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SL(2,C) Floer Homology for Knots and Knot Surgeries

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SL$(2, \mathbb{C})$ Floer Homology for Knots and Knot Surgeries

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by

Ikshu Neithalath

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ABSTRACT OF THE DISSERTATION

SL$(2,\mathbb{C})$ Floer Homology for Knots and Knot Surgeries

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2021
Professor Ciprian Manolescu, Chair

We investigate the sheaf-theoretic SL$(2,\mathbb{C})$ Floer cohomology for knots [CM] and 3-manifolds [AM20] presented as surgeries on knots in S^3. We establish a relationship between the 3-manifold invariants, $HP(Y)$ and $HP_{\#}(Y)$ for Y a surgery on a small knot in S^3, and the SL$(2,\mathbb{C})$ Casson invariant defined in [Cur01]. We use this to compute HP for surgeries on the trefoil and the figure-eight knots. We also compute HP for surgeries on two non-small knots, the granny and square knots. For the knot invariant, we prove that the (τ-weighted, sheaf-theoretic) SL$(2,\mathbb{C})$ Casson-Lin invariant introduced in [CM] is generically independent of the parameter τ and additive under connected sums of knots in integral homology 3-spheres. This addresses two questions posed in [CM].

The dissertation of Ikshu Neithalath is approved.

Sucharit Sarkar

Ko Honda

Raphaël Rouquier

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2021

To my parents

TABLE OF CONTENTS

1 Introduction . . . **1**

1.1 Introduction . . . 1

1.1.1 Results on knot surgeries . . . 2

1.1.2 Results on the $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant . . . 4

2 Background . . . **8**

2.1 The 3-manifold invariants $HP(Y)$ and $HP_{\#}(Y)$ . . . 8

2.1.1 Construction . . . 8

2.1.2 Smooth schemes . . . 10

2.1.3 The $\mathrm{SL}(2,\mathbb{C})$ Casson invariant . . . 11

2.2 The knot invariant $HP_{\tau}(K)$ . . . 12

2.2.1 Overview . . . 12

2.2.2 Notation and conventions . . . 12

2.2.3 Construction . . . 13

2.2.4 Stratifications and constructible objects . . . 14

2.2.5 The Behrend function . . . 16

3 $\mathrm{SL}(2,\mathbb{C})$ Floer Homology for surgeries on some knots . . . **19**

3.1 Surgeries on Small Knots and the $\lambda_{SL(2,\mathbb{C})}$ Casson Invariant . . . 19

3.1.1 Surgeries on small knots . . . 19

3.1.2 Large surgeries on small knots . . . 20

3.1.3 HP for surgeries on the trefoil . . . 21

3.1.4 HP for surgeries on the figure-eight knot . . . 21

3.2 $HP_{\#}$ for surgeries on small knots . . . 22

3.3 The character variety of $S^3\backslash(3_1\#3_1)$ 24
3.4 The A-polynomials of the square and granny knots 28
3.5 Surgeries on the Granny and Square knots 30
3.5.1 Character scheme of a composite knot 30
3.5.2 Irreducible representations in the character scheme of a composite knot 32
3.5.3 Character scheme of a connected sum of two trefoils 34
3.5.4 Character scheme for granny knot surgeries 35
3.5.5 Character scheme for square knot surgeries 38
3.5.6 Smoothness of the Character Schemes 40
3.6 Further Discussion 44
3.6.1 Exact triangles 44
3.6.2 A conjecture 44

4 The sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant 46
4.1 Generic independence of the weight 46
4.1.1 Setup and algebraic preliminaries 46
4.1.2 Passage to a cover 49
4.1.3 Stratification theory 51
4.1.4 Completion of the argument 53
4.1.5 Proof of Theorem 1.1.8 55
4.2 Additivity of (sheaf-theoretic) $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant 57
4.2.1 The structure of the character variety 58
4.2.2 A holomorphic $\mathbb{C}^*$-action 61
4.3 The $\mathrm{SL}(2,\mathbb{C})$-Floer homology of a knot in families 65
4.3.1 Overview of the argument 66

4.3.2 Application of a Theorem of Verdier 67
4.3.3 Moduli spaces of $K(D)$-pairs 69
4.3.4 $\mathrm{SL}(2,\mathbb{C})$ Floer homology in families 73
4.4 Appendix 74
4.4.1 A new convention 75
4.4.2 Equality of Euler characteristics 76
References **78**

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PUBLICATIONS

On the Sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ *Casson-Lin Invariant* (2019) (with Laurent Côté), submitted to the Journal of the Mathematical Society of Japan. (arXiv:1907.02593)

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CHAPTER 1

Introduction

1.1 Introduction

In [AM20], the authors defined a new invariant of closed, connected, orientable 3-manifolds Y called sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ Floer cohomology, denoted $HP(Y)$. It is defined as the hypercohomology of the perverse sheaf on the character scheme of irreducible representations, $\mathscr{X}_{\mathrm{irr}}(Y)$, coming from a description of this space as a complex Lagrangian intersection. They also define a framed version of this invariant, denoted $HP_{\#}(Y)$, which takes into account the reducible representations. The motivation for these invariants is to develop instanton Floer homology using the gauge group $\mathrm{SL}(2,\mathbb{C})$, rather than $\mathrm{SU}(2)$. Unfortunately, various analytical difficulties arise in such a construction. However, for a complex gauge group, one expects no instanton corrections to the Floer differential, suggesting that the construction of Floer homology in this case can be done algebraically, without counting solutions to PDE. The work of [AM20] realizes this program, using techniques from algebraic geometry and the theory of perverse sheaves to define the invariants.

In [CM], the authors define a related invariant for knots. Given a knot K in a closed, orientable 3-manifold Y and a real parameter $\tau \in (-2,2)$, they produce a sequence of abelian groups $HP^*_\tau(K)$ which are knot invariants. As is the case with the 3-manifold invariants, these groups are constructed using tools from derived algebraic geometry, but they can morally be interpreted as the Morse homology of the $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons action functional on $Y-K$, restricted to the space of connections with trace $\tau \in (-2,2)$ along the knot meridian.

The Euler characteristic of the knot invariant, $\chi_\tau(K) := \sum_{n\in\mathbb{Z}}(-1)^n \operatorname{rk}_{\mathbb{Z}} HP^n_\tau(K)$, is of independent interest since it can be viewed as an $\mathrm{SL}(2,\mathbb{C})$ analog of the Casson-Lin invariant.

The Casson-Lin invariant, which is defined using gauge theory, counts SU(2) connections with trace zero along the knot meridian and has been well-studied in the literature; cf. [Lin92, Her97].

This thesis combines work from two papers. In the first paper, largely contained in Chapter 3, we investigate the 3-manifold invariant for surgeries on some knots. Chapter 4 contains the second paper, which is joint work with Laurent Côté. The main goal of that paper is to establish new properties of the τ-weighted $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant $\chi_\tau(-)$ and also partly answer some questions which were stated in [CM].

1.1.1 Results on knot surgeries

Let K be a knot in S^3 and $S^3_{p/q}(K)$ its p/q Dehn surgery. When $S^3\backslash K$ contains no closed, incompressible surfaces, we say that K is a small knot. The calculation of $HP(S^3_{p/q}(K))$ for K a small knot and generic values of p/q reduces to the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant $\lambda_{SL(2,\mathbb{C})}$ as defined by Curtis [Cur01] and explored in her joint work with Boden [BC16]. Specifically, we have

Theorem 1.1.1. *Let $K \subset S^3$ be a small knot, and let $Y = S^3_{p/q}(K)$ denote p/q surgery on K. Then, for all but finitely many values of p, q, we have $HP(Y) \cong \mathbb{Z}_{(0)}^{\lambda_{\mathrm{SL}(2,\mathbb{C})}(Y)}$.*

Remark 1.1.2. We will often use the notation $A_{(k)}$ to denote a graded abelian group with A in degree k. A more common notation for this is $A[-k]$.

For example, when K is the right-handed trefoil, we have the following explicit formula:

Theorem 1.1.3. *Let $S^3_{p/q}(3_1)$ denote the 3-manifold obtained from p/q Dehn surgery on the right-handed trefoil in S^3. Then we have the following formula for the sheaf-theoretic Floer cohomology:*

$$HP(S^3_{p/q}(3_1)) = \begin{cases} \mathbb{Z}_{(0)}^{\frac{1}{2}|p-6q|-\frac{1}{2}} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{(0)}^{\frac{1}{2}|p-6q|} & \text{if } p \text{ is even, } 12 \not| p \\ \mathbb{Z}_{(0)}^{\frac{1}{2}|p-6q|-2} & \text{if } 12|p \end{cases}$$

Similarly for the figure-eight knot,

Theorem 1.1.4. *Let $S^3_{p/q}(4_1)$ denote the 3-manifold obtained from p/q Dehn surgery on the figure-eight knot in S^3. Then we have the following formula for the sheaf-theoretic Floer cohomology:*

$$HP(S^3_{p/q}(4_1)) = \begin{cases} \mathbb{Z}_{(0)}^{\frac{1}{2}(|p-4q|+|p+4q|)-1} & \text{if } p \text{ is odd} \\ \\ \mathbb{Z}_{(0)}^{\frac{1}{2}(|p-4q|+|p+4q|)} & \text{if } p \text{ is even, } p \neq \pm 4 \\ \\ \mathbb{Z}_{(0)}^{2} & \text{if } p = \pm 4 \end{cases}$$

In [AM20], the authors also define a framed version of sheaf-theoretic Floer cohomology denoted $HP_\#(Y)$. It is defined as the hypercohomology of a certain perverse sheaf on the representation scheme of Y, $\mathrm{Hom}(\pi_1(Y), \mathrm{SL}(2,\mathbb{C}))$. We would like to compute the framed sheaf-theoretic Floer homology, $HP_\#(S^3_{p/q}(K))$, for surgeries on knots. However, the representation schemes are usually not zero-dimensional and are often singular. So, we only give a formula for $HP_\#$ for surgeries where the character scheme is zero-dimensional, smooth, and does not contain non-abelian reducible representations.

Theorem 1.1.5. *Let K be a small knot and let $Y = S^3_{p/q}(K)$ denote the 3-manifold obtained from p/q Dehn surgery on K. Let $p' = p$ for p odd and $p' = \frac{p}{2}$ for p even. Assume that the character scheme $\mathscr{X}_{irr}(Y)$ is zero-dimensional and smooth and no p'^{th} root of unity is a root of the Alexander polynomial of K. Then,*

$$HP^*_\#(Y) = H^*(pt)^{\oplus 2-\sigma(p)} \oplus H^{*+2}(\mathbb{CP}^1)^{\oplus \frac{1}{2}(|p|-2+\sigma(p))} \oplus H^{*+3}(\mathrm{PSL}(2,\mathbb{C}))^{\oplus \lambda_{\mathrm{SL}(2,\mathbb{C})}(Y)}$$

where $\sigma(p) \in \{0,1\}$ is the parity of p.

We use this result to show that there does not exist an exact triangle relating $HP_\#$ for surgeries on the trefoil.

In light of Theorem 1.1.1, we are interested in computing $HP(S^3_{p/q}(K))$ when K is not a small knot. The character schemes of such manifolds may have positive dimensional components, in which case the calculation of the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant is insufficient to determine HP. In fact, when $K = K_1 \# K_2$ is a composite knot, we are guaranteed to have positive dimensional components. We provide a calculation of HP with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients for surgeries on the square and granny knots. Recall that the granny knot is the connected sum of two right-handed trefoils, whereas the square knot is a composite of a trefoil with its mirror.

Theorem 1.1.6. *Let $S^3_{p/q}(G)$ denote the 3-manifold obtained from p/q Dehn surgery on the granny knot, $G = 3_1^r \# 3_1^r$. Then we have the following formula for the sheaf-theoretic Floer cohomology:*

$$HP(S^3_{p/q}(G); \mathbb{F}) = \begin{cases} \mathbb{F}_{(0)}^{|6q-p|+\frac{1}{2}|12q-p|-\frac{3}{2}} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|12q-p|-\frac{1}{2}} & \text{if } p \text{ is odd} \\ \mathbb{F}_{(0)}^{|6q-p|+\frac{1}{2}|12q-p|-1} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|12q-p|-1} & \text{if } p \text{ is even, } p \neq 12k \\ \mathbb{F}_{(0)}^{|6q-p|+\frac{1}{2}|12q-p|-5} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|12q-p|+1} & \text{if } p = 12k, p/q \neq 12 \\ \mathbb{F}^4_{(1)} \oplus \mathbb{F}^4_{(0)} \oplus \mathbb{F}_{(-2)} & \text{if } p/q = 12 \end{cases}$$

Theorem 1.1.7. *Let $S^3_{p/q}(Q)$ denote the 3-manifold obtained from p/q Dehn surgery on the square knot, $Q = 3_1^r \# 3_1^l$. Then we have the following formula for the sheaf-theoretic Floer cohomology:*

$$HP(S^3_{p/q}(Q); \mathbb{F}) = \begin{cases} \mathbb{F}_{(0)}^{\frac{1}{2}|6q-p|+\frac{1}{2}|6q+p|+\frac{1}{2}|p|-\frac{3}{2}} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|p|-\frac{1}{2}} & \text{if } p \text{ is odd} \\ \mathbb{F}_{(0)}^{\frac{1}{2}|6q-p|+\frac{1}{2}|6q+p|+\frac{1}{2}|p|-1} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|p|-1} & \text{if } p \text{ is even, } p \neq 12k \\ \mathbb{F}_{(0)}^{\frac{1}{2}|6q-p|+\frac{1}{2}|6q+p|+\frac{1}{2}|p|-5} \oplus \mathbb{F}_{(-1)}^{\frac{1}{2}|p|+3} & \text{if } p = 12k, p \neq 0 \\ \mathbb{F}^4_{(1)} \oplus \mathbb{F}^4_{(0)} \oplus \mathbb{F}_{(-2)} & \text{if } p = 0 \end{cases}$$

1.1.2 Results on the $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant

Our first result related to the $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant is that this quantity is generically independent of the parameter τ.

Theorem 1.1.8. *Let K be an oriented knot in a closed, oriented 3-manifold Y. Then the (sheaf-theoretic, τ-weighted)* $\mathrm{SL}(2,\mathbb{C})$ *Casson-Lin invariant $\chi_\tau(K)$ is constant as a function of τ on a Zariski open subset of the complex plane.*

Theorem 1.1.8 answers a weaker form of Question 1.5 in [CM], which asked whether $HP^*_\tau(-)$ is generically independent of τ. The statement of this theorem merits some clarification due to the fact that $HP^*_\tau(-)$ was only defined in [CM] for $\tau \in (-2,2)$. In fact, we will show in Section 4.3 that the construction of $HP^*_\tau(-)$ can be generalized to all $\tau \in \mathbb{C} - \{\pm 2\}$. Moreover, there is an alternative definition of $\chi_\tau(-)$ which makes sense for all $\tau \in \mathbb{C}$ and agrees with the Euler characteristic of $HP^*_\tau(-)$ when these groups are defined. Our proof of Theorem 1.1.8 actually works with this alternative definition of $\chi_\tau(-)$.

As a result of Theorem 1.1.8, we can introduce the following definition.

Definition 1.1.9. Let $\chi_{CL}(K) \in \mathbb{Z}$ be defined as the generic value of $\chi_\tau(K)$ for $\tau \in \mathbb{C}$. We say that $\chi_{CL}(-)$ is the *(sheaf-theoretic)* $\mathrm{SL}(2,\mathbb{C})$ *Casson-Lin invariant.*

Although $\chi_{CL}(-)$ contains less information than $HP^*_\tau(-)$, our next theorem shows that it has the advantage of being additive under connected sums of knots.

Theorem 1.1.10. *For $i = 1, 2$, let Y_i be a closed, orientable integral homology 3-sphere and let $K_i \subset Y_i$ be a knot. Letting $K_1\#K_2$ denote the connected sum of K_1 and K_2 (see* [CM, Sec. 7.1]*), we have*

$$\chi_{CL}(K_1\#K_2) = \chi_{CL}(K_1) + \chi_{CL}(K_2).$$

Theorem 1.1.10 affirmatively answers Question 1.6 of [CM] for generic $\tau \in \mathbb{C}$. This question asks whether $\chi_\tau(-)$ is additive for knots in S^3 and all $\tau \in (-2,2)$, so there are always finitely many cases for which it remains open. However, one could reasonably argue that $HP^*_\tau(-)$ is not a meaningful invariant for certain non-generic choices of τ, and that Theorem 1.1.10 therefore addresses the most interesting part of Question 1.6. This is because $HP^*_\tau(-)$ only counts irreducible representations, and families of irreducibles can sometimes converge to a reducible representation at certain exceptional points. If $\rho_{red} : \pi_1(Y - K) \to \mathrm{SL}(2,\mathbb{C})$ is such a representation and has trace $\tau_0 \in \mathbb{C} - \{\pm 2\}$ along the meridian of K, then $HP^*_{\tau_0}(K)$

does not see ρ_{red} and therefore gives the "wrong" count. This situation could hopefully be corrected by defining an invariant which also takes into account reducibles.

We remark that a weaker version of Theorem 1.1.10 was proved in [CM, Thm. 7.17]. They showed, for $K_i \subset Y_i$ a knot in an integral homology 3-sphere, that $\chi_\tau(\#_{i=1}^n K_i) = \sum_{i=1}^n \chi_\tau(K_i)$ for generic $\tau \in \mathbb{C}$ under the assumption that the character schemes $\mathscr{X}^\tau_{irr}(K_i)$ are smooth. In principle, Theorem 1.1.10 is a much stronger result since character schemes of knot complements can be singular in general (in fact, singularities of 3-manifold groups can in some sense be arbitrarily bad; see [KM17]). On the other hand, from a purely computational perspective, Theorem 1.1.10 may turn out not to be particularly useful: for a knot $K \subset Y$, one needs to understand $\mathscr{X}^\tau_{irr}(K)$ very well in order to compute $\chi_\tau(K)$. The only examples that we have been able to handle turn out to be smooth.

The proofs of these theorems rely on our study of the so-called *Behrend function*, introduced by Behrend [Beh09]. The Behrend function is a constructible function which can be associated to any scheme (or complex-analytic space) over $\mathbb{C}$. Roughly speaking, it keeps track of singular or non-reduced behavior on the scheme; in particular, it is identically equal to $(-1)^n$ on the locus of smooth points of an n-dimensional scheme. The precise definition is explained in Section 2.2.5.

Given a knot $K \subset Y$, the $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant $\chi_\tau(K)$ can be defined in two ways. The first definition is just the one above, namely the alternating sum of the ranks of the groups $HP^*_\tau(K)$ constructed in [CM]. The second one defines $\chi_\tau(K)$ as the Euler characteristic of the character scheme $\mathscr{X}^\tau_{irr}(K)$ *weighted* by the Behrend function (see Definition 2.2.4). The fact that these definitions agree is essentially built into Joyce's theory of critical loci, which underlies the construction of the groups $HP^*_\tau(K)$ in [CM].

The perspective we take is to work almost entirely with the second definition, in terms of Behrend functions. This is in contrast to [CM], which only considers the first definition.

The usefulness of this perspective is illustrated by our proof of Theorem 1.1.8, Indeed, Theorem 1.1.8 is deduced as a straightforward corollary of the following result, which is purely a statement about complex algebraic geometry and may be of independent interest.

Theorem 1.1.11. *Fix a complex affine variety X and a morphism $X \to \mathbb{A}^1$. Then the function $\tau \mapsto \chi_B(X_\tau)$ is Zariski-locally constant (here $\chi_B(-)$ denotes the Euler characteristic weighted by the Behrend function).*

The statement of Theorem 1.1.11 appears plausible in light of the general philosophy that “bad behavior should only occur in codimension ≥ 1”. However, the proof crucially exploits deep work of Verdier [Ver76] on stratifications of complex algebraic varieties.

Our proof of Theorem 1.1.10 also relies on $\chi_\tau(K)$ being defined via the Behrend function. As mentioned above, Theorem 1.1.10 generalizes Theorem 7.17 in [CM]. The proof of Theorem 1.1.10 can in fact be carried out along similar lines as the original argument in [CM], treating the Euler characteristic weighted by the Behrend function as a replacement for the topological Euler characteristic which was considered in the proof of Theorem 7.17 in [CM]. However, there were some technical challenges to this generalization: the proof of Theorem 4.2.11 relies on nontrivial results about triangulations of algebraic subsets, and we also needed to clarify some ambiguities in the literature concerning the definition of the Behrend function, which is done in the appendix.

In Section 4.3, we prove that the invariant $HP^*_\tau(K)$ constructed for $\tau \in (-2, 2)$ in [CM] can be defined for all $\tau \in \mathbb{C} - \{\pm 2\}$. This actually boils down to proving that a certain character variety $X^\tau_{irr}(\Sigma)$ is connected and simply-connected. For $\tau \in (-2, 2)$, this character variety is homeomorphic to a certain moduli space of Higgs bundles. This homeomorphism was exploited in [CM] to analyze their topology.

For $\tau \in \mathbb{C} - \{\pm 2\}$, one needs to consider so-called $K(D)$-pairs, which are a mild generalization of Higgs bundles. Using again a result of Verdier on stratifications of complex varieties [Ver76] and a “variation of weights” argument originally due to Thaddeus [Tha02] which also played a role in [CM], we deduce that $X^\tau_{irr}(\Sigma)$ is connected and simply connected for all $\tau \in \mathbb{C} - \{\pm 2\}$ as a consequence of the fact, proved in [CM], that this holds for $\tau \in (-2, 2)$.

CHAPTER 2

Background

2.1 The 3-manifold invariants $HP(Y)$ and $HP_{\#}(Y)$

2.1.1 Construction

For a topological space X, let $\mathscr{R}(X)$ denote the $\mathrm{SL}(2,\mathbb{C})$ representation scheme of $\pi_1(X)$, defined as

$$\mathscr{R}(X) = \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2,\mathbb{C}))$$

Assuming $\pi_1(X)$ is finitely generated, this set is naturally identified as the $\mathbb{C}$ points of an affine scheme. The character scheme $\mathscr{X}(X)$ is the GIT quotient of $\mathscr{R}(X)$ by the conjugation action of $\mathrm{SL}(2,\mathbb{C})$.

A representation $\rho \in \mathscr{R}(X)$ is irreducible if the image of ρ is not contained in any proper Borel subgroup. The irreducible representations comprise the stable locus for the GIT action. Let $\mathscr{R}_{\mathrm{irr}}(X) \subset \mathscr{R}(X)$ denote the open subscheme corresponding to irreducible representations, and similarly $\mathscr{X}_{\mathrm{irr}}(X) \subset \mathscr{X}(X)$. When X is a closed surface of genus $g > 1$, $\mathscr{X}_{\mathrm{irr}}(X)$ is a holomorphic symplectic manifold of dimension $6g - 6$ [Gol04].

To investigate character schemes of 3-manifolds, we take the perspective of [AM20] using Heegaard splittings. Let $Y = U_0 \cup_\Sigma U_1$ be a Heegaard splitting of a closed, orientable, 3-manifold Y into two handlebodies U_0 and U_1 with Heegaard surface Σ. Then $\mathscr{X}_{\mathrm{irr}}(U_i)$ is a complex Lagrangian in $\mathscr{X}_{\mathrm{irr}}(\Sigma)$ and $\mathscr{X}_{\mathrm{irr}}(Y) = \mathscr{X}_{\mathrm{irr}}(U_0) \cap \mathscr{X}_{\mathrm{irr}}(U_1)$ is a Lagrangian intersection [AM20].

In [Bus], the author applies the work of [Joy15] to define a perverse sheaf of vanishing cycles associated to any Lagrangian intersection in a holomorphic symplectic manifold. A perverse sheaf on a scheme X is a certain type of object in $D^b_c(X)$, the bounded derived category of complexes of constructible sheaves on X. The category of perverse sheaves, $\mathrm{Perv}(X)$, is an abelian subcategory of $D^b_c(X)$. Perverse sheaves have wide application in algebraic geometry and are often used to study the topology of complex varieties. Given a function $f : U \to \mathbb{C}$ on a smooth scheme U, we can define a perverse sheaf of vanishing cycles, $\mathcal{PV}_f \in \mathrm{Perv}(U)$, with the property that the cohomology of the stalk of $\mathcal{PV}_f$ at a point x is the cohomology of the Milnor fiber of f at x (up to a degree shift). The perverse sheaf associated to a Lagrangian intersection in [Bus] is modeled on perverse sheaves of vanishing cycles.

In [AM20], the authors use Bussi's construction to associate a perverse sheaf to a Heegaard splitting of a 3-manifold. Moreover, they show that the perverse sheaf is independent of the Heegaard splitting:

Theorem 2.1.1 ([AM20])**.** *Let Y be a closed, connected, oriented 3-manifold with a Heegaard splitting $Y = U_0 \cup_\Sigma U_1$. Define the Lagrangians $L_i = \mathscr{X}_{irr}(U_i) \subset \mathscr{X}_{irr}(\Sigma)$. Apply the construction of [Bus] to obtain a perverse sheaf $P_{L_0,L_1} \in \mathrm{Perv}(\mathscr{X}_{irr}(Y))$ associated to the Lagrangian intersection $\mathscr{X}_{irr}(Y) = L_0 \cap L_1$. Then $P(Y) := P_{L_0,L_1}$ is an invariant of the 3-manifold Y up to canonical isomorphism in $\mathrm{Perv}(\mathscr{X}_{irr}(Y))$.*

We call its hypercohomology $HP^*(Y) = \mathbb{H}^*(P(Y))$ the sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ Floer homology of Y. They also define an invariant using the representation scheme that takes into account the reducibles, called the framed sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ Floer cohomology of Y, $HP_\#(Y)$. To define this invariant, we use the notion of the twisted character variety.

Definition 2.1.2. Let Σ be a closed surface with a basepoint w. Let D be a small disc neighborhood of w. We define the *twisted character variety* as

$$\mathscr{X}_{\mathrm{tw}}(\Sigma, w) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma - \{w\}), \mathrm{SL}(2,\mathbb{C})) | \rho(\partial D) = -I\} // \mathrm{SL}(2,\mathbb{C})$$

The twisted character variety is a smooth, holomorphic symplectic manifold.

Given a Heegaard splitting $Y = U_0 \cup_\Sigma U_1$ and a base point $z \in \Sigma$, we define $Y^\# = Y\#(T^2 \times [0,1])$, where the connected sum is performed in a neighborhood of z, arranged so that $T^2\times[0,1/2]$ is attached to U_0 and $T^2\times[1/2,1]$ is attached to U_1. Let $\Sigma^{\#}=\Sigma\#(T^2\times[1/2])$ be the new splitting surface and $U_i^{\#}$ be the resulting compression bodies. Then, choose a basepoint $w\in T^2\times\{1/2\}$ away from the connected sum region. Let $\ell_0=w\times[0,1/2]$ and $\ell_1=w\times[1,1/2]$ be lines in each compression body.

In the holomorphic symplectic manifold $\mathscr{X}_{\text{tw}}(\Sigma^{\#},w)$, the subspaces $L_i^{\#}$ consisting of twisted representations that factor through $\pi_1(U_i^{\#}-\ell_i)$ are complex Lagrangian submanifolds. Furthermore, their intersection $L_0^{\#}\cap L_1^{\#}$ can be identified with the representation variety $\mathscr{R}(Y)$ [AM20]. Analogously to the previous situation, this leads to a perverse sheaf invariant of the 3-manifold, $\mathcal{P}_{\#}(Y)\in\text{Perv}(\mathscr{R}(Y))$. We denote its hypercohomology $HP_{\#}(Y)$, the framed sheaf-theoretic $\text{SL}(2,\mathbb{C})$ Floer homology of Y.

2.1.2 Smooth schemes

To compute the invariants $HP(Y)$ and $HP_{\#}(Y)$, we can use the following proposition:

Proposition 2.1.3. *Let $X\subset\mathscr{X}_{irr}(Y)$ (resp. $X\subset\mathscr{R}_{irr}(Y)$) be a smooth topological component of the character scheme (resp. representation scheme) of complex dimension d. Then the restriction of the perverse sheaf $\mathcal{P}(Y)$ (resp. $\mathcal{P}_{\#}(Y)$) to X is a local system with stalks isomorphic to $\mathbb{Z}[d]$. In particular, if X is simply connected, then $HP(Y)$ (resp. $HP_{\#}(Y)$) contains $H^*(X)[d]$ as a direct summand.*

Furthermore, if $[\rho]$ is an isolated irreducible character and $X\cong\text{PSL}(2,\mathbb{C})$ is the orbit of $[\rho]$ in the representation scheme, then the local system $P_{\#}(Y)|_X$ is trivial.

Proof. The first part is Proposition 6.2 in [AM20]. The second part is Lemma 8.3 of [AM20]. □

When X is smooth but not simply connected, then there is some ambiguity over the local system $P(Y)|_X$. This can be circumvented by using $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Corollary 2.1.4. *Assume $\mathscr{X}_{irr}(M)$ is smooth with topological components X_i of complex dimensions d_i. Then $HP(Y;\mathbb{Z}/2\mathbb{Z})=\bigoplus_i H^*(X_i;\mathbb{Z}/2\mathbb{Z})[d_i]$.*

Proof. This follows from the fact that all local systems with $\mathbb{Z}/2\mathbb{Z}$ coefficients are trivial, since $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})$ is trivial. $\square$

2.1.3 The $\mathrm{SL}(2,\mathbb{C})$ Casson invariant

Morally, $HP(Y)$ should be a version of instanton Floer homology using the gauge group $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{SU}(2)$. Pursuing this analogy, the Euler characteristic of $HP(Y)$, denoted $\lambda^P(Y)$, should be a type of Casson invariant, just as the Euler characteristic of instanton Floer homology is related to the original Casson invariant, which is a count of irreducible $\mathrm{SU}(2)$ characters. There is another invariant called the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant defined in [Cur01] that counts isolated, irreducible $\mathrm{SL}(2,\mathbb{C})$ characters. To distinguish it from this invariant, $\lambda^P(Y)$ is called the full Casson invariant since it takes into account the positive dimensional components of the character scheme. When $\mathscr{X}_{\mathrm{irr}}(Y)$ is zero-dimensional, λ^P and $\lambda_{\mathrm{SL}(2,\mathbb{C})}$ agree. In fact, we have

Theorem 2.1.5. *Let Y be a 3-manifold such that $\mathscr{X}_{irr}(Y)$ is zero-dimensional. Then $HP(Y) \cong \mathbb{Z}^{\lambda}_{(0)}$, where $\lambda = \lambda_{SL(2,\mathbb{C})}(Y)$ is the $SL(2,\mathbb{C})$ Casson invariant as defined in* [Cur01].

Proof. The definition of $HP(Y)$ uses the characterization of $\mathscr{X}_{\mathrm{irr}}(Y)$ as a complex Lagrangian intersection $L_0 \cap L_1$ in the character scheme of a Heegaard surface for Y. The stalk of the perverse sheaf $P^\bullet(Y)$ at a point $p \in \mathscr{X}_{\mathrm{irr}}(Y)$ is the the degree-shifted cohomology of the Milnor fiber of some function $f : U \to \mathbb{C}$, for U an open neighborhood in one of the Lagrangians, such that the graph $\Gamma_{df} \subset T^*U$ is identified with L_1 in an appropriate polarization of the symplectic manifold near p. Since $\mathscr{X}_{\mathrm{irr}}(Y)$ is zero-dimensional, we know that f has an isolated singularity at p. Thus, the Milnor fiber has the homotopy type of a bouquet of spheres. The number of spheres in the bouquet is the Milnor number, denoted μ_p. Then, the stalk is given by $(P^\bullet(Y))_p \cong \mathbb{Z}^{\mu_p}_{(0)}$. The hypercohomology is $HP(Y) \cong \mathbb{Z}^{\sum \mu_p}_{(0)}$, where the sum is over all components of $\mathscr{X}_{\mathrm{irr}}(Y)$. The definition of the Casson invariant in terms of intersection cycles given in [Cur01] is $\lambda_{SL(2,\mathbb{C})}(Y) = \sum_p n_p$, where the sum is over all zero-dimensional components of $X_{\mathrm{irr}}(Y)$, and n_p is the intersection multiplicity of L_0 with L_1. But the Milnor number μ_p is equal to the intersection multiplicity of Γ_{df} with L_0, hence the result follows. $\square$

2.2 The knot invariant $HP_\tau(K)$

2.2.1 Overview

Fix the data of an oriented knot K in a closed, orientable 3-manifold Y and a real parameter $\tau \in (-2, 2)$. Let E_K denote the knot exterior and fix a suitable Heegaard splitting (Σ, U_0, U_1) of E_K, where Σ is a Riemann surface with two disks removed and the U_i are handlebodies. Now consider the moduli space $X^\tau_{irr}(\Sigma)$ of irreducible $\mathrm{SL}(2,\mathbb{C})$ representations with trace τ along the boundary circles. As before, the moduli space turns out to admit a natural holomorphic symplectic structure and the natural inclusions $\iota_j : L_j := X^\tau_{irr}(U_j) \hookrightarrow X^\tau_{irr}(\Sigma)$ are Lagrangian embeddings.

We again apply the methods of [Bus] to obtain a perverse sheaf $P^\bullet_{L_0,L_1}$ on (the complex-analytification of) $X^\tau_{irr}(E_K)$. It can be shown that this perverse sheaf is independent of the Heegaard splitting (see [CM, Prop. 3.9]), and $HP^*_\tau(K)$ is defined to be its hypercohomology.

2.2.2 Notation and conventions

As a general rule, we always use the same notation and follow the same conventions as in [CM].

- All schemes are assumed to be separated and of finite-type over $\mathbb{C}$. Since we are exclusively dealing with subschemes of affine varieties, these hypotheses are automatically satisfied.

- All subschemes are assumed to be locally closed.

- As in [CM], a variety is a (not necessarily irreducible) reduced scheme. A subvariety of a scheme is a subscheme which is a variety.

- Following [Ful98, Sec. 1.3], an *algebraic cycle* on a scheme X is a finite formal sum of irreducible subvarieties with integer coefficients. These form a group under addition which is denoted $Z_*(X)$.

- When we refer to a point of a $\mathbb{C}$-scheme, we mean a closed point unless otherwise indicated. To lighten the notation, we will not distinguish between $\mathbb{C}$-schemes and their associated set of closed points $X(\mathbb{C})$ in situations where the intended meaning seems clear. Thus, if X is a subscheme of Y, we sometimes write $X \subset Y$ as shorthand for $X(\mathbb{C}) \subset Y(\mathbb{C})$.

- If $X_1, \ldots, X_n$ are subschemes of X, then $\sqcup X_i \subset X$ is the *set-theoretic* union of the points of X_i. In particular, $\sqcup X_i$ should be viewed as a topological subspace of X endowed with the subspace topology – not with the disjoint union topology.

- Given a scheme X, a *partition* is a collection of pairwise disjoint subschemes $\{X_i\}$ such that $X = \sqcup X_i$.

2.2.3 Construction

We now give a brief overview of some constructions and objects described in more detail in [CM] and which we will also be using.

We will be considering Heegaard splittings $U_0 \cup_\Sigma U_1$ of a knot complement $Y - K$ or knot exterior E_K. In case $U_0 \cup_\Sigma U_1 = Y - K$, we always assume as in [CM] that Σ has genus at least six, that K intersects $\overline{\Sigma}$ in two points, and that the arcs $K \cap \overline{U_i}$ are isotopic rel endpoints to arcs contained in $\overline{\Sigma} = \partial \overline{U_i}$. There are analogous conditions for Heegaard splittings of knot exteriors; cf. [CM, Def. 3.2.]. Concretely, such Heegaard splittings can be constructed by choosing a Morse function on Y with a single minimum and maximum on K, and such that K is preserved by the gradient flow for an auxiliary metric.

To define the knot invariant, we need the notion of a *relative* character variety. More precisely, given a finitely presented group Γ and some conjugacy classes $\mathfrak{c}_1, \ldots, \mathfrak{c}_j \subset \Gamma$, one can consider relative representation schemes which parametrize representations with fixed trace τ on the $\mathfrak{c}_i$. These behave essentially like ordinary representation schemes, and a detailed account is provided in [CM, Sec. 2.1]. For $Y - K = U_0 \cup_\Sigma U_1$, let $\mathscr{R}^\tau(\Sigma)$ be the relative representation scheme parametrizing representations with fixed trace along the two boundary punctures. Let $\mathscr{R}^\tau(U_i)$ be the scheme of representations having fixed trace along the knot meridian. Let $\mathscr{R}^\tau(K) = \mathscr{R}^\tau(E_K) = \mathscr{R}^\tau(Y - K)$ be the scheme of representations of $\pi_1(Y - K) = \pi_1(E_K)$ having fixed trace along the knot meridian. We use analogous notation to denote relative representation varieties, and to denote relative character schemes and varieties.

All of the schemes and varieties described above have an open locus consisting of irreducible representations. We denote them by $\mathscr{R}_{irr}(\Gamma) \subset \mathscr{R}(\Gamma)$, and similarly for the other cases. We remark that the character varieties $X(\Sigma), X(U_i), X(K)$ (resp. $X^\tau(\Sigma), X^\tau(U_i), X^\tau(K)$) can also be viewed as moduli spaces of flat $\mathrm{SL}(2,\mathbb{C})$ connections (resp. flat connections with holonomy having trace τ along the knot meridian). Our arguments do not rely on this interpretation, but we have used it informally in the introduction.

2.2.4 Stratifications and constructible objects

Given two vector subspaces $F, G \subset \mathbb{R}^n$, we let

$$\delta(F,G) := \sup_{\substack{x \in F \\ \|x\|=1}} \operatorname{dist}(x, G),$$

where $\|\cdot\|$ is the standard Euclidean metric and the distance is also measured using this metric.

Let M and M' be smooth, locally closed submanifolds of $\mathbb{R}^n$ such that $M \cap M' = \emptyset$ and $y \in \overline{M} \cap M'$. Following [Ver76, Sec. 1], we say that the pair (M, M') satisfies property w) at y if there is a neighborhood U of y in $\mathbb{R}^n$ and a positive constant C such that for all $q' \in U \cap M'$ and $x \in U \cap M$ we have $\delta(T_{M',y'}, T_{M,x}) \leq C\|x - y'\|$. We say that the pair (M, M') satisfies property w) if it satisfies this property at all points $y \in \overline{M} \cap M'$.

Let M, M' be locally closed submanifolds of a complex algebraic variety V such that $M \cap M' = \emptyset$ and $y \in \overline{M} \cap M'$. We say that the pair (M, M') satisfies condition w) at y if there is a local real analytic embedding $\phi : U \cap V \to \mathbb{R}^n$ so that $(\phi(M), \phi(M'))$ satisfies property w) at $\phi(y)$. We say that (M, M') satisfies property w) if it satisfies this condition for all $y \in \overline{M} \cap M$.

We now introduce the notion of a w-stratification.

Definition 2.2.1 (see (2.1) in [Ver76]). A *w-stratification* of a variety X (i.e. a reduced scheme) over $\mathbb{C}$ is a partition $X = \sqcup_{i=1}^{n} X_i$, where the $X_i \subset X$ are smooth, connected subschemes, which satisfies the following axioms:

(i) $X_i \cap X_j = \emptyset$ if $i \neq j$.

(ii) If $\overline{X}_i \cap X_j \neq \emptyset$, then $X_j \subset \overline{X}_i$. (One gets the same notion using the analytic or Zariski topology.)

(iii) If $X_i \subset \overline{X}_j$ and $i \neq j$, then the pair (X_j, X_i) satisfies the condition w).

A w-stratification of a $\mathbb{C}$-scheme just means a w-stratification of the associated variety. The notion of a w-stratification is introduced by Verdier in [Ver76, (2.1)]. Unless otherwise specified, we only consider w-stratifications. We will therefore usually omit the prefix and refer to w-stratifications simply as *stratifications*.

Remark 2.2.2. It is shown in [Ver76] that w-stratifications are Whitney stratifications (i.e. they satisfy Whitney's so-called (b) condition). The converse is in general not true. We have chosen to work with w-stratifications simply for consistency with [Ver76] since we quote results of this paper throughout. However, we don't use any properties of w-stratifications which aren't also satisfied by Whitney stratifications, so we could just as easily have worked with ordinary Whitney stratifications.

Definition 2.2.3. Given a scheme X, a subset $C \subset X(\mathbb{C})$ is said to be *constructible* if it is a finite union of subschemes, i.e. $C = \sqcup_{i=1}^{n} X_i(\mathbb{C})$, where $X_i(\mathbb{C}) \subset X(\mathbb{C})$ is a subscheme. A function $f : X(\mathbb{C}) \to \mathbb{Z}$ is said to be *constructible* if $f(X)$ is finite and if, for every $n \in \mathbb{Z}$, $f^{-1}(n) \subset X(\mathbb{C})$ is constructible; see [Joy06].

It follows from [Ver76, (2.2)] that we can always refine our partition to be a stratification; in particular, we can assume that the X_i are smooth.

Definition 2.2.4 (see [Beh09] or Sec. 3.3 of [JT17])**.** Let $f : X \to \mathbb{Z}$ be a constructible function. We define the Euler characteristic of X *weighted* by the constructible function f as

$$\chi(X, f) := \sum_{n \in \mathbb{Z}} n\chi(f^{-1}(n)), \tag{2.2.1}$$

where $\chi(-)$ is the topological Euler characteristic.

We warn the reader that there is considerable ambiguity in the literature concerning the definition of the Euler characteristic weighted by a constructible function. First of all, some authors define $\chi(X, f)$ as in Definition 2.2.4 but replacing the topological Euler characteristic with the Euler characteristic with compact support. Other authors adopt the following definition. Given a constructible set $C \subset X(\mathbb{C})$ and a partition $C = \sqcup_{i=1}^{m} X_i(\mathbb{C})$ where the $X_i \subset X$ are subschemes, one defines $\chi_{an}(C) := \sum_{i=1}^{n} \chi_c(X_i(\mathbb{C}))$, where $\chi_c(X_i(\mathbb{C})) := \sum_{k \in \mathbb{Z}} (-1)^k H_c^k(X_i(\mathbb{C}))$ is the Euler characteristic with compact support of $X_i(\mathbb{C})$; see [Joy06, Def. 3.7]. One can then show (see [Joy06, (2)]) that $\chi_{an}(C)$ is independent of the chosen partition. One then sets $\chi(X, f) := \sum_{n \in \mathbb{Z}} n\chi_{an}(f^{-1}(c))$.

It is not at all obvious that these definitions all agree in the present context. We have therefore provided a proof of their equivalence in the appendix. We wish to emphasize that the argument in the appendix uses the fact that we are dealing with varieties over $\mathbb{C}$ (the analogous equivalences would be false over $\mathbb{R}$).

2.2.5 The Behrend function

Given a $\mathbb{C}$-scheme X, there exists a constructible function $\nu_X : X(\mathbb{C}) \to \mathbb{Z}$ introduced by Behrend in [Beh09] which is usually called the *Behrend function*; see also [JS12, Sec. 4.1]. Let us state the definition for affine subschemes of $\mathbb{A}^n$ as this is sufficient for our purposes.

Let $X \subset \mathbb{A}^n$ be an affine scheme over $\mathbb{C}$ defined by the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. We can consider the $\mathbb{C}$-algebra $R = \bigoplus_{m \geq 0} I^m/I^{m+1}$ (where $I^0 := \mathbb{C}[x_1, \ldots, x_n]$) and let

$$C_{X/\mathbb{A}^n} = \operatorname{Spec} R.$$

The $\mathbb{C}$-algebra inclusion $\mathbb{C}[x_1, \ldots, x_n]/I \hookrightarrow R$ induces a projection map $\pi : C_{X/\mathbb{A}^n} \to X$. We say that $C_{X/\mathbb{A}^n}$ is the *normal cone* of $X \subset \mathbb{A}^n$; see [Ful98, B.6]. This is a generalization of the normal bundle (and coincides with it for smooth schemes).

We define

$$\mathfrak{c}_{X/\mathbb{A}^n} := \sum_{C'} (-1)^{\dim \pi(C')} \operatorname{mult}(C') \pi(C') \in Z_*(X), \tag{2.2.2}$$

where the sum is over all irreducible components $C' \subset C_{X/\mathbb{A}^n}$; see [JS12, Sec. 4.1]. Here $\pi(C')$ denotes the underlying reduced closed subscheme which is the image of C' under π. The multiplicity $\operatorname{mult}(C')$ is the length of $C_{X/\mathbb{A}^n}$ at the generic point of C'. This is often referred to as the geometric multiplicity, for instance in [Ful98, Sec. 1.5].

It turns out that the cycle $\mathfrak{c}_{X/\mathbb{A}^n}$ depends only on X, i.e. it is independent of the embedding of X into $\mathbb{A}^n$. Letting $\mathrm{CF}(X)$ denote the group of constructible functions on X, there is a well-known group morphism $\mathrm{Eu} : Z_*(X) \to \mathrm{CF}(X)$ called the *(local) Euler obstruction* which was originally introduced by MacPherson in [Mac74]. We now define the Behrend function $\nu_X : X \to \mathbb{Z}$ by letting $\nu_X := \mathrm{Eu}(\mathfrak{c}_{X/\mathbb{A}^n})$.

If $\nu_X : X \to \mathbb{Z}$ is the Behrend function, then we write

$$\chi_B(X) := \chi(X, \nu_X).$$

We refer to this quantity the Euler characteristic of X weighted by the Behrend function.

The Behrend function plays an essential role in this work. This is mainly due to the fact that it can be computed in two ways on the schemes which we will be considering. On the one hand, the Behrend function can be defined for any finite-type $\mathbb{C}$-scheme in terms of normal cones and Euler obstructions, as explained above for affine schemes. On the other hand, if we consider a $\mathbb{C}$-scheme X whose complex-analytification is locally the critical locus of a holomorphic function $f : V \subset \mathbb{C}^n \to \mathbb{C}$, then for $x \in V \cap \operatorname{crit}(f) \hookrightarrow X$, the Behrend function can be computed in terms of the Euler characteristic of the sheaf of vanishing cycles of f.

More precisely, we have the following formula, due to Parusiński-Pragacz [JS12, Thm. 4.7]:

$$\nu_X(x) = (-1)^{\dim V}(1 - \chi(\mathrm{MF}_f(x))), \tag{2.2.3}$$

where $\mathrm{MF}_f(-)$ is the Milnor fiber of f and $x \in V \cap \mathrm{crit}(f) \hookrightarrow X$.

Now, let us consider the perverse sheaf $P^\bullet_\tau(K)$ introduced in [CM], for a knot $K \subset Y$. Recall that $P^\bullet_\tau(K)$ is a perverse sheaf on (the complex analytification of) the relative chararacter scheme $\mathscr{X}^\tau_{irr}(K)$ of K, for some $\tau \in (-2, 2)$. Recall from the discussion in Section 2.2.3 (see also [AM20, p. 17]) that a choice of Heegaard splitting (Σ, U_0, U_1) allows one to write locally $\mathscr{X}^\tau_{irr}(K)$ as the critical locus of a holomorphic function $f : V \to \mathbb{C}$ for some neighborhood $V \subset X^\tau(U_0)$. In this case, we have that $P^\bullet_\tau(K)|_V$ is isomorphic to the perverse sheaf of vanishing cycles of f. Hence, one can compute as in [AM20, p. 20] that, for $x \in V \cap \mathrm{crit}(f) \hookrightarrow \mathscr{X}^\tau_{irr}(K)$, we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{rk} \mathcal{H}^i(P^\bullet_\tau(K))_x &= \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{rk} \mathcal{H}^i(\phi[-1](\mathbb{Z}[\dim V]))_x \\ &= (-1)^{\dim V - 1}(\chi(\mathrm{MF}_f(x)) - 1) \\ &= (-1)^{\dim V}(1 - \chi(\mathrm{MF}_f(x))).\end{aligned}$$

According to [Dim04, Thm. 4.1.22], if $\mathcal{F}^\bullet$ is a complex of sheaves constructible with respect to a Whitney stratification $\mathcal{S}$, then

$$\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{rk} \mathbb{H}^i(X, \mathcal{F}^\bullet) = \chi(X, \mathcal{F}^\bullet) = \sum_{S\in\mathcal{S}} \chi(S)\chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{x_S}), \tag{2.2.4}$$

where $x_S \to S$ in the inclusion of an arbitrary point in a stratum $S \in \mathcal{S}$ and $\mathbb{H}(-)$ is hypercohomology. In particular, letting $\mathcal{F}^\bullet = P^\bullet_\tau(K)$, letting $X = \mathscr{X}^\tau_{irr}(K)$ and fixing a stratification $\mathcal{S}$ with respect to which $P^\bullet_\tau(K)$ is constructible, we find that

$$\begin{aligned}\chi_\tau(K) := \sum(-1)^i\mathbb{H}^i(X, P^\bullet_\tau(K)) &= \sum_{S\in\mathcal{S}} \chi(S)\chi(\mathcal{H}^\bullet(P^\bullet_\tau(K))_{x_S}) \\ &= \sum_{S\in\mathcal{S}} \chi(S)(-1)^{\dim V}(1 - \mathrm{MF}_f(x_S)) \\ &= \sum_{S\in\mathcal{S}} \chi(S)\nu_X(x_S) \\ &= \chi(X, \nu_X) = \chi_B(X) =: \chi_B(\mathscr{X}^\tau_{irr}(K))\end{aligned} \tag{2.2.5}$$

CHAPTER 3

$\mathrm{SL}(2,\mathbb{C})$ Floer Homology for surgeries on some knots

The goal of this chapter is prove Theorems 1.1.1, 1.1.3, 1.1.4, 1.1.5, 1.1.6, and 1.1.7. The proofs of several of these theorems involve direct computations of the representation and character schemes of knot surgeries. In the conclusion, we apply our calculations to demonstrate the non-existence of a surgery exact triangle for $HP_{\#}$.

3.1 Surgeries on Small Knots and the $\lambda_{SL(2,\mathbb{C})}$ Casson Invariant

3.1.1 Surgeries on small knots

By applying Theorem 2.1.5, we can establish the connection between $HP(Y)$ for Y a surgery on a small knot in S^3 and the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant, $\lambda_{\mathrm{SL}(2,\mathbb{C})}(Y)$ as given in Theorem 1.1.1.

Proof of Theorem 1.1.1. The group $\pi_1(Y)$ is a quotient of $\pi_1(S^3 \backslash K)$ by the subgroup normally generated by the class of the peripheral curve $m^p\ell^q$, where m is the meridian and ℓ the longitude. Thus, $\mathscr{X}_{\mathrm{irr}}(Y)$ is a closed subscheme of $\mathscr{X}_{\mathrm{irr}}(S^3\backslash K)$. However, $\dim \mathscr{X}_{\mathrm{irr}}(S^3\backslash K) = 1$ when K is a small knot [CCG+94]. Thus, if $\dim \mathscr{X}_{\mathrm{irr}}(Y) > 0$, then we must have that the reduced scheme $\mathscr{X}_{\mathrm{irr}}(Y)_{\mathrm{red}}$ appears as one of the irreducible components of $\mathscr{X}_{\mathrm{irr}}(S^3\backslash K)_{\mathrm{red}}$. Observe that the $\mathscr{X}_{\mathrm{irr}}(S^3_{p/q}(K))$ are disjoint for different values of p/q, since if $m^p\ell^q = m^{p'}\ell^{q'} = 1$ for distinct ratios p/q and p'/q', then we would have $m = 1$ and the representation would be trivial because m normally generates the fundamental group. Then, as $\mathscr{X}_{\mathrm{irr}}(S^3\backslash K)_{\mathrm{red}}$ has only finitely many components, we see that $\dim \mathscr{X}_{\mathrm{irr}}(S^3_{p/q}(K)) = 0$ for all but finitely many p/q. The result then follows from Theorem 2.1.5. $\square$

The invariant $\lambda_{SL(2,\mathbb{C})}$ has been computed for a range of 3-manifolds, including surgeries on many families of knots [Cur01][BC16][BC06]. We provide a few examples of how those results yield formulae for the sheaf-theoretic Floer homology of surgeries on knots.

3.1.2 Large surgeries on small knots

We review the results of [Cur01]. Let $M = S^3 \backslash N(K)$ be a knot exterior. Let $i : \partial M \to M$ denote the inclusion and $r : \mathscr{X}(M) \to \mathscr{X}(\partial M)$ denote the restriction map.

Definition 3.1.1. A slope $\gamma \in \partial M$ is *irregular* if there exists an irreducible representation ρ of $\pi_1(M)$ such that:

(i) the character $[\rho]$ is in a one-dimensional component $\mathcal{X}_i$ of $\mathscr{X}_{\mathrm{irr}}(M)$ such that $r(\mathcal{X}_i)$ is also one-dimensional;

(ii) $\mathrm{tr}(\rho(\alpha)) = \pm 2$ for all $\alpha \in \partial M$;

(iii) $\ker(\rho \circ i_*)$ is cyclic, generated by $[\gamma]$.

Definition 3.1.2. A slope p/q is *admissible* if:

(i) It is regular and not a strict boundary slope;

(ii) No p'-th root of unity is a root of the Alexander polynomial of K, where $p' = p$ for p odd and $p' = \frac{p}{2}$ for p even.

With these definitions, we can state Theorem 4.8 of [Cur01]:

Theorem 3.1.3 ([Cur01])**.** *Let K be a small knot in S^3 with complement M. Let $\{\mathcal{X}_i\}$ be the collection of one-dimensional components of $\mathscr{X}(M)$ such that $r(\mathcal{X}_i)$ is one-dimensional and such that $\mathcal{X}_i$ contains an irreducible representation. Then there exist integral weights $m_i > 0$ depending only on $\mathcal{X}_i$ and non-negative $E_0, E_1 \in \frac{1}{2}\mathbb{Z}$ depending only on K such that for every admissble $\frac{p}{q}$ we have*

$$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(K)) = \frac{1}{2}\sum_i m_i ||p\mathscr{M} + q\mathscr{L}||_i - E_{\sigma(p)}$$

where $\sigma(p) \in \{0, 1\}$ is the parity, and $||-||_i$ is the Culler-Shalen seminorm associated to $\mathcal{X}_i$.

There are only finitely many inadmissible slopes and only finitely many strict boundary slopes [Cur01]. Provided p is not chosen so that some p'-th root of unity is a root of the Alexander polynomial, where p' is as in Definition 3.1.2(ii), the above theorem only excludes finitely many slopes p/q. Thus, by combining Theorem 2.1.5 with Theorem 3.1.3 we obtain a formula for the sheaf-theoretic Floer homology for most surgeries on small knots.

3.1.3 *HP* for surgeries on the trefoil

The character schemes of all surgeries on the trefoil are zero-dimensional. They have been explicitly computed and the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant determined in [BC06]. From their computation and Theorem 2.1.5 we obtain Theorem 1.1.3.

Proof of Theorem 1.1.3. This follows from the calculation of $\lambda_{\mathrm{SL}(2,\mathbb{C})}(Y)$ in Theorem 5.9 of [BC06]. □

3.1.4 *HP* for surgeries on the figure-eight knot

Proof of Theorem 1.1.4. Since the figure-eight knot is small, its admissible surgeries have zero-dimensional character varieties and we can apply Theorem 2.1.5 in conjunction with Theorem 3.1.3. By the results of [BC12], the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant of surgeries on the figure-eight knot is

$$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(4_1)) = \frac{1}{2}(|p-4q|+|p+4q|) - E_{\sigma(p)}$$

for all admissible slopes p/q, where $E_0 = 0$ and $E_1 = 1$. This proves the theorem for all but the inadmissible slopes, which are the strict boundary slopes ± 4.

To compute the Casson invariant for these surgeries, we begin by computing the character variety of the figure-eight knot. Using the presentation of the knot group given by

$$\pi_1(S^3 \backslash 4_1) \cong \langle x, y | (x^{-1}yxy^{-1})x = y(x^{-1}yxy^{-1})\rangle$$

where x is the class of a meridian and

$$L = y^{-1}xy^{-1}x^{-1}yxyx^{-1}y^{-1}x$$

is the associated longitude. We can compute the character variety to be

$$\mathscr{X}(S^3\backslash 4_1) = \{(a,b) \in \mathbb{C}^2 | (b-2)(b^2-(a^2-1)b+a^2-1) = 0\}$$

where $a = \mathrm{tr}(\rho(x)) = \mathrm{tr}(\rho(y))$ is the meridional trace and $b = \mathrm{tr}(\rho(xy^{-1}))$. The line $b = 2$ corresponds to the abelian representations. For the irreducibles, the trace of the longitude is given by

$$\mathrm{tr}(\rho(L)) = a^4 - 5a^2 + 2$$

The surgery equation $L = M^{\pm 4}$ becomes:

$$\begin{aligned} \mathrm{tr}(\rho(L)) =& \mathrm{tr}(\rho(M^{\pm 4})) \\ a^4 - 5a^2 + 2 =& a^4 - 4a^2 + 2 \\ a^2 =& 0 \end{aligned}$$

From this, we see that the solution is $a = 0$ with multiplicity two. Hence, $\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{\pm 4}(4_1)) = 2$. Since the character variety is zero-dimensional, we can use this data with Theorem 2.1.5 to obtain the result. □

3.2 $HP_\#$ for surgeries on small knots

In this section, we prove Theorem 1.1.5 computing $HP_\#(Y)$ when $Y = S^3_{p/q}(K)$ is a surgery on a small knot K.

Proof of Theorem 1.1.5. We are assuming that no p'^{th} root of unity is a root of the Alexander polynomial of K, where $p' = p$ for p odd and $p' = \frac{p}{2}$ for p even. By Lemma 3.3.2, this condition ensures that there are no non-abelian reducibles. We first consider the abelian representations. These representations are those which factor through $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. So, we have

$$\mathscr{R}_{\mathrm{ab}}(Y) \cong \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$$

Letting a denote the generator of $\mathbb{Z}/p\mathbb{Z}$, we see that $\rho(a)$ can be any matrix with eigenvalue a p^{th} root of unity. There are $|p|$ such roots. When p is even, two of these roots are ± 1, and $\pm I$ is the unique matrix with that eigenvalue. For the other $\frac{1}{2}(|p|-2)$ roots, there is a conjugation orbit worth of choices, giving a copy of $T\mathbb{CP}^1$. Thus, we obtain 2 points and $\frac{1}{2}(|p|-2)$ copies of $T\mathbb{CP}^1$ in the representation variety. Similarly, for p odd, there is only one central representation and $\frac{1}{2}(|p|-1)$ copies of $T\mathbb{CP}^1$.

For the irreducible representations, the Casson invariant $\lambda_{\mathrm{SL}(2,\mathbb{C})}(Y)$ gives the count of isolated points with multiplicity in the character variety. Since we are assuming that $\mathscr{X}_{\mathrm{irr}}(Y)$ is zero-dimensional, the isolated points account for all irreducible. Since we assume the scheme is smooth, the multiplicities of the points are all 1 and the Casson invariant gives an honest count of points. The conjugation orbit of each isolated irreducible representation is a copy of $\mathrm{PSL}(2,\mathbb{C})$.

For an irreducible representation ρ, the character scheme is smooth at $[\rho]$ if and only if the representation scheme is smooth at ρ by Lemma 2.4 in [AM20]. Since we are assuming that the character scheme is smooth, we conclude that the representation scheme is smooth. Hence, we can apply Proposition 2.1.3 to compute $HP_{\#}$. □

For the right-handed trefoil knot, the slopes exlcuded by the hypothesis of this theorem are those for which p is a multiple of 12. In all cases, the character scheme is smooth and zero-dimensional [BC06]. Using the value of the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant for the trefoil, we obtain the following corollary

Corollary 3.2.1. *Let $S^3_{p/q}(3_1)$ denote the 3-manifold obtained from p/q Dehn surgery on the right-handed trefoil in S^3. Then for p not a multiple of 12, we have the following formula for the framed sheaf-theoretic Floer cohomology:*

$$HP^*_{\#}(S^3_{p/q}(3_1)) = H^*(pt)^{\oplus d_1} \oplus H^{*+2}(\mathbb{CP}^1)^{\oplus d_2} \oplus H^{*+3}(\mathrm{PSL}(2,\mathbb{C}))^{\oplus d_3}$$

where the multiplicities are given by

$$(d_1,d_2,d_3) = \begin{cases} (1, \frac{1}{2}(|p|-1), \frac{1}{2}|6q-p| - \frac{1}{2}), & \textit{if } p \textit{ is odd} \\ (2, \frac{1}{2}(|p|-2), \frac{1}{2}|6q-p|) & \textit{if } p \textit{ is even and not a multiple of 12} \end{cases}$$

In Section 8, we use this partial calculation to show that there does not exist an exact triangle relating $HP_{\#}$ for surgeries on the trefoil.

Remark 3.2.2. When p is a multiple of 12, the smoothness of the character scheme does not imply the smoothness of the representation scheme because of the presence of non-abelian reducible representations. One can check that the representation scheme is in fact singular in these cases.

3.3 The character variety of $S^3\backslash(3_1\#3_1)$

The knot group of the trefoil has the presentations

$$\begin{aligned}\pi_1(S^3\backslash 3_1) =&\langle a,b|a^3=b^2\rangle\\ \cong&\langle rsr=srs\rangle\end{aligned}$$

The character scheme is

$$\mathscr{X}(S^3\backslash 3_1)\cong\{(y-2)(x^2-y-1)=0\}\subset\mathbb{C}^2$$

where $x=\operatorname{tr}\rho(r)$ and $y=\operatorname{tr}(rs^{-1})$. The line $\{y=2\}$ is $\mathscr{X}_{\text{red}}$ and $\{x^2-y=1, y\neq 2\}$ is $\mathscr{X}_{\text{irr}}$.

The fundamental group of the complement of the knot $3_1\#3_1$ has the presentation

$$\Gamma=\langle a,b,c,d|a^3=b^2,c^3=d^2,d=ba^{-2}c^2\rangle$$

where the subgroup Γ_0 generated by a and b corresponds to a copy of $\pi_1(S^3\backslash 3_1)$ and similarly the subgroup Γ_1 generated by c,d corresponds to the knot group of the other 3_1 summand. The relation $a^2b^{-1}=c^2d^{-1}$ comes from setting the meridian in Γ_0 equal to the meridian in Γ_1. Consider the following closed subsets of $\mathscr{X}(\Gamma)$,

$$\begin{aligned}\mathscr{X}_{\text{red}} =&\{[\rho]\mid\rho\text{ is abelian}\}\\ \mathscr{X}_i =&\{[\rho]\mid\rho|_{\Gamma_{|1-i|}}\text{ is abelian}\}\end{aligned}$$

where clearly $\mathscr{X}_{\text{red}}\subset\mathscr{X}_i$. Since the abelianization of the knot group is generated by the meridian, we have that $\mathscr{X}_{\text{red}}=\mathscr{X}(\mathbb{Z})\cong\mathbb{C}$, where the meridional trace is a coordinate for $\mathbb{C}$.

Lemma 3.3.1. *Let $\mathscr{X}(\Gamma) \xrightarrow{r} \mathscr{X}(\Gamma_i)$ denote the natural restriction map. Then the composite $\mathscr{X}_i \hookrightarrow \mathscr{X}(\Gamma) \xrightarrow{r} \mathscr{X}(\Gamma_i)$ is an isomorphism $\mathscr{X}_i \cong \mathscr{X}(\Gamma_i)$.*

Proof. If $\rho|_{\Gamma_{|1-i|}}$ is abelian, then it is determined by its value on the meridian. But the value of ρ on the meridian is determined by its restriction to Γ_i, since the meridian lies in the intersection $\Gamma_0 \cap \Gamma_1$, establishing injectivity.

For surjectivity, we observe that for any representation $\rho \in \mathscr{X}(\Gamma_i)$, there exists an extension of ρ to a representation of Γ given by setting $\rho|_{\Gamma_{|1-i|}}$ to be the abelian representation of $\Gamma_{|1-i|}$ with the required meridional value. This lies in $\mathscr{X}_i$ by construction. $\square$

Recall the following fact:

Lemma 3.3.2. [CCG+94] *Let ρ be a representation of $\pi_1(S^3 \backslash K)$ with $[\rho] \in \mathscr{X}_{\text{red}} \cap \overline{\mathscr{X}_{\text{irr}}}$. Then the following equivalent conditions hold:*

- $\Delta(\mu^2) = 0$*, where Δ is the Alexander polynomial of K and μ is an eigenvalue of $\rho(m)$, for m the meridian of the knot.*
- *There exists a non-abelian reducible representation ρ' with the same character as ρ.*

The Alexander polynomial of the trefoil is the sixth cyclotomic polynomial, $\Delta_{3_1}(t) = t^2 - t + 1$. Thus, the above lemma guarantees non-abelian reducibles at meridional trace $\pm\sqrt{3}$. The same holds for $3_1\#3_1$ since $\Delta_{3_1\#3_1} = (\Delta_{3_1})^2$. This allows us to establish the following proposition:

Proposition 3.3.3. *Let $\mathscr{X}_{i,\text{irr}} = \mathscr{X}_i \backslash \mathscr{X}_{\text{red}}$ and let $S = \mathscr{X}(\Gamma) \backslash (\mathscr{X}_0 \cup \mathscr{X}_1)$. Then the four irreducible components of $\mathscr{X}(\Gamma)$ are $\mathscr{X}_{\text{red}}, \overline{\mathscr{X}}_{0,\text{irr}}, \overline{\mathscr{X}}_{1,\text{irr}}$ and $\overline{S}$. Moreover, these four components pairwise intersect in the same two points, corresponding to characters of non-abelian reducibles.*

Proof. That none of the four closed sets share any irreducible components follows from the description of the intersections. If $[\rho] \in \overline{S} \cap \overline{\mathscr{X}}_{0,\text{irr}}$, then by restricting to $\mathscr{X}(\Gamma_1)$, we see that

$[\rho|_{\Gamma_1}] \in \mathscr{X}_{\text{red}}(\Gamma_1) \cap \overline{\mathscr{X}_{\text{irr}}(\Gamma_1)}$. Thus, Lemma 3.3.2 implies that $[\rho]$ is one of two points in $\mathscr{X}_{\text{red}}$ corresponding to non-abelian reducibles. The other intersections follow similarly.

It only remains to check that each of the four pieces is in fact irreducible. From the coordinate description of $\mathscr{X}(S^3 \backslash 3_1)$, we see that $\mathscr{X}_{\text{red}} = \{y = 2\}$ and the $\overline{\mathscr{X}}_{i,\text{irr}}$ are equal to $\{x^2 - y = 1\}$. In either case, they are isomorphic to $\mathbb{C}$. The irreducibility of $\overline{S}$ follows from Proposition 3.3.4 below. $\square$

Proposition 3.3.4. *$\overline{S}$ is an affine cubic surface with precisely two A_1 singularities at the points $S_{\text{sing}} = \overline{S} \backslash S = \mathscr{X}_{nar}$, the two characters of non-abelian reducible representations.*

Proof. Let $A = \rho(a)$, $B = \rho(b)$, etc. If $[\rho] \in S$, then $\rho|_{\Gamma_i}$ is non-abelian. But since $a^3 = b^2$ is a central element of Γ_0, we must have $A^3 = B^2 = \pm I$. However, if $B^2 = I$, then $B = \pm I$ and $\rho|_{\Gamma_0}$ would be abelian. Thus, we must have $A^3 = B^2 = -I$, and similarly $C^3 = D^2 = -I$ and $A, C \neq -I$. These equations are equivalent to $\text{tr}(A) = \text{tr}(C) = 1$ and $\text{tr}(B) = \text{tr}(D) = 0$. Now, since $d = ba^{-2}c^2$, we see that $D = BAC^{-1}$. Thus, we have the inclusion

$$S \subset \mathscr{S} = \{[\rho] \in \mathscr{X}(F_3) \mid \text{tr}(A) = \text{tr}(C) = 1, \text{tr}(B) = \text{tr}(BAC^{-1}) = 0\}$$

where F_3 is the free group generated by a, b, c. Also, any representation of F_3 that lies in $\mathscr{S}$ is a representation of Γ, so that $\mathscr{S} \subset \mathscr{X}(\Gamma)$. Since S is open in $\mathscr{X}(\Gamma)$, $\overline{S}$ is a union of the irreducible components meeting S. Thus, $\overline{S} = \mathscr{S}$ provided $\mathscr{S}$ is irreducible.

So, we now turn to describing the algebraic set $\mathscr{S}$. Regarding $\mathscr{X}(F_3)$ as the character variety of the four-holed sphere, we see that $\mathscr{S}$ is a relative character variety; $\mathscr{S}$ is the locus of characters of $\pi_1(S^2 - \{p_0, p_2, p_3, p_4\})$ with fixed traces along the four boundary circles. This relative character variety can be computed [FK65] to be the affine cubic hypersurface in $\mathbb{C}^3$ given by the equation

$$f = x^2 + y^2 + z^2 + xyz - z - 2 = 0$$

where $x = \text{tr}(AB), y = \text{tr}(B^{-1}C)$ and $z = \text{tr}(A^{-1}C)$. Furthermore, the reducible representations, which are the points in $\overline{S} \backslash S$, correspond to $(x, y, z) = (\pm\sqrt{3}, \mp\sqrt{3}, 2)$. These are precisely the singular points of the affine cubic surface $\overline{S}$. Since the Tjurina number, $\dim \widehat{\mathcal{O}}_{(x,y,z)}/(f, \partial_x f, \partial_y f, \partial_z f)$, is equal to 1 at the singularities, they are A_1 singularities. $\square$

We record a calculation of the singular cohomology groups of S for use in Section 7.

Proposition 3.3.5. *The singular cohomology groups of S are*

$$H^*(S;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z}^4 & i = 3 \\ 0 & i \geq 4 \end{cases}$$

Proof. Let Q denote the projective closure of $\overline{S}$ inside of $\mathbb{P}^3$. One can check that Q is smooth at infinity, meaning that Q_{sm}, the smooth locus of Q, is the complement of the two singularities at $\overline{S} \backslash S$. By Theorem 4.3 in [Dim92], the homology groups of Q are

$$H_*(Q;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^5 & i = 2 \\ 0 & i = 3 \\ \mathbb{Z} & i = 4 \end{cases}$$

By Poincaré duality,

$$H_n(Q_{\mathrm{sm}};\mathbb{Z}) \cong H_c^{4-n}(Q_{\mathrm{sm}};\mathbb{Z})$$

And we can equate the compactly supported cohomology with a relative cohomology group,

$$H_c^n(Q_{\mathrm{sm}};\mathbb{Z}) \cong H^n(Q, Q_{\mathrm{sing}};\mathbb{Z})$$

which can be determined from the long exact sequence

$$\cdots \to H^n(Q, Q_{\mathrm{sing}};\mathbb{Z}) \to H^n(Q;\mathbb{Z}) \to H^n(Q_{\mathrm{sing}};\mathbb{Z}) \to \ldots$$

In particular, since Q_{sing} is zero-dimensional, we see that $H_n(Q_{\mathrm{sm}};\mathbb{Z}) \cong H^{4-n}(Q;\mathbb{Z})$ for $n \leq 2$. And

$$\mathrm{rk}\, H_3(Q_{\mathrm{sm}};\mathbb{Z}) = \mathrm{rk}\, H^1(Q, Q_{\mathrm{sing}};\mathbb{Z})$$

$$= \operatorname{rk} H^1(Q;\mathbb{Z}) + |Q_{\text{sing}}| - 1$$

So, the homology groups of Q_{sm} are

$$H_*(Q_{\text{sm}};\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^5 & i = 2 \\ \mathbb{Z} & i = 3 \\ 0 & i \geq 4 \end{cases}$$

Let $Q_\infty = Q \backslash \overline{S}$. Then $S = Q_{\text{sm}} \backslash Q_\infty$. We have $Q_\infty = \{xyz = 0\} \subset \mathbb{P}^2$, which is a triangular arrangement of three lines. The normal bundle of each of these three copies of $\mathbb{P}^1$ has degree -1. So, a neighborhood of each sphere inside of S is diffeomorphic to the D^2 bundle over S^2 with Euler number -1. The boundary of this neighborhood is diffeomorphic to S^3. Hence, the boundary of a neighborhood of Q_∞, $\partial N(Q_\infty)$, is a necklace of three copies of S^3. We can then apply the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_*(\partial N(Q_\infty)) \longrightarrow H_*(Q_\infty) \oplus H_*(S) \longrightarrow H_*(Q_{\text{sm}}) \longrightarrow \dots$$

to compute the stated cohomology groups. $\square$

3.4 The A-polynomials of the square and granny knots

We wish to describe the image of the natural map $r : \mathscr{X}(\Gamma) \to \mathscr{X}(\partial(S^3 \backslash (3_1 \# 3_1)))$ given by restriction to the boundary torus. Coordinates on $\mathscr{X}(\partial S^3 \backslash N(3_1 \# 3_1)) = \mathscr{X}(T^2)$ are given by the traces of the meridian and longitude. One may consider the double branched cover $d : \mathbb{C}^* \times \mathbb{C}^* \to \mathscr{X}(T^2)$ where the coordinates on the cover are given by the eigenvalues of the meridian and longitude, M and L. The defininining polynomial for the closure of the pull-back of the image of r to $\mathbb{C}^* \times \mathbb{C}^*$ is called the A-polynomial [CCG+94].

For the right-handed trefoil, the A-polynomial is $M^{-6} + L = 0$, whereas for the left-handed trefoil it is $M^6 + L = 0$ [CCG+94]. These equations define the image under r of the components $\overline{\mathscr{X}_{i,irr}}$. $\mathscr{X}_{\text{red}}$ is mapped to the line $L = 1$.

Lemma 3.4.1. *Let S be as in Proposition 3.3.3. Then the defining equation of the algebraic set $d^{-1}(\overline{r(S)})$ in eigenvalue coordinates is $L - M^{-12} = 0$ for the granny knot (the composite of two right-handed trefoils) and $L = 1$ for the square knot (the composite of oppositely oriented trefoils).*

Proof. Let ℓ_i denote the longitude of the i^{th} summand of $3_1\#3_1$. Then, the longitude of $3_1\#3_1$ is $\ell = \ell_0\ell_1$. Also, each of the ℓ_i commutes with the meridian μ in Γ. Since $\rho(\mu)$ is non-central, this means that $\rho(\ell_0)$ and $\rho(\ell_1)$ must commute with each other. In fact, for the irreducible representations of the right-handed trefoil, we have $\rho(\ell_i) = -\rho(m)^{-6}$ and similarly $\rho(\ell_i) = -\rho(m)^6$ for the left-handed trefoil.

For $\rho \in S$, we have that ρ restricted to either summand is irreducible. So for the granny knot, we then have $\rho(\ell) = \left(-\rho(m)^{-6}\right)^2 = \rho(m)^{-12}$ and for the square knot we obtain $\rho(\ell) = 1$. These matrix equations give the desired eigenvalue equations. ☐

Proposition 3.4.2. *The A-polynomial of the granny knot, $3_1^r\#3_1^r$, is*

$$A_{3_1^r\#3_1^r} = (L-1)(L+M^{-6})(L-M^{-12})$$

The A-polynomial of the square knot, $3_1^r\#3_1^l$, is

$$A_{3_1^r\#3_1^l} = (L-1)(L+M^{-6})(L+M^6)$$

Proof. The A-polynomial is a product (omitting repeated factors) of the the defining polynomials for the images of the four components of $\mathscr{X}(\Gamma)$. Two of the components are copies of $\mathscr{X}(3_1)$, and therefore contribute factors corresponding to the A-polynomial of right or left-handed trefoil. The reducibles give the factor of $L-1$. The factor coming from the two-dimensional component $\overline{S}$ was determined in Lemma 3.4.1. ☐

We will also be interested in the defining equation for the image of the map $r : \mathscr{X}_{\mathrm{irr}}(\Gamma) \to \mathscr{X}(T^2)$, where we only consider the irreducibles. Let us call the defining polynomial for this curve $A_K^{\mathrm{irr}}(M, L)$. Then, by the above discussion, we find

$$A^{\mathrm{irr}}_{3_1^r\#3_1^r} = (L+M^{-6})(L-M^{-12})$$

$$A^{\mathrm{irr}}_{3_1^r\#3_1^l} = (L+M^{-6})(L+M^{6})(L-1)$$

3.5 Surgeries on the Granny and Square knots

In this section, we prove Theorems 1.1.6 and 1.1.7. We proceed by calculating the relevant character schemes, showing they are smooth, and then computing their singular cohomology groups so that we can apply Corrolary 2.2 to write HP as the (degree shifted) singular cohomology of the character scheme.

3.5.1 Character scheme of a composite knot

First, we establish a general procedure for computing the (set-theoretic) characters of the exterior of a composite knot. Although the character variety of $3_1\#3_1$ was computed in Section 5, the description given here will be particularly amenable for computing the character varieties of the surgeries. The description from Section 5 will also be useful.

Let K_1 and K_2 be two knots in S^3 and set $K = K_1\#K_2$, $M_i = S^3\backslash K_i$, $M = S^3\backslash K$. We have the following pushout diagram of spaces:

$$\begin{tikzcd} M & M_1 \arrow[l, hook'] \\ M_2 \arrow[u, hook] & S^1 \arrow[l, hook', "i_2"'] \arrow[u, hook, "i_1"'] \end{tikzcd}$$

where $i_j(S^1) = m_j$, a meridian for K_j, $j = 1, 2$. By the Van Kampen theorem, we have the pushout diagram of groups:

$$\begin{tikzcd} \pi_1(M) & \pi_1(M_1) \arrow[l, hook'] \\ \pi_1(M_2) \arrow[u, hook] & \pi_1(S^1) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

That is, $\pi_1(M) \cong \pi_1(M_1)*\pi_1(M_2)/\langle m_1 = m_2\rangle$. We have a pullback diagram of representation spaces:

$$\begin{tikzcd}
\mathscr{R}(M) \arrow[r, two heads] \arrow[d, two heads] & \mathscr{R}(M_1) \arrow[d, two heads, "r_1"] \\
\mathscr{R}(M_2) \arrow[r, two heads, "r_2"] & \mathscr{R}(S^1)
\end{tikzcd}$$

To analyze $\mathscr{X}(M) = \mathscr{R}(M)//G$, we can compare it to a simpler object: the fiber product of the character schemes $\mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$. We have the diagram

$$\begin{tikzcd}
\mathscr{X}(M) \arrow[dr, "\varphi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2) \arrow[r, two heads] \arrow[d, two heads] & \mathscr{X}(M_1) \arrow[d, two heads, "\overline{r}_1"] \\
& \mathscr{X}(M_2) \arrow[r, two heads, "\overline{r}_2"] & \mathscr{X}(S^1)
\end{tikzcd}$$

where $\overline{r}_1([\rho_1]) = \mathrm{tr}(\rho_1(m_1))$.

3.5.1.1 Pullbacks and quotients

In order to understand the character scheme of M from the fiber product of the character schemes of M_1 and M_2, we must determine the pre-images of points under φ. We establish the following lemma:

Lemma 3.5.1. *Let $\varphi : \mathscr{X}(M) \to \mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$ denote the natural map as above. Then for any $p = ([\rho_1], [\rho_2]) \in \mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$, we have*

$$\varphi^{-1}(p) \cong \mathrm{Stab}(m)/\langle \mathrm{Stab}(\rho_1), \mathrm{Stab}(\rho_2)\rangle$$

where $m = r_1(\rho_1) = r_2(\rho_2)$

Proof. The pre-image of p in $\mathscr{R}(M_1) \times \mathscr{R}(M_2)$ is $\mathrm{Orb}(\rho_1) \times \mathrm{Orb}(\rho_2)$. The pair (ρ_1, ρ_2) is a point here that is also in $\mathscr{R}(M_1) \times_{\mathscr{R}(S^1)} \mathscr{R}(M_2)$. All other such points can be obtained by using the action of $\mathrm{Stab}(m)$ on each factor, or else using the diagonal action of G. This gives the set

$$(\mathscr{R}(M_1) \times_{\mathscr{R}(S^1)} \mathscr{R}(M_2)) \cap (\mathrm{Orb}(\rho_1) \times \mathrm{Orb}(\rho_2)) = G \cdot (\mathrm{Stab}(m) \cdot \rho_1 \times \mathrm{Stab}(m) \cdot \rho_2)$$

$$=G \cdot (\mathrm{Stab}(m) \cdot \rho_1 \times \rho_2)$$

Reducing modulo the diagonal action of G,

$$\begin{aligned} &G \cdot (\mathrm{Stab}(m) \cdot \rho_1 \times \rho_2)/G \\ =&\,\mathrm{Stab}(m)/\langle \mathrm{Stab}(\rho_1), \mathrm{Stab}(\rho_2)\rangle \end{aligned}$$

Thus, $\varphi^{-1}(p) \cong \mathrm{Stab}(m)/\langle \mathrm{Stab}(\rho_1), \mathrm{Stab}(\rho_2)\rangle$. □

3.5.2 Irreducible representations in the character scheme of a composite knot

To determine the locus of irreducible representations $\mathscr{X}_{\mathrm{irr}}(M)$, we first describe $\mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$ and then use Lemma 3.5.1 to understand the fibers of φ over the various components.

Recall that $\mathscr{X}(M)$ has a stratification $\mathscr{X}_{\mathrm{nar}} \subset \mathscr{X}_{\mathrm{red}} \subset \mathscr{X}$, where $\mathscr{X}_{\mathrm{nar}}$ is the locus of characters of non-abelian reducible representations. The complement $\mathscr{X}_{\mathrm{irr}} = \mathscr{X} \setminus \mathscr{X}_{\mathrm{red}}$ is the locus of irreducibles. The scheme $\mathscr{X}_{\mathrm{nar}}$ can be identified from Lemma 4.2. The characters of non-abelian reducibles are also the characters of abelian reducibles. That is, every reducible character has an associated orbit of abelian representations, but for those characters in $\mathscr{X}_{\mathrm{nar}}$, there is an additional orbit corresponding to non-abelian reducible representations.

Taking the product stratification on $\mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$ gives nine different strata of six essentially different types. The following proposition states which strata intersect the image $\varphi(\mathscr{X}_{\mathrm{irr}}(M))$ and also identifies the set of irreducible representations in the fiber of φ over a point in a given stratum.

Proposition 3.5.2. *Using the previously established notation, $\varphi(\mathscr{X}_{irr}(M))$ consists of the following pieces*

- $\mathscr{X}_{irr}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{irr}(M_2)$
- $\mathscr{X}_{irr}(M_i)$
- $\mathscr{X}_{nar}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{nar}(M_2)$

The fibers of φ are copies of:

- $\mathbb{C}^*$ *over points in* $\mathscr{X}_{irr}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{irr}(M_2)$ *with meridional eigenvalue* $\mu \neq \pm 1$.

- $\mathbb{C}$ *over points in* $\mathscr{X}_{irr}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{irr}(M_2)$ *with meridional eigenvalue* $\mu = \pm 1$.

- *A single point over points in* $\mathscr{X}_{irr}(M_i)$ *with* $\Delta(\mu^2) \neq 0$.

- $\mathbb{C}$ *over points in* $\mathscr{X}_{irr}(M_i)$ *with* $\Delta(\mu^2) = 0$.

- $\mathbb{C}^* - \{1\}$ *over points in* $\mathscr{X}_{nar}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{nar}(M_2)$

Proof. First, we identify the copy of $\mathscr{X}_{\text{irr}}(M_1)$ that appears in $\mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$. A reducible character is the character of an abelian representation, and the meridian generates the abelianization of the knot group. Thus, the isomorphism $H_1(M_1) \cong \pi_1(S^1)$, where S^1 a meridional circle, yields an isomorphism $\mathscr{X}_{\text{red}}(M_1) \cong \mathscr{X}(S^1)$. And taking fiber products, $\mathscr{X}_{\text{irr}}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{\text{red}}(M_2) \cong \mathscr{X}_{\text{irr}}(M_1)$.

Now, we show that image of φ consists of the stated pieces. Indeed, the only strata not included in the list are contained in $(\mathscr{X}_{\text{red}}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{\text{red}}(M_2)) \backslash (\mathscr{X}_{\text{nar}}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}_{\text{nar}}(M_2))$. These correspond to representations of the form $\rho_1 * \rho_2$ where (e.g.) ρ_1 is abelian and ρ_2 is reducible. However, for an abelian representation, $\text{im}(\rho_1) = \text{im}(\rho_1|_{m_1})$ since the meridian m_1 generates the abelianization of $\pi_1(M_1)$. Thus, since the ρ_i agree on m_i, we see that $\text{im}(\rho_1 * \rho_2) = \text{im}(\rho_2)$, so that the composite representation is also reducible. Thus, none of these pairings provide irreducible representations.

For $p = ([\rho_1], [\rho_2]) \in \mathscr{X}(M_1) \times_{\mathscr{X}(S^1)} \mathscr{X}(M_2)$, if both $[\rho_1], [\rho_2] \in \mathscr{X}_{\text{irr}}$, then $\text{Stab}(\rho_i) = \{\pm 1\}$. Furthermore, $r_1(\rho_1)$ is an abelian, non-central representation (if $\rho_1(m) = \pm I$, then the entire representation is central because m_1 normally generates $\pi_1(M_1)$). Thus, $\text{Stab}(r_1(\rho_1)) \cong \mathbb{C}^*$ for meridional trace not ± 2, and $\text{Stab}(r_1(\rho_1)) \cong \mathbb{C} \times \mathbb{Z}/2$ otherwise. So, $\varphi^{-1}(p) \cong \mathbb{C}^*/\{\pm 1\} \cong \mathbb{C}^*$ or $\varphi^{-1}(p) \cong \mathbb{C}$ by Lemma 3.5.1.

If $[\rho_1]$ is irreducible but $[\rho_2]$ is reducible, then we can find an abelian lift ρ_2, so that $\text{Stab}(\rho_2) = \text{Stab}(r_2(\rho))$, and the fiber $\varphi^{-1}(p)$ is a point. For a non-abelian lift of ρ_2, $\text{Stab}(\rho_2)$ is trivial. Moreover, the trace of the meridian cannot be ± 1 for a non-abelian reducible because $\Delta(\pm 1) \neq 0$. Therefore, the stabilizer of the meridian must be $\mathbb{C}^*$. The abelian lies in the closure of the orbit of non-abelian reducibles, so that $\varphi^{-1}(p) = \mathbb{C}$ for such a point.

If both are reducible and at least one is abelian, then the overall representation is reducible. If both are non-abelian reducibles, then the stabilizers of each representation are trivial and the stabilizer of the meridian is $\mathbb{C}^*$, giving that the fiber of φ is $\mathbb{C}^*$. However, not all of these representations are irreducible. We have that $\operatorname{im}(\rho_i) \subset B_i$, for B_1, B_2 Borel subgroups. For some $d \in \operatorname{Stab}(r_1(\rho_1))$, the composite representation corresponding to d has image generated by $\langle \operatorname{im}(\rho_1), d^{-1} \operatorname{im}(\rho_2) d \rangle$. If this image were contained in some Borel subgroup B, then $\operatorname{im}(\rho_1)$ would be contained in two Borel subgroups, so either it is contained in a diagonal subgroup (but then ρ_1 is abelian), or else $B = B_1$. Then, we have $d^{-1} \operatorname{im}(\rho_2) d \subset B_1$, and so by the same argument we conclude $B_1 = d^{-1} B_2 d$. Thus, $d \in \operatorname{Stab}(B_2)$, which is trivial in G^{ad}. Hence, precisely one point in $\operatorname{Stab}(r_1(\rho_1))$ corresponds to a reducible — the rest are irreducible. So, the irreducibles in $\varphi^{-1}(p)$ form a copy of $\mathbb{C}^* - \{1\}$. □

3.5.3 Character scheme of a connected sum of two trefoils

We now focus on the case when $K_1 = K_2 = 3_1^r$. The character scheme of the trefoil can be described as a plane curve:

$$\mathscr{X}(3_1) \cong \{(y-2)(x^2 - y - 1) = 0\} \subset \mathbb{C}^2$$

where x is the trace of the meridian. In terms of the Wirtinger presentation, we have $x = \operatorname{tr}(\rho(r)) = \operatorname{tr}(\rho(s))$ and $y = \operatorname{tr}(rs^{-1})$. The line $\{y = 2\}$ is $\mathscr{X}_{\mathrm{red}}$ and $\{x^2 - y = 1, y \neq 2\}$ is $\mathscr{X}_{\mathrm{irr}}$. The map $\overline{r}_1$ is projection onto the x coordinate. The longitude for 3_1^r is $\ell = sr^2 sr^{-4}$, and its trace in the x, y coordinates is given by the polynomial

$$L(x,y) = x^6 y - 2x^6 - x^4 y^2 - 2x^4 y + 8x^4 + 2x^2 y^2 + x^2 y - 10x^2 + 2$$

The restriction of $L(x,y)$ to $y = 2$ is the constant function 2, as expected. On this component, $\rho(\ell) = I$. The restriction of $L(x,y)$ to $y = x^2 - 1$ is $L = -x^6 + 6x^4 - 9x^2 + 2$, which can be deduced from the fact that for the irreducible representations, we have $\rho(\ell) = -\rho(m)^{-6}$.

The Alexander polynomial has roots that are primitive 6th roots of unity. So, non-abelian reducibles occur at the points $(\pm\sqrt{3}, 2) \in \mathscr{X}_{\mathrm{red}}$. Observe that this is precisely $\overline{\mathscr{X}_{\mathrm{irr}}} - \mathscr{X}_{\mathrm{irr}}$.

The fiber product of the character varieties over the meridional trace map is

$$\mathscr{X}(3_1) \times_{\mathbb{C}} \mathscr{X}(3_1) \cong \{(y-2)(x^2-y-1)=0, (z-2)(x^2-z-1)=0\} \subset \mathbb{C}^3$$

Applying Proposition 3.5.2, we have the following explicit descriptions of the fibers of φ over points in the various strata of $\varphi(\mathscr{X}_{\mathrm{irr}}(K_1 \# K_2))$:

- $\mathscr{X}_{\mathrm{irr}} \times_{\mathbb{C}} \mathscr{X}_{\mathrm{irr}} = \{x^2-y-1=0, x^2-z-1=0, y \neq 2, z \neq 2\}$. The fibers of φ are $\mathbb{C}^*$ unless $x = \pm 2$, in which case they are $\mathbb{C}$.
- $\mathscr{X}_{\mathrm{irr}}(M_1) = \{z=2, x^2-y-1=0, y \neq 2\}$. Note that since $y \neq 2$, we have $x \neq \pm\sqrt{3}$ and $\Delta(m^2) \neq 0$. So, the fibers of φ are just points. The same holds for $\mathscr{X}_{\mathrm{irr}}(M_2) = \{y=2, x^2-z-1=0, z \neq 2\}$.
- $\mathscr{X}_{\mathrm{nar}} \times_{\mathbb{C}} \mathscr{X}_{\mathrm{nar}} = \{(\pm\sqrt{3}, 2, 2)\}$. The fibers of φ are $\mathbb{C}^* - \{1\}$.

Remark 3.5.3. To compare this description with that of Proposition 3.3.3, we see that

- $\varphi^{-1}(\mathscr{X}_{\mathrm{irr}} \times_{\mathbb{C}} \mathscr{X}_{\mathrm{irr}} \cup \mathscr{X}_{\mathrm{nar}} \times_{\mathbb{C}} \mathscr{X}_{\mathrm{nar}}) = S$
- $\mathscr{X}_{\mathrm{irr}}(M_i) = \mathscr{X}_{i,irr}$.

Since $\pi_1(3_1^r \# 3_1^r)$ and $\pi_1(3_1^r \# 3_1^l)$ are isomorphic, the same description applies to $\mathscr{X}_{\mathrm{irr}}(3_1^r \# 3_1^l)$.

3.5.4 Character scheme for granny knot surgeries

Let G denote the connected sum of two right-handed trefoils, and $S^3_{p/q}(G)$ the p/q surgery. We have the following description of $\mathscr{X}_{\mathrm{irr}}(S^3_{p/q}(G))$.

Proposition 3.5.4. *$\mathscr{X}_{irr}(S^3_{p/q}(G))$ consists of $2\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1))$ points and*

- *$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/2q}(3_1))$ copies of $\mathbb{C}^*$ when p is odd*
- *$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/2q}(3_1)) - 1$ copies of $\mathbb{C}^*$ when p is even, $p \neq 12k$.*
- *$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/2q}(3_1)) - 3$ copies of $\mathbb{C}^*$ and 2 copies of $\mathbb{C}^* - \{1\}$ when $p = 12k, p/q \neq 12$.*

- $S = \varphi^{-1}(\mathscr{X}_{nar} \times_{\mathbb{C}} \mathscr{X}_{nar} \cup \mathscr{X}_{irr} \times_{\mathbb{C}} \mathscr{X}_{irr})$ *when* $p/q = 12$

We will describe $\mathscr{X}_{\text{irr}}(S^3_{p/q}(G))$ as a closed subscheme of $\mathscr{X}_{\text{irr}}(S^3\backslash G)$. First, we have the following lemma.

Lemma 3.5.5. *Let* $\varphi : \mathscr{X}_{irr}(S^3\backslash G) \to \mathscr{X}(S^3\backslash 3_1) \times_{\mathbb{C}} \mathscr{X}(S^3\backslash 3_1)$ *denote the map to the fiber product over the meridional trace. Then*

$$\mathscr{X}_{irr}(S^3_{p/q}(G)) = \varphi^{-1}(\varphi(\mathscr{X}_{irr}(S^3_{p/q}(G))))$$

Proof. A character $[\rho] = [(\rho_1, \rho_2)] \in \mathscr{X}_{\text{irr}}(S^3\backslash G)$ is in the character scheme for the p/q surgery if the surgery equation $\rho(m^p\ell^q) = I$ is satisfied. For a composite knot, the longitude ℓ is the product of the two longitudes for the constituent knots. Thus, the surgery equation is

$$\rho_1(m)^p\,(\rho_1(\ell_1)\rho_2(\ell_2))^q = I$$

If $[\rho'] \in \varphi^{-1}(\varphi([\rho]))$, then it is of the form $[\rho'] = [(\rho_1, g^{-1}\rho_2 g)]$ for some $g \in \text{Stab}(\rho(m))$. For an irreducible representation, we cannot have $\rho(m) = \pm I$. Thus, $\text{Stab}(\rho(m))$ is one-dimensional. Furthermore, since ℓ_2 and m commute, we must have $\text{Stab}(\rho(m)) \subset \text{Stab}(\rho(\ell_2))$. Therefore, $g^{-1}\rho_2(\ell_2)g = \rho_2(\ell_2)$, verifying the surgery equation for $[\rho']$. □

Thanks to this lemma, it suffices to describe $\varphi(\mathscr{X}_{\text{irr}}(S^3_{p/q}(G)))$. We consider each of the three different types of points in $\mathscr{X}_{\text{irr}}(3_1) \times_{\mathbb{C}} \mathscr{X}_{\text{irr}}(3_1)$ separately.

Lemma 3.5.6. *The locus of characters of* $\pi_1(S^3_{p/q}(G))$ *that restrict to an irreducible in* $\pi_1(S^3\backslash K_1)$ *and an abelian in* $\pi_1(S^3\backslash K_2)$ *is* $\mathscr{X}_{irr}(S^3_{p/q}(G)) \cap \varphi^{-1}(\mathscr{X}_{irr}(M_i))$. *This space consists of* $\lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1))$ *points .*

Proof. For $([\rho_1], [\rho_2]) \in \mathscr{X}_{\text{irr}}(M_1) \subset \mathscr{X}(3_1) \times_{\mathbb{C}} \mathscr{X}(3_1)$, ρ_2 is an abelian representation. Thus, $\rho_2(\ell_2) = I$. The surgery equation then reduces to $\rho(m^p\ell_1^q) = I$, which is just the condition for p/q surgery on the trefoil. So,

$$|\varphi(\mathscr{X}_{\text{irr}}(S^3_{p/q}(G))) \cap \mathscr{X}_{\text{irr}}(M_1)| = \lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1))$$

Since the fibers of φ over these types of characters are just points, we obtain the result. □

Lemma 3.5.7. *The set of characters that restrict to an irreducible representation on both factors is given by $\mathscr{X}_{irr}(S^3_{p/q}(3_1)) \cap \varphi^{-1}(\mathscr{X}_{irr} \times_{\mathbb{C}} \mathscr{X}_{irr})$, which consists of*

$$\begin{cases} \lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/2q}(3_1)) = \frac{1}{2}|p-12q| - \frac{1}{2} \text{ copies of } \mathbb{C}^* & \text{if } p \text{ is odd} \\ \frac{1}{2}|p-12q| - 1 \text{ copies of } \mathbb{C}^* & \text{if } p \text{ is even}, p \neq 12k \\ \frac{1}{2}|p-12q| - 3 \text{ copies of } \mathbb{C}^* & \text{if } p = 12k, p/q \neq 12 \\ \varphi^{-1}(\mathscr{X}_{irr} \times_{\mathbb{C}} \mathscr{X}_{irr}) & \text{if } p/q = 12 \end{cases}$$

Proof. For irreducible representations of $\pi_1(S^3 \backslash 3_1)$, $\rho(\ell)$ is determined by $\rho(m)$. In fact, we have $\rho(\ell) = -\rho(m)^{-6}$. For a point $\varphi([\rho]) = ([\rho_1],[\rho_2]) \in \mathscr{X}_{\mathrm{irr}} \times_{\mathbb{C}} \mathscr{X}_{\mathrm{irr}}$, we have $\rho_1(m_1) = \rho_2(m_2)$, so that $\rho_1(\ell_1) = \rho_2(\ell_2)$. Thus,

$$\rho(m^p \ell^q) = \rho_1(m^p \ell_1^{2q})$$

For p odd, the equation $\rho_1(m^p \ell_1^{2q}) = I$ is just the defining equation for $p/2q$ surgery on the trefoil. Thus, we obtain $\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/2q}(3_1))$ points. None of these occur at meridional trace ± 2, so that the fiber of φ is a copy of $\mathbb{C}^*$ for all of these points.

For p even, $p \neq 12k$, the surgery equation

$$\rho(m)^{p-12q} = I$$

has an even exponent. Thus, we obtain

$$\frac{1}{2}(|12q-p| - 2)$$

distinct characters, where the -2 term serves to discount the roots at $\rho(m) = \pm I$. For $p = 12k, p/q \neq 12$, two of the characters in this count occur at meridional trace $\pm\sqrt{3}$, so we subtract 2 in this case. Again, all of the fibers of φ are $\mathbb{C}^*$.

For $p/q = 12$, the surgery equation is trivial, so that every representation of this form provides a representation of the surgery. □

Lemma 3.5.8. *The set of irreducible representations formed from a composite of non-abelian reducible representations is*

$$\mathscr{X}_{irr}(S^3_{p/q}(G)) \cap \varphi^{-1}(\mathscr{X}_{nar} \times_{\mathbb{C}} \mathscr{X}_{nar}) = \begin{cases} 2 \text{ copies of } \mathbb{C}^* - \{1\} & \text{if } p = 12k \\ \emptyset & \text{else} \end{cases}$$

Proof. For $([\rho_1], [\rho_2]) \in \mathscr{X}_{\text{nar}} \times_{\mathbb{C}} \mathscr{X}_{\text{nar}}$, we have $\text{tr}(\rho_i(m)) = \pm\sqrt{3}$ and $\rho_i(\ell_i) = I$. Thus, the surgery equation becomes $\rho(m)^p = I$. This holds if and only if $p = 12k$. □

For the remaining case of $p/q = 12$, we have found that the character scheme of 12 surgery on the granny knot, $\mathscr{X}_{\text{irr}}(S^3_{12}(G))$, consists of 2 points coming from the irreducible representation in each of the two copies of $\mathscr{X}_{\text{irr}}(S^3_{12}(3_1))$ and the surface

$$S = \varphi^{-1}(\mathscr{X}_{\text{nar}} \times_{\mathbb{C}} \mathscr{X}_{\text{nar}} \cup \mathscr{X}_{\text{irr}} \times_{\mathbb{C}} \mathscr{X}_{\text{irr}})$$

Putting this and the preceding lemmas together, we obtain Proposition 3.5.4.

Remark 3.5.9. 12 surgery on the granny knot yields a Seifert fiber space fibered over the orbifold base $S^2(2, 2, 3, 3)$ [KT90]. Thus,

$$\pi_1(S^3_{12}(G)) \cong \langle a, b, c | a^3 = b^3 = c^2 = (abc)^{-2} \rangle$$

3.5.5 Character scheme for square knot surgeries

Let Q denote the square knot, a connected sum of two mirror trefoils, and $S^3_{p/q}(Q)$ the p/q surgery. We have the following description of $\mathscr{X}_{\text{irr}}(S^3_{p/q}(Q))$

Proposition 3.5.10. *The character scheme $\mathscr{X}_{irr}(S^3_{p/q}(Q))$ consists of $\lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1)) + \lambda_{\text{SL}(2,\mathbb{C})}(S^3_{-p/q}(3_1))$ points and*

- $\frac{1}{2}|p| - \frac{1}{2}$ *copies of $\mathbb{C}^*$ when p is odd*
- $\frac{1}{2}|p| - 1$ *copies of $\mathbb{C}^*$ when p is even, $p \neq 12k$*
- $\frac{1}{2}|p| - 3$ *copies of $\mathbb{C}^*$ and 2 copies of $\mathbb{C}^* - \{1\}$ when $p = 12k \neq 0$*

- $S = \varphi^{-1}(\mathscr{X}_{nar} \times_{\mathbb{C}} \mathscr{X}_{nar} \cup \mathscr{X}_{irr} \times_{\mathbb{C}} \mathscr{X}_{irr})$ *when* $p = 0$

Proof. The proof is analogous to that of Proposition 3.5.4. The essential difference is that we also need to consider the representations of the left-handed trefoil. Since $S^3_{p/q}(3_1^R) \cong S^3_{-p/q}(3_1^L)$, we can relate the Casson invariants by

$$\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1^R)) = \lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{-p/q}(3_1^L))$$

Thus, the intersection of $\mathscr{X}_{\mathrm{irr}}(S^3_{p/q}(Q))$ with the two copies of $\mathscr{X}_{\mathrm{irr}}(3_1)$ give contributions of $\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(3_1))$ and $\lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{-p/q}(3_1))$ points, depending on whether the copy of $\mathscr{X}_{\mathrm{irr}}(3_1)$ corresponds to the right or left-handed trefoil.

For irreducible representations of the right-handed trefoil, we have $\rho_1(\ell_1) = -\rho_1(m)^{-6}$, whereas for the left-handed trefoil we have $\rho_2(\ell_2) = -\rho_2(m)^6$. So, for a representation of the composite that restricts to irreducibles on either factor, we find that $\rho(\ell) = \rho(\ell_1\ell_2) = I$. The equation for p/q surgery reduces to

$$\rho(m)^p = I$$

Throwing away the solutions $\rho(m) = \pm I$ and counting solutions up to conjugacy (i.e. dividing by the equivalence $\rho(m) \sim \rho(m)^{-1}$), we find $\frac{1}{2}|p| - \frac{1}{2}$ solutions for p odd, and $\frac{1}{2}|p| - 1$ solutions for p even, $p \neq 12k$. For $p = 12k \neq 0$, we omit the two solutions with $\mathrm{tr}(\rho(m)) = \pm\sqrt{3}$, as these correspond to non-abelian reducible representations rather than irreducibles. The case of irreducibles formed from the composite of non-abelian reducible representations, which only occurs when $p = 12k$, is the same as in Lemma 3.5.8. When $p = 0$, the surgery equation is trivial, and we have the same situation as for $p = 12$ for the granny knot. □

Remark 3.5.11. 0 surgery on the square knot yields a Seifert fiber space fibered over the orbifold base $S^2(-2, 2, 3, 3)$ [KT90]. Thus,

$$\pi_1(S^3_0(Q)) \cong \langle a, b, c | a^3 = b^3 = c^2 = (abc)^2 \rangle$$

3.5.6 Smoothness of the Character Schemes

Proposition 3.5.12. *Let G and Q denote the granny and square knots, respectively. The schemes $\mathscr{X}_{irr}(S^3_{p/q}(G))$ and $\mathscr{X}_{irr}(S^3_{p/q}(Q))$ are smooth schemes for all p and q.*

Proof. The sets of complex points of these schemes were computed in the previous section. They consisted of components of dimensions zero, one, and, in the cases of $S^3_{12}(G)$ and $S^3_0(Q)$, two. To establish the smoothness of the character scheme near some irreducible representation ρ, we must show that the local dimension of the set of complex points at ρ equals the dimension of the tangent space to the scheme at ρ. Recall that for an irreducible representation ρ the tangent space is computed by $T_{[\rho]}\mathscr{X}_{\mathrm{irr}}(\Gamma) = H^1(\Gamma; \mathrm{ad}\,\rho)$. Thus, the proposition follows from the calculation of these H^1 groups in Lemma 3.5.14 below. □

Lemma 3.5.13. *Let ρ be an irreducible representation of $\pi_1(S^3\backslash(3_1\#3_1))$ (where $3_1\#3_1$ is either the square or granny knot, which have isomorphic fundamental groups). Let ρ_1 and ρ_2 be the restrictions of ρ to each of the two copies of $\pi_1(S^3\backslash 3_1)$. Then,*

$$\dim H^1(\pi_1(S^3\backslash(3_1\#3_1)); \mathrm{ad}\,\rho) = \begin{cases} 2 & \textit{if neither of the } \rho_i \textit{ are abelian} \\ 1 & \textit{if either of the } \rho_i \textit{ are abelian} \end{cases}$$

Proof. We can compute $H^1(\pi_1(S^3\backslash(3_1\#3_1)); \mathrm{ad}\,\rho)$ (we will suppress the π_1 from this notation without confusion, as all spaces in consideration are aspherical) from the following portion of the Mayer-Vietoris sequence:

$$\begin{aligned} 0 \to H^0(S^3\backslash 3_1; \mathrm{ad}\,\rho_1) \oplus H^0(S^3\backslash 3_1; \mathrm{ad}\,\rho_2) \to H^0(S^1\ \mathrm{ad}\,\rho) \to H^1(S^3\backslash(3_1\#3_1)\ \mathrm{ad}\,\rho) \to \\ \to H^1(S^3\backslash 3_1; \mathrm{ad}\,\rho_1) \oplus H^1(S^3\backslash 3_1; \mathrm{ad}\,\rho_2) \to H^1(S^1; \mathrm{ad}\,\rho) \to \dots \end{aligned} \tag{3.5.1}$$

The ρ_i are the restrictions of ρ to the two copies of $S^3\backslash 3_1$, and the S^1 refers to the meridional annulus along which the connected sum operation is performed. Technically, ρ restricts to the complement of the meridional annulus inside of $S^3\backslash 3_1$, but since removing a subset of the boundary of a manifold does not change its homotopy type, this is homotopy equivalent to $S^3\backslash 3_1$ so we ignore the distinction.

Observe that $H^1(S^1; \operatorname{ad}\rho) \cong H^0(S^1; \operatorname{ad}\rho) \cong \mathbb{C}$. The first isomorphism follows from Poincaré duality. The second follows from the fact that since ρ is an irreducible representation of $\pi_1(S^3\backslash(3_1\#3_1))$, it restricts to a non-central abelian representation on the meridian and the invariants of such a representation are a one-dimensional subspace of $\operatorname{ad}\rho$.

The last map in (3.5.1) is the sum of two maps, each of the form $H^1(S^3\backslash 3_1; \operatorname{ad}\rho_i) \to H^1(S^1; \operatorname{ad}\rho)$. When ρ_i is irreducible, this is the derivative at $[\rho_i]$ of the natural map $\mathscr{X}_{\mathrm{irr}}(S^3\backslash 3_1; \operatorname{ad}\rho_i) \to \mathscr{X}(S^1; \operatorname{ad}\rho)$, where S^1 refers to the meridional circle. From our description of $\mathscr{X}_{\mathrm{irr}}(S^3\backslash 3_1)$ as a plane curve, we observe that the meridional trace map is non-singular at all points. Thus, the map on tangent spaces is surjective.

We now consider the case when the ρ_i are both irreducible or both non-abelian reducibles. In this case, $H^0(S^3\backslash 3_1; \operatorname{ad}\rho_i) = 0$. When ρ_i is an irreducible representation, we observe that $\dim H^1(S^3\backslash 3_1; \operatorname{ad}\rho_i) = 1$ because the character scheme is smooth of dimension 1. When ρ_i is a non-abelian reducible, we can compute $\dim H^1(S^3\backslash 3_1; \operatorname{ad}\rho_i) = 1$ directly, as there are only finitely many non-abelian reducible representations up to conjugacy. From this data, (3.5.1) yields $\dim H^1(S^3\backslash(3_1\#3_1); \operatorname{ad}\rho) = 2$.

When ρ_1 is abelian and ρ_2 is irreducible, $H^0(S^3\backslash 3_1; \operatorname{ad}\rho_2) = 0$ and the map $H^0(S^3\backslash 3_1; \operatorname{ad}\rho_1) \to H^0(S^1; \operatorname{ad}\rho)$ at the start of (3.5.1) is an isomorphism. For an abelian representation, $\dim H^1(S^3\backslash 3_1; \operatorname{ad}\rho_1) = 1$. Thus, we compute $\dim H^1(S^3\backslash(3_1\#3_1); \operatorname{ad}\rho) = 1$. □

Lemma 3.5.14. *Let G and Q denote the square and granny knots (and let $3_1\#3_1$ denote either). Let ρ be an irreducible representation of $\pi_1(S^3_{p/q}(3_1\#3_1))$. Let ρ_1 and ρ_2 be the restrictions of ρ to each of the two copies of $\pi_1(S^3\backslash 3_1)$. Then,*

$$\dim H^1(\pi_1(S^3_{p/q}(3_1\#3_1)); \operatorname{ad}\rho) = \begin{cases} 2 & \textit{if both of the } \rho_i \textit{ are non-abelian and p/q=12 for the granny knot o} \\ & \textit{p/q=0 for the square knot} \\ 1 & \textit{if both of the } \rho_i \textit{ are irreducible and we are not in the above case} \\ 0 & \textit{if either of the } \rho_i \textit{ are abelian} \end{cases}$$

Proof. We can compute $H^1(S^3_{p/q}(3_1\#3_1); \operatorname{ad}\rho)$ from the following Mayer-Vietoris sequence:

$$\dots \xrightarrow{0} H^1(S^3_{p/q}(3_1\#3_1); \operatorname{ad}\rho) \to H^1(S^3\backslash(3_1\#3_1); \operatorname{ad}\rho) \oplus H^1(D^2 \times S^1, \operatorname{ad}\rho) \xrightarrow{f} H^1(T^2; \operatorname{ad}\rho) \to \dots \tag{3.5.2}$$

Since ρ must restrict to a non-central abelian representation on the boundary torus, we have $H^2(T^2; \operatorname{ad}\rho) \cong H^0(T^2; \operatorname{ad}\rho) \cong \mathbb{C}$. From the Euler characteristic, we compute $\dim H^1(T^2; \operatorname{ad}\rho) = 2$. Similarly, ρ restricts to a non-central abelian representation on the solid torus (if it sent the core of the solid torus to a central element, then in fact ρ would be central on the entire boundary torus, and in particular on the meridian). So, $\dim H^1(D^2 \times S^1; \operatorname{ad}\rho) = 1$.

We claim that f has rank 1 when $p/q = 12$ for the granny knot and $p/q = 0$ for the square knot and neither of the ρ_i are abelian representations, and in all other cases, f has rank 2.

Let $s : \mathscr{X}(D^2 \times S^1) \to \mathscr{X}(T^2)$ be the restriction map. The map on cohomology groups $H^1(D^2 \times S^1; \operatorname{ad}\rho) \to H^1(T^2; \operatorname{ad}\rho)$ can be identified with $ds_{[\rho]}$, the derivative of s at $[\rho]$. Similarly, we can identify the map $H^1(S^3\backslash(3_1\#3_1); \operatorname{ad}\rho) \to H^1(T^2; \operatorname{ad}\rho)$ with the derivative at $[\rho]$ of the restriction map $r : \mathscr{X}_{\text{irr}}(S^3\backslash(3_1\#3_1)) \to \mathscr{X}(T^2)$. Thus, we can write f as $f = (dr \oplus ds)_{[\rho]}$.

By a standard application of Lefschetz duality and the long exact sequence of the pair $(Y, \partial Y)$, where here $Y = D^2 \times S^1$ or $S^3\backslash(3_1\#3_1)$, we know that $\operatorname{rank}(dr) = \operatorname{rank}(ds) = 1$ [Sik12]. Thus, the rank of f is 2 unless the images of r and s have the same tangent spaces at $[\rho]$, in which case the rank of f is 1. We claim that this equality of tangent spaces occurs only when $p/q = 12$ for the granny knot and $p/q = 0$ for the square knot and neither of the ρ_i are abelian representations.

Let $t : \mathbb{C}^* \times \mathbb{C}^* \to \mathscr{X}(T^2)$ be the map from the eigenvalue variety to the character variety. With the coordinates (M, L) on $\mathbb{C}^* \times \mathbb{C}^*$ for the meridional and longitudinal eigenvalues, $t(M, L)$ is the class of a representation with $\rho(m) = \operatorname{diag}(M, M^{-1})$ and $\rho(\ell) = \operatorname{diag}(L, L^{-1})$. Away from the central representations, t is a degree two covering map. Thus, we can consider the tangent spaces to $t^{-1}(\operatorname{im}(s))$ and $t^{-1}(\operatorname{im}(r))$ in order to prove the claim.

The curve $t^{-1}(\operatorname{im}(s))$ is the surgery curve $\{M^pL^q = 1\}$. The closure of the curve

$t^{-1}(\mathrm{im}(r))$ is the vanishing locus of the A-polynomial of the knot (ignoring the factor coming from reducibles). Recall our calculation of the A-polynomials from Section 6,

$$
\begin{aligned}
A^{\mathrm{irr}}_{3_1^r\#3_1^r}(M,L) =& (L+M^{-6})(L-M^{-12})\\
A^{\mathrm{irr}}_{3_1^r\#3_1^l}(M,L) =& (L+M^{-6})(L+M^{6})(L-1)
\end{aligned}
$$

The factor of $L+M^{-6}$ (which is the A-polynomial of the right-handed trefoil) comes from representations that are irreducible on a 3_1^r summand and abelian on the other summand. Similarly, $L+M^6$ is the A-polynomial of the left-handed trefoil. The last factors come from the composites of two non-abelian representations. For such representations of the the granny knot, we have $L_1 = L_2 = -M^{-6}$ and $L = L_1L_2$, so that $L = M^{-12}$. For the square knot, $L_1 = L_2^{-1}$, so that this component is mapped to the line $L = 1$.

Now we see that the only situations in which the tangent space to the vanishing locus of the A-polynomial coincides with the tangent space to the surgery curve are when $p = 12, q = 1$ for the granny knot or $p = 0, q = 1$ for the square knot and ρ is a composite of two non-abelian representations ρ_i. This proves the claim.

From (3.5.2), we see that

$$\dim H^1(S^3_{p/q}(3_1\#3_1);\mathrm{ad}\,\rho) = \dim H^1(S^3\backslash(3_1\#3_1);\mathrm{ad}\,\rho) + 1 - \mathrm{rank}(f)$$

The result follows from combining the above formula, our computations of the rank of f, and Lemma 3.5.13. □

Theorems 1.1.6 and 1.1.7 now follow from applying Corollary 2.1.4 to the calculation of the respective character varieties in Propositions 3.5.4 and 3.5.10 and the determination of the singular cohomology of these character schemes from Proposition 3.3.5.

Remark 3.5.15. We use HP with $\mathbb{Z}/2\mathbb{Z}$ coefficients in Theorems 1.1.6 and 1.1.7 only to avoid determining the relevant local system. Indeed, the character schemes of surgeries on $3_1\#3_1$ include some components isomorphic to $\mathbb{C}^*$ and $\mathbb{C}^* - \{1\}$, while the other topological types of components that appear are simply connected. We conjecture that the local systems are in fact trivial on all of the components and that Theorems 1.1.6 and 1.1.7 hold over $\mathbb{Z}$.

3.6 Further Discussion

3.6.1 Exact triangles

In analogy with other Floer theories [OS04][Sca15][Flo90], one may conjecture the existence of a surgery exact triangle for $HP_\#$. That is, one may hope that there exists a long exact sequence

$$HP_\#(S^3)[1] \to HP_\#(S^3_{p+1}(K)) \to HP_\#(S^3_p(K)) \to HP_\#(S^3)$$

However, since $HP_\#(S^3)$ is supported in degree zero, such a long exact sequence would imply that $HP_\#(S^3_p(K))$ and $HP_\#(S^3_{p+1}(K))$ are isomorphic except possibly in degrees $-1, 0$, and 1. Yet the data from Corollary 3.2.1 shows that this is not the case. For example, $HP_\#(S^3_2(3_1))$ has rank 2 in degree -3, whereas $HP_\#(S^3_3(3_1))$ has rank 1 in degree -3.

One can also ask whether a surgery exact triangle exists for HP. The data in Theorem 1.1.3 can be used to show that such a triangle cannot exist for the trefoil. However, one would not even expect such a surgery exact triangle for HP since for other Floer theories such exact triangles are not usually formulated for the versions that exclude reducibles. For example, there is no surgery exact triangle for HF°_{red} in Heegaard Floer homology.

3.6.2 A conjecture

In [BC16], the authors define an $\mathrm{SL}(2,\mathbb{C})$ Casson knot invariant by

$$\lambda'_{\mathrm{SL}(2,\mathbb{C})}(K) = \lim_{q\to\infty} \frac{1}{q} \lambda_{\mathrm{SL}(2,\mathbb{C})}(S^3_{p/q}(K))$$

where p is fixed and the limit is taken over all q relatively prime to p. In particular, this quantity is independent of p. We can make the analogous conjecture for HP and $HP_\#$.

Conjecture 3.6.1. *Let $K \subset S^3$ be a knot and $S^3_{p/q}(K)$ it p/q surgery. Then the quantities*

$$\lim_{q\to\infty} \frac{1}{q} \operatorname{rk}(HP^n(S^3_{p/q}(K)))$$

and

$$\lim_{q\to\infty} \frac{1}{q} \operatorname{rk}(HP^n_\#(S^3_{p/q}(K)))$$

are well-defined invariants of the knot K.

For example, by Theorems 1.1.6 and 1.1.7 we can verify this conjecture for HP of surgeries on the granny and square knots. We obtain the numerical data

$$\lim_{q\to\infty} \frac{1}{q}\,\mathrm{rk}(HP^0(S^3_{p/q}(G))) = 12$$
$$\lim_{q\to\infty} \frac{1}{q}\,\mathrm{rk}(HP^{-1}(S^3_{p/q}(G))) = 6$$

and

$$\lim_{q\to\infty} \frac{1}{q}\,\mathrm{rk}(HP^0(S^3_{p/q}(Q))) = 6$$
$$\lim_{q\to\infty} \frac{1}{q}\,\mathrm{rk}(HP^{-1}(S^3_{p/q}(Q))) = 0$$

CHAPTER 4

The sheaf-theoretic $\mathrm{SL}(2,\mathbb{C})$ Casson-Lin invariant

The goal of this chapter is to prove Theorems 1.1.11, 1.1.8, and 1.1.10 and to extend the definition of $HP_\tau(K)$ to all $\tau \in \mathbb{C} - \{\pm 2\}$. Theorem 1.1.11 is purely about algebraic geometry, and our proof follows the work of Verdier in [Ver76]. We deduce Theorem 1.1.8 as a consequence of Theorem 1.1.11. We prove Theorem 1.1.10 by analyzing a certain $\mathbb{C}^*$ action on the character scheme of the connected sum of two knots. To extend the definition of $HP_\tau(K)$, we check that $X^\tau_{irr}(\Sigma)$ is connected and simply connected for all $\tau \in \mathbb{C} - \{\pm 2\}$ and for Σ with genus at least 6. We prove this by first applying a theorem of Verdier that allows us to extend the same result from the case of $\tau \in (-2, 2)$, which was shown in [CM], to generic values of τ, and then an analysis of moduli of $K(D)$ pairs to extend to all τ.

4.1 Generic independence of the weight

The purpose of this section is to prove Theorem 1.1.8 from the introduction. In fact, most of the effort is directed at proving Theorem 1.1.11, which is purely a statement about algebraic geometry. We deduce Theorem 1.1.8 as an easy corollary in Section 4.1.5.

4.1.1 Setup and algebraic preliminaries

Given a scheme X over $\mathbb{C}$, recall that $\chi_B(X)$ is the Euler characteristic of X weighted by the Behrend function.

Let $X \subset \mathbb{A}^n$ be an affine $\mathbb{C}$-scheme corresponding to the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. Let $X^0 \subset X$ be an open embedding. For $\tau \in \mathbb{A}^1$ a (closed) point, let $X^0_\tau \subset X_\tau$ be the (scheme-theoretic) fiber of the morphism to $\mathbb{A}^1$ induced by the projection $(x_1, \ldots, x_n) \mapsto x_1$.

The proof of Theorem 1.1.11 will occupy the next three sections.

Let us view $\mathbb{C}[x_1, \dots, x_n]/I$ as a $\mathbb{C}[x]$-algebra via the natural map taking $x \mapsto x_1$. Observe that there are then isomorphisms

$$\begin{aligned}\left(\mathbb{C}[x_1, \dots, x_n]/I \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-\tau)\right) &= \mathbb{C}[x_1, \dots, x_n]/(I, x_1 - \tau) \\ &= \mathbb{C}[z_2, \dots, z_n]/I_\tau,\end{aligned} \tag{4.1.1}$$

where the second map takes $(x_1, x_2, \dots, x_n) \mapsto (\tau, z_2, \dots, z_n)$ and I_τ is the image of I in $\mathbb{C}[z_2, \dots, z_n]$.

Hence we have that

$$X_\tau = \operatorname{Spec}(\mathbb{C}[z_2, \dots, z_n]/I_\tau) \subset \mathbb{A}^{n-1}.$$

Let us set

$$R = \bigoplus_{m \geq 0} I^m / I^{m+1}$$

and observe that R is naturally a $\mathbb{C}[x_1, \dots, x_n]/I$-algebra via the inclusion of the zero-graded piece.

Let

$$C = C_{X/\mathbb{A}^n} = \operatorname{Spec} R,$$

be the normal cone of $X \subset \mathbb{A}^n$ and consider the map

$$\phi : \bigoplus_{m \geq 0} I^m / I^{m+1} \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-\tau) \to \bigoplus_{m \geq 0} I_\tau^m / I_\tau^{m+1}. \tag{4.1.2}$$

Proposition 4.1.1. *Suppose that $x_1 - \tau$ is not a zero-divisor in $\mathbb{C}[x_1, \dots, x_n]/I^l$ for any l. Then ϕ is an isomorphism.*

Proof. We have a natural isomorphism

$$\bigoplus_{m \geq 0} I^m / I^{m+1} \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-\tau) = \bigoplus_{m \geq 0} I^m / I^m(I + (x_1 - \tau)), \tag{4.1.3}$$

which allows us to rewrite ϕ as the natural projection map

$$\phi : \bigoplus_{m \geq 0} I^m / I^m(I + (x_1 - \tau)) \to \bigoplus_{m \geq 0} I_\tau^m / I_\tau^{m+1}. \tag{4.1.4}$$

This map is clearly surjective. To check injectivity, choose $\alpha \in I^m$ and suppose that the composition

$$I^m \to I^m/I^m(I + (x_1 - \tau)) \to I_\tau^m/I_\tau^{m+1}$$

annihilates α. Then there exists $\beta \in I^{m+1}$ such that $\alpha - \beta \in (x_1 - \tau)$. Hence $\alpha - \beta` \in (x_1 - \tau) \cap I^m$. The proposition now follows from Lemma 4.1.2 below. $\square$

Lemma 4.1.2. *Suppose that $x_1 - \tau$ is not a zero-divisor in $\mathbb{C}[x_1, \ldots, x_n]/I^l$ for any $l \geq 0$. Then in $\mathbb{C}[x_1, \ldots, x_n]$, we have $(x_1 - \tau) \cap I^m = (x_1 - \tau)I^m$ for all $m \geq 0$.*

Proof. We only need to check the nontrivial inclusion. Any element $(x_1 - \tau) \cap I^m$ is of the form $(x_1 - \tau)\gamma$ for some $\gamma \in \mathbb{C}[x_1, \ldots, x_n]$. But $(x_1 - \tau)\gamma \in I^m$ implies that $(x_1 - \tau)\gamma = 0$ in $\mathbb{C}[x_1, \ldots, x_n]/I^m$. By hypothesis, this implies $\gamma \in I^m$ and therefore $(x_1 - \tau)\gamma \in (x_1 - \tau)I^m$. $\square$

Our next task is to check that the assumptions of Proposition 4.1.1 and Lemma 4.1.2 are satisfied generically.

Lemma 4.1.3. *Suppose that $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a zero-divisor of $\mathbb{C}[x_1, \ldots, x_n]/I^l$ for some $l \geq 0$. Then f (viewed as an element of the 0-graded piece of R) is a zero-divisor in R.*

Proof. By hypothesis, there exists $g \in \mathbb{C}[x_1, \ldots, x_n]$ such that $g \notin I^l$ but $fg \in I^l$. Let $0 \leq k < l$ be the largest integer such that $g \in I^k$ but $g \notin I^{k+1}$. Observe that g can be viewed as a non-zero element of the k-graded piece of R. Viewing f as an element of the 0-graded piece of R, we have $0 = fg \in R$. Hence f is a zero-divisor in R. $\square$

Corollary 4.1.4. *For all but finitely many $\tau \in \mathbb{C}$, the element $x_1 - \tau \in \mathbb{C}[x_1, \ldots, x_n]$ is not a zero divisor in the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I^l$ for any $l \geq 0$.*

Proof. Observe that R is generated in degrees 0 and 1 as a $\mathbb{C}$-algebra, so R is in particular a finitely generated $\mathbb{C}$-algebra. In particular, R is a Noetherian ring and it therefore has finitely many associated prime ideals whose union is precisely the set of zero-divisors of R. If we suppose for contradiction that the corollary is false, then it follows from Lemma 4.1.3 that $(x_1 - \tau)$ is a zero-divisor in R for infinitely many values of $\tau \in \mathbb{C}$. Hence there exist

$\tau_1, \tau_2 \in \mathbb{C}$ with $\tau_1 \neq \tau_2$ such that $(x_1 - \tau_1)$ and $(x_1 - \tau_2)$ are both elements of the same associated prime ideal. Since $(x_1 - \tau_1) - (x_1 - \tau_2) = (\tau_2 - \tau_1)$ is a unit, this gives the desired contradiction. □

Corollary 4.1.5. *There is a Zariski open set $\mathcal{U}_1 \subset \mathbb{A}^1$ such that (4.1.2) is an isomorphism for all $\tau \in \mathcal{U}_1$. We therefore have the following commutative diagram of schemes, for all $\tau \in \mathcal{U}_1$:*

$$\begin{tikzcd} C_\tau \arrow[r, "\sim"] \arrow[d] & C_{X_\tau/\mathbb{A}^{n-1}} \arrow[d] \\ X_\tau \arrow[r, "\sim"] & X_\tau. \end{tikzcd} \tag{4.1.5}$$

Proof. The fact that (4.1.2) is an isomorphism for $\tau \in \mathcal{U}_1$ follows from Proposition 4.1.1 and Theorem 4.1.4. The restriction of (4.1.2) to the zero-graded piece is just the isomorphism (4.1.1), so we get the above diagram of schemes by taking $\mathrm{Spec}(-)$. □

4.1.2 Passage to a cover

In general, the irreducible components of C are not in bijection with the irreducible components of the fibers C_τ associated to the projection $C \to X \to \mathbb{A}^1$. However, the next proposition shows that this property becomes true in an open subset after passing to a suitable branched cover of $\mathbb{A}^1$, and that the fibers can moreover be assumed to have generically constant multiplicity and dimension. These facts are well-known in algebraic geometry, but we provide a detailed argument for completeness.

Proposition 4.1.6. *There exists an open set $\mathcal{U} \subset \mathbb{A}^1$ and a finite étale cover $\psi : \tilde{\mathcal{U}} \to \mathcal{U}$ such that the following holds: if we let $Q^1, \ldots, Q^q$ be the irreducible components of $C_{\tilde{\mathcal{U}}}$, then for all $p \in \tilde{\mathcal{U}}$, the irreducible components of C_p are precisely $Q^1_p, \ldots, Q^q_p$. Moreover, the multiplicity and dimension of the Q^i_p is independent of $p \in \tilde{\mathcal{U}}$.*

The proof is an immediate consequence of the next three lemmas. Before stating these lemmas, it will be useful to make the following remark.

Remark 4.1.7. If $p : \tilde{\mathcal{U}} \to \mathcal{U}$ is a finite étale map between smooth $\mathbb{C}$-schemes of dimension 1, then given an open subset $\mathcal{U}' \subset \mathcal{U}$, the map p restricts to an étale map $p|_{\mathcal{U}'} : p^{-1}(\mathcal{U}') \to \mathcal{U}'$.

If $\mathcal{V} \subset \tilde{\mathcal{U}}$ is an open subset, then after taking a possibly smaller open $\mathcal{V}' \subset \mathcal{V} \subset \tilde{\mathcal{U}}$, we can assume that p restricts to an étale map onto its image. Indeed, observe that $\tilde{\mathcal{U}} - \mathcal{V}$ is closed. Hence $p(\tilde{\mathcal{U}} - \mathcal{V})$ is closed (since finite morphisms are closed) and $p^{-1}(p(\tilde{\mathcal{U}} - \mathcal{V}))$ is also closed. We can then set $\mathcal{V}' = \tilde{\mathcal{U}} - p^{-1}(p(\tilde{\mathcal{U}} - \mathcal{V}))$.

Lemma 4.1.8. *There exists an open set $\mathcal{U} \subset \mathbb{A}^1$ and a finite étale cover $\psi : \tilde{\mathcal{U}} \to \mathcal{U}$ such that the irreducible components of the generic fiber of $C_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ are geometrically irreducible.*

Proof. Let $\eta \in \mathbb{A}^1$ be the generic point and note that $k(\eta) \simeq \mathbb{C}(x)$. By applying [TSPA18, Tag 054R], there is a finite extension $K/k(\eta)$ such that C_K is geometrically irreducible over K. Observe that K has transcendence degree one over $\mathbb{C}$. Hence, according to [TSPA18, Tag 0BY1], this extension is induced by a dominant rational map $f : \Sigma \to \mathbb{A}^1$, where Σ is an algebraic curve over $\mathbb{C}$. By the theorem on generic smoothness on the target [Vak, 25.3.3] (and the fact that f is dominant), there is an open $\mathcal{U} \subset \mathbb{A}^1$ such that $f|_{f^{-1}(\mathcal{U})}$ is smooth. Since f is evidently of relative dimension zero, $f|_{f^{-1}(\mathcal{U})}$ is étale. The lemma follows with $\tilde{\mathcal{U}} := f^{-1}(\mathcal{U})$ and $\psi := f$. □

Lemma 4.1.9. *Let $C_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ be as in Lemma 4.1.8. After possibly shrinking $\tilde{\mathcal{U}}$ (cf. Theorem 4.1.7), we can assume that the following holds: if we let $Q^1, \ldots, Q^q$ be the irreducible components of $C_{\tilde{\mathcal{U}}}$, then for all $p \in \tilde{\mathcal{U}}$, the irreducible components of C_p are precisely $Q^1_p, \ldots, Q^q_p$.*

Proof. Let $C^1_K, \ldots, C^q_K$ be the irreducible components of the generic fiber C_K. For $1 \le i \le q$, let C^i be the smallest closed irreducible subscheme of $C_{\tilde{\mathcal{U}}}$ whose generic fiber is C^i_K. According to [TSPA18, Tag 054Y], we can assume after possibly shrinking $\tilde{\mathcal{U}}$ that the irreducible components of the fiber C_p for any $p \in \tilde{\mathcal{U}}$ are precisely $\{C^{i,j}_p\}_{j=1}^{n_i}$ where $1 \le i \le q$. It now follows from Lemma 4.1.8 and [TSPA18, Tag 0559] (geometric irreducibility spreads out) that $n_i = 1$ for all i. This completes the proof. (Note that geometric irreducibility plays a crucial role, since the analog of [TSPA18, Tag 0559] is false for irreducible schemes which are not geometrically irreducible). □

Lemma 4.1.10. *Let $C_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ satisfy the conditions of Lemma 4.1.8 and Lemma 4.1.9. After possibly shrinking $\tilde{\mathcal{U}}$, we can assume that the multiplicity and dimension of the Q_p^i is independent of $p \in \tilde{\mathcal{U}}$.*

Proof. Lemma 4.1.9 gives a bijection between the irreducible components of the generic fiber C_K and the irreducible components of the fibers C_p for $p \in \tilde{\mathcal{U}}$. The present lemma is simply a consequence of the fact that both dimension and multiplicity "spread out"; that is, after possibly further shrinking $\tilde{\mathcal{U}}$, we can assume that the bijection constructed in Lemma 4.1.9 preserves dimension and multiplicity. The relevant reference for dimension is [TSPA18, Tag 02FZ]; for multiplicity, one can apply [Gro64, III, 9.8.6] to the structure sheaf of C. Note that the notion of geometric multiplicity in [Gro64, III, 9.8.6] agrees with our notion of multiplicity since we are in characteristic zero; see [Gro64, II, 4.7.5]. □

Proof of Proposition 4.1.6. Combine Lemma 4.1.9 and Lemma 4.1.10. □

Note that the image of each Q^i under the map $C_{\tilde{\mathcal{U}}} \to X_{\tilde{\mathcal{U}}}$ is irreducible (since the image of an irreducible set under a continuous map is irreducible). It is also closed: this follows by combining [Ful98, B.5.3.] and the fact that $\psi : \tilde{\mathcal{U}} \to \mathcal{U}$ is an étale cover. We let V^i be the image of C^i and conclude that V^i is an irreducible subvariety of $X_{\tilde{\mathcal{U}}}$ when endowed with the canonical reduced closed subscheme structure. Observe also that the fibers V_p^i are irreducible for $p \in \tilde{\mathcal{U}}$. Indeed, the Q_p^i are irreducible, so this follows from the fact that $V_p^i = \pi(Q_p^i)$.

4.1.3 Stratification theory

All stratifications which we consider in this section will be assumed to be w-stratifications in the sense of Definition 2.2.1. In particular, this implies that our stratifications are Whitney stratifications and that the strata are smooth, connected, locally closed subvarieties.

Definition 4.1.11 (cf. (3.2) in [Ver76])**.** Given a morphism $f : X \to Y$ of complex algebraic varieties and a stratification $\mathcal{S}$ of X, we say that f is transverse to $\mathcal{S}$ if f restricts to a smooth morphism on each stratum.

As observed in [Ver76, (3.6)], if $f : X \to Y$ is transverse to a stratification $\mathcal{S}$, then given any $y \in Y$, the fiber $f^{-1}(y)$ inherits a stratification by restriction of the strata.

It will be convenient to record the following lemma, whose proof is a routine verification.

Lemma 4.1.12. *Suppose that $V' \subset V$ is a (locally closed) subvariety of V. Suppose that $\mathcal{S}$ is a stratification of V such that V' is a union of strata. Then $\mathcal{S}|_{V'}$ is a stratification of V' (in particular, $\mathcal{S}$ also satisfies the axioms of Definition 2.2.1).*

We consider $\tilde{\pi} : X_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ satisfying the properties of Proposition 4.1.6.

Proposition 4.1.13. *After possibly replacing $\mathcal{U}$ with a smaller open $\mathcal{U}_2 \subset \mathcal{U}$ (cf. Theorem 4.1.7), we can assume that $X_{\tilde{\mathcal{U}}}$ admits a stratification $\mathcal{S}$ with the following properties:*

(i) The stratification $\mathcal{S}$ is transverse to $\tilde{\pi}$.

(ii) The subvarieties $X^0_{\tilde{\mathcal{U}}}, V^1, \ldots, V^q$ of $X_{\tilde{\mathcal{U}}}$ are a union of strata.

(iii) For each $x \in \tilde{\mathcal{U}}$, there exists a ball $B_x \subset \tilde{\mathcal{U}}$ such that $\tilde{\pi}^{-1}(B_x)$ is homeomorphic to $X_x \times B_x$. Moreover, this homeomorphism is compatible with the projection and preserves the natural product stratification.

Proof. For ease of notation, we write $f = \tilde{\pi} : X_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$. We will argue exactly as in the proof of Proposition 5.1 in [Ver76]. Applying the Nagata compactification theorem, we can factor $f : X_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ as an open embedding $i : X_{\tilde{\mathcal{U}}} \to \overline{X_{\tilde{\mathcal{U}}}}$ followed by a proper map $\overline{f} : \overline{X_{\tilde{\mathcal{U}}}} \to \tilde{\mathcal{U}}$. Given an open set $V \subset \tilde{\mathcal{U}}$, we write $f|_V$ or $\overline{f}|_V$ for the restriction of f or $\overline{f}$ to $f^{-1}(V)$ or $\overline{f}^{-1}(V)$ respectively.

According to (2.2) in [Ver76], we can choose a Whitney stratification $\mathcal{S}$ of $\overline{X_{\tilde{\mathcal{U}}}}$ so that $X^0_{\tilde{\mathcal{U}}}, V^1, \ldots, V^q$ and $X_{\tilde{\mathcal{U}}}$ are a union of strata. Next, (3.3) in [Ver76] shows that one can find an open $\mathcal{V} \subset \tilde{\mathcal{U}}$ so that $\overline{f}$ is transverse on $\overline{f}^{-1}(\mathcal{V})$ to $\mathcal{S} \cap \overline{f}^{-1}(\mathcal{V})$. Finally, Verdier shows in (4.14) of [Ver76] that there are trivializations of $\overline{f}|_{\mathcal{V}}$ with the desired properties, i.e. which are compatible with projection and preserve the stratifications. Since $X_{\tilde{\mathcal{U}}}$ is a union of strata, it follows that these also give local trivializations for $f|_{\mathcal{V}}$, as desired. $\square$

Since $X^0_{\tilde{\mathcal{U}}}, V^1, \ldots, V^s$ are each a union of strata (and since $\mathring{V}^i := X^0_{\tilde{\mathcal{U}}} \cap V^i$ is therefore also a union of strata), we obtain the following corollary of (ii) and (iii):

Corollary 4.1.14. *If we let $\tilde{\pi}^0 : X^0_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$ be the composition $X^0_{\tilde{\mathcal{U}}} \hookrightarrow X_{\tilde{\mathcal{U}}} \to \tilde{\mathcal{U}}$, then (iii) holds for $X^0_{\tilde{\mathcal{U}}}, \tilde{\pi}^0, W^0_x$ in place of $X_{\tilde{\mathcal{U}}}, \tilde{\pi}, X_x$. If, for $i = 1, \ldots, v$, we let $\tilde{\pi}^i : V^i \to \mathcal{U}$ be the composition $V^i \hookrightarrow X_{\tilde{\mathcal{U}}} \to \mathcal{U}$, then (iii) holds for $V^i, \tilde{\pi}^i, V^i_x$ in place of $X_{\tilde{\mathcal{U}}}, \tilde{\pi}, X_x$. Finally, letting $\mathring{V}^i := X^0_{\tilde{\mathcal{U}}} \cap V^i$ and letting $\tilde{\pi}^{i,0} : \mathring{V}^i \hookrightarrow X_{\tilde{\mathcal{U}}} \to \mathcal{U}$ be the obvious composition, then (iii) holds for $\mathring{V}^i, \tilde{\pi}^{i,0}, \mathring{V}^i_x$ in place of $X_{\tilde{\mathcal{U}}}, \tilde{\pi}, X_x$.*

4.1.4 Completion of the argument

We now have the ingredients in place to prove Theorem 1.1.11. For $\tau \in \mathbb{A}^1$, it follows from [Beh09, Prop. 1.5(i)] that $\nu_{X^0_\tau} = \nu_{X_\tau}|_{X^0_\tau}$. Hence, we have

$$\chi_B(X^0_\tau) := \chi(X^0_\tau, \nu_{X^0_\tau}) = \chi(X^0_\tau, \nu_{X_\tau}|_{X^0_\tau}). \tag{4.1.6}$$

According to Theorem 4.1.5 and Proposition 4.1.6, up to replacing $\mathcal{U}_1$ and $\mathcal{U}$ by a possibly smaller open set $\mathcal{U}_3 \subset \mathcal{U}_1 \cap \mathcal{U}$, we can assume that for all $\tau \in \mathcal{U}$ and $p \in \tilde{\mathcal{U}}$ satisfying $\psi(p) = \tau$ there is a diagram

$$\begin{array}{ccccc} C_p & \xrightarrow{\sim} & C_\tau & \xrightarrow{\sim} & C_{X_\tau/\mathbb{A}^{n-1}} \\ \downarrow & & \downarrow & & \downarrow \\ X_p & \xrightarrow{\sim} & X_\tau & \xrightarrow{\sim} & X_\tau. \end{array} \tag{4.1.7}$$

We then have, by (2.2.2) and Proposition 4.1.6,

$$\nu_{X_p} = \mathrm{Eu}\left(\sum_{i=1}^{q} a_i(p) V^i_p\right),$$

where

$$a_i(p) = (-1)^{\dim \pi(Q^i_p)} \operatorname{mult}(Q^i_p) = (-1)^{\dim V^i_p} \operatorname{mult}(Q^i_p). \tag{4.1.8}$$

We conclude that

$$\chi_B(X^0_\tau) = \chi\left(X^0_p, \mathrm{Eu}(\sum_{i=1}^{q} a_i(p) V^i_p)\big|_{X^0_p}\right). \tag{4.1.9}$$

By appealing to the complex-analytic definition of the local Euler obstruction (see [Mac74, Sec.3] or [JS12, p. 32]), we see that the local Euler obstruction of a cycle at some point x

only depends on an analytic neighborhood of x. Hence we have

$$\begin{aligned}\mathrm{Eu}(\sum a_i(p)V_p^i)|_{X_p^0} &= \mathrm{Eu}(\sum a_i(p)(V_p^i \cap X_p^0)) \\ &= \sum_{\{j\in\Sigma\}} a_j(p)\,\mathrm{Eu}(\mathring{V}_p^j),\end{aligned} \tag{4.1.10}$$

where $\Sigma \subset \{1, 2 \ldots, q\}$ and $j \in \Sigma$ iff $V^j \cap X^0_{\tilde{\mathcal{U}}} = \emptyset$, and where we let $\mathring{V}_p^j = V_p^i \cap X_p^0$. Note that the second line follows from the fact that $\mathrm{Eu}(-)$ is a homomorphism from the group of algebraic cycles on X_p^0 to the group of constructible functions; see [Ful98, p. 376].

Next, it follows from [Beh09, 1.3(ii)] that

$$\chi(X_p^0, \sum_{j\in\Sigma} a_j(p)\,\mathrm{Eu}(\mathring{V}_p^j)) = \sum_{j\in\Sigma} a_j(p)\chi(X_p^0, \mathrm{Eu}(\mathring{V}_p^j)) = \sum_{j\in\Sigma} a_j(p)\chi(\mathring{V}_p^j, \mathrm{Eu}(\mathring{V}_p^j)). \tag{4.1.11}$$

Proposition 4.1.15. *After possibly replacing $\tilde{\mathcal{U}}$ with a smaller open subset $\mathcal{U}_4$, we can assume that the function $p \mapsto a_i(p)$ is constant for $p \in \tilde{\mathcal{U}}$ for all $i = 1, \ldots, q$.*

Proof. Noting that $V_p^i = \pi(Q_p^i) = \pi(Q^i)_p$, it follows from [TSPA18, Tag 05F7] that $\dim V_p^i$ is constant on an open subset of $\tilde{\mathcal{U}}$. The result now follows by combining (4.1.8) and Proposition 4.1.6. □

We will also need the following lemma:

Lemma 4.1.16 (Lem. 1.1(3) in [PP95]). *Assume that an irreducible variety Y is embedded in $\mathbb{C}^n$ and a nonsingular subvariety Z intersects a Whitney stratification of Y transversally. Then $\mathrm{Eu}(Z \cap Y)(x) = \mathrm{Eu}(Y)(x)$ for all $x \in Z \cap Y$.*

Proposition 4.1.17. *After possibly replacing $\tilde{\mathcal{U}}$ with a smaller open $\mathcal{U}_5 \subset \tilde{\mathcal{U}}$, we may assume that the function $p \mapsto \chi(\mathring{V}_p^i, \mathrm{Eu}(\mathring{V}_p^i))$ is locally (and hence globally) constant for $p \in \tilde{\mathcal{U}}$ and $i = 1, \ldots, v$.*

Proof. It follows from Proposition 4.1.13(ii) that V^i is a union of strata of the stratification $\mathcal{S}$ of $X_{\tilde{\mathcal{U}}}$. Lemma 4.1.12 then implies that the restriction of $\mathcal{S}$ to $\mathring{V}^i$ is a stratification which we call $\mathring{\mathcal{S}}^i$. According to Proposition 4.1.13(i) and the comment following Definition 4.1.11, the fiber $\mathring{V}^i_p$ inherits a stratification from $\mathring{\mathcal{S}}^i$ which we call $\mathring{\mathcal{S}}^i_p$. Note that the strata of $\mathring{\mathcal{S}}^i_p$ are of the form S_p for $S \in \mathring{\mathcal{S}}^i$.

According to [Bra00, Prop. 2], the constructible function $\mathrm{Eu}(\mathring{V}^i)$ is constant on each stratum of $\mathring{\mathcal{S}}^i$. It follows that $\mathrm{Eu}(\mathring{V}^i_p)|_{S'}$ is constant for any $S' \in \mathring{\mathcal{S}}^i_p$. If $S' = S_p$ for some stratum $S \in \mathring{\mathcal{S}}^i$, then Proposition 4.1.13(i) and Lemma 4.1.16 imply that $\mathrm{Eu}(\mathring{V}^i_p)|_{S'} = \mathrm{Eu}(\mathring{V}^i)|_S$ is constant.

It now follows that

$$\begin{aligned} \chi(\mathring{V}^i_p, \mathrm{Eu}(\mathring{V}^i_p)) &:= \sum_{S' \in \mathring{\mathcal{S}}^i_p} \chi(S')\,\mathrm{Eu}(\mathring{V}^i_p)|_{S'} \\ &= \sum_{S \in \mathring{\mathcal{S}}^i} \chi(S_p)\,\mathrm{Eu}(\mathring{V}^i)|_S. \end{aligned} \tag{4.1.12}$$

According to Theorem 4.1.14, the topological Euler characteristic of the fiber $\chi(S_p)$ is independent of $p \in \tilde{\mathcal{U}}$, for $S \in \mathring{\mathcal{S}}^i$. It follows that (4.1.12) is independent of $p \in \tilde{\mathcal{U}}$, which is what we wanted to show. $\square$

Proof of Theorem 1.1.11. By combining (4.1.9), (4.1.10) and (4.1.11), we find that

$$\chi_B(X^0_\tau) = \sum_{j \in \Sigma} a_j(p)\chi(\mathring{V}^j_p, \mathrm{Eu}(\mathring{V}^j_p)). \tag{4.1.13}$$

Setting $\mathcal{V} = \tilde{\mathcal{U}}$, it follows from Proposition 4.1.15 and Proposition 4.1.17 that this expression is constant for $p \in \mathcal{V}$. $\square$

4.1.5 Proof of Theorem 1.1.8

Fix a presentation $\Gamma = \langle g_1, \ldots, g_m \mid r_1, \ldots, r_l \rangle$. As explained for example in [CM, Sec. 2.1], one can associate to this presentation a $\mathbb{C}$-algebra $\mathcal{A}(\Gamma)$ with generators $x^{g_1}_{11}, x^{g_1}_{12}, x^{g_1}_{21}, x^{g_1}_{22}, \ldots, x^{g_m}_{11}, x^{g_m}_{12}, x^{g_m}_{21}, x$ whose spectrum is the representation scheme $\mathscr{R}(\Gamma)$. The group scheme SL_2 acts on $\mathscr{R}(\Gamma)$ by conjugation. Letting R be the ring of functions of SL_2, this action is induced by a map $\mu : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma) \otimes R$. The ring of invariants is $\mathcal{A}(\Gamma)^{\mathrm{SL}_2}$ is the character scheme is $\operatorname{Spec} \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$.

Note that $\mathcal{A}(\Gamma)^{\mathrm{SL}_2}$ is a finitely-generated $\mathbb{C}$-algebra, so we can fix a generating set $X_1, \ldots, X_n$ and we can moreover assume that $X_1 = x_{11}^{g_1} + x_{22}^{g_1}$. The surjective ring map $\mathbb{C}[x_1, \ldots, x_n] \to \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$ sending $x_i \mapsto X_i$ induces an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]/I \to \mathcal{A}(\Gamma)^{\mathrm{SL}_2}, \tag{4.1.14}$$

where I is the kernel of the surjection. This gives an embedding of schemes

$$\mathscr{X}(\Gamma) \hookrightarrow \mathbb{A}^n.$$

There is also an open embedding $\mathscr{X}_{irr}(\Gamma) \hookrightarrow \mathscr{X}(\Gamma)$; see [AM20, p. 7]. After composing with the projection $\mathbb{A}^n \to \mathbb{A}^1$ sending $(x_1, \ldots, x_n) \mapsto x_1$, we get a morphism $\mathscr{X}(\Gamma) \to \mathbb{A}^1$.

According to Theorem 1.1.11 applied to $\mathscr{X}_{irr}(\Gamma) \subset \mathscr{X}(\Gamma) \subset \mathbb{A}^n$, there is a Zariski open $\mathcal{V} \subset \mathbb{A}^1$ such that $\chi_B(\mathscr{X}_{irr}(\Gamma)_\tau)$ is constant over all $\tau \in \mathcal{V}$. To complete the proof of Theorem 1.1.8, it follows from (2.2.5) that it is enough to prove that $\mathscr{X}_{irr}(\Gamma)_\tau = \mathscr{X}_{irr}^\tau(\Gamma)$ for all but finitely many $\tau \in \mathbb{A}^1$. This is the content of the following proposition.

Proposition 4.1.18. *(i) $\mathscr{X}(\Gamma)_\tau = \mathscr{X}^\tau(\Gamma)$ for all but finitely many values of $\tau \in \mathbb{A}^1$.*

(ii) $\mathscr{X}_{irr}(\Gamma)_\tau = \mathscr{X}_{irr}^\tau(\Gamma)$ for all but finitely many values of $\tau \in \mathbb{A}^1$.

Proof of (i). For $\tau \in \mathbb{A}^1$, let $h_t = x_{11}^{g_1} + x_{22}^{g_1} - t$. Thus h_τ is the image of $x_1 - \tau$ under the isomorphism $\mathbb{C}[x_1, \ldots, x_n]/I \to \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$ described above. Consider the surjective map

$$\mathcal{A}(\Gamma)^{\mathrm{SL}_2} \to (\mathcal{A}(\Gamma)/h_\tau\mathcal{A}(\Gamma))^{\mathrm{SL}_2} = \mathcal{A}(\Gamma)^{\mathrm{SL}_2}/(h_\tau\mathcal{A}(\Gamma))^{\mathrm{SL}_2},$$

which induces the quotient map

$$\mathcal{A}(\Gamma)^{\mathrm{SL}_2}/(h_\tau\mathcal{A}(\Gamma)^{\mathrm{SL}_2}) \to \mathcal{A}(\Gamma)^{\mathrm{SL}_2}/(h_\tau\mathcal{A}(\Gamma))^{\mathrm{SL}_2}. \tag{4.1.15}$$

We wish to show that the morphism in (4.1.15) is injective for all but finitely many $\tau \in \mathbb{A}^1$. We closely follow the proof of [CM, Prop. 5.3]. To this end, observe that it is enough to establish the following containment:

$$(h_\tau\mathcal{A}(\Gamma))^{\mathrm{SL}_2} \subset h_\tau\mathcal{A}(\Gamma)^{\mathrm{SL}_2}. \tag{4.1.16}$$

Let R be the coordinate ring of the group scheme SL_2. Let $\mu : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma) \otimes R$ be the $\mathbb{C}$-algebra morphism inducing the SL_2-action on $\mathscr{R}(\Gamma) = \operatorname{Spec} \mathcal{A}(\Gamma)$. By definition, $f \in \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$ if and only if $\mu(f) = f \otimes 1$.

Suppose for contradiction that (4.1.16) is false for infinitely many values of τ. Then there exists $g \in \mathcal{A}(\Gamma)$ such that $h_\tau g \in (h_\tau \mathcal{A}(\Gamma))^{\mathrm{SL}_2}$ but $g \notin \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$. Hence

$$0 = \mu(h_\tau g) - \mu(h_\tau)\mu(g) = (h_\tau g \otimes 1) - (h_\tau \otimes 1)\mu(g) = (h_\tau \otimes 1)(g \otimes 1 - \mu(g)). \tag{4.1.17}$$

Since $g \notin \mathcal{A}(\Gamma)^{\mathrm{SL}_2}$, we have that $g \otimes 1 - \mu(g) \neq 0$ which implies that $(h_\tau \otimes 1)$ is a zero-divisor in the ring $\mathcal{A}(\Gamma) \otimes R$.

Since $\mathcal{A}(\Gamma) \otimes R$ is Noetherian, it has finitely many associated prime ideals. Moreover, it is a general fact that every zero-divisor must be contained in one of these ideals; see [Vak, (5.5.10)]. By combining this fact with the previous paragraph, it follows that we can find $s, s' \in \mathbb{A}^1$ with $s \neq s'$ such that $(h_s \otimes 1)$ and $(h_{s'} \otimes 1)$ are contained in the same associated prime ideal. However, observe that $(h_s \otimes 1) - (h_{s'} \otimes 1) = (h_s - h_{s'} \otimes 1) = (s' - s \otimes 1) = (s' - s)(1 \otimes 1)$. This is a contradiction since $(s' - s)(1 \otimes 1)$ is a unit. □

Proof of (ii). The argument is similar to the proof of [CM, Prop. 5.5]. It follows from (i) that, for all but finitely many values of $\tau \in \mathbb{A}^1$, the composition $\mathscr{X}^{\tau}_{irr}(\Gamma) \hookrightarrow \mathscr{X}^{\tau}(\Gamma) = \mathscr{X}(\Gamma)_\tau$ is an open embedding. On the other hand, $\mathscr{X}_{irr}(\Gamma)_\tau \subset \mathscr{X}(\Gamma)_\tau$ is also an open embedding. It's clear that both open embeddings have the same closed points (corresponding to irreducible representations ρ such that $\mathrm{Tr}(\rho(g)) = \tau$). Hence the claim follows from the fact that any two open subschemes which have the same closed points coincide. □

4.2 Additivity of (sheaf-theoretic) $\mathrm{SL}(2, \mathbb{C})$ Casson-Lin invariant

The goal of this section is to prove Theorem 1.1.10, which states that the $\mathrm{SL}(2, \mathbb{C})$ Casson-Lin invariant $\chi_{CL}(K)$ is additive under connected sums of knots in integral homology 3-spheres.

4.2.1 The structure of the character variety

Following [AM20, p. 5], we will partition the $\mathrm{SL}(2,\mathbb{C})$ representations of a finitely-presented group Γ into five classes. Let $B \subset \mathrm{SL}(2,\mathbb{C})$ be the Borel subgroup of upper-triangular matrices and let $B_P \subset B$ be the subgroup of matrices of the form $\pm\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for $a \in \mathbb{C}$. Let $D \subset \mathrm{SL}(2,\mathbb{C})$ be the group of diagonal matrices. Every representation $\Gamma \to \mathrm{SL}(2,\mathbb{C})$ is of exactly one of the following types:

(a) The irreducible representations. These representations have trivial stabilizer.

(b) Representations which are conjugate to one in B but not in B_P or in D.

(c) Representations which are conjugate to one in B_P but not in $\{\pm \mathrm{Id}\}$.

(d) Representations which are conjugate to one in D but not in $\{\pm \mathrm{Id}\}$.

(e) Representations with image in $\{\pm \mathrm{Id}\}$.

We say that representations of types (b)-(e) are *reducible* and that representations of types (d) and (e) are *abelian*.

Let us now specialize to the case where $\Gamma = \pi_1(K)$ for $K \subset Y$ a knot in an integral homology 3-sphere. We fix $\tau \in \mathbb{C} - \{\pm 2\}$ and consider the relative character variety $X^\tau(K)$. The points of $X^\tau(K)$ correspond to irreducible representations $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ with $\mathrm{Tr}(\rho(m)) = \tau$ for $m \in \Gamma$ a meridian.

It will be useful to introduce the following terminology.

Definition 4.2.1. Let $K \subset Y$ be as above. Let $\mathcal{G}(K) \subset \mathbb{C}$ be the set of values $\tau \in \mathbb{C} - \{\pm 2\}$ with the property that $\tau \neq e^{\alpha/2} + e^{-\alpha/2}$ whenever e^α is a root of the Alexander polynomial of K, where $\alpha \in \mathbb{C}$. We say that representation $\rho : \pi_1(Y - K) \to \mathrm{SL}(2,\mathbb{C})$ is *good* if $\mathrm{Tr}\,\rho(m) \in \mathcal{G}(K)$ for $m \in \pi_1(Y - K)$ a meridian.

It is a remarkable fact first observed by de Rham (see [CCG+94, Sec. 6.1]) that a good representation is reducible if and only if it is abelian.

Lemma 4.2.2. *Let $K \subset Y$ be as above and suppose that $\tau \in \mathcal{G}(K) \subset \mathbb{C} - \{\pm 2\}$. Then there is a (scheme-theoretic) decomposition $\mathscr{R}^\tau(K) = \mathscr{R}^\tau_{ab}(K) \sqcup \mathscr{R}^\tau_{irr}(K)$ where $\mathscr{R}^\tau_{ab}(K)$ is the image of $\mathscr{R}^\tau(\mathbb{Z})$ under the abelianization map $\pi_1(Y - K) \to H_1(Y - K; \mathbb{Z}) \simeq \mathbb{Z}$.*

Proof. The surjective map $\pi_1(Y - K) \to H_1(Y - K; \mathbb{Z}) \simeq \mathbb{Z}$ induces a closed embedding of relative representation schemes $\phi^\tau : \mathscr{R}^\tau(\mathbb{Z}) \hookrightarrow \mathscr{R}^\tau(K)$. Let $C^\tau \subset \mathscr{R}^\tau(K)$ be the union of all irreducible components of $\mathscr{R}^\tau(K)$ which contain a non-abelian representation. Since $\operatorname{im}\phi^\tau$ and C^τ are closed and cover $\mathscr{R}^\tau(K)$, it's enough to show that they have empty intersection.

Suppose for contradiction that there is an abelian representation $\rho \in C^\tau$. Then ρ belongs to an irreducible component $C_0^\tau \subset C^\tau$ which contains an non-abelian representation. Since all representations in $\mathscr{R}^\tau(K)$ are good, C_0^τ contains an irreducible representation ρ'. Let $\pi : \mathscr{R}^\tau(K) \to \mathscr{X}^\tau(K)$ be the natural projection map and note that the closure $\overline{\pi(C_0^\tau)}$ is irreducible. It follows that there is an irreducible component $D_0 \subset \mathscr{X}(K)$ which contains $\overline{\pi(C_0^\tau)}$, and hence contains both ρ and ρ'. But this impossible in view of [CCG+94, Prop. 6.2], which states that if an abelian representation lies on a component of the character variety which also contains an irreducible representation, then this abelian representation is bad.

We have shown that $\mathscr{R}^\tau(K) = \mathscr{R}^\tau_{ab}(K) \sqcup C^\tau$, since both components are open and closed. It's clear that $C^\tau = \mathscr{R}^\tau_{irr}(K)$ since both open subschemes have the same closed points. Finally, the fact that the closed embedding $\mathscr{R}^\tau(\mathbb{Z}) \hookrightarrow \mathscr{R}^\tau_{ab}$ is an isomorphism (i.e. also an open embedding) is straightforward; cf. [CM, Prop. 7.6]. □

The connected sum operation for knots is described in detail in [CM, Sec. 7.1], and it will be useful to review this description in order to set our notation.

Let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be oriented knots in integral homology 3-spheres. Let $\overline{B}_1 \subset Y_1$ and $\overline{B}_2 \subset Y_2$ be small closed balls with the property that $\overline{B}_i - K_i$ is diffeomorphic to $\{(x, y, z) \mid \|(x, y, z)\| \leq 1, (x, y, z) \neq (x, 0, 0)\}$ for $i = 1, 2$. Let $B_i \subset \overline{B}_i$ be the (open) interior. Let $C_i := (Y_i - B_i)$ and let $\phi : C_1 \to C_2$ be an orientation reversing diffeomorphism which sends $\{C_1 \cap K_1\} \to \{C_2 \cap K_2\}$ and preserves the orientation on the sets $\{C_i \cap K_i\}$ induced by the orientation of K_i.

Definition 4.2.3. Let $Y = Y_1 \# Y_2 := Y_1 \cup_\phi Y_2$ be obtained by gluing $Y_1 - B_1$ to $Y_2 - B_2$ via ϕ and let $K = K_1 \# K_2 := (K_1 - B_i) \cup_\phi (K_2 - B_i)$ be the induced knot. We say that $K \subset Y$ is the *connected sum* of K_1 and K_2. While this construction appears to depend on choices, it can be shown that $K \subset Y$ is well-defined up to equivalence of knots.

For the remainder of this section, we assume that $K_i \subset Y_i$ are fixed and let $K = K_1 \# K_2$. By van Kampen's theorem, we have

$$\begin{aligned}\pi_1(Y - K) &= \pi_1(Y_1 - K_1 - B_1) *_{\pi_1(S^2 - p_1 - p_2)} \pi_1(Y_2 - K_2 - B_2) \\ &= \pi_1(Y_1 - K_1) *_{\pi_1(S^2 - p_1 - p_2)} \pi_1(Y_2 - K_2).\end{aligned} \tag{4.2.1}$$

Here, we have identified $\partial(Y_1 - B_1) = \partial(Y_2 - B_2) = (S^2 - p - q)$ via ϕ, for p, q a pair of distinct points on S^2. We can assume that the above fundamental groups are computed with respect to some reference basepoint $x \in S^2 - p - q$.

Since the class of the meridian generates $\pi_1(S^2 - p - q)$, we find that the representations of $\pi_1(Y - K)$ are pairs of representations $(\rho_1, \rho_2) \in \mathscr{R}^\tau(\pi_1(Y_1 - K_1)) \times \mathscr{R}^\tau(\pi_1(Y_2 - K_2))$ such that ρ_1 and ρ_2 agree on the meridian. In fact, by combining Lemma 4.2.2 and (4.2.1), we get the following fiber product presentation for the relative representation scheme:

$$\begin{aligned}\mathscr{R}^\tau(K) &= (\mathscr{R}^\tau_{ab}(K_1) \sqcup \mathscr{R}^\tau_{irr}(K_1)) \times_{\mathscr{R}^\tau(\mathbb{Z})} (\mathscr{R}^\tau_{ab}(K_2) \sqcup \mathscr{R}^\tau_{irr}(K_2)) \\ &= \left(\mathscr{R}^\tau_{ab}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{ab}(K_2)\right) \sqcup \left(\mathscr{R}^\tau_{irr}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{ab}(K_2)\right) \\ &\sqcup \left(\mathscr{R}^\tau_{ab}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{irr}(K_2)\right) \sqcup \left(\mathscr{R}^\tau_{irr}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{irr}(K_2)\right).\end{aligned} \tag{4.2.2}$$

Lemma 4.2.4. *Suppose that $\tau \in \mathcal{G}(K_1) \cap \mathcal{G}(K_2) \subset \mathbb{C} - \{\pm 2\}$. Then the open subscheme of irreducibles $\mathscr{R}^\tau_{irr}(K) \subset \mathscr{R}^\tau(K)$ consists precisely of the union of the second, third and fourth components in the above decomposition.*

Proof. It's clear that the second, third and fourth components consist of irreducible representations. Hence we only need to show that the first component does not contain an irreducible representations. Equivalently, we need to argue that an irreducible representation of K cannot restrict to an abelian representation on both K_1 and K_2. This property was proved in [CM, Prop. 7.3]. □

Following [CM, Def. 7.4], we introduce the following definition:

Definition 4.2.5. Let $K = K_1 \# K_2$ be as above. An irreducible representation $\rho : \pi_1(Y - K) \to \mathrm{SL}(2,\mathbb{C})$ is said to be of Type I if it restricts to an irreducible representation on K_i and to an abelian representation on K_j for $i, j \in \{1, 2\}$, $i \neq j$. An irreducible representation is said to be of Type II if it restricts to an irreducible representation on both factors. We also refer to a connected component of $\mathscr{R}^\tau_{irr}(K)$ or $\mathscr{X}^\tau_{irr}(K)$ as being of Type I or Type II if its closed points are all of Type I or Type II respectively. So for instance, in the decomposition (4.2.2), the second and third terms are of Type I and the fourth term is of Type II.

The Type I locus of $\mathscr{X}^\tau(K)$ admits the following description:

Lemma 4.2.6. *The image of the Type I locus* $\big(\mathscr{R}^\tau_{irr}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{ab}(K_2)\big) \sqcup \big(\mathscr{R}^\tau_{ab}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{irr}(K_2)\big)$ *under the projection map* $\pi : \mathscr{R}^\tau(K) \to \mathscr{X}^\tau(K)$ *is isomorphic to the disjoint union of* $\mathscr{X}^\tau_{irr}(K_1)$ *and* $\mathscr{X}^\tau_{irr}(K_2)$.

Proof. We only show that the image of $\mathscr{R}^\tau_{ab}(K_1) \times_{\mathscr{R}^\tau(\mathbb{Z})} \mathscr{R}^\tau_{irr}(K_2)$ is isomorphic to $\mathscr{X}^\tau_{irr}(K_2)$ since the other case is analogous. According to Lemma 4.2.2 and (4.2.1), the map $\mathscr{R}^\tau(\mathbb{Z}) \to \mathscr{R}^\tau_{ab}(K_1)$ is induced by the composition $\pi_1(S^2 - p - q) \to \pi_1(Y_1 - K_1) \to H_1(Y_1 - K_1; \mathbb{Z}) \simeq \mathbb{Z}$, which is an isomorphism. The desired claim now reduces to a straightforward algebraic fact: let A, B, B' be $\mathbb{C}$-algebras and suppose that $\mathrm{SL}(2,\mathbb{C})$ acts on the underlying vector spaces. Given a morphism $B \to A$ and an isomorphism $B \to B'$ which both commute with the $\mathrm{SL}(2,\mathbb{C})$ action, there is an induced action of $\mathrm{SL}(2,\mathbb{C})$ on the tensor product $A \otimes_B B'$ and an isomorphism of invariant rings $(A \otimes_B B')^{\mathrm{SL}(2,\mathbb{C})} \to A^{\mathrm{SL}(2,\mathbb{C})}$. $\square$

4.2.2 A holomorphic $\mathbb{C}^*$-action

Let $(\Sigma, p_i, q_i, U_i, U_i')$ be a Heegaard splitting for $Y_i - K_i$. Following [CM, Sec. 7.4], we can construct a Heegaard splitting for $Y - K = (Y_1 \# Y_2) - (K_1 \# K_2)$ as follows. Let $D_q \subset \Sigma$ and $D_{q_1} \subset \Sigma'$ be open balls around q_1, p_2 having smooth boundary and closure diffeomorphic to the unit disk. Fix a diffeomorphism $\psi : D_{q_1} \to D_{p_2}$ which extends smoothly to the boundary. We let $U_1 \#_b U_2$ and $U_1' \#_b U_2'$ be obtained by gluing the handlebodies via ψ. Let

$\Sigma \# \Sigma'$ be obtained by gluing $\Sigma - D_{q_1}$ and $\Sigma - D_{p_2}$. Observe that $\underset{.}{(}D_{q_1}) =_\psi \underset{.}{(}D_{p_2}) \subset \Sigma \# \Sigma'$ is a separating, simple closed curve which we call c.

The goal of this section is to establish the following proposition.

Proposition 4.2.7. *For $\tau \in \mathcal{G}(K_1) \cap \mathcal{G}(K_2) \subset \mathbb{C} - \{\pm 2\}$, there exists a holomorphic action of $\mathbb{C}^*$ on an open subset $\mathcal{U} \subset \mathscr{X}^\tau_{irr}(\Sigma)$ such that the Lagrangians L_0 and L_1 are contained in $\mathcal{U}$ and preserved by the action. The induced action on the Type II locus of $\mathscr{X}^\tau_{irr}(K) = L_0 \cap L_1$ is free.*

The action which we will exhibit was already considered in [CM, Sec. 7.4], but it will be useful to give a more detailed construction following [Gol04].

4.2.2.1 A holomorphic action of $(\mathbb{C}, +)$

We assume throughout this section that $\tau \in \mathcal{G}(K_1) \cap \mathcal{G}(K_2) \subset \mathbb{C} - \{\pm 2\}$. We begin with the following lemma.

Lemma 4.2.8. *There exists a holomorphic,* $\mathrm{SL}(2, \mathbb{C})$*-equivariant action of the additive group $(\mathbb{C}, +)$ on $\mathscr{R}^\tau_{irr}(\Sigma)$, which therefore induces a holomorphic action on $\mathscr{X}^\tau_{irr}(\Sigma)$.*

Proof. From van Kampen's theorem, we have the following description of $\pi_1(\Sigma)$:

$$\pi_1(\Sigma - \{p_1, q_2\}) = \pi_1(\Sigma_1 - \{p_1, q_1\}) *_{\pi_1(\partial D_{q_1})} \pi_1(\Sigma_2 - \{p_2, q_2\}).$$

We fix an isomorphism $\mathbb{Z} = \pi_1(\partial D_{q_1})$ be sending $1 \mapsto c$. For $i = 1, 2$, we let $\iota_i : \mathbb{Z} \to (\Sigma_i - \{p_i, q_i\})$ be the maps inducing above pushout diagram.

Identifying c with its image under ι_i, note that c is a meridian for K_i. The points of $\mathscr{R}^\tau(\Sigma)$ can therefore be viewed as pairs $(\rho_1, \rho_2) \in \mathscr{R}^\tau(\Sigma_1) \times \mathscr{R}^\tau(\Sigma_2)$ such that $\rho_1(c) = \rho_2(c)$.

Let $F : \mathrm{SL}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$ be the projection onto the trace-free part. That is, $F(A) = A - \frac{1}{2}\operatorname{tr}(A) I$ for $A \in \mathrm{SL}(2, \mathbb{C})$. The additive group $(\mathbb{C}, +)$ acts on $\mathscr{R}^\tau(\Sigma)$ by

$$t * (\rho_1, \rho_2) = (\exp(tF(\rho_1([\mu_1])))\rho_1 \exp(-tF(\rho_1([\mu_1]))), \rho_2)\,.$$

This action is evidently holomorphic. It is well-defined on $\mathscr{R}^\tau(\Sigma)$ due to the fact that $\exp(tF(\rho_1(c)) \in \mathrm{Stab}(\rho_1(c))$ for all $t \in \mathbb{C}$.

We claim that the action also restricts to $\mathscr{R}^\tau_{irr}(\Sigma)$. To prove this, it suffices to check that it sends reducibles to reducibles. If (ρ_1, ρ_2) is reducible, then let $0 \neq v \in \mathbb{C}^2$ be a generator of the line preserved by this representation, i.e. v is an eigenvector of every matrix in the image. In particular, it is an eigenvector of $\rho_1(c)$, which means it is an eigenvector of $\exp(tF(\rho_1(c)))$. Thus, the line is also preserved by the representation $\exp(tF(\rho_1(c)))\rho_1 \exp(-tF(\rho_1(c)))$.

The $\mathrm{SL}(2,\mathbb{C})$-equivariance of the action follows from the conjugation equivariance of exp and the projection F. It follows from the equivariance of the action that it passes to the quotient $\mathscr{X}^\tau_{irr}(\Sigma)$. $\square$

4.2.2.2 The induced $\mathbb{C}^*$ action

Given a representation $\rho \in \mathscr{R}^\tau_{irr}(\Sigma)$, the function $\zeta(\rho) := \det F(\rho(c))$ is clearly algebraic and invariant under conjugation. It therefore defines a function on $\mathscr{X}^\tau_{irr}(\Sigma)$ (i.e. an element of the ring of functions of this scheme).

It's straightforward to check that $\zeta(\rho) \in \mathbb{C}^*$ due to our assumption that $\tau \neq \pm 2$. We can therefore choose a ball $B_\epsilon \subset \mathbb{C}^*$ centered at $\zeta(\rho) \in \mathbb{C}^*$ and let $\mathcal{U} := \zeta^{-1}(B_\epsilon) \subset \mathscr{X}^\tau_{irr}(\Sigma)$. We also choose a square root on $B_\epsilon \subset \mathbb{C}^*$ which will be fixed for the remainder of this section.

Observe that the $(\mathbb{C}, +)$ action described in the previous section preserves $\mathcal{U} \subset \mathscr{X}^\tau_{irr}(\Sigma)$. Let us now analyze the stabilizer of its restriction to $\mathcal{U}$.

Lemma 4.2.9. *Given $[\rho] \in \mathcal{U}$, the stabilizer of $[\rho]$ under the $(\mathbb{C}, +)$ action contains $(\pi/\sqrt{\zeta(\rho)})\mathbb{Z}$. If $\rho = (\rho_1, \rho_2) \in \mathscr{R}^\tau(\Sigma_1) \times \mathscr{R}^\tau(\Sigma_2)$ and ρ_1, ρ_2 are both irreducible, then the stabilizer is exactly $(\pi/\sqrt{\zeta(\rho)})\mathbb{Z}$.*

Proof. By direct computation, one checks that $\exp(tF(\rho(c))) = \pm\,\mathrm{Id}$ if and only if $t \in (\pi/\sqrt{\zeta(\rho)})\mathbb{Z}$.

As we noted at the beginning of Section 4.2.1, the irreducibility of ρ_1 and ρ_2 implies that their stabilizer under the conjugation action of $\mathrm{SL}(2,\mathbb{C})$ is $\pm\,\mathrm{Id}$. Suppose now that

$t * [(\rho_1, \rho_2)] = [t^*(\rho_1, \rho_2)] = [(\rho_1, \rho_2)]$. It follows by irreducibility of ρ_2 that $t * (\rho_1, \rho_2) = (\rho_1, \rho_2)$. This then implies, by irreducibility of ρ_1, that $t \in (\pi/\sqrt{\zeta(\rho)})\mathbb{Z}$. $\square$

Corollary 4.2.10. *The restriction of the* $(\mathbb{C}, +)$ *action to* $\mathcal{U}$ *induces a holomorphic* $\mathbb{C}^*$ *action.*

Proof. Given $\lambda \in \mathbb{C}^*$, define

$$\lambda \cdot [\rho] = \frac{1}{2\sqrt{\zeta(\rho)}} \log(\lambda) * [\rho].$$

It follows from the previous lemma that this action is well-defined (i.e. independent of the choice of logarithm). $\square$

Proof of Proposition 4.2.7. We will treat the case of L_0 as the other one is analogous. If $\rho \in L_i$, then $\operatorname{Tr} \rho(c) = \tau$ because c is a meridian of K. It follows that $\rho \in \mathcal{U}$. To see that the action preserves L_0, observe that a representation $\rho \in L_0$ can be viewed as a pair $\rho = (\rho_1, \rho_2)$ where $\rho_i \in \mathscr{R}^\tau(U_i)$ and $\rho_1(c) = \rho_2(c)$. Evidently, for $t * (\rho_1, \rho_2)$ is of the same form for $t \in \mathbb{C}$ so the claim follows. Finally, the fact that the action is free on the Type II locus of $\mathscr{X}^\tau_{irr}(K)$ is an immediate consequence of Lemma 4.2.9. $\square$

Let us now consider the inclusion $\mathbb{Z}/n \hookrightarrow \mathbb{C}^*$ sending $[k] \mapsto e^{2\pi i k/n}$. It follows from Proposition 4.2.7 that the $\mathbb{C}^*$ action we have described is free on the Type II locus. It follows that the induced $\mathbb{Z}/n$ action is free.

Corollary 4.2.11. *The Euler characteristic of the Type II locus, weighted by the Behrend function, is zero.*

Proof. Let us write $X_{\mathrm{II}} \subset \mathscr{X}^\tau_{irr}(K_1 \# K_2)$ for the Type II locus. By definition we have $\chi_B(X_{\mathrm{II}}) = \sum_{m \in \mathbb{Z}} m \chi(\nu^{-1}(m))$. According to [JS12, Prop. 4.2], the Behrend function depends only on the complex-analytic structure and is in particular preserved under isomorphisms of complex-analytic spaces. Hence, the $\mathbb{Z}/n$-action preserves the sets $X_{\mathrm{II}}(m) := \nu^{-1}(m)$; in particular, $\mathbb{Z}/n$ acts on each $X_{\mathrm{II}}(m)$.

The $\mathbb{Z}/n$ action on $X_{\mathrm{II}}(m)$ is evidently free and properly discontinuous. Hence the quotient projection $X_{\mathrm{II}}(m) \to X_{\mathrm{II}}(m)/(\mathbb{Z}/n)$ is a covering map. Note that $X_{\mathrm{II}}(m) = \nu^{-1}(m)$ is a pre-stratified subset of $\mathbb{C}^n$, in the sense of [Mat70]. It follows by the main result of [Gor78] that $X_{\mathrm{II}}(m)$ admits a triangulation of dimension at most n. Hence $X_{\mathrm{II}}(m)$ is naturally a countable, locally finite CW complex. It then follows by an argument due to Belegradek [IB] that the quotient $X_{\mathrm{II}}(m)/(\mathbb{Z}/n)$ is homotopy-equivalent to a CW complex. Hence by [McC01, Sec. 5.1], we have a Serre spectral sequence for homology associated to the fibration $(\mathbb{Z}/n) \to X_{\mathrm{II}}(m) \to X_{\mathrm{II}}(m)/(\mathbb{Z}/n)$. It follows from the existence of this spectral sequence that $\chi(X_{\mathrm{II}}(m))$ is divisible by $\chi(\mathbb{Z}/n) = n$ for every n. Hence $\chi(X_{\mathrm{II}}(m)) = 0$. Hence $\chi_B(X_{\mathrm{II}}) = 0$. □

We now have the necessary tools to prove Theorem 1.1.10.

Proof of Theorem 1.1.10. We can assume that $\tau \in \mathcal{G}(K_1) \cap \mathcal{G}(K_2) \subset \mathbb{C} - \{\pm 2\}$. Note that the quotient map $\mathscr{R}^\tau(K) \to \mathscr{X}^\tau(K)$ preserves the decomposition (4.2.2). It now follows from Lemma 4.2.4 that we have a scheme theoretic decomposition

$$\mathscr{X}^\tau_{irr}(K) = \mathscr{X}^\tau_{irr}(K_1) \sqcup \mathscr{X}^\tau_{irr}(K_2) \sqcup X_{\mathrm{II}}.$$

It follows from the definition of $\chi_B(-)$ that it is additive under disjoint unions of schemes. Hence

$$\chi_B(\mathscr{X}^\tau_{irr}(K)) = \chi_B(\mathscr{X}^\tau_{irr}(K_1)) + \chi_B(\mathscr{X}^\tau_{irr}(K_2)) + \chi_B(X_{\mathrm{II}}) = \chi_B(\mathscr{X}^\tau_{irr}(K_1)) + \chi_B(\mathscr{X}^\tau_{irr}(K_2),$$

where we have used Theorem 4.2.11. The desired conclusion now follows from (2.2.5). □

4.3 The $\mathrm{SL}(2,\mathbb{C})$-Floer homology of a knot in families

In this section, we prove that the τ-weighted sheaf-theoretic Floer homology groups $HP^*_\tau(-)$ constructed for $\tau \in (-2,2)$ in [CM] can in fact be defined for all $\tau \in \mathbb{C} - \{\pm 2\}$. As discussed in the introduction, the statement of Theorem 1.1.8 implicitly relies on this fact. As another application, we observe that the groups $HP^*_\tau(K)$ are canonically the stalks of a constructible sheaf $\mathcal{F}(K) \in D^b(\mathbb{C} - \{\pm 2\})$ for a wide class of knots $K \subset Y$. We describe $\mathcal{F}(K)$ explicitly when $K \subset S^3$ is the figure-eight knot.

4.3.1 Overview of the argument

Given data $K \subset Y$ and a choice of Heegaard splitting (Σ, U_0, U_1) for the knot exterior E_K as in Section 2.2.3, the construction of $P^\bullet_{L_0,L_1}$ in [CM, Sec. 3] works for all $\tau \in \mathbb{C} - \{\pm 2\}$. (When $\tau = \pm 2$, the arguments used to show that the Heegaard splitting gives smooth symplectic manifolds and smooth Lagrangians break down; see for instance Propositions 3.3 and 3.4 in [CM].) The only place where one uses the assumption that $\tau \in (-2, 2)$ is in proving that $P^\bullet_{L_0,L_1}$ is independent of choice of Heegaard splitting. This is done in [CM, Prop. 3.9], where one crucially needs the fact that $X^\tau_{irr}(\Sigma)$ is connected and simply-connected if Σ has genus at least 6. This fact is established in the appendix of [CM] by exploiting a correspondence between character varieties of punctured surfaces and appropriate moduli spaces of parabolic Higgs bundles, whose topology is easier to analyze.

Most of this section is devoted to proving that $X^\tau_{irr}(\Sigma)$ is connected and simply-connected for all $\tau \in \mathbb{C}$ under the same assumption that Σ has genus at least 6. This is the content of Proposition 4.3.6. As explained above, it then follows immediately from the arguments in [CM] that the perverse sheaf $P^\bullet_\tau(K)$ is well-defined for all $\tau \in \mathbb{C} - \{\pm 2\}$; see Theorem 4.3.7.

The proof begins with the observation that the varieties $X^\tau_{irr}(\Sigma)$ are the fibers of the projection map $X_{irr}(\Sigma) \to \mathbb{C}$ described in Section 2.2.3 which takes a representation to the trace of its holonomy along the boundary circles. It was already proved in [CM] that $X^\tau_{irr}(\Sigma)$ is connected and simply-connected for $\tau \in (-2, 2)$. The key step is then to appeal to a theorem of Verdier (Theorem 4.3.2) which implies that all but finitely many fibers are homeomorphic. It particular, all but finitely many fibers are connected and simply-connected.

The conclusion can be extended to the remaining finitely many fibers by essentially repeating the arguments of the appendix of [CM] in a slightly more general setting. Specifically, one exploits the correspondence between $X^\tau_{irr}(\Sigma)$ and the moduli space of so-called $K(D)$-

pairs, which are a slight generalization of the parabolic Higgs bundles considered in [CM]. These moduli spaces depend on a choice of "weight data", which determine the stability conditions but do not affect the underlying bundles. One then shows by varying the weights that each fiber is homeomorphic to infinitely many other ones, and the conclusion follows. (For a few special cases, one needs to be more careful, as one only gets a homeomorphism in the complement of a set of large codimension. This sort of phenomenon also occurred in [CM].)

4.3.2 Application of a Theorem of Verdier

Let $\mathcal{S}$ be a compact and Riemann surface with empty boundary of genus $g \geq 2$ and let $p, q \in \mathcal{S}$ be a pair of distinct points. Let $\Sigma := \mathcal{S} - p - q$. Let c_p and c_q be loops around p and q respectively, with respect to some arbitrary fixed basepoint.

We choose a presentation

$$\pi_1(\Sigma) = \{a_1, \dots, a_g, b_1, \dots, b_g, c_1, c_2 \mid \prod_i [a_i, b_i] c_1 c_2 = \mathrm{Id}\}.$$

Let $\mathscr{R}(\Sigma)$ be the representation scheme of $\mathcal{S}$. Using the above presentation for $\pi_1(\Sigma)$, the ring of functions of $\mathscr{R}(\Sigma)$ can be constructed as follows (see [CM, Sec. 2.1]): start with the polynomial ring in $2g+2$ variables

$$\mathbb{C}[x_{11}^{a_1}, x_{12}^{a_1}, x_{21}^{a_1}, x_{22}^{a_1}, \dots, x_{11}^{c_2}, x_{12}^{c_2}, x_{21}^{c_2}, x_{22}^{c_2}],$$

then mod out by the relations $\{x_{11}^{\alpha} x_{21}^{\alpha} - x_{12}^{\alpha} x_{22}^{\alpha} = 1\}$ where α ranges over the generators of $\pi_1(\Sigma)$. Finally, mod out by an additional polynomial coming from the relation $\prod_i [a_i, b_i] c_1 c_2 = \mathrm{Id}$.

Let A be the resulting ring. The complex algebraic group SL_2 acts by conjugation. Let us call the ring of invariants A^G.

Using the fact that SL_2 is linearly reductive, the coordinate ring of $\mathscr{X}^{\tau}(\Sigma)$ is exactly

$$\begin{aligned} A^G/(x_{11}^{c_1} + x_{22}^{c_1} - \tau, x_{11}^{c_2} + x_{22}^{c_2} - \tau) &= A^G/(x_{11}^{c_1} + x_{22}^{c_1} - \tau, x_{11}^{c_1} + x_{22}^{c_1} - (x_{11}^{c_2} + x_{22}^{c_2})) \\ &= A^G/(x_{11}^{c_1} + x_{22}^{c_1} - (x_{11}^{c_2} + x_{22}^{c_2})) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x - \tau), \end{aligned}$$

where the tensor product is formed via the map $\mathbb{C}[x] \to A$ sending $x \mapsto x_{11}^{c_1} + x_{22}^{c_1}$.

On the other hand, using again the fact that SL_2 is linearly reductive, the ring $A^G/(x_{11}^{c_1} + x_{22}^{c_1} - (x_{11}^{c_2} + x_{22}^{c_2}))$ is the ring of invariants of $A/(x_{11}^{c_1} + x_{22}^{c_1} - (x_{11}^{c_2} + x_{22}^{c_2}))$. Let

$$W = \mathrm{Spec}\left(A^G/(x_{11}^{c_1} + x_{22}^{c_1} - (x_{11}^{c_2} + x_{22}^{c_2}))\right).$$

There is an open locus $W_{irr} \subset W$ corresponding to irreducible representations.

Lemma 4.3.1. *We have $\mathscr{X}_{irr}^{\tau}(\mathcal{S}) = W_{irr} \times_{\mathbb{A}^1} \{\tau\} := (W_{irr})_\tau$.*

Proof. As demonstrated above, the coordinate rings of $\mathscr{X}^{\tau}(\mathcal{S})$ and $W \times_{\mathbb{A}^1} \{\tau\}$ are identical, so they are the same scheme. The irreducible representations in either scheme are precisely those that are irreducible as representations of $\pi_1(\mathcal{S})$. That is, $W_{irr} = W \cap \mathscr{X}_{irr}(\mathcal{S})$. Taking fibers over τ, we find that $(W_{irr})_\tau = W_\tau \cap \mathscr{X}_{irr}(\mathcal{S}) = \mathscr{X}^{\tau}(\mathcal{S}) \cap \mathscr{X}_{irr}(\mathcal{S}) = \mathscr{X}_{irr}^{\tau}(\mathcal{S})$. □

We will need the following result of Verdier.

Theorem 4.3.2 ([Ver76], Cor. 5.1)**.** *Let X, Y be complex algebraic varieties (separated and of finite type). Let $f : X \to Y$ be a morphism. Then there is a dense Zariski open set $U \subset Y$ such that $f|_U := f^{-1}(U) \to U$ is a locally trivial topological fibration (in the analytic topology).*

We are led to the following corollary.

Corollary 4.3.3. *There is a Zariski open set $\mathcal{V} \subset \mathbb{A}^1$ such that $X_{irr}^{\tau}(\mathcal{S})$ is connected simply-connected, and of dimension $6g-2$ for all $\tau \in \mathcal{V}$.*

Proof. Since W_{irr} is an open subset of a finitely-generated complex algebraic variety, it is separated and of finite type. So the theorem implies that it is a locally trivial topological fibration over some Zariski open subset $\mathcal{V} \subset \mathbb{C}$.

It was shown in [CM, Appendix I] that $X_{irr}^{\tau}(\mathcal{S})$ is connected, simply-connected and of dimension $6g-2$ for all $\tau \in (-2,2)$. Since $(-2,2) \cap \mathcal{V}$ must be nonempty, it follows by Theorem 4.3.2 that $X_{irr}^{\tau}(\mathcal{S})$ is connected and simply-connected for all $\tau \in U \subset \mathcal{C}$. □

4.3.3 Moduli spaces of $K(D)$-pairs

We will now upgrade Theorem 4.3.3 by proving that $X^{\tau}_{irr}(\Sigma)$ is in fact connected and simply-connected for all $\tau \in \mathbb{C} - \{\pm 2\}$. The argument makes use of a diffeomorphism between $X^{\tau}_{irr}(\Sigma)$ and certain moduli spaces of so-called $K(D)$-pairs, which are a modest generalization of Higgs bundles.

We note that Theorem 4.3.3 already allows us to make sense of $HP^*_{\tau}(-)$ for generic τ, which is all that one needs for the purpose of Theorem 1.1.8 and Theorem 1.1.10. However, we have chosen to include this section for completeness and in view of the possibility of studying $HP^*_{\tau}(-)$ in families that was alluded to in the introduction.

Throughout this section, we need to appeal to the general theory of parabolic vector bundles and Higgs bundles. The relevant definitions are introduced in Section 8.1 of [CM] and we have chosen not to repeat them here for the sake of concision. We now introduce a class of objects which are very similar to Higgs bundles and were not considered in [CM]. A good reference for these is [Mon16] (but the reader should be warned that the objects which we refer to as $K(D)$-pairs are just called "parabolic Higgs bundles" in [Mon16]).

Definition 4.3.4 (cf. Sec. 8.2 of [CM])**.** Fix a Riemann surface $\mathcal{S}$ and let $D = p_1 + \cdots + p_n$ for some collection of n distinct points. A $K(D)$-pair on $(\mathcal{S}, D)$ is the data of a pair (E_*, Φ) consisting of a parabolic vector bundle E_* and a (not necessarily strongly) parabolic morphism $\Phi : E_* \to E_* \otimes K(D)$, where K is the canonical bundle. The morphism Φ is often called a *Higgs field*. We usually denote $K(D)$-pairs by boldface letters $\mathbf{E} = (E_*, \Phi)$.

We remind the reader that parabolic Higgs bundles are defined in the same way as $K(D)$-pairs, except that one requires Φ to be a strongly parabolic morphism; see [CM, Sec. 8.2]. In fact many authors including [Mon16] refer to $K(D)$-pairs as parabolic Higgs bundles. There are many other inconsistent conventions in this theory, so we remind the reader that we will always follow the conventions of [CM, Sec. 8].

For the remainder of this section, we specialize to the case of a Riemann surface $\mathcal{S}$ and a divisor $D = p + q$ for two distinct points $p, q \in \mathcal{S}$. Let $\boldsymbol{\omega}$ denote the data of weights

$0 \leq \alpha_1(p) \leq \alpha_2(p) < 1$ and $0 \leq \alpha_1(q) \leq \alpha_2(q) < 1$. Let $\mathfrak{c}$ denote the data of a pair of matrices $\nu_p, \nu_q \in \mathfrak{sl}(2, \mathbb{C})$.

We consider the moduli space $\mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}, 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c})$ which parametrizes isomorphism classes of stable $K(D)$-pairs (E_*, Φ) satisfying the following conditions:

- (E_*, Φ) has rank 2,
- $\mathrm{Res}_p \Phi = \nu_p$ and $\mathrm{Res}_q \Phi = \nu_q$,
- the weights are given by $\boldsymbol{\omega}$,
- $\det(E_*) \simeq \mathcal{O}_{\mathcal{S}}$,
- $\mathrm{Tr}\, \Phi = 0$.

We define $\mathrm{wHiggs}^{ss}(\mathcal{S}, \boldsymbol{\omega}, 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c})$ analogously, though we warn the reader that this is not in general a fine moduli space. We remark that the notation $\mathrm{wHiggs}(-)$ is intended to be compatible with the notation of [CM, Sec. 8.2] and [Mon16]. The prefix "w" stands for "weak" and reflects the fact that the Higgs field of a $K(D)$-pairs satisfies a weaker condition than for an ordinary Higgs bundle.

For $\alpha \in (0, 1/2)$, it will be convenient to let $\boldsymbol{\omega}(\alpha)$ denote the data of weights $0 < \alpha < 1 - \alpha < 1$ at p and q. If $\alpha = 0$, we let $\boldsymbol{\omega}(\alpha) = \boldsymbol{\omega}(0)$ denote the weights $0 = \alpha_1(p) = \alpha_2(p) = \alpha_1(q) = \alpha_2(q)$. If $\alpha = 1/2$, we let $\boldsymbol{\omega}(\alpha) = \boldsymbol{\omega}(1/2)$ denote the weights $1/2 = \alpha_1(p) = \alpha_2(p) = \alpha_1(q) = \alpha_2(q)$. For $t \in [0, \infty)$ we let $\mathfrak{c}(t)$ be the data of weights $\nu_p = \nu_q$ having t as an eigenvalue.

For our purposes, the importance of $K(D)$-pairs is mainly due to the following theorem.

Theorem 4.3.5 (see Thm. 4.12 in [Mon16] and c.f. Thm. 8.4. in [CM])**.** *For $\tau \in \mathbb{C}$, choose $0 \leq \alpha \leq 1/2$ and $t > 0$ so that $\tau = \mathrm{Tr}(\mathrm{diag}(e^t e^{2\pi i \alpha}, e^{-t} e^{-2\pi i \alpha})) = e^t e^{2\pi i \alpha} + e^{-t} e^{-2\pi i \alpha}$. Then there is a real-analytic diffeomorphism*

$$X^{\tau}_{irr}(\Sigma) \simeq \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t)).$$

In the proof of the next proposition, it will be convenient to view the choice weights as an additional piece of data on a fixed $K(D)$-pair. From this perspective, when one changes the weights, one does not change the underlying set of $K(D)$-pairs but one changes their slopes; i.e. one changes the stability conditions.

Proposition 4.3.6. *For all $\tau \in \mathbb{C}$, the relative character variety $X^{\tau}_{irr}(\Sigma)$ is connected and simply-connected.*

Proof of Proposition 4.3.6. According to Theorem 4.3.3, the proposition is already proved for all values of τ contained in a (Zariski) open set $\mathcal{V} \subset \mathbb{A}^1$ which contains the interval $(-2, 2)$.

Fix $\tau \in \mathbb{C}$ and choose $(t, \alpha) \in \mathbb{R}_{\geq 0} \times [0, 1/2]$ so that $e^t e^{2\pi i\alpha} + e^{-t} e^{-2\pi i\alpha} = \tau$. Observe that there exists $\alpha' \in (0, 1/2)$ so that $\tau' = e^t e^{2\pi i\alpha'} + e^{-t} e^{-2\pi i\alpha'} \in \mathcal{V}$. We now consider three possibilities.

Case I: $\alpha \in (0, 1/2)$. Given a $K(D)$ pair (E_*, Φ) of rank 2, parabolic degree 0 and weights $\boldsymbol{\omega}(\alpha)$, it's not hard to check that the stability conditions are constant under varying $\alpha \in (0, 1/2)$. We can therefore define a map

$$\psi_\alpha : \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t)) \to \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha'), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t))$$

by sending (E_*, Φ) to itself and replacing the weights (α, α) by (α', α'). Since this map is evidently invertible, it is an isomorphism. We conclude that the left hand side is connected and simply connected since the right hand side is.

Case II: $\alpha = 0$. Fix $z \in \Sigma - p - q$. We define a map

$$\psi_0 : \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(0), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t)) \to \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha'), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t))$$

by sending $(E_*, \Phi) \mapsto (E_* \otimes \mathcal{O}(-z), \Phi)$ and replacing the weights $(0, 0)$ by (α', α').

This map is an embedding, but it fails to be surjective. Indeed, (E_*, Φ) may admit a sub-bundle (E^-, Φ^-) of parabolic degree $-1 + 2\alpha < 0$, having weights $0 < \alpha < 1$ at p, q. Such a bundle is not in the image of ψ_0, and one can easily check that these are the only bundles which can fail to be in the image of ψ_0.

Let $B = \mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t)) - \mathrm{im}(\psi_0)$. We just saw that B is contained in the locus $\mathcal{E}$ of the moduli space which consists of extensions of $K(D)$ pairs $0 \to \mathbf{E}^- \to \mathbf{E} \to \mathbf{E}^+ \to 0$ where E^- has parabolic degree $-1 + 2\alpha$. Since $\mathrm{wHiggs}^s(\mathcal{S}, \boldsymbol{\omega}(\alpha), 2, \mathcal{O}_{\mathcal{S}}, \mathfrak{c}(t))$ has dimension $6g - 2$ by Theorem 4.3.3, it follows that $\mathrm{im}(\psi_0)$ also has this property whenever $\dim(B) \leq 6g - 5$.

The dimension of the space of extensions $K(D)$ pairs can be computed as in [CM, Sec. 8.4], so we only sketch the details. Let $\mathcal{E}$ be the space of extensions. Let $\mathcal{X}$ be the set of pairs $(E^-, \Phi^-), (E^+, \Phi^+)$ where E^+, E^- have rank 1 and the underlying line bundles have degree -1. There is a natural forgetful map $\mathcal{E} \to \mathcal{X}$ and the dimension of $\mathcal{E}$ is bounded above by the sum of the dimension of $\mathcal{X}$ and of the fibers.

The dimension of $\mathcal{X}$ is computed in [BY96, p. 3] to be $2g + 1$. The fiber over a fixed pair $(E^-, \Phi^-), (E^+, \Phi^+)$ is the space of extensions of this pair. Since each pair has different weights, this space is parametrized by the first homology group of an appropriate double complex as in [CM, Prop. 8.12]. The dimension can be computed as in [CM, Sec. 8.4] and one gets an upper bound of $4g+1$ on the dimension of $\mathcal{E}$. (This is essentially the same answer as in [CM, Thm. 8.9], up to an additive constant which is independent of g). In particular, our assumption that $g \geq 6$ implies that $\dim(B) \leq \dim(\mathcal{E}) \leq 6g - 5$ as desired. (This is in fact true once $g \geq 4$, but the requirement that $g \geq 6$ is needed in [CM, Prop. 3.14]).

Case III: $\alpha = 1/2$. We use the same map as in Case I. The map is an embedding, but fails to be a surjective as in Case II. The problem occurs again with sub-bundle of parabolic degree $-1 + 2\alpha$, and the subsequent argument is then the same as in Case II. □

Corollary 4.3.7. *Given a knot K in an oriented, closed 3-manifold Y, the perverse sheaf $P^\bullet_\tau(K)$ constructed in* [CM] *is well-defined (i.e. independent of the choice of Heegaard splitting) for $\tau \in \mathbb{C} - \{\pm 2\}$.*

Proof. As explained in Section 4.3.1, the proof is entirely similar to the construction in Section 3 of [CM]. □

4.3.4 $\mathrm{SL}(2,\mathbb{C})$ Floer homology in families

As an consequence of Theorem 4.3.7, it makes sense to study the behavior of $HP^*_\tau(K)$ in families for a given knot $K \subset Y$. As discussed in the introduction, one expects that these groups should restrict to a local system on a Zariski open subset of the complex plane. This expectation can already be verified for a wide class of knots considered in [CM].

More precisely, for a fixed 3-manifold Y, one considers [CM, Sec. 5.2] the class of all knots $K \subset Y$ whose character scheme $\mathscr{X}(K)$ is reduced and of dimension at most 1; see Assumptions A.1 and A.2 in [CM, Sec. 5.2]. For $Y = S^3$, this includes all two-bridge knots, torus knots and many pretzel knots. Letting now $\pi : \mathscr{X}(K) \to \mathbb{A}^1$ be the map taking a representation onto its trace, it follows from the discussion in Section 5.3 of [CM] that there is an open set $U \subset \mathbb{A}^1$ such that π restricts to a smooth and proper morphism on the preimage of U. (More precisely, one can take U is the set of points $\tau \in \mathbb{A}^1$ satisfying Assumptions B.1–B.4, which is shown to be a cofinite set).

Letting $X_{irr}(K)_\tau$ be the fiber of $\tau \in \mathbb{A}^1$ under π, there is then a canonical identification $HP^*_\tau(K) = H^*(X_{irr}(K)_\tau)$; see [CM, Cor. 5.8]. It is well-known that the cohomology of a smooth a proper map forms a local system on the base, so we conclude that there is a local system $\mathcal{F}_1(K) \in D^b(U)$ with $F(K)_\tau = HP^*_\tau(K)$.

Let $E(K) = \mathbb{C} - \{\pm 2\} - U$ (a finite set) and define $\mathcal{F}_2(K) \in D^b(E(K))$ to be the unique sheaf whose stalk at each point $\tau \in E(K)$ is $HP^*_\tau(K)$. We then find that

$$\mathcal{F}(K) := j_! \mathcal{F}_1 \oplus i_* \mathcal{F}_2 \in D^b(\mathbb{C} - \{\pm 2\})$$

is a constructible sheaf whose stalks compute $HP^*_\tau(K)$, where $i : E(K) \to \mathbb{C}$ and $j : U \to \mathbb{C}$ are the inclusion maps.

It would be interesting to construct $\mathcal{F}(K)$ systematically, and for all knots, as the push-forward of a suitable sheaf on the character scheme $\mathscr{X}_{irr}(K)$ under the projection map considered above.

An example: the figure-eight

According to [Por], the character variety (which agrees with the character scheme) of the figure-eight knot is

$$X(4_1) = \{(x, y) \mid (y-2)(y^2 - (x^2-1)y + x^2 - 1) = 0\}, \tag{4.3.1}$$

where $\{(y-2) = 0\}$ is the component of reducible representations and x is the trace of a meridian.

Let $E(4_1) = \{\pm 1, \pm\sqrt{5}\}$. For $\tau \in \mathbb{C} - E(4_1)$, we have

$$\mathscr{X}^{\tau}_{irr}(4_1) = X^{\tau}_{irr}(4_1) = \{(\tau, y) \mid y^2 - (\tau^2 - 1)y + (\tau^2 - 1) = 0\} \subset \mathbb{C}. \tag{4.3.2}$$

For $\theta \in [0, 2\pi]$ and $0 < \epsilon \ll 1$, let us consider the loop $\tau(\theta) = 1 + \epsilon e^{i\theta}$. The points of $X^{\tau(\theta)}_{irr}(4_1) \subset \mathbb{C}$ move around as θ goes from 0 to 2π and can be computed by the quadratic formula. Noting that $\tau(\theta)^2 - 1 = 2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta} \sim 2\epsilon e^{i\theta}$, we compute that the relevant roots are approximately

$$\frac{-2\epsilon e^{i\theta} \pm \sqrt{4\epsilon^2 e^{i\theta} - 8\epsilon e^{i\theta}}}{2} \sim -\epsilon e^{i\theta} \pm \sqrt{2\epsilon} i e^{i\theta/2}.$$

This implies that the points of $X^{\tau(\theta)}_{irr}(4_1) \subset \mathbb{C}$ get interchanged as θ varies from 0 to 2π. It follows from the above discussion that the local system $\mathcal{F}_1(4_1) \in D^b(\mathbb{C} - \{\pm 1, \pm\sqrt{5}\})$ has fibers $\mathbb{Z}^2$ concentrated in degree 0. Fixing a reference fiber, we have seen that the monodromy around $+1$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By a similar argument, one can check that the monodromy around $\{-1, \pm\sqrt{5}\}$ is given by the same matrix up to conjugacy.

4.4 Appendix

As explained in Section 2.2.4, there are many definitions in the literature of the Euler characteristic *weighted* by a constructible function. The purpose of this appendix is to prove that these definitions are all equivalent. This is the content of Theorem 4.4.8.

4.4.1 A new convention

It will be convenient in this appendix to consider ordinary Whitney stratifications (which we will call b-stratifications), rather than the more restrictive w-stratifications defined in Definition 2.2.1 which we have mainly considered. The reader may note that we only defined w-stratifications in the complex algebraic category (in particular, the strata were required to be complex algebraic). In contrast, we will define b-stratifications in the smooth category.

Remark 4.4.1. Vedier's condition w) also makes sense in the smooth category, so we could also have defined w-stratifications in the smooth category in Definition 2.2.1. However, since this appendix is the only place in which we want to consider non-algebraic stratifications, it seemed more natural to include the requirement that the strata be algebraic as part of our definition.

For convenience, we only consider b-stratifications of subsets of $\mathbb{R}^n$, for $n \geq 1$.

Definition 4.4.2 (see Sec. 1 in [Ver76])**.** Let (M', M) be a pair of locally closed, smooth submanifolds of $\mathbb{R}^n$ for some $n \geq 1$ such that $M \cap M' = \emptyset$ and $y \in \overline{M} \cap M'$. We say that the pair (M', M) satisfies Whitney's condition b) at y if the following holds: for any sequence $(x_n, y_n) \in M \times M'$ such that

- $x_n \to y$ and $y_n \to y$,
- the sequence of lines $\rho(x_n - y_n)$ has a limit L in $\mathbb{P}(\mathbb{R}^n)$,
- the sequence of tangent planes T_{M,x_n} has a limit T in $\mathrm{Grass}(\mathbb{R}^n)$,

then we have $L \subset T$.

We now state our notion of a b-stratification. We emphasize that this is not the most general definition (for instance, could allow locally-finite strata), but it is sufficient for our purposes.

Definition 4.4.3 (cf. (2.1) in [Ver76] and Sec. 1.2 in [GM88])**.** A *b-stratification* of a subset $X \subset \mathbb{R}^n$ is a partition $X = \sqcup_{i=1}^n X_i$, where the $X_i \subset X$ are locally closed, smooth submanifolds, which satisfies the following axioms:

(i) $X_i \cap X_j = \emptyset$ if $i \neq j$.

(ii) If $\overline{X}_i \cap X_j \neq \emptyset$, then $X_j \subset \overline{X}_i$. (One gets the same notion using the analytic or Zariski topology.)

(iii) If $X_i \subset \overline{X}_j$ and $i \neq j$, then the pair (X_j, X_i) satisfies the condition b).

Recall that the w-stratifications introduced in Definition 2.2.1 are in particular b-stratifications.

4.4.2 Equality of Euler characteristics

Let X be a closed subset of $\mathbb{R}^n$. Let $X = \sqcup_{i=1}^n S_i$ be a b-stratification which we call $\mathcal{S}$. We say that an arbitrary subset $C \subset X$ is $\mathcal{S}$-constructible if $C = \cup_{i\in\Sigma} S_i$ for some subset $\Sigma \subset \{1, 2, \ldots, n\}$. Let $D = D(C) = \max_{i\in\Sigma} \dim(S_i)$ and let $d = d(C) = \min_{i\in\Sigma} \dim(S_i)$. We say that $D - d$ is the *length* of C.

For $j = d, d+1, \ldots, D$, let $C_j = \cup_{\dim(S_k)=j} S_k$. It follows from the second axiom of Definition 4.4.3 that the subspace topology on C_j coincides with the natural topology on C_j as a disjoint union of smooth manifolds S_k.

Definition 4.4.4 (see p. 41 in [GM88])**.** Fix a point x in a b-stratified set $X \subset \mathbb{R}^n$. Let S denote some stratum and let T be a smooth submanifold which is transverse to every stratum of X, which intersects S at x and nowhere else, and such that $\dim S + \dim T = n$. Let $B_\partial(x) := \{z \mid \|x - z\| \leq \delta\}$ with the distance measured in the standard Euclidean metric.

For $0 < \delta \ll 1$, let $N(x) := T \cap X \cap B_\delta(x)$ and let $\ell k(x) := T \cap X \cap \partial B_\delta(x)$. For δ small enough, the homeomorphism type of these spaces is independent of T, δ, and of the choice of x. Moreover, they are canonically b-stratified as transverse intersections of b-stratified spaces.

Lemma 4.4.5. *There is a closed neighborhood $C_j \subset U_j$ with $U_j \subset C$ and a locally trivial projection map $\pi : U_j \to C_j$. The fiber F_j over a point $x \in C_j$ is naturally a subspace of $N(x)$ and is $\mathcal{S}'$-constructible, where $\mathcal{S}'$ is the induced Whitney stratification on $N(x)$.*

Proof. According to [GM88, p. 41], there is a closed neighborhood $\tilde{U}_j \subset X$ of C_j and a locally trivial projection map $\tilde{\pi}_j : \tilde{U}_j \to C_j$ whose fibers are homeomorphic to $N(x)$ for $x \in C_j$. Moreover, this fibration is locally homeomorphic to $\mathbb{R}^j \times N(x)$ by a stratification-preserving homeomorphism. The lemma now follows simply by letting $U_j = \tilde{U}_j \cap C$. $\square$

Corollary 4.4.6. *The projection $U_j \to C_j$ is a (weak) homotopy equivalence (and hence an ordinary homotopy equivalence, since these are all CW complexes).*

Proposition 4.4.7. *Suppose that one of the following two hypotheses holds:*

(i) All the strata have vanishing Euler characteristic,

(ii) all the strata are even-dimensional.

Then for any $\mathcal{S}$-constructible subset $C = \cup_{i\in\Sigma} S_i$, we have

$$\chi(C) = \sum_{i\in\Sigma} \chi(S_i) = \sum_{i\in\Sigma} \chi_c(S_i) = \chi_c(C), \tag{4.4.1}$$

where $\chi_c(C) := \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(C)$. (In case (i), all these numbers are all zero!)

Proof. We work by induction on the length of C. If C has length 1, then C is a union of smooth manifolds, which have vanishing Euler characteristic in case (i) and are even dimensional in case (ii). The desired result then follows from Poincaré duality.

Suppose now that the result has been proved for all $\mathcal{S}$-constructible sets C of length $n - 1 = D(C) - d(C) = D - d$.

Let $C' = \cup_{j>d} C_j$. Then $C = C_d \cup C'$. By Lemma 4.4.5, C_d has a closed neighborhood U_j (in general non-compact) such that $U_d - C_d$ is a locally trivial fibration over C_d. Since $\mathrm{int}(U_d) \cup \mathrm{int}(C') = C$, we have a Mayer-Vietoris long exact sequence for singular homology

$$\cdots \to H_k(U_d - C_d) \to H_k(C_d) \oplus H_k(C') \to H_k(C) \to \dots$$

Suppose that (i) holds. Then $\chi(C_d) = 0$. Hence $\chi(U_d - C_d) = 0$ since $U_d - C_d$ is a locally trivial fibration. Since $\chi(C') = 0$ by induction hypothesis, we conclude that $\chi(C) = 0$. This proves the first equality of (4.4.1) in case (i).

Suppose now that (ii) holds. We first claim that $\chi(U_d - C_d) = 0$. Indeed, according to Lemma 4.4.5, $U_d - C_d$ is a locally trivial fibration and according to [Sul71], the fiber F_d satisfies (i). We have already shown that this implies that $\chi(F_d) = 0$. It follows that $\chi(U_d - C_d) = 0$. The desired claim follows again from Mayer-Vietoris. This proves the first equality of (4.4.1) in case (ii).

The second equality of (4.4.1) is a direct consequence of Poincaré duality. The third equality can be proved by the same argument as the first. One now needs to use a version of Mayer-Vietoris for compactly-supported cohomology (see for instance [Mat, Sec. 3.4]), as well as a multiplicativity property for Euler characteristic with compact support of locally trivial fibrations which can be deduced from the argument of [Dim04, Cor. 2.5.5] using [Dim04, Cor. 2.3.24]. (We note that there is a typo in the description of the E_2 page of the spectral sequence of [Dim04, Cor. 2.3.24] which should be $E_2^{p,q} = H_c^p(Y, R^q f_! \mathcal{F}^\bullet)$). □

Corollary 4.4.8. *Let X be a complex-algebraic variety which admits an embedding into $\mathbb{C}^n$. Let $f : X \to \mathbb{Z}$ be a constructible function. Then all notions of the Euler characteristic of X weighted by f mentioned in Section 2.2.4 coincide. More precisely, we have:*

$$\chi(X, f) := \sum_{n\in\mathbb{Z}} n \cdot \chi(\{f^{-1}(n)\}) = \sum_{S\in\mathcal{S}} f|_S \cdot \chi(S) = \sum_{S\in\mathcal{S}} f|_S \cdot \chi_c(S) = \sum_{n\in\mathbb{Z}} n \cdot \chi_c(\{f^{-1}(n)\}). \tag{4.4.2}$$

Proof. Since f is constructible, it follows from Definition 2.2.3 and the comment following it that we can choose a w-stratification $X = \sqcup_{i=1}^n X_i$ (which is hence a b-stratification) by subvarieties which we call $\mathcal{S}$, and such that the sets $\{f^{-1}(m)\}_{m\in\mathbb{Z}}$ are $\mathcal{S}$-constructible. The second and fourth equalities then follow from Proposition 4.4.7. The third equality is a consequence of Poincaré duality, since the strata are even-dimensional manifolds. □

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