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UNIVERSITY OF CALIFORNIA,
IRVINE

Geometric Curve Flows

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Hsiao-Fan Liu

Dissertation Committee:
Professor Chuu-Lian Terng, Chair
Professor Peter Li
Professor Zhiqin Lu

2014

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DEDICATION

To my parents...

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CURRICULUM VITAE

Hsiao-Fan Liu

EDUCATION

Doctor of Philosophy in Mathematics **2014**
University of California, Irvine *Irvine, California*

Master of Science in Applied Mathematics **2007**
National Chiao Tung University *Hsinchu, Taiwan*

Bachelor of Science in Mathematics **2004**
National Taiwan Normal University *Taiwan, Taiwan*

ABSTRACT OF THE DISSERTATION

Geometric Curve Flows

By

Hsiao-Fan Liu

Doctor of Philosophy in Mathematics

University of California, Irvine, 2014

Professor Chuu-Lian Terng, Chair

We study geometric curves flows whose invariants flow according to some soliton equations. We discuss the correspondences between the Schödinger flows on Hermitian symmetric spaces and equations of the nonlinear Schrödinger(NLS) type. And we use these correspondences to construct Bäcklund transformations for these curve flows. We also study the geometric Airy curve flows on space forms whose invariants satisfy the vector modified KdV(vmKdV) type equations. The existence of solutions to the Cauchy problems of curve flows for periodic boundary conditions follows from the correspondence. We then obtain geometric algorithms to solve periodic Cauchy problems numerically.

Introduction

A geometric curve flow is called *integrable* if the local invariants satisfy some soliton equations. The most famous example is the vortex filament equation (VFE), i.e., the following curve flow in $\mathbb{R}^3$:

$$\gamma_t = \gamma_x \times \gamma_{xx}, \tag{1}$$

modeled by Da Rios for a self-induced motion of vortex lines in an incompressible fluid. It can be checked that $\frac{\partial}{\partial t}(\|\gamma_x\|^2) = 0$. So we may assume $\|\gamma_x(x,t)\| = 1$ for all t, i.e., $\gamma(\cdot, t)$ is parametrized by its arc-length. Hence the VFE can be rewritten as

$$\gamma_t = k\vec{n}$$

under the Frenet frame, where k is the curvature and $\vec{n}$ is the bi-normal. Hasimoto showed in [4] that if γ is a solution of the VFE, then there exists a function $\theta : \mathbb{R} \to \mathbb{R}$

such that

$$q(x,t) = k(x,t)e^{i(\theta(t)+\int_0^x \tau(s,t)ds)} \tag{2}$$

is a solution of the non-linear Schrödinger equation (NLS)

$$q_t = i(q_{xx} + 2|q|^2 q), \tag{3}$$

where $\tau(\cdot,t)$ is the torsion for $\gamma(\cdot,t)$ and x is the arc-length parameter. For example, $\gamma(x,t) = (\cos x, \sin x, t)$ is a solution of the VFE with $k(x,t) = 1$, $\tau(x,t) = 0$, and $q(x,t) = e^{2it}$ is the corresponding solution of the NLS. Due to this transform, VFE is regarded as a completely integrable curve flow.

The NLS is a soliton equation. In other words, it has properties shared by other integrable equations: (i) the NLS has infinitely many commuting conservation laws, (ii) there are infinitely many families of explicit soliton solutions, (iii) the Cauchy problem with rapidly decaying initial data can be solved by the inverse scattering method or by group factorization method (cf. [17]). Therefore, VFE has similar properties.

One interesting modification for NLS equation is the derivative nonlinear Schrödinger equation (DNLS) [2]:

$$q_t = -\frac{i}{2}(q_{xx} + (|q|^2 q)_x). \tag{4}$$

The difference between the NLS and the DNLS is the nonlinear term and it will lead to some new properties. Also, the DNLS is highly connected to Hermitian symmetric space [2]. It turns out to be a better and more precise equation in simulating the deep water waves, such as waves in the ocean.

Terng and Uhlenbeck in [15] first proposed the Schrödinger flow

$$\gamma_t = [\gamma, \gamma_{xx}]$$

on the complex Grassmannian manifold. Later Terng and Thorbergsson generalized in [12] to classical Hermitian symmetric spaces. It has a Lax pair that is gauge equivalent to the Lax pair of the NLS-type equations associated to each Hermitian symmetric space. One can construct solutions to the Schrödinger flow from solutions to the NLS-type equation and vice versa, so that the Schrödinger flow on a Hermitian symmetric space corresponds to the NLS-type equation.

Langer and Perline considered in [7] the following curve flow

$$\gamma_t = -(\gamma_{xxx} + \frac{3}{2}||\gamma_{xx}||^2\gamma_x). \tag{5}$$

They proved that (5) preserves arc-length (so we may assume $||\gamma_x(x,t)|| = 1$), and there exists an orthonormal frame $(e_1, \ldots, e_n)$ along γ with $e_1 = \gamma_x$ and $(e_2, \ldots, e_n)$ a parallel normal frame for the normal bundle of the curve $\gamma(\cdot, t)$ at each time-level t, such that $u = (k_2, \ldots, k_n)$ satisfies the vector modified KdV equation (vmKdV)

$$u_t = -\left(u_{xxx} + \frac{3}{2}\|u\|^2 u_x\right), \tag{6}$$

where $k_2, \ldots, k_n$ are the principal curvatures. Under the parallel frame, (5) can be written as

$$\gamma_t = -\frac{1}{2}\sum_{i=1}^{n-1} k_i^2 e_0 - \sum_{i=1}^{n-1}(k_i)_x e_i,$$

or equivalently,

$$\gamma_t = -(\frac{1}{2}||H||^2 e_0 + \nabla_{e_0}^{\perp} H),$$

where ∇ is the Levi-Civita connection and H is the mean curvature vector. When it comes to change the tangential coefficient, it means reparametrizations of the flow. Therefore, we consider the *geometric Airy curve flow*

$$\gamma_t = -\nabla^{\perp}_{e_0} H,$$

which is geometric as the curve velocity γ_t can be expressed as geometric quantities of γ.

In this thesis(joint work with Dr. Chuu-Lian Terng), we will discuss Schrödinger flows for the Hermitian symmetric spaces and the Airy curve flows on the space forms. We construct Bäcklund transformations for theses curve flows via the correspondences between geometric curve flows and soliton equations and use these transformations to construct recursively infinitely many families of explicit solutions for these flows. Moreover, we solve the periodic Cauchy problems for the Schrödinger flow on 2-sphere, the VFE, and the geometric Airy curve flow on $\mathbb{R}^2$. This provides a geometric approach to obtain numerical solutions for these flows using MatLab. The advantage of this geometric algorithm is that we reduce the curve PDE to one soliton equation and ODE systems.

The paper is organized as follows: In Section 1, we review three hierarchies that give soliton equations including NLS, derivative NLS and vmKdV. In Section 2, we will discuss the Schrödinger flow on Hermitian symmetric spaces. The Airy curve flows on space forms will be discussed in Section 3. In Section 4, we construct Bäcklund transformation for the curve flow solutions and several examples are given. Section 5 is dedicated to solve periodic Cauchy problems of the Schrödinger flow on 2-sphere, the VFE, and the geometric Airy curve flow on $\mathbb{R}^2$ for periodic boundary conditions. In the last section, we explain the geometric algorithms and demonstrate numerical experiments, including error estimates and the behaviors of numerical solutions starting with different initial curves.

Chapter 1

Soliton Hierarchies from Splittings of Lie Algebra

1.1 Soliton Hierarchy constructed from $\mathcal{L}_{\pm}^{\tau}(\mathcal{G})$ and $\{a\lambda^j \mid j \geq 1\}$

In this section, we review the construction of soliton hierarchies from splittings of loop algebras given in [15].

Let G be a complex simple Lie group and τ an involution on G. Suppose the induced involution $d\tau_e$ (still denoted by τ), the differential of τ at the identity on $\mathcal{G}$, is conjugate linear, i.e., $\tau(z\xi) = \bar{z}\tau(\xi)$ for all $\xi \in \mathcal{G}$ and $z \in \mathbb{C}$. Let U denote the fixed point set of τ and $\mathcal{U}$ the corresponding Lie sub algebra, i.e., the fixed point set of $d\tau_e$. Such $\mathcal{U}$ is a *real form* of $\mathcal{G}$.

Let $L(G) = C^{\infty}(\mathbb{S}^1, G)$, $L_+(G)$ the subgroup of $f \in L(G)$ such that f can be extended holomorphically to $\{ \mid \lambda \mid \ \leq 1\}$, and $L_-(G)$ the subgroup of $f \in L(G)$ that can be extended holomorphically to $\{\infty \geq \mid \lambda \mid > 1\}$ and having value e at the infinity.

The Lie algebras written in terms of Fourier series are

$$
\begin{cases}
\mathcal{L}(\mathcal{G}) = \{\xi(\lambda) = \Sigma_i \xi_i \lambda^i \mid \xi_i \in \mathcal{G}\}, \\
\mathcal{L}_+(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \xi(\lambda) = \Sigma_{i \geq 0} \xi_i \lambda^i\}, \\
\mathcal{L}_-(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \xi(\lambda) = \Sigma_{i < 0} \xi_i \lambda^i\}.
\end{cases} \tag{1.1}
$$

Note that $\mathcal{L}(\mathcal{G}) = \mathcal{L}_+(\mathcal{G}) \oplus \mathcal{L}_-(\mathcal{G})$ and $L_+(G) \cap L_-(G) = \{e\}$. Such $(L_+(G), L_-(G))$ is called *a splitting*.

Theorem 1.1.1 (Birkhoff Factorization Theorem [9])**.** *The multiplication maps*

$$\mu_1 : L_+(G) \times L_-(G) \to L(G), \quad \mu_2 : L_-(G) \times L_+(G) \to L(G),$$

defined by $\mu_1(f_+, f_-) = f_+ f_-, \mu_2(f_-, f_+) = f_- f_+$ *respectively are injective and* $\mathrm{Im}(\mu_1), \mathrm{Im}(\mu_2)$ *are open dense subsets of* $L(G)$.

Definition 1.1.2. Let $\mathcal{U}$ be the real form defined by $\tau : G \to G$. We say that an element $f(\lambda)$ in $L(G)$ satisfies the *U-reality condition* if

$$\tau(f(\bar{\lambda})) = f(\lambda), \tag{1.2}$$

Let $L^\tau(G) = \{f \in L(G) \mid \tau(f(\bar{\lambda})) = f(\lambda)\}$, and $L^\tau_\pm(G) = L^\tau(G) \cap L_\pm(G)$.

In addition, the Lie algebras are

$$
\begin{cases}
\mathcal{L}^\tau(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \tau(\xi(\bar{\lambda})) = \xi(\lambda)\}, \\
\mathcal{L}^\tau_+(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}^\tau(\mathcal{G}) \mid \xi(\lambda) = \Sigma_{i \geq 0} \xi_i \lambda^i\}, \\
\mathcal{L}^\tau_-(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}^\tau(\mathcal{G}) \mid \xi(\lambda) = \Sigma_{i < 0} \xi_i \lambda^i\}.
\end{cases} \tag{1.3}
$$

Note that $\xi(\lambda) = \sum_i \xi_i \lambda^i$. Then $\xi \in L^\tau(G) \iff \xi_i \in \mathcal{U} \quad \forall\, i$.

Definition 1.1.3. The rational elements in $L^\tau_-(G)$ with the minimal number of poles are called the *simple element*, which we use later to construct Bäcklund transformation.

Let $a \in \mathcal{U}$ be regular, $\mathcal{U}_a = \{y \in \mathcal{U} \mid [y, a] = 0\}$ the centralizer of a in $\mathcal{U}$. We will construct a family of evolution equations for $u \in C^\infty(\mathbb{R}, \mathcal{U}_a^\perp)$ from the splitting of $L^\tau_\pm(G)$ of $L^\tau(G)$ and $\{a\lambda^j \mid j \geq 1\}$.

Theorem 1.1.4 ([11], [13])**.** *Given $u \in C^\infty(\mathbb{R}, \mathcal{U}_a^\perp)$, then*

1. *there exists unique $Q(u, \lambda) = a\lambda + \sum_{i=0}^\infty Q_{-i}(u)\lambda^{-i} \in L^\tau(\mathcal{G})$ satisfying*

$$\begin{cases} [\partial_x + a\lambda + u, Q(u, \lambda)] = 0, \\ Q(u, \lambda) \text{ is conjugate to } a\lambda. \end{cases} \tag{1.4}$$

2. *Recursive formula: $(Q_{-j}(u))_x + [u, Q_{-j}(u)] = [Q_{-j-1}(u), a]$.*

3.

$$u_{t_j} = (Q_{-(j-1)}(u))_x + [u, Q_{-(j-1)}(u)] = [Q_{-j}(u), a], \tag{1.5}$$

gives a flow on $C^\infty(\mathbb{R}, \mathcal{U}_a^\perp)$ and the equation (1.5) is called the j-th flow.

Proof. Let $Q = a\lambda + \sum_{i=0}^\infty Q_{-i}\lambda^{-i}$ be a power series of λ. From the first equation in (1.4), we have $Q_x + [a\lambda + u, Q] = 0$. Compare coefficients of λ^j of this equation to get

$$(Q_{-j})_x + [u, Q_{-j}] = [Q_{-(j+1)}, a]. \tag{1.6}$$

For $j = 0$, we get $[a, Q_0] + [u, a] = 0$. Write $Q_0 = T_0 + P_0 \in \mathcal{U}_a \oplus \mathcal{U}_a^\perp$. Note that $\mathrm{ad}(a) : \mathcal{U}_a^\perp \to \mathcal{U}_a^\perp$ is a linear isomorphism and $\mathcal{U}_a^\perp = [a, \mathcal{U}]$. Then $[a, Q_0] + [u, a] = 0$ gives $P_0 = -\mathrm{ad}(a)^{-1}([a, u])$. So P_0 is a polynomial of u. Let $p(x)$ be the minimal polynomial of a and we write

$$p(x) = x^d + c_1 x^{d-1} + \cdots + c_{d-1} x + c_d.$$

Since $Q(u, \lambda)$ is conjugate to $a\lambda$, $p(Q(u, \lambda)\lambda^{-1}) = 0$, i.e.,

$$p(a + Q_0 \lambda^{-1} + Q_{-1} \lambda^{-2} + \cdots) = 0. \tag{1.7}$$

Compare the coefficient of λ^j, we can obtain a formula for the $\mathcal{U}_a$ components of Q. First, the coefficient of λ^{d-1} gives $\sum_{k=0}^{d-1} a^{(d-1)-k} Q_0 a^k = 0$. Since T_0 commutes with a, we have $d a^{d-1} T_0 = 0$. So $T_0 = 0$ and hence Q_0 is a polynomial of u.

We prove Q_{-j} is a polynomial differential operator in u by induction. Suppose Q_{-i} is a polynomial differential operator in u for $i \leq j$. Write

$$Q_{-i} = P_{-i} + T_{-i} \in \mathcal{U}_a^\perp + \mathcal{U}_a.$$

Let π and $\pi^\perp$ be the projection to $\mathcal{U}_a$ and $\mathcal{U}_a^\perp$, respectively. (1.6) shows that

$$(P_{-j})_x + \pi^\perp([u, Q_{-j}]) = [P_{-(j+1)}, a],$$

that is, $P_{-(j+1)} = -\mathrm{ad}(a)^{-1}((P_{-j})_x + \pi^\perp([u, Q_{-j}]))$, i.e., $P_{-(j+1)}$ is a polynomial in u and its x-derivatives.

Note that $p'(a\lambda)$ is invertible and $T_{-(j+1)}$ commutes with a. Compare the coefficient of λ^j in (1.7), we get that $T_{-(j+1)}$ can be written in terms of $a, Q_0, Q_{-1}, \cdots, Q_{-j}$.

This proves that $Q_{-(j+1)}$ is a polynomial of u and its x derivatives. □

The second condition of (1.4) can be replaced by $p(Q(u,\lambda)) = 0$, where p is the minimal polynomial of $a\lambda$. Hence we can replace system (1.4) by

$$\begin{cases} [\partial_x + a\lambda + u, Q(u,\lambda)] = 0, \\ p(Q(u,\lambda)) = 0, \end{cases} \tag{1.8}$$

where p is the minimal polynomial of $a\lambda$.

It follows from (1.6) that we have the following:

Theorem 1.1.5. *The following statements are equivalent for $u \in C^\infty(\mathbb{R}^2, \mathcal{U}_a^\perp)$:*

1. *u is a solution of* (1.5).

2. *$\theta_\lambda = (a\lambda + u)dx + (a\lambda^j + u\lambda^{j-1} + \cdots + Q_{-j}(u))dt$ is a flat connection on the (x,t) plane for all complex parameter λ.*

3. *$[\frac{\partial}{\partial x} + (a\lambda + u), \frac{\partial}{\partial t} + (a\lambda^j + u\lambda^{j-1} + \cdots + Q_{-j}(u))] = 0, \quad \forall \lambda \in \mathbb{C}$.*

4. $$\begin{cases} E_x = E(a\lambda + u) \\ E_t = E(a\lambda^j + u\lambda^{j-1} + \cdots + Q_{-j}(u)) \end{cases}$$

   is solvable. We call $E(x,t,\lambda)$ an extended frame if $\tau(E(x,t,\bar{\lambda})) = E(x,t,\lambda)$, and (2) *or* (3) *the Lax pair of the j-th flow* (1.5).

Next, we give some known examples.

Example 1.1.6. The $SU(2)$-hierarchy defined by $a = \mathrm{diag}(i, -i)$

Let $\tau(g) = (\bar{g}^t)^{-1}$ be an involution on $G = SL(2, \mathbb{C})$. Then $SU(2)$ is the fixed point of τ. Let $a = \mathrm{diag}(i, -i) \in \mathfrak{su}(2)$. Note that

$$\mathcal{U}_a = \left\{ \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}, \mathcal{U}_a^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \,\middle|\, q \in \mathbb{C} \right\}.$$

We can solve $Q = a\lambda + Q_0 + Q_{-1}\lambda^{-1} + \cdots$ from (1.5), i.e., solve

$$\begin{cases} [\partial_x + a\lambda + u, Q] = 0, \\ Q^2 = (a\lambda)^2 = -\lambda^2 I, \end{cases}$$

to get

$$Q_0 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix},$$

$$Q_{-1} = \frac{i}{2} \begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix},$$

$$Q_{-2} = \frac{1}{4} \begin{pmatrix} q_x\bar{q} - q\bar{q}_x & -q_{xx} - 2|q|^2 q \\ \bar{q}_{xx} + 2|q|^2 q & q\bar{q}_x - q_x\bar{q} \end{pmatrix}.$$

So the first three flows in the $SU(2)$-hierarchy are

$$q_t = q_x,$$

$$q_t = \tfrac{i}{2}(q_{xx} + 2 \mid q \mid {}^2 q),$$

$$q_t = -\tfrac{1}{4}(q_{xxx} + 6 \mid q \mid {}^2 q_x).$$

The second flow is the nonlinear Schrödinger equation(NLS). This is the NLS-hierarchy.

Next we describe the NLS-type of hierarchy associated to each compact irreducible

Hermitian symmetric spaces given in [3]. Let G be a simple complex Lie group, and τ the involution that gives the maximal compact subgroup U. It is known that there exists $a \in \mathcal{U}$ such that $\text{ad}(a)^2|_{\mathcal{U}_a^\perp} = -\text{Id}_{\mathcal{U}_a^\perp}$, where $\mathcal{U} = \mathcal{U}_a \oplus \mathcal{U}_a^\perp$.Then the Adjoint U-orbit at a in $\mathcal{U}$ is diffeomorphic to $\frac{U}{U_a}$ and is a compact irreducible Hermitian symmetric space.

Below is the list of the element a for each classical Hermitian symmetric space:

1. $\text{Gr}(k, \mathbb{C}^n) = \frac{U(n)}{U(k)\times U(n-k)} = U(n)\cdot a$, where $a = \frac{i}{2}\begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$, and the involution that gives the Hermitian symmetric space is $\sigma(g) = aga^{-1} = I_{k,n-k}gI_{k,n-k}^{-1}$, where $I_{k,n-k} = \text{diag}(I_k, -I_{n-k})$.
2. $\text{Gr}(2, \mathbb{R}^{n+2}) = \frac{SO(n+2)}{S(O(2)\times O(n))} = SO(n+2)\cdot a$, where $a = e_{21} - e_{12}$, and the involution that gives the Hermitian symmetric space is $\sigma(g) = I_{2,n+2}gI_{2,n+2}^{-1}$.
3. $\frac{SO(2n)}{U(n)} = SO(2n)\cdot a$, where $a = \frac{1}{2}\begin{pmatrix} 0 & -\text{I}_n \\ \text{I}_n & 0 \end{pmatrix}$, and the involution that gives the Hermitian symmetric space is $\sigma(g) = aga^{-1}$.
4. $\frac{Sp(n)}{U(n)} = Sp(n)\cdot a$, where $a = \frac{1}{2}\begin{pmatrix} 0 & -\text{I}_n \\ \text{I}_n & 0 \end{pmatrix}$, and the involution that gives the Hermitian symmetric space is $\sigma(g) = \bar{g}$.

Note that the minimal polynomial for a in cases (1), (3), and (4) are $p(x) = x^2 + \frac{1}{4}$ and for case (2) is $p(x) = x^3 + x$.

We consider the hierarchy constructed from the splitting $\mathcal{L}_\pm^\tau(\mathcal{G})$ of $\mathcal{L}^\tau(\mathcal{G})$ and the sequence $\{a\lambda^j \mid j \geq 1\}$. A direct computation implies that

$$Q_{-1}(u) = [a, u_x] - \frac{1}{2}[u, [a, u]], \tag{1.9}$$

where $Q(u) = a\lambda + u + \sum_{j\geq 1} Q_{-j}\lambda^{-j}$ is the solution of (1.8). So the first two flows in the hierarchy constructed from $L_{\pm}^{\tau}(G)$ and $\{a\lambda^j \mid j \geq 1\}$ are

$$u_t = u_x, \tag{1.10}$$

$$u_t = [a, u_{xx}] - \frac{1}{2}[u, [u, [a, u]]], \tag{1.11}$$

where $u : \mathbb{R}^2 \to \mathcal{U}_a^{\perp}$. This is the $\frac{U}{K}$-NLS heirarchy constructed in [3]. In particular, we have the Lax pair.

Proposition 1.1.7 ([3])**.** Given $u \in C^{\infty}(\mathbb{R}^2, \mathcal{U}_a^{\perp})$, let θ_λ denote the following $\mathcal{G}_{\mathbb{C}}$-valued connection 1-form on the (x,t)-plane with complex parameter λ:

$$\theta_\lambda = (a\lambda + u)\mathrm{d}x + (a\lambda^2 + u\lambda + Q_{-1}(u))\mathrm{d}t, \tag{1.12}$$

where

$$Q_{-1}(u) = [a, u_x] - \frac{1}{2}[u, [a, u]].$$

Then the following statements are equivalent for $u \in C^{\infty}(\mathbb{R}^2, \mathcal{U}_a^{\perp})$:

1. u is a solution of (1.11),
2. θ_λ is flat for all $\lambda \in \mathbb{C}$,
3. $\theta_0 = u\mathrm{d}x + Q_{-1}(u)\mathrm{d}t$ is flat,

We call such θ_λ a Lax pair for the solution u of(1.11).

Remark 1.1.8. Note that the Lax pair θ_λ (1.12) of the solution u of the $\frac{U}{K}$-NLS satisfies the linear equation-reality condition, i.e.,

1. $\theta_{\bar{\lambda}}^* + \theta_\lambda = 0$ when $G = U(n)$.

2. $\theta_\lambda^t + \theta_\lambda = 0$, $\overline{\theta_{\bar\lambda}} = \theta_\lambda$ when $G = O(n)$.

3. $\theta_\lambda^t J_n + J_n\theta_\lambda = 0$, $\theta_{\bar\lambda}^* + \theta_\lambda = 0$ when $G = Sp(n)$, where $J_n = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}$.

Remark 1.1.9. Let u be a solution of the $\frac{U}{K}$-NLS, and $C_0 : \mathbb{C} \to GL(n, \mathbb{C})$ is holomorphic and satisfy the U-reality condition, then there exists a unique $E(x, t, \lambda)$ satisfying $E^{-1}dE = \theta_\lambda$ and $E(0, 0, \lambda) = C_0(\lambda)$.

Example 1.1.10 ($\frac{U(n)}{U(k)\times U(n-k)}$-NLS)**.** Let $\tau : GL(n, \mathbb{C}) \to GL(n, \mathbb{C}), \tau(g) = (\bar g^t)^{-1}, \mathcal{U} = \mathfrak{u}(n)$, and $a = \mathrm{diag}(\frac{i}{2} I_k, -\frac{i}{2} I_{n-k})$. So,

$$\mathcal{K} = \mathcal{U}_a = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \;\middle|\; X_1 \in \mathfrak{u}(k), X_2 \in \mathfrak{u}(n-k) \right\},$$
$$\mathcal{P} = \mathcal{U}_a^\perp = \left\{ \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} \;\middle|\; X \in M_{k\times(n-k)} \right\}.$$

We solve (1.8) for

$$u = Q_0 = \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}, \quad X \in M_{k\times(n-k)}$$

to get

$$Q_{-1}(u) = \begin{pmatrix} -iXX^* & iX_x \\ iX_x^* & iX^*X \end{pmatrix}.$$

The hierarchy constructed from the splitting $\mathcal{L}_\pm^\tau(\mathfrak{gl}(n, \mathbb{C}))$ of $\mathcal{L}^\tau(\mathfrak{gl}(n, \mathbb{C}))$ and $\{a\lambda^j \mid j \geq 1\}$ is

$$X_{t_1} = X_x,$$
$$X_{t_2} = iX_{xx} + 2iXX^*X,$$
$$X_{t_3} = -X_{xxx} - 3X_xX^*X - 3XX^*X_x.$$

Example 1.1.11 ($\frac{O(n+2)}{O(2)\times O(n)}$-NLS)**.** Let $\tau : SO(n+2,\mathbb{C}) \to SO(n+2,\mathbb{C}), \tau(g) = \bar{g}$, and $a = \mathrm{diag}(J_1, O_n)$, where $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So,

$$\mathcal{K} = \mathcal{U}_a = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(2), X_2 \in \mathfrak{so}(n) \right\},$$

$$\mathcal{P} = \mathcal{U}_a^{\perp} = \left\{ \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ -X^t & -Y^t & 0 \end{pmatrix} \mid X = (x_1, \cdots, x_n), Y = (y_1, \cdots, y_n) \right\}.$$

We solve (1.8) for

$$u = Q_0 = \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ -X^t & -Y^t & 0 \end{pmatrix}, \text{ where } X = (x_1, \cdots, x_n), Y = (y_1, \cdots, y_n)$$

and get

$$Q_{-1}(u) = \begin{pmatrix} 0 & \frac{1}{2}(\sum_i x_i^2 + y_i^2) & -Y_x \\ -\frac{1}{2}(\sum_i x_i^2 + y_i^2) & 0 & X_x \\ -Y_x^t & X_x^t & 0 \end{pmatrix}.$$

The hierarchy constructed from the splitting $\mathcal{L}_{\pm}^{\tau}(\mathfrak{so}(n+2,\mathbb{C}))$ of $\mathcal{L}^{\tau}(\mathfrak{so}(n+2,\mathbb{C}))$ and

$\{a\lambda^j \mid j \geq 1\}$ is

$$\begin{cases} X_{t_1} = X_x, \\ Y_{t_1} = Y_x, \\ X_{t_2} = -Y_{xx} + (X \cdot Y)X - \frac{1}{2}(3X \cdot X + Y \cdot Y)Y, \\ Y_{t_2} = X_{xx} + \frac{1}{2}(X \cdot X + 3Y \cdot Y)X - (X \cdot Y)Y, \end{cases}$$

$$\begin{cases} X_{t_3} = -X_{xxx} - \frac{3}{2}(X \cdot X + Y \cdot Y)X_x + (X \cdot X_x + 2X \cdot Y_x)Y - (X \cdot X_x + 3Y \cdot Y_x)X, \\ Y_{t_3} = -Y_{xxx} - \frac{3}{2}(X \cdot X + Y \cdot Y)Y_x + (2X \cdot X_x + X \cdot Y_x)X - (3X_x \cdot X + Y \cdot Y_x)Y. \end{cases}$$

Example 1.1.12 ($\frac{O(2n)}{U(n)}$-NLS)**.** Let $\tau : O(2n, \mathbb{C}) \to O(2n, \mathbb{C}), \tau(g) = \bar{g}, \mathcal{U} = \mathfrak{so}(2n)$, and

$$a = \frac{1}{2} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

So,

$$\mathcal{K} = \mathcal{U}_a = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X \in \mathfrak{so}(n), Y = Y^t \right\},$$
$$\mathcal{P} = \mathcal{U}_a^{\perp} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X, Y \in \mathfrak{so}(n) \right\}.$$

We solve (1.8) for

$$u = Q_0(u) = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, \quad X, Y \in \mathfrak{so}(n),$$

and get

$$Q_{-1}(u) = \begin{pmatrix} -Y_x + XY - YX & X_x - (X^2 + Y^2) \\ X_x + (X^2 + Y^2) & Y_x + XY - YX \end{pmatrix}.$$

The hierarchy constructed from the splitting $\mathcal{L}^\tau_\pm(\mathfrak{so}(2n,\mathbb{C}))$ of $\mathcal{L}^\tau(\mathfrak{so}(2n,\mathbb{C}))$ and $\{a\lambda^j \mid j \geq 1\}$ is

$$\begin{cases} X_{t_1} = X_x, \\ Y_{t_1} = Y_x, \end{cases}$$

$$\begin{cases} X_{t_2} = (-Y)_{xx} + [X,[X,Y]] + 2Y^3 + YX^2 + X^2Y, \\ Y_{t_2} = X_{xx} + [Y,[X,Y]] - 2X^3 - XY^2 - Y^2X. \end{cases}$$

$$\begin{cases} X_{t_3} = E_x + XE - YF^t - EX - FY, \\ Y_{t_3} = F_x + XF + YG - EY + FX, \end{cases}$$

where

$$\begin{aligned} E &= 2([X,Y] - Y_x)Y - 2(X_x - (X^2+Y^2))X - (X_x - (X^2+Y^2))_x, \\ F^t &= 2Y(-X_x - (X^2+Y^2)) - 2X([X,Y] - Y_x) - ([X,Y] - Y_x)_x, \\ F &= ([X,Y] - Y_x)_x - 2([X,Y] - Y_x)X - 2(X_x - (X^2+Y^2))Y, \\ G &= (X_x - (X^2+Y^2))_x + 2X(X_x - (X^2+Y^2)) + 2Y([X,Y] + Y_x). \end{aligned}$$

Example 1.1.13 ($\frac{SP(n)}{U(n)}$-NLS)**.** Let $\tau : SP(n,\mathbb{C}) \to SP(n,\mathbb{C}), \tau(g) = (\bar{g}^t)^{-1}$, and

$$a = \frac{1}{2}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

So,

$$\mathcal{K} = \mathcal{U}_a = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X \in \mathfrak{o}(n), Y = Y^t \right\},$$
$$\mathcal{P} = \mathcal{U}_a^\perp = \left\{ i\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X = X^t, Y = Y^t \text{ are real} \right\}.$$

We solve (1.8) for

$$u = Q_0(u) = i \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, \quad X = X^t, Y = Y^t \text{ real}$$

to get

$$Q_{-1}(u) = \begin{pmatrix} -iY_x - XY + YX & iX_x + (X^2 + Y^2) \\ iX_x - (X^2 + Y^2) & iY_x - XY + YX \end{pmatrix}.$$

The hierarchy constructed from the splitting $\mathcal{L}^\tau_\pm(\mathfrak{sp}(n, \mathbb{C}))$ of $\mathcal{L}^\tau(\mathfrak{sp}(n, \mathbb{C}))$ and $\{a\lambda^j \mid j \geq 1\}$ is

$$\begin{cases} X_{t_1} = X_x, \\ Y_{t_1} = Y_x, \end{cases}$$

$$\begin{cases} X_{t_2} = (-Y)_{xx} - [X, [X, Y]] - 2Y^3 - YX^2 - X^2 Y, \\ Y_{t_2} = X_{xx} - [Y, [X, Y]] + 2X^3 + XY^2 + Y^2 X, \end{cases}$$

$$\begin{cases} X_{t_3} = -iE_x + [X, E] + YG - FY, \\ Y_{t_3} = -iF_x + XF - YE^t - EY + FX, \end{cases}$$

where

$$\begin{aligned}
E &= -(iX_x + (X^2 + Y^2))_x + 2i([Y, X] - iY_x)Y - 2i((X^2 + Y^2) + iX_x)X, \\
E^t &= -(iX_x + (X^2 + Y^2))_x - 2iY([Y, X] + iY_x) - 2i(iX_x + (X^2 + Y^2))X, \\
F &= ([Y, X] - iY_x)_x - 2i([Y, X] - iY_x)X - 2i(iX_x + (X^2 + Y^2))Y, \\
G &= ([Y, X] - iY_x)_x + 2iX([Y, X] - iY_x) + 2iY(iX_x - (X^2 + Y^2)).
\end{aligned}$$

1.2 Soliton Hierarchy constructed from $\mathcal{L}_{\pm}^{\tau,\sigma}(\mathcal{G})$ and $\{a\lambda^{2j} \mid j \geq 1\}$

Let $\tau : G \to G$ be an involution that gives the real form $\mathcal{U}$, and $\sigma : G \to G$ another involution such that $d\sigma_e$ is complex linear and $\sigma\tau = \tau\sigma$.

Let K denote the fixed point set of σ in U. Then $\frac{U}{K}$ is a symmetric space. Let $\mathcal{P}$ denote the -1 eigenspace of σ in $\mathcal{U}$. Then $\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$ is a Cartan decomposition,

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K},$$

and the tangent space of $\frac{U}{K}$ at eK can be identified as $\mathcal{P}$.

Definition 1.2.1. Let $L^{\tau,\sigma}(G)$ be the subgroup of $f \in L(G)$ that satisfies

$$\tau(f(\bar{\lambda})) = f(\lambda), \quad \sigma(f(-\lambda)) = f(\lambda). \tag{1.13}$$

We call the condition (1.13) the $\frac{U}{K}$*-reality condition*.

Note that $L_{\pm}^{\tau,\sigma}(G) = L^{\tau,\sigma}(G) \cap L_{\pm}(G)$ gives a splitting of $L^{\tau,\sigma}(G)$. It follows from the definition that $\xi(\lambda) = \sum_{i \leq n_0} \xi_i \lambda^i$ with $\xi_i \in \mathcal{G}$ satisfies the $\frac{U}{K}$-reality condition if and only if $\xi_i \in \mathcal{K}$ for i even and $\xi_i \in \mathcal{P}$ for i odd.

Let σ, τ be the involutions of G that give the Hermitian symmetric space $\frac{U}{K}$ as in Section 1.1. The derivative $\frac{U}{K}$-NLS hierarchy given by Fordy [2] is the hierarchy constructed from the splitting $\mathcal{L}_{\pm}(\mathcal{G})$ of $\mathcal{L}^{\tau,\sigma}(\mathcal{G})$ and $\{a\lambda^{2j} \mid j \geq 1\}$, where $\mathcal{L}_{+}(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \xi(\lambda) = \sum_{\mathrm{i} \geq 1} \xi_i \lambda^i\}$ and $\mathcal{L}_{-}(\mathcal{G}) = \{\xi(\lambda) \in \mathcal{L}(\mathcal{G}) \mid \xi(\lambda) = \sum_{\mathrm{i} < 1} \xi_i \lambda^i\}$.

In 1984, Fordy showed in [2] that there is a *derivative NLS hierarchy* associated to each classical Hermitian symmetric space $\frac{U}{K}$. The derivative $\frac{U}{K}$-NLS hierarchy can be constructed from $L^{\tau,\sigma}(G)$ and $\{a\lambda^{2j} \mid j \geq 1\}$, where $a \in \mathcal{K}$ and $\frac{U}{K}$ is a Hermitian symmetric space as given in Section 1.1.

Below we give splittings of derivative $\frac{U}{K}$-NLS hierarchy on classical Hermitian symmetric spaces.

Example 1.2.2 (derivative $\frac{U(n)}{U(k)\times U(n-k)}$-NLS)**.**

Let τ, σ be the involutions defined by $\tau(g) = (\bar{g}^t)^{-1}, \sigma(g) = I_{k,n-k}gI_{k,n-k}^{-1}$, and $a = \mathrm{diag}(iI_k, -iI_{n-k})$ as in Example 1.1.10.

$$\mathcal{U} = \mathfrak{u}(n), \quad \mathcal{K} = \mathcal{U}_a, \quad \mathcal{P} = \mathcal{U}_a^{\perp}.$$

We use the similar technique in Theorem 1.3.1 to get the 4-th flow

$$Q_t = -\frac{i}{2}Q_{xx} - \frac{1}{2}(QQ^*Q)_x, \tag{1.14}$$

where $Q \in M_{k\times(n-k)}(\mathbb{C})$.

Example 1.2.3 (derivative $\frac{U(2)}{U(1)\times U(1)}$-NLS)**.** When $n = 2$, the 4-th flow is the derivative NLS

$$q_t = -\frac{i}{2}q_{xx} - \frac{1}{2}(q^2q^*)_x,$$

and the 6-th flow is

$$q_t = -\frac{1}{4}q_{xxx} + \frac{3i}{4}(|q|^2q_x)_x + \frac{3}{8}(q|q|^4)_x,$$

where $q \in \mathbb{C}$.

Example 1.2.4 (derivative $\frac{SO(n+2)}{SO(2)\times SO(n)}$-NLS)**.**

Let τ, σ be the involutions of G defined by $\tau(g) = \bar{g}, \sigma(g) = I_{2,n+2} g I_{2,n+2}^{-1}$, and

$$a = \mathrm{diag}(-J_1, O_n),$$

as in Example 1.1.11. The 4-th flow is

$$\begin{cases} X_t = Y_{xx} + \frac{1}{2}(4(X \cdot Y)Y + (X \cdot X - 3Y \cdot Y)X)_x, \\ Y_t = -X_{xx} + \frac{1}{2}(4(X \cdot Y)X + (-3X \cdot X + Y \cdot Y)Y)_x, \end{cases} \tag{1.15}$$

where $X, Y \in \mathbb{R}^{1\times n}$.

Example 1.2.5 (derivative $\frac{SO(2n)}{U(n)}$-NLS)**.**

Let τ, σ be the involutions of G defined by $\tau(g) = \bar{g}, \sigma(g) = J_n g J_n^{-1}$, and $a = -\frac{1}{2} J_n$ as in Example 1.1.12, where

$$J_n = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

The 4-th flow is

$$\begin{cases} Q_t = 2(R_x + 2Q(R^2 + Q^2) - 2R(QR - RQ))_x, \\ R_t = 2(-Q_x + 2Q(QR - RQ) + 2R(R^2 + Q^2))_x, \end{cases}$$

where $Q \in \mathfrak{so}(n)$ and $R = -R^t$.

Example 1.2.6 (derivative $\frac{Sp(n)}{U(n)}$-NLS)**.**

Let τ, σ be the involutions of G defined by $\tau(g) = (\bar{g}^t)^{-1}, \sigma(g) = \bar{g}$, and $a = -\frac{1}{2} J_n$ as in Example 1.1.13.

The 4-th flow is

$$\begin{cases} Q_t = 2(R_x - 2Q(R^2 + Q^2) + 2R[Q, R])_x, \\ R_t = 2(-Q_x + 2[Q, R]Q - 2(R^2 + Q^2)R)_x, \end{cases}$$

where $Q = Q^t$ and $R = R^t$ are $n \times n$ real matrices.

1.3 Soliton Hierarchy Constructed from $\mathcal{L}_{\pm}^{\tau,\sigma}(\mathcal{G})$ and $\{a\lambda^{2j+1} \mid j \geq 1\}$

Let τ, σ be the involutions of G that give the Hermitian symmetric spaces $\frac{U}{K}$ and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. Recall that $\xi(\lambda) = \sum_i \xi_i \lambda^i \in \mathcal{L}^{\tau,\sigma}(\mathcal{G})$ if and only if $\xi_i \in \mathcal{K}$ for i even and $\xi_i \in \mathcal{P}$ for i odd. So given $a \in \mathcal{P}$, $a\lambda^{2j+1} \in \mathcal{L}_{+}^{\tau,\sigma}(\mathcal{G})$ for $j \geq 0$.

Theorem 1.3.1 ([11], [13])**.** *Let $\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$ as above, $a \in \mathcal{P}$, and $u : \mathbb{R} \mapsto \mathcal{K}_a^{\perp} \cap \mathcal{K}$ a smooth map. Let $Q(u, \lambda) = a\lambda + \sum_{j=0}^{\infty} Q_{-j}(u)\lambda^{-j}$ be the solution of* (1.8)*. Then $Q(u, \lambda)$ satisfies the $\frac{U}{K}$-reality condition.*

The hierarchy constructed from the splitting $\mathcal{L}_{\pm}^{\tau,\sigma}(\mathcal{G})$ of $\mathcal{L}^{\tau,\sigma}(\mathcal{G})$ and the sequence $\{a\lambda^{2j+1} \mid j \geq 0\}$ is the $\frac{U}{K}$ hierarchy defined by a given in [14]. For example, the flow in this hierarchy generated by $a\lambda^{2j+1}$ is

$$\frac{\partial u}{\partial t} = (Q_{-2j}(u))_x + [u, Q_{-2j}(u)] = [Q_{-(2j+1)}(u), a]. \tag{1.16}$$

Given $f \in L(G)$ satisfies the $\frac{U}{K}$-reality condition, then there exists a unique solution

$E(x,t,\lambda) \in L_+^{\tau,\sigma}(G)$ of the following initial value problem:

$$(1.17) \qquad \begin{cases} E^{-1}E_x = a\lambda + u, \\ E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1}(u), \\ E(0,0,\lambda) = f(\lambda). \end{cases}$$

We call such solution E (1.17) an *extended frame* for u.

Example 1.3.2. Let $G = O(n+1,\mathbb{C}) = \{g \in GL(n+1,\mathbb{C}) \mid g^t g = \mathrm{I}\}$, and τ, σ involutions on $O(n+1,\mathbb{C})$ defined by

$$\tau(g) = \bar{g}, \quad \sigma(g) = \mathrm{I}_{1,n} g \mathrm{I}_{1,n}^{-1}.$$

So the Cartan decomposition of the symmetric space $S^n = \frac{SO(n+1)}{SO(n)}$ is $so(n+1) = \mathcal{K} + \mathcal{P}$, where

$$\mathcal{K} = 0 \times so(n), \quad \mathcal{P} = \oplus_{i=2}^{n+1} \mathbb{R}(e_{i1} - e_{1i}).$$

Let $a = e_{21} - e_{12} \in \mathcal{P}$, and

$$V = \mathcal{K}_a^{\perp} \cap \mathcal{K} = \oplus_{i=1}^{n-1} \mathbb{R}(e_{i+2,2} - e_{2,i+2}) \in \mathcal{K}.$$

The $(2j+1)$-th flow in the $\frac{SO(n+1)}{SO(n)}$-hierarchy is the following PDE for $u : \mathbb{R}^2 \to V$:

$$(1.18) \qquad u_t = (Q_{2j}(u))_x + [u, Q_{2j}(u)].$$

The third flow is the vmKdV:

$$(1.19) \qquad k_t = -\left(k_{xxx} + \frac{3}{2}\|k\|^2 k_x\right),$$

where $k = (k_1, \cdots, k_{n-1})$.

Given $u = \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2}) \in C^\infty(\mathbb{R}, V)$, then u is a solution of the third flow (1.18) of the $\frac{SO(n+1)}{SO(n)}$-hierarchy if and only if

$$\theta_\lambda = (a\lambda + u)\mathrm{d}x + (a\lambda^3 + u\lambda^2 + Q_{-1}(u)\lambda + Q_{-2}(u))\mathrm{d}t \tag{1.20}$$

is flat for all λ. In other words, θ_λ is the Lax pair for the solution u of the third flow (or the vmKdV).

Note that the Lax pair θ_λ of a solution u of the third flow (1.18) satisfies the following reality condition

$$\begin{cases} \theta_\lambda^t + \theta_\lambda = 0, \\ \overline{\theta_{\bar{\lambda}}} = \theta_\lambda, \\ \mathrm{I}_{1,n}\theta_{-\lambda}\mathrm{I}_{1,n}^{-1} = \theta_\lambda. \end{cases} \tag{1.21}$$

Hence an extended frame $E(x, t, \lambda)$ for a solution u of the third flow satisfies the following $\frac{SO(n+1)}{SO(n)}$-reality condition:

$$\begin{cases} E(x,t,\lambda)^t E(x,t,\lambda) = \mathrm{I}_n, \\ \overline{E(x,t,\bar{\lambda})} = E(x,t,\lambda), \\ \mathrm{I}_{1,n}E(x,t,-\lambda)\mathrm{I}_{1,n} = E(x,t,\lambda). \end{cases} \tag{1.22}$$

Example 1.3.3. [$\frac{O(1,n)}{O(n)}$-hierarchy] Let $O(1, n, \mathbb{C}) = \{g \in GL(n+1, \mathbb{C}) \mid g^t \mathrm{I}_{1,n} g = \mathrm{I}_{1,n}\}$, and τ, σ involutions on $O(1, n, \mathbb{C})$ defined by

$$\tau(g) = \bar{g}, \quad \sigma(g) = \mathrm{I}_{1,n} g \mathrm{I}_{1,n}^{-1}.$$

Then the Cartan decomposition is $o(1,n) = \mathcal{K} + \mathcal{P}$, where

$$\mathcal{K} = 0 \times o(n), \quad \mathcal{P} = \oplus_{i=2}^{n+1} \mathbb{R}(e_{i1} + e_{1i}).$$

Let $a = e_{21} + e_{12} \in \mathcal{P}$, and

$$V = \mathcal{K}_a^{\perp} \cap \mathcal{K} = \oplus_{i=1}^{n-1} \mathbb{R}(e_{i+2,2} - e_{2,i+2}) \in \mathcal{K}.$$

The $(2j+1)$-th flow in the $\frac{O(1,n)}{O(n)}$-hierarchy is the following PDE for $u : \mathbb{R}^2 \to V$:

$$u_t = (Q_{-2j}(u))_x + [u, Q_{-2j}(u)].$$

The third flow is:

$$k_t = k_{xxx} + \frac{3}{2}\|k\|^2 k_x, \tag{1.23}$$

where $k = (k_1, \cdots, k_{n-1})$.

Given $u = \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2}) \in C^\infty(\mathbb{R}, V)$, then u is a solution of the third flow (1.23) of the $\frac{O(1,n)}{O(n)}$-hierarchy if and only if

$$\theta_\lambda = (a\lambda + u)\mathrm{d}x + (a\lambda^3 + u\lambda^2 + Q_{-1}(u)\lambda + Q_{-2}(u))\mathrm{d}t \tag{1.24}$$

is flat for all λ, where

$$\begin{aligned}
Q_{-1}(u) &= \frac{1}{2}\|k\|^2(e_{21}+e_{12}) + \sum_{i=1}^{n-1} k_i(e_{i+2,2}+e_{2,i+2}),\\
Q_{-2}(u) &= \sum_{i=1}^{n-1}(\frac{1}{2}\|k\|^2 k_i + (k_i)_{xx})(e_{i+2,2}-e_{2,i+2})\\
&\quad + \sum_{i\geq 2, j\geq 3} (k_{j-1}(k_{i-1})_x - (k_{j-1})_x k_{i-1})(e_{j+1,i+1}-e_{i+1,j+1}).
\end{aligned}$$

In other words, θ_λ is the Lax pair for the solution u of the third flow of $\frac{O(1,n)}{O(n)}$-hierarchy.

Chapter 2

Schrödinger Flows on Hermitian Symmetric Spaces

2.1 Schrödinger Flows on Compact Kähler Manifolds

Let (M, J, g, w) be a compact Kähler manifold, i.e., J is a complex structure, g the Riemannian metric, and w a symplectic form on M satisfying $w(X, Y) = g(JX, Y)$. The energy functional $E : C^\infty(\mathbb{S}^1, M) \to \mathbb{R}$ is defined by

$$E(\gamma) = \frac{1}{2}\int_{-\infty}^{\infty} g(\gamma_x, \gamma_x)\mathrm{d}x. \tag{2.1}$$

By calculus of variation, the gradient of E is

$$\nabla E(\gamma) = \nabla_{\gamma_x}\gamma_x,$$

where ∇ is the Levi-Civita connection of the metric g.

The *Schrödinger flow* on M (cf. [12]) is the following evolution equation on $C^\infty(\mathbb{R}, M)$:

$$\gamma_t = J_\gamma(\nabla_{\gamma_x}\gamma_x), \tag{2.2}$$

where ∇ is the Levi-Civita connection of M.

Example 2.1.1. When $M = \mathbb{C}^n$, (2.2) is the linear Schrödinger equation $\gamma_t = i\gamma_{xx}$.

Example 2.1.2 (The Schrödinger flow on $\mathbb{S}^2$)**.**

The complex structure of $\mathbb{S}^2$ $J_\gamma : T\mathbb{S}^2_\gamma \mapsto T\mathbb{S}^2_\gamma$ at γ is $v \mapsto \gamma \times v$, where $\times$ is the cross product in $\mathbb{R}^3$. Note that

$$\nabla_{\gamma_x}\gamma_x = \gamma_{xx}^T = \gamma_{xx} - (\gamma_{xx}, \gamma)\gamma,$$

so

$$\gamma \times \nabla_{\gamma_x}\gamma_x = \gamma \times \gamma_{xx}.$$

It follows that (2.2) is

$$\gamma_t = \gamma \times \gamma_{xx}, \tag{2.3}$$

which is the Heisenberg ferromagnetic model (HFM) for $\gamma : \mathbb{R}^2 \to \mathbb{S}^2$.

The symplectic form w on M induces a symplectic form $\hat{\omega}$ on the space $C^\infty(\mathbb{S}^1, M)$:

$$\hat{\omega}_\gamma(v_1, v_2) = \int_{-\infty}^{\infty} \omega_{\gamma(x)}(v_1(x), v_2(x))dx = \int_{-\infty}^{\infty} g_x(J_x(v_1(x)), v_2(x))\mathrm{d}x.$$

Then we have the following:

Proposition 2.1.3 ([12])**.** The Schrödinger flow (2.2) is a Hamiltonian equation for E with respect to $\hat{\omega}$.

Note that the critical points of E are geodesics of (M, g). So the stationary solutions of the Schrödinger flow on M are closed geodesics of M.

2.2 Schrödinger Flows on Hermitian Symmetric Spaces $\frac{U}{K}$ and Relation with $\frac{U}{K}$-NLS Equations

Proposition 2.2.1 ([12],[15])**.** Under the embedding of the Hermitian symmetric space $\frac{U}{K}$ as the Adjoint orbit $U \cdot a$ in $\mathcal{U}$, the Schrödinger flow on $\frac{U}{K}$ is

$$\gamma_t = [\gamma, \gamma_{xx}]. \tag{2.4}$$

Proof. Let $M = U \cdot a$, TM and $\nu(M)$ denote the tangent and normal bundles of M in $\mathcal{U}$. Note that $T_aM = \{[a, y] \mid y \in \mathcal{U}\}$, and the complex structure on T_aM is $J_a = \mathrm{ad}(a)$. If $g \in U$, then

$$T_{gag^{-1}}M = g\mathcal{P}g^{-1}, \nu_{gag^{-1}}(M) = g\mathcal{K}g^{-1} = gT_aMg^{-1}.$$

Let π and $\pi^{\perp}$ be the orthogonal projections onto TM and $\nu(M)$ respectively. Since the metric on M is the induced metric,

$$\nabla_{\gamma_x}\gamma_x = \pi(\gamma_{xx}) = \gamma_{xx} - \pi^{\perp}(\gamma_{xx}).$$

Since $\gamma = gag^{-1}$ and $\nu(M)_\gamma = g\mathcal{K}g^{-1}$, $[\gamma, v] = 0$ for all $v \in \nu(M)_\gamma$. This implies that $[\gamma, \pi^{\perp}(\gamma_{xx})] = 0$. Hence the Schrd̈ingier flow on $\frac{U}{K}$ is (2.4). □

Terng and Uhlenbeck in [15] give a Lax pair for (2.4).

Proposition 2.2.2 ([15])**.** Let $U \cdot a$ be the Adjoint orbit at a in $\mathcal{U}$ as in Proposition 2.2.1. Then γ satisfies equation (2.4) if and only if

$$\tau_\lambda = \gamma\lambda dx + (\gamma\lambda^2 + [\gamma, \gamma_x]\lambda)dt \text{ is flat for all } \lambda \in \mathbb{C}. \tag{2.5}$$

Proof. Note that the flatness of (2.5) is equivalent to

$$(\gamma\lambda)_t - (\gamma\lambda^2 + [\gamma, \gamma_x]\lambda)_x = [\gamma\lambda, \gamma\lambda^2 + [\gamma, \gamma_x]\lambda]. \tag{2.6}$$

We compare coefficients of λ^j's in (2.6) to have

$$\gamma_t - [\gamma, \gamma_x]_x = 0, \ \ \gamma_x + \mathrm{ad}(\gamma)^2(\gamma_x) = 0.$$

The first equation implies (2.4) and the second equation is true since $\mathrm{ad}(\gamma)^2 = -\mathrm{Id}$ on $T_\gamma M$. ☐

The following Theorem was proved by Terng and Uhlenbeck in [15] for $\frac{U}{K} = \mathrm{Gr}(k, \mathbb{C}^n)$ and by Terng and Thorbergsson in [12] for the other three classical Hermitian symmetric spaces.

Theorem 2.2.3 ([14], [12])**.** *Let $\gamma : \mathbb{R}^2 \to \frac{U}{K}$ be a solution of the Schrödinger flow on the Hermitian symmetric space $\frac{U}{K} = U \cdot a \subset \mathcal{U}$. Then there exists $g : \mathbb{R}^2 \to U$ satisfying*

(i) $\gamma = gag^{-1}$,

(ii) $u = g^{-1}g_x : \mathbb{R}^2 \to \mathcal{U}_a^\perp$ *satisfies the $\frac{U}{K}$-NLS equation:*

$$u_t = [a, u_{xx}] - \frac{1}{2}[u, [u, [a, u]]], \tag{2.7}$$

(iii) $g^{-1}g_t = [a, u_x] - \frac{1}{2}[u, [a, u]]$.

Moreover, $\tilde{g}$ satisfies (i) and (ii) if and only if there is a constant $C \in U_a$ such that $\tilde{g} = gC$.

Proof. We recall that $K = U_a, P = U_a^\perp$ and $\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$. Suppose $\gamma(x, t)$ is a solution of (2.4). Then there exists $h : \mathbb{R}^2 \to U$ such that $\gamma(x, t) = h(x, t)ah(x, t)^{-1}$. Let π_0, π_1 be orthogonal projections of $\mathcal{U}$ onto $\mathcal{K}, \mathcal{P}$, respectively. We choose $k : \mathbb{R}^2 \to K$ such that $k_x k^{-1} = -\pi_0(h^{-1}h_x)$. Set $f(x, t) = h(x, t)k(x, t)$, then $\gamma = faf^{-1}$. Moreover

$$f^{-1}f_x = (hk)^{-1}(hk)_x = k^{-1}\pi_1(h^{-1}h_x)k \in \mathcal{P}. \tag{2.8}$$

A direct computation shows that

$$\gamma_x = f[f^{-1}f_x, a]f^{-1} = f[u, a]f^{-1} \text{ and } [\gamma, \gamma_x] = fuf^{-1}.$$

Since $\tau_\lambda = \gamma\lambda dx + (\gamma\lambda^2 + [\gamma, \gamma_x]\lambda)dt$ is flat for all $\lambda \in \mathbb{C}$, $f * \tau_\lambda$ is flat, i.e. the following connection is flat for all $\lambda \in \mathbb{C}$:

$$f^{-1}\tau_\lambda f + f^{-1}df = (a\lambda + u)dx + (a\lambda^2 + u\lambda + f^{-1}f_t)dt. \tag{2.9}$$

Therefore, $(a\lambda^2 + u\lambda + f^{-1}f_t)_x - (a\lambda + u)_t + [a\lambda + u, f^{-1}f_t] = 0$.

$$u_x + [a, f^{-1}f_t] = 0 \tag{2.10}$$

$$(f^{-1}f_t)_x - u_t + [u, f^{-1}f_t] = 0. \tag{2.11}$$

Write

$$f^{-1}f_t = P + T,$$

where $P \in \mathcal{P}$ and $T \in \mathcal{K}$, respectively. From (2.10), we have

$$P = [a, u_x], \ T_x = -\frac{1}{2}[u, [a, u]]_x. \tag{2.12}$$

So, $T = -\frac{1}{2}[u, [a, u]] + c(t)$ for some function $c(t)$.

Define $g = fy(t)$, where $y(t) \in \mathcal{K}$ such that $y_t y^{-1} = -c(t)$.

Next, we will show that g defined above satisfies the conditions $(i) - (iii)$. Since $y(t)$ and a commute, it is easy to see that $gag^{-1} = \gamma$. In particular,

$$g^{-1}g_x = y^{-1}f^{-1}f_x y = y^{-1}uy \in \mathcal{P}, \tag{2.13}$$

$$g^{-1}g_t = y^{-1}(f^{-1}f_t + y_t y^{-1})y = -\frac{1}{2}[y^{-1}uy, [a, y^{-1}uy]], \tag{2.14}$$

which means $y^{-1}uy$ is a solution of (2.7).

For the uniqueness, suppose $\tilde{g}$ satisfies $(1) - (2)$, and set $C = g^{-1}\tilde{g}$. Then

$$\tilde{g}^{-1}\tilde{g}_x = C^{-1}g^{-1}g_x C + C^{-1}C_x.$$

Since $\tilde{g}^{-1}\tilde{g}_x$ and $C^{-1}g^{-1}g_x C$ are in $\mathcal{P}$ while $C^{-1}C_x \in \mathcal{K}$,

$$C^{-1}C_x = 0.$$

Similarly, $C^{-1}C_t = 0$. So C is constant.

□

Theorem 2.2.4 ([14], [12])**.** *Let* $u : \mathbb{R}^2 \to \mathcal{U}_a^\perp$ *be a smooth solution of* (2.7)*. Then* *given any $c_0 \in U$, the following linear system for $g : \mathbb{R}^2 \to U$,*

$$\begin{cases} g^{-1}g_x = u, \\ g^{-1}g_t = [a, u_x] - \frac{1}{2}[u, [a, u]], \\ g(0,0) = c_0 \end{cases}$$

has a unique smooth solution $g : \mathbb{R}^2 \to U$. Moreover, $\gamma(x,t) = g(x,t)ag(x,t)^{-1}$ is a solution of the Schrödinger flow (2.4) on $\frac{U}{K}$.

Proof. Since u is a solution of (2.7), the corresponding Lax pair (1.12) is $\theta_0 = udx + ([a, u_x] - \frac{1}{2}[u, [a, u]])dt = 0$ when $\lambda = 0$. So there exists g satisfying

$$g^{-1}g_x = u, \; g^{-1}g_t = [a, u_x] - \frac{1}{2}[u, [a, u]], g(0,0) = c_0.$$

Let $\gamma = gag^{-1}$. We gauge the Lax pair (1.12) by g to get

$$g * \theta_\lambda = g\theta_\lambda g^{-1} - (g_x g^{-1} + g_t g^{-1}) = \gamma\lambda dx + (\gamma\lambda^2 + gug^{-1}\lambda)dt.$$

Since $\gamma_x = g[u, a]g^{-1}$, $[\gamma, \gamma_x] = g[a, [u, a]]g^{-1}$. As $\mathrm{ad}(a)^2 = -\mathrm{Id}$ on $\mathcal{P}$, $[\gamma, \gamma_x] = gug^{-1}$. So γ satisfies the Lax pair (2.5), i.e., γ is a solution of (2.4). □

In fact, when $\lambda = \lambda_0$ is any arbitrary real number, a shift of $\gamma = gag^{-1}$ by $2\lambda_0$ is also a solution of (2.4).

Proposition 2.2.5. Let $u : \mathbb{R}^2 \to \mathcal{U}_a^\perp$ be a solution of (2.7) and E an extended frame for q. If $\lambda_0 \in \mathbb{R}$ and $g(x,t) = E(x,t,\lambda_0)$, then $\gamma = gag^{-1}(x - 2\lambda_0 t, t)$ is a solution of (2.4).

Proof. Let $\eta(x,t) = gag^{-1}(x,t)$. It can be checked that

$$\eta_x = g[u,a]g^{-1},$$
$$\eta_t = g[u\lambda_0 + Q_{-1}]g^{-1}.$$

Direct computations show that

$$\gamma_{xx} = g[a\lambda_0 + u,[u,a]]g^{-1} + g[u_x,a]g^{-1},$$

and therefore we obtain

$$\gamma \times \gamma_{xx} = g[a,u\lambda_0]g^{-1} + gu_xg^{-1}.$$

We see that $\gamma_t = -2\lambda_0\eta_x + \eta_t$, which gives

$$g[-u\lambda_0,a]g^{-1} + g[Q_{-1},a]g^{-1}.$$

Here, since $[Q_{-1},a] = u_x$, $\gamma_t = [\gamma,\gamma_{xx}]$. □

Chapter 3

Geometric Airy Curve Flows

3.1 Equivalence of Geometric Airy Equations on $\mathbb{R}^n$ and Vector Modified KdV Equation

Suppose $\gamma(\cdot, t)$ be a smooth curve in $\mathbb{R}^n$ with $||\gamma_x|| \neq 0$ and ∇ the Levi-Civita connection on $\mathbb{R}^n$. Let T_{x_0} be the tangent space of $\gamma(\cdot, t)$ at point x_0 and $\nu(T_{x_0}) = T_{x_0}^{\perp}$. Let $v \in C^{\infty}(\nu(T_{x_0}))$ and the shape operator $A_v : T_{x_0} \to T_{x_0}$ is defined by $A_v(u) = -(\nabla_u(v)(x_0))^T$, the projection of $\nabla_u(v)(x_0)$ onto T_{x_0}. Recall that the normal connection $\nabla^{\perp}$ of $\gamma(\cdot, t)$ in $\mathbb{R}^n$ is defined by the orthogonal projection of the connection of $\mathbb{R}^n$ onto the normal bundle $\nu(T_{x_0})$. Below we write it in terms of moving frames.

Let $e_0(\cdot, t)$ be the unit tangent vector of $\gamma(\cdot, t)$ and $(e_0, e_1, \cdots, e_{n-1})$ a local orthonormal frame in $\mathbb{R}^n$. Let $\omega_0, \cdots, \omega_{n-1}$ be the dual coframe on $\mathbb{R}^n$. It follows from the definition of $\nabla^{\perp}$ that $\nabla^{\perp}(e_i) = \sum_{j=1}^{n-1} \omega_{ij} \otimes e_j$ for $i = 1, \cdots, n-1$. It is known that there exist $(e_0, \tilde{e}_1, \cdots, \tilde{e}_{n-1})$ along $\gamma(\cdot, t)$ such that

$$\begin{cases} (e_0)_x = k_1 e_1 + \cdots + k_{n-1} e_{n-1}, \\ (\tilde{e}_i)_x = -k_i e_0, \quad 1 \leq i \leq n-1, \end{cases}$$

for some smooth functions $k_1, \cdots, k_{n-1}$. We call such $(e_0, \tilde{e}_1, \cdots, \tilde{e}_{n-1})$ a parallel frame of $\gamma(\cdot, t)$ and $k_1, \cdots, k_{n-1}$ the principal curvatures of $\gamma(\cdot, t)$ along a normal vector $e_i, i = 1, \cdots, n-1$. The mean curvature vector of $\gamma(\cdot, t)$ in $\mathbb{R}^n$ is defined by $H(\gamma(\cdot, t)) = \sum_{i=1}^{n-1} k_i e_i$.

In this section, we consider the following curve flow on $\mathbb{R}^n$:

$$\gamma_t = -\nabla^{\perp}_{e_0} H. \tag{3.1}$$

Therefore (3.1) is a geometric curve flow, i.e., the velocity vector γ_t can be expressed by geometric quantity of $\gamma(\cdot, t)$.

Let $e_0 = \frac{\gamma_x}{||\gamma_x||}, (e_1, \ldots, e_{n-1})$ be a parallel normal frame for γ and $k_1, \ldots, k_{n-1}$ the normal principal curvatures along $e_1, \ldots, e_{n-1}$. Under the parallel frame, we can rewrite (3.1) in terms of $k_1, \ldots, k_{n-1}$:

$$\gamma_t = -\sum_{i=1}^{n-1} (k_i)_s e_i, \tag{3.2}$$

where s is the arc length parameter and $\frac{\partial}{\partial x} = \frac{\partial}{\partial s} ||\gamma_x||$.

Proposition 3.1.1. If γ satisfies (3.1) and is periodic in x with period L, then

$$\int_0^L (\gamma_x, \gamma_x)^{\frac{1}{2}} \, dx$$

is independent of t, i.e., the total arc length of $\gamma(\cdot, t)$ is preserved.

Proof. A direct computation shows

$$\frac{d}{dt}\int_0^L (\gamma_x,\gamma_x)^{\frac{1}{2}}\,dx = \int_0^L \frac{(\gamma_{xt},\gamma_x)}{||\gamma_x||}\,dx. \tag{3.3}$$

Note that

$$\frac{(\gamma_{xt},\gamma_x)}{||\gamma_x||} = (\sum_{i=1}^{n-1} -(k_i)_{ss}e_i + k_i(k_i)_s e_0, e_0) = -(k_0)_{ss} + \sum_{i=1}^{n-1} k_i(k_i)_s,$$

which is total derivative. As k_i is periodic, (3.3) is equal to 0. This proves that the total arc length of $\gamma(\cdot,t)$ is independent of t. □

So we may reparametrize each $\gamma(\cdot,t)$ by its arc-length parameter.

Proposition 3.1.2. Suppose x is arc-length parameter. Then the flow (3.1) can be written as

$$\gamma_t = -(\frac{1}{2}||H||^2 e_0 + \nabla^{\perp}_{e_0} H). \tag{3.4}$$

Or equivalently,

$$\gamma_t = -\frac{1}{2}\sum_{i=1}^{n-1} k_i^2 e_0 - \sum_{i=1}^{n-1}(k_i)_x e_i. \tag{3.5}$$

Proof. We reparametrize the curve so that it preserves the arc length parameter. Consider $\alpha_t = \zeta_0 e_0 - \sum_{i=1}^{n-1}(k_i)_x e_i$ because changing the coefficient of e_0 is equivalent to reparametrizing the curve. We compute to get

$$< \alpha_{tx}, \alpha_x >=< (\zeta_0)_x e_0 + \zeta_0(e_0)_x - \sum_{i=1}^{n-1}(k_i)_x(e_i)_x, e_0 >= (\zeta_0)_x + \sum_{i=1}^{n-1}(k_i)_x k_i.$$

So we choose $\zeta_0 = -\frac{1}{2}\sum_{i=1}^{n-1} k_i^2$ to make $< \alpha_{tx}, \alpha_x >= 0$, i.e., α preserves the arc length. Hence, the flow (3.1) can be written as (3.4) and (3.5). □

From now on, we may assume x is the arc-length parameter. In particular, $\gamma_{xx} = H$. Then

$$\gamma_{xxx} = (\sum_{i=1}^{n-1} k_i e_i)_x = \sum_{i=1}^{n-1}(k_i)_x e_i - \sum_{i=1}^{n-1} k_i^2 e_0. \tag{3.6}$$

Adding (3.5) and (3.6) to get

$$\gamma_t = -\gamma_{xxx} - \frac{3}{2}\sum_{i=1}^{n-1} k_i^2 e_0 = -(\gamma_{xxx} + \frac{3}{2}||\gamma_{xx}||^2\gamma_x). \tag{3.7}$$

Theorem 3.1.3 ([7]). *Let γ be a solution of the geometric Airy curve flow* (3.4) *on $\mathbb{R}^n$ parametrized by arc-length, $h = (e_0, \ldots, e_{n-1})$ the moving frame along γ, and $g = \mathrm{diag}(1, h) \in SO(n+1)$. Then $u = g^{-1}g_x$ is a solution of the third flow* (1.19) *in the $\frac{SO(n+1)}{SO(n)}$-hierarchy, i.e., $k_t = -\left(k_{xxx} + \frac{3}{2}\|k\|^2 k_x\right)$, where $k_1, \ldots, k_{n-1}$ are the principal curvatures along $e_1, \cdots, e_{n-1}$.*

Proof. Let $k = (k_1, \ldots, k_{n-1})$, where k_i is the principal curvature along e_i for all $i = 1, \ldots, n-1$. Note that $u = g^{-1}g_x = \mathrm{diag}(0, h^{-1}h_x)$, where

$$h^{-1}h_x = \begin{pmatrix} 0 & -k \\ k^t & 0 \end{pmatrix}, \text{ denoted by } (A_{ij})$$

We now compute $(B_{ij}) := h^{-1}h_t$. Note that $(e_0)_t = \gamma_{xt} = -\frac{1}{2}|k|^2 \sum_{i=1}^{n-1} k_i e_i$, so

$$B_{11} = 0, \quad B_{1i} = B_{i1} = -\frac{1}{2}||k||^2 k_{i-1}, i = 2, \cdots, n.$$

The rest of B_{ij} can be obtained from $(e_{i+1})_{tx} \cdot e_{j+1} = (e_{i+1})_{xt} \cdot e_{j+1}, i > j$. Compute to get

$$\partial_x B_{ij} = ((k_i)_x k_j - k_i(k_j)_x)_x,$$

so we may change frame so that $B_{ij} = (k_i)_x k_j - k_i(k_j)_x$. Then $(A_{ij})_t = (B_{ij})_x + [A, B]$ implies

$$(k_i)_t = -(k_i)_{xxx} - \frac{3}{2}||k||^2(k_i)_x, \quad 1 \le i \le n-1.$$

Hence u is a solution of (3.8). □

We use the standard Sym-Pohlmeyer techniques to construct solutions of the geometric Airy curve flows on $\mathbb{R}^n$ from solutions of the third flow (1.19).

Example 3.1.4. When $n = 2$, the curvature k of a curve γ satisfying (3.7) is the mKdV

$$k_t = -(k_{xxx} + \frac{3}{2}k^2 k_x). \tag{3.8}$$

Theorem 3.1.5. *Let $k = (k_1, \ldots, k_{n-1})$ be a solution of vmKdV, i.e.,*

$$u = \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2})$$

is a solution of the third flow (1.19). *Let $E(x, t, \lambda)$ be an extended frame of the solution u of the third flow* (1.19). *We identify $\mathbb{R}^{n\times 1}$ with $\mathcal{P} = \{\begin{pmatrix} 0 & -y^t \\ y & 0 \end{pmatrix} | y \in \mathbb{R}^{n\times 1}\}$ by $y \mapsto \begin{pmatrix} 0 & -y^t \\ y & 0 \end{pmatrix}$. Then $\frac{\partial E}{\partial \lambda}E^{-1} \mid_{\lambda=0} = \begin{pmatrix} 0 & -\gamma^t \\ \gamma & 0 \end{pmatrix}$ and γ is a solution of* (3.7).

Proof. Set $g(x, t) = E(x, t, 0)$. Since u is a solution of (1.19) and E satisfies the $\frac{SO(n+1)}{SO(n)}$ reality condition, we have $g \in K$ and $\gamma \in \mathcal{P}$. So $g = \mathrm{diag}(1, h)$ for some $h \in SO(n)$. Let $\hat{\gamma} = \frac{\partial E}{\partial \lambda}E^{-1} \mid_{\lambda=0}$. We use the Lax pair (1.20) to compute directly to get

$$\hat{\gamma}_x = gag^{-1},$$
$$\hat{\gamma}_t = gQ_{-1}(u)g^{-1},$$

where $Q_{-1}(u)$ is as in (1.20). Let $v_0, \ldots, v_{n-1}$ denote the standard basis of $\mathbb{R}^n$, and

$$e_i = hv_i,\ 0 \leq i \leq n-1.$$

Use Theorem 1.3.1 to get

$$Q_{-1}(u) = \begin{pmatrix} 0 & \frac{||k||^2}{2} & (k_1)_x & \cdots & (k_{n-1})_x \\ -\frac{||k||^2}{2} & 0 & \cdots & & 0 \\ -(k_1)_x & 0 & \cdots & & 0 \\ & & & & \\ -(k_{n-1})_x & 0 & \cdots & & 0 \end{pmatrix}.$$

So, $\gamma_t = h\phi$, where $\phi = -(\frac{||k||^2}{2}, (k_1)_x, \cdots, (k_{n-1})_x)^t$. A direct computation shows

$$\begin{aligned} -\gamma_t &= h\frac{||k||^2}{2}v_0 + \textstyle\sum_{i=1}^{n-1} h(k_i)_x v_i \\ &= \frac{||k||^2}{2}e_0 + \textstyle\sum_{i=1}^{n-1}(k_i)_x e_i \end{aligned},$$

which is (3.5). This proves that γ is a solution of (3.7). □

Theorem 3.1.6. *Let γ be a solution of geometric Airy curve flow on $\mathbb{R}^n$ such that k is the solution of the vmKdV* (1.19) *constructed in Theorem 3.1.3. If E is an extended frame of k and $\tilde{\gamma} = \frac{\partial E}{\partial \lambda}E^{-1} \mid_{\lambda=0}$, then γ and $\tilde{\gamma}$ differ by a rigid motion.*

Proof. By Theorem 3.1.3, there is an extended frame F of k such that $F^{-1}F_x$ is a solution of the vector mKdV, and

$$\frac{\partial E}{\partial \lambda}E^{-1} \mid_{\lambda=0} = \begin{pmatrix} 0 & -\tilde{\gamma}^t \\ \tilde{\gamma} & 0 \end{pmatrix}, \qquad \frac{\partial F}{\partial \lambda}F^{-1}\Big|_{\lambda=0} = \begin{pmatrix} 0 & -\gamma^t \\ \gamma & 0 \end{pmatrix}.$$

Since $F^{-1}\mathrm{d}F = E^{-1}\mathrm{d}E$, there exists $f : \mathbb{C} \to SO(n+1)$ satisfying the $\frac{SO(n+1)}{SO(n)}$-reality condition such that $F(x,t,\lambda) = f(\lambda)E(x,t,\lambda)$. But f satisfies the $\frac{SO(n+1)}{SO(n)}$-reality condition implies that $\frac{\mathrm{d}f}{\mathrm{d}\lambda}f^{-1}\big|_{\lambda=0}$ lies in $\mathcal{P}$, hence is of the form $\begin{pmatrix} 0 & -\xi_0^t \\ \xi_0 & 0 \end{pmatrix}$ for some $\xi_0 \in \mathbb{R}^n$. Reality condition also implies that $f(0) = \begin{pmatrix} 1 & 0 \\ 0 & h_0 \end{pmatrix}$, where $h_0 \in SO(n)$. But $F = fE$ implies that

$$\frac{\partial F}{\partial \lambda}F^{-1} = \frac{\mathrm{d}f}{\mathrm{d}\lambda}f^{-1} + f\frac{\partial E}{\partial \lambda}E^{-1}f^{-1},$$

which implies γ and $\tilde{\gamma}$ differ by a rigid motion. □

3.2 Geometric Airy Curve Flows on $\mathbb{S}^n$

Suppose $\gamma(x,t)$ is a smooth curve on $\mathbb{S}^n$, we consider the geometric Airy curve flow on $\mathbb{S}^n$, i.e., $\gamma_t = -\nabla_{e_1}^{\perp} H(\gamma)$, where $e_1 = \frac{\gamma_x}{||\gamma_x||}$. Since the geometric Airy curve flow preserves the total arc length, we can reparametrize the flow such that $||\gamma_x||$ is independent of t. Throughout this section, we assume that x is the arc length parameter.

Let $e_0 = \gamma, e_1 = \gamma_x$, and $(e_2, \cdots, e_n)$ an orthonormal frame along γ. Then we have

$$(3.9) \qquad (e_0, e_1, \cdots, e_n)_x = (e_0, e_1, \cdots, e_n) \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -k_1 & \cdots & -k_{n-1} \\ 0 & k_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & k_{n-1} & \cdots & & 0 \end{pmatrix},$$

where $k_1, \cdots, k_{n-1}$ are principal curvatures along $e_2, \cdots, e_n$. Under this parallel frame, we rewrite $\gamma_t = -\nabla^{\perp}_{e_1} H(\gamma)$ as

$$\gamma_t = -(\frac{1}{2}||k||^2 e_1 + \sum_{j=1}^{n-1} (k_j)_x e_{j+1}), \tag{3.10}$$

where $k = (k_1, \cdots, k_{n-1})$.

Theorem 3.2.1. *Let γ be a solution of the geometric Airy curve flow* (3.10) *on $\mathbb{S}^n$ parametrized by arc-length and $e_0 = \gamma, e_1 = \gamma_x$. Then there is a frame $g = (e_0, e_1, ..., e_n)$ along γ such that $g^{-1}g_x = (e_{21} - e_{12}) + \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2})$, where $k_1, \cdots, k_{n-1}$ are the principal curvatures along $e_2, \cdots, e_n$. Moreover, $k = (k_1, \cdots, k_{n-1})$ is a solution of the following evolution*

$$k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x) - k_x. \tag{3.11}$$

Proof. Let $g^{-1}g_t = (B_{ij}) \in O(n+1)$. Note that

$$(e_0)_t = \gamma_t = -(\frac{1}{2}||k||^2 e_1 + \sum_{j=1}^{n-1} (k_j)_x e_{j+1}),$$
$$(e_1)_t = \gamma_{tx} = \frac{1}{2}||k||^2 e_0 - \sum_{i=1}^{n-1} (\frac{1}{2}||k||^2 k_i + (k_i)_{xx}) e_{i+1},$$

so we have

$$B_{21} = -\frac{1}{2}||k||^2 = -B_{12},$$
$$B_{j1} = -(k_{j-2})_x, \quad B_{j2} = -(\frac{1}{2}||k||^2 k_{j-2} + (k_{j-2})_{xx}), \quad 3 \le j \le n+1.$$

Since $(e_i)_{xt} \cdot e_l = (e_i)_{tx} \cdot e_l$ for $2 \le i \le n, 3 \le l \le n$, we compute to get

$$k_{i-1}(\frac{1}{2}||k||^2 k_{l-1} + (k_{l-1})_{xx}) = (B_{l+1,i+1})_x + (\frac{1}{2}||k||^2 k_{i-1} + (k_{i-1})_{xx}) k_{l-1}.$$

So, $(B_{l+1,i+1})_x = (k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1})_x$, which implies $B_{l+1,i+1} = k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1} + c(t)$. We may change frames to have

$$B_{l+1,i+1} = k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1}.$$

Then there is a frame g satisfying the ODE system

$$\begin{cases} g^{-1}g_x = (e_{21} - e_{12}) + \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2}), \\ g^{-1}g_t = (B_{ij}). \end{cases}$$

The compatibility implies

$$(k_j)_t = -(k_j)_{xxx} - \frac{3}{2}||k||^2(k_j)_x - (k_j)_x, \quad 1 \leq j \leq n-1.$$

Let $k = (k_1, \cdots, k_{n-1})$. Then we obtain $k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x) - k_x$. □

Proposition 3.2.2. If $q(x,t)$ is a solution of the vector mKdV (1.19), then $k(x,t) = q(x-t,t)$ satisfies (3.11).

Conversely, given a solution of vmKdV q, we use the Lax pair of vmKdV to construct a solution of (3.10).

Theorem 3.2.3. *Suppose $q(x,t) = (q_1, \cdots, q_{n-1})$ is a solution of vmKdV* (1.19) *and $E(x,t,\lambda)$ is an extended frame of q. Let $E(x,t,1) = (e_0, e_1, \cdots, e_n)$. Then $\gamma(x,t) = e_0(x-t,t)$ is a solution of* (3.10).

Proof. Let $g(x,t) = E(x,t,1)$. So,

$$g^{-1}g_x = (e_{21} - e_{12}) + \sum_{i=1}^{n-1} q_i(e_{i+2,2} - e_{2,i+2}),$$
$$g^{-1}g_t = (e_{21} - e_{12}) + \sum_{i=1}^{n-1} q_i(e_{i+2,2} - e_{2,i+2}) + Q_{-1}(q) + Q_{-2}(q),$$

where

$$Q_{-1}(q) = -\frac{||q||^2}{2}(e_{21} - e_{12}) - \sum_{i=2}^{n}(q_{i-1})_x (e_{i+1,1} - e_{1,i+1}),$$
$$Q_{-2}(q) = \sum_{i=2}^{n} z_i (e_{i+1,2} - e_{2,i+1}) + \sum_{i,j=2}^{n} \eta_{i+1,j+1}\, e_{i+1,j+1},$$

and

$$\begin{cases} z_i = -\left((q_{i-1})_{xx} + \frac{1}{2}q_{i-1}||q||^2\right), & 2 \leq i \leq n, \\ \eta_{i+1,j+1} = -q_{i-1}(q_{j-1})_x + (q_{i-1})_x q_{j-1}, & 2 \leq i,j \leq n. \end{cases}$$

Let $k(x,t) = q(x-t,t)$. From Proposition 3.2.2, k is a solution of (3.11). Note that $k_x = q_x$ and a direct computation shows

$$\gamma(x,t)_t = -(e_0)_x + (e_0)_t = -e_1 + \left(e_1 - \frac{||k||^2}{2}e_1 - \sum_{i=2}^{n}(k_{i-1})_x e_i\right).$$

□

3.3 Geometric Airy Curve Flows on $\mathbb{H}^n$

In this section, we consider the geometric Airy curve flow,

$$\gamma_t = -\nabla^{\perp}_{e_1} H$$

on $\mathbb{H}^n = \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_n^2 = -1\}$. Here e_1 denotes the unit tangent vector along γ. Since this flow preserves the total arc length, we reparametrize such that $||\gamma_x|| = 1$, i.e., we may assume x is the arc length parameter.

Let $e_0 = \gamma, e_1 = \gamma_x$, and $(e_2, \cdots, e_n)$ an orthonormal frame along γ. Then we have

$$(e_0, e_1, \cdots, e_n)_x = (e_0, e_1, \cdots, e_n) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & -k_1 & \cdots & -k_{n-1} \\ 0 & k_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & k_{n-1} & \cdots & & 0 \end{pmatrix}, \tag{3.12}$$

where $k_1, \cdots, k_{n-1}$ are principal curvatures along $e_2, \cdots, e_n$. Under this parallel frame, we rewrite $\gamma_t = -\nabla^{\perp}_{e_1} H(\gamma)$ as

$$\gamma_t = -(\frac{1}{2}||k||^2 e_1 + \sum_{j=1}^{n-1} (k_j)_x e_{j+1}), \tag{3.13}$$

where $k = (k_1, \cdots, k_{n-1})$.

Theorem 3.3.1. *Let γ be a solution of the geometric Airy curve flow* (3.13) *on $\mathbb{H}^n$ parametrized by arc-length and $e_0 = \gamma, e_1 = \gamma_x$. Then there is a frame $g = (e_0, e_1, ..., e_n)$ along γ such that $g^{-1}g_x = (e_{21} + e_{12}) + \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2})$, where $k_1, \cdots, k_{n-1}$ are the principal curvatures along $e_2, \cdots, e_n$. Moreover, $k = (k_1, \cdots, k_{n-1})$ is a solution of the following evolution*

$$k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x) + k_x. \tag{3.14}$$

Proof. Let $g^{-1}g_t = (B_{ij}) \in O(1,n)$. Note that

$$(e_0)_t = \gamma_t = -(\frac{1}{2}||k||^2 e_1 + \sum_{j=1}^{n-1}(k_j)_x e_{j+1}),$$
$$(e_1)_t = \gamma_{tx} = -\frac{1}{2}||k||^2 e_0 - \sum_{i=1}^{n-1}(\frac{1}{2}||k||^2 k_i + (k_i)_{xx})e_{i+1},$$

so we have

$$B_{21} = -\frac{1}{2}||k||^2 = B_{12},$$
$$B_{j1} = -(k_{j-2})_x, \quad B_{j2} = -(\frac{1}{2}||k||^2 k_{j-2} + (k_{j-2})_{xx}), \quad 3 \le j \le n+1,$$

and $B_{i1} = B_{1i}, B_{j2} = -B_{2j}, 1 \le i \le n+1, 3 \le j \le n+1$. Since $(e_i)_{xt} \cdot e_l = (e_i)_{tx} \cdot e_l$ for $2 \le i \le n, 3 \le l \le n$, we compute to get

$$k_{i-1}(\frac{1}{2}||k||^2 k_{l-1} + (k_{l-1})_{xx}) = (B_{l+1,i+1})_x + (\frac{1}{2}||k||^2 k_{i-1} + (k_{i-1})_{xx})k_{l-1}.$$

So, $(B_{l+1,i+1})_x = (k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1})_x$, which implies $B_{l+1,i+1} = k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1} + c(t)$. We may change frames to have

$$B_{l+1,i+1} = k_{i-1}(k_{l-1})_x - (k_{i-1})_x k_{l-1}.$$

Then there is a frame g satisfying the ODE system

$$\begin{cases} g^{-1}g_x = (e_{21} + e_{12}) + \sum_{i=1}^{n-1} k_i(e_{i+2,2} - e_{2,i+2}), \\ g^{-1}g_t = (B_{ij}). \end{cases}$$

The compatibility implies

$$(k_j)_t = -(k_j)_{xxx} - \frac{3}{2}||k||^2(k_j)_x + (k_j)_x, \quad 1 \le j \le n-1.$$

Let $k = (k_1, \cdots, k_{n-1})$. Then we obtain $k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x) + k_x$. $\square$

Proposition 3.3.2. If $q(x,t)$ is a solution of (1.23), i.e., $q_t = q_{xxx} + \frac{3}{2}\|q\|^2 q_x$, then $k(x,t) = q(x+t, -t)$ satisfies (3.14).

We then use the Lax pair (1.24) of the third flow of $\frac{O(1,n)}{O(n)}$-hierarchy to construct a solution of (3.13) on $\mathbb{H}^n$.

Theorem 3.3.3. *Suppose $q(x,t) = (q_1, \cdots, q_{n-1})$ is a solution of the third flow (1.23) in $\frac{O(1,n)}{O(n)}$-hierarchy and $E(x,t,\lambda)$ is an extended frame of q. Let $E(x,t,1) = (e_0, e_1, \cdots, e_n)$. Then $\gamma(x,t) = e_0(x+t,-t)$ is a solution of* (3.13).

Proof. Let $g(x,t) = E(x,t,1)$. So,

$$g^{-1}g_x = (e_{21} + e_{12}) + \sum_{i=1}^{n-1} q_i(e_{i+2,2} - e_{2,i+2}),$$
$$g^{-1}g_t = (e_{21} + e_{12}) + \sum_{i=1}^{n-1} q_i(e_{i+2,2} - e_{2,i+2}) + Q_{-1}(q) + Q_{-2}(q),$$

where

$$Q_{-1}(q) = \frac{1}{2}\|q\|^2(e_{21} + e_{12}) + \sum_{i=1}^{n-1} q_i(e_{i+2,2} + e_{2,i+2}),$$
$$Q_{-2}(u) = \sum_{i=1}^{n-1}(\frac{1}{2}\|q\|^2 q_i + (q_i)_{xx})(e_{i+2,2} - e_{2,i+2})$$
$$+ \sum_{i\geq 2, j\geq 3}(q_{j-1}(q_{i-1})_x - (q_{j-1})_x q_{i-1})(e_{j+1,i+1} - e_{i+1,j+1}).$$

In other words, $(e_0)_t = (1 + \frac{1}{2}\|q\|^2)e_1 + \sum_{i=1}^{n-1}(q_i)_x e_{i+1}$.

Let $k(x,t) = q(x+t,-t)$. From Proposition 3.3.2, k is a solution of (3.14). Note that

$k_x = q_x$ and a direct computation shows

$$\begin{aligned}\gamma(x,t)_t &= (e_0)_x - (e_0)_t \\ &= e_1 - \left((1 + \frac{1}{2}\|q\|^2)e_1 + \sum_{i=1}^{n-1} (q_i)_x e_{i+1} \right) \\ &= -\left(\frac{\|k\|^2}{2} e_1 + \sum_{i=1}^{n-1} (k_i)_x e_{i+1} \right).\end{aligned}$$

□

Chapter 4

Bäcklund Transformations

4.1 Bäcklund Transformations for $\frac{U}{K}$-NLS and Schrödinger Flows

We review the general method given in [14] for constructing Bäcklund transformations of soliton equations:

Let u be a solution of the $\frac{U}{K}$-NLS equation, and E an extended frame of u.

Step 1: Find *simple elements*, i.e., rational maps $f : \mathbb{C} \cup \{\infty\} \to GL(n, \mathbb{C})$ that satisfy the U-reality condition, $f(\infty) = I_n$, and have minimal number of zeros and poles.

Step 2: Given a simple element f, use residue calculus to factor

$$f(\lambda)E(x,t,\lambda) = \tilde{E}(x,t,\lambda)\tilde{f}(x,t,\lambda)$$

such that $\tilde{E}(x,t,\lambda)$ is holomorphic for $\lambda \in \mathbb{C}$ and $\tilde{f}(x,t,\lambda)$ is rational in λ and

$\tilde{f}$ satisfies the U-reality condition.

Step 3: Prove that $\tilde{E} = fE\tilde{f}^{-1}$satisfies the following system

$$\begin{cases} \tilde{E}^{-1}\tilde{E}_x = a\lambda + \tilde{u}, \\ \tilde{E}^{-1}\tilde{E}_t = a\lambda^2 + \tilde{u}\lambda + Q_{-1}(\tilde{u}) \end{cases}$$

for some $\tilde{u} : \mathbb{R}^2 \to \mathcal{P}$ and $\tilde{u}$ is given by an algebraic formula in terms of u and $\tilde{f}$. Then $\tilde{u}$ is a new solution of $\frac{U}{K}$-NLS. The transformation $u \to \tilde{u}$ is usually called a Bäcklund or Darboux transformation for $\frac{U}{K}$-NLS.

Step 4: Let $\tilde{\theta}_\lambda = \tilde{E}^{-1}d\tilde{E}$, $\theta_\lambda = E^{-1}dE$. Then

$$\tilde{\theta}_\lambda = -d\tilde{f}\tilde{f}^{-1} + \tilde{f}\theta_\lambda\tilde{f}^{-1},$$

or equivalently,

$$d\tilde{f} = \tilde{f}\theta_\lambda - \tilde{\theta}_\lambda\tilde{f}. \tag{4.1}$$

Then this system (4.1) gives rise to a system of first oder PDEs for $\tilde{f}$. Substituting the formula for $\tilde{u}$ interns of u and $\tilde{f}$. If we can prove this system is compatible if θ_λ is flat then we obtain Bäcklund transformations by solving this nonlinear system (4.1) for $\tilde{f}$.

We recall the U-reality conditions below.

Definition 4.1.1 (U-Reality Conditions)**.**

Given $f(\lambda) \in GL(n, \mathbb{C})$, we say that

1. f satisfies the *$U(n)$-reality condition* if

$$f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_n, \tag{4.2}$$

2. f satisfies the *$O(n)$-reality condition* if

$$f(\lambda)^t f(\lambda) = \mathrm{I}_n, \quad \overline{f(\bar{\lambda})} = f(\lambda). \tag{4.3}$$

This is equivalent to $f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_n, \quad \overline{f(\bar{\lambda})} = f(\lambda)$.

3. $n = 2m$, f satisfies the *$Sp(n)$-reality condition* if

$$f(\lambda)^t J_m f(\lambda) = J_m, \quad f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_n. \tag{4.4}$$

We review simple elements in [16].

Proposition 4.1.2 ([16])**.** Let π be a Hermitian projection of $\mathbb{C}^n$, $z \in \mathbb{C} \setminus \mathbb{R}$, and $k : \mathbb{C} \cup \{\infty\} \to GL(n, \mathbb{C})$ a rational map defined by

$$k_{z,\pi}(\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar{z}} \pi^{\perp}. \tag{4.5}$$

Then $k_{z,\pi}$ satisfies the $U(n)$-reality condition and $k_{z,\pi} \in L_{-}^{\tau}(GL(n, \mathbb{C}))$, where $\tau(g) = (g^*)^{-1}$.

Proof. It follows from $\pi^* = \pi$ and $\pi\pi^{\perp} = 0$ that

$$k_{z,\pi}(\bar{\lambda})^* k_{z,\pi}(\lambda) = (\pi + \frac{\bar{\lambda} - z}{\bar{\lambda} - \bar{z}} \pi^{\perp})^* (\pi + \frac{\lambda - z}{\lambda - \bar{z}} \pi^{\perp}) = (\pi + \frac{\lambda - \bar{z}}{\lambda - z} \pi^{\perp})(\pi + \frac{\lambda - z}{\lambda - \bar{z}} \pi^{\perp}) = I_n.$$

□

Proposition 4.1.3 ([1])**.** Let π be a Hermitian projection of $\mathbb{C}^n$ onto V. If $\overline{V} \perp V$, then

$$p_{z,\pi}(\lambda) = (I + \frac{z - \bar{z}}{\lambda - z}\pi^{\perp})(I + \frac{\bar{z} - z}{\lambda - \bar{z}}\overline{\pi}^{\perp}). \tag{4.6}$$

satisfies the $O(n)$-reality condition.

Proof. Since $\bar{V} \perp V$, π and $\pi^{\perp}$ commute. Note that

$$\begin{aligned} p_{z,\pi}(\bar{\lambda}) &= (I + \tfrac{z-\bar{z}}{\bar{\lambda}-z}\pi^{\perp})(I + \tfrac{\bar{z}-z}{\bar{\lambda}-\bar{z}}\overline{\pi}^{\perp}) \\ &= \overline{(I + \tfrac{\bar{z}-z}{\lambda-\bar{z}}\overline{\pi}^{\perp})}\ \overline{(I + \tfrac{z-\bar{z}}{\lambda-z}\pi^{\perp})} \\ &= \overline{p_{z,\pi}(\lambda)}. \end{aligned}$$

∎

Example 4.1.4. Let $V = \mathbb{C}\begin{pmatrix} r \\ \mathrm{i}\, s \end{pmatrix}$, where $r, s \in \mathbb{R}^{n\times 1}$ with $||r|| = ||s||$. Then $V \perp \bar{V}$.

Lemma 4.1.5. Suppose V is a complex subspace of $\mathbb{C}^n$ such that $< V, \bar{V} >= 0$ and $g(\lambda)$ satisfies the $O(n)$-reality condition (4.3). Let $\tilde{V} = g(\lambda)^*(V)$. Then $\tilde{V}$ is perpendicular to $\overline{\tilde{V}}$.

Proof.

$$\begin{aligned} < g(\lambda)^*(V), \overline{g(\lambda)^*(V)} > &=< g(\lambda)^*(V), \overline{g(\lambda)^*}(\bar{V}) > \\ &=< V, g(\lambda)g(\bar{\lambda})^*(\bar{V}) > \\ &=< V, \bar{V} >= 0. \end{aligned}$$

So $\tilde{V}$ is perpendicular to $\overline{\tilde{V}}$. ∎

Proposition 4.1.6. Let π be a Hermitian projection of $\mathbb{C}^n$, and $\pi_2 = J_m \bar{\pi} J_m^{-1}$ such that $\pi\pi_2 = \pi_2\pi$, where $n = 2m$. Then

$$f_{z,\pi}(\lambda) = (\pi + \frac{\lambda - z}{\lambda - \bar{z}}\pi^{\perp})(\pi_2 + \frac{\lambda - \bar{z}}{\lambda - z}\pi_2^{\perp}). \tag{4.7}$$

satisfies the $Sp(n)$-reality condition.

Proof. Since $f_{z,\pi}(\lambda)$ is a product of simple elements with one pole, $f(\bar{\lambda})^* f(\lambda) = I_n$, i.e., $f(\lambda)^{-1} = f(\bar{\lambda})^*$. Note that $J_m^{-1} = -J_m$, then we have

$$\begin{aligned}
f_{z,\pi}(\lambda)^t J_m &= (\pi + \tfrac{\lambda-z}{\lambda-\bar{z}}\pi^{\perp})(J_m\bar{\pi}J_m^{-1} + \tfrac{\lambda-\bar{z}}{\lambda-z}J_m\bar{\pi}^{\perp}J_m^{-1})J_m \\
&= J_m(-J_m\pi J_m\bar{\pi} - \tfrac{\lambda-\bar{z}}{\lambda-z}J_m\pi J_m\bar{\pi}^{\perp} - \tfrac{\lambda-z}{\lambda-\bar{z}}J_m\pi^{\perp}J_m\bar{\pi} - J_m\pi^{\perp}J_m\bar{\pi}^{\perp}) \\
&= J_m(J_m\pi J_m^{-1}\bar{\pi} + \tfrac{\lambda-\bar{z}}{\lambda-z}J_m\pi J_m^{-1}\bar{\pi}^{\perp} + \tfrac{\lambda-z}{\lambda-\bar{z}}J_m\pi^{\perp}J_m^{-1}\bar{\pi} + J_m\pi^{\perp}J_m^{-1}\bar{\pi}^{\perp}) \\
&= J_m(\bar{\pi}_2 + \tfrac{\lambda-z}{\lambda-\bar{z}}\bar{\pi}_2^{\perp})(\bar{\pi} + \tfrac{\lambda-\bar{z}}{\lambda-z}\bar{\pi}^{\perp}) = J_m f_{z,\pi}(\bar{\lambda})^*.
\end{aligned}$$

This proves that $f_{z,\pi}(\lambda)$ satisfies the first equation in (4.4). □

Example 4.1.7. Let $V_1 = \mathbb{C}\begin{pmatrix} r \\ \mathrm{i}\, s \end{pmatrix}$, where $r, s \in \mathbb{R}^{n\times 1}$ with $||r|| = ||s||$ and $V_2 = J_n(\overline{V_1})$. Then $V_1 \perp V_2$.

Lemma 4.1.8. Suppose V_1 is a complex subspace of $\mathbb{C}^{2n}$, $V_2 = J_n(\bar{V}_1)$ such that $< V_1, V_2 >= 0$, and $g(\lambda)$ satisfies the $Sp(n)$-reality condition (4.4). Let $\tilde{V}_1 = g(\lambda)^*(V_1)$ and $\tilde{V}_2 = J_n(\bar{\tilde{V}}_1)$. Then $\tilde{V}_2 \perp \tilde{V}_1$.

Proof.

$$\begin{aligned}
< \tilde{V}_1, \tilde{V}_2 > &=< g(\lambda)^*(V_1), g(\bar{\lambda})^* J_n(\bar{V}_1) > \\
&=< V_1, g(\lambda)g(\bar{\lambda})^* J_n(\bar{V}_1) > \\
&=< V_1, V_2 >= 0.
\end{aligned}$$

So $\tilde{V}_2$ is perpendicular to $\tilde{V}_1$.

□

Theorem 4.1.9. [BT for $\frac{U}{K}$-NLS]

Let $E(x,t,\lambda)$ be the extended frame of a solution u of $\frac{U}{K}$-NLS and $z \in \mathbb{C} \setminus \mathbb{R}$. Let $h_{z,\pi}$ be a simple element for $U = U(n), O(n),$ and $Sp(n)$ with the Hermitian projection π onto a complex vector subspace V satisfying

1. $h_{z,\pi} = k_{z,\pi}$ *defined by* (4.5) *for* $U = U(n)$,
2. $h_{z,\pi} = p_{z,\pi}$ *defined by* (4.6) *and* $V \perp \bar{V}$ *for* $U = O(n)$,
3. $h_{z,\pi} = f_{z,\pi}$ *defined by* (4.7) *and* $V \perp J_n(\bar{V})$ *for* $U = Sp(n)$.

Set

$$\begin{aligned} \tilde{V}(x,t) &= E(x,t,z)^*(V), \\ \tilde{\pi}(x,t) &= \text{ the Hermitian projection of } \mathbb{C}^n \text{ onto } \tilde{V}(x,t), \\ \tilde{E}(x,t,\lambda) &= h_{z,\pi}(\lambda)E(x,t,\lambda)h_{z,\tilde{\pi}(x,t)}(\lambda)^{-1}. \end{aligned}$$

Then $\tilde{E}$ is holomorphic for $\lambda \in \mathbb{C}$ and

1. **(i)** $\tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a]$ *is a new solution of* $\frac{U}{K}$*-NLS for* $U = U(n)$.

   (ii) $\tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a] + (\bar{z} - z)[\bar{\tilde{\pi}}, a]$ *is a new solution of* $\frac{U}{K}$*-NLS for* $U = O(n)$.

   (iii) $\tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a] + (\bar{z} - z)[\bar{\tilde{\pi}}_2, a]$ *is a new solution of* $\frac{U}{K}$*-NLS for* $U = Sp(n)$*, where* $\tilde{\pi}_2 = J_n \bar{\tilde{\pi}} J_n^{-1}$.

2. $\tilde{E}(x,t,\lambda)$ *is an extended frame of* $\tilde{u}$.

Proof. We will prove this theorem for the case $U = Sp(n)$ and similar arguments prove the other two cases. We first claim that $\tilde{E}$ is holomorphic for $\lambda \in \mathbb{C}$. From (4.7), we see that

$$h_{z,\pi} = (\mathrm{I} + \frac{\bar{z} - z}{\lambda - \bar{z}}\pi_2 + \frac{z - \bar{z}}{\lambda - z}\pi).$$

The residue of $\tilde{E}$ at $\lambda = z$ is

$$R_z = (z - \bar{z})(\pi E(x,t,z)(\mathrm{I} - \tilde{\pi}) + (\mathrm{I} - \pi_2)E(x,t,z)\tilde{\pi}_2).$$

Note that $\tilde{V}_1 = E(x,t,z)^*(V_1)$ is equivalent to $V_1 = E(x,t,\bar{z})(\tilde{V}_1)$, we have the inner product

$$\begin{aligned}
< V_1, E(x,t,z)(\tilde{V}_1^{\perp}) > &=< E(x,t,\bar{z})(\tilde{V}_1), E(x,t,z)(\tilde{V}_1^{\perp}) > \\
&=< E(x,t,z)^* E(x,t,\bar{z})(\tilde{V}_1), \tilde{V}_1^{\perp} > \\
&=< \tilde{V}_1, \tilde{V}_1^{\perp} > \\
&= 0.
\end{aligned}$$

This says that V_1 is perpendicular to $E(x,t,z)(\tilde{V}_1^{\perp})$, i.e., $\pi E(x,t,z)(\mathrm{I} - \tilde{\pi}) = 0$. Similarly,

$$\begin{aligned}
< V_2^{\perp}, E(x,t,z)(\tilde{V}_2) > &=< V_2^{\perp}, E(x,t,z)E(x,t,\bar{z})^* J_n(\bar{V}_1) > \\
&=< V_2^{\perp}, J_n(\bar{V}_1) > \\
&=< V_2^{\perp}, V_2 > \\
&= 0.
\end{aligned}$$

This implies $(\mathrm{I} - \pi_2)E(x,t,z)\tilde{\pi}_2) = 0$. So, $R_z = 0$, i.e., $\tilde{E}$ is holomorphic at $\lambda = z$. Since $\tilde{E}$ satisfies the $Sp(n)$-reality condition, it is also holomorphic at $\lambda = \bar{z}$. Let $\tilde{h}$ denote $h_{z,\tilde{\pi}(x,t)}$ and we expand $\tilde{h}$ at $\lambda = \infty$ as follows:

$$\tilde{h} = \mathrm{I} + \tilde{m}_1(x,t)\lambda^{-1} + \tilde{m}_2(x,t)\lambda^{-2} + \cdots.$$

Since $\tilde{E}$ is holomorphic for $\lambda \in \mathbb{C}$, so is $\tilde{E}^{-1}\tilde{E}_x$. A direct computation shows

$$\begin{aligned}\tilde{E}^{-1}\tilde{E}_x &= \tilde{h}E^{-1}E_x\tilde{h}^{-1} - \tilde{h}_x\tilde{h}^{-1} \\ &= (a\lambda + u + [\tilde{m}_1, a]) + O(\lambda^{-1}).\end{aligned}$$

So, $\tilde{E}^{-1}\tilde{E}_x - (a\lambda + u + [\tilde{m}_1, a])$ is holomorphic, bounded in $\lambda \in \mathbb{C}$, and tends to 0 as $\lambda \to \infty$. By Liouville Theorem,

$$\tilde{E}^{-1}\tilde{E}_x = (a\lambda + u + [\tilde{m}_1, a]).$$

So, $\tilde{E}$ is an extended frame for $\tilde{u} = u + [\tilde{m}_1, a]$, where $\tilde{m}_1$ can be computed as

$$(z - \bar{z})[\tilde{\pi}, a] + (\bar{z} - z)[\tilde{\pi}_2, a].$$

□

As a consequence of Theorems 2.2.3 and 4.1.9, we have:

Corollary 4.1.10. [BT for Schrödinger flow on $\mathrm{Gr}(k, \mathbb{C}^n)$]

Suppose γ is a solution of Schrödinger flow on $Gr(k, \mathbb{C}^n)$, g, u as in Theorem 2.2.3. Let $E(x, t, \lambda)$ be the extended frame of u such that $E(0, 0, \lambda) = g(0, 0)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, V a complex vector subspace of $\mathbb{C}^n$, $\tilde{\pi}(x, t)$ the Hermitian projection of $\mathbb{C}^n$ onto $\tilde{V}(x, t) = E(x, t, z)^*(V)$, and $g_1(x, t) = g(x, t)(\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}^{\perp})$. Then

1. $\tilde{\gamma} := h_{z,\pi} * \gamma = g_1 a g_1^{-1} = \gamma + (1 - \frac{z}{\bar{z}})g[\tilde{\pi}a, \tilde{\pi}]g^{-1} + (1 - \frac{\bar{z}}{z})g[\tilde{\pi}, a\tilde{\pi}]g^{-1}$ is again a solution of the Schrödinger flow on $Gr(k, \mathbb{C}^n)$.

2. $\tilde{u} = g_1^{-1}(g_1)_x = u + (z - \bar{z})[\tilde{\pi}, a]$ is a solution of $\frac{U(n)}{U(k)\times(n-k)}$-NLS associated to $\tilde{\gamma}$.

Proof. (2) follows from Theorem 4.1.9. Use $\tilde{\pi}^2 = \tilde{\pi}$ to compute a new solution

$$\begin{aligned}
\tilde{\gamma} &:= g_1 a g_1^{-1} \\
&= g(\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}^{\perp}) a (\tilde{\pi} + \frac{z}{\bar{z}}\tilde{\pi}^{\perp}) g^{-1} \\
&= g(a + (\frac{\bar{z}}{z} - 1) a\tilde{\pi} + (\frac{z}{\bar{z}} - 1)\tilde{\pi} a + (2 - \frac{\bar{z}}{z} - \frac{z}{\bar{z}})\tilde{\pi} a \tilde{\pi}) g^{-1} \\
&= \gamma + (1 - \frac{z}{\bar{z}}) g[\tilde{\pi} a, \tilde{\pi}] g^{-1} + (1 - \frac{\bar{z}}{z}) g[\tilde{\pi}, a\tilde{\pi}] g^{-1}.
\end{aligned}$$

□

Example 4.1.11. [1-soliton of the Schrödinger flow on $\mathbb{CP}^{n-1}$]

We start with the constant solution $\gamma = a$, then the corresponding solution of $\mathbb{CP}^{n-1}$-NLS is $u = 0$. The frame of $u = 0$ is

$$E(x, t, \lambda) = e^{a\lambda x + a\lambda^2 t}.$$

Let $z = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, $w = \begin{pmatrix} 1 \\ v \end{pmatrix}$ a complex vector with$v \in \mathbb{C}^{n\times 1}$, $v^* v = 1$, and $V = \mathbb{C}\begin{pmatrix} 1 \\ v \end{pmatrix}$. Let π be a Hermitian projection of $\mathbb{C}^n$ onto V, i.e.,

$$\pi = \frac{1}{2}\begin{pmatrix} 1 & v^* \\ v & vv^* \end{pmatrix}.$$

Then

$$h_{z,\pi}(\lambda) = I + \frac{\bar{z} - z}{\lambda - \bar{z}}\pi^{\perp}$$

satisfies the reality condition $h_{z,\pi}(\bar{\lambda})^* h_{z,\pi}(\lambda) = I$. Set

$$\tilde{w}(x,t) = (e^{azx+az^2t})^* \begin{pmatrix} 1 \\ v \end{pmatrix}.$$

Then the Hermitian projection $\tilde{\pi}(x,t)$ of $\mathbb{C}^n$ onto $\mathbb{C}\tilde{w}(x,t)$ is

$$\tilde{\pi}(x,t) = \frac{e^{-a\bar{z}x - a\bar{z}^2t} \begin{pmatrix} 1 & v^* \\ v & vv^* \end{pmatrix} e^{azx+az^2t}}{e^{-\beta x - 2\alpha\beta t} + e^{\beta x + 2\alpha\beta t}}.$$

and

$$g_1(x,t) = (\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}^\perp).$$

So,

$$\tilde{\gamma} = (\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}^\perp) a (\tilde{\pi} + \frac{z}{\bar{z}}\tilde{\pi}^\perp).$$

is a solution of the Schrödinger flow on $\mathbb{CP}^{n-1}$. Since

$$\tilde{E}(x,t,\lambda) = e^{a(\lambda x + \lambda^2 t)} h_{z,\tilde{\pi}(x,t)}(\lambda)^{-1}$$

is an extended frame for $\tilde{u}$, we can apply Theorem 4.1.10 again to get another family of solutions of the Schrödinger flow on $\mathbb{CP}^{n-1}$. Repeat this process to get an infinitely many families of explicit solutions of the Schrödinger flow on $\mathbb{CP}^{n-1}$.

Next we state our results for Bäcklund transformations of the Schrödinger flow on $\mathrm{Gr}(2, \mathbb{R}^{n+2})$.

Corollary 4.1.12. [Bäcklund transformations for $Gr(2, \mathbb{R}^{n+2})$]

Let γ be a solution of the Schrödinger flow on $Gr(2, \mathbb{R}^{n+2})$ with $\gamma = gag^{-1}$, $u = g^{-1}g_x$, a solution of the $Gr(2, \mathbb{R}^{n+2})$-NLS as in Theorem 2.2.3, and E an extended frame of u with $E(0,0,\lambda) = g(0,0)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, V a complex vector subspace of $\mathbb{C}^{n+2}$ satisfying $V \perp \bar{V}$, $\tilde{\pi}(x,t)$ the Hermitian projection of $\mathbb{C}^{n+2}$ onto $\tilde{V}(x,t) = E(x,t,z)^*(V)$ and $g_1(x,t) = g(x,t)(\frac{z}{\bar{z}}\bar{\tilde{\pi}} + \frac{\bar{z}}{z}\tilde{\pi})$. Then

1. $\tilde{\gamma} := p_{z,\pi} * \gamma = g_1 a g_1^{-1} = 2g\mathrm{Re}(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2\bar{\tilde{\pi}}a\tilde{\pi})g^{-1}$ is a new solution of the Schrödinger flow on $Gr(2,\mathbb{R}^{n+2})$.

2. $\tilde{u} = g_1^{-1}(g_1)_x = u + (z-\bar{z})[\tilde{\pi}, a] + (\bar{z}-z)[\bar{\tilde{\pi}}, a]$ is a solution of the $\mathrm{Gr}(2,\mathbb{R}^{n+2})$-NLS associated to $\tilde{\gamma}$.

Proof. (2) follows from Theorem 4.1.9. Use $\tilde{\pi}^2 = \tilde{\pi}$ to compute a new solution

$$\begin{aligned}
\tilde{\gamma} &:= g_1 a g_1^{-1} \\
&= g(\frac{z}{\bar{z}}\bar{\tilde{\pi}} + \frac{\bar{z}}{z}\tilde{\pi})a(\frac{z}{\bar{z}}\tilde{\pi} + \frac{\bar{z}}{z}\bar{\tilde{\pi}}t)g^{-1} \\
&= g(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2\bar{\tilde{\pi}}a\tilde{\pi} + \bar{\tilde{\pi}}a\bar{\tilde{\pi}} + (\frac{\bar{z}}{z})^2\tilde{\pi}a\bar{\tilde{\pi}})g^{-1} \\
&= 2g\mathrm{Re}(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2\bar{\tilde{\pi}}a\tilde{\pi})g^{-1}.
\end{aligned}$$

□

Corollary 4.1.13. [Bäcklund transformations for $\frac{SO(2n)}{U(n)}$]

Let γ be a solution of the Schrödinger flow on $\frac{SO(2n)}{U(n)}$ with $\gamma = gag^{-1}$, $u = g^{-1}g_x$ the corresponding solution of the $\frac{SO(2n)}{U(n)}$-NLS as in Theorem 2.2.3, and $E(x,t,\lambda)$ the extended frame of u with $E(0,0,\lambda) = g(0,0)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, V a complex vector subspace of $\mathbb{C}^{2n}$ satisfying $V \perp \bar{V}$, $\tilde{\pi}(x,t)$ denote the Hermitian projection of $\mathbb{C}^{2n}$ onto $\tilde{V}(x,t) = E(x,t,z)^*(V)$ and $g_1(x,t) = g(x,t)(\frac{z}{\bar{z}}\bar{\tilde{\pi}} + \frac{\bar{z}}{z}\tilde{\pi})$. Then

1. $\tilde{\gamma} := p_{z,\pi} * \gamma = g_1 a g_1^{-1} = 2\mathrm{Re}(g(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2\bar{\tilde{\pi}}a\tilde{\pi})g^{-1})$ is a new solution of the Schrödinger flow on $\frac{SO(2n)}{U(n)}$.

2. $\tilde{u} = g_1^{-1}(g_1)_x = u + (z - \bar{z})[\tilde{\pi}, a] + (\bar{z} - z)[\bar{\tilde{\pi}}, a]$ is a solution of the $\frac{SO(2n)}{U(n)}$-NLS associated to $\tilde{\gamma}$.

The proof follows from a similar argument of the proof in Theorem 4.1.12.

Example 4.1.14. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $V = \mathbb{C}\begin{pmatrix} r \\ is \end{pmatrix}$, where $r \in \mathbb{R}^{n\times 1}$ and $s \in \mathbb{R}^{n\times 1}$ are unit vectors. Set

$$\tilde{V}(x,t) = \text{span}\{e^{-(a\bar{z}x + a\bar{z}^2 t)}\begin{pmatrix} r \\ is \end{pmatrix}\},$$

$$\tilde{\pi}(x,t) = \frac{e^{a(-\bar{z}x - \bar{z}^2 t)}\begin{pmatrix} rr^t & -irs^t \\ isr^t & ss^t \end{pmatrix} e^{a(zx + z^2 t)}}{2(\cos w) - i(\sin w) r^t s},$$

$$g_1(x,t) = \frac{z}{\bar{z}}\bar{\tilde{\pi}} + \frac{\bar{z}}{z}\tilde{\pi},$$

where $w = \frac{(z-\bar{z})x + (z^2 - \bar{z}^2)t}{2}$.

Then, $\tilde{\gamma} = g_1 a g_1^{-1}$ is a new solution of the Schrödinger flow on $\frac{SO(2n)}{U(n)}$.

Corollary 4.1.15. [Bäcklund transformations for $\frac{Sp(n)}{U(n)}$]

Let γ be a solution of the Schrödinger flow on $\frac{Sp(n)}{U(n)}$ with $\gamma = gag^{-1}$, $u = g^{-1}g_x$, a solution of the $\frac{Sp(n)}{U(n)}$-NLS as in Theorem 2.2.3. Let $E(x,t,\lambda)$ be the extended frame of u such that $E(0,0,\lambda) = g(0,0)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, V a complex subspace satisfying $V \perp J_n(\bar{V})$, $\tilde{\pi}(x,t)$ be the Hermitian projection of $\mathbb{C}^n$ onto $\tilde{V}(x,t) = E(x,t,z)^*(V)$, $\tilde{\pi}_2(x,t) = J_n \bar{\tilde{\pi}}(x,t) J_n^{-1}$, and $g_1(x,t) = g(x,t)(\frac{z}{\bar{z}}\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}_2)$. Then

1. $\tilde{\gamma} = f_{z,\pi} * \gamma = g_1 a g_1^{-1} = g(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2 \tilde{\pi}a\tilde{\pi}_2 + \tilde{\pi}_2 a \tilde{\pi}_2 + (\frac{\bar{z}}{z})^2 \tilde{\pi}_2 a \tilde{\pi})g^{-1}$ is again a solution of the Schrödinger flow on $\frac{Sp(n)}{U(n)}$.

2. $\tilde{u} = g_1^{-1}(g_1)_x = u + (z - \bar{z})[\tilde{\pi}, a] + (\bar{z} - z)[\tilde{\pi}_2, a]$ is a solution of $\frac{Sp(n)}{U(n)}$-NLS associated to $\tilde{\gamma}$.

Proof. (2) follows from Theorem 4.1.9. We compute to get

$$\begin{aligned}
\tilde{\gamma} &:= g_1 a g_1^{-1} \\
&= g(\frac{z}{\bar{z}}\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}_2)a(\frac{z}{\bar{z}}\tilde{\pi}_2 + \frac{\bar{z}}{z}\bar{\tilde{\pi}})g^{-1} \\
&= g(\tilde{\pi}a\tilde{\pi} + (\frac{z}{\bar{z}})^2\tilde{\pi}a\tilde{\pi}_2 + \tilde{\pi}_2 a\tilde{\pi}_2 + (\frac{\bar{z}}{z})^2\tilde{\pi}_2 a\tilde{\pi})g^{-1}.
\end{aligned}$$

□

Example 4.1.16. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $V = \mathbb{C}\begin{pmatrix} r \\ is \end{pmatrix}$, where $r \in \mathbb{R}^{n\times 1}$ and $s \in \mathbb{R}^{n\times 1}$ are unit vectors. Set

$$\begin{aligned}
\tilde{V}_1(x,t) &= \text{span}\{\exp(-(a\bar{z}x + a\bar{z}^2 t))(r, is)^t\}, \\
\tilde{\pi}(x,t) &= \frac{e^{a(-\bar{z}x - \bar{z}^2 t)}\begin{pmatrix} rr^t & -irs^t \\ isr^t & ss^t \end{pmatrix} e^{a(zx + z^2 t)}}{2(\cos w) - i(\sin w)r^t s}, \\
\tilde{\pi}_2(x,t) &= J\bar{\tilde{\pi}}(x,t)J^{-1}, \\
g_1(x,t) &= \frac{z}{\bar{z}}\tilde{\pi} + \frac{\bar{z}}{z}\tilde{\pi}_2,
\end{aligned}$$

where $w = \frac{(z-\bar{z})x + (z^2 - \bar{z}^2)t}{2}$.

Then, $\tilde{\gamma} = g_1 a g_1^{-1}$ is a new solution of Schrödinger flow on $\frac{Sp(n)}{U(n)}$.

4.2 Bäcklund Transformations for Derivative $\frac{U}{K}$-NLS

Let u be a solution of the derivative $\frac{U}{K}$-NLS, and E an extended frame of u. We follow the steps in previous section to construct Bäcklund transformations for derivative $\frac{U}{K}$-NLS, which has been obtained using a different approach in [10].

The reality conditions for derivative $\frac{U}{K}$-NLS on Hermitian symmetric space $\frac{U}{K}$ are stated below.

Definition 4.2.1 (Reality Conditions)**.**

Given $f(\lambda) \in GL(n, \mathbb{C})$, we say that

1. f satisfies the $\frac{U(n)}{U(k)\times U(n-k)}$-*reality condition* if

$$f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_n, \quad f(-\lambda) = I_{k,n-k} f(\lambda) I_{k,n-k}^{-1}. \tag{4.8}$$

2. f satisfies the $\frac{O(n+2)}{O(2)\times O(n)}$-*reality condition* if

$$f(\lambda)^t f(\lambda) = \mathrm{I}_{n+2}, \quad \overline{f(\bar{\lambda})} = f(\lambda), \quad f(-\lambda) = I_{2,n} f(\lambda) I_{2,n}^{-1}. \tag{4.9}$$

3. $n = 2m$, f satisfies the $\frac{O(n)}{U(m)}$-*reality condition* if

$$f(\lambda)^t f(\lambda) = \mathrm{I}_n, \quad \overline{f(\bar{\lambda})} = f(\lambda), \quad f(-\lambda) = J_m f(\lambda) J_m^{-1}. \tag{4.10}$$

4. $n = 2m$, f satisfies the $\frac{Sp(n)}{U(m)}$-*reality condition* if

$$f(\lambda)^t J_m f(\lambda) = J_m, \quad f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_n, \quad f(\lambda)^t f(-\lambda) = I_n. \tag{4.11}$$

Below we construct simple elements in $L_{-}^{\tau,\sigma}(G)$. Note that a simple lament with only one pole does not satisfies the $\frac{U}{K}$-reality conditions, so we consider simple elements with two poles, i.e.,

$$g_{\alpha,\beta,\pi_1,\pi_2}(\lambda) = (I + \frac{\alpha - \bar{\alpha}}{\lambda - \alpha}\pi_1^{\perp})(I + \frac{\beta - \bar{\beta}}{\lambda - \beta}\pi_2^{\perp}).$$

We also note that $g_{\alpha,\beta,\pi_1,\pi_2}(\bar{\lambda})^* g_{\alpha,\beta,\pi_1,\pi_2}(\lambda) = I_n$, provided that $\pi_1\pi_2 = \pi_2\pi_1$.

Proposition 4.2.2. Let π be a Hermitian projection of $\mathbb{C}^n$, $s \in \mathbb{R}$ nonzero, $\pi_2 = I_{k,n-k}\pi I_{k,n-k}^{-1}$ such that $\pi\pi_2 = \pi_2\pi$, and $k : \mathbb{C} \cup \{\infty\} \to GL(n,\mathbb{C})$ a rational map defined by

$$k_{is,\pi}(\lambda) = (I + \frac{-2is}{\lambda + is}\pi^{\perp})(I + \frac{2is}{\lambda - is}\pi_2^{\perp}). \tag{4.12}$$

Then $k_{is,\pi}$ satisfies the $\frac{U(n)}{U(k)\times U(n-k)}$-reality condition.

Proof. Since $\pi_2 I_{k,n-k} = I_{k,n-k}\pi$,

$$\begin{aligned}
I_{k,n-k}k_{is,\pi}(\lambda) &= I_{k,n-k}(I + \tfrac{-2is}{\lambda+is}\pi^{\perp} + \tfrac{2is}{\lambda-is}\pi_2^{\perp} + \tfrac{-2is}{\lambda+is}\pi^{\perp}\tfrac{2is}{\lambda-is}\pi_2^{\perp}) \\
&= (I + \tfrac{-2is}{\lambda+is}\pi_2^{\perp} + \tfrac{2is}{\lambda-is}\pi^{\perp} + \tfrac{-2is}{\lambda+is}\pi_2^{\perp}\tfrac{2is}{\lambda-is}\pi^{\perp})I_{k,n-k} \\
&= (I + \tfrac{-2is}{-\lambda+is}\pi^{\perp})(I + \tfrac{2is}{-\lambda-is}\pi_2^{\perp})I_{k,n-k} \\
&= k_{is,\pi}(-\lambda)I_{k,n-k},
\end{aligned}$$

as desired. ∎

Proposition 4.2.3. Let π be a Hermitian projection of $\mathbb{C}^{n+2}$ onto V and $s \in \mathbb{R}\setminus\{0\}$. If $\overline{V} = I_{2,n}V$ and $\overline{V}\perp V$, then

$$p_{is,\pi}(\lambda) = (I + \frac{2is}{\lambda - is}\pi^{\perp})(I + \frac{-2is}{\lambda + is}\bar{\pi}^{\perp}).$$

satisfies the $\frac{O(n+2)}{O(2)\times O(n)}$-reality condition.

Proof. Since $V \perp \bar{V}$, $\pi\bar{\pi} = \bar{\pi}\pi$. Then

$$\begin{aligned} p_{is,\pi}(\bar{\lambda}) &= (I + \tfrac{2is}{\bar{\lambda}-is}\pi^{\perp})(I + \tfrac{-2is}{\bar{\lambda}+is}\bar{\pi}^{\perp}) \\ &= \overline{(I + \tfrac{-2is}{\lambda+is}\bar{\pi}^{\perp})}\ \overline{(I + \tfrac{2is}{\lambda-is}\pi^{\perp})} \\ &= \overline{p_{is,\pi}(\lambda)}. \end{aligned}$$

In particular, $p_{is,\pi}(\lambda)^t p_{is,\pi}(\lambda) = I_{n+2}$ because $p_{is,\pi}(\bar{\lambda})^* p_{is,\pi}(\lambda) = I_{n+2}$. The last equation in (4.9) follows from the same computation as in the proof of Proposition 4.2.2. □

Proposition 4.2.4. Let π_1 be a Hermitian projection of $\mathbb{C}^n$, $s \in \mathbb{R} \setminus \{0\}$, and $\pi_2 = J_m \pi_1 J_m^{-1}$ such that $\pi_1\pi_2 = \pi_2\pi_1$. Then

$$f_{is,\pi}(\lambda) = (I + \frac{2is}{\lambda - is}\pi_1^{\perp})(I + \frac{-2is}{\lambda + is}\pi_2^{\perp}). \tag{4.13}$$

satisfies the $\frac{O(n)}{U(m)}$-reality condition with $n = 2m$.

Proof. The first two equations in (4.10) follow from Proposition 4.2.3.We prove the last one in (4.10). Note that $J_m^{-1} = -J_m$, then we have

$$\begin{aligned} f_{is,\pi}(\lambda)J_m &= (I + \tfrac{2is}{\lambda-is}\pi_1^{\perp})(I + \tfrac{-2is}{\lambda+is}\pi_2^{\perp})J_m \\ &= (I + \tfrac{2is}{\lambda-is}\pi_1^{\perp} + \tfrac{-2is}{\lambda+is}\pi_2^{\perp} + \tfrac{2is}{\lambda-is}\pi_1^{\perp}\tfrac{-2is}{\lambda+is}\pi_2^{\perp})J_m \\ &= J_m(I + \tfrac{2is}{\lambda-is}\pi_2^{\perp} + \tfrac{-2is}{\lambda+is}\pi_1^{\perp} + \tfrac{2is}{\lambda-is}\pi_2^{\perp}\tfrac{-2is}{\lambda+is}\pi_1^{\perp}) \\ &= J_m(I + \tfrac{2is}{\lambda-is}\pi_2^{\perp})(I + \tfrac{-2is}{\lambda+is}\pi_1^{\perp}) = J_m f_{is,\pi}(-\lambda). \end{aligned}$$

□

Proposition 4.2.5. Let π be a Hermitian projection of $\mathbb{C}^n$, $s \in \mathbb{R} \setminus \{0\}$, and $\pi_2 =$

$J_m\pi J_m^{-1}$ such that $\bar{\pi} = \pi, \pi\pi_2 = \pi_2\pi$. Then

$$g_{is,\pi}(\lambda) = (I + \frac{2is}{\lambda - is}\pi^{\perp})(I + \frac{-2is}{\lambda + is}\pi_2^{\perp}). \tag{4.14}$$

satisfies the $\frac{Sp(n)}{U(m)}$-reality condition with $n = 2m$.

Proof. It suffices to show $g_{is,\pi}(\lambda)^t J_m g_{is,\pi}(\lambda) = J_m$ and $g_{is,\pi}(\lambda)^t g_{is,\pi}(-\lambda) = I_n$. Since $\pi^* = \pi$ and $\bar{\pi} = \pi$, $\pi^t = \pi$, and so does π_2. Then

$$g_{is,\pi}(\lambda)^t g_{is,\pi}(-\lambda) = (I + \frac{-2is}{\lambda + is}\pi_2^{\perp})(I + \frac{2is}{\lambda - is}\pi^{\perp})(I + \frac{2is}{-\lambda - is}\pi^{\perp})(I + \frac{-2is}{-\lambda + is}\pi_2^{\perp}) = I_n.$$

And a direct computation implies

$$\begin{aligned} g_{is,\pi}(\lambda)^t J_m &= (I + \tfrac{-2is}{\lambda+is}\pi_2^{\perp})^t (I + \tfrac{2is}{\lambda-is}\pi^{\perp})^t J_m \\ &= (I + \tfrac{2is}{\lambda-is}\pi^{\perp} + \tfrac{-2is}{\lambda+is}\pi_2^{\perp} + \tfrac{2is}{\lambda-is}\pi^{\perp}\tfrac{-2is}{\lambda+is}\pi_2^{\perp}) J_m \\ &= J_m (I + \tfrac{2is}{\lambda-is}\pi_2^{\perp} + \tfrac{-2is}{\lambda+is}\pi^{\perp} + \tfrac{2is}{\lambda-is}\pi_2^{\perp}\tfrac{-2is}{\lambda+is}\pi^{\perp}) \\ &= J_m (I + \tfrac{2is}{\lambda-is}\pi_2^{\perp})(I + \tfrac{-2is}{\lambda+is}\pi^{\perp}) = J_m g_{is,\pi}(-\lambda)^t. \end{aligned}$$

So, $J_m = g_{is,\pi}(\lambda)^t J_m (g_{is,\pi}(-\lambda)^t)^{-1} = g_{is,\pi}(\lambda)^t J_m g_{is,\pi}(\lambda)$. □

Theorem 4.2.6. [BT for Derivative $\frac{U}{K}$-NLS]

Let $E(x, t, \lambda)$ be the extended frame of a solution u of the derivative $\frac{U}{K}$-NLS and $s \in \mathbb{R} \setminus \{0\}$. Let $h_{\mathrm{is},\pi}$ be a simple element for $U = U(n), O(n)$, and $Sp(n)$ with the Hermitian projection π onto a complex vector subspace V satisfying

1. *$\mathrm{his}, \pi = k_{\mathrm{is},\pi}$ defined by (4.12) for $U = U(n)$,*
2. *$\mathrm{his}, \pi = f_{\mathrm{is},\pi}$ defined by (4.13) and $V \perp \bar{V}$ for $U = O(n)$,*
3. *$h_{\mathrm{is},\pi} = g_{\mathrm{is},\pi}$ defined by (4.14) and $V \perp J_n(\bar{V})$ for $U = Sp(n)$.*

Set

$$\begin{aligned}\tilde{V}(x,t) &= E(x,t,is)^*(V),\\ \tilde{\pi}(x,t) &= \text{ the Hermitian projection of } \mathbb{C}^n \text{ onto } \tilde{V}(x,t),\\ \tilde{E}(x,t,\lambda) &= h_{is,\pi}(\lambda)E(x,t,\lambda)h_{is,\tilde{\pi}(x,t)}(\lambda)^{-1}.\end{aligned}$$

Write

$$h_{is,\tilde{\pi}}(\lambda) = I + \tilde{m}_1(,x,t)\lambda^{-1} + \cdots .$$

Then

1. $\tilde{u} = u + [\tilde{m}_1, a]$ *is a new solution of the derivative* $\frac{U}{K}$*-NLS.*

2. $\tilde{E}(x,t,\lambda)$ *is an extended frame of* $\tilde{u}$*.*

Proof. We will prove this theorem for the case $U = Sp(n)$ and similar arguments prove the other two cases. We first claim that $\tilde{E}$ is holomorphic for $\lambda \in \mathbb{C}$. From (4.14), we see that

$$h_{\mathrm{i}\,s,\pi} = (\mathrm{I} + \frac{-2\mathrm{i}\,s}{\lambda + \mathrm{i}\,s}\pi_2 + \frac{2\mathrm{i}\,s}{\lambda - \mathrm{i}\,s}\pi).$$

The residue of $\tilde{E}$ at $\lambda = \mathrm{i}\,s$ is

$$R_{\mathrm{i}\,s} = 2\mathrm{i}\,s(\pi E(x,t,\mathrm{i}\,s)(\mathrm{I} - \tilde{\pi}) + (\mathrm{I} - \pi_2)E(x,t,\mathrm{i}\,s)\tilde{\pi}_2).$$

Note that $\tilde{V}_1 = E(x,t,\mathrm{i}\,s)^*(V_1)$ is equivalent to $V_1 = E(x,t,-\mathrm{i}\,s)(\tilde{V}_1)$, we have the inner product

$$\begin{aligned}< V_1, E(x,t,\mathrm{i}\,s)(\tilde{V}_1^{\perp}) > &=< E(x,t,-\mathrm{i}\,s)(\tilde{V}_1), E(x,t,\mathrm{i}\,s)(\tilde{V}_1^{\perp}) >\\ &=< E(x,t,\mathrm{i}\,s)^*E(x,t,-\mathrm{i}\,s)(\tilde{V}_1), \tilde{V}_1^{\perp} >\\ &=< \tilde{V}_1, \tilde{V}_1^{\perp} >\\ &= 0.\end{aligned}$$

This says that V_1 is perpendicular to $E(x,t,\mathrm{i}\,s)(\tilde{V}_1^{\perp})$, i.e., $\pi E(x,t,\mathrm{i}\,s)(\mathrm{I}-\tilde{\pi}) = 0$. Similarly,

$$\begin{aligned} < V_2^{\perp}, E(x,t,\mathrm{i}\,s)(\tilde{V}_2) > &=< V_2^{\perp}, E(x,t,\mathrm{i}\,s)E(x,t,-\mathrm{i}\,s)^* J_n(\bar{V}_1) > \\ &=< V_2^{\perp}, J_n(\bar{V}_1) > \\ &=< V_2^{\perp}, V_2 > \\ &= 0. \end{aligned}$$

This implies $(\mathrm{I}-\pi_2)E(x,t,\mathrm{i}\,s)\tilde{\pi}_2) = 0$. So, $R_{\mathrm{i}\,s} = 0$, i.e., $\tilde{E}$ is holomorphic at $\lambda = \mathrm{i}\,s$. Since $\tilde{E}$ satisfies the $\frac{Sp(n)}{U(m)}$-reality condition, it is also holomorphic at $\lambda = -\mathrm{i}\,s$.

Note that $E^{-1}E_x = a\lambda^2 + u\lambda$, so

$$\begin{aligned} (\tilde{E}^{-1}\tilde{E}_x) &= h_{is,\tilde{\pi}}(a\lambda^2 + u\lambda)h_{is,\tilde{\pi}}^{-1} - \partial_x h_{is,\tilde{\pi}} h_{is,\tilde{\pi}}^{-1} \\ &= a\lambda^2 + (u + [\tilde{m}_1, a])\lambda + O(\lambda^{-1}). \end{aligned}$$

Since $\tilde{E}$ is holomorphic for $\lambda \in \mathbb{C}$, so is $\tilde{E}^{-1}\tilde{E}_x$. So, $\tilde{E}^{-1}\tilde{E}_x - a\lambda^2 - (u + [\tilde{m}_1, a])\lambda$ is holomorphic, bounded in $\lambda \in \mathbb{C}$, and tends to 0 as $\lambda \to \infty$. By Liouville Theorem,

$$\tilde{E}^{-1}\tilde{E}_x = a\lambda^2 + (u + [\tilde{m}_1, a])\lambda.$$

So, $\tilde{E}$ is an extended frame for $\tilde{u}$, where $\tilde{u} = u + [\tilde{m}_1, a]$. □

Corollary 4.2.7. [BT for Derivative $\frac{U(n)}{U(k)\times U(n-k)}$-NLS]

Let $E(x,t,\lambda)$ be the extended frame of a solution q of the derivative $\frac{U(n)}{U(k)\times U(n-k)}$-NLS. Let $s \in \mathbb{R}\setminus\{0\}$, $V = \mathbb{C}\begin{pmatrix} v \\ w \end{pmatrix}$ with unit vectors $v \in \mathbb{C}^k, w \in \mathbb{C}^{n-k}$, $\pi(=\pi_1)$ the

Hermitian projection of $\mathbb{C}^n$ onto V, and $\pi^\perp = I - \pi$. Set

$$\begin{aligned}
\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} &= \begin{pmatrix} \frac{\tilde{v}}{||\tilde{v}||} \\ \frac{\tilde{w}}{||\tilde{w}||} \end{pmatrix}, \text{ where } \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}(x,t) = E(x,t,is)^* \begin{pmatrix} v \\ w \end{pmatrix}, \\
\tilde{\pi}(x,t) &= \text{ the Hermitian projection of } \mathbb{C}^n \text{ onto } \tilde{V}(x,t), \\
\tilde{E}(x,t,\lambda) &= k_{is,\pi}(\lambda) E(x,t,\lambda) k_{is,\tilde{\pi}(x,t)}(\lambda)^{-1}.
\end{aligned}$$

Then

1. $\tilde{q} = q - 4s\hat{v}\hat{w}^*$ is a new solution of the derivative $\frac{U(n)}{U(k)\times U(n-k)}$-NLS.

2. $\tilde{E}(x,t,\lambda)$ is an extended frame of $\tilde{q}$.

Corollary 4.2.8. [BT for Derivative $\frac{SO(n+2)}{SO(2)\times SO(n)}$-NLS]

Let $E(x,t,\lambda)$ be the extended frame of a solution q of the derivative $\frac{SO(n+2)}{SO(2)\times SO(n)}$-NLS. Let $s \in \mathbb{R}\setminus\{0\}$, $V = \mathbb{C}\begin{pmatrix} v \\ iw \end{pmatrix}$ with unit vectors $v \in \mathbb{R}^2, w \in \mathbb{R}^n$, $\pi (= \pi_1)$ the Hermitian projection of $\mathbb{C}^{n+2}$ onto V, and $\pi^\perp = I - \pi$. Set

$$\begin{aligned}
\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} &= \begin{pmatrix} \frac{\tilde{v}}{||\tilde{v}||} \\ \frac{\tilde{w}}{||\tilde{w}||} \end{pmatrix}, \text{ where } \begin{pmatrix} \tilde{v} \\ i\tilde{w} \end{pmatrix}(x,t) = E(x,t,is)^* \begin{pmatrix} v \\ iw \end{pmatrix}, \\
\tilde{\pi}(x,t) &= \text{ the Hermitian projection of } \mathbb{C}^{n+2} \text{ onto } \tilde{V}(x,t), \\
\tilde{E}(x,t,\lambda) &= p_{is,\pi}(\lambda) E(x,t,\lambda) p_{is,\tilde{\pi}(x,t)}(\lambda)^{-1}.
\end{aligned}$$

Then

1. $\tilde{q} = q - sJ_1\hat{v}\hat{w}^t$ is a new solution of the derivative $\frac{SO(n+2)}{SO(2)\times SO(n)}$-NLS.

2. $\tilde{E}(x,t,\lambda)$ is an extended frame of $\tilde{q}$.

Corollary 4.2.9. [BT for Derivative $\frac{SO(2n)}{U(n)}$-NLS]

Let $E(x,t,\lambda)$ be the extended frame of a solution $\begin{pmatrix} q \\ r \end{pmatrix}$ of the derivative $\frac{SO(2n)}{U(n)}$-NLS.

Let $s \in \mathbb{R} \setminus \{0\}$, $V = \mathbb{C}\begin{pmatrix} v \\ w \end{pmatrix}$ with unit vectors $v \in \mathbb{R}^n, w \in \mathbb{R}^n$, $\pi(= \pi_1)$ the Hermitian projection of $\mathbb{C}^{2n}$ onto V, and $\pi^\perp = I - \pi$. Set

$$\begin{aligned}
\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} &= \begin{pmatrix} \frac{\tilde{v}}{||\tilde{v}||} \\ \frac{\tilde{w}}{||\tilde{w}||} \end{pmatrix}, \text{ where } \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}(x,t) = E(x,t,is)^* \begin{pmatrix} v \\ w \end{pmatrix}, \\
\tilde{\pi}(x,t) &= \text{the Hermitian projection of } \mathbb{C}^{2n} \text{ onto } \tilde{V}(x,t), \\
\tilde{E}(x,t,\lambda) &= f_{is,\pi}(\lambda)E(x,t,\lambda)f_{is,\tilde{\pi}(x,t)}(\lambda)^{-1}.
\end{aligned}$$

Then

1. $\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix} + is \begin{pmatrix} -(\hat{w}\hat{v}^t + \hat{v}\hat{w}^t) \\ \hat{v}\hat{v}^t - \hat{w}\hat{w}^t \end{pmatrix}$ is a new solution of the derivative $\frac{SO(2n)}{U(n)}$-NLS.

2. $\tilde{E}(x,t,\lambda)$ is an extended frame of $\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}$.

Corollary 4.2.10. [BT for Derivative $\frac{Sp(n)}{U(n)}$-NLS]

Let $E(x,t,\lambda)$ be the extended frame of a solution $\begin{pmatrix} q \\ r \end{pmatrix}$ of the derivative $\frac{Sp(n)}{U(n)}$-NLS.

Let $s \in \mathbb{R} \setminus \{0\}$, $V = \mathbb{C}\begin{pmatrix} v \\ w \end{pmatrix}$ with unit vectors $v \in \mathbb{R}^n, w \in \mathbb{R}^n$, $\pi(= \pi_1)$ the

Hermitian projection of $\mathbb{C}^{2n}$ onto V, and $\pi^\perp = I - \pi$. Set

$$\begin{aligned}
\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} &= \begin{pmatrix} \frac{\tilde{v}}{||\tilde{v}||} \\ \frac{\tilde{w}}{||\tilde{w}||} \end{pmatrix}, \text{ where } \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix}(x,t) = E(x,t,is)^* \begin{pmatrix} v \\ w \end{pmatrix}, \\
\tilde{\pi}(x,t) &= \text{ the Hermitian projection of } \mathbb{C}^{2n} \text{ onto } \tilde{V}(x,t), \\
\tilde{E}(x,t,\lambda) &= g_{is,\pi}(\lambda) E(x,t,\lambda) g_{is,\tilde{\pi}(x,t)}(\lambda)^{-1}.
\end{aligned}$$

Then

1. $\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix} + s \begin{pmatrix} -(\hat{w}\hat{v}^t + \hat{v}\hat{w}^t) \\ \hat{v}\hat{v}^t - \hat{w}\hat{w}^t \end{pmatrix}$ is a new solution of the derivative $\frac{Sp(n)}{U(n)}$-NLS.

2. $\tilde{E}(x,t,\lambda)$ is an extended frame of $\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}$.

4.3 Bäcklund Transformations for vector mKdV and Geometric Airy Curve Flow on $\mathbb{R}^n$

Definition 4.3.1 ($\frac{O(n+1)}{O(n)}$-Reality Condition)**.** A map $g : \mathbb{C} \to GL(n+1, \mathbb{C})$ satisfies the $\frac{O(n+1)}{O(n)}$-reality condition if and only if

$$\begin{cases} g(\lambda)^t g(\lambda) = \mathrm{I}, \\ \overline{g(\bar{\lambda})} = g(\lambda), \\ \mathrm{I}_{1,n} g(\lambda) \mathrm{I}_{1,n}^{-1} = g(-\lambda). \end{cases}$$

This is equivalent to

$$
(4.15) \qquad \begin{cases} g(\bar{\lambda})^* g(\lambda) = \mathrm{I}, \\ \mathrm{I}_{1,n} g(\lambda) \mathrm{I}_{1,n}^{-1} = g(-\lambda), \\ g(\lambda)^t g(\lambda) = \mathrm{I}. \end{cases}
$$

We now construct simple elements and show that they satisfy the $\frac{O(n+1)}{O(n)}$-reality condition.

Proposition 4.3.2. Let $s \in \mathbb{R}$, $v \in \mathbb{C}^{n+1}$, π_1 the Hermitian projection of $\mathbb{C}^{n+1}$ onto $\mathbb{C}v$, and π_2 the Hermitian projection onto $\mathbb{C}\mathrm{I}_{1,n}v$. Set

$$
(4.16) \qquad \phi_{\mathrm{i}\,s,v} := g_{\mathrm{i}\,s,\pi_2} g_{-\mathrm{i}\,s,\pi_1} = \left(\mathrm{I} + \frac{2\mathrm{i}\,s}{\lambda - \mathrm{i}\,s}\pi_2^{\perp}\right)\left(\mathrm{I} - \frac{2\mathrm{i}\,s}{\lambda + \mathrm{i}\,s}\pi_1^{\perp}\right).
$$

If v satisfies

$$
(4.17) \qquad v^* \mathrm{I}_{1,n} v = 0, \quad \bar{v} = \mathrm{I}_{1,n} v,
$$

then $\phi_{\mathrm{i}\,s,v}$ satisfies the $\frac{O(n+1)}{O(n)}$-reality condition.

Proof. Condition (4.17) implies that $\pi_1 \pi_2 = \pi_2 \pi_1 = 0$ and $\bar{\pi}_1 = \pi_2$. Then the condition (4.15) follows from a direct computation. □

Theorem 4.3.3 (Bäcklund transformation for vmKdV)**.**

Let u be a solution of vmKdV (1.19) *of the $\frac{SO(n+1)}{SO(n)}$ hierarchy, and $E(x,t,\lambda)$ the extended frame of u. Given $s \in \mathbb{R} \setminus 0$, $v \in \mathbb{C}^{n+1}$ satisfying* (4.17)*, then*

1. *$\tilde{v}(x,t) := E(x,t,-\mathrm{i}\,s)^{-1} v$ satisfying* (4.17)*,*

2. *let* $\tilde{\pi}_1(x,t)$ *denote the Hermitian projection of* $\mathbb{C}^{n+1}$ *onto* $\mathbb{C}\tilde{v}$ *and* $\tilde{\pi}_2(x,t)$ *the Hermitian projection onto* $\mathbb{C}\mathrm{I}_{1,n}\tilde{v}(x,t)$*, then*

$$\tilde{u} = u + 2\mathrm{i}\, s[\tilde{\pi}_1 - \tilde{\pi}_2, a],$$

where $a = e_{21} - e_{12}$.

3.
$$\tilde{E}(x,t,\lambda) := \phi_{\mathrm{i}\, s,v}(\lambda) E(x,t,\lambda) \phi^{-1}_{\mathrm{i}\, s,\tilde{v}(x,t)}$$

is an extended frame of the solution $\tilde{u}$.

Proof. If $\bar{v} = \mathrm{I}_{1,n} v$, then

$$\begin{aligned} \mathrm{I}_{1,n}\tilde{v} &= \mathrm{I}_{1,n} E(x,t,-\mathrm{i}\, s)^{-1} v = E(x,t,\mathrm{i}\, s)^{-1} \mathrm{I}_{1,n} v, \\ \bar{\tilde{v}} &= \overline{E(x,t,-\mathrm{i}\, s)^{-1}}\bar{v} = E(x,t,\mathrm{i}\, s)^t \bar{v} = E(x,t,\mathrm{i}\, s)^{-1}\bar{v} = E(x,t,\mathrm{i}\, s)^{-1}\mathrm{I}_{1,n} v. \end{aligned}$$

So $\mathrm{I}_{1,n}\tilde{v} = \bar{\tilde{v}}$. So, (1) is true. The rest follows from Theorem 4.2.6. □

Theorem 4.3.4. [BT for geometric Airy curve flow on $\mathbb{R}^n$]

Let γ *be a solution of the geometric Airy curve flow* (3.7) *on* $\mathbb{R}^n$, $h = (e_0, \ldots, e_{n-1})$, $g = \mathrm{diag}(1,h)$ *as in Theorem 3.1.3,* $u = g^{-1} g_x$ *the solution of the third flow* (1.19), *and* E *an extended frame of* u. *Let* $s \in \mathbb{R}$ *be a non-zero constant,* $v = (1, ic_1)^t$ *with* $c_1 \in \mathbb{R}^{n\times 1}$ *a unit vector,* $\tilde{v}(x,t) = E(x,t,-is)^{-1} v$, *and* $\tilde{\pi}(x,t)$ *the Hermitian projection onto* $\mathbb{C}\tilde{v}(x,t)$. *Then*

1. $\tilde{v}$ *is of the form* $(c_0, iy_0, \ldots, iy_{n-1})^t$ *for some real valued functions* c_0 *and* y_i *for* $0 \le i \le n-1$,

2. $\phi_{is,\tilde{v}} * \gamma := \gamma_1 = \gamma - \frac{2}{sc_0}\sum_{i=0}^{n-1} y_i e_i$ *is again a solution of* (3.7).

Proof. By Theorem 4.3.3, $\tilde{v}$ satisfies the condition (4.17). Condition $\overline{\tilde{v}} = I_{1,n}\tilde{v}$ implies (1). Let $\psi = \phi^{-1}_{\mathrm{i}\, s,\tilde{v}}$. By Theorem 4.3.3, $E_1 = E\psi$ is an extended frame for the new solution $\tilde{u}$ of vmKdV. Since $\pi_1\pi_2 = \pi_2\pi_1 = 0$, we have

$$\phi_{\mathrm{i}\, s,\tilde{v}}(\lambda) = \mathrm{I} + \frac{2\mathrm{i}\, s}{\lambda - \mathrm{i}\, s}\pi_1 - \frac{2\mathrm{i}\, s}{\lambda + \mathrm{i}\, s}\pi_2.$$

Since $\phi^{-1}_{\mathrm{i}\, s,\tilde{v}}(\lambda) = (\phi_{\mathrm{i}\, s,\tilde{v}}(\bar{\lambda}))^*$, we have

$$\psi = \mathrm{I} - \frac{2\mathrm{i}\, s}{\lambda + \mathrm{i}\, s}\pi_1 + \frac{2\mathrm{i}\, s}{\lambda + \mathrm{i}\, s}\pi_2.$$

A direct computation gives

$$\left.\frac{\partial\psi}{\partial\lambda}\psi^{-1}\right|_{\lambda=0} = \frac{2\mathrm{i}}{s}(\tilde{\pi}_1 - \tilde{\pi}_2).$$

Set $g(x,t) = E(x,t,0)$. Then we have

$$\begin{aligned}
\hat{\gamma}_1 := \left.\frac{\partial E_1}{\partial\lambda}E_1^{-1}\right|_{\lambda=0} &= \begin{pmatrix} 0 & -\gamma_1^t \\ \gamma_1 & 0 \end{pmatrix} \\
&= \left.\frac{\partial E}{\partial\lambda}E^{-1}\right|_{\lambda=0} + \frac{2}{s}g(\tilde{\pi}_1 - \tilde{\pi}_2)g^{-1} \\
&= \begin{pmatrix} 0 & -\gamma^t \\ 0 & \gamma \end{pmatrix} + \frac{2\mathrm{i}}{s}g(\tilde{\pi}_1 - \tilde{\pi}_2)g^{-1} \\
&= \hat{\gamma} + \frac{2\mathrm{i}}{s}g(\tilde{\pi}_1 - \tilde{\pi}_2)g^{-1}.
\end{aligned}$$

Note the projection $\tilde{\pi}_1 = \frac{1}{||\tilde{v}||^2}\tilde{v}\tilde{v}^*$. So, the above formula for $\hat{\gamma}_1$ becomes

$$\hat{\gamma}_1 = \hat{\gamma} - \frac{2}{sy_0}g\begin{pmatrix} 0 & -y^t \\ y & 0 \end{pmatrix}g^{-1}.$$

But $g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ and $(e_1, \ldots, e_n) = h$. So (2) follows.

$\square$

Chapter 5

Periodic Cauchy Problems

5.1 Periodic Cauchy Problems for Schrödinger Flow on $\mathbb{S}^2$

In this section, we consider the periodic Cauchy problem for Schrödinger Flow on $\mathbb{S}^2$, i.e.,

$$\begin{cases} \gamma_t &= \gamma \times \gamma_{xx} \\ \gamma(x,0) &= \gamma_0(x) \end{cases}, \tag{5.1}$$

where $\gamma_0 : [0, 2\pi] \to \mathbb{S}^2$ is smooth and periodic in x with period 2π.

We recall that by Theroem 2.2.3, given $\gamma_0 : [0, 2\pi] \to \mathbb{S}^2$, there is $f : \mathbb{R} \to SU(2)$ such that $\gamma_0 = faf^{-1}$, where $a = \mathrm{diag}(\frac{i}{2}, -\frac{i}{2})$, satisfying $f(0) = I_2$ and $f^{-1}f_x = u_0$, where u_0 is of the form

$$u_0 = \begin{pmatrix} 0 & q_0 \\ -\bar{q_0} & 0 \end{pmatrix}.$$

We notice that $f(x)$ may not be periodic, so we change frames to find a periodic one. Below we construct a periodic frame of γ_0.

Since γ_0 is periodic, $\gamma_0(2\pi) = \gamma_0(0)$. It yields that $f(2\pi)a = af(2\pi)$. That is, $f(2\pi)$ lies in the centralizer $SU(2)_a = \{\mathrm{diag}(e^{\mathrm{i}\theta}, e^{-\mathrm{i}\theta}) \mid \theta \in [0, 2\pi)\}$ and hence we may write

$$f(2\pi) = e^{2\pi c_0 a},$$

for some constant c_0. A direct computation gives the following proposition.

Proposition 5.1.1. Define

$$\tilde{f}(x) = f(x)e^{-c_0 a x}.$$

Then $\tilde{f}(x)$ has the following properties:

1. $\gamma_0 = \tilde{f} a \tilde{f}^{-1}$
2. $\tilde{f}(x)$ is periodic in x
3. $\tilde{f}^{-1}\tilde{f}_x = \begin{pmatrix} -\frac{i}{2}c_0 & \tilde{q}_0 \\ -\bar{\tilde{q}}_0 & \frac{i}{2}c_0 \end{pmatrix}$, where $\tilde{q}_0(x) = q_0(x)e^{ic_0x}$.
4. $\tilde{q}_0$ is periodic.

Proposition 5.1.2. Suppose $\gamma(x,t) : [0, 2\pi] \to \mathbb{S}^2$ solves $\gamma_t = \gamma \times \gamma_{xx}$ and is periodic in x with periodic 2π. By Theorem 2.2.3, there exists $f : \mathbb{R}^2 \to SU(2)$ such that $\gamma = faf^{-1}$, $f^{-1}f_x = u$, and $f^{-1}f_t = Q_{-1}$, where

$$a = \mathrm{diag}(\frac{i}{2}, -\frac{i}{2}), \quad u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, Q_{-1} = \frac{i}{2}\begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix}.$$

Define $c_0(t)$ to be a function of t satisfying

$$f^{-1}(0,t)f(2\pi,t) = e^{2\pi c_0(t)a}. \tag{5.2}$$

Then $c_0(t)$ is independent of t.

Proof. Taking t-derivative of (5.2) gives

$$\begin{aligned} e^{2\pi c_0(t)a} 2\pi c_0'(t) &= -f^{-1}(0,t) f_t(0,t) f(2\pi,t) + f^{-1}(0,t) f_t(2\pi,t) \\ &= e^{2\pi c_0(t)a} Q_{-1}(2\pi,t) - Q_{-1}(0,t) e^{2\pi c_0(t)a}. \end{aligned}$$

So, $2\pi c_0'(t)a = Q_{-1}(2\pi,t) - e^{-2\pi c_0(t)a} Q_{-1}(0,t) e^{2\pi c_0(t)a}$. A direct computation shows

$$e^{-2\pi c_0(t)a} Q_{-1}(0,t) e^{2\pi c_0(t)a} = \frac{i}{2} \begin{pmatrix} -|q|^2 & q_x e^{-4\pi i c_0(t)} \\ \bar{q}_x e^{4\pi i c_0(t)} & |q|^2 \end{pmatrix}.$$

Note that $Q_{-1}(0,t) = Q_{-1}(2\pi,t)$. So, $c_0'(t) = 0$, as desired. □

Next, we consider the periodic Cauchy problem for NLS. Suppose that $q : \mathbb{R}^2 \to \mathbb{C}$ is a solution of

$$\text{(5.3)} \qquad \begin{cases} q_t &= \frac{i}{2}(q_{xx} + 2|q|^2 q) \\ q(x,0) &= \tilde{q}_0(x). \end{cases},$$

Let E be an extended frame for q, i.e., E satisfies

$$\text{(5.4)} \qquad \begin{cases} E^{-1}E_x = \begin{pmatrix} \frac{i}{2}\lambda & q \\ -\bar{q} & -\frac{i}{2}\lambda \end{pmatrix}, \\ E^{-1}E_t = \begin{pmatrix} \frac{i}{2}\lambda^2 - i|q|^2 & q\lambda + iq_x \\ -\bar{q}\lambda + i\bar{q}_x & -\frac{i}{2}\lambda^2 + i|q|^2 \end{pmatrix}, \\ E(0,0,\bar{\lambda})^* = E(0,0,\lambda)^{-1}. \end{cases}$$

Then we will claim that there is a periodic frame for a solution q of NLS periodic in

x in the proof of the following theorem.

Theorem 5.1.3. *Given a smooth and periodic curve $\gamma_0 : [0, 2\pi] \to \mathbb{S}^2$ with $\gamma_0(0) = a$. Then there exists a unique $\gamma(x,t)$ periodic in x with period 2π satisfying* (5.1)*.*

Proof. We know that there is $f \in SU(2)$ such that $\gamma_0 = faf^{-1}$ and $f^{-1}f_x = \begin{pmatrix} 0 & q_0 \\ -\bar{q}_0 & 0 \end{pmatrix}$. Since γ_0 is periodic, $f(2\pi)$ commutes with a. So

$$f(2\pi) = e^{2\pi c_0 a}$$

for some $c_0 \in \mathbb{R}$. Define

$$\tilde{f}(x) = f(x)e^{-c_0 a x}.$$

By Proposition 5.1.1, $\tilde{f}$ is periodic and $\gamma_0 = \tilde{f}a\tilde{f}^{-1}$. In particular,

$$\tilde{f}^{-1}\tilde{f}_x = \begin{pmatrix} -\frac{i}{2}c_0 & q_0(x)e^{ic_0 x} \\ -\bar{q}_0(x)e^{-ic_0 x} & \frac{i}{2}c_0 \end{pmatrix}.$$

Let $q(x,t)$ be the solution of

$$\begin{cases} q_t &= \frac{i}{2}(q_{xx} + 2|q|^2 q) \\ q(x,0) &= q_0(x)e^{ic_0 x}, \end{cases},$$

periodic in x, and $E(x,t,\lambda)$ the extended frame for q satisfying

$$\begin{cases} E^{-1}E_x = a(-c_0) + u, \\ E^{-1}E_t = ac_0^2 + u(-c_0) + Q_{-1}(u), \\ E(0,0,-c_0) = \tilde{f}(0). \end{cases} \tag{5.5}$$

We claim that $g(x,t) = E(x,t,-c_0)$ is periodic in x with period 2π. Let $y(t) = g(2\pi,t) - g(0,t)$. We know $g^{-1}g_t = c_0^2 a - c_0 u + Q_{-1}(u)$ and $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ is periodic. Then

$$\begin{aligned} y'(t) &= g(2\pi,t)(c_0^2 a - c_0 u + Q_{-1}(u))|_{x=2\pi} - g(0,t)(c_0^2 a - c_0 u + Q_{-1}(u))|_{x=0} \\ &= (g(2\pi,t) - g(0,t))(c_0^2 a - c_0 u + Q_{-1}(u))|_{x=0} \\ &= y(t)A(t), \end{aligned}$$

where $A(t) = (c_0^2 a - c_0 u + Q_{-1}(u))|_{x=0}$.

Since $y(0) = 0$ solves the ODE $y'(t) = y(t)A(t)$, the uniqueness theorem of ODE shows that $y(t) \equiv 0$. The claim follows. Let $\eta = gag^{-1}$. Then $\gamma(x,t) = \eta(x + 2c_0 t, t)$ is a solution of $\gamma_t = \gamma \times \gamma_{xx}$ by Proposition 2.2.5.

It remains to verify the initial condition. Note that Proposition 5.1.1 implies

$$\gamma(x,0) = \eta(x,0) = \tilde{f}(x) a \tilde{f}^{-1}(x) = \gamma_0(x).$$

In particular, that γ is periodic in x follows from the periodicity of $E(x,t,-c_0)$. Finally, the uniqueness of γ follows from the uniqueness of $E(x,t,-c_0)$. $\square$

5.2 Periodic Cauchy Problems for VFE on $\mathbb{R}^3$

This section is dedicated to the periodic Cauchy problem of VFE (1), i.e.,

$$\begin{cases} \gamma_t = \gamma_x \times \gamma_{xx}, \\ \gamma(x,0) = \gamma_0, \end{cases} \tag{5.6}$$

where $\gamma_0 : [0, 2\pi] \to \mathbb{R}^3$ is a smooth arc-length parametrized curve periodic in x with period 2π.

Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be a closed curve, and $e_0 = \gamma_x$. $(e_0, \vec{n}_1, \vec{n}_2)$ is orthonormal and periodic. Then we have

$$(e_0, \vec{n}_1, \vec{n}_2)_x = (e_0, \vec{n}_1, \vec{n}_2) \begin{pmatrix} 0 & -\nu_1 & -\nu_2 \\ \nu_1 & 0 & -\omega \\ \nu_2 & \omega & 0 \end{pmatrix}, \tag{5.7}$$

for some smooth functions ν_1, ν_2, ω.

Let (e_0, e_1, e_2) a parallel frame for γ such that

$$(e_0, e_1, e_2)_x = (e_0, e_1, e_2) \begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix}.$$

Let θ denote the angle from $\vec{n}_1$ to e_1. Then we have the following relations:

$$\theta_x = -\omega, \tag{5.8}$$

$$\begin{cases} e_1 = \cos\theta \vec{n}_1 + \sin\theta \vec{n}_2, \\ e_2 = -\sin\theta \vec{n}_1 + \cos\theta \vec{n}_2. \end{cases} \tag{5.9}$$

Note that although γ is periodic, (e_1, e_2) need not be periodic.

Definition 5.2.1. The normal holonomy of γ is

$$\theta(2\pi) - \theta(0) = -\int_0^{2\pi} \omega(x,t)\, dx. \tag{5.10}$$

Proposition 5.2.2. Suppose $\gamma(x,t)$ is a solution of the VFE (1) which is periodic in x with period 2π and $\| \gamma_x \| = 1$. Let $e_0 = \gamma_x$. Then the normal holonomy is independent of t.

Proof. Let $(e_0, \vec{n}_1, \vec{n}_2)$ be a periodic orthonormal frame along γ such that $e_0 = \gamma_x$ and

$$(e_0, \vec{n}_1, \vec{n}_2)_x = (e_0, \vec{n}_1, \vec{n}_2) \begin{pmatrix} 0 & -\nu_1 & -\nu_2 \\ \nu_1 & 0 & -\omega \\ \nu_2 & \omega & 0 \end{pmatrix}. \tag{5.11}$$

Since $(e_0, \vec{n}_1, \vec{n}_2)$ is periodic, ν_1, ν_2, and ω are periodic. Use the fact that γ is a solution of the VFE (1), we compute to get

$$(e_0, \vec{n}_1, \vec{n}_2)_t = (e_0, \vec{n}_1, \vec{n}_2) \begin{pmatrix} 0 & \nu_1\omega + (\nu_2)_x & \nu_2\omega - (\nu_1)_x \\ -\nu_1\omega - (\nu_2)_x & 0 & -\xi \\ -\nu_2\omega + (\nu_1)_x & \xi & 0 \end{pmatrix}, \tag{5.12}$$

where ξ is a smooth function such that $\xi_x = \omega_t - \frac{1}{2}(\nu_1^2 + \nu_2^2)_x$. The compatibility condition of (5.11) and (5.12) implies

$$\begin{cases} (\nu_1)_t = -(\nu_1\omega + (\nu_2)_x)_x - (-\nu_2\omega^2 + (\nu_1)_x\omega) + \nu_2\xi, \\ (\nu_2)_t = -(\nu_2\omega - (\nu_1)_x)_x - (\nu_1\omega^2 + (\nu_2)_x\omega) - \nu_1\xi, \\ \omega_t = \xi_x + \frac{1}{2}(\nu_1^2 + \nu_2^2)_x. \end{cases} \tag{5.13}$$

We take t derivative and substitute by the third equation of(5.13) to get

$$\begin{aligned}
\tfrac{d}{dt}\int_0^{2\pi}\omega(x,t)\ dx &= \int_0^{2\pi}\omega_t(x,t)\ dx \\
&= \int_0^{2\pi}\xi_x+\tfrac{1}{2}(\nu_1^2+\nu_2^2)_x\ dx \\
&= 0.
\end{aligned}$$

□

Next, we rotate the normal frame to get a periodic frame for γ. Let

$$c_0=\frac{1}{2\pi}\int_0^{2\pi}\omega(x,t)\ dx, \tag{5.14}$$

and $u_0=e_0=\gamma_x$. Let

$$\begin{cases} u_1=\cos(c_0x)e_1+\sin(c_0x)e_2, \\ u_2=-\sin(c_0x)e_1+\cos(c_0x)e_2. \end{cases} \tag{5.15}$$

We call this frame (u_0,u_1,u_2) the *h-frame*. As a corollary of Proposition 5.2.2, we have

Corollary 5.2.3. c_0 defined in (5.14) is independent of t.

Lemma 5.2.4. The h-frame (u_0,u_1,u_2) defined by (5.15) is periodic and

$$(u_0,u_1,u_2)_x=(u_0,u_1,u_2)\begin{pmatrix} 0 & -\mu_1 & -\mu_2 \\ \mu_1 & 0 & -2c_0 \\ \mu_2 & 2c_0 & 0 \end{pmatrix}. \tag{5.16}$$

Proof. A direct computation will lead us to (5.16). It suffices to show that (u_0,u_1,u_2) is periodic. We may assume $\theta(0)=0$. Then (5.10) and (5.14) imply $\theta(2\pi)=-2\pi c_0$.

Note that u_0 is the tangent, so it is periodic. Let $R_\phi(v)$ denote the rotation of a vector v by angle ϕ. Note that by (5.15) we have

$$\begin{cases} u_1(0) = e_1(0), \\ u_1(2\pi) = R_{2\pi c_0} e_1(2\pi). \end{cases} \tag{5.17}$$

In particular,

$$e_1(2\pi) = R_{\theta(2\pi)} e_1(0) = R_{\theta(2\pi)} u_1(0).$$

We substitute $e_1(2\pi)$ in the second equation of (5.17) to get

$$u_1(2\pi) = R_{2\pi c_0} R_{\theta(2\pi)} u_1(0) = u_1(0).$$

Similarly, one can show that $u_2(2\pi) = u_2(0)$. $\square$

In other words, we have proved the following theorem.

Theorem 5.2.5. *Let $\gamma(x,t)$ be a solution of the VFE (1) that is periodic in x with period 2π and $\| \gamma_x \| = 1$. Suppose $(e_0, \vec{n}_1, \vec{n}_2)$ is orthonormal along γ such that $e_0 = \gamma_x$. Let $\omega = (\vec{n}_1)_x \cdot \vec{n}_2$. Then $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \omega(x,t)\, dx$ is constant for all t, and there is $g = (u_0, u_1, u_2)(x,t)$ such that*

1. *$g(\cdot, t)$ is a periodic h-frame along $\gamma(\cdot, t)$,*
2. $g^{-1} g_x = \begin{pmatrix} 0 & -\zeta_1 & -\zeta_2 \\ \zeta_1 & 0 & -2c_0 \\ \zeta_2 & 2c_0 & 0 \end{pmatrix}$,
3. *$q = \frac{1}{2}(\zeta_1 + i\zeta_2)$ is a solution of the NLS.*

Proof. Let (e_0, e_1, e_2) be the parallel frame along γ such that

$$(5.18) \qquad (e_0, e_1, e_2)_x = (e_0, e_1, e_2) \begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix}.$$

So, $k = k_1 + ik_2$ satisfies $k_t = \frac{i}{2}(k_{xx} + \frac{1}{2}|k|^2 k)$. Note that $u = \frac{k}{2}$ satisfies the NLS. From the construction of the h-frame defined by (5.16), we know that $\mu = \frac{1}{2}(\mu_1 + i\mu_2) = \frac{1}{2}(k_1 + ik_2)e^{-2ic_0 x}$, i.e., $\mu = u^{-2ic_0 x}$ and there is a h-frame f such that

$$f^{-1}\tilde{f}_x = \begin{pmatrix} 0 & -\mu_1 & -\mu_2 \\ \mu_1 & 0 & -2c_0 \\ \mu_2 & 2c_0 & 0 \end{pmatrix}.$$

We compute to see that μ satisfies the equation $\mu_t = \frac{i}{2}(\mu_{xx} + 2|\mu|^2\mu) - 2c_0\mu_x - 2ic_0^2\mu$. Let $\tilde{g} = fC$, where

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(c_0^2 t) & \sin(c_0^2 t) \\ 0 & -\sin(c_0^2 t) & \cos(c_0^2 t) \end{pmatrix}.$$

Then

$$(5.19) \qquad \tilde{g}^{-1}\tilde{g}_x = C^{-1}f^{-1}f_x C = \begin{pmatrix} 0 & -\zeta_1 & -\zeta_2 \\ \zeta_1 & 0 & -2c_0 \\ \zeta_2 & 2c_0 & 0 \end{pmatrix},$$

where $\zeta := \frac{1}{2}(\zeta_1 + i\zeta_2) = \mu e^{2ic_0^2 t}$. It is easy to see that ζ satisfies the equation $\zeta_t = \frac{i}{2}(\zeta_{xx} + 2|\zeta|^2\zeta) - 2c_0\zeta_x$. Let $g(x,t) = \tilde{g}(x + 2c_0 t, t)$. Then $g^{-1}g_x$ is of the form of (5.19) and $q(x,t) = \zeta(x + 2c_0 t, t)$ is a solution of NLS. $\square$

Proposition 5.2.6. Let q be a solution of NLS periodic in x with period 2π, $\lambda_0 \in \mathbb{R}$, and $E(x,t,\lambda)$ the extended frame of q. If $E(x,0,\lambda_0)$ is periodic in x with period 2π, then so is $E(x,t,\lambda_0)$.

Proof. Let $y(t) = E(2\pi,t,\lambda_0) - E(0,t,\lambda_0)$. Note that $E(x,0,\lambda_0)$ is periodic in x, so $y(0)=0$. As $E^{-1}E_t = a\lambda_0^2 + u\lambda_0 + Q_{-1}(u)$ and $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ is periodic, we have

$$\begin{aligned} y'(t) &= E(2\pi,t,\lambda_0)(a\lambda_0^2 + u\lambda_0 + Q_{-1}(u))|_{x=2\pi} - E(0,t,\lambda_0)(a\lambda_0^2 + u\lambda_0 + Q_{-1}(u))|_{x=0} \\ &= (E(2\pi,t,\lambda_0) - E(0,t,\lambda_0))(a\lambda_0^2 + u\lambda_0 + Q_{-1}(u))|_{x=0} = y(t)A(t), \end{aligned}$$

where $A(t) = (a\lambda_0^2 + u\lambda_0 + Q_{-1}(u))|_{x=0}$.

Since $y(0)=0$ solves the ODE $y'(t) = y(t)A(t)$, the uniqueness theorem of ODE shows that $y(t) \equiv 0$. The desired follows.

□

Proposition 5.2.7. Let q be a solution of NLS periodic in x with period 2π and E the extended frame for q. Let $\lambda_0 \in \mathbb{R}$ and

$$\eta = E_\lambda E^{-1} \mid_{\lambda=\lambda_0}, \quad \gamma(x,t) = \eta(x - 2\lambda_0 t, t). \tag{5.20}$$

Then $\gamma(x,t)$ in (5.20) is a solution of VFE $\gamma_t = \gamma_x \times \gamma_{xx}$.

Proof. It can be checked that

$$\eta_x = E(x-2\lambda_0 t,t)aE(x-2\lambda_0 t,t)^{-1}, \eta_t = E(x-2\lambda_0 t,t)(2a\lambda_0 + u)E(x-2\lambda_0 t,t)^{-1},$$

where $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, and $\gamma_t = -2\lambda_0\eta_x + \eta_t = EuE^{-1}$. On the other hand,

$$\gamma_x \times \gamma_{xx} = [\gamma_x, \gamma_{xx}] = E[a, [u, a]]E^{-1} = EuE^{-1}.$$

□

Now, we consider the periodic Cauchy problem of VFE (5.6). Given a closed curve $\gamma_0(x) : [0, 2\pi] \to \mathbb{R}^3$, there is a periodic h-frame $f = (u_0^0, u_1^0, u_2^0)$ such that $u_0^0 = \gamma_0'(x)$ and

$$f^{-1}f_x = \begin{pmatrix} 0 & -2q_1^0 & -2q_2^0 \\ 2q_1^0 & 0 & -2c_0 \\ 2q_2^0 & 2c_0 & 0 \end{pmatrix}.$$

Let $(\zeta, \eta) = -\frac{1}{2}\mathrm{tr}(\zeta\eta)$ for $\zeta, \eta \in \mathfrak{su}(2)$, and

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form an orthonormal basis of $\mathfrak{su}(2)$. We identify $\mathfrak{su}(2)$ as $\mathbb{R}^3$ by mapping a, b, c to the standard basis of $\mathbb{R}^3$. Then there is $\phi \in SU(2)$ such that

$$(u_0^0, u_1^0, u_2^0) = (\phi a \phi^{-1}, \phi b \phi^{-1}, \phi c \phi^{-1}).$$

In particular, ϕ is periodic in x with period 2π.

Proposition 5.2.8. Suppose $\gamma_0(x) : \mathbb{R}^2 \to \mathbb{R}^3$ is a periodic curve parametrized by arc-length with period 2π. Let (u_0^0, u_1^0, u_2^0) be a h-frame $2q_1^0, 2q_2^0$ and $\phi \in SU(2)$ such that

$$(u_0^0, u_1^0, u_2^0) = (\phi a \phi^{-1}, \phi b \phi^{-1}, \phi c \phi^{-1}).$$

Suppose $q : \mathbb{R}^2 \to \mathbb{C}$ is a periodic solution of

$$(5.21) \qquad \begin{cases} q_t = \frac{i}{2}(q_{xx} + 2 \mid q \mid {}^2 q), \\ q(x,0) = q_1^0 + i q_2^0. \end{cases}$$

Let E be the extended frame with initial data $E(0,0,c_0) = \phi, \eta = E_\lambda E^{-1} \mid_{\lambda = c_0}$, and $\gamma(x,t) = \eta(x - 2c_0 t, t)$. Then $\tilde{\gamma}(x,t) = \gamma(x,t) - \eta(0,0) + \gamma_0(0)$ solves (5.6) and is periodic in x with period 2π.

Proof. By Theorem 5.2.7, we know that $\tilde{\gamma}$ satisfies the VFE (1). In particular, $\gamma(x,0) = \eta(x,0)$ from (5.20). We claim that $\gamma(x,0) = \gamma_0(x) + \eta(0,0) - \gamma_0(0)$. In this case, one obtains $\tilde{\gamma}(x,0) = \gamma_0(x)$. Note that

$$(5.22) \qquad \eta_x(x,0) = E(x,0)aE(x,0)^{-1} = \phi a \phi^{-1} = u_0^0 = \gamma_0'(x),$$

which implies

$$\eta(x,0) = \gamma_0(x) + c,$$

for some constant c. So $c = \eta(0,0) - \gamma_0(0)$. Since $E(x,t,c_0) = \phi(x)$ is periodic in x, the periodicity of $\tilde{\gamma}$ follows from Proposition 5.2.6. □

So, we have proved the following:

Theorem 5.2.9. *Let* $\gamma_0 : [0, 2\pi] \to \mathbb{R}^3$ *be a closed curve and arc-length parametrized. Then there exists a unique periodic solution* $\gamma(x,t)$ *of* (5.6).

5.3 Periodic Cauchy Problems for Geometric Airy Curve Flow on $\mathbb{R}^2$

Recall that a geometric Airy curve flow on $\mathbb{R}^2$ (3.1) can be reparametrized by its arc-length as follows:

$$\gamma_t = -(\frac{1}{2}k^2 e_0 + k_x e_1), \tag{5.23}$$

where $e_0 = \gamma_x$ and k is the curvature. And (5.23) preserves the arc length. By Theorem 3.1.3, the curvature k satisfies the mKdV

$$k_t = -(k_{xxx} + \frac{3}{2}k^2 k_x).$$

Let $\gamma(x,t)$ satisfy (5.23). Note that if θ is the tangent angle, then

$$\begin{cases} e_0(x,t) & = (\cos\theta(x,t), \sin\theta(x,t)), \\ e_1(x,t) & = (-\sin\theta(x,t), \cos\theta(x,t)). \end{cases} \tag{5.24}$$

Direct computations show that

$$\theta_x = k,\ \ \theta_t = -(\frac{1}{2}k^3 + k_{xx}).$$

By integration with respect to t, we see that

$$\eta(x,t) = \eta(x,0) + \int_0^t -(\frac{1}{2}k^2 e_0(x,\tau) + k_x e_1(x,\tau))\ d\tau, \tag{5.25}$$

is a solution of (5.23).

Proposition 5.3.1. Suppose k is a solution of mKdV periodic in x with period 2π and $E(x,t,\lambda)$ an extended frame of k. Let $g(x,t) = E(x,t,0)$ satisfy

$$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \\ g^{-1}g_t = \begin{pmatrix} 0 & -(k_{xx} + \frac{1}{2}k^3) \\ k_{xx} + \frac{1}{2}k^3 & 0 \end{pmatrix}. \end{cases}$$

If $g(x,0)$ is periodic in x with period 2π, then so is $g(x,t)$ for all t.

Proof. Let $y(t) = g(2\pi,t) - g(0,t)$. Then $y'(t) = g(2\pi,t)B(2\pi,t) - g(0,t)B(0,t)$, where

$$B(x,t) = \begin{pmatrix} 0 & -(k_{xx} + \frac{1}{2}k^3) \\ k_{xx} + \frac{1}{2}k^3 & 0 \end{pmatrix}.$$

Note that $B(2\pi,t) = B(0,t)$, so $y'(t) = y(t)B(0,t)$. Since $y(0) = 0$ solves the ODE $y'(t) = y(t)B(0,t)$, the uniqueness theorem of ODE implies that $y(t) \equiv 0$. This completes the proof. □

Now, we consider the periodic Cauchy problem of the geometric Airy curve flow on $\mathbb{R}^2$.

Theorem 5.3.2. *Let $\gamma_0(x) : [0,2\pi] \to \mathbb{R}^2$ be periodic with period 2π and arc-length parametrized. Then there exists a unique periodic solution of the Cauchy problem*

$$\begin{cases} \gamma_t = -(\frac{1}{2}k^2 e_0 + k_x e_1), \\ \gamma(x,0) = \gamma_0(x). \end{cases} \tag{5.26}$$

Proof. Let $e_0^0(x) = \gamma_0'(x)$ and $e_1^0(x)$ is the rotation of $e_0^0(x)$ by $90°$. Since $\gamma_0'(x)$ is periodic in x with period 2π, so is $(e_0^0(x), e_1^0(x))$. In addition, the curvature $k_0(x)$ of

$\gamma_0(x)$ is periodic in x. Suppose k is a solution of the mKdV periodic in x with period 2π, i.e., k solves

$$\begin{cases} k_t = -(k_{xxx} + \frac{3}{2}k^2 k_x), \\ k(x,0) = k_0(x). \end{cases} \tag{5.27}$$

Let $E(x,t,\lambda)$ the extended frame of k with initial data $E(0,0,\lambda) = (e_0^0(0), e_1^0(0))$ and $g(x,t) = E(x,t,0)$ satisfy

$$g^{-1}g_x = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad g^{-1}g_t = \begin{pmatrix} 0 & -(k_{xx} + \frac{1}{2}k^3) \\ k_{xx} + \frac{1}{2}k^3 & 0 \end{pmatrix}, \quad g(0,0) = (e_0^0(0), e_1^0(0)).$$

It follows from Proposition 5.3.1 that $g(x,t)$ is periodic in x with period 2π. Let $(e_0(x,t), e_1(x,t)) = g(x,t)$ and

$$\gamma(x,t) = \gamma_0(x) + \int_0^t -(\frac{1}{2}k^2 e_0(x,\tau) + k_x e_1(x,\tau))\, d\tau.$$

It is easy to see that γ is a solution of (5.26). We then claim that γ is periodic in x with period 2π.

Let $y(t) = \gamma(2\pi,t) - \gamma(0,t)$. Note that $\gamma_t = -(\frac{1}{2}k^2 e_0(x,t) + k_x e_1(x,t))$. Proposition 5.3.1 implies that (e_0, e_1) is periodic in x with period 2π, hence so is γ_t. So we have

$$y'(t) = \gamma_t(2\pi,t) - \gamma_t(0,t) = 0.$$

As $y(0) = 0$ solves the ODE $y'(t) = 0$, the uniqueness of ODE theorem gives $y(t) \equiv 0$. This proves that γ is a periodic solution of (5.26). The uniqueness follows from the uniqueness of solutions for the system (5.27). □

Chapter 6

Numerics

In 1998, Hou, Klapper, and Si provided a formulation in [5] for calculating numerical solutions of VFE. This method is a generalization of the previous 2-D work [6] of Hou *et al.*. This $\theta - L$ formulation method for two dimensional curves has none of the high order time step stability constraints that are usually induced when an explicit method is used, where θ is the tangent angle and L is the total arc length. However, this $\theta - L$ frame cannot be generalized to three dimensional Euclidean spaces since θ is not always defined. Hou, Klapper, and Si then proposed to use the normal principal curvatures k_1, k_2 as new variables to compute the motion of the curve in 3D.

Hasimoto gives a way in [4] to obtain solutions of NLS from solutions of VFE, that is, if γ is a solution of the VFE, then there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$q(x,t) = k(x,t)e^{i(\theta(t)+\int_0^x \tau(s,t)ds)} \tag{6.1}$$

is a solution of NLS where $\tau(\cdot,t)$ is the torsion for $\gamma(\cdot,t)$ and x is the arc-length parameter.

Using geometry, we have already showed the converse in previous sections, i.e., constructing solutions of VFE by making use of solutions of NLS. This construction actually provides an algorithm of computing solutions of VFE numerically.

One advantage of this method is that we reduce the curve PDE to one soliton equation and a compatible ODE systems in x and t. The most famous soliton equations such as NLS and mKdV can be computed numerically using the pseudo spectral method in [8], which provides a good accuracy for periodic solutions. As for the compatible ODE systems, there are several schemes that solve ODE well. For example, ode solvers in MatLab and Runge-Kutta method.

We will present geometric algorithms for computing numerical solutions of the VFE, the Schrdinger flow on $\mathbb{S}^2$, and the Airy geometric curve flow on $\mathbb{R}^2$.

6.1 Geometric Algorithms

Below we state steps of implementing codes for calculating numerical solutions.

Numerical Solutions of VFE:

Given a closed curve $\gamma_0(x) : [0, 2\pi] \to \mathbb{R}^3$. We consider the following periodic Cauchy problem

$$\begin{cases} \gamma_t = \gamma_x \times \gamma_{xx}, \\ \gamma(x, 0) = \gamma_0(x), \end{cases} \tag{6.2}$$

In order to solve (6.2) numerically, we

Step 1. compute the tangent unit vector $e_0 = \frac{\gamma_0'(x)}{||\gamma_0'(x)||}$, so there exist e_1, e_2 such that

$$(e_0, e_1, e_2)_x = ||\gamma_0'(x)||(e_0, e_1, e_2) \begin{pmatrix} 0 & -k_1^0 & -k_2^0 \\ k_1^0 & 0 & -w \\ k_2^0 & w & 0 \end{pmatrix}.$$

In particular,

$$w = \frac{1}{||\gamma_0'(x)||}(e_1)_x \cdot e_2, \quad k_i^0 = \frac{1}{||\gamma_0'(x)||}(e_0)_x \cdot e_i, \quad i = 1, 2.$$

Step 2. integrate $\psi_x = -||\gamma_0'(x)||w$ to get ψ, so we obtain a parallel frame $(e_0, \tilde{e}_1, \tilde{e}_2)$ of $\gamma_0(x)$ such that

$$(e_0, \tilde{e}_1, \tilde{e}_2)_x = ||\gamma_0'(x)||(e_0, \tilde{e}_1, \tilde{e}_2) \begin{pmatrix} 0 & -\tilde{k}_1^0 & -\tilde{k}_2^0 \\ \tilde{k}_1^0 & 0 & 0 \\ \tilde{k}_2^0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{e}_1 &= \cos\psi e_1 + \sin\psi e_2, \\ \tilde{e}_2 &= -\sin\psi e_1 + \cos\psi e_2, \\ \tilde{k}_1^0 &= k_1^0 \cos\psi, \\ \tilde{k}_2^0 &= k_2^0 \sin\psi. \end{aligned}$$

Step 2. compute c_0 of γ_0 defined by (5.14) and get the $h-$frame f defined by (5.15). And convert f to be elements $\phi a\phi^{-1}, \phi b\phi^{-1}, \phi c\phi^{-1}$ in $\mathfrak{su}(2)$ identified with $\mathbb{R}^3$.

Step 3. use the pseudo spectral method in [8] to solve the periodic Cauchy problem of NLS with the initial data $q_0 = \tilde{k}_1^0 + \mathrm{i}\,\tilde{k}_2^0$.

Step 4. Solving the following system

$$\begin{cases} E^{-1}E_x = \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}, \\ E^{-1}E_t = \begin{pmatrix} i\lambda^2 - \frac{i}{2}|q|^2 & q\lambda + \frac{i}{2}q_x \\ -\bar{q}\lambda + \frac{i}{2}\bar{q}_x & -i\lambda^2 + \frac{i}{2}|q|^2 \end{pmatrix}, \\ E(0,0,\lambda) = \phi(0), \end{cases}$$

at different values of λ which are sufficiently close to c_0, where the data on the right hand side is given by solutions q of the periodic Cauchy problem of NLS. Here we use the second order Runge-Kutta method to approximate.

Step 5. use definition of η, γ in (5.20) and Proposition 5.2.8 to construct solutions of (6.2) in $\mathfrak{su}(2)$. Map them back to $\mathbb{R}^3$.

Numerical Solutions of Schrd̈inger flow on 2-sphere:

Given a closed curve $\gamma_0(x) : [0, 2\pi] \to \mathbb{S}^2$. We consider the following periodic Cauchy problem

$$\begin{cases} \gamma_t = \gamma \times \gamma_{xx}, \\ \gamma(x,0) = \gamma_0(x), \end{cases} \tag{6.3}$$

In order to solve (6.3) numerically, we

Step 1. write γ_0 as an element in $\mathfrak{su}(2)$ and diagonalize γ_0 to find $f \in SU(2)$ such that $\gamma_0 = faf^{-1}$ and

$$f^{-1}f_x = \begin{pmatrix} 0 & q_0 \\ -\bar{q}_0 & 0 \end{pmatrix}.$$

Step 2. compute c_0 by solving $f(2\pi) = e^{2\pi c_0 a}$.

Step 3. use the pseudo spectral method in [8] to solve the periodic Cauchy problem of NLS (5.3) with the initial data $q_0(x)e^{ic_0 x}$.

Step 4. compute E by solving the ODE system (5.5) with the right hand side given by solutions q of (5.3) and the initial data $fe^{-c_0 ax}$.

Step 5. calculate $\gamma = EaE^{-1}$ in terms of elements in $\mathfrak{su}(2)$ and then we map them back to $\mathbb{R}^3$, which is the numerical solution to (6.3).

Numerical Solutions of the Geometric Airy Curve Flow on $\mathbb{R}^2$:

Given a closed curve $\gamma_0(x) : [0, 2\pi] \to \mathbb{R}^2$. We consider the periodic Cauchy problem (5.26), i.e.,

$$\begin{cases} \gamma_t = -(\frac{1}{2}k^2 e_0 + k_x e_1), \\ \gamma(x, 0) = \gamma_0(x), \end{cases} \tag{6.4}$$

In order to solve (5.26) numerically, we

Step 1. find the Frènet frame, and the curvature k_0 of γ_0.

Step 2. compute the tangent angle θ_0 of γ_0 using the inverse trigonometric function of cos since

$$e_0^0 = \gamma_0' = (\cos\theta_0, \sin\theta_0).$$

Step 3. use the pseudo spectral method in [8] to solve the periodic Cauchy problem of mKdV (5.27) with the initial data k_0.

Step 4. update the tangent angle θ at each (x, t) by the evolutions

$$\theta_x = k, \;\; \theta_t = -(\frac{1}{2}k^3 + k_{xx}).$$

So, we update frames $g = (e_0, e_1) = (\cos\theta, \sin\theta)$.

Step 5. recover solutions γ by integrating

$$\gamma(x,t) = \gamma_0(x) + \int_0^t -(\frac{1}{2}k^2 e_0(x,\tau) + k_x e_1(x,\tau)) \; d\tau.$$

6.2 Numerical Experiments

This section includes three testing examples for computing numerical solutions of the VFE, Schrödinger flow on $\mathbb{S}^2$, and the geometric Airy curve flow on $\mathbb{R}^2$. The errors are provided to verify the accuracy of this geometric scheme. Note that the NLS and mKdV have infinitely many conserved quantities, so we would also like to see if numerical solutions obtained from geometric algorithms preserve these conserved quantities. For example,

Conserved Quantities for NLS:

1. $H_1 = \oint |q|^2 \; dx$

2. $H_2 = \oint \bar{q} q_x \; dx$

3. $H_3 = \oint |q_x|^2 - |q|^4 \; dx$

4. $H_4 = \oint q \bar{q}_x - \bar{q} q_x \; dx$

Conserved Quantities for mKdV:

1. $H_1 = \oint q \, dx$

2. $H_2 = \oint q^2 \, dx$

3. $H_3 = \oint 3q_x^2 - q^4 \, dx$

Example 6.2.1 (VFE)**.**

We consider the initial curve is a circle, then the solution to the VFE is

$$\gamma(x,t) = (\cos x, \sin x, t).$$

The following errors are estimated between the true solution γ and numerical solutions γ_n.

Table 6.1: Errors with the true solution γ and the numerical solution γ_n

$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$	$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$
10^{-6}	2^6	7.8089×10^{-1}	10^{-4}	2^{10}	6.2257×10^{-2}
	2^8	1.1071×10^{-2}	10^{-5}		1.7759×10^{-2}
	2^{10}	5.5132×10^{-3}	10^{-6}		5.5132×10^{-3}
10^{-7}	2^6	2.4694×10^{-1}	10^{-4}	2^{11}	4.2486×10^{-2}
	2^8	3.5011×10^{-3}	10^{-5}		1.3946×10^{-2}
	2^{10}	1.7504×10^{-3}	10^{-6}		3.7876×10^{-3}

Remark 6.2.2. For these errors are more or less the same, when N is smaller, $\triangle t$ needs to be much smaller in order to get a better estimate. This may be caused by solving $E^{-1}E_x$ first. In other words, the precision of E depends on how accurate the solution to $E^{-1}E_x$ is. Moreover, if one wishes to improve the accuracy, using another ODE solver in Step 4 is more intuitive.

Remark 6.2.3. Fixed $\triangle t = 10^{-6}$. When $N = 2^6, 2^8, 2^{10}$, $\triangle x = 9.817 \times 10^{-2}, \dot{2}.45 \times 10^{-2}, \dot{6}.14 \times 10^{-3}$, respectively. The errors seem to be of order $\triangle x$. On the other hand, when $N = 2^{10}$, errors look like to have order $\sqrt{\triangle t}$.

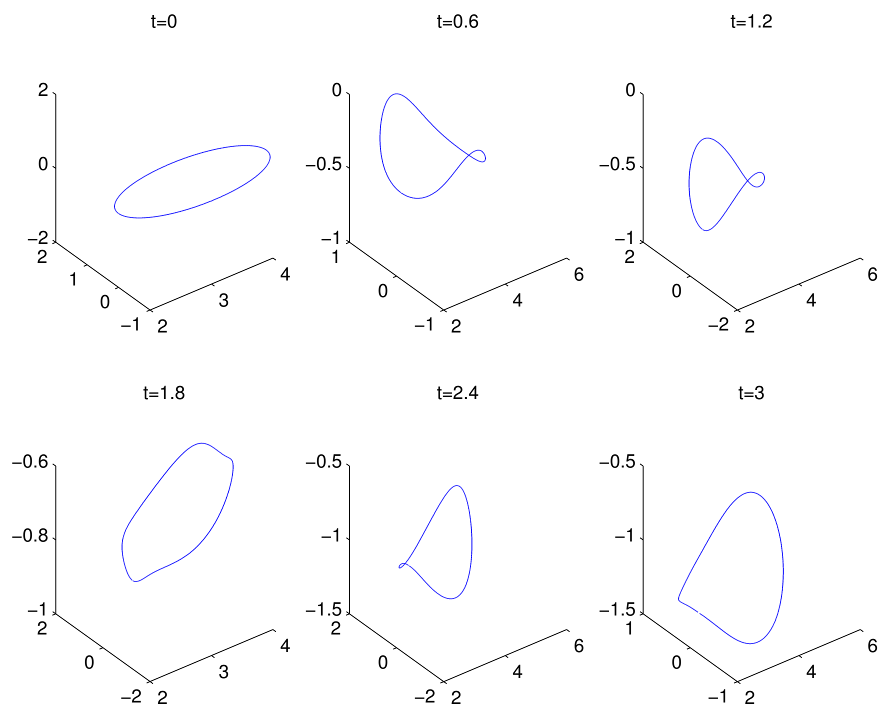

Figure 6.1: Numerical solution $\gamma(x,t)$ of VFE where $\gamma(x,0) = (2\cos x, \sin x, 0)$

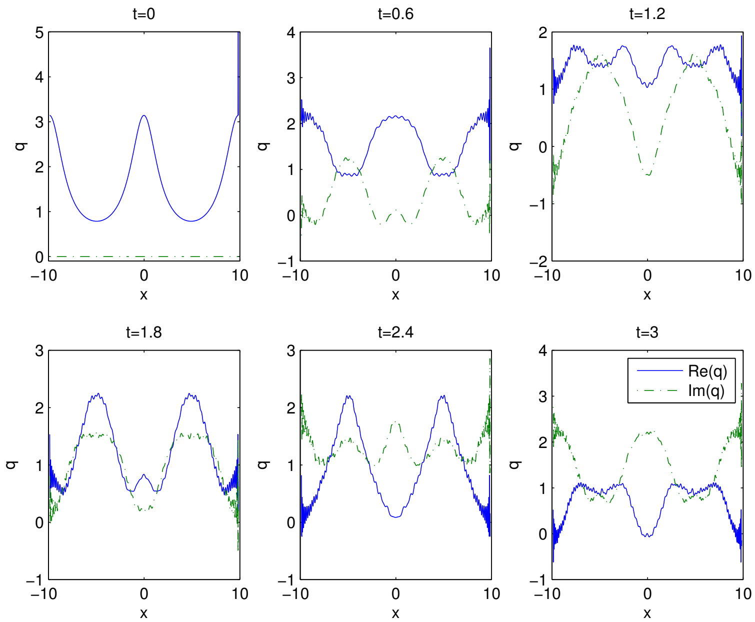

Figure 6.2: Solution q of NLS corresponding to $\gamma(x, t)$ in Figure 6.1

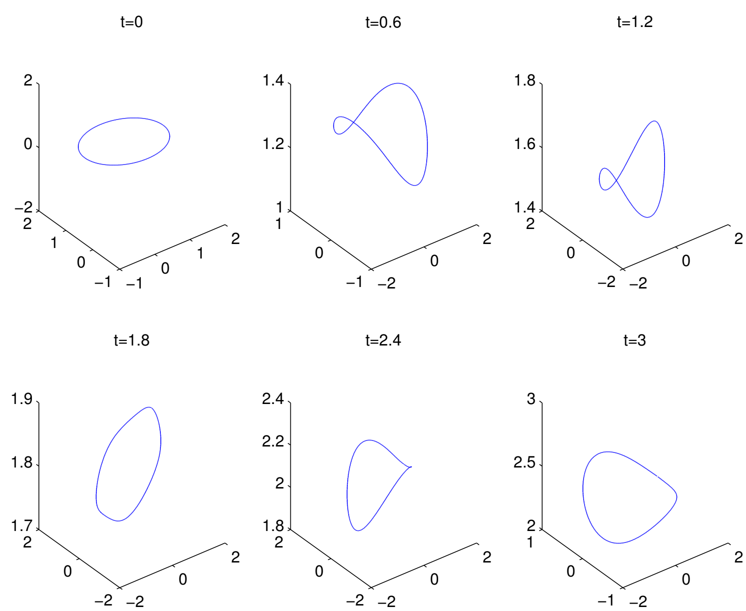

Figure 6.3: Numerical solution $\gamma(x, t)$ of VFE where $\gamma(x, 0) = (\cos x, \sin x, \cos x)$

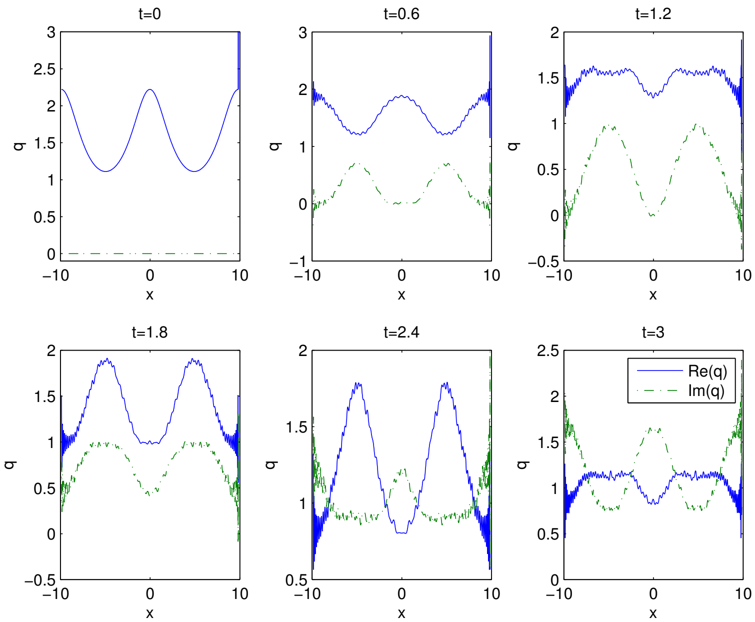

Figure 6.4: Solution q of NLS corresponding to $\gamma(x,t)$ in Figure 6.3

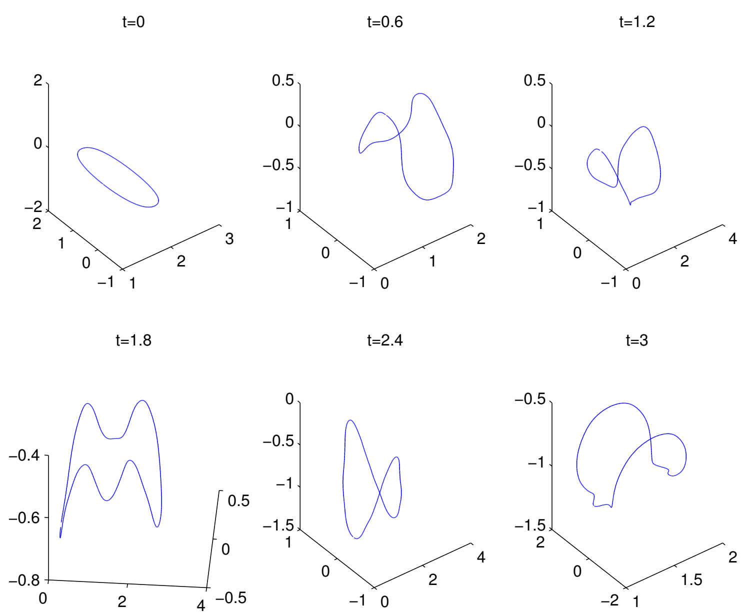

Figure 6.5: Numerical solution $\gamma(x,t)$ of VFE where $\gamma(x,0) = (e^{0.1\cos x}, \sin x, 0)$

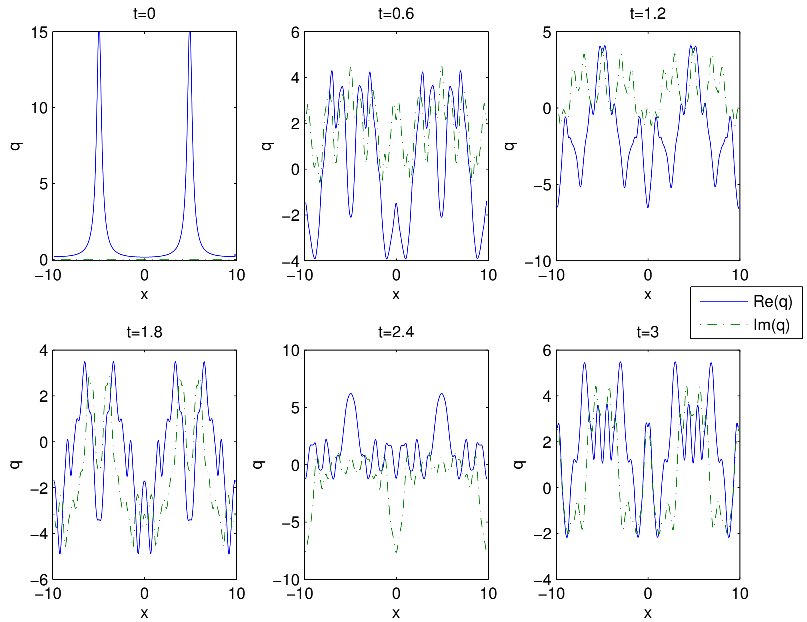

Figure 6.6: Solution q of NLS corresponding to $\gamma(x,t)$ in Figure 6.5

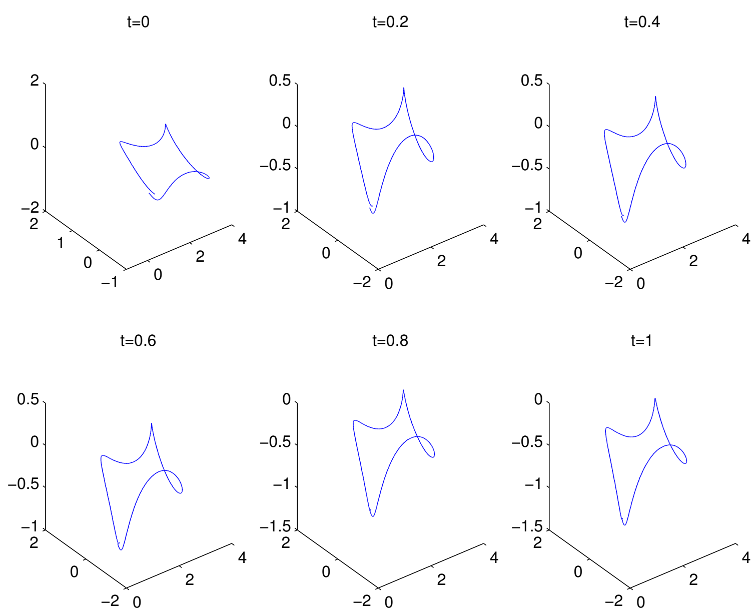

Figure 6.7: Apply BT to $\gamma(x,t) = (\cos x, \sin x, t)$

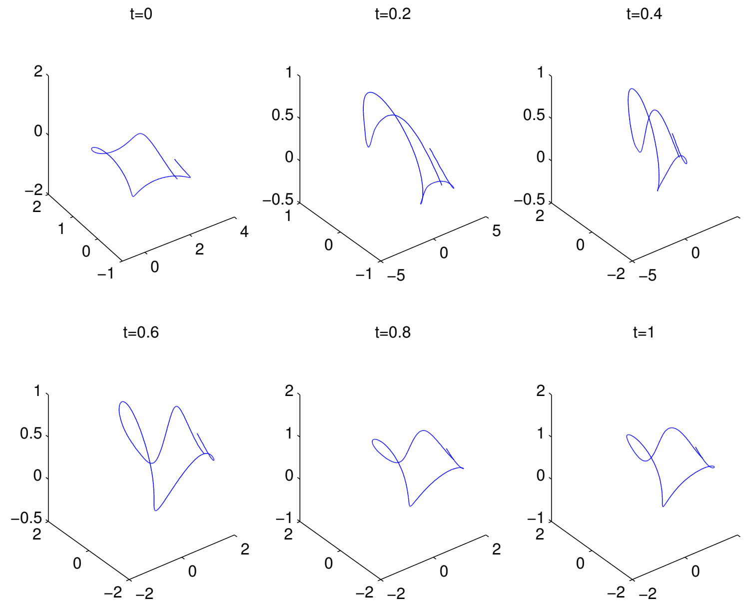

Figure 6.8: Apply BT to $\gamma(x,t)$ where $\gamma(x,0) = (2\cos x, \sin x, 0)$

Below we consider errors for conserved quantities of NLS. Since H_i's are constant with respect time t, we consider the L^p-error as follows:

$$(\sum_k \triangle t |H_i(t_k) - H_i(t_1)|^p)^{\frac{1}{p}},$$

where t_1 is the initial time. Below we give errors of each conserved quantity for different initial curves. Here N is the step size of the spatial parameter x.

Table 6.2: Conserved Quantities Error for VFE with initial data $\gamma_0 = (\cos x, \sin x, 0)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$	$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-3}	2^6	1.6×10^{-3}	10^{-3}	2^8	4.8892×10^{-4}
	2^8	4.1058×10^{-4}	10^{-4}		3.4008×10^{-4}
	2^9	2.0169×10^{-4}	10^{-5}		3.3779×10^{-4}
10^{-4}	2^8	2.4564×10^{-4}	10^{-3}	2^9	3.5137×10^{-4}
	2^9	8.0498×10^{-5}	10^{-4}		1.1703×10^{-4}
	2^{10}	4.1524×10^{-5}	10^{-5}		1.1137×10^{-4}
10^{-5}	2^8	2.4170×10^{-4}	10^{-3}	2^{10}	2.8024×10^{-4}
	2^9	6.7006×10^{-5}	10^{-4}		4.6202×10^{-5}
	2^{10}	1.8534×10^{-5}	10^{-5}		3.6804×10^{-5}

Remark 6.2.4. Because q is computed numerically from the pseudo spectral method, $\triangle t$ and $\triangle x$ need not be too small to obtain good estimates.

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-1}	2^8	1.642×10^{-1}
	2^9	1.532×10^{-1}
	2^{10}	1.488×10^{-1}
10^{-3}	2^7	1.204×10^{-1}
	2^8	1.177×10^{-1}
	2^9	1.163×10^{-1}
10^{-4}	2^8	1.175×10^{-1}
	2^9	1.161×10^{-1}
	2^{10}	1.153×10^{-1}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^2}$
10^{-5}	2^8	4.378×10^{-1}
	2^9	4.3223×10^{-1}
	2^{10}	4.3220×10^{-1}
10^{-6}	2^8	2.6090×10^{-1}
	2^9	2.6072×10^{-1}
	2^{10}	2.6070×10^{-1}
10^{-7}	2^8	1.5739×10^{-1}
	2^9	1.5730×10^{-1}
	2^{10}	1.5637×10^{-1}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-3}	2^6	2.19×10^{-2}
	2^8	1.91×10^{-2}
	2^{10}	8.1×10^{-3}
10^{-4}	2^7	1.75×10^{-2}
	2^8	9.2×10^{-3}
	2^9	6.01×10^{-3}
10^{-5}	2^8	9.0×10^{-3}
	2^9	4.5×10^{-3}
	2^{10}	2.4×10^{-3}

Remark 6.2.5. The other three conservation laws give bigger errors than the first one does. One reason might be that these three contain the first derivatives as conserved quantities. Therefore, extra errors occur during the process of finite difference of

computing derivatives. Moreover, we expect the same situation happens to the later conservation laws since they consist of further derivatives of q. We also provide the conserved quantities errors for VFE with different initial curves in Appendices.

Example 6.2.6 (Schrödinger Flow on $\mathbb{S}^2$)**.**

We consider the initial curve is a circle, then the real solution is stationary, i.e.,

$$\gamma(x,t) = (\cos x, \sin x, 0).$$

The following errors are estimated between the numerical solution γ_n, and γ.

Table 6.3: Errors with the true solution γ and the numerical solution γ_n

$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$	$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$
10^{-2}	2^{10}	2.7357×10^{-2}	10^{-3}	2^{7}	2.3999×10^{-1}
	2^{11}	1.3833×10^{-2}	10^{-4}		2.0235×10^{-1}
	2^{12}	7.1351×10^{-3}	10^{-5}		2.0190×10^{-1}
10^{-3}	2^{10}	2.7111×10^{-2}	10^{-3}	2^{8}	1.1683×10^{-1}
	2^{11}	1.3282×10^{-2}	10^{-4}		1.0705×10^{-1}
	2^{12}	6.5692×10^{-3}	10^{-5}		1.0247×10^{-1}

Below we give errors of each conserved quantity for Schrödinger flow with initial curve as a circle and more error estimates with other initial curves can be found in Appendices.

Table 6.4: Conserved Quantities Error for Schrödinger flow with initial data $\gamma_0 = (\cos x, \sin x, 0)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-3}	2^8	4.8897×10^{-4}
	2^9	3.5145×10^{-4}
	2^{10}	2.8128×10^{-4}
10^{-4}	2^8	3.4041×10^{-4}
	2^9	1.1705×10^{-4}
	2^{10}	4.6203×10^{-5}

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-5}	2^9	5.9648×10^{-3}
	2^{10}	4.5897×10^{-3}
	2^{11}	4.1896×10^{-3}
10^{-6}	2^9	5.9646×10^{-3}
	2^{10}	4.5645×10^{-3}
	2^{11}	4.1173×10^{-3}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-1}	2^7	3.7428×10^{-2}
	2^8	2.2321×10^{-2}
	2^9	1.0517×10^{-2}
10^{-2}	2^7	1.2461×10^{-2}
	2^8	1.2392×10^{-2}
	2^9	8.8145×10^{-3}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-5}	2^8	1.5485×10^{-2}
	2^9	7.8836×10^{-3}
	2^{10}	4.1293×10^{-3}
10^{-6}	2^8	1.5490×10^{-2}
	2^9	7.8567×10^{-3}
	2^{10}	3.934×10^{-3}

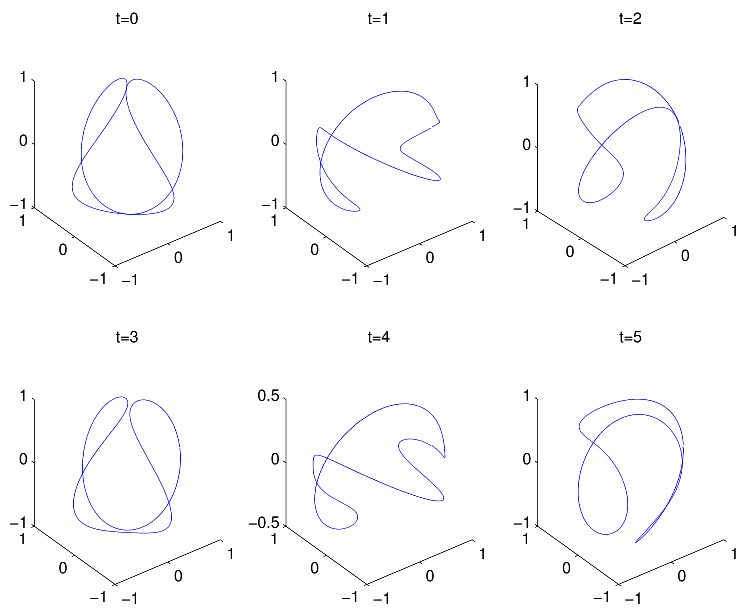

Figure 6.9: $\gamma(x,t)$: the solution of Schrödinger flow on $\mathbb{S}^2$ with initial data $\gamma_0 = (\cos x \cos(2x), \sin x \cos(2x), \sin(2x))$

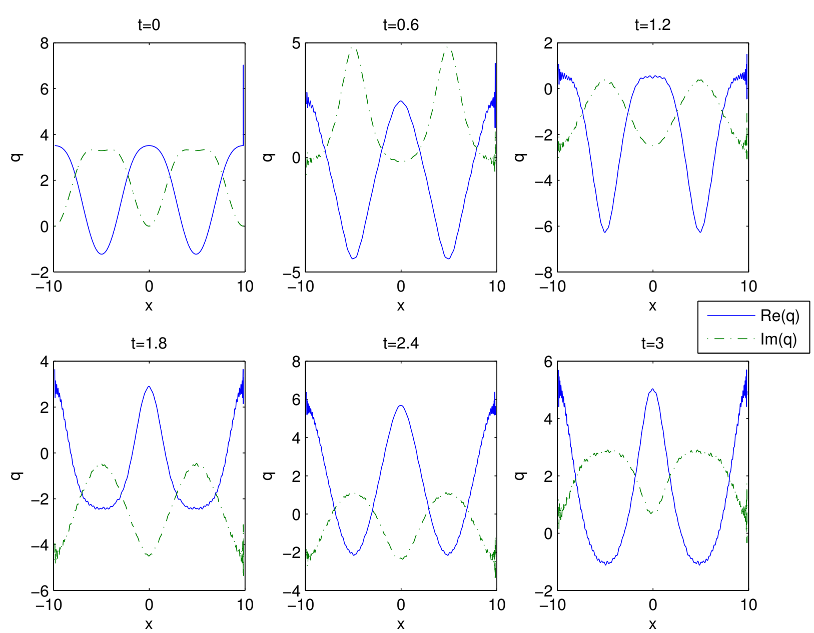

Figure 6.10: Solution q of NLS corresponding to $\gamma(x,t)$ in Figure 6.9. The solid line represents the real part of q while the dashed one shows the imaginary part of q.

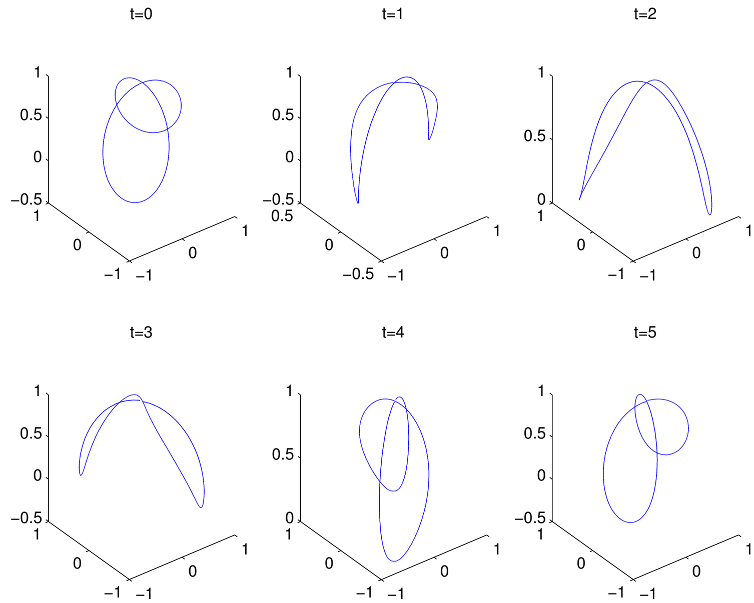

Figure 6.11: $\gamma(x,t)$: the solution of Schr̈dingier flow on $\mathbb{S}^2$ with initial data $\gamma_0 = (0.5\sin(2x), \sin x, \sqrt{1 - 0.25\sin^2(2x) - \sin^2 x})$

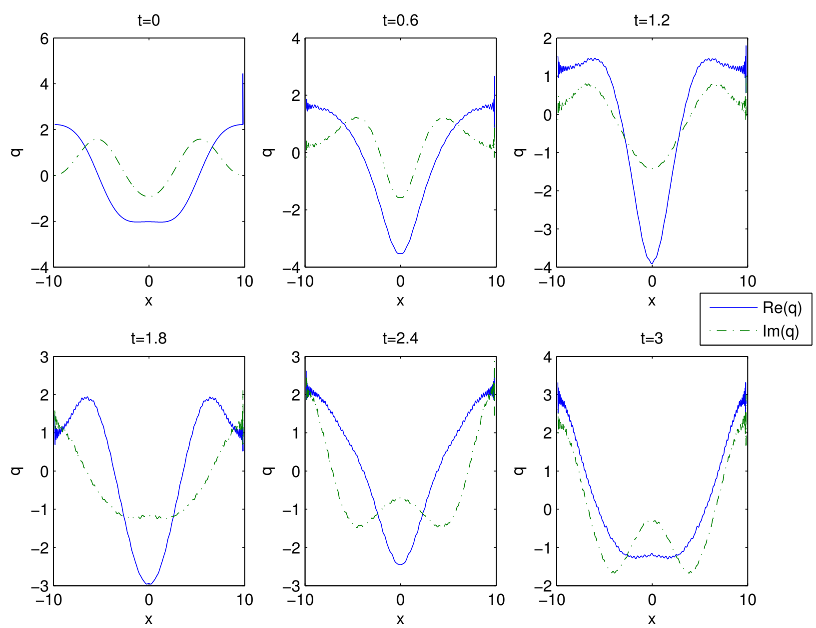

Figure 6.12: Solution q of NLS corresponding to $\gamma(x,t)$ in Figure 6.11. The solid line represents the real part of q while the dashed one shows the imaginary part of q.

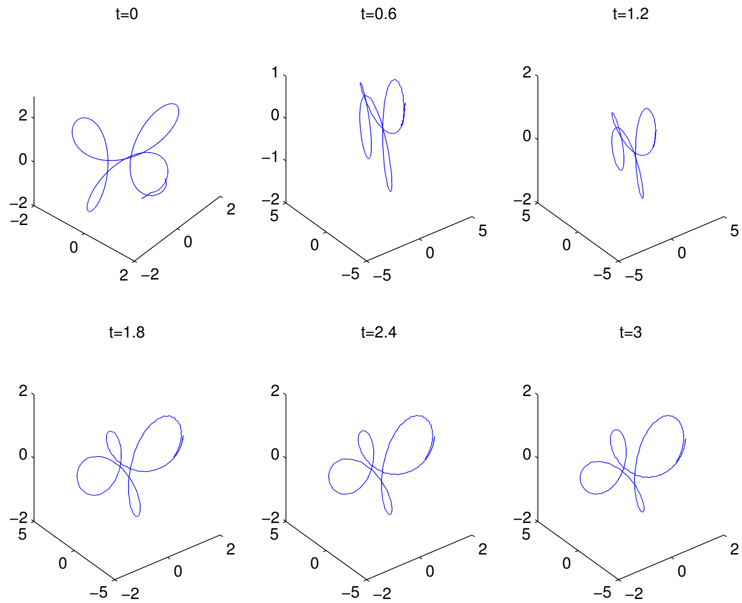

Figure 6.13: Apply BT to $\gamma(x, t) = (\cos x, \sin x, 0)$

Remark 6.2.7. Note that new solutions are not periodic in x, which is also shown on the graphs.

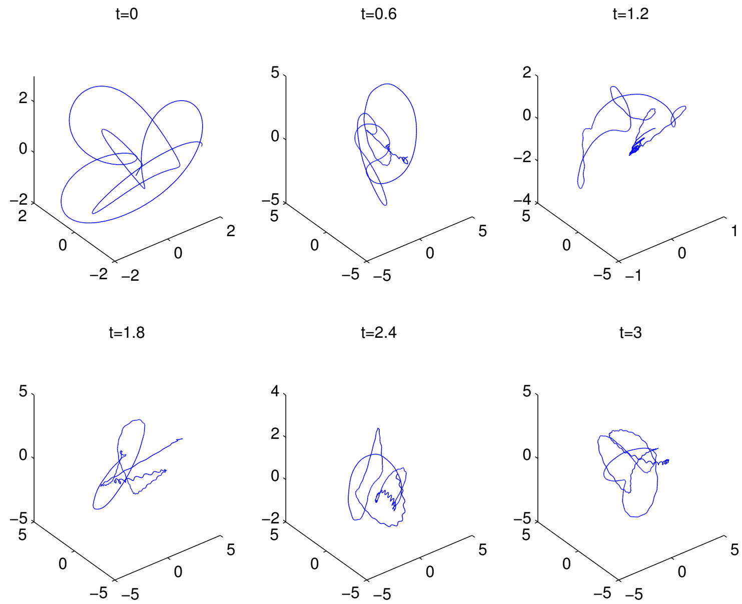

Figure 6.14: Apply BT to $\gamma(x,t)$ in Fig. 6.9

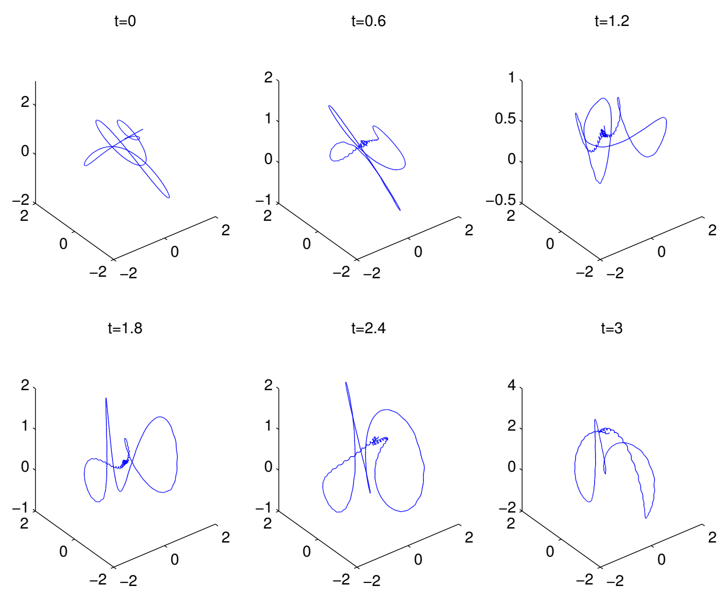

Figure 6.15: Apply BT to $\gamma(x,t)$ in Fig. 6.11

Example 6.2.8 (Geometric Airy Curve flow on $\mathbb{R}^2$).

We consider the initial curve is a circle, then the real solution is just reparametrizing and moves along the same circle, i.e.,

$$\gamma(x,t) = (\cos(x + \frac{t}{2}), \sin(x + \frac{t}{2}))$$

with the corresponding solution $q(x,t) = 1$ of mKdV. The following errors are estimated between the numerical solution γ_n, and real solution γ.

Table 6.5: Errors with the true solution γ and the numerical solution γ_n

$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$	$\triangle t$	N	$\|\|\gamma_n - \gamma\|\|_{L^2}$
10^{-3}	2^8	1.1146×10^{-1}	10^{-3}	2^8	1.1146×10^{-1}
	2^9	1.1145×10^{-1}	10^{-4}		3.6249×10^{-3}
	2^{10}	1.1144×10^{-1}	10^{-5}		1.1463×10^{-4}
10^{-4}	2^8	3.6249×10^{-3}	10^{-3}	2^9	1.1145×10^{-1}
	2^9	3.6213×10^{-3}	10^{-4}		3.6213×10^{-3}
	2^{10}	3.6195×10^{-3}	10^{-5}		1.1451×10^{-4}

Below we give errors of each conserved quantity for Airy curve flow on $\mathbb{R}^2$ with initial curve as a circle and more error estimates with other initial curves can be found in Appendices.

Table 6.6: Conserved Quantity Errors for Airy curve Flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (\cos x, \sin x)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^1}$	$\|\|H_2^n - H_2\|\|_{L^1}$	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-2}	2^7	1.6600×10^{-16}	1.3776×10^{-17}	6.9475×10^{-20}
	2^8	1.6470×10^{-16}	7.3299×10^{-17}	8.7895×10^{-21}
	2^9	7.7670×10^{-17}	1.1154×10^{-18}	3.8586×10^{-22}
10^{-3}	2^7	1.6666×10^{-17}	1.4087×10^{-18}	6.6345×10^{-21}
	2^8	1.5985×10^{-17}	6.6010×10^{-19}	7.9663×10^{-22}
	2^9	1.2772×10^{-17}	2.8760×10^{-19}	8.7462×10^{-23}

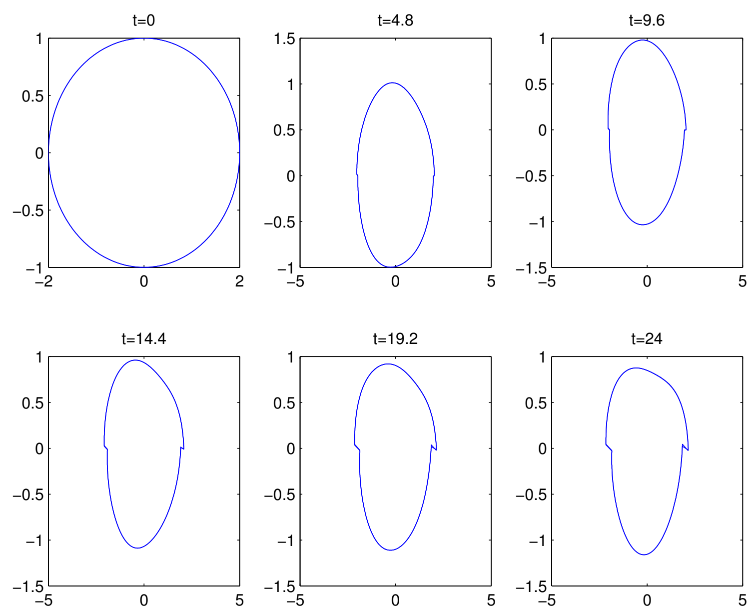

Figure 6.16: Solution of Geometric Airy curve flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (2\cos(2x), \sin(2x))$

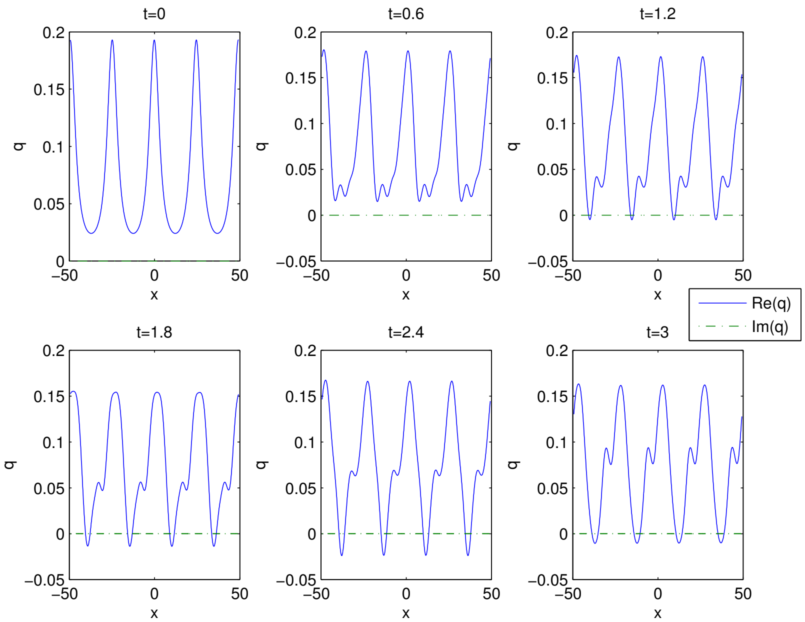

Figure 6.17: Solution q of mKdV corresponding to $\gamma(x,t)$ in Figure 6.16. The solid line represents the real part of q and the dashed one shows the imaginary part of q.

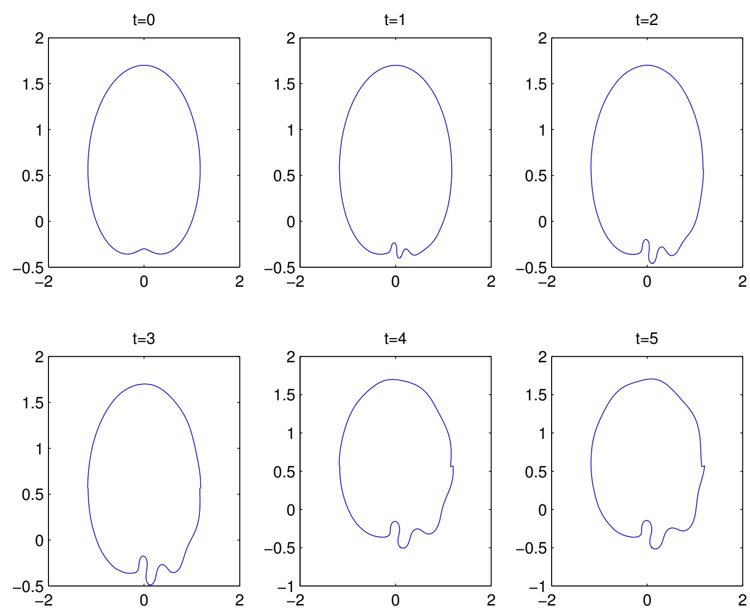

Figure 6.18: Solution of Geometric Airy curve flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (\cos x + 0.7\sin x\cos x, \sin x + 0.7\sin x\sin x)$

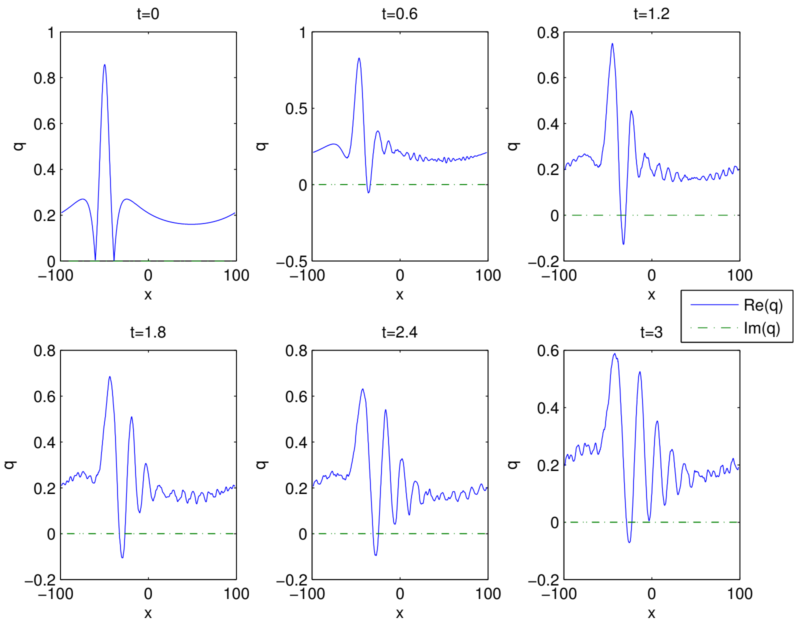

Figure 6.19: Solution q of mKdV corresponding to $\gamma(x,t)$ in Figure 6.18. The solid line shows the real part of q while the dashed one is the imaginary part of q.

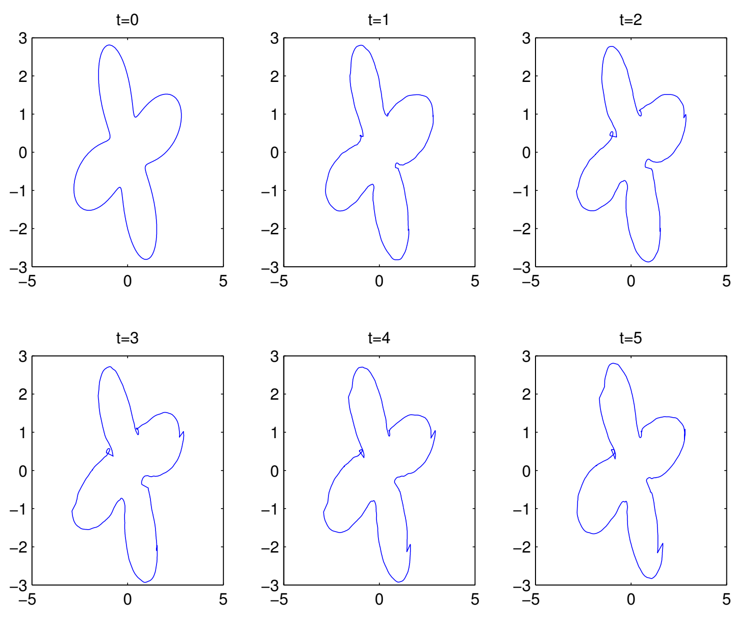

Figure 6.20: Solution of Geometric Airy curve flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (\sqrt{\sin(2x)}\cos x, \sqrt{\sin(2x)}\sin x)$

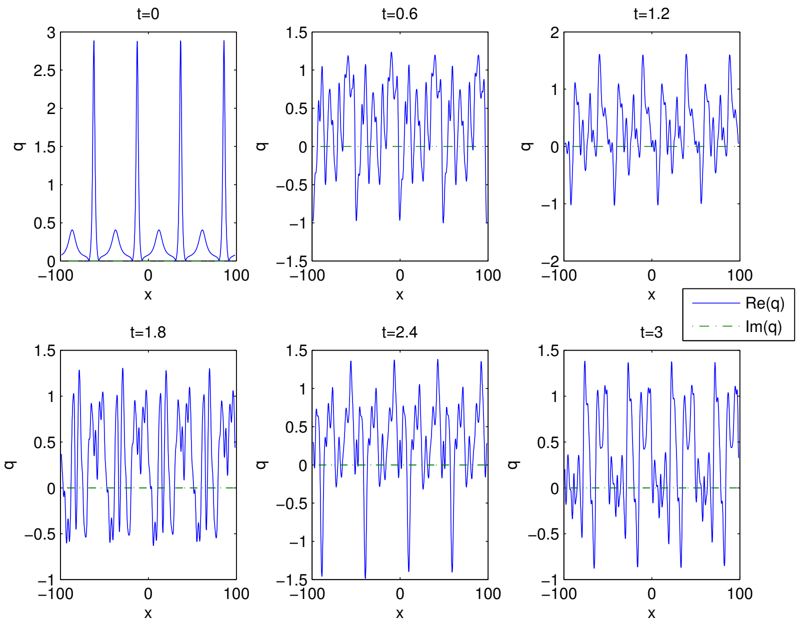

Figure 6.21: Solution q of mKdV corresponding to $\gamma(x,t)$ in Figure 6.20. The solid line shows the real part of q while the dashed one is the imaginary part of q.

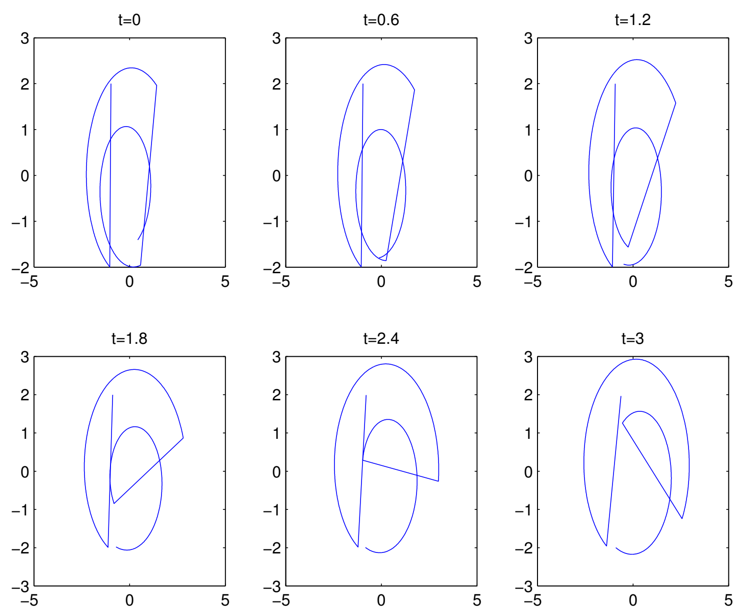

Figure 6.22: Apply BT to $\gamma(x,t) = (\cos(x + \frac{t}{2}), \sin(x + \frac{t}{2}))$

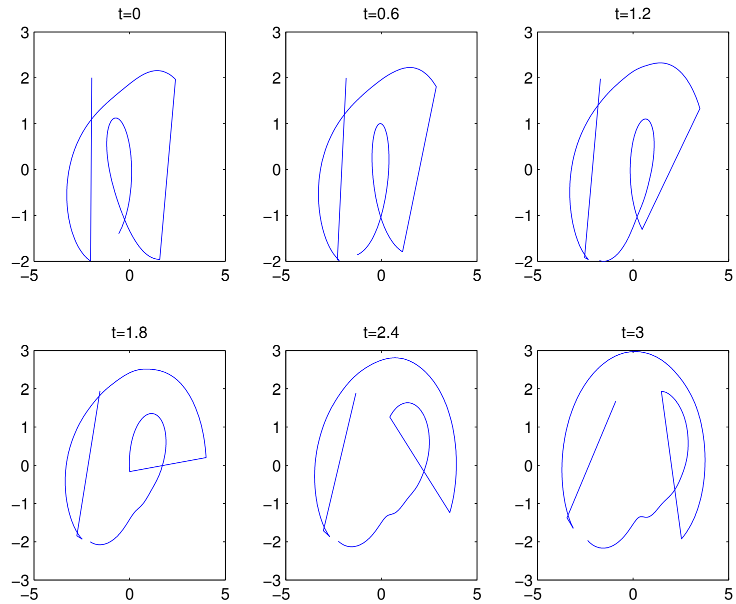

Figure 6.23: Apply BT to the solution $\gamma(x,t)$ of geometric Airy curve flow on $\mathbb{R}^2$ with initial data $\gamma(x,0) = (2\cos x, \sin x)$

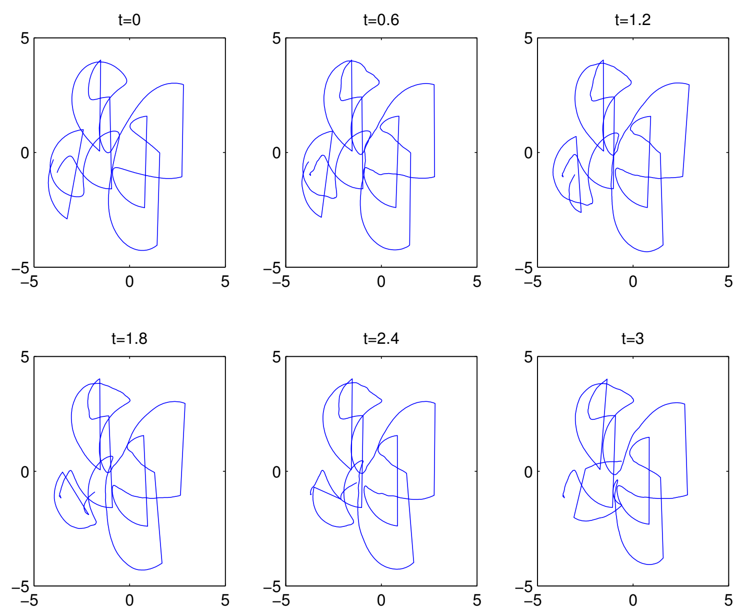

Figure 6.24: Apply BT to $\gamma(x,t)$ in Fig. 6.20

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Appendices

A Appendix A

Table A.7: Conserved Quantity Errors for Schrödinger Flow on $\mathbb{S}^2$ with initial data $\gamma_0 = (0.5\sin(2x), \sin x, \sqrt{1 - 0.25\sin(2x)^2 - \sin^2 x})$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-3}	2^8	2.6628×10^{-3}
	2^9	1.2077×10^{-3}
	2^{10}	6.3759×10^{-4}
10^{-4}	2^8	8.1342×10^{-4}
	2^9	4.5558×10^{-4}
	2^{10}	2.0489×10^{-4}

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-5}	2^{11}	8.095×10^{-3}
10^{-6}		8.0533×10^{-3}
10^{-7}		8.0526×10^{-3}
10^{-5}	2^{12}	8.0481×10^{-3}
10^{-6}		7.9491×10^{-3}
10^{-7}		7.9470×10^{-3}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-1}	2^7	7.7837×10^{-2}
	2^8	4.7026×10^{-2}
	2^9	2.2704×10^{-2}
10^{-2}	2^7	2.5912×10^{-2}
	2^8	2.5221×10^{-2}
	2^9	1.7832×10^{-2}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-3}	2^9	2.3233×10^{-2}
10^{-4}		1.2536×10^{-2}
10^{-5}		1.1030×10^{-2}
10^{-3}	2^{10}	2.6471×10^{-2}
10^{-4}		9.0316×10^{-3}
10^{-5}		5.7436×10^{-3}

Table A.8: Conserved Quantity Errors for Schrödinger Flow on $\mathbb{S}^2$ with initial data $\gamma_0 = (\cos x \cos(2x), \sin x \cos(2x), \sin(2x))$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-3}	2^8	1.3961×10^{-2}
	2^9	6.6850×10^{-3}
	2^{10}	3.4997×10^{-3}
10^{-4}	2^8	1.3802×10^{-2}
	2^9	6.3873×10^{-3}
	2^{10}	3.0693×10^{-3}

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-3}	2^6	2.1511×10^{-1}
	2^7	2.0501×10^{-1}
	2^8	2.0116×10^{-1}
10^{-4}	2^6	2.1494×10^{-1}
	2^7	2.0454×10^{-1}
	2^8	1.9985×10^{-3}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-1}	2^8	1.8063×10^{-1}
	2^9	1.1076×10^{-1}
	2^{10}	7.0834×10^{-2}
10^{-2}	2^8	6.6871×10^{-2}
	2^9	4.6219×10^{-2}
	2^{10}	2.6477×10^{-2}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-3}	2^9	2.3233×10^{-2}
10^{-4}		1.2536×10^{-2}
10^{-5}		1.1030×10^{-2}
10^{-3}	2^{10}	2.6471×10^{-2}
10^{-4}		9.0316×10^{-3}
10^{-5}		5.7436×10^{-3}

Table A.9: Conserved Quantities Error for VFE with initial data $\gamma_0 = (\sin x + \cos x, \sin x, \cos x)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-4}	2^9	3.1668×10^{-5}
	2^{10}	1.5662×10^{-5}
	2^{11}	1.2218×10^{-5}
10^{-5}	2^9	2.7050×10^{-5}
	2^{10}	7.3382×10^{-6}
	2^{11}	2.4698×10^{-6}

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-1}	2^7	1.216×10^{-1}
10^{-3}		7.1093×10^{-2}
10^{-4}		7.1078×10^{-2}
10^{-1}	2^8	1.094×10^{-1}
10^{-3}		6.5688×10^{-2}
10^{-4}		6.5662×10^{-2}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-2}	2^9	9.9085×10^{-3}
10^{-3}		4.3464×10^{-3}
10^{-4}		4.0556×10^{-3}
10^{-2}	2^{11}	2.8921×10^{-3}
10^{-3}		2.5636×10^{-3}
10^{-4}		1.6970×10^{-3}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-5}	2^9	1.6855×10^{-3}
	2^{10}	8.9906×10^{-4}
	2^{11}	7.0293×10^{-4}
10^{-6}	2^9	1.679855×10^{-3}
	2^{10}	8.4861×10^{-4}
	2^{11}	4.2816×10^{-4}

Table A.10: Conserved Quantities Error for solution $\gamma(x,t)$ of VFE where $\gamma(x,0) = (\sin x \cos(2x), \sin x, \cos x)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^2}$
10^{-4}	2^{10}	8.2571×10^{-6}
	2^{11}	3.9310×10^{-6}
	2^{12}	5.4828×10^{-7}
10^{-6}	2^{10}	3.9851×10^{-6}
	2^{11}	1.0877×10^{-6}
	2^{12}	3.1094×10^{-7}

$\triangle t$	N	$\|\|H_2^n - H_2\|\|_{L^2}$
10^{-3}	2^9	5.322×10^{-3}
	2^{10}	4.927×10^{-3}
	2^{11}	4.299×10^{-3}
10^{-4}	2^9	3.172×10^{-3}
	2^{10}	3.112×10^{-3}
	2^{11}	3.1094×10^{-7}

$\triangle t$	N	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-1}	2^{10}	1.2835
10^{-2}		1.5646×10^{-1}
10^{-3}		6.8974×10^{-2}
10^{-1}	2^{11}	1.2634
10^{-2}		1.4403×10^{-1}
10^{-3}		4.5752×10^{-2}

$\triangle t$	N	$\|\|H_4^n - H_4\|\|_{L^2}$
10^{-4}	2^{9}	2.2613×10^{-3}
10^{-5}		2.2005×10^{-3}
10^{-6}		2.0036×10^{-3}
10^{-4}	2^{10}	1.8837×10^{-3}
10^{-5}		9.7956×10^{-4}
10^{-6}		9.5864×10^{-5}

Table A.11: Conserved Quantity Errors for Airy curve Flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (2\cos x, \sin x)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^1}$	$\|\|H_2^n - H_2\|\|_{L^1}$	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-2}	2^{7}	7.9435×10^{-6}	1.5482×10^{-6}	1.7585×10^{-5}
	2^{8}	3.5573×10^{-7}	3.4980×10^{-8}	7.4414×10^{-7}
	2^{9}	2.8868×10^{-8}	1.4197×10^{-9}	3.1467×10^{-8}
10^{-3}	2^{7}	3.7674×10^{-7}	7.3404×10^{-8}	3.9588×10^{-7}
	2^{8}	5.0565×10^{-8}	4.9641×10^{-9}	5.3145×10^{-8}
	2^{9}	2.8957×10^{-9}	1.4232×10^{-10}	2.5394×10^{-9}

Table A.12: Conserved Quantity Errors for Airy curve Flow on $\mathbb{R}^2$ with initial data $\gamma_0 = (\cos x + 0.7 \sin x \cos x, \sin x + 0.7 \sin x \cos x)$

$\triangle t$	N	$\|\|H_1^n - H_1\|\|_{L^1}$	$\|\|H_2^n - H_2\|\|_{L^1}$	$\|\|H_3^n - H_3\|\|_{L^1}$
10^{-2}	2^8	5.2020×10^{-7}	2.7472×10^{-8}	3.3087×10^{-6}
	2^9	7.3991×10^{-8}	1.9636×10^{-9}	2.0813×10^{-7}
	2^{10}	9.8127×10^{-9}	1.3045×10^{-10}	1.3987×10^{-8}
10^{-3}	2^8	4.1061×10^{-8}	2.1729×10^{-9}	1.9064×10^{-7}
	2^9	6.4102×10^{-9}	1.7067×10^{-10}	6.4224×10^{-9}
	2^{10}	9.1354×10^{-10}	1.2181×10^{-11}	1.0128×10^{-9}