \mapsto D \cap Z$ given in Proposition 3.2.3), one can check that the $M$-divisors containing $Y \cap Z$ are precisely the intersections $D \cap Z$ for any $G$-divisor $D$ containing $Y$. Moreover, for any such $G$-divisor $D$, Proposition 3.2.3 implies that $\varphi_D = \varphi_{D \cap Z}$ as elements of $N(X) = N(Z)$. This proves that $\mathcal{C} = \mathcal{C}_Y(X)$, as desired. □

**Remark 3.5.7.** In the above theorem, the complement $X \setminus \Delta$ intersects every $G$-orbit of $X$ (because $X$ is toroidal). Thus, we have $G \cdot (X \setminus \Delta) = X$. It follows that $X$ is covered by open subsets (namely, the sets $g \cdot (X \setminus \Delta)$ for $g \in G$) which are isomorphic to $P_u \times Z$ for some toric variety $Z$. We can thus think of the above theorem as saying that any toroidal variety $X$ is in some sense "locally toric."

Even though cones in colored fans need not generally behave like the cones in fans of toric varieties (see Remark 3.3.23), the cones in the colored fans of toroidal varieties actually do form a fan. This follows immediately from Theorem 3.5.6d. Alternately, one can check from the definitions that, given finitely many cones $\mathcal{C}_1, \ldots, \mathcal{C}_m \subset \mathcal{V}(X)$, the set $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ is a fan if and only if the set $\{(\mathcal{C}_1, \varnothing), \ldots, (\mathcal{C}_m, \varnothing)\}$ is a colored fan. It follows that toroidal varieties are relatively easy to construct: all we need to do is take a homogeneous spherical variety $G/H$ and pick a fan $\mathscr{F}$ consisting of cones contained in $\mathcal{V}(G/H)$. Then, the colored cones $(\mathcal{C}, \varnothing)$ for $\mathcal{C} \in \mathscr{F}$ will actually form a colored fan and hence define an embedding $G/H \hookrightarrow X$ with $X$ toroidal. In particular, a toroidal embedding of $G/H$ always exists.

The following proposition does even better: it tells us that for any spherical variety $X$, there exists a projective $G$-equivariant birational morphism $\tilde{X} \to X$ with $\tilde{X}$ toroidal. In particular, not only do toroidal varieties exist, but they are relatively abundant. Thanks to Luna–Vust theory, the proof is completely combinatorial: we essentially show that we can “take the colors out” of any given colored fan to make another colored fan.

**Proposition 3.5.8** ([Bri97, Proposition 2.4.2], [Kno91, Lemma 5.1])**.**

*(a) Let $X$ be a spherical $G$-variety. Then, there exists a toroidal $G$-variety $\tilde{X}$ and a projective $G$-equivariant birational morphism $\pi : \tilde{X} \to X$.*

*(b) Let $G/H$ be a homogeneous spherical $G$-variety. There exists an embedding $G/H \hookrightarrow X$ such that $X$ is toroidal and projective.*

*Proof.* It is possible to prove (a) by a constructing the blow-up of $X$ along some subvariety of the colors of $X$. For a proof using this sort of argument, see [Bri97, Proposition 2.4.2]. Alternately, we can avail ourselves of Luna–Vust theory in the following way. Let $G/H$ be the open $G$-orbit of $X$, and let $\mathscr{F} = \mathscr{F}_X$ be the colored fan of $X$. For any colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}$, we check that $(\mathcal{C} \cap \mathcal{V}(G/H), \varnothing)$ is a strictly convex colored cone. First, $\mathcal{C} \cap \mathcal{V}(G/H)$ is a polyhedral cone because both $\mathcal{C}$ and $\mathcal{V}(G/H)$ are (this follows formally from the definition of a polyhedral cone as an intersection of finitely many half-spaces — see [Ful93, Section 1.2, Property (9)]). This implies that $\mathcal{C} \cap \mathcal{V}(G/H)$ is generated as a cone by finitely many elements of $\mathcal{V}(G/H)$. Moreover, the interior of $\mathcal{C} \cap \mathcal{V}(G/H)$ is contained in $\mathcal{V}(G/H)$, so the pair $(\mathcal{C} \cap \mathcal{V}(G/H), \varnothing)$ is a colored cone. Finally, $\mathcal{C} \cap \mathcal{V}(G/H)$ is strictly convex because $\mathcal{C}$ is.

Now, we define a set of colored cones $\tilde{\mathscr{F}}$ by taking every face of the colored cone $(\mathcal{C} \cap \mathcal{V}(G/H), \varnothing)$ for all $(\mathcal{C}, \Delta) \in \mathscr{F}$. Note that every element of $\tilde{\mathscr{F}}$ has the form $(\mathcal{C}, \varnothing)$ (because it is the face of a colored cone having this form). Moreover, $\tilde{\mathscr{F}}$ contains every face of each of its elements by definition, and for every $v \in \mathcal{V}(G/H)$, there is at most one colored cone $(\tilde{\mathcal{C}}, \varnothing) \in \tilde{\mathscr{F}}$ with $v \in \tilde{\mathcal{C}}^\circ$ (because the same is true of $\mathscr{F}$). Thus, $\tilde{\mathscr{F}}$ is a strictly convex colored fan and so defines an embedding $G/H \hookrightarrow \tilde{X}$ with $\tilde{X}$ toroidal. Moreover, for every $(\tilde{\mathcal{C}}, \varnothing) \in \tilde{\mathscr{F}}$, there is by definition some $(\mathcal{C}, \Delta) \in \mathscr{F}$ such that $\tilde{\mathcal{C}} \subset \mathcal{C}$. It follows that the identity map $G/H \to G/H$ maps $\tilde{\mathscr{F}}$ into $\mathscr{F}$, so it extends to a $G$-equivariant morphism

$$\pi : \tilde{X} \to X.$$

Note that $\pi$ is birational because it is an extension of the identity map on $G/H$. Moreover, we have $\operatorname{Supp}(\mathscr{F}) = \operatorname{Supp}(\tilde{\mathscr{F}})$ by construction of $\tilde{\mathscr{F}}$, so $\pi$ is proper by Theorem 3.3.28.

It remains to show that we can choose $\pi$ to be quasi-projective as well (hence projective). This is essentially an application of the “equivariant Chow lemma” of Sumihiro (Theorem 2.6.14). More precisely, that theorem gives us a $G$-equivariant projective birational morphism $\pi' : \tilde{X}' \to \tilde{X}$ with $\tilde{X}'$ quasi-projective. Recall that the normalization of $\tilde{X}'$ is $G$-equivariantly birational to $\tilde{X}'$ and is quasi-projective because the normalization morphism is finite (see Lemma 2.6.15). Thus, after replacing $\tilde{X}'$ by its normalization, we may assume that $\tilde{X}'$ is normal. Since $\pi'$ is $G$-equivariant and birational, $\tilde{X}'$ is a spherical variety with open orbit $G/H$, and $\pi'$ is an extension of the identity morphism $G/H \to G/H$. By Theorem 3.3.28, this means that $\pi'$ maps the colored fan $\tilde{\mathscr{F}}'$ of $\tilde{X}'$ to $\tilde{\mathscr{F}}$. It follows that no colored cone in $\tilde{\mathscr{F}}'$ has any colors (because the same is true of $\tilde{\mathscr{F}}$), so that $\tilde{X}'$ is toroidal. Finally, the composition $\pi \circ \pi'$ is $G$-equivariant, birational, and proper because $\pi$ and $\pi'$ are, and it is quasi-projective because $\tilde{X}'$ is (any ample line bundle on $\tilde{X}$' is $(\pi \circ \pi')$-ample). So, $\pi \circ \pi'$ is projective. Thus, we may replace $\pi$ by $\pi \circ \pi'$ and so take $\pi$ to be projective. This choice of $\pi$ satisfies (a).

As for (b), there again exists a proof by an explicit, geometric construction; see [Kno91, Lemma 5.1]. Alternately, we can deduce (b) from (a) in the following way. Since $G/H$ is quasi-projective and normal, Theorem 2.6.12 gives us a $G$-equivariant immersion $G/H \hookrightarrow \mathbb{P}(M)$ for some finite-dimensional $G$-module $M$. Let $X$ be the normalization of the closure of $G/H$ in $\mathbb{P}(M)$. Then, $X$ is a spherical variety, and since $G/H$ is contained in the smooth locus of $X$, we have an open immersion $G/H \hookrightarrow X$. Let $\pi : \tilde{X} \to X$ be the morphism given by applying (a) to $X$. Then, $\tilde{X}$ is a toroidal variety and is projective (because $X$ is projective by construction and $\pi$ is projective). Moreover, since $\pi$ is $G$-equivariant and birational, it is an isomorphism on open $G$-orbits, so the open $G$-orbit of $\tilde{X}$ is $G/H$. Thus, $\tilde{X}$ is the desired variety. □

In Theorem 3.5.6, we applied the local structure theorem to the complement of all colors of $X$, instead of to the sets $X_{B,Y}$ that one typically uses for the local structure theorem on spherical varieties. However, all of the arguments in the proof of that theorem apply just as well to $X_{B,Y}$ (which we can apply the local structure theorem to by Theorem 3.2.7a). Thus, repeating essentially the same proof (but with $X_{B,Y}$ in place of $X \setminus \Delta$ in that theorem), we obtain the following theorem.

**Theorem 3.5.9.** *Let $X$ be a toroidal variety, let $Y \subset X$ be a $G$-orbit, and let $P$, $M$ and $Z$ be as in the application of the local structure theorem to the set $X_{B,Y}$ (see Theorem 3.2.2). Then, $Z$ is an affine toric variety for a quotient of $M$, and the cone corresponding to $Z$ is $\mathcal{C}_Y$.*

In the theory of toric varieties, there is a standard procedure for constructing a resolution of singularities by "subdividing" cones in a fan. (For details on this technique of subdividing cones, see [Ful93, Section 2.6], which gives examples and exercises that lead up to a proof, or [Oda88, Corollary 1.18 and following discussion], which gives a formal statement and references to proofs.) Since toroidal varieties are in some sense "locally toric" and their colored fans determine fans, we can apply the same subdivision procedure to obtain resolutions of singularities for toroidal varieties. The precise statement is as follows.

**Theorem 3.5.10.** *Let $X$ be a toroidal variety.*

*(a) The variety $X$ is smooth if and only if for every colored cone $(\mathcal{C}, \varnothing) \in \mathscr{F}_X$, the cone $\mathcal{C}$ is generated by a part of a basis for the lattice* $\mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Z}) \subset N(X)$.

(b) *There exists a toroidal resolution of singularities of $X$. In other words, there exists a smooth toroidal variety $X$ and proper $G$-equivariant birational morphism $\pi : \tilde{X} \to X$.*

*Proof.* For (a), we note that the sets $G \cdot X_{B,Y}$ cover $X$. So, $X$ is smooth if and only if $G \cdot X_{B,Y}$ is smooth for all $G$-orbits $Y$, or equivalently, if and only if $X_{B,Y}$ is smooth. The local structure theorem gives us

$$X_{B,Y} \cong R_u(P) \times Z,$$

where $Z$ is an affine toric variety whose corresponding cone is $\mathcal{C}_Y$ (see Theorem 3.5.9). Since $R_u(P)$ is smooth, we see that $X_{B,Y}$ is smooth if and only if $Z$ is smooth, and it is a standard fact about affine toric varieties (see e.g. [Ful93, Section 2.1, Proposition 1]) that $Z$ is smooth if and only if the corresponding cone $\mathcal{C}_Y$ is generated by a part of a basis for the lattice in $N(X)$ dual to $\Lambda(X)$. This gives us (a).

As for (b), we note that $X$ is toroidal, so the cones in the colored fan $\mathscr{F}_X$ define a fan $\mathcal{F}$ in the sense of toric varieties. As mentioned above, the theory of toric varieties provides a way to subdivide each cone in $\mathcal{F}$ into a union of cones, each of which is generated by a part of a basis for the lattice $\mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Z})$. This is how one typically demonstrates the existence of a resolution of singularities for toric varieties. For our purposes, such a subdivision will define a new fan $\tilde{\mathcal{F}}$ consisting of cones contained in $\mathcal{V}(G/H)$, and we can then define a new colored fan $\tilde{\mathscr{F}}$ by

$$\tilde{\mathscr{F}} = \{(\mathcal{C}, \varnothing) \mid \mathcal{C} \in \tilde{\mathcal{F}}\}.$$

The colored fan $\tilde{\mathscr{F}}$ defines a toroidal variety $\tilde{X}$ whose open $G$-orbit is the same as the open $G$-orbit of $X$. Since $\tilde{\mathcal{F}}$ was obtained by subdividing cones in $\mathcal{F}$, every cone in the colored fan $\tilde{\mathscr{F}}$ is contained in a cone in $\mathscr{F}_X$, and we have $\mathrm{Supp}(\mathscr{F}) = \mathrm{Supp}(\tilde{\mathscr{F}})$. So, Theorem 3.3.28 gives us a proper $G$-equivariant morphism $\tilde{X} \to X$ which is the identity on open $G$-orbits. Finally, we note that by (a) and the construction of the fan $\mathcal{F}$, the toroidal variety $\tilde{X}$ is smooth. □

### 3.5.b Sober Subgroups and Standard Embeddings

Corollary 3.4.10 tells us that the valuation cone $\mathcal{V}(G/H)$ is strictly convex if and only if $H$ has finite index in its normalizer $N_G(H)$. When this is the case, there is a particularly spherical variety that we can consider.

**Definition 3.5.11.** Let $H \subset G$ be a spherical subgroup.

1. We say that $H$ is *sober* if the group $N_G(H)/H$ is finite (equivalently, if $H$ has finite index in $N_G(H)$).

2. If $H$ is sober, we define the *standard embedding* (or *canonical embedding*) of $G/H$ to be the embedding $G/H \hookrightarrow X$ defined by the colored fan $\{(\mathcal{V}(G/H), \varnothing)\}$ (which is strictly convex because $\mathcal{V}(G/H)$ is, see Corollary 3.4.10).

Since we are working with the normalizer $N_G(H)$, we will need a few technical algebraic statements about the interaction of $N_G(H)$ and the Borel subgroup $B$.

**Lemma 3.5.12** ([Bri97, Theorem 4.3(iii)], [Tim11, Lemma 30.2])**.** *Let $H \subset G$ be a spherical subgroup.*

*(a) There exists a choice of Borel subgroup $B$ such that the set*

$$BH = \{bh \mid b \in B, h \in H\}$$

*is open in $G$.*

*(b) Consider the action of $N_G(H)/H$ on $G/H$ defined by $nH * gH = gnH$. For any choice of $B$ as in (a), the natural action of $N_G(H)/H$ on $G/H$ fixes the open $B$-orbit.*

*(c) For any intermediate subgroup $H \subseteq H' \subseteq N_G(H)$, we have $N_G(H') = N_G(H)$. In particular, $N_G(N_G(H)) = N_G(H)$.*

*Proof.* For any Borel subgroup $B \subset G$, the set $BH$ is open in $G$ if and only if the open $B$-orbit of $G/H$ is the orbit containing the coset $H$. So, pick any Borel subgroup $B$, let $gH$ is be any coset in the open $B$-orbit of $G/H$, and define $B' = g^{-1}Bg$. One can check that if $\mathcal{O}$ is the open $B$-orbit of $G/H$, then $g^{-1}\mathcal{O}$ is the open $B'$-orbit of $G/H$. In particular, $H \in g^{-1}\mathcal{O}$, so $B'H$ is open in $G$. This proves (a).

For (b), the open $B$-orbit in $G/H$ is the image of $BH$ under the quotient map. Thus, it suffices to show that the normalizer $N_G(H)$ fixes the subgroup $BH \subset G$ under the action given by multiplication on the right. For this, we follow part of the proof of [Bri97, Theorem 4.3(iii)]. For any $g \in N_G(H)$, the set $BHg$ is an open subset of $G$ and so intersects $BH$. On the other hand, we have $BHg = BgH$, so $BgH$ intersects $BH$. Thus, there exist some $b_1, b_2 \in B$ and $h_1, h_2 \in H$ such that $b_1gh_1 = b_2h_2$. This gives us $g = b_1^{-1}b_2h_2h_1^{-1} \in BH$ and hence $BHg = BgH = BH$.

The proof of (c) is slightly more technical; see [Tim11, Lemma 30.2] for details. ☐

When it exists, the standard embedding enjoys many nice properties.

**Lemma 3.5.13.** *Let $H \subset G$ be a sober subgroup, and let $i : G/H \hookrightarrow X$ be the standard embedding.*

*(a) The standard embedding is the unique (up to $G$-isomorphism) complete embedding of $G/H$ that is both simple and toroidal. Moreover, the standard embedding exists (i.e. $H$ is sober) if and only if there exists a simple complete embedding of $G/H$.*

*(b) The standard embedding is the "maximal" simple completion, i.e. it satisfies the following universal property: for any embedding $G/H \hookrightarrow X'$ with $X'$ simple and complete, there exists a unique birational $G$-equivariant morphism $X \to X'$ which extends the identity map on $G/H$.*

(c) *The standard embedding is the "minimal" toroidal completion, i.e. it satisfies the following universal property: for any embedding $G/H \hookrightarrow X'$ with $X'$ toroidal and complete, there exists a unique birational $G$-equivariant morphism $X' \to X$ which extends the identity map on $G/H$.*

(d) *The standard embedding $X$ is smooth if and only if the spherical roots $\Psi_{G,X}$ form a basis for the lattice $\Lambda(X)$.*

*Proof.* First, note that any embedding of $G/H$ is simple if and only if it is given by a colored fan consisting of a single colored cone, and it is toroidal if and only if this colored cone has the form $(\mathcal{C}, \varnothing)$ with $\mathcal{C} \subset \mathcal{V}(G/H)$. By Theorem 3.3.28, the embedding is then complete if and only if $\mathcal{V}(G/H) = \mathcal{C}$, i.e. if and only if the embedding is the standard embedding. If the standard embedding exists, it is a simple complete embedding. Conversely, if a simple complete embedding exists and is given by a colored cone $(\mathcal{C}, \Delta)$, then Theorem 3.3.28 gives us $\mathcal{V}(G/H) \subset \mathcal{C}$. So, $\mathcal{V}(G/H)$ is strictly convex because $\mathcal{C}$ is, and this implies that the standard embedding exists. This proves (a).

Next, let $G/H \hookrightarrow X'$ be any embedding with $X'$ complete. Suppose that $X'$ is simple, and let $(\mathcal{C}, \Delta)$ be the corresponding colored cone. Since $X$ is complete, Theorem 3.3.28 implies that $\mathcal{V}(G/H) \subset \mathcal{C}$, and the same theorem then gives us a morphism of $G$-varieties $f : X \to X'$ extending the identity morphism on $G/H$. If $X'$ is toroidal instead of simple, then let $\mathscr{F}$ be the corresponding colored fan. For every colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}$, we have $\Delta = \varnothing$ and hence $\mathcal{C} \subset \mathcal{V}(G/H)$. So, the identity map on $G/H$ maps $\mathscr{F}$ into the colored fan $\{(\mathcal{V}(G/H), \varnothing)\}$ defining the standard embedding, and Theorem 3.3.28 gives us a morphism of $G$-varieties $f : X' \to X$ extending the identity morphism on $G/H$. In both the simple and toroidal cases, the morphism $f$ is birational (since it extends the identity map $G/H \to G/H$), and $f$ is unique because it is determined by its restriction to $G/H$ (see [Har80, Chapter II, Exericse 4.2]). This proves (b) and (c).

For (d), Theorem 3.5.10 tells us that $X$ is smooth if and only if the cone $\mathcal{V}(X)$ is generated by a part of a basis for the dual lattice $\Lambda(X)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Z})$. This in turn holds if and only if $-\mathcal{V}(X)^\vee$ is generated by a part of a basis for $\Lambda(X)$ (this follows from a standard procedure for computing generators of the dual cone, see [Ful93, Section 1.2, Property (8)]). The spherical roots $\Psi_{G,X}$ are the unique set of minimal generators of $-\mathcal{V}(X)^\vee$ which are indivisible elements of the lattice $\Lambda(X)$, so $-\mathcal{V}(X)^\vee$ is generated by part of a basis for $\Lambda(X)$ if and only if the $\Psi_{G,X}$ are part of a basis for $\Lambda(X)$. Finally, we note that since $X$ is the standard embedding, $\mathcal{V}(X)$ must be strictly convex, so its dual cone is full-dimensional (see e.g. [Ful93, Section 1.2, Property (13)]). It follows that $\Psi_{G,X}$ is part of a basis for $\Lambda(G/H)$ if and only if it is a basis for $\Lambda(G/H)$. ☐

**Remark 3.5.14.** The set of spherical roots $\Psi_{G,X}$ is always linearly independent in $\Lambda(X)_{\mathbb{Q}}$ (see Remark 3.4.7), and by our arguments for part (d) in the above proof, $\Psi_{G,X}$ spans $\Lambda(X)_{\mathbb{Q}}$ when $X$ is the standard embedding. So, in part (d) of the above lemma, we already know that $\Psi_{G,X}$ is a basis for the vector space $\Lambda(X)_{\mathbb{Q}}$. The question of smoothness of the standard embedding $X$ is thus a question of whether this basis for $\Lambda(X)_{\mathbb{Q}}$ generates the lattice $\Lambda(X)$. In equations: we necessarily have $\Psi_{G,X} \subset \Lambda(X) \subset \mathbb{Q}\Psi_{G,X}$; the question is whether $\Lambda(X) = \mathbb{Z}\Psi_{G,X}$.

We will primarily be interested in the case where the standard embedding is smooth. However, it is very difficult to find interesting conditions which are sufficient for the standard embedding to be smooth. There is one main condition which is known to be sufficient, thanks to a result of Knop. The setup is as follows. Let $H \subset G$ be a spherical subgroup. By Lemma 3.5.12, we may pick the Borel subgroup $B$ such that the natural action of $N_G(H)/H$ on $G/H$ fixes the open $B$-orbit of $G/H$. It follows that $N_G(H)/H$ permutes the colors of $G/H$, so we have an action of $N_G(H)/H$ on the set of colors $\Delta(G/H)$. Knop's result says that the standard embedding is smooth when this action is particularly nice.

**Theorem 3.5.15** ([Kno96, Corollaries 7.2, 7.6])**.** *Let $G/H$ be a homogeneous spherical $G$-variety. If $N_G(H)/H$ acts effectively on $\Delta(G/H)$ (i.e. if the only element of $N_G(H)/H$ that fixes every color is the identity), then the standard embedding of $G/H$ exists and is smooth. In particular, if $N_G(H) = H$, then the standard embedding of $G/H$ exists and is smooth.*

**Remark 3.5.16.** Note that since the set $\Delta(G/H)$ is finite, the assumption that $N_G(H)/H$ acts effectively on $\Delta(G/H)$ forces $N_G(H)/H$ to be finite, so the standard embedding must exist. The deep part of the above theorem is not that the standard embedding is actually smooth (which Knop proves by using the criterion of Theorem 3.5.10).

In light of the above theorem, we make the following definition.

**Definition 3.5.17.** Let $H \subset G$ be a spherical subgroup.

1. We say that $H$ is *very sober* (or *spherically closed*) if $N_G(H)/H$ acts on $\Delta(G/H)$ effectively. (This in particular implies that $H$ is sober, see Remark 3.5.16 above.)

2. We define the *very sober hull* (or the *spherical closure*) of $H$, denoted $\overline{H}$, to be the subgroup of $N_G(H)$ consisting of all elements whose images in $N_G(H)/H$ act trivially on $\Delta(G/H)$.

Note that $H \subset \overline{H}$ by definition, and Lemma 3.5.12 tells us that $N_G(\overline{H}) = N_G(H)$. It follows that $N_G(\overline{H})/\overline{H}$ does act effectively on $\Delta(G/H)$. In particular, the very sober hull $\overline{H}$ is indeed very sober, so by Theorem 3.5.15, the standard embedding of $G/\overline{H}$ exists and is smooth. Moreover, most of the interesting combinatorial invariants of $G/H$ are preserved when passing to $G/\overline{H}$ (see [Lun01, Sections 6.1 and 7.1]). Thus, one can sometimes pass from $G/H$ to $G/\overline{H}$ in order to reduce questions about homogeneous spherical varieties to the case where the standard embedding is smooth.

It turns out that the standard embedding has especially nice geometric properties when it is smooth. In fact, the smooth standard embeddings of homogeneous spherical varieties are precisely the so-called *wonderful varieties*, which are the next "nice" type of spherical variety that we will discuss.

### 3.5.c Wonderful Varieties

**Definition 3.5.18.** We say that a $G$-variety $X$ is *wonderful* if the following properties hold.

1. $X$ is smooth and proper over $k$.
2. $X$ contains an open $G$-orbit $X_G^\circ$, and if $X_1, \ldots, X_r$ are the irreducible components of $X \setminus X_G^\circ$, then the $X_i$ are smooth and have normal crossings, and $\bigcap_i X_i \neq \varnothing$.
3. For all $x, y \in X$, we have $Gx = Gy$ if and only if
$$\{i \mid x \in X_i\} = \{j \mid y \in X_j\}.$$

In this case, the integer $r$ is called the *rank* of the wonderful variety $X$.

**Remark 3.5.19.** It follows immediately from the above definition that if $X$ is wonderful, the $G$-orbits of $X$ are precisely the subsets of the form
$$\bigcap_{i \in I} X_i \setminus \bigcup_{i \notin I} X_i$$
for any subset $I \subset \{1, \ldots, r\}$ (where we take the convention that the empty intersection is $X$, so that we get $X_G^\circ$ when $I = \varnothing$.) In particular, a wonderful variety of rank $r$ has exactly $2^r$ $G$-orbits and is simple (the unique closed orbit is $\bigcap_i X_i$).

**Example 3.5.20.** Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, with the action of $G = \mathrm{SL}_2$ from Example 3.1.16. We saw in that example that the two colors of $X$ are
$$D_1 = \mathbb{P}^1 \times \{[1:0]\}, \text{ and } D_2 = \{[1:0]\} \times \mathbb{P}^1,$$
and the two $G$-orbits of $X$ are the diagonal $\Delta$ and its complement $X_G^\circ$. Note that $X$ is smooth and complete. Moreover, $\Delta$ is the unique irreducible component of $X \setminus X_G^\circ$, and $\Delta$ is also the unique $G$-orbit of $X$ besides the open orbit $X_G^\circ$. It follows from the definition that $X$ is a wonderful variety of rank 1.

Compare this to some other properties of $X$ that we are already familiar with. First of all, we saw in Example 3.1.16 that $\Lambda(X) = \mathbb{Z} \cdot \alpha_1$, where $\alpha_1$ is the unique simple root of $G$. In particular, the rank of $X$ (as a $G$-variety) is 1, which is the same as the rank of $X$ as a wonderful variety. Moreover, neither of the $G$-orbits of $X$ is contained in $D_1$ or $D_2$, so $X$ is a toroidal variety. Also, the unique closed $G$-orbit of $X$ is $\Delta$, so $X$ is simple, and $X$ is also complete. Thus, $X$ is the standard embedding of its open $G$-orbit (see Lemma 3.5.13), and this embedding is smooth.

The above example gives us one wonderful variety that is also a smooth standard embedding. Remarkably, it turns out that the wonderful $G$-varieties are precisely the standard embeddings of homogeneous spherical $G$-varieties that are smooth. Thus, the definition of a wonderful variety, which depends only on the structure of $G$-orbits, actually characterizes all the nice properties of smooth standard embeddings.

The proof of this fact largely revolves around the local structure theorem. The key difficulty is to prove that a wonderful variety is even spherical to begin with. This was first proven by Luna in [Lun96].

**Theorem 3.5.21** ([Lun96]; see also [Pez10, Proposition 3.3.1], [Tim11, Theorem 30.15])**.** *Let $X$ be a $G$-variety. The following are equivalent.*

*(i) $X$ is a wonderful $G$-variety.*

*(ii) $X$ is a spherical $G$-variety, $X$ is the standard embedding of its open $G$-orbit, and $X$ is smooth.*

*Moreover, if these conditions are satisfied, then the rank of $X$ as a wonderful variety is equal to the rank $r(X)$ of $X$ as a $G$-variety.*

*Proof.* Suppose that (ii) holds, and let $G/H$ be the open $G$-orbit of $X$. Then, $X$ is toroidal, so by Theorem 3.5.6a, we obtain an isomorphism of $P$-varieties $P_u \times Z \xrightarrow{\sim} X \setminus \Delta$ (with notation as in the theorem). The same theorem (along with Proposition 3.2.3) tells us that $Z$ is a smooth toric variety for a quotient $T'$ of $M$, and the corresponding fan has only one cone, namely $\mathcal{V}(G/H)$. Because this fan contains only one cone, $Z$ is affine. Moreover, it is a standard fact from the theory of toric varieties (see e.g. [Ful93, Section 2.1, Proposition 1]) that the only smooth affine toric varieties are $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$, with the action of the torus given by the natural action of $\mathbb{G}_m^n \subset \mathbb{A}^r \times \mathbb{G}_m^{n-r}$, and that the cone corresponding to $\mathbb{A}^r \times \mathbb{G}_m^{n-r}$ has dimension $r$ . By Theorem 3.4.1, the cone $\mathcal{V}(G/H)$ is full-dimensional in $N(G/H)$, so we have $Z \cong \mathbb{A}_k^r$, where $r = r(X) = \dim(N(G/H))$ is the rank of $X$.

Now, pick coordinates $Z \cong \operatorname{Spec}(k[x_1, \dots, x_r])$. The irreducible components of the complement of the dense $T'$-orbit $Z^\circ \subset Z$ are precisely the vanishing loci $Z_i$ of $x_i$ for $i \in \{1, \dots, r\}$. These are smooth and have normal crossings, and $\cap_i Z_i \neq \varnothing$. Moreover, the $T'$-orbits (hence also the $M$-orbits) of $Z$ are the sets of the form

$$Z_I = \bigcap_{i \in I} Z_i \setminus \bigcup_{i \notin I} Z_i$$

for any subset $I \subset \{1, \dots, r\}$. Thus, $Z$ satisfies conditions 2 and 3 in the definition of a wonderful variety (Definition 3.5.18). On the other hand, by Proposition 3.2.3, the map $D \mapsto D \cap Z$ is a bijection between $T$-divisors of $Z$ and $B$-divisors of $X$ intersecting $X \setminus \Delta$ (which are precisely the $G$-divisors of $X$ by definition of $\Delta$), and by Theorem 3.5.6, the map $Y \mapsto Y \cap Z$ is a bijection between $G$-orbits of $X$ and $M$-orbits of $Y$. Using these bijections and the fact that every $G$-orbit of $X$ intersects $Z$ (by definition of $\Delta$), one can prove that $X$ satisfies conditions 2 and 3 in the definition of a wonderful variety because $Z$ does. Since $X$ is smooth and proper by assumption, this implies that $X$ is a wonderful variety. Moreover, the irreducible components of the complement of the open $G$-orbit of $X$ are the $G$-divisors of

$X$ (Lemma 3.1.17), and as noted above, these are in bijection with the $Z_i$. Since the number of $Z_i$ is $r = r(X)$, we conclude that the rank of $X$ as a wonderful variety is equal to $r(X)$. This proves that (ii) $\Rightarrow$ (i), and it also proves the desired equality on ranks.

Conversely, suppose that $X$ is a wonderful variety. We omit the proof that $X$ is spherical, which is by far the most technical part of the proof. The strategy is to assume that $X$ is wonderful but not spherical, reduce first to the case where $X$ is wonderful of rank 2 and then to the case where $G = \mathrm{SL}_2$, and finally to prove that there cannot exist a wonderful variety which is not spherical in this case. For details, see [Lun96] or [Tim11, Theorem 30.15] (which uses the same strategy but gives a different argument than the original proof in [Lun96]).

In short, $X$ is spherical by [Lun96]. Also, the definition of a wonderful variety implies that $X$ is smooth, complete, and simple. In light of Lemma 3.5.13a, it remains to prove that $X$ is toroidal. In [Pez10, Proposition 3.3.1], Pezzini gives an argument for this that involves the combinatorics of Luna–Vust theory. However, we were unable to verify this combinatorial argument. So, we provide another approach, which hinges on the local structure theorem and one combinatorial fact about smooth spherical varieties. For an alternative, more geometric approach, see [Tim11, Theorem 30.15].

Let $r$ be the rank of the wonderful variety $X$. We claim that $r(X) \le r$. For this, let $Y \subset X$ be the unique closed $G$-orbit. Since $X$ (hence also $Y$) is complete, $Y$ has a unique $B^-$-fixed point $y$ (see Lemma 2.1.3). With $P$, $M$, and $Z$ as in the local structure theorem (Theorem 3.2.2) applied to $X_{B,Y}$, we have $Y \cap Z = \{y\}$ by Lemma 3.2.9. Since $X_{B,Y} \cong R_u(P) \times Z$ and $Y \cap X_{B,Y}$ is a $B$-orbit (Theorem 3.2.7), we see that $Y \cap X_{B,Y} \cong R_u(P) \times (Y \cap Z)$ and hence that

$$\dim(Y) = \dim(Y \cap X_{B,Y}) = \dim(R_u(P)) + \dim(Y \cap Z) = \dim(R_u(P)).$$

On the other hand, $Y$ is the intersection of the $r$ $G$-divisors of $X$, and these have simple normal crossings, so $\dim(Y) = \dim(X) - r$. The fact that $X_{B,Y} \cong R_u(P) \times Z$ then gives us

$$\begin{aligned} \dim(X) = \dim(X_{B,Y}) &= \dim(R_u(P)) + \dim(Z) \\ &= \dim(Y) + \dim(Z) \\ &= \dim(X) - r + \dim(Z), \end{aligned}$$

which implies that $\dim(Z) = r$. Moreover, we have $r(Z) \le \dim(Z)$ by Lemma 3.2.10 and $r(Z) = r(X)$ by Proposition 3.2.3. Putting this all together gives us

$$r(X) = r(Z) \le \dim(Z) = r,$$

as desired.

Now, consider the set $\mathcal{D}_Y \subset \mathcal{D}_{G,X}$ consisting of $B$-divisors containing $Y$. Since every $G$-divisor contains a $G$-orbit, all $r$ $G$-divisors of $X$ are contained in $\mathcal{D}_Y$. On the other hand, since $X$ is smooth, Proposition 3.7.9 below implies that the valuations $\varphi_D$ for $D \in \mathcal{D}_Y$ form part of a basis for the dual lattice $\Lambda(X)^\vee$. Since $\Lambda(X)^\vee$ has rank $r(X) \le r$, this is only possible if $r(X) = r$ and the $G$-divisors of $X$ are the only elements of $\mathcal{D}_Y$. In other words, no color of $X$ lies in $\mathcal{D}_Y$, which implies that $X$ is toroidal. This completes the proof. We remark that, although Proposition 3.7.9 does not come until Section 3.7, nothing in that section hinges on anything related to wonderful varieties (or indeed on any material appearing after Section 3.4). Thus, there is nothing circular about our use of Proposition 3.7.9 in this proof. $\square$

Wonderful varieties are generally the nicest of all spherical varieties. As a result, the local structure theorem gives us particularly strong statements when we apply it to wonderful varieties. The following theorem gives a few such statements.

**Theorem 3.5.22.** *Let $X$ be a wonderful variety of rank $r$, let $X_B^\circ \subset X$ be the open $B$-orbit, and set*

$$P = \{g \in G \mid gX_B^\circ = X_B^\circ\}.$$

*Let $P^-$ be the opposite parabolic subgroup to $P$ containing $T$, and let $M = P \cap P^-$. Let $Y \subset X$ be the unique closed $G$-orbit, and let $y \in Y$ be the unique $B^-$-fixed point.*

*(a) There exists some $M$-stable closed subvariety $Z \subset X_{B,Y}$ such that the map*

$$P_u \times Z \to X_{B,Y}$$

*given by $(p, z) \mapsto pz$ is a $P$-equivariant isomorphism. Moreover, $Z \cap Y = \{y\}$ is the unique closed $M$-orbit of $Z$, and $GX_{B,Y} = X$.*

*(b) If $M_0$ is the stabilizer of a point in the open $M$-orbit of $Z$, then $M/M_0 \cong \mathbb{G}_m^r$, and we have $Z \cong \mathbb{A}_k^r$ as $\mathbb{G}_m^r$-varieties.*

*(c) Because $Y$ and $Z$ are $T$-stable and $T$ fixes $y$, the torus $T$ acts on $T_yZ$ and on the quotient $T_yX/T_yY$. Moreover, we have an isomorphism of $T$-modules*

$$T_yZ \cong T_yX/T_yY.$$

*(d) The spherical roots of $X$ are the $T$-weights of $T_X/T_yY$, and they form a basis for the lattice $\Lambda(X)$.*

*Proof.* Statement (a) is essentially a combination of general facts about the local structure theorem: see Theorem 3.2.2, Theorem 3.2.7, and Lemma 3.2.9. By Theorem 3.5.9, the variety $Z$ is an affine toric variety for some quotient $T'$ of $M$, and the cone corresponding to this variety is the cone $\mathcal{C}_Y$. By arguing as in the proof of Theorem 3.5.21 above, we get that $T' \cong \mathbb{G}_m^r$ and that $Z \cong \mathbb{A}_k^r$ as $\mathbb{G}_m^r$-varieties. To obtain (b) from this, we need only note that $T' \subset Z$ is the open $T'$-orbit (which is also the open $M$-orbit), so $T'$ must be the quotient $M/M_0$, where $M_0$ is the stabilizer of any point in the open $M$-orbit of $Z$.

As noted above, the cone corresponding to the toric variety $Z$ is $\mathcal{C}_Y$. Since $X$ is the standard embedding of its open $G$-orbit, we have $\mathcal{C}_Y = \mathcal{V}(X)$. So, the construction of an affine toric variety from its cone tells us that

$$Z \cong \operatorname{Spec}(k[z_1, \ldots, z_r]),$$

where the weights of the $T$-eigenvectors $z_i$ are precisely the set of indivisible elements of $\Lambda(X)$ that are minimal generators of the cone $-\mathcal{V}(X)^\vee$ (see Example 3.3.4 and the surrounding discussion). In other words, the weights of the $z_i$ are the spherical roots of $X$. Since $y$ is fixed by $T$, we see that $y = 0$ under the isomorphism $Z \cong \mathbb{A}_k^r$. The tangent space $T_yZ \cong T_0\mathbb{A}_k^r$ is a $k$–vector space with basis given by the $dz_i$, and the $T$-module structure on $T_yZ$ is given by

$$t \cdot dz_i = d(t \cdot z_i) = d(\gamma_i(t)z_i) = \gamma_i(t)dz_i,$$

where $\gamma_i \in \mathcal{X}(T)$ is the weight of $z_i$. It follows that the only $T$-eigenvectors in $T_yZ$ are scalar multiples of the $dz_i$, and these have the same weights as the $z_i$. Thus, the $T$-weights of $T_yZ$ are precisely the spherical roots of $X$. Moreover, since $X$ is the standard embedding of its open $G$-orbit and is smooth, the spherical roots form a basis for $\Lambda(X)$ by Lemma 3.5.13. (Alternately, one can check this by an explict computation involving the cones associated to the toric variety $\mathbb{A}_k^r \cong Z$.) Statement (d) follows immediately from these facts along with (c).

It remains to prove (c). By (a), we have $Y \cap Z = \{y\}$, so the isomorphism $X_{B,Y} \cong P_u \times Z$ identifies $Y \cap X_{B,Y}$ with $P_u \times \{y\}$ and identifies $Z$ with $\{e\} \times Z$. It follows that the tangent spaces $T_yY$ and $T_yZ$ are $T$-submodules of the tangent space $T_yX \cong T_y(P_u \times Z)$ that intersect only in 0 (this is a general fact about the tangent space of the fiber product $P_u \times Z$). Thus, it will suffice to prove that

$$\dim(T_yX) = \dim(T_yY) + \dim(T_yZ). \tag{3.5.1}$$

If this equation holds, then we have $T_yX = T_yY \oplus T_yZ$, and statement (c) follows. To prove (3.5.1), note that $Y \cap X_{B,Y}$ is a $B$-orbit (Theorem 3.2.7), so we have

$$Y \cap X_{B,Y} \cong P_u \times (Y \cap Z) = P_u \times \{y\}.$$

It follows that $\dim(Y) = \dim(Y \cap X_{B,Y}) = \dim(P_u)$. On the other hand, the isomorphism $X_{B,Y} \cong P_u \times Z$ gives us

$$\dim(X) = \dim(X_{B,Y}) = \dim(P_u) + \dim(Z) = \dim(Y) + \dim(Z).$$

Since $X$, $Y$, and $Z$ are all smooth, this equation implies (3.5.1). $\square$

### 3.5.d Classifications of Some "Nice" Spherical Varieties

Before we turn to the general classification of homogeneous spherical varieties in the next section, we briefly discuss a few classifications of "nice" types of spherical varieties that can be done more easily. These classifications all involve considering varieties of low rank, which greatly restricts the possible varieties. Some of these classifications also play a key role in the proof of the the classification of homogeneous spherical varieties.

First, we consider spherical varieties of rank 0. Note that if $r(G/H) = 0$, then $\Lambda(G/H) = 0$, so for any embedding $G/H \hookrightarrow X$, we have $\varphi_D = 0$ for every $B$-divisor $D$ of $X$. In particular, if $D$ is a $G$-divisor, then $D$ agrees with the trivial valuation (which is also $G$-invariant) on all $B$-eigenvectors. It follows that $v_D$ is the trivial valuation (see Corollary 3.1.14), which is absurd. So, $X$ cannot have any $G$-divisors. Thus, the image of $G/H \hookrightarrow X$ is surjective (see Lemma 3.1.17), so we have $X = G/H$. In other words, every spherical variety of rank 0 is homogeneous.

Which homogeneous spherical varieties have rank 0 is determined by the following proposition. The proof is quite general and involves only the theory of reductive groups; in particular, the statement of the proposition holds in arbitrary characteristic.

**Proposition 3.5.23** ([Tim11, Proposition 10.1]). *Let $H \subset G$ be any subgroup. Then, $r(G/H) = 0$ if and only if $H$ is parabolic, i.e. if and only if $G/H$ is projective.*

Notice that for any parabolic subgroup $P \subset G$, the quotient $G/P$ is a spherical variety: indeed, any quotient of $G$ is smooth, any parabolic subgroup contains some Borel subgroup $B \subset P$, and since $B^-P \supset B^-B$ is an dense subset of $G$, it follows that the opposite Borel subgroup $B^-$ has a dense orbit in $G/P$. Thus, we have completely classified the rank-0 spherical varieties: they are all homogeneous, smooth, and projective, and they are precisely the varieties of the form $G/P$ for some parabolic subgroup $P \subset G$.

Next, we consider spherical varieties of rank 1. Luna–Vust theory reduces the classification of all spherical varieties to the homogeneous case. Moreover, for any homogeneous spherical variety $G/H$ of rank 1, we have $N(G/H) \cong \mathbb{Q}$. Since the valuation cone $\mathcal{V}(G/H)$ is a full-dimensional cone in $N(G/H)$, there are only a few possibilities for $\mathcal{V}(G/H)$.

1. $\mathcal{V}(G/H)$ is the single ray $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}_{\leq 0}$ in $N(G/H) \cong \mathbb{Q}$. In particular, $\mathcal{V}(G/H)$ is strictly convex, so the standard embedding $G/H \hookrightarrow X$ exists. Moreover, some generator for the cone $\mathcal{V}(G/H)$ necessarily generates the lattice $\Lambda(G/H) \cong \mathbb{Z}$ (this is a nice quirk of the rank-1 case), so Theorem 3.5.10 implies that $X$ is smooth and hence is a wonderful variety.

2. $\mathcal{V}(G/H)$ is all of $N(G/H)$, i.e. $G/H$ is horospherical.

In short, we just need to classify $G/H$ in the case where a (rank-1) wonderful embedding $G/H \hookrightarrow X$ exists and the case where $G/H$ is horospherical. The classifications of both cases were accomplished by Akhiezer [Akh83] in an analytic setting and by Brion [Bri89b] in an algebraic setting. We refer the interested reader also to [Tim11, Section 30.8] for a more detailed discussion of this classification.

For our purposes, the more interesting part of this classification is the case where a wonderful embedding $G/H \hookrightarrow X$ exists. Since every homogeneous spherical variety has at most one wonderful embedding (namely, the standard embedding, when this is smooth), this part of the classification boils down to a classification of all rank-1 wonderful varieties. It turns out that every such wonderful variety arises from the so-called *prime* (or *primitive*) rank-1 wonderful varieties via some standard constructions. There are only 15 prime rank-1 wonderful varieties; they can be listed out explicitly, and all their standard combinatorial invariants can be computed. (See [Was96, Section 2 and Table 1] for details and a list of combinatorial invariants, or [Tim11, Table 30.1] for explicit constructions of the wonderful varieties in question.)

In a similar way, rank-2 wonderful varieties are all induced by the same standard operations from so-called *prime* rank-2 wonderful varieties. Thus, the classification of rank-2 wonderful varieties reduces to the case of prime rank-2 wonderul varieties, which were classified by Wasserman [Was96]. In particular, we refer the reader to [Was96, Tables A-G] for a full list of all prime rank-2 wonderful varieties and their combinatorial invariants.

Unlike in the rank-1 case, Wasserman's classification does not yield a classification of all rank-2 spherical varieties. However, these classifications of wonderful varieties of rank $\leq 2$ play a crucial role in developing and proving the classification of all homogeneous spherical varieties, which is our next topic of discussion.

## 3.6 The Classification of Homogeneous Spherical Varieties

In this section, we develop the combinatorial invariants needed to classify homogeneous spherical varieties. We then briefly discuss the proof that these invariants classify homogeneous spherical varieties up to $G$-equivariant isomorphism. Combined with the classification of $G$-equivariant open embeddings $G/H \hookrightarrow X$ provided by Luna–Vust theory (see Section 3.3 above), this completely classifies all spherical varieties up to $G$-equivariant isomorphism.

Throughout this section, we largely omit proofs, both because they become increasingly technical and because the statements themselves will be more useful to us than their proofs. Instead, we briefly discuss the ideas behind the proofs and then give references to more rigorous presentations in the literature.

### 3.6.a Localizations at Sets of Simple Roots

In order to understand the combinatorial invariants needed to classify homogeneous spherical varieties, we first require an important construction called the *localization* of a spherical variety $X$ at a set $I \subset \Pi_G$ of simple roots of $G$. The main significance of this construction is that we can often use it to reduce to the case where $G$ has $\leq 2$ simple roots. In such a case, the lattice $\Lambda_G$ has rank 2, so any spherical $G$-variety has rank at most 2. Combined with certain classifications of spherical varieties in low rank (see Section 3.5.d), this will allow us to greatly constraint certain behavior of combinatorial invariants on spherical varieties. Our discussion of the localization at simple roots will be brief; we refer the reader to [Kno14a, Section 4] for a more thorough and rigorous treatment (see also [Per18, Section 3.5.1] and [Tim11, Section 30.9]).

Recall from Section 2.2.d that for any dominant one-parameter subgroup $\lambda : \mathbb{G}_m \to T$, we can use limits to describe a parabolic subgroup $P_\lambda$ containing $B$, a Levi subgroup $M_\lambda \subset P_\lambda$ containing $T$, the unipotent radical $U_\lambda = R_u(P_\lambda)$, and the opposite parabolic subgroup $P_\lambda^- = P_{\lambda^{-1}}$. For the current construction, we wish to decompose a $G$-variety $X$ in a nice way using limits of the form $\lim_{t\to 0}\lambda(t)x$ for $x \in X$. We can do this in the following way. Let $X$ be a complete normal $G$-variety, and let $\mathbb{G}_m$ act on $X$ via its image under $\lambda$. For any connected component $Y$ of the set $X^{\mathbb{G}_m}$ of $\mathbb{G}_m$-fixed points, we define

$$X_Y = \{x \in X \mid \lim_{t\to 0}\lambda(t)x \in Y\}.$$

Note that since $X$ is complete, the limit $\lim_{t\to 0}\lambda(t)x$ always exists. Moreover, this limit always lies in $X^{\mathbb{G}_m}$, because one can check by definition of the limit that for any $t' \in \mathbb{G}_m$, we have

$$\lambda(t')\cdot\lim_{t\to 0}\lambda(t)x = \lim_{t\to 0}\lambda(t')\lambda(t)x = \lim_{t\to 0}\lambda(t't)x = \lim_{t\to 0}\lambda(t)x.$$

It follows that

$$X = \bigsqcup_{Y} X_Y,$$

where the union is over all the connected components $Y$ of $X^{\mathbb{G}_m}$. This disjoint union is called the *Białynicki-Birula decomposition* of $X$, and the $X_Y$ are called the *cells* of the decomposition.

In general, the cells $X_Y$ will not always be well-behaved. However, because $X$ is normal, there will always be one choice of $Y$ such that $X_Y$ is nice. The following theorem makes this statement precise.

**Theorem 3.6.1** ([Kno14a, Proposition 4.1, Lemmas 4.2 and 4.3])**.** *Let $X$ be a complete normal $G$-variety, and let $\lambda : \mathbb{G}_m \to T$ be a one-parameter subgroup. There exists a unique connected component $S \subset X^{\mathbb{G}_m}$ such that $X_S$ is an open subset of $X$. Moreover, the following hold.*

*(a) The cell $X_S$ is a $P_\lambda$-stable subset of $X$, and $P_\lambda^-$ acts trivially on $X_S$.*

*(b) The map $x \mapsto \lim_{t\to 0}\lambda(t)x$ defines an $M_\lambda$-equivariant morphism*

$$\pi_S : X_S \to S,$$

*and for any $u \in R_u(P_\lambda)$ and any $x \in X_S$, we have $\pi_S(u\cdot x) = \pi_S(x)$.*

*(c) The morphism $\pi_S$ is affine and is a categorical quotient by $\mathbb{G}_m$, and general fibers of $\pi_S$ are irreducible. In particular, $S$ is a normal variety.*

**Definition 3.6.2.** With $X$ and $\lambda$ as Theorem 3.6.1 above, we call $S$ the *source* of $X$, and we call $X_S$ The *big cell* of the Białynicki-Birula decomposition for $X$.

Our intuition for the big cell $X_S$ is that the map $\pi_S : X_S \to S$ acts as a sort of "retraction" morphism from the open subset $X_S \subset X$ to the subvariety $S$ of $X$. Our hope is to use this construction to reduce from considering $X$ to considering the smaller variety $S$. In order to do this, we first need to understand the combinatorial invariants on $S$ in terms of those on $X$.

To begin, we note that $S$ is spherical (resp. toroidal) whenever $X$ is, so we really can consider our usual combinatorial invariants on $S$.

**Proposition 3.6.3** (cf. [Kno14a, Proposition 4.5]). *Let $X$ be a complete normal $G$-variety, let $\lambda : \mathbb{G}_m \to T$ be a one-parameter subgroup, and let $S \subset X^{\mathbb{G}_m}$ be the source of $X$. Then, $S$ is a complete normal $G$-variety. Moreover, if $X$ is a spherical (resp. toroidal) $G$-variety, then $S$ is a spherical (resp. toroidal) $M_\lambda$-variety.*

We are primarily interested in the case where $X$ is toroidal. So, let $X$ be a complete toroidal $G$-variety, and let $\lambda : \mathbb{G}_m \to T$ be a dominant one-parameter subgroup. For convenience, we write $X_\lambda = X_S$ for the big cell, $X^\lambda = S$ for the source, and

$$\pi_\lambda : X_\lambda \to X^\lambda$$

for the $M_\lambda$-equivariant map given by Theorem 3.6.1.

Note that $X^\lambda$ is a complete toroidal $M_\lambda$-variety by Proposition 3.6.3 above. In order to describe the combinatorial invariants of $X^\lambda$ in terms of those of $X$, we first require a little setup. Let $\mathcal{F}$ be the fan consisting of all the cones in the colored fan of $X$ (this defines a fan because $X$ is toroidal, see e.g. Theorem 3.5.6). Via the usual pairing between one-parameter subgroups and characters of $T$ (see the discussion of $P_\lambda$ in Section 2.2.d), we may view $\lambda$ as an element of $N_G = \mathrm{Hom}_{\mathbb{Z}}(\Lambda_G, \mathbb{Q})$. The inclusion map $\Lambda(X) \hookrightarrow \Lambda_G$ induces a map $N_G \to N(X)$ given by restricting maps $\Lambda_G \to \mathbb{Q}$ to $\Lambda(X)$, so we may consider the restriction of $\lambda$ to an element $\lambda^r \in N(X)$. Since $\lambda$ is dominant, it lies in the dominant Weyl chamber in $N_G$, so $-\lambda^r$ lies in $\mathcal{V}(X)$ by Theorem 3.4.1. Because $X$ is complete, Theorem 3.3.28 implies that $-\lambda^r$ lies in some cone in $\mathcal{F}$. In fact, since $\mathcal{F}$ is a fan, there exists a unique cone $\mathcal{C}(\lambda) \in \mathcal{F}$ containing $-\lambda^r$ in its relative interior.

Now, we define

$$V(\lambda) = \mathcal{C}(\lambda) + \mathbb{Q}_{\geq 0}\lambda^r.$$

Because $-\lambda^r \in \mathcal{C}(\lambda)^\circ$, the set $V(\lambda)$ is the subspace of $N(X)$ generated by the elements of $\mathcal{C}(\lambda)$. We then define

$$\mathcal{F}^\lambda = \{(\mathcal{C} + \mathbb{Q}_{\geq 0}\lambda^r)/V(\lambda) \mid \mathcal{C} \in \mathcal{F},\ -\lambda^r \in \mathcal{C}\}.$$

Note that $-\lambda^r \in \mathcal{C}$ implies $\mathcal{C}(\lambda) \subset \mathcal{C}$ (since $\mathcal{F}$ is a fan and $-\lambda^r \in \mathcal{C}(\lambda)^\circ$). It follows that $\mathcal{C} + \mathbb{Q}_{\geq 0}\lambda^r$ is a cone in $N(X)$ containing $V(\lambda)$, so it is not a strictly convex cone. However, the quotient $(\mathcal{C} + \mathbb{Q}_{\geq 0}\lambda)/V(\lambda)$ is a strictly convex cone in the vector space $N(X)/V(\lambda)$. Moreover, one can check that $\mathcal{F}^\lambda$ is a strictly convex fan in the vector space $N(X)/V(\lambda)$. (This follows formally from the fact that $\mathcal{F}$ is a strictly convex fan.)

With this construction of the fan $\mathcal{F}^\lambda$ in hand, we are now ready to describe the combinatorial invariants on $X^\lambda$.

**Theorem 3.6.4** ([Kno14a, Theorem 4.6 and Corollary 4.7]). *Let $X$ be a complete toroidal $G$-variety with corresponding fan $\mathcal{F}$, and let $\lambda : \mathbb{G}_m \to T$ be a dominant one-parameter subgroup.*

*(a) We have $\Lambda(X^\lambda) = \Lambda(X) \cap V(\lambda)^\perp$ and hence $N(X^\lambda) \cong N(X)/V(\lambda)$.*

(b) *The fan corresponding to the toroidal $G$-variety $X^\lambda$ is the fan $\mathcal{F}^\lambda$ in $N(X)/V(\lambda) \cong N(X^\lambda)$ defined above.*

(c) *We have*
$$\Psi_{M_\lambda, X^\lambda} = \Psi_{G,X} \cap V(\lambda)^\perp = \Psi_{G,X} \cap \lambda^\perp$$
*as subsets of $\Lambda(X^\lambda)$.*

**Corollary 3.6.5** (cf. [Tim11, Lemma 30.19 and following discussion])**.** *Let $X$ be a complete toroidal $G$-variety, and let $\lambda : \mathbb{G}_m \to T$ be a dominant one-parameter subgroup. If $X$ is smooth, then $X^\lambda$ is smooth as well.*

*Proof.* Since $X$ is smooth, any cone $\mathcal{C} \in \mathcal{F}$ is generated by a part of a basis for $\Lambda(X)$ (Theorem 3.5.10). A cone $(\mathcal{C}+\mathbb{Q}_{\geq 0}\lambda)/V(\lambda)$ in $\mathcal{F}^\lambda$ is generated by the images of the generators of $\mathcal{C}$ under the quotient map $N(X) \to N(X)/V(\lambda)$. This map induces an isomorphism between $\Lambda(X^\lambda)$ and the lattice $\Lambda(X)/V(\lambda)$, and the generators of $(\mathcal{C} + \mathbb{Q}_{\geq 0}\lambda)/V(\lambda)$ are a part of a basis for the lattice $\Lambda(X)/V(\lambda)$. Since $\mathcal{F}^\lambda$ is the fan corresponding to $X^\lambda$, we conclude that $X^\lambda$ is smooth by Theorem 3.5.10. □

We can also pass information about the valuations of colors from $X$ to $X^\lambda$.

**Proposition 3.6.6** ([Kno14a, Proposition 4.9]; cf. [Tim11, Lemma 30.19])**.** *Let $X$ be a complete toroidal variety, and let $\lambda : \mathbb{G}_m \to T$ be a dominant one-parameter subgroup. The map $D \mapsto \pi_\lambda^{-1}(D)$ defines a bijection between colors of $X^\lambda$ and $P_\lambda$-unstable colors of $X$. Moreover, for any colors $D \subset X^\lambda$, the valuation $\varphi_D$ is the restriction of $\varphi_{\pi_\lambda^{-1}(D)}$ to $\Lambda(X^\lambda) \subset \Lambda(X)$.*

Recall that $\Pi = \Pi(G,T)$ denotes the set of simple roots of $G$. For any subset $I \subset \Pi$, there exists a parabolic subgroup $P_I$ of $G$ containing $B$. We saw in Section 2.2.d that $P_I = P_\lambda$, where $\lambda : \mathbb{G}_m \to T$ is any dominant one-parameter subgroup such that $\lambda(\alpha) = 0$ for all $\alpha \in I$ and $\lambda(\beta) > 0$ for all $\beta \notin I$. Such a choice of $\lambda$ always exists; moreover, using the local structure theorem on $X$ and $X^\lambda$, one can show that for a fixed set $I$, we obtain the same $M_\lambda$-variety $X^\lambda$ for a general choice of $\lambda$ such that $P_I = P_\lambda$ (see e.g. [Tim11, Lemma 30.19 and following discussion]). For such a general choice of $\lambda$, we write $X^I = X^\lambda$ and call $X^I$ the *localization of $X$ at the set $I$*. In what follows, we will generally use the notation $X^I$ instead of $X^\lambda$, since we are typically more interested in the set of simple roots $I$ than in the one-parameter subgroup $\lambda$.

### 3.6.b Combinatorial Properties of Simple Roots

Note that any homogeneous spherical variety $G/H$ has no $G$-divisors and so consists of an open $B$-orbit and a finite set of colors (see Lemma 3.1.17). Our main task in classifying $G/H$ is to find a way to encode the information of the colors of $G/H$ into combinatorial data. As we have seen in Luna–Vust theory, the valuation $\varphi_D$ for a color $D \in \Delta(G/H)$ is a useful piece of combinatorial data that captures much of the information about this divisor. However, we will also need to know "just how $G$-unstable" a color $D$ is. More precisely, we wish to understand the subgroup $G_D = \{g \in G \mid gD = D\}$ of $G$. Since $D$ is a $B$-divisor, we have $B \subset G_D$, so $G_D$ is parabolic. This implies that $G_D = P_{I_D}$ for some set of simple roots $I_D \subset \Pi$. We can thus reduce the question of finding the subgroup $G_D$ to the more combinatorial question of describing the set of simple roots $I_D$ for any given color $D$.

In practice, it is more convenient to rephrase this question as follows: for any given simple root $\alpha$, which colors $D$ have $\alpha \in I_D$? For this, we note that $\alpha \in I_D$ if and only if $P_{\{\alpha\}} \subset P_{I_D} = G_D$ (see Section 2.2.d). We are thus interested in which divisors $D$ satisfy $P_{\{\alpha\}} \cdot D = D$ for any given simple root $\alpha$. We now introduce some terminology to describe this situation. For convenience, we write $P_\alpha$ for the parabolic subgroup $P_{\{\alpha\}}$.

**Definition 3.6.7.** Let $X$ be a spherical variety, let $D \in \mathcal{D}_{G,X}$ be a $B$-divisor, and let $\alpha \in \Pi$ be a simple root.

1. We say that $\alpha$ *moves* $D$ if $P_\alpha \cdot D \neq D$.
2. We define
$$\mathcal{D}_{G,X}(\alpha) = \{D \in \mathcal{D}_{G,X} \mid \alpha \text{ moves } D\}.$$

**Remark 3.6.8.** Note that any $B$-divisor $D \in \mathcal{D}_{G,X}$ is moved by some simple root if and only if $D$ is a color. Indeed, using our above notation, a $B$-divisor $D$ is a color of $X$ if and only if $G_D \neq G$, i.e. if and only if $I_D \neq \Pi$; on the other hand, the simple roots that move $D$ are the elements of $\Pi \setminus I_D$, so such a simple root exists if and only if $I_D \neq \Pi$. In terms of equalities on sets, we have
$$\mathcal{D}_{G,X} = \mathcal{D}_{G,X}^G \sqcup \bigcup_{\alpha \in \Pi} \mathcal{D}_{G,X}(\alpha).$$
Note that the union of the $\mathcal{D}_{G,X}(\alpha)$ here is typically not a disjoint union, as one color may generally be moved by multiple simple roots.

**Remark 3.6.9.** For any $\alpha \in \Pi$, the set $\mathcal{D}_{G,X}(\alpha)$ is in some sense a "$G$-equivariant birational invariant" of $X$. More precisely, elements of $\mathcal{D}_{G,X}(\alpha)$ are colors, which are uniquely determined by their intersections with the open $G$-orbit $G/H$ of $X$. Moreover, continuity of the action morphism $G \times X \to X$ implies that $P_\alpha \cdot D = D$ if and only if $P_\alpha \cdot (D \cap G/H) = D \cap G/H$. So, we see that $\alpha$ moves a color $D$ of $X$ if and only if $\alpha$ moves the color $D \cap G/H$ of $G/H$.

The set $\mathcal{D}_{G,X}(\alpha)$ turns out to enjoy many remarkably nice properties related to the simple root $\alpha$. Many of these properties were first proven by Luna in [Lun97], and they later played a large role in the classification of homogeneous spherical varieties, which was first proposed by Luna in [Lun01]. Luna's proofs in [Lun97] relied heavily on the localization at sets of $\leq 2$ simple roots (see Section 3.6.a) and the classification of spherical varieties of rank $\leq 1$ and of wonderful varieties of rank 2 (see Section 3.5.d). On the other hand, Knop has since provided an alternative presentation of these properties of $\mathcal{D}_{G,X}(\alpha)$, which relies more on the theory of reductive groups and which holds in arbitrary characteristic ([Kno14a]).

The following theorem gives us the first (and probably most important) property of the sets $\mathcal{D}_{G,X}(\alpha)$. The theorem states that there are only 4 different possible types of behavior for $\mathcal{D}_{G,X}(\alpha)$, which correspond to 4 different possible localizations $(X')^{\{\alpha\}}$ for some "nice" embedding $X'$ of the open $G$-orbit $G/H \subset X$.

**Theorem 3.6.10** ([Tim11, Section 30.10]; cf. [Kno14a, Section 2])**.** *Let $G/H$ be a homogeneous spherical variety, let $X$ be a spherical embedding of $G/H$, and let $X'$ be a smooth complete toroidal embedding of $G/H$ (such a choice of $X'$ exists by Proposition 3.5.8 and Theorem 3.5.10). For any $\alpha \in \Pi$, the localization $(X')^\alpha$ of $X'$ at the set $\{\alpha\}$ is a smooth complete toroidal variety under the action of $S_\alpha = [M_\alpha, M_\alpha]$. Moreover, exactly one of the following possibilities takes place.*

*(a)* $\mathcal{D}_{G,X}(\alpha) = \varnothing$. *In this case,* $(X')^\alpha = \operatorname{Spec}(k)$ *(with the trivial action of* $S_\alpha$*).*

*(b)* $\alpha \in \Psi_{G,X}$. *In this case, we have* $\#\mathcal{D}_{G,X}(\alpha) = 2$ *and* $S_\alpha \cong \mathrm{SL}_2$ *or* $\mathrm{PSL}_2$. *Moreover, we have* $(X')^\alpha \cong \mathbb{P}^1 \times \mathbb{P}^1$*, with the action of* $S_\alpha$ *given by Example 3.1.16 (the group* $\mathrm{SL}_2$ *is used in that example, but the action of* $\mathrm{SL}_2$ *induces an action of* $\mathrm{PSL}_2 = \mathrm{SL}_2/\{\pm I\}$ *as well).*

*(c)* $2\alpha \in \Psi_{G,X}$. *In this case, we have* $\#\mathcal{D}_{G,X}(\alpha) = 1$ *and* $(X')^\alpha \cong \mathbb{P}(\mathfrak{sl}_2)$*, with the action of* $S_\alpha \cong \mathrm{PSL}_2$ *induced by conjugation of matrices in* $\mathfrak{sl}_2$.

*(d)* $\mathcal{D}_{G,X}(\alpha) \neq \varnothing$ *and* $\mathbb{Q}\alpha \cap \Psi_{G,X} = \varnothing$. *In this case, we have* $\#\mathcal{D}_{G,X}(\alpha) = 1$ *and* $S_\alpha \cong \mathrm{SL}_2$ *or* $\mathrm{PSL}_2$. *Moreover, either* $(X')^\alpha \cong \mathbb{P}^1$*, with the action of* $S_\alpha$ *given by Example 2.4.19, or* $(X')^\alpha \cong S_\alpha \times^{B\cap S_\alpha} \mathbb{P}^1$*, with* $B \cap S_\alpha$ *acting on* $\mathbb{P}^1$ *via a character.*

*Sketch of proof.* Everything about $\mathcal{D}_{G,X}(\alpha)$ and $\Psi_{G,X}$ in the statement depends only on the open $G$-orbit $G/H$ of $X$ (see Remark 3.6.9 above). Thus, it will suffice to consider the case where $X = X'$.

The localization $X^\alpha$ is a smooth complete toroidal $M_\alpha$-variety by Proposition 3.6.3 and Corollary 3.6.5. The fact that we can use the action of $S_\alpha$ on $X^\alpha$ in place of $M_\alpha$ follows from certain facts about the construction of the localization $X^\alpha$. More precisely, for a general choice of the one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ in the construction of $X^\alpha$, the localization $X^\alpha$ consists of points fixed by the torus $Z(M_\alpha)^0$ (see [Tim11, Lemma 30.19 and surrounding discussion]). So, the action of $M_\alpha$ on $X^\alpha$ descends to an action of the quotient $M_\alpha/Z(M_\alpha)^0$. On the other hand, the composition $[M_\alpha, M_\alpha] \hookrightarrow M_\alpha \to M_\alpha/Z(M_\alpha)^0$ is an isogeny (see [Mil17, proof of Proposition 21.60]), so replacing $M_\alpha/Z(M_\alpha)^0$ by $S_\alpha = [M_\alpha, M_\alpha]$ does not change anything. Note that $S_\alpha$ is a semisimple group whose only simple root is $\alpha$, so the classification of reductive groups implies that $S_\alpha \cong \mathrm{SL}_2$ or $S_\alpha \cong \mathrm{PSL}_2$. In particular, any maximal torus of $S_\alpha$ has rank 1, so $r(X^\alpha) \leq 1$, and since $\dim(S_\alpha) = 3$ and $X^\alpha$ has an open $S_\alpha$-orbit, we have $\dim(X^\alpha) \leq 3$.

First, consider the case where $\mathcal{D}_{G,X}(\alpha) = \varnothing$. In this case, we have $P_\alpha \cdot D = D$ for every $B$-divisor $D$ of $X$, i.e. every $B$-divisor is $P_\alpha$-stable. Proposition 3.6.6 implies that $X^\alpha$ has no colors, so Proposition 3.1.19 implies that $X^\alpha$ is toric for a quotient of $S_\alpha/[S_\alpha, S_\alpha]$. Since the dimension and rank of a torus are equal, we have $\dim(X^\alpha) = r(X^\alpha) \leq 1$. The only complete toric varieties of dimension $\leq 1$ are $\mathrm{Spec}(k)$ and $\mathbb{P}^1$ (with the torus $T'$ acting on $\mathbb{P}^1$ by a nontrivial character); however, the only $S_\alpha$-variety structure on $\mathbb{P}^1$ such that $\mathbb{P}^1$ is a spherical $S_\alpha$-variety is the one in Example 2.4.19, and the quotient $S/[S_\alpha, S_\alpha]$ does not act by a character on $\mathbb{P}^1$ in this action. So, we must have $X^\alpha = \mathrm{Spec}(k)$.

Next, suppose that we have $c\alpha \in \Psi_{G,X}$ for some $c \in \mathbb{Q}^\times$. By definition, $X^\alpha$ is the variety $X^\lambda$ for some one-parameter subgroup $\lambda$ such that $\alpha \in \lambda^\perp$, so Theorem 3.6.4 implies that $c\alpha \in \Psi_{S_\alpha, X^\alpha}$. In particular, we have $r(X^\alpha) \geq 1$ and hence $r(X^\alpha) = 1$. Moreover, since $X^\alpha$ has a spherical root, the cone $\mathcal{V}(X^\alpha)$ is not all of $N(X^\alpha) \cong \mathbb{Q}$ and hence is a single ray (either $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}_{\leq 0}$). It follows that there is at most one colored cone in the colored fan $\mathscr{F}_{X^\alpha}$, so $X^\alpha$ is simple. But $X^\alpha$ is smooth, complete and toroidal as well, so $X^\alpha$ must be a smooth standard embedding, hence wonderful. Using the fact that $c\alpha \in \Psi_{S_\alpha, X^\alpha}$, The classification of rank-1 wonderful varieties gives us only two possibilities for $X^\alpha$.

1. $X^\alpha \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the variety of Example 3.1.16. In this case, $\alpha$ is the unique spherical root of $X^\alpha$, and $X^\alpha$ has 2 colors.

2. $X^\alpha \cong \mathbb{P}(\mathfrak{sl}_2) \cong \mathbb{P}^2$, with the action of $S = \mathrm{PSL}_2$ given by conjugation of matrices in the Lie algebra $\mathfrak{sl}_2$. In this case, $2\alpha$ is the unique spherical root of $X^\alpha$, and $X^\alpha$ has a unique color.

Since the colors of $X^\alpha$ are in bijection with $\mathcal{D}_{G,X}(\alpha)$, these two options give us the situations of (b) and (c). Notice that since $\mathcal{D}_{G,X}(\alpha) \neq \varnothing$ in each of these cases, the situations of (b) and (c) are mutually exclusive with (a).

The only remaining possibility is that we have both $\mathcal{D}_{G,X}(\alpha) \neq \varnothing$ and $\mathbb{Q}\alpha \cap \Psi_{G,X} = \varnothing$. By definition, $X^\alpha$ is the variety $X^\lambda$, where $\alpha$ is the unique simple root lying in $\lambda^\perp \subset \Lambda_G$. Since every spherical root is a sum of positive roots (this follows from the statement about the antidominant Weyl chamber in Theorem 3.4.1), we see that $\lambda$ is positive on every spherical root, so Theorem 3.6.4 gives us $\Psi_{S_\alpha, X^\alpha} = \varnothing$. We are left with two cases, depending on the rank of $X^\alpha$.

1. If $r(X^\alpha) = 0$, the classification of rank-0 varieties (along with the fact that $X^\alpha$ has at least one color, since $\mathcal{D}_{G,X}(\alpha) \neq \varnothing$) implies that $X \cong S_\alpha/(B \cap S_\alpha)$, which is isomorphic to $\mathbb{P}^1$ with the action of $S_\alpha \cong \mathrm{SL}_2$ or $\mathrm{PSL}_2$ given by Example 2.4.19. In this case, the unique color of $X^\alpha$ is the $B$-fixed point $[1:0] \in \mathbb{P}^1$.

2. If $r(X^\alpha) = 1$, then $\Psi_{S_\alpha, X^\alpha} = \varnothing$ implies that that $X^\alpha$ is horospherical. Since $X^\alpha$ is smooth and complete, it follows from the classification of rank-1 spherical varieties that $X^\alpha \cong S_\alpha \times^Q \mathbb{P}^1$ for some parabolic subgroup $Q \subset S_\alpha$. The only parabolic subgroups of $S_\alpha$ containing $B \cap S_\alpha$ are $B \cap S_\alpha$ and $S_\alpha$, and $Q \neq S_\alpha$ because otherwise $X^\alpha$ would not be a spherical $S_\alpha$-variety. So, $Q = B \cap S_\alpha$, and one can check that the unique color of $X^\alpha$ is $\{1\} \times \mathbb{P}^1$.

In either case, $X^\alpha$ has a unique color, so $\mathcal{D}_{G,X}(\alpha)$ has a unique element. $\square$

**Definition 3.6.11.** Let $X$ be a spherical variety.

1. We say that a simple root $\alpha \in \Pi$ is *of type a* (resp. of type $b$, $c$, $d$) for $X$ if possibility $a$ (resp. $b$, $c$, $d$) in Theorem 3.6.10 takes place for $\alpha$.

2. We denote by $\Pi_X^a$ (resp. $\Pi_X^b$, $\Pi_X^c$, $\Pi_X^d$) the set of simple roots of type $a$ (resp. $b$, $c$, $d$) for $X$.

3. When $\alpha \in \Pi$ is of type c or d, we write $D_\alpha$ for the unique element of $\mathcal{D}_{G,X}(\alpha)$. When $\alpha$ is of type b, we write $D_\alpha^+$ and $D_\alpha^-$ for the two elements of $\mathcal{D}_{G,X}(\alpha)$.

One remarkable consequence of Theorem 3.6.10 is that any simple root $\alpha$ moves at most two colors of any spherical variety. Conversely, the following proposition greatly constrains when we can have multiple simple roots moving the same color. The proposition was originally proven by Luna in [Lun01] by passing to a homogeneous spherical variety $G/H$, then to the quotient by the very sober hull $G/\overline{H}$, then considering the wonderful embedding of $G/\overline{H}$, and finally using localization at a set of 2 simple roots along with the classification of wonderful varieties of rank 2. Knop has given an alternative proof in [Kno14a] that relies mainly on the classification of spherical varieties of rank $\leq 1$ (along with a few technical arguments from the theory of reductive groups).

**Proposition 3.6.12** ([Lun01, Proposition 3.2], [Kno14a, Lemma 2.5 and Proposition 5.4])**.** *Let $X$ be a spherical variety, and let $\alpha, \beta \in \Pi$.*

*(a) If $\alpha, \beta \in \Pi_X^b$ are both roots of type b, then there exists at most one color of $X$ moved by both $\alpha$ and $\beta$.*

*(b) If $\langle \alpha^\vee, \beta \rangle = 0$ and $\alpha + \beta = \gamma$ or $2\gamma$ for some $\gamma \in \Psi_{G,X}$, then $\alpha, \beta \in \Pi_X^d$ are both roots of type d which move the same color of $X$, and $\alpha^\vee|_{\Lambda(X)} = \beta^\vee|_{\Lambda(X)}$.*

*Conversely, if there exists a color of $X$ moved by both $\alpha$ and $\beta$, then either the assumptions of (a) are satisfied, or the assumptions of (b) are satisfied.*

Using the localization $(X')^\alpha$ given by Theorem 3.6.10, we can also obtain some strong statements about the valuations of colors moved by a root $\alpha$. As with Theorem 3.6.10, Knop has provided an alternative proof that does not use localizations in [Kno14a].

**Proposition 3.6.13** ([Tim11, Lemma 30.20], [Kno14a, Proposition 2.3])**.** *Let $X$ be a spherical variety, and let $\alpha \in \Pi$ be a simple root not of type a. Depending on the type of $\alpha$, the following relations hold.*

*(Type b) We have*

$$\varphi_{D_\alpha^+} + \varphi_{D_\alpha^-} = \alpha^\vee|_{\Lambda(X)}$$

*and $\varphi_{D_\alpha^\pm}(\alpha) = 1$ (which makes sense because $\alpha$ is a spherical root, so in particular, $\alpha \in \Lambda(X)$).*

*(Type c) We have* $\varphi_{D_\alpha} = \frac{1}{2}\alpha^\vee|_{\Lambda(X)}$.

*(Type d) We have* $\varphi_{D_\alpha} = \alpha^\vee|_{\Lambda(X)}$.

*Sketch of proof.* Since the valuations of colors will be the same for any embedding of the open $G$-orbit $G/H$ of $X$, we may replace $X$ by a smooth complete toroidal embedding of $G/H$. In this case, Proposition 3.6.6 implies that the valuations of colors moved by $\alpha$ can be computed on the localization $X^\alpha$. In particular, when $\alpha$ is type $b$, the statement that $\varphi_{D_\alpha^\pm}(\alpha) = 1$ can be checked directly using the fact that $X^\alpha \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Now, let $S_\alpha$ be as in Theorem 3.6.10. fix $\mu \in \Lambda(X^\alpha)$, fix a closed $S_\alpha$-orbit $Y^\alpha \subset X^\alpha$, and let $\delta = \sum_{D \in \mathcal{D}_{M,X^\alpha}} \varphi_D(\mu) D$. Note that if $\alpha$ is type $d$ and $X^\alpha \cong \mathbb{P}^1$, then $\Lambda(X^\alpha) = 0$, so there is nothing to prove. In every other case, $X^\alpha$ is a smooth projective surface, and $Y^\alpha$ is a smooth projective curve. So, we may consider the intersection pairing $\langle Y^\alpha, \delta \rangle$. We compute this intersection pairing in two different ways. One way to compute it is to explicitly calculate it for each of the possibilities of $X^\alpha$. This gives $\varphi_{D_\alpha^+} + \varphi_{D_\alpha^-}$, $2\varphi_{D_\alpha}$, or $\varphi_{D_\alpha}$ for $\alpha$ of type $b$, $c$, or $d$ (respectively). The other way to compute the intersection pairing is to compute the degree of $\mathcal{O}_{X^\alpha}(\delta)|_{Y^\alpha}$ using some facts about the geometry of spherical varieties. This yields $\langle \alpha^\vee, \mu \rangle$ in all 3 cases. See [Tim11, Lemma 30.20] for details. $\square$

The above proposition tells us that when $\alpha$ is of type $c$ or $d$, the valuation $\varphi_{D_\alpha}$ is determined by the coroot $\alpha^\vee$. This in some sense tells us the entire geometry of the color $D_\alpha$ in terms of the combinatorial data of $\alpha^\vee$. Because of this, we will not need to keep track of the valuations of colors moved by roots of type $c$ or $d$ in the classification of homogeneous spherical varieties. However, we will need to keep track of the valuations of colors moved by roots of type $b$, because the above proposition shows that there may in general be multiple possibilities for these valuations.

So far, we have not said much about roots of type $a$. From the perspective of roots moving divisors, there is not much to say. However, there are a couple standard facts about roots of type $a$ that bear mentioning. Let $X$ be a spherical variety with open $B$-orbit $X_B^\circ$. Then, the roots of type $a$ for $X$ are related to the parabolic subgroup

$$P_X = \{g \in G \mid gX_B^\circ = X_B^\circ\}$$

of Lemma 3.2.10. Indeed, for any $\alpha \in \Pi$, since $X \setminus X_B^\circ$ is the union of all $B$-divisors of $X$, we have $P_\alpha \subset P_X$ if and only if $P_\alpha \cdot D = D$ for all $D \in \mathcal{D}_{G,X}$, i.e. if and only if $\alpha$ has type $a$ for $X$. This implies that $P_X = P_{\Pi_X^a}$.

There is one other interesting thing that we can say about roots of type $a$: namely, that $\alpha^\vee|_{\Lambda(X)} = 0$ for all $\alpha \in \Pi_X^a$. One can see this as a sort of analogue of Proposition 3.6.13: that proposition relates $\alpha^\vee|_{\Lambda(X)}$ to the valuations of colors moved by $\alpha$, but when $\alpha$ is of type $a$, our claim here is that both of these concepts are trivial (that is, there are no divisors moved by $\alpha$, and $\alpha^\vee$ is just 0 on $\Lambda(X)$). In fact, the proof of Proposition 3.6.13 given in [Kno14a, Proposition 2.3] also yields the statement that $\alpha^\vee|_{\Lambda(X)} = 0$. However, we instead give an argument using the local structure theorem. We do not know of a reference for this proof in the literature, but it seems to be well-known (for instance, it is mentioned without any details in [Los09c]).

**Lemma 3.6.14** ([Los09c, Lemma 3.5.7], [Kno14a, Proposition 2.3])**.** *Let $X$ be a spherical variety, and let $\alpha \in \Pi_X^a$. We have $\alpha^\vee|_{\Lambda(X)} = 0$.*

*Proof.* Let $Y$ be the open $G$-orbit of $X$. Then, $X_{B,Y}$ is the open $B$-orbit of $X$ by definition (see Theorem 3.2.7 and Lemma 3.1.17). Let $P$, $M$, and $Z$ be as in the local structure theorem applied to $X_{B,Y}$ (see Theorem 3.2.2, which applies by Theorem 3.2.7a). Since every $B$-divisor of $X$ is in the compliment $X \setminus X_{B,Y}$, Proposition 3.2.3f implies that $Z$ has no $(B \cap M)$-divisors at all. In particular, $Z$ has no colors, so Proposition 3.1.19 implies that $Z$ is a toric $T'$-variety for some quotient $T'$ of $M/[M, M]$. Moreover, since $Z$ also has no $M$-divisors, $Z$ is a single $M$-orbit, i.e. $Z = T'$.

Now, Proposition 3.2.3d gives us $\Lambda(X) = \Lambda(Z) = \Lambda(T')$. So, we wish to prove that $\alpha^\vee|_{\Lambda(T')} = 0$, where we view $\Lambda(T')$ as a subgroup of $\mathcal{X}(T) = \Lambda(T)$ via the inclusion $i : \Lambda(T') \hookrightarrow \Lambda(T)$ induced by the quotient map $\rho : T \to T'$ (explicitly, the map $i$ sends any character $\mu : T' \to \mathbb{G}_m$ to the composition $\mu \circ \rho$). This is essentially a matter of tracing through some theory of reductive groups to see what $\alpha^\vee$ actually is. We saw in Theorem 2.2.6 that $\alpha^\vee$ is constructed in the following way. We first define a certain element $s_\alpha$ in the Weyl group $W(G, T) = N_G(T)/C_G(T)$. We then lift $s_\alpha$ to an element $n_\alpha \in N_G(T)$, and conjugation by $n_\alpha$ induces an automorphism $j_\alpha : T \to T$, which in turn induces an automorphism $\iota_\alpha : \Lambda(T) \xrightarrow{\sim} \Lambda(T)$ that sends any character $\mu : T \to \mathbb{G}_m$ to the composition $\mu \circ j_\alpha$. Theorem 2.2.6d states that there is a unique $\alpha^\vee \in \Lambda(T)^\vee$ such that for all $\mu \in \Lambda(T)$, we have

$$\iota_\alpha(\mu) = \mu - \langle \alpha^\vee, \mu \rangle \alpha.$$

This is taken as the definition of the coroot $\alpha^\vee$.

Now, let $n_\alpha \in N_G(T)$ be as above. Since the quotient map $\rho : T \to T'$ extends to a map $G \to G/[G, G] \to T'$, we may consider the element $\rho(n_\alpha) \in T'$. Since $\rho$ is a homomorphism, we see that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\rho"] \arrow[d, "t \mapsto n_\alpha t n_\alpha^{-1}"'] & T' \arrow[d, "t' \mapsto \rho(n_\alpha) t' \rho(n_\alpha)^{-1}"] \\ T \arrow[r, "\rho"] & T' \end{tikzcd}$$

Passing to character groups, this commutativity statement says that the automorphism $\iota_\alpha$ restricts to the automorphism of $\Lambda(T')$ induced by conjugation by $\rho(n_\alpha)$. But $T'$ is commutative and $\rho(n_\alpha) \in T'$, so conjugation by $\rho(n_\alpha)$ is the identity on $T'$. So, for any $\mu \in \Lambda(T')$, we have

$$\mu = \iota_\alpha(\mu) = \mu - \langle \alpha^\vee, \mu \rangle \alpha,$$

which implies that $\langle \alpha^\vee, \mu \rangle = 0$. $\square$

### 3.6.c Homogeneous Spherical Data

We are now almost ready to present the classification of homogeneous spherical varieties. We just require a few more facts about the behavior of spherical roots and simple roots. The proofs of these facts make use of a construction called "the localization at a set of spherical roots," which is similar to (though somewhat less technical than) the localization at a set of simple roots. While the localization at a set of simple roots allows us to reduce to the case where $G$ has only one or two simple roots, the localization at a set of spherical roots allows us to reduce to the case where a toroidal variety $X$ has only one or two spherical roots, which is sometimes more useful. We refer the reader to [Kno14a, Section 6] for details.

**Theorem 3.6.15.** *Any spherical root of a spherical variety $X$ is a spherical root of some rank-1 wonderful variety $X'$. Moreover, we may coose $\Pi^a_X = \Pi^a_{X'}$.*

*Proof.* Since spherical roots and $\Pi^a_X$ are $G$-equivariant birational invariants, we may as well consider the case of a spherical root $\gamma$ of a homogeneous spherical variety $G/H$. Pick a complete toroidal embedding $G/H \hookrightarrow X$ (one exists by Proposition 3.5.8). By the definition of a spherical root, the intersection $\gamma^\perp \cap \mathcal{V}(X)$ is a face of the cone $\mathcal{V}(X)$. Since $X$ is complete, some cone $\mathcal{C}$ in the fan corresponding to $X$ intersects $\gamma^\perp \cap \mathcal{V}(X)$. We have $\mathcal{C} \subset \mathcal{V}(X)$ (because $X$ is toroidal), so $\gamma^\perp \cap \mathcal{C}$ is a face of the cone $\mathcal{C}$. This face is another cone in the fan corresponding to $X$, so it corresponds to a $G$-orbit $Y$ of $X$. Now, $Y$ is itself a (homogeneous) spherical $G$-variety (Theorem 3.1.8); moreover, $Y$ is actually the so-called localization of $X$ at the set of spherical roots $\{\gamma\}$.

Using standard properties of this "localization at a set of spherical roots" (see [Kno14a, Proposition 6.1]), we find that $r(Y) = 1$, that $\Pi^a(Y) = \Pi^a(X)$, and that $\gamma$ is a spherical root of $Y$. Moreover, we have $N(Y) \cong \mathbb{Q}$, and since there exists a spherical root of $Y$, the cone $\mathcal{V}(Y)$ is either $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}_{\leq 0}$. In particular, $\mathcal{V}(Y)$ is strictly convex, so we may consider the standard embedding $Y'$ of $Y$. Since $\gamma \in \Lambda(Y)$ is an indivisible element of the lattice and this lattice has rank 1, we see that $\gamma$ generates $\Lambda(Y)$. So, $Y'$ is smooth by Lemma 3.5.13 and hence is a wonderful variety. Finally, we note that $r(Y') = r(Y) = 1$ and $\gamma \in \Psi_{G,Y} = \Psi_{G,Y'}$. Thus, $Y'$ is the desired wonderful variety. □

Recall from Section 3.5.d that all wonderful varieties of rank 1 have been completely classified. The above theorem tells us that this classification completely determines all possible spherical roots of any spherical $G$-variety. By consulting the list of rank-1 wonderful varieties, one in particular finds that for any choice of $G$, the set of all possible spherical roots is finite. A list of all of them (for every reductive group $G$) can be found in [Tim11, Table 30.2].

**Definition 3.6.16.** We denote by $\Sigma_G$ the (finite) set consisting of all elements of $\Lambda_G$ which are a spherical root of some spherical $G$-variety.

The list of all spherical roots for all groups $G$ in [Tim11, Table 30.2] is rather long, mainly because there exist constructions that take a spherical $G$-variety $X$ and turn it into a spherical $G'$-variety $X'$ for some other group $G'$ while "preserving" spherical roots in some sense. This makes it so that a spherical root of the same form appears in $\Sigma_G$ for many different reductive groups $G$. We can shorten the list in [Tim11, Table 30.2] by considering only the so-called "prime" (or "primitive") rank-1 wonderful varieties, a list of which can be found in [Was96, Table 1] (or in [Tim11, Table 30.1]). The prime wonderful varieties are the ones that cannot be obtained from any other wonderful variety using two specific constructions, namely: taking fiber products, and a construction known as "parabolic induction." Parabolic induction does not change either $\Psi_{G,X}$ or $\Pi^a_X$, but taking a fiber product can make both of these sets bigger (see e.g. [Lun01, Sections 3.4, 3.5]). As such, every spherical root comes from a prime rank-1 wonderful variety, but passing to that wonderful variety may make the set $\Pi^a_X$ smaller.

The following corollary makes this statement precise. The corollary itself is somewhat technical, and it is not needed for the classification of homogeneous spherical varieties; we prove it here primarily so we can use it later (specifically, in Section 4.6). As such, the reader can safely skip this corollary and its proof and refer back to it as needed.

**Corollary 3.6.17.** *Let $X$ be a spherical variety, and let $\gamma \in \Psi_{G,X}$ be a spherical root. There exists a semisimple simply connected group $G'$ with $\Pi_{G'} \subset \Pi_G$ and a "prime" rank-1 wonderful $G'$-variety $X'$ (i.e. a wonderful variety on the list in [Was96, Table 1]) such that $\Psi_{G,X'} = \{\gamma\}$ and $\Pi^a_{X'} \subset \Pi^a_X$.*

*sketch of proof.* By Theorem 3.6.15, it suffices to prove the statement when $X$ is a wonderful variety. By definition, a "prime" wonderful $G'$-variety (see [Was96, Definition 2.3]) is a wonderful variety $X'$ such that

1. $X'$ cannot be obtained by a construction called "parabolic induction" (see [Tim11, Definition 5.9]) from some spherical $G''$-variety $X''$, and

2. the open $G$-orbit $G'/H'$ of $X'$ cannot be written as $G'/H' = G_1/H_1 \times G_2/H_2$ for some subgroups $G_1, G_2 \subset G'$ and $H_i \subset G_i$.

Let $G' = G$, let $G'/H'$ be the open $G$-orbit of $X$, and let $X' = X$. Our plan is to repeatedly replace the pair $(G', H')$ and replace $X'$ by the wonderful embedding of $G'/H'$ until we can no longer obtain $G'$, $H'$, and $X'$ from either parabolic induction or fiber products as above. Then, $X'$ will have the desired properties provided we can check that $\Psi_{G',G'/H'}$ and $\Pi^a_{G'/H'}$ behave nicely under these replacements. Note that (except in trivial cases), both fiber products and parabolic induction increase $\dim(G')$, so every one of our replacements will decrease $\dim(G')$. We will thus require only finitely many such replacements. Once we have an acceptable choice of $X'$, we can replace $G'$ by the universal cover of $G'/Z(G')^\circ$ and so take $G'$ to be semisimple and simply connected (see [Was96, Remark 1.5]).

First, we consider parabolic induction. Whether $X'$ can be obtained by parabolic induction is actually a condition on $G'$ and $H'$ (see [Was96, Lemma 2.2]). Moreover, if $X'$ is obtained by parabolic induction from some spherical $G''$-variety $X''$, then $\Pi_{G''} \subset \Pi_{G'}$

by definition of parabolic induction. Also, $X''$ is wonderful because $X'$ is, and we have $\Psi_{G'',X''} = \Psi_{G',X'}$ and $\Pi^a_{G'',X''} = \Pi^a_{G',X'}$ (see [Lun01, Section 3.4]). Thus, we may replace $(G', H')$ by $(G'', H'')$.

As for fiber products, suppose that $G'/H' = G_1/H_1 \times G_2/H_2$. Then, we have $N_{G'}(H') = N_{G_1}(H_1) \times N_{G_2}(H_2)$, so both $H_1$ and $H_2$ are sober because $H'$ is. Moreover, if $X_i$ is the wonderful embedding of $G_i/H_i$, then $X_1 \times X_2$ is a smooth complete spherical $G'$-variety which is simple and toroidal, and the open orbit of $X_1 \times X_2$ is $G'/H'$. So, $X_1 \times X_2$ is the wonderful embedding $X'$ of $G'/H'$, and this implies that $\Psi_{G',X'} = \Psi_{G_1,X_1} \sqcup \Psi_{G_2,X_2}$ and $\Pi^a_{G',X'} = \Pi^a_{G_1,X_1} \sqcup \Pi^a_{G_2,X_2}$ (see [Lun01, Section 3.1, discussion of "factorizations"]). Thus, we may replace the pair $(G', H')$ by whichever pair $(G_i, H_i)$ has $\gamma \in \Psi_{G',X_i}$. $\square$

The following proposition is the last combinatorial fact that we need before we can present the classification of homogeneous spherical varieties.

**Proposition 3.6.18** ([Kno14a, Proposition 6.5], [Kno14b, Theorem 4.5])**.** *Let $X$ be a spherical variety, and let $\alpha \in \Pi$.*

*(a) If $\alpha \in \Pi^b_X$, then for any $D \in \mathcal{D}_{G,X}(\alpha)$ and any $\gamma \in \Psi_{G,X}$ such that $\gamma \neq \alpha$, we have $\varphi_D(\gamma) \leq 0$.*

*(b) If $\alpha \in \Pi^c_X$, then for any $\gamma \in \Psi_{G,X}$ such that $\gamma \neq 2\alpha$, we have $\langle \alpha^\vee, \gamma \rangle \leq 0$.*

*Sketch of proof.* For statement (a), one uses the "localization at spherical roots" to reduce to the case where $\Psi_{G,X} = \{\alpha, \gamma\}$. In this case, the statement follows from some combinatorial facts about $\alpha$ and $\varphi_D$ that we have already seen (specifically, Proposition 3.6.12 and Proposition 3.6.13) along with some technical arguments about reductive groups.

As for (b), the spherical root $\gamma$ is a linear combination of simple roots with nonnegative coefficients (see Remark 3.4.8). Since $\alpha$ is a simple root, it is a standard fact about root systems that $\langle \alpha^\vee, \beta \rangle \leq 0$ for any simple root $\beta \in \Pi$. It follows that $\langle \alpha^\vee, \gamma \rangle \leq 0$ as well. $\square$

We now give a name to the combinatorial objects that will classify homogeneous spherical varieties.

**Definition 3.6.19.** A *homogeneous spherical datum* for $G$ is a tuple $(\Lambda, \Pi^a, \Psi, \mathcal{D}^b)$ consisting of

1. a sublattice $\Lambda \subset \Lambda_G$,
2. a subset $\Pi^a \subseteq \Pi_G$,
3. a linearly independent subset $\Psi \subseteq \Psi_G \cap \Lambda$ consisting of indivisible vectors in $\Lambda$, and
4. a finite set $\mathcal{D}^b$ equipped with a map $\varphi : \mathcal{D}^b \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$

such that the following axioms are satisfied.

(B1) For any $D \in \mathcal{D}^b$ and any $\gamma \in \Psi$, we have $\langle \varphi(D), \gamma \rangle \leq 1$, and if $\langle \varphi(D), \gamma \rangle = 1$, then $\gamma \in \Psi \cap \Pi_G$ is a simple root of $G$.

(B2) For any $\alpha \in \Psi \cap \Pi_G$, the set

$$\mathcal{D}^b(\alpha) = \{D \in \mathcal{D}^b \mid \langle \varphi(D), \alpha \rangle = 1\}$$

contains exactly two elements, $D_\alpha^+$ and $D_\alpha^-$, and we have

$$\varphi(D_\alpha^+) + \varphi(D_\alpha^-) = \alpha^\vee|_\Lambda.$$

(B3) $\mathcal{D}^b$ is the (not necessarily disjoint) union of the $\mathcal{D}^b(\alpha)$ for all $\alpha \in \Pi \cap \Psi$.

($\Psi$1) For any $\alpha \in \frac{1}{2}\Psi \cap \Pi_G$, we have $\langle \alpha^\vee, \Lambda \rangle \subseteq 2\mathbb{Z}$, and for any $\gamma \in \Psi \setminus \{2\alpha\}$, we have $\langle \alpha^\vee, \gamma \rangle \leq 0$.

($\Psi$2) For any $\alpha, \beta \in \Pi_G$, if $\langle \alpha^\vee, \beta \rangle = 0$ and $\alpha + \beta \in \Psi \sqcup 2\Psi$, then $\alpha^\vee|_\Lambda = \beta^\vee|_\Lambda$.

($\Pi$1) For any $\gamma \in \Psi$, there exists some rank-1 wonderful variety $X$ such that $\Psi_{G,X} = \{\gamma\}$ and $\Pi_X^a = \Pi^a$.

($\Pi$2) For any $\alpha \in \Pi^a$, we have $\alpha^\vee|_\Lambda = 0$.

A *spherical system* for $G$ is a triple $(\Pi^a, \Psi, \mathcal{D}^b)$ such that $(\Lambda, \Pi^a, \Psi, \mathcal{D}^b)$ is a homogeneous spherical datum for $\Lambda = \mathbb{Z}\Psi$.

**Remark 3.6.20.** Almost all of the data in a homogeneous spherical datum is defined in terms of the sets $\Lambda_G$, $\Pi_G$, and $\Psi_G$, which depend only on $G$. The only exception to this is the set $\mathcal{D}^b$, which is an arbitrary finite set equipped with a map to $\mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$. This introduces a slight technicality about what it means for two homogeneous spherical data to be "the same." The precise definition is: we say that $(\Lambda_1, \Pi_1^a, \Psi_1, \mathcal{D}_1^b)$ and $(\Lambda_2, \Pi_2^a, \Psi_2, \mathcal{D}_2^b)$ are *equivalent* if $\Lambda_1 = \Lambda_2$, $\Pi_1^a = \Pi_2^a$, $\Psi_1 = \Psi_2$, and there exists a bijection $\mathcal{D}_1^b \to \mathcal{D}_1^b$ which identifies the given maps $\mathcal{D}_i^b \to \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$.

One can avoid dealing with bijections on the sets $\mathcal{D}_i^b$ by simply identifying $\mathcal{D}_i^b$ with its image in $\mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$. However, it is sometimes more convenient to view the $\mathcal{D}_i^b$ as abstract sets, and doing so has become the convention in the literature. It is also common in the literature to say that two homogeneous spherical data are equal when they are really equivalent. Technically, this means we are often working implicitly with equivalence classes of homogeneous spherical data; however, this technicality is rarely important in practice.

For any homogeneous spherical variety $G/H$, let $\mathcal{D}_{G,G/H}^b$ be the set of all colors of $G/H$ moved by a root of type $b$. We claim that the tuple $(\Lambda(G/H), \Pi_{G/H}^a, \Psi_{G,G/H}, \mathcal{D}_{G,G/H}^b)$ is a homogeneous spherical datum for $G$. We check one-by-one that the relevant axioms are satisfied.

(B1) For any $D \in \mathcal{D}^b_{G,G/H}$ and any $\gamma \in \Psi$, Proposition 3.6.18 gives us $\langle \varphi(D), \gamma \rangle \leq 0$ if $\gamma$ is not a simple root moving $D$. If $\gamma = \alpha \in \Psi \cap \Pi$ does move $D$, then $\langle \varphi(D), \gamma \rangle = 1$ by Proposition 3.6.13.

(B2) For any $\alpha \in \Pi \cap \Psi$, we know that $\alpha$ is a root of type $b$ for $G/H$ (see Theorem 3.6.10), and we have
$$\{D^+_\alpha, D^-_\alpha\} = \mathcal{D}_{G,G/H}(\alpha) = \{D \in \mathcal{D}^b \mid \varphi_D(\alpha) = 1\}$$
by Proposition 3.6.13 (applied to any divisor $D$ moved by $\alpha$) and Proposition 3.6.18 (applied to any divisor $D \in \mathcal{D}^b$ not moved by $\alpha$ and taking $\gamma = \alpha$). The required equation on $\varphi_{D^\pm_\alpha}$ is Proposition 3.6.13.

(B3) As in the check for (B2), we have $\mathcal{D}^b(\alpha) = \mathcal{D}_{G,G/H}(\alpha)$ for all $\alpha \in \Pi \cap \Psi$. Since $\mathcal{D}^b_{G,G/H}$ is the set of all colors moved by some such $\alpha$, this axiom is immediate from the definitions.

($\Psi$1) For any $\alpha \in \Pi \cap \frac{1}{2}\Psi$, we have $\alpha \in \Pi^c_{G/H}$, so $\alpha^\vee$ is nonpositive on $\Psi \setminus \{2\alpha\}$ by Proposition 3.6.18. Moreover, Proposition 3.6.13 implies that $\frac{1}{2}\alpha^\vee|_{\Lambda(G/H)} = \varphi_{D_\alpha}$, and since $v_{D_\alpha}$ is a discrete valuation, the map $\varphi_{D_\alpha}$ takes values in $\mathbb{Z}$. It follows that $\alpha^\vee$ takes values in $2\mathbb{Z}$ on $\Lambda(G/H)$.

($\Psi$2) If $\alpha, \beta \in \Pi$ are such that $\langle \alpha^\vee, \beta \rangle = 0$ and $\alpha + \beta \in \Psi \sqcup 2\Psi$, then $\alpha$ and $\beta$ are roots of type $d$ moving the same divisor $D$ by Proposition 3.6.12, and that proposition also gives us $\alpha^\vee|_\Lambda = \beta^\vee|_\Lambda$.

($\Pi$1) For any $\gamma \in \Psi_{G,G/H}$, Theorem 3.6.15 give us a rank-1 wonderful variety $X'$ such that $\Psi_{G,X'} = \{\gamma\}$ and $\Pi^a_{X'} = \Pi^a_{G/H}$.

($\Pi$2) For any $\alpha \in \Pi^a$, we have $\alpha^\vee|_{\Lambda(G/H)} = 0$ by Lemma 3.6.14.

Similarly, if $X$ is a wonderful $G$-variety, then $X$ is a smooth standard embedding, so Lemma 3.5.13 implies that $\Lambda(X) = \mathbb{Z}\Psi_{G,X}$. The same arguments as for $G/H$ above tell us that $(\mathbb{Z}\Psi_{G,X}, \Pi^a_X, \Psi_{G,X}, \mathcal{D}^b_{G,X})$ is a homogeneous spherical datum. In other words, the triple $(\Pi^a_X, \Psi_{G,X}, \mathcal{D}^b_{G,X})$ is a spherical system.

It turns out that homogeneous spherical data completely classify homogeneous spherical varieties up to $G$-isomorphism. The proof hinges on proving that spherical systems classify wonderful varieties up to $G$-isomorphism.

**Theorem 3.6.21.**

*(a) The map $X \mapsto (\Pi^a_X, \Psi_{G,X}, \mathcal{D}^b_{G,X})$ is a bijection between isomorphism classes of wonderful $G$-varieties and spherical systems for $G$.*

*(b) The map $G/H \mapsto (\Lambda(G/H), \Pi^a_{G/H}, \Psi_{G,G/H}, \mathcal{D}^b_{G,G/H})$ induces a bijection between isomorphism classes of homogeneous spherical $G$-varieties and homogeneous spherical data for $G$.*

It was Luna [Lun01] who first proposed the classification of Theorem 3.6.21, along with a program to prove it. His idea was as follows. The first step is to reduce Theorem 3.6.21b to Theorem 3.6.21a. To do this, we start with a homogeneous spherical variety $G/H$ and consider the variety $G/\overline{H}$, where $\overline{H}$ is the very sober hull of $H$. Since $\overline{H}$ is very sober, $G/\overline{H}$ admits a wonderful embedding, which has a corresponding spherical system $(\Pi^a, \Psi, \mathcal{D}^b)$. Luna showed in [Lun01, Section 6] that the spherical subgroups of $G$ with very sober hull $\overline{H}$ are in bijection with homogeneous spherical data of the form $(\Lambda, \Pi^a, \Psi', \mathcal{D}^b)$ such that $\Lambda \supset \Psi$ and $\Psi'$ is obtained from $\Psi$ by replacing any $\gamma \in \Psi \setminus (\Pi \sqcup 2\Pi)$ with $\frac{1}{2}\gamma$ whenever $\frac{1}{2}\gamma \in \Lambda$. Note that such a homogeneous spherical datum $(\Lambda, \Pi^a, \Psi', \mathcal{D}^b)$ is completely determined by the spherical system $(\Pi^a, \Psi, \mathcal{D}^b)$ and the lattice $\Lambda$. Using this fact, Luna showed that if wonderful $G$-varieties are classified by their spherical systems (i.e. if Theorem 3.6.21a holds), then homogeneous spherical $G$-varieties are classsified by their homogeneous spherical data (i.e. Theorem 3.6.21b holds).

Proving the classification of wonderful $G$-varieties by spherical systems is quite difficult. Luna's plan was to understand certain standard geometric constructions (such as fiber products and localizations at sets of simple roots) in terms of the relevant combinatorial data. Then, all wonderful $G$-varieties (resp. all spherical systems) could be obtained from a certain list of so-called *primitive* wonderful varieties (resp. *primitive* spherical systems) via a set of understood geometric (resp. combinatorial) constructions. The existence and uniqueness of primitive wonderful varieties with primitive spherical systems would then be proven case by case.

Luna himself carried out this plan in [Lun01] when $G$ is a reductive group of type A (see Definition 2.2.17). The same methods were later extended to other reductive groups, and some more general arguments were also found for parts of the classification. Thus, the culmination of many papers by several different researchers has led to a complete proof of Theorem 3.6.21. We refer the interested reader to the Introduction for slightly more details and to [Tim11, Section 30.11] for a detailed proof sketch as well as a very nice description (with references) of the various researchers and papers involved.

## 3.7 Picard Groups of Spherical Varieties

When working with divisors on spherical varieties, we will sometimes need to be able to answer questions like: when is a divisor Cartier or ample? In [Bri89a], Brion proved that such questions can be answered purely in terms of combinatorial data related to the classification of spherical varieties. This theory of divisors is not a part of the classification of spherical varieties, but it will be very useful for certain examples that we intend to consider later on (see Section 4.9). As such, we give a brief exposition of the theory here.

Our exposition largely follows [Bri97, Section 5] and [Per18, Section 3], but with certain results and proofs omitted. However, there are a few small details in both of these references that are imprecise (see Remark 3.7.8 below). Part of our goal in summarizing the theory here is to make these details precise, as they will be relevant to two important examples later (namely, Examples 4.9.3 and 4.9.4). Apart from explaining these details, our treatment is relatively brief; for the reader interested in a more thorough exposition, we recommend either [Bri97] or [Per18].

First, we establish some notation. Let $X$ be a $G$-variety. Recall that any $f \in K(X)$ determines a principal Cartier divisor $\operatorname{div}(f)$ on $X$. If $X$ is spherical, then for any $B$-eigenvector $f \in K(X)^{(B)}$, the only $B$-eigenvectors of $K(X)$ with the same weight as $f$ are those of the form $cf$ for some $c \in k$ (for any $f' \in K(X)^{(B)}$ of the same weight as $f$, we have $f/f' \in K(X)^B = k$, see Theorem 3.1.4). It follows that the divisor $\operatorname{div}(f)$ depends only on the weight of $f$. So, for any $\mu \in \Lambda(X)$, we write $\operatorname{div}(\mu)$ for the divisor $\operatorname{div}(f)$, where $f \in K(X)^{(B)}$ is any nonzero $B$-eigenvector of weight $\mu$.

Similarly, suppose $X$ is spherical, and let $L$ be a $G$-linearized invertible sheaf on $X$. Any global section $s \in H^0(X, L^{\otimes n})$ cuts out an effective Cartier divisor $\operatorname{div}(s)$ on $X$. Moreover, for every $n \geq 0$, the $G$-module $H^0(X, L^{\otimes n})$ is multiplicity-free (see Remark 3.1.5). It follows that for any $B$-eigenvector $s \in H^0(X, L^{\otimes n})^{(B)}$, the divisor $\operatorname{div}(s)$ depends only on the weight of $s$ and the integer $n$. So, for any $(\mu, n) \in \Lambda^+(X, L)$, we denote by $\operatorname{div}(\mu)$ the divisor cut out by any nonzero $B$-eigenvector of $H^0(X, L^{\otimes n})$ of weight $\mu$. Since $X_s$ is the complement of the support of $\operatorname{div}(s)$, the set $X_s$ also depends only on the weight of $s$ and $n$. So, for any $(\mu, n) \in \Lambda^+(X, L)$, we denote by $X_\mu \subset X$ the open subset $X_s$ for any $s \in H^0(X, L^{\otimes n})^{(B)}$ of weight $\mu$. Equivalently, we have $X_\mu = X \setminus \operatorname{Supp}(\operatorname{div}(\mu))$.

We first consider Weil divisors on spherical varieties. It turns out that, up to linear equivalence, we can compute these divisors just by using $B$-divisors and $B$-eigenvectors in the function field. In particular, the class group of a spherical variety is determined by pieces of combinatorial data that we understand well.

**Proposition 3.7.1** (cf. [Per18, Theorem 3.2.1])**.** *Let $X$ be a spherical variety. There exists an exact sequence*

$$\Lambda(X) \xrightarrow{\alpha} \bigoplus_{D \in \mathcal{D}_{G,X}} \mathbb{Z} \cdot D \xrightarrow{\beta} \operatorname{Cl}(X) \to 0,$$

*where $\operatorname{Cl}(X)$ is the Weil divisor class group of $X$, the map $\alpha$ is given by $\alpha(\mu) = \operatorname{div}(\mu) = \sum_{D \in \mathcal{D}_{G,X}} \varphi_D(\mu) D$, and the map $\beta$ is given by sending a divisor to its associated class in $\operatorname{Cl}(X)$. If $X$ is complete, this sequence is short exact.*

*sketch of proof.* First, we remark that the map $\alpha$ is well-defined because $\operatorname{div}(\mu)$ is a $B$-stable divisor for any $\mu \in \Lambda(X)$, see Corollary 2.5.5.

Exactness at $\operatorname{Cl}(X)$ is the statement that every Weil divisor on $X$ is linearly equivalent to a sum of $B$-divisors. For this, one applies the local structure theorem to the open $B$-orbit $X_B^\circ$ of $X$ to show that $\operatorname{Cl}(X_B^\circ) = 0$. Then, for any Weil divisor $D$ on $X$, the intersection $X_B^\circ \cap D$ is a Weil divisor of $X_B^\circ$ and hence is trivial, so $D$ is linearly equivalent to a divisor whose support lies in $X \setminus X_B^\circ$. Since $X \setminus X_B^\circ$ is the union of the $B$-divisors of $X$, this implies that $\beta$ is surjective.

For exactness in the middle, the divisor $\operatorname{div}(\mu)$ has image 0 in $\operatorname{Cl}(X)$ by definition, so $\operatorname{Im}(\alpha) \subset \ker(\beta)$. For the reverse containment, let $E = \sum_{D \in \mathcal{D}_{G,X}} n_D D$, and suppose that $E$

is linearly equivalent to 0. Then, $E = \mathrm{div}(f)$ for some $f \in K(X)$, and since $E$ is $B$-stable, $f$ is a $B$-eigenvector, see Corollary 2.5.5. It follows that $E = \mathrm{div}(\mu)$, where $\mu$ is the weight of $f$.

Finally, if $X$ is complete, then the union of the cones in the colored fan of $X$ contains $\mathcal{V}(X)$ (Theorem 3.3.28). Since $\mathcal{V}(X)$ is full-dimensional (Theorem 3.4.1), at least one of the cones in the colored fan of $X$ must be full-dimensional. By definition, this cone is generated by some subset of the $\varphi_D$ for $D \in \mathcal{D}_{G,X}$ (specifically, by the $\varphi_D$ for $D$ containing some fixed $G$-orbit $Y \subset X$). In particular, the $\varphi_D$ for $D \in \mathcal{D}_{G,X}$ span $N(X)$. So, for any $\mu \in \Lambda(X)$, if $\alpha(\mu) = \sum_D \varphi_D(\mu) D = 0$, then $\varphi_D(\mu) = 0$ for all $D$ implies that $\mu = 0$. In other words, $\alpha$ is injective. ☐

Next, we turn to the Picard group of a spherical variety. Since any spherical variety $X$ is a reduced scheme, the Picard group $\mathrm{Pic}(X)$ is isomorphic to the group of Cartier divisors on $X$. Thus, our first question is: which Weil divisors on a spherical variety are Cartier? Note that Proposition 3.7.1 implies that every Weil divisor is linearly equivalent to a sum of $B$-divisors. For the purposes of understanding $\mathrm{Pic}(X)$, then, it suffices to determine which $B$-stable Weil divisors are Cartier. For this, we have the following lemma.

**Lemma 3.7.2** ([Per18, Lemma 3.3.1], [Bri97, Lemma 5.2])**.** *Let $X$ be a spherical variety, and let $E = \sum_{D \in \mathcal{D}_{G,X}} n_D D$ be a $B$-stable Weil divisor. Then, $E$ is Cartier if and only if for any $G$-orbit $Y \subset X$, there exists some $\mu_{E,Y} \in \Lambda(X)$ such that $\varphi_D(\mu_{E,Y}) = n_D$ for all $B$-divisors $D$ containing $Y$.*

*sketch of proof.* Recall that $X$ is covered by the $G$-stable open subsets $G \cdot X_{B,Y}$ as $Y$ ranges over all the $G$-orbits of $X$ (see Theorem 3.2.7). Since being Cartier is a local property, it suffices to consider the case where $X = G \cdot X_{B,Y}$ for some $Y$, i.e. where $X$ is simple with unique closed $G$-orbit $Y$. In this case, one can use the local structure theorem to show that $\mathrm{Pic}(X_{B,Y}) = 0$ for all $Y$ (see the proof of [Per18, Theorem 3.1.3]). So, if $E$ is Cartier, then $E \cap X_{B,Y}$ is a principal Cartier divisor of $X_{B,Y}$. Since $E$ is $B$-stable, Corollary 2.5.5 then implies that

$$E = \mathrm{div}(\mu_{E,Y}) + \sum_{D \cap X_{B,Y} = \varnothing} m_D D$$

for some $\mu_{E,Y} \in \Lambda(X)$ and some $m_D \in \mathbb{Z}$. The $B$-divisors that don't intersect $X_{B,Y}$ are precisely those that don't contain $Y$, so comparing coefficients in the above equation gives us $\varphi_D(\mu_{E,Y}) = n_D$ for all $D \supset Y$. Conversely, if a choice of $\mu_{E,Y}$ as in the lemma statement exists, then $E$ satisfies the above equation for some choice of $m_D \in \mathbb{Z}$. Since the $B$-divisors $D$ that don't contain $Y$ are Cartier by Proposition 3.1.20 and $\mathrm{div}(\mu_{E,Y})$ is also Cartier, it follows that $E$ is Cartier. ☐

We now introduce some terminology that will help us refer to the weights $\mu_{E,Y}$ in the above lemma.

**Definition 3.7.3.** Let $X$ be a spherical variety.

1. For any $G$-orbit $Y \subset X$, we denote by $\mathcal{D}_Y$ the set of $B$-divisors of $X$ that contain $Y$.

2. For any $G$-orbit $Y \subset X$, a *linear function on* $\mathcal{D}_Y$ is a map $\ell_Y : \mathcal{D}_Y \to \mathbb{Z}$ such that there exists some $\mu_Y \in \Lambda(X)$ satisfying $\ell_Y(D) = \varphi_D(\mu_Y)$ for all $D \in \mathcal{D}_Y$. In other words, we have
$$\ell_Y = (\mu_Y \circ \varphi)|_{\mathcal{D}_Y},$$
where $\varphi : \mathcal{D}_{G,X} \to N(X)$ is the map $D \mapsto \varphi_D$ and we view $\mu_Y \in \Lambda(X)$ as a map $N(X) = \Lambda(X)^\vee_{\mathbb{Q}} \to \mathbb{Z}$.

3. By a *piecewise linear function on* $\cup_Y \mathcal{D}_Y$ we mean a family $(\ell_Y)_Y$ consisting of a linear function on $\mathcal{D}_Y$ for each $G$-orbit $Y \subset X$ such that $\ell_Y(D) = \ell_{Y'}(D)$ for any $G$-orbits $Y, Y' \subset G$ and any $D \in \mathcal{D}_Y \cap \mathcal{D}_{Y'}$.

4. We write $\mathrm{PL}(X)$ for the set of all piecewise linear functions on $\cup_Y \mathcal{D}_Y$.

5. We define $\mathrm{L}(X) \subset \mathrm{PL}(X)$ to be the subset consisting of piecewise linear functions $(\ell_Y)$ such that for some $\mu \in \Lambda(X)$, we have $\ell_Y = (\mu \circ \varphi)|_{\mathcal{D}_Y}$ for all $G$-orbits $Y$.

6. We denote by $\Delta^\circ(X)$ the set of all $B$-divisors of $X$ which do not contain any $G$-orbit of $X$ (these are necessarily colors, since any $G$-divisor contains a $G$-orbit). In an equation:
$$\Delta^\circ(X) = \mathcal{D}_{G,X} \setminus \bigcup_Y \mathcal{D}_Y.$$

**Remark 3.7.4.** We view $\mathrm{PL}(X)$ as an abelian group, with the operation given by adding linear functions on $\mathcal{D}_Y$ in the natural way. More precisely, we define $(\ell_Y)_Y + (\ell'_Y)_Y = (\ell_Y + \ell'_Y)_Y$, where $\ell_Y + \ell'_Y$ is the linear function on $\mathcal{D}_Y$ given by $D \mapsto \ell_Y(D) + \ell'_Y(D)$. Note that with this abelian group structure, the set $\mathrm{L}(X)$ is a subgroup of $\mathrm{PL}(X)$.

To give a piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$, it suffices to specify for each $G$-orbit $Y \subset X$ an element $\mu_Y \in \Lambda(X)$ such that for any two $G$-orbits $Y$ and $Y'$ and any $D \supset Y, Y'$, we have $\varphi_D(\mu_Y) = \varphi_D(\mu_{Y'})$. In particular, if $E = \sum_{D \in \mathcal{D}_{G,X}} n_D D$ is a $B$-stable Cartier divisor on $X$, then the weights $\mu_{E,Y}$ of Lemma 3.7.2 define a piecewise linear function $\ell = (\ell_Y)_Y \in \mathrm{PL}(X)$, because for any $D \supset Y, Y'$, we have
$$\varphi_D(\mu_{E,Y}) = n_D = \varphi_D(\mu_{E,Y'}).$$
Note that the $\varphi_D(\mu_{E,Y})$ for various $D$ and $Y$ will give us the coefficients $n_D$ for any $D$ containing a $G$-orbit $Y$. Thus, we can recover $E$ from $\ell$ and the set $\{n_D\}_{D \in \Delta^\circ(X)}$ in the following way. For any $B$-divisor $D \in \mathcal{D}_{G,X} \setminus \Delta^\circ(X)$, we define $\ell(D)$ to be $\ell_Y(D)$ for any $G$-orbit $Y \subset D$. (Note that this does not depend on the choice of $Y$ by definition of a piecewise linear function.) We then have
$$E = \sum_{D \in \mathcal{D}_{G,X} \setminus \Delta^\circ(X)} \ell(D) D + \sum_{D \in \Delta^\circ(X)} n_D D. \tag{3.7.1}$$

On the other hand, Lemma 3.7.2 implies that the righthand side of the above equation is a $B$-stable Cartier divisor of $X$ for any choice of piecewise linear function $\ell \in \mathrm{PL}(X)$ and any integers $n_D$. Thus, the $B$-stable Cartier divisors of $X$ are precisely the divisors which have the form of (3.7.1) for some choice of $(\ell_Y) \in \mathrm{PL}(X)$ and some $n_D \in \mathbb{Z}$.

Using the form of $B$-stable Cartier divisors given in (3.7.1), it is relatively straightforward to characterize the Picard group of $X$ in terms of combinatorial invariants of $X$. The precise statement is the following theorem, which can be proven using our above discussion along with some formal algebraic arguments.

**Theorem 3.7.5** ([Per18, Theorem 3.3.4], [Bri97, Theorem 5.2])**.** *Let $X$ be a spherical variety, and let $\mathcal{C}_X = \bigcup_Y \mathcal{C}_Y$ (with the union taken over all $G$-orbits $Y \subset X$). We have an exact sequence*

$$\mathcal{C}_X^\perp \xrightarrow{\alpha} \bigoplus_{D \in \Delta^\circ(X)} \mathbb{Z} \cdot D \xrightarrow{\beta} \mathrm{Pic}(X) \xrightarrow{\gamma} \mathrm{PL}(X)/L(X) \to 0.$$

*Here, the map $\alpha$ is given by $\mu \mapsto \mathrm{div}(\mu)$, the map $\beta$ sends a (Cartier) divisor to its class in $\mathrm{Pic}(X)$, and the map $\gamma$ sends a Cartier divisor $E$ to the image of the piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$ determined by the weights $\mu_{E,Y}$ given by Lemma 3.7.2.*

**Remark 3.7.6.** Note that $\mathrm{PL}(X)$ is finitely generated because $X$ has finitely many $G$-orbits. Since $X$ also has finitely many $B$-divisors, the exact sequence in the above theorem implies that $\mathrm{Pic}(X)$ is finitely generated as well.

**Corollary 3.7.7.** *Let $X$ be a complete spherical variety. Then, we have a short exact sequence*

$$0 \to \bigoplus_{D \in \Delta^\circ(X)} \mathbb{Z} \cdot D \to \mathrm{Pic}(X) \to \mathrm{PL}(X)/L(X) \to 0.$$

*In particular, $\mathrm{Pic}(X)$ is a free finitely generated abelian group.*

*Proof.* With $\mathcal{C}_X$ as in Theorem 3.7.5, we have $\mathcal{V}(X) \subset \mathcal{C}_X$ because $X$ is complete (see Theorem 3.3.28). Since $\mathcal{V}(X)$ is full-dimensional, we see that $\mathcal{C}_X^\perp = 0$, so the desired short exact sequence is the exact sequence of Theorem 3.7.5. Moreover, it follows from the definitions that for any piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$, if $n \cdot (\ell_Y)_Y \in \mathrm{L}(X)$ for some $n \in \mathbb{Z}$, then $(\ell_Y)_Y \in \mathrm{L}(X)$. In other words, the quotient $\mathrm{PL}(X)/\mathrm{L}(X)$ is torsion-free and hence is a free finitely generated abelian group. It follows that $\mathrm{Pic}(X)$ is free and finitely generated as well. □

**Remark 3.7.8.** As our terminology suggests, we view an element $(\ell_Y)_Y \in \mathrm{PL}(X)$ as a piecewise linear function on the set $\cup_Y \mathcal{D}_Y$. The elements of $L(X)$ are then the piecewise linear functions on $\cup_Y \mathcal{D}_Y$ which are in fact "linear." (This also explains the notation $\mathrm{PL}(X)$ and $\mathrm{L}(X)$.) On the other hand, recall that for any $G$-orbit $Y$, the cone $\mathcal{C}_Y$ is generated by the $\varphi_D$ for $D \in \mathcal{D}_Y$. Since the definition of linear functions on $\mathcal{D}_Y$ uses the maps $\varphi_D$ rather than the divisors $D$ themselves, it is tempting to think of an element $(\ell_Y)_Y \in \mathrm{PL}(X)$ as a piecewise linear function on the cones $\mathcal{C}_X = \cup_Y \mathcal{C}_Y$. Indeed, this is the approach taken in both [Per18, Section 3.3] and [Bri97, Section 5.2]. This approach is equivalent to our approach when the cones $\mathcal{C}_Y$ form a fan (e.g. when $X$ is toroidal or horospherical, see Remark 3.3.23). In general, however, when we tried to define elements of $\mathrm{PL}(X)$ as "piecewise linear functions on $\cup_Y \mathcal{C}_Y$" as in [Per18] and [Bri97], we were not able to verify that all $B$-stable Cartier divisors have the form given in (3.7.1), though both [Per18] and [Bri97] assert that they do.

For these various definitions of a piecewise linear function $\ell = (\ell_Y)_Y$, the difference between using functions $\ell_Y : \mathcal{D}_Y \to \mathbb{Z}$ and using functions $\ell_Y : \mathcal{C}_Y \to \mathbb{Z}$ is very subtle. However, it becomes clearer when we consider situations stemming from the two pathological behaviors of colored cones mentioned in Remark 3.3.23:

(1) It is possible that for some $G$-orbits $Y, Y' \subset X$, the intersection $F = \mathcal{C}_Y \cap \mathcal{C}_{Y'}$ is a face of both $\mathcal{C}_Y$ and $\mathcal{C}_{Y'}$, but we have $F^\circ \cap \mathcal{V}(X) = \varnothing$. Then, $F$ contains the cone generated by the $\varphi_D$ for $D \in \mathcal{D}_Y \cap \mathcal{D}_{Y'}$. So, if we wish to use functions $\ell_Z : \mathcal{C}_Y \to \mathbb{Z}$ and still have $\ell(D)$ be well-defined in (3.7.1), we need to ensure that $\ell_Y(\varphi_D) = \ell_{Y'}(\varphi_D)$ for every $D \in \mathcal{D}_Y \cap \mathcal{D}_{Y'}$. However, because $F^\circ \cap \mathcal{V}(X) \neq \varnothing$, there is no $G$-orbit $Y''$ such that $\mathcal{C}_{Y''} = F$, so there is no nice way to state the condition that $\ell_Y(\varphi_D) = \ell_{Y'}(\varphi_D)$ for the necessary choices of $D$ except to say that $\ell_Y|_F = \ell_{Y'}|_F$. In particular, the definition in [Bri97] only asserts that $\ell_Y$ and $\ell_{Y'}$ agree on $\mathcal{C}_{Y''}$ for any $Y''$ such that $Y, Y' \subset \overline{Y''}$. Since there is not necessarily any such choice of $Y''$ with $F \subset \mathcal{C}_{Y''}$, it seems that under the definition of $(\ell_Y)_Y \in \mathrm{PL}(X)$ given in [Bri97], the value of $\ell(D)$ in (3.7.1) may not be well-defined. The definition in [Per18] avoids this issue by requiring that the functions $\ell_Z$ glue to a map $\ell : \cup_Y \mathcal{C}_Y \to \mathbb{Z}$.

(2) It is possible that the intersection $\mathcal{C}_Y \cap \mathcal{C}_{Y'}$ is not even contained in a face of either cone, i.e. that $\mathcal{C}_Y^\circ \cap \mathcal{C}_{Y'}^\circ \neq \varnothing$. In this case, there may be some $D \in \mathcal{D}_Y$ such that $\varphi_D \in \mathcal{C}_{Y'}^\circ$, and $D$ need not contain $Y'$ in this situation. If $Y' \not\subset D$, then there is no reason that $\ell_Y(\varphi_D)$ and $\ell_{Y'}(\varphi_D)$ should be equal. Indeed, in the situation of Lemma 3.7.2, the coefficient of $D$ in the divisor $E$ is given by $\varphi_D(\mu_{E,Y})$ but not necessarily by $\varphi_D(\mu_{E,Y'})$. Moreover, it is possible to construct examples of a $B$-stable divisor $E$ such that weights $\mu_{E,Z}$ as in Lemma 3.7.2 do exist, so $E$ is Cartier, but for any choice of the $\mu_{E,Z}$ as in the lemma, we have $\varphi_D(\mu_{E,Y}) \neq \varphi_D(\mu_{E,Y'})$. For such an example, the family of functions $\ell_Z : \mathcal{C}_Z \to \mathbb{Z}$ determined by the weights $\mu_{E,Z}$ has $\ell_Y(\varphi_D) \neq \ell_{Y'}(\varphi_D)$, so the $\ell_Z$ do not glue to a function $\cup_Y \mathcal{C}_Y \to \mathbb{Z}$. However, the definition of $(\ell_Z)_Z \in \mathrm{PL}(X)$ given in [Per18] requires the $\ell_Z$ to glue like this. It follows that for certain examples, the definition in [Per18] will not allow us to associate an element $(\ell_Z)_Z \in \mathrm{PL}(X)$ to every $B$-stable Cartier divisor. In other words, there may be some $B$-stable Cartier divisors $E$ which do not have the form in (3.7.1) for some $\ell \in \mathrm{PL}(X)$.

We remark that these concerns are not new in the literature; in fact, an explicit example like the one described in (2) above can be found in [Tim11, Example 17.7], and this example is mentioned in the context of showing that the weights $\mu_{E,Y} \in \Lambda(X)$ from Lemma 3.7.2 need not generally define a function $\cup_Y \mathcal{C}_Y \to \mathbb{Z}$. Also, by taking different choices of cones in

[Tim11, Example 17.7], one can obtain an explicit example of the situation described in (1) above. The workaround that [Tim11] uses for these issues is to think of $\mathrm{PL}(X)$ as a family of functions $\ell_Y : \mathcal{C}_Y \to \mathbb{Q}$ such that $\ell_Y = \ell_Z|_{\mathcal{C}_Y}$ whenever $\mathcal{C}_Y$ is a face of $\mathcal{C}_Z$. Our discussion surrounding (3.7.1) indicates that the definition we have given in Definition 3.7.3 is another suitable solution.

These technicalities aside, we note that all of the arguments and intuition are correct in both [Per18] and [Bri97], provided we use a definition of a piecewise linear function $(\ell_Y)_Y$ such that the $B$-stable Cartier divisors of $X$ are precisely the divisors of the form in (3.7.1).

Another nice consequence of Lemma 3.7.2 is the following combinatorial criterion for when a spherical variety is locally factorial.

**Proposition 3.7.9** ([Per18, Theorem 3.2.3])**.** *Let $X$ be a spherical variety. Then, $X$ is locally factorial if and only if for every $G$-orbit $Y \subset X$, the valuations $\varphi_D$ for all $D \in \mathcal{D}_Y$ form part of a basis for the dual lattice $\Lambda(X)^\vee \subset N(X)$.*

*Proof.* Since $X$ is normal, $X$ is locally factorial if and only if every Weil divisor is Cartier. (see e.g. [Sta20, Tag 0BE9]). Since every Weil divisor is linearly equivalent to a $B$-stable divisor (Proposition 3.7.1) and sums of Cartier divisors are Cartier, we see that $X$ is locally factorial if and only if every $D \in \mathcal{D}_{G,X}$ is Cartier. By Lemma 3.7.2, a $B$-divisor $D$ is Cartier if and only if for every $G$-orbit $Y$ contained in $D$, there exists some $\mu_Y \in \Lambda(X)$ such that $\varphi_D(\mu_Y) = 1$ and $\varphi_{D'}(\mu_Y) = 0$ for every $D' \in \mathcal{D}_Y \setminus \{D\}$. The existence of such a $\mu_Y$ for every $D \in \mathcal{D}_Y$ is equivalent to the statement that the $\varphi_D$ for $D \in \mathcal{D}_Y$ form part of a basis for $\Lambda(X)^\vee$. ☐

Theorem 3.7.5 gives us a nice combinatorial description of the Cartier divisors on a spherical variety $X$. We are interested in using this description to characterize when a Cartier divisor is either globally generated or ample. It is not too difficult to give a general characterization of being globally generated (see [Per18, Theorem 3.3.6]), but this characterization involves picking weights $\mu_Y \in \Lambda(X)$ such that $\ell_Y = (\mu_Y \circ \varphi)|_{\mathcal{D}_Y}$ for a given piecewise linear funtion $(\ell_Y)_Y \in \mathrm{PL}(X)$. For any $G$-orbit $Y$, such a choice of $\mu_Y$ does determine the linear function $\ell_Y : \mathcal{D}_Y \to |Z$, but $\ell_Y$ does not uniquely determine a choice of the $\mu_Y$ in general, because the valuations $\varphi_D$ for $D \in \mathcal{D}_Y$ may not span $N(X)$.

If $X$ is complete, we can avoid this ambiguity in the choice of $\mu_Y$ thanks to the following lemma.

**Lemma 3.7.10.** *Let $X$ be a complete spherical variety. For any closed $G$-orbit $Y \subset X$, the cone $\mathcal{C}_Y$ is full-dimensional.*

*sketch of proof.* By Proposition 3.5.8, there exists a projective $G$-equivariant birational morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ toroidal. Since $\pi$ is $G$-equivariant, dominant, and complete, any closed $G$-orbit $Y$ of $X$ is the image of some closed $G$-orbit $\tilde{Y}$ of $\tilde{X}$. It follows that $\pi(G \cdot \tilde{X}_{B,\tilde{Y}}) \subset G \cdot X_{B,Y}$, so Theorem 3.3.28 (applied to the restriction of $\pi$ to a morphism

$G \cdot \tilde{X}_{B,\tilde{Y}} \to G \cdot X_{B,Y}$) implies that $\mathcal{C}_{\tilde{Y}} \subset \mathcal{C}_Y$. Thus, it will suffice to prove the statement for $\tilde{X}$. After replacing $X$ by $\tilde{X}$, we may assume that $X$ is toroidal.

In this case, the set $\mathcal{F} = \{\mathcal{C}_Y \mid Y \text{ a } G\text{-orbit of } X\}$ forms a strictly convex fan, and the cone $\mathcal{C}_X = \bigcup_Y \mathcal{C}_Y$ contains $\mathcal{V}(X)$ and hence is full-dimensional (see Theorems 3.5.6, 3.3.28 and 3.4.1). Moreover, the closed $G$-orbits $Y$ are precisely the $G$-orbits such that $\mathcal{C}_Y$ is maximal in $\mathcal{F}$ with respect to the partial order given by one cone being a face of another (see Proposition 3.3.24). We have thus reduced the question to the following statement: for any fan $\mathcal{F}$ such that the cone $C_{\mathcal{F}} = \bigcup_{C \in \mathcal{F}} C$ is full-dimensional, every maximal cone in $\mathcal{F}$ is full-dimensional. This can be proven using standard topological arguments about cones.

□

Let $X$ be a spherical variety, and let $(\ell_Y)_Y \in \mathrm{PL}(X)$. Then, every $G$-orbit $Y'$ of $X$ contains some closed orbit $Y$ in its closure, and $\mathcal{D}_{Y'} \subset \mathcal{D}_Y$ implies that $\ell_{Y'} = \ell_Y|_{\mathcal{D}_{Y'}}$. Thus, the piecewise linear function $(\ell_Y)_Y$ is completely determined by the functions $\ell_Y$ for closed $G$-orbits $Y$. When $X$ is complete and $Y$ is closed, Lemma 3.7.10 implies that the $\varphi_D$ for $D \in \mathcal{D}_Y$ span $N(X)$; it follows that for any linear function $\ell_Y : \mathcal{D}_Y \to \mathbb{Z}$, there is a unique $\mu_Y \in \Lambda(X)$ such that

$$\ell_Y = (\mu_Y \circ \varphi)|_{\mathcal{D}_Y}.$$

Since $\ell_Y$ determines $\mu_Y$, we may identify $\ell_Y$ with the function $\mu_Y \circ \varphi$ and so view $\ell_Y$ as a function on all of $\mathcal{D}_{G,X}$. This identification allows us to state relatively clean criteria for when a Cartier divisor is globally generated or ample.

**Definition 3.7.11.** Let $X$ be a complete spherical variety. We say that a piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$ is *convex* if for any two closed $G$-orbits $Y$ and $Y'$ and any $D \in \mathcal{D}_{Y'}$, we have

$$\ell_{Y'}(D) \geq \ell_Y(D).$$

(Here we define $\ell_Y(D)$ by viewing $\ell_Y$ as a function on all of $\mathcal{D}_{G,X}$, as discussed above.) If this inequality is strict for all $D \in \mathcal{D}_{Y'} \setminus \mathcal{D}_Y$, then we say that $(\ell_Y)_Y$ is *strictly convex*.

**Remark 3.7.12.** In [Per18] and [Bri97], the definition of "convex" and "strictly convex" is the same as in the above definition, except that the orbit $Y'$ is not assumed to be closed. However, as noted above, the value of $\ell_{Y'}$ is always equal to that of $\ell_{Y''}$ for any closed $G$-orbit $Y'' \subset \overline{Y'}$. This implies that our definition is equivalent to the one given in [Per18] and [Bri97].

**Theorem 3.7.13** ([Per18, Corollary 3.3.8], [Bri97, Corollary 5.2.1])**.** *Let $X$ be a complete spherical variety, and let*

$$E = \sum_{D \in \mathcal{D}_{G,X} \setminus \Delta^\circ(X)} \ell(D) D + \sum_{D \in \Delta^\circ(X)} n_D D$$

*be a $B$-stable Cartier divisor on $X$ for some $(\ell_Y)_Y \in \mathrm{PL}(X)$ and some $n_D \in \mathbb{Z}$.*

(a) *The divisor $E$ is globally generated if and only if $(\ell_Y)$ is convex and for any closed $G$-orbit $Y \subset X$ and any $D \in \Delta^\circ(X)$, we have $\ell_Y(D) \leq n_D$.*

(b) *The divisor $E$ is ample if and only if $(\ell_Y)$ is strictly convex and for any closed $G$-orbit $Y \subset X$ and any $D \in \Delta^\circ(X)$, we have $\ell_Y(D) < n_D$.*

*Proof.* For the proof of statement (a), see [Per18, Theorem 3.3.6] or [Bri97, Theorem 5.2(iii)]. There is also a proof of statement (b) in these references (see [Per18, Corollary 3.3.8] and [Bri97, Corollary 5.2.1]). The proofs in these references rely on certain technical arguments about divisors; alternately, we give a proof of (b) here that relies instead on certain facts about the sets $X_{B,Y}$ for closed $G$-orbits $Y$.

Suppose that $E$ is ample, and let $Y$ be a closed $G$-orbit. There exists some $f \in H^0(X, \mathcal{O}_X(E))$ such that $X_f = X_{B,Y}$ (see Theorem 3.2.7). Let $E' = \operatorname{div}(f)$. Since $E$ and $E'$ are $B$-stable and linearly equivalent, we have

$$E' = E + \operatorname{div}(\mu_Y) \tag{3.7.2}$$

for some $\mu_Y \in \Lambda(X)$ (see Proposition 3.7.1). Moreover, the divisor $E'$ is effective, and $\operatorname{Supp}(E') = X \setminus X_{B,Y}$ is the set of $B$-divisors of $X$ which do not contain $Y$. So, for any $D \in \mathcal{D}_Y$, comparing coefficients of $D$ in (3.7.2) gives us

$$0 = \ell_Y(D) + \varphi_D(\mu_Y).$$

This implies that $\ell_Y = (-\mu_Y) \circ \varphi$ as functions on $\mathcal{D}_{G,X}$, or equivalently, that

$$\ell_Y(D) = -\varphi_D(\mu_Y) \tag{3.7.3}$$

for all $D \in \mathcal{D}_{G,X}$. On the other hand, any $D \in \mathcal{D}_{G,X} \setminus \mathcal{D}_Y$ lies in $\operatorname{Supp}(E')$, so comparing coefficients of $D$ in (3.7.2) gives

$$\ell_{Y'}(D) + \varphi_D(\mu) > 0$$

if $D \in \mathcal{D}_{Y'}$ for some $Y' \neq Y$ and

$$n_D + \varphi_D(\mu) > 0$$

otherwise. In the latter case, the equation for $\ell_Y(D)$ in (3.7.3) gives us the inequality on $n_D$ in (b), and in the former case, the equation for $\ell_Y(D)$ in (3.7.3) implies that $(\ell_Y)$ is strictly convex.

Conversely, suppose that $(\ell_Y)_Y$ is strictly convex and that $\ell_Y(D) \leq n_D$ for all closed $G$-orbits $Y$ and $D \in \Delta^\circ(X)$. Recall that $\mathcal{O}_X(E)$ is ample if and only if $X$ is covered by affine open subsets of the form $X_f$, where $f \in H^0(X, \mathcal{O}_X(nE))$ for some $n \geq 1$. (This is the definition of ampleness used in both EGA and the Stacks Project, and it is equivalent to all other standard definitions under mild hypotheses; see [Sta20, Tag 01PR] and [Sta20, Tag 02NO] for details.) On the other hand, $X$ is covered by the open subsets of the form $G \cdot X_{B,Y}$ for $Y$ a closed $G$-orbit. (Proof: every $G$-orbit $Y'$ of $X$ has $Y \subset \overline{Y'}$ for some closed

$G$-orbit of $Y$, and Theorem 3.2.7 then implies that $Y' \subset G \cdot X_{B,Y}$.) Thus, it will suffice to show that each such set $G \cdot X_{B,Y}$ is covered by affine open subsets of the form $X_f$ for some $f \in H^0(X, \mathcal{O}_X(E))$. For this, we only need to find some $f \in H^0(X, \mathcal{O}_X(E))$ such that $X_f = X_{B,Y}$; then, $X_f$ is affine by Theorem 3.2.7, and $G \cdot X_{B,Y}$ is covered by the affine open subsets $g \cdot X_f = X_{gf}$ for any $g \in G$.

So, let $Y \subset X$ be a closed $G$-orbit, and let $\mu_Y \in \Lambda(X)$ be the weight determined by the function $\ell_Y$. Consider the divisor $E - \operatorname{div}(\mu_Y)$. For any $B$-divisor $D$ such that $D \supset Y$, the coefficient of $D$ in $E$ is $\ell_Y(D) = \varphi_D(\mu_Y)$ by definition, so the coefficient of $D$ in $E - \operatorname{div}(\mu_Y)$ is 0. On the other hand, suppose $D \not\supset Y$. If $D$ contains some other $G$-orbit, then $D$ contains a closed $G$-orbit $Y'$. The fact that $(\ell_Y)_Y$ is strictly convex then gives us

$$\ell_{Y'}(D) > \ell_Y(D) = \varphi_D(\mu_Y),$$

so the coefficient of $D$ in $E - \operatorname{div}(\mu_Y)$ is positive (by definition of $E$). If instead $D$ contains no $G$-orbit, then our assumptions give us

$$n_D > \ell_Y(D) = \varphi_D(\mu_Y),$$

so the coefficient of $D$ in $E - \operatorname{div}(\mu_Y)$ is again positive. This proves that $E - \operatorname{div}(\mu_Y)$ is an effective Cartier divisor whose support is the union of the $B$-divisors not containing $Y$. In other words, we have $\operatorname{Supp}(E - \operatorname{div}(\mu_Y)) = X \setminus X_{B,Y}$. By Lemma 2.5.3, the effective divisor $E - \operatorname{div}(\mu_Y)$ corresponds to a section $f \in H^0(X, \mathcal{O}_X(E))$ such that $\operatorname{div}(f) = E - \operatorname{div}(\mu_Y)$. Then, we have

$$\operatorname{Supp}(\operatorname{div}(f)) = \operatorname{Supp}(E - \operatorname{div}(\mu_Y)) = X \setminus X_{B,Y}.$$

This implies that $X_f = X_{B,Y}$, so $f$ is the desired global section of $\mathcal{O}_X(E)$. □

**Corollary 3.7.14.** *Let $X$ be a complete spherical variety. Then, $X$ is projective if and only if there exists a strictly convex element of* $\mathrm{PL}(X)$.

*Proof.* Because $X$ is a complete variety, $X$ is projective if and only if it is quasi-projective, i.e. if and only if there exists an ample (Cartier) divisor on $X$. By Proposition 3.7.1, every divisor is linearly equivalent to a $B$-stable one, so $X$ is projective if and only if there exists a $B$-stable ample divisor on $X$. By Theorem 3.7.13, giving a $B$-stable ample divisor is equivalent to giving a strictly convex piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$ and coefficients $n_D$ for each $D \in \Delta^\circ(X)$ such that $n_D > \varphi_D(\ell_Y)$ for all closed $G$-orbits $Y$. So, if a $B$-stable ample divisor exists, there must exist a strictly convex element of $\mathrm{PL}(X)$; conversely, given a strictly convex element $(\ell_Y)_Y \in \mathrm{PL}(X)$, all we have to do is choose the $n_D$ to be large enough, and we will obtain a $B$-stable ample Cartier divisor. □

**Remark 3.7.15.** In analogy with the above corollary, is also possible to use Theorem 3.7.13 to prove a combinatorial criterion for when a simple spherical variety is affine. See [Bri97, Corollary 5.2.2] or [Per18, Theorem 3.3.14] for details.

# Chapter 4

# Weight Monoids on Smooth Projective Spherical Varieties

In the previous chapter, we introduced many different invariants on spherical varieties and saw how these invariants classify spherical varieties up to $G$-isomorphism (specifically, see Theorem 3.3.26, Theorem 3.3.28, and Theorem 3.6.21 for the main classification statements). The invariants used in this classification are primarily related to divisors (and the simple roots that move them) and valuations (either $G$-invariant valuations or valuations of colors). There is another interesting type of invariant we can consider: namely, the monoid of weights of $B$-eigenvectors $\Lambda^+(X, L)$ for a $G$-linearized line bundle $L$ on $X$ (or the monoid $\Lambda^+(X)$, which is just a special case of $\Lambda^+(X, L)$). These weight monoids are representation-theoretic in nature, so it is interesting to ask how we might relate weight monoids to the more geometric invariants that classify spherical varieties.

In this chapter, we focus on the following question: to what extent (and under what conditions) does a weight monoid $\Lambda^+(X, L)$ determine the invariants on $X$ that arise from the classification of spherical varieties? When $X$ is affine, this question is answered by the so-called *Knop conjecture*, which was proven by Losev in [Los09a]. In Section 4.1, we phrase the classification of spherical varieties from the previous chapter in terms that are more suitable for discussing these types of questions. We then discuss some basic background to the Knop conjecture in Section 4.2. In Section 4.3, we prove a projective analog of a result that Losev used in his proof of the Knop conjecture (see Corollary 4.3.5). In Section 4.4, we use the Knop conjecture and the local structure theorem to obtain a "local isomorphism" result (Theorem 4.4.6), which will allow us to compare combinatorial invariants on spherical varieties provided they can be captured locally, in an appropriate sense. In Sections 4.5, 4.6, and 4.7, we use this "local isomorphism" to attempt to compare each relevant type of combinatorial invariant in turn. Our main results are Theorem 4.5.5, which gives us an "equality" on the combinatorial data of $B$-divisors under certain conditions, and Theorem 4.6.8, which gives us an equality on "most" spherical roots. Finally, in Section 4.9, we give some examples in which the weight monoid $\Lambda^+(X, L)$ does *not* determine certain combinatorial invariants. These examples show that our results in Sections 4.5, 4.6, and 4.7 are relatively optimal.

## 4.1 Reworking the Classification of Spherical Varieties

We are interested in viewing the classification of spherical varieties as a way to tell when two spherical varieties are $G$-equivariantly isomorphic by looking at combinatorial data. In this section, we reframe the classification of spherical varieties in order to better use it for this purpose. In order to do this, we will need to make precise what it means to have an "equality" of the combinatorial data of two spherical varieties.

Everything in this section is almost certainly known to experts. For instance, the main notion of "equality" that we need to introduce has already been mostly established in Losev's proof that homogeneous spherical data classify homogeneous spherical varieties up to $G$-equivariant isomorphism (compare the discussion preceding [Los09c, Theorem 1] with our definition of a "$\mathcal{D}$-equivalence" in Definition 4.1.1 below). However, we are not aware of anywhere in the literature where the classification of spherical varieties has been written out as a criterion for being $G$-equivariantly isomorphism in a precise way, as it is in Theorem 4.1.9 below. Our goal in this section is to provide a reference for these facts and to introduce some terminology for certain "equalities" of combinatorial data that will be useful to us later.

### 4.1.a $\mathcal{D}$-Equivalences

We have seen in Section 3.6 that the classification of homogeneous spherical varieties hinges on data related to $B$-divisors (more precisely, their valuations and the simple roots that move them). Because we are interested in using this classification to tell when two spherical varieties are isomorphic, it will be useful to formalize what is means for two spherical varieties to have "the same" data on their $B$-divisors. The following definition provides this formalism.

**Definition 4.1.1.** Let $X_1$ and $X_2$ be spherical $G$-varieties such that $\Lambda(X_1) = \Lambda(X_2)$. A $\mathcal{D}$-*equivalence* is a bijection $\iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$ such that for all $D \in \mathcal{D}_{G,X_1}$, we have $\varphi_D = \varphi_{\iota(D)}$, and a root $\alpha \in \Pi_G$ moves $D$ if and only if $\alpha$ moves $\iota(D)$. If a $\mathcal{D}$-equivalence exists, we say that $X_1$ and $X_2$ are $\mathcal{D}$-*equivalent*.

The notion of $\mathcal{D}$-equivalence will be essential in much of what follows. As such, we use this section to collect a few basic results about $\mathcal{D}$-equivalences. To begin, we note that $\mathcal{D}$-equivalences automatically allow us to match up types of simples roots.

**Lemma 4.1.2.** *Let $X_1$ and $X_2$ be $\mathcal{D}$-equivalent spherical varieties. Then, every root $\alpha \in \Pi_G$ has the same type for $X_1$ as it does for $X_2$.*

*Proof.* We have $\alpha \in \Pi^a_{X_1}$ if and only if $\alpha$ moves no $B$-divisor of $X_1$, and $\alpha \in \Pi^b_{X_1}$ if and only if $\alpha$ moves exactly 2 $B$-divisors of $X_1$. But any $\mathcal{D}$-equivalence induces a bijection $\mathcal{D}_{G,X_1}(\alpha) \xrightarrow{\sim} \mathcal{D}_{G,X_1}(\alpha)$, so we immediately get $\Pi^a_{X_1} = \Pi^a_{X_2}$ and $\Pi^b_{X_1} = \Pi^b_{X_2}$. Now let $\alpha \in \Pi^c_{X_1} \cup \Pi^d_{X_1}$. Then, $\alpha \in \Pi^c_{X_2} \cup \Pi^d_{X_2}$, so for $i \in \{1, 2\}$, we know that $\alpha$ moves a unique $B$-divisor $D_i$ of $X_i$. Moreover, we have $\alpha \in \Pi^c_{X_i}$ if and only if $\varphi_{D_i} = \frac{1}{2}\alpha^\vee|_{\Lambda(X_i)}$, and $\alpha \in \Pi^d_{X_i}$ if and only if

$\varphi_{D_i} = \alpha^\vee|_{\Lambda(X_i)}$. Any $\mathcal{D}$-equivalence will map $D_1$ to $D_2$ and preserve valuations, so it follows that $\alpha \in \Pi^c_{X_1}$ if and only if $\alpha \in \Pi^c_{X_2}$, and $\alpha \in \Pi^d_{X_1}$ if and only if $\alpha \in \Pi^d_{X_2}$. □

In general, a $\mathcal{D}$-equivalence need not be unique. However, the following lemma describes the extent to which two $\mathcal{D}$-equivalences can differ.

**Lemma 4.1.3.** *Let $X_1$ and $X_2$ be spherical varieties, and let $\iota, \iota' : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be two $\mathcal{D}$-equivalences. If $D \in \mathcal{D}_{G,X_1}$ satisfies $\iota(D) \neq \iota'(D)$, then $D$ is moved by a unique root $\alpha \in \Pi^b_{X_1}$. Moreover, if $D'$ is the other $B$-divisor of $X_1$ moved by $\alpha$, then $\alpha$ is also the unique root moving $D'$, and we have $\varphi_D = \varphi_{D'}$, $\iota(D) = \iota'(D')$ and $\iota(D') = \iota'(D)$.*

*Proof.* Let $D \in \mathcal{D}_{G,X_1}$. Then, $\iota(D)$ and $\iota'(D)$ are both moved by precisely the same roots as $D$. We consider 3 cases, based on which roots move $D$.

1. If $D$ is $G$-stable, then $\iota(D)$ and $\iota'(D)$ are also $G$-stable. But a $G$-divisor of $X_2$ is determined by its valuation as an element of $N(X_2)$ (see Corollary 3.1.14). So, the equality
$$\varphi_{\iota(D)} = \varphi_D = \varphi_{\iota'(D)}$$
implies that $\iota(D) = \iota'(D)$

2. Suppose $D$ is moved by a root $\alpha$ of type $c$ or $d$ for $X_1$. Then, $\alpha$ has type $c$ or $d$ for $X_2$ as well (Lemma 4.1.2). So, both $\iota(D)$ and $\iota'(D)$ are the unique $B$-divisor of $X_2$ moved by $\alpha$, which implies that $\iota(D) = \iota'(D)$.

3. The only remaining option is that $D$ is moved by a root $\alpha \in \Pi^b_{X_1} = \Pi^b_{X_2}$. In this case, write $\mathcal{D}_{G,X_1}(\alpha) = \{D, D'\}$. Since both $\iota$ and $\iota'$ preserve the property of being moved by $\alpha$, we have
$$\mathcal{D}_{G,X_2}(\alpha) = \{\iota(D), \iota(D')\} = \{\iota'(D), \iota'(D')\}.$$
So, if $\iota(D) \neq \iota'(D)$, then the only possibility is that $\iota(D) = \iota'(D')$ and that $\iota(D') = \iota'(D)$. Moreover, since $\iota$ and $\iota'$ preserve valuations, we have
$$\varphi_D = \varphi_{\iota(D)} = \varphi_{\iota'(D')} = \varphi_{D'}.$$

□

**Remark 4.1.4.** We note that the above lemma has a natural converse, which follows immediately from the definition of a $\mathcal{D}$-equivalence. Let $X_1$ and $X_2$ be spherical varieties, and suppose we have a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$. Let $\alpha \in \Pi^b_{X_1}$, and write $\mathcal{D}_{G,X_1}(\alpha) = \{D, D'\}$. If $\varphi_D = \varphi_{D'}$ and $D$ and $D'$ are moved by no root other than $\alpha$, then we can define another $\mathcal{D}$-equivalence $\iota' : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ by setting $\iota'(D) = \iota(D')$, $\iota'(D') = \iota(D)$, and $\iota = \iota'$ on every $B$-divisor besides $D$ and $D'$. Combined with the above lemma, this shows that we can construct all $\mathcal{D}$-equivalences between $X_1$ and $X_2$ from a single $\mathcal{D}$-equivalence $\iota$ just by swapping the values of $\iota$ on pairs of divisors $D$ and $D'$ satisfying the conditions mentioned in the lemma statement.

We will be interested in understanding the relationship between $\mathcal{D}$-equivalences and weight monoids of $G$-linearized invertible sheaves. However, we run into the problem that not every invertible sheaf is $G$-linearizable until we replace $G$ by $\tilde{G}$ for some central isogeny $\tilde{G} \to G$ (see Theorem 2.6.11). This replacement is typically safe thanks to the following lemma.

**Lemma 4.1.5.** *Let $X_1$ and $X_2$ be spherical varieties, and let $\pi : \tilde{G} \to G$ be an isogeny of reductive groups. Then, there exists a Borel subgroup $\tilde{B} \subset \tilde{G}$ and a maximal torus $\tilde{T} \subset \tilde{B}$ such that the following hold.*

*(a) $X_1$ and $X_2$ are spherical $\tilde{G}$-varieties (with $\tilde{G}$ acting via its image in $G$).*

*(b) $X_1$ and $X_2$ are $\mathcal{D}$-equivalent as spherical $G$-varieties if and only if $X_1$ and $X_2$ are $\mathcal{D}$-equivalent as spherical $\tilde{G}$-varieties.*

*(c) The valuation cones $\mathcal{V}(X_i)$ and the colored fans $\mathscr{F}_{X_i}$ are the same regardless of whether we view the $X_i$ as spherical $G$-varieties or spherical $\tilde{G}$-varieties.*

*(d) For any $G$-linearized divisorial sheaf $\mathcal{O}_{X_i}(D)$ on $X_i$, there exists a $\tilde{G}$-linearization on $\mathcal{O}_{X_i}(D)$ such that $\Lambda^+(X_i, \mathcal{O}_{X_i}(D))$ is the same whether we use the $G$-linearization or the $\tilde{G}$-linearization.*

*Proof.* We can pick a Borel subgroup $\tilde{B}$ of $\tilde{G}$ and a maximal torus $\tilde{T} \subset \tilde{B}$ such that $\tilde{B}$ and $\tilde{T}$ map surjectively onto $B$ and $T$, respectively (see e.g. [Mil17, Proposition 17.20]). In particular, the $\tilde{B}$-orbits of $X_i$ are precisely the $B$-orbits, so the $X_i$ are spherical $\tilde{G}$-varieties. Moreover, passing from the action of $G$ to the action of $\tilde{G}$ does not change any of the following data:

- Which divisors of the $X_i$ are $B$-stable or $G$-stable, (hence also which divisors are colors). More precisely: any divisor $D \subset X_i$ is $\tilde{B}$-stable (resp. $\tilde{G}$-stable) if and only if it is $B$-stable (resp. $G$-stable).
- The lattices $\Lambda(X_1)$ and $\Lambda(X_2)$ (as subgroups of $\Lambda_G \subset \Lambda_{\tilde{G}}$).
- The valuation $\varphi_D$ for any $B$-divisor $D$ of $X_1$ or $X_2$ (as a function on $\Lambda(X_1)$ or $\Lambda(X_2)$).
- The set of simple roots $\Pi_G$. This follows from the classification of reductive groups, since $\pi : \tilde{G} \to G$ is a central isogeny (for an explanation of why $\pi$ is central, see Remark 2.2.10).
- Which simple roots move a given $B$-divisor of $X_1$ or $X_2$. This follows from the definitions and the fact that $\pi(\tilde{P}_\alpha) = P_\alpha$ for any simple root $\alpha$. Alternately, it can be proven using the fact that the valuations $\varphi_D$ do not change, along with certain combinatorial facts about the $\varphi_D$ (cf. Lemma 4.4.2 and Corollary 4.4.3).

- The valuation cones $\mathcal{V}(X_1)$ and $\mathcal{V}(X_2)$. This is because $\tilde{G}$ acts on the $\tilde{G}$-module $K(X_i)$ via its image in $G$, so any valuation is $\tilde{G}$-equivariant if and only if it is $G$-equivariant.

- The colored fans $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$. This is because the $\tilde{G}$-orbits of $X_i$ are precisely the $G$-orbits, along with the fact that the $B$-divisors and $G$-divisors are the same and have the same valuations (as noted above).

This already proves (a), (b), and (c). As for (d), we note that pulling back any $G$-linearization by the map $(\pi, \mathrm{id}_{X_i}) : \tilde{G} \times X_i \to G \times X_i$ induces a $\tilde{G}$-linearization on $\mathcal{O}_{X_i}(D)$. Since $\tilde{G}$ acts on $X_i$ via its image under $\pi$, the resulting $\tilde{G}$-module structure on $\Gamma_*(X_i, \mathcal{O}_{X_i}(D))$ will be given by letting $\tilde{G}$ acts via its image under $\pi$. It follows that the $\tilde{G}$-linearization will have the same weight monoid as the $G$-linearization. □

## 4.1.b Combinatorial Data and Isomorphisms

The notion of a $\mathcal{D}$-equivalence will make it easy for us to describe when the classification of spherical varieties gives us $G$-equivariant isomorphisms between two spherical varieties. We first describe the homogeneous case.

**Theorem 4.1.6.** *Let $G/H_1$ and $G/H_2$ be two homogeneous spherical varieties. The following are equivalent.*

*(i) $G/H_1 \cong G/H_2$ as $G$-varieties (i.e. $H_1$ and $H_2$ are conjugate subgroups of $G$).*

*(ii) $G/H_1$ and $G/H_2$ are $\mathcal{D}$-equivalent, and $\Psi_{G,G/H_1} = \Psi_{G,G/H_2}$.*

*Moreover, if these conditions hold, then for any $\mathcal{D}$-equivalnece $\iota : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$, there exists a $G$-equivariant isomorphism $i : G/H_1 \xrightarrow{\sim} G/H_2$ such that $\iota(D) = i(D)$ for all $D \in \mathcal{D}_{G,G/H_1}$.*

*Proof.* If $i : G/H_1 \xrightarrow{\sim} G/H_2$ is a $G$-equivariant isomorphism, then $i$ induces a $G$-equivariant isomorphism on function fields, and this isomorphism identifies the valuation cones $\mathcal{V}(G/H_1)$ and $\mathcal{V}(G/H_2)$ (it identifies all valuations because it is a ring isomorphism and identifies the $G$-invariant ones because it is $G$-equivariant). This gives us $\Lambda(G/H_1) = \Lambda(G/H_2)$ and $\Psi_{G,G/H_1} = \Psi_{G,G/H_2}$. Moreover, the map $D \mapsto i(D)$ defines a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$.

Conversely, suppose that (ii) holds. Let $(\Lambda_i, \Pi_i^a, \Psi_i, \mathcal{D}_i^b)$ be the homogeneous spherical datum of $G/H_i$. We show that these homogeneous spherical data are equal, so that $G/H_1$ and $G/H_2$ are $G$-equivariantly isomorphic by Theorem 3.6.21. Let $\iota : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$ be a $\mathcal{D}$-equivalence. The definition of a $\mathcal{D}$-equivalence gives us $\Lambda(G/H_1) = \Lambda(G/H_2)$, i.e. $\Lambda_1 = \Lambda_2$, and Lemma 4.1.2 gives us $\Pi_1^a = \Pi_2^a$. Also, we have $\Psi_1 = \Psi_2$ by assumption. Finally, we also have $\Pi_{G/H_1}^b = \Pi_{G/H_2}^b$ by Lemma 4.1.2, and $\iota$ induces a bijection on colors moved by roots of type $b$, hence a bijection $\iota_b : \mathcal{D}_1^b \xrightarrow{\sim} \mathcal{D}_2^b$. The maps $\mathcal{D}_i^b \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_i, \mathbb{Z})$ in the homogeneous spherical data are the maps $D \mapsto \varphi_D$, so $\iota_b$ identifies these maps with one another because $\varphi_D = \varphi_{\iota(D)}$ for all $D$.

For the "moreover" statement, we note that there exists some $G$-equivariant isomorphism $i_0 : G/H_1 \xrightarrow{\sim} G/H_2$, and we can then define a $\mathcal{D}$-equivalence $\iota_0 : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$ by setting $\iota_0(D) = i_0(D)$ for all $D$. Thus, it suffices to show that for any other $\mathcal{D}$-equivalnece $\iota : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$, there exists some automorphism $a : G/H_2 \xrightarrow{\sim} G/H_2$ such that $\iota(D) = a(\iota_0(D))$ for all $D$. This is a (nontrivial) consequence of Losev's proof that homogeneous spherical data determine homogeneous spherical varieties up to $G$-isomorphism (see e.g. the discussion following [Los09b, Theorem 3]).

$\square$

Recall that the classification of all spherical varieties is simply a combination of the classification of homogeneous spherical varieties and Luna–Vust theory, which classifies open embeddings of spherical varieties of the form $G/H \hookrightarrow X$. Thus, to upgrade the above theorem into a statement about all spherical varieties, we need to understand what should be meant by an "equality" on the combinatorial invariants of Luna–Vust theory, i.e. colored fans. The following definition gives us this notion of equality.

**Definition 4.1.7.** Let $X_1$ and $X_2$ be spherical varieties. We say that a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ *preserves colored fans* if

$$\mathscr{F}_{X_2} = \{(\mathcal{C}, \iota(\Delta)) \mid (\mathcal{C}, \Delta) \in \mathscr{F}_{X_1}\}.$$

**Remark 4.1.8.** We would like to make explicit an implicit identification made in the above definition. Given a colored fan $(\mathcal{C}, \Delta) \in \mathscr{F}_{X_i}$, the set $\Delta$ contains colors of the open $G$-orbit $G/H_i$ of $X_i$, so we cannot literally apply $\iota$ to elements of $\Delta$. However, the map $D \mapsto D \cap G/H_i$ is a bijection $\mathcal{D}_{G,X_i} \to \mathcal{D}_{G,G/H_i}$ whose inverse is $D_0 \mapsto \overline{D_0}$ (with the closure taken in $X_i \supset G/H_i$). Thus, we will typically identify colors of $G/H_i$ with their closures in $X_i$ and so view them as colors of $X_i$. This explains what we mean by $\iota(\Delta)$ in the above equation.

We are now ready to formulate precisely what the classification of spherical varieties has to say about when two spherical varieties are $G$-equivariantly isomorphic.

**Theorem 4.1.9.** *Let $X_1$ and $X_2$ be spherical varieties. The following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) There exists a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ that preserves colored fans, and $\Psi_{G,X_1} = \Psi_{G,X_2}$.*

*Proof.* Let $G/H_i$ be the open $G$-orbit of $X_i$. Any $G$-equivariant isomorphism $i : X_1 \xrightarrow{\sim} X_2$ restricts to a $G$-equivariant isomorphism $G/H_1 \xrightarrow{\sim} X_2$, so $\Psi_{G,X_1} = \Psi_{G,X_2}$ by Theorem 4.1.6. Moreover, setting $\iota(D) = i(D)$ defines a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$, and one can check from the definitions that this $\mathcal{D}$-equivalence preserves colored fans.

Conversely, suppose that (ii) holds, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence preserving colored fans. Then, $\iota$ induces a $\mathcal{D}$-equivalence $\iota_0 : \mathcal{D}_{G,G/H_1} \xrightarrow{\sim} \mathcal{D}_{G,G/H_2}$ (explicitly, we set $\iota_0(D_0) = \iota(\overline{D_0}) \cap G/H_2$, see Remark 4.1.8 above). Thus, Theorem 4.1.6 gives us a $G$-equivariant isomorphism $i_0 : G/H_1 \xrightarrow{\sim} G/H_2$ such that $\iota(D_0) = \iota(D_0)$ for all $D_0 \in \mathcal{D}_{G,G/H_1}$. The statement that $\iota$ preserves colored fans is thus the statement that the isomorphism $i_0$ identifies the two colored fans $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$. It follows from Theorem 3.3.26 that $X_1$ and $X_2$ are $G$-equivariantly isomorphic. Or, more precisely: since $\iota$ preserves colored fans, we see that $i_0$ maps $\mathscr{F}_{X_1}$ into $\mathscr{F}_{X_2}$ and that $i_0^{-1}$ maps $\mathscr{F}_{X_2}$ into $\mathscr{F}_{X_1}$. So, $i_0$ and $i_0^{-1}$ extend to $G$-equivariant morphisms $i : X_1 \to X_2$ and $i^{-1} : X_2 \to X_1$ (respectively) by Theorem 3.3.28, and $i^{-1} \circ i$ and $i \circ i^{-1}$ are both equal to the identity morphism because they agree with the identity on a dense open subset (namely, the open $G$-orbit) and because $X_1$ and $X_2$ are separated and reduced (see e.g. [Har77, Chapter II, Exercise 2.4.2]). □

In this chapter, we investigate to what extent weight monoids of the form $\Lambda^+(X, L)$ determine these pieces of combinatorial data. Theorem 4.1.9 is the key starting point for this task, for two reasons: first, the theorem summarizes which pieces of combinatorial data show up in the classification of spherical varieties; and second, the theorem gives us the language and formalism needed in order to understand what an "equality" on this combinatorial data really means.

## 4.2 On Losev's Proof of the Knop Conjecture

In this section, we provide some introductory background to the Knop Conjecture and its proof by Losev in [Los09a]. This is provided for the benefit of the reader and can be skipped without any loss of continuity. In what follows, the only material from this section that will be necessary is Theorem 4.2.2, Definition 4.2.4, and Question 4.2.5. The reader who wishes to skip this section can either read through these statements briefly now or refer back to them as they arise later on.

We first recall some notation from Section 2.5. For any $G$-variety $X$, we denote by $\Lambda^+(X)$ the set of all weights of $B$-eigenvectors of the global sections $\Gamma(X, \mathcal{O}_X)$. Note that $\Lambda^+(X)$ is a commutative monoid; we will refer to it as the *weight monoid* of $X$. Similarly, given a $G$-linearized invertible sheaf $L$ on $X$, we denote by $\Lambda^+(X, L)$ the monoid of all weights of $(B \times \mathbb{G}_m)$-eigenvectors in the ring $\Gamma_*(X, L) = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$ (with $B$ acting via the $G$-linearization on $L^{\times n}$ and $\mathbb{G}_m$ acting according to the grading). We will refer to $\Lambda^+(X, L)$ as the *weight monoid* of the pair $(X, L)$. We remark that when $\mathcal{O}_X$ has the canonical $G$-linearization of Lemma 2.4.13a, we get

$$\Lambda^+(X, \mathcal{O}_X) = \Lambda^+(X) \times \mathbb{N}.$$

Thus, the weight monoid $\Lambda^+(X, L)$ is in some sense a generalization of $\Lambda^+(X)$.

Consider the case where $X$ is an affine spherical $G$-variety and $G = T$ is a torus. Then, $X$ is by definition a toric variety for a quotient $T'$ of $T$ (namely, $T' = T/G_x$, where $x$ is any point in the open $T$-orbit of $X$). We claim that the weight monoid $\Lambda^+(X)$ completely determines $X$ up to $T$-equivariant isomorphism. Note that $\Lambda(X) = \Lambda^+(X)^{gp}$ (Proposition 2.5.9), so $\Lambda^+(X)$ determines the lattice $\Lambda(X)$ as a subgroup of $\Lambda(T)$. By standard facts about tori (see e.g. [Mil17, Theorem 12.9]), the inclusion $\Lambda(X) = \Lambda(T') \subset \Lambda(T)$ determines the torus $T'$ and the quotient map $T \to T'$ up to $T$-equivariant isomophism. Moreover, the global sections $\Gamma(X, \mathcal{O}_X)$ are $T'$-equivariantly isomorphic to the $k$-algebra $k[-\Lambda^+(X)]$ (see Proposition 3.3.5 and the preceding discussion), so the weight monoid determines $X$ up to $T'$-equivariant isomorphism, hence also up to $T$-equivariant isomorphism.

Thus, in the toric case, the weight monoid $\Lambda^+(X)$ completely classifies $X$. Even better, one can impose nice combinatorial conditions such that any monoid in a given lattice $\Lambda$ satisfying those conditions is $\Lambda^+(X)$ for some toric variety $X$ such that the torus $T \subset X$ has $\Lambda(T) \cong \Lambda$ (see e.g. [Ful93, Section 1.3, Exercise 3]). In the general spherical case, one cannot hope for quite such a nice statement. For starters, the weight monoid does not even classify all affine spherical varieties up to $G$-isomorphism, as the following example shows.

**Example 4.2.1.** Let $A = k[x_1, x_2, x_3]$, and let $q = 2x_1x_3 + x_2^2 \in A$. Then, $G = \mathrm{SO}_3$ is the subgroup of $\mathrm{SL}_3$ that fixes the quadratic form $q$, so we obtain actions of $G$ on the varieties

$$X = \mathrm{Spec}(A/(1-q)), \quad Y = \mathrm{Spec}(A/(q)).$$

Both $X$ and $Y$ are spherical varieties. Moreover, $X$ and $Y$ are both fibers of the flat family

$$\mathrm{Spec}(A[t]/(t-q)) \to \mathrm{Spec}(k[t]),$$

and weight monoids are locally constant in flat families, so we have $\Lambda^+(X) = \Lambda^+(Y)$. (This is a consequence of a "sheafified" version of semisimplicity of $G$-modules, along with cohomology and base change; see [AB06, Section 3.1] for details). However, $X$ and $Y$ are not $G$-isomorphic, because $X$ is smooth and $Y$ is not (as a Jacobian calculation will show).

We can also see that $X$ and $Y$ are not $G$-isomorphic by looking at valuation cones. For this, note that the maximal torus of $G$ is 1-dimensional, and $x_1$ is a $B$-eigenvector of both $A/(q)$ and $A/(1-q)$ (a Borel subgroup for $G$ with our choice of $q$ is given by the upper triangular matrices in $G$). So, we have

$$\Lambda(Y) = \Lambda(X) = \Lambda_G = \mathbb{Z}$$

and hence $N(X) = N(Y) = N = \mathbb{Q}$. Moreover, Since $G$ contains the automorphism of $A$ that fixes $x_2$ and swaps $x_1$ and $x_3$, any $G$-invariant valuation $v$ of either $X$ or $Y$ must have $v(x_1) = v(x_3)$. In the case of $Y$, we can define a $G$-invariant valuation $v : K(Y)^\times \to \mathbb{Q}$ by setting $v(x_1) = v(x_2) = v(x_3) = 1$, and $v(k^\times) = 0$, then extending to all of $K(Y)^\times$ by defining $v(ab) = v(a)+v(b)$ and $v(a+b) = \min\{v(a)+v(b)\}$ for all $a, b \in K(Y)^\times$. Noting that $K(Y) \cong k(x_1, x_3, \sqrt{x_1x_3})$, we see that $K(Y)$ is a $\mathbb{Z}$-graded ring (taking $\deg(\sqrt{x_1x_3}) = 1$), and for any $a \in K(Y)^\times$, the value of $v(a)$ is the smallest integer $d$ such that the homogeneous part of $a$ in degree $d$ is nonzero. This is certainly a valuation, since $G$ acts by graded ring homomorphisms on $K(Y)$ and $v$ is constant on each homogeneous part of $K(Y)$, we see that $v$ is $G$-invariant. We have thus found a nonzero element $v \in \mathcal{V}(Y)$. In fact, the same argument would have worked if we set $v(x_1) = v(x_2) = v(x_3)$ to be any value in $\mathbb{Q}$, so we see that $\mathcal{V}(Y) = N(Y) = \mathbb{Q}$. In other words, $Y$ is a horospherical variety.

As for $X$, let $v : K(X)^\times \to \mathbb{Q}$ be a $G$-invariant valuation. As noted above, we must have $v(x_1) = v(x_3)$. Since $2x_1x_3 = 1 - x_2^2$ on $X$, this implies that

$$2v(x_1) = v(x_1) + v(x_3) = v(2x_1x_3) = v(1 - x_2^2) \geq \min\{v(1), 2v(x_2)\},$$

with equality if $v(1) \neq 2v(x_2)$ (this "with equality" statement is a general fact about valuations which follows from the fact that $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a$ and $b$). So, we have three cases:

1. If $v(x_2) > v(1) = 0$, then we have $v(x_1) = v(x_3) = 0$ by the above and hence

   $$2v(x_2) = v(1 - 2x_1x_3) \geq \min\{v(1), v(x_1x_3)\} = 0.$$

   One can check that the $k$-linear map sending $x_1 \mapsto x_1$, $x_2 \mapsto x_1 + x_2$, and $x_3 \mapsto -\frac{1}{2}x_1 - x_2 + x_3$ is an element of $G$. Since $v$ is $G$-invariant, this implies that

   $$v(x_2) = v(x_1x_2) = v(x_1(x_1 + x_2)) = v(x_1^2 + x_1x_2) \geq \min\{2v(x_1), v(x_1x_2)\},$$

   with equality if $v(x_1x_2) \neq 2v(x_1)$. Since we have $v(x_1) = 0$ and we have assumed $v(x_1x_2) = v(x_2) \geq 0$, this implies that $v(x_2) = \min\{2v(x_1), v(x_1x_2)\} = 0$, a contradiction.

2. If $v(x_2) = 0$, then the above equation gives us $2v(x_1) \geq 0$ and hence $v(x_1) = v(x_3) \geq 0$. Suppose that $v(x_1) \neq 0$. Then, we have

   $$v(x_1 + x_2) \geq \min\{v(x_1), 0\},$$

   and since $v(x_1) \neq 0$, we get $v(x_1 + x_2) = 0$. Using the element of $G$ given in case 1, which sends $x_3 \mapsto -\frac{1}{2}x_1 - x_2 + x_3$, we then get

   $$v(x_3) = v(-x_1/2 - x_2 + x_3) \geq \min\{v(x_1 + x_2), v(x_3)\} = \min\{0, v(x_3)\},$$

   and since $v(x_3) \neq 0$, we have equality, i.e. $v(x_3) = 0$, which is a contradiction. So in fact, $v(x_1) \neq 0$ is impossible, which means $v$ must be the trivial valuation.

3. If $v(x_2) < 0$, then the above inequality gives us $2v(x_1) = 2v(x_2)$ and hence $v(x_1) = v(x_2) = v(x_3)$. On the other hand, $\mathcal{V}(X)$ is a full-dimensional cone in $N(X)$ (Theorem 3.4.1), so in particular, we cannot have $\mathcal{V}(X) = \{0\}$. Since the above cases only gave us the trivial valuation, there must exist some $G$-invariant valuation $v$ with $v(x_1) = v(x_2) = v(x_3) < 0$.

It follows that $\mathcal{V}(X) = \mathbb{Q}_{\leq 0} \subset N = \mathbb{Q}$. In particular, we have $\mathcal{V}(X) \neq \mathcal{V}(Y)$. (Alternately, one can explicitly compute the $G$-orbits of $X$ to show that $X$ is a wonderful variety. Since wonderful varieties are standard embeddings, their valuation cones must be strictly convex, so this is another way to see that $\mathcal{V}(X) \neq \mathcal{V}(Y)$.)

In the above example, the variety $X$ is smooth, but $Y$ is not. Knop conjectured that the weight monoid actually does classify *smooth* affine spherical varieties up to $G$-equivariant isomorphism. This was proven by Losev, who in fact proved a somewhat stronger statement: affine spherical varieties are determined by their weight monoid and their valuation cone, and smooth affine spherical varieties with the same weight monoid have the same valuation cone.

**Theorem 4.2.2** (Knop Conjecture; [Los09a, Theorems 1.2, 1.3])**.** *Let $X_1$ and $X_2$ be two affine spherical $G$-varieties.*

*(a) If $\Lambda^+(X_1) = \Lambda^+(X_2)$ and $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ (as cones in $N(X_1) = N(X_2)$), then $X_1$ and $X_2$ are $G$-isomorphic.*

*(b) If $X_1$ and $X_2$ are smooth and $\Lambda^+(X_1) = \Lambda^+(X_2)$, then $X_1$ and $X_2$ are $G$-isomorphic.*

**Remark 4.2.3.** As with toric varieties, Pezzini and van Steirteghem [PV19] have given a combinatorial characterization of the monoids that occur as the weight monoid of some smooth affine spherical variety.

Losev's proof of Theorem 4.2.2 hinges on his proof in [Los09c] that homogeneous spherical varieties are classified by their homogeneous spherical data, which is part of the classification statement in Theorem 3.6.21. Indeed, given two affine spherical varieties $X_1$ and $X_2$ such that $\Lambda^+(X_1) = \Lambda^+(X_2)$, if both $X_1$ and $X_2$ have the same open $G$-orbit $G/H$, then both $\Gamma(X_1, \mathcal{O}_{X_1})$ and $\Gamma(X_2, \mathcal{O}_{X_2})$ are $G$-submodules of $K(G/H)$. Since $G$-modules are determined by their $B$-eigenvectors and $K(G/H)$ is multiplicity-free (because $K(G/H)^B = k$, see Theorem 3.1.4), the statement that $\Lambda^+(X_1) = \Lambda^+(X_2)$ says that $\Gamma(X_1, \mathcal{O}_{X_1})$ and $\Gamma(X_2, \mathcal{O}_{X_2})$ are the same submodule of $K(G/H)$, i.e. that $X_1 \cong X_2$ as $G$-varieties. Thus, the difficulty in the proof is showing that $X_1$ and $X_2$ have the same open $G$-orbit, which Losev does by showing that the two open orbits have the same homogeneous spherical data.

We are interested in investigating to what extent a weight monoid of the form $\Lambda^+(X, L)$ classifies the spherical variety $X$ in cases besides the smooth affine case. The most natural other case to consider is the case where $X$ is projective and where we consider $\Lambda^+(X, L)$ for a fixed $G$-linearized ample invertible sheaf $L$. To this end, we make the following definition.

**Definition 4.2.4.** A *polarized spherical variety* is a pair $(X, L)$, where $X$ is a projective spherical variety and $L$ is a $G$-linearized ample invertible sheaf on $X$. A *morphism* of polarized spherical varieties $(X, L) \to (Y, M)$ is a $G$-equivariant morphism $f : X \to Y$ such that $f^*M \cong L$.

Motivated by Losev's proof of the Knop conjecture, we ask the following question.

**Question 4.2.5.** *Let $(X, L)$ and $(Y, M)$ be two polarized spherical varities.*

*(a) If $\Lambda^+(X, L) = \Lambda^+(Y, M)$ and $\mathcal{V}(X) = \mathcal{V}(Y)$, is it the case that $(X, L) \cong (Y, M)$ as polarized $G$-varieties (i.e. is there a $G$-equivariant isomorphism $i : X \to Y$ such that $i^*M \cong L$?)*

(b) *If $\Lambda^+(X, L) = \Lambda^+(Y, M)$ and $X$ and $Y$ are smooth, is it the case that $(X, L) \cong (Y, M)$ as polarized spherical varieties?*

We will see that the answer to Question (a) is "yes." The proof revolves around passing to the affine cone of $X$. The main challenge is to pass the statement that $\mathcal{V}(X) = \mathcal{V}(Y)$ to the affine cone, for which we will need an understanding of $G$-invariant valuations on affine cones. Once this is done, we will be able to reduce Question (a) to Theorem 4.2.2a. This is the subject of Section 4.3; specifically, the proof of Question (a) is Corollary 4.3.5.

Question (b) is significantly more subtle. There is no easy reduction to the affine case here, because the affine cones in question need not be smooth. Ultimately, we will see that Question (b) is not true as stated: we will give an explicit counterexample in Examples 4.9.1 and 4.9.2 in which the spherical varieties are even rank-1 and toroidal, as well as other counterexamples in Examples 4.9.3 and 4.9.4. So instead, we rephrase Question (b) as follows:

**Question 4.2.6.** *Let $(X, L)$ and $(Y, M)$ be two polarized spherical varities.*

(b') *If $\Lambda^+(X, L) = \Lambda^+(Y, M)$ and $X$ and $Y$ are smooth, are there any other combinatorial invariants of the spherical varieties $X$ and $Y$ that must be equal?*

Our main strategy for answering Question (b') will be to use the local structure theorem to obtain affine varieties which we can apply the Knop conjecture to. This will allow us to "match up" pieces of combinatorial data between two spherical varieties $X$ and $Y$ as in Quetion (b'), provided that this combinatorial data can be detected locally. Specifically, we will be interested in the three pieces of combinatorial data specified in Theorem 4.1.9, which yields three different questions:

1. Does there exist a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X} \xrightarrow{\sim} \mathcal{D}_{G,Y}$?

2. If a $\mathcal{D}$-equivalence exists, does there exist a $\mathcal{D}$-equivalence that preserves colored fans?

3. Do we have $\Psi_{G,X} = \Psi_{G,Y}$?

In attempting to answer these questions, we will prove several interesting results, which have already been discussed at length in Section 1.2.

We note that both smoothness and projectivity are essential conditions in Question (b'). Indeed, the following two examples show that it is relatively easy to pick at least some combinatorial invariants of $X$ and $Y$ to be unequal when either smoothness or projectivity is dropped.

**Example 4.2.7.** To show that smoothness is necessary in Question (b), we give what is essentially a projective version of Example 4.2.1. Let $G = \mathrm{SO}_3$ be the subgroup of $\mathrm{GL}_3$ stabilizing $q = 2x_1x_3 + x_2^2$ in $k[x_1, x_2, x_3]$, and let

$$X = \operatorname{Proj}(k[w, x_1, x_2, x_3]/(w^2 - q)), \quad Y = \operatorname{Proj}(k[w, x_1, x_2, x_3]/(q)),$$

with $G$ acting on $k[x_1, x_2, x_3]$ in the natural way and fixing $w$. As before, $X$ is smooth, and $Y$ is smooth except at $[1:0:0:0]$ by a Jacobian calculation. In particular, $X$ and $Y$ are both regular in codimension 1, hence normal. Also, $X$ and $Y$ contain the varieties of Example 4.2.1 above as the $G$-stable open subset $\mathrm{D}_+(w)$. So, $X$ and $Y$ are spherical because they contain an open subset which is spherical.

As in Example 4.2.1, we have a flat family

$$\mathrm{Proj}(k[t, w, x_1, x_2, x_3]/(tw^2 - q)) \to \mathrm{Spec}(k[t])$$

in which $X$ and $Y$ appear as fibers, and this implies that $\Lambda^+(X, \mathcal{O}_X(1)) = \Lambda^+(Y, \mathcal{O}_Y(1))$. However, we know that $X \ncong Y$, since $X$ is smooth and $Y$ is not, so we cannot possibly have $(X, \mathcal{O}_X(1)) \cong (Y, \mathcal{O}_Y(1))$. In fact, the valuation cones of $X$ and $Y$ are the same as those of Example 4.2.1 (because the valuation cone of $X$ is that of its $G$-stable open subset $\mathrm{D}_+(w)$, and likewise for $Y$). So we again have $\mathcal{V}(X) \neq \mathcal{V}(Y)$.

**Example 4.2.8.** The following example shows that projectivity is necessary in Question (b). Let $X$ be a smooth projective spherical variety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. Let $Y \subset X$ be the union of all $G$-orbits of codimension $\leq 1$, and set $M = L|_Y$. Since there are finitely many closures of orbits of codimension $> 2$, and $Y$ is the complement of the union of these orbit closures, so $Y$ is a $G$-stable open subset of $X$. Moreover, since the inclusion $i : Y \hookrightarrow X$ is $G$-equivariant, the $G$-linearization on $L$ induces a $G$-linearization on $M = i^*L$. In particular, $Y$ is smooth because $X$ is, and pulling back by $i$ induces a $G$-equivariant isomorphism $H^0(X, L^{\otimes d}) \cong H^0(Y, M^{\otimes d})$ for all $d \geq 0$ (because $X$ is normal and $\mathrm{codim}(X \setminus Y) \geq 2$). This implies that $\Lambda^+(X, L) = \Lambda^+(Y, M)$.

On the other hand, note that $Y$ will not be projective unless we have $X = Y$. (If $Y$ is projective, then the inclusion $i$ is a projective morphism and hence closed, so $Y$ is open and closed in $X$.) It follows that $X \cong Y$ as $G$-varieties if and only if $X = Y$, which is rarely the case. Indeed, let $\varphi_1, \ldots, \varphi_m$ be the valuations of $G$-divisors of $X$ (viewed as elements of $N(X)$). Note that the open $G$-orbit $G/H$ of $Y$ is the same as that of $X$, and the codimension-1 $G$-orbits of $X$ are precisely the orbits which are dense in the $G$-divisors of $X$. It follows that $Y$ is the embedded of $G/H$ corresponding to the colored fan

$$\mathscr{F}_Y = \{(\mathbb{Q}_{\geq 0}\varphi_1, \varnothing), \ldots, (\mathbb{Q}_{\geq 0}\varphi_m, \varnothing)\}.$$

The colored cones in this colored fan are necessarily also in the colored fan $\mathscr{F}_X$ (they are the colored cones corresponding to the same $G$-orbits of $Y$, just viewed as $G$-orbits of $X$). To ensure that $X = Y$, we just need to pick $\mathscr{F}_X$ to have at least one other colored cone besides those in $\mathscr{F}_Y$. For instance, note that $Y$ is toroidal, i.e. no colored cone in $\mathscr{F}_Y$ contains a color. Thus, if $X$ is any smooth projective spherical variety which is not toroidal, then $\mathscr{F}_X$ contains a colored cone which has a color and hence is not in $\mathscr{F}_Y$, so we have $X \neq Y$ and hence $X \ncong Y$.

## 4.3 Valuations on Affine Cones

In this section, we analyze the relationship between the valuation cone $\mathcal{V}(X)$ of a projective spherical variety $X$ and the valuations on the affine cone of $X$. This will culminate in Corollary 4.3.5, which is the projective equivalent of Theorem 4.2.2a and so answers Question 4.2.5a in the affirmative. At the end of Section 4.5, we will give an alternate proof of Corollary 4.3.5 that makes use of the local structure theorem instead of affine cones. Throughout the rest of this chapter, our focus will be on the approach using the local structure theorem rather than the approach using affine cones. As such, the reader who is not interested in valuations on affine cones can safely skip this section.

We begin by recalling a few facts about affine cones (see Appendix A). Given a polarized $G$-variety $(X, L)$, we let $G$ act on $\Gamma_*(X, L)$ via the $G$-linearization on $L$, and we let $\mathbb{G}_m$ act on the degree-$d$ part of $\Gamma_*(X, L)$ via the character $d \in \mathbb{Z} \cong \mathcal{X}(T)$. Writing $\tilde{G} = G \times \mathbb{G}_m$, this gives us an action of $\tilde{G}$ on the affine cone $\tilde{X} = \operatorname{Spec}(\Gamma_*(X, L))$. As discussed in Section 2.5, we use the Borel subgroup $\tilde{B} = B \times \mathbb{G}_m$ of $\tilde{G}$ and the maximal torus $\tilde{T} = T \times \mathbb{G}_m \subset \tilde{B}$, and these give us

$$\Lambda_{\tilde{G}} \cong \Lambda_G \times \mathbb{Z} \quad \text{and} \quad \Lambda^+(X, L) = \Lambda^+(\tilde{X}).$$

Write $A = \Gamma_*(X, L)$, so that $X \cong \operatorname{Proj}(A)$ and $\tilde{X} = \operatorname{Spec}(A)$. Since $X$ is a projective variety and $k = \overline{k}$, we have $\Gamma(X, \mathcal{O}_X) = k$, so the vertex of the affine cone $\tilde{X}$ is a single point 0. Moreover, we have a principal $\mathbb{G}_m$-bundle

$$\pi : \tilde{X} \setminus \{0\} \to X$$

which is given locally by the morphism of affine schemes $\tilde{X}_f \to X_f$ corresponding to the inclusion $(A_f)_0 \hookrightarrow A_f$ for any homogeneous element $f \in A$. Note that if $f \in A_1$, then the map

$$(A_f)_0 \otimes_k k[t^{\pm}] \to A_f \tag{4.3.1}$$

given by $g/f^d \otimes t^n \mapsto g/f^{d-n}$ is an isomorphism of graded rings with inverse given by $g/f^m \mapsto g/f^{\deg g} \otimes t^{d-m}$. We thus have $\tilde{X}_f \cong \mathbb{G}_m \times X_f$. This gives us a nice local trivialization of the principal $\mathbb{G}_m$-bundle $\pi$ whenever $A$ is generated in degree 1 (which always holds after replacing $L$ by some suitable power $L^{\otimes n}$).

Now, if $X$ is a spherical $G$-variety, then Corollary A.6 implies that $\tilde{X}$ is a spherical $\tilde{G}$-variety. Our goal is to relate the valuation cone $\mathcal{V}(X)$ to the valuation cone $\mathcal{V}(\tilde{X})$. Note that $\pi$ is surjective and so gives an inclusion on function fields $K(X) \hookrightarrow K(\tilde{X})$. Since $\pi$ is also $G$-equivariant, we may view $K(X)$ as a $G$-submodule of $K(\tilde{X})$. A $\tilde{G}$-invariant valuation $v : K(\tilde{X})^\times \to \mathbb{Q}$ is one which is both $G$-invariant and $\mathbb{G}_m$-invariant, so the restriction $v|_{K(X)^\times}$ is a $G$-invariant valuation of $K(X)/k$. To understand the relationship between $\mathcal{V}(X)$ and $\mathcal{V}(\tilde{X})$, then, we want to know which elements of $\mathcal{V}(X)$ we get by restricting in this way, and to what extent this restriction determines the original valuation $v$.

Since we know that $\tilde{X}$ is Zariski-locally isomorphic to $\mathbb{G}_m \times X$, and since $v$ must be $\mathbb{G}_m$-invariant, the first step is to characterize the $\mathbb{G}_m$-invariant valuations of $\mathbb{G}_m \times X$. We do this algebraically, by proving that $\mathbb{G}_m$-invariance forces valuations on $K(\mathbb{G}_m \times X) \cong K(X)(t)$ to have a particularly nice form.

**Proposition 4.3.1.** *Let $k$ be an infinite field, and let $K_0/k$ be a field extension.*

*(a) For any $\mathbb{G}_{m,k}$-invariant valuation $v : K_0(t)^\times \to \mathbb{Q}$ over $k$ and any element $\sum_{i=-m}^{n} a_i t^i \in K(t)$ (here $a_i \in K_0$ for each $i$), we have*

$$v\left(\sum_{i=-m}^{n} a_i t^i\right) = \min_i \{v(a_i t^i)\} = \min_i \{v(a_i) + iv(t_i)\}.$$

*(b) The map*

$$\{\mathbb{G}_{m,k}\text{-}invariant\ valuations\ v : K_0(t)^\times \to \mathbb{Q}\} \to \left\{(v_0, r) \;\middle|\; \substack{r\in\mathbb{Q},\ v_0 : K_0^\times \to \mathbb{Q} \\ a\ valuation\ of\ K_0/k}\right\}$$

*given by $v \mapsto (v|_{K_0^\times}, v(t))$ is a bijection.*

*Proof.* Note that (a) immediately implies that the map in (b) is injective; conversely, given any $(v_0, r)$ in the image of this map, one can check that defining $v$ according to the rule in (a) and setting $v|_{K_0} = v_0$ and $v(t) = r$ gives a $\mathbb{G}_m$-invariant valuation. Thus, (b) will follows directly from (a). To prove (a), we note that since

$$v\left(\sum_{i=-m}^{n} a_i t^i\right) = -mv(t) + v\left(\sum_{i=0}^{m+n} a_{i-m} t^i\right),$$

(just factor out $t^{-m}$ from the element of $K_0(t)$ on the lefthand side), we may assume that $m = 0$. The proof is by induction on $n$. The base case $n = 0$ is trivial, so assume the statement holds for $0 \le n < N$, and write $f = \sum_{i=0}^{N} a_i t^i$. Then, the definition of a valuation gives us

$$v(f) = v\left(\sum_{i=0}^{N} a_i t^i\right) \ge \min_i \{v(a_i t^i)\},$$

so we just need to prove the opposite inequality. The idea is to use $\mathbb{G}_m$-invariance of $v$ to cancel out first the constant term $a_0$ and then the highest-power term $a_N t^N$. After each cancellation, we will be able to use the induction hypothesis to get an inequality, and combining these two inequalities will give us the inequality we want.

For any $c \in k^\times$ such that $c \ne 1$, the action of $\mathbb{G}_m$ is multiplication by $c^i$ on $t^i$. Since $v$ is $\mathbb{G}_m$-invariant, we have $v(f) = v(c \cdot f) = v\left(\sum_{i=0}^{N} c^i a_i t^i\right)$, so that

$$v\left(\sum_{i=0}^{N} (c^i a_i t^i - a_i t^i)\right) \ge \min\left\{v\left(\sum_{i=0}^{N} c^i a_i t^i\right), v\left(a_i t^i\right)\right\} = \min\{v(c \cdot f), v(f)\} = v(f).$$

But
$$\sum_{i=0}^{N}(c^i a_i t^i - a_i t^i) = \sum_{i=1}^{N}(c^i - 1)a_i t^i = t\sum_{i=0}^{N-1}(c^{i+1} - 1)a_{i+1}t^i,$$
so the above two equations and the induction hypothesis give us
$$\begin{aligned} v(f) \le v\left(\sum_{i=0}^{N}(c^i a_i t^i - a_i t^i)\right) &= v(t) + v\left(\sum_{i=0}^{N-1}(c^{i+1} - 1)a_{i+1}t^i\right) \\ &= v(t) + \min_{0\le i\le N-1}\{v((c^{i+1} - 1)a_{i+1}t^i)\} \\ &= \min_{0\le i\le N-1}\{v(t) + v(a_{i+1}t^i)\} \\ &= \min_{1\le i\le N}\{v(a_i t^i)\}. \end{aligned} \tag{4.3.2}$$
(Note that we have used $c \in k^\times$ here to conclude that $v(c^{i+1} - 1) = 0$.)

This is almost the inequality that we want, except that the $v(a_0)$ term is missing. To get it back, we make an analogous computation but cancel out $a_N t^N$ this time instead of $a_0$. We may assume that $a_N \neq 0$ (otherwise we are done by induction hypothesis), and we may pick some $c$ such that $c^N \neq 1$ (there are only finitely many $c \in k^\times$ such that $c^N = 1$, and $k$ is infinite, so some such $c$ exists). By acting by $c$ again and then multiplying by $c^{-N}$, we have
$$v(f) = v(c^{-N}f) = v(c\cdot(c^{-N}f)) = v\left(\sum_{i=0}^{N} c^{i-N}a_i t^i\right),$$
so that
$$\begin{aligned} v(f) = \min\{v(f), v(c\cdot(c^{-N}f))\} &= \min\{v\left(\sum_{i=0}^{N} a_i t^i\right), v\left(\sum_{i=0}^{N}(c^{i-N}a_i t^i\right)\} \\ &\le v\left(\sum_{i=0}^{N} a_i(1 - c^{i-N})t^i\right) \\ &= v\left(\sum_{i=0}^{N-1} a_i(1 - c^{i-N})t^i\right) \\ &= \min_{0\le i\le N-1}\{v(a_i(1 - c^{i-N}t^i)\} \\ &= \min_{0\le i\le N-1}\{v(a_i t^i)\}. \end{aligned}$$
(Here, the penultimate equality follows from the induction hypothesis.) Combining this with (4.3.2) gives us
$$v(f) \le \min_{0\le i\le N}\{v(a_i t^i)\},$$
as desired. $\square$

Write $A = \Gamma_*(X, L)$ and $K_0 = K(X)$. The above proposition tells us what the $\mathbb{G}_m$-invariant valuations on $K(\tilde{X}) \cong K(\mathbb{G}_m \times X) \cong K_0(t)$ are. To understand what the $(G \times \mathbb{G}_m)$-invariant valuations are, the main issue is to find an appropriate local trivialization of the principal $\mathbb{G}_m$-bundle $\pi : \tilde{X} \setminus \{z\} \to X$. We already have a good local trivialization most of the time: namely, the isomorphism $\tilde{X}_f \cong X_f \times \mathbb{G}_m$ of (4.3.1) for any $f \in A_1$. If we pick $f \in A_1^{(B)}$ to be a $B$-eigenvector, we can also keep track of the $\tilde{B}$-eigenvectors of $A_f$ under this trivialization. Eigenvectors for $\mathbb{G}_m$ are homogeneous elements, so the $\tilde{B}$-eigenvectors of $A_f$ are just the $B$-eigenvectors of the homogeneous parts of $A_f$. Since the local trivialization $\tilde{X}_f \cong X_f \times \mathbb{G}_m$ defined by (4.3.1) identifies $f$ with $t$, we see that the $\tilde{B}$-eigenvectors in $(A_f)_0 \otimes_k k[t^\pm] \cong (A_f)_0[t^\pm]$ are the elements of the form $f_0 t^n$ for some $f_0 \in (A_f)_0^{(B)}$ and some $n \in \mathbb{Z}$. Moreover, if $\lambda \in \Lambda(X)$ (resp. $\lambda_0 \in \Lambda(X)$) is the character corresponding to $f$ (resp. $f_0$), then $f_0 t^n$ has corresponding character

$$(\lambda_0 + n\lambda, n) \in \Lambda(\tilde{X}) \subset \Lambda_{\tilde{G}} = \Lambda_G \times \mathbb{Z}.$$

In summary, we can drop locally on $X$ and $\tilde{X}$ to get a trivialization $\tilde{X}_f \cong X_f \times \mathbb{G}_m$ in such a way that we can keep track of $B$-eigenvectors and their weights. We have characterized above what it means to be a $\mathbb{G}_m$-invariant valuation on $X_f \times \mathbb{G}_m$ above, and a $\mathbb{G}_m$-invariant valuation will turn out to be $\tilde{G}$-invariant if and only if its restriction to $K_0 = K(X)$ is $G$-invariant. Since we can keep track of $B$-eigenvectors, we can then trace out what each $\tilde{G}$-invariant valuation on $\tilde{X}$ is and what its image in $N(\tilde{X})$ is. Putting this all together allows us to characterize $\mathcal{V}(\tilde{X})$ in terms of $\mathcal{V}(X)$.

There is just one more technicality: in general, we will not be able to pick $f$ to have degree 1, because $A = \Gamma_*(X, L)$ may not be generated in degree 1. For $f$ of higher degree, the above trivialization will not work, but we can generalize it to keep track of all the $B$-eigenvectors in degrees dividing $f$. Since any valuation must satisfy $v(t^d) = dv(t)$, where $d = \deg(f)$, the values of the valuation on degrees dividing $f$ will actually uniquely determine the valuation everywhere, so the arguments above will still go through (just with a little more work to translate between valuations on $K(\tilde{X})$ and those on $K_0(t)$).

Putting this all together (and with one more minor scheme-theoretic lemma first), we are now ready to characterize $\mathcal{V}(\tilde{X})$ in terms of $\mathcal{V}(X)$. For this statement, we will frequently switch between viewing $\mathcal{V}(X)$ as a set of valuations and viewing it as a subset of $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Z})$ (this identification is allowed by Corollary 3.1.14).

**Lemma 4.3.2.** *Let $X = \mathrm{Proj}(A)$, let $v_0 : K(X)^\times \to \mathbb{Q}$ be a valuation whose center on $X$ is the closed subscheme $Z_{v_0} \subset X$, and let $f_1, f_2 \in A$ be homogeneous elements. Suppose that*

$$Z_{v_0} \cap X_{f_1} \neq \varnothing.$$

*Then, we have $Z_{v_0} \cap X_{f_2} \neq \varnothing$ if and only if $v_0(f_2^{\deg(f_1)}/f_1^{\deg(f_2)}) = 0$ (here viewing $f_2^{\deg(f_1)}/f_1^{\deg(f_2)}$ as an element of $K(X) \cong \mathrm{Frac}(A)_0$).*

*Proof.* Write $d_i = \deg(f_i)$, and let $\mathfrak{p} \subset A$ be the homogeneous prime ideal corresponding to the generic point $\eta \in Z_{v_0}$. Then, we have $f_1 \notin \mathfrak{p}$ by assumption, and $\mathfrak{p}_0 = \mathfrak{p}A_{f_1} \cap (A_{f_1})_0$ is the prime ideal of $(A_{f_1})_0$ corresponding to $\eta \in X_{f_1} \cong \operatorname{Spec}((A_{f_1})_0)$. We have

$$\mathcal{O}_{X,x} \cong ((A_{f_1})_0)_{\mathfrak{p}_0},$$

so we see that $v_0(f_2^{d_1}/f_1^{d_2}) = 0$ if and only if $f_2^{d_1}/f_1^{d_2} \notin \mathfrak{p}_0$, which is true if and only if $f_2^{d_1}/f_1^{d_2} \notin \mathfrak{p}A_f$ and hence if and only if $f_2 \notin \mathfrak{p}$. But $f_2 \notin \mathfrak{p}$ is equivalent to $\eta \in X_{f_2}$, which in turn is equivalent to $Z_{v_0} \cap X_{f_1} \neq \varnothing$. □

**Theorem 4.3.3.** *Let $(X, L)$ be a polarized spherical variety, and let $\tilde{X} = \operatorname{Spec}(\Gamma_*(X, L))$ be the affine cone over $X$. Let $f \in \Gamma_*(X, L)^{(\tilde{B})}$ be an element of positive degree (i.e. $f$ is in $H^0(X, L^{\otimes d})^{(B)}$ for some $d \geq 1$).*

*(a) We have*
$$\Lambda(\tilde{X}) = (\Lambda(X) \times \{0\}) + \mathbb{Z}\lambda_f$$
*as subgroups of $\Lambda_{\tilde{G}} \cong \Lambda_G \times \mathbb{Z}$.*

*(b) A valuation $v : K(\tilde{X})^\times \to \mathbb{Q}$ is $\tilde{G}$-invariant if and only if it is $\mathbb{G}_m$-invariant and its restriction $v|_{K(X)^\times} : K(X)^\times \to \mathbb{Q}$ is $G$-invariant.*

*(c) The map*
$$\rho_f : \mathcal{V}(\tilde{X}) \to \mathcal{V}(X) \times \mathbb{Q}$$
*given by $v \mapsto (v|_{K(X)^\times}, v(f))$ is a bijection. Its inverse is the map which assigns to any pair $(v_0, r) \in \mathcal{V}(X) \times \mathbb{Q}$ the element $v \in \mathcal{V}(\tilde{X})$ which, as a map $\Lambda(\tilde{X}) \to \mathbb{Q}$, is given by*
$$v(\mu) = \frac{1}{\deg(f)} \left[ v_0 \left( \deg(f)\mu - \deg(\mu)\lambda_f \right) + \deg(\mu) r \right],$$
*where for any $\mu \in \Lambda(\tilde{X}) \subset \Lambda_G \times \mathbb{Z}$, we write $\deg(\mu)$ for the projection of $\mu$ to $\{0\} \times \mathbb{Z}$ (and $\deg(f)$ is the degree of $f$ in the graded ring $\Gamma_*(X, L)$, which is equal to $\deg(\lambda_f)$).*

*Proof.* Notice that $\mathbb{Z} \cdot \lambda_f \subset \Lambda(\tilde{X})$ because $K(\tilde{X}) = \operatorname{Frac}(\Gamma_*(X, L))$, and $\Lambda(X) \times \{0\} \subset \Lambda(\tilde{X})$ because $K(X)^{(B)} \subset K(\tilde{X})^{(\tilde{B})}$ is the set of $\tilde{B}$-eigenvectors fixed by $\mathbb{G}_m$. Conversely, for any $h \in K(\tilde{X})^{(\tilde{B})}$, we have $h' = h/f^{\deg(h)} \in K(X)^{(B)}$ and $\lambda_{h'} = \lambda_h - \deg(h) \cdot \lambda_f$, with $\lambda_{h'} \in \Lambda(X) \times \{0\}$. This proves (a).

Now, write $A = \Gamma_*(X, L)$ and $K_0 = K(X)$, and let $d = \deg(f)$. The map

$$\varphi_f : (A_f)_0 \otimes_k k[t^{\pm d}] \to A_f^{(d)} = \bigoplus_{n \in \mathbb{Z}} (A_f)_{nd}$$

given by $g/f^{\deg(g)/d} \otimes t^{dn} \mapsto g/f^{(\deg(g)/d - n)}$ is an isomorphism of graded rings with inverse given by $g/f^m \mapsto g/f^{\deg(g)/d} \otimes t^{\deg(g) - md}$. Note that the inclusion

$$A' = (A_f)_0 \otimes_k k[t^{\pm d}] \cong (A_f)_0[t^{\pm d}] \hookrightarrow (A_f)_0[t^{\pm}]$$

gives us an inclusion

$$\mathrm{Frac}(A') = K_0(t^d) \subset K_0(t) = \mathrm{Frac}((A_f)_0[t^{\pm}])$$

Similarly, the inclusion $A_f^{(d)} \hookrightarrow A_f$ gives an inclusion

$$\mathrm{Frac}(A_f^{(d)}) = K(\tilde{X})^{(d)} \subset K(\tilde{X}) = \mathrm{Frac}(A_f).$$

(Here, $K(\tilde{X})^{(d)}$ denotes the subfield of $K(\tilde{X})$ consisting of all elements whose degree is $nd$ for some $n \in \mathbb{Z}$.) We define an action of $G$ on $K_0(t^d)$ by the rule

$$g \cdot (at^{dn}) = (g \cdot a)\left(\frac{g \cdot f^n}{f^n}\right) t^{dn}$$

for any $g \in G$, any $a \in K_0$, and any $n \in \mathbb{Z}$. (Note that this action makes sense because $g$ acts on each graded piece of $A$ separately, so that $\deg(g \cdot f^n) = \deg(f^n)$. Also, $g \cdot a$ denotes the usual action of $G$ on $K_0 = K(X)$, and $g \cdot f^n$ denotes the action of $G$ on $H^0(X, L^{\otimes d})$.) With this definition and the above inclusions, $\varphi_f$ induces an isomorphism

$$\iota : K_0(t^d) \xrightarrow{\sim} K(\tilde{X})^{(d)}$$

which is $\tilde{G}$-equivariant ($\iota$ is $\mathbb{G}_m$-equivariant because $\varphi_f$ is graded and is $G$-equivariant by our definition of the $G$-action on $K_0(t)$, since $\varphi_f(t^d) = f \in (A_f)_1^{(d)}$.)

We claim that $\iota$ induces a canonical bijection between valuations $v$ of $K(\tilde{X})$ and valuations $v'$ of $K_0(t)$ in such a way that

1. $v|_{K(X)^\times} = v'|_{K_0^\times}$ and $v(f) = \frac{1}{d}v(t)$, and
2. $v$ is $\mathbb{G}_m$-invariant if and only if $v'$ is, and
3. $v$ is $G$-invariant if and only if the restriction of $v'$ to $K_0(t^d)$ is.

Given any valuation $v : K(\tilde{X})^\times \to \mathbb{Q}$, we note that $v$ restricts to a valuation $v_d$ of $K(\tilde{X})^{(d)}$. Conversely, given a valuation $v_d$ of $K(\tilde{X})^{(d)}$, we can define a valuation $v$ on $K(\tilde{X})$ as follows. For any $a \in K(\tilde{X})^\times$, let $n \in \mathbb{N}$ be the minimal number such that $af^n$ has positive degree in $K(\tilde{X}) \cong \mathrm{Frac}(A_f)$ (with the $\mathbb{Z}$-graded induced by the usual grading on $A_f$, i.e. any $a/b \in K(\tilde{X})$ with $a, b \in A_f$ has degree $\deg(a) - \deg(b)$). We define

$$v(a) = \frac{1}{d}v_d((af^n)^d) - dv_d(f^n).$$

This definition makes sense because the fact that $af^n$ has positive degree gives us $(af^n)^d \in K(\tilde{X})^{(d)}$. Moreover, one can check formally that the map $v$ really is a valuation whose restriction to $K(\tilde{X})^{(d)}$ is $v_d$, and that any such $v$ can be recovered from $v_d = v|_{K(\tilde{X})^{(d)}}$ using this formula. Thus, we have a bijection between valuations of $K(\tilde{X})$ and valuations of $K(\tilde{X})^{(d)}$, and one can check that $v|_{K(X)^\times} = v_d|_{K(X)^\times}$ and that $v$ is $\mathbb{G}_m$-invariant (resp. $G$-invariant) if and only if $v_d$ is. Then, the isomorphism $\iota$ identifies $v_d$ with a valuation $v'_d$ of $K_0(t^d)$ in a way that preserves the restriction to $K(X) = K_0$ and preserves the notions of $\mathbb{G}_m$-invariance and $G$-invariance (because $\iota$ is $\tilde{G}$-equivariant). Moreover, since $\varphi_f(t^d) = f$, we see that $v'_d(t^d) = v_d(f)$. Finally, arguing as with $K(\tilde{X})^{(d)}$ above, valuations on $K_0(t^d)$ can be identified with valuations on $K_0(t)$ in a way that preserves the necessary properties, and our construction of $v$ from $v_d$ above tells us that the valuation $v' : K_0(t)^\times \to \mathbb{Q}$ induced by $v'_d$ satisfies

$$v'(t) = \frac{1}{d} v'_d(t^d) - d v'_d(1) = \frac{1}{d} v_d(f) - 0 = \frac{1}{d} v(f).$$

So, putting all these identifications together gives us the claim.

Now, suppose for the moment that Statement (b) holds. We prove Statement (c). First, the above claim allows us to think of $\mathcal{V}(\tilde{X})$ as the set of $\tilde{G}$-invariant valuations of $K_0(t)$, which is a subset of the $\mathbb{G}_m$-invariant valuations of $K_0(t)$. We can thus consider the restriction $\rho$ of the bijection in Proposition 4.3.1b to $\mathcal{V}(\tilde{X})$, which is a bijection onto its image. The image of $\rho$ is precisely the set of pairs $(v_0, r)$, where $v_0$ is the restriction of some element of $\mathcal{V}(\tilde{X})$ and $r \in \mathbb{Q}$. Any such $v_0$ lies in $\mathcal{V}(X)$ by (b). Conversely, for any $G$-invariant valuation on $v_0 \in \mathcal{V}(X)$, Proposition 4.3.1b says that there exists some extension of $v_0$ to a $\mathbb{G}_m$-invariant valuation $v$ of $K_0(t)$ (in fact, there is such an extension for any possible value of $v(t)$). By the above claim, we may view $v$ as a $\mathbb{G}_m$-invariant valuation on $K(\tilde{X})$. Since $v_0$ is $G$-invariant, we see that $v \in \mathcal{V}(\tilde{X})$ by (b), so $v_0$ is in the image of $\rho$. Thus the image of $\rho$ is precisely $\mathcal{V}(X) \times \mathbb{Q}$, and $\rho$ is simply the map $\rho_f$ of (c). On the other hand, for any $v \in \mathcal{V}(\tilde{X})$ and any $\mu \in \Lambda(\tilde{X})$, write $v_0 = v|_{\Lambda(X)}$. The inclusion $\Lambda(X) \subset \Lambda(\tilde{X})$ identifies $\Lambda(X)$ with $\Lambda(X) \times \{0\} \subset \Lambda(\tilde{X})$, so we have $\deg(f)\mu - \deg(\mu)\lambda_f \in \Lambda(X)$ and hence

$$\begin{aligned} v(\mu) = \frac{1}{\deg(f)} v(\deg(f)\mu) &= \frac{1}{\deg(f)} v(\deg(f)\mu - \deg(\mu)\lambda_f + \deg(\mu)\lambda(f)) \\ &= \frac{1}{\deg(f)} \left[ v_0(\deg(f)\mu - \deg(\mu)\lambda_f) + \deg(\mu) v(\lambda_f) \right]. \end{aligned}$$

Since $\rho_f$ maps $v$ to the pair $(v_0, v(\lambda_f))$, this is the description of the inverse of $\rho_f$ given in (c).

It remains to prove (b). For this, let $v : K(\tilde{X})^\times \to \mathbb{Q}$ be any valuation, and let $v_0 = v|_{K(X)^\times}$. One direction is immediate, so we assume that $v$ is $\mathbb{G}_m$-invariant and that $v_0$ is $G$-invariant, and we prove that $v$ is $G$-invariant (hence also invariant under $\tilde{G} = G \times \mathbb{G}_m$). Since $X$ is complete, the center $Z_{v_0} \subset X$ of $v_0$ on $X$ exists. $v_0$ is $G$-invariant of $v$ is $\tilde{G}$-invariant; conv is $G$-invariant. In this case, $Z_{v_0}$ is $G$-stable closed subscheme, so there exists some $B$-eigenvector $f_0 \in A^{(B)}$ of positive degree such that $Z_{v_0} \cap X_{f_0} \neq \varnothing$ (by arguing similarly to the proof of Lemma 3.2.6; see e.g. [Kno91, Theorem 1.3]). Since nothing in Statement (b) depends on $f$ in any way, we may as well replace $f$ by $f_0$. Then, we can use the claim above to view $v$ as a valuation on $K_0(t)$ in such a way that $v$ is $G$-invariant if and only if its restriction $v_d$ to $K_0(t^d)$ is $G$-equivariant (using the $G$-action on $K_0(t)$ defined above).

Moreover, since $v$ is $\mathbb{G}_m$-invariant, the equality in Proposition 4.3.1a holds for $v$ and hence also for $v_d$. It follows that $v_d$ is $G$-invariant if and only it is $G$-invariant on monomials in $K_0(t^d)$, i.e. if and only if $v_d(g \cdot (at^{dn})) = v_d(at^{dn})$ for all $g \in G$, $a \in K_0$ and $n \in \mathbb{Z}$. By definition of the $G$-action on $K_0(t^d)$ given above, we have

$$v_d(g \cdot (at^{nd})) = v_d\left((g \cdot a)\left(\frac{g \cdot f}{f}\right)t^{nd}\right) = v_0(g \cdot a) + v_0\left(\frac{g \cdot f}{f}\right) + v_d(t^{nd}).$$

Becuase $v_0$ is $G$-invariant, we have $v_0(g \cdot a) = v_0(a)$. Moreover, since $Z_{v_0}$ is $G$-stable and $Z_{v_0} \cap X_f \neq \varnothing$, we see that

$$Z_{v_0} \cap X_{gf} = g \cdot Z_{v_0} \cap g \cdot X_f = g \cdot (Z_{c_0} \cap X_f) \neq \varnothing.$$

Lemma 4.3.2 then gives us $v_0((g \cdot f)/f) = 0$, so the above equation becomes

$$v_d(g \cdot (at^{nd})) = v_d(a) + v_d(t^{nd}) = v_d(at^{nd}).$$

Thus, $v_d$ is $G$-invariant, as desired. □

**Remark 4.3.4.** The assumptions of the above theorem are general enough for our purposes. However, it seems possible that these assumptions could be weakend in a few different ways:

1. We have only used the assumption that $X$ is spherical to make sense of $\mathcal{V}(X)$ and to say that $\mathcal{V}(X)$ injects into $N(X)$ (i.e. that $G$-invariant valuations are determined by their values on $B$-eigenvectors). Thus, some appropriate generalization of Theorem 4.3.3 may hold for more general $X$ if we replace $\mathcal{V}(X)$ by a suitable more general invariant (for instance, the cone of "central valuations," see [Tim11, Section 21]).

2. We have used the assumption that $X$ is projective in only one place, to claim that every valuation has a center on $X$. If $X$ is instead only quasi-projective, we might be able to replace $X$ by $\mathrm{Proj}(\Gamma_*(X, L))$ to reduce to the projective case. This replacement would require $\Gamma_*(X, L)$ to be of finite type, so that $\mathrm{Proj}(\Gamma_*(X, L))$ is a projective variety. However, we suspect (though we have not checked it) that this might be true for any spherical variety $X$, for instance because spherical varieties have finitely generated class groups (see Proposition 3.7.1).

3. We have only used the assumption that $k$ is algebraically closed to apply Proposition 4.3.1. Spherical varieties are almost always considered over algebraically closed fields, anyway, but if some suitable generalization called for it, taking $k$ to be any infinite field would most likely be good enough (since Proposition 4.3.1 still applies in this setting).

The above theorem allows us to apply Losev's results to affine cones to obtain a projective analog of Theorem 4.2.2a

**Corollary 4.3.5.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. If $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ (as cones in $N(X_1) = N(X_2)$), then $(X_1, L_1) \cong (X_2, L_2)$, i.e. there exists a $G$-equivariant isomorphism $i : X_1 \xrightarrow{\sim} X_2$ such that $i^* L_2 \cong L_1$ as $G$-linearized invertible sheaves.*

*Proof.* First, we note that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ implies that $\Lambda(X_1) = \Lambda(X_2)$ as subgroups of $\Lambda_G$ (by Proposition 2.5.2) and $\Lambda(\tilde{X_1}) = \Lambda(\tilde{X_2})$ (by the same proposition, applied to affine schemes and the ample line bundles $\mathcal{O}_{X_i}$). So, it will suffice to show that the affine cones $\tilde{X_1}$ and $\tilde{X_2}$ have the same valuation cone. If this is true, then Theorem 4.2.2a implies that $\tilde{X_1} \cong \tilde{X_2}$. This isomorphism corresponds to a $G$-equivariant isomorphism $\Gamma_*(X_1, L_1) \cong \Gamma_*(X_2, L_2)$, which induces a $G$-equivariant isomorphism

$$X_1 \cong \operatorname{Proj}(\Gamma_*(X_1, L_1)) \cong \operatorname{Proj}(\Gamma_*(X_2, L_2)) \cong X_2$$

that identifies $L_1$ with $L_2$.

To prove that $\mathcal{V}(\tilde{X_1}) = \mathcal{V}(\tilde{X_2})$, it will suffice to show that the sets of valuations extending a given valuation $v_0 \in \mathcal{V}(X_1) = \mathcal{V}(X_2)$ are equal. Pick any $f_1 \in \Gamma_*(X_1, L_1)^{(B)}$ and any $f_2 \in \Gamma_*(X_2, L_2)^{(B)}$ of positive degrees which have the same weight $\lambda$ in $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. By Theorem 4.3.3c, the $\tilde{G}$-invariant valuations $v$ on $K(\tilde{X_1})$ extending $v_0$ are uniquely determined by $v(\lambda)$, and any choice of $v(\lambda) \in \mathbb{Q}$ will work. Moreover, viewing $v$ as an element of $N(\tilde{X_1})$, Theorem 4.3.3c explicitly describes $v$ as an element of $N(\tilde{X_1}) = N(\tilde{X_2})$ in terms of $v_0$ and $v(\lambda)$. The same exact description holds for valuations of $K(\tilde{X_2})$ extending $v_0$, so we conclude that the elements of $N(\tilde{X_1}) = N(\tilde{X_2})$ extending $v_0$ are the same for both $X_1$ and $X_2$, as desired. $\square$

In particular, the above corollary gives us the Knop conjecture for projective varieties in a very special case.

**Corollary 4.3.6.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polaried spherical varieties. Suppose that $X_1$ and $X_2$ are smooth, that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and that there is some element $(\lambda, d) \in \Lambda^+(X_1, L_1)$ such that the weight $\lambda$ extends to a nontrivial character of $G$ (equivalently, the simple $G$-module $V(\lambda)$ has dimension 1). Then, we have $(X_1, L_1) \cong (X_2, L_2)$ as polarized $G$-varieties.*

*Proof.* For any $i \in \{1, 2\}$, let $f_i \in H^0(X_i, L_i^{\otimes d})$ be an element whose weight is $\lambda$. Then $f_i$ generates a simple $G$-module isomorphic to $V(\lambda)$, which by assumption is a character. In other words, for all $g \in G$, the section $g \cdot f_i$ is just a scalar multiple of $f_i$. It follows that

$$g \cdot (X_i)_{f_i} = (X_i)_{gf_i} = (X_i)_{f_i}$$

for all $g \in G$, i.e. that $(X_i)_{f_i}$ is $G$-stable.

Now, the $G$-varieties $Y_i = (X_i)_{f_i}$ are are smooth (because the $X_i$ are), affine (because $X_i \cong \operatorname{Proj}(\Gamma_*(X_i, L_i)))$, and spherical (because $(X_i)_{f_i}$ contains the open $B$-orbit of $X_i$). Write $A_i = \Gamma_*(X_i, L_i)$. Then, we have $Y_i \cong \operatorname{Spec}(((A_i)_{f_i})_0)$, and the $B$-eigenvectors of

$((A_i)_{f_i})_0$ are the elements of the form $h/f_i^n$, where $h \in A_i^{(B)}$ and $h/f_i^n$ has degree 0 in $(A_i)_{f_i}$. We then have

$$\Lambda(X_1) = \{(\mu, 0) \in \Lambda^+(X_1, L_1)^{gp}\} = \Lambda(X_2)$$

and

$$\Lambda^+((X_1)_{f_1}) = \{\mu \in \Lambda(X) \mid (\mu + n\lambda, nd) \in \Lambda^+(X, L) \text{ for some } n \geq 0\} = \Lambda^+((X_2)_{f_2})$$

(see Lemma 4.4.4 below). By the Knop conjecture (Theorem 4.2.2b), we conclude that $(X_1)_{f_1} \cong (X_2)_{f_2}$ as $G$-varieties. In particular,

$$\mathcal{V}(X_1) = \mathcal{V}((X_1)_{f_1}) = \mathcal{V}((X_2)_{f_2}) = \mathcal{V}(X_2),$$

so Corollary 4.3.5 gives us $(X_1, L_1) \cong (X_2, L_2)$. □

## 4.4 The Local Structure Theorem and the Knop Conjecture

In the previous section, the key idea was to utilize the Knop conjecture (or more precisely, Losev's related statement in Theorem 4.2.2a) on affine cones to deduce new statements in the (quasi)-projective case. In this section, we discuss a different way to make use of the Knop conjecture. This time, we will use the local structure theorem to obtain affine varieties to which we can apply Theorem 4.2.2. To do this, we will also require a few more general facts about the local structure theorem and about combinatorial invariants on spherical varieties.

To begin, we already know (Proposition 3.2.3) that most nice invariants of a spherical variety $X$ are passed along to the variety $Z$ obtained from the local structure theorem. However, there are a few invariants arising from the classification of homogeneous spherical varieties that we have not yet discussed in the context of the local structure theorem. The following proposition says that these are passed along to $Z$ in a nice way as well.

**Proposition 4.4.1** ([Los09a, Proposition 5.3])**.** *Let $X$ be a spherical variety, and let $U \subset X$ be a $B$-stable open subset. Let $P = \{g \in G \mid g \cdot U = U\}$, and let $M$ be the standard Levi subgroup of $P$. Suppose that, as in the local structure theorem (Theorem 3.2.2), there exists some $M$-stable closed subvariety $Z \subset U$ such that the morphism*

$$R_u(P) \times Z \to U$$

*given by $(u, z) \mapsto uz$ is a $P$-equivariant isomorphism.*

*(a) Let $\mathcal{D} \subset \mathcal{D}_{G,X}$ be the $B$-divisors in the complement $X \setminus U$, and let $\iota : \mathcal{D}_{M,Z} \to \mathcal{D}_{G,X} \setminus \mathcal{D}$ be the bijection of Proposition 3.2.3e. For any $D \in \mathcal{D}_{M,Z}$ and any $\alpha \in \Pi_M \subset \Pi_G$, the root $\alpha$ moves $D$ if and only if $\alpha$ moves $\iota(D)$.*

*(b) We have $\Psi_{M,X} = \Psi_{G,X} \cap \operatorname{Span}_{\mathbb{Q}}(\Pi_M)$*

*(c) If $U$ is affine, then*

$$\Pi_M = \{\alpha \in \Pi_G \mid D \cap U \neq \varnothing \ \forall\ D \in \mathcal{D}_{G,X}(\alpha)\}.$$

*Proof.* Note that the parabolic subgroup of $M$ associated to the set $\{\alpha\}$ is $P_\alpha \cap M$ (by the construction of standard parabolic subgroups). Moreover, $\alpha \in \Pi_M$ implies that $P_\alpha \subset P$ (by the construction of standard Levi subgroups), so we have $R_u(P) \cdot (P_\alpha \cap M) = P_\alpha$ (because $P = R_u M$ and $R_u(P) \subset P_\alpha$). So, if $\alpha$ does not move $D$, i.e. $(P_\alpha \cap M) \cdot D = D$, then the product $R_u(P) \times D \subset R_u(P) \times Z$ is stable under $P_\alpha$. Since $\iota(D) = \overline{R_u(P) \times D}$, we conclude that $\alpha$ does not move $\iota(D)$. Conversely, if $\alpha$ does not move $\iota(D)$, then $D = \iota(D) \cap Z$ is the intersection of two sets that are stable under $P_\alpha \cap M$, so $D$ is also stable under $P_\alpha \cap M$. In other words, $D$ is not moved by $\alpha$.

Statement (c) also boils down to a fact about standard parabolic subgroups and their standard Levi subgroups: namely, that $P = P_{\Pi_M}$. Since $P_\alpha \subset P_I$ for any subset $I \subset \Pi_G$ if and only if $\alpha \in I$, this gives us

$$\Pi_M = \{\alpha \in \Pi_G \mid P_\alpha \subset P\}.$$

Now, $P_\alpha \subset P$ if and only if $U$ is stable under $P_\alpha$ (by definition of $P$). Since $U$ is $B$-stable affine and open, its complement is a union of $B$-divisors (cf. the proof of Lemma 3.1.17a), so $U$ is stable under $P_\alpha$ if and only if every $B$-divisor in $X \setminus U$ is stable under $P_\alpha$. This is equivalent to saying that every element of $\mathcal{D}_{G,X}(\alpha)$ intersects $U$.

For (b), Losev gives a proof in [Los09c, Lemma 3.5.5] when $X$ is smooth and quasi-projective by reducing to an existing argument in the analytic setting of Hamiltonian manifolds. On the other hand, in [Gag15, Proposition 3.2], Gagliardi gives an argument in a more general setting using the algebraic theory of spherical varieties. We sketch Gagliardi's argument here. Let $G/H \subset X$ be the open $G$-orbit, and consider the wonderful embedding $G/\overline{H} \hookrightarrow Y$, where $\overline{H}$ is the very sober hull of $H$. Let $I \subset \Pi_G$ be such that $P = P_I$, and consider the localization $Y^I$ of $Y$ at $I$. The composition

$$G/H \to G/\overline{H} \dashrightarrow Y_I \xrightarrow{\pi_I} Y^I$$

induces a rational map $Z \dashrightarrow Y^I$ which is defined on the open $M$-orbits of $Z$ and $Y^I$. Let $M/H_1$ and $M/H_2$ be the open $M$-orbits of $Z$ and $Y^I$ (respectively), and consider the morphism $f : M/H_1 \to M/H_2$ just described. Then, $f$ is an $M$-equivariant morphism of homogeneous spherical $M$-varieties. Moreover, the retraction map $\pi_I : Y_I \to Y^I$ induces a bijection between colors of $Y^I$ and $P$-unstable colors of $Y$ (Proposition 3.6.6), and the inclusion map $i : Z \hookrightarrow X$ induces a bijection between colors of $Z$ and $P$-unstable colors of $X$ (because the inverse to the bijection $\iota$ is $D' \mapsto D' \cap Z = i^{-1}(D')$, see Proposition 3.2.3). It follows that $f$ induces a bijection on colors of $M/H_1$ and colors of $M/H_2$ An algebraic characterization of the spherical closure as a subgroup of $M$ now implies that $\overline{H_1} = \overline{H_2}$. On the other hand, the localization $Y^I$ is wonderful because $Y$ is (cf. Proposition 3.6.3), so $\overline{H_2} = H_2$, and $f$ is the usual quotient map $M/H_1 \to M/\overline{H_1}$.

Now, the spherical roots of $M/H_1$ (which are the spherical roots of $Z$) are all multiples of $M/\overline{H_1}$ and are indivisible elements of the lattice $\Lambda(Z) = \Lambda(M/H_1)$; these two statements completely determine $\Psi_{M,Z}$ in terms of $\Psi_{M,M/\overline{H_1}} = \Psi_{M,Y^I}$. Theorem 3.6.4 can be used to show that

$$\Psi_{M,Y^I} = \Psi_{G,Y} \cap \mathrm{Span}_{\mathbb{Q}}(\Pi_M)$$

as subsets of $\Lambda(Y)$. On the other hand, the spherical roots of $Y$ and those of $X$ have the same relationship as those of $Z$ and $Y^I$. Putting all of these identifications together and using the fact that $\Lambda(X) = \Lambda(Z)$ now gives the desired result. $\square$

We will also need a couple more combinatorial results about roots moving divisors. These could just as easily have been proven in Section 3.6, but we have deferred their proofs until now because they were not needed for the classification of homogeneous spherical varieties in that section.

**Lemma 4.4.2** ([Los09a, Lemma 3.3])**.** *Let $X$ be a spherical $G$-variety, and let $D \in \mathcal{D}_{G,X}$.*

*(a) For any $\alpha \in \Pi_X^b$ such that $\alpha$ does not move $D$, we have $\varphi_D(\alpha) \leq 0$.*

*(b) For any $\alpha in \Pi_X^c$ such that $\alpha$ does not move $D$, we have $\varphi_D(2\alpha) \leq 0$.*

*(c) Let $\alpha_1, \alpha_2 \in \Pi_X^d$ be such that $\langle \alpha_1^\perp, \alpha_2 \rangle = 0$ and $\alpha_1 + \alpha_2 = \gamma$ or $2\gamma$ for some $\gamma \in \Psi_{G,X}$. If $D$ is not the unique element of $\mathcal{D}_{G,X}(\alpha_1) = \mathcal{D}_{G,X}(\alpha_2)$ (see Proposition 3.6.12), then $\varphi_D(\gamma) \leq 0$.*

*Proof.* There are 3 cases, depending on the divisor $D$:

Case 1: If $D$ is a $G$-divisor, then $\varphi_D \in \mathcal{V}(X)$. In Statements (a), (b), and (c), we are considering $\varphi_D$ applied to a spherical root, so the statements are immediate from the definition of spherical roots.

Case 2: If $D$ is moved by a root $\beta$ of type $c$ or $d$, then $\varphi_D$ is proportional to the dual root $\alpha^\vee$. It it is a standard fact about root systems that $\langle \beta^\vee, \beta' \rangle \leq 0$ for all simple roots $\beta' \neq \beta$. In Statements (a), (b), and (c), we are considering $\varphi_D$ applied to a sum of simple roots not equal to $\beta$ with positive coefficients, so this implies the result.

Case 3: Suppose that $D$ is moved by a root of type $b$. Then we are applying $\varphi_D$ to a spherical root not moving $D$ in all three statements (note that $\alpha_1$ and $\alpha_2$ cannot move $D$ in Statement (c) by Proposition 3.6.12). Thus, the statement is immediate from Proposition 3.6.18. $\square$

**Corollary 4.4.3.** *Let $X$ be a spherical variety, let $D$ be a color of $X$, and let $\alpha \in \Pi$ be a root moving $D$.*

*(a) If $\alpha \in \Pi_X^b$, then any root $\beta \in \Pi$ moves $D$ if and only if $\beta \in \Pi_X^b$ and $\varphi_D(\beta) = 1$.*

*(b) If $\alpha \in \Pi_X^c$, then $\alpha$ is the unique root moving $D$.*

(c) *Suppose $\alpha \in \Pi_X^d$. If $\alpha$ is not the unique root moving $D$, then there is one other root $\beta$ moving $D$. In this case, we have $\alpha + \beta \in \Lambda(X)$. Moreover, $D$ is the unique $B$-divisor of $X$ such that $\varphi_D(\alpha + \beta) > 0$, and we have $\varphi_D(\alpha + \beta) = 2$.*

*Proof.* For (a), given any $\beta \in \Pi$ moving $D$, we have $\varphi_D(\beta) = 1$ by Proposition 3.6.13, and $\beta \in \Pi_X^b$ by Proposition 3.6.12. Conversely, if $\beta \in \Pi_X^b$ and $\varphi_D(\beta) = 1$, then $\beta$ must move $D$ because we would otherwise have $\varphi_D(\beta) \leq 0$ by Lemma 4.4.2.

Statement (b) is immediate from Proposition 3.6.12. As for (c), suppose that another root $\beta$ moving $D$ exists. We first show that $\alpha$ and $\beta$ are the only roots moving $D$. Suppose $\gamma$ is a third root moving $D$. Then, $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta) = \mathcal{D}_{G,X}(\gamma)$, so Proposition 3.6.12 gives us

$$\langle \alpha^\vee, \beta \rangle = \langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle = 0.$$

The same proposition implies that $\beta + \gamma \in \Lambda(X)$ (because this sum is a multiple of a spherical root), so we have

$$\langle \alpha^\vee - \beta^\vee, \beta + \gamma \rangle = \langle \beta^\vee, \beta \rangle = 2 \neq 0,$$

contradicting the fact that $\alpha^\vee = \beta^\vee$ on $\Lambda(X)$ (by Proposition 3.6.12 again).

Now, much as above, Proposition 3.6.12 implies that $\alpha + \beta \in \Lambda(X)$, and that proposition also says that $D$ is the unique element of $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$. So, if $D' \neq D$ is any other $B$-divisor of $X$, then we have $\varphi_{D'}(\alpha + \beta) \leq 0$ by Lemma 4.4.2. On the other hand, since $\varphi_D = \alpha^\vee|_{\Lambda(X)}$ (by Proposition 3.6.13) and $\langle \alpha^\vee, \beta \rangle = 0$ (by Proposition 3.6.12), we have

$$\varphi_D(\alpha + \beta) = \langle \alpha^\vee, \alpha \rangle + \langle \alpha^\vee, \beta \rangle = 2 + 0 = 2.$$

□

From the perspective of the Knop conjecture, we are interested in determining geometric data from the combinatorial data of a monoid of the form $\Lambda^+(X, L)$. The following lemma helps us do that in the situation of the local structure theorem. To state it, we first recall that for any $G$-linearized invertible sheaf $L$ on a spherical variety $X$ and any $B$-eigenvector $f \in H^0(X, L)$ of weight $\mu$, we denote by $X_\mu$ the open subset $X_f$, or equivalently, the complement $X \setminus \operatorname{div}(\mu)$ (see the discussion preceding Proposition 3.7.1).

**Lemma 4.4.4.** *Let $X$ be a spherical variety, Let $L$ be a $G$-linearized globally generated invertible sheaf on $X$, and let $f \in H^0(X, L^{\otimes d})^{(B)}$ for some $d \geq 1$. Let $\mu$ be the weight of $f$, and let $P$, $M$, and $Z$ be as in the local structure theorem (Theorem 3.2.2) applied to $X_\mu$. Suppose that $X_\mu$ (hence also $Z$) is affine.*

(a) *$P$ and $M$ are uniquely determined (as subgroups of $G$) by $\Lambda^+(X, L)$ and the element $(\mu, d) \in \Lambda^+(X, L)$, and $Z$ is determined up to $M$-equivariant isomorphism by $X$, $L$, and $(\mu, d)$.*

(b) *We have*

$$\Lambda^+(Z) = \{\lambda \in \Lambda(X) \mid (\lambda + n\mu, nd) \in \Lambda^+(X, L) \text{ for some } n \geq 0\}.$$

*Proof.* By Corollary 2.5.5, $P$ is the subgroup of $G$ fixing the line $k \cdot f \subset \Gamma_*(X, L)$, so $P$ depends only on the $G$-module structure of $\Gamma_*(X, L)$ (which is determined by $\Lambda^+(X, L)$) and the line $k \cdot f$ (which is determined by the pair $(\mu, d)$ because $H^0(X, L^{\otimes d})$ is multiplicity-free, see Remark 3.1.5). Then, $M$ is determined from $P$ as the unique Levi subgroup containing $T$. As for $Z$, the data of $X$ and $L$ determines $\Lambda^+(X, L)$, and as above, this monoid along with $(\mu, d)$ determines the line $k \cdot f \subset \Gamma_*(X, L)$. The line $k \cdot f$ determines $X_\mu$, and the data of $X$, $L$, and $(\mu, d)$ determine $P$ (hence also $R_u(P)$) by our above arguments; thus, this data also determines the quotient $X_\mu/R_u(P) \cong Z$. This proves (a).

Write $A = \Gamma_*(X, L)$. It is a general fact (see e.g. [Sta20, Tag 01PW]) that $\Gamma(X_f, \mathcal{O}_X) = (A_f)_0$; since $X_\mu = X_f$ is affine, this gives us $X_\mu \cong \operatorname{Spec}((A_f)_0)$. In particular, we have $\Lambda^+(X_\mu) \subset \Lambda^+(X_\mu)^{gp} = \Lambda(X)$ (see Proposition 2.5.9). Now, the $B$-eigenvectors of $(A_f)_0$ are the elements of the form $h/f^n$, where $h \in A^{(B)}$ and $\deg(h) = nd$. For any such $h/f^n$, let $\lambda$ (resp. $\mu_h$) be the weight of $h/f^n$ (resp. $h$). Then, we have $(\lambda + n\mu, nd) = (\mu_h, \deg(h)) \in \Lambda^+(X, L)$. Conversely, suppose $\lambda \in \Lambda(X)$ is such that $(\lambda+n\mu, nd) \in \Lambda^+(X, L)$. Let $h \in A_{nd}^{(B)}$ be a $B$-eigenvector of weight $\lambda + n\mu_f$. Then, we see that $h/f^n$ has degree 0 in $A_f$ and is a $B$-eigenvector of weight $\lambda$. This proves that

$$\Lambda^+(X_f) = \{\lambda \in \Lambda(X) \mid (\lambda + n\mu, nd) \in \Lambda^+(X, L) \text{ for some } n \geq 0\}.$$

On the other hand, the isomorphism $Z \cong X_\mu/R_u(P)$ identifies the inclusion $Z \hookrightarrow X_\mu$ with a section of the quotient map $X_\mu \to X_\mu/R_u(P)$. It follows that restriction from $X_\mu$ to $Z$ gives a bijection between $B$-eigenvectors in $\Gamma(X_\mu, \mathcal{O}_X)$ and $(B \cap M)$-eigenvectors in $\Gamma(Z, \mathcal{O}_Z)$ (see the proof of Proposition 3.2.3 for details). It follows that $\Lambda^+(X_f) = \Lambda^+(Z)$, and this along with our above arguments implies (b). $\square$

Statement (a) in the above lemma will allow us to compare the local structure theorem on two spherical varieties with the same weight monoid $\Lambda^+(X, L)$ in a very useful way. We now define some notation to help us make such a comparison.

**Definition 4.4.5.** In the situation of the above lemma, we denote by $P_\mu$ (resp. $M_\mu$, $X_\mu$, $X(\mu)$) the subgroup $P$ (resp. the subgroup $M$, the subvariety $X_f$, the subvariety $Z$).

The following theorem is the main application of the Knop conjecture that we will use in the context of the local structure theorem. It is stated in a somewhat general way, in order to cover all the use cases that will be interesting to us.

**Theorem 4.4.6.** *Let $X_1$ and $X_2$ be two spherical varieties, and let $L_1$ and $L_2$ be $G$-linearized globally generated sheaves on $X_1$ and $X_2$ (respectively). Suppose that*

1. *$\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$,*
2. *$\mu \in \Lambda^+(X_1, L_1)$ is such that $(X_1)_\mu$ and $(X_2)_\mu$ are both affine, and*
3. *either $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ or $X_1$ and $X_2$ are smooth.*

*Then, we have $X_1(\mu) \cong X_2(\mu)$ as $M_\mu$-varieties and $(X_1)_\mu \cong (X_2)_\mu$ as $P_\mu$-varieties*

*Proof.* The $X_i(\mu)$ are spherical $M_\mu$-varieties (see Proposition 3.2.3) and are affine because they are closed subvarieties of the $(X_i)_\mu$. Moreover, Assumption 1 gives us $\Lambda(X_1) = \lambda(X_2)$ (see Proposition 2.5.2), and Lemma 4.4.4 then implies that $\Lambda^+(X_1(\mu)) = \Lambda^+(X_2(\mu))$. We conclude that that $X_1(\mu) \cong X_2(\mu)$ as $M_\mu$-varieties by the Knop conjecture (see Theorem 4.2.2, which contains one statement for the smooth case and one statement for the $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ case). The local structure theorem then gives us

$$(X_1)_\mu \cong R_u(P_\mu) \times X_1(\mu) \cong R_u(P_\mu) \times X_2(\mu) \cong (X_2)_\mu,$$

and all of these isomorphisms are $P_\mu$-equivariant. $\square$

## 4.5 Comparing $B$-Divisors

Let $(X_1, L_1)$ and $(X_2, L_2)$ be two polarized spherical varieties such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. In this section, we will use Theorem 4.4.6 repeatedly to compare all combinatorial invariants related to divisors of $X_1$ and $X_2$.

To begin, we can deduce that $X_1$ and $X_2$ have the same open $B$-orbits, which implies that several of their combinatorial invariants must be the same. To state this, recall that given a spherical variety $X$, we write $P_X$ for the parabolic subgroup $\{g \in G \mid gX_B^\circ = X_B^\circ\}$, where $X_B^\circ \subset X$ is the dense $B$-orbit. Equivalently, $P_X$ is the parabolic subgroup $P_{\Pi_X^a}$ corresponding to the set of all roots of type $a$ for $X$.

**Lemma 4.5.1.** *Let $X_1$ and $X_2$ be two spherical varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively).*

1. *$\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and*
2. *either $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ or $X_1$ and $X_2$ are smooth.*

*Then, the following hold.*

*(a) $\Pi^a_{X_1} = \Pi^a_{X_2}$, or equivalently, $P_{X_1} = P_{X_2}$.*

*(b) The open $B$-orbits of $X_1$ and $X_2$ are $P_{X_1}$-equivariantly isomorphic. In particular, $\dim(X_1) = \dim(X_2)$.*

*Proof.* Let $Y_i$ be the open $G$-orbit of $X_i$. Then, the subset $(X_1)_{B,Y_1}$ of Theorem 3.2.7 is precisely the open $B$-orbit of $X_1$. In particular, since $L_1$ is ample, that theorem states that there exists some weight $\mu$ of a $B$-eigenvector of $\Gamma_*(X_1, L_1)$ such that $(X_1)_{B,Y_1} = (X_1)_\mu$. By Theorem 4.4.6, we have $(X_1)_\mu \cong (X_2)_\mu$ as $P_\mu$-varieties. In particular, $B \subset P_\mu$ implies that $(X_2)_\mu$ is a $B$-orbit. In fact, $(X_2)_\mu$ is the open $B$-orbit of $X_2$ (since $(X_2)_\mu$ is open by definition). Moreover, for any $i \in \{1,2\}$ the definition of $P_\mu$ via the local structure theorem (Theorem 3.2.2) applied to $X_i$ tells us that

$$P_\mu = \{g \in G \mid g(X_i)_\mu = (X_i)_\mu\} = P_{X_i},$$

so we have $P_{X_1} = P_\mu = P_{X_2}$. Finally, the equality on dimensions in (b) comes from the fact that $\dim((X_i)_\mu) = \dim(X_i)$. □

Now that we know $\Pi^a_{X_1} = \Pi^a_{X_2}$, it is interesting to ask whether the same is true for roots of other types. We will see that roots of types $b$ and $c$ are easier to compare than roots of type $d$. This is in large part thanks to Lemma 4.4.2, from which we obtain the following lemma.

**Lemma 4.5.2** (cf. [Los09a, Proposition 6.4])**.** *Let $X$ be a spherical variety, let $L$ be a $G$-linearized invertible sheaf on $X$ such that $H^0(X,L) \neq 0$, and let $\alpha \in \Pi$. Under any one of the following 3 assumptions, there exists some $B$-eigenvector $f \in H^0(X, L^{\otimes n})^{(B)}$ for some $1 \le n \le 2$ such that $D \cap X_f \neq \varnothing$ for some $D \in \mathcal{D}_{G,X}(\alpha)$.*

*(a) $\alpha \in \Pi^b_X$.*

*(b) $\alpha \in \Pi^c_X$.*

*(c) $\alpha \in \Pi^d_X$ and there exists another root $\beta \in \Pi^d_X$ such that $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$.*

*Proof.* Consider first the case where $\alpha \in \Pi^b_X$. Since $H^0(X,L)$ is a nontrivial $G$-module, it contains some $B$-eigenvector $f_0$. Since $X$ is a variety, $f_0$ cuts out an effective Cartier divisor $E$ of $X$. Since $X$ is normal, $E$ is also an effective Weil divisor, so we may ask about the coefficients $c_+$ and $c_-$ of the $B$-divisors $D^+_\alpha$ and $D^-_\alpha$ (respectively) in $E$. The support of $E$ is the complement of $X_f$, so we have $D \cap X_f \neq \varnothing$ for any $B$-divisor $D$ if and only if the coefficient of $D$ in $E$ is 0. Thus, our goal is to pick $f$ such that one of $c_+$ and $c_-$ is 0.

For this, after swapping $D^+_\alpha$ and $D^-_\alpha$ if necessary, we may assume that $c_+ \le c_-$. Because $\alpha$ is a spherical root and in particular lies in $\Lambda(X)$, there exists a $B$-eigenvector $f_{-\alpha} \in K(X)^{(B)}$ whose weight is $-\alpha$. By Proposition 3.6.13 and Lemma 4.4.2, the principal Cartier divisor $\mathrm{div}(f_{-\alpha})$ has coefficient $-1$ for $D^+_\alpha$ and $D^-_\alpha$ and has nonnegative coefficients for every other $B$-divisor. It follows that $E + \mathrm{div}(f^{c_+}_{-\alpha})$ is effective (because $c_+ \le c_-$) and has coefficient 0 for $D^+_\alpha$. So, there exists some $B$-eigenvector $f \in H^0(X,L)^{(B)}$ such that $\mathrm{div}(f) = E + \mathrm{div}(f^{c_+}_{-\alpha})$ (explicitly, $f$ is determined by the equation $f/f_0 = f_{-\alpha}$ in the function field $K(X)$; see Lemma 2.5.3 for details). Since the coefficient of $D^+_\alpha$ in $\mathrm{div}(f)$ is 0, we have $D^+_\alpha \cap X_f \neq \varnothing$.

The case where $\alpha \in \Pi^c_X$ is almost identical. We pick $f_0$ and $E$ as above. There is a single divisor $D_\alpha$ moved by $\alpha$; let $c$ be its coefficient in $E$. After replacing $f_0$ by $f_0 \otimes f_0 \in H^0(X, L^{\otimes 2})$ (which replaces $E$ by $2E$), we may assume that $c$ is even. Let $f_{-2\alpha} \in K(X)^{(B)}$ be a $B$-eigenvector of weight $-2\alpha$ (which exists because $2\alpha$ is a spherical root). By Proposition 3.6.13 and Lemma 4.4.2, we see that the divisor $\mathrm{div}(f_{-2\alpha})$ has nonnegative coefficients on every $B$-divisor except $D_\alpha$, and the coefficient of $D_\alpha$ is

$$\varphi_{D_\alpha}(-2\alpha) = \frac{1}{2}\langle \alpha^\vee, -2\alpha\rangle = -\langle \alpha^\vee, \alpha\rangle = -2.$$

Thus, $E + \mathrm{div}(f_{-2\alpha}^{c/2})$ is an effective Cartier divisor with coefficient 0 for $D_\alpha$. As above, Lemma 2.5.3 gives us a $B$-eigenvector $f \in H^0(X, L^{\otimes 2})$ such that $\mathrm{div}(f) = E + \mathrm{div}(f_{-2\alpha}^{c/2})$, and we then have $D_\alpha \cap X_f \neq \varnothing$.

Finally, consider the case of Statement (c). Let $D$ be the unique element of $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$. Corollary 4.4.3 tells us that $D$ is the unique $B$-divisor of $X$ such that $\varphi_D(\alpha+\beta) > 0$, and that $\varphi_D(\alpha+\beta) = 2$. So, pick $f_0$ and $E$ as above, let $c$ be the coefficient of $D$ in $E$, and let $f_{-\alpha-\beta} \in K(X)^{(B)}$ be a $B$-eigenvector of weight $-\alpha-\beta$. After replacing $f_0$ by $f_0 \otimes f_0$ as in the case $\alpha \in \Pi_X^c$ above, we may assume that $c$ is even. Then, $E + \mathrm{div}(f_{-\alpha-\beta}^{c/2})$ is an effective Cartier divisor with coefficient 0 on $D$, and Lemma 2.5.3 again gives us the desired section $f \in H^0(X, L^{\otimes 2})$. $\square$

**Remark 4.5.3.** The above lemma is an analog of [Los09a, Proposition 6.4], which Losev used in his proof of the Knop conjecture. As stated, the above lemma does allow for the trivial behavior of $X_f = X$. In this case, one of $L$ or $L^{\otimes 2}$ has a global section which vanishes nowhere, hence it is the trivial line bundle. This is important to deal with in the case where $X$ is affine and $L = \mathcal{O}_X$ (which is essentially the case Losev considers). However, our main application will be when $X$ is projective and $L$ is ample, so if $L \cong \mathcal{O}_X$, then everything is trivial: $X$ is quasi-affine because $\mathcal{O}_X$ is ample, and the only quasi-affine complete variety over $k$ is $\mathrm{Spec}(k)$.

The following lemma is the most fundamental technical tool that we will use for comparing divisors on spherical varieties.

**Lemma 4.5.4.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that*

1. *$\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and*
2. *either $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ or $X_1$ and $X_2$ are smooth.*

*Let $(\mu, n) \in \Lambda^+(X_1, L_1)$ be such that $n > 0$. There exists a bijection*

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\}$$

*such that for any $D$ in the domain of $\iota_\mu$, the following hold.*

(a) *We have $\varphi_D = \varphi_{\iota_\mu(D)}$.*

(b) *For any $\alpha \in \Pi_{M_\mu} \subset \Pi_G$ moves $D$ if and only if $\alpha$ moves $\iota_\mu(D)$. If $\Pi_{X_1}^b = \Pi_{X_2}^b$, then the same holds for any $\alpha \in \Pi_G$.*

(c) *For any isomorphism $f : X_1(\mu) \xrightarrow{\sim} X_2(\mu)$, the map $\iota_\mu$ can be taken to be the composition $\iota_2 \circ f_* \circ \iota_1^{-1}$, where $f_* : \mathcal{D}_{M_\mu, X_1} \to \mathcal{D}_{M_\mu, X_2}$ is the map $D \mapsto f(D)$ and $\iota_1$ and $\iota_2$ are the bijections given by Proposition 3.2.3e.*

*Proof.* Since the $X_i$ are projective and the $L_i$ are ample, the $(X_i)_\mu$ are affine, so Theorem 4.4.6 gives us an isomorphism of $M_\mu$-varieties $f : X_1(\mu) \xrightarrow{\sim} X_2(\mu)$. In particular, this isomorphism gives us a bijection

$$f_* : \mathcal{D}_{M_\mu, X_1(\mu)} \to \mathcal{D}_{M_\mu, X_2(\mu)}, \quad D \mapsto f(D).$$

Since $f$ is an $M_\mu$-equivariant isomorphism, for all $D \in \mathcal{D}_{M_\mu, X_1(\mu)}$, we have $\varphi_D = \varphi_{f_*(D)}$, and any $\alpha \in \Pi_{M_\mu}$ moves $D$ if and only if it moves $f_*(D)$. On the other hand, for $i \in \{1, 2\}$, we have bijections $\iota_i$ between $\mathcal{D}_{M_\mu, X_i(\mu)}$ and the set of $B$-divisors of $X_i$ intersecting $(X_i)_\mu$ which satisfy the same conditions on the $\varphi_D$ and the roots moving $D$ as $f_*$ does (see Proposition 3.2.3e and Proposition 4.4.1). Putting all these bijections together, we obtain a bijection

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\}$$

such that for all $D$, we have $\varphi_D = \varphi_{\iota_\mu(D)}$, and any $\alpha \in \Pi_{M_\mu}$ moves $D$ if and only if $\alpha$ moves $\iota_\mu(D)$.

This proves everything except for the "moreover" statement in (b). So, suppose that $\Pi^b_{X_1} = \Pi^b_{X_2}$, let $D$ be in the domain of $\iota_\mu$, and let $\alpha \in \Pi_G$. We show that if $\alpha$ moves $D$, then $\alpha$ moves $\iota_\mu(D)$; the reverse implication follows by the same argument with $X_1$ and $X_2$ swapped. If $\alpha \in \Pi^c_{X_1} \cup \Pi^d_{X_1}$, then we have $\mathcal{D}_{G,X_1}(\alpha) = \{D\}$. In particular, every element of $\mathcal{D}_{G,X_i}(\alpha)$ intersects $(X_i)_\mu$ for some $i$, so Proposition 4.4.1 gives us $\alpha \in \Pi_{M_\mu}$. For such a choice of $\alpha$, we already know that (b) holds. Suppose instead that $\alpha \in \Pi^b_{X_1} = \Pi^b_{X_2}$. Since $\alpha$ moves $D$, Proposition 3.6.13 gives us

$$\varphi_{\iota_\mu(D)}(\alpha) = \varphi_D(\alpha) = 1,$$

and since $\alpha \in \Pi^b_{X_2}$, this implies that $\alpha$ moves $\iota(D)$ (see Corollary 4.4.3). $\square$

With the above results in hand, we can now match up not only root types of $X_1$ and $X_2$ but also all $B$-divisors. Losev proved a similar statement as a stepping stone to the proof of the Knop conjecture, and his method of proof was very similar to ours. The theorem below can thus be thought of as an analog of Losev's result in the projective case.

**Theorem 4.5.5** (cf. [Los09a, Theorem 4.8])**.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that*

1. *$\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and*
2. *either $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ or $X_1$ and $X_2$ are smooth.*

*Then, the following hold.*

*(a) We have $\Pi^c_{X_1} = \Pi^c_{X_2}$.*

*(b) If $\Pi^b_{X_1} = \Pi^b_{X_2}$, then $X_1$ and $X_2$ are $\mathcal{D}$-equivalent.*

**Remark 4.5.6.** Notice that $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ is equivalent to saying $\Psi_{G,X_1} = \Psi_{G,X_2}$. In particular, since the roots of type $b$ are exactly the elements of $\Pi_G$ which are spherical roots, it follows that $\Pi^b_{X_1} = \Pi^b_{X_2}$. So if we use the hypothesis $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ in the above theorem, then the hypothesis $\Pi^b_{X_1} = \Pi^b_{X_2}$ in Statement (b) automatically holds.

**Remark 4.5.7.** In our proof of this theorem, the only place we have used that the $X_i$ are projective and the $L_i$ are ample is to claim that sets of the form $(X_i)_f$ for some $f \in H^0(X, L^{\otimes n})$ are always affine. This is of course also true when the $X_i$ are affine and $L_i = \mathcal{O}_{X_i}$, which is essentially the case that Losev considers in [Los09a, Theorem 4.8]. The primary difference between our argument here and Losev's argument is that Losev had to use an induction argument along with the local structure theorem, which required quite a bit of extra care in the case where $X_1$ and $X_2$ are smooth. By constrast, we are using the Knop conjecture in conjunction with the local structure theorem, as in Theorem 4.4.6, so we do not need to deal with the same technicalities.

*Proof.* For any $(\mu, n) \in \Lambda^+(X_1, L_1)$ be such that $n > 0$, let

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\}$$

be the bijection of Lemma 4.5.4. We will use $\iota_\mu$ for different choices of $\mu$ repeatedly throughout the proof.

We first prove (a) using this construction. Let $\alpha \in \Pi^c_{X_1}$; we show that $\alpha \in \Pi^c_{X_2}$. Swapping $X_1$ and $X_2$ and applying the same argument will then give us statement (a). By Lemma 4.5.2, we may pick $(\mu, n) \in \Lambda^+(X_1, L_1)$ such that the divisor $D_1$ moved by $\alpha$ satisfies $D_1 \cap (X_1)_\mu \neq \varnothing$. Since $D_1$ is the only element of $\mathcal{D}_{G,X_1}(\alpha)$, we have $\alpha \in \Pi_{M_\mu}$ by Proposition 4.4.1, and the same proposition implies that every element of $\mathcal{D}_{G,X_2}(\alpha)$ intersects $(X_2)_\mu$. It follows that the image of $\iota_\mu$ contains every element of $\mathcal{D}_{G,X_2}(\alpha)$, so $\iota_\mu$ restricts to a bijection $\mathcal{D}_{G,X_1}(\alpha) \to \mathcal{D}_{G,X_2}(\alpha)$. In particular, the set $\mathcal{D}_{G,X_2}(\alpha)$ has a unique element $D_2$, and $D_2 = \iota_\mu(D_1)$ implies that $\varphi_{D_1} = \varphi_{D_2}$. Since there is exactly one divisor moved by $\alpha$ in $X_2$, we must have $\alpha \in \Pi^c_{X_2} \cup \Pi^d_{X_2}$. On the other hand, we have

$$\varphi_{D_2} = \varphi_{D_1} = \frac{1}{2}\alpha^\vee|_{\Lambda(X_1)},$$

but the valuation of a divisor moved by a root of type $d$ would be $\alpha^\vee|_{\Lambda(X_2)}$ (Proposition 3.6.13). We conclude that $\alpha \in \Pi^c_{X_2}$, which proves (a).

Now, assume that $\Pi^b_{X_1} = \Pi^b_{X_2}$. Since $\Pi^a_{X_1} = \Pi^a_{X_2}$ by Lemma 4.5.1 and $\Pi^c_{X_1} = \Pi^c_{X_2}$ by (a), we also have $\Pi^d_{X_1} = \Pi^d_{X_2}$. As such, we will mainly refer to roots in this proof as being "of type $b$" (or $c$ or $d$), by which we mean of type $b$ (or $c$ or $d$) for both $X_1$ and $X_2$, and we will write $\Pi^a$, $\Pi^b$, $\Pi^c$, and $\Pi^d$ (dropping the subscripts $X_1$ and $X_2$) for the sets of roots of type $a$, $b$, $c$, and $d$ (respectively).

We will define the desired bijection $\iota$ of Statement (b) by splitting the sets $\mathcal{D}_{G,X_i}$ for $i \in \{1, 2\}$ into three disjoint subsets:

1. The set $\mathcal{D}_1^i$ of $B$-divisors that are either $G$-stable or moved by multiple roots of type $d$.

2. The set $\mathcal{D}_2^i$ of $B$-divisors that are moved by a root of type $b$.

3. The set $\mathcal{D}_3^i$ of $B$-divisors that are either moved by a root of type $c$ or are moved by a unique root of type $d$.

We define $\iota$ on each of these three subsets in three different steps.

Step 1: We define $\iota$ on the set $\mathcal{D}_1^1$ of $B$-divisors $D_1$ of $X_1$ such that either (1) $D_1$ is $G$-stable, or (2) $D_1$ is moved by multiple roots of type $d$. In either case, we claim that there exists some $(\mu, n) \in \Lambda^+(X_1, L_1)$ such that $n > 0$ and $(X_1)_\mu \cap D_1 \neq \varnothing$. If $D_1$ is moved by multiple roots of type $d$, this is Lemma 4.5.2c. If instead $D_1$ is $G$-stable, then $D_1$ contains some $G$-orbit $Y_1 \subset X_1$. By Theorem 3.2.7a, we have $(X_1)_{B,Y_1} = (X_1)_\mu$ for some weight $\mu$ of a $B$-eigenvector of $\Gamma_*(X, L)$, so $(\mu, n) \in \Lambda^+(X_1, L_1)$ for some $n$. The definition of $(X_1)_{B,Y_1}$ gives us $D_1 \cap (X_1)_\mu \neq \varnothing$. As for the statement that $n > 0$, suppose instead that $n = 0$. Since $\Gamma(X_1, \mathcal{O}_{X_1}) = k$ only has sections which vanish nowhere, this implies that $(X_1)_\mu = X_1$. On the other hand, $(X_1)_{B,Y_1}$ is always affine, so $X_1$ is both affine and projective, hence $X_1 = \mathrm{Spec}(k)$. It follows that $L_1 \cong \mathcal{O}_{X_1}$, so that

$$\Lambda^+(X_2, L_2) = \Lambda^+(X_1, L_1) = \{(0, d) \mid d \in \mathbb{N}\}$$

(here 0 denotes the trivial character in $\Lambda_G$). There is only one $k$-algebra $\Gamma_*(X_2, L_2)$ yielding this weight monoid: namely, $\Gamma_*(X_2, L_2) \cong k[x]$, with $G$ acting trivially on $X$. Thus $X_2 \cong \mathrm{Proj}(k[x]) \cong \mathrm{Spec}(k)$ as well, so everything is trivial. In particular, $X_1 \cong X_2$ as $G$-varieties, and the bijection $\iota$ of Statement (b) certainly exists. So, we may assume that $n > 0$, in which case $(\mu, n)$ is as claimed.

Our choice of $(\mu, n)$ implies that $D_1$ is in the domain of the bijection $\iota_\mu$ constructed above. So, we set $\iota(D_1) = \iota_\mu(D_1)$. The construction of $\iota_\mu$ immediately gives us $\varphi_{D_1} = \varphi_{\iota(D_1)}$, and we showed above that any $\alpha \in \Pi_G$ moves $D_1$ if and only if $\alpha$ moves $\iota(D_1)$. In particular, $\iota(D_1)$ is $G$-stable (i.e. moved by no roots of $\alpha$) if and only if $D_1$ is, and $\iota(D_1)$ is moved by multiple roots of type $d$ if and only if $D_1$ is. Thus, we have defined $\iota$ as a map $\mathcal{D}_1^1 \to \mathcal{D}_1^2$ satisfying the necessary properties. It remains to check that $\iota$ is a bijection on these sets. Injectivity on $G$-divisors follows from the fact that any $G$-invariant valuation is determined by its image under the map $\varphi$ (see Corollary 3.1.14). More explicitly: if $\iota(D_1) = \iota(D_1')$ for two $G$-divisors $D_1$ and $D_1'$, then we have

$$\varphi_{D_1} = \varphi_{\iota(D_1)} = \varphi_{\iota(D_1')} = \varphi_{D_1'}.$$

This gives us $v_{D_1} = v_{D_1'}$, and the valuative criterion of separatedness then implies that $D_1 = D_1'$. Injectivity on divisors moved by multiple roots of type $d$ is analogous, but instead using the fact that such a divisor is the unique $B$-divisor $D$ satisfying $\varphi_D(\alpha + \beta) > 0$, where $\alpha$ and $\beta$ are the two roots moving $D$ (see Corollary 4.4.3. Surjectivity follows from the symmetry in $X_1$ and $X_2$ of our construction of $\iota(D_1)$. That is, for any $D_2 \in \mathcal{D}_1^2$, we can repeat our construction of $\iota(D_1)$ but with $X_1$ and $X_2$ swapped to obtain a divisor $D_1 \in \mathcal{D}_1^1$ such that $\varphi_{D_1} = \varphi_{D_2}$ and $D_1$ is moved by the same roots as $D_2$. Then, we have $\varphi_{\iota(D_1)} = \varphi_{D_2}$, and the arguments we used to prove injectivity then imply that $\iota(D_1) = D_2$.

Step 2: We define $\iota$ on the set $\mathcal{D}_2^1$ of $B$-divisors of $X_1$ moved by some root $\alpha$ of type $b$. We will do this by defining $\iota$ on $\mathcal{D}_{G,X_1}(\alpha)$ for each $\alpha \in \Pi^b$ in turn. For such a root $\alpha$, write $\mathcal{D}_{G,X_1}(\alpha) = \{D_1, D_1'\}$. By Lemma 4.5.2, there exists some $(\mu, n) \in \Lambda^+(X_1, L_1)$ with $n \geq 1$ such that either $(X_1)_\mu \cap D_1 \neq \varnothing$ or $(X_1)_\mu \cap D_1' \neq \varnothing$. After swapping $D_1$ and $D_1'$ if necessary, we may assume that $D_1 \cap (X_1)_\mu \neq \varnothing$. We define $\iota(D_1) = \iota_\mu(D_1)$. This immediately gives us that $\varphi_{D_1} = \varphi_{\iota(D_1)}$ and that any root $\beta$ moves $D_1$ if and only if $\beta$ moves $\iota(D_1)$. We further define $\iota(D_1')$ to be the unique $B$-divisor of $X_2$ moved by $\alpha$ that is not equal to $\iota(D_1)$. In this case, Proposition 3.6.13 gives us

$$\varphi_{D_1} + \varphi_{D_1'} = \alpha^\vee|_{\Lambda(X_1)} = \varphi_{\iota(D_1)} + \varphi_{\iota(D_1')}.$$

Since $\varphi_{D_1} = \varphi_{\iota(D_1)}$, we see that $\varphi_{D_1'} = \varphi_{\iota(D_1')}$. Moreover, for any $B$-divisor $D$ of $X_i$ moved by a root of type $b$, the set of roots moving $D$ is completely determined by the set $\Pi^b_{X_i}$ and the valuation $\varphi_D$, see Corollary 4.4.3. Since $\Pi^b_{X_1} = \Pi^b_{X_2}$ and $\varphi_{D_1'} = \varphi_{\iota(D_1')}$, it follows that any root $\beta$ moves $D_1'$ if and only if $\beta$ moves $\iota(D_1')$.

We claim that our definition of the $\mathcal{D}_{G,X_1}(\alpha)$ here glues to a definition on all of $\mathcal{D}_2^1$. We just need to check that if either $D_1$ or $D_1'$ is moved by another root $\beta$, then defining $\iota(D_1)$ or $\iota(D_1')$ using $\beta$ instead of $\alpha$ gives us the same divisor. After swapping $D_1$ and $D_1'$ if necessary, we may assume that $D_1$ is moved by $\beta$. Whether we use $\alpha$ or $\beta$ to define $\iota(D_1)$, the $B$-divisor $\iota(D_1)$ will be moved by both $\alpha$ and $\beta$. But there is at most one such divisor, see Proposition 3.6.12. So, $\iota(D_1)$ is the same whether we define it using $\alpha$ or $\beta$. We remark that our definition of $\iota$ may depend on the choice of $(\mu, n)$. That is, if there exists some $(\mu, n)$ and some $(\mu', n')$ such that $(X_1)_\mu \cap D_1 \neq \varnothing$ and $(X_1)_{\mu'} \cap D_1' \neq \varnothing$, then it is possible that choosing $(\mu', n')$ instead of $(\mu, n)$ will swap $\iota(D_1)$ and $\iota(D_1')$. However, this choice does not affect whether $\iota$ is well-defined.

It remains to check bijectivity of the map $\iota : \mathcal{D}_2^1 \to \mathcal{D}_2^2$ we have defined. Surjectivity is immediate from the construction, because $\iota$ maps $\mathcal{D}_{G,X_1}(\alpha)$ surjectively onto $\mathcal{D}_{G,X_1}(\alpha)$ for all $\alpha \in \Pi^b$. On the other hand, for any divisors $D$ and $D'$ moved by roots of type $b$ such that $\iota(D) = \iota(D')$, both $D$ and $D'$ must be moved by the same root $\alpha$ of type $b$ (namely, $\alpha$ is a root moving both $\iota(D)$ and $\iota(D')$). If $D \neq D'$, then $\mathcal{D}_{G,X_1}(\alpha) = \{D, D'\}$ and, by construction, $\iota(D) \neq \iota(D')$ are the two distinct elements of $\mathcal{D}_{G,X_2}(\alpha)$. But we know that $\iota(D) = \iota(D')$, so $D$ and $D'$ must be the same element of $\mathcal{D}_{G,X_1}(\alpha)$.

Step 3: We define $\iota$ on the set $\mathcal{D}_3^1$ of $B$-divisors $D_1$ of $X_1$ such that either (1) $D_1$ is moved by a root of type $c$, or (2) $D_1$ is moved by a unique root of type $d$. In either case, $D_1$ is the unique $B$-divisor of $X_1$ moved by some root $\alpha$, so we set $\iota(D_1)$ to be the unique $B$-divisor of $X_2$ moved by $\alpha$, and this is well-defined because $\alpha$ is the unique root moving $D_1$ (see Corollary 4.4.3). The valuations of $D_1$ and $\iota(D_1)$ are completely determined by the coroot $\alpha^\vee$ (see Proposition 3.6.13), so we have $\varphi_{D_1} = \varphi_{\iota(D_1)}$. We claim that $\alpha$ is the only root moving moving $\iota(D_1)$. If $\alpha \in \Pi^c$, this is Corollary 4.4.3b. On the other hand, suppose that $\alpha \in \Pi^d$ and that $\iota(D_1)$ is moved by some other root $\beta$. Then, $\beta \in \Pi^d$, and by Step 1, $\iota(D_1)$ is the image under $\iota$ of some divisor $D_1'$ which is moved by both $\alpha$ and $\beta$. But $D_1$ is the unique divisor moved by $\alpha$, so $D_1 = D_1'$, which contradicts the fact that $D_1$ is moved by a unique root of type $d$. Thus, no such $\beta$ can exist.

It remains to check bijectivity of the map $\iota : \mathcal{D}_3^1 \to \mathcal{D}_3^2$ we have defined. Injectivity follows from the fact that any divisor $D_1 \in \mathcal{D}_3^1$ is moved by a unique root $\alpha$ and is mapped to the unique divisor in $\mathcal{D}_3^2$ moved by $\alpha$. As for surjectivity, any $D_2 \in \mathcal{D}_3^2$ moved by a root of type $c$ is the image of the unique $B$-divisor of $X_1$ moved by the same root. So, suppose that $D_2 \in \mathcal{D}_3^2$ is moved by a unique root $\alpha$ of type $d$, and let $D_1$ be the unique divisor of $X_1$ moved by $\alpha$. If $D_1$ is moved by another root $\beta$, then in Step 1, we have already shown that there exists a divisor $D_2'$ of $X_2$ moved by both $\alpha$ and $\beta$. But $D_2$ is the unique divisor of $X_2$ moved by $\alpha$, so $D_2 = D_2'$, contradicting the fact that $D_2 \in \mathcal{D}_3^2$. Thus, $D_1$ is moved only by $\alpha$, so we have $D_1 \in \mathcal{D}_3^1$ and $\iota(D_1) = D_2$. □

**Remark 4.5.8.** The assumption $\Pi^b_{X_1} = \Pi^b_{X_2}$ is essential in part (b) of Theorem 4.5.5. Indeed, we will see an explicit counterexample to the theorem without $\Pi^b_{X_1} = \Pi^b_{X_2}$ in Examples 4.9.1 and 4.9.2. In other words, the sets $\Pi^b_{X_1}$ and $\Pi^b_{X_2}$ are actually not entirely captured by the weight monoids $\Lambda^+(X_1, L_1)$ and $\Lambda^+(X_2, L_2)$, even in the smooth projective setting. However, there are ways to obtain the conclusion of Theorem 4.5.5b just using the data of weight monoids, without having to keep track of the set of roots of type $b$. We will study these alternatives in Chapter 5.

As an application of Theorem 4.5.5, if we take the assumption $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ in the theorem and add in a small argument about affine cones, we can give an alternate proof of Corollary 4.3.5 that does not use any of our our results about valuations on affine cones from Section 4.3.

*Alternate proof of Corollary 4.3.5.* By Theorem 4.5.5, we know that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent. This implies that the open $G$-orbits of $X_1$ and $X_2$ are $\mathcal{D}$-equivalent (cf. the proof of Theorem 4.1.9), so Theorem 4.1.6 implies that the open $G$-orbits of $X_1$ and $X_2$ are $G$-equivariantly isomorphic. In particular, there is a $G$-equivariant ring isomorphism $K(X_1) \cong K(X_2)$. Pick $B$-eigenvectors $f_i \in H^0(X_i, L_i)^{(B)}$ of the same weight $\mu$ for $i \in \{1, 2\}$, and write $A_i = \Gamma_*(X_i, L_i)$ and $K_i = K(X_i)$. We have seen in (4.3.1) (see also the our general discussion in Section 4.3) that the principal $\mathbb{G}_m$-bundle $\tilde{X}_i \setminus \{0\} \to X_i$ admits a local trivialization $(\tilde{X}_i)_{f_i} \cong (X_i)_{f_i} \times \mathbb{G}_m$ which maps $t \in \Gamma(\mathbb{G}_m, \mathcal{O}_{\mathbb{G}_m}) \cong k[t^{\pm}]$ to $f_i \in \Gamma((\tilde{X}_i)_{f_i}, \mathcal{O}_{\tilde{X}_i})$. On function fields, this trivialization gives a $G$-equivariant isomorphism $K(\tilde{X}_i) \cong K_i(t)$ which maps $f_i$ to $t$. In particular, $t$ is a $B$-eigenvector of weight $\mu$. Since $\mathbb{G}_m$ acts on $K_i(t)$ according to the grading, it follows that the $\tilde{B}$-eigenvectors of $K_i(t)$ are the monomials $at^n$ for some $a \in K_i^{(B)}$ and $n \in \mathbb{Z}$, and such a monomial has weight $\mu_a + n\mu$.

Now, the $G$-equivariant ring isomorphism $K_1 \cong K_2$ induces a ring isomorphism

$$\iota : K_1(t) \xrightarrow{\sim} K_2(t)$$

by setting $t \mapsto t$. By our above arguments, $\iota$ and its inverse map $B$-eigenvectors to $B$-eigenvectors of the same weight. It follows that $\iota$ is $G$-equivariant, and since $\iota$ is graded, it is even $\tilde{G}$-equivariant. Since the $\tilde{X}_i$ are spherical $\tilde{G}$-varieties, the function fields $K_i(t)$ are multiplicity-free (because $K_i(t)^B = k$, see Theorem 3.1.4). Thus, the weight monoid $\Lambda^+(X_i, L_i)$ determines the ring $A_i$ as a $G$-submodule of $K_i(t)$. Since $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, we see that $\iota$ restricts to a $\tilde{G}$-equivariant isomorphism $A_1 \cong A_2$. This implies that $\tilde{X}_1 \cong \tilde{X}_2$ as $\tilde{G}$-varieties, and arguing as in the proof of Corollary 4.3.5, we obtain $(X_1, L_1) \cong (X_2, L_2)$ as polarized spherical $G$-varieties. $\square$

Corollary 4.3.5 is essentially as nice a result as possible under the assumption that $\Psi_{G,X_1} = \Psi_{G,X_2}$; in particular, it is a direct projective analog of the behavior in the affine case proven by Losev, see Theorem 4.2.2a. In the remainder of this chapter, then, we will mainly concern ourselves with the case where $X_1$ and $X_2$ are smooth, in the hopes of proving some projective analogs of Theorem 4.2.2b.

## 4.6 Comparing Spherical Roots

Let $(X_1, L_1)$ and $(X_2, L_2)$ be two polarized spherical varieties such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. In this section, we will use Theorem 4.4.6 repeatedly to attempt to equate spherical roots of $X_1$ with those of $X_2$. This seems like a reasonable task, because Proposition 4.4.1 tells us that we can sometimes detect spherical roots of $X_1$ and $X_2$ using the local structure theorem. The main issue is: can we find an open subset of $X_1$ such that a given spherical root $\gamma \in \Psi_{G,X_1}$ will be detected by the local struture theorem? Or, more precisely: does there exist $(\mu, d) \in \Lambda^+(X_1, L_1)$ such that, writing $(X_1)_\mu \cong R_u(P) \times Z$ and $P = R_u(P) \cdot M$ as in the local structure theorem, we have $\gamma \in \Psi_{M,Z}$? In order to answer this question, we will first need a few more combinatorial facts about spherical roots.

### 4.6.a Spherical Roots and Root Systems

We begin with a couple useful definitions.

**Definition 4.6.1.** Let $X$ be a spherical variety, and let $\gamma \in \Psi_{G,X}$.

1. We define the *support* of $\gamma$ to be the subset $\mathrm{Supp}(\gamma) \subset \Pi_G$ consisting of simple roots whose coefficients in $\gamma$ are positive. This is well-defined because simple roots are always linearly independent, and $\mathrm{Supp}(\gamma) \neq \varnothing$ because $\gamma$ is a linear combination of simple roots with nonnegative coefficients, see Remark 3.4.8.

2. We say that any $\alpha \in \Pi_G$ is *adjacent to* $\mathrm{Supp}(\gamma)$ if $\alpha$ is adjacent to any simple root in $\mathrm{Supp}(\gamma)$ (in the sense of Definition 2.2.18).

3. For any $D \in \mathcal{D}_{G,X}$, we say that $D$ *is moved by* $\mathrm{Supp}(\gamma)$ if $D$ is moved by some element of $\mathrm{Supp}(\gamma)$ (i.e. if $D \in \mathcal{D}_{G,X_1}(\alpha)$ for some $\alpha \in \mathrm{Supp}(\gamma)$).

The following lemma gives us some general behavior of spherical roots and simple roots of $G$.

**Lemma 4.6.2.** *Let $X$ be a spherical variety, let $\gamma \in \Psi_{G,X}$ be a spherical root, and let $\alpha \in \Pi_G$.*

*(a) We have $\langle \alpha^\vee, \gamma \rangle \leq 0$ if and only if $\varphi_D(\gamma) \leq 0$ for all $D \in \mathcal{D}_{G,X}(\alpha)$ and $\langle \alpha^\vee, \gamma \rangle = 0$ if and only if $\varphi_D(\gamma) = 0$ for all $D \in \mathcal{D}_{G,X}(\alpha)$.*

*(b) If $\alpha \notin \operatorname{Supp}(\gamma)$, we have $\langle \alpha^\vee, \gamma \rangle \leq 0$.*

*(c) If $\alpha \in \Pi_X^a$, either $\alpha \in \operatorname{Supp}(\gamma)$ or $\alpha$ is not adjacent to $\operatorname{Supp}(\gamma)$.*

*Proof.* By definition, $\gamma$ is a positive linear combination of the elements of $\operatorname{Supp}(\gamma)$. Moreover, it is a general fact about root systems that $\langle \alpha^\vee, \beta \rangle \leq 0$ for any simple roots $\alpha \neq \beta$. In particular, it follows that $\alpha \notin \operatorname{Supp}(\gamma)$ implies $\langle \alpha^\vee, \gamma \rangle \leq 0$, which is statement (b).

As for statement (c), let $\alpha \in \Pi_X^a$. If $\alpha \notin \operatorname{Supp}(\gamma)$, then our above arguments give us $\langle \alpha^\vee, \beta \rangle \leq 0$ for all $\beta \in \operatorname{Supp}(\gamma)$. If in addition $\alpha$ is adjacent to $\operatorname{Supp}(\gamma)$, then $\langle \alpha^\vee, \beta \rangle < 0$ for some $\beta \in \operatorname{Supp}(\gamma)$, which implies that $\langle \alpha^\vee, \gamma \rangle < 0$. This contradicts the fact that $\langle \alpha^\vee, \gamma \rangle = 0$ by Lemma 3.6.14.

For statement (a), we argue as in Lemma 4.4.2. There are 3 cases, depending on the type of the root $\alpha$ for $X$.

1. Suppose $\alpha \in \Pi_X^a$. Then, $\mathcal{D}_{G,X}(\alpha) = \varnothing$, and $\langle \alpha^\vee, \gamma \rangle = 0$ by Lemma 3.6.14, so all the conditions in (a) are always true.

2. Suppose $\alpha \in \Pi_X^c \cup \Pi_X^d$, and write $\mathcal{D}_{G,X}(\alpha) = \{D\}$. We have $\varphi_D = c\alpha^\vee|_{\Lambda(X)}$ for some $c \in \{1, 1/2\}$ (see Proposition 3.6.13). Since $\gamma \in \Lambda(X)$ and $c > 0$, statement (a) follows immediately.

3. Suppose $\alpha \in \Pi_X^b$, and write $\mathcal{D}_{G,X}(\alpha) = \{D, D'\}$. Note that $\alpha$ is a spherical root of $X$ and that $\langle \alpha^\vee, \alpha \rangle = 2 > 0$. In particular, if $\langle \alpha^\vee, \alpha \rangle \leq 0$, then $\gamma \neq \alpha$, so we have $\varphi_D(\gamma), \varphi_{D'}(\gamma) \leq 0$ by Proposition 3.6.18. $\varphi_D(\gamma), \varphi_{D'}(\gamma) \leq 0$, then Proposition 3.6.13 gives us
$$\langle \alpha^\vee, \gamma \rangle = \varphi_D(\gamma) + \varphi_{D'}(\gamma) \leq 0.$$
The above equation also gives us $\langle \alpha^\vee, \gamma \rangle = 0$ if $\varphi_D(\gamma) = \varphi_{D'}(\gamma) = 0$. On the other hand, if $\langle \alpha^\vee, \gamma \rangle = 0$, then we already saw that $\varphi_D(\gamma), \varphi_{D'}(\gamma) \leq 0$. But the above equation gives us $\varphi_D(\gamma) + \varphi_{D'}(\gamma) = 0$, so in fact we must have $\varphi_D(\gamma) = \varphi_{D'}(\gamma) = 0$.

□

The next lemma gives us a useful combinatorial characterization of certain "nice" types of spherical roots. These types of spherical roots are relatively easy to "match up" between two (nice enough) spherical varieties, so generally, we will use this lemma to separate out this "nice" case from the more difficult possibilities for spherical roots.

**Lemma 4.6.3.** *Let $X$ be a spherical variety, let $\gamma \in \Psi_{G,X}$, and let $\alpha \in \Pi_G$ be such that $\langle \alpha^\vee, \gamma \rangle > 0$.*

*(a) If $\alpha \in \Pi_X^b$, then $\gamma = \alpha$.*

*(b) If $\alpha \in \Pi_X^c$, then $\gamma = 2\alpha$.*

*(c) If $\alpha \in \Pi_X^d$ and there exists some $\beta \neq \alpha$ with $\mathcal{D}_{G,X}(\alpha) = \mathcal{D}_{G,X}(\beta)$, then $\gamma = c(\alpha + \beta)$ for some $c \in \{1, 1/2\}$.*

**Remark 4.6.4.** Lemma 4.4.2 is a sort of converse to the above lemma. Indeed, combined with Lemma 4.6.2a above, Lemma 4.4.2 implies that whenever $\gamma$ is as in (a), (b), or (c) of the above lemma, then $\alpha$ (and $\beta$ in statement (c)) is the only simple root such that $\langle \alpha^\vee, \gamma \rangle > 0$.

*Proof.* Statement (b) is the contrapositive of Proposition 3.6.18b, and part (a) of the same proposition along with Lemma 4.6.2a gives us (a). As for (c), Proposition 3.6.12 gives us $\langle \alpha^\vee, \beta \rangle = 0$ (i.e. $\alpha$ and $\beta$ are not adjacent) and $\alpha^\vee|_{\Lambda(X)} = \beta^\vee|_{\Lambda(X)}$. In particular, we have

$$\langle \beta^\vee, \gamma \rangle = \langle \alpha^\vee, \gamma \rangle > 0,$$

so Lemma 4.6.2b implies that $\alpha, \beta \in \mathrm{Supp}(\gamma)$.

Now, we will need to consult the list of possible spherical roots. More precisely, Corollary 3.6.17 gives us some semisimple simply connected group $G'$ with $\Pi_{G'} \subset \Pi_G$ and some prime rank-1 wonderful $G'$-variety $X'$ such that $\Psi_{G',X'} = \{\gamma\}$ and $\Pi_{X'}^a \subset \Pi_X^a$. The possible choices for $G'$ and $X'$ can be found in Table 1 of [Was96]. In particular, notice that any element of $\mathrm{Supp}(\gamma)$ must be an element of $\Pi_{G'}$, so the combinatorial properties of roots in $\mathrm{Supp}(\gamma)$ are determined by Table 1. We remark that while Table 1 does not explicitly describe the set $\Pi_{X'}^a$, we can find it using this table in the following way: for each color $D$, of $X'$, there is a tuple in Column 4 of the table, and the first element of this tuple is the weight $\mu_D$ of Proposition 5.2.5. In particular, this element will be linear combination of the fundamental weights whose corresponding roots move $D$. So, checking the first element of each tuple in Column 4 of the table tells us every simple root of $G'$ that moves some color of $X'$, and every root other than those is a root of type $a$ for $X'$.

Now since $\alpha, \beta \in \mathrm{Supp}(\gamma) \setminus \Pi_X^a$, the fact that $\Pi_{X'}^a \subset \Pi_X^a$ gives us $\alpha, \beta \in \Pi_{G'} \setminus \Pi_{X'}^a$. So, $X'$ is an entry in Table 1 of [Was96] with 2 roots not of type $a$ whose coroots are both $> 0$ on the single spherical root $\gamma$. There are only 3 entries in the table satisfying these conditions:

1. $\Pi_{G'} = A_n$ for some $n \geq 2$ and $\gamma = \alpha_1 + \cdots + \alpha_n$, in which case $\{\alpha, \beta\} = \{\alpha_1, \alpha_n\}$ are the two roots not of type $a$.

2. $\Pi_{G'} = A_1 \times A_1$ and $\gamma = c(\alpha_1 + \alpha_1')$ for some $c \in \{1, 1/2\}$, in which case $\{\alpha, \beta\} = \{\alpha_1, \alpha_1'\}$.

3. $\Pi_{G'} = B_n$ for some $n \geq 2$ and $\gamma = \alpha_1 + \cdot + \alpha_n$, in which case $\{\alpha, \beta\} = \{\alpha_1, \alpha_n\}$ are the two roots not of type $a$.

4. $\Pi_{G'} = D_3$ and $\gamma = \alpha_1 + \alpha_2 + \alpha_3$, in which case $\{\alpha, \beta\} = \{\alpha_2, \alpha_3\}$ are the two roots not of type $a$.

All we need to do is show that Case 2 is the only possible one. Notice that in Cases 1 and 3, we must have $n \geq 3$, as otherwise $\alpha$ and $\beta$ would be adjacent. So, in cases 1, 3, or 4, there must exist some $\alpha_0 \in \Pi^a_{X'} \cap \mathrm{Supp}(\gamma)$ which is adjacent to $\alpha$. In particular, we have $\alpha_0 \in \Pi^a_X$. On the other hand, Proposition 3.6.12 tells us that $\gamma' = c(\alpha + \beta)$ is a spherical root of $X$ for some $c \in \{1, 1/2\}$. So, $\alpha_0$ is adjacent to $\mathrm{Supp}(\gamma')$ but $\alpha_0 \notin \mathrm{Supp}(\gamma')$, which contradicts Lemma 4.6.2. $\square$

We now turn to classifying the possible behavior of spherical roots.

**Proposition 4.6.5.** *Let $X$ be a spherical variety, and let $\gamma \in \Psi_{G,X}$ be a spherical root.*

*(a) We have*
$$1 \leq \#(\mathrm{Supp}(\gamma) \cap (\Pi_G \setminus \Pi^a_X)) \leq 2.$$
*Moreover, there is at least one $\alpha \in \mathrm{Supp}(\gamma) \cap (\Pi_G \setminus \Pi^a_X)$ such that $\langle \alpha^\vee, \gamma \rangle > 0$.*

*(b) If $\mathrm{Supp}(\gamma)$ is not connected, then $\gamma = c(\alpha + \alpha')$ for some $c \in \{1, 1/2\}$, and $\alpha, \alpha' \in \Pi^d_X$ both move the same color of $X$.*

*(c) If $\mathrm{Supp}(\gamma)$ is connected and $\#(\mathrm{Supp}(\gamma) \cap (\Pi_G \setminus \Pi^a_X)) = 2$, then viewing $\mathrm{Supp}(\gamma)$ as a root subsystem of $\Pi_G$, exactly one of the following possibilities occurs.*

*(1) $\mathrm{Supp}(\gamma) \cong A_n$ for some $n \geq 2$, $\gamma = \alpha_1 + \cdots + \alpha_n$, and $\mathrm{Supp}(\gamma) \cap \Pi^a_X = \{\alpha_2, \ldots, \alpha_{n-1}\}$.*

*(2) $\mathrm{Supp}(\gamma) \cong B_n$ for some $n \geq 2$, $\gamma = \alpha_1 + \cdots + \alpha_n$, and $\mathrm{Supp}(\gamma) \cap \Pi^a_X = \{\alpha_2, \ldots, \alpha_{n-1}\}$.*

*(3) $\mathrm{Supp}(\gamma) \cong C_n$ for some $n \geq 2$, $\gamma = \alpha_1 + \alpha_n + 2\sum_{i=2}^{n-1} \alpha_i$, and $\mathrm{Supp}(\gamma) \cap \Pi^a_X = \{\alpha_3, \ldots, \alpha_n\}$.*

*(4) $\mathrm{Supp}(\gamma) \cong G_2$, $\gamma = \alpha_1 + \alpha_2$, and $\mathrm{Supp}(\gamma) \cap \Pi^a_X = \varnothing$.*

*Proof.* We will need to consult the list of possible spherical roots. More precisely, Corollary 3.6.17 gives us some semisimple simply connected group $G'$ with $\Pi_{G'} \subset \Pi_G$ and some prime rank-1 wonderful $G'$-variety $X'$ such that $\Psi_{G',X'} = \{\gamma\}$ and $\Pi^a_{X'} \subset \Pi^a_X$. The possible choices for $G'$ and $X'$ can be found in Table 1 of [Was96]. In particular, notice that any $\alpha \in \mathrm{Supp}(\gamma)$ must be an element of $\Pi_{G'}$, so we can read off combinatorial properties of $\alpha$ and $\gamma$ from Table 1. We remark that while Table 1 does not explicitly describe the set $\Pi^a_{X'}$, we can find it using this table in the following way: for each color $D$, of $X'$, there is a tuple in Column 4 of the table, and the first element of this tuple is the weight $\mu_D$ of Proposition 5.2.5. In particular, this element will be linear combination of the fundamental weights whose corresponding roots move $D$. So, checking the first element of each tuple in

Column 4 of the table tells us every simple root of $G'$ that moves some color of $X'$, and every root other than those is a root of type $a$ for $X'$.

Now, inspecting Table 1 of [Was96], we immediately see that $\mathrm{Supp}(\gamma)$ contains either 1 or 2 roots not of type $a$ for $X'$. Since $\Pi^a_{X'} \subset \Pi^a_X$, this implies that $\mathrm{Supp}(\gamma)$ contains at most 2 roots not of type $a$ for $X$. On the other hand, the table also shows that there is at least one $\alpha \in \mathrm{Supp}(\gamma)$ with $\langle \alpha^\vee, \gamma \rangle > 0$. By Lemma 3.6.14, we have $\alpha \notin \Pi^a_X$, so $\mathrm{Supp}(\gamma)$ contains at least one root not of type $a$. These observations prove (a). Moreover, if there are 2 roots in $\mathrm{Supp}(\gamma)$ that are not of type $a$ for $X$, then these must both be not of type $a$ for $X'$ as well (because $\Pi^a_{X'} \subset \Pi^a_X$). By inspecting Table 1 of [Was96], one finds that the 4 options in (c) are the only possibilities for $\mathrm{Supp}(\gamma)$ and $\gamma$ such that $\mathrm{Supp}(\gamma)$ is connected. (There is actually a 5th entry in the table that would work, namely Row 1D, in which $\mathrm{Supp}(\gamma) \cong D_3$ and $\gamma = \alpha_1 + \alpha_2 + \alpha_3$; but since $D_3 \cong A_3$, this case has exactly the same combinatorial properties as the case where $\mathrm{Supp}(\gamma) \cong A_n$ and $\gamma = \alpha_1 + \cdots + \alpha_n$ when $n = 3$.) In any of these cases, we have

$$\mathrm{Supp}(\gamma) \cap (\Pi_{G'} \setminus \Pi^a_{X'}) = \mathrm{Supp}(\gamma) \cap (\Pi_G \setminus \Pi^a_X) = 2,$$

and since $\mathrm{Supp}(\gamma) \subset \Pi_{G'}$ and $\Pi^a_{X'} = \Pi^a_X$, this implies that $\mathrm{Supp}(\gamma) \cap \Pi^a_X = \mathrm{Supp}(\gamma) \cap \Pi^a_{X'}$. So, the set $\mathrm{Supp}(\gamma) \cap \Pi^a_X$ can also be read off from the form of $\gamma$ in Table 1 of [Was96].

As for (c), the only possibilities with $\mathrm{Supp}(\gamma)$ not connected in [Was96, Table 1] are $G' = \mathrm{SL}_2 \times \mathrm{SL}_2$ and $\gamma = c(\alpha_1 + \alpha_1')$ for some $c \in \{1, 1/2\}$ (these are Rows 3 or 4 of Wasserman's table). In this case, Proposition 3.6.12 implies that both $\alpha_1$ and $\alpha_1'$ are roots of type $d$ moving the same divisor $D$.

□

The following is a sort of "classification" result for spherical roots satisfying some very particular conditions. These conditions may seem ad hoc, but they will arise naturally once we consider all the information that we can easily match up using the local structure theorem.

**Proposition 4.6.6.** *Let $X_1$ and $X_2$ be spherical varieties, and let $\gamma_h \in \Psi_{G,X_1}$ and $\gamma_2 \in \Psi_{G,X_2}$ be spherical roots. Suppose that the following conditions hold.*

1. *We have $\Pi^a_{X_1} = \Pi^a_{X_2}$.*

2. *The intersection $\mathrm{Supp}(\gamma_h) \cap \mathrm{Supp}(\gamma_2)$ contains some root $\alpha$ such that $\langle \alpha, \gamma_h \rangle > 0$ and $\langle \alpha, \gamma_2 \rangle > 0$.*

3. *The pair $(\alpha, \gamma_2)$ is not the pair $(\alpha, \gamma)$ in any of the statements from Lemma 4.6.3 applied to $X_2$. (In other words, $\alpha \in \Pi^d_{X_2}$, and there does not exist any $\beta \neq \alpha$ with $\mathcal{D}_{G,X_2}(\alpha) = \mathcal{D}_{G,X_2}(\beta)$.)*

4. *We have $\#(\mathrm{Supp}(\gamma_h) \cap (\Pi_G \setminus \Pi^a_{X_1})) = 2$, and $\mathrm{Supp}(\gamma_h)$ is connected.*

5. *We have $\gamma_2 \neq \gamma_h$.*

*Then, we have* $\alpha \in \Pi^d_{X_1} \cap \Pi^d_{X_2}$ *and* $\#(\mathrm{Supp}(\gamma_h) \cap (\Pi_G \setminus \Pi^a_{X_1})) = 2$, *and* $\mathrm{Supp}(\gamma_h)$ *is connected. Moreover, viewing* $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2)$ *as a root subsystem of* $\Pi_G$, *one of the four following possibilities holds.*

*(1)* $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2) \cong A_3$, $\gamma_2 = \alpha_1 + \alpha_2$, $\gamma_h = \alpha_2 + \alpha_3$, $\alpha = \alpha_2$.

*(2)* $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2) \cong D_n$ *for some* $n \geq 4$, $\gamma_2 = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}$, $\gamma_h = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$, $\alpha = \alpha_1$.

*(3)* $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2) \cong B_3$, $\gamma_2 = \alpha_1 + \alpha_2$, $\gamma_h = \alpha_2 + \alpha_3$, $\alpha = \alpha_2$.

*(4)* $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2) \cong B_3$, $\gamma_2 = \alpha_2 + \alpha_3$, $\gamma_h = \alpha_1 + \alpha_2$, $\alpha = \alpha_2$.

*Proof.* As in the proof of Lemma 4.6.3, we check the possibilities for $\gamma_h$ given in Table 1 of [Was96]. Corollary 3.6.17 gives us a semisimple simply connected group $G'_h$ and a prime rank-1 wondervul $G'_h$-variety $X'_h$ with $\Psi_{G'_h, X'_h} = \{\gamma_h\}$ and $\Pi^a_{X'_h} \subset \Pi^a_{X_1}$. Using Assumption 4 to apply Proposition 4.6.5c, there are only 4 possibilities for $G'_h$, $\Pi^a_{X'_h}$, and $\gamma_h$.

(A) $\Pi_{G'_h} = A_n$, $\gamma_h = \alpha_1 + \cdots + \alpha_n$, $\Pi^a_{X'_h} = \{\alpha_2, \ldots, \alpha_{n-1}\}$. Since there is an automorphism of $A_n$ given by $\alpha_i \mapsto \alpha_{n+1-i}$, it does not affect any of the conclusions of the proposition to assume that $\alpha = \alpha_1$ in this case.

(B) $\Pi_{G'_h} = B_n$, $\gamma_h = \alpha_1 + \cdots + \alpha_n$, $\Pi^a_{X'_h} = \{\alpha_2, \ldots, \alpha_{n-1}\}$. In this case, $\langle \alpha_n^\vee, \gamma_h \rangle = 0$, so we must have $\alpha = \alpha_1$.

(C) $\Pi_{G'_h} = C_n$, $\gamma_h = \alpha_1 + \alpha_n + 2\sum_{i=2}^{n-1} \alpha_i$, $\Pi^a_{X'_h} = \{\alpha_3, \ldots, \alpha_n\}$. In this case, $\langle \alpha_1^\vee, \gamma_h \rangle = 0$, so we must have $\alpha = \alpha_2$.

(G) $\Pi_{G'_h} = G_2$, $\gamma_h = \alpha_1 + \alpha_2$, $\Pi^a_{X'_h} = \varnothing$.

Step 1: We make a few preliminary observations about the root $\alpha$. First, Assumption 2 (along with Lemma 3.6.14) gives us $\alpha \notin \Pi^a_{X_1} = \Pi^a_{X_2}$, so Assumption 3 implies that $\alpha \in \Pi^d_{X_2}$. In particular, we have $\mathrm{Supp}(\gamma_2) \neq \{\alpha\}$ (otherwise $\gamma_2$ is either $\alpha$ or $2\alpha$ and $\alpha$ is either type $b$ or $c$ for $X_2$, see Theorem 3.6.10). Similarly, we have $\alpha \notin \Pi^b_{X_1} \cup \Pi^c_{X_1}$ (otherwise $\gamma_h$ is either $\alpha$ or $2\alpha$ by Lemma 4.6.3, contradicting the fact that $\mathrm{Supp}(\gamma_h)$ has at least 2 roots by Assumption 4). Thus, we also have $\alpha \in \Pi^d_{X_1}$. In particular, $\alpha$ is one of the elements of $\mathrm{Supp}(\gamma_h)$ which is not of type $a$ for $X_1$. By Assumption 2, there is another such element $\alpha' \in \mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X_1}$.

Step 2: We show that

$$\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2} = \mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1} = \mathrm{Supp}(\gamma_h) \cap \Pi^a_{X'_h}. \tag{4.6.1}$$

For the equality on the right, we note that $\Pi^a_{X'_h} \subset \Pi^a_{X_1}$ immediately gives $\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X'_h} \subset \mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1}$ and hence

$$\mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X_1} \subset \mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X'_h}.$$

On the other hand, the set $\mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X'_h}$ has 2 elements in each of Cases (A), (B), (C), and (G), and there are exactly 2 elements of $\mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X_1}$ (namely, $\alpha$ and $\alpha'$). So in fact, we have $\mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X_1} = \mathrm{Supp}(\gamma_h) \setminus \Pi^a_{X'_h}$ and hence $\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1} = \mathrm{Supp}(\gamma_h) \cap \Pi^a_{X'_h}$.

As for the left equality in (4.6.1), note that in each of Cases (A), (B), (C), and (G), the set

$$S_\alpha = \{\alpha'\} \bigcup (\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X'_h}$$

is connected. We claim that

$$S_\alpha \subset \mathrm{Supp}(\gamma_2).$$

Suppose this is false. Since $\alpha \in S_\alpha \cap \mathrm{Supp}(\gamma_2)$ and $S_\alpha$ is connected, must exist some $\alpha_0 \in S_\alpha \setminus \mathrm{Supp}(\gamma_2)$ which is adjacent to $\mathrm{Supp}(\gamma_2)$. But then $\alpha_0 \neq \alpha$, so the definition of $S_\alpha$ gives us

$$\alpha_0 \in \Pi^a_{X'_h} \subset \Pi^a_{X_1} = \Pi^a_{X_2},$$

which contradicts Lemma 4.6.2c. Thus, we have shown that

$$\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X'_h} \subset S_\alpha \subset \mathrm{Supp}(\gamma_2).$$

An analogous argument will also give us

$$\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2} \subset \mathrm{Supp}(\gamma_h)$$

if $\mathrm{Supp}(\gamma_2)$ is connected and $\#(\mathrm{Supp}(\gamma_2) \cap (\Pi_G \setminus \Pi^a_{X_2})) = 2$ (so that $\gamma_2$ satisfies Assumption 4 and hence also comes from one of Cases (A), (B), (C), or (G).) Note that Lemma 4.6.2b and Assumption 3 imply that $\mathrm{Supp}(\gamma_2)$ is connected, so the only other possibility (see Lemma 4.6.2a) is $\#(\mathrm{Supp}(\gamma_2) \cap (\Pi_G \setminus \Pi^a_{X_2})) = 1$. In this case, $\alpha$ is the unique root not of type $a$ in $\mathrm{Supp}(\gamma_2)$, so we have

$$\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2} = \mathrm{Supp}(\gamma_2) \setminus \{\alpha\}.$$

Since $\mathrm{Supp}(\gamma_2)$ is connected, the set on the righthand side here has at most 2 connected components, and $\alpha$ is adjacent to each of them. So, repeating the above arguments with these connected components in place of $S_\alpha$ will give the desired result.

Step 3: The root $\gamma_2$ comes from Table 1 of [Was96] as well. Indeed, Corollary 3.6.17 again gives us a semisimple simply connected group $G'_2$ and a prime rank-1 wondervul $G'_2$-variety $X'_2$ with $\Psi_{G'_2, X'_2} = \{\gamma_2\}$ and $\Pi^a_{X'_2} \subset \Pi^a_{X_2}$. We claim that the only possibility with $\Pi_{G'_2} \cong A_{n'}$ for some $n'$ is Row 1A of the table, where $\gamma_2 = \alpha'_1 + \cdots + \alpha'_n$ and $\Pi^a_{X'_2} = \{\alpha'_2, \ldots, \alpha'_{n-1}\}$. (This is the same as Case (A) above, but for $\gamma_2$ instead of $\gamma_h$.) Indeed, notice that for every choice of group $G'$ and spherical root $\gamma$ in Table 1 of [Was96], we have $\mathrm{Supp}(\gamma) = \Pi_{G'}$. It follows that if $\Pi_{G'_2} \cong A_{n'}$, then $G'$ is a group of type $A$ (the only other group on the table with root system isomorphic to $A_{n'}$ is $\mathrm{Sp}_6$ in Row 1D of the table, which is isomorphic to Row 1A of the table with $n = 3$, see our comments on Case (D) above). Besides Row 1A, the only other entries in the table for groups of type $A$ are Rows 2, 3, 4, 5A, and 6A.

Row 2 has $\gamma_2 = 2\alpha_1$, which forces $\alpha = \alpha_1$ and hence $\alpha \in \Pi^c_{X_2}$, contradicting Assumption 3. Rows 3 and 4 have $\gamma = c(\alpha + \beta)$ for some $c \in \{1, 1/2\}$ and some $\beta \in \Pi_{G'_2} \subset \Pi_G$ such that $\langle \alpha^\vee, \beta \rangle = 0$. It follows from Proposition 3.6.12 that $\mathcal{D}_{G,X_2}(\alpha) = \mathcal{D}_{G,X_2}(\beta)$ in this case, contradicting Assumption 3. Finally, in Rows 5A and 6A, the only root not of type $a$ in $\mathrm{Supp}(\gamma_2)$ (which is $\alpha$ by Assumption 2) is adjacent to two roots of type $a$. By (4.6.1), both of these roots of type $a$ are in $\mathrm{Supp}(\gamma_h)$. However, none of the Cases (A), (B), (C), (D), or (G) for $\gamma_h$ has any possibility for $\alpha$ (or indeed any root not of type $a$) adjacent to two roots of type $a$. So this is also impossible.

Step 4: We claim that

$$\mathrm{Supp}(\gamma_2) \not\subset \mathrm{Supp}(\gamma_h).$$

Suppose instead that $\mathrm{Supp}(\gamma_2) \subset \mathrm{Supp}(\gamma_h)$. We will check that Cases (A), (B), (C), and (G) are all impossible, so that there is no possible choice of $\gamma_h$. Note that

$$\mathrm{Supp}(\gamma_h) = \{\alpha, \alpha'\} \bigcup (\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1}).$$

Since $\alpha \in \mathrm{Supp}(\gamma_2)$, Assumption 5 along with (4.6.1) gives us

$$\mathrm{Supp}(\gamma_2) = \{\alpha\} \cup (\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1}).$$

Moreover, we have $\mathrm{Supp}(\gamma_2) \neq \{\alpha\}$, so there must exist some roots of type $a$ in $\mathrm{Supp}(\gamma_h)$. This immediately rules out Case (G) (which has no roots of type $a$). In Case (C), we have $\mathrm{Supp}(\gamma_h) \cong C_n$ for some $n > 2$ and $\mathrm{Supp}(\gamma_2) = \{\alpha_2, \ldots, \alpha_n\} \cong C_{n-1}$, so $X'_2$ must come from Rows 7C or 8C of [Was96, Table 1]. But for either of these rows, we would get $\alpha_2 \in \Pi^a_{X'_2} \subset \Pi^a_{X_2}$ is type $a$, contradicting the fact that $\alpha_2 = \alpha$ is type $d$ for $X_2$. Finally, in Cases (A) and (B), we have $\mathrm{Supp}(\gamma_h) \cong A_n$ for $n > 2$ and $\mathrm{Supp}(\gamma_2) = \{\alpha_1, \ldots, \alpha_{n-1}\} \cong A_{n-1}$. By Step 3, this implies that $\gamma_2 = \alpha_1 + \cdots + \alpha_{n-1}$, so that $\langle \alpha^\vee_{n-1}, \gamma_2 \rangle = 1 \neq 0$. This contradicts the fact that

$$\alpha_{n-1} \in \Pi^a_{X'_h} \subset \Pi^a_{X_1} = \Pi^a_{X_2}.$$

Step 5: In light of Step 4, there exists some $\beta \in \mathrm{Supp}(\gamma_2) \setminus \mathrm{Supp}(\gamma_h)$, and (4.6.1) (plus the fact that $\Pi^a_{X'_2} \subset \Pi^a_{X_2}$) gives us $\beta \notin \Pi^a_{X'_2}$. In particular, both $\alpha$ and $\beta$ are elements of $\mathrm{Supp}(\gamma_2)$ which are not of type $a$ for $X_2$. We noted in Step 2 that $\mathrm{Supp}(\gamma_2)$ is connected, so the conditions of Assumption 4 also hold for $\gamma_2$. Thus, $\gamma_2$ must also come from one of Cases (A), (B), (C), (D), or (G). As with $\gamma_h$, we need not consider Case (D) separately, since it is isomorphic to Case (A) with $n = 3$. Moreover, we can now rule out Case (G) for both $\gamma_h$ and $\gamma_2$. Indeed, the union $\mathrm{Supp}(\gamma_2) \cup \mathrm{Supp}(\gamma_h)$ is now a connected root subsystem of $\Pi_G$ containing at least three distinct simple roots (namely, $\alpha$, $\alpha'$, and $\beta$). If either $\gamma_2$ or $\gamma_h$ comes from Case (G), then $\mathrm{Supp}(\gamma_2) \cup \mathrm{Supp}(\gamma_h)$ contains a copy of $G_2$. But $G_2$ is not properly contained in any connected Dynkin diagram, so we must have $\mathrm{Supp}(\gamma_2) \cup \mathrm{Supp}(\gamma_h) \cong G_2$. This contradicts the fact that $G_2$ has only 2 simple roots. We conclude that each of $\gamma_2$ and $\gamma_h$ must come from either Cases (A), (B), or (C). For clarity, when viewing these cases for $\gamma_2$, we will write $n'$ in place of $n$ and $\alpha'_i$ in place of $\alpha_i$.

Step 6: We are left with 3 cases for each of $\gamma_2$ and $\gamma_h$, hence 9 total possibilities for the pair $(\gamma_2, \gamma_h)$. We will go through each of these in turn. First, we make a couple observations that will simplify this casework. Notice that Assumption 4 now applies to $\gamma_2$ (see Step 5) and Assumption 3 holds for $\gamma_h$ (by Assumption 4, Proposition 4.6.5, and Lemma 4.6.3). Thus, all of our assumptions and conditions on $\gamma_h$ and $\gamma_2$ are completely identical now, so swapping the cases for $\gamma_h$ and $\gamma_2$ will give us all the same combinatorial possibilities, just with $\gamma_2$ and $\gamma_h$ swapped. This cuts the number of cases we have to consider in half: for instance, if we take all the possibilities with $\gamma_h$ in Case (A) and $\gamma_2$ in Case (B) and swap $\gamma_2$ and $\gamma_h$, we will get all the the possibilities with $\gamma_h$ in Case (B) and $\gamma_2$ in Case (A). Also, note that each of $\mathrm{Supp}(\gamma_h)$ and $\mathrm{Supp}(\gamma_2)$ consists of type $a$ roots along with 2 roots not of type $a$. Since the type $a$ roots are the same in both cases by (4.6.1), we conclude that $\#\,\mathrm{Supp}(\gamma_h) = \#\,\mathrm{Supp}(\gamma_2)$. In all three possible cases, the number $n$ (resp. $n'$) is precisely $\#\,\mathrm{Supp}(\gamma_h)$ (resp. $\#\,\mathrm{Supp}(\gamma_2)$). Thus, we will have $n = n'$ no matter which cases we are in. Finally, we remark that Cases (B) and (C) are isomorphic when $n = 2$. We will see that $n$ must be 2 whenever one of $\gamma_h$ or $\gamma_2$ is in Case (B) or (C). This implies, for instance, that the possibilities for $\gamma_h$ in Case (A) and $\gamma_2$ in Case (B) must be isomorphic to the possibilities for $\gamma_h$ in Case (A) and $\gamma_2$ in Case (C).

Step 7: We go through each of the remaining cases one by one.

(A) Suppose that $\gamma_h$ is in Case (A) for some $n \geq 2$. Then, $\gamma_h = \alpha_1 + \cdots + \alpha_n$, and we may take $\alpha = \alpha_1$.

(AA) Suppose that $\gamma_2$ is in Case (A). Then, $\gamma_2 = \alpha'_1 + \cdots + \alpha'_n$, and we may take $\alpha = \alpha'_1$. Comparing the root systems $\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1}$ and $\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2}$ (which are equal by (4.6.1)), we see that $\alpha'_i = \alpha_i$ for all $1 \leq i \leq n-1$. Thus, both $\alpha'$ and $\beta$ are adjacent to $\alpha_{n-1}$. This gives us possibility (1) of the proposition statement if there are no roots of type $a$ in either $\mathrm{Supp}(\gamma_h)$ or $\mathrm{Supp}(\gamma_2)$ and possibility (2) of the proposition statement otherwise.

(AB) Suppose that $\gamma_2$ is in Case (B). Then, we have $\gamma_2 = \alpha'_1 + \cdots + \alpha'_n$ and $\alpha = \alpha'_1$. Suppose that $n > 2$. As in (AA) above, comparing the root systems $\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1}$ and $\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2}$, we obtain $\alpha'_i = \alpha_i$ for all $1 \leq i \leq n-1$. Thus, both $\alpha' = \alpha_n$ and $\beta = \alpha'_n$ are adjacent to $\alpha_{n-1}$, and $\alpha_{n-1}$ is also adjacent to $\alpha_{n-2}$ (which exists because $n > 2$). So, the Dynkin diagram of $\mathrm{Supp}(\gamma_h) \cup \mathrm{Supp}(\gamma_2)$ is connected but not simply laced (which only occurs for the diagrams of $B_m$, $C_m$, $F_4$, and $G_2$) and yet has a node (corresponding to $\alpha_{n-1}$) which is adjacent to 3 other nodes (which only occurs for the diagrams of $D_m$, $E_6$, $E_7$, and $E_8$). No such Dynkin diagram exists, so we must have $n = 2$. In this case, the only roots appearing in $\mathrm{Supp}(\gamma_2) \cup \mathrm{Supp}(\gamma_h)$ are $\alpha$, $\alpha'$, and $\beta$, and we know that $\mathrm{Supp}(\gamma_h) \cong A_2$ and $\mathrm{Supp}(\gamma_2) \cong B_2$. So, we obtain possibility (4) of the proposition statement.

(AC) Suppose that $\gamma_2$ is in Case (C). Then, we have $\gamma_2 = \alpha'_1 + 2\alpha'_2 + \cdots + 2\alpha'_{n-1} + \alpha_n$ and $\alpha = \alpha'_2$. Suppose that $n > 2$. Then, we have

$$\mathrm{Supp}(\gamma_2) \cap \Pi^a_{X_2} = \mathrm{Supp}(\gamma_2) \cap \Pi^a_{X'_2} \cong C_{n-2}.$$

(For the first equality here, one can argue just as we did with $\gamma_h$ in Step 2 to get the right equality in (4.6.1).) But $\mathrm{Supp}(\gamma_h) \cap \Pi^a_{X_1} \cong A_{n-2}$, which is not isomorphic to $C_{n-2}$, so this contradicts (4.6.1). We conclude that $n = 2$, so as noted in Step 6, we must get a possibility isomorphic to that for (AC) above.

(B) Suppose that $\gamma_h$ is in Case (B) for some $n \geq 2$. Then, $\gamma_h = \alpha_1 + \cdots + \alpha_n$ and $\alpha = \alpha_1$, so we must have $\alpha' = \alpha_n$.

(BA) By Step 6, this is the same as (AB) above but with $\gamma_2$ and $\gamma_h$ swapped. Since (AB) gave us possibility (4) in the proposition statement, this will give us possibility (3) in the proposition statement.

(BB) In this case, the Dynkin diagram for $\mathrm{Supp}(\gamma_1) \cup \mathrm{Supp}(\gamma_h)$ is connected, and has 2 edges between $\alpha'$ and $\alpha_{n-1}$ and 2 edges between $\beta$ and $\alpha'_{n-1}$. Since $\{\beta, \alpha'_{n-1}\} \neq \{\alpha', \alpha_{n-1}\}$ (both sets contain only one root of type $a$, and we know that $\beta \neq \alpha'$), these are two distinct intances of multiple edges, which never occurs on a connected Dynkin diagram. Thus, this case is impossible.

(BC) The same argument as in (BB) shows that this case is impossible. Alternately, using Step 6, any possibility here would have to be isomorphic to a possibility for (BB), and no such possibilities exist.

(C) Suppose that $\gamma_h$ is in Case (C) for some $n \geq 2$. Using our arguments in Step 6, there is almost nothing to do here. If $\gamma_2$ is in Case (A), then by Step 6, we obtain a possibility isomorphic to (AC) above, but with $\gamma_2$ and $\gamma_h$ swapped. Since (AC) gave us possibility (4) of the proposition statement, this gives us possibility (3). If $\gamma_2$ is in Case (B), then we obtain the same as (BC) above but with $\gamma_h$ and $\gamma_2$ swapped. Since (BC) is impossible, we conclude that $\gamma_2$ cannot be in Case (B) here. The only remaining option is $\gamma_2$ in Case (C). For this, we can argue exactly as we did in (BB) to see that this case is impossible.

□

### 4.6.b Matching up Spherical Roots

We are now ready to prove our main results in this section.

**Proposition 4.6.7.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that $X_1$ and $X_2$ are smooth and that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Let $\gamma \in \Psi_{G,X_1}$, and suppose that $\gamma \notin \Pi^b_{X_1}$. If either* $\mathrm{Supp}(\gamma)$ *is disconnected or*

$$\#(\mathrm{Supp}(\gamma) \cap (\Pi_G \setminus \Pi^a_X)) = 1,$$

*then we have $\gamma \in \Psi_{G,X_2}$.*

*Proof.* We reduce to the case where there is a unique color $D_0$ moved by $\mathrm{Supp}(\gamma)$, and that $\varphi_{D_0}(\gamma) > 0$. This condition is immediate from Proposition 4.6.5b if $\mathrm{Supp}(\gamma)$ is not connected. Suppose instead that there is a unique $\alpha \in \mathrm{Supp}(\gamma)$ not of type $a$. Proposition 4.6.5a tells us that $\langle \alpha^\vee, \gamma \rangle > 0$, and Lemma 4.6.3 implies that either $\alpha \in \Pi_X^d$ or $\mathrm{Supp}(\gamma) = \{\alpha\}$. In the former case, the color $D_0$ moved by $\alpha$ is the unique color moved by $\mathrm{Supp}(\gamma)$ (since every element of $\mathrm{Supp}(\gamma)$ other than $\alpha$ moves no colors), and we have $\varphi_{D_0}(\gamma) = \langle \alpha^\vee, \gamma \rangle > 0$. If instead $\mathrm{Supp}(\gamma) = \{\alpha\}$, then we have either $\alpha \in \Pi_{X_1}^b$ and $\gamma = \alpha$ or $\alpha \in \Pi_{X_1}^c$ and $\gamma = 2\alpha$ (see Theorem 3.6.10). The former case is impossible by assumption, and in the latter case, we have $\Pi_{X_1}^c = \Pi_{X_2}^c$ by Theorem 4.5.5, so $\alpha \in \Pi_{X_2}^c$ implies that $\gamma = 2\alpha \in \Psi_{G,X_2}$. In summary, we are done in the case $\mathrm{Supp}(\gamma) = \{\alpha\}$, and the desired color $D_0$ exists in every other case.

Now, Corollary 5.5.11 gives us some $G$-linearized ample invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Let $D_1$ be a $B$-stable ample effective divisor of $X_i$ such that $\mathcal{O}_{X_i}(D_1) = L_1$ (equivalently, let $D_1$ be the divisor cut out by some $B$-eigenvector of $H^0(X_1, L_1)$), and let and let $n_0$ be the coefficient of $D_0$ in $D_1$. For any $B$-divisor $D \in \mathcal{D}_{G,X_1}$, if $D$ is $G$-stable, then $\varphi_D \in \mathcal{V}(X_1)$ implies that $\varphi_D(-\gamma) \geq 0$. If instead $D$ is a color and $D \neq D_0$, then $D$ is moved by some root but is not moved by any element of $\mathrm{Supp}(\gamma)$, so Lemma 4.6.2c implies that $\varphi_D(-\gamma) \geq 0$ as well. On the other hand, we know that $\varphi_{D_0}(-\gamma) < 0$. Since the valuation $v_{D_0}$ is a discrete valuation, we have $\varphi_{D_0}(-\gamma) \in \mathbb{Z}$. After replacing $L_1$, $L_2$, and $D_1$ by some positive multiple, we may assume that

$$n_\alpha = -m\varphi_{D_\alpha}(-\gamma)$$

for some $m \in \mathbb{N}$. It follows that $D_1' = D_1 + m\,\mathrm{div}(-\gamma)$ is an effective divisor whose coefficient of $D_0$ is 0. Then, $D_1'$ corresponds to a $B$-eigenvector in $H^0(X_1, L_1)$ of some weight $\mu$, and $(X_1)_\mu \cap D_0 \neq \varnothing$. Since $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, we know that $\mu$ is a weight of a $B$-eigenvector in $H^0(X_2, L_2)$ as well. Moreover, we have $X_1(\mu) \cong X_2(\mu)$ by Theorem 4.4.6. On the other hand, the only color moved by $\mathrm{Supp}(\gamma)$ is $D_0$, which intersects $(X_1)_\mu$, so Proposition 4.4.1 implies that $\mathrm{Supp}(\gamma) \subset \Pi_{M_\mu}$. The same proposition then implies that

$$\gamma \in \Psi_{M_\mu, X_1(\mu)} = \Psi_{M_\mu, X_2(\mu)} \subset \Psi_{G,X_2}.$$

□

**Theorem 4.6.8.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that $X_1$ and $X_2$ are smooth and that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Let $\gamma \in \Psi_{G,X_1}$. One of the following possibilities must hold.*

*(1)* $\gamma \in \Psi_{G,X_2}$.

*(2)* $\gamma \in \Pi_{X_1}^b$.

*(3)* $\mathrm{Supp}(\gamma)$ *and* $\gamma$ *are given by one of the 4 possibilities in Proposition 4.6.5c.*

*Proof.* In light of Proposition 4.6.5c, this is just a rephrasing of Proposition 4.6.7. □

If we are willing to assume that $X_1$ and $X_2$ have "the same" colored fans, we can make a slight improvement on the above theorem.

**Theorem 4.6.9.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that $X_1$ and $X_2$ are smooth, that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and that there exists a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ that preserves colored fans. For any $\gamma_h \in \Psi_{G,X_1}$, one of the following holds.*

*(1)* $\gamma_h \in \Psi_{G,X_2}$.

*(2) There exists some $\gamma_2 \in \Psi_{G,X_2}$ such that $\gamma_h$ and $\gamma_2$ satisfy the assumptions of Proposition 4.6.6, so $\gamma_h$ and $\gamma_2$ are given by one of the 4 possibilities in that proposition.*

*Proof.* By Corollary 4.3.5 then, it suffices to show that $\mathcal{V}(X_1) = \mathcal{V}(X_2)$, or equivalently, that $\Psi_{G,X_1} = \Psi_{G,X_2}$. Let $\gamma_h \in \Psi_{G,X_1}$, and suppose that $\gamma_h \notin \Psi_{G,X_2}$.

Step 1: We claim that there exists some colored cone in $\mathscr{F}_{X_2}$ containing a color moved by $\operatorname{Supp}(\gamma_h)$. Since $\gamma_h \notin \Psi_{G,X_2}$, there exists some $v \in \mathcal{V}(X_2)$ such that $v(\gamma_h) > 0$. Because $X_2$ is complete, we have $v \in \mathcal{C}$ for some colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}_{X_2}$ (see Theorem 3.3.28). The cone $\mathcal{C}$ is generated by the $\varphi_D$ for $D$ in some subset of $\mathcal{D}_{G,X_2}$. It follows that there must be some choice of $D$ such that $\varphi_D(\gamma_h) > 0$. If $D$ is $G$-stable, then $\iota^{-1}(D)$ is also $G$-stable, so $\varphi_D = \varphi_{\iota^{-1}(D)} \in \mathcal{V}(X_1)$ and $\gamma \in \Psi_{G,X_1}$ implies that $\varphi_D(\gamma_h) \leq 0$. So $D$ must be a color of $X_2$, and if $\alpha$ is any root moving $D$, then Lemma 4.6.2 implies that $\langle \alpha^\vee, \gamma \rangle > 0$ and hence that $\alpha \in \operatorname{Supp}(\gamma)$.

Step 2: We claim that $\alpha \in \Pi^d_{X_2}$ and that $\alpha$ is the unique root moving $D$. If this is not the case, then we have either $\alpha \in \Pi^b_{X_2} \cup \Pi^c_{X_2}$, or $\alpha \in \Pi^d_{X_2}$ and there exists some $\beta \neq \alpha$ also moving $D$. In the former case, then we have $\gamma_h = c\alpha$ for some $c \in \{1, 2\}$ by Lemma 4.6.3. But Lemma 4.1.2 tells us that $\Pi^b_{X_1} = \Pi^b_{X_2}$ and $\Pi^c_{X_1} = \Pi^c_{X_2}$, so we would have $\gamma_h \in \Psi_{G,X_2}$, contradicting our assumptions. If instead $\alpha \in \Pi^d_{X_2}$ and there exists some $\beta \neq \alpha$ also moving $D$, then Lemma 4.6.3 gives us $\gamma_h = c(\alpha + \beta)$ for some $c \in \{1, 1/2\}$. In particular, $\operatorname{Supp}(\gamma_h)$ is not connected, so Proposition 4.6.7 implies that $\gamma_h \in \Psi_{G,X_2}$ again.

Step 3: We claim that there exists some $\gamma_2 \in \Psi_{G,X_2}$ such that $\varphi_D(\gamma_2) > 0$. If this is not the case, then $\varphi_D \in \mathcal{V}(X_2)$ implies that $(\mathbb{Q}_{\geq 0}\varphi_D, \{D\})$ is a face of the colored cone $(\mathcal{C}, \Delta)$, so we have $(\mathbb{Q}_{\geq 0}\varphi_D, \{D\}) \in \mathscr{F}_{X_2}$. Since $\iota$ preserves colored fans, it follows that $(\mathbb{Q}_{\geq 0}\varphi_D, \{\iota^{-1}(D)\}) \in \mathscr{F}_{X_1}$. In particular, $(\mathbb{Q}_{\geq 0}\varphi_D, \{\iota^{-1}(D)\})$ is a colored cone for the open $G$-orbit of $X_1$, which implies that $\mathbb{Q}_{>0}\varphi_D \cap \mathcal{V}(X_1) \neq \varnothing$, or equivalently, $\varphi_D \in \mathcal{V}(X_1)$. So, we must have $\varphi_D(\gamma) \leq 0$ for all $\gamma \in \Psi_{G,X_1}$, contradicting the fact that $\varphi_D(\gamma_h) > 0$.

Step 4: To complete the proof, we check that the assumptions of Proposition 4.6.6 apply to $\gamma_2$ and $\gamma_h$. Since $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, we see that $\Pi^a_{X_1} = \Pi^a_{X_2}$, that $\alpha \in \Pi^d_{X_1}$, and that $\alpha$ is the unique root moving $\iota^{-1}(D)$. In other words, Assumptions 1 and 3 of the proposition are satisfied. If Assumption 4 of the proposition did not hold, then Proposition 4.6.7 would imply that $\gamma_h \in \Psi_{G,X_2}$, and Assumption 5 also holds, because otherwise $\gamma_2 = \gamma_h$ would give $\gamma_h \in \Psi_{G,X_2}$. Finally, Assumption 2 of the proposition holds for $\alpha$ by Lemma 4.6.2a,b along with the fact that $\varphi_D(\gamma_h) > 0$ and $\varphi_D(\gamma_2) > 0$. $\square$

## 4.7 Comparing Colored Fans

In this section, we consider using our proof techniques involving the local structure theorem and the Knop conjecture to compare the colored fans of two smooth polarized spherical varieties. Unfortunately, these techniques do not get us very far for this task, because the colored cones in the colored fan of a spherical variety are typically not detectable using the local structure theorem.

More precisely, let $X$ be a spherical variety, and let $Y \subset X$ be a $G$-orbit corresponding to the colored cone $(\mathcal{C}, \Delta)$. We can apply the local structure theorem to $X_{B,Y}$ to get an isomorphism $X_{B,Y} \cong R_u(P) \times Z$, where $P \subset G$ is a parabolic subgroup, $M \subset P$ is the standard Levi subgroup, and $Z \subset X_{B,Y}$ is an $M$-stable closed subvariety. One can show that $Z$ has a unique closed $M$-orbit $Y'$, and that every $(B \cap M)$-divisor of $Z$ contains $Y'$ (cf. the proof of Proposition 4.7.1 below). By Proposition 3.2.3, the map $D \mapsto D \cap Z$ gives a bijection between $B$-divisors of $X$ intersecting $X_{B,Y}$ (i.e. $B$-divisors of $X$ containing $Y$, see Theorem 3.2.7) and the $(B \cap M)$-divisors of $Z$, and this bijection preserves $\varphi_D$. It follows that $(\mathcal{C}, \Delta') \in \mathscr{F}_Z$ is the colored cone corresponding to $Y'$, where

$$\Delta' = \{D \cap Z \mid D \in \Delta\} \setminus \mathcal{D}_{M,Z}^{M}.$$

(We note that removing elements of $\mathcal{D}_{M,Z}^{M}$ is necessary here, since some colors $D$ of $X$ containing $Y$ may actually be $M$-stable, in which case $D \cap Z$ contains $Y'$ but is not a color and so is not contained in $\Delta'$ by definition.)

This shows that colored cones on $X$ descend to colored cones on $Z$ in a nice way. However, the converse is not necessarily true. Indeed, given any $M$-orbit $Y_2'$ of $Z$ corresponding to a colored cone $(\mathcal{C}', \Delta')$, the natural attempt to relate $(\mathcal{C}', \Delta')$ to a colored cone of $X$ is to consider the $G$-orbit $Y_2 = G \cdot Y_2'$ (here viewing $Y_2'$ as a subscheme of $X \supset Z$). Every $(B \cap M)$-divisor $D'$ of $Z$ containing $Y_2'$ corresponds to a $B$-divisor $D = \overline{R_u(P) \cdot D'}$ of $X$ (see Proposition 3.2.3), but in general, there is no easy way to tell if $D$ actually contains $Y_2$. If we knew that $D$ was $G$-stable, then $D' \supset Y_2'$ would imply that $D \supset Y_2$, but it may well be the case that $D$ is not $G$-stable, even if $D'$ is $M$-stable.

Because of these issues, there is only one special case where we can use the local structure theorem to "match up" a colored cone between two spherical varieties: namely, the case where all the divisors we are dealing with are $G$-stable, or equivalently, when the colored cone $(\mathcal{C}, \Delta)$ in question has $\Delta = \varnothing$.

**Proposition 4.7.1.** *Let $X_1$ and $X_2$ be smooth spherical varieties such that $\Pi^b_{X_1} = \Pi^b_{X_2}$, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. For any colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}_{X_1}$ such that $\Delta = \varnothing$, we have $(\mathcal{C}, \varnothing) \in \mathscr{F}_{X_2}$.*

*Proof.* Let $Y_1 \subset X_1$ be the $G$-orbit corresponding to the colored cone $(\mathcal{C}, \varnothing)$. By Theorem 3.2.7, there exists some $(\mu, d) \in \Lambda^+(X_1, L_1)$ with $d > 0$ such that $(X_1)_\mu = (X_1)_{B,Y_1}$. Since $(X_1)_{B,Y_1}$ is affine, Theorem 4.4.6 gives us $X_1(\mu) \cong X_2(\mu)$ as $M_\mu$-varieties.

Now, let

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_2)_\mu\}$$

be the bijection of Lemma 4.5.4. Since $\Pi^b_{X_1} = \Pi^b_{X_2}$, any element $D$ of the domain of $\iota_\mu$ is a $G$-divisor if and only if $\iota_\mu(D)$ is. By the description of $(X_1)_{B,Y_1}$ in Theorem 3.2.7, the elements of the domain of $\iota_\mu$ are the $B$-divisors containing $Y_1$, and since $\Delta = \varnothing$, these are all $G$-divisors. It follows that the target of $\iota_\mu$ also consists entirely of $G$-divisors. On the other hand, $X_1(\mu)$ is affine and so has a unique closed $M_\mu$-orbit $Y_1'$ (Lemma 2.5.8). By arguing as in the proof of Theorem 3.2.7, we see that $Y_1' = Y_1 \cap X_1(\mu)$. Since the $(B \cap M_\mu)$-divisors of $X_1(\mu)$ are precisely those of the form $D \cap X_1(\mu)$ for $D \in \mathcal{D}_{G,X_1}$ (Proposition 3.2.3), it follows that every $(B \cap M_\mu)$-divisor of $X_1(\mu)$ contains $Y_1'$. The isomorphism $X_1(\mu) \cong X_2(\mu)$ then tells us that every $(B \cap M_\mu)$-divisor of $X_2(\mu)$ contains the unique closed $M_\mu$-orbit $Y_2'$ of $X_2(\mu)$.

Let $Y_2 = G \cdot Y_2'$ (here viewing $Y_2'$ as a subscheme of $X_2$). We claim that the $B$-divisors of $X_2$ containing $Y_2$ are precisely those in the target of $\iota_\mu$. For any $D \supset Y_2$, we have $D \cap (X_2)_\mu \supset Y_2' \neq \varnothing$. Conversely, if $D \cap (X_2)_\mu \neq \varnothing$, then $D = \overline{R_u(P_\mu) \cdot D'}$ for some $(B \cap M_\mu)$-divisor $D' \subset X_2(\mu)$ (see Proposition 3.2.3). We showed above that $D' \supset Y_2'$, so $D \supset Y_2'$. We also showed above that $D$ is $G$-stable, so this implies that $D \supset Y_2$.

Consider the colored cone $(\mathcal{C}_{Y_2}, \Delta_{Y_2})$ corresponding to $Y_2$. Our above claim in particular tells us that every $B$-divisor containing $Y_2$ is $G$-stable, so $\Delta_{Y_2} = \varnothing$. Moreover, $\mathcal{C}_{Y_2}$ is the cone in $N(X_1) = N(X_2)$ generated by the $\varphi_D$ for all $D \supset Y_2$, i.e. for all $D$ in the target of $\iota_\mu$. Since $\varphi_D = \varphi_{\iota_\mu(D)}$ for all $D$, this is equal to the cone generated by $\varphi_D$ for all $D$ in the domain of $\iota_\mu$, which by the description of $(X_1)_\mu = (X_1)_{B,Y_1}$ in Theorem 3.2.7 is precisely the cone $\mathcal{C}$. Thus, we have

$$(\mathcal{C}, \varnothing) = (\mathcal{C}_{Y_2}, \Delta_{Y_2}) \in \mathscr{F}_{X_2},$$

as desired. □

If $X_1$ and $X_2$ are toroidal in the situation of Proposition 4.7.1, then the proposition immediately implies that any $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ must preserve colored fans (if it exists). However, $X_i$ being toroidal is equivalent to the condition that $\Delta = \varnothing$ for all $(\mathcal{C}, \Delta) \in \mathscr{F}_{X_i}$. So, whenever one of the $X_i$ is not toroidal, there is no way to use Proposition 4.7.1 to obtain a $\mathcal{D}$-equivalence that preserves colored fans. In other words, our proof techniques have failed to give us the "equality" on colored fans that we were looking for. We do not expect that any other proof techniques will fare better: indeed, we will see in Examples 4.9.3 and 4.9.4 many examples of $X_1$ and $X_2$ as in Proposition 4.7.1 such that no $\mathcal{D}$-equivalence preserves colored fans. In fact, in these examples, Proposition 4.7.1 does apply to some of the colored cones in the relevant colored fans (just not to all of the colored cones), and the colored cones it does not apply to have just a single color in them (i.e. we have $\#\Delta = 1$ instead of $\Delta = \varnothing$). Thus, while Proposition 4.7.1 is a relatively limited result, we do not expect that a better result is possible.

There is one other attempt at comparing colored fans that we have considered. Instead of using the local structure theorem and the Knop conjecture, the idea is to use the combinatorial criterion for ampleness in Theorem 3.7.13. This criterion depends only on the valuations of $B$-divisors of a spherical variety $X$ and on the maximal colored cones in $\mathscr{F}_X$ (under the partial order given by the relation of "being a face"), so we obtain the following corollary.

**Corollary 4.7.2.** *Let $X_1$ and $X_2$ be spherical varieties, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence which* preserves maximal colored cones*, in the following sense: the maximal colored cones of $\mathscr{F}_{X_2}$ are precisely the pairs $(\mathcal{C}, \iota(\Delta))$, where $(\mathcal{C}, \Delta)$ is a maximal colored cone of $\mathscr{F}_{X_1}$. Let $E_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ be a $B$-stable (Weil) divisor on $X_1$, and let $E_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$.*

- *(a) $E_1$ is Cartier if and only if $E_2$ is Cartier.*
- *(b) If $X_1$ and $X_2$ are complete, then $E_1$ is ample (and Cartier) if and only if $E_2$ is ample (and Cartier).*

**Remark 4.7.3.** It may be the case that completeness is unnecessary in part (b) of the above corollary. We have included it mainly because completeness allows for a clear presentation of the combinatorial criterion for ampleness, and we have not attempted to remove it because we are mainly interested in the projective case, anyway.

*Proof.* Suppose that $E_1$ is Cartier. For each closed $G$-orbit $Y_1 \subset X_1$, let $\mu_{E_1,Y_1} \in \Lambda(X_1) = \Lambda(X_2)$ be as in Lemma 3.7.2 applied to the Cartier divisor $E_1$. Since $\iota$ preserves maximal colored cones, which are the ones corresponding to closed $G$-orbits (Proposition 3.3.24), for every closed $G$-orbit $Y_2 \subset X_2$, there exists some closed $G$-orbit $Y_1 \subset X_1$ such that $\iota$ restricts to a bijection

$$\mathcal{D}_{Y_1} \xrightarrow{\sim} \mathcal{D}_{Y_2}.$$

(To be precise: $Y_1$ is the closed $G$-orbit such that $(\mathcal{C}_{Y_2}, \Delta_{Y_2}) = (\mathcal{C}_{Y_1}, \iota(\Delta_{Y_1}))$.) Set $\mu_{E_2,Y_2} = \mu_{E_1,Y_1}$ for all such $Y_2$ and $Y_1$. Then, for any $G$-orbit $Y \subset X_2$, we define $\mu_{E_2,Y} \in \Lambda(X_2)$ by picking any some closed $G$-orbit $Y_2 \subset X_2$ such that $Y_2 \subset \overline{Y}$ and setting $\mu_{E_2,Y} = \mu_{E_2,Y_2}$. Since $\mathcal{D}_Y \subset \mathcal{D}_{Y_2}$ for such a choice of $Y$ and $Y_2$, it follows immediately that $\mu_{E_2,Y}$ satisfies the condition in Lemma 3.7.2 for $E_2$ and any choice of $G$-orbit $Y \subset X$, so the lemma implies that $E_2$ is Cartier. The proof that $E_1$ is Cartier if $E_2$ is Cartier is the same, but with $X_1$ and $X_2$ swapped.

For the proof of (b), we may assume that $E_1$ and $E_2$ are both Cartier. In this case, let $(\ell_Y) \in \mathrm{PL}(X_1)$ be the piecewise linear function corresponding to $E_1$. Since $\iota$ preserves maximal colored cones and every $B$-divisor in a colored cone is contained in a maximal colored cone (say, by Proposition 3.3.24), we see that $\iota$ induces a bijection $\Delta^\circ(X_1) \xrightarrow{\sim} \Delta^\circ(X_2)$ on divisors contained in no colored cone. Moreover, as above, for each closed $G$-orbit $Y_1 \subset X_1$, there exists a closed $G$-orbit $Y_2 \subset X_2$ such that $\iota$ induces a bijection $\mathcal{D}_{Y_1} \xrightarrow{\sim} \mathcal{D}_{Y_2}$, and this bijection identifies $\ell_{Y_1}$ with some linear function $\ell_{Y_2} : \mathcal{D}_{Y_2} \to \mathbb{Z}$. By arguing as in the proof of (a) above, we conclude that the $\ell_{Y_2}$ determine an element of $\mathrm{PL}(X_2)$ which corresponds to $E_2$. Since $\iota$ identifies the families $\ell_{Y_1}$ with the families $\ell_{Y_2}$ and identifies $\Delta^\circ(X_1)$ with $\Delta^\circ(X_2)$, and all of these identifications preserve $\varphi_D$, the statement now follows immediately from the ampleness criterion in Theorem 3.7.13. □

We conjecture that the above corollary might admit a nice converse.

**Conjecture 4.7.4.** *Let $X_1$ and $X_2$ be locally factorial complete spherical varieties, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence. Suppose that $\iota$* preserves ample cones*, in the following sense: for any $B$-stable divisor $E_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ on $X_1$, the divisor $E_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$ on $X_2$ is ample if and only if $E_1$ is ample. Then, $\iota$ preserves maximal colored cones.*

The main content of Corollary 4.7.2 is that the maximal colored cones of a complete spherical variety $X$ determine the ample cone of $X$; conversely, Conjecture 4.7.4 asserts that the ample cone determines the maximal colored cones. Since we have assumed $X_1$ and $X_2$ are locally factorial in the conjecture, every Weil divisor is Cartier. Thus, under the assumptions of the conjecture, the ampleness criterion in Theorem 3.7.13 gives many combinatorial conditions that must "match up" for $X_1$ and $X_2$. The main difficulty is that what these conditions are actually depends on which divisors are contained in the same (maximal) colored cone, so it is a subtle combinatorial question to ask whether these constraints actually force the maximal cones on $X_1$ and $X_2$ to be equal.

As far as examples go, we know of know conterexample to Conjecture 4.7.4. However, we will see in Example 4.9.3 that even for two smooth polarized spherical varieties $(X_1, L_1)$ and $(X_2, L_2)$ such that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, it may be the case that no $\mathcal{D}$-equivalence preserves ample cones. Moreover, we will see in Example 4.9.4 that even if a $\mathcal{D}$-equivalence that preserves ample cones does exist, it may be the case that this $\mathcal{D}$-equivalence preserves maximal colored cones but does not preserve colored fans. In summary, while Conjecture 4.7.4 would, if true, be an interesting way to "equate" some colored cones on $X_1$ and $X_2$, the conjecture still would not give us an "equality" on the entire colored fans, and the conjecture would not even necessarily apply in the situations of interest to us.

## 4.8 Proving Two Spherical Varieties Are Isomorphic

Now that we have considered each of the combinatorial invariants in the classification of spherical varieties, we are ready to combine our results to prove certain results about when two spherical varieties (or polarized spherical varieties) are isomorphic. These results will largely parallel Theorem 4.1.9, which tells us what the general classification of spherical varieties says about when two spherical varieties are isomorphic. The difference in our results is that we will replace some of the equalities on combinatorial data in Theorem 4.1.9 with an equality on weight monoids of the form $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.

We have actually already seen a result of this form: namely, Corollary 4.3.5, which tells us that the weight monoid $\Lambda^+(X, L)$ and the valuation cone $\mathcal{V}(X)$ determine a polarized spherical variety $(X, L)$ up to $G$-equivariant isomorphism. When $X$ is smooth, we can actually detect some spherical roots of $X$ from the weight monoid $\Lambda^+(X, L)$ using Theorem 4.6.8, which leads to a slightly nicer statement than that of Corollary 4.3.5.

**Corollary 4.8.1.** *Let $X_1$ and $X_2$ be smooth projective spherical varieties, and let $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ be the set of all $\gamma \in \Psi_{G,X_i}$ such that either $\gamma \in \Pi^b_{X_i}$ or $\gamma$ satisfies one of the 4 possibilities in Proposition 4.6.5c. The following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$, and there exist $G$-linearized invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $L_1$ and $L_2$ are both ample and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

*Proof.* Given a $G$-equivariant isomorphism $i : X_1 \xrightarrow{\sim} X_2$, we may pick any $G$-linearized ample invertible sheaf $L_2$ on $X_2$ (which exists by Theorem 2.6.11), and then $L_1 = i^*L_2$ is a $G$-linearized ample invertible sheaf with $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Moreover, we have $\Psi_{G,X_1} = \Psi_{G,X_2}$, and since the subset $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ is just the subset of spherical roots that satisfy certain combinatorial conditions, this implies that $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$.

Conversely, suppose that (ii) holds. Theorem 4.6.8 implies that

$$\Psi_{G,X_1} \setminus \Psi^{exc}_{G,X_1} = \Psi_{g,X_2} \setminus \Psi^{exc}_{G,X_2},$$

so we have $\Psi_{G,X_1} = \Psi_{G,X_2}$, or equivalently, $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. Then, Corollary 4.3.5 implies that $(X_1, L_1) \cong (X_2, L_2)$ as polarized spherical varieties, which in particular means that $X_1 \cong X_2$ as $G$-varieties. □

As discussed in Section 1.2, Corollary 4.8.1 is essentially optimal, in the following sense: for any $\gamma$ that lies in $\Psi^{exc}_{G,X}$ for some $X$, there exists some choice of smooth polarized spherical varieties $(X_1, L_1)$ and $(X_2, L_2)$ with $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ such that $\gamma \in \Psi^{exc}_{G,X_1}$ but $\gamma \notin \Psi_{G,X_2}$. For the case where $\gamma = \alpha \in \Pi_G$ (so that $\gamma$ is necessarily a root of type $b$), see Examples 4.9.1 and 4.9.2. For the case where $\gamma$ has any of the forms in Proposition 4.6.5c, see Example 4.9.4. In the first two examples, we see that $X_1$ and $X_2$ are not $\mathcal{D}$-equivalent, and in the third example, we see that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, but that no $\mathcal{D}$-equivalence preserves colored fans. Thus, there is some piece of combinatorial data besides the spherical roots $\Psi_{G,X_i}$ that is unequal in all of these examples, which suggests that Corollary 4.3.5 is also relatively optimal.

In the remainder of this section, we consider a few special cases in which we can improve on Corollary 4.8.1. Since the main issue in Corollary 4.8.1 is the possibility that $\Psi^{exc}_{G,X_1} \neq \Psi^{exc}_{G,X_2}$, these special cases will be ones in which we can get the equality $\Psi_{G,X_1} = \Psi_{G,X_2}$ some other way. We first do this for a couple of standard "nice" types of spherical varieties (namely, horospherical, and toroidal varieties).

**Theorem 4.8.2.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties, and suppose that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

*(a) If $X_1$ and $X_2$ are horospherical, then $(X_1, L_1) \cong (X_2, L_2)$ as polarized $G$-varieties.*

*(b) If $X_1$ and $X_2$ are toroidal and $\mathcal{D}$-equivalent, then $(X_1, L_1) \cong (X_2, L_2)$ as polarized $G$-varieties.*

*(c) If $X_1$ and $X_2$ are smooth and toroidal and $\Pi^b_{X_1} = \Pi^b_{X_2}$, then $(X_1, L_1) \cong (X_2, L_2)$ as polarized $G$-varieties.*

**Remark 4.8.3.** The statement on horospherical varieties in the above theorem is in some sense a generalization of a result on projective toric varieties (which are a particular type of horospherical variety, see the discussion following Definition 3.4.11). It is a standard result that projective toric varieties are classified by a convex polytope, which is an invariant of the variety (see e.g. [Oda88, Theorem 2.22]). One can generalize the construction of a convex polytope from a toric variety to the case of projective spherical varieties, see [Bri97, Section 5.3], and in this generalization, the polytope is completely determined by a weight monoid $\Lambda^+(X, L)$ for some $G$-linearized ample invertible sheaf $L$ on $X$. Thus, the the statement in Theorem 4.8.2 that horospherical varieties are determined up to $G$-equivariant isomorphism by a weight monoid $\Lambda^+(X, L)$ is essentially a generalization of this classification of projective toric varieties by convex polytopes.

*Proof.* For (a), we have $\Psi_{G,X_1} = \Psi_{G,X_2} = \varnothing$, hence $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. So, Corollary 4.3.5 immediately gives us the result.

For (b), the fact that $X_1$ and $X_2$ are toroidal implies that the cones in their colored fan are made up of their $G$-divisors. The valuations of these $G$-divisors are in bijection because $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, and since $X_1$ and $X_2$ are complete, these divisors generate their valuation cones (see Theorem 3.3.28). We conclude that $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. Corollary 4.3.5 now tells us that $(X_1, L_1)$ and $(X_2, L_2)$ are isomorphic.

The hypotheses of statement (c) imply that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, see Theorem 4.5.5. Thus, statement (c) follows immediately from (b). □

**Remark 4.8.4.** The method of proof of (b) in the above theorem also shows that two complete toroidal varieties which are $\mathcal{D}$-equivalent are $G$-equivariantly birational.

We can also obtain slightly nicer results in low rank, primarily because $\Lambda(X_1) = \Lambda(X_2)$ greatly constrains what the spherical roots of $X_1$ and $X_2$ can be in this case.

**Theorem 4.8.5.** *Let $X_1$ and $X_2$ be spherical varieties. Suppose that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent.*

*(a) If $r(X_1) = 0$ (equivalently, $r(X_2) = 0$), then $X_1 \cong X_2$ as $G$-varieties.*

*(b) If $r(X_1) = 1$ (equivalently, $r(X_2) = 1$) and $\mathcal{V}(X_1)$ and $\mathcal{V}(X_2)$ are strictly convex, then $X_1$ and $X_2$ are $G$-equivariantly birational.*

*(c) If $r(X_1) = 1$ (equivalently, $r(X_2) = 1$), the $X_i$ are complete, and there exists a $\mathcal{D}$-equivalence $\iota : X_1 \xrightarrow{\sim} X_2$ preserving colored fans, then $X_1 \cong X_2$ as $G$-varieties.*

*Proof.* Note that $r(X_1) = r(X_2)$ because $\Lambda(X_1) = \Lambda(X_2)$ (this is part of the definition of a $\mathcal{D}$-equivalence). If $r(X_1) = r(X_2) = 0$, then $\mathcal{V}(X_i) = N(X_i) = 0$ in particular gives $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. So $X_1$ and $X_2$ are $G$-equivariantly birational. Moreover, there is no nontrivial colored fan for the open $G$-orbits of $X_1$ and $X_2$ (because $N(X_i) = 0$), so both $X_1$ and $X_2$ are equal to their own open $G$-orbit, and $X_1 \cong X_2$.

Now suppose that $r(X_1) = r(X_2) = 1$. Then, $\#\Psi_{G,X} \leq 1$. There are three possible cases:

1. If $\Psi_{G,X_1} = \Psi_{G,X_2} = \varnothing$, then $\mathcal{V}(X_1) = N(X_1) = N(X_2) = \mathcal{V}(X_2)$, so $X_1$ and $X_2$ are $G$-equivariantly birational. If in addition the assumptions of (c) hold, then $X_1$ and $X_2$ also have the same colored fan, so they are $G$-equivariantly isomorphic. Note that this case is impossible under the assumptions of (b), since neither $\mathcal{V}(X_1)$ nor $\mathcal{V}(X_2)$ is strictly convex.

2. If $\Psi_{G,X_i} \neq \varnothing$ for $i \in \{1, 2\}$, then let $\gamma_i$ be the unique spherical root of $X_i$. The $\gamma_i$ are indivisible elements of the rank-1 lattice $\Lambda(X_1) = \Lambda(X_2)$, so we must have $\gamma_1 = \pm\gamma_2$. But spherical roots are always linear combinations of simple roots with nonnegative coefficients (see Remark 3.4.8), so we can never have $\gamma_1 = -\gamma_2$. So $\gamma_1 = \gamma_2$ implies that $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ and hence that $X_1$ and $X_2$ are $G$-equivariantly birational. Once again, under the assumptions of (c), both $X_1$ and $X_2$ have the same colored fan, so we get $X_1 \cong X_2$.

3. After swapping $X_1$ and $X_2$ if necessary, we are left with the case where $\Psi_{G,X_1} \neq \varnothing$ but $\Psi_{G,X_2} = \varnothing$. Note that this case is impossible under the assumptions of (b), since $\mathcal{V}(X_2) = N(X_2)$ is not strictly convex. So, suppose that the assumptions of (c) hold. Since $N(X_1) = N(X_2) \cong \mathbb{Q}$, the cone $\mathcal{V}(X_1)$ must be one of the rays $\mathbb{Q}_{\geq 0}$ or $\mathbb{Q}_{\leq 0}$. Since $X_2$ is complete and $\mathcal{V}(X_2) = N(X_2)$, there must exist some colored cone of the form $(-\mathcal{V}(X_1), \Delta)$ in the colored fan $\mathscr{F}_{X_2}$. Our assumptions then imply that $-\mathcal{V}(X_1)$ is a cone for some colored cone in $\mathscr{F}_{X_1}$. But $-\mathcal{V}(X_1)$ contains no element of $\mathcal{V}(X_1)$ in its interior, so this contradicts the definition of a colored cone. Thus, this case is impossible.

In summary: under the assumptions of (b), only Case 2 is possible, and in this case, $X_1$ and $X_2$ are $G$-equivariantly birational; and under the assumptions of (c), Cases 1 and 2 are possible, and $X_1 \cong X_2$ as $G$-varieties in both cases under the assumptions of (c). □

**Corollary 4.8.6.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that $X_1$ and $X_2$ are smooth, that $\mathcal{V}(X_1)$ and $\mathcal{V}(X_2)$ are strictly convex, and that one of $r(X_1)$ and $r(X_2)$ is $\leq 1$. The following are equivalently.*

*(i) $(X_1, L_1) \cong (X_2, L_2)$ as polarized spherical varieties.*

*(ii) We have $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\Pi^b_{X_1} = \Pi^b_{X_2}$.*

*Proof.* If $(X_1, L_1) \cong (X_2, L_2)$, then $\Pi^b_{X_1} = \Pi^b_{X_2}$ because $X_1 \cong X_2$ as $G$-varieties, and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ because there exists a $G$-equivariant isomorphism $i : X_1 \to X_2$ such that $i^* L_2 \cong L_1$ as $G$-linearized ample invertible sheaves. Conversely, if (ii) holds, then $\Lambda(X_1) = \Lambda(X_2)$ (see Proposition 2.5.2) implies that $r(X_1) = r(X_2) \leq 1$. Moreover, $X_1$ and $X_2$ are $\mathcal{D}$-equivalent by Theorem 4.5.5, hence $G$-equivariantly birational by Theorem 4.8.5, so $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. Then, Corollary 4.3.5 implies that $(X_1, L_1) \cong (X_2, L_2)$. □

## 4.9 Counterexamples

In this section, we give a few examples of polarized spherical varieties $(X_1, L_1)$ and $(X_2, L_2)$ such that $X_1$ and $X_2$ are smooth and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, but where some of the combinatorial invariants that classify spherical varieties (see Theorem 4.1.9) are not equal for $X_1$ and $X_2$. These examples indicate that our results in this chapter are essentially the best that one can hope for.

Recall from Theorem 4.1.9 that there are three pieces of combinatorial data which, together, classify a spherical variety up to $G$-equivariant isomorphism: the $B$-divisors (for which "equality" of combinatorial data means a $\mathcal{D}$-equivalence), the spherical roots (for which equality is a literal equality of sets), and colored fans (for which "equality" means a $\mathcal{D}$-equivalence that preserves colored fans). If any of these pieces of data is not equal between $X_1$ and $X_2$, then we must necessarily have $\Psi_{G,X_1} \neq \Psi_{G,X_2}$, as otherwise $(X_1, L_1) \cong (X_2, L_2)$ by Corollary 4.3.5. Moreover, it makes no sense to have colored fans be "equal" but $B$-divisors not be "equal," as our notion of "equality" on colored fans relies on the existence of a $\mathcal{D}$-equivalence. With these constraints in mind, we are left with the following possibilities for examples in which $(X_1, L_1) \not\cong (X_2, L_2)$ (or equivalently, in which some of the combinatorial invariants in Theorem 4.1.9 are not equal):

1. An example in which $X_1$ and $X_2$ are not $\mathcal{D}$-equivalent and $\Psi_{G,X_1} \neq \Psi_{G,X_2}$.

2. An example in which $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, but $\Psi_{G,X_1} \neq \Psi_{G,X_2}$ and no $\mathcal{D}$-equivalence preserves colored fans.

3. An example in which there exsits a $\mathcal{D}$-equivalence between $X_1$ and $X_2$ that preserves colored fans, but $\Psi_{G,X_1} \neq \Psi_{G,X_2}$.

Of these 3 possibilities, we will give examples satisfying possibilities 1 and 2.

### 4.9.a Examples With Different $B$-Divisors

We begin with examples in which the spherical varieties $X_1$ and $X_2$ are not $\mathcal{D}$-equivalent. These examples have $\#\Pi^b_{X_1} = 1$ and $\Pi^b_{X_2} = \varnothing$, and $X_1$ and $X_2$ are even rank-1 toroidal varieties. By contrast, we have seen in Theorem 4.5.5 that when $\Pi^b_{X_1} = \Pi^b_{X_2}$, then $X_1$ and $X_2$ are necessarily $\mathcal{D}$-equivalent. Our examples thus indicate that, even for "very nice" spherical varieties, the assumption $\Pi^b_{X_1} = \Pi^b_{X_2}$ seems to be necessary in Theorem 4.5.5.

**Example 4.9.1.** Let $G = \mathrm{SL}_2$ acting on $X = \mathbb{P}^1 \times \mathbb{P}^1$ diagonally via the action on $\mathbb{P}^1$ given in Example 2.4.19. In other words, the action of $G$ on $X$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot ([x : y], [w : z]) = ([ax + by : cx + dy], [aw + bz : cw + dz]).$$

We take $B \subset G$ to be the Borel subgroup of upper triangular matrices and $T \subset B$ to be the subgroup of diagonal matrices. We have seen that $X$ is wonderful (Example 3.5.20). Moreover, we calculated all the $B$-divisors of $X$ and their valuations in Example 3.1.16. The two colors of $X$ are

$$D_1 = \mathbb{P}^1 \times \{[1 : 0]\} \quad \text{and} \quad D_2 = \{[1 : 0]\} \times \mathbb{P}^1,$$

and the two $G$-orbits of $X$ are the diagonal $Y$ and its complement. Moreover, we have $\Lambda(X) = \mathbb{Z}\cdot\alpha_1$, where $\alpha_1 \in \Pi_G$ is the unique simple root, and we have $\varphi_{D_1}(\alpha_1) = \varphi_{D_2}(\alpha_1) = 1$ but $\varphi_Y(\alpha_1) = -1$. The colored fan of $X$ contains a single colored cone with no colors in it (because $X$ is wonderful, hence simple and toroidal), and the cone in this fan is generated by the valuations of $G$-divisors containing the unique closed $G$-orbit $Y$. But $Y$ itself is the only such divisor, so the colored fan of $X$ is

$$\mathscr{F}_X = \{(\mathbb{Q}_{\geq 0}\varphi_Y, \varnothing)\}.$$

Since $X$ is a standard embedding, we must have $\mathcal{V}(X) = \mathbb{Q}_{\geq 0}\varphi_Y$. Also, $X$ has two colors, each of which must be moved by some simple root. But the only simple root of $G$ is $\alpha_1$. So, we have $\mathcal{D}_{G,X}(\alpha_1) = \{D_1, D_2\}$, which implies that $\Pi_X^b = \{\alpha_1\}$.

For our example, we wish to construct a certain line bundle $L$ on $X$ and write down the weight monoid $\Lambda^+(X, L)$. In fact, we will compute $\Lambda^+(X, L)$ for all $G$-linearized ample invertible sheaves $L$ on $X$. First, we determine what choices of $L$ there are. Note that the global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ contain a $B$-eigenvector $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{(B)}$ which vanishes at the $B$-fixed point $[1 : 0]$. Write $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for the projections maps. The pullback $\mathrm{pr}_1^* s$ is a global section of $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ which vanishes on $D_2$, so we have $\mathcal{O}_X(D_2) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1)$. Considering $\mathrm{pr}_2$ instead in this argument, we get $\mathcal{O}_X(D_1) = \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(1)$. It follows that every line bundle on $X$ is isomorphic to $\mathcal{O}_X(aD_1 + bD_2)$ for some $a, b \in \mathbb{Z}$. Such a line bundle is ample if and only if its pullbacks to $D_1$ and $D_2$ are ample, and since these pullbacks are $\mathcal{O}_{\mathbb{P}^1}(a)$ and $\mathcal{O}_{\mathbb{P}^1}(b)$ (respectively), this holds if and only if $a, b > 0$. For $i \in \{1, 2\}$, the map $\mathrm{pr}_i$ is $G$-equivariant, so the $G$-linearization of $\mathcal{O}_{\mathbb{P}^1}(n)$ given in Example 2.4.19 induces a $G$-linearization on $\mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(n)$ for any $n$ such that pulling back global sections defines a $G$-equivariant map

$$\mathrm{pr}_i^* : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \to H^0(X, \mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(n)).$$

This, in turn, induces a $G$-linearization on $\mathcal{O}_X(aD_1 + bD_2)$ for all $a, b \in \mathbb{Z}$ (see Lemma 2.4.13b. Because no nontrivial character of $G$ exists, Corollary 2.6.9 implies that any line bundle on $X$ has at most one $G$-linearization, so this is the unique $G$-linearization of $\mathcal{O}_X(aD_1 + bD_2)$.

Now, the isomorphism $T \cong \mathbb{G}_m$ given by $\mathrm{diag}(t, t^{-1}) \mapsto t$ induces an isomorphism $\Lambda_G \cong \mathbb{Z}$ which identifies $\alpha_1$ with $2 \in \mathbb{Z}$. We will think of weights of $B$-eigenvectors as elements of $\mathbb{Z}$ using this isomorphism. Fix $a, b > 0$. The section

$$f_0 = \mathrm{pr}_2^*(s^{\otimes a}) \otimes \mathrm{pr}_1^*(s^{\otimes b}) \in H^0(X, \mathcal{O}_X(aD_1 + bD_2))$$

is a $B$-eigenvector of weight $a+b$ whose vanishing locus is the divisor $aD_1 + bD_2$. On the other hand, there is a natural bijection between global sections $H^0(X, \mathcal{O}_X(aD_1 + bD_2))$ and effective divisors linearly equivalent to $aD_1 + bD_2$ (given by sending a global section $f$ to the divisor $aD_1 + bD_2 + \mathrm{div}(f/f_0)$), and this bijection identifies $B$-eigenvectors with $B$-stable divisors, see Lemma 2.5.3 and Corollary 2.5.5. To find these $B$-stable divisors, the same proposition gives us a split short exact sequence

$$0 \to \Lambda(X) \to \bigoplus_{D \in \mathcal{D}(X)} \mathbb{Z} \cdot D \to \mathrm{Cl}(X) \to 0,$$

where the first map is given by $\lambda \mapsto \sum_{D \in \mathcal{D}(X)} \varphi_D(\lambda) D$ and the second map is given by taking the divisor class of a given Weil divisor. By our above arguments, we have $\Lambda(X) = \mathbb{Z} \cdot \alpha_1$ and $\mathrm{Cl}(X) \cong \mathbb{Z} \cdot D_1 \oplus \mathbb{Z} \cdot D_2$. It follows that the $B$-stable divisors linearly equivalent to $aD_1 + bD_2$ are those of the form $aD_1 + bD_2 + \mathrm{div}(f_{\alpha_1}^n)$, where $n \in \mathbb{Z}$ and $f_{\alpha_1} \in K(X)^{(B)}$ is an eigenvector with character $\alpha_1$. Our above computations give us

$$\mathrm{div}(f_{\alpha_1}) = \varphi_{D_1}(f_{\alpha_1}) D_1 + \varphi_{D_2}(f_{\alpha_1}) D_2 + \varphi_\Delta(f_{\alpha_1}) \Delta = D_1 + D_2 - \Delta,$$

so the divisors linearly equivalent to $aD_1 + bD_2$ are

$$(a+n)D_1 + (b+n)D_2 - n\Delta$$

for any $n \in \mathbb{Z}$. This divisor is effective precisely when $n \leq 0$ and $a+n, b+n \geq 0$, and for any such choice of $n$, the corresponding $B$-eigenvector in $H^0(X, \mathcal{O}_X(aD_1 + bD_2))$ has weight $a + b + 2n$. Thus, the highest weights of the $G$-representation $H^0(X, \mathcal{O}_X(aD_1 + bD_2))$ are

$$\Lambda^+(H^0(X, \mathcal{O}_X(aD_1 + bD_2))) = \{a+b, a+b-2, \ldots, a+b-2\min\{a,b\}\}.$$

These are just the elements of $\Lambda^+(X, \mathcal{O}_X(aD_1 + bD_2))$ in degree 1. However, the elements in degree $d$ are given by the above equation with $ad$ and $bd$ in place of $a$ and $b$. It follows that

$$\Lambda^+(X, \mathcal{O}_X(aD_1 + bD_2)) = \bigcup_{d \geq 1} \{(da+db, d), (da+db-2, d), \ldots, (d\max\{a,b\} - d\min\{a,b\}, d)\}.$$

**Example 4.9.2.** As in the above example, let $G = \mathrm{SL}_2$, let $B \subset G$ to be the Borel subgroup of upper triangular matrices, and let $T \subset B$ to be the subgroup of diagonal matrices. We consider the ruled surface $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ for some $e > 0$. We saw in Example 3.5.4 that $Y$ can be given a $G$-action in such a way that $Y$ is a toroidal $G$-variety and the structure morphism $\pi : Y \to \mathbb{P}^1$ is $G$-equivariant (where $G$ acts on $\mathbb{P}^1$ via the action of Example 2.4.19). We also computed all $B$-divisors of $Y$ in Example 3.5.4; we summarize their description here. The unique color of $Y$ is the fiber $C = \pi^{-1}([1:0])$. As for $G$-divisors, writing $\mathbb{P}^1 \setminus \{[1:0]\} \cong \mathbb{A}^1$, we have $\pi^{-1}(\mathbb{A}^1) \cong \mathbb{A}^1 \times \mathbb{P}^1$. Under this isomorphism, the two $G$-divisors of $Y$ are given by

$$E_1 = \overline{\mathbb{A}^1 \times [1:0]}, \quad E_2 = \overline{\mathbb{A}^1 \times [0:1]}.$$

We will also need to compute $\Lambda(Y)$ and the valuations $\varphi_{E_1}$, $\varphi_{E_2}$, and $\varphi_C$. For this, we will use the isomorphism $T \cong \mathbb{G}_m$ given by $\mathrm{diag}(t, t^{-1}) \mapsto t$ to identify $\Lambda_G$ with $\mathbb{Z}$. The isomorphism $\pi^{-1}(\mathbb{A}^1) \cong \mathbb{A}^1 \times \mathbb{P}^1$ above induces the following $B$-action on $\mathbb{A}^1 \times \mathbb{P}^1$: the action on $\mathbb{A}^1$ is given by the action on $\mathbb{A}^1 \subset \mathbb{P}^1$, and the action on $\mathbb{P}^1 \cong \mathrm{Proj}(k[w, z])$ is given by letting $w$ be fixed by $B$ and $z$ be a $B$-eigenvector of weight $-e$. (This follows from the definition of the $G$-action on $Y$, which was induced by $G$-linearizations on $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-e)$; cf. the arguments in Example 3.5.4.) In particular, $f_e = w/z \in K(Y)^{(B)}$ is a $B$-eigenvector of weight $e$ which has a zero at $[0:1]$ and a pole at $[1:0]$, so we have $\varphi_{E_1}(e) = -1$ and $\varphi_{E_2}(e) = 1$. Also, since $f_e \in K(Y)$ is an eigenvector of weight $e$, we have $\mathbb{Z} \cdot e \subset \Lambda(Y)$. On the other hand, we have $\Lambda(Y) = \mathbb{Z} \cdot \ell$ for some $1 \leq \ell \leq e$. Since $\varphi_{E_1}(\ell) = \ell/e$ must be an integer, we conclude that $\ell = e$ and hence that $\Lambda(Y) = \mathbb{Z} \cdot e$.

We claim that $\varphi_C(e) = e$. We can compute this using intersection numbers. First, we have an isomorphism $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{E}$, and under this isomorphism, the section $1 \oplus 0 \in H^0(\mathbb{P}^1, \mathcal{E})$ has vanishing locus on $\mathbb{P}(\mathcal{E})$ equal to $E_2$. It follows that $\mathcal{O}_Y(E_2) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and a standard result about ruled surfaces then tells us that $E_2^2 = -e$ (see [Har77, Chapter V, Proposition 2.9]). On the other hand, since $E_1$ and $E_2$ are the images of sections of $\pi$ and $C$ is a fiber of $\pi$, so another general fact about ruled surfaces (see e.g. [Har77, Chapter V, Proposition 2.3]) gives us

$$C.E_1 = C.E_2 = 1, \quad C^2 = E_1.E_2 = 0.$$

Since $\mathrm{div}(f_e)$ is a $B$-stable divisor, the only prime divisors with nonzero coefficient in $\mathrm{div}(f_e)$ are $B$-divisors. Thus, we have

$$\mathrm{div}(f_e) = \varphi_C(e)C + \varphi_{E_1}(e)E_1 + \varphi_{E_1}(e)E_2 = \varphi_C(e)C - E_1 + E_2.$$

Taking intersections of both sides with $E_2$, we see that $0 = \varphi_C(e) - e$ and hence that $\varphi_C(e) = e$, as claimed.

We can now also compute all the other combinatorial invariants of $Y$ that interest us. First, since $E_1$ and $E_2$ are both $G$-orbits and each $E_i$ is the only $B$-divisor of $Y$ containing itself, the colored fan corresponding to $Y$ is

$$\mathscr{F}_Y = \{(\mathbb{Q}_{\geq 0}\varphi_{E_1}, \varnothing), (\mathbb{Q}_{\geq 0}\varphi_{E_2})\} = \{(\mathbb{Q}_{\leq 0}, \varnothing), (\mathbb{Q}_{\geq 0}, \varnothing)\}.$$

We know that $Y$ is complete and toroidal, so the union of the cones in this fan is precisely $\mathcal{V}(Y)$. Hence $\mathcal{V}(Y) = \mathbb{Q} = N(Y)$, which implies that $\Psi_{G,Y} = \varnothing$ (in other words, $Y$ is horospherical). In particular, we have $\Pi_Y^b = \varnothing$. Notice that with $X = \mathbb{P}^1 \times \mathbb{P}^1$ as in Example 4.9.1 above, we have $\mathscr{F}_X \neq \mathscr{F}_Y$, $\mathcal{V}(X) \neq \mathcal{V}(Y)$, and $\Pi_X^b \neq \Pi_Y^b$. However, we will see that there do exist $G$-linearized ample invertible sheaves $L$ on $X$ and $M$ on $Y$ such that $\Lambda^+(X, L) = \Lambda^+(Y, M)$.

As in Example 4.9.1 above, we have a split short exact sequence

$$0 \to \Lambda(Y) \to \bigoplus_{D \in \mathcal{D}(Y)} \mathbb{Z} \cdot D \to \mathrm{Cl}(Y) \to 0.$$

Using the fact that $\mathrm{div}(f_e) = eC - E_1 + E_2$, we see that $\mathrm{Cl}(Y) \cong \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot E_2$ and that the divisors linearly equivalent to the divisor $mC + nE_2$ are the divisors

$$mC + nE_2 - r\,\mathrm{div}(f_e) = (m - er)C + rE_1 + (n - r)E_2.$$

This is an effective divisor precisely when $n \geq r \geq 0$ and $m \geq er$. In particular, it follows that any divisor of the form $mC + nE_2$ is effective if and only if $m, n \geq 0$.

As in Example 4.9.2, we will compute the weight monoid $\Lambda^+(Y, M)$ for every $G$-linearized ample line bundle $M$ on $Y$. First, we identify the possible choices of $M$. It is a general fact about ruled surfaces that $\mathrm{Pic}(Y)$ is freely generated by $\mathcal{O}_Y(C)$ and $\mathcal{O}_Y(E_2)$ (see e.g. [Har77, Chapter 5, Proposition 2.3]). For any $m, n \in \mathbb{Z}$, we claim that $\mathcal{O}_Y(mC + nE_2)$ is ample if and only if $m > ne > 0$. Since the ample cone is the interior of the nef cone for projective varieties, it will suffice to show that $mC + nE_2$ is nef if and only if $m \geq ne \geq 0$. As noted above, the effective divisors on $Y$ are of the form $aC + bE_2$ with $a, b \geq 0$, so $mC + nE_2$ is nef if and only if

$$0 \leq (aC + bE_2).(mC + nE_2) = an + bm - bne = an + b(m - ne)$$

for all $a, b \geq 0$. If $m \geq ne \geq 0$, this equation certainly holds; conversely, if the equation holds, taking $a = 0$ and $b = 1$ gives us $m \geq ne$, and taking $b = 0$ and $a = 1$ gives us $n \geq 0$ and hence $ne \geq 0$. (Alternately, one can deduce the ample cone from a general numerical criterion for ampleness on spherical varieties, see e.g. [Per18, Theorem 3.3.8].)

Now, assuming $m > ne > 0$, we compute $\Lambda^+(Y, \mathcal{O}_Y(mC+nE_2))$. First, if $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{(B)}$ is a nonzero $B$-eigenvector, it vanishes at the $B$-fixed point $[1 : 0] \in \mathbb{P}^1$, so $\pi^* s^{\otimes m}$ cuts out the divisor $mC$. In particular, $\mathcal{O}_Y(mC) = \pi^*\mathcal{O}_{\mathbb{P}^1}(m)$. On the other hand, we have an isomorphism $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \cong \mathrm{Sym}^d(\mathcal{E})$ for any $d \geq 1$. Since $1 \oplus 0 \in H^0(\mathbb{P}^1, \mathcal{E}) \cong H^0(Y, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ vanishes on $E_2$, as noted above, we have a section

$$t = (1 \oplus 0) \otimes \cdots \otimes (1 \oplus 0) \in H^0(\mathbb{P}^1, \mathrm{Sym}^n(\mathcal{E})) \cong H^0(Y, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n))$$

which is fixed by $G$ and cuts out the divisor $nE_2$. It follows that

$$f_0 = \pi^* s^{\otimes m} \otimes t \in \pi^*\mathcal{O}_{\mathbb{P}^1}(m) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \cong \mathcal{O}_Y(mC + nE_2)$$

is a $B$-eigenvector of weight $m$ which cuts out the divisor $mC + nE_2$. As noted above, the effective divisors linearly equivalent to $mC + nE_2$ are those of the form $mC + nE_2 - r\,\mathrm{div}(f_e)$, where $n \geq r \geq 0$ and $m \geq er$. Since $m > ne$, the inequality $m \geq er$ follows from $n \geq r$, so the only inequality on $r$ that we need is $n \geq r \geq 0$. On the other hand, Lemma 2.5.3 (plus the fact that $\Gamma(Y, \mathcal{O}_Y) = k$) implies that the effective divisors linearly equivalent to $mC + nE_2$ are in bijection with lines of $B$-eigenvectors in $H^0(Y, \mathcal{O}_Y(mC + nE_2))$, with the divisor $mC + nE_2 - r\operatorname{div}(f_e)$ corresponding to a line of $B$-eigenvectors of weight $m - re$. Putting this all together, we have

$$\Lambda^+(H^0(Y, \mathcal{O}_Y(mC + nE_2))) = \{m, m - e, \ldots, m - ne\}.$$

Replacing $m$ and $n$ by $md$ and $nd$ for any $d \geq 1$ in this equation gives us the $B$-eigenvectors of $\mathcal{O}_Y(mC + nE_2)^{\otimes d}$, so we see that

$$\Lambda^+(Y, \mathcal{O}_Y(mC + nE_2)) = \bigcup_{d \geq 1} \{(dm, d), (dm - e, d), \ldots, (dm - dne, d)\}.$$

Comparing this with our equation for $\Lambda^+(X, \mathcal{O}_X(aD_1 + bD_2))$ in Example 4.9.1, we see that

$$\Lambda^+(X, \mathcal{O}_X(aD_1 + bD_2)) = \Lambda^+(Y, \mathcal{O}_Y(mC + nE_2))$$

if and only if $e = 2$, $m = a + b$ and $n = \min\{a, b\}$. Since $a, b > 0$, this choice of $m$ and $n$ defines an ample line bundle on $Y$ if and only if $a \neq b$ (otherwise, we have $m = 2n$, not $m > 2n = en$). However, we have $X \ncong Y$ as varieties, so we certainly cannot have $(X, \mathcal{O}_X(aD_1 + bD_2)) \cong (Y, \mathcal{O}_Y(mC + nE_2))$ as polarized $G$-varieties. Since $X$ and $Y$ are smooth projective spherical $G$-varieties, this in particular provides a counterexample to Question 4.2.5b, but only when $a \neq b$. On the other hand, when $a = b$, there is no $G$-linearized ample line bundle $M$ on $Y$ such that $\Lambda^+(X, \mathcal{O}_X(aD_1 + bD_2)) = \Lambda^+(Y, M)$.

## 4.9.b Examples with Different Spherical Roots and Colored Fans

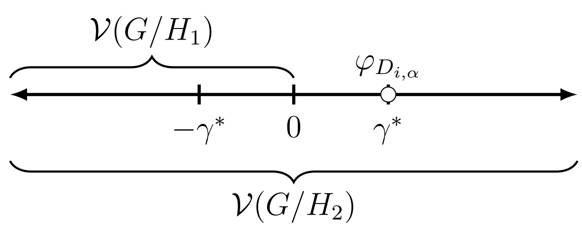
Next, we give examples where $X_1$ and $X_2$ are $\mathcal{D}$-equivalent, but $\Psi_{G,X_1} \neq \Psi_{G,X_2}$, and there does not exist a $\mathcal{D}$-equivalence between $X_1$ and $X_2$ that preserves colored fans. These examples show that our above results on comparing spherical roots and comparing colored fans (namely, Theorem 4.6.8 and Proposition 4.7.1) are relatively optimal. The work in these examples also seems to intuitively indicate that the only thing that being $\mathcal{D}$-equivalent and have $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ controls is the maximal colored cones of the colored fans $\mathscr{F}_{X_1}$ and $\mathscr{F}_{X_2}$. This informs our discussion of ample cones in Section 4.7, and it indicates that the weight monoid $\Lambda^+(X, L)$ is most closely related to the existence of a $\mathcal{D}$-equivalence, which motivates our study of weight monoids and divisors in Chapter 5.

**Example 4.9.3.** Let $\gamma$ be any of the spherical roots in [Was96, Table 1] such that $\operatorname{Supp}(\gamma)$ is connected and contains two roots not of type $a$, and let $G$ be the corresponding group. (See Proposition 4.6.5 for a list of all such choices of $\gamma$; each choice comes from a unique entry in [Was96, Table 1].) Let $\Lambda = \mathbb{Z} \cdot \gamma$, let $\Pi^a$ be the set of roots of type $a$ in $\operatorname{Supp}(\gamma)$, and set

$$\Psi_1 = \{\gamma\}, \quad \Psi_2 = \varnothing.$$

The tuples $S_1 = (\Lambda, \Pi^a, \Psi_1, \varnothing)$ and $S_2 = (\Lambda, \Pi^a, \Psi_2, \varnothing)$ both satisfy the definition of a homogeneous spherical datum for $G$. Indeed, axioms (B1-3) and ($\Psi$1-2) are vacuous, and ($\Pi$1-2) follows immediately from the fact that $\Lambda$ is generated by $\gamma$ and that we chose $\gamma$ and $\Pi^a$ to be that of a wonderful rank-1 variety. So, the classification of homogeneous spherical varieties (see Theorem 3.6.21) gives us two homogeneous spherical varieties $G/H_1$ and $G/H_2$ corresponding to $S_1$ and $S_2$ (respectively). We remark that $G/H_1$ is precisely the open $G$-orbit of the wonderful variety in [Was96, Table 1] that $\gamma$ comes from, since they both have the same homogeneous spherical data. By contrast, $G/H_2$ is horospherical, since it has no spherical roots.

Let $\gamma^* : \Lambda \to \mathbb{Z}$ be the map determined by $\gamma \mapsto 1$, so that $N(G/H_1) = N(G/H_2) = \mathbb{Q}\cdot\gamma^*$. Let $\alpha, \beta \in \Pi_G$ be the two simple roots not of type $a$ for the variety that gives $\gamma$ in [Was96, Table 1]. By our definition of $\Pi^a = \Pi^a_{G/H_1} = \Pi^a_{G/H_2}$, both $\alpha$ and $\beta$ are not of type $a$ for either $G/H_1$ or $G/H_2$, and since the $G/H_i$ have no type $b$ or $c$ roots, we must have $\alpha, \beta \in \Pi^d_{G/H_i}$. By inspecting the possibilities in [Was96, Table 1], we see that after swapping $\alpha$ and $\beta$ if necessary, we may assume that $\langle \alpha^\vee, \gamma \rangle = 1$, in which case $\langle \beta^\vee, \gamma \rangle$ is either 1, 0, or -1, depending on the choice of $\gamma$. For $i \in \{1, 2\}$. Let $D_{i,\alpha}$ and $D_{i,\beta}$ be the colors of $G/H_i$ moved by $\alpha$ and $\beta$ (respectively). Proposition 3.6.13 gives us $\varphi_{D_{i,\alpha}} = \alpha^\vee|_{\Lambda(X)}$ and $\varphi_{D_{i,\beta}} = \beta^\vee|_{\Lambda(X)}$. The following diagram summarizes the combinatorial data of $G/H_i$ appearing in $N(G/H_i)$ (except for $\varphi_{D_{i,\beta}}$, which depends on our choice of $\gamma$).




Consider the following strictly convex colored fans $\mathscr{F}_1$ and $\mathscr{F}_2$ for $G/H_1$ and $G/H_2$ respectively:

$$\mathscr{F}_1 = \{(\mathbb{Q}_{\le 0}\gamma^*, \varnothing), (0, \varnothing)\}, \quad \mathscr{F}_2 = \{(\mathbb{Q}_{\le 0}\gamma^*, \varnothing), (\mathbb{Q}_{\ge 0}\gamma^*, \{D_{2,\alpha}\}), (0, \varnothing)\}.$$

By Theorem 3.3.26, the colored fan $\mathscr{F}_i$ determines an open embedding $G/H_i \hookrightarrow X_i$ for some spherical $G$-variety $X_i$. We claim that the $X_i$ are smooth and complete. In fact, $X_1$ is the standard embedding of $G/H_1$ and hence is the wonderful variety that $\gamma$ comes from in [Was96, Table 1], so $X_1$ is in particular smooth and projective. As for $X_2$, since the union of the cones in $\mathscr{F}_2$ covers $\mathcal{V}(G/H_2)$, Theorem 3.3.28 tells us that $X_2$ is complete, and we will see below that there exists an ample line bundle on $X_2$, so $X_2$ is in fact projective. Moreover, the structure of the cones in $\mathscr{F}_2$ implies that $X_2$ is locally factorial, see Proposition 3.7.9. For horospherical varieties, one can add an extra condition to this criterion for being locally factorial to obtain a criterion for smoothness. This was first done in certain cases by Pauer [Pau83] and then generalized by Pasquier [Pas06, Theorem 2.6] (see also [Tim11, Theorem 28.10] for a treatment of the generalization due to Pasquier). The extra condition (beyond being locally factorial) in this smoothness criterion is: for each colored cone $(\mathcal{C}, \Delta)$ in the colored fan of $X_2$, every connected component of the Dynkin diagram for the root subsystem

$$\Pi^a_{X_2} \cup \{\beta \in \Pi_G \mid \beta \text{ moves an element of } \Delta\} \subset \Pi_G$$

appears among a certain list of admissible Dynkin diagrams. For the cone $(\mathbb{Q}_{\leq 0}\gamma, \varnothing) \in \mathscr{F}_2$, this root subsystem consists entirely of roots of type $a$, and any such diagram is in the list of admissible ones. For the cone $(\mathbb{Q}_{\geq 0}\gamma, \{D_{2,\alpha}\})$, no matter which option for $\gamma$ we choose, one can check that this root subsystem is isomorphic to $A_n$ for some $n \geq 1$ in such a way that $\alpha = \alpha_1$ and $\alpha_i \in \Pi^a_{X_2}$ for all $i > 1$. This is also in the list of admissible Dynkin diagrams. So, this smoothness criterion for horospherical varieties implies that $X_2$ is smooth.

Next, we consider the ample line bundles on $X_1$ and $X_2$ and their weight monoids. By definition of the colored fan $\mathscr{F}_i$, the only $B$-divisors of $X_i$ are the colors $D_{i,\alpha}$ and $D_{i,\beta}$ and a single $G$-divisor $E_i$ such that $\varphi_{E_i} = -\gamma^*$. Evidently, we can define a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ by setting

$$\iota(D_{1,\alpha}) = D_{2,\alpha},\ \iota(D_{1,\beta}) = D_{2,\beta},\ \iota(E_1) = E_2.$$

For any $B$-divisor $D_1 = m_\alpha D_{1,\alpha} + m_\beta D_{1,\beta} + m_E E_1$ on $X_1$, let $D_2 = m_\alpha D_{2,\alpha} + m_\beta D_{2,\beta} + m_E E_2$. Then, Corollary 5.4.3 (plus the fact that $\mathrm{Pic}(G) = 0$) implies that there exist $G$-linearizations on $L_1 = \mathcal{O}_{X_1}(D_1)$ and $L_2 = \mathcal{O}_{X_2}(D_2)$ such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.

We claim that we can pick $m, q, r \in \mathbb{Z}$ such that both $L_1$ and $L_2$ are ample. For this, we use the criterion for ampleness in Theorem 3.7.13 to compute the ample cones of $X_1$ and $X_2$. By definition of $\mathscr{F}_1$, the variety $X_1$ consists of one open $G$-orbit and one closed $G$-orbit, which is contained in (in fact equal to) the $G$-divisor $E_1$. Since the open $G$-orbit is contained in no divisors, a piecewise linear function in $\mathrm{PL}(X_1)$ is given by a single weight $\ell \in \Lambda$, and for the function corresponding to $D_1$ (see Lemma 3.7.2 and the discussion that follows it), we have

$$m_E = \varphi_{E_1}(\ell) = -\gamma^*(\ell),$$

or equivalently, $\ell = -m_E\gamma$. Since both $D_{1,\alpha}$ and $D_{1,\beta}$ contain no $G$-orbits, Theorem 3.7.13 implies that $D_1$ is ample if and only if

$$m_\alpha > \varphi_{D_{1,\alpha}}(\ell) = -m_E \quad \text{and} m_\beta > \varphi_{D_{1,\beta}}(\ell) = -m_E\langle \beta^\vee, \gamma\rangle.$$

As for $X_2$, the definition of $\mathscr{F}_2$ implies that there are two non-open $G$-orbits of $X_2$, one contained in $E_2$ and the other contained in $D_{2,\alpha}$. So, a piecewise linear function in $\mathrm{PL}(X_2)$ is given by two weights $\ell_\alpha, \ell_E \in \Lambda$, and for the function corresponding to $D_2$, we have

$$m_\alpha = \varphi_{D_{2,\alpha}}(\ell_\alpha) = \gamma^*(\ell_\alpha), \quad m_E = \varphi_{E_2}(\ell_E) = -\gamma^*(\ell_E),$$

or equivalently, $\ell_\alpha = m_\alpha\gamma$ and $\ell_E = -m_E\gamma$. By definition, the piecewise linear function consisting of $\ell_\alpha$ and $\ell_E$ is strictly convex if and only if

$$\varphi_{D_{2,\alpha}}(\ell_\alpha) > \varphi_{D_{2,\alpha}}(\ell_E) \quad \text{and} \quad \varphi_{E_2}(\ell_E) > \varphi_{E_2}(\ell_\alpha).$$

Plugging everything into these equations gives us

$$m_\alpha > -m_E \quad \text{and} \quad m_E > -m_\alpha,$$

which are equivalent inequalities. So, Theorem 3.7.13 implies that $D_2$ is ample if and only if $m_\alpha > -m_E$ and we have

$$m_\beta > \varphi_{D_{1,\beta}}(\ell_\alpha) = m_\alpha\langle\beta^\vee, \gamma\rangle \quad \text{and} \quad m_\beta > \varphi_{D_{1,\beta}}(\ell_E) = -m_E\langle\beta^\vee, \gamma\rangle.$$

We remark that two of these inequalities are the precisely the two inequalities we got for ampleness of $D_1$ above, but the third inequality here does not arise for $D_1$. Wading through the combinatorial data here, one can check that one of the inequalities that arises for both $X_1$ and $X_2$ comes from the fact that $(\mathbb{Q}_{\geq 0}\gamma, \varnothing)$ is a colored cone in both $\mathscr{F}_1$ and $\mathscr{F}_2$, and the other inequality that arises for both $X_1$ and $X_2$ comes from the fact that $D_{i,\beta}$ contains no $G$-orbit of $X_i$ for both $i = 1$ and $i = 2$. On the other hand, the only difference between $\mathscr{F}_1$ and $\mathscr{F}_2$ is that the latter has an extra colored cone in it, and this extra cone is essentially where the additional third inequality for $D_2$ comes from.

In summary, we see that both $D_1$ and $D_2$ are ample if and only if we pick $m_\alpha, m_\beta, m_E \in \mathbb{Z}$ such that the following inequalities hold.

$$\begin{gathered} m_\alpha > -m_E \\ m_\beta > -m_E\langle\beta^\vee, \gamma\rangle \\ m_\beta > m_\alpha\langle\beta^\vee, \gamma\rangle \end{gathered}$$

On the other hand, we noted above that $\langle\beta^\vee, \gamma\rangle \in \{1, 0, -1\}$. No matter which value $\langle\beta^\vee, \gamma\rangle$ takes, one can check that there exists some choice of $m_\alpha$, $m_\beta$, and $m_E$ such that all of the above inequalities hold. For instance, choosing $m_E = 0$ and $m_\beta > m_\alpha > 0$ works for every possibility on $\langle\beta^\vee, \gamma\rangle$.

In summary, we have shown that there exist ample invertible sheaves $L_1 = \mathcal{O}_{X_1}(D_1)$ and $L_2 = \mathcal{O}_{X_2}(D_2)$ and $G$-linearizations on $L_1$ and $L_2$ such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Moreover, $X_1$ and $X_2$ are smooth and projective, and we even know that $X_1$ is wonderful and that $X_2$ is horospherical. However, by construction, we see that

$$\Psi_{G,X_1} = \{\gamma\} \neq \varnothing = \Psi_{G,X_2}.$$

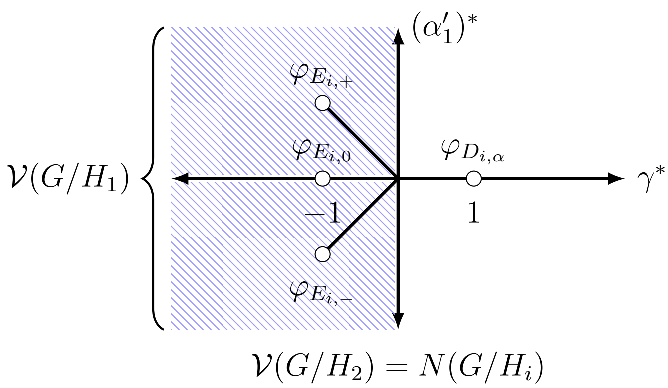
Moreover, is no reasoable way to "identify" $\mathscr{F}_{X_1} = \mathscr{F}_1$ and $\mathscr{F}_{X_2} = \mathscr{F}_2$, since one of these fans contains two colored cones and the other contains only one. In particular, no $\mathcal{D}$-equivalence between $X_1$ and $X_2$ preserves colored fans.

**Example 4.9.4.** This time, we give an example similar to that of Example 4.9.3, but in which the entire ample cones of $X_1$ and $X_2$ are identified. Let $\gamma$ be any of the spherical roots in [Was96, Table 1] such that $\mathrm{Supp}(\gamma)$ is connected and contains two roots not of type $a$, and let $G_0$ be the corresponding group, and let $G = G_0 \times \mathrm{SL}_2$. (See Proposition 4.6.5 for a list of all such choices of $\gamma$; each choice comes from a unique entry in [Was96, Table 1].) Note that $\Pi_G = \Pi_{G_0} \cup \{\alpha_1'\}$, where $\alpha_1'$ is the unique simple root of $\mathrm{SL}_2$. Let $\Lambda = \mathbb{Z} \cdot \gamma \oplus \mathbb{Z} \cdot \alpha_1'$, let $\Pi^a$ be the set of roots of type $a$ in $\mathrm{Supp}(\gamma)$, and set

$$\Psi_1 = \{\gamma\}, \quad \Psi_2 = \varnothing.$$

As in Example 4.9.3, it is almost trivial to check that the tuples $(\Lambda, \Pi^a, \Psi_1, \varnothing)$ and $(\Lambda, \Pi^a, \Psi_2, \varnothing)$ both satisfy the definition of a homogeneous spherical datum for $G$. By Theorem 3.6.21, these homogeneous spherical data correspond to homogeneous spherical varieties $G/H_1$ and $G/H_2$ for $G$ (respectively).

Let $\alpha, \beta \in \Pi_G$ be the two simple roots not of type $a$ for the variety that gives $\gamma$ in [Was96, Table 1]. As in Example 4.9.3, we have $\alpha, \beta \in \Pi^d_{G/H_i}$ for both $i$, and after swapping $\alpha$ and $\beta$ if necssary, we may assume that $\langle \alpha^\vee, \gamma \rangle = 1$ and that $\langle \beta^\vee, \gamma \rangle$ is either 1, 0, or -1, depending on the choice of $\gamma$. For $i \in \{1, 2\}$. Let $D_{i,\alpha}$ and $D_{i,\beta}$ be the colors of $G/H_i$ moved by $\alpha$ and $\beta$ (respectively). Proposition 3.6.13 gives us $\varphi_{D_{i,\alpha}} = \alpha^\vee|_{\Lambda(X)}$ and $\varphi_{D_{i,\beta}} = \beta^\vee|_{\Lambda(X)}$. Let $\gamma^* : \Lambda \to \mathbb{Z}$ (resp. $(\alpha_1')^* : \Lambda \to \mathbb{Z}$) be the map sending $\gamma$ to 1 (resp. 0) and $\alpha_1'$ to 0 (resp. 1). Then, $\gamma^*$ and $(\alpha_1')^*$ form a basis for $N(G/H_1) = N(G/H_2)$. To define an embedding $G/H_i \hookrightarrow X_i$, we intend to use $G$-divisors $E_{i,+}$, $E_{i,0}$, and $E_{i,-}$ whose valuations are $-\gamma^* + (\alpha_1')^*$, $-\gamma^*$, and $-\gamma^* - (\alpha_1')^*$, respectively. The following diagram summarizes all of this combinatorial data that appears in $N(G/H_i)$ (except for $\varphi_{D_{i,\beta}}$, which depends on our choice of $\gamma$).




To define the embeddings $G/H_i \hookrightarrow X_i$, consider the set

$$\begin{aligned}\mathscr{F}_{i,max} = \{&(\mathrm{Cone}(\varphi_{D_{i,\alpha}}, \varphi_{E_{i,+}}), \{D_{i,\alpha}\}), (\mathrm{Cone}(\varphi_{D_{i,\alpha}}, \varphi_{E_{i,-}}), \{D_{i,\alpha}\}),\\ &(\mathrm{Cone}(\varphi_{E_{i,+}}, \varphi_{E_{i,0}}), \varnothing), (\mathrm{Cone}(\varphi_{E_{i,0}}, \varphi_{E_{i,-}}), \varnothing)\}.\end{aligned}$$

Note that $\mathscr{F}_{i,max}$ is a set of strictly convex colored cones for $G/H_i$. So, we can define a strictly convex colored fan $\mathscr{F}_i$ for $G/H_i$ by taking $\mathscr{F}_i$ by taking $\mathscr{F}_{i,max}$ and repeatedly adding in all faces of colored cones until every face of every element of $\mathscr{F}_i$ is contained in $\mathscr{F}_i$. Explicitly, this gives us

$$\mathscr{F}_1 = \mathscr{F}_{1,max} \bigcup \{(\mathrm{Cone}(\varphi_{E_{1,+}}), \varnothing), (\mathrm{Cone}(\varphi_{E_{1,0}}), \varnothing), (\mathrm{Cone}(\varphi_{E_{1,-}}), \varnothing), (0, \varnothing)\}$$

and

$$\mathscr{F}_2 = \mathscr{F}_{2,max} \bigcup \{(\mathrm{Cone}(\varphi_{E_{2,+}}), \varnothing), (\mathrm{Cone}(\varphi_{E_{2,0}}), \varnothing), (\mathrm{Cone}(\varphi_{E_{2,-}}), \varnothing), (0, \varnothing)\} \\ \bigcup \{(\mathrm{Cone}(\varphi_{D_{2,\alpha}}), \{D_{2,\alpha}\})\}.$$

Since the $\mathscr{F}_i$ are strictly convex colored fans for $G/H_i$, they define $G$-equivariant embeddings $G/H_i \hookrightarrow X_i$. Since the cones in $\mathscr{F}_{i,max}$ cover $\mathcal{V}(G/H_i)$, the spherical varieties $X_i$ are complete (Theorem 3.3.28). We will see below that both $X_1$ and $X_2$ admit ample line bundles, so they are in fact projective. We claim that the $X_i$ are also smooth. Since the maximal colored cones in $\mathscr{F}_i$ are those in $\mathscr{F}_{i,max}$, the variety $X_i$ is covered by the open subsets $(X_i)_{G,Y} = G \cdot (X_i)_{B,Y}$ for $Y$ an orbit corresponding to a colored cone in $\mathscr{F}_{i,max}$ (see Theorem 3.2.7 and Proposition 3.3.24). It thus suffices to show that each such $(X_i)_{B,Y}$ is smooth. Fix any colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}_{i,max}$, and let $Y \subset X_i$ be the corresponding $G$-orbit. Applying the local structure theorem to $(X_i)_{B,Y}$ gives us an isomorphism

$$(X_i)_{B,Y} \cong R_u(P) \times Z,$$

where $P$ is the parabolic subgroup of $G$ given in Theorem 3.2.2, $M \subset P$ is the standard Levi subgroup, and $Z \subset (X_i)_{B,Y}$ is some $M$-stable closed subvariety. Since $R_u(P)$ is an affine algebraic group in characteristic 0, it is smooth, so it will suffice to show that $Z$ is smooth. Now, $Z$ is a spherical $M$-variety (see Proposition 3.2.3), and since $\beta \in \mathrm{Supp}(\gamma)$ but no choice of $(\mathcal{C}, \Delta)$ has $D_{i,\beta} \in \Delta$, Proposition 4.4.1 tells us that $\Psi_{M,Z} = \varnothing$. In other words, $Z$ is horospherical, so we may use the same criterion for smoothness on horospherical varieties used in Example 4.9.3 above.

Note that $Z$ is affine (since $(X_i)_{B,Y}$ is affine, see Proposition 3.2.3), and by arguing as in the proof of Theorem 3.2.7, one can show that the unique closed $M$-orbit of $Z$ is $Z \cap Y$. It follows from the description of the $(B \cap M)$-divisors of $Z$ in Proposition 3.2.3 that the unique maximal cone in the colored fan $\mathscr{F}_Z$ is $(\mathcal{C}, \Delta')$, where $\Delta'$ is the set of all $D \cap Z$, where $D \in \Delta$ and $D \cap Z$ is $M$-unstable. For the cones in $\mathscr{F}_{i,max}$, Proposition 4.4.1 tells us that if $D_{i,\alpha} \in \Delta$, then $D_{i,\alpha} \cap Z$ is $M$-unstable (because $\alpha \in \Pi_M$ in this case, and $\alpha$ moves $D_{i,\alpha} \cap Z$). So, $\Delta' = \{D_{i,\alpha} \cap Z\}$ if $D_{i,\alpha} \in \Delta$, and $\Delta' = \varnothing$ if $\Delta = \varnothing$. Moreover, $\mathcal{V}(Z) = N(Z) = N(X_i)$, every face of the cone $\mathcal{C}$ determines a face of the colored cone $(\mathcal{C}, \Delta')$, so the colored cones in $\mathscr{F}_Z$ are in bijection with the faces of $\mathcal{C}$. In particular, for every $(\mathcal{C}_0, \Delta_0) \in \mathscr{F}_Z$, we have either $\Delta_0 = \{D_{i,\alpha} \cap Z\}$ or $\Delta_0 = \varnothing$. In the former case, the Dynkin diagram for the root subsystem

$$\Pi_Z^a \cup \{\alpha' \in \Pi_G \mid \alpha' \text{ moves an element of } \Delta_0\} \subset \Pi_G$$

is isomorphic to $A_n$ for some $n \geq 1$ in such a way that $\alpha = \alpha_1$ and $\alpha_i \in \Pi_Z^a$ for all $i > 1$. In the case where $\Delta_0 = \varnothing$, this Dynkin diagra minstead only has roots of type $a$ for $Z$. This is exactly the same behavior we saw in Example 4.9.3, so the criterion for smoothness referenced in that example implies that $Z$ is smooth. This proves the claim that $X_i$ is smooth.

Now, we consider the combinatorial data on $X_1$ and $X_2$. By definition of the colored fans $\mathscr{F}_i$, the $B$-divisors of $X_i$ are the colors $D_{i,\alpha}$ and $D_{i,\beta}$ and the $G$-divisors $E_{i,+}$, $E_{i,0}$, and $E_{i,-}$. So, we can define a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ by:

$$\iota(D_{1,\alpha}) = D_{2,\alpha}, \quad \iota(D_{1,\beta}) = D_{2,\beta},$$
$$\iota(E_{1,+}) = \iota(E_{2,+}), \quad \iota(E_{1,0}) = E_{2,0}, \quad \iota(E_{1,-}) = E_{2,-}.$$

It follows immediately that for every colored cone $(\mathcal{C}, \Delta) \in \mathscr{F}_1$, we have $(\mathcal{C}, \iota(\Delta)) \in \mathscr{F}_2$, and in fact that

$$\mathscr{F}_{2,max} = \{(\mathcal{C}, \iota(\Delta)) \mid (\mathcal{C}, \Delta) \in \mathscr{F}_{1,max}\}.$$

However, since $\varphi_{D_{2,\alpha}} \in \mathcal{V}(G/H_2) \setminus \mathcal{V}(G/H_1)$, we have $(\mathrm{Cone}(\varphi_{D_{2,\alpha}}), \{D_{2,\alpha}\}) \in \mathscr{F}_2$ but $(\mathrm{Cone}(\varphi_{D_{1,\alpha}}), \{D_{1,\alpha}\}) \notin \mathscr{F}_1$. So, we see that

$$\mathscr{F}_2 = \{(\mathrm{Cone}(\varphi_{D_{2,\alpha}}), \{D_{2,\alpha}\})\} \bigcup \{(\mathcal{C}, \iota(\Delta)) \mid (\mathcal{C}, \Delta) \in \mathscr{F}_1\}.$$

So, $\iota$ does not preserve colored fans, but $\iota$ does preserve the maximal colored cones of these colored fans. In particular, since the combinatorial criterion for ampleness in Theorem 3.7.13 depends only on maximal colored cones, we conclude that any $B$-stable divisor $D_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ is ample if and only if the divisor $D_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$ is ample. For any such choice of $D_1$ and $D_2$, Corollary 5.4.3 tells us that there exist $G$-linearizations on the sheaves $L_i = \mathcal{O}_{X_i}(D_i)$ such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. (Note that $\mathrm{Pic}(G) = 0$ because $G$ is a product of two affine varieties whose picard groups are trivial, see e.g. [Isc74, Theorem 1.7].) In summary, $\iota$ is a $\mathcal{D}$-equivalence which preserves all maximal colored cones and hence “preserves” ample cones (in a sense made precise above), but $\iota$ does not preserve colored fans, and moreover, we have $\mathcal{V}(X_1) \neq \mathcal{V}(X_2)$.

It remains to check that there actually exists a choice of $D_1$ and $D_2$ as above which are ample, or equivalently, that $X_1$ and $X_2$ are actually quasi-projective (hence projective). By Corollary 3.7.14, it will suffice to show that there exists a strictly convex piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X_i)$. Let $Y_1$, $Y_2$, $Y_3$, and $Y_4$ be the closed $G$-orbits of $X_i$ corresponding to the maximal colored cones whose cones are $\mathrm{Cone}(\varphi_{D_{i,\alpha}}, \varphi_{E_{i,+}})$, $\mathrm{Cone}(\varphi_{D_{i,\alpha}}, \varphi_{E_{i,-}})$, $\mathrm{Cone}(\varphi_{E_{i,+}}, \varphi_{E_{i,0}})$, and $\mathrm{Cone}(\varphi_{E_{i,0}}, \varphi_{E_{i,-}})$, respectively. Given any piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X_i)$, we write $\ell_j = \ell_{Y_j}$, and we set $\ell_j = (x_j, y_j)$ in the basis of $\Lambda$ given by $\gamma$ and $\alpha_1'$, i.e. $x_j, y_j \in \mathbb{Z}$ are such that $\ell_j = x_j\gamma + y_j\alpha_1'$. By definition, any choice of the $\ell_j$ (or equivalently, of the $x_j$ and $y_j$) constitutes a piecewise linear function if and only if for any $j$ and $j'$ and any $D \in \mathcal{D}_{G,X_i}$ containing both $Y_j$ and $Y_{j'}$, we have $\ell_j(D) = \ell_{j'}(D)$. Considering the maximal colored cones in $\mathscr{F}_{i,max}$, these equalities for each choice of $D$, $j$, and $j'$ are

$$x_1 = x_4, \quad y_1 - x_1 = y_2 - x_2, \quad -x_2 = -x_3, \quad -x_3 - y_3 = -x_4 - y_4,$$

or equivalently,

$$x_1 = x_4, \quad x_2 = x_3, \quad y_2 = y_1 + x_2 - x_1, \quad y_4 = y_3 + x_2 - x_1. \tag{4.9.1}$$

The equalities $\ell_j(D) = \ell_{j'}(D)$ in this situation also imply that, in the definition of "strictly convex" for $(\ell_Y)_Y$ (see Definition 3.7.11), we only need to consider the inequalities arising from one choice of $Y_j \subset D$ for each $D \in \mathcal{D}_{G,X_i}$. So, the piecewise linear function $(\ell_Y)_Y$ is strictly convex if and only if the following inequalities hold:

$$\begin{array}{llll} \ell_1(D_{i,\alpha}) > \ell_2(D_{i,\alpha}), & \ell_1(D_{i,\alpha}) > \ell_3(D_{i,\alpha}), & \ell_1(E_{i,+}) > \ell_3(E_{i,+}), & \ell_1(E_{i,+}) > \ell_4(E_{i,+}) \\ \ell_2(E_{i,0}) > \ell_1(E_{i,0}), & \ell_2(E_{i,0}) > \ell_4(E_{i,0}), & \ell_3(E_{i,-}) > \ell_1(E_{i,-}), & \ell_3(E_{i,-}) > \ell_2(E_{i,-}), \end{array}$$

Plugging in $\ell_i = (x_i, y_i)$ and the various valuations of divisors here, these equalities become the following:

$$\begin{array}{cccc} x_1 > x_2, & x_1 > x_3, & -x_1 + y_1 > -x_3 + y_3, & -x_1 + y_1 > -x_4 + y_4 \\ -x_2 > -x_1, & -x_2 > -x_4, & -x_3 - y_3 > -x_1 - y_1, & -x_3 - y_3 > -x_2 - y_2 \end{array}$$

Coincidentally, several of these inequalities turn out to be redundant, thanks to the particular valuations we are using here. More precisely, we have $x_1 = x_4$ and $x_2 = x_3$ from (4.9.1), so the inequalities $x_1 > x_2$, $x_1 > x_3$, $-x_2 > -x_1$, and $-x_2 > -x_4$ are all equivalent. Substituting the other equalities in (4.9.1) into the other 4 inequalities as well, we are left with the following 5 inequalities:

$$\begin{array}{rll} x_1 > x_2, & -x_1 + y_1 > -x_2 + y_3, & -x_1 + y_1 > -x_1 + y_3 + x_2 - x_1 \\ & -x_2 - y_3 > -x_1 - y_1, & -x_2 - y_3 > -x_2 - (y_1 + x_2 - x_1). \end{array}$$

Simplifying each of these inequalities gives us

$$\begin{array}{rll} x_1 > x_2, & y_1 > y_3 + x_1 - x_2, & y_1 > y_3 + x_2 - x_1 \\ & y_1 > y_3 + x_2 - x_1, & y_1 > y_3 + x_1 - x_2 \end{array}.$$

Note that two more of these inequalities are redundant, so we are left with only 3 inequalities:

$$x_1 > x_2, \quad y_1 > y_3 + x_1 - x_2, \quad y_1 > y_3 + x_2 - x_1. \tag{4.9.2}$$

In summary, picking a strictly convex piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X_i)$ amounts to picking integers $x_i, y_i \in \mathbb{Z}$ for $i \in \{1, 2, 3, 4\}$ satisfying the equalities in (4.9.1) (so that the $x_i$ and $y_i$ determine an element of $\mathrm{PL}(X_i)$) and the inequalities in (4.9.2) (so that the element of $\mathrm{PL}(X_i)$ that we get is strictly convex). One can check directly that there are choices of the $x_i$ and $y_i$ satisfying these conditions. For instance, to satisfy the inequalities in (4.9.2), we may take $x_1 = 1$, $x_2 = 0$, $y_1 = 2$, and $y_3 = 0$, and the equalities in (4.9.1) then give us:

$$\begin{array}{ll} \ell_1 = (x_1, y_1) = (1, 2), & \ell_2 = (x_2, y_2) = (0, 1), \\ \ell_3 = (x_3, y_3) = (0, 0), & \ell_4 = (x_4, y_4) = (1, -1). \end{array}$$

This choice of the $\ell_i$ thus determines a strictly convex piecewise linear function in $\mathrm{PL}(X_i)$, so Corollary 3.7.14 implies that $X_i$ is quasi-projective. Translating back into the language of ample divisors (cf. the proof of Corollary 3.7.14), the strictly convex function $(\ell_Y)_Y \in \mathrm{PL}(X_i)$

that we have given, along with any choice of coefficient $m \in \mathbb{Z}$ for $D_{i,\beta}$, determines a $B$-stable (Cartier) divisor

$$\begin{aligned} D_i &= \varphi_{D_{i,\alpha}}(\ell_1)D_{i,\alpha} + \varphi_{E_{i,+}}(\ell_1)E_{i,+} + \varphi_{E_{i,0}}(\ell_2)E_{i,0} + \varphi_{E_{i,-}}(\ell_3)E_{i,-} + mD_{i,\beta} \\ &= D_{i,\alpha} + E_{i,+} + mD_{i,\beta}. \end{aligned}$$

(For details on this equation, see Lemma 3.7.2 and the discussion that follows it.) If we choose $m$ to be large enough (more precisely, to be greater thatn $\varphi_{D_{i,\beta}}(\ell_j)$ for all $j$), then the divisor $D_i$ will be ample by the criterion for ampleness in Theorem 3.7.13.

# Chapter 5

# Weight Monoids and $\mathcal{D}$-Equivalences

In the previous chapter, we set out to understand what combinatorial data of a spherical variety $X$ is controlled by a weight monoid $\Lambda^+(X, L)$ of a $G$-linearized ample invertible sheaf $L$ on $X$. We found that $\Lambda^+(X, L)$ need not determine the colored fan of $X$ (see Section 4.7), and while $\Lambda^+(X, L)$ can determine many spherical roots of $X$, it cannot generally determine the elements $\alpha \in \Psi_{G,X} \cap \Pi_G$ (i.e. the elements $\alpha \in \Pi^b_X$) or certain other "exceptional types" of spherical roots (see Theorem 4.6.8). As for the data of $B$-divisors on $X$, we saw in Theorem 4.5.5 that the data of $\Lambda^+(X, L)$ and $\Pi^b_X$ together determines all of the combinatorial information about $B$-divisors on $X$. Including $\Pi^b_X$ in with this data is essential, as counterexamples to Theorem 4.5.5 do exist when $\Pi^b_{X_1} \neq \Pi^b_{X_2}$, see Examples 4.9.1 and 4.9.2.

These results suggest that, if we could somehow capture the data of both $\Lambda^+(X, L)$ and $\Pi^b_X$ using weight monoids, then we would be able to use those weight monoids to better determine the combinatorial invariants of a spherical variety (especially those invariants pertaining to $B$-divisors). In this chapter, we consider a couple different ways to do that. In Section 5.1, we show that if we choose a "nice" ample line bundle $L$, then the weight monoid $\Lambda^+(X, L)$ actually does determine the set $\Pi^b_X$. Such a nice line bundle does not always exist, but it does in many cases. In Section 5.2, we introduce an alternative idea, which is to use the weight monoids $\Lambda^+(X, L)$ for every line bundle $L$ at once. This leads to the notion of a $\Lambda^+$-*equivalence* between $X_1$ and $X_2$. In Sections 5.3 and 5.4, we show that, under some technical not terribly restrictive assumptions, two spherical varieties $X_1$ and $X_2$ are $\Lambda^+$-equivalent if and only if they are $\mathcal{D}$-equivalent. In other words, the data of all the $\Lambda^+(X, L)$ is actually equivalent to the combinatorial data of $B$-divisors on $X$. In order to translate the results about spherical roots in Chapter 4 to this perspective of "using all weight monoids at once," we need some notion of compatibility with ample line bundles (because ampleness is a key assumption for our results in Chapter 4). In Section 5.5, we introduce such a compatibility condition in the form of a *strong* $\Lambda^+$-*equivalence*, and we use it to combine our results in Chapter 4 with our results in this chapter.

## 5.1 Level Line Bundles

Motivated by Examples 4.9.1 and 4.9.2 from the previous section, we define a certain nice condition on line bundles which rules out the undesirable behavior in those examples.

**Definition 5.1.1.** Let $X$ be a spherical variety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. Let $\alpha \in \Pi_X^b$ be a simple root of type $b$ for $X$ (see Definition 3.6.11). and write $\mathcal{D}_{G,X}(\alpha) = \{D_1, D_2\}$.

1. We say that $L$ is *level with respect to* $\alpha$ if there exists a $B$-stable effective Cartier divisor $D = \sum_{E \in \mathcal{D}_{G,X}} n_E E$ such that $\mathcal{O}_X(D) \cong L^{\otimes d}$ for some $d \geq 1$ and $n_{D_1} = n_{D_2}$.

2. We say that $L$ is *level* if $L$ is level with respect to every simple root of type $b$ for $X$.

**Example 5.1.2.** In the notation of Examples 4.9.1 and 4.9.2, we have $\Pi_X^b = \{\alpha_1\}$ and $\mathcal{D}_{G,X}(\alpha_1) = \{D_1, D_2\}$, while $\Pi_Y^b = \varnothing$. It follows that every line bundle on $Y$ is trivially level, and since $\operatorname{div}(f_{\alpha_1}) = D_1 + D_2 - \Delta$ on $X$, we see that a line bundle $L$ on $X$ is level if and only if every $B$-stable divisor $D = aD_1 + bD_2 + c\Delta$ such that $\mathcal{O}_X(D) = L$ has $a = b$. Thus, the line bundle $\mathcal{O}_X(aD_1 + bD_2)$ on $X$ is level if and only if $a = b$. Our computations of weight monoids in these examples thus show that $\Lambda^+(X, L) \neq \Lambda^+(Y, M)$ for any choices of $L$ and $M$ which are both ample and level.

To understand level line bundles, we are first of all interested in when they exist. The following results tells us that we can find a $G$-linearized ample invertible sheaf which is level with respect to "many" roots of type $b$.

**Lemma 5.1.3.** *Let $X$ be a spherical $G$-variety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. Let $\alpha \in \Pi_X^b$, and write $\mathcal{D}_{G,X}(\alpha) = \{D_1, D_2\}$. If both $D_1$ and $D_2$ contain a $G$-orbit of $X$, then $L$ is level with respect to $\alpha$.*

*Proof.* Let $D$ be a $B$-stable effective Cartier divisor of $X$ such that $\mathcal{O}_X(D) = L$ (for instance, take $D$ to be the divisor cut out by any $B$-eigenvector in $H^0(X, L)$). Let $Y_1 \subset D_1$ and $Y_2 \subset D_2$ be $G$-orbits. After replacing $D$ by some positive multiple if necessary, we may find some $B$-eigenvectors $s_1, s_2 \in H^0(X, L)^{(B)}$ such that $X_{B,Y_i} = X_{s_i}$ (see Theorem 3.2.7a). Then, $s_1$ and $s_2$ cut out effective Cartier divisors $\delta_1$ and $\delta_2$, and $D_i$ is not in the support of $\delta_i$ (by definition of $X_{B,Y_i}$). Let $n_1$ (resp. $n_2$) be the coefficient of $D_2$ (resp. $D_1$) in $\delta_1$ (resp. $\delta_2$). Then, the divisor $n_2\delta_1 + n_1\delta_2$ has coefficient $n_1 n_2$ for both $D_1$ and $D_2$, and since $\delta_1 \equiv \delta_2 \equiv D$, we have

$$\mathcal{O}_X(n_2\delta_1 + n_1\delta_2) \cong \mathcal{O}_X((n_1 + n_2)D) \cong L^{\otimes(n_1+n_2)}.$$

This proves that $L$ is level with respect to $\alpha$. □

**Proposition 5.1.4.** *Let $X$ be a quasi-projective spherical $G$-variety, and define*

$$\tilde{\Pi}_X^b = \{\alpha \in \Pi_X^b \mid \forall D \in \mathcal{D}_{G,X}(\alpha),\ D \text{ contains no } G\text{-orbit of } X\}$$

*There exists a $G$-linearized ample invertible sheaf $L$ on $X$ which is level with respect to every element of $\tilde{\Pi}^b_X$.*

*Proof.* Let $L$ be any $G$-linearized ample invertible sheaf on $X$ (one exists by Theorem 2.6.12), and let $\delta = \sum_{D \in \mathcal{D}_{G,X}} n_D D$ be any $B$-stable effective Cartier divisor on $X$ such that $\mathcal{O}_X(\delta) \cong L$. We will repeatedly "adjust" the coefficients $n_D$ until $L$ has the desired properties. The main subtlety is that for any root $\alpha \in \tilde{\Pi}^b_X$, it may be that some divisor moved by $\alpha$ is moved by $\beta$, and that some divisor moved by $\beta$ is also moved by some simple root $\gamma \in \tilde{\Pi}^b_X$, etc. In order to deal with this phenomenon more clearly, we write $\alpha \perp \beta$ if $\alpha, \beta \in \tilde{\Pi}^b_X$ are simple roots such that $\mathcal{D}_{G,X}(\alpha) \cap \mathcal{D}_{G,X}(\beta) \neq \varnothing$, and we write $\alpha \sim \beta$ if either $\alpha \perp \beta$ or if there exist $\alpha_1, \ldots, \alpha_n \in \tilde{\Pi}^b_X$ for some $n \geq 1$ such that

$$\alpha \perp \alpha_1 \perp \cdots \perp \alpha_n \perp \beta.$$

Then, $\sim$ is an equivalence relation on $\tilde{\Pi}^b_X$; for any $\alpha \in \tilde{\Pi}^b_X$, we denote by $\tilde{\Pi}_\alpha$ the equivalence class of $\alpha$. By definition of $\sim$, if we change any coefficient in $D$ of any $B$-divisor moved by an element of $\tilde{\Pi}_\alpha$, this cannot affect whether $L$ is level with respect to any element of $\tilde{\Pi}_\beta$ for any $\beta \not\sim \alpha$ (because no such $\beta$ moves any divisor moved by an element of $\tilde{\Pi}_\alpha$). Our plan is to"adjust" the coefficients of $\delta$ to make $L$ level with respect to every element of $\tilde{\Pi}_\alpha$. Repeating this process for each choice $\alpha \in \tilde{\Pi}^b_X$ in turn will then yield a choice of $L$ which is level with respect to every element of $\tilde{\Pi}^b_X$.

Fix $\alpha \in \tilde{\Pi}^b_X$, and define

$$\tilde{\mathcal{D}}_\alpha = \{D \in \mathcal{D}_{G,X} \mid D \text{ is moved by an element of } \tilde{\Pi}_\alpha\}.$$

Notice that by definition of $\tilde{\Pi}_\alpha$, the sheaf $L$ is level with respect to every element of $\tilde{\Pi}_\alpha$ if and only if there exists a $B$-stable effective Cartier divisor $\delta'$ such that $\mathcal{O}_D(\delta') = L^{\otimes n}$ for some $n \geq 1$ and every element $\tilde{\mathcal{D}}_\alpha$ has the same coefficient in $\delta'$. In light of this, define $n = \max_{D \in \tilde{\mathcal{D}}_\alpha} n_D$, and set

$$\delta' = \delta + \sum_{D \in \tilde{\mathcal{D}}_\alpha} (n - n_D) D.$$

Then, every element of $\tilde{\mathcal{D}}_\alpha$ has the same coefficient in $\delta'$. Moreover, every element $D \in \tilde{\mathcal{D}}_\alpha$ contains no $G$-orbit of $X$ (by definition of $\tilde{\Pi^b_X}$), so $D$ is an effective Cartier divisor, and $\mathcal{O}_X(D)$ is globally generated (Proposition 3.1.20). Thus, the sheaf

$$L' = \mathcal{O}_X(\delta') \cong L \otimes \bigotimes_{D \in \tilde{\mathcal{D}}_\alpha} \mathcal{O}_X(D)$$

is ample. There exists some $m \geq 1$ such that $(L')^{\otimes m}$ is $G$-linearizable (see Theorem 2.6.11). Pick a $G$-linearization on $(L')^{\otimes m}$; then, $(L')^{\otimes m}$ is a $G$-linearized ample invertible sheaf, and $(L')^{\otimes m} \cong \mathcal{O}_X(m\delta')$ implies that $(L')^{\otimes m}$ is level with respect to every element of $\tilde{\Pi}_\alpha$.

□

**Corollary 5.1.5.** *Any quasi-projective toroidal variety admits a level line bundle.*

*Proof.* This is immediate from the above proposition, since any divisor moved by a root must be a color and no color contains a $G$-orbit on a toroidal variety. □

**Remark 5.1.6.** Let $X$ be a quasi-projective spherical variety. Combining Lemma 5.1.3 and Proposition 5.1.4, we see that there exists an ample line bundle which is level with respect to all $\alpha \in \Pi_X^b$ such that either (1) both elements of $\mathcal{D}_{G,X}(\alpha)$ contain a $G$-orbit, or (2) neither element of $\mathcal{D}_{G,X}(\alpha)$ contains a $G$-orbit. On the other hand, write $\mathcal{D}_{G,X}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$, and suppose that $D_\alpha^+$ contains a $G$-orbit but $D_\alpha^-$ does not. Let $L$ be any $G$-linearized ample invertible sheaf, and let $\delta$ be a $B$-stable effective Cartier divisor such that $L = \mathcal{O}_X(\delta)$. Suppose that $X$ is complete, and consider the combinatorial descriptions of ampleness and Cartier divisors on $X$ given by Lemma 3.7.2 and Theorem 3.7.13. Lemma 3.7.2 implies that the coefficient of $D_\alpha^+$ in $\delta$ is $\varphi_{D_\alpha^+}(\mu)$ for some $\mu \in \Lambda(X)$, and since $L$ is ample, Theorem 3.7.13 implies that the coefficient of $D_\alpha^-$ in $\delta$ is $> \varphi_{D_\alpha^-}(\mu)$. In particular, if $\varphi_{D_\alpha^-} = \varphi_{D_\alpha^+}$, then no matter what $\delta$ and $\mu$ are, it is impossible for $D_1$ and $D_2$ to have the same coefficient in $\delta$. It follows that for projective spherical varieties $X$, there is no ample line bundle that is level for any $\alpha$ such that (1) $\varphi_{D_\alpha^-} = \varphi_{D_\alpha^+}$, and (2) exactly one of $D_\alpha^+$ and $D_\alpha^-$ contains a $G$-orbit. Such choies of $X$ and $\alpha$ do exist.

**Remark 5.1.7.** It seems likely that if $X$ is complete, the situation of Remark 5.1.6 is the only situation in which there does not exist a line bundle that is level for $\alpha$. More precisely, we expect that the combinatorial characterization of ampleness in Theorem 3.7.13 can be used to show that there does exist a line bundle which is level with respect to $\alpha$ whenever $\varphi_{D_\alpha^-} \neq \varphi_{D_\alpha^+}$. After a few reductions, one can boil this down to a completely combinatorial question of whether there exists a strictly convex piecewise linear function $(\ell_Y)_Y \in \mathrm{PL}(X)$ (see Definitions 3.7.3 and 3.7.11) satisfying the following condition: for some (equivalently, for any) closed $G$-orbit $Y_\alpha$ contained in $D_\alpha^+$, we have

$$\varphi_{D_\alpha^-}(\ell_Y) < \varphi_{D_\alpha^+}(\ell_{Y_\alpha})$$

for all $G$-orbits $Y$. (Here we assume that $D_\alpha^+$ is the element of $\mathcal{D}_{G,X}(\alpha)$ containing some $G$-orbit, and $D_\alpha^-$ is the element that contains no $G$-orbit.)

That a strictly convex piecewise linear function $(\ell_Y)_Y$ exists is just the statement that $X$ is quasi-projective (see Corollary 3.7.14). It is not clear whether there exists some $(\ell_Y)_Y$ that is strictly convex and satisfies the above condition, but it seems likely that such a choice of $(\ell_Y)_Y$ exists in most cases, provided that $\varphi_{D_\alpha^-} \neq \varphi_{D_\alpha^+}$ (as otherwise the above condition would always fail for $Y = Y_\alpha$).

**Remark 5.1.8.** One possible way to cope with the non-existence of level line bundles in the situation of Remark 5.1.6 is to replace $X$ by another embedding of its open $G$-orbit $G/H$. We sketch one conceivable such approach (with many details missing and perhaps not actually possible). Suppose as above that $D_\alpha^+$ contains a $G$-orbit but $D_\alpha^-$ does not. If one can show that there exists some colored cone $(\mathcal{C}, \Delta)$ in the colored fan $\mathscr{F}_X$ of $X$ such that replacing $(\mathcal{C}, \Delta)$ by

$$(\mathbb{Q}_{\geq 0}\mathcal{C} + \mathbb{Q}_{\geq 0}\varphi_{D_\alpha}^-, \Delta \cup \{D_\alpha^-\}$$

still yields a colored fan $\mathscr{F}'$ for $G/H$, then $\mathscr{F}'$ determines an embedding $G/H \hookrightarrow X'$, and since $\mathrm{id}_{G/H}$ maps $\mathscr{F}_X$ into $\mathscr{F}'$, we have a $G$-equivariant birational map $f : X \to X'$. Because $X$ and $X'$ have all the same $G$-divisors (by our construction of $\mathscr{F}'$), it follows that $f$ is an isomorphism in codimension 1. Taking an open subset $U$ of $X$ whose codimension is $\geq 2$, we may be able to extend the line bundle $f_*(L|_U)$ to a line bundle $L'$ on $X'$ (this is for instance possible if $X'$ happens to be smooth by Hartog's extension theorem, but it might work in general using arguments similar to Proposition 3.1.20, since $f$ also induces a bijection on $G$-orbits). In this case, since sections of line bundles on normal schemes are determined by a codimension-1 subset, we should be able to define a $G$-linearization of $L'$ such that $f^*$ induces a $G$-equivariant isomorphism $H^0(X, L) \xrightarrow{\sim} H^0(X', L')$. Then, $L'$ with be level with respect to any $\beta \in \Pi_X^b$ if $L$ is. However, since both colors of $X'$ moved by $\alpha$ contain a closed $G$-orbit, the line bundle $L'$ is also level with respect to $\alpha$.

As the above remarks suggest, there is still a lot of room for further study regarding the existence of level line bundles. For the remainder of this section, we turn to the main reason that we have introduced these types of line bundles: in order to "avoid" the behavior of Examples 4.9.1 and 4.9.2. In fact, we will show that in the presence of level line bundles $L_i$ on complete spherical varieties $X_i$, the condition $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ implies that $\Pi_{X_1}^b = \Pi_{X_2}^b$, so that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent by Theorem 4.5.5. This is much nicer than the behavior of the varieties $X$ and $Y$ of Examples 4.9.1 and 4.9.2, where we have $\Pi_X^b \neq \Pi_Y^b$, and where $X$ has only 1 $G$-divisor but $Y$ has 2.

Our proof techniques here will largely resemble those of Theorem 4.5.5a, which says that $\Pi_{X_1}^c = \Pi_{X_2}^c$ under certain conditions. The main reason that this argument does not also generally give $\Pi_{X_1}^b = \Pi_{X_2}^b$ is that the auxiliary Lemma 4.5.2 gives some $(\mu, n) \in \Lambda^+(X_1, L_1)$ such that $(X_1)_\mu$ intersects the unique element of $\mathcal{D}_{G,X}(\alpha)$ when $\alpha \in \Pi_{X_1}^c$, but when $\alpha \in \Pi_{X_1}^b$, the same lemma only tells us that $(X_1)_\mu$ intersects one of the two elements of $\mathcal{D}_{G,X}(\alpha)$. In the case where $L_1$ is level, we can actually get $(X_1)_\mu$ to intersect both elements of $\mathcal{D}_{G,X}(\alpha)$ (which we do in Lemma 5.1.9 below). With this fact, the proof of Theorem 4.5.5a goes through just fine for roots of type $b$.

**Lemma 5.1.9.** *Let $X$ be a spherical variety, let $L$ be a $G$-linearized ample invertible sheaf on $X$, and let $\alpha \in \Pi_X^b$. If $L$ is level with respect to $\alpha$, then there exists some $f \in H^0(X, L^{\otimes n})^{(B)}$ for some $n \geq 1$ such that $X_f$ intersets both elements of $\mathcal{D}_{G,X}(\alpha)$.*

*Proof.* The proof is almost identical to that of Lemma 4.5.2, except that we pick a divisor using levelness of $L$ with respect to $\alpha$. Write $\mathcal{D}_{G,X}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$. Since $L$ is level with respect to $\alpha$, there exists some $f_0 \in H^0(X, L^{\otimes n})^{(B)}$ for some $n \geq 1$ such that the effective Cartier divisor $D$ cut out by $f_0$ has the same coefficient $c$ for both $D_\alpha^+$ and $D_\alpha^-$. Because $\alpha$ is a spherical root and in particular lies in $\Lambda(X)$, there exists a $B$-eigenvector $f_{-\alpha} \in K(X)^{(B)}$

whose weight is $-\alpha$. By Proposition 3.6.13 and Lemma 4.4.2, the principal Cartier divisor $\operatorname{div}(f_{-\alpha})$ has coefficient $-1$ for $D_\alpha^+$ and $D_\alpha^-$ and has nonnegative coefficients for every other $B$-divisor. It follows that $D' = D + \operatorname{div}(f_{-\alpha}^c)$ is effective and has coefficient 0 for both $D_\alpha^+$ and $D_\alpha^-$. The divisor $D'$ is $B$-stable and linearly equivalent to $D$, so it corresponds to a $B$-eigenvector $f \in H^0(X, L^{\otimes n})^{(B)}$, and since the support of $D'$ is $X \setminus X_f$, we have $D_\alpha^\pm \cap X_f \neq \varnothing$. □

With the above lemma in hand, we now give a refinement of Theorem 4.5.5 for the case of level line bundles.

**Theorem 5.1.10.** *Let $X_1$ and $X_2$ be smooth projective spherical $G$-varieties. Let $\alpha \in \Pi_{X_1}^b$, and suppose there exists some $G$-linearized ample line bundles $L_1$ on $X_1$ and $L_2$ on $X_2$ such that $L_1$ is level with respect to $\alpha$ and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Then, $\alpha \in \Pi_{X_2}^b$.*

*Proof.* We argue as in the proof of Theorem 4.5.5a. By Lemma 5.1.9, there exists some $(\mu, n) \in \Lambda^+(X_1, L_1)$ such that $n > 0$ and both elements of $\mathcal{D}_{G,X_1}(\alpha)$ intersect $(X_1)_\mu$. Since the $X_i$ are projective and the $L_i$ are ample, the $(X_i)_\mu$ are affine, so Theorem 4.4.6 gives us an isomorphism of $M_\mu$-varieties $X_1(\mu) \cong X_2(\mu)$. In particular, this isomorphism gives us a bijection

$$\iota_0 : \mathcal{D}_{M_\mu, X_1(\mu)} \to \mathcal{D}_{M_\mu, X_2(\mu)}$$

such that for all $D$, we have $\varphi_D = \varphi_{\iota_0(D)}$, and any $\alpha \in \Pi_{M_\mu}$ moves $D$ if and only if it moves $\iota_0(D)$. On the other hand, for $i \in \{1, 2\}$, we have bijections $\iota_i$ between $\mathcal{D}_{M_\mu, X_i(\mu)}$ and the set of $B$-divisors of $X_i$ intersecting $(X_i)_\mu$ which satisfy the same conditions on the $\varphi_D$ and the roots moving $D$ as $\iota_0$ does (see Proposition 3.2.3e and Proposition 4.4.1). Putting all these bijections together, we obtain a bijection

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\}$$

such that for all $D$, we have $\varphi_D = \varphi_{\iota_\mu(D)}$, and any $\alpha \in \Pi_{M_\mu}$ moves $D$ if and only if $\alpha$ moves $\iota_\mu(D)$.

Now, since both elements of $\mathcal{D}_{G,X_1}(\alpha)$ intersect $(X_1)_\mu$, we have $\alpha \in \Pi_{M_\mu}$ by Proposition 4.4.1, and the same proposition then implies that every element of $\mathcal{D}_{G,X_2}(\alpha)$ intersects $(X_2)_\mu$. It follows that $\iota_\mu$ contains every element of $\mathcal{D}_{G,X_2}(\alpha)$ in its image and so restricts to a bijection $\mathcal{D}_{G,X_1}(\alpha) \to \mathcal{D}_{G,X_2}(\alpha)$. In particular, there must be two elements of $\mathcal{D}_{G,X_2}(\alpha)$, so $\alpha \in \Pi_{X_2}^b$. □

**Corollary 5.1.11.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that*

1. *$\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$,*
2. *$X_1$ and $X_2$ are smooth, and*
3. *$L_1$ and $L_2$ are level.*

*Then, $X_1$ and $X_2$ are $\mathcal{D}$-equivalent.*

*Proof.* Since $L_1$ is level with respect to every $\alpha \in \Pi^b_{X_1}$, we have $\Pi^b_{X_1} \subset \Pi^b_{X_2}$ by Theorem 5.1.10. On the other hand, $L_2$ is also level with respect to every $\alpha \in \Pi^b_{X_2}$, so the theorem also gives us $\Pi^b_{X_2} \subset \Pi^b_{X_1}$. So $\Pi^b_{X_1} = \Pi^b_{X_2}$, and now we conclude by Theorem 4.5.5. □

We can also use Theorem 5.1.10 to provide a refinement of Corollary 4.8.1 for the case of level line bundles.

**Corollary 5.1.12.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be smooth polarized spherical varieties, and let $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ be the set of all $\gamma \in \Psi_{G,X_i}$ such that either $\gamma \in \Pi^b_{X_i}$ or $\gamma$ satisfies one of the 4 possibilities in Proposition 4.6.5c. Suppose that $L_1$ and $L_2$ are level. The following are equivalent.*

*(i) There exists a $G$-equivariant isomorphism $i : X_1 \to X_2$ such that $i^*L_2 \cong L_1$ as $G$-linearized invertible sheaves.*

*(ii) $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$ and $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

**Remark 5.1.13.** We note that the above corollary is stated for a fixed choice of $L_1$ and $L_2$, whereas Corollary 4.8.1 is not. This is primarily due to the fact that level line bundles need not exist in general, which makes the phrasing of Corollary 4.8.1 less convenient for the level case. However, one could rephrase the above corollary in a way that more closely resembles Corollary 4.8.1 by adding in the assumption that one of $X_1$ and $X_2$ admits a level line bundle.

*Proof.* If (i) holds, then $i^*L_2 \cong L_1$ implies that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Moreover, we have $\Psi_{G,X_1} = \Psi_{G,X_2}$, and since the subset $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ is just the subset of spherical roots that satisfy certain combinatorial conditions, this implies that $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$.

Conversely, suppose that (ii) holds. Theorem 5.1.10 implies that $\Pi^b_{X_1} = \Pi^b_{X_2}$, and Theorem 4.6.8 implies that

$$\Psi_{G,X_1} \setminus \Psi^{exc}_{G,X_1} = \Psi_{g,X_2} \setminus \Psi^{exc}_{G,X_2},$$

so we have $\Psi_{G,X_1} = \Psi_{G,X_2}$, or equivalently, $\mathcal{V}(X_1) = \mathcal{V}(X_2)$. Corollary 4.3.5 then implies that $(X_1, L_1) \cong (X_2, L_2)$ as polarized spherical varieties. □

## 5.2 $\Lambda^+$-Equivalences

As we saw in Corollary 5.1.11, the assumption that line bundles are level implies that $\Pi^b_{X_1} = \Pi^b_{X_2}$ and hence gives us a $\mathcal{D}$-equivalence. However, we also saw in Remark 5.1.6 that level line bundles need not exist. Moreover, for some applications, one may not wish to choose a line bundle at all. In this section, we introduce an alternative condition on weight monoids, which we will see also yields a $\mathcal{D}$-equivalence. The essential idea is to consider all the weight monoids $\Lambda^+(X, L)$ for every $G$-linearized invertible sheaf $L$ at once.

For the theory, it is convenient to consider not just invertible sheaves but rather so-called *divisorial sheaves*, which are a slight generalization. Indeed, to study the relationship between $\mathcal{D}$-equivalences and weight monoids, it is crucial for us to be able to translate between divisors and sheaves. Cartier divisors correspond to invertible sheaves in great generality (for instance, on any Noetherian reduced scheme, see [Liu02, Corollary 7.1.19]), but Weil divisors do not necessarily correspond to invertible sheaves in general. However, it turns out that any Weil divisor $D$ on a normal variety $X$ does have an associated divisorial sheaf $\mathcal{O}_X(D)$. When $X$ is locally factorial, every Weil divisor $D$ is Cartier, and the divisorial sheaf $\mathcal{O}_X(D)$ is isomorphic to the usual invertible sheaf associated to a Cartier divisor. Working with divisorial sheaves thus allows us to make all the arguments we wish to make with divisors and invertible sheaves, even without assuming that $X$ is locally factorial.

An overview of the theory of divisorial sheaves is provided in Appendix B. The upshot is that all of the data we are interested in ($G$-linearizations, weight monoids, divisors cut out by global sections, etc.) work the same for divisorial sheaves as they do for invertible sheaves. As such, the reader uninterested in divisorial sheaves will lose very little by simply imagining that $X$ is locally factorial and that all of the sheaves $\mathcal{O}_X(D)$ are invertible. The only material from Appendix B that is essential is the "$G$-equivariant class group" $\mathrm{Cl}_G(X)$, whose definition we repeat here.

**Definition 5.2.1** (cf. Definition B.16)**.** Let $X$ be a normal $G$-variety.

1. We denote by $\mathrm{Cl}_G(X)$ the abelian group of $G$-equivariant isomorphism classes of $G$-linearized divisorial sheaves on $X$.
2. We denote by $\mathrm{Div}_B(X)$ the group of all $B$-stable Weil divisors of $X$. In other words: $\mathrm{Div}_B(X) = \bigoplus_{D\in\mathcal{D}_{G,X}} \mathbb{Z}\cdot D$.
3. We denote by $\mathrm{Div}_B^G(X)$ the subgroup of $\mathrm{Div}_B(X)$ consisting of the divisors $D$ such that $\mathcal{O}_X(D)$ is $G$-linearizable.

Our primary object of study for the next couple sections will be a notion of "equality on all weight monoids," which we now define.

**Definition 5.2.2.** Let $X_1$ and $X_2$ be $G$-varieties with finitely many orbits. A $\Lambda^+$-*equivalence* from $X_1$ to $X_2$ is an isomorphism of abelian groups $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ such that for any $G$-linearized divisorial sheaf $\mathcal{F}$ on $X_1$, we have

$$\Lambda^+(X_1,\mathcal{F}) = \Lambda^+(X_2,\theta(\mathcal{F})).$$

If a $\Lambda^+$-equivalence from $X_1$ to $X_2$ exists, we say that $X_1$ and $X_2$ are $\Lambda^+$-*equivalent*.

**Remark 5.2.3.** The notion of $\Lambda^+$-equivalence can also be expressed using the so-called *$G$-equivariant total coordinate ring* introduced by Brion in [Bri07]. More precisely, given a spherical variety $X$, we define

$$R(X) = \bigoplus_{\mathcal{O}_X(D)\in\mathrm{Cl}_G(X)} \Gamma(X,\mathcal{O}_X(D)).$$

Note that $R(X)$ comes equipped with a $G$-module structure induced by that on the global sections $\Gamma(X, \mathcal{O}_X(D))$ for a $G$-linearized divisorial sheaf $\mathcal{O}_X(D)$. Brion showed ([Bri07, Proposition 4.2.2]) that $R(X)$ has the natural ring structure such that multiplication commutes with the $G$-action. Thus, we may view $R(X)$ as a $\mathrm{Cl}_G(X)$-graded algebra with a compatible $G$-module structure. One can check from the definitions that two spherical varieties $X_1$ and $X_2$ are $\Lambda^+$-equivalent if and only if there is an isomorphism $\mathrm{Cl}_G(X_1) \cong \mathrm{Cl}_G(X_2)$ such that $R(X_1)$ and $R(X_2)$ are isomorphic as $\mathrm{Cl}_G(X_1)$-graded $G$-modules, or equivalently, if the monoids

$$\Lambda^+(X_i) = \{(\mu, \mathcal{F}) \mid \mathcal{F} \in \mathrm{Cl}_G(X_1) \cong \mathrm{Cl}_G(X_2),\ \mu \in \Gamma(X_i, \mathcal{F})\}$$

are equal for $i \in \{1, 2\}$.

Our primary goal in the following sections is to relate $\Lambda^+$-equivalences to $\mathcal{D}$-equivalences. One of the key elements of this relationship is understanding how the data of simple roots moving certain colors (which is part of the data of a $\mathcal{D}$-equivalence) relates to the weight monoids of divisorial sheaves. Proposition 5.2.5 below will give us this relationship between simple roots and weight monoids. In order to prove this proposition, we first need a technical lemma.

**Lemma 5.2.4.** *Let $X$ be a spherical variety, let $G/H \subset X$ be the open $G$-orbit, and let $D \in \mathcal{D}_{G,X}$ be a $B$-divisor. If $\mathcal{O}_X(D)$ is $G$-linearizable, then there exists some $f \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ such that $D \cap G/H$ is the vanishing locus of $f$.*

*Proof.* The lemma is essentially a combination of several standard facts about $\Gamma(G, \mathcal{O}_G)$ and $\Gamma(G/H, \mathcal{O}_{G/H})$. Any $G$-linearization of $\mathcal{O}_X(D)$ induces a $G$-linearization of $\mathcal{O}_{X^{\leq 1}}(D \cap X^{\leq 1})$ (see Appendix B). Thus, after replacing $X$ by $X^{\leq 1}$ and $D$ by $D \cap X^{\leq 1}$, we may assume that $D$ is Cartier (or equivalently, that $\mathcal{O}_X(D)$ is invertible).

Now, let $\sigma \in H^0(X, \mathcal{O}_X(D))$ be the canonical section, and consider the map $\pi : G \to X$ given by the composition

$$G \to G/H \hookrightarrow X$$

of the quotient map followed by the inclusion. Since $D$ is Cartier, the pullback $D'$ of $D$ by $\pi$ is well-defined: it is the vanishing locus of the section $\pi^*\sigma \in H^0(G, \pi^*\mathcal{O}_X(D))$. On the other hand, $\pi$ factors as

$$G \overset{j}{\hookrightarrow} G \times X \overset{\rho}{\to} X,$$

where $\rho$ is the action morphism and $j$ is given by $g \mapsto (g, x)$ for some $x \in G/H \subset X$. Any $G$-linearization on $\mathcal{O}_X(D)$ gives us an isomorphism $\rho^*\mathcal{O}_X(D) \cong \mathrm{pr}_X^* \mathcal{O}_X(D)$, and pulling this isomorphism back by $j$ gives us an isomorphism $\pi^*\mathcal{O}_X(D) \cong \mathcal{O}_X$. This isomorphism identifies $\pi^*\sigma$ with a global section $f \in \Gamma(G, \mathcal{O}_G)$ whose vanishing locus is $D'$. Since $\pi$ is $G$-equivariant, we see that $D'$ is $B$-stable, hence $f \in \Gamma(G, \mathcal{O}_G)^{(B)}$.

Now, the quotient map $q : G \to G/H$ gives a map of sheaves $q^\# : \mathcal{O}_{G/H} \to q_*\mathcal{O}_G$ which, on global sections, identifies $\Gamma(G/H, \mathcal{O}_{G/H})$ as the subring of $\Gamma(G, \mathcal{O}_G)$ consisting of all sections that are $H$-eigenvectors under the right regular representation (i.e. under the action of $H$ on $G$ given by right multiplication by $H$). So, if we can show that $f$ is an $H$-eigenvector, then we have $f \in \Gamma(G/H, \mathcal{O}_{G/H})$, and $f$ will be the desired section. When $H$ is connected, this follows from Proposition 2.5.4, since $D' = q^{-1}(D \cap G/H)$ is stable under right multiplication by $H$. In general, we instead need a certain characterization of $G$-linearized line bundles on $G/H$ in terms of characters of $H$; see [Tim11, Remark 13.4 and discussion following Proposition 2.4] for details. □

**Proposition 5.2.5** ([Bri07, Section 4.1], [Tim11, Lemma 30.24])**.** *Let $X$ be a spherical variety, and let $D \in \mathcal{D}_{G,X}$ be a $B$-divisor. Let $\alpha_1, \dots, \alpha_r$ be the simple roots of $G$, and for each $i$, let $\omega_i$ be the fundamental weight corresponding to $\alpha_i$ (i.e. $\omega_i$ is the element of $\Lambda_G$ dual to $\alpha_i^\vee$).*

*(a) If $D$ is a $G$-divisor, then there exists a canonical $G$-linearization of $\mathcal{O}_X(D)$ such that the canonical section has weight 0.*

*(b) Suppose that $\mathcal{O}_X(D)$ is $G$-linearizable. There exists a canonical $G$-linearization of $\mathcal{O}_X(D)$ such that the canonical section has weight $\mu_D$, where*

$$\mu_D = \begin{cases} \sum_{D \in \mathcal{D}_{G,X}(\alpha_i)} \omega_i, & D \text{ moved by a root of type } b \text{ or } d \ , \\ 2\omega_i, & D \text{ moved by } \alpha_i \in \Pi^c_{X_1} \end{cases}$$

*(Note that the sum in Case 1 is over all roots $\alpha_i \in \Pi_G$ such that $D$ is moved by $\alpha_i$.)*

*Proof.* First, suppose that $D$ is a $G$-divisor. By our discussion on $G$-linearizations of divisorial sheaves at the end of Appendix B, it will suffice to define a $G$-linearization on the restriction of $\mathcal{O}_X(D)$ to $X^{\leq 1}$, which is simply $\mathcal{O}_{X^{\leq 1}}(D)$. Thus, after replacing $X$ by $X^{\leq 1}$, we may assume that $\mathcal{O}_X(D)$ is invertible. In this case, $\mathcal{O}_X(-D)$ is the ideal sheaf corresponding to the closed subscheme $D$ of $X$. Since $D$ is $G$-stable, then sections of $\mathcal{O}_X(-D)$ define a $G$-submodule of the sections of $\mathcal{O}_X$, and one can check that the canonical $G$-linearization of $\mathcal{O}_X$ restricts to a $G$-linearization of $\mathcal{O}_X(-D)$, and this induces a $G$-linearization of the sheaf $\mathcal{O}_X(-D)^{-1} \cong \mathcal{O}_X(D)$ (see Lemma 2.4.13). The isomorphism $\mathcal{O}_X(-D) \otimes \mathcal{O}_X(D) \cong \mathcal{O}_X$ induces a map

$$m : H^0(X, \mathcal{O}_X(-D)) \otimes_k H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X),$$

and the $G$-linearizations we've defined here are such that $m$ is $G$-equivariant (again by Lemma 2.4.13). But writing $\mathcal{O}_X(D) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X(-D), \mathcal{O}_X)$, the map $m$ is just given by evaluation of sheaf morphisms $\mathcal{O}_X(-D) \to \mathcal{O}_X$ on global sections of $\mathcal{O}_X(-D)$. The canonical section corresponds to the inclusion $i : \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$, so $G$-equivariance of $m$ implies that for any $s \in H^0(X, \mathcal{O}_X(-D))$ and any $g \in G(k)$, the element $g \cdot i \in H^0(X, \mathcal{O}_X(D))$ is given by

$$(g \cdot i)(g \cdot s) = g \cdot i(s) = i(g \cdot s).$$

(Here we have also used the fact that $i$ is $G$-equivariant, which follows from the $G$-linearization we defined on $\mathcal{O}_X(-D)$.) So, we have $g \cdot i = i$, whence $i$ is invariant under $G(k)$. It now follows formally that $i$ is $G$-invariant (see e.g. [Mil17, Proposition 4.6]).

Next, suppose that $D$ is a color and that $\mathcal{O}_X(D)$ is $G$-linearizable. To define a $G$-linearization of $\mathcal{O}_X(D)$, we follow a somewhat standard construction, see e.g. [Bri07, Section 4.1]. By Lemma 5.2.4, the divisor $D \cap G/H \subset G/H$ is trivial, and there exists some section $f \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ whose vanishing locus is $D \cap G/H$. Let $N$ be the $G$-submodule of $\Gamma(G/H, \mathcal{O}_{G/H})$ generated by $f$. Note that $N$ is a finite-dimensional vector space, generated over $k$ by $f$ and finitely many translates $g_1 f, \ldots, g_n f$ for some $g_i \in G(k)$. Since $D \cap G/H$ is trivial, we have

$$\mathcal{O}_X(D)|_{G/H} = \mathcal{O}_{G/H}(D \cap G/H) \cong \mathcal{O}_{G/H},$$

so we may view $f$ and the $g_i f$ as sections of $\mathcal{O}_X(D)|_{G/H}$. Moreover, if $\sigma \in H^0(X, \mathcal{O}_X(D)$ is the canonical section, then the above isomorphism identifies $\sigma|_{G/H}$ with $f$.

Now, $f$ and the $g_i f$ determine a $G$-equivariant morphism $\varphi : U \to \mathbb{P}(N)$ such that $\varphi^* \mathcal{O}_{\mathbb{P}(N)}(1) \cong \pi^* \mathcal{O}_X(D)|_U$, where $U \subset G/H$ is the locus on which not all of $f$ and the $g_i f$ vanish. On the other hand, the set of points in $G/H$ where $f$ and all of the $g_i f$ vanish is $G$-stable (it is the vanishing locus of the $G$-submodule $N \subset \Gamma(G/H, \mathcal{O}_{G/H})$) and contains no $G$-orbit (since $G/H$ is itself a single $G$-orbit), so this locus is empty. In other words, $U = G/H$. Moreover, $\sigma$ does not vanish anywhere in $X$ except on $D$. In particular, $\sigma$ does not vanish on every codimension-1 $G$-orbit, so the morphism $\varphi$ extends to a $G$-equivariant morphism $\varphi' : X^{\leq 1} \to \mathbb{P}(N)$. Since $\varphi'$ is $G$-equivariant, the natural $G$-linearization on $\mathcal{O}_{\mathbb{P}(N)}(1)$ (see Proposition 2.4.17) induces a $G$-linearization on $(\varphi')^* \mathcal{O}_{\mathbb{P}(N)}(1) \cong \mathcal{O}_X(D)|_{X^{\leq 1}}$, and this $G$-linearization induces a $G$-linearization on $\mathcal{O}_X(D)$. This is the canonical $G$-linearization on $\mathcal{O}_X(D)$ that we wanted.

It remains to check what the weight of the canonical section $\sigma$ is. For this, we follow the argument of [Tim11, Lemma 30.24]. We begin with a few reductions to a nice case. Since $\pi^* \sigma = f$ and $\pi$ is $G$-equivariant, the weight of $\sigma$ is the same as the weight of $f$. In particular, this weight depends only on $G/H$, and the type of every root for $X$ also only depends on $G/H$. By definition of the very sober hull $\overline{H}$, the quotient map $G/H \to G/\overline{H}$ induces a bijection on colors. It follows that the image $\overline{D}$ of $D'$ in $G/\overline{H}$ is cut out by a $B$-eigenvector of the same weight as $f$. Also, every simple root has the same type for $G/\overline{H}$ as it does for $G/H$ (see [Lun01, Section 7.1]). So, we may replace $X$ by the wonderful embedding of $G/\overline{H}$ to reduce to the case where $X$ is wonderful. Moreover, note that any very sober hull contains the center $Z(G)$ by construction. It follows that $Z(G)$ acts trivially on $G/\overline{H}$ and hence on $X$. So, the $G/Z(G)$ acts on $X$, and since $G/Z(G)$ is a reductive group with trivial center, $G/Z(G)$ is actually semisimple (see Proposition 2.2.21). Note that replacing $\sigma$ has the same weight for $G/Z(G)$ as it does for $G$, hence it has the same weight for the universal cover $\tilde{G}$ of $G/Z(G)$ (acting on $X$ via the covering map $\tilde{G} \to G/Z(G)$), and it is a general fact about universal covers that $\mathrm{Pic}(\tilde{G}) = 0$ (see e.g. [Mil17, Corollary 18.24]). Thus, after replacing $G$ by $\tilde{G}$, we may assume that $G$ is semisimple and simply connected.

In particular, since $G$ is semisimple, the lattice $\Lambda_G$ is generated by the simple roots $\alpha_i$

of $G$, hence also by the fundamental weights $\omega_i$. Our plan is to use the localization $X^{\alpha_i}$ at the simple root $\alpha_i$ to compute the coefficient of $\omega_i$ in $\mu_D$. For any $\alpha_i \in \Pi_G$, let $P = P_{\alpha_i}$, let $M$ be the standard Levi subgroup of $P$, and let $S = [M, M]$. Then, Theorem 3.6.10 tells us that $S \cong \mathrm{SL}_2$ or $\mathrm{PSL}_2$ and and that $X^{\alpha_i}$ is a toroidal $S$-variety; moreover, the theorem gives a list of all possible toroidal varieties that $X^{\alpha_i}$ could be. Let $D' = D \cap X^{\alpha_i}$, and consider the restriction

$$\sigma|_{X^{\alpha_i}} \in H^0(X^{\alpha_i}, \mathcal{O}_X(D)|_{X^{\alpha_i}}).$$

The $G$-linearization of $\mathcal{O}_X(D)$ induces an $S$-linearization of $\mathcal{O}_X(D)|_{X^{\alpha_i}}$ (via pullback along the $S$-equivariant inclusion morphism $X^{\alpha_i} \hookrightarrow X$), and since $\Lambda_S = \mathbb{Z} \cdot \omega_i$, the weight of $\sigma|_{X^{\alpha_i}}$ under this $S$-linearization is the projection of $\mu_D$ to $\mathbb{Z} \cdot \omega_i \subset \Lambda_G$.

In particular, suppose $D' = \varnothing$. Then, we have $\mathcal{O}_X(D)|_{X^{\alpha_i}} \cong \mathcal{O}_{X^{\alpha_i}}$. Since $X^{\alpha_i}$ is complete, this implies that $H^0(X^{\alpha_i}, \mathcal{O}_X(D)|_{X^{\alpha_i}}) \cong k$ as $G$-modules, so $S$ acts on $\sigma|_{X^{\alpha_i}}$ by a character. But $\mathrm{SL}_2$ and $\mathrm{PSL}_2$ have no nontrivial characters, so $\sigma|_{X^{\alpha_i}}$ has weight 0, hence the coefficient of $\omega_i$ in $\mu_D$ is 0. Since $D$ is a color and intersection with $X^{\alpha_i}$ induces a bijection between colors of $X^{\alpha_i}$ and colors of $X$ moved by $\alpha_i$ (Proposition 3.6.6), we see that $D' = \varnothing$ if and only if $\alpha_i$ does not move $D$. Thus, the coefficient of $\omega_i$ in $\mu_D$ is 0 whenever $\alpha_i$ does not move $D$.

On the other hand, suppose that $\alpha_i$ does move $D$. Then, $D'$ is a color of $X^{\alpha_i}$ such that $\mathcal{O}_{X^{\alpha_i}}(D') \cong \mathcal{O}_X(D)|_{X^{\alpha_i}}$, and the canonical section of $D'$ is $\sigma|_{X^{\alpha_i}}$. All possible options for the $S$-variety $X^{\alpha_i}$ are listed in Theorem 3.6.10, and which one we get is determined by the type of $\alpha_i$. Also, since $S$ has no nontrivial characters, the $S$-linearization on $\mathcal{O}_{X^{\alpha_i}}(D')$ here is the unique $S$-linearization on this sheaf (see Corollary 2.6.9). Thus, one can simply take each of the possible choices of $X^{\alpha_i}$ from Theorem 3.6.10, compute any $S$-linearization of $\mathcal{O}_{X^{\alpha_i}}(E)$ for any color $E$ of this variety, then compute the weight of the canonical section under this $S$-linearization, and this will be the weight of $\sigma|_{X^{\alpha_i}}$. We have already done these computations for almost all the possibilities. More specifically:

1. The case $X^{\alpha_i} \cong \mathbb{P}^1 \times \mathbb{P}^1$ follows from our computations in Example 4.9.1.

2. The case $X^{\alpha_i} \cong S \times^{B \cap S} \mathbb{P}^1$ follows from our computations in Example 4.9.2. Actually, the variety in Example 4.9.2 is the ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ for some $e > 0$. However, one can show that this variety is $S$-equivariantly isomorphic to $S \times^{B \cap S} \mathbb{P}^1$. One way to do this is to check that the ruled surface satisfies the universal property of the homogeneous fiber bundle $S \times^{B \cap S} \mathbb{P}^1$. Alternately, one can check that these two spherical $S$-varieties have the same homogeneous spherical data and the same colored fans.

3. In the case $X^{\alpha_i} \cong \mathbb{P}^1$, the unique color is the $B$-fixed point $[1 : 0] \in \mathbb{P}^1$, and the corresponding line bundle is $\mathcal{O}_{\mathbb{P}}^1(1)$ with the $G$-linearization given in Example 2.4.19, so everything for this case follows from that example.

4. Since we know that $X^{\alpha_i}$ has at least one color, the only remaining case is $X^{\alpha_i} \cong \mathbb{P}(\mathrm{sl}_2)$, which is the case where $\alpha_i \in \Pi_X^c$. We omit the verification in this case.

Finally, we remark that only the case $X^{\alpha_i} \cong \mathbb{P}^1 \times \mathbb{P}^1$ has more than one color, and in that case, both colors turn out to have the same weight for their canonical sections, so there is no need to worry about which color is $D'$ in these computations. □

**Corollary 5.2.6.** *For any toroidal variety $X$, we have $\mathrm{Div}_B^G(X) = \mathrm{Div}_B(X)$.*

*Proof.* The above proposition say that $\mathcal{O}_X(D)$ is $G$-linearizable when $D \in \mathcal{D}_{G,X}^G$. On the other hand, any color $D$ of $X$ contains no $G$-orbit, so $\mathcal{O}_X(D)$ is $G$-linearizable by Proposition 3.1.20. Taking tensor products and inverses of the $\mathcal{O}_X(D)$ then induces $G$-linearizations on every divisorial sheaf. □

One key use of the above proposition is the following corollary, which allows us to detect which $G$-linearization is the canonical one

**Corollary 5.2.7.** *Let $X$ be a spherical variety, and let $D_0 = \sum_{D \in \mathcal{D}_{G,X}} n_D D$ be a $B$-stable (Weil) divisor on $X$, and let $\sigma_0 \in H^0(X, \mathcal{O}_X(D_0))$ be the canonical section. Suppose that* $\mathrm{Pic}(G) = 0$. *There exists a unique $G$-linearization of $\mathcal{O}_X(D_0)$ such that the weight of $\sigma_0$ is a linear combination of fundamental weights of $G$. For this $G$-linearization, the weight of $\sigma_0$ is*

$$\mu_{D_0} = \sum_{D \in \mathcal{D}_{G,X}} n_D \mu_D,$$

*where $\mu_D$ is as in Proposition 5.2.5 for any color $D$ and $\mu_D = 0$ for any $D \in \mathcal{D}_{G,X}^G$.*

*Proof.* Since $\mathrm{Pic}(G) = 0$, every divisorial sheaf on $X$ is $G$-linearizable (see Theorem 2.6.11). So, the $G$-linearization of Proposition 5.2.5 exists for every $D \in \mathcal{D}_{G,X}$. Taking tensor products of these $G$-linearizations gives us a $G$-linearization on $D_0$. Since $\sigma_0$ is the tensor prouct of the canonical sections of the $\mathcal{O}_X(D)$ for $D \in \mathcal{D}_{G,X}$, the weight $\mu_{D_0}$ of $\sigma_0$ can be read off from the weights $\mu_D$ of the canonical sections of the $\mathcal{O}_X(D)$, which are given by Proposition 5.2.5. Every other $G$-linearization of $\mathcal{O}_X(D)$ is obtained by tensoring by $\mathcal{O}_X(\lambda)$ for some $\lambda \in \mathcal{X}(G)$, and in that $G$-linearization, the weight of $\sigma_0$ will then be $\mu_{D_0} + \lambda$ (see Corollary 2.6.8). Since the fundamental weights of $G$ are linearly independent from the characters of $G$ (see Lemma 2.2.25) and $\mu_{D_0}$ is a linear combination of fundamental weights, we cannot possibly have $\mu_{D_0} + \lambda$ be a linear combination of fundamental weights for any $\lambda \neq 0$. □

## 5.3 From $\Lambda^+$-Equivalences to $\mathcal{D}$-Equivalences

In this section, we start with a $\Lambda^+$-equivalence and attempt to construct a $\mathcal{D}$-equivalence. It turns out that this is possible under some relatively mild assumptions (which are satisfied, for instance, whenever $X_1$ and $X_2$ are projective). The main idea is to use equalities on weight monoids to "lift" the $\Lambda^+$-equivalence from the level of isomorphism classes of divisorial sheaves to the level of divisors, i.e. to take our isomorphism $\theta : \mathrm{Cl}_G(X_1) \to\xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ and "lift" it to an isomorphism $\tilde{\theta} : \mathrm{Div}_B^G(X_1) \xrightarrow{\sim} \mathrm{Div}_B^G(X_2)$. The isomorphism $\tilde{\theta}$ will satisfy some nice compatibility properties, and these properties along with the essential Proposition 5.2.5 will allow us to construct a bijection $\mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ and show that this bijection is a $\mathcal{D}$-equivalence.

First of all, a $\mathcal{D}$-equivalence between $X_1$ and $X_2$ by definition only exists in the situation where $\Lambda(X_1) = \Lambda(X_2)$ (otherwise we cannot compare the $\varphi_D$ for $D \in \mathcal{D}_{G,X_i}$). So if we want to get a $\mathcal{D}$-equivalence from a $\Lambda^+$-equivalence, we first of all need to get $\Lambda(X_1) = \Lambda(X_2)$.

**Lemma 5.3.1.** *Let $X_1$ and $X_2$ be quasi-projective normal $G$-varieties with finitely many orbits. If $X_1$ and $X_2$ are $\Lambda^+$-equivalent, then $\Lambda(X_1) = \Lambda(X_2)$.*

*Proof.* Let $\mathcal{O}_{X_1}(D_1)$ be any $G$-linearized ample invertible sheaf on $X_1$, and let $D_2 \in \mathrm{Div}_B^G(X_2)$ be such that $\theta([\mathcal{O}_{X_1}(D_1)]) = [\mathcal{O}_{X_2}(D_2)]$. Any element of $\Lambda(X_1)$ has the form $\mu - \mu'$, where $\mu, \mu' \in \Lambda_G$ and $(\mu, n), (\mu', n) \in \Lambda^+(X_1, \mathcal{O}_{X_1}(D_1))$ for some $n \geq 0$. Then, we have $(\mu, n), (\mu', n) \in \Lambda^+(X_2, \mathcal{O}_{X_2}(D_2))$ as well, so we may pick nonzero $B$-eigenvectors $f, f' \in H^0(X_2, \mathcal{O}_{X_2}(nD_2))$ of weights $\mu$ and $\mu'$ (respectively). Then, we have an isomorphism $\mathcal{O}_{(X_2)_{f'}} \xrightarrow{\sim} \mathcal{O}_{X_2}(nD_2)|_{(X_2)_{f'}}$ given by sending $1 \mapsto f'$. The $G$-linearization on $\mathcal{O}_{X_2}(nD_2)$ then induces a $G$-linearization on $\mathcal{O}_{(X_2)_{f'}}$ such that $f|_{(X_2)_{f'}}$ is identified with a nonzero $B$-eigenvector of weight $\mu - \mu'$ in $\Gamma((X_2)_{f'}, \mathcal{O}_{X_2})$. The restriction of this $B$-eigenvector to the function field $K(X_2)$ is again a $B$-eigenvector of weight $\mu - \mu'$, so that $\mu - \mu' \in \Lambda(X_2)$. This proves that $\Lambda(X_1) \subset \Lambda(X_2)$, and swapping the roles of $X_1$ and $X_2$ and repeating this argument gives us the opposite containment. $\square$

Recall that, given any character $\lambda \in \mathcal{X}(G)$, we denote by $\mathcal{O}_X(\lambda)$ the structure sheaf equipped with the $G$-linearization induced by $\lambda$. By Theorem 2.6.5 (applied to divisorial sheaves instead of invertible sheaves, which is allowed by our arguments in Appendix B), we have an exact sequence

$$\mathcal{X}(G) \xrightarrow{\sigma_X} \mathrm{Cl}_G(X) \xrightarrow{\tau_X} \mathrm{Cl}(X),$$

where $\sigma_X$ is given by $\lambda \mapsto \mathcal{O}_X(\lambda)$ and $\tau_X$ is the "forgetful map," i.e. the map that sends the $G$-equivariant isomorphism class of a $G$-linearized divisorial sheaf $\mathcal{O}_X(D)$ to the isomorphism class of $\mathcal{O}_X(D)$ (ignoring the $G$-linearization). Moreover, by Theorem 2.6.11 the image of $\tau_X$ has finite index in $\mathrm{Cl}(X)$, and there exists an isogeny of algebraic groups $\tilde{G} \to G$ such that the "forgetful map" $\mathrm{Cl}_{\overline{G}}(X) \to \mathrm{Cl}(X)$ is surjective.

Using these facts, we first show that a $\Lambda^+$-equivalence induces an isomorphism $\mathrm{Cl}(X_1) \xrightarrow{\sim} \mathrm{Cl}(X_2)$.

**Lemma 5.3.2.** *Let $X_1$ and $X_2$ be normal $G$-varieties with finitely many $G$-orbits, and let $\theta : \mathrm{Cl}_G(X_1) \to \mathrm{Cl}_G(X_2)$ be a $\Lambda^+$-equivalence. Suppose that $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.*

*(a) For any $\lambda \in \hat{G}$, we have $\alpha([\mathcal{O}_{X_1}(\lambda)]) = [\mathcal{O}_{X_2}(\lambda)]$.*

(b) *There exists an isomorphism* $\overline{\theta} : \mathrm{Cl}(X_1) \xrightarrow{\sim} \mathrm{Cl}(X_2)$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Cl}_G(X_1) \arrow[r, "\theta"] \arrow[d, "\tau_{X_1}"] & \mathrm{Cl}_G(X_2) \arrow[d, "\tau_{X_2}"] \\ \mathrm{Cl}(X_1) \arrow[r, "\overline{\theta}"] & \mathrm{Cl}(X_2) \end{tikzcd}$$

*If $X_2$ is a complete spherical variety (or more generally, if $\mathrm{Cl}(X_2)$ is free), then $\overline{\theta}$ is the unique isomorphism fitting into this diagram.*

*Proof.* First, we claim that $\theta$ induces a bijection between classes containing $\mathcal{O}_{X_1}$ and those containing $\mathcal{O}_{X_2}$. Since any $D \in \mathrm{Div}_B(X_i)$ is effective if and only if $\Lambda^+(X_i, \mathcal{O}_{X_i}(D)) \neq \{(0,d) \mid d \in \mathbb{N}\}$, we see that $\theta$ is a bijection between classes of divisorial sheaves of the form $\mathcal{O}_{X_i}(D)$ for some effective divisor $D$. Since $\theta$ is a group homomorphism, it also induces a bijection on sheaves $\mathcal{O}_{X_i}(D)$ such that such that $D$ is effective and $\mathcal{O}_{X_i}(-D) \cong \mathcal{O}_{X_i}(E)$ for some effective divisor $E$ (equivalently, $-D$ is linearly equivalent to some effective divisor $E$). To prove the claim, then, it suffices to show that if $D \in \mathrm{Div}_B(X_i)$ is effective and $-D+\mathrm{div}(f)$ is effective for some $f \in K(X)$, then $D = 0$. For this, we note that the divisor

$$D - D + \mathrm{div}(f) = \mathrm{div}(f)$$

is a sum of effective divisors and so is effective. In other words, for every point $x \in X_i$ such that $\overline{\{x\}}$ has codimension 1, we have $f \in \mathcal{O}_{X_i,x}$. Because $X$ is normal, we have

$$\Gamma(X_i, \mathcal{O}_{X_i}) = \bigcap_{\mathrm{codim}(\overline{\{x\}})=1} \mathcal{O}_{X_i,x}.$$

So, we have $f \in \Gamma(X_i, \mathcal{O}_{X_i}) = k$ and hence $\mathrm{div}(f) = 0$. In other words, both $D$ and $-D$ are effective, which implies that $D = 0$.

Now, for any $\lambda \in \mathcal{X}(G)$, we have

$$\Lambda^+(X_i, \mathcal{O}_{X_i}(\lambda)) = \{(d\lambda, d) \mid d \in \mathbb{N}\}.$$

In particular, the class $[\mathcal{O}_{X_i}(\lambda)] \in \mathrm{Cl}_G(X_i)$ is uniquely determined among classes containing some $G$-lienarization of $\mathcal{O}_{X_i}$ by its weight monoid $\Lambda^+(X_i, \mathcal{O}_{X_i}(\lambda))$. Statement (a) now follows immediately from this fact, the above claim, and the definition of a $\Lambda^+$-equivalence. As for (b), we have

$$\ker(\tau_i) = \{[\mathcal{O}_{X_i}(\lambda)] \mid \lambda \in \hat{G}\}.$$

So, Statement (a) is exactly the statement that $\theta(\ker(\rho_1)) = \ker(\rho_2)$, and (b) follows formally from this fact. Finally, let $I_1$ of $\tau_1$. Then, $\overline{\theta}|_{I_1}$ is determined by commutativity of the diagram in Statement (b). On the other hand, we noted above that $I_1$ is a subgroup of finite index in $\mathrm{Cl}(X_1)$. It follows that $\overline{\theta}$ is completely determined by its restriction to $I_1$. More explicitly:

for any $[D] \in \mathrm{Cl}(X_1)$, we have $n[D] \in I_1$ for some $n > 0$. Since $\overline{\theta}$ is a homomorphism of abelian groups, the image $\overline{\theta}([D]) = [D']$ is an element of $\mathrm{Cl}(X_2)$ such that

$$n[D'] = \overline{\theta}|_{I_1}(n[D]).$$

If $X_2$ is complete, then $\mathrm{Cl}(X_2)$ is free by Proposition 3.7.1, so there is at most one such choice of $D'$ (namely: writing $\mathrm{Cl}(X_2) \cong \mathbb{Z}^r$ for some $r$ and $\overline{\theta}|_{I_1}(n[D]) = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, we have $[D'] = (m_1/n, \ldots, m_r/n)$.) ☐

Next, we lift the isomorphism on class groups $\mathrm{Cl}(X_1) \xrightarrow{\sim} \mathrm{Cl}(X_2)$ in the above lemma to an isomorphism on the level of divisors.

**Proposition 5.3.3.** *Let $X_1$ and $X_2$ be normal $G$-varieties with finitely many orbits such that $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$, and let $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ be a $\Lambda^+$-equivalence. There exists an isomorphism $\tilde{\theta} : \mathrm{Div}_B^G(X_1) \xrightarrow{\sim} \mathrm{Div}_B^G(X_2)$ such that the following hold.*

*(a) For all $D_1 \in \mathrm{Div}_B^G(X_1)$ and any $G$-linearization on $\mathcal{O}_{X_1}(D_1)$, we have $\theta([\mathcal{O}_{X_1}(D_1)]) = [\mathcal{O}_{X_2}(\tilde{\theta}(D_1))]$ for some $G$-linearization on $\mathcal{O}_{X_2}(\tilde{\theta}(D_1))$. In particular, for these $G$-linearizations, we have*

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(D_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(\tilde{\theta}(D_1))).$$

*(b) Any $D_1 \in \mathrm{Div}_B^G(X_1)$ is effective if and only if $\tilde{\theta}(D_1)$ is. Moreover, if $D_1$ and $\tilde{\theta}(D_1)$ are effective, then for any $G$-linearizations on $\mathcal{O}_{X_1}(D_1)$ and $\mathcal{O}_{X_2}(\tilde{\theta}(D_1))$ as in (a), the canonical sections of $\mathcal{O}_{X_1}(D_1)$ and $\mathcal{O}_{X_2}(\tilde{\theta}(D_1))$ are $B$-eigenvectors of the same weight.*

*(c) We have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Div}_B^G(X_1) & \longrightarrow & \mathrm{Cl}(X_1) \\ \big\downarrow{\scriptstyle\tilde{\theta}} & & \big\downarrow{\scriptstyle\overline{\theta}} \\ \mathrm{Div}_B^G(X_2) & \longrightarrow & \mathrm{Cl}(X_2) \end{array}$$

*Here, the horizontal arrows send a $B$-stable divisor to its linear equivalence class, and $\overline{\theta}$ is the isomorphism of Lemma 5.3.2.*

*Proof.* We first define $\tilde{\theta}$ on effective divisors. Let $D \in \mathrm{Div}_B^G(X_1)$ be an effective divisor, and let $s_1 \in H^0(X_1, \mathcal{O}_{X_1}(D))$ be the canonical section. Pick any $G$-linearization of $\mathcal{O}_{X_1}(D)$. Then, $s_1$ is a $B$-eigenvector (because $D$ is $B$-stable). So, we may pick some $B$-eigenvector $s_2 \in H^0(X_2, \theta([\mathcal{O}_{X_1}(D)]))$ of the same weight as $s_2$. We define $\tilde{\theta}(D) = \mathrm{div}(s_2)$. We will check in a moment that this is well-defined. First, however, we note that this definition of $\tilde{\theta}$ on effective divisors commutes with addition: that is, for any $D, D' \in \mathrm{Div}_B^G(X_1)$ both effective, we have

$$\tilde{\theta}(D + D') = \tilde{\theta}(D) + \tilde{\theta}(D').$$

Indeed, if $s_1$ and $s_1'$ are the canonical sections of $\mathcal{O}_{X_1}(D)$ and $\mathcal{O}_{X_1}(D')$, respectively, then $s_1 \otimes s_1'$ is the canonical section of $\mathcal{O}_{X_1}(D + D')$, and the weight of $s_1 \otimes s_1'$ is the sum of the weights of $s_1$ and $s_1'$. It follows that $s_2 \otimes s_2'$ is a $B$-eigenvector of the same weight as $s_1 \otimes s_1'$, so we have

$$\operatorname{div}(s_2 \otimes s_2') = \operatorname{div}(s_2) + \operatorname{div}(s_2') = \tilde{\theta}(D) + \tilde{\theta}(D').$$

We claim that the above construction of $\tilde{\theta}$ on effective divisors is well-defined. Note that for any $i \in \{1, 2\}$, any $D' \in \operatorname{Div}_B^G(X_i)$, and any $G$-linearization of $\mathcal{O}_{X_1}(D')$, the $G$-module $H^0(X_i, \mathcal{O}_{X_i}(D'))$ is multiplicity-free. (This follows immediately from Theorem 3.1.4 applied to the restriction of $\mathcal{O}_{X_i}(D')$ to the open subset $X^{\leq 1}$ from Appendix B.) In particular, under our chosen $G$-linearization, $s_1$ is the unique (up to scalar) $B$-eigenvector of some weight, say $\mu$, and $s_2$ is the unique (up to scalar) $B$-eigenvector of $H^0(X_2, \theta([\mathcal{O}_{X_1}(D)]))$ of weight $\mu$. So, we just need to check that the line $k \cdot s_2$ does not depend on the choice of $G$-linearization on $\mathcal{O}_{X_1}(D)$. The exact sequence of Theorem 2.6.5 tells us that any other $G$-linearization of $\mathcal{O}_{X_1}(D)$ can be obtained as the $G$-linearization of $\mathcal{O}_{X_1}(D) \otimes \mathcal{O}_{X_1}(\lambda)$ for a unique $\lambda \in \mathcal{X}(G)$, and for this $G$-linearization, the proof of Corollary 2.6.8 implies that $s_1$ has weight $\mu + \lambda$ On the other hand, Lemma 5.3.2 gives

$$\theta([\mathcal{O}_{X_1}(D) \otimes \mathcal{O}_{X_1}(\lambda)]) = \theta([\mathcal{O}_{X_1}(D)]) \otimes [\mathcal{O}_{X_2}(\lambda)].$$

Using Corollary 2.6.8 again, we see that the unique line of $B$-eigenvectors of weight $\mu + \lambda$ in $H^0(X_2, \theta([\mathcal{O}_{X_1}(D_1) \otimes \mathcal{O}_{X_1}(\lambda)]))$ is the unique line of $B$-eigenvectors of weight $\mu$ in $H^0(X_2, \theta([\mathcal{O}_{X_1}(D_1)])$, which is precisely $k \cdot s_2$ by definition. This proves the claim.

We now extend our definition of $\tilde{\theta}$ from effective divisors in $\operatorname{Div}_B^G(X_1)$ to all of $\operatorname{Div}_B^G(X_1)$. Any divisor $D \in \operatorname{Div}_B^G(X_1)$ can be written in a unique way as a difference $D = E_+ - E_-$ for some effective divisors $E_+, E_- \in \operatorname{Div}_B(X_1)$. Let $m$ be the smallest positive integer such that $mE_- \in \operatorname{Div}_B^G(X_1)$ (such an integer exists by Theorem 2.6.11). Then, we have $D + mE_- \in \operatorname{Div}_B^G(X_1)$ as well, and $D + mE_- = E_+ + (m-1)E_-$ is effective, so we have already defined $\tilde{\theta}$ on both $mE_-$ and $D + mE_-$. We can thus define $\tilde{\theta}(D)$ by

$$\tilde{\theta}(D) = \tilde{\theta}(D + mE_-) - \tilde{\theta}(mE_-). \tag{5.3.1}$$

This definition of $\tilde{\theta}$ agrees with our original definition of $\tilde{\theta}$ on effective divisors (since $E_- = 0$ if $D$ is effective). Moreover, since $\tilde{\theta}$ commutes with addition on effective divisors, it is formal to check that the above definition commutes with addition on all of $\operatorname{Div}_B^G(X_1)$. Thus, we have defined a homomorphism $\tilde{\theta} : \operatorname{Div}_B^G(X_1) \to \operatorname{Div}_B^G(X_2)$.

Now, we can define an inverse to $\tilde{\theta}$ by repeating the same construction but with $X_1$ and $X_2$ swapped (which amounts to replacing $\theta$ by $\theta^{-1}$. So, $\tilde{\theta}$ is in fact an isomorphism. Since $\tilde{\theta}$ sends effective divisors to effective divisors by construction, and likewise for $\theta^{-1}$, we immediately see that $\tilde{\theta}(D_1)$ is effective if and only if $D_1$ is. The rest of (b) is immediate from the way we constructed $\tilde{\theta}$ on effective divisors.

As for (c), using (5.3.1) and the fact that all maps in the given diagram are homomorphisms, one sees that it will suffice to check commutativity of the diagram on effective divisors. So, let $D \in \mathrm{Div}_B^G(X_1)$ be an effective divisor. For any $G$-linearization on $\mathcal{O}_{X_1}(D)$, our construction of $\tilde{\theta}$ for effective divisors gives us

$$\theta([\mathcal{O}_{X_1}(D)]) = [\mathcal{O}_{X_2}(\tilde{\theta}(D))]$$

for some $G$-linearization on $\mathcal{O}_{X_2}(\tilde{\theta}(D))$. On the other hand, the commutative diagram for $\overline{\theta}$ in Lemma 5.3.2 tells us that every $G$-linearization of the sheaf $\overline{\theta}(\mathcal{O}_{X_1}(D))$ lies in the class $\theta([\mathcal{O}_{X_1}(D)]) \in \mathrm{Cl}_G(X_2)$. It follows that

$$\overline{\theta}(\mathcal{O}_{X_1}(D)) \cong \mathcal{O}_{X_2}(\tilde{\theta}(D)),$$

which is exactly the statement that the diagram in (c) commutes for $D$.

Finally, we prove (a). Fix any $G$-linearization on $\mathcal{O}_{X_1}(D_1)$. Statement (c) tells us that $\mathcal{O}_{X_2}(\tilde{\theta}(D_1))$ is in the isomorphism class $\overline{\theta}(\mathcal{O}_{X_1}(D_1))$ in $\mathrm{Cl}(X_2)$. The commutative diagram for $\overline{\theta}$ in Lemma 5.3.2 then tells us that $\theta([\mathcal{O}_{X_1}(D_1)])$ is the class of some $G$-linearization on $\mathcal{O}_{X_2}(\tilde{\theta}(D_1))$, as desired. □

There is one more property of the isomorphism $\tilde{\theta}$ in the above proposition that we will want.

**Lemma 5.3.4.** *Let $X_1$ and $X_2$ be spherical varieties. Suppose that*

1. $\mathrm{Div}_B^G(X_i) = \mathrm{Div}_B(X_i)$ *for* $i \in \{1, 2\}$,
2. $\Lambda(X_1) = \Lambda(X_2)$, *and*
3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*Let $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ be a $\Lambda^+$-equivalence, and let $\tilde{\theta} : \mathrm{Div}_B^G(X_1) \xrightarrow{\sim} \mathrm{Div}_B^G(X_2)$ be the isomorphism of Proposition 5.3.3. For any $\mu \in \Lambda(X_1)$, if $f_i \in K(X_i)^{(B)}$ is a $B$-eigenvector of weight $\mu$, then $\tilde{\theta}(\mathrm{div}(f_1)) = \mathrm{div}(f_2)$.*

*Proof.* By Assumption 1, the isomorphsim $\tilde{\theta}$ is a map $\mathrm{Div}_B(X_1) \xrightarrow{\sim} \mathrm{Div}_B(X_2)$ and so restricts to an isomorphism on monoids of effective divisors. The (unique) minimal sets of generators for these monoids of effective divisors are $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$ (respectively). It follows that $\tilde{\theta}$ induces a bijection between $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$. So, for $i \in \{1, 2\}$, we may write $\mathcal{D}_{G,X_i} = \{D_{i,1}, \ldots, D_{i,r_i}\}$ in such a way that $\tilde{\theta}(D_{1,j}) = D_{2,j}$ for all $j$. With $\mu$ and the $f_i$ as in the lemma statement, Write $\mathrm{div}(f_i) = \sum_j m_{i,j} D_{i,j}$ for some $m_{i,j} \in \mathbb{Z}$. We will use the properties of $\tilde{\theta}$ given in Proposition 5.3.3 to show that $m_{1,j} = m_{2,j}$ for all $j$.

Fix any $j$. Define

$$E_1 = \sum_{j' \neq j} \max\{|m_{1,j'}|, |m_{2,j'}|\} D_{1,j'}, \quad E_2 = \tilde{\theta}(E_1) = \sum_{j' \neq j} \max\{|m_{1,j'}|, |m_{2,j'}|\} D_{2,j'}.$$

Notice that $m_{i,j} \geq 0$ if and only if $E_i$ is effective. Since $\tilde{\theta}$ induces a bijection on effective divisors, it follows that $m_{1,j} \geq 0$ if and only if $m_{2,j} \geq 0$. Thus, either $m_{i,j} \geq 0$ for all $i$ or

$m_{i,j} \leq 0$ for all $i$. After replacing the $f_i$ by $f_i^{-1}$ (which amounts to replacing all the $m_{i,j'}$ by $-m_{i,j'}$), we may assume that $m_{i,j} \leq 0$. In this case, we have

$$m_{i,j} = -\min\{m \in \mathbb{N} \mid E_i + mD_{i,j} + \mathrm{div}(f_i) \geq 0\}.$$

So, it will suffice to show that for any $m \geq 0$, the divisor $E_1 + mD_{1,j} + \mathrm{div}(f_1)$ is effective if and only if $E_2 + mD_{2,j} + \mathrm{div}(f_2)$ is effective.

For this, write $E_i' = E_i + mD_{i,j}$. Since $\tilde{\theta}(E_1') = E_2'$, we we may pick $G$-linearizations on the $\mathcal{O}_{X_i}(E_i')$ as in Proposition 5.3.3a. Then, statement (b) of the proposition says that the canonical sections of $\mathcal{O}_{X_1}(E_1')$ and $\mathcal{O}_{X_2}(E_2')$ are $B$-eigenvectors of the same weight, say $\mu'$. The divisor $E_1' + \mathrm{div}(f_1)$ is effective, so the $G$-module $H^0(X_1, \mathcal{O}_{X_1}(E_1'))$ contains a nonzero $B$-eigenvector of weight $\mu + \mu'$. It follows (again from Proposition 5.3.3a) that $H^0(X_2, \mathcal{O}_{X_2}(E_2'))$ also contains a nonzero $B$-eigenvector of weight $\mu + \mu'$, so that $E_2' + \mathrm{div}(f)$ is effective for some eigenvector $f \in K(X)^{(B)}$ of weight $\mu$. But $f$ must be proportional to the eigenvector $f_2$ (because both have weight $\mu$), so we have $\mathrm{div}(f) = \mathrm{div}(f_2)$. Thus, the sum $E_2' + \mathrm{div}(f_2)$ is effective. Swapping $X_1$ and $X_2$ and repeating this argument gives the converse, that $E_2' + \mathrm{div}(f_2)$ effective implies $E_1' + \mathrm{div}(f_1)$ effective. $\square$

We now use the isomorphism $\tilde{\theta}$ from the above proposition, along with the relationship between simple roots and weights of canonical sections in Proposition 5.2.5, to construct a $\mathcal{D}$-equivalence.

**Definition 5.3.5.** Let $X_1$ and $X_2$ be spherical varieties. Let $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ be a $\Lambda^+$-equivalence, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence. We say that $\iota$ and $\theta$ are *compatible* if for any $B$-stable divisor $D_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ and any $G$-linearization of $\mathcal{O}_{X_1}(D_1)$, there exists a $G$-linearization on $\mathcal{O}_{X_2}(D_2)$ for $D_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$ such that

$$\theta([\mathcal{O}_{X_1}(D_1)]) = [\mathcal{O}_{X_2}(D_2)].$$

**Theorem 5.3.6.** *Let $X_1$ and $X_2$ be spherical varieties, and suppose that*

1. $\mathrm{Pic}(G) = 0$,
2. $\Lambda(X_1) = \Lambda(X_2)$, *and*
3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*For any $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$, there exists a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ which is compatible with $\theta$.*

*Proof.* Since $\mathrm{Pic}(G) = 0$, we have $\mathrm{Div}_B^G(X_i) = \mathrm{Div}_B(X_i)$ (see Theorem 2.6.11). So, the isomorphism of Proposition 5.3.3 is a map $\tilde{\theta} : \mathrm{Div}_B(X_1) \xrightarrow{\sim} \mathrm{Div}_B(X_2)$ that restricts to an isomorphism on monoids of effective divisors, and the (unique) minimal sets of generators for these monoids of effective divisors are $\mathcal{D}_{G,X_1}$ and $\mathcal{D}_{G,X_2}$ (respectively). It follows that $\tilde{\theta}$ restricts to a bijection $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$. We will show that $\iota$ is a $\mathcal{D}$-equivalence. The statement that $\iota$ and $\theta$ are compatible is then almost immediate from Proposition 5.3.3a. Indeed, for any $B$-stable divisor $D_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ on $X_1$, since $\tilde{\theta}$ is a homomorphism and $\tilde{\theta}(D) = \iota(D)$ for all $D \in \mathcal{D}_{G,X_1}$, the divisor $D_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$ satisfies $D_2 = \tilde{\theta}(D_1)$. Thus, the definition of compatibility for $D_1$ is precisely the statement of Proposition 5.3.3a.

First, we claim that for any $D_1 \in \mathcal{D}_{G,X_1}$, we have $\varphi_{D_1} = \varphi_{\iota(D_1)}$ (as maps from $\Lambda(X_1) = \Lambda(X_2)$ to $\mathbb{Z}$). Let $\mu \in \Lambda(X_1)$. Lemma 5.3.4 tells us that $\tilde{\theta}(\mathrm{div}(f_1)) = \mathrm{div}(f_2)$, where $f_i \in K(X)^{(B)}$ is a $B$-eigenvector of weight $\mu$. This gives us

$$\sum_{D \in \mathcal{D}_{G,X_2}} \varphi_D(\mu) D = \mathrm{div}(f_2) = \tilde{\theta}\left(\sum_{D \in \mathcal{D}_{G,X_1}} \varphi_D(\mu) D\right) = \sum_{D \in \mathcal{D}_{G,X_1}} \varphi_D(\mu)\iota(D).$$

The coefficient of $\iota(D_1)$ on the lefthand side of this equation is $\varphi_{\iota(D_1)}(\mu)$, and the coefficient of $\iota(D_1)$ on the righthand side of the equation is $\varphi_{D_1}(\mu)$. Thus, we have $\varphi_{D_1}(\mu) = \varphi_{\iota(D_1)}(\mu)$ for any $\mu \in \Lambda(X_1)$, as desired.

Next, we claim that for any $D_1 \in \mathcal{D}_{G,X_1}$ and any $\alpha \in \Pi_G$, we have $D_1 \in \mathcal{D}_{G,X_1}(\alpha)$ if and only if $\iota(D_1) \in \mathcal{D}_{G,X_2}(\alpha)$. Write $D_2 = \iota(D_1)$. Since $\iota$ is the restriction of $\tilde{\theta}$, Proposition 5.3.3 implies that, for any $G$-linearization on $\mathcal{O}_{X_1}(D_1)$, there exists a $G$-linearization on $\mathcal{O}_{X_2}(D_2)$ such that

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(D_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(D_2)),$$

and moreover, that the canonical sections of $\mathcal{O}_{X_1}(D_1)$ and $\mathcal{O}_{X_2}(D_2)$ have the same weight. Consider the $G$-linearization on $\mathcal{O}_{X_1}(D_1)$ given by Corollary 5.2.7. This is the unique $G$-linearization such that the weight $\mu$ of the canonical section of $\mathcal{O}_{X_1}(D_1)$ is a linear combination of fundamental weights of $G$. By applying the same corollary to $D_2$ (whose canonical section also has weight $\mu$), we conclude that the $G$-linearization on $\mathcal{O}_{X_2}(D_2)$ satisfying the above conditions must also be that of Corollary 5.2.7. In other words, we have we have $\mu_{D_1} = \mu = \mu_{D_2}$, where $\mu_{D_1}$ and $\mu_{D_2}$ are the weights given in the corollary. Since the $D_i$ are prime divisors, the weights $\mu_{D_i}$ in this case are precisely given by Proposition 5.2.5. More precisely, we have:

$$\mu_{D_i} = \begin{cases} \sum_{D \in \mathcal{D}_{G,X(\alpha_j)}} \omega_j, & D_i \text{ moved by a root of type } b \text{ or } d \text{ ,} \\ 2\omega_i, & D_i \text{ moved by } \alpha_j \in \Pi^c_{X_i} \\ 0, & D \text{ is } G\text{-stable} \end{cases}$$

(Here, $\omega_j$ denotes the fundamental weight corresponding to the simple root $\alpha_j$, and the sum in Case 1 is over all roots $\alpha_j \in \Pi_G$ such that $D$ is moved by $\alpha_j$.) In particular, for any $\alpha \in \Pi_G$, the fundamental weight $\omega$ corresponding to $\alpha$ has nonzero coefficient in $\mu_{D_i}$ if and only if $\alpha$ moves $D_i$. Since $\mu_{D_1} = \mu_{D_2}$, we conclude that $\alpha$ moves $D$ if and only if it moves $\iota(D)$. $\square$

**Corollary 5.3.7.** *Let $X_1$ and $X_2$ be projective spherical varieties, and suppose that* $\mathrm{Pic}(G) = 0$. *For any $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$, there exists a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ which is compatible with $\theta$.*

*Proof.* Since $X_1$ and $X_2$ are complete, we have $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$, and since $X_1$ and $X_2$ are quasi-projective and $\Lambda^+$-equivalent, we have $\Lambda(X_1) = \Lambda(X_2)$ by Lemma 5.3.1. Thus, the corollary is immediate from Theorem 5.3.6. $\square$

**Remark 5.3.8.** Assumptions 1-3 in Theorem 5.3.6 seem to be necessary for the theory of $\Lambda^+$-equivalences to work as well as one might hope. Indeed, we will see more key results below that require similar assumptions (e.g. Theorem 5.4.1 and Corollary 5.4.2). As the above corollary indicates, only Assumption 1 (that $\mathrm{Pic}(G) = 0$) is essential in the projective case. In general, all three assumptions are sometimes necessary, but they are both standard and not terribly restrictive, see Remark 1.3.5. Moreover, there is some hope for weakening these assumptions for certain applications: for instance, see Corollary 5.4.3 below.

The above theorem allows us to replace $\mathcal{D}$-equivalences by $\Lambda^+$-equivalences in our phrasing of the classification of spherical varieties. However, our definition of "equality on colored fans" is a $\mathcal{D}$-equivalence that preserves colored fans. So, to remove $\mathcal{D}$-equivalences entirely, we need to define what an "equality on colored fans" means in the context of $\Lambda^+$-equivalences

**Definition 5.3.9.** Let $X_1$ and $X_2$ be spherical varieties, and let $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ be a $\Lambda^+$-equivalence. Suppose that $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

1. For any $B$-divisors $D_1 \in \mathcal{D}_{G,X_1}$ and $D_2 \in \mathcal{D}_{G,X_2}$, we say that $\theta$ *maps* $D_1$ *to* $D_2$ if there exists some $m > 0$ such that $mD_i \in \mathrm{Div}_B^G(X_i)$ and $\tilde{\theta}(mD_1) = mD_2$ (where $\tilde{\theta}$ is the map of Proposition 5.3.3).

2. Let $Y_1 \subset X_1$ be a $G$-orbit with corresponding colored cone $(\mathcal{C}_1, \Delta_1) \in \mathscr{F}_{X_1}$. Suppose that every $B$-divisor $D_1 \in \mathcal{D}_{G,X_1}$ containing $Y_1$ maps to some $B$-divisor of $X_2$. We denote by $\theta(\Delta_1)$ the set
$$\theta(\Delta_1) = \{D_2 \in \mathcal{D}_{G,X_2} \mid \exists D_1 \supset Y_1 \text{ such that } \theta \text{ maps } D_1 \text{ to } D_2\}.$$

3. We say that $\theta$ *preserves colored fans* if for every $(\mathcal{C}, \Delta) \in \mathscr{F}_X$, the set $\theta(\Delta)$ is defined, and
$$\mathscr{F}_{X_2} = \{(\mathcal{C}, \theta(\Delta)) \mid (\mathcal{C}, \Delta) \in \mathscr{F}_{X_1}.$$

Following arguments as in proof of Theorem 5.3.6 above, we saw that when $\mathrm{Pic}(G) = 0$, the map $\tilde{\theta}$ induces a bijection $\mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$. In particular, this implies that every $B$-divisor of $X_1$ maps to some $B$-divisor of $X_2$ (and in fact, we may take $m = 1$ in the above definition of "maps to"). This readily implies the following lemma.

**Lemma 5.3.10.** *In the situation of Theorem 5.3.6, the $\Lambda^+$-equivalence $\theta$ preserves colored fans if and only if the $\mathcal{D}$-equivalence $\iota$ preserves colored fans.*

*Proof.* Our above comments imply that the set $\theta(\Delta)$ is defined for all $(\mathcal{C}, \Delta) \in \mathscr{F}_{X_1}$. Moreover, our construction of $\iota$ is as the restriction of $\tilde{\theta}$. It follows that for any choice of $(\mathcal{C}, \Delta)$, the set $\theta(\Delta)$ is precisely the set $\iota(\Delta)$ in the definition of a $\mathcal{D}$-equivalence "preserving colored fans" (see Definition 4.1.7 and the remark that follows). The lemma now follows immediately from the definitions. □

We now give two variations on the classification of spherical varieties using $\Lambda^+$-equivalences in place of $\mathcal{D}$-equivalences. The first variation is as general as possible, while the second variation assumes projectivity to give a nicer statements. Both statements are made in analogy to our statement of the classification of spherical varieties in Theorem 4.1.9

**Corollary 5.3.11.** *Let $X_1$ and $X_2$ be spherical varieties. Suppose that*

1. $\mathrm{Pic}(G) = 0$ *and*
2. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*Then, the following are equivalent.*

*(i)* *$X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii)* *$\Lambda(X_1, L_1) = \Lambda^+(X_2, L_2)$, $\Psi_{G,X_1} = \Psi_{G,X_2}$, and there exists a $\Lambda^+$-equivalence between $X_1$ and $X_2$ that preserves colored fans.*

*Proof.* That (i) implies (ii) is formal, since everything in (ii) is essentially an "equality" on a combinatorial invariant. If (ii) holds, then Theorem 5.3.6 and Lemma 5.3.10 imply that there exists a $\mathcal{D}$-equivalence between $X_1$ and $X_2$ that preserves colored fans, so (i) follows from Theorem 4.1.9. □

**Corollary 5.3.12.** *Let $X_1$ and $X_2$ be projective spherical varieties, and suppose that* $\mathrm{Pic}(G) = 0$. *The following are equivalent.*

*(i)* *$X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii)* *$\Psi_{G,X_1} = \Psi_{G,X_2}$, and there exists a $\Lambda^+$-equivalence between $X_1$ and $X_2$ that preserves colored fans.*

*Proof.* We have $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$ because $X_1$ and $X_2$ are complete, and assuming (ii), we have $\Lambda(X_1) = \Lambda(X_2)$ because $X_1$ and $X_2$ are quasi-projective (see Lemma 5.3.1). Thus, the statement follows immediately from Corollary 5.3.11 above. □

## 5.4 From $\mathcal{D}$-Equivalences to $\Lambda^+$-Equivalences

In the previous section, we saw how to obtain a $\mathcal{D}$-equivalence from a $\Lambda^+$-equivalence. In this section, we aim to obtain a $\Lambda^+$-equivalence from a $\mathcal{D}$-equivalence. Recall from Lemma 2.5.3 that for any $B$-stable Weil divisor $D$ on a spherical variety $X$ whose canonical section on $\mathcal{O}_X(D)$ has weight $\mu_0$ (for some $G$-linearization of $\mathcal{O}_X(D)$), we have

$$\Lambda^+(H^0(X, \mathcal{O}_X(D))) = \{\mu_0 + \mu \in \Lambda(X) \mid D + \mathrm{div}(f_\mu) \geq 0\}.$$

The question of whether $D + \mathrm{div}(f_\mu) \geq 0$ is entirely determined by the valuations $\varphi_{D'}$ of $B$-divisors $D' \in \mathcal{D}_{G,X}$. Thus, a $D$-equivalence should be able to equate all of the information in the above equation except for the weight $\mu_0$ of the canonical section. Fortunately, for the canonical $G$-linearization of Corollary 5.2.7, the weight of the canonical section can be determined entirely by the data of which simple roots move which colors, which is another piece of data that a $\mathcal{D}$-equivalence can equate.

**Theorem 5.4.1.** *Let $X_1$ and $X_2$ be spherical varieties, and suppose that*

1. $\mathrm{Pic}(G) = 0$ *and*
2. $\Gamma(X_1, \mathcal{O}_{X_1}) \cong \Gamma(X_2, \mathcal{O}_{X_2})$ *as $G$-modules.*

*For any $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$, there exists a $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ which is compatible with $\iota$.*

*Proof.* The assumption $\mathrm{Pic}(G) = 0$ implies that every divisorial sheaf on $X_1$ and $X_2$ is $G$-linearizable, i.e. that $\mathrm{Div}^G_B(X_i) = \mathrm{Div}_B(X_i)$ (see Theorem 2.6.11). We will use this fact throughout the proof without further mention.

To begin, we can define an isomorphism $\tilde{\theta} : \mathrm{Div}_B(X_1) \xrightarrow{\sim} \mathrm{Div}_B(X_2)$ from $\iota$ by setting $\tilde{\theta}(D) = \iota(D)$ for all $D \in \mathcal{D}_{G,X_1}$ and extending linearly. Using the fact that $\varphi_D = \varphi_{\iota(D)}$ for all $D$, one can check that $\tilde{\theta}$ fits into the following commutative diagram:

$$\begin{tikzcd}
\Lambda(X_1) \arrow[r] \arrow[d, equal] & \mathrm{Div}_B(X_1) \arrow[r] \arrow[d, "\tilde{\theta}"] & \mathrm{Cl}(X_1) \arrow[r] \arrow[d, dashed, "\overline{\theta}"] & 0 \\
\Lambda(X_2) \arrow[r] & \mathrm{Div}_B(X_2) \arrow[r] & \mathrm{Cl}(X_2) \arrow[r] & 0
\end{tikzcd}$$

Here, the rows are the exact sequences of Proposition 3.7.1. Exactness of the rows (along with commutativity) implies that $\tilde{\theta}$ descends to an isomorphism $\overline{\theta}$.

We will define a $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ from $\overline{\theta}$ in the following way. Let $\mathcal{F} \in \mathrm{Cl}(X_1)$ be any divisorial sheaf, and let $\tau_i : \mathrm{Cl}_G(X_i) \to \mathrm{Cl}(X)$ be the "forgetful" map (i.e. the one that sends a $G$-linearized divisorial sheaf to the isomorphism class of that sheaf). We will define a bijection

$$\theta_{\mathcal{F}} : \tau_1^{-1}(\mathcal{F}) \to \tau_2^{1}(\overline{\theta}(\mathcal{F}))$$

such that $\Lambda^+(X_1, c) = \Lambda^+(X_2, \theta_{\mathcal{F}}(c))$ for every class $c \in \tau_1^{-1}(\mathcal{F})$. Since $\mathrm{Cl}_G(X_1)$ (resp. $\mathrm{Cl}_G(X_2)$) is the disjoint union of the preimages $\tau_1^{-1}(\mathcal{F})$ (resp. $\tau_2^{-1}(\overline{\theta}(\mathcal{F}))$) as $\mathcal{F}$ varies over every element of $\mathrm{Cl}(X_1)$, taking the $\theta_{\mathcal{F}}$ for every choice of $\mathcal{F}$ will give us the desired $\Lambda^+$-equivalence $\theta$.

Now, any $\mathcal{F} \in \mathrm{Cl}(X_1)$ is represented by a sheaf of the form $\mathcal{O}_{X_1}(E_1)$, where $E_1 \in \mathrm{Div}_B(X_1)$. Write $E_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$. Proposition 5.2.5 gives us a canonical $G$-linearizations of $\mathcal{O}_X(D)$ for all $D \in \mathcal{D}_{G,X_1}$, and tensoring these together gives a $G$-linearization of $\mathcal{O}_{X_1}(E_1)$ such that the canonical section of $\mathcal{O}_{X_1}(E_1)$ has weight

$$\mu_0 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \mu_D,$$

where $\mu_D$ is as in Proposition 5.2.5. Note in particular that $\mu_D$ depends only on the fundamental weights corresponding to simple roots (which are invariants of $G$) and on which simple roots move $D$ (which is a property preserved by $\iota$). It follows that $\mu_D = \mu_{\iota(D)}$ for all $D \in \mathcal{D}_{G,X_1}$. Thus, Proposition 5.2.5 gives us a $G$-linearization of $\mathcal{O}_{X_2}(\tilde{\theta}(E_1))$ whose canonical section also has weight $\mu_0$.

Write $E_2 = \tilde{\theta}(E_1)$. We claim that

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(E_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(E_2)).$$

In degree 0, we have $H^0(X_1, \mathcal{O}_{X_1}) \cong H^0(X_2, \mathcal{O}_{X_2})$ as $G$-modules by assumption. For any $n \geq 1$, Lemma 2.5.3 tells us that the weights of $B$-eigenvectors in $H^0(X_i, \mathcal{O}_{X_i}(nE_i))$ are precisely the weights $n\mu_0 + \mu$, where $\mu \in \Lambda(X_1)$ is such that $nE_i + \mathrm{div}(f_\mu)$ is effective (here $f_\mu \in K(X_i)^{(B)}$ is an element of weight $\mu$). Commutativity of the above diagram with $\tilde{\theta}$ in it is precisely the statement that $\tilde{\theta}(nE_1 + \mathrm{div}(f_\mu)) = nE_2 + \mathrm{div}(f_\mu)$, and we know that $\tilde{\theta}$ restricts to a bijection on effective divisors. It follows that $nE_1 + \mathrm{div}(f_\mu)$ is an effective divisor on $X_1$ if and only if $nE_2 + \mathrm{div}(f_\mu)$ is an effective divisor on $X_2$. So, $H^0(X_1, \mathcal{O}_{X_1}(nE_1))$ and $H^0(X_2, \mathcal{O}_{X_2}(nE_2))$ have the same weights of $B$-eigenvectors for all $n$, which proves the claim.

We wish to define $\theta_{\mathcal{F}}$ by sending by sending $\mathcal{O}_{X_1}(E_1)$ to $\mathcal{O}_{X_2}(E_2)$, both with the above $G$-linearizations. Any other $G$-linearization on $\mathcal{O}_{X_i}(E_i)$ is obtained from this one by tensoring by $\mathcal{O}_{X_i}(\lambda)$ for some $\lambda \in \mathcal{X}(G)$, and the weight monoid for this $G$-linearization is completely determined by $\lambda$ and the weight monoid for our bove $G$-linearization of $\mathcal{O}_{X_i}(E_i)$, see Corollary 2.6.8. So, we define $\theta_{\mathcal{F}}$ by setting

$$\mathcal{O}_{X_1}(E_1) \otimes \mathcal{O}_{X_1}(\lambda) \mapsto \mathcal{O}_{X_2}(E_2) \otimes \mathcal{O}_{X_2}(\lambda).$$

From the description of weight monoids of different $G$-linearizations in Corollary 2.6.8, one can check that for any two characters $\lambda, \lambda' \in \Lambda_G$, we have

$$\Lambda^+(X_i, \mathcal{O}_{X_i}(E_i) \otimes \mathcal{O}_{X_i}(\lambda)) = \Lambda^+(X_i, \mathcal{O}_{X_i}(E_i) \otimes \mathcal{O}_{X_i}(\lambda'))$$

if and only if $\lambda = \lambda'$. It follows that $\theta_{\mathcal{F}}$ is the unique bijection $S_1 \to S_2$ such that $\Lambda^+(X_1, s) = \Lambda^+(X_2, \theta_{S_1}(s))$ for every class $s \in S_1$. In particular, $\theta_{\mathcal{F}}$ must not depend on our choice of divisor $E_1$ such that $\mathcal{F} \cong \mathcal{O}_{X_1}(E_1)$, so $\theta_{\mathcal{F}}$ is well-defined and satisfies the necessary properties.

Finally, we note that by definition of $\tilde{\theta}$, we have $\tilde{\theta}(E_1) = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$ in the above construction. Thus, the definition of $\theta$ and $\iota$ being compatible follows immediately from the above construction (specifically, from the fact that we defined $\theta([\mathcal{O}_{X_1}(E_1)])$ for any $G$-linearization on $\mathcal{O}_{X_1}(E_1)$ to be the class of some $G$-linearization on $\mathcal{O}_{X_2}(E_2)$). $\square$

**Corollary 5.4.2.** *Let $X_1$ and $X_2$ be spherical varieties, and suppose that*

1. $\mathrm{Pic}(G) = 0$,
2. $\Lambda(X_1) = \Lambda(X_2)$, *and*
3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*(For instance, these 3 assumptions hold if* $\mathrm{Pic}(G) = 0$ *and* $X_1$ *and* $X_2$ *are projective, cf. Corollary 5.3.7.) Then,* $X_1$ *and* $X_2$ *are* $\mathcal{D}$*-equivalent if and only if they are* $\Lambda^+$*-equivalent, and if this is true, then there exist a* $\mathcal{D}$*-equivalence and a* $\Lambda^+$*-equivalence which are compatible.*

*Proof.* One direction is Theorem 5.3.6. The other direction is Theorem 5.4.1. □

Since $\iota$ and $\theta$ are compatible in the above theorem, we can take any $B$-stable divisor $D_1$ on $X_1$ that is interesting to us and obtain an equality on weight monoids of the form $\Lambda^+(X_1, \mathcal{O}_{X_1}(D_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(D_2))$. The ability to get this equality on weight monoids can be very useful in practice. However, when we do this using Theorem 5.4.1, we do need to assume that $\mathrm{Pic}(G) = 0$. The following corollary allows us to get a nice equality on weight monoids even without the assumption $\mathrm{Pic}(G) = 0$, which can be useful for certain technical applications.

**Corollary 5.4.3.** *Let* $X_1$ *and* $X_2$ *be spherical varieties such that* $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$*, and let* $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ *be a* $\mathcal{D}$*-equivalence. Let* $D_1 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D D$ *be a* $B$*-stable divisor on* $X_1$*, and let* $D_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_D \iota(D)$*. There exists some* $m > 0$ *and some* $G$*-linearizations on* $\mathcal{O}_{X_1}(mD_1)$ *and* $\mathcal{O}_{X_2}(mD_2)$ *such that*

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(mD_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(mD_2)).$$

*Moreover, we may pick* $m$ *to be any integer such that the sheaves* $\mathcal{O}_{X_i}(mD_i)$ *are* $G$*-linearizable.*

*Proof.* Let $\pi : \tilde{G} \to G$ be a central isogeny such that $\mathrm{Pic}(\tilde{G}) = 0$. Let $m_i > 0$ be such that $\mathcal{O}_{X_1}(m_i D_i)$ is $G$-linearizable, and set $m = m_1 m_2$. Then, both $\mathcal{O}_{X_1}(mD_1)$ and $\mathcal{O}_{X_2}(mD_2)$ are $G$-linearizable, and pulling back any $G$-linearization of $\mathcal{O}_{X_i}(mD_i)$ by the map $\rho_i = (\pi, \mathrm{id}_{X_i}) : \tilde{G} \times X_i \to G \times X_i$ induces a $\tilde{G}$-linearization of $\mathcal{O}_{X_i}(mD_i)$ with the same weight monoid (see Lemma 4.1.5). Moreover, precomposing characters of $G$ by $\pi$ yields a homomorphism $\iota : \mathcal{X}(G) \to \mathcal{X}(\tilde{G})$ which is is injective because $\pi$ is surjective, and for any $\lambda \in \mathcal{X}(G)$, pulling back $\mathcal{O}_{X_i}(\lambda)$ by $\rho$ yields the $\tilde{G}$-linearized sheaf $\mathcal{O}_{X_i}(\iota(\lambda))$. So, picking any $G$-linearization on $\mathcal{O}_{X_i}(mD_i)$, we see that the $\tilde{G}$-linearizations on $\mathcal{O}_{X_i}(mD_i)$ that come from $G$-linearizations are precisely those of the form $\mathcal{O}_{X_i}(mD_i) \otimes \mathcal{O}_{X_i}(\iota(\lambda))$ for some $\lambda \in \mathcal{X}(G)$, or equivalently, those such that the degree-0 part of $\Lambda^+(X_i, \mathcal{O}_{X_i}(mD_i))$ has a single weight which is a character of $G$ (see Corollary 2.6.8, and note that we are using the fact that the global sections of $X_1$ and $X_2$ are both $k$ here).

Now, pick a $G$-linearization on $\mathcal{O}_{X_1}(mD_1)$, and consider the $\tilde{G}$-linearization induced by it. Theorem 5.4.1 gives us a $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_{\tilde{G}}(X_1) \xrightarrow{\sim} \mathrm{Cl}_{\tilde{G}}(X_2)$ such that $\theta(\mathcal{O}_{X_1}(mD_1)) = \mathcal{O}_{X_2}(mD_2)$ for some $G$-linearization on the $\mathcal{O}_{X_2}(mD_2)$. In particular, we have

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(mD_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(mD_2)).$$

Since the $\tilde{G}$-linearization on $\mathcal{O}_{X_1}(mD_1)$ comes from a $G$-linearization, the monoid $\Lambda^+(X_1, \mathcal{O}_{X_1}(mD_1))$ has a single character of $G$ in degree 0, hence so does $\Lambda^+(X_2, \mathcal{O}_{X_2}(mD_2))$. Thus, our $\tilde{G}$-linearization on $\mathcal{O}_{X_2}(mD_2)$ comes from a $G$-linearization. Since passing between $G$-linearizations and $\tilde{G}$-linearizations does not change weight monoids, we now have

$$\Lambda^+(X_1, \mathcal{O}_{X_1}(mD_1)) = \Lambda^+(X_2, \mathcal{O}_{X_2}(mD_2))$$

using $G$-linearizations on these sheaves, which is exactly what we wanted. $\square$

## 5.5 Strong Equivalences

Let $X_1$ and $X_2$ be projective spherical varieties, and suppose that $X_1$ and $X_2$ are $\Lambda^+$-equivalent. If $\mathrm{Pic}(G) = 0$, then $X_1$ and $X_2$ are $\mathcal{D}$-equivalent by Theorem 5.3.6, which in particular gives us $\Pi^b_{X_1} = \Pi^b_{X_2}$. We thus recover the conclusion of Theorem 4.5.5, our main result on weight monoids and $\mathcal{D}$-equivalences from Chapter 4, and we avoid having to make the extra assumption that $\Pi^b_{X_1} = \Pi^b_{X_2}$ in that theorem. In other words, working with a $\Lambda^+$-equivalence instead of with a single equality $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ avoids the main technical issue we had in Theorem 4.5.5 and gives us as nice a statement as we could hope for.

We are thus interested in translating the other results of Chapter 4 to the setting of $\Lambda^+$-equivalences. At face value, this seems easy: a $\Lambda^+$-equivalence is essentially like having many equalities $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, which seems better than only having one such equality. However, our proof techniques in Chapter 4 hinged on being able to take both $L_1$ and $L_2$ to be ample: indeed, ampleness guarantees that when we apply the local structure theorem, we get affine varieties, so we can apply the Knop conjecture to them (see Theorem 4.4.6). One might hope that another proof technique would circumvent this need for ampleness. However, the data of a $\Lambda^+$-equivalence is actually equivalent to that of a $\mathcal{D}$-equivalence under our assumptions here (see Corollary 5.4.2). So, we would not expect any other proof technique to allow us to use a $\Lambda^+$-equivalence to obtain results about other combinatorial data, such as our results on spherical roots in Theorem 4.6.8. Intuitively, it seems that having $L_1$ and $L_2$ be ample is a critical piece of data in most of our results of Chapter 4, and a $\Lambda^+$-equivalence has no way of capturing that data.

To rectify this issue and relate our results in Chapter 4 to the setting of $\Lambda^+$-equivalences, we want to consider $\Lambda^+$-equivalences which also carry some information about ampleness. To this end, we make the following definition.

**Definition 5.5.1.** Let $X_1$ and $X_2$ be quasi-projective spherical varieties.

1. Let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $G$-linearized divisorial sheaves on $X_1$ and $X_2$ (respectively). We say that $\iota$ *maps* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ if there exists an effective $B$-stable Weil divisor $\sum_{D\in\mathcal{D}_{G,X_1}} n_D D$ on $X_1$ and isomorphisms $\mathcal{F}_1 \cong \mathcal{O}_{X_1}(\sum_D n_D D)$ and $\mathcal{F}_2 \cong \mathcal{O}_{X_2}(\sum_D n_D \iota(D))$ such that the canonical sections of $\mathcal{O}_{X_1}(\sum_D n_D D)$ and $\mathcal{O}_{X_2}(\sum_D n_D \iota(D))$ have the same weights (under the given $G$-linearizations on $\mathcal{F}_1$ and $\mathcal{F}_2$).

2. We say that a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ is *strong* if there exist $G$-linearized ample invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\iota$ maps $L_1$ to $L_2$. If a strong $\mathcal{D}$-equivalence exists, we say that $X_1$ and $X_2$ are *strongly $\mathcal{D}$-equivalent*.

3. We say that a $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ is *strong* if there exists a $G$-linearized ample invertible sheaf $L_1$ such that $\theta(L_2)$ is also invertible and ample. If a strong $\Lambda^+$-equivalence exists, we say that $X_1$ and $X_2$ are *strongly $\Lambda^+$-equivalent*.

Corollary 5.4.2 tells us that $\mathcal{D}$-equivalences and $\Lambda^+$-equivalences are "often" the same. It follows almost immediately that strong $\mathcal{D}$-equivalences and strong $\Lambda^+$-equivalences are equivalent under the same conditions.

**Corollary 5.5.2.** *Let $X_1$ and $X_2$ be spherical varieties. Suppose that*

1. $\mathrm{Pic}(G) = 0$,

2. $\Lambda(X_1) = \Lambda(X_2)$, *and*

3. $\Gamma(X_1, \mathcal{O}_{X_1}) = \Gamma(X_2, \mathcal{O}_{X_2}) = k$.

*(For instance, this conditions hold if* $\mathrm{Pic}(G) = 0$ *and $X_1$ and $X_2$ are projective, cf. Corollary 5.3.7.) Then, $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent if and only if they are strongly $\Lambda^+$-equivalent. Moreover, in this case, there exists a strong $\mathcal{D}$-equivalence and a strong $\Lambda^+$-equivalence which are compatible with each other (in the sence of general $\mathcal{D}$-equivalences and $\Lambda^+$-equivalences).*

*Proof.* Corollary 5.4.2 implies that $X_1$ and $X_2$ are $\mathcal{D}$-equivalent if and only if they are $\Lambda^+$-equivalent, and if this is the case, then we have a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ and a $\Lambda^+$-equivalence $\theta : \mathrm{Cl}_G(X_1) \xrightarrow{\sim} \mathrm{Cl}_G(X_2)$ which are compatible. In other words, for any $B$-stable divisor $D_1 = \sum_{D\in\mathcal{D}_{G,X_1}} n_D D$ and any $G$-linearization on $\mathcal{O}_{X_1}(D_1)$, we may pick some $G$-linearization on $\mathcal{O}_{X_2}(D_2)$ with $D_2 = \sum_{D\in\mathcal{D}_{G,X_1}} n_D \iota(D)$ such that

$$\theta([\mathcal{O}_{X_1}(D_1)]) = [\mathcal{O}_{X_2}(D_2)].$$

This equation (along with the fact that $D_i$ is Cartier if and only if $\mathcal{O}_{X_i}(D_i)$ is invertible) implies that $\iota$ is strong if and only if $\theta$ is strong. $\square$

Suppose $X_1$ and $X_2$ are smooth, projective spherical varities. If $X_1$ and $X_2$ are strongly $\Lambda^+$-equivalent (or strongly $\mathcal{D}$-equivalent), then there exist $G$-linearized ample invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, so the assumptions of most of our main results in Chapter 4 are satisfied. These results thus readily translate over to the setting of a *strong* ($\Lambda^+$- or $\mathcal{D}$-) equivalence. For instance, our main "classification result," Corollary 4.8.1, almost immediately yields the following statement:

**Corollary 5.5.3.** *Let $X_1$ and $X_2$ be smooth projective spherical varieties, and let $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ be the set of all $\gamma \in \Psi_{G,X_i}$ such that either $\gamma \in \Pi^b_{X_i}$ or $\gamma$ satisfies one of the 4 possibilities in Proposition 4.6.5c. The following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$, and $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent.*

*(iii) $\Psi^{exc}_{G,X_1} = \Psi^{exc}_{G,X_2}$, and $X_1$ and $X_2$ are strongly $\Lambda^+$-equivalent.*

*Proof.* It is formal to check that any $G$-equivariant isomorphism induces a strong $\Lambda^+$-equivalence and a strong $\mathcal{D}$-equivalence, so (i) implies both (ii) and (iii). Conversely, if either (ii) or (iii) holds, then Condition (ii) of Corollary 4.8.1 holds by definition of a strong ($\Lambda^+$- or $\mathcal{D}$-) equivalence, so that corollary implies (i). □

We can alter the above corollary slightly by removing the condition on type $b$-roots, since that condition is captured by a $\mathcal{D}$-equivalence (or a $\Lambda^+$-equivalence when $\mathrm{Pic}(G) = 0$). A similar alteration will also work for any other results in Chapter 4 that involve type $b$ roots (for instance, Theorem 4.6.8) when translating to the setting of strong equivalences.

**Corollary 5.5.4.** *Let $X_1$ and $X_2$ be smooth projective spherical varieties, and let $\Psi^{exc}_{G,X_i} \subset \Psi_{G,X_i}$ be the set of all $\gamma \in \Psi_{G,X_i}$ such that either $\gamma \in \Pi^b_{X_i}$ or $\gamma$ satisfies one of the 4 possibilities in Proposition 4.6.5c. The following are equivalent.*

*(i) $X_1$ and $X_2$ are $G$-equivariantly isomorphic.*

*(ii) $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$, and $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent.*

*If* $\mathrm{Pic}(G) = 0$*, these are also equivalent to the following condition.*

*(iii) $\Psi^{exc}_{G,X_1} \setminus \Pi^b_{X_1} = \Psi^{exc}_{G,X_2} \setminus \Pi^b_{X_2}$, and $X_1$ and $X_2$ are strongly $\Lambda^+$-equivalent.*

*Proof.* If either (i) or (ii) holds, we have $\Pi^b_{X_1} = \Pi^b_{X_2}$ (see Lemma 4.1.2), so equivalence of (i) and (ii) follows from Corollary 5.5.3. If $\mathrm{Pic}(G) = 0$, then (ii) and (iii) are equivalent by Corollary 5.5.2. □

In the remainder of this section, we seek to understand more directly what the relationship is between the data of a strong ($\Lambda^+$- or $\mathcal{D}$-) equivalence and the data of two $G$-linearized ample invertible sheaves $L_1$ and $L_2$ such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. For this, we will work primarily with $\mathcal{D}$-equivalences, noting that Corollary 5.5.2 typically allows one to translate between strong $\Lambda^+$- and $\mathcal{D}$-equivalences.

The key will be to understand what conditions allow us to conclude that a given $\mathcal{D}$-equivalence maps some $G$-linearized ample line bundle $L_1$ to some $G$-linearized ample line bundle $L_2$. For this, we actually introduce another compatibility condition between $\mathcal{D}$-equivalences and $G$-linearized ample line bundles. Intuitively, this condition says that the $\mathcal{D}$-equivalence "plays nicely" with the sorts of arguments we made in Chapter 4 using the local structure theorem and the Knop conjecture.

**Definition 5.5.5.** Let $X_1$ and $X_2$ be spherical varieties, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. We say that a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ is *adapted to* $L_1$ and $L_2$ if for any $(\mu, n) \in \Lambda^+(X_1, L_1)$ with $n > 0$ and any $D \in \mathcal{D}_{G,X_1}$, we have $D \cap (X_1)_\mu \neq \varnothing$ if and only if $\iota(D) \cap (X_2)_\mu \neq \varnothing$.

We wish to prove that, under nice enough circumstances, we can always find some $\mathcal{D}$-equivalence that is adapted to $L_1$ and $L_2$. We will first need an auxiliary lemma.

**Lemma 5.5.6.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence. Suppose that $X_1$ and $X_2$ are smooth and that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. For any $(\mu, d) \in \Lambda^+(X_1, L_1)$ such that $d > 0$ and any $D \in \mathcal{D}_{G,X_1}$ such that $D \cap (X_1)_\mu \neq \varnothing$, one of the following is true.*

*(1) We have $\iota(D) \cap (X_2)_\mu \neq \varnothing$.*

*(2) $D$ is moved by $\alpha \in \Pi^b_{X_1}$, and if $D'$ is the other $B$-divisor of $X_1$ moved by $\alpha$, then $\varphi_D = \varphi_{D'}$ and $\iota(D') \cap (X_2)_\mu \neq \varnothing$. Moreover, neither $D$ nor $D'$ is moved by any root other than $\alpha$.*

*Proof.* Lemma 4.5.4 gives us a bijection

$$\iota_\mu : \{B\text{-divisors of } X_i \text{ intersecting } (X_1)_\mu\} \to \{B\text{-divisors of } X_i \text{ intersecting } (X_2)_\mu\}$$

such that for all $D$, we have $\varphi_D = \varphi_{\iota_\mu(D)}$, and any $\alpha \in \Pi_{M_\mu}$ moves $D$ if and only if $\alpha$ moves $\iota_\mu(D)$. Let $D \in \mathcal{D}_{G,X_1}$ be a $B$-divisor intersecting $(X_1)_\mu$. Notice that if $\iota(D) = \iota_\mu(D)$, then $\iota(D)$ is in the target of $\iota_\mu(D)$, hence $\iota(D) \cap (X_2)_\mu \neq \varnothing$. Thus, we can argue in essentially the same way as the proof of Lemma 4.1.3, but with $\iota_\mu$ in place of $\iota'$. There are three cases.

1. If $D$ is $G$-stable, then $\iota(D)$ and $\iota_\mu(D)$ are both $G$-stable divisors of $X_2$. But a $G$-divisor of $X_2$ is determined by its valuation as an element of $N(X_2)$ (see Corollary 3.1.14). So, the equality
$$\varphi_{\iota(D)} = \varphi_D = \varphi_{\iota_\mu(D)}$$
implies that $\iota(D) = \iota_\mu(D)$ and hence that possibility (1) in the lemma statement holds.

2. Suppose $D$ is moved by a root $\alpha$ of type $c$ or $d$ for $X_1$. Then, $\alpha$ has type $c$ or $d$ for $X_2$ as well (Lemma 4.1.2). Moreover, since $D$ is the only element of $\mathcal{D}_{G,X_1}(\alpha)$ and $D \cap (X_1)_\mu \neq \varnothing$, we have $\alpha \in \Pi_{M_\mu}$ (see Proposition 4.4.1). It follows that both $\iota(D)$ and $\iota_\mu(D)$ are the unique $B$-divisor of $X_2$ moved by $\alpha$, so that $\iota(D) = \iota_\mu(D)$.

3. The only remaining option is that $D$ is moved by a root $\alpha \in \Pi^b_{X_1} = \Pi^b_{X_2}$. In this case, write $\mathcal{D}_{G,X_1}(\alpha) = \{D, D'\}$. Then, $\iota(D)$ and $\iota(D')$ are the unique $B$-divisors of $X_2$ moved by $\alpha$, hence the unique $B$-divisors of $X_2$ whose valuations are $> 0$ on $\alpha$ (see Proposition 3.6.13 and Lemma 4.4.2). On the other hand, we have

$$\varphi_{\iota_\mu(D)}(\alpha) = \varphi_{\iota(D)}(\alpha) = 1.$$

It follows that $\iota_\mu(D)$ is either $\iota(D)$ or $\iota(D')$. If $\iota_\mu(D) = \iota(D)$, then we have possibility (1) again. If instead $\iota_\mu(D) = \iota(D')$, then we have $\iota(D') \cap (X_2)_\mu \neq \varnothing$ and

$$\varphi_D = \varphi_{\iota_\mu(D)} = \varphi_{\iota(D')} = \varphi_{D'}.$$

Finally, the roots moving $D$ are precisely the elements $\alpha' \in \Pi^b_{X_1}$ such that $\varphi_D(\alpha') = 1$, and likewise for $D'$ (see Corollary 4.4.3). So, $\varphi_D = \varphi_{D'}$ implies that any root $\alpha'$ moves $D$ if and only if it moves $D'$. If this is the case, then $\mathcal{D}_{G,X}(\alpha') \cap \mathcal{D}_{G,X}(\alpha) = \{D, D'\}$ contains 2 elements, which is only possible if $\alpha' = \alpha$ (see Proposition 3.6.12).

□

**Proposition 5.5.7.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties, and suppose that $X_1$ and $X_2$ are smooth and $\mathcal{D}$-equivalent and that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. There exists a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ which is adapted to $L_1$ and $L_2$.*

*Proof.* Let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be any $\mathcal{D}$-equivalence. We need to show that, for any $(\mu, n) \in \Lambda^+(X_1, L_1)$ with $n > 0$ and any $D \in \mathcal{D}_{G,X_1}$, we have $D \cap (X_1)_\mu \neq \varnothing$ if and only if $\iota(D) \cap (X_2)_\mu \neq \varnothing$. Lemma 5.5.6 tells us that the desired statement already holds for any $(\mu, n)$ with $n > 0$ and any $D \in \mathcal{D}_{G,X_1}$ not moved by a root of type $b$. Our plan is to change the definition of $\iota$ on $\mathcal{D}_{G,X_1}(\alpha)$ for each $\alpha \in \Pi^b_{X_1}$ in turn to make the desired statement hold for every $D \in \mathcal{D}_{G,X_1}(\alpha)$.

Let $\alpha \in \Pi^b_{X_1} = \Pi^b_{X_2}$ be any root of type $b$. Write $\mathcal{D}_{G,X_1}(\alpha) = \{D_1^+, D_1^-\}$, and let $D_2^+ = \iota(D_1^+)$ and $D_2^- = \iota(D_2^-)$. By Lemma 4.5.2, there exists some $(\mu, n) \in \Lambda^+(X_1, L_1)$ such that at least one of $D_1^+$ and $D_1^-$ intersects $(X_1)_\mu$. After swapping $D_1^+$ and $D_1^-$ if necessary, we may assume that $D_1^+ \cap (X_1)_\mu \neq \varnothing$. If $D_2^+ = \iota(D_1^+)$ does not intersect $(X_2)_\mu$, then Lemma 5.5.6 tells us that $D_2^-$ does intersect $(X_2)_\mu$, that $\varphi_{D_i^+} = \varphi_{D_i^-}$, and that $D_i^+$ and $D_i^-$ are moved by $\alpha$ and by no other simple root. Thus, redefining $\iota$ by setting $\iota(D_1^+) = D_2^-$ and $\iota(D_1^-) = D_2^+$ still gives us a $\mathcal{D}$-equivalence. After making this redefinition and swapping $D_2^+$ and $D_2^-$, we may assume that $\iota(D_1^\pm) = D_2^\pm$ and that $D_i^+$ intersects $(X_i)_\mu$ for $i \in \{1, 2\}$.

Now, we claim that the desired statement holds when $D = D_1^+$. Let $(\mu', n') \in \Lambda^+(X_1, L_1)$ be such that $n' > 0$. For $i \in \{1, 2\}$, let $E_i$ (resp. $E_i'$) be the $B$-divisor of $X_i$ cut out by a nonzero $B$-eigenvector in $H^0(X_i, L_i^{\otimes n})$ (resp. $H^0(X_i, L_i^{\otimes n'})$) of weight $\mu$ (resp. $\mu'$), and let $n_{i,+}$ (resp. $n'_{i,+}$) be the coefficient of $D_i^+$ in $E_i$ (resp. $E'_i$). The divisors $n'E_i$ and $nE'_i$ are cut out by $B$-eigenvectors in $H^0(X_i, L_i^{\otimes nn'})$ of weights $n'\mu$ and $n\mu'$ (respectively). So, we have $n\mu' - n'\mu \in \Lambda(X_i)$ (Proposition 2.5.2) and

$$nE'_i = n'E_i + \operatorname{div}(n\mu' - n'\mu).$$

Comparing coefficients of $D_i^+$ on both sides of this equation gives

$$n \cdot n'_{i,+} = n' \cdot n_{i,+} + \varphi_{D_i^+}(n_0\mu - n\mu_0)$$

On the other hand, since $\operatorname{Supp}(E_i) = X_i \setminus (X_i)_\mu$ (and likewise for $E'_i$), we see that $D_i^+$ intersects $(X_i)_\mu$ (resp. $(X_i)_{\mu'}$) if and only if $n_{i,+} = 0$ (resp. $n'_{i,+} = 0$). In particular, since $D_i^+$ intersects $(X_i)_\mu$, we have $n_i = 0$, and since $\varphi_{D_1^+} = \varphi_{\iota(D_1^+)} = \varphi_{D_2^+}$, the above equation implies that

$$n \cdot n'_{1,+} = n \cdot n'_{2,+}.$$

This gives us $n'_{1,+} = 0$ if and only if $n'_{2,+} = 0$, i.e. $D_1^+$ intersects $(X_1)_{\mu'}$ if and only if $D_2^+$ intersects $(X_2)_{\mu'}$.

It remains to show that the desired statement holds when $D = D_1^-$. As above, let $(\mu', n') \in \Lambda^+(X_1, L_1)$ be such that $n' > 0$. We show that $D_1^- \cap (X_1)_\mu \neq \varnothing$ implies $D_2^- \cap (X_2)_\mu \neq \varnothing$; the reverse implication will then follow from the same argument with $X_1$ and $X_2$ swapped. Suppose that $D_1^-$ intersects $(X_1)_{\mu'}$. There are two possible cases.

1. Suppose that $D_1^+ \cap (X_1)_{\mu'} \neq \varnothing$. Then, Both elements of $\mathcal{D}_{G,X_1}(\alpha)$ intersect $(X_1)_{\mu'}$, so Proposition 4.4.1 implies that $\alpha \in M_{\mu'}$ and hence that every element of $\mathcal{D}_{G,X_2}(\alpha)$ intersects $(X_2)_{\mu'}$. In particular, since $D_2^- = \iota(D_1^-)$ is moved by $\alpha$, we see that $D_2^- \cap (X_2)_{\mu'} \neq \varnothing$, as desired.

2. Suppose that $D_1^+ \cap (X_1)_{\mu'} = \varnothing$. Then, our above arguments give us $D_2^+ \cap (X_2)_{\mu'} = \varnothing$. But if $D_2^-$ does not intersect $(X_2)_{\mu'}$, then Lemma 5.5.6 implies that $D_2^+$ must intersect $(X_2)_{\mu'}$. Since this is impossible, we conclude that $D_2^-$ does intersect $(X_2)_{\mu'}$, as desired.

□

We now turn to the task of relating the notion of a $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ being adapted to two line bundles and the notion of $\iota$ mapping one line bundle to the other. First of all, we show that mapping one ample line bundle to another is a stronger notion than that of being adapted.

**Lemma 5.5.8.** *Let $X_1$ and $X_2$ be spherical varieties, and let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a strong $\mathcal{D}$-equivalence. Let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\iota$ maps $L_1$ to $L_2$. Then, $\iota$ is adapted to $L_1$ and $L_2$.*

*Proof.* The proof is similar to part of our proof that adapted $\mathcal{D}$-equivalences exist (see Proposition 5.5.7). Since $\iota$ maps $L_1$ to $L_2$, we have $B$-stable effective Cartier divisors $D_1 = \sum_D n_D D$ and $D_2 = \sum_D n_D \iota(D)$ and isomorphisms $L_i \cong \mathcal{O}_{X_i}(D_i)$ such that the canonical sections of the $\mathcal{O}_{X_i}(D_i)$ have the same weight $\mu_0$. Let $(\mu, n) \in \Lambda^+(X_1, L_1)$ be any element with $n > 0$. Then, $\mu - n\mu_0 \in \Lambda(X_i)$ (see Proposition 2.5.2), and $D_i' = D_i + \operatorname{div}(\mu - n\mu_0)$ is the divisor cut out by an eigenvector in $H^0(X_i, L_i^{\otimes d})$ of weight $\mu$. In particular, we have $\operatorname{Supp}(D_i') = X_i \setminus (X_i)_\mu$, so for any $D \in \mathcal{D}_{G,X_1}$, the divisor $D$ (resp. $\iota(D)$) intersects $(X_1)_\mu$ (resp. $(X_2)_\mu$) if and only if the coefficient of $D$ (resp. $\iota(D)$) in $D_1'$ (resp. $D_2'$) is 0. By definition of the $D_i$ and the $D_i'$, these two coefficients are

$$n_D + \varphi_D(\mu - n\mu_0) \quad \text{and} \quad n_D + \varphi_{\iota(D)}(\mu - n\mu_0),$$

respectively. Since $\varphi_D = \varphi_{\iota(D)}$, the statement of the lemma now follows. $\square$

Our main result about strong $\mathcal{D}$-equivalences is the following theorem, which is essentially a converse to the above lemma in the smooth projective case. The key idea is to utilize the canonical $G$-linearizations given by Corollary 5.2.7, which allow us to read off coefficients of $B$-divisors from the weight of the canonical section of their associated line bundles.

**Theorem 5.5.9.** *Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized spherical varieties. Suppose that $X_1$ and $X_2$ are smooth and $\mathcal{D}$-equivalent and that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Then, any $\mathcal{D}$-equivalence $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ which is adapted to $L_1$ and $L_2$ maps $L_1$ to $L_2$. In particular, $\iota$ is strong, and $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent.*

*Proof.* Let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence which is adapted to $L_1$ and $L_2$ (one exists by Proposition 5.5.7), and let $\mu_0$ be any nonzero weight of a nonzero section $f_1 \in H^0(X_1, L_1)^{(B)}$. (If no such weight exists, then $H^0(X_1, L_1) \cong k$. Since $L_1$ is ample, this implies that $H^0(X_1, L_1)$ has a secton vanishing nowhere on $X_1$, so that $L_1 \cong \mathcal{O}_{X_1}$. So $X_1$ is projective and quasi-affine, hence $X_1 = \operatorname{Spec}(k)$. Since $L_2$ has the same weight monoid as $L_1$, we likewise have $X_2 = \operatorname{Spec}(k)$, and the whole proposition now becomes trivial.) By assumption, $\mu_0$ is also the weight of some $f_2 \in H^0(X_2, L_2)^{(B)}$. Let $D_i$ be the divisor on $X_i$ cut out by $f_i$, and write $D_i = \sum_{D \in \mathcal{D}_{G,X_i}} n_{i,D} D$. It will suffice to prove that for all $D \in \mathcal{D}_{G,X_1}$, we have $n_{1,D} = n_{2,\iota(D)}$. If this is the case, then we have $D_2 = \sum_{D \in \mathcal{D}_{G,X_1}} n_{1,D} \iota(D)$. Moreover, the definition of the $D_i$ gives us isomorphisms $L_i \cong \mathcal{O}_{X_i}(D_i)$ which identify $f_i$ with the canonical section of $\mathcal{O}_{X_i}(D_i)$. In particular, these canonical sections both have weight $\mu_0$, so $\iota$ maps $L_1$ to $L_2$ by definition.

First, Corollary 2.6.8 implies that every $G$-linearization of $L_i$ is the canonical $G$-linearization of Corollary 5.2.7 tensored by $\mathcal{O}_{X_i}(\lambda)$ for some $\lambda \in \mathcal{X}(G)$, and the weight $\mu_0$ will be $\mu_{D_i} + \lambda$ for this $G$-linearization (here $\mu_{D_i}$ is the weight given in Corollary 5.2.7). Letting $\lambda_i$ be $\lambda$ for the given $G$-linearization on $L_i$, we have

$$\mu_{D_1} + \lambda_1 = \mu_0 = \mu_{D_2} + \lambda_2.$$

But $\mu_{D_i}$ is a linear combination of fundamental weights of $G$, which are all linearly independent from the characters of $G$ (see Lemma 2.2.25). So, the above equation implies that $\mu_{D_1} = \mu_{D_2}$ and that $\lambda_1 = \lambda_2$. Writing out the weights $\mu_{D_i}$ explicitly, we get

$$\mu_0 = \mu_{D_i} = \sum_{D \in \mathcal{D}_{G,X_1}} n_{1,D}\mu_D, \tag{5.5.1}$$

where

$$\mu_D = \begin{cases} \sum_{D \in \mathcal{D}_{G,X_i}(\alpha_j)} \omega_j, & D \text{ moved by a root of type } b \text{ or } d \text{ ,} \\ 2\omega_j, & D \text{ moved by } \alpha_j \in \Pi^c_{X_i} \\ 0, & D \text{ is } G\text{-stable} \end{cases}$$

(Here, $\omega_j$ denotes the fundamental weight corresponding to the simple root $\alpha_j$, and the sum in Case 1 is over all roots $\alpha_j \in \Pi_G$ such that $D$ is moved by $\alpha_j$.)

Now, let $D \in \mathcal{D}_{G,X_1}$. We will show that $n_{1,D} = n_{2,\iota(D)}$. There are three cases to consider, depending on what roots move $D$.

Case 1: Suppose that $D$ is moved by a root $\alpha_1$ of type $d$. By Theorem 3.6.10 and Corollary 4.4.3, $D$ is moved by at most one other root $\alpha_2$, and $D$ and $\iota(D)$ are the unique $B$-divisors of $X_1$ and $X_2$ (respectively) that are moved by $\alpha_1$ (and likewise for $\alpha_2$, if it exists). Let $\omega_1$ and $\omega_2$ be the fundamental weights corresponding to $\alpha_1$ and $\alpha_2$, respectively. It follows from (5.5.1) that the coefficient of $\omega_1$ (or of $\omega_1 + \omega_2$ if $\omega_2$ exists) in $\mu_{D_1}$ (resp. $\mu_{D_2}$) is $n_{1,D}$ (resp. $n_{2,\iota(D)}$). Since the fundamental weights are linearly independent, this coefficient is uniquely determined by the weight $\mu_{D_i}$, so the fact that $\mu_{D_1} = \mu_0 = \mu_{D_2}$ gives us $n_{1,D} = n_{2,\iota(D)}$. We remark that one can also handle the case where $D$ is moved by a root of type $c$ in the same manner. However, we will instead handle this possibility another way in Case 2.

Case 2: Suppose that $D$ is either $G$-stable or is moved by a root of type $c$. In this case, our argument is similar to that of Lemma 5.5.8 above. If $D$ is $G$-stable, then $D$ contains some $G$-orbit $Y$, and $(X_1)_{B,Y} = (X_1)_\mu$ for some $(\mu, d) \in \Lambda^+(X_1, L_1)$ with $d > 0$ (see Theorem 3.2.7). In particular, $D \cap (X_1)_\mu \neq \varnothing$. If instead $D$ is moved by a root $\alpha$ of type $c$, then we still obtain such a pair $(\mu, d)$ by Lemma 4.5.2. Since $\iota$ is adapted to $L_1$ and $L_2$, we have $\iota(D) \cap (X_2)_\mu \neq \varnothing$. Now, let $D'_i$ be the $B$-stable divisor of $X_i$ cut out by some nonzero $B$-eigenvector if $H^0(X_i, L_i^{\otimes d})$ of weight $\mu$. Note that $H^0(X_i, L_i^{\otimes d})$ also has a $B$-eigenvector of weight $d\mu_0$, and this section cuts out the divisor $d \cdot D_i$. It follows that

$$D'_i = d \cdot D_i + \operatorname{div}(\mu - d\mu_0).$$

Let $n_1$ (resp. $n_2$) be the weight of $D$ (resp. $\iota(D)$) in $D_1$ (resp. $D_2$). Since $D \cap (X_1)_\mu \neq \varnothing$ and $\iota(D) \cap (X_2)_\mu \neq \varnothing$, the coefficient of $D$ (resp. $\iota(D)$) in $D'_1$ (resp. $D'_2$) is 0. The above equation thus gives us

$$dn_1 = 0 + \varphi_D(d\mu_0/\mu), \quad dn_2 = 0 + \varphi_{\iota(D)}(d\mu_0/\mu).$$

Since $\varphi_D = \varphi_{\iota(D)}$, we obtain $dn_1 = dn_2$ and hence $n_1 = n_2$, as desired.

<u>Case 3</u>: It remains to consider the case where $D$ is moved by a root $\alpha$ of type $b$. Let $D'$ be the other $B$-divisor of $X_1$ moved by $\alpha$, and let $\omega$ be the fundamental weight corresponding to $\alpha$. Since $D$ and $D'$ (resp. $\iota(D)$ and $\iota(D')$) are the only $B$-divisors of $X_1$ (resp. $X_2$) moved by $\alpha$, (5.5.1) implies that $n_{1,D} + n_{1,D'}$ and $n_{2,\iota(D)} + n_{2,\iota(D')}$ are both coefficient of $\omega$ in $\mu_0$. This gives us

$$n_{1,D} + n_{1,D'} = n_{2,\iota(D)} + n_{2,\iota(D')}. \tag{5.5.2}$$

Now, by Lemma 4.5.2, there exists some $(\mu, d) \in \Lambda^+(X_1, L_1)$ with $d > 0$ such that either $D \cap (X_1)_\mu \neq \varnothing$ or $D' \cap (X_1)_\mu \neq \varnothing$. After swapping $D$ and $D'$ if necessary (which changes nothing, since we intend to show that both $n_{1,D} = n_{2,\iota(D)}$ and $n_{1,D'} = n_{2,\iota(D')}$), we may assume that $D \cap (X_1)_\mu \neq \varnothing$. Since $\iota$ is adapted to $L_1$ and $L_2$, this implies that $\iota(D) \cap (X_2)_\mu \neq \varnothing$ as well. So, we may argue exactly as in Case 2 to show that $n_{1,D} = n_{2,\iota(D)}$, and (5.5.2) then gives $n_{1,D'} = n_{2,\iota(D')}$ as well.

□

The following result summarizes what we've proven about the relationship between being adapted to two ample line bundles and mapping one ample line bundle to another.

**Corollary 5.5.10.** *Let $X_1$ and $X_2$ be smooth projective spherical varieties, let $\iota : \mathcal{D}_{G,X_1} \xrightarrow{\sim} \mathcal{D}_{G,X_2}$ be a $\mathcal{D}$-equivalence, and let $L_1$ and $L_2$ be $G$-linearized ample invertible sheaves on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$. Then, $\iota$ maps $L_1$ to $L_2$ if and only if $\iota$ is adapted to $L_1$ and $L_2$.*

*Proof.* One direction is Lemma 5.5.8, and the other direction is Theorem 5.5.9. □

The above results also allow us to compare the data of a strong $\mathcal{D}$-equivalence with the combined data of a weight monoid $\Lambda^+(X, L)$ and the set $\Pi^b_X$.

**Corollary 5.5.11.** *Let $X_1$ and $X_2$ be smooth projective spherical varieties. The following are equivalent:*

*(i) $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent.*

*(ii) $\Pi^b_{X_1} = \Pi^b_{X_2}$, and there exist $G$-linearized ample invertible sheaves $L_1$ and $L_2$ on $X_1$ and $X_2$ (respectively) such that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$.*

*Proof.* The implication (i) $\Rightarrow$ (ii) is immediate from the definition of a strong $\mathcal{D}$-equivalnece (and Lemma 4.1.2). The implication (ii) $\Rightarrow$ (i) follows immediately from Theorem 4.5.5 and Theorem 5.5.9.

□

The above corollary tells us that in Theorem 4.5.5, the two assumptions $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$ and $\Pi^b_{X_1} = \Pi^b_{X_2}$ are equivalent to saying that $X_1$ and $X_2$ are strongly $\mathcal{D}$-equivalent. This is interesting because the data of $\Lambda^+(X_i, L_i)$ and $\Pi^b_{X_i}$ is the main data that appears everywhere in our results in Chapter 4. For instance, in the classification statement of Corollary 4.8.1 (cf. Corollary 5.5.3 above for the $\Lambda^+$-equivalence setting), we assume that $\Lambda^+(X_1, L_1) = \Lambda^+(X_2, L_2)$, and we also assume $\Pi^b_{X_1} = \Pi^b_{X_2}$ under condition (ii) (because $\Pi^b_{X_i} \subset \Psi^{exc}_{G,X_i}$). Thus, Corollary 5.5.11 says that a strong $\mathcal{D}$-equivalence (or when $\mathrm{Pic}(G) = 0$, a strong $\Lambda^+$-equivalence, see Corollary 5.5.2 above) is actually equivalent to the data needed for our main results in Chapter 4. This is somewhat surprising: earlier in this section, we saw very readily that strong equivalences were sufficient to obtain our main results in Chapter 4; but at first glance, it seems very possible that assuming the existence of a strong equivalence might be stronger than the assumption on our results in Chapter 4. However, Corollary 5.5.11 indicates that this is not the case. Put another way, the data of a strong ($\Lambda^+$- or $\mathcal{D}$-) equivalence is essentially the same as the data we used for our results in Chapter 4.

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# Appendix A

# Affine Cones over Projective Varieties

Here we briefly review the construction of an affine cone over a projective variety. This construction is standard and appears in many places in the literature, but often without proofs of certain elementary facts. For completeness, we provide proofs of some of these facts here. Some of the material in this appendix involves $G$-linearizations, and one statement at the end (Corollary A.6) involves the definition of a spherical variety. The reader unfamiliar with these ideas may wish to consult Section 2.4 and Definition 3.1.1.

Let $X$ be a projective $k$-scheme, and let $L$ be an ample line bundle on $X$. Write $A = \Gamma_*(X, L)$. Recall that we have a canonical isomorphism $X \cong \mathrm{Proj}(A)$. Indeed, since $L$ is ample, the canonical map $f : X \to \mathrm{Proj}(A)$ is a dominant open immersion (see e.g. [Sta20, Tag 01Q1]). On the other hand, $f$ is proper (because $X$ is proper over $k$ and $\mathrm{Proj}(A)$ is separated over $k$), so $f$ is surjective and hence an isomorphism. Because of this, we will frequently identify $X$ with $\mathrm{Proj}(A)$ in what follows.

Now, let $\tilde{X} = \mathrm{Spec}(A)$. For any homogeneous element $f \in A$, consider the morphism of affine schemes

$$\pi_f : \tilde{X}_f \cong \mathrm{Spec}(A_f) \to X_f \cong \mathrm{Spec}((A_f)_0)$$

which, on global sections, is given by the inclusion $(A_f)_0 \hookrightarrow A_f$ of the degree-0 part of $A_f$. One can check that that the $\pi_f$ glue to a morphism

$$\pi : \tilde{X} \setminus Z \to X,$$

where $Z \subset \tilde{X}$ is the closed subscheme corresponding to the ideal $A_+ \subset A$ of positive-degree elements.

**Definition A.1.** Let $X$ be a projective $k$-scheme, and let $L$ be an ample line bundle on $X$.

1. The *affine cone* over $X$ with respect to $L$ is the scheme $\tilde{X}$ defined above. When we wish to consider the morphism $\pi : \tilde{X} \setminus Z \to X$ constructed above, we sometimes say that $\pi : \tilde{X} \setminus Z \to X$ is "the affine cone over $X$" (by which we really mean that $\tilde{X}$ is the affine cone and $\pi : \tilde{X} \setminus Z \to X$ is the morphism constructed above).

2. If $\Gamma(X, \mathcal{O}_X) = k$ (which occurs, for instance, if $X$ is projective and geometrically integral over $k$), then we have

$$A/A_+ \cong A_0 = H^0(X, \mathcal{O}_X) = k.$$

It follows that $Z$ is a single point. In this case, we denote the unique point in $Z$ by 0 and call this point the *vertex* of the affine cone $\tilde{X}$.

**Example A.2.** When $X = \mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \ldots, x_n])$, we have $\tilde{X} = \mathrm{Spec}(k[x_0, \ldots, x_n]) = \mathbb{A}^{n+1}_k$. In this case, the vertex of $\tilde{X}$ is the point 0 corresponding to the maximal ideal $(x_0, \ldots, x_n)$, and the map $\pi : \tilde{X} \setminus \{0\} \to X$ is classically thought of as the "quotient map"

$$\mathbb{A}^{n+1}_k \setminus \{0\} \to (\mathbb{A}^{n+1}_k \setminus \{0\})/k^\times \cong \mathbb{P}^n_k.$$

The affine cone is essentially a generalization of this classifical construction. Indeed, the map $\pi : \tilde{X} \setminus Z \to X$ turns out to be a GIT quotient in general, see Theorem A.3 below.

With notation as above, we can define a $\mathbb{G}_m$-module structure on $\Gamma_*(X, L)$ as follows: for any $S = \mathrm{Spec}(R)$ and any point $r \in \mathbb{G}_m(R) = R^\times$, we let $r$ act on $H^0(X, L^{\otimes d}) \otimes_k R$ via multiplication by $r^d$. In other words, $\mathbb{G}_m$ acts on the degree-$d$ part $H^0(X, L^{\otimes d})$ of $\Gamma_*(X, L)$ via the character $d \in \mathbb{Z} \cong \mathcal{X}(\mathbb{G}_m)$. This $\mathbb{G}_m$-module structure on $\Gamma_*(X, L)$ induces an action of $\mathbb{G}_m$ on $\tilde{X}$ (see Lemma 2.4.4). Moreover, the action of $\mathbb{G}_m$ fixes the ideal of positive-degree elements of $\Gamma_*(X, L)$, so the subscheme $Z$ cut out by this ideal is $\mathbb{G}_m$-stable. We may thus consider the action of $\mathbb{G}_m$ on the complement $\tilde{X} \setminus Z$. The following theorem gives us a few important properties of this action.

**Theorem A.3.** *Let $X$ be a projective $k$-scheme, let $L$ be an ample line bundle on $X$, and let $\pi : \tilde{X} \setminus Z \to X$ be the affine cone over $X$.*

*(a) $\pi$ is affine, faithfully flat and of finite presentation.*

*(b) $\pi$ is a principal $\mathbb{G}_m$-bundle (see [Bri18, Definition 2.3.1]).*

*(c) $\pi$ is the geometric GIT quotient of $\tilde{X} \setminus Z$ by the action of $\mathbb{G}_m$.*

*sketch of proof.* Write $A = \Gamma_*(X, L)$. Then, $\pi$ is locally given by the morphism of affine schemes $\tilde{X}_f \to X_f$ corresponding to the inclusion $(A_f)_0 \hookrightarrow A_f$ for some homogeneous element $f \in A$. This morphism is faithfully flat and finitely presented, which immediately implies (a). Statement (b) can be checked locally as well: thanks to (a), we just have to show that the morphism

$$\tilde{X}_f \times \mathbb{G}_m \to \tilde{X}_f \times_{X_f} \tilde{X}_f$$

given on rings by the map

$$A_f \otimes_{(A_f)_0} A_f \to A_f \otimes k[t^\pm], \qquad x \otimes y \mapsto xy \otimes t^{\deg(x)}$$

is an isomorphism. One can check that the inverse of this ring map is given by $y \otimes t^n \mapsto f^n \otimes \frac{y}{f^n}$. Finally, it follows from the definition of the $\mathbb{G}_m$-action on $A$ that $(A_f)^{\mathbb{G}_m} = (A_f)_0$ for any $f \in A$ homogeneous. So, $\pi$ is Zariski-locally a categorical GIT quotient (see [MF82, Theorem 1.1]), hence also globally a categorical GIT quotient (by some formal arguments involving universal properties). One can then check from the definitions that $\pi$ is in fact a geometric GIT quotient. □

Many nice properties of $X$ can be transferred over to the affine cone $\tilde{X}$. For the sake of brevity, we discuss this for only a couple important properties.

**Proposition A.4.** *Let $X$ be a projective variety over $k$, let $L$ be an ample invertible sheaf on $X$, and let $\pi : \tilde{X} \setminus Z \to X$ be the affine cone over $X$.*

*(a) $\tilde{X}$ is a $k$-variety.*

*(b) $\tilde{X}$ is normal if and only if $X$ is normal.*

*sketch of proof.* Let $A = \Gamma_*(X, L)$. We claim that $A$ is a domain. It will suffice to prove that for any $s \in H^0(X, L^{\otimes m})$ and $t \in H^0(X, L^{\otimes n})$ such that $s \otimes t = 0$, one of $s$ or $t$ is 0. dropping to the stalk at any point $x \in X$ and picking an isomorphism $L_x \cong \mathcal{O}_{X,x}$, the equality $s \otimes t = 0$ becomes $s_x t_x = 0$ (here identifying $s_x$ and $t_x$ with their images in $\mathcal{O}_{X,x}$ under the isomorphisms $L_x^{\otimes m} \cong L_x^{\otimes n} \cong \mathcal{O}_{X,x}$). Since $X$ is integral, we conclude that for all $x \in X$, either $s_x = 0$ or $t_x = 0$. In other words, we have

$$X = (X \setminus X_s) \cup (X \setminus X_t).$$

Since $X$ is irreducible, this implies that $X = X \setminus X_s$ or $X = X \setminus X_t$, so one of $s$ and $t$ vanishes everywhere. In other words, we have $s = 0$ or $t = 0$, as desired.

This proves that $A$ is a domain, so $\tilde{X}$ is integral. Moreover, $\tilde{X}$ is separated because it is affine, so to prove (a), we just need to show that $A$ is finitely generated. This fact was first proven by Zariski (who in fact proved something much more general). The proof is rather technical, so we omit it here; see [Laz04, Example 2.1.30] for a proof when $X$ is normal and [Băd01, Theorem 9.14] for a proof of Zariski's original result. (Note that Zariski's result requires some tensor power of $L$ to be globally generated, which is true here because $L$ is ample.) Finally, for a proof of (b), see [AB04, Lemma 2.1]. □

We are mainly interested in the case where $X$ is a $G$-variety and $L$ is a $G$-linearized invertible sheaf on $X$. In this case, we can obtain a group action on the affine cone $\tilde{X}$ in the following way. Let $\tilde{G} = G \times \mathbb{G}_m$. The ring $\Gamma_*(X, L)$ has the structure of a $\tilde{G}$-module, where $G$ acts via the given $G$-linearization on $L$ and $\mathbb{G}_m$ acts as above. This $\tilde{G}$-module structure induces a $\tilde{G}$-action on $\tilde{X}$ (see Lemma 2.4.4). Moreover, one can check that $\pi : \tilde{X} \setminus Z \to X$ is $G$-equivariant and $\mathbb{G}_m$-invariant, in the sense that for any $(g, t) \in G \times \mathbb{G}_m$ and any $\tilde{x} \in \tilde{X}$, we have

$$\pi((g, t) \cdot \tilde{x}) = g \cdot \pi(\tilde{x}).$$

It follows that $\pi$ identifies $\tilde{G}$-orbits of $\tilde{X} \setminus Z$ with $G$-orbits of $X$. More precisely:

**Proposition A.5.** *Let $G$ be an algebraic group, let $X$ be a projective $G$-scheme, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. Let $\pi : \tilde{X} \setminus Z \to X$ be the affine cone over $X$.*

(a) *The map $\mathcal{O} \mapsto \pi(\mathcal{O})$ is a bijection between $G$-orbits of $X$ and $\tilde{G}$-orbits of $\tilde{X} \setminus Z$. Its inverse is the map $\mathcal{O} \mapsto \pi^{-1}(\mathcal{O})$.*

(b) *$Z$ is a $\tilde{G}$-stable subvariety of $\tilde{X}$.*

(c) *If $\Gamma(X, \mathcal{O}_X) = k$, then $Z = \{0\}$ is the unique closed $\tilde{G}$-orbit of $\tilde{X}$.*

*Proof.* Write $A = \Gamma_*(X, L)$, so that $\tilde{X} = \mathrm{Spec}(A)$ and $X \cong \mathrm{Proj}(A)$. Given any $\tilde{G}$-orbit $\mathcal{O} \subset \tilde{X}$, the fact that $\pi$ is $G$-equivariant implies that $\pi(\mathcal{O})$ is a $G$-orbit of $\tilde{X}$. Conversely, for any $G$-orbit $\mathcal{O} \subset \tilde{X}$, the preimage $\pi^{-1}(\mathcal{O})$ is $\tilde{G}$-stable because $\pi$ is $G$-equivariant and $\mathbb{G}_m$-invariant. More precisely: for any $(g, t) \in G \times \mathbb{G}_m$ and any $\tilde{x} \in \pi^{-1}(\mathcal{O})$, we have

$$\pi((g, t) \cdot \tilde{x}) = g \cdot \pi(\tilde{x}) \in \mathcal{O},$$

so $(g, t) \cdot \tilde{x} \in \pi^{-1}(\mathcal{O})$. On the other hand, For any $x_1, x_2 \in \pi^{-1}(\mathcal{O})$, we know that $\mathcal{O}$ is a $G$-orbit, so there exists some $g \in G$ such that $\pi(x_2) = g\pi(x_1) = \pi(gx_1)$. Then, $gx_1$ and $x_2$ are in the same fiber of $\pi$ and hence are in the same $\mathbb{G}_m$-orbit (because $\pi$ is a geometric GIT quotient and all schemes are of finite-type over the algebraically closed field $k$, see Theorem A.3 and [MF82, Definition 0.6]). It follows that $x_1$ and $x_2$ are in the same $\tilde{G}$-orbit, so $\pi^{-1}(\mathcal{O})$ is a single $\tilde{G}$-orbit. One can check from the definitions that the maps $\mathcal{O} \mapsto \pi(\mathcal{O})$ and $\mathcal{O} \mapsto \pi^{-1}(\mathcal{O})$ are inverses, so this proves (a).

Now, the subscheme $Z \subset \tilde{X}$ corresponds to the maximal ideal $A_+ \subset A$ generated by all homogeneous elements of positive degree. Since $\tilde{G}$ acts on each graded piece of $A$ separately, we immediately see that $Z$ is $\tilde{G}$-stable, which proves (b). In particular, if $A_0 = \Gamma(X, \mathcal{O}_X) = k$, then $Z = \{0\}$ consists of a single $\tilde{G}$-stable point, so $Z$ is a $\tilde{G}$-orbit. Moreover, for any $G$-orbit $\mathcal{O} \subset X$, the generic point of $\pi^{-1}(\mathcal{O})$ is the preimage of the generic point of $\mathcal{O}$ and hence is a homogeneous prime ideal $\mathfrak{p} \subset A$. The fact that $A_0 = k$ gives us $\mathfrak{p} \subset A_+$, which implies that $0 \in \overline{\pi^{-1}(\mathcal{O})}$. By (a), this means that none of the $\tilde{G}$-orbits in $\tilde{X} \setminus Z$ is closed, so $Z = \{0\}$ is the unique closed $\tilde{G}$-orbit. □

Note that if $G$ is reductive, then so is $\tilde{G} = G \times \mathbb{G}_m$, and if $B$ is a Borel subgroup of $G$, then $\tilde{B} = G \times \mathbb{G}_m$ is a Borel subgroup of $\tilde{G}$. Our above results now readily imply the following useful statement about spherical varieties.

**Corollary A.6.** *Let $G$ be a reductive group, let $X$ be a projective $G$-variety, and let $L$ be a $G$-linearized ample invertible sheaf on $X$. Let $\tilde{X}$ be the affine cone over $X$. Then, $\tilde{X}$ is a spherical $\tilde{G}$-variety if and only if $X$ is a spherical $G$-variety.*

*Proof.* Proposition A.4 implies that $\tilde{X}$ is a variety and that $\tilde{X}$ is normal if and only if $X$ is. Moreover, let $\pi : \tilde{X} \setminus Z \to X$ be the affine cone over $X$. Proposition A.5 implies that $Z$ is $\tilde{B}$-stable, so any open $\tilde{B}$-orbit of $\tilde{X}$ would have to intersect the open set $\tilde{X} \setminus Z$ and hence be contained in $\tilde{X} \setminus Z$. On the other hand, Proposition A.5 also implies that $\tilde{X} \setminus Z$ contains an open $\tilde{B}$-orbit if and only if $X$ contains an open $B$-orbit. □

**Remark A.7.** For the reader who has already seen most of the theory of spherical varieties, it may be interesting to ask whether any "special" types of spherical varieties have correspondingly special affine cones. We briefly discuss this question for the various types of spherical varieties considered throughout this thesis.

Suppose that $X$ is a projective spherical variety. The affine cone $\tilde{X}$ can never be wonderful without being trivial, since $\tilde{X}$ is affine and wonderful varieties are projective. Moreover, $\tilde{X}$ is rarely toroidal. Indeed, since $k$ is algebraically closed and $X$ is projective and integral, we have $\Gamma(X, \mathcal{O}_X) = k$ and hence $Z = \{0\}$. So, Proposition A.5 implies that $Z$ is a closed $\tilde{G}$-orbit. On the other hand, by arguing as in the proof of that proposition, one can show that every $\tilde{B}$-divisor of $\tilde{X}$ has the form $\pi^{-1}(D)$ for some $B$-divisor of $X$ and that $0 \in \pi^{-1}(D)$. It follows that $\tilde{X}$ is not toroidal unless it has no colors (in which case it is a toric variety by Proposition 3.1.19).

On the other hand, the affine cone is always simple, as are all affine spherical varieties (see Lemma 2.5.8). As for horospherical varieties, one of our main results is Theorem 4.3.3, which relates the valuation cone of $X$ to that of $\tilde{X}$. Since horospherical varieties are characterized by their valuation cones (see Corollary 3.4.10), one can use Theorem 4.3.3 to show that $\tilde{X}$ is horospherical if and only if $X$ is.

# Appendix B

# Divisorial Sheaves

Here we give an overview of the theory of divisorial sheaves. This theory generalizes the usual correspondence between Cartier divisors and invertible sheaves to the setting of Weil divisors on a normal variety. For our purposes, the most important parts of the theory are the main results and the notion of $G$-linearizations of divisorial sheaves. As such, we omit many technical proofs in what follows. The reader interested in these proofs may wish to consult [Sta20, Tag 0EBK] and [Har80, Section 1].

We begin by recalling a few standard definitions for coherent sheaves.

**Definition B.1.** Let $\mathcal{F}$ be a coherent sheaf on a $k$-variety $X$.

1. We say that $\mathcal{F}$ is *torsion-free* if the stalk $\mathcal{F}_x$ is a torsion-free $\mathcal{O}_{X,x}$-module for all $x \in X$.
2. We define the *dual sheaf* of $\mathcal{F}$ to be $\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We often write $\mathcal{F}^{\vee\vee}$ for the dual of the dual sheaf, i.e. the sheaf $(\mathcal{F}^\vee)^\vee$.
3. Note that evaluation of sheaf morphisms on sections defines a canonical map of coherent $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{F}^{\vee\vee}$. We say that $\mathcal{F}$ is *reflexive* if this canonical map is an isomorphism.
4. By generic freeness (see e.g. [Sta20, Tag 051S]), there exists a nonempty open subset $U \subset X$ such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X$-module of finite rank. We define the *rank* of $\mathcal{F}$ to be the rank of this free $\mathcal{O}_X$-module. (This does not depend on the choice of $U$, because any two nonempty open subsets intersect, and the rank can be read off from any stalk.)

**Remark B.2.** If $X$ is projective, the rank of $\mathcal{F}$ can also be computed as $a_{\mathcal{F}}/a_{\mathcal{O}_X}$, where $a_{\mathcal{F}}$ denotes the leading coefficient of the Hilbert polynomial of $\mathcal{F}$.

Intuitively, we think of locally free sheaves of finite rank as an analog of vector bundles, and we think of reflexive sheaves as "vector bundles with singularities." In particular, every locally free sheaf is reflexive (one can check this from the definitions by looking at the morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ on stalks, where it is a morphism of free modules). On the other hand, if $X$ is locally factorial, the converse is also true for rank-1 sheaves:

**Proposition B.3** ([Har80, Proposition 1.9])**.** *Let $X$ be a locally factorial variety. Then, any reflexive sheaf of rank 1 is invertible.*

There is a whole algebraic theory of reflexive sheaves, which is interesting in its own right. For a discussion of this theory, see [Har80, Section 1]. For our purposes, however, there are only two key facts that we need about reflexive sheaves. The first is Proposition B.3 above; the second is the following proposition.

**Proposition B.4** ([Har80, Proposition 1.6])**.** *Let $\mathcal{F}$ be a coherent sheaf on a normal variety $X$. The following are equivalent.*

*(i) $\mathcal{F}$ is reflexive.*

*(ii) $\mathcal{F}$ is torsion-free, and for any open subset $U \subset X$ and any closed subset $Y \subset U$ of codimension $\geq 2$, we have $i_*\mathcal{F}|_{U\setminus Y} \cong \mathcal{F}|_U$, where $i : U \setminus Y \to U$ is the inclusion morphism.*

We will restrict our attention to reflexive sheaves of rank 1. These are important enough to merit their own name.

**Definition B.5.** We say that a coherent sheaf $\mathcal{F}$ on a normal variety $X$ is a *divisorial sheaf* if $\mathcal{F}$ is reflexive and the rank of $\mathcal{F}$ is 1.

Our interest in divisorial sheaves is primarily due to the following theorem.

**Theorem B.6** ([Sta20, Tags 0EBL, 0EBM])**.** *Let $X$ be a normal varietiy.*

*(a) The set of isomorphism classes of divisorial sheaves on $X$ is an abelian group under the operation $\mathcal{F} * \mathcal{G} = (\mathcal{F} \otimes \mathcal{G})^{\vee\vee}$. Moreover, the class of $\mathcal{O}_X$ is the identity element in this group, and the inverse of the class of a divisorial sheaf $\mathcal{F}$ in this group is the class of the dual sheaf $\mathcal{F}^\vee$.*

*(b) There exists a canonical isomorphism of abelian groups*

$$\alpha : \{\textit{isomorphism classes of divisorial sheaves on } X\} \xrightarrow{\sim} \mathrm{Cl}(X)$$

*sketch of proof.* The proof of (a) is essentially a technical, sheaf-theoretic argument; see [Sta20, Tag 0EBL] for details. As for (b), we explain the construction of the map $\alpha$; the proof that this map is an isomorphism can be found in [Sta20, Tag 0EBM]. Let $U \subset X$ be the regular locus of $X$. For any divisorial sheaf $\mathcal{F}$, the restriction $\mathcal{F}|_U$ is a reflexive rank-1 sheaf on the regular variety $U$, so Proposition B.3 impies that $\mathcal{F}|_U$ is invertible. Thus, $\mathcal{F}|_U$ represents an element of $\mathrm{Pic}(U)$. On the other hand, we have $\mathrm{Pic}(U) \cong \mathrm{Cl}(U)$ (because $U$ is regular) and $\mathrm{Cl}(U) \cong \mathrm{Cl}(X)$ (proof: the Weil divisor class group depends only on $K(X)$

and the stalks $\mathcal{O}_{X,x}$ of any point $x \in X$ such that $\operatorname{codim}(\overline{\{x\}}) \leq 1$; but any such $x$ lies in $U$ because $X$ is normal). Under these isomorphisms, the invertible sheaf $\mathcal{F}|_U$ determines a class $[D_{\mathcal{F}}] \in \mathrm{Cl}(X)$. We define $\alpha$ by sending the isomorphism class of $\mathcal{F}$ to $[D_{\mathcal{F}}]$. □

Let $\mathcal{K}_X$ denote the constant sheaf of $K(X)$ on $X$. Recall that for a Cartier divisor $D$ on $X$, we may view $\mathcal{O}_X(D)$ as the subsheaf of $\mathcal{K}_X$ defined by

$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K(X) \mid V \cap (D + \operatorname{div}(f)) \geq 0\}. \tag{B.0.1}$$

for any open subset $V \subset X$ (cf. [GW10, Section 11.12]. Also, note that $\Gamma(V, \mathcal{K}_X) = K(X)$ for all $V$ because $X$ is irreducible.)

We can use the above fact to explicitly describe the inverse of the isomorphism $\alpha$ from Theorem B.6. Let $\mathcal{F}$ be a divisorial sheaf, and let $D$ be any Weil divisor in the class $\alpha(\mathcal{F}) \in \mathrm{Cl}(X)$. Let $U \subset X$ be the regular locus, and let $i : U \hookrightarrow X$ is the inclusion map. The intersection $D \cap U$ is Cartier because $U$ is regular, and it follows from the construction of $\alpha$ that $\mathcal{F}|_U \cong \mathcal{O}_U(D \cap U)$. Since $X$ is normal, the complement $X \setminus U$ has codimension $\geq 2$, so Proposition B.4 (applied to the reflexive sheaf $\mathcal{F}$ and the closed subset $X \setminus U \subset X$) implies that

$$i_*\mathcal{O}_U(D \cap U) \cong i_*\mathcal{F}|_U \cong \mathcal{F}.$$

For any open subset $V \subset X$, the above isomorphism gives us

$$\begin{aligned} \Gamma(V, \mathcal{F}) &\cong \Gamma(U \cap V, \mathcal{O}_{U\cap V}(D \cap U \cap V)) \\ &\cong \{f \in K(X) \mid (D \cap U \cap V) + \operatorname{div}(f) \geq 0\} \\ &\cong \{f \in K(X) \mid V \cap (D + \operatorname{div}(f)) \geq 0\}. \end{aligned}$$

(The second line here is (B.0.1) applied to the sheaf $\mathcal{O}_{U\cap V}(D \cap U \cap V)$, and the third line follows from the fact that $U$ intersects every prime Weil divisor of $X$.) This equation tells us exactly how to recover the sheaf $\mathcal{F} = \alpha^{-1}([D])$ from the divisor $D$. In practice, we are much more interested in this description of $\alpha^{-1}$ than we are in the map $\alpha$. As such, we define some notation to help us refer to this description of $\alpha^{-1}$.

**Definition B.7.** Let $X$ be a normal variety, and let $D$ be a Weil divisor on $X$. We define a sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ by setting

$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K(X) \mid V \cap (D + \operatorname{div}(f)) \geq 0\}$$

for any open subset $V \subset X$. (Here, the action of $\Gamma(V, \mathcal{O}_X)$ is given by multiplication in $K(X) \supset \Gamma(V, \mathcal{O}_X)$.)

**Remark B.8.** Notice that our definition of $\mathcal{O}_X(D)$ for a Weil divisor $D$ is identical to the description of $\mathcal{O}_X(D)$ for a Cartier divisor $D$ in (B.0.1). It follows that when a Weil divisor $D$ is actually Cartier, the divisorial sheaf $\mathcal{O}_X(D)$ is invertible and is equal to the usual invertible sheaf corresponding to a Cartier divisor.

Our above discussion (along with a couple formal arguments) readily implies several nice properties about the sheaf $\mathcal{O}_X(D)$.

**Corollary B.9.** *Let $X$ be a normal variety, and let $D_1$ and $D_2$ be Weil divisors on $X$.*

*(a) $D_1$ and $D_2$ are linearly equivalent if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.*

*(b) We have*
$$\mathcal{O}_X(D_1 + D_2) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee},$$
*and for any $i$, we have $\mathcal{O}_X(-D_i) \cong \mathcal{O}_X(D_i)^{\vee}$.*

*(c) The map*
$$\beta : \mathrm{Cl}(X) \to \{\textit{isomorphism classes of divisorial sheaves on } X\}$$
*given by $[D] \mapsto \mathcal{O}_X(D)$ is the inverse to the map $\alpha$ of Theorem B.6. In particular, $\beta$ is an isomorphism.*

*Proof.* It follows immediately from the definitions (and the equation $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ for any $f, g \in K(X)$) that $D_1 \equiv D_2$ implies $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$. This is one direction of (a), and it also implies that the map $\beta$ is well-defined. Our above discussion shows that $\beta \circ \alpha = \mathrm{id}$; since $\alpha$ is an isomorphism, this gives us $\beta = \alpha^{-1}$. In particular, $\beta$ is a homomorphism, which implies (b). Also, the fact that $\beta$ is injective implies that if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, then $D_1$ and $D_2$ are linearly equivalent, which is the other direction of (a). $\square$

**Remark B.10.** In light of the above corollary, we will typically think of divisorial sheaves as being the sheaves which (up to isomorphism) have the form $\mathcal{O}_X(D)$ for some Weil divisor $D$. This is much more useful in practice than the (somewhat abstract and algebraic) definition of a divisorial sheaf.

Remark B.8 above indicates that the sheaves $\mathcal{O}_X(D)$ for Weil divisors $D$ are a generalization of the sheaves $\mathcal{O}_X(D)$ for Cartier divisors $D$. The above corollary thus generalizes the usual correspondence $D \leftrightarrow \mathcal{O}_X(D)$ between Cartier divisors and invertible sheaves. Indeed, when $X$ is locally factorial, all Weil divisors are Cartier, and Remark B.8 implies that the map $\beta$ in the above corollary is precisely the usual correspondence between Cartier divisors and invertible sheaves.

The relationship between a Weil divisor $D$ and the divisorial sheaf $\mathcal{O}_X(D)$ is generally analogous to the relationship between Cartier divisors and their corresponding invertible sheaves. For instance, we have the following lemma, which generalizes a standard fact about $\mathcal{O}_X(D)$ when $D$ is an effective Cartier divisor (see e.g. [GW10, Remark 11.25]).

**Lemma B.11.** *Let $X$ be a normal variety, and let $D$ be an effective Weil divisor.*

*(a) The sheaf $\mathcal{O}_X(-D)$ is an ideal sheaf of $\mathcal{O}_X$.*

*(b) The closed subscheme defined by* $\mathcal{O}_X(-D)$ *has support equal to* $\operatorname{Supp}(D)$*.*

*(c) Viewing* $D$ *as the closed subscheme defined by the ideal sheaf* $\mathcal{O}_X(-D)$*, and letting* $i : D \to X$ *be the inclusion map, we have the following short exact sequence of* $\mathcal{O}_X$*-modules:*

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

*Proof.* For any open subet $V \subset X$, the definition of $\mathcal{O}_X(-D)$ gives us

$$\Gamma(V, \mathcal{O}_X(-D)) = \{f \in K(X) \mid V \cap (\operatorname{div}(f) - D) \geq 0\}.$$

The righthand side of the above equation is contained in $\Gamma(V, \mathcal{O}_X)$ for all $V$ if and only if $\operatorname{div}(f) - D \geq 0$ implies $f \in \Gamma(X, \mathcal{O}_X)$. For any $f \in K(X)$ such that $\operatorname{div}(f) - D \geq 0$, the fact that $D$ is effective gives us $\operatorname{div}(f) \geq D \geq 0$ and hence $f \in \Gamma(X, \mathcal{O}_X)$, so this proves (a). Moreover, we have $\Gamma(V, \mathcal{O}_X(-D)) = \Gamma(V, \mathcal{O}_X)$ if and only if $1 \in \Gamma(V, \mathcal{O}_X(-D))$, or equivalently, if and only if

$$V \cap (-D) \geq 0.$$

Since $D$ is effective, this is equivalent to the condition $D \cap V = \varnothing$. We conclude that the closed subscheme cut out by $\mathcal{O}_X(-D)$ has complement $X \setminus \operatorname{Supp}(D)$; in other words, the underlying set of this closed subscheme is $\operatorname{Supp}(D)$. Finally, the short exact sequence in (c) follows formally from the fact that $\mathcal{O}_X(-D)$ is the ideal sheaf of the closed subscheme $D$. □

**Corollary B.12.** *Let* $X$ *be a normal variety, and let* $D$ *be an effective Weil divisor on* $X$*. Then,* $D$ *is Cartier if and only if* $\mathcal{O}_X(D)$ *is invertible.*

*Proof.* We already know that $\mathcal{O}_X(D)$ is invertible if $D$ is Cartier, see Remark B.8. Conversely, suppose that $\mathcal{O}_X(D)$ is invertible. Then, the ideal sheaf $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$ is invertible as well. Picking an open cover $\{U_i\}_i$ of $X$ such that $\mathcal{O}_X(-D)|_{U_i} \cong \mathcal{O}_{U_i}$ for all $i$, we see that the ideal sheaf $\mathcal{O}_X(-D)|_{U_i}$ is generated as a $\mathcal{O}_{U_i}$-module by a single section $f_i \in \Gamma(U_i, \mathcal{O}_X(-D))$. In other words, $D$ is a divisor which is locally cut out by a single equation, which is exactly what it means for $D$ to be Cartier. (More precisely, the data $\{(U_i, f_i)\}_i$ defines an effective Cartier divisor, and one can check that this divisor is precisely the divisor $D$.) □

For our purposes, the main reason for using the divisorial sheaf $\mathcal{O}_X(D)$ instead of the divisor $D$ is to consider the notion of a $G$-linearization of $\mathcal{O}_X(D)$. Note that nothing in the definition of a $G$-linearization depends on the sheaf being invertible; thus, we can define a $G$-linearization of a divisorial sheaf in exactly the same way as for an invertible sheaf. More precisely:

**Definition B.13.** Let $X$ be a normal $G$-variety, let $\rho : G \times X \to X$ be the action morphism, and let $\mathcal{F}$ be a divisorial sheaf on $X$. A *$G$-linearization* of $\mathcal{F}$ is an isomorphism

$$\phi : \rho^*\mathcal{F} \xrightarrow{\sim} \operatorname{pr}_X^* \mathcal{F}$$

that satisfies the cocyle condition of Definition 2.4.9.

**Remark B.14.** In the literature, the notion of a $G$-linearization on a general (i.e. not necessarily invertible or even divisorial) sheaf is sometimes referred to as an "equivariant sheaf." We prefer to use the term "$G$-linearization" here in order to emphasize the analogy between divisorial sheaves and invertible sheaves.

Let $X$ be a normal variety, and suppose that $X$ has finitely many $G$-orbits. Let $U$ be the union of all the $G$-orbits of $X$ of codimension $\leq 1$. Then, $U$ is a $G$-stable open subset, $U$ is regular because $X$ is normal, and $X \setminus U$ has codimension $\geq 2$. Let $i : U \to X$ be the inclusion map. We claim that the pullback by $i$ induces a bijection

$$i^* : \left\{ \substack{\text{isomorphism classes of} \\ \text{divisorial sheaves on } X} \right\} \xrightarrow{\sim} \left\{ \substack{\text{isomorphism classes of} \\ \text{invertible sheaves on } U} \right\}$$

whose inverse is the pushforward $i_*$. For any Weil divisor $D$ on $X$, the pullback $i^*\mathcal{O}_X(D) = \mathcal{O}_X(D)|_U$ is invertible by Proposition B.3, so the map $i^*$ is well-defined. Moreover, we have $i_*\mathcal{O}_X(D)|_U \cong \mathcal{O}_X(D)$ by Proposition B.4, so $i^*$ is injective, and $i_*$ is a left inverse of $i^*$. On the other hand, every Cartier divisor on $U$ is a Weil divisor and so has the form $D \cap U$ for some Weil divisor $D$ on $X$ (see our discussion of the isomorphisms $\mathrm{Pic}(U) \cong \mathrm{Cl}(U) \cong \mathrm{Cl}(X)$ in the proof of Theorem B.6 above). Comparing the definition of $\mathcal{O}_X(D)$ with (B.0.1) then gives us $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U(D \cap U)$. This proves that $i^*$ is surjective, hence bijective, so the left inverse $i_*$ is in fact the inverse of $i^*$.

We can use the bijection $i^*$ to relate $G$-linearizations of divisorial sheaves on $X$ with $G$-leariations of invertible sheaves on $U$. Because $i$ is $G$-equivariant, any $G$-linearization on a divisorial sheaf $\mathcal{O}_X(D)$ induces a $G$-linearization on the restriction $\mathcal{O}_X(D)|_U = i^*\mathcal{O}_X(D)$ (via pullback by the map $(\mathrm{id}_G, i) : G \times U \to G \times X$, see Lemma 2.4.12). On the other hand, any $G$-linearization of $\mathcal{O}_X(D)|_U$ induces a $G$-linearization on $i_*\mathcal{O}_X(D)|_U \cong \mathcal{O}_X(D)$ (by pushing forward by $(\mathrm{id}_G, i)$). One can check that these constructsion of $G$-linearizations are inverses, so the $G$-linearizations on $\mathcal{O}_X(D)$ are in bijection with those on $\mathcal{O}_X(D)|_U$. It follows that all of our existence and uniqueness results in Section 2.6 apply just as well to $G$-linearizations of divisorial sheaves on $X$, even when these sheaves are not invertible. This fact will be very important for our purposes; as such, we define some notation for the subset $U$ to help us refer to this situation.

**Definition B.15.** Let $X$ be a normal variety, and suppose that $X$ has finitely many $G$-orbits. We define $X^{\leq 1}$ to be the union of all the $G$-orbits of $X$ of codimension $\leq 1$.

In summary, we may view $\mathrm{Cl}(X)$ as the abelian group of isomorphism classes of divisorial sheaves, or equivalently, as the abelian group of isomorphism classes of sheaves of the form $\mathcal{O}_X(D)$ for some Weil divisor $D$. Moreover, $G$-linearizations work essentially the same way on these sheaves as they do on invertible sheaves (at least when $X$ has finitely many $G$-orbits). We can even restrict ($G$-linearized) divisorial sheaves to $X^{\leq 1}$ to obtain ($G$-linearized) invertible sheaves, which allows us to work only with invertible sheaves if necessary.

Since $G$-linearizations on divisorial sheaves behave just like $G$-linearizations on invertible sheaves, we can also generalize the $G$-equivariant Picard group $\mathrm{Pic}_G(X)$ to the setting of divisorial sheaves. Recall that $\mathrm{Pic}_G(X)$ is the abelian group of $G$-equivariant isomorphism classes of invertible sheaves (see Definition 2.4.14). In order to define the group action on $\mathrm{Pic}_G(X)$, the key fact is Lemma 2.4.13, which states that $G$-linearizations on invertible sheaves induce $G$-linearizations on tensor products and inverses in a canonical way. The entire proof of Lemma 2.4.13 also works for $G$-linearizations of divisorial sheaves, provided we consider the dual sheaf $\mathcal{F}^\vee$ in place of the inverse $L^{-1}$ of an invertible sheaf. This allows us to define an analog of $\mathrm{Pic}_G(X)$, but with divisorial sheaves instead of invertible sheaves.

**Definition B.16.** Let $X$ be a normal $G$-variety. We define the *$G$-equivariant class group* of $X$, denoted $\mathrm{Cl}_G(X)$, to be the abelian group of $G$-equivariant isomorphism classes of $G$-linearized divisorial sheaves on $X$. Here a "$G$-equivariant isomorphism" means the same as it does for $G$-linearized invertible sheaves (see Definition 2.4.14), and the group structure is given by the analog of Lemma 2.4.13 discussed above.

# Index of Terminology

# Index of Notation