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Author
Oh, Jihwan

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by

Jihwan Oh

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Abstract

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This dissertation has two parts. The first part is devoted to the study of a line(ray) operator in 5d SCFTs with exceptional group global symmetry. We construct an index for BPS operators supported on a ray in five dimensional superconformal field theories with exceptional global symmetries. We compute the E_n representations (for $n = 2, \ldots, 7$) of operators of low spin, thus verifying that while the expression for the index is only $\mathrm{SO}(2n-2)\times\mathrm{U}(1)$ invariant, the index itself exhibits the full E_n symmetry (at least up to the order we expanded). The ray operators we studied in 5d can be viewed as generalizations of operators constructed in a Yang-Mills theory with fundamental matter by attaching an open Wilson line to a quark. For $n \leq 7$, in contrast to local operators, they carry nontrivial charge under the $\mathbb{Z}_{9-n} \subset E_n$ center of the global symmetry. The representations that appear in the ray operator index are therefore different, for $n \leq 7$, from those appearing in the previously computed superconformal index. For $3 \leq n \leq 7$, we find that the leading term in the index is a character of a minuscule representation of E_n. We also discuss the case $n = 8$, which presents a unique technical challenge, and remains an open problem.

The second part discusses line defects in 5d non-commutative Chern-Simons theory. We studied aspects of a topologically twisted supergravity under Omega background and its interpretation as the bulk side of topological subsector of AdS/CFT correspondence. The field theory side is a protected sub-sector of a specific 3d $\mathcal{N} = 4$ gauge theory(from M2-branes), especially its Higgs branch, whose chiral ring deformation-quantizes into an algebra via the $\Omega-$background. The line defect comes from this 3d system. The bulk side is interestingly captured by a field theory again, a 5d Chern-Simons theory, which is topological in 1 dimension and holomorphic in 4 dimensions. The statement of topological holography is an isomorphism between the operator algebras. It is possible to introduce M5-brane to decorate the relation. It acts like a module of the algebra(of M2-brane) in the field theory side and a chiral algebra that interacts with 5d Chern-Simons in the gravity side.

To my mother

Contents

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Chapter 1

Introduction

The main theme of this thesis is about exact calculations that probe non-perturbative aspects of supersymmetric quantum field theory. Especically, we will focus on physical observables associated to line operators in two types of 5d gauge theories.

The first part of this thesis calculates a generating function (akin to a character of a Lie algebra) for (a basis of) quantum states in a subspace of a Hilbert space(that is carved out by the insertion of the line operator) of states of a certain 5d QFT. The 5d QFT will be described below, and while a concrete Lagrangian definition is not known at present, evidence has accumulated from string theory about its existence and properties. In particular, it has a superconformal symmetry algebra of $F(4)$ and the subspace of quantum states, whose generating function is described in this work, is the space of certain "short representations" of the superconformal algebra. Technically, we consider 8 theories labeled by an integer $0 \leq N_f \leq 7$ and the result presented is a function of $N_f + 3$ complex variables, which we denote by $q, t, u, m_1, \ldots, m_{N_f}$, and whose physical meaning will be elucidated below. We provided an explicit expansion as a Taylor series in t up to $O(t^6)$. The coefficients of that Taylor expansion are rational functions which can be interpreted as characters of exceptional groups. The complete generating function is expressed as an infinite sum of contour integrals of rational functions. Before describing the particular theory, we would like to provide some general background, which is needed to understand the technique we used.

[91] showed that in the presence of supersymmetry, whose generator will be denoted by Q, there is a way to simplify the path integral by adding a suitable Q-exact term to the action, and taking the limit whereby the coefficient of the Q-exact term is infinite. The technique is called *supersymmetric localization*, and the principle can be traced back to the Duistermaat-Heckman formula. In [91], Witten reproduced Donaldson invariants of a four-manifold by computing the correlation functions of certain observables in 4d $\mathcal{N} = 2$ gauge theory on the four-manifold. Another representative example is [16], where Vafa and Witten computed a partition function of 4d $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) on various four-manifolds, and showed the partition function is given by the Euler characteristic of a certain instanton moduli space. Importantly, by examining the modular property of the partition function, they could show evidence of S-duality, which inverts the gauge coupling, and at the same

time changes the gauge group, with the modular parameter being the inverse gauge coupling square.

An important example is an exact computation of the partition function of 4d $\mathcal{N} = 2$ theory on $\mathbb{R}^4$, where a certain background called $\Omega-$background was turned on in $\mathbb{R}^4$. The partition function, or a pre-potential of the theory, is a series in $q = \exp 2\pi i\tau$, where τ is an inverse gauge coupling square. The coefficient of the series is computed exactly by performing localization in the instanton moduli space [98]. The partition function is sometimes called Nekrasov partition function.

In recent two decades, there was a tremendous effort to compute partition functions of supersymmetric field theories on various manifold of diverse dimensions. The program was started in [31], where the author considered $4d$ $\mathcal{N} = 2$ gauge theory on S^4. Supersymmtric localization tells us that the field configuration space of the gauge field, which is a part of the integral domain of the path integral, is localized into instanton and anti-instanton moduli space at the $SO(4)$-fixed points of S^4. As a result, the integrand is simply given by a product of Nekrasov partition function with its conjugate. On the other hand, the vector multiplet scalar, which is a part of the bosonic path integral measure, localizes on a constant matrix, so the path integral that computes the partition function is simply a matrix integral, with the integral measures being the Haar measure of the gauge group.

Another important development, which unified the above theme and opened entirely new directions, is the so called AGT (Alday, Gaiotto, Tachikawa) conjecture [99]. The conjecture is about the equality between the Nekrasov partition function of a certain class of $\mathcal{N} = 2$ SCFTs, so called class-S theories, and a conformal block of Liouville theory. This equality is drawn between four dimensional physical quantity and two dimensional quantity, so sometimes called as 4d/2d correspondence. Arguably, the most natural way to understand the correspondence is to think of the 6d origin of the 4d theory, and think of the 4d theory as a compactification of the 6d $(2, 0)$ SCFT(superconformal field theory) on a Riemann surface; the 6d $(2, 0)$ theory is defined as a woroldvolume theory of M5-brane in M-theory on flat 11 dimensional background [18]. Following this example, many analogous correspondences between D-dimensional theory and $6 - D$ dimensional theory were discovered; all of them have the same origin, 6d $(2, 0)$ SCFT.

The first part of the thesis applies ideas and techniques developed within the localization program to a specific (but novel) counting problem in the context of 5d superconformal field theory. The 5d superconformal field theory, although introduced 25 years ago as an infrared fixed point of a certain 5d supersymmetric gauge theory [13], is still poorly understood, and its existence itself is a conjecture. Nevertheless, proposed techniques for solving certain counting problems of operators in these types of theories have been developed, and have passed nontrivial consistency checks. This first part of the thesis is based on work [1].

During the “second string revolution” 25 years ago, a new (indirect) technique has been introduced for constructing (often strongly interacting) quantum field theories. Shortly after the discovery that a gauge theory describes the low-energy degrees of freedom of coincident D-branes, it was realized that D-branes that are near (or on top of) certain types of singularities (where either the curvature or other fields blow up) carry additional low-energy degrees of freedom. Even though the singularity itself can not be analyzed directly, using current tools, the idea that a space with a singularity is consistent has been central to the "second string revolution", and it can often be analyzed using a dual weakly coupled description.

The 5d SCFT with E_8 global symmetry was first realized by Seiberg [10] as a D4 brane worldvolume theory that probes an E_8 singularity. The E_8 singularity can be equally thought of as 8 D8 branes and O8-plane located at a fixed point of $S^1/\mathbb{Z}_2$ in type I' string theory background. At the singularity, the 5d gauge coupling formally diverges and the naive flavor symmetry $SO(14)$ enhances into E_8, and the theory is believed to flow into non-trivial interacting quantum field theory. However, it is not known how to renormalize 5d supersymmetric gauge theories, and similar problem happens in the higher dimensional gauge theory, e.g. 6d $\mathcal{N} = (1, 1)$ theory [28]. Nevertheless, 1-loop integrals that correct the gauge coupling are well-defined, as [13] calculated. More evidences that support the existence of the 5d SCFT are the following.

- M-theory on Calabi-Yau construction [13]: it is generally believed that 11 dimensional M-theory can be compactified on Calabi-Yau three-fold to yield 5d gauge theory with some amount of supersymmetry. The non-trivial fixed points with global symmetry E_n introduced above correspond to M-theory compactified on singular Calabi-Yau three-fold, where there is a collapse of a del-Pezzo surface with $n - 1$ points blown-up.

- E-string/S^1 construction [14]: In fact, the 5d SCFT with E_n global symmetry can be derived from certain 6d SCFT, called E-string theory. E-string theory is defined as a worldvolume theory of M5 brane probing the end of the world M9 brane. It is 6d $\mathcal{N} = (1, 0)$ SCFT with a global symmetry E_8; it looks like a local quantum field theory, but does not have a Lagrangian description. Its existence is also conjectural. By compactifying the theory on a circle S^1, and reduce it, we can obtain some 5d SCFT. Moreover, by turning on a Wilson line on the circle S^1, taking values in E_8, we can systematically reduce the global symmetry from E_8 to E_n where $n = 1, \dots, 7$. Those theories are the 5d SCFT described above.

In each case above the existence of a well-defined 5d QFT with $F(4)$ superconformal symmetry as the low-energy limit is a conjecture, but is backed by an argument that is widely believed to be well-defined in physics. This theory is the subject of the first part of this thesis. Assuming the theory's existence, we will explain some properties and observables of the theory. For each value $N_f \in \{0, \dots, 7\}$, Part I is studying a function (called the "ray operator superconformal index") of $N_f + 3$ real variables, and the results presented here are the first 5 terms in a Taylor expansion with respect to the first variable, which we denote by t. As will be explained below, the remaining $N_f + 2$ variables can be identified with a parameterization of the Cartan subalgebra of $su(2) \oplus E_{N_f+1}$, where E_n is the exceptional Lie algebra of rank n (which is isomorphic to a classical algebra for $n \leq 5$). The point of the calculation is to check that the coefficients of the Taylor expansion are characters of finite dimensional representations of $su(2) \oplus E_{N_f+1}$, even though the technique that is used to calculate these coefficients is not manifestly $SU(2) \times E_{N_f+1}$ invariant.

The "ray operator superconformal index" – an object that was defined in our work – can be thought of as a character of a certain module of the 5d superconformal algebra $F(4)$. To define the ray operator superconformal index, we therefore, first need to understand the symmetry algebra of 5d SCFTs. The superconformal algebra under the 5d $\mathcal{N} = 1$ superconformal field theories consists of bosonic algebra and fermionic algebra. The bosonic symmetry algebra is $SO(5,2) \times SU(2)_R$, where the first factor is a conformal group, whose subgroup can be $SO(2)_{Dilatation} \times SO(5)_{Lorentz} \subset SO(5,2)$, and the second factor is R-symmetry group. Let us denote Δ, ${M_m}^n$, ${R_B}^A$ to be the $SO(2)$ dilatation, $SO(5)$ rotation, and $SU(2)$ R-symmetry generators. The fermionic symmetry generators Q_m^A, S_B^n are charged under the bosonic symmetry algebra, where m, n are $SO(5)$ vector indices and A, B are $SU(2)_R$ spinor indices. One of the most important anti-commutation relation of the algebra that contains every generators of the superconformal algebra is

$$\{Q_m^A, S_B^n\} = \delta_m^n \delta_B^A \Delta + 2\delta_B^A {M_m}^n - 3\delta_m^n {R_B}^A \tag{1.1}$$

For later use, let us denote $J_+ \equiv {M_1}^1$, $J_- \equiv {M_3}^3$, $J_R \equiv {R_1}^1$ to be the Cartan generators of $SU(2)_+ \times SU(2)_- \subset SO(5)$, $SU(2)_R$. We are especially interested in some states that are annihilated by Q_2^1, which we will denote as simply Q. Denoting S_1^2 as S, since S is a Hermitian conjugate of Q, the anti-commutator gives a unitarity bound:

$$\{Q, S\} = \Delta - 2J_+ - 3J_R \geq 0 \tag{1.2}$$

Formally, one can regard Q as an exterior derivative d acting on a formal complex, which we will define soon, and S as its conjugate $d^\star$. The anti-commutator and the states with $\Delta = 0$ can be then thought of a Laplacian, and harmonic forms. By Hodge theorem, the space of harmonic forms is isomorphic to the cohomology with a differential d. Hence, we can identify the Hilbert space, which is the space of 1/8 BPS(Bogomol'nyi-Prasad-Sommerfield) operators, with the cohomology of Q. The superconformal index is an Euler characteristic of the Q-cohomology.

Formally, the superconformal index is defined as the character of the Hilbert space $\mathcal{H}$ (viewed as a module over the superconformal algebra $F(4)$ whose Cartan subalgebra is generated by $\Delta, J_\pm, J_R, F_1, \ldots, F_{N_f}$):

$$\mathcal{I}_{SCI}(\epsilon_+, \epsilon_-, m_i) = \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta\Delta} e^{-2\epsilon_+(J_+ + J_R) - 2\epsilon_- J_-} e^{-\sum F_i m_i} \right], \tag{1.3}$$

where F is the fermion number (i.e., the standard $\mathbb{Z}_2$ grading of a module of a superalgebra). $\epsilon_\pm$, m_i are chemical potentials for $J_\pm$, F_i are flavor symmetry generators, and β is the radius of the time circle S^1 where the trace is taken over. β is the parameter associated with the generator Δ, but note that $\mathcal{I}_{SCI}$ is independent of β, as follows from the representation theory of the superconformal algebra. (Only "short representations", which are eigenspaces with $\Delta = 0$, contribute nontrivially to $\mathcal{I}_{SCI}$.) $\mathcal{H}$ is the Hilbert space of states on S^4, and by the standard state-operator correspondence of conformal field theories, it is isomorphic to the space of local operators. In other words, the Index counts the local operators of the theory defined on a flat spacetime, but it can be computed as the path integral of the theory defined on $S^4 \times S^1$, invoking the state-operator correspondence.

The 5d superconformal field theory with an exceptional global symmetry E_n does not have a known Lagrangian description. Hence, it is hard to compute physical quantities directly. However, for observables that are protected by supersymmetry, we can study more explicitly by recalling the RG flow invariance of the observables, which is believed to exist. The 5d SCFT with E_n global symmetry is believed to be a UV fix point of a particular 5d $\mathcal{N}=1$ supersymmetric gauge theory, which has a Lagrangian description. However, it was conjectured by [24] that $\mathcal{I}_{SCI}$ can be effectively computed in terms of a partition function of a certain gauge theory on $S^4 \times S^1$. The gauge theory that corresponds to a given 5d SCFT is the one conjectured by Intriligator, Morrison, and Seiberg [24] as the IR limit under RG flow of the 5d SCFT. Even though a 5d gauge theory is not renormalizable, it can still produce a well-defined superconformal index, via localization. In the work described below, we adapted the technique introduced by [24] for local operators to the case of ray-operators, which are nonlocal operators that also form a module of the superconformal algebra, but a different one from the module of local operators, and adapting the technique of [24] involved some subtleties.

I will now describe the technique developed by [24] and the localization procedure in greater detail. The reader who is not interested in the details is advised to skip to a sample of the results, reported in (1.14)-(1.15).

The 5d $\mathcal{N}=1$ supersymmetric gauge theory has a gauge group $G = Sp(1)$ with a vector multiplet $\mathcal{V}$, and $n-1$ fundamental hypermultiplets $\mathcal{H}_a$, which are charged under the flavor symmetry $SO(2(n-1))$. The obvious flavor symmetry of the gauge theory $SO(2(n-1))$ is enhanced into E_n by combining with the $U(1)$ topological symmetry. The topological $U(1)$ symmetry is relevant for an instanton particle, which is also BPS. The instanton is a particle with a worldline in 5d, different from that of 4d. It is defined as a field configuration that satisfies $F = \star F$, where F is a field strength of the gauge field. Given the explicit field content of the theory, we can perform supersymmetric localization(we will review it briefly soon) on a particular background $S^4 \times S^1$, and compute the partition function. As the partition function counts only BPS operators, it is invariant under the RG flow. Therefore, it can be thought of as the partition function of the 5d $\mathcal{N}=1$ SCFT with E_n global symmetry; we define the partition function of D-dimensional SCFT on $S^{D-1} \times S^1$ as superconformal index.

Let us describe the supersymmetric localization briefly. Let $S^4 \times S^1$ be the 5-dimensional space-time for the 5d $\mathcal{N}=1$ IR gauge theory, and let $\mathcal{L}(\mathcal{V}, \mathcal{H}_a)$ be a Lagrangian of the theory. We deform the Lagrangian by some Q-exact term, with a chosen supercharge Q: $\mathcal{L}' = \mathcal{L} + t\{Q, V\}$, where $t \in \mathbb{C}$, and V is a function of fields in the supermultiplets $\mathcal{V}$, $\mathcal{H}_a$. As the theory is supersymmetric, the result of the path integral does not change by the Q-exact deformation. Also, we can take $t \to \infty$. As a result, the path integral is localized to the saddle point equation $\{Q, V\} = 0$. It turns out that the field configurations localize on (1) the two poles of S^4, which are $SO(4)-$fixed points and (2) other part of S^4.

The field configuration on each pole is related to instanton particles. More precisely, the saddle points of the bosonic part of $\{Q, V\}$ term consist of $F = \star F$ at $\theta = 0$, and $F = -\star F$

at $\theta = \pi$, where θ is one of the spherical coordinates of S^4. Note that in the S^4 background, we can explicitly quantify the $U(1)$ topological symmetry charges, up to some normalization, explicitly from

$$\int_{S^4} F \wedge F = k. \tag{1.4}$$

The set of BPS operators are then graded by the number k, which is we call as instanton number. We will denote the partition function from each pole as Z_{inst} and $Z_{anti-inst}$. Each of Z collects all the contribution from (anti)instanton moduli space. For a concise presentation, let us discuss Z_{inst} only, as $Z_{anti-inst}$ can be obtained by a complex conjugation. Z_{inst} can be expressed as a generating series in q, where each term is relevant for the subsector of the set of BPS operators with charge k under $U(1)$ topological symmetry.

$$Z_{inst} = 1 + \sum_{k=1}^{\infty} q^k Z_{inst}^k \tag{1.5}$$

It is well known that the partition function Z_{inst}^k can be computed by k D0-brane quantum mechanical partition function, where the D0 branes probe the k-instanton moduli space. The fields configurations on the other part of S^4 is perturbative modes of 5d fields. We will denote the partition function of the 5d perturbative modes as Z_{pert}; as a function, Z_{pert} is a rational function of sum of exponentials with their exponents being polynomials of various fugacities conjugate to Cartan generators of superconformal algebra.

Other than the north pole and the south pole, the saddle point equation $\{Q, V\} = 0$ reduces to $F = 0$, whose solution set is a space of flat connections. The space of flat connection can also be interpreted as a space of holonomy. As a result, the path integral domain related to the gauge connection reduces into $Sp(1)$ group manifold, parametrized by α.

$$\mathcal{I} = \int [d\alpha] Z_{inst}(\epsilon_+, \epsilon_-, \alpha, \phi_k, m_i) Z_{anti-inst}(\epsilon_+, \epsilon_2, \alpha, \phi_k, m_i) Z_{pert}(\epsilon_+, \epsilon_-, \alpha, \phi_k, m_i) \tag{1.6}$$

where the new fugacity ϕ_k is that of the Cartan generators of $O(k)$.

Let us now introduce the line defect on $\{P\} \times \mathbb{R}_+ \subset S^4 \times \mathbb{R}$, where P is a $SO(4)$ fixed point of S^4. We will first define the infra-red formula of the line defect in the IR gauge theory, and proceed to its realization in the UV string theory. Let A_0 be the time component of the gauge field A_μ, and Φ be its scalar component of the vector multiplet, where A_μ lives. The line defect in IR is given by a Wilson ray, which is a path-ordered exponential of gauge holonomy around the time circle S^1:

$$P \exp\left[i \int_0^\infty (A_0 + \Phi)\, dx^0\right] \tag{1.7}$$

However, this line defect by itself is not gauge invariant, and we need to neutralize the gauge charge by capping the end $\{0\}$ with a quark $\mathcal{O}$. We then define the ray operator as

$$\mathcal{R}_{\mathcal{O}} = P \exp\left[i \int_0^\infty (A_0 + \Phi)\, dx^0\right] \mathcal{O}(0) \tag{1.8}$$

In other words, in the presence of the line defect, the index counts $\mathcal{R}$, which is labeled by $\mathcal{O}$. Given this definition, we now want to show how to modify (1.6) for it to compute the index in the presence of the line defect.

The defect effectively produces a one-dimensional quantum mechanical fields χ, $\chi^\dagger$ that interact with the pullback of 5d dynamical and non-dynamical gauge fields A_t, $\tilde{A}_t$ and the vector multiplet scalar Φ. The action is given by

$$S^{1d} = \int dt \chi^\dagger_{i,\rho} \left(\delta_{\rho\sigma}(\delta_{ij}\partial_t - iA_{t,ij} + \Phi_{ij}) + \delta_{ij}\tilde{A}_{t,\rho\sigma} \right) \chi_{j,\sigma} \tag{1.9}$$

where i, j are 5d gauge indices and ρ, σ are 5d flavor group indices. As we will elaborate in the next paragraph, this action modifies the k D0 brane quantum mechanics.

It is helpful to recall string theory picture to compute Z^K_{inst} explicitly, as K D0-brane quantum mechanical partition function. In string theory, instantons in 5d gauge theory can be realized as K D0 branes in the worldvolume of D4 brane, which realizes the 5d gauge theory. D0 brane worldvolume theory is 1d $\mathcal{N} = (0, 4)$ $G = O(K)$ supersymmetric quantum mechanics with its fields obtained from the quantization of all D0-brane related strings– D0-D0 and D0-D4. On the other hand, the line operator is realized by an orthogonal D4' brane to D4 brane. Hence, the inclusion of the line operator induces D0-D4' strings– a quantization of those strings yield χ, $\chi^\dagger$ of (1.9). Hence, the problem of computing Z_{inst} is reduced into computing the partition function of $\mathcal{N} = (0, 4)$ supersymmetric quantum mechanics of D0 branes, modified by (1.9).

$$Z_{inst} = \oint \prod_{i=1}^{[K/2]} [d\phi_i] Z_{D0-D0} Z_{D0-D4} Z_{D0-D4'} \tag{1.10}$$

where $\prod_{i=1}^{[K/2]} [d\phi_i]$ is a Haar measure of $O(K)$. Z_{D0-D0} and Z_{D0-D4} are 1-loop determinants of quantum mechanical fields obtained by the quantization of D0-D0, D0-D4 strings; they depend on $\epsilon_+, \epsilon_-, \alpha, \phi_k, m_i$. $Z_{D0-D4'}$ has an additional parameter M, which is a chemical potential conjugate to the Cartan of gauge group $Sp(1)$ of $D4'$ brane worldvolume theory.

All the Z-functions in the integrand are rational functions of $\sinh(\dots)$ functions with their argument being a polynomial of $\epsilon_\pm$, ϕ_k, α, m_i. In other words, it takes a following form:

$$Z_{Da-Db} = \frac{\prod_{j=1}^{n_1} \sinh(\vec{\rho}_j \cdot \vec{\phi} + f_j(\epsilon_+, \epsilon_-, \alpha, m_i))}{\prod_{k=1}^{n_2} \sinh(\vec{\rho}_k \cdot \vec{\phi} + f_k(\epsilon_+, \epsilon_-, \alpha, m_i))} \tag{1.11}$$

where $\vec{\rho}_j$, $\vec{\rho}_k$ are in the root lattice of $G = O(K)$, $\vec{\phi} = (\phi_1, \dots, \phi_k)$, and $n_i \in \mathbb{Z}^+$. Hence, there are many poles in the integrand. We use a contour prescription called Jeffrey-Kirwan(JK) residue formula [42] to evaluate the integral. Let us briefly review the JK prescription.

In JK prescription, one first picks a vector η; we picked it as $(1, 3, \dots, 2i-1, \dots, 2k-1)$; in general the integral result does not depend on the choice of the η vector. Then, we pick k hyperplanes, which are arguments of sinh functions in the denominator, taking the following form:

$$\vec{\rho}_l \cdot \vec{\phi} + f_l = 0, \text{ where } l = 1, \dots, k. \tag{1.12}$$

We are supposed to use the usual residue formula at the solution of the linear system of the equations. JK prescription directs us only evaluate the residue if the set of vectors $\vec{\rho}_l$ spans the η-vector. One practical way to test it is to form a $k \times k$ matrix $Q = Q_{li} = (\rho_l)_i$, where $\vec{\rho}_l = ((\rho_l)_1, \ldots, (\rho_l)_k)$, and test if all the components of ηQ^{-1} are positive.

After evaluation, we obtain a rational function of a sum of exponentials with their arguments being polynomials of ϵ_+, ϵ_-, α, ϕ_k, m_i. After plugging (1.10) into (1.6), and we perform the α integral. As this integral is complex one dimensional integral, we apply the normal residue formula. After making following substitutions,

$$e^{\epsilon_+} \to t, \quad e^{\epsilon_-} \to u, \quad e^{m_i} \to f_i, \tag{1.13}$$

and expanding in t up to fifth order, we obtain a nice polynomial of t with its coefficients being a Weyl character formula for various representations of E_{N_f+1}. For instance, let us look at $N_f = 5, 6$ cases. The final expression from the above computation is

$$\begin{aligned} q^{-\frac{2}{3}}\mathcal{I}_{E_6} = {}& \chi^{E_6}_{[0,0,0,0,0,1]} t + \chi^{E_6}_{[0,1,0,0,0,1]} t^3 + \chi_2(u)\left[\chi^{E_6}_{[0,0,0,0,0,1]} + \chi^{E_6}_{[0,1,0,0,0,1]} + \chi^{E_6}_{[0,0,1,0,0,0]}\right] t^4 \\ & + \left[\chi_3(u)\big(2\chi^{E_6}_{[0,0,0,0,0,1]} + \chi^{E_6}_{[0,1,0,0,0,1]} + \chi^{E_6}_{[0,0,1,0,0,0]}\big) + \chi^{E_6}_{[0,2,0,0,0,1]} + \chi^{E_6}_{[0,0,0,0,0,1]}\right] t^5 + O(t^6) \end{aligned} \tag{1.14}$$

$$\begin{aligned} q^{-1} I_{E_7} = {}& \chi^{E_7}_{[0,0,0,0,0,0,1]} t + \chi^{E_7}_{[1,0,0,0,0,0,1]} t^3 + \chi_2(u)\left[\chi^{E_7}_{[0,0,0,0,0,0,1]} + \chi^{E_7}_{[0,1,0,0,0,0,0]} + \chi^{E_7}_{[1,0,0,0,0,0,1]}\right] t^4 \\ & + \left[\chi_3(u)\big(2\chi^{E_7}_{[0,0,0,0,0,0,1]} + \chi^{E_7}_{[0,1,0,0,0,0,0]} + \chi^{E_7}_{[1,0,0,0,0,0,1]}\big) + \chi^{E_7}_{[2,0,0,0,0,0,1]} + \chi^{E_7}_{[0,0,0,0,0,0,1]}\right] t^5 + O(t^6) \end{aligned} \tag{1.15}$$

Here, $\chi^{E_6}_{[\bullet]}$, $\chi^{E_7}_{[\bullet]}$ denote the E_6, E_7 character formula of a highest weight representation, labeled by the Dynkin labels $[\bullet]$, and $\chi_n(u)$ is $SU(2)$ character formula of a dimension n highest weight representation.

In general, $\chi^{E_{N_f+1}}_{[\bullet]}$ is a rational function of f_i's and q, where $i = 1, \ldots, N_f$. However, we substitute $f_i = 1$ for all i for a simple presentation. In this case, the formula becomes a rational function of q only, and the coefficients of q-series are dimension of $SO(2N_f)$ highest weight representations. For instance, the relevant E_7 characters are listed as follows

$$\begin{aligned} \chi^{E_7}_{[0,0,0,0,0,0,1]} &= \frac{12}{q} + 32 + 12q, \\ \chi^{E_7}_{[0,1,0,0,0,0,0]} &= \frac{32}{q^2} + \frac{232}{q} + 384 + 232q + 32q^2, \\ \chi^{E_7}_{[1,0,0,0,0,0,1]} &= \frac{12}{q^3} + \frac{384}{q^2} + \frac{1596}{q} + 2496 + 1596q + 384q^2 + 12q^3, \\ \chi^{E_7}_{[2,0,0,0,0,0,1]} &= \frac{12}{q^5} + \frac{384}{q^4} + \frac{6348}{q^3} + \frac{31008}{q^2} + \frac{73536}{q} + 97536 \\ &\quad + 73536q + 31008q^2 + 6348q^3 + 384q^4 + 12q^5. \end{aligned} \tag{1.16}$$

All the weights that appear in the characters above do not belong to the root lattice. Since the root lattice is a sub-lattice of index 2 in the weight lattice of E_7, we see that all the E_7

representations of ray operators are odd under the $\mathbb{Z}_2$ center. In other words, the highest weight representation labeled by [$\bullet$]'s that appear in the above formulas transform non-trivially under the center of E_{N_f+1}. This is to be contrasted with the neutral operators under the center of E_{N_f+1} counted by the usual superconformal index.

The second part of this thesis is on the line operator in a non-commutative deformation of 5d $U(1)$ Chern-Simons theory with a particular topological twist. We first derive the operator algebra of the quantum mechanics living on the line. Then, we insert a surface operator in 5d Chern-Simons theory and show that the operator algebra on the surface operator forms a bi-module of the algebra of the quantum mechanics. Before providing the detail of the problem, let us start with a general background used in the main body of the work– *cohomological field theories from topological twisting[91, 93] and Ω-background[98].* Along the way, we will also provide particular applications.

The topological twist of supersymmetric quantum field theories was introduced by Witten in 1988 [91] and applied to define partition functions of two-dimensional sigma-models [93] and famously applied to four-dimensional gauge theories [92] where it simplified the construction of Donaldson invariants on manifolds with Kähler structure. The topological twist allows one to define and in some cases exactly compute partition functions of certain supersymmetric field theories on compact manifolds. The topological twist relies on the supersymmetry algebra which contains a (classical group) R-symmetry subalgebra R as well as a subspace of supersymmetry (SUSY) generators on which the R-symmetry acts nontrivially (i.e., the SUSY generators subspace is a module of the R-symmetry). The supersymmetry algebra also contains a Lorentz subalgebra M (a copy of $so(d)$ where d is the dimension of the manifold on which the theory is formulated) which commutes with the R-symmetry, and the supersymetry generators subspace is a module of the tensor product $M \otimes R$ of the Lorentz algebra and the R-symmetry. The R-symmetry algebra can be exponentiated to an R-symmetry group, and the Lorentz subalgebra can be exponentiated to the Lorentz group.

To define a topological twist, we need a map from the holonomy algebra of the manifold to the R-symmetry algebra. Given the map, we re-define the Lorentz symmetry algebra in such a way that a component of the spinor supercharge becomes a scalar Q in the new Lorentz algebra. The embedding, which we assume lifts to an embedding of the holonomy group into the R-symmetry group, allows one to construct an R-symmetry principal bundle with a connection over the manifold, and the Lagrangian of the twisted theory is defined by modifying the covariant derivatives of the various original fields to include both the spin connection and the R-symmetry-valued connection.

The SUSY generators subalgebra form a module of the tensor product $M \otimes R$, and with a suitably chosen embedding, the SUSY generators subspace contains a nontrivial invariant element, which is called the BRST operator and will be denoted by Q. The BRST operator squares to zero and acts on the various operators of the QFT, and the spectrum of the topologically twisted theory is defined as the Q-cohomology, since it can be shown that correlation functions of Q-closed operators are independent of their representative in Q-cohomology.

Twisting can also be performed when the QFT is formulated on flat space, but it is then trivially equivalent to the untwisted theory, since the holonomy group is trivial. Nevertheless, if we choose an embedding of the holonomy group of a curved manifold into the Lorentz group of flat space, we can identify the BRST operator Q associated with the curved manifold compactification with a generator of the supersymmetry algebra in the flat case, and we can define the associated Q-cohomology in the flat case as well. We refer to it as the BPS sector of the theory. The operators in the Q-cohomology consist of the spectrum of a topological field theory in the sense that the metric deformation of the theory yields only a Q-exact deformation. Twisting is essential to apply localization or compactification on a curved manifold and to focus on a particular BPS sector in the configuration space.

The supersymmetry algebra also contains elements that can be identified with translations, which act as ordinary partial derivatives on local fields. If a translation generator P can be written as a Q-commutator $P = [Q, K]$, for some K in the algebra, then the BPS operators are independent of the position, in the sense that the difference between the operator at two different points is Q-exact. We will also consider cases where only a subspace of the full (complexified) translation algebra is Q-exact, and in particular, with a choice of complex structure on the manifold, only anti-holomorphic translations (derivatives) are Q-exact, while the holomorphic translations are not. We call this type of twist as holomorphic-twist. If the dimension of spacetime d is greater than or equal to 3, and number of supercharges is more than 4, one can define a hybrid type of twist. With respect to Q, translations along some spacetime directions are Q-exact and anti-holomorphic translations along some directions are Q-exact; the representative example is Kapustin twist of 4d $\mathcal{N} = 2$ supersymmetric gauge theory on $\Sigma_1 \times \Sigma_2$, where Σ_i are Riemann surfaces.

Ω-deformation gives a further deformation in the supersymmetry algebra of the topologically twisted field theory with a scalar supercharge Q. Outstanding feature of the deformation is that the Q becomes $Q^\hbar$ such that $(Q^\hbar)^2 = L_V$, where L_V is the conserved charge acting on fields as the Lie derivative $\mathcal{L}_V$ by V. Here V is proportional to $\hbar$. Hence, we should pick a Killing vector field V to define $\Omega-$background. Before the introduction of Ω-background, the states in the Q-cohomology of the topological field theory satisfied $Q^2 = 0$ relation. After applying the Ω-background, the Q-cohomology is deformed to be $Q^\hbar$-cohomology, which again satisfies $(Q^\hbar)^2 = L_V = 0$. In other words, the operators in $Q^\hbar$-cohomology is localized in the fixed point of V.

In the famous example of Donaldson-Witten twist, it is possible to apply double Ω-background on each 2-planes in 4d spacetime. This was used to derive the prepotential of 4d $\mathcal{N} = 2$ gauge theory [98]. Similarly, in topological-holomorphic twist, we can apply Ω-deformation on one 2-plane in 4d spacetime, and localize the theory on the other 2-plane, where the theory becomes holomorphic. Holomorphic field theory in 1 complex dimension is special in the sense that the symmetry algebra becomes infinite dimensional. Until recently, it was not known that there is an infinite symmetry enhancement in 4d(2 complex dimension) holomorphic field theory [95, 133, 97], which can be obtained by the pure holomorphic twist in four dimensions. It is expected that one can find more fascinating algebraic structures in supersymmetric quantum field theory using various twists.

In [3], with Junya Yagi, we proposed a different derivation of the celebrated

SCFT/VOA correspondence (proposed by [125]) via Ω-background in some particular topological twist, so called Kapustin twist [135], of $4d$ $\mathcal{N}=2$ SCFTs. The SCFT/VOA correspondence is a correspondence between a 4d $\mathcal{N}=2$ superconformal field theory and a chiral algebra. More precisely, by picking a well designed supercharge $\mathbb{Q}$ in 4d $\mathcal{N}=2$ superconformal algebra by combining a certain supercharge with a certain conformal supercharge, one can define $\mathbb{Q}$-cohomology. The operators in the $\mathbb{Q}$-cohomology turn out to form a chiral algebra; they satisfy the usual holomorphic Operator Product Expansion(OPE) of Vertex Operator Algebra(VOA). In our work, the first key observation was that the 4d $\mathcal{N}=2$ SCFTs on a product of two Riemann surfaces can be seen as 2d $\mathcal{N}=(2,2)$ theories on one of the Riemann surfaces. Here, we do not integrate out all the KK modes, but keep those as a form of the superpotential of the 2d $\mathcal{N}=(2,2)$ theories. It is known that a localization can be performed [102], which directly gives a chiral algebra Lagrangian from the superpotential of the 2d $\mathcal{N}=(2,2)$ theory. Second idea was to prove that the supercharge $Q^{\hbar}$(or $Q^{\hbar}$-cohomology) of the Omega deformed theory corresponds to the one used in the original paper, which is $\mathbb{Q}$. By showing the equivalence of $\mathbb{Q}$ and $Q^{\hbar}$-cohomology, we re-derived the SCFT/VOA correspondence.

Continuing in the same line of idea, in [6], with Junya Yagi, we identified a classical limit of Vertex Operator Algebra(VOA) in supersymmetric field theories in various dimensions with topological-holomorphic twist, which is a more general version than a pure topological twist. The topological-holomorphic twist is a hybrid twist that renders the operators in the Q-cohomology invariant under the full translation in the topological directions and invariant under an anti-holomorphic translation in the holomorphic directions. The classical limit is meant to turn off a deformation parameter, which previously made the VOA noncommutative. Upon the twist, we get a scalar supercharge Q and an one-form supercharge Q_μ. Acting the one-form supercharge $\mathbb{Q}$ n-times on the operator $\mathcal{O}$ in the Q-cohomology, we can build another set of Q-closed operators $\mathcal{O}^{(n)}$, which are non-local. Given the new ingredients, we define a new algebraic structure with a new bi-linear operation between two algebra elements $[\![\mathcal{O}_1]\!]$, $[\![\mathcal{O}_2]\!]$, so called $\lambda-$bracket.

$$\{[\![\mathcal{O}_1]\!]\,{}_\lambda\,[\![\mathcal{O}_2]\!]\}(z_2) = (-1)^{F_1 d}\left[\!\!\left[\left(\int_{S^{d+1}_{x_2}} e^{\lambda(z_1-z_2)}\mathrm{d}z_1 \wedge \mathcal{O}_1^{(d)}(x_1)\right)\mathcal{O}_2(x_2)\right]\!\!\right]. \tag{1.17}$$

where F_1 is a Fermion number of $\mathcal{O}_1$ and d is the number of topological directions, which is 2. Here $\mathcal{O}_1^{(d)}$ is obtained by applying the 1-form supercharge on the local operator $\mathcal{O}_1$ d-times. The λ-bracket satisfies three properties: sesquilinearity, symmetry, and Jacobi identity. First, sesquilinearity:

$$\begin{aligned}\{\partial_z[\![\mathcal{O}_1]\!]\,{}_\lambda\,[\![\mathcal{O}_2]\!]\} &= -\lambda\{[\![\mathcal{O}_1]\!]\,{}_\lambda\,[\![\mathcal{O}_2]\!]\}\,,\\ \{[\![\mathcal{O}_1]\!]\,{}_\lambda\,\partial_z[\![\mathcal{O}_2]\!]\} &= (\lambda+\partial_z)\{[\![\mathcal{O}_1]\!]\,{}_\lambda\,[\![\mathcal{O}_2]\!]\}\,.\end{aligned} \tag{1.18}$$

Second, symmetry:

$$\{[\![\mathcal{O}_1]\!]\,{}_\lambda\,[\![\mathcal{O}_2]\!]\} = -(-1)^{(F_1+d)(F_2+d)}\{[\![\mathcal{O}_2]\!]\,{}_{-\lambda-\partial_z}\,[\![\mathcal{O}_1]\!]\}\,. \tag{1.19}$$

Finally, Jacobi identity:

$$\begin{aligned}&\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \{\llbracket \mathcal{O}_2 \rrbracket_\mu \llbracket \mathcal{O}_3 \rrbracket\}\}\\ &\qquad = \{\{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_2 \rrbracket\}_{\lambda+\mu} \llbracket \mathcal{O}_3 \rrbracket\} + (-1)^{(F_1+d)(F_2+d)} \{\llbracket \mathcal{O}_2 \rrbracket_\mu \{\llbracket \mathcal{O}_1 \rrbracket_\lambda \llbracket \mathcal{O}_3 \rrbracket\}\}\,. \end{aligned} \tag{1.20}$$

The algebraic structure we have identified is called the Poisson Vertex Algebra in the literature. The physical implication of this algebra is not known yet, however.

The application of Ω-background does not restrict to the field theory, but also apply to supergravity. It gives a new path to rigorously understand a topological subsector of AdS/CFT correspondence. The celebrated AdS/CFT correspondence [120, 121, 122] lacks a rigorous mathematical proof; however, in a particular topological sector the correspondence can be proven. Recently, Costello formed a precise mathematical definition of topological twists and Ω−deformation of M-theory and coined a notion of *Topological Holography*[106]: an isomorphism between the algebra of observables in both sides of the duality. In this context, one can discover a nice interplay among seemingly different, but well-established concepts like 2d chiral algebra, 3d $\mathcal{N} = 4$ Coulomb and Higgs branch algebra. Those algebraic structures are all realized in Ω-deformed worldvolume theories of branes. Moreover, both sides of the duality turn out to have a surprising triality structure. There seem to be many interesting aspects to be uncovered in this exact holography.

In [5], with Davide Gaiotto, we studied aspects of a topologically twisted supergravity [90] under Omega background and its interpretation as the bulk side of topological subsector of AdS/CFT correspondence. The topologically twisted supergravity is similarly defined as a topologically twisted field theory. In contrast to fixing a scalar supercharge in the topologically twisted field theory, we turn on the non-zero vacuum expectation value for a bosonic ghost for a gravitino in the supergravity to turn on the topologically twisted background. The background equally applies to the branes located in the spacetime and gives a twist in the field theory of the worldvolume theory of the brane. We are especially interested in M2-brane probing in this twisted background.

We consider 11d supergravity with a particular topological twist that makes 4 directions ($\mathbb{C}^2_{NC}$) holomorphic and 7 directions($\mathbb{R} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$) topological and turn on Ω_{ϵ_i} background on independent 2-planes labeled by i, with $i = 1, 2, 3$, where NC stands for non-commutative, and $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ with $\epsilon_i \in \mathbb{R}$, with $\epsilon_1 \ll \epsilon_2 \ll 1$. The non-commutativity originates from the non-zero 3-form $C = V^d \wedge d\bar{z}_1 \wedge d\bar{z}_2$, where V^d is a 1-form, which is a Poincare dual of the vector field V on $\mathbb{C}_{\epsilon_2}$, and z_i are coordinates of $\mathbb{C}^2_{NC}$. We place N M2 branes on $\mathbb{R} \times \mathbb{C}_{\epsilon_1}$.

The field theory side is a protected sub-sector of 3d $\mathcal{N} = 4$ $G = U(N)$ gauge theory(from M2-branes) with adjoint hypermultiplet and fundmental hypermultiplet, especially its Higgs branch, which effectively forms 1d topological quantum mechanics. It is possible to focus on the Higgs branch as the topologically twisted background gives a certain supercharge, whose cohomology exactly corresponds to the Higgs branch chiral ring for the 3d $\mathcal{N} = 4$ theory.

3d $\mathcal{N} = 4$ supersymmetry algebra is generated by the Lorentz symmetry $SU(2)_{Lor}$ and R-symmetry $SU(2)_H \times SU(2)_C$. The fermionic symmetry generators $Q^\alpha_{A\dot{A}}$ are charged under

$SU(2)_{Lor} \times SU(2)_H \times SU(2)_C$, as $(2,2,2)$, where $\alpha, A\dot{A}$ are the spinor indices of various $SU(2)$'s. The topologically twisted supergravity background induces Rozansky-Witten twist on the M2-brane worldvolume theory. The Rozansky-Witten twist is defined as a map from $SU(2)_{Lor}$ to $SU(2)_C$. Under this embedding map, the Cartan of the Lorentz symmetry generator M is combined with the Cartan R_C of $SU(2)_C$ generator to become $M' = M + R_C$. As a result, the resulting scalar supercharge Q that defines the Q-cohomology is a linear combinations of supercharges $Q = Q^{+}_{1\dot{1}} + Q^{-}_{1\dot{2}}$. It is known that the Q-cohomology coincides with the Higgs branch of the 3d $\mathcal{N} = 4$ theory.

The Higgs branch is parametrized by the scalar components of fundamental, and adjoint hypermultiplets, which are $\{I_i, J^i, X^i_j, Y^i_j\}$, where i, j are $U(N)$ gauge indices. Those scalars parametrize the hyper-Kahler target manifold $\mathcal{M}$, which has non-degenerate holomorphic symplectic structure. This structure turns the ring of holomorphic functions on $\mathcal{M}$ into a Poisson algebra with Poisson brackets between I and J, X and Y. Upon the Ω_{ϵ_1}-deformation, the Poisson bracket of the ring is quantized to become a commutator so that the Poisson algebra becomes a non-commutative algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$, with the deformation parameter given by the parameter ϵ_1 of the $\Omega-$background, and ϵ_2 enters due to the F-term relation to be explained. Another effect of the $\Omega-$deformation is the localization of the 3d theory into the fixed point of the $\Omega-$background, which is $\mathbb{R} \times \{0\} \in \mathbb{R} \times \mathbb{C}$, so eventually we have 1d quantum mechanics. The observables of the quantum mechanics consist of $t[m,n]$, which will be defined below. Keeping the notations for each 3d fields, we can write down the path integral of the quantum mechanics as follows:

$$Z = \int [DI][DJ]DX][DY] \exp\left[\int_{\mathbb{R}} dt \left(I(\partial_t + \sigma)J + \mathrm{Tr}\left(X\partial_t Y + X[\sigma, Y]\right)\right)\right] \tag{1.21}$$

where σ is the 1d avatar of 3d $\mathcal{N} = 4$ vector multiplet scalar. Due to the topologically twisted supergravity background, this quantum mechanics is topological.

The defining commutation relations of $\mathcal{A}_{\epsilon_1,\epsilon_2}$, which are descended from the Poisson brackets, are given by

$$\left[I_i, J^j\right] = \epsilon_1 \delta^i_j, \quad \left[X^i{}_j, Y^k{}_l\right] = \epsilon_1 \delta^i_l \delta^k_j \tag{1.22}$$

The gauge invariant operators in the Higgs branch form a chiral ring. The elements of the ring are

$$t[m,n] = \frac{1}{\epsilon_2} I X^m Y^n J = STr X^m Y^n \tag{1.23}$$

with F-term relation imposed on it:

$$\left[X^i{}_j, Y^j{}_k\right] + I_k J^i = \epsilon_2 \delta^i_k \tag{1.24}$$

Here, STr means the symmetrized trace; for example, $STrXY = TrXY + TrYX$.

The elements $t[m,n]$ and the commutation relations define the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$. In [7], with Yehao Zhou, we derived the most simplest commutation relation of $\mathcal{A}_{\epsilon_1,\epsilon_2}$:

$$[t[2,1], t[1,2]]_{\epsilon_1} = \epsilon_1 \epsilon_2 t[0,0] + \epsilon_1 \epsilon_2^2 t[0,0] t[0,0] \tag{1.25}$$

where $[\bullet]_{\epsilon_1}$ is the $\mathcal{O}(\epsilon_1)$ part of $[\bullet]$, $t[m,n] \in \mathcal{A}_{\epsilon_1,\epsilon_2}$.

The gravity side is interestingly captured by a field theory again, a 5d Chern-Simons theory [94], which is topological in 1 dimension and holomorphic in 4 dimensions, where there is a non-commutativity in the holomorphic directions, encoded as $[z_1, z_2] = \epsilon_2$. The action is given by

$$S = \frac{1}{\epsilon_1} \int_{\mathbb{R}_t \times \mathbb{C}^2_{NC}} dz_1 dz_2 \left(A \star_{\epsilon_2} dA + \frac{2}{3} A \star_{\epsilon_2} A \star_{\epsilon_2} A \right) \tag{1.26}$$

with $|\epsilon_1| \ll |\epsilon_2| \ll 1$; here $\star$ is a standard Moyal product, defined as

$$f \star_{\epsilon_2} g = fg + \epsilon_2 \frac{1}{2} \epsilon_{ij} \frac{\partial}{\partial z_i} f \frac{\partial}{\partial z_j} g + \epsilon_2^2 \frac{1}{2^2 2!} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \left(\frac{\partial}{\partial z_{i_1}} \frac{\partial}{\partial z_{i_2}} f \right) \left(\frac{\partial}{\partial z_{j_1}} \frac{\partial}{\partial z_{j_2}} g \right) + \dots \tag{1.27}$$

where $f, g \in \mathcal{O}(\mathbb{C}^2)$ and ϵ_{ij} is the alternating symbol.

In components, the 5d gauge field A can be written as

$$A = A_t dt + A_{\bar{z}_1} d\bar{z}_1 + A_{\bar{z}_2} d\bar{z}_2 \tag{1.28}$$

with all the component functions A_n are smooth in $\mathbb{R}^1$ and holomorphic in $\mathbb{C}^2_{NC}$, where NC means non-commutative. As $\mathbb{R}$ direction is topological, we can lift the dependence on t from the components A_n, and write it as a power series in z_1 and z_2.

$$A_n = \sum_{i,j=0} z_1^i z_2^j \partial_{z_1}^i \partial_{z_2}^j A_n \tag{1.29}$$

As the equation of motion for 5d CS theory is $F = 0$, where F is a field strength of the gauge field A, there is no physical degree of freedom; rather the operator algebra in the theory consists of the Fourier modes of the gauge field.

As a Lie algebra, the algebra of holomorphic functions on $\mathbb{C}^2_{NC}$ is isomorphic to a universal enveloping algebra of $Diff_{\epsilon_2}(\mathbb{C})$, which is an algebra of differential operators on $\mathbb{C}$ with $[z, \partial_z] = \epsilon_2$. Tensored with the apparent gauge symmetry algebra $gl(K)$ of the theory, the gauge symmetry algebra is

$$\mathfrak{g} = gl_K \otimes Diff_{\epsilon_2}(\mathbb{C}) \tag{1.30}$$

Hence, the classical observables in the 5d CS theory is

$$Obs^{cl}(5d\ CS) = (Sym^*(\mathfrak{g}^\vee[1]), d) = C^*(gl_K \otimes Diff_{\epsilon_2}\mathbb{C}) \tag{1.31}$$

where $C^*(\mathfrak{g})$ is the Lie algebra cohomology of $\mathfrak{g}$. Here, $\mathfrak{g}^\vee$ is a dual of $\mathfrak{g}$, [1] indicates ghost number 1, and d is the differential of the complex. The Koszul dual of $Obs^{cl}(5d\ CS)$ is isomorphic to $U(\mathfrak{g})$, where $U(g)$ is a universal enveloping algebra for g.

As there is a non-trivial deformation induced by the coupling constant $1/\epsilon_1$ in the action (1.26), the operator algebra is deformed accordingly. So, we will denote the operator algebra of 5d CS theory as $\mathcal{A}'_{\epsilon_1,\epsilon_2} = U_{\epsilon_1}(Diff_{\epsilon_2}(\mathbb{C}) \otimes gl_1)$, where K was specified as 1.

The statement of topological holography is an isomorphism between two operator algebras, $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and $\mathcal{A}'_{\epsilon_1,\epsilon_2}$. The isomorphism is manifested by the following interaction Lagrangian between 5d CS modes and 1d quantum mechanics, which is realized as an action defined on a line.

$$\int_{\mathbb{R}} t[m,n]\partial^m_{z_1}\partial^n_{z_2}A_t(t,0,0)dt \tag{1.32}$$

The detailed proof of the isomorphism argued by the the uniqueness of the deformation of the algebra $U(Diff_{\epsilon_2}(\mathbb{C})\otimes gl_1)$ can be found in [109].

As a result of this interaction Lagrangian, some correlation functions in the 5d-1d coupled system get a gauge anomaly. The 5d gauge theory itself is renormalizable as proved in [94]. The presence of the 5d-1d interaction Lagrangian introduces a new vertex in various Feynman diagrams, and some of those Feynman diagrams are not invariant under the gauge symmetry of the 5d gauge theory, at least superficially. Even though some Feynman diagrams have non-zero variations under the gauge transformation, what is actually important is to make sure that the sum of gauge variations of particular combinations of Feynman diagrams is zero. In [7], with Yehao Zhou, by identifying some Feynman diagrams whose variations are all proportional to ϵ_1, we showed how to cancel the gauge anomaly. The cancellation of the 5d gauge anomaly itself enabled us to reproduce the algebra commutation relation of $\mathcal{A}_{\epsilon_1,\epsilon_2}$, (1.25).

It is possible to introduce M5-brane to decorate the relation.

	0	1	2	3	4	5	6	7	8	9	10
Geometry	$\mathbb{R}_t$	$\mathbb{C}$		$\mathbb{C}^2_{NC}$				$\mathbb{C}$		$\mathbb{R}$	S^1
$M2(D2)$	$\times$	$\times$	$\times$								
$M5$		$\times$	$\times$	$\times$	$\times$					$\times$	$\times$

Table 1.1: M2, M5-brane

The gauge invariant operators in the M5-brane form a bi-module of the algebra(of M2-brane) in the field theory side and a chiral algebra that interacts with 5d Chern-Simons in the gravity side. One of the results in [5] is a formulation of the module in both sides of twisted holography.

In the field theory side, it is a collection of gauge invariant operators in 2d $\mathcal{N}=(2,2)$ field theory on M5-brane, which we call as $\mathcal{M}_{\epsilon_1,\epsilon_2}$. The field content of the 2d theory is chiral and anti-chiral superfields, whose bottom components are φ, $\tilde{\varphi}$. The interaction between the 2d $\mathcal{N}=(2,2)$ system and the 3d $\mathcal{N}=4$ system is encoded in the 2d superpotential

$$\mathcal{W}=\tilde{\varphi}X\varphi \tag{1.33}$$

The gauge invariant operators that consist of $\mathcal{M}_{\epsilon_1,\epsilon_2}$ are then given by

$$b[z^m]=IY^m\varphi, \quad c[z^m]=\tilde{\varphi}Y^mJ \tag{1.34}$$

To claim that $\mathcal{M}_{\epsilon_1,\epsilon_2}$ is a bi-module of the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$, we need to establish more commutation relations between the set of letters $\{\varphi, \tilde{\varphi}\}$ and $\{X, Y, I, J\}$. Those are given by

$$\begin{aligned} IP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})I \\ JP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})J \\ X^i_jP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})X^i_j \\ Y^i_jP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})(Y^i_j+\tilde{\varphi}^i\varphi_j) \\ X^i_j\varphi_iP(\varphi,\tilde{\varphi}) &= -\epsilon_1\partial_{\tilde{\varphi}_j}P(\varphi,\tilde{\varphi}) \\ X^i_j\tilde{\varphi}^jP(\varphi,\tilde{\varphi}) &= -\epsilon_1\partial_{\varphi_i}P(\varphi,\tilde{\varphi}) \end{aligned} \tag{1.35}$$

where $P(\varphi,\tilde{\varphi})$ is a polynomial of φ, $\tilde{\varphi}$. Here, φ, $\tilde{\varphi}$ carry gauge indices i, as they are charged under the 3d gauge group $U(N)$ too. From these ingredients, we show that the simplest algebra($\mathcal{A}_{\epsilon_1,\epsilon_2}$)-bi-module($\mathcal{M}_{\epsilon_1,\epsilon_2}$) commutator is

$$\left[t[2,1], b[z^1]c[z^0]\right]_{\epsilon_1} = \epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0] + \epsilon_1\epsilon_2 b[z^0]c[z^0] \tag{1.36}$$

where $b[z^m]$, $c[z^m] \in \mathcal{M}_{\epsilon_1,\epsilon_2}$, and the notation $[\bullet]_{\epsilon_1}$ refers to the ϵ_1 order term of $[\bullet]$.

In the gravity side, the corresponding observables arise as a certain Vertex Operator Algebra, called $\beta-\gamma$ system, whose Lagrangian can be written as

$$\int_{\mathbb{C}} \beta(z)(\partial_{\bar{z}} - A_{\bar{z}}\star_{\epsilon_2})\gamma(z) \tag{1.37}$$

where $\mathbb{C} \subset \mathbb{C}^2_{NC}$. β and γ can be expanded in z, and we call the modes of β and γ as β_m and γ_m.

	0	1	2	3	4	5	6	7	8	9
Geometry	$\mathbb{R}_t$	$\mathbb{C}$		$\mathbb{C}^2_{NC}$				$\mathbb{C}$		$\mathbb{R}$
1d TQM	$\times$									
2d $\beta\gamma$				$\times$	$\times$					
5d CS	$\times$			$\times$	$\times$	$\times$	$\times$			

Table 1.2: Bulk perspective

M5-brane realized the module $\mathcal{M}_{\epsilon_1,\epsilon_2}$; however, we can not simply let M5 brane to enter into the theory. For the combined system to be quantum mechanically consistent, we need to cancel all the potential gauge anomaly. Similar to (1.32), β_m and γ_m couple to (1.34) in the following way

$$b[z^m]\beta_m, \quad c[z^m]\gamma_m \tag{1.38}$$

These couplings lead to potential gauge anomalies that are involved in Feynman diagrams of the coupled system of $\beta\gamma$, 5d CS theory. In [7], with Yehao Zhou, we showed how to cancel the gauge anomaly. The cancellation of the 5d gauge anomaly itself enabled us to reproduce the algebra-bi-module commutation relation, (1.36).

1.1. A list of papers

I have pursued other topics including

- Superconformal blocks of 2d small $\mathcal{N} = 4$ chiral algebra(work with Filip Kos) [2],
- Chiral algebra(or quantum mechanics as a dimensional reduction of chiral algebra) as a protected subsector of higher dimensional superconformal field theory(work with Junya Yagi) [3],
- Non-relativistic string theory(work with Jaume Gomis, Ziqi Yan) [4],
- Aspects of $\Omega-$deformed M-theory(work with Davide Gaiotto) [5],
- Poisson Vertex Algebra as a protected subsector of supersymmetric field theories in diverse dimensions(work with Junya Yagi) [6],
- Double quantization of Seiberg-Witten curve(work with Nathan Haouzi) [8]

However, I will focus on the two topics elaborated more explicitly in the introduction, which are

- D0-brane quantum mechanics, instanton counting and superconformal index(work with Chi-Ming Chang, Ori Ganor) [1].
- Feynman diagram and $\Omega-$deformed M-theory(work with Yehao Zhou), [7].

The content below is largely taken from [1] and [7].

Chapter 2

An Index for Ray Operators in 5d E_n SCFTs

2.1. Introduction

There is strong evidence for an interacting 5d superconformal field theory (SCFT) with E_8 global symmetry and a one-dimensional Coulomb branch [10]. A few of its (dual) realizations in string theory are the low energy limits of the systems listed below:

(i) A D4-brane probing a 9d E_8 singularity in type-I' string theory [10]; the latter is realized by the infinitely strong coupling limit of seven coincident D8-branes and an orientifold (O8) plane [11].

(ii) M-theory on a certain degenerate Calabi-Yau manifold [12, 13, 14]; the Calabi-Yau threefold can be taken as the canonical line bundle of a del Pezzo surface B_8 (which can be constructed as the blow-up of $\mathbb{CP}^2$ at 8 points) in the limit that the volume of B_8 goes to zero. (See also [15] where the study of such a limit of M-theory was initiated, and [17, 19] where the F-theory version of this degeneration was described.)

(iii) The 6d E_8 SCFT [20, 22] compactified on S^1 [14].

(iv) Webs of (p, q) 5-branes [23].

The E_8 theory can be deformed by relevant operators to 5d SCFTs with smaller E_n global symmetries ($n = 0, \ldots, 7$). One of the remarkable achievements of the last few years has been the construction of a supersymmetric index that counts local operators that preserve (at least) $\frac{1}{8}$ of the supersymmetry of the E_n theories [24, 25, 26, 27, 29, 30].

Technically, this superconformal index is constructed by computing the partition function on $\mathrm{S}^4 \times \mathrm{S}^1$ of a 5d supersymmetric gauge theory with gauge group SU(2) and $N_f = n - 1$ hypermultiplets. The global flavor symmetry is $\mathrm{SO}(2N_f)$, which combines with the U(1) instanton charge to form the $\mathrm{SO}(2n-2) \times \mathrm{U}(1) \subset E_n$, as predicted in [10]. The partition function is computed using the techniques developed in [31], with insight from string theory for the proper treatment of zero size instantons [34]. The partition function is presented as an integral over a product of Nekrasov partition functions [98], and the resulting index is expressed as an infinite sum of monomials in fugacities that capture spin and R-charge, with coefficients that are linear combinations of characters of $\mathrm{SO}(2n-2)\times\mathrm{U}(1)$. It is remarkable that these linear combinations of characters match representations of $E_n \supset \mathrm{SO}(2n-2)\times\mathrm{U}(1)$.

The string-theory or M-theory realizations of the E_n theories allow for a construction of BPS line operators akin to Wilson lines as follows. In the type-I' setting (i), we introduce a semi-infinite fundamental string (F1) perpendicular to the plane of the D8-branes with one of its endpoints at infinity and the other on the D4-brane. In the M-theory setting (ii), we add an M2-brane that fills the $\mathbb{C}$ fiber of the canonical bundle above a point of the del Pezzo base. In the 6d setting (iii), the line operator is the low-energy limit of a surface operator on S^1, and in the type-IIB setting (iv), it is realized by an open (p, q) string. In addition, the 5d E_n theories also possess BPS operators supported on a line with an endpoint, which we will refer to as Ray operators. They are analogous to a Wilson line along a ray, capped by a quark field at the endpoint. The aim of this paper is to study these 5d ray operators and extend the results of [24, 30] by constructing an index for $\frac{1}{8}$BPS ray operators.

Calculating the index for ray operators again requires a careful treatment of zero-size instantons and additional insight from string theory. The index can again be written in terms of characters of $\mathrm{SO}(2n-2)\times\mathrm{U}(1)$ which combine into characters of E_n. Unlike local operators, the ray operators are charged under the center of E_n, in the cases where it is nontrivial ($n < 8$). The appearance of complete E_n characters is a nontrivial check of the validity of the assumptions behind the computation of the index. Moreover, for $n < 8$ the weight lattice is larger than the root lattice of E_n, and we find E_n representations that do not appear in the superconformal index. For example, for E_6 we find the representations 27, 1728, etc., consistent with the $\mathbb{Z}_3$ charge of the ray operator.

This chapter is organized as follows. In §2.2 we review the construction of 5d SCFTs with E_n global symmetry and their superconformal indices. In §2.3 we introduce ray operators into the 5d SCFTs, and we compute their indices in §2.4 (with our final results in §2.4). We conclude with a summary and discussion in §2.5.

2.2. Review of the 5d E_n SCFTs and their superconformal indices

Following the discovery of [11] that in the infinite string coupling limit of type I' string theory, a 9d E_n gauge theory describes the low-energy limit of $N_f = n - 1$ D8-branes coincident with an O8-plane, Seiberg constructed a 5d SCFT with E_n global symmetry by probing the D8/O8 singularity with N D4-branes [10]. The brane directions are listed in the table below.

	0	1	2	3	4	5	6	7	8	9
D8/O8	×	×	×	×	×	×	×	×	×	
D4	×	×	×	×	×					

Denoting by ϵ_L, ϵ_R left and right 10d Majorana-Weyl spinors, the configuration preserves those SUSY parameters that satisfy $\Gamma^{012345678}\epsilon_L = \Gamma^{01234}\epsilon_L = \epsilon_R$. The rotations in directions $5\ldots8$ act on spinors as $\mathrm{SU}(2)^R_+\times\mathrm{SU}(2)^R_-$. The second factor acts trivially on the supercharges, while the first factor acts nontrivially and is identified with the R-symmetry $\mathrm{SU}(2)_R$ of the 5d theory. The theory has an N-dimensional Coulomb branch $(\mathbb{R}_+)^N/S_N$ that can be identified with the D4-branes moving away from the E_n singularity, and a Higgs branch that can be identified with the moduli space of E_n instantons at instanton number N. The supermultiplet associated with the center of mass of the D4-branes in directions $5\ldots8$ decouples and is not considered part of the SCFT.

Raising the value of the inverse string coupling constant $1/g_{\mathrm{st}}$ at the common position of the D8-branes and O8-plane from zero (formally $g_{\mathrm{st}}=\infty$) to a nonzero value breaks the global E_n symmetry to $\mathrm{SO}(2N_f)\times\mathrm{U}(1)$. The $\mathrm{SO}(2N_f)$ factor comes from the gauge symmetry of the D8-branes, while the U(1) factor is associated with D0-brane charge. At low-energy the D4-brane probe theory is then described by $\mathrm{Sp}(N)$ SYM coupled to N_f hypermultiplets in the fundamental representation $2N$ of the gauge group, and also a single hypermultiplet in the antisymmetric representation. $\mathrm{SO}(2N_f)$ is the global flavor symmetry, while U(1) is the symmetry associated with instanton number whose conserved current is

$$J = \frac{1}{8\pi^2}\,\mathrm{tr}\star(F\wedge F). \tag{2.1}$$

Here F is the $\mathrm{Sp}(N)$ field strength.

In this paper, we will focus on the Sp(1) gauge theories, which have no antisymmetric hypermultiplet. The vector multiplet consists of a gauge field A_μ, a real scalar Φ, and symplectic-Majorana fermions λ^A_m, where $A=1,2$ denotes the $\mathrm{SU}(2)_R$ R-symmetry doublet index, and $m=1,\ldots,4$ denotes the SO(1,4) spinor index. The hypermultiplet consists of complex scalars q^A and fermions ψ_m.

Superconformal indices of 5d SCFTs

We will now discuss some general aspects of the superconformal index [36] of 5d SCFTs. The superconformal algebra of 5d SCFTs is $F(4)$, which in Euclidean signature has the bosonic subgroup SO(1,6)$\times\mathrm{SU}(2)_R$. One of the most important consequences of the superconformal symmetry is the BPS bound, which follows from the anticommutator [36],

$$\{Q^A_m, S^n_B\} = \delta^n_m\delta^A_B D + 2\delta^A_B {M_m}^n - 3\delta^n_m {R_B}^A, \tag{2.2}$$

where D is the dilatation generator, ${M_m}^n$ are the SO(5) rotation generators, and ${R_B}^A$ are the $\mathrm{SU}(2)_R$ R-symmetry generators. Q^A_m and S^m_A are the supercharge and superconformal charge. Note that ${M_m}^n$ are components of an Sp(2) matrix. We will work in the basis where ${M_1}^1 = -{M_2}^2$ and ${M_3}^3 = -{M_4}^4$, and we will denote $J_+ \equiv {M_1}^1$, $J_- \equiv {M_3}^3$, which are the Cartan generators of $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \subset \mathrm{SO}(5)$. We also denote $J_R \equiv {R_1}^1$, which is the Cartan generator of $\mathrm{SU}(2)_R$. In radial quantization, the superconformal generator is the hermitian conjugate of the supercharge, i.e., $S^m_A = (Q^A_m)^\dagger$. The anticommutator (2.2) implies positivity conditions on linear combinations of the dilatation, rotations, and R-symmetry generators. For instance, in the case of $m = 2$ and $A = 1$, (2.2) implies

$$\Delta \equiv \{Q, S\} = D - 2J_+ - 3J_R \geq 0, \tag{2.3}$$

where for simplicity we denote $Q \equiv Q^1_2$, $S \equiv S^2_1$. The operators that saturate the BPS bound (2.3) are called $\frac{1}{8}$BPS operators. These operators are annihilated by both Q and S. By the state-operator correspondence, the space of local operators is isomorphic to the Hilbert space H of the (radially quantized) theory on S^4. The number of $\frac{1}{8}$BPS operators with given quantum numbers (counted with $\pm$ signs according to whether they are bosonic or fermionic) is captured by the superconformal index,

$$I_{\mathrm{SCI}}(\epsilon_+, \epsilon_-, m_i) = \mathrm{Tr}_H \left[(-1)^F e^{-\beta\Delta} e^{-2\epsilon_+(J_+ + J_R) - 2\epsilon_- J_-} e^{-\sum F_i m_i}\right], \tag{2.4}$$

where F is the fermion number operator, and F_i denote the generators of other global symmetries.[1] Only the states that saturate the BPS bound (with $\Delta = 0$) contribute to the trace, and the contributions from states with nonzero Δ pairwise cancel out due to $(-1)^F$, since Q and S commute with the other operators inside the trace.

The $\frac{1}{8}$BPS operators are annihilated by both the supercharge Q and one superconformal charge S. Formally, if we regard Q as an exterior derivative d and S as its Hermitian conjugate $d^\star$, then $\{Q, S\}$ corresponds to the Laplacian $\Delta = d^\star d + d d^\star$. Hodge theorem states that the space of harmonic forms (states with $\Delta = 0$) is isomorphic to the cohomology of d. Analogous arguments, formulated in terms of Q, S, show that the Hilbert space H of $\frac{1}{8}$BPS operators is isomorphic to the cohomology of Q [37], which will be referred to as Q-cohomology. The superconformal index can be interpreted as the Euler characteristic of the Q-cohomology.[2]

Consider an $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R$ multiplet of operators with $\mathrm{SU}(2)_+$ spin j_+ and $\mathrm{SU}(2)_R$ spin j_R. Out of the $(2j_+ + 1)(2j_R + 1)$ states, at most one can saturate the BPS bound (2.3) — this is the state with maximal $J_+ = j_+$ and $J_R = j_R$. Thus, a $\frac{1}{8}$BPS state that contributes to the index (2.4) has maximal J_+ and J_R charge in its $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R$ multiplet. Consider a $\frac{1}{8}$BPS state with $J_R = 0$. According to the above discussion, it must be a singlet of $\mathrm{SU}(2)_R$. The algebra of $\mathrm{SU}(2)_R$ is generated by ${R_1}^1$, ${R_1}^2$ and ${R_2}^1$, and since the state is annihilated by both ${R_1}^2$ and $Q \equiv Q^1_2$, it must be annihilated by $[{R_1}^2, Q^1_2]$ which is proportional to Q^2_2. Similarly, we see that it is annihilated by S^1_1 as well. It therefore has enhanced supersymmetry, being annihilated by Q^1_2, Q^2_2, S^2_1, S^2_2, and is in fact a $\frac{1}{4}$BPS. Similarly, a $\frac{1}{8}$BPS state with $J_+ = 0$ is

[1] In this paper, we will sometimes parametrize the indices using the fugacities $t = e^{-\epsilon_+}$ and $u = e^{-\epsilon_-}$.

[2] The Q-cohomology contains more information than its Euler characteristic. For example, the Hilbert-Poincaré polynomial of the Q-cohomology gives the partition function of the BPS operators [37, 38].

a singlet of SU(2)$_+$ and is annihilated by ${M_1}^1$, ${M_1}^2$ and ${M_2}^1$, and therefore also by $[{M_1}^2, Q_2^1]$ and $[{M_2}^1, S_1^2]$. It is thus annihilated by Q_1^1, Q_2^1, S_1^1, S_1^2. Moreover, since it saturates the BPS bound $D = 3J_R$, it must be an SU(2)$_-$ singlet as well. This is because similarly to the BPS bound (2.3), we also have in general

$$\{Q_4^1, S_1^4\} = D - 2J_- - 3J_R \geq 0. \tag{2.5}$$

If the state in question had nonzero SU(2)$_-$ spin j_-, then it would be part of a multiplet of $(2j_- + 1)$ states with $D = 3J_R$, but the state with maximal $J_- = j_-$ in that multiplet would then violate the bound (2.5). It follows that a $\frac{1}{8}$BPS state with $J_+ = 0$ must also have $J_- = 0$ and is in fact $\frac{1}{2}$BPS, being annihilated by all Q_m^1 and all S_1^m $(m = 1, \ldots, 4)$. For example, a $\frac{1}{8}$BPS state with $J_+ + J_R = \frac{1}{2}$ must have either $J_+ = 0$ or $J_R = 0$ and is therefore at least $\frac{1}{4}$BPS. If it is not an SU(2)$_-$ singlet, then it must have $J_R = 0$ and $J_+ = \frac{1}{2}$ and is $\frac{1}{4}$BPS. As another example, a $\frac{1}{8}$BPS state that has $J_+ + J_R = 0$ is a singlet of both SU(2)$_+$ and SU(2)$_R$. It is therefore annihilated by all Q_m^A and S_A^m and must be the vacuum state. Thus in an expansion of the index (2.4) in $e^{-2\epsilon_+}$, the only term that is ϵ_+-independent is the contribution of the identity operator 1. Other terms can be expanded in characters of SU(2)$_-$,

$$\chi_{2j+1}(e^{-\epsilon_-}) = \sum_{m=-j}^{j} e^{-2m\epsilon_-} = \frac{\sinh(2j+1)\epsilon_-}{\sinh \epsilon_-}.$$

The terms linear in $e^{-\epsilon_+}$ and proportional to $\chi_{2j+1}(e^{-\epsilon_-})$, with $j > 0$, are the contributions of $\frac{1}{4}$BPS states. Note that even when both J_+ and J_R are nonzero, the $\frac{1}{8}$BPS states preserve half of the supercharges, namely Q_2^1 as well as Q_1^1, Q_3^1, and Q_4^1, since the latter three lower Δ (but the hermitian conjugates of Q_1^1, Q_3^1, and Q_4^1 are in general not preserved).

In Euclidean signature, the space $\mathbb{R}^5$ can be conformally mapped to $\mathbb{R}\times\mathrm{S}^4$, and the superconformal index (2.4) can be interpreted as a twisted partition function of the theory on $\mathrm{S}^1\times\mathrm{S}^4$. For theories with a Lagrangian description, it can be computed by a path integral with the fields satisfying periodic boundary conditions along S^1, further twisted by the various fugacities.

Superconformal indices from 5d SYM

Now, let us focus on the E_n SCFTs. It has been shown that the superconformal indices of them can be computed using the IR 5d SYM with fundamental matters [24, 30].

In the IR theory, the superconformal algebra is not defined, because the Yang-Mills coupling constant is dimensionful, but we can consider the Q-cohomology on gauge invariant operators.

A large class of gauge invariant operators can be constructed from the 'letters' of the 5d $N = 1$ gauge theory, given by the fields listed in the following table, acted on by an arbitrary numbers of derivatives, modulo the free field equations of motion.

	$F_{\mu\nu}$	λ^A_m	Φ	q^A	ψ_m	∂_μ	Q
$\widetilde{E}$	2	$\frac{3}{2}$	1	$\frac{3}{2}$	2	1	$\frac{1}{2}$

In this table, we also introduced a new quantum number $\widetilde{E}$ for the fields, the derivative symbol and the supercharge Q, and we define

$$\widetilde{\Delta} = \widetilde{E} - 2J_+ - 3J_R, \tag{2.6}$$

so that $\widetilde{\Delta}(Q) = 0$[3], which will help with computing the Q-cohomology. One should not confuse $\widetilde{E}$ and $\widetilde{\Delta}$ with the dimension D and the radial Hamiltonian Δ that appear in the BPS bound formula (2.3). We emphasize that $\widetilde{E}$, which measures the classical dimension of the corresponding field in the SYM theory, is purely a bookkeeping device and the Q-cohomology does not depend on the assignment of the $\widetilde{E}$. Let us first consider the single-letter Q-cohomology. The supersymmetry transformation on the component fields in the vector and hypermultiplets can be found in (2.10) and (2.14) of [24]. It is not hard to see that the operators with $\widetilde{\Delta} \geq 1$ have trivial Q-cohomology. On the other hand, for $\widetilde{\Delta} = -1$ and 0, we have nontrivial cohomology generated by

$$\begin{aligned} \widetilde{\Delta} = -1: &\quad \lambda_{+0+}, \\ \widetilde{\Delta} = 0: &\quad \lambda_{0\pm+}, \quad q^+, \end{aligned} \tag{2.7}$$

and also the two derivatives $\partial_{+\pm}$ acting on them. The subscripts of $\partial_{\pm\pm}$ and the first two subscripts of $\lambda_{\pm0\pm}$, $\lambda_{0\pm\pm}$ denote their $2J_+$ and $2J_-$ charges. The last subscripts of $\lambda_{\pm0\pm}$, $\lambda_{0\pm\pm}$ and also the superscript of q^+ denote their $2J_R$ charges.[4] Note that only those components with maximal J_+ and J_R in an SU(2)$_+\times$SU(2)$_R$ multiplet, for each field, can be generators of a nontrivial Q-cohomology.

The single-letter operators are subject to an equation of motion,

$$\partial_{++}\lambda_{0-+} + \partial_{+-}\lambda_{0++} = -\partial_5\lambda_{+0+}, \tag{2.8}$$

and it is not hard to check that $\partial_5\lambda_{+0+}$ is Q-exact, and therefore vanishes in the Q-cohomology. We compute the single-letter index by summing over the letters (F, λ, Φ, q, ψ):

$$f = \sum_{letters} (-1)^F t^{2(J_+ + J_R)} u^{2J_-} e^{-\sum F_i m_i} = f_{\text{adj}} + f_{\text{fund}}, \tag{2.9}$$

where t and u are related to ϵ_+ and ϵ_- by $t = e^{-\epsilon_+}$ and $u = e^{-\epsilon_-}$, the m_i ($i = 1, \dots, N_f$) are the chemical potentials of the flavor charges of a Cartan subalgebra U(1)$^{N_f} \subset$ O($2N_f$), and the single-letter index for the vector multiplet and fundamental hypermultiplet are given by

$$\begin{aligned} f_{\text{adj}} &= \frac{-t^2 - t(u + u^{-1}) + t^2}{(1-tu)(1-tu^{-1})} = -\frac{t(u+u^{-1})}{(1-tu)(1-tu^{-1})}, \\ f_{\text{fund}} &= \frac{t}{(1-tu)(1-tu^{-1})} \sum_{\ell=1}^{N_f} 2\cosh m_\ell. \end{aligned} \tag{2.10}$$

[3]Recall $Q \equiv Q^1_2$ has $J_+ = -1/2$ and $J_R = 1/2$.

[4]$q^\pm = q^1$.

The multi-letter index can be computed by the following formula [39, 40],

$$
\begin{aligned}
I(t,u,m_i) &= \int_{Sp(1)} Z_{\text{1-loop}}(t,u,w,m_i)dU, \\
Z_{\text{1-loop}}(t,u,w,m_i) &= \exp\Big[\sum_{\substack{R \\ \{\text{fund, adj}\}}} \sum_{n=1}^{\infty} \frac{1}{n} f_R(t^n,u^n,m_i^n)\chi_R(w^n)\Big],
\end{aligned}
\tag{2.11}
$$

where the integral is over the Sp(1) matrices U, and w is one of the eigenvalues of U. The U-integral can be simplified to a one-dimensional integral $dU = \frac{1}{\pi}\sin^2\alpha d\alpha$ where $w = e^{i\alpha}$. The representations R that appear in the sum are the fundamental and adjoint representations. $\chi_R(w)$ are the Sp(1) characters, for example $\chi_{\text{adj}}(w) = w^{-2} + 1 + w^2$ and $\chi_{\text{fund}}(w) = w^{-1} + w$. One can recognize that the integrand $Z_{\text{1-loop}}$ is a multi-letter index that counts gauge covariant operators, and the integration over the gauge group imposes the gauge invariance. The single-letter indices (2.10) and the formula (2.11) can also be derived by evaluating a path integral of the 5d SYM on $\mathrm{S}^1\times\mathrm{S}^4$ [31, 24], where the matrix U is identified with the Sp(1) holonomy along the S^1.

The index (2.11) cannot be the full superconformal index, because all the gauge invariant operators that contribute to (2.11) do not carry the topological U(1) charge associated with the conserved current (2.1).

The contributions of the operators with n units of the topological U(1) charge to the superconformal index can be computed in the path integral on $\mathrm{S}^1\times\mathrm{S}^4$ with the field strength restricted to the n-th instanton sector,

$$
\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F\wedge F) = n. \tag{2.12}
$$

In [31, 24], using supersymmetric localization, it was shown that the path integral localizes at the singular instanton solution at the south pole and anti-instanton solution at the north pole. Near the south (north) poles, the spacetime looks like $\mathrm{S}^1\times\mathbb{R}^4$, and the path integral over the solutions to the instanton (anti-instanton) equation reduces to the the Nekrasov instanton partition function $Z_{\text{inst}}(t,u,m_i,q)$ in the Ω-background on $\mathbb{R}^4$. The superconformal index is then computed by the formula

$$
I_{\text{SCI}}(t,u,m_i,q) = \int_{Sp(1)} Z_{\text{1-loop}}(t,u,w,m_i)|Z_{\text{inst}}(t,u,w,m_i,q)|^2 dU. \tag{2.13}
$$

where $Z_{\text{inst}}(t,u,w,m_i,q)^* = Z_{\text{inst}}(t,u,w^{-1},-m_i,q^{-1})$ is the contribution from the anti-instantons at the north pole.

In [30], it has been argued that the Nekrasov instanton partition function can be computed by the Witten indices of certain D0-brane quantum mechanics. In the next subsection, we review the D0-brane quantum mechanics, and compute their Witten indices.

strings	$N=4$ multiplets	fields	$\mathrm{SU}(2)_-\times\mathrm{SU}(2)_+\times\mathrm{SU}(2)^R_-\times\mathrm{SU}(2)^R_+$
D0-D0 strings	vector	gauge field	$(1,1,1,1)$
		scalar	$(1,1,1,1)$
		fermions	$(1,2,1,2)$
	Fermi	fermions	$(2,1,2,1)$
	twisted hyper	scalars	$(1,1,2,2)$
		fermions	$(1,2,2,1)$
	hyper	scalars	$(2,2,1,1)$
		fermions	$(2,1,1,2)$
D0-D4 strings	hyper	scalars	$(1,2,1,1)$
		fermions	$(1,1,1,2)$
	Fermi	fermions	$(1,1,2,1)$
D0-D8 strings	Fermi	fermions	$(1,1,1,1)$

Table 2.1: The field content of the D0-D4-D8/O8 quantum mechanics.

The D0-D4-D8/O8 system

The instantons in the IR 5d Sp(1) SYM of the E_n theory are described by the D0-branes moving in the background of one D4-brane and N_f D8-branes coincident with an O8-plane [30]. The low energy theory on k D0-branes is a $N=4$ $\mathrm{O}(k)$ gauged quantum mechanics, whose field content is listed in Table 2.1, where the last column lists the representations of various fields under the R-symmetry $\mathrm{SU}(2)_+\times\mathrm{SU}(2)^R_+$ and global symmetry $\mathrm{SU}(2)_-\times\mathrm{SU}(2)^R_-$. The $\mathrm{SU}(2)_+\times\mathrm{SU}(2)_-$ and $\mathrm{SU}(2)^R_+\times\mathrm{SU}(2)^R_-$ are the rotation groups of the four-planes $\mathbb{R}^{1234}$ and $\mathbb{R}^{5678}$, respectively. The vector and Fermi multiplets from the D0-D0 strings are in the antisymmetric representation of the gauge group $\mathrm{O}(k)$. The hyper- and twisted hypermultiplets are in the symmetric representation of $\mathrm{O}(k)$. The D0-D4 (D0-D8) strings are in the bifundamental representation of the gauge group $\mathrm{O}(k)$ and flavor group Sp(1) ($\mathrm{SO}(2N_f)$).

Consider a $N=2$ subalgebra with supercharges Q and $Q^\dagger$ inside the $N=4$ supersymmetry algebra. The Witten index is defined as

$$Z^k_{\text{D0-D4-D8/O8}}(t,u,w,m_i)=\mathrm{Tr}_{H_{\mathrm{QM}}}\left[(-1)^F e^{-\beta\{Q^\dagger,Q\}}t^{2(J_++J_R)}u^{2J_-}v^{2J'_R}w^{2\Pi}e^{-\sum F_i m_i}\right], \quad (2.14)$$

where $J_\pm$, J_R, J'_R and Π are the Cartan generators of the $\mathrm{SU}(2)_\pm$, $\mathrm{SU}(2)^R_+$, $\mathrm{SU}(2)^R_-$ and the Sp(1) flavor symmetry. We give a very brief description of how this Witten index is computed, following [30, 41], by applying supersymmetric localization. The index is invariant under continuous deformations that preserve the supercharges Q and $Q^\dagger$. One can consider the free field limit, and the path integral over nonzero modes reduces to the product of one-loop determinants, which depend on the fixed background of bosonic zero modes. One then integrates over the zero modes exactly.

The 1d gauge field is non-dynamical, but its holonomy on S^1 is a bosonic zero mode, which combines with the zero mode of the scalar in the vector multiplet to form a complex variable ϕ taking values in the maximal torus of the complexified gauge group. There are fermionic zero modes coming from the fermions (gauginos) in the vector multiplet, which are absorbed by the Yukawa coupling terms in the action involving the gauginos, the scalars, and the fermions in the charged matter multiplets (as in (2.21) of [30]). This contributes additional terms to the integrand as the free correlators of the scalars and the fermions, which in turn combine with the previous one-loop determinants to a total derivative of $\partial/\partial\bar{\phi}$. The zero mode integral becomes a contour integral over ϕ. The contour can be determined by a careful regularization of the divergences on the complex ϕ-plane [42, 43], by reintroducing the auxiliary field D of the vector multiplet.

The gauge group $\mathrm{O}(k)$ has two disjoint components. The group element in one component, denoted by $\mathrm{O}(k)_+$, has determinant $+1$, and in the other component, denoted by $\mathrm{O}(k)_-$, has determinant -1. The index is a sum of the ϕ-contour integrals in each of the components,

$$\begin{aligned} Z^k_{\text{D0-D4-D8/O8}} &= \frac{1}{2}(Z^k_+ + Z^k_-), \\ Z^k_\pm &= \frac{1}{|W|}\oint [d\phi] Z^{\pm,k}_{\text{D0-D0}} Z^{\pm,k}_{\text{D0-D4}} Z^{\pm,k}_{\text{D0-D8}}, \end{aligned} \tag{2.15}$$

where $|W|$ is the Weyl factor, i.e., the order of the Weyl group of $\mathrm{O}(k)$, and the integrands are given by the one-loop determinants of the fields listed in Table 2.1. They are computed in [24, 30], and we summarize them in Appendix 2.6.

For $k = 1$ there is no integral, and the integrand directly gives the Witten index. For $k = 2$ and 3, the integrals are one-dimensional. The contour prescription is such that the integrals pick up the residues of the poles coming from the terms in the denominator of the integrand (2.54)-(2.58) with + sign in front of the ϕ. For $k \geq 4$, the integrals are multi-dimensional, and the precise contour prescriptions are provided in [30] in terms of Jeffrey-Kirwan residues. (See Appendix 2.7 for more details about which poles contribute to the integral in the cases $k = 2$ and $k = 4$.)

We can combine the Witten indices for theories with different k into a generating function

$$Z_{\text{D0-D4-D8/O8}}(t, u, v, w, m_i, q) = 1 + \sum_{k=1}^{\infty} q^k Z^k_{\text{D0-D4-D8/O8}}(t, u, v, w, m_i). \tag{2.16}$$

The Nekrasov instanton partition function can be expressed as a ratio of the generating functions [30],

$$Z_{\text{inst}}(t, u, w, m_i, q) = \frac{Z_{\text{D0-D4-D8/O8}}(t, u, v, w, m_i, q)}{Z_{\text{D0-D8/O8}}(t, u, v, m_i, q)}, \tag{2.17}$$

where $Z_{\text{D0-D8/O8}}(t, u, v, m_i, q)$ is the generating function of the Witten indices of the system with only D0-branes and N_f D8-branes coincident with an O8-plane. It can be obtained by decoupling the D4-brane from our original system,

$$Z_{\text{D0-D8/O8}}(t, u, v, m_i, q) = \lim_{w\to 0} Z_{\text{D0-D4-D8/O8}}(t, u, v, w, m_i, q). \tag{2.18}$$

	0	1	2	3	4	5	6	7	8	9	Comments
D8/O8	×	×	×	×	×	×	×	×	×		$N_f \equiv (n-1)$ D8's at $x_9 = 0$
D4	×	×	×	×	×						
F1	×									×	$x_0 \geq 0$
D4′	×					×	×	×	×		$x_9 = L > 0$
D2								×	×	×	$x_0 = 0$

Table 2.2: The directions of the various branes.

Notice that the Nekrasov instanton partition function on the left hand side of (2.17) is independent of the fugacity v associated with the Cartan of $\mathrm{SU}(2)^R_-$. As shown in [30], the v-dependences of the Witten indices in the numerator and the denominator on the right hand side of (2.17) cancel each other.

2.3. Line and Ray Operators

The E_n theories possess BPS line operators that preserve half the supersymmetries. They can be realized in the type I' brane construction by probing the D4 brane with a fundamental string along directions 9 and, say, 0, in analogy with the way a BPS Wilson line was introduced into the low-energy $\mathcal{N} = 4$ SYM on D3-branes in [44, 45]. The configuration preserves an $\mathrm{SO}(4) \subset \mathrm{SO}(4,1)$ rotation group, as well as those SUSY parameters that satisfy

$$\epsilon_L = \Gamma^{1234}\epsilon_L = \Gamma^{5678}\epsilon_L\,, \qquad \epsilon_R = \Gamma^9\epsilon_L\,. \tag{2.19}$$

The position of the line operator can be fixed at $x_1 = \cdots = x_4 = 0$ by introducing an additional D4-brane (which we denote by D4') at $x_9 = L > 0$ and requiring the fundamental string to end on it. This D4' brane does not break any additional SUSY. Note that the orientation of the string and D4'-brane must be correlated in order to preserve SUSY. Reversing the orientation of the string changes the subspace of (ϵ_L, ϵ_R) that are preserved by inserting a $(-)$ in front of Γ^{1234} in (2.19). The directions of the various branes so far are summarized in the first four rows of Table 2.2. As we reviewed in §2.2, the E_n SCFT can be deformed by an operator of dimension 4 to a theory that flows in the IR to a weakly coupled $\mathrm{Sp}(1)$ gauge theory with flavor group $\mathrm{SO}(2N_f)$, by lowering the string coupling constant g_{st} to a finite value at the location of the D8/O8. Denoting by A_0 the time component of the gauge field and by Φ its scalar superpartner (i.e., the scalar component of the vector multiplet), the line operator reduces to a supersymmetric Wilson line

$$P\exp\Big[i\int_{-\infty}^{\infty}(A_0+\Phi)dx^0\Big],$$

which can be made gauge invariant by compactifying time on S^1 and taking the trace. The operator preserves half of the supercharges.[5] The gauge theory also has hypermultiplets with fields in $(2, 2N_f)$ of Sp(1)×SO($2N_f$). Denoting the scalar component of these fields by $q_{\mathbf{i}}^A$ (with $\mathbf{i} = 1, \cdots, 2N_f$ and $A = 1, 2$), we can construct gauge invariant operators

$$\bar{q}_{2,\mathbf{j}}(t_f) P \exp\left[i \int_{t_i}^{t_f} (A_0 + \Phi) dx^0\right] q_{\mathbf{i}}^1(t_i), \tag{2.20}$$

which preserve half of the SUSY (2.19) that the Wilson line preserves, that is two of the eight supercharges of the 5d theory.[6] The fields at each endpoint can be locally modified, for example by replacing $q_{\mathbf{i}}^1(t_i)$ with $D_\mu q_{\mathbf{i}}^1(t_i)$ ($D = \partial - iA$). Most kinds of insertions will break all of the SUSY, but there are some operators that preserve the same amount of SUSY as (2.20) does. We are interested in counting the number of such BPS operators, with given spin and R-charge, and in testing whether they can be collected into complete E_n multiplets. We therefore consider ray operators of the form

$$\mathcal{R}_{\mathcal{O}} = P \exp\big[i \int_0^\infty (A_0 + \Phi) dx^0\big] \mathcal{O}(0), \tag{2.21}$$

where $\mathcal{O}(0)$ is a local operator at $x_0 = 0$ in the 2 of Sp(1).

In order to make the case that (2.21) descend from ray operators in the E_n SCFT, it is useful to modify the type-I' construction by letting the worldsheet of the string end on a (Euclidean) D2-brane at $x^0 = 0$, extending in directions $7, 8, 9$, as listed in Table 2.2. Introducing a Euclidean brane, which behaves like an instanton, requires us to switch to Euclidean signature (similarly to Yang-Mills theory for which there are no real instanton solutions in Minkowski signature). We therefore Wick rotate $x^0 \to -ix^{\bar{0}}$. In Euclidean signature, the Weyl spinor condition is[7]

$$i\Gamma^{\bar{0}123456789}\epsilon_R = \epsilon_R, \quad i\Gamma^{\bar{0}123456789}\epsilon_L = -\epsilon_L. \tag{2.22}$$

The SUSY generators preserved by a Euclidean Dp-brane in directions $\bar{0}, \ldots, p$ satisfy the condition $\epsilon_R = i\Gamma^{\bar{0}1\cdots p}\epsilon_L$. The generators preserved by a fundamental string in the 9^{th} direction satisfy the conditions $\epsilon_R = i\Gamma^{\bar{0}9}\epsilon_R$ and $\epsilon_L = -i\Gamma^{\bar{0}9}\epsilon_L$. Combining these conditions we obtain the same conditions on the supersymmetry generators as those imposed by the Euclidean D4-D8/O8 configuration, which is given by the same equations as (2.19).

[5] The supersymmetry transformation on A_0 and Φ can be found in (2.10) of [24]: $\delta A_\mu = i\bar{\lambda}\gamma_\mu\epsilon$, $\delta\Phi = \bar{\lambda}\epsilon$, where the index A for the SU(2)$_R$ is implicit. The combination $A_0 + \Phi$ is preserved by the transformations that satisfy the condition $i\gamma_0\epsilon + \epsilon = 0$, which is the condition imposed by fundamental strings. The 5d spinor and 10d spinor are related by $\epsilon = \epsilon_L$ and $i\gamma^\mu = \Gamma^9\Gamma^\mu$.

[6] The supersymmetry transformation of $q_{\mathbf{i}}^A$ and $\bar{q}_{A,\mathbf{i}}$ can be found in (2.14) of [24]: $\delta q_{\mathbf{i}}^A = \sqrt{2}i\bar{\epsilon}^A\psi_{\mathbf{i}}$ and $\delta\bar{q}_{A,\mathbf{i}} = \sqrt{2}i\bar{\psi}_{\mathbf{i}}\epsilon_A$. By the symplectic-Majorana condition $\bar{\epsilon}^A = \varepsilon^{AB}(\epsilon^T)_B\gamma^2\gamma^4$, the transformations preserving $q_{\mathbf{i}}^1$ and $\bar{q}_{2,\mathbf{i}}$ satisfy the condition $\epsilon_2 = 0$, which is equivalent to $\frac{i}{2}(\Gamma^{56} - \Gamma^{78})\epsilon_L = \epsilon_L$ for the 10d spinor.

[7] There is also a reality condition $\epsilon_L^\star = C\epsilon_R$, where C is the charge conjugation matrix.

The D2-brane provides an anchor for the F1 to end on. It breaks half of the remaining supersymmetries, preserving only those SUSY parameters that satisfy

$$\epsilon_L = \Gamma^{1234}\epsilon_L = -i\Gamma^{56}\epsilon_L = i\Gamma^{78}\epsilon_L\,, \qquad \epsilon_R = \Gamma^9\epsilon_L\,. \tag{2.23}$$

The F1-D2-D4-D8/O8 configuration thus preserves only two linearly independent supercharges, and we can take one of them to coincide with $Q = Q^1_2$ of (2.3). Indeed, a generator of a Cartan subalgebra of the $\mathrm{SU}(2)_R$ that acts on the index A of Q^A_m can be identified with $\frac{i}{2}(\Gamma^{78} - \Gamma^{56})$, and the index $A = 1$ can be defined to label the eigenspace of $\frac{i}{2}(\Gamma^{78} - \Gamma^{56})$ with eigenvalue $+1$. Similarly, $\frac{i}{2}(\Gamma^{12} + \Gamma^{34})$ can be identified with J_-, which was defined below (2.2), and the index $m = 2$ is one of the two indices that generate an eigenspace of J_- with eigenvalue (i.e., spin) zero. The insertion of an F1 ending on a Euclidean D2-brane thus preserves two out of the eight supercharges Q^A_m of the 5d SCFT that describes the low-energy of the D4-brane in the background of the D8/O8 system.

We will be interested in extending the superconformal index (2.4) to ray operators of the form (2.21), and the operators that will contribute to our index need to preserve only one supercharge (Q^1_2). Such operators can appear on the F1 $\cap$ D4 intersection by coupling to operators on the two dimensional D2 $\cap$ (D8/O8) intersection. More precisely, the low-energy action of the brane configuration includes factors schematically of the form

$$\begin{aligned}
&S(F1 - D2 - D4 - (D8/O8)) \sim \\
&\quad S_{9d}(D8/O8) + S_{5d}(D4) + S_{3d}(D2) + S_{2d}(F1) \\
&\quad + S_{2d}(D2 \cap (D8/O8)) + S_{1d}(F1 \cap D2) + S_{1d}(F1 \cap D4) + S_{0d}(D2 \cap D4 \cap (D8/O8))\,.
\end{aligned}$$

The 2d intersection D2 $\cap$ (D8/O8) supports an E_n chiral current algebra at level $k = 1$ [denoted $(\widehat{E_n})_1$], and the exponent of the terms $-S_{1d}(F1 \cap D4)$ and $-S_{0d}$ is expected to be a sum of products of a ray operator of the E_n SCFT and a local operator of the $F1 \cap (D8/O8)$ theory, of the form $\sum_\alpha \mathcal{R}_\alpha \mathfrak{V}_\alpha(0)$, where $\mathfrak{V}_\alpha$ is some local operator in a representation of $(\widehat{E_n})_1$. This suggests a connection between the multiplicities of ray operators and representations of $(\widehat{E_n})_1$, which we will explore elsewhere [46].

The states corresponding to a ray operator

The state-operator correspondence of 5d SCFTs assigns to a local operator a gauge invariant state in the Hilbert space of the theory on S^4 via a conformal transformation that acts on the radial coordinate as $r \to \tau = \log r$. This state-operator correspondence converts the ray operator to a state in the Hilbert space of the theory on S^4 with an impurity at one point of S^4, which we shall refer to as the South Pole (SP). After flowing to the Sp(1) gauge theory, the impurity is replaced with an external quark at SP in the fundamental representation of Sp(1). Let $\mathbf{G}$ be the (infinite dimensional) group of Sp(1) gauge transformations on S^4, and let $\widetilde{\mathbf{G}} \subset \mathbf{G}$ be the group of gauge transformations that are trivial at SP. Then $\mathbf{G}/\widetilde{\mathbf{G}} \cong \mathrm{Sp}(1)$ and the states that correspond to ray operators are those that are invariant under $\widetilde{\mathbf{G}}$ but are doublets of $\mathbf{G}/\widetilde{\mathbf{G}}$. We will "count" them, or rather calculate their supersymmetric index, by inserting a Wilson loop at SP into the partition function of the Sp(1) gauge theory on $S^4 \times S^1$, as will be explained in detail in §2.4.

As we argued above, the "impurity" at SP preserves the SUSY generators with parameters restricted by (2.23). In the notation of §2.2, these are the generators Q_m^A with $A = 1$ and $m = 1, 2$ (and the generators with $m = 3, 4$ or $A = 2$ are generally not preserved). The impurity preserves the inversion $\tau \to -\tau$; hence, also preserves the superconformal generators S_A^m with $A = 1$ and $m = 1, 2$. The bosonic subalgebra that preserves a ray is generated by the dilatation operator D, the generators ${M_1}^1 = -{M_2}^2$, ${M_1}^2$, ${M_2}^1$, ${M_3}^3 = -{M_4}^4$, ${M_3}^4$, ${M_4}^3$ of the rotation subgroup $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \cong \mathrm{Spin}(4) \subset \mathrm{Spin}(5)$, and the R-symmetry generators ${R_1}^1$. The above bosonic and fermionic generators form a closed subalgebra of $F(4)$. (See [48] for a discussion of the subalgebra preserved by a ray in 4d superconformal theories.)

In order to extend the discussion of §2.2 to states on S^4 with impurity at SP, we need to establish first which subalgebra of the superconformal algebra $F(4)$ acts on the Hilbert space. Such subalgebra properly contains the subalgebra preserved by a ray.[8] It also contains the translation operator $P_{\overline{0}}$ and conformal generator $K_{\overline{0}}$. Note that neither of these preserve the ray — $P_{\overline{0}}$ does not preserve the origin, while $K_{\overline{0}}$ does not preserve the endpoint at infinity. Nevertheless, if we define the Hilbert space at, say, $r = 1$, both generators preserve the location of the impurity on S^4. We can now obtain the full set of Q_m^A and S_A^m with $m = 1, 2$ and $A = 1, 2$ by starting with $Q = Q_2^1$ and its hermitian conjugate $S = S_1^2$ and successively calculating commutators with the bosonic generators $K_{\overline{0}}$, $P_{\overline{0}}$, ${M_1}^2$ and ${M_2}^1$.

The BPS bound (2.3) is therefore valid for ray operators, too. Note, however, that Q_3^A and Q_4^A are not preserved by the line impurity at SP, and there is no way to get them from commutators of Q with $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ generators. We therefore cannot assume (2.5). Nevertheless, the parts of the discussion at the end of §2.2 that do not rely on (2.5) are still valid. In particular, we can define an index similarly to (2.4), and it receives contributions only from nontrivial elements of the Q-cohomology. Moreover, states that contribute to the index have maximal $J_+ + J_R$ in their $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R$ multiplet. It follows that no state that contributes to the index can have $J_+ + J_R = 0$, because if it did it would be a singlet of $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R$ and thus would be annihilated by ${R_1}^2$ and ${M_1}^2$, and therefore also by the commutator $[{M_1}^2, [{R_1}^2, {Q_2}^1]] \propto {Q_1}^2$. But $\{{Q_1}^2, {Q_2}^1\} \propto P_{\overline{0}}$, which does not preserve a ray operator. This observation will become relevant in §2.4 when we preserve our result for the index of the E_8 theory.

The calculation of the index of ray operators that will follow makes the $\mathrm{SO}(2n-2) \times \mathrm{U}(1) \subset E_n$ global symmetry explicit, but in order to properly combine the $\mathrm{SO}(2n-2) \times \mathrm{U}(1)$ characters into E_n characters it is important to first explain a shift in the U(1) charge.

[8] This distinction also occurs in the case of local operators, where the origin is only preserved by dilatations and by $\mathrm{SO}(5) \times \mathrm{SU}(2)_R$, but translations P_μ and conformal transformations K_μ are good operators on the Hilbert space, defined as the space of states on S^4.

Shifted instanton number

Denote the U(1) charge by $\mathbf{Q}$. On local operators that correspond to gauge invariant states on S^4, the U(1) charge is simply the integer instanton number

$$\mathbf{Q} = \frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F\wedge F) = \mathbf{k}.$$

However, on states that correspond to a ray operator, the $U(1)$ charge receives an anomalous contribution and reads

$$\mathbf{Q} = \mathbf{k} + \frac{2}{N_f - 8}\,, \qquad (N_f = 2,\ldots,7). \tag{2.24}$$

The correction $2/(N_f - 8)$, which is fractional for $N_f < 6$, will be borne out by the index that we will present in §2.4. Below, we will review the physical origin of this shift. Our discussion is similar to the arguments presented in [49, 50].

The shift (2.24) is easy to explain on the Coulomb branch of the E_n theory by using the D4-D8/O8 brane realization. The space $x^9 > 0$ is described by massive type-IIA supergravity at low-energy with mass parameter proportional to $\mathbf{m} \stackrel{\mathrm{def}}{=} 8 - N_f$ [11]. In Appendix 2.8, we review the D8-brane solution in the massive type-IIA supergravity, and the D-branes worldvolume actions in that background. The Coulomb branch corresponds to the D4-brane moving away from the D8/O8 plane in the positive x^9 direction. The low-energy description is a free U(1) vector multiplet. Denote the vector field by $\mathbf{a}$, the field strength by $\mathbf{f} = d\mathbf{a}$, and scalar component by φ. The scalar component has a nonzero VEV $v = \langle\varphi\rangle > 0$ (proportional to the x^9 coordinate of the D4-brane).

We can understand the shift (2.24) in the U(1) charge after reviewing the peculiar interaction terms that are part of the low-energy description of a D-brane in massive type-IIA supergravity [51]. As we review in Appendix 2.8, the super Yang-Mills effective action on a Dp-brane includes an additional Chern-Simons term proportional to $\mathbf{m}\mathbf{a}\wedge\mathbf{f}^{p/2}$ [52]. It implies a few modifications to the conservation of string number. For $p = 0$, we find that a net number of $\mathbf{m}$ fundamental strings must emanate from any D0-brane. As usual, a D0-brane can be absorbed by a D4-brane and convert into one unit of instanton charge. In that case, the $\mathbf{m}$ strings that are attached to the D0-brane can convert to $\mathbf{m}$ units of electric flux. Indeed, the low-energy description of the D4-brane, which is the low-energy effective action of the 5d E_n theory on the Coulomb branch [13], contains an effective Chern-Simons interaction term proportional to $\mathbf{m}\mathbf{a}\wedge\mathbf{f}\wedge\mathbf{f}$, and can be written as

$$I_{\mathrm{Coulomb}} = -\int\left[\frac{1}{8\pi^2}\mathbf{m}v\mathbf{f}\wedge{}^\star\mathbf{f} + \frac{1}{24\pi^2}\mathbf{m}\mathbf{a}\wedge\mathbf{f}\wedge\mathbf{f} - \mathbf{a}\wedge{}^\star\mathbf{j}\right], \tag{2.25}$$

where $\mathbf{j}$ is the contribution of the hypermultiplet to the $U(1)$ current. The $\mathbf{a}_0$ equation of motion can be written as

$$\frac{1}{4\pi^2}\mathbf{m}v d({}^\star\mathbf{f}) = \frac{1}{8\pi^2}\mathbf{m}\mathbf{f}\wedge\mathbf{f} - {}^\star\mathbf{j}\,, \tag{2.26}$$

Integrating (2.26) over 4d space shows that $\mathbf{m}/2$ units of electric flux accompany one unit of instanton charge.

It follows that we can measure the U(1) charge $\mathbf{Q}$ in two equivalent ways, by either (i) integrating $\frac{1}{4\pi^2}\mathbf{f}\wedge\mathbf{f}$ over all of space[9], or (ii) measuring the electric flux $\frac{\mathbf{m}v}{4\pi^2}\int {}^\star\mathbf{f}$ at infinity and dividing by $\mathbf{m}/2$. In a general situation, however, there could be a net number n_1 of open fundamental strings attached to the D4-brane and extending into the bulk x^9 direction. Their endpoints on the D4-brane behave as external charges, which contribute $n_1\delta^4(x)$ to ${}^\star\mathbf{j}$, and then methods (i) and (ii) above will give a different answer for $\mathbf{Q}$. The answers differ by $2n_1/\mathbf{m}$. To determine which method is correct, we note that an instanton can evaporate into the x^9-bulk as a D0-brane, which would then carry with it $\mathbf{m}$ open strings. Such a process reduces the instanton number $\mathbf{k}$ by 2 (the factor of 2 is the effect of the orientifold), and at the same time reduces n_1 by $\mathbf{m}$. The possibility of such a process demonstrates that instanton number alone is not conserved, and method (ii) is the correct one. This is consistent with the shift of $-2/\mathbf{m}$ in (2.24).

The center of E_n

We will now discuss the action of the center $\mathcal{Z}_n$ of the enhanced E_n flavor symmetry. For $n\geq 3$, we will use the convention that E_n is simply connected. For $n=3,\ldots,7$, E_n then has a nontrivial center given by $\mathcal{Z}_n\cong\mathbb{Z}_{9-n}=\mathbb{Z}_{8-N_f}$. For $n=2$ we have $E_2=\mathrm{SU}(2)\times\mathrm{U}(1)$ which has $\mathcal{Z}_2\cong\mathbb{Z}_2\times\mathrm{U}(1)$ as center. So far, when discussing the "$\mathrm{SO}(2N_f)\times\mathrm{U}(1)$ subgroup" of E_n, we have not been precise about the global structure, which we will now rectify.

Local operators of the E_n SCFT are neutral under $\mathbb{Z}_{9-n}$. Indeed, the only E_n representations of local operators found in [30] have weights belonging to the root lattice. In contrast, ray operators are charged under $\mathbb{Z}_{9-n}$. Let $Q^{(n)}_{\mathrm{rt}}$ be the root lattice of E_n, and let $Q^{(n)}_{\mathrm{wt}}$ be the weight lattice. We will find in §2.4 representations whose weights project to a nontrivial element of $Q^{(n)}_{\mathrm{wt}}/Q^{(n)}_{\mathrm{rt}}\cong\mathbb{Z}_{9-n}$. For $3\leq n\leq 7$, this $\mathbb{Z}_{9-n}$ can be identified with the Pontryagin dual of the center $\mathcal{Z}_n$. In other words, ray operators carry nontrivial $\mathbb{Z}_{9-n}$ charge.

Consider $N_f=6$, for example. E_7 has a subgroup $[\mathrm{Spin}(12)\times\mathrm{SU}(2)]/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ identification means the following. Denote by 32 one of the two chiral spinor representations of Spin(12), and by 32′ the other one. Spin(12) has a center $\mathbb{Z}_2'\times\mathbb{Z}_2''$, where the generator of $\mathbb{Z}_2'$ is defined to be (-1) in 12 and 32′, and the generator of $\mathbb{Z}_2''$ is defined to be (-1) in 12 and (-1) in 32; the $\mathbb{Z}_2'\times\mathbb{Z}_2''$ charges in other representations of Spin(12) are defined by requiring additivity mod 2 under tensor products. Then, when decomposing representations of E_7 into irreducible representations of $\mathrm{Spin}(12)\times\mathrm{SU}(2)$, half-integer SU(2) spins will always be paired with representations of Spin(12) that are odd under $\mathbb{Z}_2'$, while even SU(2) spin will

[9] In our convention, the SU(2) instanton number is related to the U(1) instanton number by

$$\mathbf{k}=\frac{1}{8\pi^2}\int_{S^4}\mathrm{tr}(F\wedge F)=\frac{1}{4\pi^2}\int_{S^4}\mathbf{f}\wedge\mathbf{f}, \tag{2.27}$$

where we have used $F=\mathbf{f}\sigma_3$.

be paired with zero $\mathbb{Z}_2'$ charge. So, for example, $(12,2)$, $(32',2)$, and $(32,1)$ are allowed, but neither $(12,1)$ nor $(1,2)$ nor $(32,2)$ can appear. Taking the $\mathrm{U}(1) \subset \mathrm{SU}(2)$ subgroup, we find the subgroup $[\mathrm{Spin}(12)\times\mathrm{U}(1)]/\mathbb{Z}_2 \subset E_7$ under which the fundamental representation 56 and adjoint 133 decompose as

$$56 = 12_1 + 12_{-1} + 32_0, \qquad 133 = 1_2 + 1_0 + 66_0 + 1_{-2} + \overline{32}'_1 + 32'_{-1}.$$

In our conventions, one unit of instanton number ($k = 1$) corresponds to one unit of the above U(1) charge. The generator of $\mathbb{Z}_2$ that appears in $[\mathrm{Spin}(12) \times U(1)]/\mathbb{Z}_2$ is therefore identified with $(-1)^k$ times the generator of $\mathbb{Z}_2' \subset \mathbb{Z}_2' \times \mathbb{Z}_2'' \subset \mathrm{Spin}(12)$. The center of E_7 is identified with the other factor, $\mathbb{Z}_2'' \subset \mathrm{Spin}(12)$. We will see that ray operators are odd under $\mathbb{Z}_2''$.

For E_6 ($N_f = 5$), the center is $\mathbb{Z}_3$. The representations of ray operators that we will find are 27, 1728, etc., and the ray operator has one unit of $\mathbb{Z}_3$ charge. E_6 has a subgroup $[\mathrm{Spin}(10)\times\mathrm{U}(1)]/\mathbb{Z}_4 \subset E_6$. The $\mathbb{Z}_4$ identification means the following. The center of Spin(10) is $\mathbb{Z}_4$, and a representation of Spin(10) can be assigned a $\mathbb{Z}_4$ charge by the rules that: (i) the $\mathbb{Z}_4$ charge is additive under tensor products; (ii) the left-chirality spinors 16 are assigned charge 1 mod 4. Thus the fundamental 10 is assigned 2 mod 4, the adjoint is assigned 0, and the right-chirality $\overline{16}$ is assigned charge 3 mod 4. Then, when decomposing any representation of E_6 under $\mathrm{Spin}(10)\times\mathrm{U}(1)$, a Spin(10) representation with $\mathbb{Z}_4$ charge γ will always carry U(1) charge that is γ mod 4. For example, 27 of E_6 decomposes under $\mathrm{SO}(10)\times\mathrm{U}(1)$ as

$$27 \to 10_{-2} + 16_1 + 1_4.$$

In our conventions, one unit of instanton number ($k = 1$) corresponds to 3 units of U(1) charge. Thus, any of the states of 10 carry $-2/3$ instanton number, the states of 16 carry $1/3$ instanton number and 1 carries $4/3$. The center $\mathbb{Z}_3$ is generated by the projection to $[\mathrm{Spin}(10)\times\mathrm{U}(1)]/\mathbb{Z}_4$ of the element $(1, e^{2\pi i/3}) \in \mathrm{Spin}(10)\times\mathrm{U}(1)$. Thus, for the case $N_f = 5$ we see that ray operators carry instanton charge in $\frac{1}{3} + \mathbb{Z}$ and are charged one unit under $\mathbb{Z}_3$.

For $N_f = 4$, we have $E_5 = \mathrm{Spin}(10)$ and the center of $\mathrm{Spin}(2N_f) = \mathrm{Spin}(8)$ is $\mathbb{Z}_2'\times\mathbb{Z}_2''$ with $\mathbb{Z}_2'$ nontrivial in the vector representation 8_v and the spinor representation 8_s, and trivial in the spinor representation 8_c of opposite chirality, while $\mathbb{Z}_2''$ is trivial in 8_v and nontrivial in both spinor representations. Then $E_5 = \mathrm{Spin}(10) = [\mathrm{Spin}(8)\times\mathrm{U}(1)]/\mathbb{Z}_2$, where one unit of U(1) charge is identified with instanton number k and the generator of the last $\mathbb{Z}_2$ is identified with $(-1)^k$ times the generator of $\mathbb{Z}_2'$. Thus, in decomposing a representation of Spin(10) into representations of $\mathrm{Spin}(8)\times\mathrm{U}(1)$, even U(1) charge is paired with representations of Spin(8) that appear in tensor products of 8_c, while odd U(1) charge is paired with 8_v or representations that appear in tensor products of one factor of 8_v and an arbitrary number of 8_c. The center $\mathbb{Z}_4 \subset \mathrm{Spin}(10)$ is generated by $k \pmod 4$ plus twice the $\mathbb{Z}_2''$ charge. So, for example, the fundamental representation 10 of Spin(10) decomposes under $\mathrm{Spin}(8)\times\mathrm{U}(1)$

as[10]

$$10 = (8_c)_0 + 1_2 + 1_{-2}\,.$$

The states of 10 have $\mathbb{Z}_4$ charge $2 \pmod 4$, while

$$16 = (8_v)_1 + (8_s)_{-1}$$

has charge $1 \pmod 4$, and

$$\overline{16} = (8_v)_{-1} + (8_s)_1$$

has charge $3 \pmod 4$. The trivial representation 1 and the adjoint 45 of Spin(10) have $\mathbb{Z}_4$ charge $0 \pmod 4$. We will find that ray operators have $\mathbb{Z}_4$ charge $1 \pmod 4$. (Whether it is 1 or -1 is a matter of convention.)

For $N_f = 3$, we have $E_4 = \mathrm{SU}(5)$ with center $\mathbb{Z}_5$ and subgroup $[\mathrm{Spin}(6)\times\mathrm{U}(1)]/\mathbb{Z}_4 \subset \mathrm{SU}(5)$. Here a generator of $\mathbb{Z}_4$ can be taken as a generator of the center $\mathbb{Z}_4 \subset \mathrm{Spin}(6) = \mathrm{SU}(4)$ times $i \in \mathrm{U}(1)$. A generator of the center of SU(5) can be taken as $e^{2\pi i/5} \in \mathrm{U}(1)$ [times the identity in Spin(6)]. We will find that ray operators have one unit of charge under $\mathbb{Z}_5$, meaning that only representations that have Young diagrams with number of boxes equal to $3 \pmod 5$ can appear.

For $N_f = 2$, we have $E_3 = \mathrm{SU}(3)\times\mathrm{SU}(2)$, and $\mathrm{Spin}(2N_f)\times\mathrm{U}(1) = \mathrm{SU}(2)'\times\mathrm{SU}(2)''\times\mathrm{U}(1)$ is related to $E_3 = \mathrm{SU}(3)\times\mathrm{SU}(2)$ by identifying the SU(2) factor with $\mathrm{SU}(2)'$, and noting the subgroup $[\mathrm{SU}(2)''\times \mathrm{U}(1)]/\mathbb{Z}_2 \subset \mathrm{SU}(3)$. Again, the $\mathbb{Z}_2$ identification means that in decomposing representations of SU(3) under $\mathrm{SU}(2)''\times\mathrm{U}(1)$, odd $\mathrm{SU}(2)''$ spin is paired with odd U(1) charge, and vice versa. The center of E_3 is $\mathbb{Z}_3 \times \mathbb{Z}_2'$. Referring to $\mathrm{SU}(2)'\times[\mathrm{SU}(2)''\times\mathrm{U}(1)]/\mathbb{Z}_2 \subset E_3$, the generator of $\mathbb{Z}_3 \subset E_3$ can be identified with $e^{2\pi i/3} \in \mathrm{U}(1)$, and $\mathbb{Z}_2' \subset E_3$ is identified with the center of $\mathrm{SU}(2)'$. We will see that ray operators have charge $2 \pmod 3$ under $\mathbb{Z}_3$, and they are odd under $\mathbb{Z}_2'$.

For $N_f = 1$, we have $E_2 = \mathrm{SU}(2)\times\mathrm{U}(1)$, and the center is $\mathbb{Z}_2\times\mathrm{U}(1)$. We will find that ray operators carry fractional U(1) charge in $\frac{4}{7} + \mathbb{Z}$ and their $\mathbb{Z}_2$ charge is correlated with their U(1) charge. More details on the definition of E_2 and the embedding of $\mathrm{SO}(2N_f) = \mathrm{SO}(2)$ in it are reviewed in Appendix 2.9.

2.4. The index of line and ray operators

Wilson ray indices

The spectrum of line or ray operators can be studied by computing the line/ray operator indices analogous to the superconformal indices. Let us first discuss Wilson line operators. Consider a line operator supported on a line $\mathbb{R}^1 \subset \mathbb{R}^5$, which without loss of generality we can choose to pass through the origin of $\mathbb{R}^5$. By a conformal map to $\mathrm{S}^4 \times \mathbb{R}^1$, the line

[10] Note that 8_c and not 8_v appears in the decomposition above, and the embedding of $\mathrm{Spin}(8)\times\mathrm{U}(1)$ is not the standard one, but related to it by triality.

$\mathbb{R}^1 \subset \mathbb{R}^5$ is mapped to two lines at antipodal points $\mathbf{p}, \mathbf{q} \in \mathrm{S}^4$ and along the $\mathbb{R}^1$ factor of $\mathrm{S}^4 \times \mathbb{R}^1$, with opposite orientations for the two lines $\{\mathbf{p}\} \times \mathbb{R}^1$ and $\{\mathbf{q}\} \times \mathbb{R}^1$. Similarly to the superconformal index, the Wilson line index can be computed by a path integral on $\mathrm{S}^4 \times \mathrm{S}^1$. Since the path integral localizes on solutions of constant holonomy $U = e^{i\int_{S^1} A_\mu dx^\mu}$ along the "thermal" circle S^1, the Wilson line operators simply reduce to the Sp(1) characters

$$\chi_R(w) = \mathrm{tr}_R\, U. \tag{2.28}$$

Since the Wilson lines at the antipodal points have opposite orientations, they correspond to characters of conjugate representations. For Sp(1), conjugate representations are equivalent, but since the construction generalizes to Sp(N) with any N, we will retain the distinction between a representation R and its conjugate $\bar{R}$. This discussion suggests that the Wilson line index can be calculated by inserting a pair of characters of opposite representations into the integral formula (2.13) of the superconformal index [53],

$$I_R^{\text{Wilson line}}(t,u,m_i,q) \stackrel{?}{=} \int_{Sp(1)} \chi_R(w)\chi_{\bar{R}}(w) Z_{\text{1-loop}}(t,u,w,m_i)|Z_{\text{inst}}(t,u,w,m_i,q)|^2 dU. \tag{2.29}$$

The question mark over the equality sign indicates that (2.29) is not the complete answer, as we will discuss below, in the context of ray operators.

The ray operator indices can be studied in a similar way. Consider a ray operator located on a half line $\mathbb{R}^+$, whose end point is chosen to be the origin of the $\mathbb{R}^5$ spacetime. Under the conformal map, the ray operator is mapped to a line operator along the $\mathbb{R}^1$ factor of $\mathrm{S}^4 \times \mathbb{R}^1$ and located at a point on S^4. The Wilson ray operator index therefore appears to be given by the formula

$$I_R^{\text{Wilson ray}}(t,u,m_i,q) \stackrel{?}{=} \int_{Sp(1)} \chi_R(w) Z_{\text{1-loop}}(t,u,w,m_i)|Z_{\text{inst}}(t,u,w,m_i,q)|^2 dU. \tag{2.30}$$

For example, let us consider the indices of Wilson rays in the fundamental representation. For $0 \le N_f \le 7$, the indices in the t-expansion are given by

$$\begin{aligned}
N_f = 0: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 0, \\
N_f = 1: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 2t + \left(\frac{1}{q} + q\right)t^3 + O(t^4), \\
N_f = 2: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 4t + \left(\frac{6}{q} + 16 + 6q\right)t^3 + O(t^4), \\
N_f = 3: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 6t + \left(\frac{20}{q} + 64 + 20q\right)t^3 + O(t^4), \\
N_f = 4: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 8t + \left(\frac{56}{q} + 160 + 56q\right)t^3 + O(t^4), \\
N_f = 5: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 10t + \left(\frac{144}{q} + 320 + 144q\right)t^3 + O(t^4), \\
N_f = 6: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 12t + \left(\frac{12}{q^2} + \frac{352}{q} + 560 + 352q + 12q^2\right)t^3 + O(t^4), \\
N_f = 7: &\quad I_2^{\text{Wilson ray}} \stackrel{?}{=} 14t + \left(\frac{195}{q^2} + \frac{832}{q} + 896 + 832q + 195q^2\right)t^3 + O(t^4),
\end{aligned} \tag{2.31}$$

where we have turned off the fugacities associated to the flavor $\mathrm{SO}(2N_f)$ group. A few comments on the above formulas are in order. The leading terms of the t-expansions in (2.31) correspond to the Wilson rays contracted with the scalar q^1 of the hypermultiplet [see (2.20)]. As we discussed in §2.2, the operator q^1 represents a nontrivial class of the Q-cohomology, has J_R charge 1/2, and transforms in the $(2, 2N_f)$ of $\mathrm{Sp}(1)\times\mathrm{SO}(2N_f)$. Hence, it contributes a term $2N_f t$ to the Wilson ray index. The flavor symmetry of the IR Sp(1) SYM combines with the instanton number symmetry U(1) and is enhanced to form the E_n global symmetry of the UV CFT. Under the broken generators of $E_n \to \mathrm{SO}(2N_f)\times\mathrm{U}(1)$, the Wilson ray operators are transformed to ray operators of nonzero instanton U(1) charge. However, our Wilson ray indices (2.31) did not capture those contributions. One would expect the full ray operator indices to exhibit the structure of the E_n symmetry; more precisely, the coefficients of the t-expansion must be characters of E_n, but this is not the case in (2.31). For example, for $N_f = 6$, the representation 12 that appears in the leading t-expansion of the Wilson ray index should be completed to the representation 56 of E_7. This instructs us to look for additional contributions to the Nekrasov instanton partition function Z_{inst}.

The problem with the naive prescription (2.30) is that it relies on the evaluation of the Wilson loop (2.28) at a point (the south pole) where the gauge field configuration is singular (a zero size instanton). To resolve the singularity, we have to invoke a string theory construction similar to Table 2.2. We then see that in the presence of zero-size instantons (interpreted as D0-branes) the fundamental string (F1) can end on either the D4-brane directly or on a D0-brane. If the F1 ends on the D4 it induces the Wilson loop term (2.28) in the action, but if F1 ends on a D0-brane, with k D0-branes present, it will manifest itself as an O(k) Wilson loop. The Hilbert space of the F1-D4-D8/O8 system thus has several sectors.

Instead of analyzing each string sector separately and adding up the contributions, it is much more convenient to compute a generating function for the partition function with an arbitrary number l of F1 strings. We therefore introduce a new variable x, which will play the role of fugacity for the string number, so that the terms of order x^{l-1} in the generating function will capture the partition function with l strings present. (The shift by -1 will be explained shortly below.) So, we propose a formula for the generating function of the ray operator indices,

$$\begin{aligned} &I_{\text{all}}(t,u,x,m_i,q) \\ &\quad = \int_{Sp(1)} Z_{\text{1-loop}}(t,u,w,m_i) Z_{\text{inst+line}}(t,u,w,x,m_i,q) Z_{\text{inst}}(t,u,w^{-1},-m_i,q^{-1}) dU, \end{aligned} \tag{2.32}$$

where $Z_{\text{inst+line}}$ is the instanton partition function on the background of a a line operator on $\mathbb{R}^1 \subset \mathbb{R}^5$ with the Omega background turned on (on the space $\mathbb{R}^4$ transverse to the line operator).

To compute $Z_{\text{inst+line}}$ we follow a technique developed in [54, 103, 60] and introduce a D4′-brane on which F1 can end (as in Table 2.2). The D0-D4-D4′-D8/O8 system then automatically allows for a dynamical generation of finite-mass F1-strings. The D4′ brane supports a U(1) gauge field, and x is more precisely identified as the fugacity for this U(1) charge. The presence of the D4-brane generates nontrivial RR four-form flux that induces a background of (-1) units of U(1)-charge[11], and so the sector with l F1-strings has U(1) charge $(l-1)$.

Similarly to Nekrasov's instanton partition function Z_{inst}, the modified partition function $Z_{\text{inst+line}}$ can be computed as a ratio of Witten indices of D0-brane quantum mechanics systems,

$$Z_{\text{inst+line}}(t,u,w,x,m_i,q) = \frac{Z_{\text{D0-D4-D4}'\text{-D8/O8}}(t,u,v,w,x,m_i,q)}{Z_{\text{D0-D4}'\text{-D8/O8}}(t,u,v,x,m_i,q)}. \tag{2.33}$$

The D0-D4-D4′-D8/O8 and D0-D4′-D8/O8 quantum mechanics systems will be discussed in detail in the next section. The partition function $Z_{\text{inst+line}}$ is expected to be independent of the fugacity v associated with the Cartan subgroup of the $\text{SU}(2)^R_- \subset \text{Spin}(4)_{5678}$ rotation group, since none of the gauge theory degrees of freedom are charged under it. For $N_f < 7$, we checked up to $O(t^5)$ order that the v-dependences of the numerator and denominator on the right hand side of (2.33) cancel each other. For $N_f = 7$, however, the right hand side of (2.33) does depend on the fugacity v. Thus, in this special case we see that $Z_{\text{D0-D4-D4}'\text{-D8/O8}}$ does not factorize into a decoupled E_8 SCFT contribution and $Z_{\text{D0-D4}'\text{-D8/O8}}$. We will return to this problem in §2.4.

[11]The D4-D4′ system is T-dual to a D0-D8 system, and the effect is similar to the induced charge of $\mathbf{m}$ units discussed in §2.3.

strings	$N = 4$ multiplets	fields	$\mathrm{SU}(2)_- \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)^R_- \times \mathrm{SU}(2)^R_+$
D0-D4′ strings	twisted hyper	scalar	$(1,1,1,2)$
		fermions	$(1,2,1,1)$
	Fermi	fermions	$(2,1,1,1)$
D4-D4′ strings	Fermi	fermions	$(1,1,1,1)$

Table 2.3: The quantum mechanics fields from the D0-D4′ strings and D4-D4′ strings.

Setting aside the problem of $N_f = 7$, for now, the ray operator indices are then extracted from the generating function I_{all} by expanding in x,

$$I_{\text{all}}(t,u,w,x,m_i,q) = \frac{1}{x} I_{\text{SCI}}(t,u,w,m_i,q) - I_{\text{ray}}(t,u,w,m_i,q) + O(x). \tag{2.34}$$

The first term of the expansion is the superconformal index I_{SCI}, and the second term I_{ray} is the ray operator index. The minus sign in front of I_{ray} is because the single D4−D4′ string is fermionic.

The D0-D4-D4′-D8/O8 system

In [54], the interaction of instantons and Wilson line operators was studied by introducing an extra D4-brane (referred to as a D4′-brane) to the D0-D4 quantum mechanics. The D4′-brane has no spatial direction in common with the D4-branes in the D0-D4 system. The directions of the D4- and D4′-branes are listed in Table 2.2. As we take the distance between the D4- and D4′-branes (along direction 9) to be large, the fundamental strings suspended between D4 and D4′ become non-dynamical. The boundaries of the fundamental strings on the D4-branes realize the line operators in the 5d gauge theory, whose positions are fixed by the D4′-brane. The Witten indices of the D0-D4-D4′ quantum mechanics were studied in [60]. The D4′-brane introduces additional degrees of freedom coming from the D0-D4′ strings and D4-D4′ strings, which are listed in Table 2.3. The D0-D4′ and D4-D4′ strings are charged under the U(1) symmetry associated with the D4′-brane. We denote the fugacity of the U(1) by $x = e^{-M}$. Up to normalization, the distance between the D4- and D4′-branes is identified with $\log |x|$. It was shown in [60] that the Witten indices admit a finite series expansion in the fugacity x, and the k-th order term in the expansion receives contributions from Wilson line in the k-th anti-symmetric representation of SU(N).

Following [54, 60], we consider the D0-D4-D4′-D8/O8 quantum mechanics.[12] The Witten index of this quantum mechanics can be computed by a contour integral similarly to the way the index of the D0-D4-D8/O8 was calculated, as reviewed in §2.2. The only modification required is to include the contributions from the D0-D4′ strings and D4-D4′ strings to the

[12]The D2−brane that we used in the construction of the ray operators in Table 2.2 doesn't exist anymore when we radially quantize the theory on S^4 and focus on one of the two poles of S^4, to which the ray operator is mapped. Hence, we do not include D2 brane in the index calculation on the South Pole.

integrand of the ϕ-contour integral (2.15). It is easy to obtain the one-loop determinant of the D0-D4′ strings without any additional calculations. If we exchange the four-planes $\mathbb{R}^{1234}$ and $\mathbb{R}^{5678}$, the orientations of the D4- and D4′-branes are interchanged while the orientations of the D0-branes and D8/O8 singularity remain the same. One can also see from Table 2.1 and Table 2.3 that when exchanging the $\mathrm{SU}(2)_-\times\mathrm{SU}(2)_+$ rotation symmetry of $\mathbb{R}^{1234}$ with the $\mathrm{SU}(2)^R_-\times\mathrm{SU}(2)^R_+$ rotation symmetry of $\mathbb{R}^{5678}$, the D0-D4 strings switch roles with the D0-D4′ strings while the other types of strings listed in the tables remain unchanged. To proceed, we reparametrize the chemical potentials associated to the spacetime rotations and introduce new chemical potentials $\epsilon_1,\dots,\epsilon_4$ by setting

$$\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}, \quad \epsilon_- = \frac{\epsilon_1-\epsilon_2}{2}, \quad m = \frac{\epsilon_3-\epsilon_4}{2}, \tag{2.35}$$

with the condition $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4=0$. The ϵ_1, ϵ_2, ϵ_3, ϵ_4 are the chemical potentials associated to the rotations of the two-planes $\mathbb{R}^{12}$, $\mathbb{R}^{34}$, $\mathbb{R}^{56}$, $\mathbb{R}^{78}$. The exchange $\epsilon_1 \leftrightarrow \epsilon_3$ combined with $\epsilon_2 \leftrightarrow \epsilon_4$ corresponds to the exchange $\epsilon_+ \leftrightarrow -\epsilon_+$ combined with $\epsilon_- \leftrightarrow m$. The one-loop determinant of the D0-D4′ strings is therefore obtained from the one-loop determinant of the D0-D4 strings (2.57), by performing this simple substitution. The result is

$$\begin{aligned} Z^{+,\,k=2n+\chi}_{\text{D0-D4}'} &= \left(\frac{2\sinh\frac{\pm M-\epsilon_-}{2}}{2\sinh\frac{\pm M-\epsilon_+}{2}}\right)^{\chi} \prod_{I=1}^{n} \frac{2\sinh\frac{\pm\phi_I\pm M-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm M-\epsilon_+}{2}}, \\ Z^{-,\,k=2n+1}_{\text{D0-D4}'} &= \frac{2\cosh\frac{\pm M-\epsilon_-}{2}}{2\cosh\frac{\pm M-\epsilon_+}{2}} \prod_{I=1}^{n} \frac{2\sinh\frac{\pm\phi_I\pm M-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm M-\epsilon_+}{2}}, \\ Z^{-,\,k=2n}_{\text{D0-D4}'} &= \frac{2\sinh(\pm M-\epsilon_-)}{2\sinh(\pm M-\epsilon_+)} \prod_{I=1}^{n-1} \frac{2\sinh\frac{\pm\phi_I\pm M-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm M-\epsilon_+}{2}}, \end{aligned} \tag{2.36}$$

where we have also replaced the chemical potential α associated to the Sp(1) symmetry on the D4-brane with the chemical potential M associated to the Sp(1) symmetry on the D4′-brane, and we used the shorthand notation of [30], where $\sinh(\pm A \pm B\cdots)$ represents the product of sinh's of arguments with all possible sign combinations. (See Appendix 2.6 for more details.)

The D4-D4′ string is a single fermion in the bifundamental representation of the $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ symmetry, or equivalently in the vector representation of $\mathrm{Spin}(4)\cong\mathrm{Sp}(1)\times\mathrm{Sp}(1)$. The zero modes of the four components of the fermionic field in the vector representation form a 4d Clifford algebra, and the ground states form a spinor representation of the Clifford algebra. One then easily reads off the one-loop determinant of the D4-D4′ string,

$$Z_{\text{D4-D4}'} = 2\cosh M - 2\cosh\alpha = 2\sinh\tfrac{\pm\alpha-M}{2}, \tag{2.37}$$

where the shorthand notation of [30], reviewed in Appendix 2.6, was used again.

The integrands $Z_{\text{D0-D4}'}$ and $Z_{\text{D4-D4}'}$ have the x-expansions

$$Z_{\text{D0-D4}'} = 1 + O(x), \quad Z_{\text{D4-D4}'} = \frac{1}{x} - \chi_2(w) + x. \tag{2.38}$$

Plugging this into the ϕ-contour integral and the integration formula (2.32), one can see that the leading $O(\frac{1}{x})$ order term in the expansion of the generating function I_{all} gives the superconformal index I_{SCI} and the "naive" expression (2.30) for the Wilson ray index $I_2^{\text{Wilson ray}}$ contributes to I_{ray} in the $O(1)$ order term of the expansion, as shown in (2.34). However, one should be cautious, because the x-expansion in general does not commute with the ϕ-contour integral, and the above discussion should be just taken as heuristics. In the next section, we present the results for the ray operator indices obtained by evaluating the integrals (2.15) and (2.32) and expanding the generating function I_{all}. We will demonstrate that the ray operator indices contain the "naive" Wilson ray operator indices and exhibit the E_n symmetry.

Ray operator indices

We computed the ray operator indices up to $O(t^5)$ order in the t-expansion, which receives contributions from up to instanton number five. For simplicity, except for the case $N_f = 1$, we turn off all the SO($2N_f$) fugacities, and leave only the fugacity q associated to the U(1) instanton number symmetry. We list our results for each value of N_f below, including the correction $q^{-2/(8-N_f)}$ discussed in §2.3. Note that for $1 \leq N_f \leq 6$ (i.e., $E_2, \ldots, E_7$) the leading order term in the index is $O(t)$, which corresponds to a doublet of the diagonal subgroup of SU(2)$_+\times$SU(2)$_R$, according to the discussion in §2.3. Furthermore, for $2 \leq N_f \leq 6$ the coefficient of the $O(t)$ term is a character of a minuscule representation of E_{N_f+1} (i.e., a representation whose weights form a single orbit of the Weyl group). Thus, these terms appear to capture the operators that generalize (2.20), with a minuscule representation of E_{N_f+1} playing the role of the fundamental representation of the flavor symmetry in an ordinary gauge theory coupled to quarks. For convenience, we recall that

$$t \equiv e^{-\epsilon_+}, \quad u \equiv e^{-\epsilon_-},$$

are the fugacities that couple to $J_+ + J_R$ and J_-, respectively.

Ray operator index in E_2 =SU(2)×U(1) theory

$$\begin{aligned} q^{-\frac{2}{7}} I_{\text{ray}}(t,u,m_\ell,q) &= \left[z^{\frac{4}{7}} + z^{-\frac{3}{7}}\chi_2(y)\right] t + z^{-\frac{3}{7}}\chi_4(y)t^3 \\ &+ \chi_2(u)\left[z^{-\frac{3}{7}}\left(\chi_4(y)+2\chi_2(y)\right) + z^{\frac{4}{7}}\left(\chi_3(y)+1\right)\right] t^4 \\ &+ \Big[-z^{-\frac{10}{7}}\chi_3(y) + z^{-\frac{3}{7}}\left[\chi_3(u)(\chi_4(y)+3\chi_2(y)) + \chi_6(y)+\chi_2(y)\right] \\ &\quad + z^{\frac{4}{7}}\left[\chi_3(u)(\chi_3(y)+2)+1\right] - z^{\frac{11}{7}}\chi_2(y)\Big] t^5 + O(t^6) \end{aligned} \tag{2.39}$$

Here we defined the fugacities y and z, which correspond to the SU(2) and U(1) factors of E_2 respectively. They are related to the fugacities q and y_1 [where y_1 is associated with the flavor SO(2) symmetry of the $N_f = 1$ 5d SYM] by [24],

$$y^2 = qy_1, \quad z^2 = \frac{y_1^7}{q}. \tag{2.40}$$

(See Appendix 2.9 for more details.) Note that the prefactor $q^{\frac{2}{7}}$ is $q^{2/(8-N_f)}$, in accordance with the shift in (2.24). The U(1) charge $\mathbf{Q}$ of all the ray operators (as captured by z) is in $\frac{4}{7} + \mathbb{Z}$, and the SU(2) spin j [encoded in the character $\chi_{2j+1}(y)$] is integer (half-integer) when $\mathbf{Q} - \frac{4}{7}$ is even (odd).

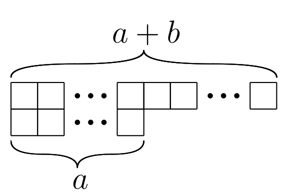
Ray operator index in $E_3 = \mathrm{SU}(3)\times\mathrm{SU}(2)$ theory

$$\begin{aligned} q^{-\frac{1}{3}} I_{\mathrm{ray}}(t, u, m_\ell, q) = \chi^{E_3}_{[1,0,1]} t + \left[\chi^{E_3}_{[2,1,1]} + \chi^{E_3}_{[1,0,3]}\right] t^3 + \chi_2(u) \left[\chi^{E_3}_{[1,0,3]} + \chi^{E_3}_{[2,1,1]} + \chi^{E_3}_{[0,2,1]}\right] t^4 \\ + \left[\chi_3(u)\big(\chi^{E_3}_{[1,0,1]} + \chi^{E_3}_{[1,0,3]} + \chi^{E_3}_{[2,1,1]} + \chi^{E_3}_{[0,2,1]}\big) + \chi^{E_3}_{[3,2,1]} + \chi^{E_3}_{[1,0,7]}\right] t^5 + O(t^6) \end{aligned} \tag{2.41}$$

The relevant E_3 characters are as follows:

$$\begin{aligned} \chi^{E_3}_{[1,0,1]} &= \frac{4}{q^{1/3}} + 2q^{2/3}, \\ \chi^{E_3}_{[1,0,3]} &= \frac{8}{q^{1/3}} + 4q^{2/3}, \\ \chi^{E_3}_{[1,0,7]} &= \frac{16}{q^{1/3}} + 8q^{2/3}, \\ \chi^{E_3}_{[2,1,1]} &= \frac{6}{q^{4/3}} + \frac{12}{q^{1/3}} + 8q^{2/3} + 4q^{5/3}, \\ \chi^{E_3}_{[0,2,1]} &= \frac{2}{q^{4/3}} + \frac{4}{q^{1/3}} + 6q^{2/3}, \\ \chi^{E_3}_{[3,2,1]} &= \frac{8}{q^{7/3}} + \frac{16}{q^{4/3}} + \frac{24}{q^{1/3}} + 18q^{2/3} + 12q^{5/3} + 6q^{8/3}. \end{aligned} \tag{2.42}$$

Our notation for E_3 characters $\chi^{E_3}_{[a,b,c]}$ is equivalent to the product $\chi^{SU(3)}_{[a,b]}\chi^{SU(2)}_{c+1}$, where $j = c/2$ is the spin of the SU(2) representation, and $[a, b]$ denotes an SU(3) representation with Young diagram



Note that the spin $j = c/2$ is always half integral, and the number of boxes in the Young diagram is always 2 (mod 3). This corresponds to charge 5 (mod 6) under the $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ center of E_3. Note also that the coefficient of the $O(t)$ term is the character of the minuscule representation $(3, 2)$ of $E_3 \cong \mathrm{SU}(3)\times\mathrm{SU}(2)$.

Ray operator index in $E_4 = \mathrm{SU}(5)$ theory

$$\begin{aligned} q^{-\frac{2}{5}} I_{\text{ray}}(t,u,m_\ell,q) &= \chi^{E_4}_{[0,1,0,0]} t + \chi^{E_4}_{[1,1,0,1]} t^3 \\ &+ \chi_2(u) \left[\chi^{E_4}_{[1,1,0,1]} + \chi^{E_4}_{[2,0,0,0]} + \chi^{E_4}_{[0,0,1,1]} + \chi^{E_4}_{[0,1,0,0]}\right] t^4 \\ &+ \Big[\chi_3(u)\big(\chi^{E_4}_{[1,1,0,1]} + \chi^{E_4}_{[2,0,0,0]} + \chi^{E_4}_{[0,0,1,1]} + 2\chi^{E_4}_{[0,1,0,0]}\big) \\ &\quad + \chi^{E_4}_{[2,1,0,2]} + \chi^{E_4}_{[1,1,0,1]} + \chi^{E_4}_{[2,0,0,0]} + \chi^{E_4}_{[0,0,1,1]} + 3\chi^{E_4}_{[0,1,0,0]}\Big] t^5 + O(t^6) \end{aligned} \tag{2.43}$$

The relevant E_4 characters are as follows:

$$\begin{aligned} \chi^{E_4}_{[0,1,0,0]} &= \frac{6}{q^{2/5}} + 4q^{3/5}, \\ \chi^{E_4}_{[0,0,1,1]} &= \frac{4}{q^{7/5}} + \frac{16}{q^{2/5}} + 20q^{3/5}, \\ \chi^{E_4}_{[1,1,0,1]} &= \frac{20}{q^{7/5}} + \frac{80}{q^{2/5}} + 60q^{3/5} + 15q^{8/5}, \\ \chi^{E_4}_{[2,0,0,0]} &= \frac{10}{q^{2/5}} + 4q^{3/5} + q^{8/5}, \\ \chi^{E_4}_{[2,1,0,2]} &= \frac{45}{q^{12/5}} + \frac{180}{q^{7/5}} + \frac{450}{q^{2/5}} + 360q^{3/5} + 144q^{8/5} + 36q^{13/5}. \end{aligned} \tag{2.44}$$

The representation $[a,b,c,d]$ corresponds to a Young diagram with rows of lengths $a+b+c+d, a+b+c, a+b, a$. Note that the representations have Young diagrams with total number of boxes $4a+3b+2c+d = 3, 8, 13, \ldots$. Thus, under the $\mathbb{Z}_5$ center they charge 3 (mod 5), as promised in §2.3. Note also that the coefficient of the $O(t)$ term is the character of the minuscule representation $\overline{10}$ of $E_4 \cong \mathrm{SU}(5)$.

Ray operator index in $E_5 = \mathrm{SO}(10)$ theory

$$\begin{aligned} q^{-\frac{1}{2}} I_{\text{ray}}(t,u,m_\ell,q) &= \chi^{E_5}_{[0,0,0,0,1]} t + \chi^{E_5}_{[0,1,0,0,1]} t^3 + \chi_2(u) \left[\chi^{E_5}_{[1,0,0,1,0]} + \chi^{E_5}_{[0,1,0,0,1]} + \chi^{E_5}_{[0,0,0,0,1]}\right] t^4 \\ &+ \left[\chi_3(u)\big(\chi^{E_5}_{[1,0,0,1,0]} + \chi^{E_5}_{[0,1,0,0,1]} + 2\chi^{E_5}_{[0,0,0,0,1]}\big) + \chi^{E_5}_{[0,2,0,1,0]} + \chi^{E_5}_{[0,0,0,0,1]}\right] t^5 + O(t^6) \end{aligned} \tag{2.45}$$

The relevant E_5 characters are as follows:

$$\begin{aligned} \chi^{E_5}_{[0,0,0,0,1]} &= \frac{8}{q^{1/2}} + 8q^{1/2}, \\ \chi^{E_5}_{[0,1,0,0,1]} &= \frac{56}{q^{3/2}} + \frac{224}{q^{1/2}} + 224q^{1/2} + 56q^{3/2}, \\ \chi^{E_5}_{[1,0,0,1,0]} &= \frac{8}{q^{3/2}} + \frac{64}{q^{1/2}} + 64q^{1/2} + 8q^{3/2}, \\ \chi^{E_5}_{[0,2,0,1,0]} &= \frac{224}{q^{5/2}} + \frac{1120}{q^{3/2}} + \frac{2688}{q^{1/2}} + 2688q^{1/2} + 1120q^{3/2} + 224q^{5/2}. \end{aligned} \tag{2.46}$$

Recall that the root lattice is a sublattice of index 4 in the weight lattice of E_5. The quotient of the weight lattice by root lattice can be identified with the Pontryagin dual of the $\mathbb{Z}_4 \subset E_5$ center, and all the weights appearing in the characters above project to the same generator of $\mathbb{Z}_4$. In other words, there is a natural assignment of an additive $\mathbb{Z}_4$ charge to every weight, with roots having charge 0, and it is not hard to check that all the weights appearing above have the same nonzero $\mathbb{Z}_4$ charge, which is ± 1 (depending on convention). As discussed in §2.3, referring to the "SO(8)×U(1)" subgroup, the value of the $\mathbb{Z}_4$ charge when taken mod 2 corresponds to the U(1) charge mod 2. The fact that all q powers in the ray operator index are half integers means that the U(1) charge is odd, and this confirms that the $\mathbb{Z}_4$ charge is $\pm 1 \pmod 4$. Note also that the coefficient of the $O(t)$ term is the character of the minuscule representation 16 of $E_5 \cong \mathrm{SO}(10)$.

Ray operator index in E_6 theory

$$\begin{aligned} q^{-\frac{2}{3}} I_{\mathrm{ray}}(t,u,m_\ell,q) &= \chi^{E_6}_{[0,0,0,0,0,1]} t + \chi^{E_6}_{[0,1,0,0,0,1]} t^3 + \chi_2(u) \left[\chi^{E_6}_{[0,0,0,0,0,1]} + \chi^{E_6}_{[0,1,0,0,0,1]} + \chi^{E_6}_{[0,0,1,0,0,0]}\right] t^4 \\ &+ \left[\chi_3(u)\big(2\chi^{E_6}_{[0,0,0,0,0,1]} + \chi^{E_6}_{[0,1,0,0,0,1]} + \chi^{E_6}_{[0,0,1,0,0,0]}\big) + \chi^{E_6}_{[0,2,0,0,0,1]} + \chi^{E_6}_{[0,0,0,0,0,1]}\right] t^5 + O(t^6) \end{aligned} \tag{2.47}$$

The relevant E_6 characters are as follows:

$$\begin{aligned} \chi^{E_6}_{[0,0,0,0,0,1]} &= \frac{10}{q^{2/3}} + 16q^{1/3} + q^{4/3}, \\ \chi^{E_6}_{[0,0,1,0,0,0]} &= \frac{16}{q^{5/3}} + \frac{130}{q^{2/3}} + 160q^{1/3} + 45q^{4/3}, \\ \chi^{E_6}_{[0,1,0,0,0,1]} &= \frac{144}{q^{5/3}} + \frac{576}{q^{2/3}} + 736q^{1/3} + 256q^{4/3} + 16q^{7/3}, \\ \chi^{E_6}_{[0,2,0,0,0,1]} &= \frac{1050}{q^{8/3}} + \frac{5712}{q^{5/3}} + \frac{13506}{q^{2/3}} + 15696q^{1/3} + 8226q^{4/3} + 2016q^{7/3} + 126q^{10/3}. \end{aligned} \tag{2.48}$$

The root lattice of E_6 is a sublattice of index 3 in the weight lattice. The quotient of the weight lattice by root lattice can be identified with the Pontryagin dual of the $\mathbb{Z}_3 \subset E_6$ center, and again all the weights appearing in the characters above project to the same generator of $\mathbb{Z}_3$. This is consistent with the discussion in §2.3, and indeed, as promised there, all the E_6 characters that appear in the index of ray operators decompose under SO(10)×U(1) in such a way that the powers of q (which are proportional to the U(1) charge) take values in $\frac{1}{3} + \mathbb{Z}$. Note also that the coefficient of the $O(t)$ term is the character of the minuscule representation 27 of E_6.

Ray operator index in E_7 theory

$$
\begin{aligned}
q^{-1} I_{\text{ray}}(t,u,m_\ell,q) &= \chi^{E_7}_{[0,0,0,0,0,0,1]} t + \chi^{E_7}_{[1,0,0,0,0,0,1]} t^3 \\
&+ \chi_2(u) \left[\chi^{E_7}_{[0,0,0,0,0,0,1]} + \chi^{E_7}_{[0,1,0,0,0,0,0]} + \chi^{E_7}_{[1,0,0,0,0,0,1]}\right] t^4 \\
&+ \Big[\chi_3(u)\big(2\chi^{E_7}_{[0,0,0,0,0,0,1]} + \chi^{E_7}_{[0,1,0,0,0,0,0]} + \chi^{E_7}_{[1,0,0,0,0,0,1]}\big) \\
&\quad + \chi^{E_7}_{[2,0,0,0,0,0,1]} + \chi^{E_7}_{[0,0,0,0,0,0,1]}\Big] t^5 + O(t^6)
\end{aligned} \tag{2.49}
$$

The relevant E_7 characters are listed as follows

$$
\begin{aligned}
\chi^{E_7}_{[0,0,0,0,0,0,1]} &= \frac{12}{q} + 32 + 12q, \\
\chi^{E_7}_{[0,1,0,0,0,0,0]} &= \frac{32}{q^2} + \frac{232}{q} + 384 + 232q + 32q^2, \\
\chi^{E_7}_{[1,0,0,0,0,0,1]} &= \frac{12}{q^3} + \frac{384}{q^2} + \frac{1596}{q} + 2496 + 1596q + 384q^2 + 12q^3, \\
\chi^{E_7}_{[2,0,0,0,0,0,1]} &= \frac{12}{q^5} + \frac{384}{q^4} + \frac{6348}{q^3} + \frac{31008}{q^2} + \frac{73536}{q} + 97536 \\
&\quad + 73536q + 31008q^2 + 6348q^3 + 384q^4 + 12q^5.
\end{aligned} \tag{2.50}
$$

It is not hard to check that all the weights that appear in the characters above do not belong to the root lattice (i.e., they are not representations of the adjoint form of E_7). Since the root lattice is a sublattice of index 2 in the weight lattice of E_7, we see that all the E_7 representations of ray operators are odd under the $\mathbb{Z}_2$ center. Note also that the coefficient of the $O(t)$ term is the character of the minuscule representation 56 of E_7.

Ray operator index in E_8 theory

The case $N_f = 7$ poses a special challenge, because we do not have a consistent result for the South Pole contribution $Z_{\text{inst+line}}$ to the partition function (2.32). The problem, as we discussed below equation (2.33), is that a direct computation of $Z_{\text{inst+line}}$, following the ideas developed in [30], yields a result that depends on the $\mathrm{SU}(2)^R_-$ fugacity v. Nevertheless, it is instructive to look at the result of the integral formula (2.32), after substituting for $Z_{\text{inst+line}}$ the problematic formula (2.33). With $\chi_2(v) \equiv v + \frac{1}{v}$, and Z_{inst} denoting the instanton partition function (2.17) without the line, we find

$$
q^{-2} I^{(\text{calculated})}_{\text{ray}}(t,u,v,m_\ell,q) = \chi_2(v) I_{\text{SCI}}(t,u,m_\ell,q) + I_{v\text{-independent}}(t,u,m_\ell,q), \tag{2.51}
$$

where I_{SCI} is the index of local operators given in (2.13), and

$$\begin{aligned} I_{v\text{-independent}} &= \left(1+\chi^{E_8}_{[0,0,0,0,0,0,0,1]}\right)t + \chi_2(u)t^2 + \bigg(\chi^{E_8}_{[0,0,0,0,0,0,0,1]} + \chi^{E_8}_{[0,0,0,0,0,0,1,0]} \\ &+ \chi^{E_8}_{[0,0,0,0,0,0,0,2]}\bigg)t^3 \\ &+ \Big\{\chi_2(u) + \chi_2(u)\big(3\chi^{E_8}_{[0,0,0,0,0,0,0,1]} + \chi^{E_8}_{[1,0,0,0,0,0,0,0]} + \chi^{E_8}_{[0,0,0,0,0,0,0,2]} + \chi^{E_8}_{[0,0,0,0,0,0,1,0]}\big)\Big\}\, t^4 \\ &+ \Big\{2 + 2\chi^{E_8}_{[0,0,0,0,0,0,0,1]} + \chi^{E_8}_{[0,0,0,0,0,0,1,1]} + \chi^{E_8}_{[0,0,0,0,0,0,0,2]} + \chi^{E_8}_{[0,0,0,0,0,0,0,3]} \\ &\quad + \chi_3(u)\big(2 + 4\chi^{E_8}_{[0,0,0,0,0,0,0,1]} + \chi^{E_8}_{[1,0,0,0,0,0,0,0]} + \chi^{E_8}_{[0,0,0,0,0,0,0,2]} + \chi^{E_8}_{[0,0,0,0,0,0,1,0]}\big)\Big\}\, t^5 + O(t^6). \end{aligned} \tag{2.52}$$

The relevant E_8 characters are listed as follows

$$\begin{aligned} \chi_{[0,0,0,0,0,0,0,1]} &= \frac{14}{q^2} + \frac{64}{q} + 92 + 64q + 14q^2, \\ \chi_{[1,0,0,0,0,0,0,0]} &= \frac{1}{q^4} + \frac{64}{q^3} + \frac{378}{q^2} + \frac{896}{q} + 1197 + 896q + 378q^2 + 64q^3 + q^4, \\ \chi_{[0,0,0,0,0,0,1,0]} &= \frac{91}{q^4} + \frac{896}{q^3} + \frac{3290}{q^2} + \frac{6720}{q} + 8386 + 6720q + 3290q^2 + 896q^3 + 91q^4, \\ \chi_{[0,0,0,0,0,0,0,2]} &= \frac{104}{q^4} + \frac{832}{q^3} + \frac{2990}{q^2} + \frac{5888}{q} + 7372 + 5888q + 2990q^2 + 832q^3 + 104q^4, \\ \chi_{[0,0,0,0,0,0,1,1]} &= \frac{896}{q^6} + \frac{11584}{q^5} + \frac{65792}{q^4} + \frac{221248}{q^3} + \frac{496768}{q^2} + \frac{791168}{q} + 921088 \\ &\quad + 791168q + 496768q^2 + 221248q^3 + 65792q^4 + 11584q^5 + 896q^6, \\ \chi_{[0,0,0,0,0,0,0,3]} &= \frac{546}{q^6} + \frac{5824}{q^5} + \frac{30394}{q^4} + \frac{98176}{q^3} + \frac{214474}{q^2} + \frac{336960}{q} + 390377 \\ &\quad + 336960q + 214474q^2 + 98176q^3 + 30394q^4 + 5824q^5 + 546q^6. \end{aligned} \tag{2.53}$$

We computed the contribution to the formula (2.52) up to instanton number five. The $O(t^5)$ order of (2.52) receives contribution from higher instanton number, and we completed the formula (2.52) "by hand" using the property $\chi^{E_8}_R(q) = \chi^{E_8}_R(q^{-1})$ of the E_8 characters.

Note that since $I_{\text{SCI}} = 1 + O(t^2)$, the expression (2.51) starts with a t-independent term $\chi_2(v)$. But a ray operator index is forbidden from having such a term, according to the discussion at the end of §2.3, since it would require the existence of a BPS ray operator with $J_+ + J_R = 0$. More generally, the first term on the RHS of (2.51) suggests that there are unwanted $\mathrm{SU}(2)^R_-$ doublet states that contribute to the partition function (2.33). It is tempting to drop the $\chi_2(v)I_{\text{SCI}}$ term entirely from the ray index and keep only the v-independent term, but we have not found a satisfactory argument for this ad-hoc prescription, and it is not clear whether any additional $\mathrm{SU}(2)^R_-$ singlets should be dropped as well, or not.

$N_f = 7$ is special, because the parameter $\mathbf{m} = 8 - N_f$ is exactly 1 in this case, which makes it possible for the F1 that appeared in the construction of the line and ray operators in §2.3 to end on a D0-brane instead of the D4-brane. The F1 can thus be "screened", and the x^9 coordinate of the D0-brane is a free parameter, which gives rise to a continuum, in the absence of the D4'-brane.[13] We do not understand why this effect creates the v-dependent terms, but we suspect that it is part of the problem.

2.5. Discussion

We have extended the analysis of [30] by calculating the index of ray operators in 5d E_n SCFTs for $n = 2, \dots, 7$. We converted the problem to a partition function of the SCFT on $\mathrm{S}^4 \times \mathrm{S}^1$ with a Wilson loop along S^1 and with twisted boundary conditions parameterized by the various fugacities. Following [30], we provided evidence that the manifest $\mathrm{SO}(2n-2)$ flavor symmetry combines with the U(1) symmetry associated with the conserved instanton charge to form a subgroup of an enhanced E_n "flavor" symmetry, as predicted in [10]. Our index reveals E_n representations that do not appear in the superconformal index of local operators. These are representations with weights that are not in the root lattice of E_n, and the ray operators are charged under the nontrivial center of E_n. For $n = 8$ we encountered a problem with the calculation of the contribution of zero-size instantons to the ray index. The prescription that we followed for calculating the Nekrasov partition function in the presence of a Wilson loop does not appear to yield a result that factorizes properly into field theory modes and modes that are decoupled from the D4-brane. We do not know the reason for this inconsistency, but we suspect it has to do with the possibility for a fundamental string (that induces the line operator) to end on a D0-brane.

As in the work of [30], a key ingredient in the calculation is the contribution of coincident zero-size instantons. In our case, the instantons are also coincident with the defect introduced by a Wilson loop, and we needed to regularize their contribution carefully. As we saw in §2.4, merely localizing the Wilson loop on BPS configurations is not the right answer. Instead, we followed [54, 103, 60] and rather than introducing the Wilson loop directly to the $\mathrm{S}^4 \times \mathrm{S}^1$ partition function, we modified the Nekrasov partition function that captures the contribution of the zero-size instantons near the Wilson loop. The modified Nekrasov partition function is an index of the quantum mechanics of D0-branes that probe a D4-D8/O8 system, and the Wilson loop was captured by introducing an additional D4-brane (denoted by D4') to the system so that after integrating out the (heavy) fermionic D4-D4' string modes, the Wilson loop is recovered. That the final result (after inserting this modified Nekrasov partition function into the 5d index formula) reveals the expected hidden E_n global symmetry lends credence to this resolution of zero-size instanton singularities in our context as well. The modified Nekrasov partition function also appeared as part of Nekrasov's larger work [103] on the qq-character.

A better understanding of how the exceptional symmetry of the E_n SCFTs arises is important both in its own right and since the E_n SCFTs describe the low-energy degrees of

[13] The mass of the D0-brane increases linearly with x^9, but this effect is canceled by the decreasing length of F1, and so there is no potential.

freedom of M-theory near degenerations of Calabi-Yau manifolds [61] and can also provide clues about the 6d $(1, 0)$ SCFT with E_8 global symmetry. The 5d ray operators presumably descend from BPS cylinder operators in 6d, that is, surface operators associated with open surfaces with $S^1 \times \mathbb{R}_+$ geometry. The AdS/CFT dual of such an operator, as well as the analysis of §2.3 suggest that the 1d boundary of these operators are "labeled" by a state of an E_8 affine Lie algebra at level 1. This is the extended symmetry of the low-energy 2d CFT that described the M2-M9 intersection [62]. It would be interesting to examine further the relationship between 5d ray operators and 6d surface operators.

On the Coulomb branch of the E_n theories the low-energy description is given by a single U(1) vector multiplet, and the ray operators that we counted in this work can act on the vacuum and create BPS states with one unit of charge. It would be interesting to explore the relation between the BPS spectrum of the E_n theories on their Coulomb branch (computed in [63, 64, 65]) and the index that we calculated in this paper [46]. Indeed, for $n = 2, 3, 4, 5, 6, 7$, the net numbers of $\frac{1}{8}$BPS operators with $J_+ + J_R = \frac{1}{2}$ and $J_- = 0$ are $3, 6, 10, 16, 27, 56$. These are precisely the numbers of isolated holomorphic curves of genus 0 embedded in the del Pezzo surface B_n [66], and are a special case of the Gopakumar-Vafa invariants of the Calabi-Yau manifolds that enter the M-theory construction of the E_n theories [67, 68]. It would be interesting [46] to explore the connection between ray operator indices generated by probing the D4-brane with more than one fundamental string and Gopakumar-Vafa invariants of higher genera and degrees, as computed in [66].

Our results are also related to the elliptic genus of an E-string near a surface operator of the 6d $(1, 0)$ E_8-theory. The E-string is the BPS string-like excitation of the 6d theory on the Coulomb branch. A single E-string is described by a left-moving E_8 chiral current algebra together with four noninteracting 2d bosons and right-moving fermions, but k coincident E-strings have nontrivial 2d CFT descriptions with $(4, 0)$ supersymmetry [69]. It was shown in [70] that the intermediate steps in the computation of a 5d index for the $N_f = 8$ case can be used to also compute the elliptic genus of k E-strings (see also [69, 71]). More precisely, the index of the 5d theory on $S^4 \times S^1$ is a contour integral over a complex variable w that can be identified with the holonomy of a $U(1) \subset SU(2)$ gauge field on S^1. The integrand is a product of terms, one of which is a Nekrasov partition function whose w^k coefficient yields the elliptic genus of k E-strings. Our computation of the index of ray operators also has a Nekrasov partition function ingredient, from which a modified E-string elliptic genus can be read off. It counts bound states of k E-strings and a 1+1d defect, introduced into the 6d theory via a BPS surface operator, and compactified on S^1.

Our calculation, which builds on the techniques developed in [24, 30], uses an ordinary super Yang-Mills theory to capture properties of a strongly interacting SCFT. It joins a growing body of work that demonstrates that the manifestly nonrenormalizable Yang-Mills theories in dimensions $d > 4$ still prove to be very useful in the right context. For example, [72] proposed that 5d super Yang-Mills theory can describe the 6d $(2, 0)$-theory, in [73] it was shown how Yang-Mills theory can be used to calculate a superconformal index for the 6d $(2, 0)$-theory and reproduce its anomaly coefficient, and in [74, 75] it was demonstrated that a 6d Yang-Mills theory can be used to calculate Little String Theory amplitudes.

The localization computation of the superconformal indices in [] requires deforming the 5d SYM in a way that keeps the indices invariant. In §2.2, we demonstrated that the perturbative part of the indices (2.11) can be reproduced by directly counting the local gauge invariant operators in the 5d SYM. One expects that the instanton contribution to the indices can be reproduced in a similar way, involving quantizing the moduli space of the instantons on S^4 and counting the instanton operators [76, 77, 78, 79] in 5d SYM. Similar problems have been studied in 3d Chern-Simons matter theories [80, 81, 82, 83], where partial success was achieved, and the superconformal indices were computed in certain monopole sectors by directly counting monopole operators.

2.6. Appendix: One-loop determinants in the D0-D4-D8/O8 quantum mechanics

The one-loop determinants of the D0-D4-D8/O8 quantum mechanics fields, listed in Table 2.1, were computed in [24, 30]. The exact forms can be found in equations (3.42)-(3.50) of [30], and we summarize them in this appendix, using the conventions of [30] whereby, for example, a term of the form $2\sinh(\pm A \pm B \pm C + D)$ should be interpreted as a product over eight terms (all combinations of $\pm$ signs):

$$\begin{aligned} 2\sinh(\pm A \pm B \pm C + D) \to\ & 256\sinh(A+B+C+D)\sinh(A+B-C+D) \\ & \sinh(A-B+C+D)\sinh(A-B-C+D)\sinh(-A+B+C+D) \\ & \sinh(-A+B-C+D)\sinh(-A-B+C+D)\sinh(-A-B-C+D). \end{aligned}$$

The one-loop determinants of the D0-D0 strings are given by

$$\begin{aligned} Z^{+,\,k=2n+\chi}_{\text{D0-D0}} =\ & \left[\Big(\prod_{I=1}^{n} 2\sinh\tfrac{\pm\phi_I}{2}\Big)^{\chi}\prod_{I<J}^{n} 2\sinh\tfrac{\pm\phi_I\pm\phi_J}{2}\right](2\sinh\epsilon_+)^n\Big(\prod_{I=1}^{n} 2\sinh\tfrac{\pm\phi_I+2\epsilon_+}{2}\Big)^{\chi} \\ & \times\prod_{I<J}^{n} 2\sinh\tfrac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}\big(2\sinh\tfrac{\pm m-\epsilon_-}{2}\big)^n\Big(\prod_{I=1}^{n} 2\sinh\tfrac{\pm\phi_I\pm m-\epsilon_-}{2}\Big)^{\chi}\prod_{I<J}^{n} 2\sinh\tfrac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2} \\ & \times\frac{1}{\big(2\sinh\frac{\pm m-\epsilon_+}{2}\big)^{n+\chi}}\Big(\prod_{I=1}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm m-\epsilon_+}{2}}\Big)^{\chi}\prod_{I=1}^{n}\frac{1}{2\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}}\prod_{I<J}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \\ & \times\frac{1}{\big(2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\big)^{n+\chi}}\Big(\prod_{I=1}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}}\Big)^{\chi}\prod_{I=1}^{n}\frac{1}{2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}}\prod_{I<J}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}}, \end{aligned} \tag{2.54}$$

and

$$\begin{aligned}
Z^{-,\,k=2n+1}_{\text{D0-D0}} &= \Big(\prod_{I}^{n} 2\cosh\tfrac{\pm\phi_I}{2}\prod_{I<J}^{n} 2\sinh\tfrac{\pm\phi_I\pm\phi_J}{2}\Big)(2\sinh\epsilon_+)^n\prod_{I=1}^{n}2\cosh\tfrac{\pm\phi_I+2\epsilon_+}{2}\\
&\times\prod_{I<J}^{n}2\sinh\tfrac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}\big(2\sinh\tfrac{\pm m-\epsilon_-}{2}\big)^n\prod_{I=1}^{n}2\cosh\tfrac{\pm\phi_I\pm m-\epsilon_-}{2}\prod_{I<J}^{n}2\sinh\tfrac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}\\
&\times\frac{1}{\big(2\sinh\frac{\pm m-\epsilon_+}{2}\big)^{n+1}}\prod_{I=1}^{n}\frac{1}{2\cosh\frac{\pm\phi_I\pm m-\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}}\prod_{I<J}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}}\\
&\times\frac{1}{\big(2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\big)^{n+1}}\prod_{I=1}^{n}\frac{1}{2\cosh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}}\prod_{I<J}^{n}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}},
\end{aligned}\tag{2.55}$$

and

$$\begin{aligned}
Z^{-,\,k=2n}_{\text{D0-D0}} &= \Big(\prod_{I<J}^{n-1} 2\sinh\tfrac{\pm\phi_I\pm\phi_J}{2}\prod_{I}^{n-1} 2\sinh(\pm\phi_I)\Big)2\cosh\epsilon_+(2\sinh\epsilon_+)^{n-1}\\
&\times\prod_{I=1}^{n-1}2\sinh(\pm\phi_I+2\epsilon_+)\prod_{I<J}^{n-1}2\sinh\tfrac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}\\
&\times 2\cosh\tfrac{\pm m-\epsilon_-}{2}\big(2\sinh\tfrac{\pm m-\epsilon_-}{2}\big)^{n-1}\prod_{I=1}^{n-1}2\sinh(\pm\phi_I\pm m-\epsilon_-)\prod_{I<J}^{n-1}2\sinh\tfrac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}\\
&\times\frac{1}{\big(2\sinh\frac{\pm m-\epsilon_+}{2}\big)^{n}2\sinh(\pm m-\epsilon_+)}\prod_{I=1}^{n-1}\frac{1}{2\sinh(\pm\phi_I\pm m-\epsilon_+)\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}}\\
&\times\frac{1}{\big(2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\big)^{n}2\sinh(\pm\epsilon_-+\epsilon_+)}\prod_{I=1}^{n-1}\frac{1}{2\sinh(\pm\phi_I\pm\epsilon_-+\epsilon_+)2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}}\\
&\times\prod_{I<J}^{n-1}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}}\prod_{I<J}^{n-1}\frac{1}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}}
\end{aligned}\tag{2.56}$$

The first to the forth lines of the equations (2.54), (2.55) and(2.56) are the one-loop determinants of the $N = 4$ vector multiplet, Fermi multiplet, twisted hypermultiplet and hypermultiplet, respectively. The one-loop determinants of the D0-D4 strings are given by

$$\begin{aligned}
Z^{+,\,k=2n+\chi}_{\text{D0-D4}} &= \Big(\frac{2\sinh\frac{m\pm\alpha}{2}}{2\sinh\frac{\pm\alpha+\epsilon_+}{2}}\Big)^{\chi}\prod_{I=1}^{n}\frac{2\sinh\frac{\pm\phi_I\pm\alpha-m}{2}}{2\sinh\frac{\pm\phi_I\pm\alpha+\epsilon_+}{2}},\\
Z^{-,\,k=2n+1}_{\text{D0-D4}} &= \frac{2\cosh\frac{m\pm\alpha}{2}}{2\cosh\frac{\pm\alpha+\epsilon_+}{2}}\prod_{I=1}^{n}\frac{2\sinh\frac{\pm\phi_I\pm\alpha-m}{2}}{2\sinh\frac{\pm\phi_I\pm\alpha+\epsilon_+}{2}},\\
Z^{-,\,k=2n}_{\text{D0-D4}} &= \frac{2\sinh(m\pm\alpha)}{2\sinh(\pm\alpha+\epsilon_+)}\prod_{I=1}^{n-1}\frac{2\sinh\frac{\pm\phi_I\pm\alpha-m}{2}}{2\sinh\frac{\pm\phi_I\pm\alpha+\epsilon_+}{2}}.
\end{aligned}\tag{2.57}$$

The one-loop determinants of the D0-D8 strings are given by

$$\begin{aligned} Z^{+,\,k=2n+\chi}_{\text{D0-D8}} &= \prod_{\ell=1}^{N_f}\Big((2\sinh\tfrac{m_\ell}{2})^\chi\prod_{I=1}^{n}2\sinh\tfrac{\pm\phi_I+m_\ell}{2}\Big),\\ Z^{-,\,k=2n+1}_{\text{D0-D8}} &= \prod_{\ell=1}^{N_f}\Big(2\cosh\tfrac{m_\ell}{2}\prod_{I=1}^{n}2\sinh\tfrac{\pm\phi_I+m_\ell}{2}\Big),\\ Z^{-,\,k=2n}_{\text{D0-D8}} &= \prod_{\ell=1}^{N_f}\Big(2\sinh m_\ell\prod_{I=1}^{n-1}2\sinh\tfrac{\pm\phi_I+m_\ell}{2}\Big). \end{aligned} \tag{2.58}$$

Finally, the Weyl factors of the $\mathrm{O}(k)_+$ and $\mathrm{O}(k)_-$ components in (2.15) are given by

$$|W|^{\chi=0}_+ = \frac{1}{2^{n-1}n!},\ |W|^{\chi=1}_+ = \frac{1}{2^n n!},\ |W|^{\chi=0}_- = \frac{1}{2^{n-1}(n-1)!},\ |W|^{\chi=1}_- = \frac{1}{2^n n!}. \tag{2.59}$$

2.7. Appendix: On the computation of North Pole and South Pole contributions

The South Pole (and similarly North Pole) contribution to the integrands (2.13) and (2.32) is evaluated by a separate index computation of a 1d field theory (Quantum Mechanics) that describes the dynamics of strings connecting D0-branes to the various D-branes in the problem (D4-branes, D4'-branes, N_f D8-branes, and the D0-branes themselves). The integrals involved have been described in great detail in [30], but for the sake of completeness we will now expand on a few of the technical details involved.

The $O(q^k)$ North Pole contribution, for $k = 2n$ or $k = 2n+1$, is given by an integral over n variables, denoted as $\phi_1, \ldots, \phi_n$. The integrand is a fraction whose numerator and denominator are both products of terms that are contributions of individual fields of the 1d field theory, with bosonic fields contributing to the denominator and fermionic fields to the numerator. Each individual term is written as $2\sinh\mathbf{X}$, with $\mathbf{X}$ a linear expression in the equivariant parameters ϵ_+, ϵ_-, the $U(1)\subset \mathrm{Sp}(1)$ chemical potential α, the $U(1)^{2N_f}\subset \mathrm{SO}(2N_f)$ chemical potentials $m_1, \ldots, m_{N_f}$ and the integration variables $\phi_1, \ldots, \phi_n$. The exact form of the integrand can be found in equations (3.42)-(3.50) of [30]. We also summarized it in Appendix 2.6. For simplicity, we will set $m_1 = \cdots = m_{N_f} = 0$ from now on.

For even $k = 2n$, the integral takes the form

$$Z^{2n}_{\text{D0-D4-D4}'\text{-D8/O8}} = \frac{1}{2^n n!}\oint Z^+_{\text{D0-D0}}Z^+_{\text{D0-D4}}Z^+_{\text{D0-D8}}Z^+_{\text{D0-D4}'}d\phi_1\cdots d\phi_n, \tag{2.60}$$

where $Z^+_{\text{D0-D0}}$, $Z^+_{\text{D0-D4}}$, $Z^+_{\text{D0-D8}}$ are all functions of $\phi_1, \ldots, \phi_n$, ϵ_+, ϵ_-, and α, and $Z^+_{\text{D0-D4}'}$ is a function of the same parameters and also M. The formulas for $Z^+_{\text{D0-D0}}$, $Z^+_{\text{D0-D4}}$, $Z^+_{\text{D0-D8}}$ are

$$\begin{aligned}
Z^+_{\text{D0-D0}} &= \left(\frac{2\sinh\frac{\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}}\right)^n \prod_{I=1}^n \frac{1}{2\sinh\frac{\pm 2\phi_I \pm m-\epsilon_+}{2}2\sinh\frac{\pm 2\phi_I\pm\epsilon_-+\epsilon_+}{2}} \\
&\quad\times\prod_{I<J}^n \frac{2\sinh\frac{\pm\phi_I\pm\phi_J}{2}2\sinh\frac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}}, \\
Z^+_{\text{D0-D4}} &= \prod_{I=1}^n \frac{2\sinh\frac{\pm\phi_I\pm\alpha-m}{2}}{2\sinh\frac{\pm\phi_I\pm\alpha+\epsilon_+}{2}}, \quad Z^+_{\text{D0-D8}} = \prod_{\ell=1}^{N_f}\prod_{I=1}^n 2\sinh\frac{\pm\phi_I+m_\ell}{2}.
\end{aligned}$$

The additional parameter m that appears in $Z^+_{\text{D0-D0}}$ and $Z^+_{\text{D0-D4}}$ represents an additional twist that can be set to $m=0$, but is kept nonzero in intermediate stages of the computation in order to regularize the integral over $\phi_1,\dots,\phi_n$, as we shall review below. The formulas for odd k are of a similar spirit, but slightly more complicated, and can be found in [30], and also copied in Appendix 2.6. The formula for $Z^+_{\text{D0-D4}'}$ is

$$Z^+_{\text{D0-D4}'} = \prod_{I=1}^n \frac{2\sinh\frac{\pm\phi_I\pm M-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm M-\epsilon_+}{2}}. \tag{2.61}$$

The integration parameters ϕ_I ($I=1,\dots,n$) live on a cylinder, with $-\infty<\text{Re}\phi_I<\infty$, and $0\le\text{Im}\phi_I<2\pi$ periodic. The integral (2.60) is performed by summing over the contributions of the poles within the integration path, which we have not described yet. A pole can arise when an argument of a sinh in the denominator of $Z^+_{\text{D0-D0}}$ or $Z^+_{\text{D0-D4}}$ equals a multiple of πi. Which poles to keep was determined in [30], using the Jeffrey-Kirwan (JK) residue technique developed in [84] and explained in [43, 42]. For $n=1$ the integration is one-dimensional and the JK prescription is to consider only the poles arising from the terms where the coefficient of ϕ_1 in the argument of sinh is positive. These are 14 poles, which we list below:

$$\phi_1\to\pm\tfrac12\epsilon_- -\tfrac12\epsilon_+,\quad \pm\tfrac12\epsilon_- -\tfrac12\epsilon_+ + i\pi,\quad \pm\alpha-\epsilon_+,\quad \pm\tfrac12 m+\tfrac12\epsilon_+,\quad \pm\tfrac12 m+\tfrac12\epsilon_+ + i\pi, \tag{2.62}$$

and

$$\phi_1\to\pm M+\epsilon_+.$$

For generic M, m, ϵ_-, ϵ_+, and α, these are all simple poles, but when we set $m\to0$, we get a double pole at $\phi_1=\frac12\epsilon_+$. For $n=1$, keeping $m\neq0$ in intermediate steps is only a convenience. It will become crucial for $n>1$. The poles are depicted in Figure 2.1 in the regime $m\ll\epsilon_+\ll\epsilon_-\ll\alpha$.

For $k=3$ the index is similarly calculated by an integral over a single parameter ϕ_1, but for $k=4$ the integral is over two parameters $d\phi_1 d\phi_2$, and the prescription is as follows. Residues of poles are evaluated at values of (ϕ_1,ϕ_2) where the arguments of at least two different sinh's in the denominator of the integrand are an integer multiple of $i\pi$. They are

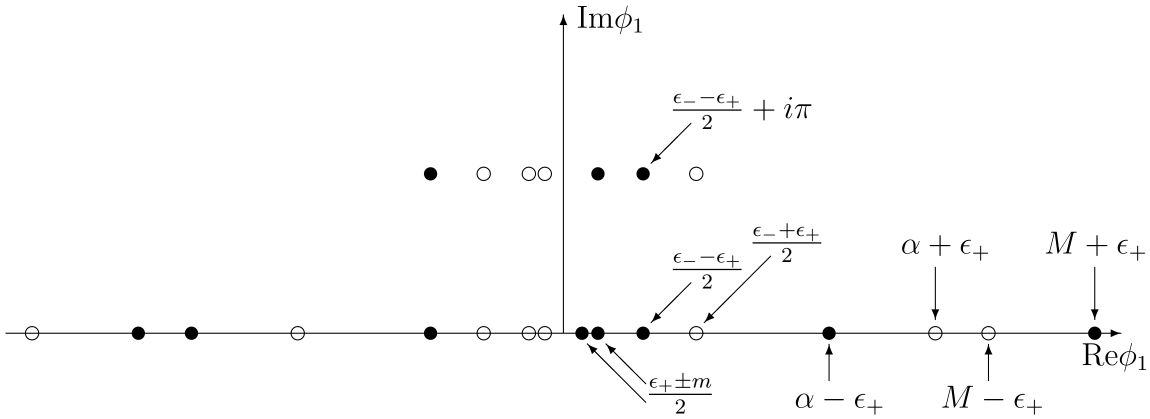


Figure 2.1: The location of the poles on the complex ϕ_1 plane for instanton number $k = 2$. The filled circles indicate the poles that are retained by the Jeffrey-Kirwan prescription, while the hollow circles indicate the poles that are ignored.

a simple pole if exactly two sinh's vanish. The argument of the i^{th} sinh ($i = 1, 2$) takes the form $\sum_I \mathbf{Q}_{iI}\phi_I + \zeta_i$, where $\mathbf{Q}_{iI}$ are constants (taking the possible values 0, $\pm 1/2$ or ± 1), and ζ_i are independent of ϕ_1 and ϕ_2 (and are linear expressions in ϵ_+, ϵ_-, m, M, α). The Jeffrey-Kirwan prescription requires us to fix an arbitrary (row) vector $\eta \equiv (\eta_1\, \eta_2)$, then calculate, for each pole, the vector $\eta\mathbf{Q}^{-1}$, and keep the residue only if all the components of $\eta\mathbf{Q}^{-1}$ are positive. (In other words, η has to be inside the cone generated by the rows of $\mathbf{Q}$.)

Double poles appear at

$$\phi_1 = \pm\phi_2 = \pm\tfrac{1}{2}m \pm \tfrac{1}{2}\epsilon_+ \qquad (\pm \text{ signs are uncorrelated}),$$

where also one of the expressions $\pm\phi_1 \pm \phi_2 \pm m - \epsilon_+$ (for the appropriate sign assignments) vanishes. These are 8 in number, and there are additional 8 double poles at

$$\phi_1 = \pm\phi_2 = \pm\tfrac{1}{2}\epsilon_- \pm \tfrac{1}{2}\epsilon_+ \qquad (\pm \text{ signs are uncorrelated}),$$

where also one of the expressions $(\pm\phi_1 \pm \phi_2 \pm \epsilon_- + \epsilon_+)/2$ vanishes. However, in these cases also $\pm\phi_1 \pm \phi_2$ vanishes (for two sign assignments), which gives the numerator of $Z^+_{\text{D0-D0}}$ a double zero, and these poles therefore do not contribute to the integral. In the above discussion, we can also add $i\pi$ to both ϕ_1 and ϕ_2 and get another set of eight double poles, but if we add $i\pi$ to only ϕ_1 or only ϕ_2, we get a simple pole. Ignoring the above mentioned double poles, for $k = 4$ there are 352 simple poles that pass the Jeffrey-Kirwan requirement.

2.8. Appendix: D-branes in massive type IIA

The massive type IIA supergravity action is given by[14]

$$
\begin{aligned}
S_{\mathrm{NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\left(R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}|H_3|^2\right),\\
S_{\mathrm{R}} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_2+\mathbf{M}B_2|^2+|\widetilde{F}_4-\frac{1}{2}\mathbf{M}B_2^2|^2\right),\\
S_{\mathrm{CS}} &= -\frac{1}{4\kappa_{10}^2}\int\left\{B_2\wedge F_4^2-\frac{1}{3}\mathbf{M}B_2^3\wedge F_4+\frac{1}{20}\mathbf{M}^2B_2^5\right\},\\
S_{\mathrm{mass}} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,\mathbf{M}^2+\frac{1}{2\kappa_{10}^2}\int\mathbf{M}F_{10},
\end{aligned}
\tag{2.63}
$$

where the $\widetilde{F}_4$ is defined by

$$
\widetilde{F}_4 = dC_3 - C_1\wedge dB_2. \tag{2.64}
$$

Consider a D8-brane localized at a constant value of x^9, say at $x^9=0$. It behaves like a domain wall that splits the spacetime into two regions $x^9<0$ and $x^9>0$. The action of the D8-brane is given by

$$
S_{\mathrm{D8}} = -\mu_8\int d^9x\,e^{-\Phi}\sqrt{-G^{(9)}}+\mu_8\int C_9. \tag{2.65}
$$

In this appendix, we use Polchinski's convention [85]. The gravitational coupling κ_{10} and the Dp-brane charge μ_p are given by

$$
\kappa_{10}^2=\frac{1}{2}(2\pi)^7\alpha'^4,\quad \mu_p^2=(2\pi)^{-2p}\alpha^{-p-1}. \tag{2.66}
$$

Varying the total action by C_9 gives the equation of motion of the Romans mass $\mathbf{M}$,

$$
\frac{\partial\mathbf{M}}{\partial x^9}=2\kappa_{10}^2\mu_8\delta(D8), \tag{2.67}
$$

which implies that the Romans mass jumps by $2\kappa_{10}^2\mu_8$ when crossing the D8-brane. Similarly, the derivative of the dilaton jumps when crossing the D8-brane[15]

$$
\partial_9\Phi\Big|_{x^9=0^+}-\partial_9\Phi\Big|_{x^9=0^-}=\frac{5}{2}\mu_8\kappa_{10}^2e^{\Phi(0)}\sqrt{G_{99}(0)}. \tag{2.69}
$$

[14]The action is invariant under the NSNS gauge transformation, where the usual B_2-field transformation $\delta B_2=d\Lambda_1$ is accompanied with the transformation of the RR-fields $\delta C_1=-\mathbf{M}\Lambda_1$ and $\delta C_3=\mathbf{M}\Lambda_1\wedge B_2$.

[15]The simplest way to derive this relation is to consider the equation of motion of the dilaton in the Einstein frame $G^E_{\mu\nu}=e^{-\frac{1}{2}\Phi}G_{\mu\nu}$,

$$
\nabla^E_\mu\partial^\mu\Phi-\frac{5}{4}\mathbf{M}^2e^{\frac{5}{2}\Phi}=\frac{5}{2}\mu_8\kappa_{10}^2(G^E_{99})^{-\frac{1}{2}}e^{\frac{5}{4}\Phi}\delta(D8). \tag{2.68}
$$

Away from the D8-brane, the equations of motion of the dilaton Φ and the metric $G_{\mu\nu}$, with all the other fields setting to zero, are given by

$$
\begin{aligned}
&R_{\mu\nu} + 2\nabla_\mu \partial_\nu \Phi - \frac{1}{2} G_{\mu\nu} \left(R + 4\nabla^\rho \partial_\rho \Phi - 4\partial^\rho \Phi \partial_\rho \Phi - \frac{1}{2} \mathbf{M}^2 e^{2\Phi} \right) = 0 \\
&R + 4\nabla^\mu \partial_\mu \Phi - 4\partial^\mu \Phi \partial_\mu \Phi = 0
\end{aligned}
\tag{2.70}
$$

Let us consider a domain wall ansatz,

$$
ds^2 = \Omega^2(x^9) \eta_{\mu\nu} dx^\mu dx^\nu, \quad \Phi = \Phi(x^9). \tag{2.71}
$$

The solution to the equations is given by

$$
\Omega(x^9) = \frac{2}{3} c_2 (c_1 \pm c_2 \mathbf{M} x^9)^{-\frac{1}{6}}, \quad e^{\Phi(x^9)} = (c_1 \pm c_2 \mathbf{M} x^9)^{-\frac{5}{6}}, \tag{2.72}
$$

c_1 and c_2 are constant away from the D8-brane. By the equations (2.67) and (2.69) and the continuity of the metric and dilaton, c_1 and c_2 still remain constant when crossing the D8-brane, and we must take the lower sign in (2.72). By a coordinate transformation $x'^9 = \frac{1}{\mathbf{M}} (c_1 - c_2 \mathbf{M} x^9)^{\frac{2}{3}}$, the solution can be put into the form as (relabel x'^9 by x^9)

$$
e^{\Phi(x^9)} = (\mathbf{M} x^9)^{-\frac{5}{4}}, \quad ds^2 = (\mathbf{M} x^9)^{-\frac{1}{2}} \left[-(dx^0)^2 + (dx^1)^2 + \cdots + (dx^8)^2 \right] + (\mathbf{M} x^9)^{\frac{1}{2}} (dx^9)^2. \tag{2.73}
$$

Now, let us focus on the case of interest: N_f D8-branes coinciding with O8 plane in the strong coupling limit. The D8/O8 singularity is located at $x^9 = 0$, where the string coupling diverges. The total RR 9-form charge is $\mathbf{m} \equiv (8 - N_f)$, and the Romans mass is given by

$$
\mathbf{M} = 2\mathbf{m} \mu_8 \kappa_{10}^2 = \frac{\mathbf{m}}{2\pi\sqrt{\alpha'}}. \tag{2.74}
$$

We introduce a D4-brane located at $y > 0$. The DBI action of the U(1) gauge theory on the D4-brane worldvolume is given by

$$
S_{DBI} = -\mu_4 \int d^5 x \, e^{-\Phi} \left[-\det(G^{(5)}_{ab} + B_{ab} + 2\pi\alpha' f_{ab}) \right]^{1/2}. \tag{2.75}
$$

In the static gauge, the induced metric $G^{(5)}_{ab}$ is given by

$$
G^{(5)}_{ab} = (\mathbf{M} x^9)^{-\frac{1}{2}} \left(\eta_{ab} + (2\pi\alpha')^2 \delta_{AB} \partial_a X^A \partial_b X^B \right) + (2\pi\alpha')^2 (\mathbf{M} x^9)^{\frac{1}{2}} \partial_a \varphi \partial_b \varphi, \tag{2.76}
$$

where $a, b = 0, 1, \cdots, 4$ and $A, B = 5, \cdots, 8$. We expand the DBI action

$$
\begin{aligned}
S_{\mathrm{DBI}} = -\mu_4 \, vol_{D4} &- \frac{1}{2g_{ym}^2(v)} \int d^5 x \left[\frac{1}{2} \big(f + \frac{1}{2\pi\alpha'} B_2 \big)_{ab} \big(f + \frac{1}{2\pi\alpha'} B_2 \big)^{ab} + \eta^{ab} \partial_a \varphi \partial_b \varphi \right] \\
&- \frac{1}{8\pi^2 \sqrt{\alpha'}} \int d^5 x \, \eta^{ab} \delta_{AB} \partial_a X^A \partial_b X^B + O(\alpha^3),
\end{aligned}
\tag{2.77}
$$

where $*_5$ is the Hodge star operator with respect to the 5-dimensional flat metric. The Yang-Mills coupling g_{ym} is determined by vev of the scalar field $v = \langle\varphi\rangle = x^9/2\pi\alpha'$ as

$$\frac{1}{g_{ym}^2(v)} = \mu_4(2\pi\alpha')^2\mathbf{M}x^9 = \frac{\mathbf{m}v}{4\pi^2}. \tag{2.78}$$

The Wess-Zumino action on D4-brane worldvolume is given by

$$S_{\mathrm{WZ}} = \mu_4\left[\int C_5 + \int(2\pi\alpha' f + B_2)\wedge C_3 + \frac{1}{2}\int(2\pi\alpha' f + B_2)^2\wedge C_1\right]. \tag{2.79}$$

There is an additional Chern-Simons term [52]

$$S_{\mathrm{CS}} = -\frac{1}{6}\mu_4\mathbf{M}(2\pi\alpha')^3\int a\wedge f^2, \tag{2.80}$$

which is required to maintain gauge invariance under the NSNS gauge transformation,

$$\delta B_2 = d\Lambda_1,\quad a = -\frac{1}{2\pi\alpha'}\Lambda_1,\quad \delta C_1 = -\mathbf{M}\Lambda_1,\quad \delta C_3 = \mathbf{M}\Lambda_1\wedge B_2,\quad \delta C_5 = -\frac{1}{2}\mathbf{M}\Lambda_1\wedge B_2^2. \tag{2.81}$$

In general, the Chern-Simons action on the Dp-brane worldvolume reads

$$S_{\mathrm{CS}} = -\frac{1}{(\frac{p}{2}+1)!}\mu_p\mathbf{M}(2\pi\alpha')^{\frac{p}{2}+1}\int_{p+1} a\wedge f^{\frac{p}{2}} = -\frac{1}{(\frac{p}{2}+1)!(2\pi)^{\frac{p}{2}}}\mathbf{m}\int_{p+1} a\wedge f^{\frac{p}{2}}. \tag{2.82}$$

2.9. Appendix: E_2 group theory

The case $N_f = 1$ corresponds to flavor group $E_2 \cong \mathrm{SU}(2)\times\mathrm{U}(1)$. In (2.40) we used a relation, given in (4.10) of [24], to convert the fugacities associated with E_2 to fugacities associated with the $\mathrm{SO}(2N_f)\times\mathrm{U}(1)_I$ subgroup that is manifest in the index formula [and we added the subscript I to distinguish $\mathrm{U}(1)_I$ from the U(1) factor of E_2]. We will now explain the origin of (2.40). Pick a Cartan subalgebra $\mathrm{U}(1)' \subset \mathrm{SU}(2)$, and consider a state with $\mathrm{U}(1)'\times\mathrm{U}(1) \subset E_2$ charges Q' and $\widetilde{Q}$. With the fugacities defined in (2.40), its contribution to the index is $y^{Q'}z^{\widetilde{Q}}$, which can also be written as $y_1^{Q_1}q^{Q_I}$, where Q_1 is the charge associated with $\mathrm{SO}(2N_f) = \mathrm{SO}(2) \cong \mathrm{U}(1)$ and Q_I is the instanton charge associated with $\mathrm{U}(1)_I$. According to (2.40), the charges are related by

$$Q_1 = \tfrac{1}{2}Q' + \tfrac{7}{2}\widetilde{Q}\,,\qquad Q_I = \tfrac{1}{2}Q' - \tfrac{1}{2}\widetilde{Q}. \tag{2.83}$$

These relations have a nice string theory interpretation in terms of the D0-D8/O8 system, following the analysis of [86, 87, 88, 89]. Consider an O8 plane with $N_f = 1$ D8-brane, and separate the D8-brane from the orientifold plane. The W-boson of $\mathrm{SU}(2) \subset \mathrm{SU}(2)\times\mathrm{U}(1)$

$\cong E_2$ can be constructed as an open fundamental string connecting the D8-brane to a D0-brane that is stuck on the O8-plane. This string has charges $Q_I = Q_1 = 1$, and since it is the W-boson of SU(2), it has charge $Q' = 2$ and $\widetilde{Q} = 0$, which is consistent with (2.83). [Our normalization has charge $Q' = \pm 1$ for the fundamental representation 2 of SU(2).] On the other hand, we can construct an SU(2) neutral state from a D0-brane connected by $8 - N_f = 7$ strings to the D8-brane. This particle has charges $Q_I = 1$, $Q_1 = -7$, $Q' = 0$ and $\widetilde{Q} = -2$, again consistent with (2.83).

Let us now turn to the algebraic description of E_2 and its SO(2)×U(1)$_I$ subgroup. For $n \geq 3$, the Lie algebra E_n corresponds to the Dynkin diagram of E_8 with simple roots $\alpha_{n+1}, \ldots, \alpha_8$ deleted, referring to the root labeling as in Figure 2.3.[16] This definition, however, is inadequate for $n = 2$, as $E_2 \cong$ SU(2)×U(1) (and not SU(2)×SU(2), as the extension of the above definition to $n = 2$ might suggest). Before we proceed to the definition of E_2, let us list for reference the simple weights of E_8,

$$
\begin{aligned}
\Lambda_1 &= 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8,\\
\Lambda_2 &= 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8,\\
\Lambda_3 &= 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8,\\
\Lambda_4 &= 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8,\\
\Lambda_5 &= 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8,\\
\Lambda_6 &= 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8,\\
\Lambda_7 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8,\\
\Lambda_8 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.
\end{aligned}
$$

They satisfy $(\Lambda_i|\alpha_j) = \delta_{ij}$, where $(\cdot|\cdot)$ is the bilinear form on the root lattice.

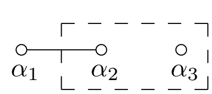
The correct definition of our Lie algebra E_n, valid for $2 \leq n \leq 8$, is as follows. First, consider the sublattice $\mathbf{Q}_{8-n}$ of the E_8 root lattice that is generated by the simple roots $\alpha_{n+1}, \ldots, \alpha_8$. The root spaces of those roots of E_8 that are in $\mathbf{Q}_{8-n}$ generate an su$(9-n)$ subalgebra. Indeed, the roots $\alpha_{n+1}, \ldots, \alpha_8$ form a subdiagram of Dynkin type A_{8-n}. The exponent of this subalgebra is a subgroup SU$(9-n) \subset E_8$, and E_n is defined as the commutant of this subgroup. Note that with this definition, the simple roots of E_n are not $\alpha_1, \ldots, \alpha_n$. For example, for E_7, the root spaces of $\pm\alpha_8$ generate an su(2) subalgebra that does not commute with the root space of α_7, so α_7 cannot be a simple root of E_7, as defined. Instead, we define

$$
\alpha_7' \equiv \Lambda_7 - \Lambda_6 = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 6\alpha_4 - 5\alpha_5 - 4\alpha_6 - 2\alpha_7 - \alpha_8\,,
$$

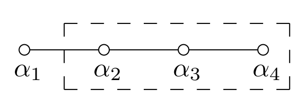
and the simple roots of E_7 can then be taken as $\alpha_1, \ldots, \alpha_6, \alpha_7'$. It is easy to verify that their inner products correspond to the Dynkin diagram of E_7, and they all have zero inner product

[16]For $n = 3, 4$, the simple roots are relabeled as

E_3 :



E_4 :



with α_8. Similarly, for $n = 5, 6$, we take the simple roots of $E_n \subset E_8$ to be $\alpha_1, \ldots, \alpha_{n-1}, \Lambda_n - \Lambda_{n-1}$. For $n = 4$, we take the simple roots of $E_4 \subset E_8$ to be $\alpha_1, \alpha_2, \alpha_3, \Lambda_4 - \Lambda_3 - \Lambda_2$. For $n = 3$ we take the simple roots of $E_3 \subset E_8$ to be $\alpha_1, \Lambda_3 - \Lambda_1 - \Lambda_2, \Lambda_3 - \Lambda_2$.

The case $E_2 \subset E_8$ requires a more careful treatment. $E_2 \simeq \mathrm{SU}(2)\times\mathrm{U}(1)_I$ is defined as the subgroup that commutes with the $\mathrm{SU}(7) \subset E_8$ generated by the root spaces of $\alpha_3, \ldots, \alpha_8$. Define the root

$$\beta \equiv \Lambda_2 - \Lambda_1 = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \,.$$

Then $\pm\beta$ are the only roots of E_8 that are orthogonal to $\alpha_3, \ldots, \alpha_8$. The root spaces of β and $-\beta$ generate an su(2) subalgebra whose exponent we identify with the SU(2) factor of E_2. The intersection of this su(2) with the Cartan subalgebra of E_8 is spanned by $\beta^\star$, which is the element of the Cartan subalgebra that assigns to a state with weight λ the charge $Q'(\lambda) \equiv (\beta|\lambda)$. Then, the generator of the U(1) factor of $E_2 \cong \mathrm{SU}(2)\times\mathrm{U}(1)$ is $\gamma^\star$, with

$$\gamma \equiv 3\Lambda_1 - \Lambda_2 = 7\alpha_1 + 7\alpha_2 + 11\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8 \,,$$

which is the unique (up to multiplication) element of the root lattice that is orthogonal to $\alpha_3, \ldots, \alpha_8$ and β.[17] Under the subgroup $E_2\times\mathrm{SU}(7) \subset E_8$ the representation 248 decomposes as

$$\begin{aligned} 248 = {} & (1,1)_0 + (3,1)_0 + (1,48)_0 + (1,7)_4 + (1,\overline{7})_{-4} \\ & + (2,7)_{-3} + (2,\overline{7})_3 + (1,35)_{-2} + (1,\overline{35})_2 + (2,21)_1 + (2,\overline{21})_{-1} \,. \end{aligned}$$

We now define "fugacities" y and z, so that the contribution of a hypothetical state with E_8 weight λ (assumed to be orthogonal to $\alpha_3, \ldots, \alpha_8$) to the $E_2 \cong \mathrm{SU}(2)\times\mathrm{U}(1)$ index will be $y^{(\beta|\lambda)} z^{\frac{1}{7}(\gamma|\lambda)}$.

Now, consider the $\mathrm{U}(1)_I\times\mathrm{SO}(2) \subset E_2$ subgroup. For a state associated to a weight λ of E_8, we can associate $\mathrm{U}(1)'\times\mathrm{U}(1) \subset \mathrm{SU}(2)\times\mathrm{U}(1) \cong E_2$ charges

$$Q'(\lambda) = (\beta|\lambda)\,, \qquad \widetilde{Q}(\lambda) = \tfrac{1}{7}(\gamma|\lambda).$$

Then their SO(2) and $\mathrm{U}(1)_I$ charges are given by

$$Q_1(\lambda) = \tfrac{1}{2}Q'(\lambda) + \tfrac{7}{2}\widetilde{Q}(\lambda) = (\tfrac{1}{2}\beta + \tfrac{1}{2}\gamma|\lambda) = (\Lambda_1|\lambda)\,,$$

and

$$Q_I(\lambda) = \tfrac{1}{2}Q'(\lambda) - \tfrac{1}{2}\widetilde{Q}(\lambda) \equiv \tfrac{1}{7}(\delta|\lambda),$$

where we defined

$$\delta \equiv \tfrac{7}{2}\beta - \tfrac{1}{2}\gamma = 7\alpha_2 + 5\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8,$$

[17] γ is not a root because $(\gamma|\gamma) = 14$.

which is the (unique up to multiplication) weight that is orthogonal to Λ_1 and $\alpha_3, \ldots, \alpha_8$. As was discovered in [24], the superconformal index for local operators is

$$I_{\mathrm{SCI}} = 1 + (2 + \frac{1}{qy_1} + qy_1)t^2 + O(t^3) = 1 + \left[1 + \left(\frac{1}{y^2} + 1 + y^2\right)\right] t^2 + O(t^3),$$

and as we have seen in §2.4, the ray operator index is

$$I_{\mathrm{ray}} = q^{-2/7}\left(\frac{1}{y_1^2} + \frac{q}{y_1} + y_1^2\right) t + O(t^2) = \left[z^{-3/7}\left(y + \frac{1}{y}\right) + z^{4/7}\right] t + O(t^2),$$

which both nicely fit into $E_2 \cong \mathrm{SU}(2)\times\mathrm{U}(1)$ representations.

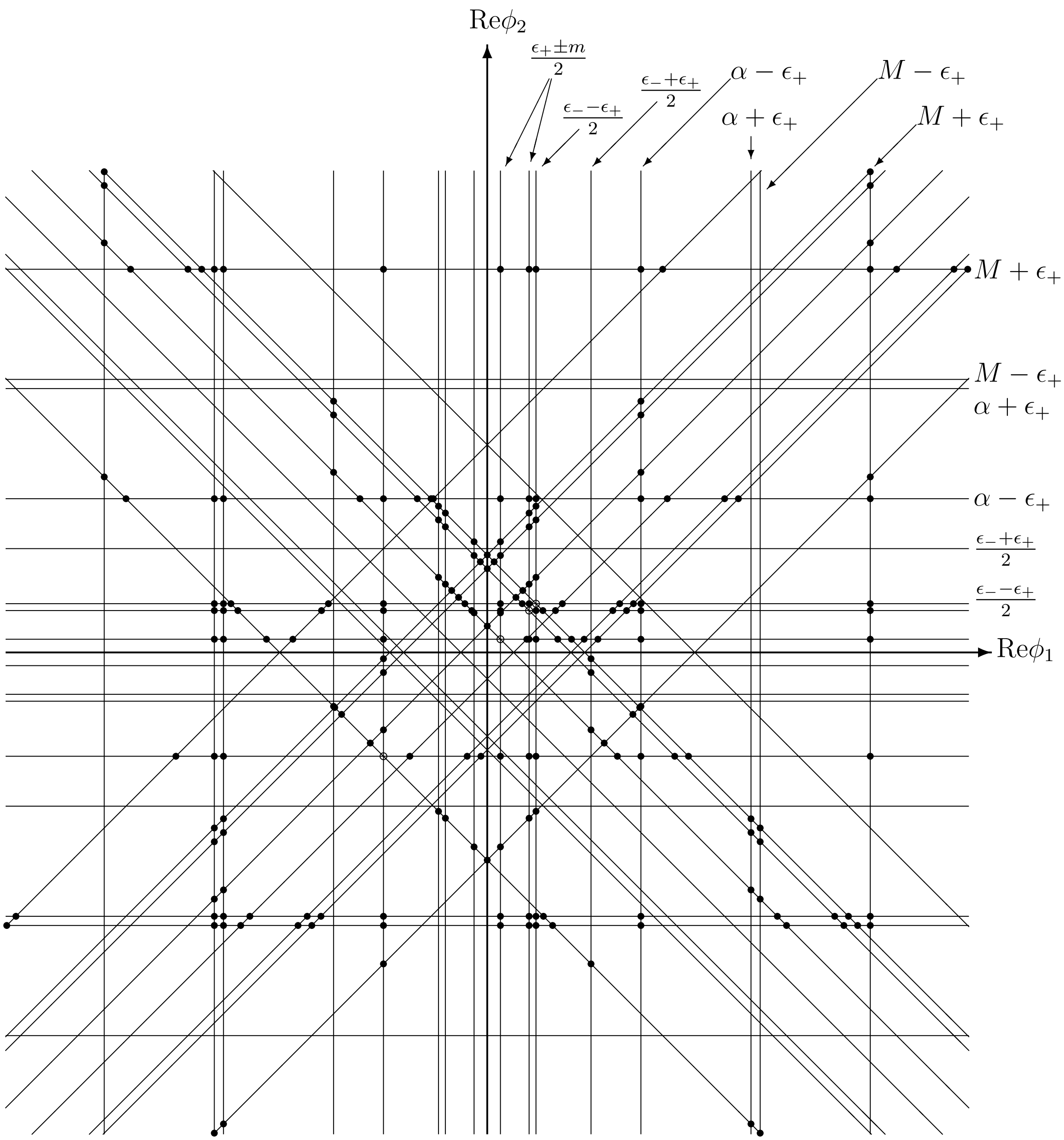


Figure 2.2: The location of the poles on the $\phi_1 - \phi_2$ real plane for instanton number $k = 4$, for $\eta = (1, 3)$. The lines are the loci where the argument of a single sinh in the denominator of the integrand vanishes. Poles are at the intersection of two lines. The solid circles indicate the poles that are retained by the Jeffrey-Kirwan prescription. (One pole, at $\phi_1 = -M - \epsilon_- - 2\epsilon_+$ and $\phi_2 = M + \epsilon_+$, is outside the frame of the picture.) The hollow circles are possible locations of non-simple poles, where three lines intersect. (Whether they are simple or non-simple depends on $\mathrm{Im}\phi_1$ and $\mathrm{Im}\phi_2$.)

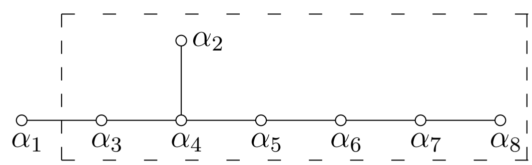


Figure 2.3: The Dynkin diagram of E_8 and its subdiagram corresponding to $\mathrm{SO}(14) \subset E_8$.

Chapter 3

Feynman diagrams and $\Omega-$deformed M-theory

3.1. Introduction

In [90], Costello and Li developed a beautiful formalism, which prescribes a way to topologically twist supergravity. Combining with the classical notion of topological twist of supersymmetric quantum field theory [91, 93], we are now able to explore a topological sector for both sides of AdS/CFT correspondence. It was further suggested in [94] a systematic method of turning an Ω-background, which plays an important roles [98, 99, 100, 101, 102, 103] in studying supersymmetric field theories, in the twisted supergravity.

Topological twist along with Ω-deformation enables us to study a particular protected sub-sector of a given supersymmetric field theory [3, 104, 105, 6], which is localized not only in the field configuration space, but also in the spacetime. Interesting dynamics usually disappear in the way, but as a payoff we can make more rigorous statement on the operator algebra.

The topological holography [106] is an exact isomorphism between the operator algebras of gravity and field theory. [94] studied $\Omega-$deformed M-theory and M2-brane inside, and proved the isomorphism between 5d non-commutative $U(K)$ CS(Chern-Simons theory) [107, 108], which consists of the topological sector of 11d supergravity, and 1d TQM(topological quantum mechanics), which is obtained from the M2-brane theory: Higgs branch of 3d $\mathcal{N}=4$ ADHM gauge theory. The isomorphism was manifested by the mathematical notion, so called Koszul duality [109].

The important first step of the proof was to impose a BRST-invariance of the 5d $U(K)$ CS theory coupled with the 1d TQM. 5d CS theory is a renormalizable, and self-consistent theory [108]. However, in the presence of the topological defect that couples 1d TQM and 5d CS theory, certain Feynman diagrams turn out to have non-zero BRST variations. For the combined, interacting theory to be quantum mechanically consistent, the BRST variations of the Feynman diagrams should combine to give zero. This procedure magically reproduces the algebra commutation relations that define 1d TQM operator algebra, $\mathcal{A}_{\epsilon_1,\epsilon_2}$. It is very intriguing that one can extract non-perturbative information in the protected operator algebra from the perturbative calculation.

In fact, both the algebra of local operators in 5d CS theory and the 1d TQM operator algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ are deformations of the universal enveloping algebra of the Lie algebra $\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K$ over the ring $\mathbb{C}[\![\epsilon_1]\!]$. Deformation theory tells us that the space of deformations of $U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K)$ is the second Hochschild cohomology $\mathrm{HH}^2(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K))$.

Let us elaborate a little more of Hochschild homology. For g an associative algebra, we define the Hochschild chain complex by $C_n(g, M) = g^{\otimes n}$, where M is a $g-$module. The complex is equipped with a boundary operator d_i with

$$
\begin{aligned}
d_0(a_1 \otimes \ldots \otimes a_n) &= a_1 \otimes \ldots \otimes a_n \\
d_i(a_1 \otimes \ldots \otimes a_n) &= a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\
d_n(a_1 \otimes \ldots \otimes a_n) &= a_n \otimes a_1 \otimes \ldots \otimes a_{n-1}
\end{aligned}
\tag{3.1}
$$

We define Hochschild homology of g with coefficients in M as $(C_n(g, M), D)$ where

$$
D = \sum_{i=0}^{n} (-1)^i d_i \tag{3.2}
$$

Although this Hochschild cohomology is known to be hard to compute, there is still a clever way of comparing these two deformations [109]: notice that both of the algebras are defined compatibly for super groups $\mathrm{GL}_{K+R|R}$, so they are actually controlled by elements in the limit

$$
H^2(\lim_{R} \mathrm{HC}^*(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_{K+R|R}))) \tag{3.3}
$$

and the limit is well-understood, it turns out that the space of all deformations is essentially one-dimensional: a free module over $\mathbb{C}[\kappa]$ where κ is the central element $1 \otimes \mathrm{Id}_K$. Hence the algebra of local operators in 5d CS theory and the 1d TQM operator algebra are isomorphic up to a κ-dependent reparametrization

$$
\hbar \mapsto \sum_{i=1}^{\infty} f_i(\kappa) \hbar^i \tag{3.4}
$$

where $f_i(\kappa)$ are polynomials in κ.

Later, in [5] the same algebra with $K = 1$ was defined using the gauge theory approach, and a combined system of M2-branes and M5-branes were studied. Especially, [5] interpreted the degrees of freedom living on M5-branes as forming a bi-module $\mathcal{M}_{\epsilon_1,\epsilon_2}$ of the M2-brane operator algebra, and suggested the evidence by going to the mirror Coulomb branch algebra [110, 111] and using the known Verma module structure of massive supersymmetric vacua [112, 113]. Appealing to the brane configuration in type IIB frame, they argued a triality in the M2-brane algebra, which can also be deduced from its embedding in the larger algebra, affine $gl(1)$ Yangian [114, 115, 116, 117].

Crucially, [5] noticed $U(1)$ CS should be treated separately from $U(K)$ CS theory with $K > 1$, since the algebras differ drastically and the ingredients of Feynman diagram are different in $U(1)$ CS, due to the non-commutativity. As a result, the operator algebra isomorphism should be re-assessed.

Our work was motivated by the observation, and we will solve the following problems in a part of this paper.

- The simplest algebra($\mathcal{A}_{\epsilon_1,\epsilon_2}$) commutator, which has ϵ_1 correction.
- Feynman diagrams whose non-trivial BRST variation lead to the simplest algebra commutator.

Next, we will make a first attempt to derive the bi-module structure from the 5d $U(1)$ CS theory, where the combined system of the M2-branes and the M5-brane is realized as the 1d TQM and the $\beta - \gamma$ system. Especially, we will answer the following problems.

- The simplest algebra($\mathcal{A}_{\epsilon_1,\epsilon_2}$), bi-module($\mathcal{M}_{\epsilon_1,\epsilon_2}$) commutator, which has ϵ_1 correction.
- Feynman diagrams whose non-trivial BRST variation lead to the simplest algebra($\mathcal{A}_{\epsilon_1,\epsilon_2}$), bi-module($\mathcal{M}_{\epsilon_1,\epsilon_2}$) commutator.

Our work is only a part of a bigger picture. The algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ is a sub-algebra of affine $gl(1)$ Yangian, and there exists a closed form formula for the most general commutators, which can be derived from affine $gl(1)$ Yangian. One can try to derive the commutators from 5d $U(1)$ CS theory Feynman diagram computation.

Going to type IIB frame, the brane configurations map to Y-algebra configuration [118]. Here, the general M2-brane algebra is formed by the co-product of three different M2-brane algebras related by the triality. M5-brane VOA is the generalized $\mathcal{W}_{1+\infty}$ algebra, whereas our M5-brane VOA is the simplest possible VOA, $\beta - \gamma$ system. Hence, we are curious if our story can be further generalized to the coupled system of the 5d $U(1)$ CS theory and the generalized $\mathcal{W}_{1+\infty}$ algebra.

Lastly, [94] argued that considering N $M5$ branes and take large N limit, $\mathcal{W}_{1+\infty}$ algebra emerges as an operator algebra on the $M5$ branes. It would be nice to revisit the argument using the technique shown in this paper, which originally came from [119].

After reviewing the general concepts in section §3.2, we show the following algebra commutator in §3.3.

$$[t[2,1], t[1,2]]_{\epsilon_1} = \epsilon_1\epsilon_2 t[0,0] + \epsilon_1\epsilon_2^2 t[0,0]t[0,0] \tag{3.5}$$

where $[\bullet]_{\epsilon_1}$ is the $\mathcal{O}(\epsilon_1)$ part of $[\bullet]$, $t[m,n] \in \mathcal{A}_{\epsilon_1,\epsilon_2}$. The detail of the proof is shown in Appendix 3.6. The commutation relation was successfully checked by 1-loop Feynman diagram associated to 5d CS theory and 1d TQM. This is the content of section §3.4. We collected some intermediate integral computations used in the Feynman diagram in Appendix 3.7.

Next, we show the following algebra-bi-module commutator in §3.3.

$$\left[t[2,1], b[z^1]c[z^0]\right]_{\epsilon_1} = \epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0] + \epsilon_1\epsilon_2 b[z^0]c[z^0] \tag{3.6}$$

where $b[z^m]$, $c[z^m] \in \mathcal{M}_{\epsilon_1,\epsilon_2}$. The detail of the proof can be found in Appendix 3.6. We reproduced the commutation relation using the 1-loop Feynman diagram computation in the 5d CS theory, 1d TQM, and 2d $\beta\gamma$ coupled system. This is the content of section §3.5. We collected some intermediate integral computations used in the Feynman diagram in Appendix 3.7 and Appendix 3.7.

3.2. Twisted holography via Koszul duality

Twisted holography is the duality between the protected sub-sectors of full supersymmetric AdS/CFT [120, 121, 122], obtained by topological twist and Ω-background both turned on in the field theory side and supergravity side. The most glaring aspect of twisted holography[1] is an exact isomorphism between operator algebra in both sides, which is manifested by a rigorous Koszul duality. Moreover, the information of physical observables such as Witten diagrams in the bulk side that match with correlation functions in the boundary side is fully captured by OPE algebra in the twisted sector [126].

This section is prepared for a quick review of twisted holography for non-experts. The idea was introduced in [90] and studied in various examples [94, 109, 127, 106, 5, 128] with or without Ω-deformation. The reader who is familiar with [94] can skip most of this section, except for §3.2, §3.2, and §3.2, where we set up the necessary conventions for the rest of this paper. These subsections can be skipped as well, if the reader is familiar with [5]. Also, see a complementary review of the formalism in the section 2 of [5].

After defining the notion of twisted supergravity in §3.2, we will focus on a particular (twisted and $\Omega-$deformed) M-theory background on $\mathbb{R}_t \times \mathbb{C}^2_{NC} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$, where NC means non-commutative, and ϵ_i stands for $\Omega-$background related to $U(1)$ isometry with a deformation parameter ϵ_i in §3.2. N $M2$ branes extending $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}$ leads to the field theory side. As we will explain in §3.2, a bare operator algebra isomorphism between twisted supergravity and twisted M2-brane worldvolume theory is given by an interaction Lagrangian between two system. Due to this interaction, a perturbative gauge anomaly appears in various Feynman diagrams, and a careful cancellation of the anomaly will give a consistent quantum mechanical coupling between two systems. Strikingly, the anomaly cancellation condition itself leads to a complete operator algebra isomorphism, by fixing algebra commutators. This will be described in §3.2. To discuss holography, it is necessary to include the effect of taking large N limit and the subsequent deformation in the spacetime geometry. We will illustrate the concepts in §3.2. In §3.2, we will explain how to introduce M5-brane in the system and describe the role of M5-brane in the gravity and field theory side. In short, the degree of freedom on M5-brane will form a module of the operator algebra of M2-brane. Similar to M2-brane case, anomaly cancellation condition for M5-brane uniquely

[1]A similar line of development was made in [123, 124], using twisted $\mathbb{Q}$-cohomology, where $\mathbb{Q}$ is a particular combination of a supercharge Q and a conformal supercharge S [125]. In the sense of [3], $\mathbb{Q}$-cohomology is equivalent to Q_V-cohomology, where Q_V is the modified scalar super charge in $\Omega-$deformed theories.

fixes the structure of the module. Lastly, in section §3.2, we will introduce more general framework where our work can be embedded using type IIb string theory and suggest some future directions.

Twisted supergravity

Before discussing the topological twist of supergravity, it would be instructive to recall the same idea in the context of supersymmetric field theory, and make an analogue from the field theory example.

Given a supersymmetric field theory, we can make it topological by redefining the global symmetry M using R-symmetry R.

$$M \quad \rightarrow \quad M' = M + R \tag{3.7}$$

As a part of Lorentz symmetry is redefined, supercharges, which were previously spinor(s), split into a scalar Q, which is nilpotent

$$Q^2 = 0, \tag{3.8}$$

and a 1-form Q_μ. Because of the nilpotency of Q, one can define the notion of Q-cohomology.

Following anti-commutator explains the topological nature of the operators in Q-cohomology– a translation is Q-exact.

$$\{Q, Q_\mu\} = P_\mu \tag{3.9}$$

To go to the particular Q-cohomology, one needs to turn off all the infinitesimal super-translation ϵ_Q except for the one that parametrizes the particular transformation δ_Q generated by Q.

More precisely, if we were to start with a gauge theory, which is quantized with BRST formalism, the physical observables are defined as BRST cohomology, with respect to some Q_{BRST}. The topological twist modifies Q_{BRST}, and the physical observables in the resulting theory are given by Q'_{BRST}-cohomology.

$$Q_{BRST} \quad \rightarrow \quad Q'_{BRST} = Q_{BRST} + Q \tag{3.10}$$

As an example, consider $3d$ $\mathcal{N} = 4$ supersymmetric field theory. The Lorentz symmetry is $SU(2)_{Lor}$ and R-symmetry is $SU(2)_H \times SU(2)_C$, where H stands for Higgs and C stands for Coulomb. There are two ways to re-define the Lorentz symmetry algebra, and we choose to twist with $SU(2)_C$, as this will be used in the later discussion. In other words, one redefines

$$M \quad \rightarrow \quad M' = M + R_C \tag{3.11}$$

The resulting scalar supercharge is obtained by identifying two spinor indices, one of Lorentz symmetry α and one of $SU(2)_C$ R-symmetry a

$$Q^\alpha_{a\dot{a}} \quad \rightarrow \quad Q^a_{a\dot{a}} \tag{3.12}$$

and taking a linear combination.

$$Q = Q^{+}_{1\bar{1}} + Q^{-}_{1\bar{2}} \tag{3.13}$$

This twist is called Rozansky-Witten twist [129], and will be used in twisting our M2-brane theory.

One way to start thinking about the topological twist of supergravity is to consider a brane in the background of the "twisted" supergravity. If one places a brane in a twisted supergravity background, it is natural to guess that the worldvolume theory of the brane should also be topologically twisted coherently with the prescribed twisted supergravity background.

Given the intuition, let us define twisted supergravity, following [90]. In supergravity, the supersymmetry is a local(gauge) symmetry, a fermionic part of super-diffeomorphism. To quantize the supergravity, one needs to introduce ghost field for the local symmetry. As supersymmetry is a fermionic symmetry, the corresponding ghost field used in the quantization is a bosonic spinor, q.

One can think the infinitesimal super-translation parameter ϵ that appears in the global supersymmetry transformation as a rigid limit of the bosonic ghost q. For instance, in 4d $\mathcal{N} = 1$ holomorphically twisted field theory [130, 131, 132, 133], with Q paired with ϵ_+, the supersymmetry transformation of the bottom component ϕ of anti-chiral superfield $\bar{\Psi} = (\bar{\phi}, \bar{\psi}, \bar{F})$ transforms as

$$\delta\phi = \bar{\epsilon}\bar{\psi}, \quad \delta\bar{\psi} = i\epsilon_+\bar{\partial}\bar{\phi} + i\epsilon_-\partial\bar{\phi} + \bar{\epsilon}\bar{F} \tag{3.14}$$

As we focus on Q-cohomology, we set $\epsilon_+ = 1$, $\epsilon_- = \bar{\epsilon} = 0$, then the equations reduce into

$$\delta\bar{\phi} = 0, \quad \delta\bar{\psi} = i\bar{\partial}\bar{\phi} \tag{3.15}$$

In the similar spirit, in the twisted supergravity, we control the twist by giving non-zero VEV to components of the bosonic ghost q.

Indeed, [90] proved that by turning on non-zero bosonic spinor vacuum expectation value $\langle q \rangle \neq 0$ with $q_\alpha \Gamma^{\alpha\beta}_\mu q_\beta = 0$ for a vector gamma matrix, one can obtain the effect of topological twisting. We can now compare with the field theory case above (3.8): $Q^2 = 0$ with $Q \neq 0$. One can think of ϵ_Q as a rigid limit of q.

The operator algebra of twisted type IIB supergravity is isomorphic to that of Kodaira-Spencer theory [134]. The following diagram gives a pictorial definition of the two theories, which turned out to be isomorphic to each other.

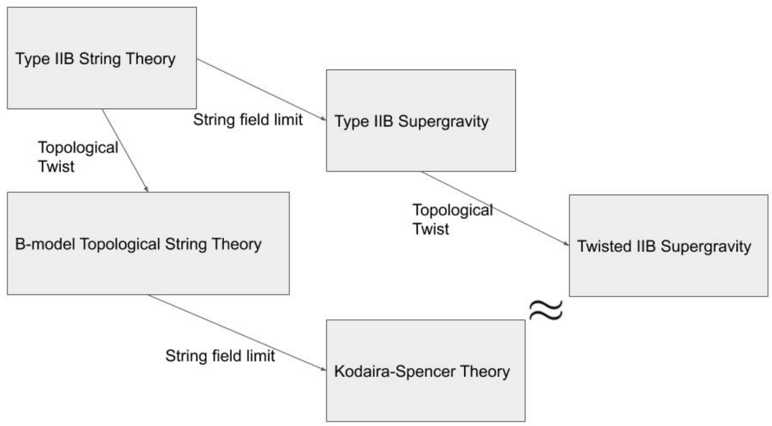


Figure 3.1: Starting from type IIB string theory, one can obtain same theory by taking two operations– 1. String field limit, 2. Topological twist– in any order.

Notice that the topological twist in the first column of the picture is the twist applied on the worldsheet string theory[2], whereas that in the second column is the twist on the target space theory.

Lastly, there are two types of twists available: a topological twist and a holomorphic twist, and it is possible to turn on the two different types of twists in the two different directions of the spacetime. The mixed type of twists is called a topological-holomorphic twist, e.g. [135]. Different from a topological twist, a holomorphic twist makes only the (anti)holomorphic translation to be Q-exact; after the twist we have Q and Q_z such that

$$\{Q, Q_z\} = P_z \tag{3.16}$$

Hence, the anti-holomorphic translation is actually physical(not Q-exact), and there exists non-trivial dynamics arising from this. [90, 94] showed that it is possible to discuss a holomorphic twist in the supergravity. We will refer a topological twist as A-twist and a holomorphic twist as B-twist. It is actually important to have a holomorphic direction to keep the non-trivial dynamics, as we will later see.

Ω-deformed M-theory

Similar to the previous subsection, we will start reviewing the notion of Ω-deformation of topologically twisted field theory. To define Ω-background, one first needs an isometry,

[2] We thank Kevin Costello, who pointed out that the arrow from Type IIb string theory to B-model topological string theory is still mysterious in the following sense. In Ramon-Ramond formalism, as the super-ghost is in the Ramond sector and it is hard to give it a VEV. In the Green-Schwarz picture surely it should work better, but there are still problems there, as the world-sheet is necessarily embedded in space-time whereas in the B model that is not allowed.

typically $U(1)$, generated by some vector field V on a plane where one wants to turn on the Ω-background. Ω-deformation is a deformation of topologically twisted field theory and physical observables are defined with respect to the modified Q_V cohomology, which satisfies

$$Q_V^2 = L_V, \quad \text{where } Q_V = Q + i_{V^\mu} Q_\mu \tag{3.17}$$

where L_V is a conserved charge associated to V, and i_{V^μ} is a contraction with the vector field V^μ, reducing the form degree by 1.

As the RHS of (3.17) is non-trivial, Q_V cohomology only consists of operators, which are fixed by the action of L_V− $\mathcal{O}$ such that $L_V\mathcal{O} = 0$. Hence, effectively, the theory is defined in two less dimensions. More generally, one can turn on Ω-background in the n planes, and the dynamics of the original theory defined on D-dimensions is localized on $D - 2n$ dimensions.

In [94], a prescription for turning Ω-background in twisted 11d supergravity was introduced; we need 3-form field ϵC, along with $U(1)$ isometry generated by a vector field ϵV, where ϵ is a constant, measuring the deformation. Similar to the field theory description, in this background($\langle q\rangle, C \neq 0$), the bosonic ghost q squares into the vector field, ϵV to satisfy the 11d supergravity equation of motion.

$$q^2 = q_\alpha(\Gamma^{\alpha\beta})_\mu q_\beta = \epsilon V_\mu \tag{3.18}$$

The Ω-background localizes the supergravity field configuration into the fixed point of the $U(1)$ isometry. More generally, one can turn on multiple Ω_{ϵ_i}-background in the separate 2-planes, which we will denote as $\mathbb{C}_{\epsilon_i}$.

The 11d background that we will focus in this paper is

$$\text{11d SUGRA: } \mathbb{R}_t \times \mathbb{C}^2_{NC} \times \mathbb{C}_{\epsilon_1} \times TN_{1;\epsilon_2,\epsilon_3} \tag{3.19}$$

where $TN_{1;\epsilon_2,\epsilon_3}$ is Taub-NUT space, which can be thought of as $S^1_{\epsilon_2} \times \mathbb{R} \times \mathbb{C}_{\epsilon_3}$. The twist is implemented with the bosonic ghost chosen such that B(holomorphic) twist in $\mathbb{C}^2_{NC}$ directions [3] and A(topological) twist in $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1} \times TN_{1;\epsilon_2,\epsilon_3}$ directions[4]. The 3-form is

$$C = V^d \wedge d\bar{z}_1 \wedge d\bar{z}_2 \tag{3.20}$$

where V^d is 1-form, which is a Poincare dual of the vector field V on $\mathbb{C}_{\epsilon_2}$ plane.

The statement of twisted holography is the duality between the protected subsector of M2(M5)-brane and the localized supergravity, due to the Ω-background. We first want to introduce $M2$ branes and establish the explicit isomorphism at the level of operator algebras. Place N M2-branes on

$$\text{M2-brane: } \mathbb{R}_t \times \{\cdot\} \times \mathbb{C}_{\epsilon_1} \times \{\cdot\} \tag{3.21}$$

[3]NC stands for Non-Commutative. This will become clear in the type IIa frame.

[4]As remarked, if one introduces branes, the worldvolume theory inherits the particular twist that is turned on in the particular direction that the branes extend.

To set up the stage for the concrete computation, it is convenient to go to type IIa frame by reducing along an M-theory circle. We pick the M-theory circle as $S^1_{\epsilon_2}$, which is in the direction of the vector field V.[5]

After reducing on $S^1_{\epsilon_2}$, the Taub-NUT geometry maps into one D6-brane and N M2-branes map to N D2-branes.

$$\begin{aligned} \text{type IIa SUGRA} : \mathbb{R}_t \times \mathbb{C}^2_{NC} \times \mathbb{C}_{\epsilon_1} \times \mathbb{R} \times \mathbb{C}_{\epsilon_3} \\ \text{D6-brane} : \mathbb{R}_t \times \mathbb{C}^2_{NC} \times \mathbb{C}_{\epsilon_1} \\ \text{D2-branes} : \mathbb{R}_t \times \qquad\quad \times \mathbb{C}_{\epsilon_1} \end{aligned} \tag{3.22}$$

and 3-form C-field reduces into a B-field, which induces a non-commutativity $[z_1, z_2] = \epsilon_2$ on $\mathbb{C}^2_{NC}$.

$$B = \epsilon_2 d\bar{z}_1 \wedge d\bar{z}_2 \tag{3.23}$$

There are two types of contributions to gravity side: 1. closed strings in type IIa string theory and 2. open strings on the D6-brane. It was shown in [94] that we can completely forget about the closed strings, so the open strings from the D6-brane entirely capture gravity side.

D6-brane worldvolume theory is 7d SYM, and it localizes on 5d non-commutative $U(1)$ Chern-Simons on $\mathbb{R}_t \times \mathbb{C}^2_{NC}$ due to Ω_{ϵ_1} on $\mathbb{C}_{\epsilon_1}$ [136]. The 5d Chern-Simons theory is not the typical Chern-Simons theory, as it inherits a topological twist in $\mathbb{R}_t$ direction and a holomorphic twist in $\mathbb{C}^2_{NC}$ direction, in addition to the non-commutativity. As a result, a gauge field only has 3 components

$$A = A_t dt + A_{\bar{z}_1} d\bar{z}_1 + A_{\bar{z}_2} d\bar{z}_2 \tag{3.24}$$

and the action takes the following form.

$$S = \frac{1}{\epsilon_1} \int_{\mathbb{R}_t \times \mathbb{C}^2_{NC}} dz_1 dz_2 \left(A \star dA + \frac{2}{3} A \star A \star A \right) \tag{3.25}$$

The star product $\star_{\epsilon_2}$ is the standard Moyal product induced from the non-commutativity of $\mathbb{C}^2_{NC}$: $[z_1, z_2] = \epsilon_2$. The Moyal product between two holomorphic functions f and g is defined as

$$f \star_\epsilon g = fg + \epsilon \frac{1}{2} \epsilon_{ij} \frac{\partial}{\partial z_i} f \frac{\partial}{\partial z_j} g + \epsilon^2 \frac{1}{2^2 2!} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \left(\frac{\partial}{\partial z_{i_1}} \frac{\partial}{\partial z_{i_2}} f \right) \left(\frac{\partial}{\partial z_{j_1}} \frac{\partial}{\partial z_{j_2}} g \right) \tag{3.26}$$

The gauge transformation $\Lambda \in \Omega^0(\mathbb{R} \times \mathbb{C}^2_{NC}) \otimes gl_1$ acting on the gauge field A is

$$A \mapsto A + d\Lambda + [\Lambda, A], \text{ where } [\Lambda, A] = \Lambda \star_{\epsilon_2} A - A \star_{\epsilon_2} \Lambda \tag{3.27}$$

[5]For a different purpose, to make contact with Y-algebra system, type IIb frame is better, but we will not pursue this direction in this paper.

The field theory side is defined on N D2-branes, which extend on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}$. This is 3d $\mathcal{N}=4$ gauge theory with 1 fundamental hypermultiplet and 1 adjoint hypermultiplet. Since the D2-branes are placed on the A-twisted background, the theory inherits the topological twist, which is Rozansky-Witten twist. We will work on $\mathcal{N}=2$ notation, then each of $\mathcal{N}=4$ hypermultiplet splits into a chiral and an anti-chiral $\mathcal{N}=2$ multiplet. We denote the scalar bottom component of the fundamental chiral and anti-chiral multiplet as I_a and J^a, and that of adjoint multiplets as X^a_b and Y^a_b, where a and b are $U(N)$ gauge indices. They satisfy following basic Poisson bracket:

$$\{I_a, J^b\} = \delta^b_a, \quad \{X^a_b, Y^c_d\} = \delta^a_d \delta^c_b \tag{3.28}$$

It is known that the Q-cohomology of Rozansky-Witten twisted $\mathcal{N}=4$ theory consists of Higgs branch chiral ring, after imposing gauge invariance. The elements of Higgs branch chiral ring are gauge invariant polynomials of I, J, X, and Y.

$$IS(X^m Y^n)J, \quad \mathrm{Tr}S(X^m Y^n) \tag{3.29}$$

where $S[\bullet]$ means fully symmetrized polynomial of the monomial $\bullet$.

Upon imposing the F-term relation

$$[X, Y] + IJ = \epsilon_2 \delta, \tag{3.30}$$

one can show two words in (3.29) are equivalent up to a factor of ${\epsilon_2}$[6], and the physical observables purely consist of one of them. Let us call them as

$$t[m, n] = \frac{1}{\epsilon_1} \mathrm{Tr} S X^m Y^n \tag{3.32}$$

Ω_{ϵ_1} quantizes the chiral ring to an algebra and the support of the operator algebra in 3d $\mathcal{N}=4$ theory also localizes to the fixed point of the Ω_{ϵ_1}. Therefore, the theory effectively becomes 1d TQM(Topological Quantum Mechanics) [137, 138, 113].

In summary, two sides of twisted holography are 5d non-commutative Chern-Simons theory and 1d TQM. Until now, we have not quite taken a large N limit and resulting back-reaction that will deform the geometry. The large N limit will be crucial for the operator algebra isomorphism to work and we will illustrate this point in the section §3.2.

Comparing elements of operator algebra

As 5d CS theory has a trivial equation of motion: $F = 0$, all the observables have positive ghost numbers. Also, since $\mathbb{R}_t$ direction is topological, the fields do not depend on t. As a result, operator algebra consist of ghosts $c(z_1, z_2)$ with holomorphic dependence on coordinates

[6] They are related by following relation:

$$IS[X^m Y^n]J = \epsilon_2 \mathrm{TrS}[X^m Y^n] \tag{3.31}$$

of $\mathbb{C}^2_{NC}$, z_1, z_2. The elements are then Fourier modes of the ghosts.

$$c[m,n] = z_1^m z_2^n \partial_{z_1}^m \partial_{z_2}^n c(0,0) \tag{3.33}$$

The non-commutativity in $\mathbb{C}^2_{NC}$ planes induces an algebraic structure in the holomorphic functions on $\mathbb{C}^2_{NC}$ defined by the Moyal product.

$$\left[z_1^a z_2^b, z_1^c z_2^d\right] = (z_1^a z_2^b) \star_{\epsilon_2} (z_1^c z_2^d) - (z_1^c z_2^d) \star_{\epsilon_2} (z_1^a z_2^b) = \sum_{m,n} f^{m,n}_{a,b;c,d} z_1^m z_2^n \tag{3.34}$$

The operator algebra $A_{\epsilon_1,\epsilon_2}$ of 5d CS theory is defined by (3.33) and (3.34). Formally, $A_{\epsilon_1,\epsilon_2} = C^*_{\epsilon_1}(g)$, where $g = Diff_{\epsilon_2}\mathbb{C} \otimes gl_1$, and $C^*_{\epsilon_1}(g)$ is a Lie algebra cohomology of g. One can understand the new factor $Diff_{\epsilon_2}\mathbb{C}$ in the gauge symmetry algebra, from the isomorphism between the algebra of holomorphic functions on $\mathbb{C}^2_{NC}$ and the algebra of differential operators on $\mathbb{C}_{\epsilon_2}$.

On the other hand, the elements of the algebra of operators in 1d TQM consist of $t[m,n]$. The defining commutation relations come from the quantization of the Poisson brackets deformed by Ω_{ϵ_1}:

$$\left[I_a, J^b\right] = \epsilon_1 \delta_a^b, \quad \left[X_b^a, Y_d^c\right] = \epsilon_1 \delta_d^a \delta_b^c \tag{3.35}$$

We will write the F-term relation with gauge indices explicit as follows.

$$X_k^i Y_j^k - X_j^k Y_k^i + I_j J^i = \epsilon_2 \delta_j^i \tag{3.36}$$

We will call the algebra defined by $t[m,n]$ and (3.35), (3.36) as ADHM algebra or $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

There is a one-to-one correspondence between $c[m,n]$ and $t[m,n]$, and [109] proved an isomorphism between ${}^!A_{\epsilon_1,\epsilon_2} = U_{\epsilon_1}(g)$ and $\mathcal{A}_{\epsilon_1,\epsilon_2}$ for 5d $U(K)$ Chern-Simons theory coupled with 1d TQM with $N > 1$, where ${}^!A_{\epsilon_1,\epsilon_2}$ is a Koszul dual of an algebra $A_{\epsilon_1,\epsilon_2}$[7]. The proof consists of two parts. First, one checks two algebras' commutation relations match in the $\mathcal{O}(\epsilon_1)$ order. Next, one proves the uniqueness of the deformation of the universal enveloping algebra $U(g)$ by ϵ_1 that ensures all order matching.

One of our goal is to extend the $\mathcal{O}(\epsilon_1)$ order matching to $K = 1$. It may seem trivial compared to higher K, but it turns out that it is actually more complicated. We will give the proof in §3.4, §3.5. The uniqueness of the deformation applies for all K including $K = 1$, so we will not try to spell out the details in this work.

Koszul duality

Let us explain why in the first place we can expect the Koszul duality between 5d and 1d operator algebra. Further details on Koszul duality can be found in [139, 140, 5, 128]

The 5d theory is defined on $\mathbb{R}_t \times \mathbb{C}^2_{NC}$, where $\mathbb{R}_t$ is topological and $\mathbb{C}^2_{NC}$, and 1d TQM couples to the 5d theory along $\mathbb{R}_t$. As explained in (3.9), there is a scalar supercharge Q and

[7] It is known that for $A_{\epsilon_1,\epsilon_2} = C^*(g)$, the Koszul dual ${}^!A_{\epsilon_1,\epsilon_2}$ is $U(g)$.

1-form supercharge δ that anti-commute to give a translation operator P_t. We can build a topological line defect action using topological descent.

$$Pexp\int_{-\infty}^{\infty}[\delta, x(t)] \tag{3.37}$$

where

$$x(t) = \sum_{m,n} c[m,n]t[m,n] \tag{3.38}$$

The BRST variation of (3.37) vanishes if $x(t)$ satisfy a Maurer-Cartan equation:

$$[Q, x] + x^2 = 0 \tag{3.39}$$

and if $x \in A \times {}^!A$ for some A, the Maurer-Cartan equation is always satisfied. Hence, it is natural to expect the Koszul duality between $A_{\epsilon_1,\epsilon_2}$ and $\mathcal{A}_{\epsilon_1,\epsilon_2}$. So, the coupling between the 5d ghosts and gauge invariant polynomials of 1d TQM is given by

$$S_{int} = \int_{\mathbb{R}_t} t[m,n]c[m,n]dt. \tag{3.40}$$

Now that we have three types of Lagrangians:

$$S_{5d\ CS} + S_{1d\ TQM} + S_{int} \tag{3.41}$$

We need to make sure if the quantum gauge invariance of 5d Chern-Simons theory remains to be true in the presence of the interaction with 1d TQM. Namely, we need to investigate if there is non-vanishing gauge anomaly in Feynman diagrams. Along the way, we will derive the isomorphism between the operator algebras, as a consistency condition for the gauge anomaly cancellation.

Anomaly cancellation

To give an idea that the cancellation of the gauge anomaly of 5d CS Feynman diagrams fixes the algebra of operators in 1d TQM that couples to the 5d CS, let us review 5d $U(K)$ Chern-Simons example shown in [109]. Consider following Feynman diagram.

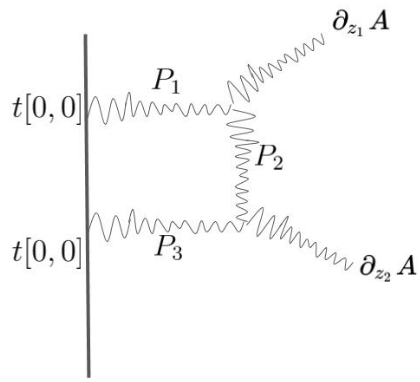


Figure 3.2: The vertical solid line represents the time axis. Internal wiggly lines stand for 5d gauge field propagators P_i, and the external wiggly lines stand for Fourier components 5d gauge field.

The BRST variation($\delta A = \partial c$) of the amplitude of the above Feynman diagram is non-zero.

$$\epsilon_1 \epsilon_{ij} (\partial_{z_i} A^a)(\partial_{z_j} c^b) K^{fe} f^c_{ae} f^d_{bf} t[0,0] t[0,0] \tag{3.42}$$

where K^{ab}, f^a_{bc} are a Killing form and a structure constant of $u(K)$, and $t[m,n]$ is an element of $G = U(N)$, $\hat{G} = U(K)$ ADHM algebra.

To have a gauge invariance, we need to cancel the anomaly, and the gauge variation of the following diagram has exactly factors like $\epsilon_{ij}(\partial_{z_i} A^a)(\partial_{z_j} c^b)$:

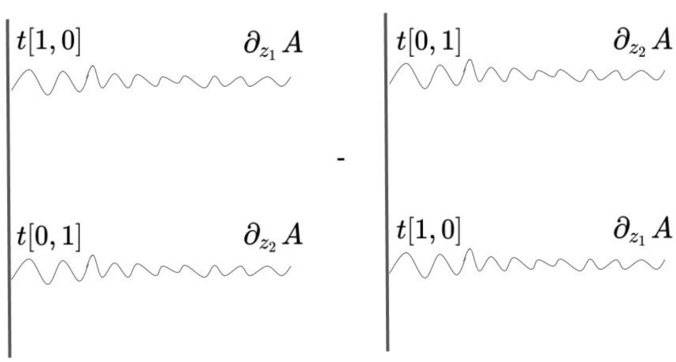


Figure 3.3

The BRST variation of the amplitude is

$$\epsilon_1 \epsilon_{ij} (\partial_{z_i} A^a)(\partial_{z_j} c^b) K^{fe} f^c_{ae} f^d_{bf} [t[1,0], t[0,1]] \tag{3.43}$$

Imposing the cancellation of the BRST variation between (3.42) and (3.43), we obtain

$$[t[1,0], t[0,1]] = \epsilon_1 t[0,0] t[0,0] \tag{3.44}$$

This is very impressive, since we obtain the ADHM algebra from 5d Chern-Simons theory Feynman diagrams!

We will see that if $K = 1$, some ingredients of Feynman diagram change, but we can still reproduce ADHM algebra with $G = U(N)$, $\hat{G} = U(1)$.

Large N limit and a back-reaction of N M2-branes

Although we have not discussed explicitly about taking large N limit, but it was being used implicitly in establishing the isomorphism between ${}^!A_{\epsilon_1,\epsilon_2}$ and $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

Here we explain some detail of taking large N limit. First notice that there are homomorphisms $\iota_N^{N'} : \mathcal{O}(T^*V_{K,N'}) \to \mathcal{O}(T^*V_{K,N})$ for all $N' > N$ induced by natural embedding $\mathbb{C}^N \hookrightarrow \mathbb{C}^{N'}$, where

$$V_{K,N} = \mathfrak{gl}_N \oplus \mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^N), \tag{3.45}$$

so that $T^*V_{K,N}$ is the linear span of single operators I, J, X, Y, and the algebra $\mathcal{O}(T^*V_{K,N})$ is the commutative (classical) algebra generated by these operators (with no relations imposed). Then we define the *admissible* sequence of weight 0 as

$$\{f_N \in \mathcal{O}(T^*V_{K,N})^{\mathrm{GL}_N} | \iota_N^{N'}(f_{N'}) = f_N\}, \tag{3.46}$$

and for integer $r \geq 0$, a sequence $\{f_N\}$ is called admissible of weight r if $\{N^{-r} f_N\}$ is admissible sequence of weight 0 (e.g. the sequence $\{N\}$ is admissible of weight 1), and define $\mathcal{O}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}$ to be the linear span of admissible sequences of all possible weights. It's easy to see that $\mathcal{O}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}$ is an algebra. Next we turn on the quantum deformation which turn the ordinary commutative product to the Moyal product $\star_{\epsilon_1}$, and it's easy to see that for admissible sequences $\{f_N\}$ and $\{g_N\}$, $\{f_N \star_{\epsilon_1} g_N\}$ is also admissible. In this way we obtained the quantized algebra $\mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}$.

Consider the moment map

$$\mu_{\epsilon_2} : \mathfrak{gl}_N \to \mathcal{O}_{\epsilon_1}(T^*V_{K,N}), \quad E_i^j \mapsto X_i^k Y_k^j - X_k^j Y_i^k + I_i J^j - \epsilon_2 \delta_i^j, \tag{3.47}$$

which is GL_N-equivaraint. Together with the Moyal product on $\mathcal{O}_{\epsilon_1}(T^*V_{K,N})$, μ_{ϵ_2} gives rise to a GL_N-equivaraint map of left $\mathcal{O}_{\epsilon_1}(T^*V_{K,N})$-modules

$$\mu_{\epsilon_2} : \mathcal{O}_{\epsilon_1}(T^*V_{K,N}) \otimes \mathfrak{gl}_N \to \mathcal{O}_{\epsilon_1}(T^*V_{K,N}). \tag{3.48}$$

Taking GL_N-invariance, we obtain the quantum moment map

$$\mu_{\epsilon_2} : (\mathcal{O}_{\epsilon_1}(T^*V_{K,N}) \otimes \mathfrak{gl}_N)^{\mathrm{GL}_N} \to \mathcal{O}_{\epsilon_1}(T^*V_{K,N})^{\mathrm{GL}_N}. \tag{3.49}$$

It's easy to varify that the image of μ_{ϵ_2} is a two-sided ideal. Similar to $\mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}$, we can define admissible sequences in $(\mathcal{O}_{\epsilon_1}(T^*V_{K,N}) \otimes \mathfrak{gl}_N)^{\mathrm{GL}_N}$ and call this space $(\mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet}) \otimes \mathfrak{gl}_\bullet)^{\mathrm{GL}_\bullet}$. Quantum moment maps for all N give rise to

$$\mu_{\epsilon_2} : (\mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet}) \otimes \mathfrak{gl}_\bullet)^{\mathrm{GL}_\bullet} \to \mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}, \tag{3.50}$$

and the image is a two-sided ideal, so we can take the quotient of $\mathcal{O}_{\epsilon_1}(T^*V_{K,\bullet})^{\mathrm{GL}_\bullet}$ by this ideal, this is by definition the large-N limit denoted by $\mathcal{O}_{\epsilon_1}(\mathcal{M}^{\epsilon_2}_{K,\bullet})$.

From the definition above, we can write down a set of generators of $\mathcal{O}_{\epsilon_1}(\mathcal{M}^{\epsilon_2}_{K,\bullet})$:

$$\{N\} \text{ and } \{I_\alpha S(X^nY^m)J^\beta\} \text{ for all integers } n,m \geq 0. \tag{3.51}$$

Note that Costello also defined a combinatorical algebra $\mathcal{A}^{\mathrm{comb}}_{\epsilon_1,\epsilon_2}$ in section 10 of [109], which depends on K but not on N. This is related to $\mathcal{O}_{\epsilon_1}(\mathcal{M}^{\epsilon_2}_{K,\bullet})$ in the sense that generators of $\mathcal{A}^{\mathrm{comb}}_{\epsilon_1,\epsilon_2}$ are

$$\{N\} \text{ and } \{\frac{1}{\epsilon_1}I_\alpha S(X^nY^m)J^\beta\} \text{ for all integers } n,m \geq 0, \tag{3.52}$$

when $\epsilon_1 \neq 0$. In the notation of [109] they corresponds to

$$D(\emptyset) \text{ and } \mathrm{Sym}(D(\alpha \Downarrow, \uparrow^n, \downarrow_m, \beta \Uparrow)) \text{ for all integers } n,m \geq 0, \tag{3.53}$$

respectively.

The general philosophy of AdS/CFT [120] teaches us that the back-reaction of N M2-branes will deform the spacetime geometry. In our case, since the closed strings completely decouple from the analysis, the back-reaction is only encoded in the interaction related to the open strings. More precisely, the back-reaction is already encoded in the 5d-1d interaction Lagrangian (3.40), a part of which we reproduce below.

$$S_{back} = \int_{\mathbb{R}_t} t[0,0]c[0,0]dt. \tag{3.54}$$

Here, we can explicitly see N in $t[0,0]$, as

$$t[0,0] = IJ/\epsilon_1 = \epsilon_2 Tr\delta^i_j/\epsilon_1 = N\frac{\epsilon_2}{\epsilon_1} \tag{3.55}$$

where in the second equality, we used the F-term relation.

After taking large N limit, N becomes an element of the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$, which is coupled to the zeroth Fourier mode of the 5d ghost, $c[0,0]$.

M5-brane in Ω−deformed M-theory

We want to include one M5(D4)-brane in the story, and review the role played by the new element(the bi-module from M5(D4)-brane) in the boundary and the bulk.

	0	1	2	3	4	5	6	7	8	9	10
Geometry	$\mathbb{R}_t$	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}^2_{NC}$				$\mathbb{C}_{\epsilon_3}$		$\mathbb{R}$	$S^1_{\epsilon_2}$
$M2(D2)$	×	×	×								
$M5$		×	×	×	×					×	×
$D4$		×	×	×	×					×	

Table 3.1: M2, M5-brane

In the boundary perspective, it intersects with the M2(D2)-brane with two directions and supports 2d $\mathcal{N} = (2, 2)$ supersymmetric field theory with two chiral superfields, whose bottom components are φ, $\tilde{\varphi}$, arising from $D2 - D4$ strings. This 2d theory interacts with the 3d $\mathcal{N} = 4$ ADHM theory with a superpotential

$$\mathcal{W} = \tilde{\varphi} X \varphi \tag{3.56}$$

where X is a scalar component of the adjoint hypermultiplet of the 3d theory.

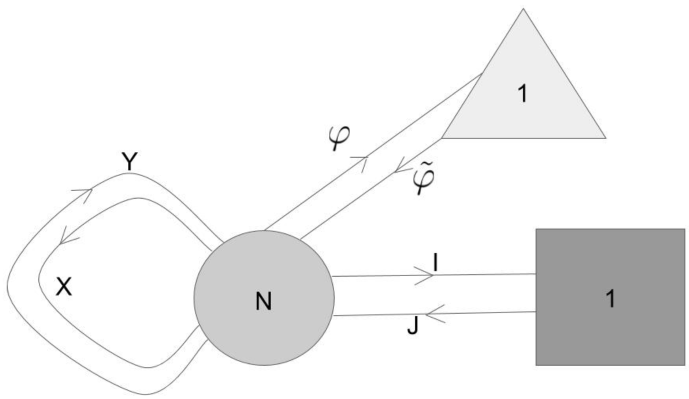


Figure 3.4: 3d $\mathcal{N} = 4$ ADHM quiver gauge theory with $G = U(N)$, $F = U(1)$, decorated with $2d$ $\mathcal{N} = (2, 2)$ field theory. X, Y are scalars of adjoint hypermultipet, and I, J are scalars of (anti)fundamental hypermultiplet. The triangle node encodes the 2d theory. φ and $\tilde{\varphi}$ are 2d scalars. In type IIA language, the circle, square, and triangle node correspond to D2, D6, D4 branes, respectively.

A naive set of gauge invariant operators living on the 2d intersection are

$$IX^mY^n\tilde{\varphi}, \quad \varphi X^mY^nJ, \quad \varphi X^mY^n\tilde{\varphi} \tag{3.57}$$

The superpotential reduces [112, 5] the above set into

$$\mathcal{M}_{\epsilon_1,\epsilon_2} = \{b[z^n] = IY^n\tilde{\varphi}, \quad c[z^n] = \varphi Y^n J\} \tag{3.58}$$

The set of 2d observables $\mathcal{M}_{\epsilon_1,\epsilon_2}$ forms a bi-module of the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

The difference between left and right actions of the algebra $\mathcal{A}$ on $\mathcal{M}_{\epsilon_1,\epsilon_2}$ is encoded in the form of a commutator:

$$[a, m] = m', \quad \text{where } a \in \mathcal{A}, \quad m, m' \in \mathcal{M}_{\epsilon_1,\epsilon_2} \tag{3.59}$$

To verify (3.59), we need to establish the commutation relations between the set of letters $\{\varphi, \tilde{\varphi}\}$ and $\{X, Y, I, J\}$. Those are given by[8]

$$\begin{aligned} IP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})I \\ JP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})J \\ X^i_jP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})X^i_j \\ Y^i_jP(\varphi,\tilde{\varphi}) &= P(\varphi,\tilde{\varphi})(Y^i_j+\tilde{\varphi}^i\varphi_j) \\ X^i_j\varphi_iP(\varphi,\tilde{\varphi}) &= -\epsilon_1\partial_{\tilde{\varphi}_j}P(\varphi,\tilde{\varphi}) \\ X^i_j\tilde{\varphi}^jP(\varphi,\tilde{\varphi}) &= -\epsilon_1\partial_{\varphi_i}P(\varphi,\tilde{\varphi}) \end{aligned} \tag{3.60}$$

Again, the non-trivial commutation relations in the last three lines originates from the effect of the particular superpotential $\mathcal{W}$.

Ω_{ϵ_1} localizes 2d $\mathcal{N} = (2,2)$ theory on a point, which is the origin of $\mathbb{R}_t$.

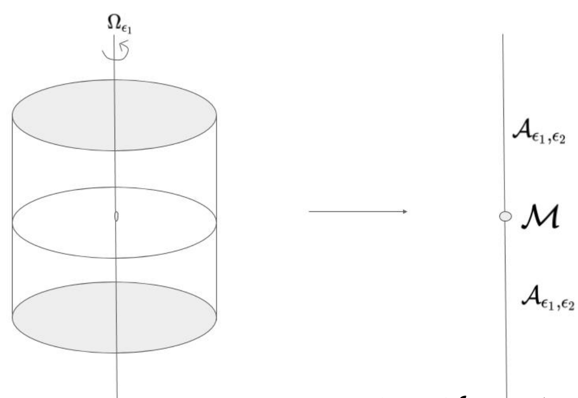


Figure 3.5: Left figure represents a coupled system of 3d $\mathcal{N} = 4$ ADHM theory(the cylinder) and 2d $\mathcal{N} = (2,2)$ theory(the middle disk in the cylinder) from D2 branes and a D4 brane. Ω_{ϵ_1} localizes the system to $1d + 0d$ system.

Hence, the resulting system is 1d ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and 0d bi-module $\mathcal{M}_{\epsilon_1,\epsilon_2}$ of the algebra.

To study the bulk perspective, we need to study what degree of freedoms that M5-brane support in the 5d spacetime $\mathbb{R}_t \times \mathbb{C}^2_{NC}$ and how the M5-brane interacts with 5d Chern-Simons theory. 5d CS theory is defined in the context of type IIa, and M5-brane is mapped to a D4-brane. The local degree of freedom comes from D4-D6 strings, which are placed on $\{\cdot\} \times \mathbb{C} \in \mathbb{R}_t \times \mathbb{C}^2_{NC}$. These 2d degrees of freedom are actually coming from 4d $\mathcal{N} = 2$ hypermultiplet, as the true intersection between D4 and D6 is $\mathbb{C} \times \mathbb{C}_{\epsilon_1}$. The Ω_{ϵ_1} reduces the 4d $\mathcal{N} = 2$ hypermultiplet into a $\beta - \gamma$ system [3]. Hence, we arrive at $\beta - \gamma$ VOA on $\mathbb{C} \subset \mathbb{C}^2_{NC}$.

[8]For the derivation, we refer the reader to [112, 5].

	0	1	2	3	4	5	6	7	8	9
Geometry	$\mathbb{R}_t$	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}^2_{NC}$				$\mathbb{C}_{\epsilon_3}$		$\mathbb{R}_{\epsilon_2}$
1d TQM	×									
2d $\beta\gamma$				×	×					
5d CS	×			×	×	×	×			

Table 3.2: Bulk perspective

The $\beta - \gamma$ system minimally couples to 5d Chern-Simons theory via

$$\int_{\mathbb{C}} \beta(\bar{\partial} + A\star)\gamma \tag{3.61}$$

The observables to be compared with those of field theory side: $b[z^n]$ and $c[z^n]$ can be naturally compared with the modes of β and γ: $\partial_z^n\beta$, $\partial_z^n\gamma$, and the Koszul duality manifests itself by the coupling between two types of observables:

$$\int_{\{0\}} \partial_{z_2}^{k_1}\beta\gamma \cdot b[z^{k_1}] + \int_{\{0\}} \partial_{z_2}^{k_2}\gamma \cdot c[z^{k_2}] \tag{3.62}$$

where $z = z_2$, and the integral on a point is merely for a formal presentation.

The following figure depicts the entire bulk and boundary system including the line and the surface defect, and describes how all the ingredients are coupled.

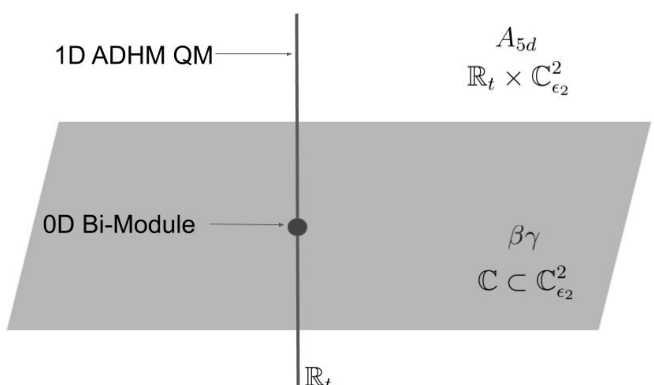


Figure 3.6: 5d Chern-Simons($\mathbb{R}_t \times \mathbb{C}^2_{NC}$), 1d generalized Wilson line defect($\mathbb{R}_t$), and 2d surface defect($\mathbb{C} \subset \mathbb{C}^2_{NC}$).

As explained in section §3.2, we need to make sure if the introduction of the 2d system is quantum mechanically consistent, or anomaly free. Imposing the anomaly cancellation condition of 5d, 2d, 1d coupled system, we should be able to derive the bi-module commutation relations defined in the field theory side. This is the content of §3.5.

The most general configuration in type IIb frame

The system we are considering in this work is the simplest configuration belong to the more general framework [5]. We will briefly sketch it; however, we will not elaborate more on this in the later sections. This can be seen as some possible future directions, related to our remark in the introduction.

We can introduce more $M2_i$-branes on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_i}$ and $M5_I$-branes on $\mathbb{C} \times \mathbb{C}_j \times \mathbb{C}_k$, where $i \in \{1,2,3\}$, $(j,k) \in \{(1,2),(2,3),(3,1)\}$, and $I = \{1,2,3\}\backslash\{j,k\}$. Using the M-theory / type IIB duality, we can map the most general configuration to "GL-twisted type IIB" theory [141], where each M2-brane maps to $(1,0),(0,1),(1,1)$ 1-brane, respectively, and each M5-brane maps to D3-brane whose boundary is provided by $(1,0),(0,1),(1,1)$ 5-branes.

At the corner of the tri-valent vertex, so-called Y-algebra [118], which comes form D3-brane boundary degree of freedom [142, 143], lives. This VOA(Vertex Operator Algebra) is the most general version of our toy model $\beta\gamma$ system, and is labeled by three integers N_1, N_2, N_3, each of which is the number of D3-branes on three corners of the trivalent graph. So, in principle, one can extend our analysis related to the M5-brane into Y-algebra VOA. The Koszul dual object of the the VOA was called as universal bi-module $\mathcal{B}^{N_1,N_2,N_3}_{\epsilon_1,\epsilon_2}$ in [5].

Moreover, our ADHM algebra from $M2_1$-brane has its triality image at $M2_2$-brane and $M2_3$-brane. It was proposed in [5] that there is a co-product structure in $M2_i$-brane algebras in the Coulomb branch algebra language[9]. Hence, one can generalize our analysis related to the M2-brane into the most general algebra, obtained by fusion of three $M2_i$-brane algebra. This was called as universal algebra $\mathcal{A}^{n_1,n_2,n_3}_{\epsilon_1,\epsilon_2}$ in [5].

3.3. M2-brane algebra and M5-brane module

In this section, we will provide a representative commutation relation for the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$

$$[a, a'] = a_0 + \epsilon_1 a_1 + \epsilon_1^2 a_2 + \dots, \quad \text{where } a, a', a_i \in \mathcal{A}_{\epsilon_1,\epsilon_2} \tag{3.63}$$

and a representative commutation relation for the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and the bi-module $\mathcal{M}_{\epsilon_1,\epsilon_2}$ for $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

$$[a, m] = m_0 + \epsilon_1 m_1 + \epsilon_1^2 m_2 + \dots, \quad \text{where } a \in \mathcal{A}_{\epsilon_1,\epsilon_2},\ m, m_i \in \mathcal{M}_{\epsilon_1,\epsilon_2} \tag{3.64}$$

We first recall the notation for a typical element of $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and $\mathcal{M}_{\epsilon_1,\epsilon_2}$:

$$\begin{aligned} t[m,n] &= \frac{1}{\epsilon_1} TrS(X^m Y^n) = \frac{1}{\epsilon_1\epsilon_2} IS(X^m Y^n)J \in \mathcal{A}_{\epsilon_1,\epsilon_2} \\ b[z^m] &= \frac{1}{\epsilon_1} IY^m \tilde{\varphi} \in \mathcal{M}_{\epsilon_1,\epsilon_2} \\ c[z^n] &= \frac{1}{\epsilon_1} \varphi Y^n J \in \mathcal{M}_{\epsilon_1,\epsilon_2} \end{aligned} \tag{3.65}$$

[9] It is equally possible to describe the M2-brane algebra in terms of Coulomb branch algebra, as the ADHM theory is a self-mirror in the sense of 3d mirror symmetry [144, 145].

For the convenience of later discussions, we also introduce the notation:

$$T[m,n] = \frac{\epsilon_2}{\epsilon_1} TrS(X^m Y^n) = \frac{1}{\epsilon_1} IS(X^m Y^n)J \in \mathcal{A}_{\epsilon_1,\epsilon_2} \tag{3.66}$$

Our final goal is to reproduce the $\mathcal{A}_{\epsilon_1,\epsilon_2}$ algebra from the anomaly cancellation of 1-loop Feynman diagrams in 5d Chern-Simons theory. So, it is important to have commutation relations that yield $\mathcal{O}(\epsilon_1)$ term in the right hand side, where ϵ_1 is a loop counting parameter in 5d CS theory.

M2-brane algebra

Since we have not provided a concrete calculation until now, let us give a simple computation to give an idea of ADHM algebra and its bi-module. It is useful to recall $G = U(N)$, $\hat{G} = U(K)$ ADHM algebra, which serves as a practice example, and at the same time as an example that explains the non-triviality of $G = U(N)$, $\hat{G} = U(1)$ ADHM algebra, compared to $K > 1$ cases.

It was shown in [109] that following commutation holds for $G = U(N)$, $\hat{G} = U(K)$ ADHM algebra.

$$[t[1,0], t[0,1]] = \epsilon_1 t[0,0]t[0,0] \quad \text{or} \quad [IXJ, IYJ] = \epsilon_1 (IJ)(IJ) \tag{3.67}$$

This does not work for $\hat{G} = U(1)$. It is instructive to see why.

$$\begin{aligned} [TrX, TrY] &= [X^i_i, Y^j_j] = \delta^i_j \delta^j_i \epsilon_1 = \delta^i_j \epsilon_1 \\ &= N\epsilon_1 \end{aligned} \tag{3.68}$$

Multiplying both sides by $\epsilon_2^2/\epsilon_1^2$, we can convert it into $T[m,n]$ basis:

$$[T[1,0], T[0,1]] = \epsilon_2 T[0,0] \tag{3.69}$$

The RHS of (3.69) is different from (3.67) crucially in its dependence on ϵ_1. The RHS of (3.69) is $\mathcal{O}(\epsilon_1^0)$, but that of (3.67) is $\mathcal{O}(\epsilon_1)$. While it was sufficient to consider this simple commutator to see the ϵ_1 deformation of the algebra for $\hat{G} = U(K)$ with $K > 1$, we need to consider a more complicated commutator to see $\mathcal{O}(\epsilon_1)$ correction in the RHS.

With the help of the computer algebra, we could identify the simplest non-trivial pairs are $(t[3,0], t[0,3])$, $(t[2,1], t[1,2])$.

$$\begin{aligned} [t[3,0], t[0,3]] &= 9t[2,2] + \frac{3}{2}\big(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]\big) \\ [t[2,1], t[1,2]] &= 3t[2,2] - \frac{1}{2}\big(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]\big) \end{aligned} \tag{3.70}$$

where

$$\sigma_2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2, \quad \sigma_3 = -\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2) \tag{3.71}$$

We gave a proof for $[t[3,0],t[0,3]]$ in Appendix §3.6.

To compare the commutation relation to that from 5d Chern-Simons calculation, we need to make sure if the parameters of ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ are the same as those in 5d CS theory. From [109], the correct parameter dictionary[10] is

$$(\epsilon_1)_{ADHM} = (\epsilon_1)_{CS}, \quad \left(\epsilon_2 + \frac{1}{2}\epsilon_1\right)_{ADHM} = (\epsilon_2)_{CS} \tag{3.72}$$

Hence, the commutation relation that we are supposed to match from the 5d computation is

$$[t[2,1],t[1,2]] = 3t[2,2] - \frac{1}{2}\Big(\big(\epsilon_2^2 + \frac{3}{4}\epsilon_1^2\big)t[0,0] + \big(\epsilon_1\epsilon_2^2 - \frac{\epsilon_1^3}{4}\big)t[0,0]t[0,0]\Big) \tag{3.73}$$

There is one term in the RHS of (3.73) that is in $\mathcal{O}(\epsilon_1)$ order:

$$[t[2,1],t[1,2]] = \mathcal{O}(\epsilon_1^0) - \frac{1}{2}\epsilon_1\epsilon_2^2 t[0,0]t[0,0] + \mathcal{O}(\epsilon_1^2) \tag{3.74}$$

We will try to recover the $\mathcal{O}(\epsilon_1)$ term from 5d Feynman diagram calculation[11] in section §3.4; the general argument that gauge anomaly cancelation leads to the Koszul dual algebra commutation relation is given in §3.2.

M5-brane module

We will use the commutation relations (3.35), (3.36), (3.60) to compute the commutators between $a \in \mathcal{A}_{\epsilon_1,\epsilon_2}$ and $m \in \mathcal{M}_{\epsilon_1,\epsilon_2}$, which are defined in (3.32), (3.58). When one tries to compute some commutators, one immediately notices some normal ordering ambiguity in a general module element m, which can be seen in following example.

$$[IXJ, (I\tilde{\varphi})(\varphi J)] = \left[I_i X^i_j J^j, I_a \tilde{\varphi}^a \varphi_b J^b\right] \tag{3.75}$$

Assuming that the order of letters is consistent with the order of fields in the real line $\mathbb{R}_t$, it is obvious that we need to place $\tilde{\varphi}^a\varphi_b$ together, as they are defined at a point $\{0\} \in \mathbb{R}_t$[12]. However, it is ambiguous whether we put I_a, J^b in the right or left of $\tilde{\varphi}^a\varphi_b$, as I_a, J^b are living on $\mathbb{R}_t$. We will try to fix this ambiguity to prepare a concrete calculation.

Considering following normal ordering when writing a module element $(IY\varphi)(\varphi J)$ will be enough to fix the ambiguity.

$$|\tilde{\varphi}^j\varphi_k| I_i J^k {Y^i}_j \tag{3.76}$$

[10]We thank Davide Gaiotto, who pointed out this subtlety.

[11]The basis used in the Feyman diagram computation is $T[m,n]$, not $t[m,n]$. However, the change of basis does not affect any computation because the $\mathcal{O}(\epsilon_1)$ term in (3.74) is quadratic in t.

[12]Recall that φ, $\tilde{\varphi}$ are chiral multiplet scalars that are localized at the interface(between the line and the surface). After Ω_{ϵ_1} deformation, the interface localizes to a point. Hence, φ, $\tilde{\varphi}$ are localized to be at a point on the line.

We simply choose other letters like X, Y, I, J to be placed on the right side of φ and $\tilde{\varphi}$.

Still, there is an ordering ambiguity. For instance between two words:

$$|\tilde{\varphi}\varphi|IJY \quad vs \quad |\tilde{\varphi}\varphi|JIY \tag{3.77}$$

We simply choose an alphabetical order to arrange letters. In other words, we use the commutation relations until the letters in the word has a alphabetical order. When the word has an alphabetical order, we contract the gauge indices to form a single-trace word, and omit the gauge indices. For instance,

$$\begin{aligned} (\tilde{\varphi}\varphi) &:= |\tilde{\varphi}^j\varphi_j| \\ (IY\tilde{\varphi})(\varphi J) &:= |\tilde{\varphi}^j\varphi_l|I_k J^l Y^k{}_j \\ (I\tilde{\varphi})(\varphi J)(IJ) &:= |\tilde{\varphi}^j\varphi_k|I_j J^k I_i J^i \end{aligned} \tag{3.78}$$

As a consequence, some more steps are needed for the following:

$$|\tilde{\varphi}^j\varphi_k|I_i I_j J^k J^i \tag{3.79}$$

That is, we need to commute I_i through J^k to contract with J^i. While doing this, we necessarily use $[I_i, J^k] = \epsilon_1\delta_i^k + J^k I_i$, which produces two terms.

Having fixed the ordering ambiguity, there is a few things to keep in mind additionally:

- We use F-term relation and the basic commutation relation between X and Y in maximum times to get rid of X's in the word, since the module only consists of φ, $\tilde{\varphi}$, I, J, Y.

- To use F-term relation, we first need to pull the target XY(or YX) pair to the right end, not to ruin the gauge invariance, and pull it back to the original position in the word.

- To use the superpotential relations($X\varphi = \epsilon_1\partial_{\tilde{\varphi}}$ or $X\tilde{\varphi} = \epsilon_1\partial_{\varphi}$), we need to bring X right next to φ or $\tilde{\varphi}$.

Given the prescription, we would like to find $a \in \mathcal{A}_{\epsilon_1,\epsilon_2}$ and $m \in \mathcal{M}_{\epsilon_1,\epsilon_2}$ such that the value of $[a, m]$ contains $\mathcal{O}(\epsilon_1)$ terms. To illustrate the prescription, let us consider following simple example, which will not produce $\mathcal{O}(\epsilon_1)$ term.

Example: $[IXJ, (IY\tilde{\varphi})(\varphi J)]$

It is much clear and convenient to use closed word version for the algebra element. We will recover the open word at the end by simply multiplying ϵ_2 on the closed words.

$$[TrX, (IY\tilde{\varphi})(\varphi J)] = (X)\cdot(IY\tilde{\varphi})(\varphi J) - (IY\tilde{\varphi})(\varphi J)\cdot(X) \tag{3.80}$$

Compute the first term:

$$\begin{aligned} X_0^0|\tilde{\varphi}^b\varphi_c|I_aY_b^aJ^c =& |\tilde{\varphi}^b\varphi|I_a(\epsilon_1\delta_b^a + Y_b^aX_0^0)J^c \\ &= \epsilon_1|\tilde{\varphi}^b\varphi_c|I_bJ^c + (IY\tilde{\varphi})(\varphi J)\cdot(X) \end{aligned} \tag{3.81}$$

So,

$$\begin{aligned} [TrX, (IY\tilde{\varphi})(\varphi J)] &= \epsilon_1|\tilde{\varphi}^b\varphi_c|I_bJ^c \\ &= \epsilon_1(I\tilde{\varphi})(\varphi J) \end{aligned} \tag{3.82}$$

After normalization, by multiplying $\frac{\epsilon_2}{\epsilon_1^3}$ both sides, we get

$$[T[1,0], b[z]c[1]] = \epsilon_2 b[1]c[1] \tag{3.83}$$

There is no $\mathcal{O}(\epsilon_1)$ correction. So, we need to work harder.

The first bi-module commutator that has an ϵ_1 correction with some non-trivial dependence on ϵ_2 is $[TrS(X^2Y), (IY\tilde{\varphi})(\varphi J)]$. After properly normalizing it, we have

$$\begin{aligned} [T[2,1], b[z]c[1]] =& \Big(-\frac{5}{3}\epsilon_2 T[0,1] + \epsilon_2^2 b[1]c[1] \Big) \\ &+ \boxed{\epsilon_1\Big(-\epsilon_2 b[1]c[1]T[0,0] + \frac{4}{3}\epsilon_2 b[1]c[1] \Big)} \\ &+ \epsilon_1^2\Big(-\frac{4}{3} b[1]c[1]T[0,0] \Big) \\ &+ \epsilon_1^3\Big(-\frac{1}{3} b[1]c[1]b[1]c[1] \Big) \end{aligned} \tag{3.84}$$

Here, we used the re-scaled basis $T[m,n]$ for $\mathcal{A}_{\epsilon_1,\epsilon_2}$. This is a better choice to be coherent with the form of the bi-module elements, since $b[z^n] = IY^n\tilde{\varphi}$ and $c[z^n] = \varphi Y^n J$ explicitly depend on I and J. [13]We have shown the proof in Appendix §3.6.

3.4. Perturbative calculations in 5d $U(1)$ CS theory coupled to 1d QM

In this section, we will provide a derivation of the $G = U(N)$, $\hat{G} = U(1)$ ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ using the perturbative calculation in 5d $U(1)$ CS. We will see the result from the perturbative calculation matches with the expectation (3.74). The strategy, which we will spell out in detail in this section, is to compute the $\mathcal{O}({\epsilon_1}^1)$ order gauge anomaly of various

[13]Similar to the algebra case, there might be a shift in parameters ϵ_1 and ϵ_2 in 5d CS side; here, we simply assumed that there is no shift: $(\epsilon_1)_{5d} = (\epsilon_1)_{1d-2d}$, $(\epsilon_2)_{5d} = (\epsilon_2)_{1d-2d}$. If there were a shift in the ϵ_2 dictionary, the tree level term may be a potential problem.

Feynman diagrams in the presence of the line defect from $M2$ brane($\mathbb{R}^1 \times \{0\} \subset \mathbb{R}^1 \times \mathbb{C}^2_{NC}$). Imposing a cancellation of the anomaly for the 5d CS theory uniquely fixes the algebra commutation relations.

Purely working in the weakly coupled 5d CS theory, we will derive the representative commutation relations of the ADHM algebra (3.74):

- Algebra commutation relation

$$[t[2,1], t[1,2]] = \ldots + \epsilon_1 \epsilon_2^2 t[0,0] t[0,0] + \ldots \tag{3.85}$$

where $t[n,m]$ is a basis element of $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

As we commented in §3.3, the algebra basis used in the Feynman diagram computation is $T[m,n]$, which is related to $t[m,n]$ by rescaling with ϵ_2. The effect of the change of basis is trivial in (3.85), so we will interchangeably use $t[m,n]$ and $T[m,n]$ without loss of generality.

Ingredients of Feynman diagrams

To set-up the Feynman diagram computations, we recall the 5d $U(1)$ Chern-Simons theory action on $\mathbb{R}_t \times \mathbb{C}^2_{NC}$.

$$S = \frac{1}{\epsilon_1} \int_{\mathbb{R}_t \times \mathbb{C}^2_{NC}} dz_1 dz_2 \left(A \star_{\epsilon_2} dA + \frac{2}{3} A \star_{\epsilon_2} A \star_{\epsilon_2} A \right) \tag{3.86}$$

with $|\epsilon_1| \ll |\epsilon_2| \ll 1$. In components, the 5d gauge field A can be written as

$$A = A_t dt + A_{\bar{z}_1} d\bar{z}_1 + A_{\bar{z}_2} d\bar{z}_2 \tag{3.87}$$

with all the components are smooth holomorphic functions on $\mathbb{R}^1 \times \mathbb{C}^2_{NC}$.

Now, we want to collect all the ingredients of the Feynman diagram computation. It is convenient to rewrite (3.86) as

$$S = \frac{1}{\epsilon_1} \int_{\mathbb{R}^1 \times \mathbb{C}^2_{NC}} dz_1 dz_2 \left(A dA + \frac{2}{3} A (A \star_{\epsilon_2} A) \right) \tag{3.88}$$

(3.88) is equivalent to (3.86) up to a total derivative. From the kinetic term of the Lagrangian, we can read off the following information:

- 5d gauge field propagator P is a solution of

$$dz_1 \wedge dz_2 \wedge dP = \delta_{t=z_1=z_2=0}. \tag{3.89}$$

That is,

$$P(v_1, v_2) = \langle A(v_1) A(v_2) \rangle = \frac{\bar{z}_{12} d\bar{w}_{12} dt_{12} - \bar{w}_{12} d\bar{z}_{12} dt_{12} + t_{12} d\bar{z}_{12} d\bar{w}_{12}}{d_{12}^5} \tag{3.90}$$

where

$$v_i = (t_i, z_i, w_i), \quad d_{ij} = \sqrt{t_{ij}^2 + |z_{ij}|^2 + |w_{ij}|^2}, \quad t_{ij} = t_i - t_j \tag{3.91}$$

From the three point coupling in the Lagrangian, we can extract 3-point vertex. This is not immediate, as the theory is defined on non-commutative background. Different from $U(N)$ CS, where the leading contribution of the 3-point vertex was AAA, the leading contribution of the 3-point coupling of the $U(1)$ gauge bosons starts from $\mathcal{O}(\epsilon_2)A\partial_{z_1}A\partial_{z_2}A$. The reason is following:

$$
\begin{aligned}
&\int dz \wedge dw \wedge A \wedge (A \star_{\epsilon_2} A) \\
=&\int A \wedge ((A_t dt + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w}) \star (A_t dt + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w})) \\
=&\int dz \wedge dw \wedge A \wedge [dt \wedge d\bar{z}\,(A_t \star A_{\bar{z}} - A_{\bar{z}} \star A_t) + \ldots] \\
=&\int dz \wedge dw \wedge A \wedge [dt \wedge d\bar{z}\,(0 + 2\epsilon_2\,(\partial_z A_t \partial_w A_{\bar{z}} - \partial_w A_t \partial_z A_{\bar{z}})) + \ldots] \\
=&\,2\epsilon_2 \int dz \wedge dw \wedge A \wedge [dt \wedge d\bar{z}(\partial_z A_t \partial_w A_{\bar{z}} - \partial_w A_t \partial_z A_{\bar{z}})] + \mathcal{O}(\epsilon_2^2)
\end{aligned}
\tag{3.92}
$$

Note that for $U(N)$ case, $SU(N)$ Lie algebra factors attached to each A prevents the $\mathcal{O}(\epsilon_2^0)$ term to vanish. Still, $U(1) \subset U(N)$ part of A contributes as $\mathcal{O}(\epsilon_2)$, but it can be ignored, since we take $\epsilon_2 \ll 1$.

Hence, in $U(1)$ CS, the 3-point $A\partial_z A\partial_w A$ coupling contributes as

- Three-point vertex $\mathcal{I}_{3pt}$:

$$
\mathcal{I}_{3pt} = \epsilon_2 dz \wedge dw \tag{3.93}
$$

Now, we are ready to introduce the line defect into the theory and study how it couples to 5d gauge fields. Classically, $t[n_1, n_2]$ couples to the mode of 5d gauge field by

$$
\int_{\mathbb{R}} t[n_1, n_2]\partial_{z_1}^{n_1}\partial_{z_2}^{n_2} A dt \tag{3.94}
$$

The last ingredient of the bulk Feynman diagram computation comes from the interaction (3.94).

- One-point vertex $\mathcal{I}_{1pt}^A$:

$$
\mathcal{I}_{1pt}^A = \begin{cases} t[n_1, n_2]\delta_{t,z_1,z_2} & \text{if } \partial_{z_1}^{n_1}\partial_{z_2}^{n_2} A \text{ is a part of an internal propagator} \\ t[n_1, n_2]\partial_{z_1}^{n_1}\partial_{z_2}^{n_2} A & \text{if } \partial_{z_1}^{n_1}\partial_{z_2}^{n_2} A \text{ is an external leg} \end{cases} \tag{3.95}
$$

Lastly, the loop counting parameter is ϵ_1. Each of the propagator is proportional to ϵ_1 and the internal vertex is proportional to ϵ_1^{-1}. Hence, 0-loop order($\mathcal{O}({\epsilon_1}^0)$) Feynman diagrams may contain the same number of internal propagators and internal vertices and 1-loop order($\mathcal{O}(\epsilon_1)$) diagrams may contain one more internal propagators than internal vertices.

Until now, we have collected all the components of the 5d perturbative computation (3.90), (3.93), (3.94), and (3.95). With these, let us see what Feynman diagrams have non-zero BRST variations and how the cancelation of BRST variations of different diagrams leads to the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

Feynman diagram

We will show that the following Feynman diagram has a non-vanishing amplitude and a non-vanishing gauge anomaly consequently, under the BRST variation:

$$Q_{BRST}A = \partial c \tag{3.96}$$

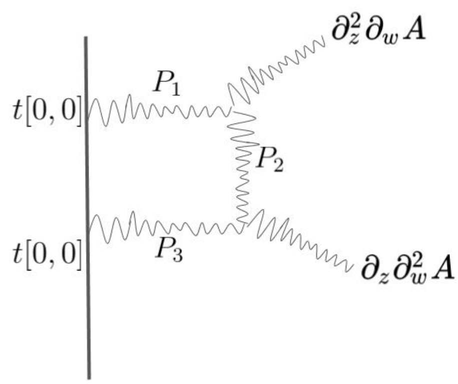


Figure 3.7: The vertical solid line represents the time axis, where 1d topological defect is supported. Internal wiggly lines stand for 5d gauge field propagators P_i, and the external wiggly lines stand for 5d gauge field A.

We will follow the approach shown in [119]. We first integrate over the first vertex $(P_1\ \partial_z^2\partial_w A\ P_2)$ and then integrate over the second vertex$(P_2\ \partial_z\partial_w^2 A\ P_3)$.

First vertex$(P_1\ \partial_z^2\partial_w A\ P_2)$

First, we focus on computing the integral over the first vertex:

$$\epsilon_1\epsilon_2^2 \int_{v_1} dw_1 \wedge dz_1 \wedge \partial_{z_1} P_1(v_0, v_1) \wedge \partial_{z_2}\partial_{w_1} P_2(v_1, v_2)(z_1^2 w_1 \partial_{z_1}^2 \partial_{w_1} A) \tag{3.97}$$

Note that ∂_{z_1} and ∂_{w_1} comes from the three point coupling at v_1:

$$\epsilon_2 A \wedge \partial_{z_1} A \wedge \partial_{w_1} A \tag{3.98}$$

And ∂_{z_2} comes from the 3-pt coupling at v_2:

$$\epsilon_2 A \wedge \partial_{z_2} A \wedge \partial_{w_2} A \tag{3.99}$$

We will consider ∂_{w_2} later when we treat the second vertex.

The factor $z_1^2 w_1 \partial_{z_1}^2 \partial_{w_1} A$ is for the external leg attached to v_1, which is $c[2,1]$. Basically, this is an ansatz, and we can start without fixing m, n in $c[m,n]$. However, we will see that the integral converges to a finite value only with this particular choice of (m,n). For a simple presentation, we will drop $\partial_{z_1}^2 \partial_{w_1} A$, and recover it later.

After some manipulation, which we defer to Lemma 1. in Appendix 3.7, (3.97) becomes

$$-\int_{v_1} dt_1 dz_1 d\bar{z}_1 dw_1 d\bar{w}_1 \frac{|z_1|^2 |w_1|^2 \bar{z}_2(\bar{w}_{12} dt_2 - t_{12} d\bar{w}_2)}{d_{01}^5 d_{12}^9} \tag{3.100}$$

This is the crucial step that shows the necessity of choosing $c[m,n]$ to be $c[2,1]$. Otherwise, the numerator of (3.100) would have holomorphic or anti-holomorphic dependence on z_1 or w_1, and this makes the z_1, w_1 integral to vanish.

The integral can be further simplified by using the typical Feynman integral technique, which can be found in Lemma 2. in Appendix 3.7. We are left with

$$\bar{z}_2(\bar{w}_2 dt_2 - t_2 d\bar{w}_2) \left(\frac{c_1}{d_{02}^5} + \frac{c_2 w_2^2}{d_{02}^7} + \frac{c_3 z_2^2}{d_{02}^7} + \frac{c_4 z_2^2 w_2^2}{d_{02}^9} \right) \tag{3.101}$$

with c_i being a constant. Note that all the terms in the parenthesis has a same order of divergence. So, it suffices to focus on the first term to check the convergence of the full integral(we still need to do v_2 integral below.)

We will explicitly show the calculation for the first term, and just present the result for the second, third and fourth term in (3.192). They are all non-zero and finite. We will denote the first term as $\mathcal{P}$, which is 1-form.

Second vertex($\mathcal{P}\ \partial_{z_1}^2 \partial_{z_2} A\ P_3$)

Now, let us do the integral over the second vertex(v_2). The remaining things are organized into

$$\int_{v_2} \mathcal{P} \wedge \partial_{w_2} P_3(v_2, v_3) \wedge dz_2 \wedge dw_2 (z_2 w_2^2 \partial_{z_2} \partial_{w_2}^2 A) \tag{3.102}$$

where we dropped forms related to v_3, as we do not integrate over it. ∂_{w_2} comes from the 3-pt coupling at v_2:

$$\epsilon_2 A \wedge \partial_{z_2} A \wedge \partial_{w_2} A \tag{3.103}$$

The factor $z_2 w_2^2 \partial_{z_2} \partial_{w_2}^2 A$ is for the external leg attached to v_2, which corresponds to $c[1,2]$. Again, this is an ansatz. We will see that only this integral converges and does not vanish below. We will drop $\partial_{z_2} \partial_{w_2}^2 A$ and recover it later.

The integral (3.102) is simplified to

$$\int_{v_2} -\frac{|z_2|^2|w_2|^4}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2 \tag{3.104}$$

The intermediate steps can be found in Lemma 3 in Appendix 3.7. We see that it was necessary to choose $c[m, n]$ to be $c[1, 2]$. Otherwise, the numerator of (3.104) would contain holomorphic or anti-holomorphic dependence on z_2 or w_2, and this makes the z_2 and w_2 integrals to vanish.

Now, it remains to evaluate the delta function at the third vertex, and use Feynman technique to evaluate the integral. By Lemma 4 in Appendix 3.7, we are left with

$$(const)\epsilon_1\epsilon_2^2 t[0,0]t[0,0]\partial_{z_1}^2\partial_{z_2}A_1\partial_{z_1}^1\partial_{z_2}^2 A_2 \tag{3.105}$$

The BRST variation of the amplitude is

$$(const)\epsilon_1\epsilon_2^2 t[0,0]t[0,0]\partial_{z_1}^2\partial_{z_2}A_1\partial_{z_1}^1\partial_{z_2}^2 c_2 \tag{3.106}$$

This indicates that the theory is quantum mechanically inconsistent, as it has a Feynman diagram that has non-zero BRST variation. However, as long as there is another diagram whose BRST variation is proportional to the same factors

$$\epsilon_1\epsilon_2^2 t[0,0]t[0,0]\partial_{z_1}^2\partial_{z_2}A_1\partial_{z_1}^1\partial_{z_2}^2 c_2, \tag{3.107}$$

we can cancel the anomaly.

Hence, imposing BRST invariance of the sum of Feynman diagrams, we bootstrap the possible 1d TQM that can couple to 5d $U(1)$ CS.

An obvious choice is the tree level diagrams where $(\partial_{z_1}A)(\partial_{z_2}A)$ appears explicitly:

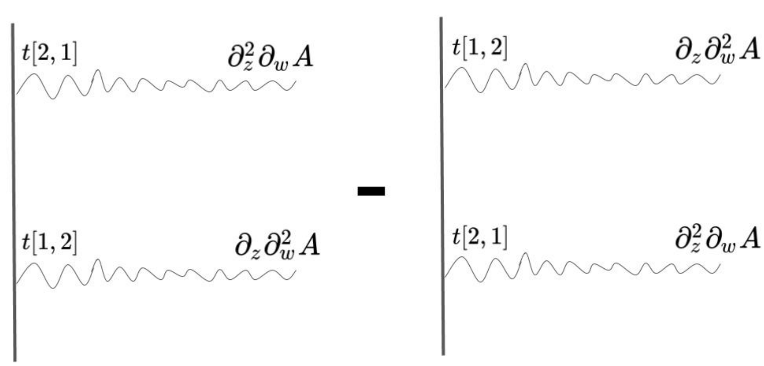


Figure 3.8: There is no internal propagators, but just external ghosts for 5d gauge fields, which directly interact with 1d QM. The minus sign in the middle literally means that we take a difference between two amplitudes. In the left diagram $t[1, 2]$ vertex is located at $t = 0$ and $t[2, 1]$ is at $t = \epsilon$. In the right diagram, $t[1, 2]$ is at $t = -\epsilon$ and $t[2, 1]$ at $t = 0$.

The amplitude of the tree level diagrams can be obtained without the above complicated calculation.

$$[t[2,1],t[1,2]]\,\partial_{z_1}^2\partial_{z_2}A_1\partial_{z_1}^1\partial_{z_2}^2A_2 \tag{3.108}$$

The BRST variation of the amplitude is proportional to

$$[t[2,1],t[1,2]]\,\partial_{z_1}^2\partial_{z_2}A_1\partial_{z_1}^1\partial_{z_2}^2c_2 \tag{3.109}$$

By equating (3.106) and (3.109), we get

$$[t[2,1],t[1,2]]=\epsilon_1\epsilon_2^2t[0,0]t[0,0]+\dots \tag{3.110}$$

So, we have reproduced the $\mathcal{O}(\epsilon_1)$ part of the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ commutation relation from the Feynman diagram computation:

$$[t[2,1],t[1,2]]_{\epsilon_1}=\epsilon_1\epsilon_2^2t[0,0]t[0,0] \tag{3.111}$$

3.5. Perturbative calculations in 5d $U(1)$ CS theory coupled to 2d $\beta\gamma$

In this section, we will provide a bulk derivation of the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ action on the bi-module $\mathcal{M}_{\epsilon_1,\epsilon_2}$ of the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ using 5d Chern-Simons theory. The strategy is similar to that of the previous section. We will compute the $\mathcal{O}({\epsilon_1}^1)$ order gauge anomaly of various Feynman diagrams in the presence of the line defect from $M2$ brane($\mathbb{R}^1\times\{0\}\subset\mathbb{R}^1\times\mathbb{C}^2_{NC}$), and at the same time the surface defect from $M5$ brane on ($\{0\}\times\mathbb{C}\subset\mathbb{R}^1\times\mathbb{C}^2_{NC}$). Imposing a cancellation of the anomaly for the 5d gauge theory uniquely fixes the algebra action on the bi-module.

We will confirm the representative commutation relation between ADHM algebra and its bi-module (3.112) using the Feynman diagram calculation in 5d Chern-Simons, 1d topological line defect, and 2d $\beta\gamma$ coupled system.

- The algebra and the bi-module commutation relation

$$\big[t[2,1],b[z^1]c[z^0]\big]|_{\epsilon_1}=\epsilon_1\epsilon_2\ t[0,0]c[z^0]b[z^0]+\epsilon_1\epsilon_2\ c[z^0]b[z^0] \tag{3.112}$$

where $c[z^n]$ and $b[z^m]$ are elements of the 0d bi-module.

Ingredients of Feynman diagrams

The generators of the 0d bi-module $b[z^n]$, $c[z^m]$ couple to the mode of β, γ through

$$\int_{\{0\}}\partial_{z_2}^{k_1}\beta\gamma\cdot b[z^{k_1}]+\int_{\{0\}}\partial_{z_2}^{k_2}\gamma\cdot c[z^{k_2}] \tag{3.113}$$

where $z = z_2$. The coupling is defined at a point, so the integral is only used for a formal presentation.

From the coupling, we learn another ingredient of the 5d-2d Feynman diagram computation:

- One-point vertices from (3.113):

$$\begin{aligned} \mathcal{I}^{\beta}_{1pt} &= \begin{cases} b[z^k]\delta_{z_2} & \text{if } \partial^k_{z_2}\beta \text{ is a part of an internal propagator} \\ b[z^k]\partial^k_{z_2}\beta & \text{if } \partial^k_{z_2}\beta \text{ is an external leg} \end{cases}, \\ \mathcal{I}^{\gamma}_{1pt} &= \begin{cases} c[z^k]\delta_{z_2} & \text{if } \partial^k_{z_2}\gamma \text{ is a part of an internal propagator} \\ c[z^k]\partial^k_{z_2}\gamma & \text{if } \partial^k_{z_2}\gamma \text{ is an external leg} \end{cases} \end{aligned} \tag{3.114}$$

  In the case of multiple β, γ internal propagators flowing out, we prescribe to keep only one δ_{z_2} function.

The $\beta\gamma$−system also couples to 5d Chern-Simons theory in a canonical way:

$$\frac{1}{\epsilon_1}\int \beta(\partial_{\bar{z}_2} - A_{\bar{z}_2}\star_{\epsilon_2})\gamma \tag{3.115}$$

from which we read off the last ingredients of the perturbative computation:

- The $\beta\gamma$ propagator $P_{\beta\gamma} = \langle\beta\gamma\rangle$ is a solution of

$$dz_2 \wedge dP_{\beta\gamma} = \delta_{z_2=0} \tag{3.116}$$

  That is,

$$P_{\beta\gamma} = \langle\beta\gamma\rangle \sim \frac{1}{z_2} \tag{3.117}$$

- The normalized three-point(β, A_{5d}, γ) vertex :

$$\mathcal{I}^{\beta A\gamma}_{3pt} = 1 \tag{3.118}$$

Note that we are taking the lowest order vertex in the Moyal product expansion of (3.115), and normalize the coefficient to 1, for simplicity, in the following computation. Each $\beta\gamma$ propagator contributes ϵ_1, and each $\beta A\gamma$ vertex contributes ${\epsilon_1}^{-1}$.

We remind the reader the universal bi-module $\mathcal{B}_{\epsilon_1,\epsilon_2}$, which we introduced in section §3.2, can couple to general Vertex Algebras at corner in the presence of N_1, N_2, N_3 M5-branes wrapping $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}$, $\mathbb{C}_{\epsilon_2}\times\mathbb{C}_{\epsilon_3}$, $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_3}$, respectively. In this subsection, we demonstrate the simplest example, a single M5-brane wrapping $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}$, where $\mathcal{M}_{\epsilon_1,\epsilon_2}$(spanned by $b[z^{n_1}]c[z^{n_2}]$) couples to a $\beta\gamma$ system. The analysis can be straightforwardly extended to bc-ghost VOA.

Feynman diagram I

Recall that there was the gauge anomaly in the 5d CS theory in the presence of the topological line defect. Similarly, the bi-module coupled with $\beta\gamma$-system provides an additional source of the 5d gauge anomaly, since $\beta\gamma$ system has the non-trivial coupling (3.115) with the 5d CS theory and is charged under the 5d gauge symmetry. For the entire 5d-2d-1d coupled system to be anomaly-free, the combined gauge anomaly should be canceled. The bulk anomaly cancellation condition beautifully fixes the action of the algebra on the bi-module.

The simplest example involving the bi-module is akin to the first example of §3.4; notice the similarity between Fig 3.2 and Fig 3.9. As a result, the calculation in this section resembles that of §3.4.

The algebra action on the bi-module, which we want to reproduce from the 5d gauge theory(with $\beta\gamma$-system) calculation, is

$$\left[t[2,1], b[z^1]c[z^0]\right]_{\epsilon_1} = \ldots + \epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0] + \ldots \tag{3.119}$$

Let us make an ansatz for the diagrams that are related to the RHS of (3.119). The diagrams should contain n interaction vertices and $n+1$ internal propagators to produce the factor ϵ_1, and there must be appropriate $\mathcal{I}^A_{1pt}$, $\mathcal{I}^\beta_{1pt}$, and $\mathcal{I}^\gamma_{1pt}$, so that each of 1-point vertex contributes $t[0,0]$, $b[z^0]$, $c[z^0]$, respectively. The answer is:

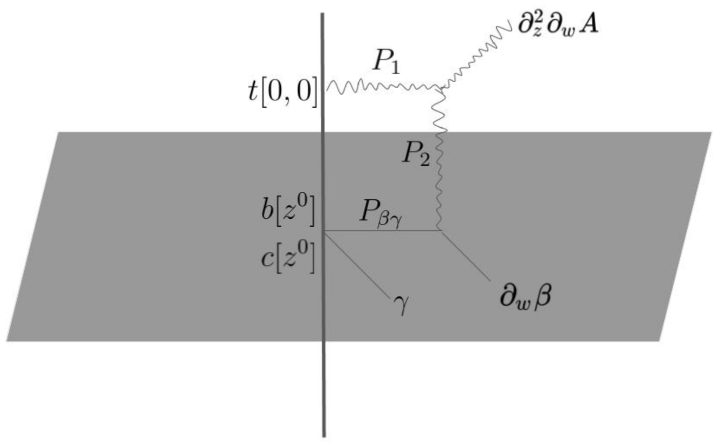


Figure 3.9: Feynman diagrams, related to the RHS of (3.119). The vertical straight lines are the time axis. The gray plane is where $\beta\gamma$-system is living. The internal horizontal straight lines are $\beta\gamma$ propagators and the external slant straight lines are modes of $\beta\gamma$. Note that no $\beta\gamma$ propagates along the time axis. The $\beta A\gamma$ three point vertex is restricted to the $\beta\gamma$-plane, but the AAA three point vertex can be anywhere in the bulk.

We will show that the amplitude for Fig 3.9 is

$$(const)\ \epsilon_1\ \partial_w^2\partial_w A\partial_z\beta\ \gamma\ c[z^0]b[z^0]t[0,0] \tag{3.120}$$

The factor $z_2^2 w_2 \partial_{z_2}^2 \partial_{w_2} A$ is for the external leg attached to the top 3-point vertex, v_2. The factor corresponds to $c[2,1]$. Again, this is an ansatz. We will see that only this integral converges and does not vanish below. We will drop $\partial_{z_2}^2 \partial_{w_2} A$ and recover it later.

We will prove that the constant factor in front of (3.120) is finite only if the external legs are $\partial_z^2 \partial_w A \partial_z \beta\gamma$. For simplicity, we will abbreviate the leg factors during the computation.

First vertex

First, we focus on computing the integral over the first vertex:

$$\int_{v_1} \partial_{z_1} P_1(v_0, v_1) \wedge (w_1 dw_1) \wedge (z_1^2 dz_1) \wedge \partial_{w_1} P_2(v_1, v_2) \tag{3.121}$$

Note that ∂_{z_1} and ∂_{w_1} comes from the three point coupling at v_1:

$$\epsilon_2 A \wedge \partial_{z_1} A \wedge \partial_{w_1} A \tag{3.122}$$

In Lemma 5 in Appendix 3.7, we showed how to evaluate (3.121) and arrive at following expression.

$$-\int_0^1 dx \sqrt{x(1-x)}^7 \int_{v_1} [dV_1] \frac{(|z_1|^2 + x^2|z_2|^2)^2(|w_1|^2 + x^2|w_2|^2)t_2 d\bar{w}_2}{(|z_1|^2+|w_1|^2 + t_1^2 + x(1-x)(|z_2|^2+|w_2|^2 + t_2^2))^7} \tag{3.123}$$

where $[dV_1]$ is an integral measure for v_1 integral. We see from (3.123) that it was necessary to choose $c[m,n]$, β_n to be $c[2,1]$, β_1. Otherwise, the numerator of (3.123) would contain holomorphic or anti-holomorphic dependence on z_1 or w_1, and this makes the z_1 or w_1 integral to vanish.

Also, we can drop terms proportional to $|z_2|^2$, since there is a delta function at the second vertex that evaluates $z_2 = 0$. So, (3.123) simplifies to

$$-\int_0^1 dx \sqrt{x(1-x)}^7 \int_{v_1} [dV_1] \frac{|z_1|^4(|w_1|^2 + x^2|w_2|^2)t_2 d\bar{w}_2}{(|z_1|^2+|w_1|^2 + t_1^2 + x(1-x)(|z_2|^2+|w_2|^2 + t_2^2))^7} \tag{3.124}$$

This is evaluated to

$$\frac{c_1 t_2}{d_{02}^3} + \frac{c_2 t_2 |w_2|^2}{d_{02}^5} \tag{3.125}$$

where c_1 and c_2 are 1-forms of v_2. Let us call them as $\mathcal{P}_{02}^1$ and $\mathcal{P}_{02}^2$ respectively.

Second vertex

Now, compute the second vertex integral, using the above computation:

$$\begin{aligned}&\int_{v_2}(\mathcal{P}^1_{02}+\mathcal{P}^2_{02})\wedge dw_2\frac{1}{w_2}(w_2)\delta(z_2=0,t_2=\epsilon)\\ &=\epsilon_1\int\left(\frac{c_1}{r^5}+\frac{c_2}{r^3}\right)rdrd\theta\\ &=4\pi^4\epsilon_1\left(\frac{1}{43200|\epsilon|}+\frac{1}{57600|\epsilon|^3}\right)\end{aligned}\tag{3.126}$$

We can re-scale ϵ to be 1, so the integral converges. Reinstating Gamma function factors, we finally obtain

$$(const)=\frac{\Gamma(7)}{\Gamma(7/2)\Gamma(7/2)}4\pi^4\left(\frac{1}{43200}+\frac{1}{57600}\right)=\frac{112\pi}{3375}\tag{3.127}$$

Hence, the amplitude for the Feynman diagram is

$$(const)\epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0](\partial_z^2\partial_w A)(\partial_w\beta)\gamma\tag{3.128}$$

Its BRST variation is

$$(const)\epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0](\partial_z^2\partial_w c)(\partial_w\beta)\gamma\tag{3.129}$$

The gauge anomaly (3.129) should be canceled by introducing another diagrams. An obvious choice is the tree level diagrams, where $\partial^2_{z_1}\partial_{z_2}A\partial_{z_2}\beta\gamma$ appears explicitly.

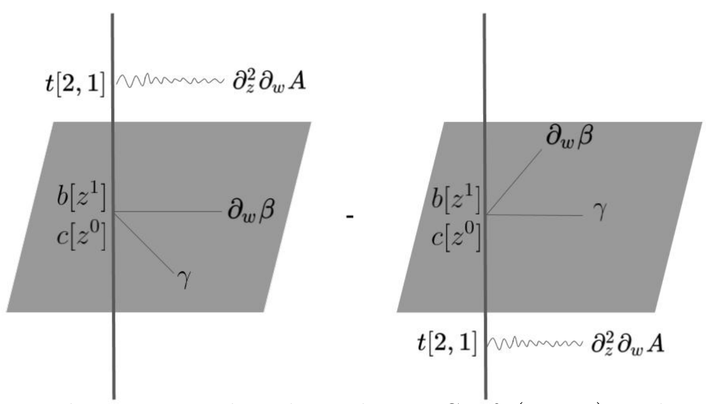


Figure 3.10: Feynman diagrams, related to the LHS of (3.119). The vertical straight lines are time axis, and $\beta\gamma$ lives on the gray planes. $\beta\gamma$ only flows out of the time axis, but not flowing along the time axis. Note that there is no internal propagators of any sort. All types of lines are external legs; they are modes of β, γ, A.

As Fig 3.10 does not involve any loops, we do not need an extra computation. The amplitude is simply

$$\left[t[2,1],b[z^1]c[z^0]\right](\partial_z^2\partial_w A)(\partial_w\beta)\gamma\tag{3.130}$$

and its BRST variation is proportional to

$$\left[t[2,1], b[z^1]c[z^0]\right](\partial_z^2\partial_w c)(\partial_w\beta)\gamma \tag{3.131}$$

By equating (3.129) and (3.131), we get

$$\left[t[2,1], b[z^1]c[z^0]\right] = \epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0] + \dots \tag{3.132}$$

We know from (3.112) that there is one more $\mathcal{O}(\epsilon_1)$ order term $\epsilon_1\epsilon_2 c[z^0]b[z^0]$, which was indicated as $\dots$ in (3.132), in the RHS of

$$[t[2,1], b[z^1]c[z^0]]_{\epsilon_1} \tag{3.133}$$

This indicates that there must be another Feynman diagram, which is proportional to $\partial_z^2\partial_w A\partial_w\beta\gamma$. We will find the Feynman diagram in the next subsection and complete the RHS of (3.133).

Feynman diagram II

We can explain the boxed term in (3.112)

$$\left[t[2,1], b[z^1]c[z^0]\right]_{\epsilon_1} = \dots + \epsilon_1\epsilon_2 b[z^0]c[z^0] + \dots \tag{3.134}$$

using the Feynman diagram below.

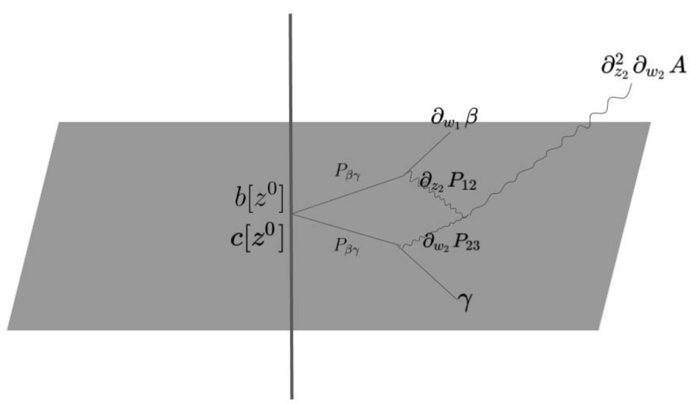


Figure 3.11: A Feynman diagram, related to RHS of (3.134).

The amplitude for the diagram is

$$(const)\epsilon_2\epsilon_1 b[z^0]c[z^0] \tag{3.135}$$

since there are 4 internal propagators(ϵ_1^4) and 3 internal vertices(ϵ_1^{-3}), one of which is $A\partial A\partial A$ type vertex(ϵ_2). We will explicitly show that $(const)$ does not vanish and hence the diagram has non-zero BRST variation, which completes the RHS of (3.133).

First vertex($P_{\beta\gamma}\ \partial_{w_1}\beta\ \partial_{z_2}P_{12}$)

First, we focus on computing the integral over the first vertex:

$$\int_{v_1} \frac{1}{w_1}(w_1 dw_1)\delta(t_1 = 0, z_1 = 0) \wedge \partial_{z_2} P_{12}(v_1, v_2) \tag{3.136}$$

Note that ∂_{w_2} comes from the three point coupling at v_2:

$$\epsilon_2 A \wedge \partial_{z_2} A \wedge \partial_{w_2} A \tag{3.137}$$

This integral evaluates to

$$-\frac{2\pi(t_2 d\bar{z}_2 + \bar{z}_2 dt_2)\bar{z}_2}{5\sqrt{t_2^2+|z_2|^2}^5} \tag{3.138}$$

We presented the details in Lemma 6. in Appendix 3.7.

Third vertex($P_{\beta\gamma}\ \gamma\ \partial_{w_2}P_{23}$)

Second, we focus on computing the integral over the third vertex:

$$\int_{v_3} \frac{1}{w_3}(dw_3)\delta(t_3 = 0, z_3 = 0) \wedge \partial_{w_2} P(v_2, v_3) \tag{3.139}$$

Note that ∂_{w_2} comes from the three point coupling at v_2:

$$\epsilon_2 A \wedge \partial_{z_2} A \wedge \partial_{w_2} A \tag{3.140}$$

This integral evaluates to

$$-(t_2 d\bar{z}_2 - \bar{z}_2 dt_2)\frac{2\pi}{15 w_2^2}\left(\frac{2}{\sqrt{t_2^2+|z_2|^2}^3} - \frac{5|w_2|^2 + 2t_2^2 + 2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+|w_2|^2}^5}\right) \tag{3.141}$$

We presented the details in Lemma 7. in Appendix 3.7.

Second vertex($\partial_{z_2}P_{12}\ \partial_{z_2}^2\partial_{w_2}A\ \partial_{w_2}P_{23}$)

Now, combine (3.138) and (3.141), and compute the second vertex integral; here $z_2^n w_2^m$

denotes the external gauge boson leg.

$$\begin{aligned}
&\int_{v_2} dw_2 \wedge dz_2 \wedge (t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \wedge (t_2 d\bar{z}_2 + \bar{z}_2 dt_2)\bar{z}_2 \\
&\times \frac{4\pi^2 z_2^n w_2^m}{75 w_2^2 \sqrt{t_2^2+|z_2|^2}^5} \left(\frac{2}{\sqrt{t_2^2+|z_2|^2}^3} - \frac{5|w_2|^2 + 2t_2^2 + 2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+|w_2|^2}^5} \right) \\
= &\int_{v_2} dw_2 \wedge dz_2 \wedge d\bar{z}_2 \wedge dt_2 \frac{4\pi^2 t_2 |z_2|^2}{75 w_2 \sqrt{t_2^2+|z_2|^2}^5} \left(\frac{2}{\sqrt{t_2^2+|z_2|^2}^3} - \frac{5|w_2|^2 + 2t_2^2 + 2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+|w_2|^2}^5} \right)
\end{aligned} \tag{3.142}$$

We inserted $(n, m) = (2, 1)$ for the external gauge boson leg. Then, z_2^2 pairs with $\bar{z}_2^2$, and w_2 combines with $1/w_2^2$ to yield $1/w_2$. Since we do not have $d\bar{w}_2$, the integral is holomorphic integral. If (n, m) were other values, the integral will simply vanish.

In Lemma 8. in Appendix 3.7, we show that (3.142) is convergent, and bounded as

$$c_1 < (3.142) < c_2 \tag{3.143}$$

where c_1, c_2 are some finite constants.

Hence, the amplitude for the Feynman diagram is

$$(const)\epsilon_1\epsilon_2 b[z^0]c[z^0](\partial_z^2\partial_w A)(\partial_w\beta)\gamma \tag{3.144}$$

Its BRST variation is therefore non-vanishing:[14]

$$(const)\epsilon_1\epsilon_2 b[z^0]c[z^0](\partial_z^2\partial_w c)(\partial_w\beta)\gamma \tag{3.145}$$

This completes the remaining part of the algebra-bi-module commutation relation (3.133):

$$\left[t[2,1], b[z^1]c[z^0]\right]_{\epsilon_1} = \epsilon_1\epsilon_2 t[0,0]b[z^0]c[z^0] + \epsilon_1\epsilon_2 b[z^0]c[z^0] \tag{3.146}$$

3.6. Appendix: Algebra and bi-module computation

We will prove the key commutation relations for the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and the bi-module $\mathcal{M}_{\epsilon_1,\epsilon_2}$.

Algebra

The simplest algebra commutator that has ϵ_1 correction in the RHS is

$$[t[3,0], t[0,3]] = 9t[2,2] + \frac{3}{2}(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]) \tag{3.147}$$

[14]We hope there is no confusion between the ghost for the 5d gauge field $\partial_z^2\partial_w c$ and the module element $c[z^0]$.

where

$$\sigma_2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2, \quad \sigma_3 = -\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2) \tag{3.148}$$

We will prove (3.147) in this section. The strategy is simple, if we notice that the first term in the RHS comes from one contraction of X and Y. While deriving $9t[2,2]$, we expect the other central terms will follow. For a simple presentation, we will abbreviate "Tr".

$$\left[X^3, Y^3\right] = (X^3)(Y^3) - (Y^3)(X^3) \tag{3.149}$$

Commute X's to the right in X^3Y^3:[15]

$$\begin{aligned}(X^3)(Y^3) &= 3\epsilon_1(X^2Y^2) + X_1^0X_2^1Y_{1'}^{0'}Y_{2'}^{1'}Y_{0'}^{2'}X_0^2 \\ &= 3\epsilon_1(X^2Y^2) + 3\epsilon_1(XYYX) + X_1^0Y_{1'}^{0'}Y_{2'}^{1'}Y_{0'}^{2'}X_2^1X_0^2 \\ &= 3\epsilon_1((X^2Y^2) + (XY^2X) + (Y^2X^2)) + (Y^3)(X^3)\end{aligned} \tag{3.150}$$

So,

$$\begin{aligned}\left[X^3, Y^3\right] &= 3\epsilon_1((X^2Y^2) + (XY^2X) + (Y^2X^2)) \\ &= \frac{3}{2}\epsilon_1((X^2Y^2) + \boxed{(X^2Y^2)} + (XY^2X) + \boxed{(XY^2X)} + (Y^2X^2) + \boxed{(Y^2X^2)})\end{aligned} \tag{3.151}$$

We would like to rearrange the boxed terms to reproduce the underlined terms in the first term of (3.147), which can be re-written as

$$\begin{aligned}9\epsilon_1 STrX^2Y^2 = \frac{9}{6}\epsilon_1\Big((X^2Y^2) &+ \underline{(XYXY)} + (XY^2X) + \underline{(YX^2Y)} \\ &+ (Y^2X^2) + \underline{(YXYX)}\Big)\end{aligned} \tag{3.152}$$

Start from the first box: To reproduce $\underline{(XYXY)}$ from $\boxed{(XXYY)}$, we may swap X and Y in the middle. I will use following F-term relation and commutation relation, same as [GO]:

$$X_b^aY_c^b - X_c^bY_b^a + I_cJ^a = \epsilon_2\delta_c^a, \quad [J^b, I_a] = \epsilon_1\delta_a^b, \quad [X_b^a, Y_d^c] = \epsilon_1\delta_d^a\delta_b^c \tag{3.153}$$

$$\begin{aligned}\boxed{(XXYY)} =& X_1^0(\epsilon_1\delta_0^1\delta_2^3 + Y_0^3X_2^1)Y_3^2 \\ =&\epsilon_1(X)(Y) + X_1^0Y_0^3(Y_2^1X_3^2 + (\epsilon_1N + \epsilon_2)\delta_3^1 - I_3J^1) \\ =&\epsilon_1(X)(Y) + (N\epsilon_1 + \epsilon_2)(XY) - I_3J^1X_1^0Y_0^3 + X_1^0Y_2^1(-\epsilon_1N\delta_0^2 + X_3^2Y_0^3) \\ =&\epsilon_1(X)(Y) + (N\epsilon_1 + \epsilon_2)(XY) - (IXYJ) - (IJ)(IJ) - N\epsilon_1(IJ) \\ &+ (\epsilon_1 + \epsilon_2)(IJ) - \epsilon_1N(XY) + \underline{(XYXY)} \\ =&\epsilon_1(X)(Y) + \epsilon_2(XY) - (IXYJ) - (IJ)(IJ) + (-N\epsilon_1 + \epsilon_1 + \epsilon_2)(IJ) \\ &+ \underline{(XYXY)}\end{aligned} \tag{3.154}$$

[15] Note: 1. When there are sub(super)scripts, they are indices, not powers, 2. (•) denotes a fully contracted word. For example, $(X) = X_i^i$, $(XY) = X_j^iY_i^j$.

The third box: To reproduce $\underline{(YXYX)}$ from $\boxed{(YYXX)}$, we may swap the middle YX.

$$\begin{aligned}
\boxed{(YYXX)} =& Y^0 Y_2^1 X_0^3 X_3^2 = Y_1^0 (X_0^3 Y_2^1 - \epsilon_1 \delta_0^1 \delta_2^3) X_3^2 \\
=& Y_1^0 X_0^3 (-\epsilon_1 N \delta_3^1 + X_3^2 Y_2^1) - \epsilon_1 (Y)(X) \\
=& -\epsilon_1 N (YX) - \epsilon_1 (Y)(X) + Y_1^0 X_0^3 (X_2^1 Y_3^2 + (I_3 J^1 - \epsilon_2 \delta_3^1)) \\
=& -\epsilon_1 N (YX) - \epsilon_1 (Y)(X) + \epsilon_1 N (YX) - \epsilon_1 N (YX) + \epsilon_1 N (YX) \\
& + \underline{(YXYX)} + (IXYJ) - \epsilon_1 N (IJ) - \epsilon_2 (YX) \\
=& -\epsilon_1 (Y)(X) + \underline{(YXYX)} + (IXYJ) - \epsilon_1 N (IJ) - \epsilon_2 (YX)
\end{aligned} \tag{3.155}$$

The second box: To reproduce $\underline{(YXXY)}$ from $\boxed{(XYYX)}$.

$$\begin{aligned}
\boxed{(XYYX)} =& X_1^0 Y_2^1 Y_3^2 X_0^3 = (\delta_3^0 \delta_1^2 \epsilon_1 + Y_3^2 X_1^0) Y_2^1 X_0^3 \\
=& \epsilon_1 (Y)(X) + Y_3^2 X_1^0 (-\epsilon_1 \delta_0^1 \delta_2^3 + X_0^3 Y_2^1) \\
=& \epsilon_1 (Y)(X) - \epsilon_1 (Y)(X) + \underline{(YXXY)} \\
=& \underline{(YXXY)}
\end{aligned} \tag{3.156}$$

Now, as we have reproduced all the desired terms in $t[2,2]$, we can collect (3.154),(3.155),(3.156), plug in to (3.151), and see if terms other than the underlined terms produce the desired central terms.

$$\begin{aligned}
& \left[STrX^3, STrY^3\right] \\
=& \frac{3}{2}\epsilon_1 \left((X^2Y^2) + (XYXY) + (XY^2X) + (YX^2Y) + (Y^2X^2) + (YXYX)\right) \\
& + \frac{3}{2}\epsilon_1 \Bigg(\epsilon_1 (X)(Y) + \epsilon_2 (XY) - (IXYJ) - (IJ)(IJ) + (-N\epsilon_1 + \epsilon_1 + \epsilon_2)(IJ) \\
& - \epsilon_1 (Y)(X) + (IXYJ) - \epsilon_1 N (IJ) - \epsilon_2 (YX) \Bigg) \\
=& 9\epsilon_1 STrX^2Y^2 + \frac{3}{2}\epsilon_1 \big((\epsilon_1 + \epsilon_2)(IJ) - (IJ)(IJ) - 2N\epsilon_1 (IJ) \\
& + \epsilon_1 [(X),(Y)] + \epsilon_2 ((XY) - (YX))\big) \\
=& 9\epsilon_1 STrX^2Y^2 + \frac{3}{2}\epsilon_1 \big((\epsilon_1 + \epsilon_2)(IJ) - (IJ)(IJ) - 2\epsilon_2\epsilon_1 N^2 + N\epsilon_1^2 + \epsilon_2\epsilon_1 N^2\big) \\
=& -9\epsilon_1 STrX^2Y^2 + \frac{3}{2}\epsilon_1 \big((\epsilon_1 + \epsilon_2)(IJ) - (IJ)(IJ) - \epsilon_2\epsilon_1 N^2 + N\epsilon_1^2\big)
\end{aligned} \tag{3.157}$$

where I used following in the last line.

$$\begin{aligned}
(XY) - (YX) &= X_b^a Y_a^b - Y_b^a X_a^b = Y_a^b X_b^a + \epsilon_1 N^2 - Y_b^a X_a^b = \epsilon_1 N^2 \\
[(X),(Y)] &= X_a^a Y_b^b - Y_b^b X_a^a = \epsilon_1 \delta_a^a + Y_b^b X_a^a - Y_b^b X_a^a = \epsilon_1 N
\end{aligned} \tag{3.158}$$

Now, we need to normalize the basis properly, recalling:

$$t_{m,n} = \frac{1}{\epsilon_1} STrX^m Y^n, \quad N = \epsilon_1 t[0,0], \quad (IJ) = t[0,0]\epsilon_1\epsilon_2 \tag{3.159}$$

So, (3.157) becomes

$$\begin{aligned}
[t[3,0], t[0,3]] &= 9t[2,2] + \frac{3}{2}\big((\epsilon_1+\epsilon_2)\frac{(IJ)}{\epsilon_1} - \epsilon_1 \frac{(IJ)}{\epsilon_1}\frac{(IJ)}{\epsilon_1} - \epsilon_2 N^2 + N\epsilon_1\big) \\
&= 9t[2,2] + \frac{3}{2}\big((\epsilon_1+\epsilon_2)\epsilon_2 t[0,0] - \epsilon_1\epsilon_2^2 t[0,0]t[0,0] - \epsilon_1^2\epsilon_2 t[0,0]t[0,0] + \epsilon_1^2 t[0,0]\big) \\
&= 9t[2,2] + \frac{3}{2}\big((\epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2)t[0,0] - \epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)t[0,0]t[0,0]\big) \\
&= 9t[2,2] + \frac{3}{2}\big(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]\big)
\end{aligned} \tag{3.160}$$

where we used (3.148) in the last equality.

Bi-module

The simplest algebra, bi-module commutator that has ϵ_1 correction in the RHS is

$$\begin{aligned}
[T[2,1], b[z]c[1]] =& \left(-\frac{5}{3}\epsilon_2 T[0,1] + \epsilon_2^2 b[1]c[1] \right) \\
&+ \epsilon_1 \left(-\epsilon_2 b[1]c[1]T[0,0] + \frac{4}{3}\epsilon_2 b[1]c[1] \right) \\
&+ \epsilon_1^2 \left(-\frac{4}{3} b[1]c[1]T[0,0] \right) \\
&+ \epsilon_1^3 \left(-\frac{1}{3} b[1]c[1]b[1]c[1] \right)
\end{aligned} \tag{3.161}$$

We will prove it in this section.

Let us expand the LHS.

$$\begin{aligned}
\big[S(X^2Y), (IY\tilde{\varphi})(\varphi J)\big] =& \frac{1}{3}(XXY + XYX + YXX)\cdot(IY\tilde{\varphi})(\varphi J) \\
&- \frac{1}{3}(IY\tilde{\varphi})(\varphi J)\cdot(XXY + XYX + YXX)
\end{aligned} \tag{3.162}$$

Compute the first term:

$$\begin{aligned}
&(XXY)\cdot(IY\tilde{\varphi})(\varphi J)=X^0_1X^1_2|\tilde{\varphi}^b\varphi_c|I_aY^a_bJ^cY^2_0+X^0_1X^1_2|\tilde{\varphi}^b\varphi_c\tilde{\varphi}^2\varphi_0|I_aY^a_bJ^c\\
=&|\tilde{\varphi}^b\varphi_c|I_aX^0_1(\epsilon_1\delta^1_b\delta^a_2+Y^a_bX^1_2)J^cY^2_0+\epsilon_1X^0_1|\tilde{\varphi}^b(\delta^1_c\varphi_0+\delta^1_0\varphi_c)|I_aY^a_bJ^c\\
=&\epsilon_1|\tilde{\varphi}^b\varphi_c|I_2X^0_bJ^cY^2_0+\epsilon_1|\tilde{\varphi}^b\varphi_c|I_a(\epsilon_1\delta^0_b\delta^a_1+Y^a_bX^0_1)X^1_2J^cY^2_0+\epsilon_1|\tilde{\varphi}^b\varphi_0|I_aX^0_cY^a_bJ^c\\
&+\epsilon_1|\tilde{\varphi}^b\varphi_c|I_a(X)Y^a_bJ^c\\
=&\epsilon_1(-\epsilon_1)(IYJ)+\epsilon_1|\tilde{\varphi}^0\varphi_c|I_1J^cX^1_2Y^2_0+\underline{(IY\tilde{\varphi})(\varphi J)(X^2Y)}+(-\epsilon_1)\epsilon_1(IYJ)\\
&+\epsilon_1|\tilde{\varphi}^b\varphi_c|I_a(\epsilon_1\delta^a_b+Y^a_b(X))J^c\\
=&-\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\underline{(IY\tilde{\varphi})(\varphi J)\cdot(XXY)}-\epsilon_1^2\epsilon_2(Y)\\
&+\epsilon_1^2(I\tilde{\varphi})(\varphi J)+\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)\\
=&-2\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\underline{(IY\tilde{\varphi})(\varphi J)\cdot(XXY)}+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)
\end{aligned}\tag{3.163}$$

So,

$$\begin{aligned}
[(XXY),(IY\tilde{\varphi})(\varphi J)]=&-2\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)
\end{aligned}\tag{3.164}$$

Next,

$$\begin{aligned}
&(XYX)\cdot(IY\tilde{\varphi})(\varphi J)=X^0_1Y^1_2|\tilde{\varphi}^b\varphi_c|I_a(\epsilon_1\delta^2_b\delta^a_0+Y^a_bX^2_0)J^c\\
=&\epsilon_1|\tilde{\varphi}^2\varphi_c|I_0X^0_1Y^1_2J^c+\epsilon_1|\tilde{\varphi}^2\varphi_c\tilde{\varphi}^1\varphi_2|I_0X^0_1J^c+|\tilde{\varphi}^b\varphi_c|I_aX^0_1Y^1_2Y^a_bX^2_0J^c\\
&+|\tilde{\varphi}^b\varphi_c\tilde{\varphi}^1\varphi_2|I_aX^0_1Y^a_bX^2_0J^c\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\epsilon_1(-\epsilon_1)((\tilde{\varphi}\varphi)(IJ)+(I\tilde{\varphi})(\varphi J))\\
&+|\tilde{\varphi}^b\varphi_c|I_a(\epsilon_1\delta^0_b\delta^a_1+Y^a_bX^0_1)J^cY^1_2X^2_0+(-\epsilon_1)(|\tilde{\varphi}^b\varphi_2|I_aY^a_bX^2_0J^0+|\tilde{\varphi}^b\varphi_c|I_aY^a_bJ^c(X))\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)+\epsilon_1|\tilde{\varphi}^0\varphi_c|I_1J^cY^1_2X^2_0\\
&+\underline{(IY\tilde{\varphi})(\varphi J)(XYX)}-\epsilon_1|\tilde{\varphi}^b\varphi_2|I_a(-\epsilon_1\delta^a_0\delta^2_b+X^2_0Y^a_b)J^0-\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)+\epsilon_1|\tilde{\varphi}^0\varphi_c|I_1J^c(-\epsilon_1N\delta^1_0+X^2_0Y^1_2)\\
&+\underline{(IY\tilde{\varphi})(\varphi J)(XYX)}+\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)-\epsilon_1(-\epsilon_1)(IYJ)\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2N(I\tilde{\varphi})(\varphi J)-\epsilon_1^2(IYJ)+\epsilon_1^2(IYJ)\\
&+\underline{(IY\tilde{\varphi})(\varphi J)(XYX)}\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2N(I\tilde{\varphi})(\varphi J)+\underline{(IY\tilde{\varphi})(\varphi J)(XYX)}
\end{aligned}\tag{3.165}$$

So,

$$[(XYX),(IY\tilde{\varphi})(\varphi J)]=\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2N(I\tilde{\varphi})(\varphi J)\tag{3.166}$$

Next,

$$\begin{aligned}
&(YXX)\cdot(IY\tilde{\varphi})(\varphi J) = Y_1^0|\tilde{\varphi}^b\varphi_c|I_aX_2^1(\epsilon_1\delta_b^2\delta_0^a+Y_b^aX_0^2)J^c\\
=&\epsilon_1Y_1^0|\tilde{\varphi}^2\varphi_c|I_0X_2^1J^c+Y_1^0|\tilde{\varphi}^b\varphi_c|I_a(\epsilon_1\delta_b^1\delta_2^a+Y_b^aX_2^1)X_0^2J^c\\
=&\epsilon_1(-\epsilon_1)(IYJ)+\epsilon_1Y_1^0|\tilde{\varphi}^1\varphi_c|I_aX_0^aJ^c+|\tilde{\varphi}^b\varphi_c\tilde{\varphi}^0\varphi_1|I_aY_b^aX_2^1X_0^2J^c+\underline{(IY\tilde{\varphi})(\varphi J)(YXX)}\\
=&-\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\epsilon_1(-N\epsilon_1)(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1|\tilde{\varphi}^1\varphi_c\varphi^0\varphi_1|I_aX_0^aJ^c+|\tilde{\varphi}^b\varphi_c\varphi^0\varphi_1|I_a(-\epsilon_1\delta_2^a\delta_b^1+X_2^1Y_b^a)X_0^2J^c+\underline{(IY\tilde{\varphi})(\varphi J)(YXX)}\\
=&-\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)-N\epsilon_1^2(I\tilde{\varphi})(\varphi J)+\epsilon_1(-\epsilon_1)(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1(-\epsilon_1)(\tilde{\varphi}\varphi)(IJ)-\epsilon_1|\tilde{\varphi}^1\varphi_c\tilde{\varphi}^0\varphi_1|I_2X_0^2J^c\\
&+(-\epsilon_1)(|\tilde{\varphi}^b\varphi_c|I_aY_b^aJ^c(X)+|\tilde{\varphi}^0\varphi_c|I_aY_2^aX_0^2J^c)+\underline{(IY\tilde{\varphi})(\varphi J)(YXX)}\\
=&-\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)-N\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)\\
&-\epsilon_1(-\epsilon_1)(I\tilde{\varphi})(\varphi J)-\epsilon_1(-\epsilon_1)(\tilde{\varphi}\varphi)(IJ)-\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)\\
&-\epsilon_1(-\epsilon_1N)(I\tilde{\varphi})(\varphi J)-\epsilon_1(-\epsilon_1)(IYJ)+\underline{(IY\tilde{\varphi})(\varphi J)(YXX)}\\
=&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)\\
&+\underline{(IY\tilde{\varphi})(\varphi J)(YXX)}
\end{aligned} \tag{3.167}$$

So,

$$\begin{aligned}
[(YXX),(IY\tilde{\varphi})(\varphi J)] =&\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\
&-\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)
\end{aligned} \tag{3.168}$$

Collecting above, we have

$$\begin{aligned}
\Big[S(X^2Y),(IY\tilde{\varphi})(\varphi J)\Big] = \frac{1}{3}\Big(&-2\epsilon_1^2\epsilon_2(Y)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)+\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2N(I\tilde{\varphi})(\varphi J)\\
&+\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\epsilon_1(IY\tilde{\varphi})(\varphi J)(X)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)-\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)\Big)\\
&=\epsilon_1(IXY\tilde{\varphi})(\varphi J)-\frac{2}{3}\epsilon_1^2\epsilon_2(Y)-\frac{1}{3}\epsilon_1^2N(I\tilde{\varphi})(\varphi J)-\frac{1}{3}\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)\\
&+\frac{1}{3}\epsilon_1^2(I\tilde{\varphi})(\varphi J)
\end{aligned} \tag{3.169}$$

We are not done yet, since $(IXY\tilde{\varphi})(\varphi J)$ is reducible by the F-term relation.

$$\begin{aligned}\epsilon_1|\tilde{\varphi}^0\varphi_c|I_1J^cX_2^1Y_0^2 =&\epsilon_1|\tilde{\varphi}^0\varphi_c|I_1J^c(X_0^2Y_2^1-(I_0J^1-\epsilon_2\delta_0^1))\\ =&\epsilon_1(-\epsilon_1)(IYJ)-\epsilon_1|\tilde{\varphi}^0\varphi_c|(J^cI_1-\epsilon_1\delta_1^c)I_0J^1+\epsilon_1\epsilon_2(I\tilde{\varphi})(\varphi J)\\ =&-\epsilon_1^2(IYJ)-\epsilon_1|\tilde{\varphi}^0\varphi_c|(I_0J^c+\epsilon_1\delta_0^c)I_1J^1+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\ &+\epsilon_1\epsilon_2(I\tilde{\varphi})(\varphi J)\\ =&-\epsilon_1^2(IYJ)-\epsilon_1(I\tilde{\varphi})(\varphi J)(IJ)-\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\ &+\epsilon_1\epsilon_2(I\tilde{\varphi})(\varphi J)\end{aligned} \tag{3.170}$$

Plugging this into (3.169), we get

$$\begin{aligned}\left[S(X^2Y),(IY\tilde{\varphi})(\varphi J)\right]=&(-\epsilon_1^2(IYJ)-\epsilon_1(I\tilde{\varphi})(\varphi J)(IJ)-\epsilon_1^2(\tilde{\varphi}\varphi)(IJ)+\epsilon_1^2(I\tilde{\varphi})(\varphi J)\\ &+\epsilon_1\epsilon_2(I\tilde{\varphi})(\varphi J))-\frac{2}{3}\epsilon_1^2\epsilon_2(Y)-\frac{1}{3}\epsilon_1^2(\tilde{\varphi}\varphi)(I\tilde{\varphi})(\varphi J)\\ &-\frac{1}{3}\epsilon_1^2N(I\tilde{\varphi})(\varphi J)+\frac{1}{3}\epsilon_1^2(I\tilde{\varphi})(\varphi J)\end{aligned} \tag{3.171}$$

After normalization, by multiplying $\frac{\epsilon_2}{\epsilon_1^3}$ both sides, and using the identity[16]

$$(\tilde{\varphi}\varphi)\epsilon_2=(I\tilde{\varphi})(\varphi J) \tag{3.173}$$

we have

$$\begin{aligned}[T[2,1],b[z]c[1]]=&\left(-\frac{5}{3}\epsilon_2T[0,1]+\epsilon_2^2b[1]c[1]\right)\\ &+\epsilon_1\left(-\epsilon_2b[1]c[1]T[0,0]+\frac{4}{3}\epsilon_2b[1]c[1]\right)\\ &+\epsilon_1^2\left(-\frac{4}{3}b[1]c[1]T[0,0]\right)\\ &+\epsilon_1^3\left(-\frac{1}{3}b[1]c[1]b[1]c[1]\right)\end{aligned} \tag{3.174}$$

[16]The identity can be derived using the F-term relation:

$$\begin{aligned}\tilde{\varphi}^i\left([X,Y]_i^j+I_iJ^j-\epsilon_2\delta_i^j\right)\varphi_j=0\\ (Y)-(Y)+(I\tilde{\varphi})(\varphi J)-\epsilon_2(\tilde{\varphi}\varphi)=0\\ (I\tilde{\varphi})(\varphi J)=\epsilon_2(\tilde{\varphi}\varphi)\end{aligned} \tag{3.172}$$

3.7. Appendix: Intermediate steps in Feynman diagram calculations

Intermediate steps in section 4.2

Lemma 1.
We will compute the following integral.

$$\epsilon_1\epsilon_2^2\int_{v_1} dw_1\wedge dz_1\wedge\partial_{z_1}P_1(v_0,v_1)\wedge\partial_{z_2}\partial_{w_1}P_2(v_1,v_2)(z_1^2w_1\partial_{z_1}^2\partial_{w_1}A) \tag{3.175}$$

Computing the partial derivatives, we can re-write it as

$$\epsilon_1\epsilon_2^2\left(\frac{\bar{z}_1}{d_{01}^2}\frac{\bar{w}_1}{d_{12}^4}(w_1z_1\bar{z}_2)\right)[P(v_0,v_1)\wedge dw_1\wedge z_1dz_1\wedge P(v_1,v_2)] \tag{3.176}$$

We see

$$\begin{aligned}P(v_0,v_1)\wedge P(v_1,v_2)=\frac{d\bar{z}_1d\bar{w}_1dt_1}{d_{01}^5d_{12}^5}&(\bar{z}_{01}\bar{w}_{12}dt_2-\bar{z}_{01}t_{12}d\bar{w}_2+\bar{w}_{01}t_{12}d\bar{z}_2\\&-\bar{w}_{01}\bar{z}_{12}dt_2+t_{01}\bar{z}_{12}d\bar{w}_2-t_{01}\bar{w}_{12}d\bar{z}_2)\end{aligned} \tag{3.177}$$

Including $\wedge dw_1\wedge(z_1dz_1)\wedge$, we can simplify it:

$$\begin{aligned}&P(v_0,v_1)\wedge P(v_1,v_2)\wedge(w_1dw_1)\wedge(z_1dz_1)=d\bar{z}_1dz_1dw_1d\bar{w}_1dt_1\left(|z_1|^2|w_1|^2\bar{z}_2\right)\times\\&\left[\partial_{\bar{z}_0}\left(\frac{\bar{z}_{01}\bar{w}_{12}dt_2-\bar{z}_{01}t_{12}d\bar{w}_2+\bar{w}_{01}t_{12}d\bar{z}_2-\bar{w}_{01}\bar{z}_{12}dt_2+t_{01}\bar{z}_{12}d\bar{w}_2-t_{01}\bar{w}_{12}d\bar{z}_{12}}{d_{01}^5d_{12}^9}\right)\right.\\&\left.-\frac{\partial_{\bar{z}_0}(\bar{z}_{01}\bar{w}_{12}dt_2-\bar{z}_{01}t_{12}d\bar{w}_2+\bar{w}_{01}t_{12}d\bar{z}_2-\bar{w}_{01}\bar{z}_{12}dt_2+t_{01}\bar{z}_{12}d\bar{w}_2-t_{01}\bar{w}_{12}d\bar{z}_{12})}{d_{01}^5d_{12}^9}\right]\end{aligned} \tag{3.178}$$

By integration by parts, the the integral over t_1, z_1, $\bar{z}_1$, w_1, $\bar{w}_1$ of all the terms in the first two lines vanishes.

So we are left with

$$-\int_{v_1} dt_1dz_1d\bar{z}_1dw_1d\bar{w}_1\frac{|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2-t_{12}d\bar{w}_2)}{d_{01}^5d_{12}^9} \tag{3.179}$$

Lemma 2.

We can use Feynman integral technique to convert (3.179) to the following:

$$\begin{aligned}&\int_{v_1}\int_0^1 dx\frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)}\frac{\sqrt{x^3(1-x)^7}|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2-t_{12}d\bar{w}_2)}{((1-x)(|z_1|^2+|w_1|^2+t_1^2)+x(|z_{12}|^2+|w_{12}|^2+t_{12}^2))^7}\\&=\int_{v_1}\int_0^1 dx\frac{(\Gamma\text{ factors})\sqrt{x^3(1-x)^7}|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2-t_{12}d\bar{w}_2)}{(|z_1-xz_2|^2+|w_1-xw_2|^2+(t_1-xt_2)^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\end{aligned} \tag{3.180}$$

Shift the integral variables as

$$z_1 \to z_1 + xz_2, \quad w_1 \to w_1 + xw_2, \quad t_1 \to t_1 + xt_2 \tag{3.181}$$

Then the above becomes

$$\begin{aligned}\int_{v_1}\int_0^1 dx \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} & \frac{\sqrt{x^3(1-x)^7}|z_1+xz_2|^2|w_1+xw_2|^2\bar{z}_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7} \\ & \times((\bar{w}_1+(x-1)\bar{w}_2)dt_2-(t_1+(x-1)t_2)d\bar{w}_2)\end{aligned} \tag{3.182}$$

Drop terms with odd number of t_1 and terms that has holomorphic or anti-holomorphic dependence on z_1 or w_1:

$$\int_{v_1}\int_0^1 dx \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} \frac{\sqrt{x^3(1-x)^9}(|z_1|^2+x^2|z_2|^2)(|w_1|^2+x^2|w_2|^2)\bar{z}_2(\bar{w}_2dt_2-t_2d\bar{w}_2)}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7} \tag{3.183}$$

After doing the v_1 integral using Mathematica with the integral measure $dt_1dz_1d\bar{z}_1dz_2d\bar{z}_2$, we get

$$\bar{z}_2(\bar{w}_2dt_2-t_2d\bar{w}_2)\left(\frac{c_1}{d_{02}^5}+\frac{c_2w_2^2}{d_{02}^7}+\frac{c_3z_2^2}{d_{02}^7}+\frac{c_4z_2^2w_2^2}{d_{02}^9}\right) \tag{3.184}$$

Lemma 3.
We will compute the integral over the second vertex.

$$\begin{aligned}&\int_{v_2}\mathcal{P}\wedge\partial_{w_2}P_3(v_2,v_3)\wedge dz_2\wedge dw_2(z_2w_2^2\partial_{z_2}\partial_{w_2}^2A) \\ =&\int_{v_2}\mathcal{P}\wedge\frac{\bar{w}_2(\bar{z}_{23}d\bar{w}_2dt_2-\bar{w}_{23}d\bar{z}_2dt_2+t_{23}d\bar{z}_2d\bar{w}_2)}{d_{23}^7}\wedge dw_2\wedge dz_2\end{aligned} \tag{3.185}$$

Now, compute the integrand:

$$\begin{aligned}&\frac{\bar{z}_2(\bar{w}_2dt_2-t_2d\bar{w}_2)\bar{w}_2(\bar{z}_{23}d\bar{w}_2dt_2-\bar{w}_{23}d\bar{z}_2dt_2+t_{23}d\bar{z}_2d\bar{w}_2)}{d_{02}^5d_{23}^7}\wedge dw_2\wedge dz_2 \\ =&\frac{|z_2|^2|w_2|^4(t_2-t_3-t_2)}{d_{02}^5d_{23}^7}dt_2d\bar{z}_2d\bar{w}_2dw_2dz_2 \\ =&-\frac{|z_2|^2|w_2|^4t_3}{d_{02}^5d_{23}^7}dt_2d\bar{z}_2d\bar{w}_2dw_2dz_2 \quad \text{substitute } t_3=\epsilon, \text{ then,} \\ =&-\frac{|z_2|^2|w_2|^4\epsilon}{d_{02}^5d_{23}^7}dt_2d\bar{z}_2d\bar{w}_2dw_2dz_2\end{aligned} \tag{3.186}$$

We can rescale $\epsilon \to 1$, without loss of generality, then it becomes

$$-\frac{|z_2|^2|w_2|^4}{d_{02}^5d_{23}^7}dt_2d\bar{z}_2d\bar{w}_2dw_2dz_2 \tag{3.187}$$

Lemma 4.
Now, it remains to evaluate the delta function at the third vertex. In other words, substitute:

$$w_3 \to 0, \quad z_3 \to 0, \quad t_3 \to \epsilon = 1 \tag{3.188}$$

Then, use Feynman technique to convert the above integral into

$$\begin{aligned}
&-\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5}|z_2|^2|w_2|^4}{(x(z_2^2+w_2^2+(t_2-1)^2)+(1-x)(z_2^2+w_2^2+t_2^2))^6} \\
&= -\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5}|z_2|^2|w_2|^4}{(z_2^2+w_2^2+(t_2-x)^2+x(1-x))^6} \\
&= -\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5}|z_2|^2|w_2|^4}{(z_2^2+w_2^2+t_2^2+x(1-x))^6}
\end{aligned} \tag{3.189}$$

In the second equality, we shifted t_2 to $t_2 + x$.

After doing v_2 integral, it reduces into

$$\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\pi}{2880}\int_0^1 dx x(1-x)^2 = \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\pi}{2880} \tag{3.190}$$

Finally, re-introduce all the omitted constants:

$$(FirstTerm) = \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)}(2\pi)^2(2\pi)^2\frac{\pi}{2880} \tag{3.191}$$

Similarly, we can compute all the others without any divergence.

$$\begin{aligned}
(\text{Second Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)}(2\pi)^2(2\pi)^2\frac{\pi}{5760} \\
(\text{Third Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)}(2\pi)^2(2\pi)^2\frac{\pi}{8640} \\
(\text{Fourth Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)}\frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)}(2\pi)^2(2\pi)^2\frac{\pi}{20160}
\end{aligned} \tag{3.192}$$

Hence, every terms in (3.184) are integrated into finite terms.

Intermediate steps in section 5.2

Lemma 5.
We want to evaluate the following integral.

$$\int_{v_1} \partial_{z_1} P_1(v_0, v_1) \wedge (w_1 dw_1) \wedge (z_1^2 dz_1) \wedge \partial_{w_1} P_2(v_1, v_2) \tag{3.193}$$

Substituting the expressions for propagators, we get

$$\begin{aligned}\int_{v_1} \frac{|z_1|^2 z_1 w_1(\bar{w}_1-\bar{w}_2)}{d_{01}^7 d_{12}^7}&(\bar{z}_{01}\bar{w}_{12}dt_2-\bar{z}_{01}t_{12}d\bar{w}_2+\bar{w}_{01}t_{12}d\bar{z}_2-\bar{w}_{01}\bar{z}_{12}dt_2\\&+t_{01}\bar{z}_{12}d\bar{w}_2-t_{01}\bar{w}_{12}d\bar{z}_2)d\bar{z}_1 d\bar{w}_1 dt_1 dz_1 dw_1\end{aligned} \tag{3.194}$$

We already know that the terms proportional to $\bar{w}_2$ will vanish in the second vertex integral, so drop them. Evaluating the delta function at v_0, the above simplifies to

$$\begin{aligned}\int_{v_1} \frac{|z_1|^2 z_1 |w_1|^2}{d_{01}^7 d_{12}^7}&(-\bar{z}_1\bar{w}_{12}dt_2+\bar{z}_1 t_{12}d\bar{w}_2-\bar{w}_1 t_{12}d\bar{z}_2+\bar{w}_1\bar{z}_{12}dt_2\\&-t_1\bar{z}_{12}d\bar{w}_2+t_1\bar{w}_{12}d\bar{z}_2)d\bar{z}_1 d\bar{w}_1 dt_1 dz_1 dw_1\end{aligned} \tag{3.195}$$

Note that the integrand with the odd number of t_1 vanishes, so

$$\int_{v_1} \frac{|z_1|^2 z_1 |w_1|^2}{d_{01}^7 d_{12}^7}(-\bar{z}_1\bar{w}_{12}dt_2-\bar{z}_1 t_2 d\bar{w}_2+\bar{w}_1 t_2 d\bar{z}_2+\bar{w}_1\bar{z}_{12}dt_2)d\bar{z}_1 d\bar{w}_1 dt_1 dz_1 dw_1 \tag{3.196}$$

Now, apply Feynman technique, and omit the Gamma functions, to be recovered at the end.

$$\begin{aligned}&\int_0^1 dx\sqrt{x(1-x)}^7\int_{v_1}\frac{|z_1|^2|w_1|^2 z_1(-\bar{z}_1\bar{w}_{12}dt_2-\bar{z}_1 t_2 d\bar{w}_2+\bar{w}_1 t_2 d\bar{z}_2+\bar{w}_1\bar{z}_{12}dt_2)}{(x(|z_1|^2+|w_1|^2+|t_1|^2)+(1-x)(|z_{12}|^2+|w_{12}|^2+|t_{12}|^2))^7}\\=&\int_0^1 dx\int_{v_1}\frac{\sqrt{x(1-x)}^7|z_1|^2|w_1|^2 z_1(-\bar{z}_1\bar{w}_{12}dt_2-\bar{z}_1 t_2 d\bar{w}_2+\bar{w}_1 t_2 d\bar{z}_2+\bar{w}_1\bar{z}_{12}dt_2)}{(|z_1-xz_2|^2+|w_1-xw_2|^2+(t_1-xt_2)^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\end{aligned} \tag{3.197}$$

Shift the integral variables as

$$z_1\to z_1+xz_2,\quad w_1\to w_1+xw_2,\quad t_1\to t_1+xt_2 \tag{3.198}$$

Then the above becomes

$$\begin{aligned}\int_0^1 dx\sqrt{x(1-x)}^7\int_{v_1}&dz_1 d\bar{z}_1 dw_1 d\bar{w}_1 dt_1(|z_1|^2+x^2|z_2|^2)(|w_1|^2+x^2|w_2|^2)(z_1+xz_2)\\&\Bigg(\frac{-(\bar{z}_1+x\bar{z}_2)(\bar{w}_1+(x-1)\bar{w}_2)dt_2-(\bar{z}_1+x\bar{z}_2)t_2 d\bar{w}_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\\&+\frac{(\bar{w}_1+x\bar{w}_2)t_2 d\bar{z}_2+(\bar{w}_1+x\bar{w}_2)(\bar{z}_1+(x-1)\bar{z}_2)dt_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\Bigg)\end{aligned} \tag{3.199}$$

The terms with (anti)holomorphic dependence on complex coordinates drop:

$$\begin{aligned}\int_0^1 dx\sqrt{x(1-x)}^7\int_{v_1}&dz_1 d\bar{z}_1 dw_1 d\bar{w}_1 dt_1(|z_1|^2+x^2|z_2|^2)(|w_1|^2+x^2|w_2|^2)\\&\Bigg(\frac{-|z_1|^2 t_2 d\bar{w}_2+x|z_1|^2\bar{w}_2 dt_2-x^2|z_2|^2(x-1)\bar{w}_2 dt_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\\&+\frac{-x^2|z_2|^2 t_2 d\bar{w}_2+x^2 z_2\bar{w}_2 t_2 d\bar{z}_2+x^2|z_2|^2\bar{w}_2(x-1)dt_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\Bigg)\end{aligned} \tag{3.200}$$

We can be prescient again; using the fact that the second vertex is tagged with a delta function $\delta(z_2 = 0, t_2 = \epsilon) \propto dz_2 d\bar{z}_2 dt_2$, we can drop most of the terms.

$$\begin{aligned}
&-\int_0^1 dx\sqrt{x(1-x)}^7 \int_{v_1} [dV_1] \frac{(|z_1|^2 + x^2|z_2|^2)(|w_1|^2 + x^2|w_2|^2)(-|z_1|^2 - x^2|z_2|^2)t_2 d\bar{w}_2}{(|z_1|^2+|w_1|^2 + t_1^2 + x(1-x)(|z_2|^2+|w_2|^2 + t_2^2))^7} \\
&= -\int_0^1 dx\sqrt{x(1-x)}^7 \int_{v_1} [dV_1] \frac{(|z_1|^2 + x^2|z_2|^2)^2(|w_1|^2 + x^2|w_2|^2)t_2 d\bar{w}_2}{(|z_1|^2+|w_1|^2 + t_1^2 + x(1-x)(|z_2|^2+|w_2|^2 + t_2^2))^7}
\end{aligned} \tag{3.201}$$

where $[dV_1]$ is an integral measure for v_1 integral.

Intermediate steps in section 5.3

Lemma 6.
We will evaluate the following integral.

$$\int_{v_1} \frac{1}{w_1}(w_1 dw_1)\delta(t_1 = 0, z_1 = 0) \wedge \partial_{z_2} P_{12}(v_1, v_2) \tag{3.202}$$

Substituting the expressions for propagators, we get

$$\begin{aligned}
&\int_{v_1} \frac{\bar{z}_1 - \bar{z}_2}{d_{12}^7}(\bar{z}_{12} d\bar{w}_{12} dt_{12} - \bar{w}_{12} d\bar{z}_{12} dt_{12} + t_{12} d\bar{z}_{12} d\bar{w}_{12}) dw_1 \delta(t_1 = z_1 = 0) \\
&= \int_{v_1} \frac{\bar{z}_1 - \bar{z}_2}{d_{12}^7}(\bar{z}_2 d\bar{w}_1 dt_2 + t_2 d\bar{z}_2 d\bar{w}_1) dw_1 \delta(t_1 = z_1 = 0) \\
&= (t_2 d\bar{z}_2 + \bar{z}_2 dt_2) \int_{v_1} \frac{\bar{z}_1 - \bar{z}_2}{\sqrt{t_{12}^2+|z_{12}|^2+|w_{12}|^2}^7} d\bar{w}_1 dw_1 \delta(t_1 = z_1 = 0) \\
&= (t_2 d\bar{z}_2 + \bar{z}_2 dt_2) \int dw_1 d\bar{w}_1 \frac{-\bar{z}_2}{\sqrt{t_2^2+|z_2|^2+|w_1 - w_2|^2}^7} \\
&= -(t_2 d\bar{z}_2 + \bar{z}_2 dt_2) \int r dr d\theta \frac{\bar{z}_2}{\sqrt{t_2^2+|z_2|^2 + r^2}^7} = -\frac{2\pi(t_2 d\bar{z}_2 + \bar{z}_2 dt_2)\bar{z}_2}{5\sqrt{t_2^2+|z_2|^2}^5}
\end{aligned} \tag{3.203}$$

where the first equality comes from the fact that $\delta(t_1 = z_1 = 0) \propto dt_1 dz_1 d\bar{z}_1$.
Lemma 7.
We will evaluate the following integral.

$$\int_{v_3} \frac{1}{w_3}(dw_3)\delta(t_3 = 0, z_3 = 0) \wedge \partial_{w_2} P(v_2, v_3) \tag{3.204}$$

Substituting the expressions for propagators, we get

$$
\begin{aligned}
&\int_{v_3} \frac{\bar{w}_2 - \bar{w}_3}{w_3 d_{23}^7}(\bar{z}_{23} d\bar{w}_{23} dt_{23} - \bar{w}_{23} d\bar{z}_{23} dt_{23} + t_{23} d\bar{z}_{23} d\bar{w}_{23}) dw_3 \delta(t_3 = z_3 = 0) \\
=&\int_{v_3} \frac{\bar{w}_2 - \bar{w}_3}{w_3 d_{23}^7}(-\bar{z}_2 d\bar{w}_3 dt_2 + t_2 d\bar{z}_2 d\bar{w}_3) dw_3 \delta(t_3 = z_3 = 0) \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int_{v_3} \frac{\bar{w}_2 - \bar{w}_3}{w_3 \sqrt{t_{23}^2 + |z_{23}|^2 + |w_{23}|^2}^7} d\bar{w}_3 dw_3 \delta(t_3 = z_3 = 0) \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int dw_3 d\bar{w}_3 \frac{(\bar{w}_2 - \bar{w}_3)/w_3}{\sqrt{t_2^2 + |z_2|^2 + |w_2 - w_3|^2}^7} \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int dw_3 d\bar{w}_3 \frac{-\bar{w}_3/(w_3 + w_2)}{\sqrt{t_2^2 + |z_2|^2 + |w_3|^2}^7} \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int_{|w_3| \le |w_2|} dw_3 d\bar{w}_3 \frac{-\bar{w}_3 \left(1 - \frac{w_3}{w_2} + \frac{1}{2!}\frac{w_3^2}{w_2^2} - \dots\right)}{w_2 \sqrt{t_2^2 + |z_2|^2 + |w_3|^2}^7} \\
&+ (t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int_{|w_3| \ge |w_2|} dw_3 d\bar{w}_3 \frac{-\bar{w}_3 \left(1 - \frac{w_2}{w_3} + \frac{1}{2!}\frac{w_2^2}{w_3^2} - \dots\right)}{w_3 \sqrt{t_2^2 + |z_2|^2 + |w_3|^2}^7} \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int_{|w_3| \le |w_2|} dw_3 d\bar{w}_3 \left(0 + \frac{-|w_3|^2}{w_2^2 \sqrt{t_2^2 + |z_2|^2 + |w_3|^2}^7} + 0 + 0 + \dots\right) \\
=&(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \int_0^{|w_2|} r dr d\theta \frac{-r^2}{w_2^2 \sqrt{t_2^2 + |z_2|^2 + r^2}^7} \\
=&-(t_2 d\bar{z}_2 - \bar{z}_2 dt_2) \frac{2\pi}{15 w_2^2} \left(\frac{2}{\sqrt{t_2^2 + |z_2|^2}^3} - \frac{5|w_2|^2 + 2t_2^2 + 2|z_2|^2}{\sqrt{t_2^2 + |z_2|^2 + |w_2|^2}^5}\right)
\end{aligned}
\tag{3.205}
$$

Lemma 8.
We will evaluate

$$
\int_{v_2} dw_2 \wedge dz_2 \wedge d\bar{z}_2 \wedge dt_2 \frac{4\pi^2 t_2 |z_2|^2}{75 w_2 \sqrt{t_2^2 + |z_2|^2}^5} \left(\frac{2}{\sqrt{t_2^2 + |z_2|^2}^3} - \frac{5|w_2|^2 + 2t_2^2 + 2|z_2|^2}{\sqrt{t_2^2 + |z_2|^2 + |w_2|^2}^5}\right). \tag{3.206}
$$

Assuming the w_2 integral domain is a contour surrounding the origin of w_2 plane or a path that can be deformed into the contour, we may use the residue theorem for the first term of (3.206). After doing w_2 integral we have

$$
\int_\epsilon^\infty dt_2 \int_{\mathbb{C}_{z_2}} d^2 z_2 \frac{4\pi^2 t_2 |z_2|^2}{75 \sqrt{t_2^2 + |z_2|^2}^5} \frac{2}{\sqrt{t_2^2 + |z_2|^2}^3} = \frac{2\pi^3}{225 \epsilon^2} \tag{3.207}
$$

Combining with the other diagram with the second vertex in the $t \in [-\infty, -\epsilon]$, we get

$$\frac{2\pi^3}{225\epsilon^2} - \left(-\frac{2\pi^3}{225\epsilon^2}\right) = \frac{4\pi^3}{225\epsilon^2} \tag{3.208}$$

Re-scaling $\epsilon \to 1$, this is finite.

For the second term of (3.206), let us choose the contour to be a constant radius circle so that $r(\theta) = R$. We need to use an unconventional version of the residue theorem, as the integrand is not a holomorphic function, depending on $|w_2|^2$. Let $w_2 = Re^{i\theta}$, then for a given integrand $f(w_2, \bar{w}_2)$, we have

$$I = \int_0^{2\pi} d(Re^{i\theta}) f(Re^{i\theta}, Re^{-i\theta}) \tag{3.209}$$

Then, w_2 integral is evaluated as

$$-\int_0^{2\pi} \frac{d(Re^{i\theta})}{Re^{i\theta}} \frac{4\pi^2 t_2|z_2|^2}{75\sqrt{t_2^2+|z_2|^2}^5} \frac{5R^2+2t_2^2+2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+R^2}^5} = -\frac{8\pi^3 i t_2|z_2|^2}{75\sqrt{t_2^2+|z_2|^2}^5} \frac{5R^2+2t_2^2+2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+R^2}^5} \tag{3.210}$$

Before evaluating z_2 integral, it is better to work without R. using the following inequality is useful to facilitate an easier integral:

$$0 < \frac{8\pi^3 i t_2|z_2|^2}{75\sqrt{t_2^2+|z_2|^2}^5} \left(\frac{5R^2+2t_2^2+2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+R^2}^5}\right) < \frac{(8\pi^3 i t_2|z_2|^2)(2t_2^2+2|z_2|^2)}{75(t_2^2+|z_2|^2)^5} \tag{3.211}$$

Here we used $R \in Real^+$. The left bound is obtained by $R \to \infty$, and the right bound is obtained by $R \to 0$. We only care the convergence of the integral. So, let us proceed with the inequalities.

$$-\frac{4\pi}{192}\frac{8\pi^3 i}{75}\frac{1}{\epsilon^3} < -\int_\epsilon^\infty dt_2 \int_{\mathbb{C}_{z_2}} d^2 z_2 \frac{8\pi^3 i t_2|z_2|^2}{75\sqrt{t_2^2+|z_2|^2}^5} \left(\frac{5R^2+2t_2^2+2|z_2|^2}{\sqrt{t_2^2+|z_2|^2+R^2}^5}\right) < 0 \tag{3.212}$$

After rescaling $\epsilon \to 1$, we have a finite answer. Combining with the other diagram with the second vertex in the $t \in [-\infty, -\epsilon]$, we get the left bound as

$$-\frac{4\pi}{192}\frac{8\pi^3 i}{75} - \left(\frac{4\pi}{192}\frac{8\pi^3 i}{75}\right) = -\frac{\pi^4 i}{225\epsilon^3} \tag{3.213}$$

After rescaling $\epsilon_1 \to 1$, this is also finite.

Hence, combining with (3.208), we get the bound

$$\frac{4\pi^3}{225\epsilon^2} - \frac{\pi^4 i}{225\epsilon^3} < (3.206) < \frac{4\pi^3}{225\epsilon^2} \tag{3.214}$$

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