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# Applications of Macdonald Ensembles

by

Shamil Shakirov

A dissertation submitted in partial satisfaction of the

requirements for the degree of

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in

Mathematics

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of the

University of California, Berkeley

Committee in charge:

Professor Nicolai Y. Reshetikhin, Chair

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Shamil Shakirov

## Abstract

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Professor Nicolai Y. Reshetikhin, Chair

Multivariate orthogonal polynomials of Macdonald are an important tool to study a variety of topics in modern mathematical physics, such as chiral algebras, three-dimensional topological theories of Chern-Simons type, five-dimensional supersymmetric Yang-Mills theories, and others. We describe several recent applications of Macdonald polynomials, based on original research contributions. Introduction gives an overview of Macdonald theory, with a view towards applications. In Chapter 2, we discuss a Macdonald deformation of three-dimensional Chern-Simons topological field theory and construct it explicitly in the case of Heegaard splitting of genus one. The resulting knot invariants turn out to be related to the recently developed theory of knot homology. In Chapter 3, we show that Macdonald ensembles are natural integral representations for the Nekrasov functions – important special functions in the context of five-dimensional supersymmetric Yang-Mills theories. This allows us to prove, in vast generality, a conjecture that Nekrasov functions are equal to the chiral blocks of  $W_{q,t}(sl_N)$  chiral algebras.

To my parents, who always believed in me.

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# Chapter 1

## Introduction

In quantitative sciences, particularly in mathematics and theoretical physics, one often uses special functions to quantitatively describe complex qualitative phenomena. These special functions typically possess a number of features, that make them useful throughout a wide range of applications:

- Have several alternative representations that complement each other;
- Closely related to interesting groups of symmetries;
- Apply to multiple different fields of science, thus facilitating connections.

An archetypical example of such a special function is the Gauss hypergeometric function  ${}_2F_1(a, b, c|z)$ . First, it can be represented in a variety of equivalent, and complementary, ways: as a series

$${}_2F_1(a, b, c|z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}, \quad (1.1)$$

as an integral

$${}_2F_1(a, b, c|z) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx, \quad (1.2)$$

or as a solution of a differential equation:

$$\left( z(1-z)\frac{d^2}{dz^2} + (c - az - bz - z)\frac{d}{dz} - ab \right) {}_2F_1(a, b, c|z) = 0. \quad (1.3)$$

Second, it ties to representation theory of Lie algebras and groups of type  $A_1$  – it describes the matrix elements of irreducible representations of the  $SU(2)$  group [3], appears in representation theory of the affine  $SU(2)$  as the solution to the Knizhnik-Zamolodchikov (KZ) equation [4], etc. Finally, it has an overwhelming number of applications, from the theory of angular momentum in quantum mechanics [3] to solutions of algebraic equations [5] to population genetics [6] and many others. This universality of the Gauss hypergeometric function makes  ${}_2F_1$  an important special function and justifies why it should be studied in detail.

## 1.1 Random matrix ensembles

This thesis is devoted to the study of an important class of generalizations of the Gauss hypergeometric function. One of the most important features of this class of special functions is replacing one integration variable  $x$  with many integration variables  $x_1, \dots, x_N$ . A specific subclass of integrals, that becomes increasingly important in modern applications, is the one where the integrand is a product of factors that depend *on at most two* variables:

$$Z = \int_{\gamma_1} dx_1 \dots \int_{\gamma_N} dx_N \prod_i V(x_i) \prod_{i \neq j} W(x_i, x_j), \quad (1.4)$$

where  $\gamma_1, \dots, \gamma_N$  are suitable contours in the complex plane. This integral has a straightforward interpretation in statistical mechanics: it represents the partition functions of an ensemble of  $N$  particles, subject to an overall potential  $\log V(x)$  and interacting pairwise with interaction potential  $\log W(x_i, x_j)$ .

While the choice of contours  $\gamma_1, \dots, \gamma_N$  and the potential  $\log V(x)$  varies very much from application to application, the interaction potential  $\log W(x_i, x_j)$  is a more robust characteristic. As a result, it is natural to classify such integrals by the form of the function  $W(x_i, x_j)$ .

A particularly important and simple special case is when the two-particle interaction function  $W$  is given by

$$W(x_i, x_j) = 1 - \frac{x_i}{x_j}. \quad (1.5)$$

This case is important because it is especially closely related to classical symmetries: the multivariate integration measure, that results from this choice, is exactly the Haar integration measure on the compact Lie group  $U(N)$ , expressed in terms of the eigenvalues  $x_1, \dots, x_N$ :

$$d\mu_{U(N)} = \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \frac{dx_1}{x_1} \cdots \frac{dx_N}{x_N}.$$

Choosing the contours  $\gamma_1, \dots, \gamma_N$  to be unit circles, the integral  $Z$  becomes a partition function of a random matrix  $X \in U(N)$  with an invariant probability distribution function  $\det V(X)$ . Not surprisingly, representation theory of  $U(N)$  plays a crucial role in this problem: for example, the orthogonal polynomials with respect to this integration measure

$$\int_{|x_1|=1} \cdots \int_{|x_N|=1} d\mu_{U(N)} \chi_\lambda(x_1, \dots, x_N) \chi_\mu(x_1^{-1}, \dots, x_N^{-1}) = \delta_{\lambda\mu} \quad (1.6)$$

are non other than Schur polynomials  $\chi_\lambda(x_1, \dots, x_N)$ , i.e. the characters of irreducible finite-dimensional representations  $V_\lambda$  of  $U(N)$  with highest weight  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$ , as a function of the conjugacy class  $(x_1, \dots, x_N)$ .

A few first Schur polynomials are given by

$$\chi_1(x_1, \dots, x_N) = p_1, \quad (1.7)$$

$$\chi_2(x_1, \dots, x_N) = \frac{1}{2} p_1^2 + \frac{1}{2} p_2, \quad (1.8)$$

$$\chi_{11}(x_1, \dots, x_N) = \frac{1}{2} p_1^2 - \frac{1}{2} p_2, \quad (1.9)$$

$$\chi_3(x_1, \dots, x_N) = \frac{1}{6} p_1^3 + \frac{1}{2} p_1 p_2 + \frac{1}{3} p_3, \quad (1.10)$$

$$\chi_{21}(x_1, \dots, x_N) = \frac{1}{3} p_1^3 - \frac{1}{3} p_3, \quad (1.11)$$

$$\chi_{111}(x_1, \dots, x_N) = \frac{1}{6} p_1^3 - \frac{1}{2} p_1 p_2 + \frac{1}{3} p_3, \quad (1.12)$$

where  $p_k = x_1^k + \dots + x_N^k$  are the Newton power sums. The study of  $Z$  and similar integrals, with two-particle logarithmic potential  $\log W(x_i, x_j) = \log(x_i - x_j)$ , is known under the name of theory of random matrix ensembles [7] or, short, matrix models.

It is now widely realized that matrix models are natural candidates to be called special functions [8], for the following reasons. First, they can be represented not only as integrals, but also as series [9] and solutions to interesting linear [10] and quadratic [11] differential equations. Second, they are closely related to representation theory of ordinary and affine Lie algebras [12]. Last but not the least, they tend to be quite universal, i.e. they have applications in many seemingly different branches of mathematics and physics, from combinatorics of two-dimensional surfaces [13] to geometry of moduli spaces [14] Laplacian growth processes [15] quantum Hall effect [16] conformal field theory [17] and many others. Many of these applications are reviewed in the book [18].

## 1.2 Macdonald ensembles and polynomials

An even more interesting class of special functions one obtains from more general two-point interactions  $W(x_i, x_j)$  and hence more general integration measures. This generalization can be summarized by the following chart:

Ensemble	Two-particle interaction	Exponential notation
Random U(N)	$\left(1 - \frac{x_i}{x_j}\right)$	$\exp\left(-\sum_{k \geq 1} \frac{(x_i/x_j)^k}{k}\right)$
Beta	$\left(1 - \frac{x_i}{x_j}\right)^\beta$	$\exp\left(-\sum_{k \geq 1} \beta \frac{(x_i/x_j)^k}{k}\right)$
Macdonald	$\prod_{m=0}^{\beta-1} \left(1 - q^m \frac{x_i}{x_j}\right)$	$\exp\left(-\sum_{k \geq 1} \frac{1-t^k}{1-q^k} \frac{(x_i/x_j)^k}{k}\right)$

The first row here is occupied by the standard  $SU(N)$  random matrix ensembles. To pass from the first to the second row, one performs a  $\beta$ -deformation, raising the measure to an arbitrary power  $\beta$ , which is a new parameter. The resulting models are known as  $\beta$ -ensembles [7, 19] and are close relatives of matrix models. In some cases, they even reduce to matrix models: namely, the cases of  $\beta = \frac{1}{2}$ ,  $\beta = 1$  and  $\beta = 2$  describe orthogonal, unitary, and symplectic matrix ensembles, respectively [7]. For general  $\beta$ , however, no random matrix interpretation is available. Finally, to pass from the second to the third row, one performs a  $q$ -deformation, splitting the  $\beta$  originally identical factors in the measure by different powers of a new parameter  $q \neq 1$ .

The result is a double deformation of random matrix ensembles, with two deformation parameters  $q$  and  $t = q^\beta$ , such that one recovers beta ensembles in the limit  $q \rightarrow 1$ , and further matrix models in the limit  $\beta \rightarrow 1$ . We will call the resulting multivariate integrals *Macdonald ensembles*, since they have been first introduced and studied in the seminal book [20] of Ian Macdonald. These multivariate integrals, and their applications in modern mathematics

and theoretical physics, constitute the main subject of this thesis. The multivariate symmetric polynomials, orthogonal w.r.t. the Macdonald measure,

$$\int_{|x_1|=1} \frac{dx_1}{x_1} \cdots \int_{|x_N|=1} \frac{dx_N}{x_N} \prod_{m=0}^{\beta-1} \left( 1 - q^m \frac{x_i}{x_j} \right) M_\lambda(x_1, \dots, x_N) M_\mu(x_1^{-1}, \dots, x_N^{-1}) = g_\lambda \delta_{\lambda\mu}, \quad (1.13)$$

are called Macdonald polynomials. They are two-parameter deformations of the Schur polynomials and, just as them, are labeled by partitions  $\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$ . A few first Macdonald polynomials are given by

$$M_1(x_1, \dots, x_N) = p_1, \quad (1.14)$$

$$M_2(x_1, \dots, x_N) = \frac{(1+q)(1-t)}{2(1-qt)} p_1^2 + \frac{(1-q)(1+t)}{2(1-qt)} p_2, \quad (1.15)$$

$$M_{11}(x_1, \dots, x_N) = \frac{1}{2} p_1^2 - \frac{1}{2} p_2, \quad (1.16)$$

$$M_3(x_1, \dots, x_N) = \frac{(1-t)^2(1+q)(1+q+q^2)}{6(1-qt)(1-q^2t)} p_1^3 + \frac{(1-t^2)(1-q^3)}{2(1-qt)(1-q^2t)} p_1 p_2 + \frac{(1-q)(1-q^2)(1+t+t^2)}{3(1-q^2t)(1-qt)} p_3, \quad (1.17)$$

$$M_{21}(x_1, \dots, x_N) = \frac{(1-t)(2qt+q+t+2)}{6(1-qt^2)} p_1^3 + \frac{(1+t)(t-q)}{2(1-qt^2)} p_1 p_2 - \frac{(1+t+t^2)(1-q)}{3(1-qt^2)} p_3, \quad (1.18)$$

$$M_{111}(x_1, \dots, x_N) = \frac{1}{6} p_1^3 - \frac{1}{6} p_1 p_2 + \frac{1}{6} p_3, \quad (1.19)$$

where, again,  $p_k = x_1^k + \dots + x_N^k$  are the Newton power sums. Note that Macdonald polynomials are orthogonal, but not orthonormal: the quadratic norm of a Macdonald polynomial  $M_\lambda$ , denoted as  $g_\lambda$  in the definition above. Explicitly,  $g_\lambda$  can be expressed in two equivalent forms.

The first one is combinatorial:

$$g_\lambda = \text{const} \prod_{(i,j) \in \lambda} \frac{1 - t^{\lambda_j^T - i} q^{\lambda_i - j + 1}}{1 - t^{\lambda_j^T - i + 1} q^{\lambda_i - j}} \frac{1 - (t/q)t^{N-i} q^j}{1 - t^{N-i} q^j},$$

where  $\text{const}$  is an inessential normalization constant, the product goes over all boxes  $(i, j)$  of the Young diagram  $\lambda$  (namely,  $1 \leq j \leq \lambda_i$ ,  $1 \leq i \leq \text{length}(\lambda)$ ) and  $\lambda^T$  is the transposed diagram to  $\lambda$  (namely,  $(\lambda^T)_j =$  the number of entries  $\leq j$  in  $\lambda$ ). The second one is rather Lie-theoretical:

$$g_\lambda = N! \prod_{m=0}^{\beta-1} \prod_{\alpha>0} \frac{q^{-\frac{1}{2}(\alpha, \lambda^*)} t^{-\frac{1}{2}(\alpha, \rho)} q^{-\frac{m}{2}} - q^{\frac{1}{2}(\alpha, \lambda^*)} t^{\frac{1}{2}(\alpha, \rho)} q^{\frac{m}{2}}}{q^{-\frac{1}{2}(\alpha, \lambda^*)} t^{-\frac{1}{2}(\alpha, \rho)} q^{\frac{m}{2}} - q^{\frac{1}{2}(\alpha, \lambda^*)} t^{\frac{1}{2}(\alpha, \rho)} q^{-\frac{m}{2}}},$$

where the product goes over all positive roots  $\alpha$  of  $SU(N)$ <sup>1</sup> (namely, over  $N(N-1)/2$  vectors  $\alpha = e_I - e_J$ ,  $I < J$  where  $e_I$  are the basis vectors  $(e_I)_j = \delta_{I,j}$ ), Weyl vector  $\rho$  is the sum of all positive roots (namely,  $\rho_j = (N+1)/2 - j$ ) the bracket is just the simple Euclidean product (namely,  $(\alpha, v) = (e_I - e_J, v) = v_I - v_J$ ) and  $\lambda^*$  is the highest weight vector in representation  $\lambda$  of  $SU(N)$  (namely,  $\lambda_j^* = \lambda_j - |\lambda|/N$ ).

Each of the two forms has its own pros and cons: the first one is better suited for actual calculations, especially with the use of computers (since it is a finite product over the cells of the Young diagram, i.e. the range of the product does not involve  $N$ ) while the second one is somewhat more conceptual and directly generalizable to arbitrary root systems (i.e. other types of Lie groups). In addition, the second definition makes explicit the property of  $g_A$  being real: it remains invariant under substitution  $(q, t) \mapsto (q^{-1}, t^{-1})$  (which for roots of unity is the same as complex conjugation).

At this point let us note that, according to taste, some people might prefer to normalize the Macdonald polynomials over the square root of the norm, thus making them not only orthogonal, but actually orthonormal polynomials. Both choices – to normalize or not to normalize – also have their pros and cons. For normalized Macdonald polynomials, the structure of formulas is often simpler, and various symmetry properties are often more transparent. However, the normalized Macdonald polynomials themselves contain square roots, and this makes them unfavorable for actual computer calculations. In this paper we choose not to normalize Macdonald polynomials.

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<sup>1</sup>From now on, we prefer to work with  $SU(N)$  rather than  $U(N)$ . It goes without saying that the second form straightforwardly generalizes to arbitrary root systems.

### 1.3 Expansions

Various functions can be expanded in the basis of Macdonald polynomials. One of the most basic such expansions is for the bilinear exponential<sup>2</sup>:

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{1-t^k}{1-q^k} p_k \tilde{p}_k\right) = \sum_{\lambda} m_{\lambda} M_{\lambda}(p) M_{\lambda}(\tilde{p}), \quad (1.20)$$

which is well-known as (generalized) Cauchy-Stanley identity. Here

$$m_{\lambda} = \prod_{(i,j) \in \lambda} \frac{1 - t^{\lambda_j^T - i + 1} q^{\lambda_i - j}}{1 - t^{\lambda_j^T - i} q^{\lambda_i - j + 1}} = \lim_{N \rightarrow \infty} g_{\lambda}^{-1} \quad (1.21)$$

is a normalization factor. Another important expansion is that of a product of two Macdonald polynomials: it gives rise to a set of "structure constants"  $N_{\lambda\mu}^{\nu}$ , the generalized Littlewood-Richardson coefficients:

$$M_{\lambda}(p) M_{\mu}(p) = \sum_{\nu} N_{\lambda\mu}^{\nu} M_{\nu}(p). \quad (1.22)$$

A few first of these coefficients are:

$$\begin{aligned} N_{\square, \square}^{\square} &= 1, & N_{\square, \square}^{\square} &= \frac{(1+t)(1-q)}{(1-tq)}, \\ N_{\square, \square}^{\square\square} &= 1, & N_{\square, \square}^{\square\square} &= \frac{(1-qt^2)(1-q^2)}{(1-tq^2)(1-tq)}, & N_{\square, \square}^{\square\square} &= 0, \\ N_{\square, \square}^{\square\square\square} &= 0, & N_{\square, \square}^{\square\square\square} &= 1, & N_{\square, \square}^{\square\square\square} &= \frac{(1+t+t^2)(1-q)}{1-qt^2}. \end{aligned}$$

---

<sup>2</sup>Note that we are viewing the Macdonald polynomials here as functions of the Newton power sums  $p_1, p_2, p_3, \dots$ ; this point of view will be often convenient, and one can always substitute the explicit formulas for the Newton power sums to obtain the actual symmetric polynomials.



## 1.4 Specializations

Macdonald polynomials generalize several previously known simpler bases of orthogonal polynomials. If one puts  $t = q^\beta$  and then takes the limit  $q \rightarrow 1$ , one recovers the basis of Jack polynomials  $J_R$

$$\lim_{q \rightarrow 1} M_\lambda \Big|_{t=q^\beta} = J_\lambda \quad \text{associated with the measure} \quad \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right)^\beta. \quad (1.23)$$

If, further, one takes  $\beta = 1$ , one recovers the Schur polynomials  $\chi_R$

$$\lim_{q \rightarrow 1} M_\lambda \Big|_{t=q} = \chi_R \quad \text{associated with the measure} \quad \prod_{i < j} \left(1 - \frac{x_i}{x_j}\right). \quad (1.24)$$

Notably, for Schur polynomials there is no need to take the  $q \rightarrow 1$  limit: in fact,  $M_\lambda|_{t=q} = \chi_\lambda$  and does not depend on  $q$ . This is completely expectable, as for  $t = q$  we have  $\beta = 1$  and the Macdonald measure is indistinguishable from the Schur one. Other, even simpler, classes of symmetric polynomials can be also recovered as particular cases of the Macdonald ones: the monomial symmetric polynomials (corresponding to the case  $t = 1$ ) and the Hall-Littlewood polynomials (corresponding to the case  $q = 0$ ).

In certain points, Macdonald polynomials take simple values, such as

$$M_\lambda(t^\rho) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{q^{\frac{\lambda_j - \lambda_i}{2}} t^{\frac{i-j}{2}} q^{-\frac{m}{2}} - q^{\frac{\lambda_i - \lambda_j}{2}} t^{\frac{j-i}{2}} q^{\frac{m}{2}}}{t^{\frac{i-j}{2}} q^{-\frac{m}{2}} - t^{\frac{j-i}{2}} q^{\frac{m}{2}}}. \quad (1.25)$$

This is a generalization (refinement) of the well-known *quantum dimension* formula. A different formula for the same value is

$$M_\lambda(t^\rho) = t^{|\lambda^T|/2 - N|\lambda|/2} \prod_{(i,j) \in \lambda} \frac{1 - (t/q)t^{N-i}q^j}{1 - t^{\lambda_j^T - i + 1}q^{\lambda_i - j}}, \quad (1.26)$$

where the product goes over all boxes  $(i, j)$  of the Young diagram  $\lambda$ .

Finally, specialization  $N = 2$  is exceptionally simple and for this reason noteworthy. In this case, it is possible to give an explicit formula for a generic Macdonald polynomial:

$$M_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} \sum_{l=0}^{\lambda_1 + \lambda_2} \left( \frac{z_2}{z_1} \right)^l \prod_{i=0}^{l-1} \frac{[\lambda_1 + \lambda_2 - i]_q [i + \beta]_q}{[\lambda_1 + \lambda_2 - i + \beta - 1]_q [i + 1]_q}, \quad (1.27)$$

where

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \quad (1.28)$$

is a  $q$ -number. In general, for higher  $N$ , such explicit formulas get more complicated (in particular, they become multiple sums instead of a single sum) but any particular Macdonald polynomial can be always computed from the first principles, by requiring orthogonality with respect to the Macdonald measure and certain simple structure properties<sup>3</sup>.

Having described the basic properties of the Macdonald measure and corresponding multivariate orthogonal polynomials, let us move on to the most interesting part – applications of this general theory. As was explained above, one of the most important properties of a good special function is its usefulness in many different applications. One of the aims of this thesis is to demonstrate that, indeed, the Macdonald ensembles appear naturally in applications of very different nature. In Chapter 2, we will describe an application to a three-dimensional topological quantum field theory. In Chapter 3, we will consider an application to chiral vertex operator algebras and certain special functions called Nekrasov functions. We believe that this is just a tip of an iceberg, and more exciting applications are to come in the near future.

---

<sup>3</sup>To be precise, Macdonald polynomials are the unique basis of symmetric functions, orthogonal w.r.t. the Macdonald measure and upper triangular in the standard monomial basis.

## Chapter 2

# Macdonald ensembles and TQFT

The first application of Macdonald polynomials that we are going to consider in this thesis is topological quantum field theory (TQFT). In exposition we will follow Atiyah's functorial approach [21]. Let  $\mathbf{Bord}_S$  be a category of oriented closed two-dimensional surfaces  $\Sigma$ , where morphisms from  $\Sigma$  to  $\Sigma'$  are oriented compact three-dimensional manifolds with boundary  $(-\Sigma) \cup \Sigma'$  and several (possibly none) trivalent ribbon graphs inside, decorated by a given set of objects  $S$ . A three-dimensional TQFT is a monoidal functor

$$Z : \mathbf{Bord}_S \rightarrow \mathbf{Vect}(\mathbb{C}), \quad (2.1)$$

where the monoidal structure on the l.h.s. is given by the disjoint union of surfaces  $\cup$  and on the r.h.s. by the tensor product of vector spaces  $\otimes$ .

Much of the interest to TQFT's is motivated by the fact that they provide topological invariants. Indeed, since an empty surface  $\emptyset$  is the unit in the bordism category ( $\emptyset \cup \emptyset = \emptyset$ ) it has to be that  $Z(\emptyset) = \mathbb{C}$ . Consequently, since any closed 3-manifold with decorated trivalent graphs inside can be regarded as a bordism between  $\emptyset$  and  $\emptyset$ , the TQFT has to associate to such manifold a complex number. By definition of TQFT, this number is an ambient isotopy invariant of collections of graphs inside the 3-manifold.

To construct a TQFT means to specify a vector space for every oriented closed two-dimensional surface and to give a linear map for every bordism, in a way consistent with functoriality. A non-trivial example of such construction is the Reshetikhin-Turaev TQFT [22, 23, 24, 25, 26], which conjecturally

provides a quantization of the classical Chern-Simons field theory. The topological invariants of knots, provided by this TQFT, coincide with the colored HOMFLY polynomials of knots. In this chapter we first briefly remind the construction of Reshetikhin-Turaev TQFT, focusing mainly on the genus one sector, and then describe a two-parametric  $q,t$ -deformation of it, related to Macdonald polynomials. We conjecture (and provide ample evidence) that the deformed TQFT invariants of knots coincide with the colored knot superpolynomials – Poincaré polynomials of the triply graded knot homology theory described by [27]. In this thesis we only formulate the deformation in the genus 1 sector: generalization to higher genus will be considered elsewhere.

## 2.1 Reshetikhin-Turaev TQFT

In this TQFT, the set of decorating objects  $S$  consists of two parts, associated to edges and vertices, respectively. The edges are decorated by irreducible representations of the quantum group  $U_q(\mathfrak{sl}_N)$ , where  $q$  is the  $(K + N)$ -th primitive root of unity. Such representations are labeled by partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$  which fit into an  $N \times K$  box, i.e.  $\text{height}(\lambda) = \lambda_1 \leq K$  and  $\text{length}(\lambda) = \text{number of non-vanishing components} < N$ . It is customary to call this a rank  $N - 1$ , level  $K$  TQFT. The vertices are deco-

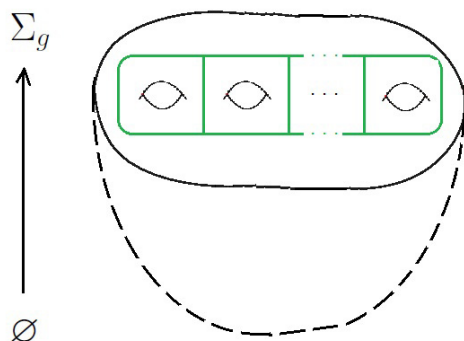


Figure 2.1: Basis vectors in the TQFT vector space.

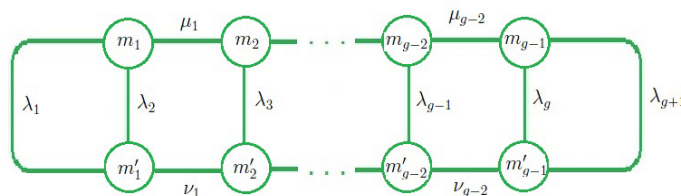


Figure 2.2: Decoration of graphs for the surface of generic genus  $g$ .

rated by intertwiners of the corresponding triple products of representations. Labeling intertwiners is a complicated combinatorial problem that we will not address explicitly: instead, we will only consider cases where there is a single intertwiner associated to every vertex, so that there is no need to label.

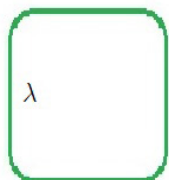


Figure 2.3: The case of the torus.

The construction of Reshetikhin-Turaev TQFT has been done in [22, 23, 24, 25] (see [26] for a comprehensive review); let us briefly describe some of its main points. To construct the vector spaces, associated by this TQFT to genus  $g$  surfaces, the following argument is useful. Consider a genus  $g$  handlebody, i.e. a solid filling of a genus  $g$  surface  $\Sigma_g$ , with any decorated trivalent ribbon graph inside. One can think of this 3-manifold as a bordism between an empty surface  $\emptyset$ , and  $\Sigma_g$ . This bordism is represented by a linear map from  $Z(\emptyset) = \mathbb{C}$  to  $Z(\Sigma_g)$ , giving a ray (one-dimensional subspace) in  $Z(\Sigma_g)$ . This allows to completely describe  $Z(\Sigma_g)$  as a span of these rays. Moreover, since in Reshetikhin-Turaev TQFT  $Z(\Sigma_g)$  is finite-dimensional, we can choose finitely many decorated trivalent ribbon graphs that will label basis vectors in  $Z(\Sigma_g)$ . One possible choice of such graphs is shown on Fig.(2.1).

The edges of these graphs have to be decorated by highest weights (partitions) and, in general, vertices have to be decorated by intertwiners, as shown on Fig.(2.2). There are exactly two cases when this complication can be avoided: first, the case of rank one ( $N = 2$ ), second, the case of genus one ( $g = 1$ ). In the first case the vector space of intertwiners is always one-dimensional. In the second case there are simply no trivalent vertices, and

the vector space is simply spanned by decorated unknots, see Fig. (2.3).

In what follows we restrict ourselves to  $g = 1$ , so the only components of boundary of 3-manifolds are tori. Despite being simple, this sector of the theory is non-trivial because, as we will see below, one can use it to compute quite a lot of knot and 3-manifold invariants. The central role in this computation is played by two particular bordisms, denoted  $S$  and  $T$ , that correspond to the automorphisms of the torus as shown on Fig.(2.4). The TQFT represents these bordisms by linear operators on the vector space  $Z(\Sigma_1)$ , that satisfy the relations of the genus 1 mapping class group,  $(ST)^3 = 1$  and  $S^4 = 1$ . By composing  $S$  and  $T$ , one can obtain an interesting class of bordisms which, in particular, contains the  $S^3$  with arbitrary torus knots.

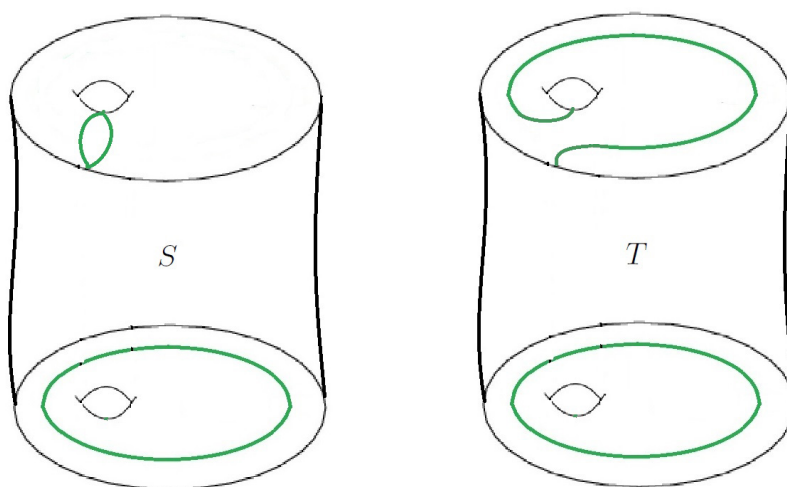


Figure 2.4: Two bordisms that come from automorphisms of the torus.

The explicit formulas for these linear operators are

$$\langle \lambda | S | \mu \rangle = S_{00} q^{-|\lambda||\mu|/N} \chi_\lambda(q^\rho) \chi_\mu(q^{\rho+\lambda}), \quad (2.2)$$

$$\langle \lambda | T | \mu \rangle = T_{00} (\sqrt{q})^{|\lambda|^2/N - \|\lambda\|} (\sqrt{q})^{\|\lambda^T\| - N|\lambda|} \delta_{\lambda\mu}, \quad (2.3)$$

where  $|\lambda\rangle$  denotes the basis vector on Fig. (2.3),  $\|\lambda\| = \sum_i \lambda_i^2$ ,  $\rho$  is a vector with components  $\rho_i = (N + 1)/2 - i$ , and  $\chi_\lambda$  are the Schur polynomials.

The normalization constants  $S_{00}$  and  $T_{00}$  ensure that the relations between  $S$  and  $T$  have the canonical  $SL(2, \mathbb{Z})$  form

$$(ST)^3 = 1, \quad S^4 = 1. \quad (2.4)$$

For generic choice of the normalization constants  $S_{00}$  and  $T_{00}$ , these relations are satisfied not literally, but only up to a scalar multiple, i.e.

$$(ST)^3 \propto 1, \quad S^4 \propto 1, \quad (2.5)$$

where  $\propto$  means that the matrices are equal up to multiplication by a scalar matrix. This is to be expected, as in general it is well-known [26] that the TQFT representation is only projective. In the case of a torus, however, there are exactly as many generators as relations – namely, two – so that it is possible to eliminate the scalar multiples from the relations by rescaling the generators. This would not be possible for higher genus.

## 2.2 Knot invariants

To compute the TQFT knot invariants, in addition to the representation of the mapping class group one also needs the *knot operators*  $O_\lambda(K)$ , that the TQFT functor associates to the bordisms inserting  $K$  colored by representation  $j$ . Fig. (2.5) illustrates this in the case of the unknot operator, that inserts the unknot along the  $A$ -cycle of the torus.

The explicit formula for this linear operator is

$$\langle \nu | O_\lambda | \mu \rangle = C_{\lambda\mu}^\nu, \quad (2.6)$$

where  $C_{\lambda,\mu}^\nu$  are the multiplication constants for the Schur polynomials,

$$\chi_\lambda(p)\chi_\mu(p) = \sum_\nu C_{\lambda\mu}^\nu \chi_\nu(p). \quad (2.7)$$

The operators that insert other torus knots can be obtained from the unknot operator by conjugation with the mapping class group operators:

$$O_j(K) = U_K O_j U_K^{-1}, \tag{2.8}$$

where  $U_K$  is a composition of operators  $S, T$  that transforms the  $A$ -cycle unknot into the knot  $K$ . Such an operator exists for any torus knot  $K$  because any torus knot can be obtained from the unknot by action of the mapping class group, which is generated by  $S$  and  $T$ .

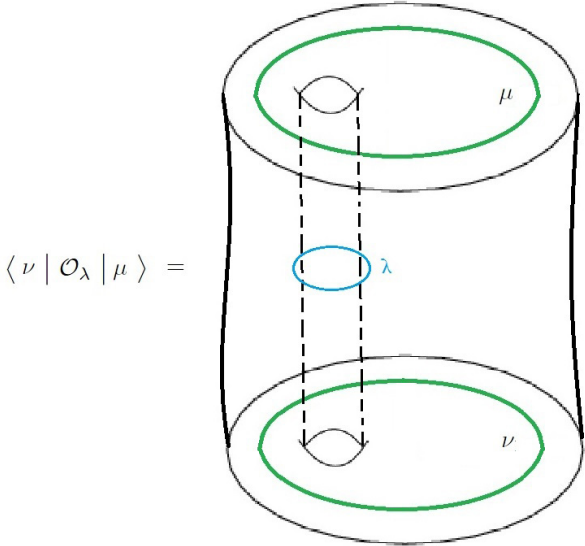


Figure 2.5: The operator that inserts an  $A$ -cycle unknot colored by  $\lambda$ .



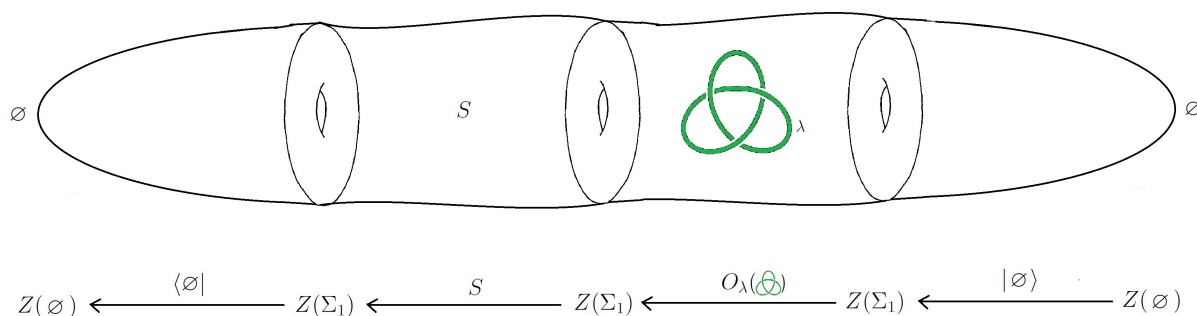


Figure 2.6: Heegaard splitting and associated contraction of TQFT operators.

Given the explicit formulas for the bordisms  $S$ ,  $T$  and  $O_\lambda$ , one can compute the TQFT invariants of any torus knots in  $S^3$  using the formula

$$Z_\lambda(K) = \langle \emptyset | S O_\lambda(K) | \emptyset \rangle, \quad (2.9)$$

that represents the geometric operation of gluing an  $S^3$  from two solid tori. One first takes a vector  $|\emptyset\rangle$  – the state corresponding to a solid torus with no knots inserted – then acts on it by the knot operator to insert a knot into it, and finally takes a scalar product with another vector  $S|\emptyset\rangle$  to glue in the second solid torus. Note that the boundaries of the solid tori are not glued identically, as this would not result in an  $S^3$ ; instead, they are glued with the help of the  $S$ -transformation. This way to obtain  $S^3$  is called Heegaard splitting [28], see Fig.(2.6).

The number  $Z_\lambda(K)$  is interesting primarily because it is a topological invariant of framed knots, i.e. for a pair of topologically equivalent knots  $K, K'$  the values of  $Z$  agree up to a framing factor:

$$K \equiv K' \implies \exists \alpha \text{ s.t. } Z_\lambda(K) = T_{\lambda\lambda}^\alpha Z_\lambda(K'). \quad (2.10)$$

It is known that the TQFT invariant coincides with the (unreduced,  $\lambda$ -colored) HOMFLY polynomial  $P_\lambda(a, q; K)$ , evaluated at  $q = e^{\frac{2\pi i}{k+N}}$ ,  $a = q^N$ .

## 2.3 Macdonald deformation

As one can see, there is a wide class of bordisms (including, in particular, the three-sphere with arbitrary torus knots inside) whose values in Reshetikhin-Turaev TQFT are completely determined by the values of three bordisms  $S, T$  and  $O_\lambda$ . The matrix elements of these three linear operators are given above in terms of the values of Schur polynomials at special points.

It is natural to ask – is there any deformation of these three operators, preserving their essential properties? This question was studied in [1], where we approached the problem from the statistical ensemble perspective, using the integral representation for the  $TST$  operator:

$$\begin{aligned} \langle \lambda | TST | \mu \rangle &= \\ &= \int_{-\infty}^{\infty} dx_1 \dots dx_N \prod_{i \neq j} (e^{x_i} - e^{x_j}) \chi_\lambda(e^{x_1}, \dots, e^{x_N}) \chi_\mu(e^{x_1}, \dots, e^{x_N}) \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2g}\right), \end{aligned} \quad (2.11)$$

where  $q = e^g$ . We then suggested, along the lines described in the Introduction, to deform/generalize this integral to the Macdonald ensemble,

$$\begin{aligned} \langle \lambda | TST | \mu \rangle &= \\ &= \int_{-\infty}^{\infty} dx_1 \dots dx_N \prod_{i \neq j} \prod_{m=0}^{\beta-1} (e^{x_i} - q^m e^{x_j}) M_\lambda(e^{x_1}, \dots, e^{x_N}) M_\mu(e^{x_1}, \dots, e^{x_N}) \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2g}\right), \end{aligned} \quad (2.12)$$

where  $q = e^g$  and  $t = e^{\beta g}$ . Evaluating the integral, we arrived at the following suggestion for deformation of the three TQFT linear operators:

$$\langle \lambda | S | \mu \rangle = S_{00} q^{-|\lambda||\mu|/N} M_\lambda(t^\rho) M_\mu(t^\rho q^\lambda), \quad (2.13)$$

$$\langle \lambda | T | \mu \rangle = T_{00} (\sqrt{q})^{|\lambda|^2/N - \|\lambda\|} (\sqrt{t})^{\|\lambda^T\| - N|\lambda|} \delta_{\lambda\mu}, \quad (2.14)$$

$$\langle \nu | O_\lambda | \mu \rangle = N_{\lambda\mu}^\nu, \quad (2.15)$$

where  $N_{\lambda,\mu}^\nu$  are the multiplication constants for the Macdonald polynomials, as described in the Introduction, and

$$q = e^{\frac{2\pi i}{K+\beta N}}, \quad t = e^{\frac{2\pi i\beta}{K+\beta N}}, \quad \forall \beta \in \mathbb{C}^*, \quad (2.16)$$

is the analogue of the fact that  $q$  in the undeformed TQFT is a root of unity. These operators share all essential properties of the original TQFT operators: for example, the deformed  $S$  and  $T$  matrices provide a representation of the mapping class group of the torus,

$$(ST)^3 = 1, \quad S^4 = 1, \quad (2.17)$$

and the deformed knot operator  $O_\lambda$  is still diagonalized by  $S$

$$S O_\lambda S^{-1} = \text{diagonal}, \quad (2.18)$$

in complete analogy with the known relation for the usual  $O_\lambda$ , originally described by Verlinde [29] and known as the Verlinde formula.

It is important to emphasize that eqs. (2.17), (2.18) have been known prior to [1]: in fact, eq. (2.17) has been proved by A.Kirillov-Jr. in [30], while the proof of eq. (2.18) is essentially unchanged from Verlinde's original proof, following directly from the fact that  $N_{\lambda\mu}^\nu$  are the multiplication constants for the Macdonald polynomials. The new step in [1] was not the identities themselves, but a new interpretation of these known identities, namely, as morphisms in some, yet unknown,  $q, t$ -deformed TQFT, which we suggested to call *refined Chern-Simons theory*.

In particular, in [1] we first suggested that the contraction

$$Z_\lambda(K) = \langle \emptyset | SO_\lambda(K) | \emptyset \rangle, \quad (2.19)$$

defined with the help of the deformed linear operators, represents some knot invariant. Moreover, direct computation and comparison to the literature allowed us to identify precisely which invariant it is, leading us to the following

**Conjecture.** For any torus knot  $K$  and any rectangular partition  $\lambda$ , the complex number  $Z_\lambda(K)/Z_\lambda(\bigcirc)$ , defined with the help of the deformed linear operators, where  $\bigcirc$  is the unknot, is equal to the reduced  $\lambda$ -colored superpolynomial  $P_\lambda(\mathbf{a}, \mathbf{q}, \mathbf{t}; K)$ , defined in [27] as a Poincare polynomial of the triply graded HOMFLY homology, evaluated at a specific point

$$\mathbf{q} = \sqrt{t}, \quad \mathbf{t} = -\sqrt{q/t}, \quad \mathbf{a}^2 = t^N \sqrt{t/q}, \quad (2.20)$$

up to an overall constant factor that does not depend on  $\mathbf{a}$ .

The conjecture we proposed is based upon many checks, and in the next section we give several examples of these.

## 2.4 Examples

One of the simplest examples is the trefoil  $K_{2,3}$ , which is a torus knot with winding numbers  $(2, 3)$ . In this case, the matrix  $U$  can be taken as  $U = S^{-1}T^{-2}S^{-1}T^{-2}$  since it is easy to check that  $U$  matrix takes the  $(1, 0)$ -cycle to the  $(2, 3)$ -cycle, as it should. The refined invariant is

$$Z_{\square}(K_{2,3}) = \langle \emptyset | SO_{\square}(K_{2,3}) | \emptyset \rangle = \langle \emptyset | S^{-1}T^{-2}O_{\square}T^2ST^2S | \emptyset \rangle, \quad (2.21)$$

where  $\square = [1]$  is the simplest non-empty partition, i.e. the highest weight of the fundamental representation. This is a finite sum of Macdonald polynomials, which can be efficiently evaluated with any computer algebra system. Using the refined  $S$  and  $T$  matrices, it is easy to compute that

$$\frac{Z_{\square}(K_{2,3})}{Z_{\square}(\bigcirc)} = t^{2-2N}q^{-3+3/N} + t^{3-2N}q^{-2+3/N} - t^{2-N}q^{-2+3/N}. \quad (2.22)$$

Making here a change of variables from  $N, t, q$  to

$$\mathbf{q} = \sqrt{t}, \quad \mathbf{t} = -\sqrt{q/t}, \quad \mathbf{a}^2 = t^N \sqrt{t/q}, \quad (2.23)$$

we find

$$\frac{Z_{\square}(K_{2,3})}{Z_{\square}(\bigcirc)} = \left(\frac{t}{q}\right)^{5/2} \frac{q^{3/N}}{t^{3N}} P_{\square}(\mathbf{a}, \mathbf{q}, \mathbf{t}; K_{2,3}), \quad (2.24)$$

where

$$P_{\square}(\mathbf{a}, \mathbf{q}, \mathbf{t}; K_{2,3}) = \mathbf{a}^2 \mathbf{q}^{-2} + \mathbf{a}^2 \mathbf{q}^2 \mathbf{t}^2 + \mathbf{a}^4 \mathbf{t}^3 \quad (2.25)$$

is the uncolored superpolynomial of the trefoil, see [27]. The prefactor here is easily related to framing: recall that

$$T_{\square} = \sqrt{\frac{t^{N+1}}{q^{1/N+1}}}, \quad (2.26)$$

so the prefactor corresponds to -6 units of framing

$$\left(\frac{t}{q}\right)^{5/2} \frac{q^{3/N}}{t^{3N}} = \sqrt{\frac{q}{t}} T_{\square}^{-6}, \quad (2.27)$$

modulo a  $\sqrt{q/t}$  factor, which vanishes in the unrefined case. This example is, of course, very simple; let us proceed to the more interesting cases, concentrating on the simplest non-empty partition  $\lambda = \square$ <sup>1</sup>. Since the results will soon become lengthy, from now on we switch to a more structured form of presenting the answers, by grouping the different terms w.r.t. their  $\mathbf{a}$ -degree. The trefoil case  $(n, m) = (2, 3)$  then takes form

$\mathbf{a}$ - degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4 \mathbf{t}^2$
$\mathbf{a}^2$	$\mathbf{q}^2 \mathbf{t}^3$

Note that from now on, for reader's convenience, we will normalize all polynomials to contain only non-negative monomials, and start with 1.

**The case**  $(n, m) = (2, 5)$

$\mathbf{a}$ - degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4 \mathbf{t}^2 + \mathbf{q}^8 \mathbf{t}^4$
$\mathbf{a}^2$	$\mathbf{q}^2 \mathbf{t}^3 + \mathbf{q}^6 \mathbf{t}^5$

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<sup>1</sup>Similar checks can be done for bigger partitions  $\lambda$ , we do not include them here.

The case  $(n, m) = (2, 7)$

$\mathbf{a}$ – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{12}\mathbf{t}^6$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^6\mathbf{t}^5 + \mathbf{q}^{10}\mathbf{t}^7$

The case  $(n, m) = (3, 4)$

$\mathbf{a}$ – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{12}\mathbf{t}^6$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^5 + \mathbf{q}^8\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^7$
$\mathbf{a}^4$	$\mathbf{t}^8\mathbf{q}^6$

The case  $(n, m) = (3, 5)$

$\mathbf{a}$ – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{10}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^6 + \mathbf{q}^{16}\mathbf{t}^8$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^5 + 2\mathbf{q}^8\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^7 + \mathbf{q}^{12}\mathbf{t}^9 + \mathbf{q}^{14}\mathbf{t}^9$
$\mathbf{a}^4$	$\mathbf{q}^6\mathbf{t}^8 + \mathbf{q}^{10}\mathbf{t}^{10}$

The case  $(n, m) = (3, 7)$

a – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{10}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^8 + \mathbf{q}^{14}\mathbf{t}^8 +$ $+\mathbf{q}^{16}\mathbf{t}^8 + \mathbf{q}^{18}\mathbf{t}^{10} + \mathbf{q}^{20}\mathbf{t}^{10} + \mathbf{q}^{24}\mathbf{t}^{12}$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^5 + 2\mathbf{q}^8\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^9 + 2\mathbf{q}^{12}\mathbf{t}^9 +$ $+\mathbf{q}^{14}\mathbf{t}^9 + \mathbf{q}^{14}\mathbf{t}^{11} + 2\mathbf{q}^{16}\mathbf{t}^{11} + \mathbf{q}^{18}\mathbf{t}^{11} + \mathbf{q}^{20}\mathbf{t}^{13} + \mathbf{q}^{22}\mathbf{t}^{13}$
$\mathbf{a}^4$	$\mathbf{q}^6\mathbf{t}^8 + \mathbf{q}^{10}\mathbf{t}^{10} + \mathbf{q}^{12}\mathbf{t}^{12} + \mathbf{q}^{14}\mathbf{t}^{12} + \mathbf{q}^{18}\mathbf{t}^{14}$

The case  $(n, m) = (3, 8)$

a – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{10}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^8 + \mathbf{q}^{14}\mathbf{t}^8 +$ $+\mathbf{q}^{16}\mathbf{t}^8 + \mathbf{q}^{16}\mathbf{t}^{10} + \mathbf{q}^{18}\mathbf{t}^{10} + \mathbf{q}^{20}\mathbf{t}^{10} + \mathbf{q}^{22}\mathbf{t}^{12} + \mathbf{q}^{24}\mathbf{t}^{12} + \mathbf{q}^{28}\mathbf{t}^{14}$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^5 + 2\mathbf{q}^8\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^9 + 2\mathbf{q}^{12}\mathbf{t}^9 + \mathbf{q}^{14}\mathbf{t}^9 + 2\mathbf{q}^{14}\mathbf{t}^{11} +$ $+2\mathbf{q}^{16}\mathbf{t}^{11} + \mathbf{q}^{18}\mathbf{t}^{11} + \mathbf{q}^{18}\mathbf{t}^{13} + 2\mathbf{q}^{20}\mathbf{t}^{13} + \mathbf{q}^{22}\mathbf{t}^{13} + \mathbf{q}^{24}\mathbf{t}^{15} + \mathbf{q}^{26}\mathbf{t}^{15}$
$\mathbf{a}^4$	$\mathbf{q}^6\mathbf{t}^8 + \mathbf{q}^{10}\mathbf{t}^{10} + \mathbf{q}^{12}\mathbf{t}^{12} + \mathbf{q}^{14}\mathbf{t}^{12} + \mathbf{q}^{16}\mathbf{t}^{14} + \mathbf{q}^{18}\mathbf{t}^{14} + \mathbf{q}^{22}\mathbf{t}^{16}$

The case  $(n, m) = (3, 10)$

a – degree	coefficient
$\mathbf{a}^0$	$1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^8\mathbf{t}^4 + \mathbf{q}^{10}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^6 + \mathbf{q}^{12}\mathbf{t}^8 + \mathbf{q}^{14}\mathbf{t}^8 +$ $+\mathbf{q}^{16}\mathbf{t}^8 + \mathbf{q}^{16}\mathbf{t}^{10} + \mathbf{q}^{18}\mathbf{t}^{10} + \mathbf{q}^{18}\mathbf{t}^{12} + \mathbf{q}^{20}\mathbf{t}^{10} + \mathbf{q}^{20}\mathbf{t}^{12} + \mathbf{q}^{22}\mathbf{t}^{12} +$ $+\mathbf{q}^{24}\mathbf{t}^{12} + \mathbf{q}^{24}\mathbf{t}^{14} + \mathbf{q}^{26}\mathbf{t}^{14} + \mathbf{q}^{28}\mathbf{t}^{14} + \mathbf{q}^{30}\mathbf{t}^{16} + \mathbf{q}^{32}\mathbf{t}^{16} + \mathbf{q}^{36}\mathbf{t}^{18}$
$\mathbf{a}^2$	$\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^5 + 2\mathbf{q}^8\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^7 + \mathbf{q}^{10}\mathbf{t}^9 + 2\mathbf{q}^{12}\mathbf{t}^9 + \mathbf{q}^{14}\mathbf{t}^9 + 2\mathbf{q}^{14}\mathbf{t}^{11} +$ $+2\mathbf{q}^{16}\mathbf{t}^{11} + \mathbf{q}^{16}\mathbf{t}^{13} + \mathbf{q}^{18}\mathbf{t}^{11} + 2\mathbf{q}^{18}\mathbf{t}^{13} + 2\mathbf{q}^{20}\mathbf{t}^{13} + \mathbf{q}^{20}\mathbf{t}^{15} + \mathbf{q}^{22}\mathbf{t}^{13} +$ $+2\mathbf{q}^{22}\mathbf{t}^{15} + 2\mathbf{q}^{24}\mathbf{t}^{15} + \mathbf{q}^{26}\mathbf{t}^{15} + \mathbf{q}^{26}\mathbf{t}^{17} + 2\mathbf{q}^{28}\mathbf{t}^{17} + \mathbf{q}^{30}\mathbf{t}^{17} + \mathbf{q}^{32}\mathbf{t}^{19} + \mathbf{q}^{34}\mathbf{t}^{19}$
$\mathbf{a}^4$	$\mathbf{q}^6\mathbf{t}^8 + \mathbf{q}^{10}\mathbf{t}^{10} + \mathbf{q}^{12}\mathbf{t}^{12} + \mathbf{q}^{14}\mathbf{t}^{12} + \mathbf{q}^{16}\mathbf{t}^{14} + \mathbf{q}^{18}\mathbf{t}^{14} + \mathbf{q}^{18}\mathbf{t}^{16} +$ $+\mathbf{q}^{20}\mathbf{t}^{16} + \mathbf{q}^{22}\mathbf{t}^{16} + \mathbf{q}^{24}\mathbf{t}^{18} + \mathbf{q}^{26}\mathbf{t}^{18} + \mathbf{q}^{30}\mathbf{t}^{20}$



**The case**  $(n, m) = (3, 11)$

a – degree	coefficient
$a^0$	$1 + q^4 t^2 + q^6 t^4 + q^8 t^4 + q^{10} t^6 + q^{12} t^6 + q^{12} t^8 + q^{14} t^8 + q^{16} t^8 + q^{16} t^{10} +$ $+ q^{18} t^{10} + q^{18} t^{12} + q^{20} t^{10} + q^{20} t^{12} + q^{22} t^{12} + q^{22} t^{14} + q^{24} t^{12} + q^{24} t^{14} +$ $+ q^{26} t^{14} + q^{28} t^{14} + q^{28} t^{16} + q^{30} t^{16} + q^{32} t^{16} + q^{34} t^{18} + q^{36} t^{18} + q^{40} t^{20}$
$a^2$	$q^2 t^3 + q^4 t^5 + q^6 t^5 + 2q^8 t^7 + q^{10} t^7 + q^{10} t^9 + 2q^{12} t^9 + q^{14} t^9 + 2q^{14} t^{11} +$ $+ 2q^{16} t^{11} + q^{16} t^{13} + q^{18} t^{11} + 2q^{18} t^{13} + 2q^{20} t^{13} + 2q^{20} t^{15} + q^{22} t^{13} +$ $+ 2q^{22} t^{15} + 2q^{24} t^{15} + q^{24} t^{17} + q^{26} t^{15} + 2q^{26} t^{17} + 2q^{28} t^{17} + q^{30} t^{17} +$ $+ q^{30} t^{19} + 2q^{32} t^{19} + q^{34} t^{19} + q^{36} t^{21} + q^{38} t^{21}$
$a^4$	$q^6 t^8 + q^{10} t^{10} + q^{12} t^{12} + q^{14} t^{12} + q^{16} t^{14} + q^{18} t^{14} + q^{18} t^{16} + q^{20} t^{16} +$ $+ q^{22} t^{16} + q^{22} t^{18} + q^{24} t^{18} + q^{26} t^{18} + q^{28} t^{20} + q^{30} t^{20} + q^{34} t^{22}$

**The case**  $(n, m) = (4, 5)$

a – degree	coefficient
$a^0$	$1 + q^4 t^2 + q^6 t^4 + q^8 t^4 + q^8 t^6 + q^{10} t^6 + q^{12} t^6 + q^{12} t^8 +$ $+ q^{14} t^8 + q^{16} t^8 + q^{16} t^{10} + q^{18} t^{10} + q^{20} t^{10} + q^{24} t^{12}$
$a^2$	$q^2 t^3 + q^4 t^5 + q^6 t^5 + q^6 t^7 + 2q^8 t^7 + q^{10} t^7 + 2q^{10} t^9 + 2q^{12} t^9 + q^{12} t^{11} +$ $+ q^{14} t^9 + 2q^{14} t^{11} + 2q^{16} t^{11} + q^{18} t^{11} + q^{18} t^{13} + q^{20} t^{13} + q^{22} t^{13}$
$a^4$	$q^6 t^8 + q^8 t^{10} + q^{10} t^{10} + q^{10} t^{12} + q^{12} t^{12} + q^{14} t^{12} + q^{14} t^{14} + q^{16} t^{14} + q^{18} t^{14}$
$a^6$	$t^{15} q^{12}$

The case  $(n, m) = (4, 7)$

a – degree	coefficient
$a^0$	$1 + q^4t^2 + q^6t^4 + q^8t^4 + q^8t^6 + q^{10}t^6 + q^{12}t^6 + 2q^{12}t^8 + q^{14}t^8 + q^{14}t^{10} + q^{16}t^8 +$ $+2q^{16}t^{10} + q^{18}t^{10} + q^{18}t^{12} + q^{20}t^{10} + 2q^{20}t^{12} + q^{22}t^{12} + q^{22}t^{14} + q^{24}t^{12} +$ $+2q^{24}t^{14} + q^{26}t^{14} + q^{28}t^{14} + q^{28}t^{16} + q^{30}t^{16} + q^{32}t^{16} + q^{36}t^{18}$
$a^2$	$q^2t^3 + q^4t^5 + q^6t^5 + q^6t^7 + 2q^8t^7 + q^{10}t^7 + 3q^{10}t^9 + 2q^{12}t^9 + 2q^{12}t^{11} + q^{14}t^9 +$ $+4q^{14}t^{11} + 2q^{16}t^{11} + 3q^{16}t^{13} + q^{18}t^{11} + 4q^{18}t^{13} + q^{18}t^{15} + 2q^{20}t^{13} + 3q^{20}t^{15} +$ $+q^{22}t^{13} + 4q^{22}t^{15} + 2q^{24}t^{15} + 2q^{24}t^{17} + q^{26}t^{15} + 3q^{26}t^{17} + 2q^{28}t^{17} + q^{30}t^{17} +$ $+q^{30}t^{19} + q^{32}t^{19} + q^{34}t^{19}$
$a^4$	$q^6t^8 + q^8t^{10} + q^{10}t^{10} + q^{10}t^{12} + 2q^{12}t^{12} + q^{14}t^{12} + 3q^{14}t^{14} + 2q^{16}t^{14} + q^{16}t^{16} +$ $+q^{18}t^{14} + 3q^{18}t^{16} + 2q^{20}t^{16} + q^{20}t^{18} + q^{22}t^{16} + 3q^{22}t^{18} + 2q^{24}t^{18} + q^{26}t^{18} +$ $+q^{26}t^{20} + q^{28}t^{20} + q^{30}t^{20}$
$a^6$	$q^{12}t^{15} + q^{16}t^{17} + q^{18}t^{19} + q^{20}t^{19} + q^{24}t^{21}$

The case  $(n, m) = (4, 9)$

a – degree	coefficient
$a^0$	$1 + q^4t^2 + q^6t^4 + q^8t^4 + q^8t^6 + q^{10}t^6 + q^{12}t^6 + 2q^{12}t^8 + q^{14}t^8 + q^{14}t^{10} +$ $+q^{16}t^8 + 2q^{16}t^{10} + q^{16}t^{12} + q^{18}t^{10} + 2q^{18}t^{12} + q^{20}t^{10} + 2q^{20}t^{12} + q^{20}t^{14} +$ $+q^{22}t^{12} + 2q^{22}t^{14} + q^{24}t^{12} + 2q^{24}t^{14} + 2q^{24}t^{16} + q^{26}t^{14} + 2q^{26}t^{16} + q^{28}t^{14} +$ $+2q^{28}t^{16} + q^{28}t^{18} + q^{30}t^{16} + 2q^{30}t^{18} + q^{32}t^{16} + 2q^{32}t^{18} + q^{32}t^{20} + q^{34}t^{18} +$ $+q^{34}t^{20} + q^{36}t^{18} + 2q^{36}t^{20} + q^{38}t^{20} + q^{40}t^{20} + q^{40}t^{22} + q^{42}t^{22} + q^{44}t^{22} + q^{48}t^{24}$
$a^2$	$q^2t^3 + q^4t^5 + q^6t^5 + q^6t^7 + 2q^8t^7 + q^{10}t^7 + 3q^{10}t^9 + 2q^{12}t^9 + 2q^{12}t^{11} + q^{14}t^9 +$ $+4q^{14}t^{11} + q^{14}t^{13} + 2q^{16}t^{11} + 4q^{16}t^{13} + q^{18}t^{11} + 4q^{18}t^{13} + 3q^{18}t^{15} + 2q^{20}t^{13} +$ $+5q^{20}t^{15} + q^{22}t^{13} + q^{20}t^{17} + 4q^{22}t^{15} + 4q^{22}t^{17} + 2q^{24}t^{15} + 5q^{24}t^{17} + q^{26}t^{15} +$ $+q^{24}t^{19} + 4q^{26}t^{17} + 4q^{26}t^{19} + 2q^{28}t^{17} + 5q^{28}t^{19} + q^{30}t^{17} + q^{28}t^{21} + 4q^{30}t^{19} +$ $+3q^{30}t^{21} + 2q^{32}t^{19} + 4q^{32}t^{21} + q^{34}t^{19} + 4q^{34}t^{21} + q^{34}t^{23} + 2q^{36}t^{21} + 2q^{36}t^{23} +$ $+q^{38}t^{21} + 3q^{38}t^{23} + 2q^{40}t^{23} + q^{42}t^{23} + q^{42}t^{25} + q^{44}t^{25} + q^{46}t^{25}$
$a^4$	$q^6t^8 + q^8t^{10} + q^{10}t^{10} + q^{10}t^{12} + 2q^{12}t^{12} + q^{14}t^{12} + 3q^{14}t^{14} + 2q^{16}t^{14} + 2q^{16}t^{16} +$ $+q^{18}t^{14} + 4q^{18}t^{16} + q^{18}t^{18} + 2q^{20}t^{16} + 3q^{20}t^{18} + q^{22}t^{16} + 4q^{22}t^{18} + 2q^{22}t^{20} +$ $+2q^{24}t^{18} + 4q^{24}t^{20} + q^{26}t^{18} + 4q^{26}t^{20} + 2q^{26}t^{22} + 2q^{28}t^{20} + 3q^{28}t^{22} + q^{30}t^{20} +$ $+4q^{30}t^{22} + q^{30}t^{24} + 2q^{32}t^{22} + 2q^{32}t^{24} + q^{34}t^{22} + 3q^{34}t^{24} + 2q^{36}t^{24} + q^{38}t^{24} +$ $+q^{38}t^{26} + q^{40}t^{26} + q^{42}t^{26}$
$a^6$	$q^{12}t^{15} + q^{16}t^{17} + q^{18}t^{19} + q^{20}t^{19} + q^{20}t^{21} + q^{22}t^{21} + q^{24}t^{21} + q^{24}t^{23} + q^{26}t^{23} +$ $+q^{28}t^{23} + q^{28}t^{25} + q^{30}t^{25} + q^{32}t^{25} + q^{36}t^{27}$

The case  $(n, m) = (4, 11)$

a – degree	coefficient
$a^0$	$1 + q^4t^2 + q^6t^4 + q^8t^4 + q^8t^6 + q^{10}t^6 + q^{12}t^6 + 2q^{12}t^8 + q^{14}t^8 + q^{14}t^{10} +$ $+q^{16}t^8 + 2q^{16}t^{10} + q^{16}t^{12} + q^{18}t^{10} + 2q^{18}t^{12} + q^{20}t^{10} + 2q^{20}t^{12} + 2q^{20}t^{14} +$ $+q^{22}t^{12} + 2q^{22}t^{14} + q^{24}t^{12} + q^{22}t^{16} + 2q^{24}t^{14} + 3q^{24}t^{16} + q^{26}t^{14} + 2q^{26}t^{16} +$ $+q^{28}t^{14} + q^{26}t^{18} + 2q^{28}t^{16} + 3q^{28}t^{18} + q^{30}t^{16} + 2q^{30}t^{18} + q^{32}t^{16} + 2q^{30}t^{20} +$ $+2q^{32}t^{18} + 3q^{32}t^{20} + q^{34}t^{18} + 2q^{34}t^{20} + q^{36}t^{18} + q^{34}t^{22} + 2q^{36}t^{20} + 3q^{36}t^{22} +$ $+q^{38}t^{20} + 2q^{38}t^{22} + q^{40}t^{20} + q^{38}t^{24} + 2q^{40}t^{22} + 2q^{40}t^{24} + q^{42}t^{22} + 2q^{42}t^{24} +$ $+q^{44}t^{22} + 2q^{44}t^{24} + q^{44}t^{26} + q^{46}t^{24} + q^{46}t^{26} + q^{48}t^{24} + 2q^{48}t^{26} + q^{50}t^{26} +$ $+q^{52}t^{26} + q^{52}t^{28} + q^{54}t^{28} + q^{56}t^{28} + q^{60}t^{30}$
$a^2$	$q^2t^3 + q^4t^5 + q^6t^5 + q^6t^7 + 2q^8t^7 + q^{10}t^7 + 3q^{10}t^9 + 2q^{12}t^9 + 2q^{12}t^{11} + q^{14}t^9 +$ $+4q^{14}t^{11} + q^{14}t^{13} + 2q^{16}t^{11} + 4q^{16}t^{13} + q^{18}t^{11} + 4q^{18}t^{13} + 4q^{18}t^{15} + 2q^{20}t^{13} +$ $+5q^{20}t^{15} + q^{22}t^{13} + 2q^{20}t^{17} + 4q^{22}t^{15} + 6q^{22}t^{17} + 2q^{24}t^{15} + 5q^{24}t^{17} + q^{26}t^{15} +$ $+4q^{24}t^{19} + 4q^{26}t^{17} + 7q^{26}t^{19} + 2q^{28}t^{17} + q^{26}t^{21} + 5q^{28}t^{19} + q^{30}t^{17} + 5q^{28}t^{21} +$ $+4q^{30}t^{19} + 7q^{30}t^{21} + 2q^{32}t^{19} + q^{30}t^{23} + 5q^{32}t^{21} + q^{34}t^{19} + 5q^{32}t^{23} + 4q^{34}t^{21} +$ $+7q^{34}t^{23} + 2q^{36}t^{21} + q^{34}t^{25} + 5q^{36}t^{23} + q^{38}t^{21} + 4q^{36}t^{25} + 4q^{38}t^{23} + 6q^{38}t^{25} +$ $+2q^{40}t^{23} + 5q^{40}t^{25} + q^{42}t^{23} + 2q^{40}t^{27} + 4q^{42}t^{25} + 4q^{42}t^{27} + 2q^{44}t^{25} + 4q^{44}t^{27} +$ $+q^{46}t^{25} + 4q^{46}t^{27} + q^{46}t^{29} + 2q^{48}t^{27} + 2q^{48}t^{29} + q^{50}t^{27} + 3q^{50}t^{29} + 2q^{52}t^{29} +$ $+q^{54}t^{29} + q^{54}t^{31} + q^{56}t^{31} + q^{58}t^{31}$
$a^4$	$q^6t^8 + q^8t^{10} + q^{10}t^{10} + q^{10}t^{12} + 2q^{12}t^{12} + q^{14}t^{12} + 3q^{14}t^{14} + 2q^{16}t^{14} + 2q^{16}t^{16} +$ $+q^{18}t^{14} + 4q^{18}t^{16} + q^{18}t^{18} + 2q^{20}t^{16} + 4q^{20}t^{18} + q^{22}t^{16} + 4q^{22}t^{18} + 4q^{22}t^{20} +$ $+2q^{24}t^{18} + 5q^{24}t^{20} + q^{26}t^{18} + q^{24}t^{22} + 4q^{26}t^{20} + 5q^{26}t^{22} + 2q^{28}t^{20} + 5q^{28}t^{22} +$ $+q^{30}t^{20} + 2q^{28}t^{24} + 4q^{30}t^{22} + 6q^{30}t^{24} + 2q^{32}t^{22} + 5q^{32}t^{24} + q^{34}t^{22} + 2q^{32}t^{26} +$ $+4q^{34}t^{24} + 5q^{34}t^{26} + 2q^{36}t^{24} + 5q^{36}t^{26} + q^{38}t^{24} + q^{36}t^{28} + 4q^{38}t^{26} + 4q^{38}t^{28} +$ $+2q^{40}t^{26} + 4q^{40}t^{28} + q^{42}t^{26} + 4q^{42}t^{28} + q^{42}t^{30} + 2q^{44}t^{28} + 2q^{44}t^{30} + q^{46}t^{28} +$ $+3q^{46}t^{30} + 2q^{48}t^{30} + q^{50}t^{30} + q^{50}t^{32} + q^{52}t^{32} + q^{54}t^{32}$
$a^6$	$q^{12}t^{15} + q^{16}t^{17} + q^{18}t^{19} + q^{20}t^{19} + q^{20}t^{21} + q^{22}t^{21} + q^{24}t^{21} + 2q^{24}t^{23} +$ $+q^{26}t^{23} + q^{26}t^{25} + q^{28}t^{23} + 2q^{28}t^{25} + q^{30}t^{25} + q^{30}t^{27} + q^{32}t^{25} +$ $+2q^{32}t^{27} + q^{34}t^{27} + q^{34}t^{29} + q^{36}t^{27} + 2q^{36}t^{29} + q^{38}t^{29} + q^{40}t^{29} +$ $+q^{40}t^{31} + q^{42}t^{31} + q^{44}t^{31} + q^{48}t^{33}$

The case  $(n, m) = (5, 6)$

a - degree	coefficient
$a^0$	$1 + q^4 t^2 + q^6 t^4 + q^8 t^4 + q^8 t^6 + q^{10} t^6 + q^{10} t^8 + q^{12} t^6 + 2q^{12} t^8 + q^{14} t^8 +$ $+ q^{14} t^{10} + q^{16} t^8 + 2q^{16} t^{10} + q^{16} t^{12} + q^{18} t^{10} + 2q^{18} t^{12} + q^{20} t^{10} + 2q^{20} t^{12} +$ $+ q^{20} t^{14} + q^{22} t^{12} + 2q^{22} t^{14} + q^{24} t^{12} + 2q^{24} t^{14} + q^{24} t^{16} + q^{26} t^{14} + q^{26} t^{16} +$ $+ q^{28} t^{14} + 2q^{28} t^{16} + q^{30} t^{16} + q^{30} t^{18} + q^{32} t^{16} + q^{32} t^{18} +$ $+ q^{34} t^{18} + q^{36} t^{18} + q^{40} t^{20}$
$a^2$	$q^2 t^3 + q^4 t^5 + q^6 t^5 + q^6 t^7 + 2q^8 t^7 + q^8 t^9 + q^{10} t^7 + 3q^{10} t^9 + 2q^{12} t^9 + 3q^{12} t^{11} +$ $+ q^{14} t^9 + 4q^{14} t^{11} + 2q^{14} t^{13} + 2q^{16} t^{11} + 4q^{16} t^{13} + q^{18} t^{11} + q^{16} t^{15} + 4q^{18} t^{13} +$ $+ 3q^{18} t^{15} + 2q^{20} t^{13} + 5q^{20} t^{15} + q^{22} t^{13} + q^{20} t^{17} + 4q^{22} t^{15} + 3q^{22} t^{17} +$ $+ 2q^{24} t^{15} + 4q^{24} t^{17} + q^{26} t^{15} + q^{24} t^{19} + 4q^{26} t^{17} + 2q^{26} t^{19} + 2q^{28} t^{17} +$ $+ 3q^{28} t^{19} + q^{30} t^{17} + 3q^{30} t^{19} + 2q^{32} t^{19} + q^{32} t^{21} + q^{34} t^{19} +$ $+ q^{34} t^{21} + q^{36} t^{21} + q^{38} t^{21}$
$a^4$	$q^6 t^8 + q^8 t^{10} + q^{10} t^{10} + 2q^{10} t^{12} + 2q^{12} t^{12} + q^{12} t^{14} + q^{14} t^{12} + 3q^{14} t^{14} +$ $+ q^{14} t^{16} + 2q^{16} t^{14} + 3q^{16} t^{16} + q^{18} t^{14} + 4q^{18} t^{16} + 2q^{18} t^{18} + 2q^{20} t^{16} +$ $+ 3q^{20} t^{18} + q^{22} t^{16} + q^{20} t^{20} + 4q^{22} t^{18} + 2q^{22} t^{20} + 2q^{24} t^{18} + 3q^{24} t^{20} +$ $+ q^{26} t^{18} + 3q^{26} t^{20} + q^{26} t^{22} + 2q^{28} t^{20} + q^{28} t^{22} + q^{30} t^{20} +$ $+ 2q^{30} t^{22} + q^{32} t^{22} + q^{34} t^{22}$
$a^6$	$q^{12} t^{15} + q^{14} t^{17} + q^{16} t^{17} + q^{16} t^{19} + q^{18} t^{19} + q^{18} t^{21} + q^{20} t^{19} + q^{20} t^{21} +$ $+ q^{22} t^{21} + q^{22} t^{23} + q^{24} t^{21} + q^{24} t^{23} + q^{26} t^{23} + q^{28} t^{23}$
$a^8$	$t^{24} q^{20}$

All of these results match the corresponding uncolored (i.e. colored by the partition  $\lambda = \square = [1]$ ) superpolynomials of torus knots, that can be found in [27]. For example, the triply graded knot homology of the trefoil, according to [27], is given by the following diagram:

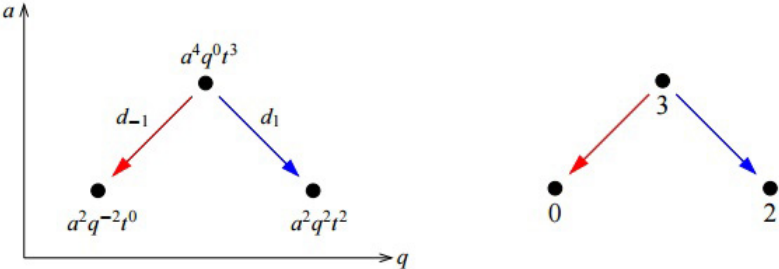


Figure 2.7: Triply graded homology for the  $(2, 3)$  torus knot. Picture courtesy of [27], see also notations therein.

Note how all the coefficients of the above polynomials are positive integers. This is to be expected from a Poincare polynomial of any knot homology theory – its coefficients are the graded Betti numbers, i.e. the dimensions of graded components of the knot homology groups, and hence positive integers.

To conclude, in this chapter we showed that Macdonald ensembles give a non-trivial deformation of TQFT invariants of torus knots, connected to knot homology. It would be interesting to generalize this to higher genus, to see if refined Chern-Simons theory actually exists as a full TQFT in the sense of Atiyah [21].

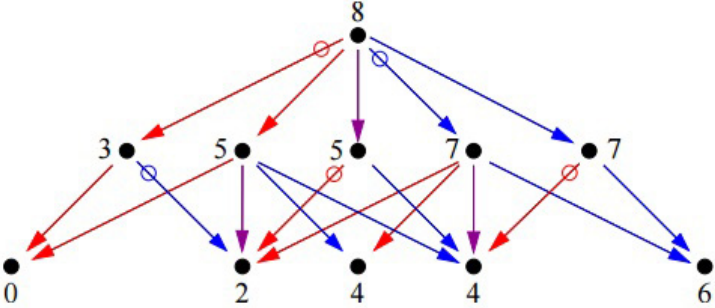


Figure 2.8: Triply graded homology for the  $(3, 4)$  torus knot. Picture courtesy of [27], see also notations therein.

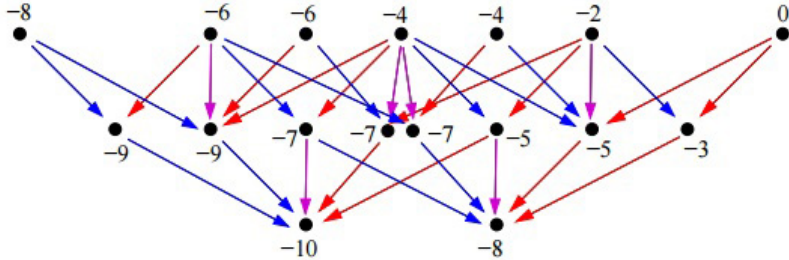


Figure 2.9: Triply graded homology for the  $(3, 5)$  torus knot. Picture courtesy of [27], see also notations therein.



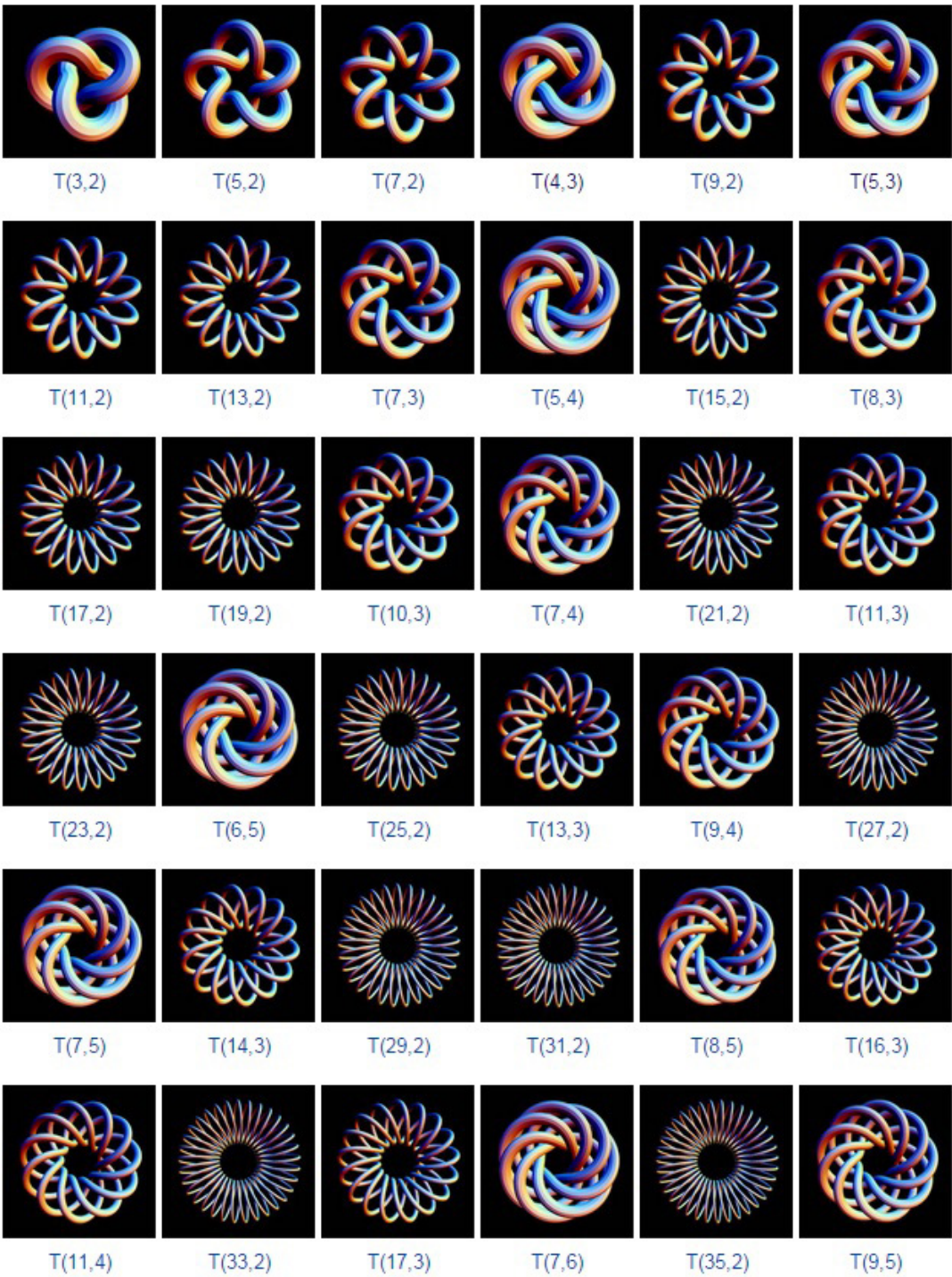


Figure 2.10: Several first torus knots. Picture courtesy of the Knot Atlas.

## Chapter 3

# Macdonald ensembles and W-algebras

The second application of Macdonald polynomials that we are going to consider in this thesis is chiral vertex operator algebras. These are the algebras generated by operators that naturally appear in two-dimensional conformal field theories and their generalizations. In this thesis we will consider a specific algebra, the quantum W-algebra of type  $gl_n$ , denoted  $W_{q,t}(gl_n)$ . This algebra has been constructed by Reshetikhin and Frenkel in [31], and can be described as the algebra of operators that commute with the given set of *screening current* operators given by, for  $a = 1, \dots, n - 1$ ,

$$\begin{aligned}
 S_a(z) = & : \exp \left( - \sum_{k>0} \frac{1-t^k}{1-q^k} h_k^a \frac{z^{-k}}{k} + \sum_{k>0} h_{-k}^a \frac{z^k}{k} \right) : \\
 & \times : \exp \left( \sum_{k>0} \frac{1-t^k}{1-q^k} v^k h_k^{a+1} \frac{z^{-k}}{k} - \sum_{k>0} v^k h_{-k}^{a+1} \frac{z^k}{k} \right) :, \quad (3.1)
 \end{aligned}$$

where  $h_k^a$  are the standard generators of  $n$  Heisenberg algebras<sup>1</sup>,

$$[h_k^a, h_m^b] = k \delta^{a,b} \delta_{k,-m}, \quad (3.2)$$

with  $a, b = 1, \dots, n$ ;  $k, m \in \mathbb{Z}$  and  $:(\dots):$  is the Heisenberg normal ordering.

---

<sup>1</sup>Note a difference in conventions between this thesis and [31]: we choose to work with the  $n$  Heisenberg generators  $h_k^a$  corresponding to simple roots of  $gl_n$ , while [31] works with  $n - 1$  generators corresponding to simple roots of  $sl_n$ . The two are related by a linear transformation.

### 3.1 Macdonald ensembles from W-algebras

In this thesis we will not use the explicit form of the generators of  $W_{q,t}(gl_n)$ , only the screening charges. One reason for this is that explicit form of the generators of  $W_{q,t}(gl_n)$  is not necessary to establish a relation to Macdonald ensembles. To see this relation it suffices to compute the two-point function

$$\langle S_a(u), S_a(v) \rangle = \frac{\varphi(u/v) \varphi(v/u)}{\varphi(tu/v) \varphi(tv/u)}, \quad (3.3)$$

where

$$\varphi(x) = \prod_{m=0}^{\infty} (1 - q^m x) \quad (3.4)$$

is a special function called *quantum dilogarithm*. If  $t = q^\beta$  with positive integer  $\beta$ , then the two-point function can be rewritten as

$$\langle S_a(u), S_a(v) \rangle = \prod_{m=0}^{\beta-1} \left(1 - q^m \frac{u}{v}\right) \left(1 - q^m \frac{v}{u}\right), \quad (3.5)$$

and is precisely the Macdonald measure with two variables  $u, v$ . To promote these variables into a statistical ensemble, one can introduce *screening charge* operators, which are integrals of the screening currents

$$Q_a = \oint dz S_a(z), \quad (3.6)$$

where the integral is taken over an appropriate integration contour in the complex plane. With this definition, a two-point function of two screening charges is a simple Macdonald ensemble – with two variables and no potential. Let us now describe, how to obtain general Macdonald ensembles.

## 3.2 Conformal blocks

The example from the previous section can be generalized to general Macdonald ensembles. To increase the number of variables, one should consider multi-point functions of several screening charges. To include potential factors, one should include more general operators called *vertex operators*. The resulting ensemble is known as the conformal (or chiral) block of  $W_{q,t}(gl_n)$ . Following [2], we will employ the following main

**Definition.** The conformal block of  $W_{q,t}(gl_n)$  is an expectation value

$$B(z_1, \dots, z_k) = \left\langle V_{\alpha_1}(z_1) \dots V_{\alpha_k}(z_k) Q_{(1)}^{N_1} \dots Q_{(n)}^{N_n} \right\rangle, \quad (3.7)$$

where  $V_\alpha(z)$  are the vertex operators, in the form

$$V_\alpha(z) =: \exp\left(\sum_{k>0} \sum_{a=1}^n \frac{1}{1-q^k} h_k^a q^{k\alpha_a} \frac{z^{-k}}{k} + \sum_{k>0} \sum_{a=1}^{n+1} \frac{q^k}{1-t^k} h_{-k}^a q^{-k\alpha_a} \frac{z^k}{k}\right) :, \quad (3.8)$$

with  $\alpha = (\alpha_1, \dots, \alpha_n)^2$ . Evaluating the block using the Wick theorem for the Heisenberg algebra, we find that the result is only non-zero<sup>3</sup>, if

$$\alpha_{1,a} + \dots + \alpha_{k,a} + \beta N_a = 0, \quad a = 1, \dots, n \quad (3.9)$$

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<sup>2</sup>Note a difference in conventions between this thesis and [31]: the vertex operators we consider are labeled by an arbitrary highest weight  $\alpha$  of  $gl_n$ , while the vertex operators of [31] are labeled by a single index  $i = 1, \dots, n$ . As explained in [31], the vertex operators we consider agree with the vertex operators of [31] for the case of  $i$ -th fundamental weight  $\alpha = (1^i, 0^{n-i})$ . More general weights  $\alpha$  can be obtained by fusion of these elementary operators.

<sup>3</sup>To simplify the presentation, here we omit the zero-mode Heisenberg generators  $\hat{p}, \hat{q}$  that enter the vertex operator as a prefactor  $\exp(\hat{q} + \hat{p} \log z)$ . We do not need to keep detailed track of these generators here, since they are only responsible for the conservation condition below, and have essentially no effect on the two-point functions.

what is known as the charge conservation condition. If this condition is satisfied, then the block is non-vanishing and is given by a multiple integral,

$$B(z_1, \dots, z_k) = \int dX_1 \dots \int dX_{n-1} J(X_1, \dots, X_{n-1}), \quad (3.10)$$

where each of the  $x_a$  is a group of  $N_a$  variables,

$$X_a = \{ x_{a,1}, \dots, x_{a,N_a} \}, \quad (3.11)$$

and the integrand is given by the product of two-point functions<sup>4</sup>,

$$J(X_1, \dots, X_{n-1}) = \text{const} \prod_{a=1}^{n-1} \prod_{i=1}^k J_a(X_a; z_i, \alpha_i) \prod_{1 \leq a \leq b < n} J_{a,b}(X_a, X_b), \quad (3.12)$$

with

$$J_{a,b}(X_a, X_b) = \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \langle S_a(x_{a,i}) S_b(x_{b,j}) \rangle, \quad (3.13)$$

$$J_a(X_a; z, \alpha) = \prod_{i=1}^{N_a} \langle V_\alpha(z) S_a(x_{a,i}) \rangle. \quad (3.14)$$

The non-vanishing two-point functions that enter the above formulas are

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<sup>4</sup>The normalization factor const is a product of two-point functions of the vertex operators with themselves,  $\langle V_{\alpha_i}(z) V_{\alpha_j}(w) \rangle$ . It does not depend on the integration variables  $X$  and is just a normalization constant, that we will not need in what follows.

$$\langle S_a(x_{a,i}) S_a(x_{a,j}) \rangle = \frac{\varphi(x_{a,i}x_{a,j}^{-1})}{\varphi(tx_{a,i}x_{a,j}^{-1})} \frac{\varphi(x_{a,j}x_{a,i}^{-1})}{\varphi(tx_{a,j}x_{a,i}^{-1})}, \quad (3.15)$$

$$\langle S_a(x_{a,i}) S_{a+1}(x_{a+1,j}) \rangle = \frac{\varphi(tv x_{a+1,j}x_{a,i}^{-1})}{\varphi(v x_{a+1,j}x_{a,i}^{-1})}, \quad (3.16)$$

$$\langle V_\alpha(z) S_a(x_{a,i}) \rangle = \frac{\varphi(vq^{\alpha_{a+1}}x_{a,i}z^{-1})}{\varphi(q^{\alpha_a}x_{a,i}z^{-1})}, \quad (3.17)$$

where  $v = (q/t)^{1/2}$ . In what follows we will need to put the first and the last  $z$ -coordinates to 0 and  $\infty$ , respectively; more precisely, to consider a limit

$$\begin{aligned} & B(0, z_1, \dots, z_\ell, \infty) \equiv \\ & \equiv \lim_{z_0 \rightarrow 0} \lim_{z_\infty \rightarrow \infty} z_0^{\alpha_0} \left\langle V_{\alpha_0}(z_0) V_{\alpha_1}(z_1) \dots V_{\alpha_k}(z_\ell) V_{\alpha_\infty}(z_\infty) Q_{(1)}^{N_1} \dots Q_{(n)}^{N_n} \right\rangle. \end{aligned} \quad (3.18)$$

Because of the charge conservation condition, the vector  $\alpha_\infty$  is fixed to be

$$\alpha_{\infty,a} = \beta N_a - \alpha_{0,a} - \dots - \alpha_{\ell,a}, \quad a = 1, \dots, n \quad (3.19)$$

and one can find that

$$B(0, z_1, \dots, z_\ell, \infty) = \int dX_1 \dots \int dX_{n-1} \prod_{a=1}^{n-1} X_a^{\zeta_a} J(X_1, \dots, X_{n-1}), \quad (3.20)$$

where  $\zeta_a = \alpha_{0,a} - \alpha_{0,a+1} + \frac{1-\beta}{2}$  and

$$J(X_1, \dots, X_{n-1}) = \text{const} \prod_{a=1}^{n-1} \prod_{i=1}^{\ell} J_a(X_a; z_i, \alpha_i) \prod_{1 \leq a \leq b < n} J_{a,b}(X_a, X_b), \quad (3.21)$$

with  $J_{a,b}$  and  $J_a$  defined above. This is the final form of the integral that we will be working with. One can see that it is a Macdonald ensemble, where  $t = q^\beta$ . If  $\beta$  is a positive integer, the ratios of quantum dilogarithms can be written in the polynomial form as finite products, as described in the Introduction. For generic  $t$ , the ratios of dilogarithms are the only way to define the Macdonald measure.

In addition to being more general, this form of notation has a fundamentally important consequence: it is no longer a polynomial, but a ratio of two infinite products and, in particular, it has poles and hence Cauchy theorem can be used to evaluate it as a sum of residues. This has been first done in [2], where we proved that the resulting sum agrees with the so-called Nekrasov partition function – an infinite series that appears in five-dimensional supersymmetric gauge theory. This provides a new proof of the relation between conformal blocks of  $W$ -algebras and Nekrasov partition functions, known as the AGT conjecture [32]. Let us now describe this proof.

### 3.3 Nekrasov partition functions

**Definition.** The Nekrasov partition function is a formal series

$$Z = \sum_{\{\lambda\}} I_{\{\lambda\}} \prod_{a,i} \Lambda_a^{|\lambda_{a,i}|},$$

in the variables  $\Lambda_1, \dots, \Lambda_{n-1}$ , given as a sum over  $n(n-1)\ell/2$  partitions

$$\lambda_{a,i}, \quad i = 1, \dots, (n-a)\ell, \quad a = 1, \dots, n-1, \quad (3.22)$$

where the summand is given by

$$I_{\{\lambda\}} = \prod_{a=1}^n \prod_{i=1}^{\ell} \prod_{j=1}^{d_1} \mathcal{N}_{\emptyset, \lambda_{1,j}} \left( \frac{v^{a-1} f_{i,a}}{e_{1,j}} \right) \quad (3.23)$$

$$\prod_{a=1}^{n-2} \prod_{i=1}^{d_a} \prod_{j=1}^{d_{a+1}} \mathcal{N}_{\lambda_{a,i}, \lambda_{a+1,j}} \left( \frac{e_{a,i}}{e_{a+1,j}} \right) \quad (3.24)$$

$$\prod_{a=1}^{n-1} \prod_{i,j=1}^{d_a} \mathcal{N}_{\lambda_{a,i}, \lambda_{a,j}} \left( \frac{e_{a,i}}{e_{a,j}} \right)^{-1} \quad (3.25)$$

$$\prod_{a=1}^{n-1} \prod_{i=1}^{d_a} (T_{\lambda_{a,i}})^{\ell}, \quad (3.26)$$

with notations  $d_a = (n - a)\ell$ , factors  $T_{\lambda} = (-1)^{|\lambda|} q^{||\lambda||/2} t^{-||\lambda^T||/2}$  and elementary building blocks given by the following infinite products

$$N_{\lambda\mu}(Q) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{\varphi(Qq^{\lambda_i - \mu_j} t^{j-i+1})}{\varphi(Qq^{\lambda_i - \mu_j} t^{j-i})} \frac{\varphi(Qt^{j-i})}{\varphi(Qt^{j-i+1})}. \quad (3.27)$$

**Physical meaning.** Let us very briefly remark on the physical meaning of this special function. It appears in five - dimensional supersymmetric gauge theory with the gauge group  $U(d_1) \times U(d_2) \times \dots U(d_{n-1})$ , as a generating function for integrals over the moduli space of instanton field configurations [33]. The first, second, third lines represent, respectively, the contributions of the fundamental, bifundamental, and gauge vector fields to the generating function, and the last line is the contribution of the five-dimensional Chern-Simons fields. Partitions  $\lambda$  label fixed points of the torus action in the moduli space of instantons. Parameters  $e$  are the Coulomb parameters of the respective gauge groups and  $\Lambda$  are the instanton parameters. Finally, parameters  $f$  are the masses of  $n\ell$  matter fields, charged fundamentally under the leftmost gauge group  $U(\ell(n - 1))$  and neutral w.r.t. all the other gauge groups.



### 3.4 The equality between the two

We will outline, following [2], the proof of the following theorem.

**Theorem.** The poles of the conformal block  $B(0, z_1, \dots, z_\ell, \infty)$  are in one-to-one correspondence with collections of partitions  $\{\lambda\}$  and the residues, corresponding to these poles, satisfy

$$I_{\{\lambda\}} = \frac{\text{res}_{X=X^{\{\lambda\}}} J(X_1, \dots, X_n)}{\text{res}_{X=X^{\{\emptyset\}}} J(X_1, \dots, X_n)}, \quad (3.28)$$

under the following identification of parameters:

$$t^{N_{a,i}} = v^{2a-1} e_{a,i} / f_{i,\tilde{a}}, \quad \tilde{i} = i \div (n-a), \quad \tilde{a} = i \bmod (n-a), \quad (3.29)$$

$$X_a^{\{\lambda\}} = \{ v^a e_{a,i} q^{\lambda_{a,j}} t^{\rho_j}, j \geq 1, i = 1, \dots, d_a \}, \quad (3.30)$$

$$X_a^{\{\emptyset\}} = \{ v^a e_{a,i} t^{\rho_j}, j \geq 1, i = 1, \dots, d_a \}, \quad (3.31)$$

$$q^{\alpha_a-1} z_i^{-1} = v^{a-1} f_{i,a}. \quad (3.32)$$

**Proof.** The fact that the poles of the conformal block are located at positions (3.30) and, in particular, are labeled by tuples of partitions was discussed in detail in [2] and we do not elaborate on this point. Here we concentrate on proving the residue formula (3.28). The starting step of the proof is the observation, that Nekrasov factor  $\mathcal{N}_{\lambda\mu}(v^2 t^{-N})$  with two indices  $\lambda, \mu$  and a non-negative integer  $N$  vanishes, unless  $l(\mu) \leq l(\lambda) + N$ . This property is a simple algebraic corollary of the explicit formulas for Nekrasov factors, and we take it for granted. It is easy to see that this property implies, if the Coloumb parameters  $e_{a,i}$  are chosen according to (3.29), that each partition  $\lambda_{a,i}$  has no more than  $N_{a,i}$  rows, i.e.,

$$l(\lambda_{a,i}) \leq N_{a,i}.$$

The next step of the proof is to rewrite the Nekrasov functions, which are a priori defined as infinite double products over all  $i$  and  $j$

$$\mathcal{N}_{\lambda,\mu}(Q) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{\varphi(Qq^{\lambda_i - \mu_j} t^{\rho_i - \rho_j + 1})}{\varphi(Qq^{\lambda_i - \mu_j} t^{\rho_i - \rho_j})} \frac{\varphi(Qt^{\rho_i - \rho_j})}{\varphi(Qt^{\rho_i - \rho_j + 1})},$$

in terms of finite products, bounded by  $l(\lambda)$  and  $l(\mu)$ . Since all the partitions now have finite length, this is possible to do: one just needs to break down the above infinite product over  $(i, j)$  into three parts: the product over  $0 \leq i \leq l(\lambda), 0 \leq j \leq l(\mu)$ ; the product over  $0 \leq i \leq l(\lambda), j \geq l(\mu)$ ; the product over  $i \geq l(\lambda), 0 \leq j \leq l(\mu)$ . The latter two products are formally infinite, but enjoy telescoping and hence are, in fact, finite products. Applying this for the Nekrasov factor  $\mathcal{N}_{\lambda,\mu}(Q)$  where partitions  $\lambda, \mu$  have corresponding Coloumb parameters  $e_1, e_2$  and lengths  $N_1, N_2$ , we obtain

$$\begin{aligned} \mathcal{N}_{\lambda,\mu}\left(\frac{e_1}{e_2}\right) &= \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{\varphi\left(\frac{e_1}{e_2} q^{\lambda_i - \mu_j} t^{\rho_i - \rho_j + 1}\right)}{\varphi\left(\frac{e_1}{e_2} q^{\lambda_i - \mu_j} t^{\rho_i - \rho_j}\right)} \frac{\varphi\left(\frac{e_1}{e_2} t^{\rho_i - \rho_j}\right)}{\varphi\left(\frac{e_1}{e_2} t^{\rho_i - \rho_j + 1}\right)}, \\ &N_{\lambda,\emptyset}\left(t^{N_2} \frac{e_1}{e_2}\right) N_{\emptyset,\mu}\left(t^{-N_1} \frac{e_1}{e_2}\right), \end{aligned}$$

Note that the right hand side, being a ratio of products of quantum dilogarithms taken at values shifted by  $t$ , is very much reminiscent of the integration measure of the conformal blocks. This observation was the main motivation of our work in [2]. It hints that the summands in the Nekrasov series  $Z$  are the residues of the conformal block  $B$ , so that the Nekrasov sum is just a residue expansion of  $B$ .

To make this observation precise, let us apply this formula to rewrite the contribution of the 5d vector multiplets as

$$\prod_{a=1}^{n-1} \prod_{i,j=1}^{d_a} N_{\lambda_{a,i}, \lambda_{a,j}} \left( \frac{e_{a,i}}{e_{a,j}} \right)^{-1} = \prod_{a=1}^{n-1} \frac{J_{a,a}(X_a^{\{\lambda\}}, X_a^{\{\lambda\}})}{J_{a,a}(X_a^{\{\emptyset\}}, X_a^{\{\emptyset\}})} \cdot V_{\text{vect}}, \quad (3.33)$$

where  $J_{a,a}$  is the contribution of the two-point functions of the screening charges of the same index, evaluated<sup>5</sup> at

$$X_a^{\{\lambda\}} = \{ v^a e_{a,i} q^{\lambda_{a,j}} t^{\rho_j}, j \geq 1, i = 1, \dots, d_a \}, \quad (3.34)$$

and  $X_a^{\emptyset}$  is the specialization of  $X_a$  to empty partitions,

$$X_a^{\{\emptyset\}} = X_a \Big|_{\lambda=\emptyset}, \quad (3.35)$$

or, more explicitly,

$$X_a^{\{\emptyset\}} = \{ v^a e_{a,i} t^{\rho_j}, j \geq 1, i = 1, \dots, d_a \}. \quad (3.36)$$

The quantity  $V_{\text{vect}}$  stands for all the remaining factors, which we leave untouched for now and will use in the last step of the proof:

$$V_{\text{vect}} = \prod_{a=1}^{n-1} \prod_{i,j=1}^{d_a} N_{\lambda_{a,i}, \emptyset} \left( t^{N_{a,j}} \frac{e_{a,i}}{e_{a,j}} \right)^{-1} N_{\emptyset, \lambda_{a,j}} \left( t^{-N_{a,i}} \frac{e_{a,i}}{e_{a,j}} \right)^{-1}. \quad (3.37)$$

---

<sup>5</sup>Note that the numerator and denominator have poles at  $X_a = X_a^{\{\lambda\}}$  resp.  $X_a^{\{\emptyset\}}$ . However, the ratio – understood as a limit at  $X_a$  approaching the poles – is regular. For the sake of brevity, throughout the proof we will simply write a ratio, not a limit, and restore full notation in the end.

In complete analogy, the contribution of 5d bifundamentals takes form

$$\prod_{a=1}^{n-2} \prod_{i=1}^{d_a} \prod_{j=1}^{d_{a+1}} N_{\lambda_{a,i}, \lambda_{a+1,j}} \left( \frac{e_{a,i}}{e_{a+1,j}} \right) = \prod_{a=1}^{n-2} \frac{J_{a,a+1}(X_a^{\{\lambda\}}, X_{a+1}^{\{\lambda\}})}{J_{a,a+1}(X_a^{\{\emptyset\}}, X_{a+1}^{\{\emptyset\}})} \cdot V_{\text{bifund}}, \quad (3.38)$$

where  $J_{a,a+1}$  is the contribution of the two-point functions of the screening charges with neighbouring indices, and  $V_{\text{bifund}}$  stands for all the remaining factors, which we similarly leave for now:

$$V_{\text{bifund}} = \prod_{a=1}^{n-2} \prod_{i=1}^{d_a} \prod_{j=1}^{d_{a+1}} N_{\lambda_{a,i}, \emptyset} \left( t^{N_{a+1,j}} \frac{e_{a,i}}{e_{a+1,j}} \right) N_{\emptyset, \lambda_{a+1,j}} \left( t^{-N_{a,i}} \frac{e_{a,i}}{e_{a+1,j}} \right). \quad (3.39)$$

At this point, putting all expressions together, we find

$$I_{\{\lambda\}} = \prod_{1 \leq a \leq b < n} \frac{J_{a,b}(X_a^{\{\lambda\}}, X_b^{\{\lambda\}})}{J_{a,b}(X_a^{\{\emptyset\}}, X_b^{\{\emptyset\}})} \cdot V_{\text{vect}} V_{\text{bifund}} V_{\text{fund}} V_{\text{CS}}, \quad (3.40)$$

where  $V_{\text{fund}}$  stands for the contribution of fundamentals

$$V_{\text{fund}} = \prod_{a=1}^n \prod_{i=1}^{\ell} \prod_{j=1}^{n\ell} N_{\emptyset, R_{1,j}} \left( \frac{v^{a-1} f_{i,a}}{e_{1,j}} \right), \quad (3.41)$$

and  $V_{\text{CS}} = \prod_{a,i} T_{\lambda_{a,i}}^{\ell}$  is the 5d Chern-Simons contribution.

The remaining product  $V_{\text{vect}}V_{\text{bifund}}V_{\text{fund}}V_{\text{CS}}$  may appear to have a lot of factors, however, there are many cancellations, implied by the identifications (3.29). After the cancellations are fully accounted for, this product matches manifestly the conformal block,

$$V_{\text{vect}}V_{\text{bifund}}V_{\text{fund}}V_{\text{CS}} = \prod_{a=1}^{n-1} \prod_{i=1}^{\ell} \frac{J_a(X_a^{\{\lambda\}}; z_i, \alpha_i)}{J_a(X_a^{\{\emptyset\}}; z_i, \alpha_i)}, \quad (3.42)$$

where

$$q^{\alpha_a-1} z_i^{-1} = v^{a-1} f_{i,a} \quad (3.43)$$

is the identification between the vertex operator parameters of the conformal blocks and the mass parameters of the Nekrasov series. This finally gives

$$I_{\{\lambda\}} = \prod_{a=1}^{n-1} \prod_{i=1}^{\ell} \frac{J_a(X_a^{\{\lambda\}}; z_i, \alpha_i)}{J_a(X_a^{\{\emptyset\}}; z_i, \alpha_i)} \prod_{1 \leq a \leq b < n} \frac{J_{a,b}(X_a^{\{\lambda\}}, X_b^{\{\lambda\}})}{J_{a,b}(X_a^{\{\emptyset\}}, X_b^{\{\emptyset\}})}. \quad (3.44)$$

Recall that throughout this proof we employed a concise notation: both the numerator and the denominator in the formula above have a pole at  $X_a$  given by (3.30) resp. (3.31), and what we write as their ratio is a concise notation for the limit, as  $X_a$  approaches (3.30) resp. (3.31). What is important, such a limit coincides with the ratio of the residues at the corresponding poles:

$$I_{\{\lambda\}} = \frac{\text{res}_{X_a=X^{\{\lambda\}}} \prod_{a=1}^{n-1} \prod_{i=1}^{\ell} J_a(X_a; z_i, \alpha_i) \prod_{1 \leq a \leq b < n} J_{a,b}(X_a, X_b)}{\text{res}_{X_a=X^{\{\emptyset\}}} \prod_{a=1}^{n-1} \prod_{i=1}^{\ell} J_a(X_a; z_i, \alpha_i) \prod_{1 \leq a \leq b < n} J_{a,b}(X_a, X_b)}. \quad (3.45)$$

This completes the proof.  $\square$

**Corollary.** The Nekrasov partition function is the power series expansion of the conformal block in the variables

$$\Lambda_a = vq^{\alpha_{0,a} - \alpha_{0,a+1}}, \quad (3.46)$$

up to an overall normalization constant, independent of these variables.

To conclude, in this chapter we used the residue properties of Macdonald ensembles at  $t = q^\beta$  with generic non-integer  $\beta$  to prove the equivalence between conformal blocks of  $W_{q,t}(gl_N)$  and Nekrasov series, what is sometimes called the AGT conjecture. It would be interesting to generalize this proof to the  $W_{q,t}$ -algebras corresponding to other root systems.

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