Title
Calibrating Determinacy Strength in Borel Hierarchies

Author
Hachtman, Sherwood J.

Publication Date
2015

Peer reviewed|Thesis/dissertation

University of California
Los Angeles

Calibrating Determinacy Strength in Borel Hierarchies

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Sherwood Julius Hachtman

2015

Abstract of the Dissertation

Calibrating Determinacy Strength in Borel Hierarchies

by

Sherwood Julius Hachtman

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2015

Professor Itay Neeman, Chair

We study the strength of determinacy hypotheses in levels of two hierarchies of subsets of Baire space: the standard Borel hierarchy $\langle \boldsymbol{\Sigma}^0_\alpha \rangle_{\alpha<\omega_1}$, and the hierarchy of sets $\langle \boldsymbol{\Sigma}^0_\alpha(\boldsymbol{\Pi}^1_1) \rangle_{\alpha<\omega_1}$ in the Borel σ-algebra generated by coanalytic sets.

We begin with $\boldsymbol{\Sigma}^0_3$, the lowest level at which the strength of determinacy had not yet been characterized in terms of a natural theory. Building on work of Philip Welch [Wel11], [Wel12], we show that Σ^0_3 determinacy is equivalent to the existence of a β-model of the axiom of $\boldsymbol{\Pi}^1_2$ monotone induction.

For the levels $\boldsymbol{\Sigma}^0_4$ and above, we prove best-possible refinements of old bounds due to Harvey Friedman [Fri71] and Donald A. Martin [Mar85, Mar] on the strength of determinacy in terms of iterations of the Power Set axiom. We introduce a novel family of reflection principles, $\Pi_1\text{-}\mathsf{RAP}_\alpha$, and prove a level-by-level equivalence between determinacy for $\Sigma^0_{1+\alpha+3}$ and existence of a wellfounded model of $\Pi_1\text{-}\mathsf{RAP}_\alpha$. For $\alpha = 0$, we have the following concise result: Σ^0_4 determinacy is equivalent to the existence of an ordinal θ so that L_θ satisfies "$\mathcal{P}(\omega)$ exists, and all wellfounded trees are ranked."

We connect our result on Σ^0_4 determinacy to work of Noah Schweber [Sch13] on higher order reverse mathematics. Schweber shows, using the method of forcing, that for games with real number moves, clopen determinacy ($\Delta^{\mathbb{R}}_1$-DET) does not imply open determinacy ($\Sigma^{\mathbb{R}}_1$-DET) over the weak base theory RCA^3_0. We show that the model L_θ is a witness to

this separation result, and furthermore, that L_θ is (in the appropriate sense) the minimal (third-order) β-model of a natural theory of projective transfinite recursion. We obtain that $\mathbf{\Sigma}^0_4$ determinacy falls strictly between the principles $\Sigma^{\mathbb{R}}_1$-DET, $\Delta^{\mathbb{R}}_1$-DET in terms of β-consistency strength.

Finally, we combine our methods with those of John Steel [Ste82] and Itay Neeman [Nee00], [Nee06] to characterize the strength of determinacy for sets in the pointclasses $\Sigma^0_{1+\alpha+3}(\Pi^1_1)$. Granted that the reals are closed under the sharp function, we show this determinacy is equivalent to the existence of an iterable mouse with a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ in which Π_1-$\mathsf{RAP}_{\kappa+1+\alpha}$ holds.

The dissertation of Sherwood Julius Hachtman is approved.

Matthias J. Aschenbrenner

John P. Carriero

Donald A. Martin

Yiannis N. Moschovakis

Itay Neeman, Committee Chair

University of California, Los Angeles

2015

For Myrna,

who understands

Table of Contents

Acknowledgments

I have had the unreasonable good fortune to meet and work under the auspices of many brilliant and giving mathematicians and friends. Without them this work would not have been possible.

Above all I owe an enormous debt of gratitude to Itay Neeman. If he had merely suggested interesting problems, it would have been enough; if he had merely met with me once a week to discuss mathematics, it would have been enough. But he has gone above and beyond, providing me with every possible opportunity for growth and experience throughout my stay in graduate school. He has been persistently generous with his time, supportive of my endeavors, reassuring in times of distress, understandingly admonishing in the many times of sloth or stagnation, and always remarkably prudent in his advice and assistance. I have been truly lucky to have him as my advisor.

It should be obvious from the title of my thesis alone that I am likewise forever indebted to Donald A. Martin. Even had I never met the man, his many ideas and writings, in their depth, clarity, and beauty, would have been a guiding light. As it stands, Tony was my first logic teacher at UCLA, an encouraging presence throughout my stay, frequently a lively sounding board, and always a generous source of good ideas. He is, with astonishing constancy, good-humored, modest, patient and forthright. To talk with and learn from him has been a great pleasure.

I should also thank Simon Thomas, who took me under his wing at Rutgers College. Before meeting Simon, I had never considered a career in mathematics as something available to me. His intervention at a pivotal moment set me down the path I'm on today.

I also thank Martin Zeman and Yiannis Moschovakis, each of whom provided clarity to material that was to become basic to my thesis.

There are many young set theorists who I have been privileged to know and learn from, and whose many accomplishments serve as an inspiration. Among them are Grigor Sargsyan, Daisuke Ikegami, Nam Trang, Spencer Unger, and Noah Schweber; I know that I will continue to learn from them as time goes on, and will consider it a great achievement if I can teach them something in return.

My time at UCLA would have been quite a different experience if not for the company and support of many wonderful friends. Without their camaraderie, there would really have been little point in doing any mathematics. Among them I would like to thank Justin Palumbo, Anush Tserunyan, William Chen, Jason Murphy, and Siddharth Bhaskar; each with their own brand of enthusiasm and discerning aesthetic and mathematical taste, they have helped me to inform and maintain my own.

I thank C. Thi Nguyen, Melissa Hughs, Jonathan Gingerich, and Tamar Weber, for guidance in the art of living, and for being the best climbing partners one could ask for.

Finally, I thank my family: Joey, Tom, Myrna, April, and Ilse. They have been a ceaseless source of love, support, and encouragement since before I can remember. Without them this work would not be possible, nor would I.

Vita

2009	B.S., Mathematics, Rutgers, the State University of New Jersey.
2011	M.A., Mathematics, University of California, Los Angeles.
2014	Teaching Assistant Instructor, Integration and Infinite Series, Winter Quarter (UCLA).
2014	Instructor: Forcing and Independence in Set Theory. Undergraduate Summer School, UCLA Logic Center.
2014-2015	Teaching Assistant Consultant. Office of Instructional Development, UCLA.

Publications

Calibrating determinacy strength in levels of the Borel hierarchy. Submitted.

Determinacy in third order arithmetic. Submitted.

CHAPTER 1

An overview of the results

1.1 Background and motivation

Let us define a game between two players, call them Player I and Player II, who take turns choosing natural numbers, $x_0, x_1, x_2, \ldots$, in sequence:

$$\begin{array}{c|cccccccc} I & x_0 & & x_2 & \ldots & x_{2n} & & \ldots \\ \hline II & & x_1 & & \ldots & & x_{2n+1} & \ldots \end{array}$$

Supposing that the players have survived through infinitely many rounds of the game, they have cooperated to produce an infinite sequence of naturals, $x = \langle x_0, x_1, \ldots \rangle$. A winner of the play x is then decided based on a priorly fixed *winning condition*: Namely, a set $A \subseteq \omega^\omega$ of sequences of naturals is fixed beforehand, called the *payoff set*. If $x \in A$, then Player I wins; if $x \notin A$, then Player II wins. We denote the game with payoff A by $G(A)$.

A strategy for Player I is a partial function $\sigma : \omega^{<\omega} \rightharpoonup \omega$ that is defined on finite sequences of even length. Intuitively, σ tells I what move to make whenever it is his turn. A strategy is *winning for I in the game* $G(A)$ if whenever x is compatible with σ, in the sense that $x_{2n+1} = \sigma(\langle x_0, \ldots, x_{2n} \rangle)$ for all n, then we have $x \in A$, that is, I wins the play x. Analogous definitions are made for Player II. If either player has a winning strategy in the game with payoff A, we say $G(A)$ is *determined*.

Such *infinite two-person games of perfect information* (henceforth simply *infinite games*) had been considered by mathematicians of the Lwów school [Mau81] and were introduced to the literature by Gale and Stewart [GS53] as a generalization of finite perfect information games, already an object of intense study in the burgeoning field of economic game theory

[NM44]. The stated hope of [GS53] was that study of infinite games would lead to the discovery of methods applicable to the study of games of finite length. Since their introduction, however, infinite games have bred a panoply of beautiful techniques and results tailored especially to a study of the infinite, elucidating an intimate connection between *axioms of large infinity* and the structure of *definable sets of reals*.

Let $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ be a collection of subsets of Baire space; adopting common set theoretic parlance, we refer to ω^ω as *the reals*. Γ determinacy, abbreviated Γ-DET, is the statement that $G(A)$ is determined for every $A \in \Gamma$. The main result in [GS53] is $\mathbf{\Sigma}^0_1$-DET: Open games are determined. On the other hand, they show there exist sets B so that *neither player* has a winning strategy in $G(B)$, but the example of such B is rather complicated: its definition makes use of a wellordering of $\mathbb{R}$, and so appeals to the axiom of choice.

This gives an inkling for the motivation behind a study of infinite games: Γ-DET is a *regularity property* for sets of reals in Γ. This had already been evident in the earliest results of the subject: if Γ is closed under continuous preimages, then all sets in Γ have the Baire property (due to Banach and Mazur [Mau81], see also [Oxt57]), the perfect set property [Dav64], and are Lebesgue measurable [MS64]. So Γ-DET implies that sets in Γ are free of the well-known pathological consequences of the axiom of choice.

The possibility emerged that a large class of sets of reals are determined, and by virtue of this are "well-behaved". In [MS62], Mycielski and Steinhaus conjectured that this is the case for all *definable* sets of reals, granting $\mathcal{P}(\omega^\omega)$-DET the status of an axiom: AD, the axiom of determinacy. Hoping the additional structure imposed by determinacy could provide an illuminating picture of the class of sets of reals, they proposed AD as an alternative to the axiom of choice for the study of analysis. This conjecture was to be dramatically validated through the 80s, as descriptive set theorists of the UCLA-Caltech Cabal Seminar [KM78, KMM81, KMM83, KMS88] were to provide a remarkably full picture of the structure of sets of reals under AD; e.g., generalizing the Kondo-Addison uniformization theorem for $\mathbf{\Pi}^1_1$ [Kon37] to all levels of the projective hierarchy [Mos71].

It remained to show determinacy holds for the largest class of sets possible. Gale-Stewart

[GS53] was the starting point; Philip Wolfe [Wol55] almost immediately proved $\mathbf{\Sigma}^0_2$-DET (determinacy for games with $F_{\sigma\delta}$ or $G_{\delta\sigma}$ payoff). Morton Davis [Dav64] made the next progress, proving determinacy for games with $\mathbf{\Sigma}^0_3$ payoff. But extending these results to $\mathbf{\Sigma}^0_4$ and beyond proved difficult.

The first indication that determinacy hypotheses possess substantial strength was Mycielski's result [MS64] that consistency of $\mathsf{ZF} + \mathsf{AD}$ implies the consistency of ZFC plus an inaccessible cardinal; Solovay (unpublished, see [Kan03]) was to later show that if the axiom of determinacy holds, then ω_1 is a measurable cardinal. Soon after, Harvey Friedman [Fri71] discovered metamathematical obstacles to proving determinacy even at low levels of the Borel hierarchy. Friedman showed that even $\mathbf{\Sigma}^0_5$-DET is unprovable in ZFC^- (ZFC with the Power Set axiom removed). The situation revealed was that any proof of Borel determinacy (if such could be effected in ZFC) would require use of the Replacement and Power Set axioms—in fact, ω_1-many iterated applications of Power Set to the reals. So even though Borel determinacy is a statement about reals and sets of reals—involving only objects in $V_{\omega+2}$—any proof would require an appeal to uncountably many levels of the von Neumann universe of sets, exhausting V_{ω_1}.

The first proof of Borel determinacy from any hypothesis was Martin's result [Mar70] that analytic determinacy, $\mathbf{\Pi}^1_1$-DET, follows from the existence of a measurable cardinal. To set theorists convinced of the consistency of measurable cardinals, then, $\mathbf{\Delta}^1_1$-DET was true. Furthermore, Borel determinacy, being a $\mathbf{\Pi}^1_3$ statement, is absolute from V to L, by Shoenfield absoluteness [Sho61]; so assuming the existence of a measurable cardinal, Borel determinacy holds in all models of set theory. The expectation was that a proof of Borel determinacy could be effected in ZFC, possibly augmented by at most some large cardinal hypothesis consistent with L (in particular, much smaller than measurable).

However, progress towards a ZFC proof was gradual. Baumgartner (unpublished) reproved $\mathbf{\Sigma}^0_3$-DET, using a variation of Davis's proof with an indiscernibles argument inspired by Martin's proof of $\mathbf{\Pi}^1_1$-DET. Martin, in turn, extended this to a proof of $\mathbf{\Sigma}^0_4$-DET assuming the existence of a weakly compact cardinal. And Paris [Par72] completed this effort by proving the existence of the required indiscernibles in ZFC—indeed, in the theory $\mathsf{ZF}^- +$ "ω_1 exists".

As far as generalizing beyond $\mathbf{\Sigma}^0_4$, the indiscernibility arguments were a dead end. Inspiration came from a slightly different quarter: Blass [Bla75] proved the equivalence of two strong forms of determinacy, $\mathsf{AD}_{\mathbb{R}}$, the determinacy of games with *real-number* moves, and $\mathsf{AD}_\omega^{\omega^2}$, the determinacy of games with natural number moves, but with ω^2 many rounds. The nontrivial implication involves the simulation of a game of length ω^2 by a game on $\mathbb{R}$ where the players play fragments of strategies in the long game on ω. As recounted in [Kan03], this development was to serve as the germ of the idea behind Martin's celebrated proof of Borel determinacy in ZFC.

Martin's proof proceeds by an induction on Borel rank. First, by defining an auxiliary game in which (in addition to the natural number moves) Player I plays reals coding sets of positions in the game of interest, an open game can be continuously reduced to a clopen game; we say that open sets can be *unraveled*. Doing this simultaneously for the countably many open sets from which a $\mathbf{\Sigma}^0_{1+\alpha}$ set A is constructed reduces the game $G(A)$ to a $\mathbf{\Sigma}^0_\alpha$ game on a tree on $\mathcal{P}(\omega)$. Iterating this process, one reduces $\mathbf{\Sigma}^0_{1+\alpha+3}$ determinacy to $\mathbf{\Sigma}^0_3$ determinacy for games with moves from $\mathcal{P}^{\alpha+1}(\omega)$, and Davis's proof of $\mathbf{\Sigma}^0_3$-DET can be carried out on the larger tree, granted a strong enough ambient theory. This level-by-level analysis corresponds very nearly exactly with Friedman's result. Refinements of these bounds by Martin (in [Mar], and discussed in subsequent sections) have remained the sharpest known prior to our work.

1.2 Reverse mathematics and Borel determinacy

A more refined study of the strength of determinacy at low levels of the Borel hierarchy was to follow. Results of this kind can be traced along two important trajectories. On the one hand, in reverse mathematics, the strength of determinacy is measured in terms of *provability*: An optimal result would be an isolation of some subsystem of second order arithmetic provably equivalent to Γ-DET over some weak base theory. On the other hand, in set theory, determinacy strength is measured in terms of *consistency strength*: An optimal result would be a characterization of some minimal model whose existence is equivalent (again over some weak base theory) to Γ-DET; such models typically take the form of minimal canonical (L-like) inner models a particular theory, and a measure of the complexity of winning strategies is provided in terms of their simplest definitions over the model in question.

On both fronts, the question of strength for the lowest levels of the Borel hierarchy has been settled. In one of the first results of reverse mathematics, Steel proved in his Ph.D. thesis [Ste77] that $\mathbf{\Sigma}^0_1$-DET is equivalent over ACA_0 to ATR_0. Blass [Bla72] showed that there exist Σ^0_1 games with no hyperarithmetical winning strategy; it follows that winning strategies in Σ^0_1 games are definable over the least wellfounded model of KP, and so are constructed at or before ω_1^{CK} in L. Tanaka [Tan91] refined Wolfe's [Wol55] original proof of Σ^0_2-DET and showed this determinacy to be equivalent to an axiom asserting the stabilization of Σ^1_1-monotone inductive operators; this in turn was inspired by work of Solovay (see [Kec78b]) from which follows a characterization of the least level of L witnessing determinacy in terms of the closure ordinal of such operators.

Already at the level of Σ^0_3-DET, a calibration of determinacy strength in terms of reverse mathematics becomes problematic. Welch [Wel11] has closely studied this strength, pushing through Davis's [Dav64] proof under minimal assumptions, and establishing that Σ^0_3-DET is provable from Π^1_3-CA_0, but not from Δ^1_3-CA_0; however, Montalbán and Shore show [MS12] that no reversal is possible, in the strong sense that Σ^0_3-DET (and indeed, any true Σ^1_4 sentence) cannot prove Δ^1_2-CA_0. However, Welch went on to characterize [Wel12] the least ordinal γ so that winning strategies for all such games belong to $L_{\gamma+1}$ as the least ordinal with an "infinite depth Σ_2-nesting".

In Chapter 2, we prove the following:

Theorem 1.2.1. *Over the base theory* $\mathbf{\Pi}^1_1$-CA_0*,* Σ^0_3-DET *is equivalent to the existence of a countably-coded β-model of* $\mathbf{\Pi}^1_2$-MI*, the axiom scheme of* $\mathbf{\Pi}^1_2$ *monotone induction.*

Let $\mathcal{M}$ be such a model. The proof of determinacy is a natural reformulation of Davis's proof [Dav64] in the language of monotone operators; however, winning strategies for Player II in these games may be proper classes from the point of view of $\mathcal{M}$, and we take some care to isolate the complexity of the definition of the winning strategies over $\mathcal{M}$. The converse implication relies heavily on Welch's characterization of the ordinal γ, and the bulk of the work is in showing that the least ordinal bearing an infinite-depth Σ_2-nesting is the ordinal height of a level of L in which $\mathbf{\Pi}^1_2$-MI holds. Our analysis has the interesting corollary that the $\mathbf{\Sigma}^1_2$ relations correctly computed in the minimal β-model of $\mathbf{\Pi}^1_2$-MI are precisely the $\Game\mathbf{\Sigma}^0_3$ relations.

We remark that Montalbán and Shore [MS12] go a bit further up, analyzing determinacy for levels of the difference hierarchy on Π^0_3, showing that $n{-}\mathbf{\Pi}^0_3$-DET lies strictly between Δ^1_{n+2}-CA_0 and Π^1_{n+2}-CA_0; again, no reversals are possible. They establish the limit of determinacy provable in second-order arithmetic as essentially $<\omega{-}\mathbf{\Pi}^0_3$-DET, and even this determinacy may fail in nonstandard models of second-order arithmetic. What is more, in recent work [MS13], they establish consistency-strength implications, obtaining the existence of $n{+}1$-admissible ordinals from $n{-}\mathbf{\Pi}^0_3$-DET, for all $n \geq 1$, though equivalences at these stages are not yet known.

This brings us to the level $\mathbf{\Sigma}^0_4$, where we confront the Martin/Friedman results and must deal with the strength of mathematics of higher types. Prior to our work, the best-known bounds on the strength of $\mathbf{\Sigma}^0_4$-DET were the following.

Theorem 1.2.2 (Martin, Friedman)**.** *(Boldface)* $\mathbf{\Sigma}^0_4$-DET *is provable in the theory* $\mathsf{Z}^- +$ Σ_1*-Replacement +* *"*$\mathcal{P}(\omega)$ *exists"; but even (lightface)* Σ^0_4-DET *is not provable in the theory* ZFC^-*.*

Here, again, the superscript "$-$" indicates removal of the Power Set axiom. Z is Zermelo set theory, i.e. ZFC with Replacement removed (but full Comprehension intact).

The table below summarizes results from the literature on the relationship between well-known axiom systems and some of the determinacy hypotheses mentioned; note that those bounds which do not give equivalences are not always the sharpest known.

Γ	Upper bound	Lower bound	$\mathsf{ZF} \vdash \Gamma$-DET	Bounds
Σ^0_1	ATR_0	ATR_0	[GS53]	[Ste77]
Σ^0_2	Σ^1_1-MI	Σ^1_1-MI	[Wol55]	[Tan91]
Σ^0_3	Π^1_3-CA_0	Δ^1_3-CA_0	[Dav64]	[Wel11]
$n{-}\Pi^0_3$	Π^1_{n+2}-CA_0	Δ^1_{n+2}-CA_0	[Mar74]	[MS12]
$<\omega{-}\Pi^0_3$	$\exists\omega$-model $\mathcal{M} \models Z_2$	Z_2	[Mar74]	[MS12]
Σ^0_4	$\mathsf{ZF}^- + \exists\mathcal{P}(\omega)$	ZF^-	[Par72]	[Mar85], [Fri71]
Σ^0_5	$\mathsf{ZF}^- + \exists\mathcal{P}^2(\omega)$	$\mathsf{ZF}^- + \exists\mathcal{P}(\omega)$	[Mar85]	[Fri71], [Mar]
$\Sigma^0_{1+\alpha+3}$	$\mathsf{ZF}^- + \exists\mathcal{P}^{\alpha+1}(\omega)$	$\mathsf{ZF}^- + \exists\mathcal{P}^{\alpha}(\omega)$	[Mar85]	[Fri71], [Mar]

The central result of this thesis is an isolation of a subtheory of ZF whose consistency strength lines up exactly with the strength of Σ^0_4-DET. A natural place to look is the theory KP + "$\mathcal{P}(\omega)$ exists", as the axiom of Σ_1-Collection can be seen as a weakening of Σ_1-Replacement. But initial investigations revealed that this theory was too weak. Indeed, extending the Martin/Friedman arguments, we found

Proposition 1.2.3. *The theory* KP + *"*$\mathcal{P}(\omega)$ *exists"* $+\Sigma_1$*-Comprehension proves* $\boldsymbol{\Sigma}^0_4$-DET*, and indeed, proves the existence of a* β*-model of* $\boldsymbol{\Sigma}^0_4$-DET*. However,* Σ^0_4-DET *is not provable in the theory* KP + *"*$\mathcal{P}(\omega)$ *exists"* $+\Sigma_1$*-Comprehension* restricted to subsets of ω*.*

If a theory with strength precisely that of Σ^0_4-DET exists, it would have to live somewhere in this narrow gap; and the arguments involved seemed to indicate that any such axiom would need to imply certain consequences of admissibility, while avoiding the full strength of Σ_1-Collection.

Chapter 3 contains the main contribution of this thesis. We isolate there (in a slightly more abstract form) the following weak reflection principle.

Definition 1.2.4. The Π_1-*Reflection to Admissibles Principle* (Π_1-RAP) is the following axiom scheme in the language of set theory: $\mathcal{P}(\omega)$ exists, and whenever $Q \subseteq \mathcal{P}(\omega)$ is a parameter and φ is a Π_1 formula so that $\varphi(Q)$ holds, then there exists an admissible set M so that

- $M \models$ "$\mathcal{P}(\omega)$ exists",
- $\bar{Q} = Q \cap M \in M$, and
- $M \models \varphi(\bar{Q})$.

We show that much in the way $\mathbf{\Pi}^1_2$-MI was "just enough" to define the winning strategies in Σ^0_3 games, Π_1-RAP is strong enough to carry out a version of this proof for the $\mathbf{\Sigma}^0_3$ games defined on the unraveled tree. However, Π_1-RAP does not imply stabilization of the monotone operators used: Rather, it guarantees that failure of a position to enter the least fixed point of a monotone operator is *reflected to an admissible set.* This reflection is just enough to construct the winning strategies for Player I; whereas for those games won by Player II, we obtain Δ_1-definable winning strategies. We have

Theorem 1.2.5. *Work over* KPI_0. *Then* Σ^0_4-DET *is equivalent to the existence of a wellfounded model of* Π_1-*RAP; moreover, if* $\mathcal{M}$ *is the smallest such model, then for every* Σ^0_4 *game, either Player I has a winning strategy in* $\mathcal{M}$, *or there is a strategy for Player II* Δ_1-*definable over* $\mathcal{M}$.

Here KPI_0 is the theory asserting "every set is contained in some transitive model of KP;" we remark that this theory has the same strength as Π^1_1-CA_0, and so we may regard this theorem as a statement of second order arithmetic proved over the latter base theory.

In the case of the Martin/Friedman results, the analysis at Σ^0_4 generalizes by a straightforward induction on both sides. Proceeding to higher pointclasses, we define the analogues Π_1-RAP_α of the Π_1-reflection to admissibles principle. Emulating the inductive Martin/Friedman arguments, we again obtain equivalences:

Theorem 1.2.6. *Work over* KPI_0. *Let* $\alpha < \omega_1^{\mathsf{CK}}$. *Then* $\Sigma^0_{1+\alpha+3}$-DET *is equivalent to the existence of a wellfounded model of* Π_1-*RAP*$_\alpha$, *and if* $\mathcal{M}$ *is the smallest such model, then for* *every $\Sigma^0_{1+\alpha+3}$ game, either Player I has a winning strategy in $\mathcal{M}$, or there is a strategy for Player II Δ_1-definable over $\mathcal{M}$.*

These results relativize to give versions for pointclasses $\Sigma^0_{1+\alpha+3}(z)$, for all reals z.

We take special note of the fact that Π_1-RAP is equivalent in certain contexts to the assertion: "$\mathcal{P}(\omega)$ exists, and all wellfounded trees on $\mathcal{P}(\omega)$ are ranked." This assertion bears a resemblance to the axiom $\mathrm{Ax}\,\beta$, a key axiom in the study of β-models of ATR_0 (see [Sim09]). In Chapter 4, we explore this connection, formulating the notion of β-model in the context of third-order arithmetic, and showing that the least level of L satisfying Π_1-RAP is, in the appropriate sense, the minimal β-model of an axiom we call *projective transfinite recursion*, Π^1_∞-$\mathsf{TR}_\mathbb{R}$. We are particularly interested in a result of Schweber [Sch13] proving that for games with real-number moves, clopen determinacy ($\Delta^\mathbb{R}_1$-DET) does not imply open determinacy ($\Sigma^\mathbb{R}_1$-DET) over the weak base theory RCA^3_0. Our analysis gives a new proof of this result using levels of L, and the following:

Theorem 1.2.7. *Work over $\mathbf{\Pi}^1_1$-CA_0. Then each of the following implies the next:*

1. *There exists a countably-coded (third order) β-model of $\Sigma^\mathbb{R}_1$-DET.*
2. *There exists a countably-coded (second order) β-model of $\mathbf{\Sigma}^0_4$-DET.*
3. Σ^0_4-DET.
4. *There exists a countably-coded (third order) β-model of $\Delta^\mathbb{R}_1$-DET.*

Moreover, the implications $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ *are not reversible; the items* $(3), (4)$ *are equivalent.*

1.3 Beyond Borel determinacy

As we mentioned above, the ZFC proof of Borel determinacy was predated by Martin's proof of $\mathbf{\Pi}^1_1$-DET from a measurable cardinal. Martin showed that even $0^\#$ suffices, and

Harrington [Har78] proved the converse implication: Thus, Π^1_1-DET is equivalent to the existence of $0^\#$. Any proof of determinacy for Wadge degrees beyond $\boldsymbol{\Delta}^1_1$ therefore requires large cardinals beyond those consistent with L.

The goal was then to extend proofs of determinacy as far as they could go from any large cardinal hypotheses available. A significant leap of ideas would turn out to be essential for even reaching as high as $\boldsymbol{\Delta}^1_2$-DET (the first proof was again Martin's [Mar80], from a rank-to-rank embedding; isolation of Woodin cardinals as the optimal hypotheses [MS88], [MS89] would require far-reaching developments in inner model theory [MS94]). In the meantime, an assortment of results were proved connecting large cardinals with determinacy for pointclasses reaching up to $\boldsymbol{\Delta}^1_2$.

A natural hierarchy to examine is $\langle \boldsymbol{\Sigma}^0_\alpha(\boldsymbol{\Pi}^1_1)\rangle_{\alpha<\omega_1}$, stratifying the sets in the Borel σ-algebra generated by $\boldsymbol{\Pi}^1_1$. In his Ph.D. thesis, Simms [Sim79] established a near-equivalence: $\Sigma^0_1(\Pi^1_1)$-DET implies the existence of an inner model with a proper class of measurable cardinals, and is implied by the existence of a sharp for such a model. Following Mitchell's introduction [Mit74] of the necessary inner model-theoretic technology, Steel [Ste82] proved lower bounds on the strength of $\boldsymbol{\Sigma}^0_\alpha(\boldsymbol{\Pi}^1_1)$-DET for $\alpha \geq 2$. The weakest hypothesis from which this determinacy had been proved was Martin and Steel's proof (unpublished) of $\mathcal{A}\boldsymbol{\Pi}^1_1$-DET from the existence of an elementary embedding $j : V \to M$ with $V_{\kappa+2} \subseteq M$; but this falls short of matching up with the lower bounds in [Ste82].

Two decades later, Neeman [Nee00] proved from a hypothesis equiconsistent with a measurable κ with Mitchell order $o(\kappa) = \kappa^{++}$ that Π^1_1 sets can be unravelled. This allowed level-by-level proofs of determinacy in $\langle \boldsymbol{\Sigma}^0_\alpha(\boldsymbol{\Pi}^1_1)\rangle_{\alpha<\omega_1}$, from the assumption of iterable mice with sufficiently many iterated Power Set operations above a measurable with maximal Mitchell order, complementing Steel's lower bounds in a fashion strikingly analogous to the Friedman/Martin results. These are summarized in the table below; there "…" indicates that the best known bounds are those in the appropriate next row.

Γ	Upper bound	Lower bound	Due to work in
Π^1_1	$0^\#$ exists	$0^\#$ exists	[Mar70], [Har78]
$\Sigma^0_1(\Pi^1_1)$	$((\forall\alpha)(\exists\kappa > \alpha)o(\kappa) = 1)^\#$	$(\forall\alpha)(\exists\kappa > \alpha)o(\kappa) = 1$	[Sim79]
$\Sigma^0_2(\Pi^1_1)$	...	$\mathsf{KP} + (\exists\kappa)o(\kappa) = \mathrm{ON} \wedge \exists\kappa^+$	[Ste82]
$\Sigma^0_3(\Pi^1_1)$	...	...	—
$\Sigma^0_4(\Pi^1_1)$	$\mathsf{ZFC}^- + (\exists\kappa)o(\kappa) = \kappa^{++}$	...	[Nee00]
$\Sigma^0_{\alpha+5}(\Pi^1_1)$	$o(\kappa) = \kappa^{++} \wedge \exists\kappa^{+(\alpha+3)}$	$o(\kappa) = \kappa^{++} \wedge \exists\kappa^{+(\alpha+2)}$	[Ste82], [Nee00]

The analogy with the Borel case turns out to be sufficiently close that we may adapt the results of Chapter 3 to this setting. Combining this work with the methods of [Ste82] and [Nee00, Nee06], we prove in Chapter 5

Theorem 1.3.1. *Assume $x^\#$ exists for all reals x. Then $\Sigma^0_{1+\alpha+3}(\Pi^1_1)$-DET is equivalent to the existence of a mouse $\mathcal{M}$ of the form $L_\alpha[\mathcal{U}]$ satisfying, for some cardinal κ of $\mathcal{M}$: $o(\kappa) = \kappa^{++}$, and Π_1-$\mathsf{RAP}_{\kappa+1+\alpha}$.*

CHAPTER 2

Σ^0_3 determinacy and $\mathbf{\Pi}^1_2$ monotone induction

In this chapter, we isolate the strength of Σ^0_3 determinacy in terms of a natural theory in second order arithmetic. Namely, we show that Σ^0_3-DET is equivalent over $\mathbf{\Pi}^1_1$-CA_0 to the existence of a countably-coded β-model of $\mathbf{\Pi}^1_2$ *monotone induction*.

There is a great deal of precedent for equivalences between determinacy in low levels of the Borel hierarchy and axioms of inductive definition. In one of the first studies in reverse mathematics, Steel [Ste77] proved over RCA_0 that ATR_0 is equivalent to both Δ^0_1-DET and Σ^0_1-DET. Tanaka [Tan90] showed over ACA_0 that Δ^0_2-DET is equivalent to Π^1_1-TR, and in [Tan91] that over ATR_0, Σ^0_2-DET is equivalent to Σ^1_1-MI. MedSalem and Tanaka [MT08] established equivalences over ATR_0 between $k{-}\Pi^0_2$-DET and $[\Sigma^1_1]^k$-ID, an axiom allowing inductive definitions using combinations of k-many Σ^1_1 operators; furthermore, they showed over Π^1_3-TI that Δ^0_3-DET is equivalent to $[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}$, an axiom allowing inductive definition by combinations of transfinitely many Σ^1_1 operators. Further results were given by Tanaka and Yoshii [YT12] characterizing the strength of determinacy for pointclasses refining the difference hierarchy on Π^0_2, again in terms of axioms of inductive definition.

Just beyond these pointclasses we have Σ^0_3, where an exact characterization of strength has been elusive. The sharpest published bounds on this strength were given by Welch [Wel11], who showed that although Σ^0_3-DET (and more) is provable in Π^1_3-CA_0, Δ^1_3-CA_0 (even augmented by AQI, an axiom allowing definition by arithmetical quasi-induction) cannot prove Σ^0_3-DET. On the other hand, Montalbán and Shore [MS12] showed that Σ^0_3-DET (and indeed, any true Σ^1_4 sentence) cannot prove Δ^1_2-CA_0. This situation is further clarified by the same authors in [MS13], where they show (among other things) that Σ^0_3-DET implies the existence of a β-model of Δ^1_3-CA_0.

Welch [Wel12] went on to give a characterization of the ordinal stage at which winning strategies in Σ^0_3 games are constructed in L. There, the least ordinal γ so that every Σ^0_3 game is determined with a winning strategy definable over L_γ is shown to be the least γ for which there exists an illfounded admissible model $\mathcal{M}$ with an infinite descending sequence of nonstandard levels of L that fully Σ_2-reflect to standard levels below γ, and so that $\mathrm{wfo}(\mathcal{M}) = \gamma$ (see Definition 2.2.1).

The question of whether Σ^0_3-DET could be connected to a natural subtheory of second order arithmetic remained open. In light of the work of Welch and Montalbán-Shore, it appeared plausible that Σ^0_3-DET could be shown to be equivalent to the existence of a β-model of some natural theory. We felt that the ordinal γ appearing as the wellfounded ordinal of Welch's nonstandard structure should be characterizable as the least so that L_γ satisfies some theory of monotone induction. This is what we show in this chapter: L_γ is the minimal model closed under $\mathbf{\Pi}^1_2$ monotone inductive definitions, and indeed, Σ^0_3-DET is equivalent over $\mathbf{\Pi}^1_1$-CA_0 to the existence of such a model.

All known proofs of Σ^0_3 determinacy trace back to Morton Davis's [Dav64] (for relevant definitions, see section 2). Let $A \subseteq \omega^\omega$ be a Σ^0_3 set, so that $A = \bigcup_{k\in\omega} B_k$ for some recursively presented sequence $\langle B_k \rangle_{k\in\omega}$ of Π^0_2 sets. The idea behind the proof that the game $G(A)$ is determined is a simple one: if I does not have a winning strategy, then player II refines to a quasistrategy W_0 so that no infinite plays in W_0 belong to B_0, and so that W_0 doesn't forfeit the game (in the sense that I has no winning strategy in $G(A; W_0)$). Having done this, player II plays inside W_0 and at all positions of length 1, refines further to a W_1 which avoids B_1 without forfeiting the game. Then refine to W_2 at positions of length 2, and so on. The ultimate refinement of the sequence $W_0, W_1, W_2 \ldots$ of quasistrategies gives a winning quasistrategy for player II in $G(A)$, since every infinite play must eventually be contained in each W_n.

The key claim that makes this proof work is Lemma 2.1.2, which asserts that whenever I does not have a winning strategy in $G(A; T)$, then for all k, there is such a quasistrategy W_k.

Welch's characterization amounts to an analysis of the way in which these W_k first appear in L. Namely, if $T \in L_\gamma$ is such that I doesn't have a winning strategy for $G(A,T)$ in L_γ, then the assumed reflection of the ordinals of $\mathcal{M}$ ensures there is a quasistrategy $W \in L_\gamma$ as in the conclusion of the lemma. Furthermore, Welch defines a game that is won by the Π^0_3 player, but for which there can be no winning strategy in L_γ. In this situation it is necessarily the case that the quasistrategies W_k from which II's winning strategy is built are constructed cofinally in L_γ, and the common refinement is only definable over the model L_γ.

Welch's proof of determinacy is difficult, and the quasistrategies of interest are obtained in something of a nonstandard way. Our present aim is to give a more constructive account of the way in which the quasistrategies W_k arise. There is a relatively straightforward way in which the quasistrategy W can obtained by iteration of a certain monotone operator. The complexity of this operator is $(\Game\boldsymbol{\Sigma}^0_3)^\vee$ in the parameter T (here $\Game$ is the game quantifier as defined in [Mos09] 6D; $\Gamma^\vee$ denotes the dual pointclass of Γ). It seemed natural to conjecture, then, that the ordinal γ is in some sense a closure ordinal for these monotone inductive definitions (indeed, Welch makes the conjecture in [Wel12] that γ is $o(\Game\Pi^0_3)$, the closure ordinal of *non*-monotone $\Game\Pi^0_3$ inductive definitions).

In what follows, we denote subsets of ω by capital Roman letters X, Y, Z, elements of ω by lowercase Roman letters from i up to n, ordinals by lowercase Greek α, β, γ... and reals (elements of $\mathbb{R} = \omega^\omega$) by w, x, y, z.

Definition 2.0.2. Let Γ be a pointclass. Γ-MI is the axiom scheme asserting, for each $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which is a Γ operator, i.e.

$$\{\langle n, X\rangle \mid n \in \Phi(X)\} \in \Gamma,$$

that is monotone, i.e.

$$(\forall X, Y) X \subseteq Y \to \Phi(X) \subseteq \Phi(Y),$$

that there exists an ordinal $o(\Phi)$ and sequence $\langle \Phi^\xi\rangle_{\xi \leq o(\Phi)}$ such that, setting $\Phi^{<\xi} = \bigcup_{\zeta<\xi} \Phi^\zeta$, we have

- for all $\xi \leq o(\Phi)$, $\Phi^\xi = \Phi(\Phi^{<\xi}) \cup \Phi^{<\xi}$,

- $\Phi^{o(\Phi)} = \Phi^{<o(\Phi)}$, and

- $o(\Phi)$ is the least ordinal with this property.

$\Phi^{o(\Phi)}$ is the *least fixed point of* Φ, denoted Φ^∞.

There is a prewellorder $\prec_\Phi$ with field $\Phi^\infty \subseteq \omega$ naturally associated with the sequence $\langle \Phi^\xi \rangle_{\xi \leq o(\Phi)}$. Namely, set $m \prec_\Phi n$ if and only if the least ξ with $m \in \Phi^\xi$ is less than the least ζ with $n \in \Phi^\zeta$.

We are interested in the case that Γ is one of $\boldsymbol{\Pi}^1_2$, $\Pi^1_2(z)$ for a real z. We regard $\Pi^1_2(z)$-MI as being formalized in the language Z_2 of second order arithmetic: It is the schema asserting the existence of the prewellorder $\prec_\Phi$, for each $\Pi^1_2(z)$ monotone operator Φ. Note for such Φ, the relation "$X = \prec_\Phi$", as a relation holding of $X \in \mathcal{P}(\omega \times \omega)$, is arithmetical in $\Sigma^1_2(z)$.

It will follow from our analysis that γ is least so that $L_\gamma \models (\Game \boldsymbol{\Sigma}^0_3)^\vee$-$\mathsf{MI}$. However, we argue to more directly show something stronger: that $L_\gamma \models \boldsymbol{\Pi}^1_2$-$\mathsf{MI}$. (We remark that, as Σ^0_3-DET fails in L_γ, $(\Game \boldsymbol{\Sigma}^0_3)^\vee$ and $\Game \Pi^0_3$ do not coincide there.)

In section 2.1, we show that winning strategies in Σ^0_3 games are definable over any β-model of $\boldsymbol{\Pi}^1_2$-MI. In section 2.2, we prove that Welch's infinite depth Σ_2-nestings furnish us with such β-models. We complete this circle of implications in section 2.3 by reproducing Welch's lower bound argument in the base theory $\boldsymbol{\Pi}^1_1$-CA_0 to show that Σ^0_3 determinacy implies the existence of an infinite depth Σ_2-nesting. We conclude with an analysis of the $\boldsymbol{\Pi}^1_2$ relations which are correctly computed in L_γ.

2.1 Proving determinacy

Let T be a (non-empty) tree with no terminal nodes; $[T]$ denotes the set of infinite branches of T, and for $p \in T$, T_p denotes the subtree of T with stem p, that is, $T_p = \{q \in T \mid q \subseteq p \vee p \subseteq q\}$. For a set $A \subseteq [T]$, the *game on* T *with payoff* A, denoted $G(A; T)$, is defined as the infinite perfect information game in which two players, I and II, alternate choosing successive nodes of a branch x of T. Player I wins if $x \in A$; otherwise, Player II wins. We write $G(A)$ for $G(A; \omega^{<\omega})$.

A *strategy for I* in a game on T is a partial function $\sigma : T \rightharpoonup X$ that assigns to an even-length position $s \in T$ a legal move x for I at s, that is, $x \in X$ so that $s^\frown\langle x\rangle \in T$. We require the domain of σ to be closed under legal moves by II as well as moves by σ; note then that due to the presence of terminal nodes in the tree, I needn't have a strategy at all. Strategies for II are defined analogously. We say a strategy σ is *winning* for I (II) in $G(A;T)$ if every play according to σ belongs to A ($[T] \setminus A$). A game $G(A;T)$ is *determined* if one of the players has a winning strategy. For a pointclass Γ, Γ-DET denotes the statement that $G(A; \omega^{<\omega})$ is determined for all $A \subseteq \omega^\omega$ in Γ.

We furthermore define a *quasistrategy for Player II in T* to be a subtree $W \subseteq T$, again with no terminal nodes, that does not restrict Player I's moves, in the sense that whenever $p \in W$ has even length, then every 1-step extension $p^\frown\langle s\rangle \in T$ belongs to W. A quasistrategy may then be thought of as a multi-valued strategy. (Similar definitions of course can be made for Player I, but at no point will we need to refer to quasistrategies for Player I.)

Quasistrategies are typically obtained in the following fashion: if Player I does not have a winning strategy in $G(A;T)$, then setting W to be the collection of $p \in T$ so that I doesn't have a winning strategy in $G(A;T_p)$, we have that W is a quasistrategy for II in T. This W is called *II's nonlosing quasistrategy in $G(A;T)$*.

Theorem 2.1.1. *Let $\mathcal{M}$ be a β-model of $\mathbf{\Pi}^1_2$-MI. Then for any $\Sigma^0_3(z)$ set A, where $z \in \mathcal{M}$, either*

1. *Player I wins $G(A)$ with a strategy $\sigma \in \mathcal{M}$; or*

2. *Player II wins $G(A)$ with a strategy $\Delta^1_3(z)$-definable over $\mathcal{M}$.*

Lemma 2.1.2. *Let z be a real and work in $\Pi^1_2(z)$-$\mathsf{MI} + \mathbf{\Pi}^1_2$-$\mathsf{CA}_0$. Suppose $T \subseteq \omega^{<\omega}$ is a tree, recursive in z, with no terminal nodes, and let $B \subseteq A \subseteq [T]$ with $B \in \Pi^0_2(z)$ and $A \in \Delta^1_1(z)$. If $p \in T$ is such that I does not have a winning strategy in $G(A;T_p)$, then there is a quasistrategy W for II in T_p so that*

- $[W] \cap B = \emptyset$, *and*
- *I does not have a winning strategy in* $G(A; W)$.

In keeping with terminology first established in [Dav64], we say a position p for which such a quasistrategy W exists is *good* and that W is a *goodness-witnessing quasistrategy* for p (relative to T, B, A).

We remark that $\mathbf{\Pi}^1_2$-CA_0 implies $\mathbf{\Delta}^1_2$-CA_0, which is equivalent to $\mathbf{\Sigma}^1_2$-AC_0 (see VII.6.9 in [Sim09]); this choice principle will be used several times in the course of the proof.

Proof of Lemma 2.1.2. Fix a set $U \subseteq \omega \times T$ recursive in z so that, setting $U_n = \{p \in T \mid (n, p) \in U\}$ and $D_n = \{x \in [T] \mid (\exists k) x \restriction k \in U_n\}$, we have $B = \bigcap_{n\in\omega} D_n$. For convenience, we may further assume that each U_n is closed under end-extension in T, i.e., if $p \subseteq q \in T$ and $p \in U_n$, then $q \in U_n$; and that $|p| > n$ whenever $p \in U_n$.

We define an operator $\Phi : \mathcal{P}(T) \to \mathcal{P}(T)$ by setting, for $X \subseteq T$,

$$\begin{aligned} p \in \Phi(X) \iff (\exists n)(\forall \sigma) &\text{ if } \sigma \text{ is a strategy for I in } T\text{, then} \\ &(\exists x) x \text{ is compatible with } \sigma, x \notin A, \text{ and} (\forall k) x \restriction k \notin U_n \setminus X. \end{aligned}$$

The operator Φ is clearly monotone on $\mathcal{P}(T)$, and the relation $p \in \Phi(X)$ is $\Pi^1_2(z)$ because this last pointclass is closed under existential quantification over ω (by $\mathbf{\Sigma}^1_2$-AC_0). We can write this more compactly by introducing an auxiliary game where Player I tries either to get into A, or to at some finite stage enter the set U_n while avoiding X. Define for $X \subseteq T$ and $n \in \omega$,

$$E^X_n = A \cup \{x \in [T] \mid (\exists k) x \restriction k \in U_n \setminus X\}.$$

Then

$$p \in \Phi(X) \iff (\exists n) \text{ I doesn't have a winning strategy in } G(E^X_n; T_p).$$

Now by $\mathbf{\Pi}^1_2$-MI let $\langle \Phi^\alpha \rangle_{\alpha \leq o(\Phi)}$ be the iteration of the operator Φ with least fixed point Φ^∞. Let $\prec_\Phi$ be the associated prewellorder of $\Phi^\infty \subseteq \omega$; formally, we regard definitions and proofs in terms of $\langle \Phi^\alpha \rangle_{\alpha \leq o(\Phi)}$ as being carried out in Z_2 with $\prec_\Phi$ as a parameter.

Claim 2.1.3. *If $p \in T \setminus \Phi^\infty$, then I has a winning strategy in $G(A; T_p)$.*

Proof. For each $q \in T \setminus \Phi^\infty$ and $n \in \omega$, we let $\sigma_{q,n}$ be a winning strategy for I in $G(E_n^{\Phi^\infty}; T_q)$, as is guaranteed to exist by the fact that $q \notin \Phi(\Phi^\infty) = \Phi^\infty$. By $\mathbf{\Sigma}^1_2$-AC_0, we may fix a real $\vec{\sigma}$ coding a sequence of such, so that $(\vec{\sigma})_{\langle q,n\rangle} = \sigma_{q,n}$ for all such pairs q, n.

Supposing now that $p \in T \setminus \Phi^\infty$, we describe a strategy σ for Player I in T_p from the parameter $\vec{\sigma}$ as follows. Set $p_0 = p$. Let n_0 be the least n so that $p_0 \notin U_n$ (such exists by our simplifying assumption that $|q| > n$ whenever $q \in U_n$). Suppose inductively that we have reached some position $p_i \notin \Phi^\infty$ and have fixed n_i such that $p_i \notin U_{n_i}$. Play according to σ_{p_i,n_i} until, if ever, we reach a position $q \in U_{n_i} \setminus \Phi^\infty$. Then set $p_{i+1} = q$, and let n_{i+1} be least such that $p_{i+1} \notin U_{n_{i+1}}$.

Note the strategy just described is arithmetical in the parameters $z, \vec{\sigma}$, and so exists; call it σ. We claim σ is winning for I in $G(A; T_p)$.

Let $x \in [T_p]$ be a play compatible with σ. Then n_0, p_0 are defined. If n_{i+1} is undefined for some i, then fixing the least such i, we must have that no initial segment of x belongs to $U_{n_i} \setminus \Phi^\infty$. So x is compatible with the strategy σ_{p_i,n_i}; since this strategy is winning in $G(E_{n_i}^{\Phi^\infty}; T_{p_i})$, we must have that $x \in E_{n_i}^{\Phi^\infty}$. But then $x \in A$, by definition of the set $E_{n_i}^{\Phi^\infty}$.

On the other hand, if n_i is defined for all i, then by definition of the strategy σ, we have $p_i \subseteq x$ for all i, and for each i, $p_i \in \bigcap_{n<n_i} U_n$ (here we use that the sets U_n are closed under end-extension in T). So $x \in \bigcap_{n\in\omega} D_n = B \subseteq A$.

We have shown σ is winning for Player I in $G(A; T_p)$. □

Claim 2.1.4. *If $p \in \Phi^\infty$, then p is good.*

Proof. The construction of a quasistrategy W^p witnessing goodness of p proceeds inductively on the ordinal rank of $p \in \Phi^\infty$, that is, on the least α so that $p \in \Phi^\alpha$. Namely, given such p, α, there is some n so that I does not have a winning strategy in the game $G(E_n^{\Phi^{<\alpha}}; T_p)$. In W^p, have II play according to II's non-losing quasistrategy in $G(E_n^{\Phi^{<\alpha}}; T_p)$ until, if ever, a position q in U_n is reached. Then since I does not have a strategy winning to reach $U_n \setminus \Phi^{<\alpha}$

in T_q, we must have $q \in \Phi^{<\alpha}$; inductively, we have some goodness-witnessing quasistrategy W^q for q, so have II switch to play according to this strategy.

Here is a more formal definition of the quasistrategy W^p. For $p \in \Phi^\infty$, define W^p to be the set of positions $q \in T_p$ for which there exists some sequence $\langle(\alpha_i, n_i)\rangle_{|p|\le i\le|q|}$ so that, whenever $|p| \le i \le |q|$,

- if $i = |p|$, or $i > |p|$ and $q \restriction i \in U_{n_{i-1}}$, then
  - α_i is the least α so that $q \restriction i \in \Phi^\alpha$;
  - n_i is the least n so that I has no winning strategy in $G(E_n^{\Phi^{<\alpha_i}}; T_{q\restriction i})$;
- if $i > |p|$ and $q \restriction i \notin U_{n_{i-1}}$, then $\alpha_i = \alpha_{i-1}, n_i = n_{i-1}$; and
- if $i < |q|$, then $q \restriction (i+1)$ is in II's non-losing quasistrategy in $G(E_{n_i}^{\Phi^{<\alpha_i}}; T_{q\restriction i})$.

Note that formally, we should regard quantification of ordinals $\alpha < o(\Phi)$ as ranging over natural number codes for such as furnished by the prewellorder $\prec_\Phi$. The most complicated clauses in the above definition are those involving assertions of the form "I has no winning strategy in the game $G(E_n^{\Phi^{<\alpha_i}}; T_{q\restriction i})$", and such are Π^1_2 in the parameter $\prec_\Phi$. So the criterion for membership in W^p is arithmetical in $\Sigma^1_2(\prec_\Phi)$ conditions, and therefore by $\mathbf{\Pi}^1_2$-CA_0 the quasistrategy W^p is guaranteed to exist.

An easy induction shows that for each $q \in W^p$, there is a unique witnessing sequence $\langle(\alpha_i, n_i)\rangle_{|p|\le i\le|q|}$ and this sequence depends continuously on q; that the α_i are non-increasing; and that I has no winning strategy in $G(E_{n_i}^{\Phi^{<\alpha_i}}; T_{q\restriction i})$ whenever $|p| \le i \le |q|$.

For $q \in W^p$, we let α^q, n^q denote the final pair (indexed by $|q|$) in the sequence witnessing this membership. By the above remarks, I has no winning strategy in $G(E_{n^q}^{\Phi^{<\alpha^q}}; T_q)$, and by the final condition for membership in W^p, the one-step extensions $q^\frown\langle l\rangle$ in W^p are exactly the one-step extensions of q in II's non-losing quasistrategy in this game. It follows that W^p is a quasistrategy for II in T_p.

We claim W^p witnesses goodness of p. We first show $[W^p] \cap B = \emptyset$. Given any play

$x \in [W^p]$, we have some least i so that $\alpha_j = \alpha_i$ for all $j \geq i$; then for all $j > i$, we have $x \upharpoonright j$ belongs to II's non-losing quasistrategy in $G(E_{n_i}^{\Phi^{<\alpha_i}}; T_{x\upharpoonright i})$. In particular, for no k do we have $x \upharpoonright k \in U_{n_i}$. Then $x \notin D_{n_i}$, so $x \notin B$ as needed.

We just need to show I has no winning strategy in $G(A; W_q^p)$, for each $q \supseteq p$ in W^p. We argue by induction on α^q. So assume that there is no winning strategy for I in $G(A; W_r^p)$ whenever $\alpha^r < \alpha^q$.

Suppose towards a contradiction that σ is a winning strategy for I in $G(A; W_q^p)$. Let j be least so that $\alpha^q = \alpha_j$. Then q is in II's non-losing quasistrategy in $G(E_{n_j}^{\Phi^{<\alpha_j}}; T_{q\upharpoonright j})$. We claim no $r \supseteq q$ compatible with σ is in U_{n_j}. For otherwise, we have $r \in \Phi^{<\alpha_j}$, so that $\alpha^r < \alpha_j = \alpha^q$, and σ is a winning strategy for I in $G(A; W_r^p)$. This contradicts our inductive hypothesis.

So σ cannot reach any position in U_{n_j}. By our definition of W^p, we have that the strategy σ stays inside II's non-losing quasistrategy for $G(E_{n_j}^{\Phi^{<\alpha_j}}; T_{q\upharpoonright j})$. But since σ is winning for I in $G(A; T_{q\upharpoonright j})$ and $A \subseteq E_{n_j}^{\Phi^{<\alpha_j}}$, this is a contradiction.

We conclude that I has no winning strategy in $G(A; W_q^p)$; inductively, the claim follows for all $q \in W^p$ extending p, so that in particular, W^p witnesses goodness of p. $\square$

For future reference, let us refer to the W^p defined in this claim as the *canonical goodness-witnessing strategy for p* (relative to T, B, A). We have the following remark, which will be important for computation of the complexity of winning strategies:

Remark 2.1.5. Since "$\prec_\Phi$ witnesses the instance of $\Pi^1_2(z)$-MI at Φ" is $\Delta^1_3(z)$, the statement "W is the canonical goodness-witnessing strategy for p relative to T, B, A" is likewise $\Delta^1_3(z)$ as a relation on pairs $\langle W, p\rangle$.

The last two claims show that every $p \in T$ is either a winning position for I in $G(A; T)$, or is good. This proves the lemma. $\square$

Proof of Theorem 2.1.1. The proof proceeds from Lemma 2.1.2 as usual (see [Dav64], [Mar]); we give a detailed account here, in order to isolate the claimed definability of II's winning strategy.

Fix a β-model $\mathcal{M}$ of $\mathbf{\Pi}^1_2$-MI. Suppose A is $\Sigma^0_3(z)$ for some $z \in \mathcal{M}$; say $A = \bigcup_{k\in\omega} B_k$. By the previous lemma, whenever $T \in \mathcal{M}$ is a tree in $\mathcal{M}$, and $p \in T$ is a position so that in $\mathcal{M}$, there is no winning strategy for I in $G(A; T_p)$, then p is good relative to T, B_k, A, for all k; that is, for each k there is W_k a quasistrategy for II in T_p so that

- $[W_k] \cap B = \emptyset$;
- I does not have a winning strategy in $G(A; W_k)$.

The idea of the proof is to repeatedly apply the lemma inside $\mathcal{M}$. At positions p of length k, II refines her present working quasistrategy W_{k-1} to one W_k witnessing goodness of p relative to W_{k-1}, B_k, A, so "dodging" each of the $\Pi^0_2(z)$ sets B_k, one at a time.

More precisely: suppose I does not win $G(A)$ in $\mathcal{M}$, where A is $\Sigma^0_3(z)$ for some $z \in \mathcal{M}$. Let $W^\emptyset$ be the canonical goodness-witnessing quasistrategy for $\emptyset$ relative to $\omega^{<\omega}, B_0, A$ as constructed in the proof of Lemma 2.1.2. Then let $H^\emptyset$ be II's non-losing quasistrategy in $G(A; W^\emptyset)$ (so that for no $p \in H^\emptyset$ do we have that I wins $G(A; W^\emptyset_p)$).

Suppose inductively that for some k, we have subtrees H^p of T, defined for a subset of $p \in T$ with length $\leq k$, so that

1. each H^p is a quasistrategy for II in T_p and belongs to $\mathcal{M}$;
2. $[H^p] \cap B_{|p|} = \emptyset$;
3. for no $q \in H^p$ does I have a winning strategy in $G(A; H^p_q)$;
4. if $p \subseteq q$, then $H^q \subseteq H^p$ whenever both are defined;
5. if $|p| < k$ and $p^\frown\langle l\rangle \in H^p$, then $H^{p^\frown\langle l\rangle}$ is defined.

In order to continue the construction, we need to define quasistrategies $H^{p^\frown\langle l\rangle}$, whenever $|p| = k$, H^p is defined, and $p^\frown\langle l\rangle \in H^p$. Given such p and l, we have that I has no winning strategy in $G(A; H^p_{p^\frown\langle l\rangle})$ by (3). So applying Lemma 2.1.2 inside $\mathcal{M}$, let $W^{p^\frown\langle l\rangle}$ be the canonical goodness-witnessing strategy for $p^\frown\langle l\rangle$ relative to H^p, B_{k+1}, A. Then let $H^{p^\frown\langle l\rangle}$

be II's non-losing quasistrategy in $G(A; W^{p^\frown\langle l\rangle})$. It is easy to see that this quasistrategy satisfies the properties (1)-(4), so we have the desired system of quasistrategies H^q satisfying (5), for $|q| = k + 1$.

Now set $p \in H$ if and only if for all $i < |p|$, $H^{p\restriction i}$ is defined and $p \in H^{p\restriction i}$. It follows from (5) that H is a quasistrategy for II, and by (4) we have $H \subseteq H^p$ for each $p \in H$. By (2) then, $[H] \cap B_k = \emptyset$ for all $k \in \omega$, so that $[H] \cap A = \emptyset$.

Observe that for each $p \in H$, we have that the sequence $\langle H^{p\restriction i}\rangle_{i<|p|}$ exists in $\mathcal{M}$, since it is obtained by a finite number of applications of $\mathbf{\Pi}^1_2$-MI and $\mathbf{\Pi}^1_2$-CA_0. Since $\mathcal{M}$ is a β-model, it really is the case (in V) that $[H] \cap B_k = \emptyset$ for all $k \in \omega$. Though H need not belong to $\mathcal{M}$, we claim it is nonetheless a $\Delta^1_3(z)$-definable class over $\mathcal{M}$. For $p \in H$ if and only if there exists a sequence $\langle W_i, H_i\rangle_{i<|p|}$, so that for all $i < |p|$,

- W_i is II's canonical goodness-witnessing strategy for $p \restriction i$, relative to H_{i-1}, B_i, A (where we set $H_{-1} = \omega^{<\omega}$);
- H_i is II's non-losing quasistrategy in $G(A; W_i)$ at $p \restriction i$;
- for all $i < |p|$, $p \in H_i$.

This is a $\Sigma^1_3(z)$ condition, by Remark 2.1.5. And note that $p \notin H$ if and only if there is a sequence $\langle H_i, W_i\rangle_{i\leq l}$, for some $l < |p|$, satisfying the first two conditions for $i \leq l$, but so that $p \notin H_l$. This is likewise $\Sigma^1_3(z)$, so that H is $\Delta^1_3(z)$-definable in $\mathcal{M}$.

Given a $\Delta^1_3(z)$ definition of the quasistrategy H, it is easy to see that the strategy τ for II obtained by taking $\tau(p)$ to be the least l so that $p^\frown\langle l\rangle \in H$ is likewise $\Delta^1_3(z)$ and winning for II in $G(A; T)$. This completes the proof of Theorem 2.1.1. $\square$

2.2 $\mathbf{\Pi}^1_2$ monotone induction from infinite depth Σ_2-nestings

In this section, the theories of KP and Σ_1-Comprehension are defined in the language of set theory as usual. We will furthermore make use of the theories KPI_0, which asserts that every set is contained in some admissible set (that is, some transitive model of KP), and KPI, which is the union of KP and KPI_0. KPI_0 is relevant largely because it is a weak theory in which Shoenfield absoluteness holds; in particular, $\mathbf{\Pi^1_2}$ expressions are equivalent over KPI_0 to Π_1 statements in the language of set theory.

We remark that KPI_0 and $\mathbf{\Pi}^1_1$-CA_0 prove the same statements of second order arithmetic. Since we primarily work with models in the language of set theory in this section, we take KPI_0 as our base theory, but all of the results proved here can be appropriately reformulated as statements about countably coded β-models in second order arithmetic (as in Chapter VII of [Sim09]).

For $\mathcal{M}$ an illfounded model in the language of set theory, we identify the wellfounded part of $\mathcal{M}$ with its transitive collapse, denote this $\mathrm{wfp}(\mathcal{M})$, and set $\mathrm{wfo}(\mathcal{M}) = \mathrm{wfp}(\mathcal{M}) \cap \mathrm{ON}$. Recall we say $\mathcal{M}$ is an ω-model if $\omega < \mathrm{wfo}(\mathcal{M})$. The following definition is due to Welch [Wel12].

Definition 2.2.1. For $\mathcal{M}$ an illfounded ω-model of KP in the language of set theory, an *infinite depth Σ_2-nesting based on $\mathcal{M}$* is a sequence $\langle \zeta_n, s_n \rangle_{n\in\omega}$ of pairs so that for all $n \in \omega$,

1. $\zeta_n \leq \zeta_{n+1} < \mathrm{wfo}(\mathcal{M})$,
2. $s_n \in \mathrm{ON}^{\mathcal{M}} \setminus \mathrm{wfo}(\mathcal{M})$,
3. $\mathcal{M} \models s_{n+1} < s_n$,
4. $(L_{\zeta_n} \prec_{\Sigma_2} L_{s_n})^{\mathcal{M}}$.

Lemma 2.2.2. *Suppose $\gamma_1 \leq \gamma_2 < \delta_2 < \delta_1$ are ordinals so that*

1. *$L_{\gamma_1} \prec_{\Sigma_1} L_{\delta_1}$;*
2. *$L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$;*
3. *δ_1 is the least admissible ordinal above δ_2;*
4. *For all $\alpha \leq \delta_2$, L_α Σ_ω-projects to ω.*

Then L_{γ_2} *satisfies* $\Pi^1_2(z)$-MI*, for all reals* $z \in L_{\gamma_1}$*.*

Item (4) simply asserts that for every α, there is a subset of ω definable over L_α that doesn't belong to L_α; this simplifying assumption ensures that every L_α is countable, as witnessed by a surjection $f : \omega \to L_\alpha$ that belongs to $L_{\alpha+1}$. Note the least level of L that does *not* Σ_ω-project to ω is a model of ZF^-; since this is far beyond the strength of the theories considered here, we don't lose anything by assuming (4).

Proof. Let $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be a $\Pi^1_2(z)$ monotone operator in L_{γ_2}; fix a $\Pi^0_1(z)$ condition T so that

$$n \in \Phi(X) \iff L_{\gamma_2} \models (\forall x)(\exists y)T(n, X, x, y, z)$$

whenever $n \in \omega$ and $X \in \mathcal{P}(\omega) \cap L_{\gamma_2}$. Notice that for such X

$$n \in \Phi(X) \iff (\forall x \in L_{\gamma_2})(\exists y)T(n, X, x, y, z),$$

by absoluteness and because γ_2 is a limit of admissibles. Regarding the operator defined in this way, "$n \in \Phi(X)$" makes sense even for sets $X \notin L_{\gamma_2}$ (though this extended Φ may fail to be monotone outside L_{γ_2}).

For each ordinal η, we define the approximation Φ_η as the operator Φ relativized to L_η,

$$n \in \Phi(X) \iff (\forall x \in L_\eta)(\exists y)T(n, X, x, y, z).$$

The point is that the operator Φ_η is then Σ^1_1 in any real parameter coding the countable set $\mathbb{R} \cap L_\eta$ (for example, $\mathrm{Th}(L_\eta)$, the characteristic function of the theory of L_η under some standard coding), and so each Φ_η will be correctly computed in, e.g., L_α for α a limit of admissibles above η.

Obviously $\Phi = \Phi_{\gamma_2}$, so is monotone in L_{γ_2}. But for $\eta \neq \gamma_2$ we may not even have that the operators Φ_η are monotone on $\mathcal{P}(\omega) \cap L_\eta$. So we instead work with the obvious "monotonizations",

$$\begin{aligned} n \in \Psi_\eta(X) &\iff (\exists X' \subseteq X) n \in \Phi_\eta(X') \\ &\iff (\exists X' \subseteq X)(\forall x \in L_\eta)(\exists y)T(n, X', x, y, z). \end{aligned}$$

These are again $\Sigma^1_1(\mathrm{Th}(L_\eta), z)$, and $\Psi_{\gamma_2}(X) = \Phi_{\gamma_2}(X) = \Phi(X)$ for $X \in L_{\gamma_2}$.

The most important properties of the sequences $\langle \Psi^\xi_\eta \rangle_{\xi \leq o(\Psi_\eta)}$ are captured in the following two claims.

Claim 2.2.3. *If $\eta < \eta'$, then $(\forall X)\Psi_\eta(X) \supseteq \Psi_{\eta'}(X)$.*

Proof. Suppose $n \in \Psi_{\eta'}(X)$; then

$$(\exists X' \subseteq X)(\forall x \in L_{\eta'})(\exists y)T(n, X', x, y, z),$$

and any such X' will likewise be a witness to $n \in \Psi_\eta(X)$, since the latter is defined the same way but with the universal quantifier bounded by the smaller set L_η. $\square$

Claim 2.2.4. *Suppose $\xi < \xi'$ and $\eta < \eta'$. Then*

1. $\Psi^\xi_\eta \subseteq \Psi^{\xi'}_\eta$;
2. $\Psi^\xi_\eta \supseteq \Psi^\xi_{\eta'}$.

Proof. (1) is by definition. (2) follows from induction and the chain of inclusions, for $X \supseteq Y$,

$$\Psi_{\eta'}(Y) \subseteq \Psi_{\eta'}(X) \subseteq \Psi_\eta(X),$$

the first by monotonicity of $\Psi_{\eta'}$, the second by the previous claim. $\square$

So the array $\langle \Psi^\xi_\eta \rangle$ is increasing in ξ and decreasing in η. Applying this claim with $\xi = \omega_1$, we have $\Psi^\xi_\eta = \Psi^\infty_\eta$, so that $\Psi^\infty_\eta \supseteq \Psi^\infty_{\eta'}$ whenever $\eta < \eta'$.

We now consider definability issues with respect to the operators Ψ_η and the associated sequences, with the aim of showing the levels of L under consideration are sufficiently closed to correctly compute these objects, and ultimately ensuring that the sequences $\langle \Psi^\xi_\eta \rangle_{\xi \leq o(\Psi_\eta)}$ converge to the sequence of interest $\langle \Psi^\xi_{\gamma_2} \rangle_{\xi \leq o(\Psi_{\gamma_2})}$ as $\eta \to \gamma_2$.

Note that the hypothesis $L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$ implies L_{γ_2} is a model of Σ_2-KP. The assumed elementarity in the lemma then implies each of $\gamma_1, \gamma_2, \delta_2$ is a limit of Σ_2-admissible ordinals.

Claim 2.2.5. *Suppose $z \in L_\alpha$ and $L_\alpha \models \mathsf{KPI}$. Then the relation "$n \in \Psi^\xi_\eta$" (as a relation on $\langle n, \xi, \eta\rangle \in \omega \times \alpha \times \alpha$) is $\Delta_1^{L_\alpha}$ in the parameter z. Consequently, for all $\eta < \alpha$ and $\nu < \alpha$, the sequence $\langle \Psi^\xi_\eta \rangle_{\xi<\nu}$ belongs to L_α.*

Proof. The relation $n \in \Psi_\eta(X)$ is, as remarked above, $\Sigma^1_1(z, \mathrm{Th}(L_\eta))$ on n, X, and so is Π_1 over the least admissible set containing z, η. Since every set is contained in some admissible set L_β with $\beta < \alpha$, we have that $n \in \Psi_\eta(X)$ is $\Delta_1(z)$ over L_α. The last part of the claim then follows from Σ_1-recursion inside L_α, using the $\Delta_1^{L_\alpha}(z)$-definability of the relation $Y = \Psi_\eta(X)$. □

Claim 2.2.6. *Suppose $z \in L_\alpha, \eta < \alpha$ and L_α is a model of Σ_1-Comprehension. Then $o(\Psi_\eta) < \alpha$, and $\langle \Psi^\xi_\eta \rangle_{\xi \le o(\Psi_\eta)} \in L_\alpha$. Moreover, the relation $n \in \Psi^\infty_\eta$ is $\Delta_1^{L_\alpha}(z)$.*

Proof. Note such L_α satisfies KPI, so by the previous claim together with Σ_1-Comprehension, $P_\eta = \{n \in \omega \mid (\exists \xi < \alpha) n \in \Psi^\xi_\eta\} \in L_\alpha$. By admissibility, the map on P_η sending n to the least ξ such that $n \in \Psi^\xi_\eta$ is bounded in α, and the claim is immediate. The last assertion holds because in L_α,

$$n \in \Psi^\infty_\eta \iff (\exists \xi) n \in \Psi^\xi_\eta \iff (\forall \xi)(\Psi^\xi_\eta = \Psi^{\xi+1}_\eta \to n \in \Psi^\xi_\eta).$$

□

Claim 2.2.7. *Suppose $z \in L_\alpha$, and that α is a limit of ordinals β so that L_β is a model of Σ_1-Comprehension. Then the relation $n \in \Psi^\infty_\eta$ is $\Delta_1^{L_\alpha}(z)$.*

Proof. Immediate from the previous claim and the fact that the sequences are correctly computed in models of KPI_0. □

Claim 2.2.8. *If $\xi < \gamma_2$, then for some $\eta_0 < \gamma_2$ we have $\Psi^\xi_{\eta_0} = \Psi^\xi_{\gamma_2}$; furthermore, $\langle \Psi^\zeta_{\gamma_2} \rangle_{\zeta<\xi} \in L_{\gamma_2}$.*

Note that then for this η_0, $\Psi^\xi_{\eta_0} = \Psi^\xi_\eta$ whenever $\eta_0 \le \eta < \delta_2$.

Proof. The set $Q_\xi = \{n \in \omega \mid (\exists \eta < \gamma_2) n \notin \Psi^\xi_\eta\}$ is a member of L_{γ_2} by Σ_1-Comprehension there. Now the map sending $n \in Q_\xi$ to the least η such that $n \notin \Psi^\xi_\eta$ is Δ_1, so by admissibility, is bounded by some $\eta_0 < \alpha$. Recall the sequence $\langle \Psi^\xi_\eta \rangle_{\eta \in \mathrm{ON}}$ is decreasing in η; so

$$n \in \Psi^\xi_{\eta_0} \iff L_{\gamma_2} \models (\forall \eta) n \in \Psi^\xi_\eta \iff L_{\delta_2} \models (\forall \eta) n \in \Psi^\xi_\eta \Longrightarrow n \in \Psi^\xi_{\gamma_2} \Longrightarrow n \in \Psi^\xi_{\eta_0}.$$

Note we have used the fact that $L_{\gamma_2} \prec_{\Sigma_1} L_{\delta_2}$. For the last part of the claim, consider the map sending $\zeta < \xi$ to the least η_0 such that $(\forall \eta > \eta_0) \Psi^\zeta_\eta = \Psi^\zeta_{\eta_0}$. This map is Π_1-definable, so by Σ_2-Collection in L_{γ_2}, we have a bound $\bar\eta < \gamma_2$, and for each $\zeta < \xi$, $\Psi^\zeta_{\bar\eta} = \Psi^\zeta_{\gamma_2}$. By Claim 2.2.5 the sequence $\langle \Psi^\zeta_{\bar\eta} \rangle_{\zeta < \xi} = \langle \Psi^\zeta_{\gamma_2} \rangle_{\zeta < \xi}$ is in L_{γ_2}. □

Claim 2.2.9. *For all $\xi < \gamma_2$, $\Psi^\xi_{\gamma_2} = \Psi^\xi_{\delta_2}$.*

Proof. By using induction on ξ and since $\Psi^{<\xi}_{\gamma_2} \in L_{\gamma_2}$ by the previous Claim, it is sufficient to show $\Psi_{\gamma_2}(X) = \Psi_{\delta_2}(X)$ whenever $X \in L_{\gamma_2}$. We already know $\supseteq$ holds.

So suppose $n \in \Psi_{\gamma_2}(X)$. Then we have $n \in \Phi_{\gamma_2}(X) = \Phi(X)$, by monotonicity of $\Phi = \Phi_{\gamma_2}$ in L_{γ_2}. So

$$L_{\gamma_2} \models (\forall x)(\exists y) T(n, X, x, y, z)$$

so that by Σ_1-elementarity (this is enough, since Π^1_2 relations are $\Pi^{\mathsf{KPl}_0}_1$), L_{δ_2} models the same. Thus $n \in \Psi_{\delta_2}(X)$ (with witness $X' = X$). □

We haven't yet used the full strength of $L_{\gamma_2} \prec_{\Sigma_2} L_{\delta_2}$, nor, for that matter, any of the assumptions on γ_1, δ_1. We appeal to the first assumption to show that in fact $o(\Psi_{\delta_2}) \leq \gamma_2$; the second will be used to show that $\Psi^\infty_{\delta_2} = \Psi^\infty_{\gamma_2}$, and it will follow that operator Ψ_{γ_2} (which is equal to Φ) stabilizes inside L_{γ_2}.

Notice that by Claim 2.2.8, $\Psi^\xi_{\gamma_2} = \bigcap_{\eta < \gamma_2} \Psi^\xi_\eta$ for all $\xi < \gamma_2$. So

$$\Psi^{<\gamma_2}_{\gamma_2} = \{n \in \omega \mid (\exists \xi < \gamma_2)(\forall \eta < \gamma_2) n \in \Psi^\xi_\eta\}.$$

This set is Σ_2-definable over L_{γ_2}. By the fact that $\Psi^{\gamma_2}_{\delta_2} \subseteq \Psi^{\gamma_2}_\eta$ for all $\eta < \delta_2$, we have

$$\Psi^{\gamma_2}_{\delta_2} \subseteq \{n \in \omega \mid (\forall \eta < \delta_2) n \in \Psi^{\gamma_2}_\eta\} \subseteq \{n \in \omega \mid (\exists \xi < \delta_2)(\forall \eta < \delta_2) n \in \Psi^\xi_\eta\}.$$

By the assumed Σ_2-elementarity, this last set is precisely $\Psi_{\gamma_2}^{<\gamma_2}$. We obtain

$$\Psi_{\delta_2}^{\gamma_2} \subseteq \Psi_{\gamma_2}^{<\gamma_2} = \Psi_{\delta_2}^{<\gamma_2} \subseteq \Psi_{\delta_2}^{\gamma_2}$$

so that $\Psi_{\delta_2}^{\gamma_2} = \Psi_{\delta_2}^{<\gamma_2}$ is the least fixed point of Ψ_{δ_2}, $\Psi_{\gamma_2}^{<\gamma_2} = \Psi_{\delta_2}^{\infty}$.

Claim 2.2.10. $\Psi_{\delta_2}^{\infty} = \Psi_{\gamma_2}^{\infty}$.

Proof. As usual, we know $\subseteq$ holds. We have $\Psi_{\delta_2}^{\infty} = \Psi_{\gamma_2}^{<\gamma_2} \in L_{\delta_2}$. Suppose $n \notin \Psi_{\delta_2}^{\infty}$. Then

$$L_{\delta_1} \models (\exists \eta)(\exists P)(\forall m \in \omega)(m \in \Psi_\eta(P) \to m \in P) \wedge n \notin P,$$

with $\eta = \delta_2$ and $P = \Psi_{\delta_2}^{\infty}$. Recall "$m \in \Psi_\eta(P)$" is Π_1 over any admissible set containing η, z, P, so that the relation above is Σ_1 in L_{δ_1}. It therefore reflects to L_{γ_1} (recall that z, the parameter from which everything is defined, is assumed to belong to L_{γ_1}). But then $n \notin \Psi_\eta^{\infty}$ for some $\eta < \gamma_1$; hence $n \notin \Psi_{\gamma_2}^{\infty}$. $\square$

So the least fixed points $\Phi^\infty = \Psi_{\gamma_2}^{\infty}$ and $\Psi_{\delta_2}^{\infty}$ are equal. The argument just given shows the relation $n \notin \Phi^\infty$ is Σ_1 over L_{δ_1}, hence over L_{γ_1}; in any event, the set Φ^∞ belongs to L_{γ_2} (using Σ_1-Comprehension in L_{γ_2} in the case that $\gamma_1 = \gamma_2$).

Finally, we claim $o(\Phi) < \gamma_2$. The map defined in L_{γ_2} that takes $n \in \Phi^\infty = \Psi_{\gamma_2}^{\infty}$ to the least ξ such that $(\exists \eta_0)(\forall \eta > \eta_0) n \in \Psi_\eta^\xi$ is Σ_2-definable, and so by Σ_2-Collection is bounded in γ_2. Since for each $\xi < \gamma_2$ we have $\Phi^\xi = \Psi_{\gamma_2}^\xi = \Psi_{\eta_0}^\xi$ for some $\eta_0 < \gamma_2$, this implies $o(\Phi) < \gamma_2$.

That $\langle \Phi^\xi \rangle_{\xi \leq o(\Phi)}$ belongs to L_{γ_2} now follows from the last assertion of Claim 2.2.8. This completes the proof that the desired instance of $\Pi_2^1(z)$-MI holds in L_{γ_2}. $\square$

Theorem 2.2.11. *Suppose M is an illfounded ω-model of* KP *with $\langle \zeta_n, s_n \rangle_{n \in \omega}$ an infinite depth Σ_2-nesting based on $\mathcal{M}$, and that $\mathcal{M}$ is locally countable, in the sense that every $L_a^{\mathcal{M}}$ has ultimate projectum ω in $\mathcal{M}$. Then if $\beta = \sup_{n \in \omega} \zeta_n$, we have $L_\beta \models \mathbf{\Pi}_2^1$-$\mathsf{MI}$.*

Proof. If $\beta = \zeta_n$ for some $n \in \omega$, then we obtain the result immediately by applying the lemma in M to the tuple $\langle \zeta_n, \zeta_{n+1}, s_{n+1}, s_n \rangle$. So we can assume $\langle \zeta_n \rangle_{n \in \omega}$ is strictly increasing. Let $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be $\Pi_2^1(z)$ and monotone in L_β for some $z \in L_\beta$, and let ζ_n be sufficiently large that $z \in L_{\zeta_n}$. Now $L_{\zeta_{n+1}} \prec_{\Sigma_1} L_\beta$ and both models satisfy KPI_0, so that whenever $X \subseteq \omega$ is in $L_{\zeta_{n+1}}$, we have

$$L_\beta \models n \in \Phi(X) \iff L_{\zeta_{n+1}} \models n \in \Phi(X).$$

In particular, $L_{\zeta_{n+1}}$ believes Φ is $\Pi^1_2(z)$ and monotone, so that by the lemma applied to the tuple $\langle \zeta_n, \zeta_{n+1}, s_{n+1}, s_n \rangle$, we have $o(\Phi) < \zeta_{n+1}$, and the sequence $\langle \Phi^\xi \rangle_{\xi \leq o(\Phi)}$ (which is computed identically in $L_{\zeta_{n+1}}$ and L_β) belongs to $L_{\zeta_{n+1}}$. □

2.3 Infinite depth Σ_2-nestings from determinacy

In this section we show in the base theory $\mathbf{\Pi}^1_1$-CA_0 that Σ^0_3-DET implies the existence of models bearing infinite depth Σ_2 nestings. The arguments are mostly cosmetic modifications of those given in Welch's [Wel11]. The most significant adjustment is to the Friedman-style game, Welch's G_ψ, which is here tailored to allow the proof of the implication to be carried out in $\mathbf{\Pi}^1_1$-CA_0.

For α an ordinal, let T_2^α denote the lightface Σ_2-theory of L_α, i.e.,

$$T_2^\alpha = \{\sigma \mid \sigma \text{ is a } \Sigma_2 \text{ sentence without parameters, and } L_\alpha \models \sigma\}.$$

We will also abuse this notation slightly by applying it to nonstandard ordinals b, so that if $b \in \mathrm{ON}^{\mathcal{M}} \setminus \mathrm{wfo}(\mathcal{M})$, T_2^b denotes the Σ_2-theory of $(L_b)^{\mathcal{M}}$. It will always be clear from context which illfounded model $\mathcal{M}$ this b comes from.

Lemma 2.3.1. *Suppose $\mathcal{M}$ is an illfounded ω-model of KP such that $(L_a)^{\mathcal{M}} \models$ "all sets are countable", for every $a \in \mathrm{ON}^{\mathcal{M}}$. Set $\beta = \mathrm{wfo}(\mathcal{M})$. Suppose for all nonstandard ordinals a of $\mathcal{M}$, there exists some $<^{\mathcal{M}}$-smaller nonstandard $\mathcal{M}$-ordinal b so that $T_2^b \subseteq T_2^\beta$. Then there is an infinite depth Σ_2 nesting based on $\mathcal{M}$.*

Proof. This is essentially shown in Claim (5) in section 3 of [Wel11]. We outline the shorter approach suggested there.

Suppose b is a nonstandard $\mathcal{M}$-ordinal with $T_2^b \subseteq T_2^\beta$. By the assumption of local countability in levels of $L^{\mathcal{M}}$, we have a uniformly Σ_2-definable Σ_2 Skolem function, see [Fri08]. The set $H = h_2^b{}``\omega^{<\omega}$ is transitive in $\mathcal{M}$, since for any $x \in H$, the $<_L^{\mathcal{M}}$-least surjection of ω onto x is in H, and since $\mathcal{M}$ is an ω-model, the range of this surjection is a subset of H. Since $H \models V = L$, we have by condensation in $\mathcal{M}$ that $H = L_{\gamma_b} \prec_{\Sigma_2} L_b$ for some $\gamma_b \leq^{\mathcal{M}} b$.

We claim that $\gamma_b < \beta$. For suppose not, so there is some nonstandard ordinal c of L_b in L_{γ_b}. Let f be the $<_L^{\mathcal{M}}$-least surjection from ω onto c. Then $f = h_2^b(k)$ for some $k \in \omega$, and for $m, n \in \omega$, the sentences "$h_2(k)$ exists, is a function from ω onto some ordinal, and $h_2(k)(m) \in h_2(k)(n)$" are Σ_2. But since $T_2^b \subseteq T_2^\beta$, this would imply $h_2^\beta(k)(m) \in h_2^\beta(k)(n)$ whenever $f(m) \in f(n)$ in $(L_b)^{\mathcal{M}}$. This contradicts the wellfoundedness of β.

The lemma now follows by choosing some descending sequence $\langle b_n \rangle_{n\in\omega}$ of nonstandard ordinals of $\mathcal{M}$ with $T_2^{b_n} \subseteq T_2^\beta$ for all n, and setting $\gamma_n = \sup h_2^{b_n}{}``\omega^{<\omega} < \beta$. Since the γ_n are true ordinals, we can choose some non-decreasing subsequence $\langle \gamma_{n_k} \rangle_{k\in\omega}$, and $\langle \gamma_{n_k}, b_{n_k} \rangle_{k\in\omega}$ is the desired infinite depth Σ_2-nesting. □

Theorem 2.3.2. *Work in $\mathbf{\Pi}_1^1$-CA_0. If Σ_3^0-determinacy holds, then there is a model $\mathcal{M}$ for which there exists an infinite depth Σ_2-nesting based on $\mathcal{M}$.*

Corollary 2.3.3. *Work in $\mathbf{\Pi}_1^1$-CA_0. Σ_3^0-determinacy implies the existence of a β-model of $\mathbf{\Pi}_2^1$-MI; indeed, $L_\gamma = \mathbf{\Pi}_2^1$-$\mathsf{MI}$ for some countable ordinal γ.*

Proof. Immediate, combining Theorem 2.3.2 with Theorem 2.2.11. □

Proof of Theorem 2.3.2. We define a variant of Welch's game G_ψ from [Wel11]. Players I and II play complete consistent theories in the language of set theory, $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively, extending

$$V = L \; + \; KP \; + \; \rho_\omega = \omega. \tag{$*$}$$

These theories uniquely determine term models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$. Player I loses if $\mathcal{M}_{\mathrm{I}}$ has nonstandard ω; similarly, if $\mathcal{M}_{\mathrm{I}}$ is an ω-model and $\mathcal{M}_{\mathrm{II}}$ is not, then Player II loses. (Note that this is a Boolean combination of Σ_2^0 conditions on $f_{\mathrm{I}}, f_{\mathrm{II}}$.)

The remainder of the winning condition assumes $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are both ω-models. Player I wins if any of the following hold.

1. $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$, or $f_{\mathrm{I}} = f_{\mathrm{II}}$.

2. $(\exists \beta \leq \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}})(\exists a \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})(\forall n \in \omega)(\exists \langle a_i, s_i \rangle_{i \leq n})$ so that, for all $i < n$,

   - $a_0 = a$ and $a_i \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$,
   - $(a_{i+1} < a_i)^{\mathcal{M}_{\mathrm{II}}}$,
   - σ_i is the first Σ_2 formula (in some fixed recursive list of all formulas in the language of set theory) so that $L_\beta^{\mathcal{M}_{\mathrm{I}}} \not\models \sigma_i$ and $L_{a_i}^{\mathcal{M}_{\mathrm{II}}} \models \sigma_i$;
   - if a_i is a successor ordinal in $\mathcal{M}_{\mathrm{II}}$, then a_{i+1} is the largest limit ordinal of $\mathcal{M}_{\mathrm{II}}$ below a_i;
   - if a_i is a limit ordinal in $\mathcal{M}_{\mathrm{II}}$, and σ_i is the formula $\exists u \forall v \psi(u, v)$, then a_{i+1} is least in $\mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$ so that $(\exists u \in L_{a_{i+1}})(L_{a_i} \models \forall v \psi(u, v))$ in $\mathcal{M}_{\mathrm{II}}$.

Note that if (2) holds, then $\mathcal{M}_{\mathrm{II}}$ must be an illfounded model, because if β, a witness the condition, then the sequences $\langle a_i, \sigma_i \rangle_{i \leq n}$ are uniquely determined for each n, and so must all be end extensions of one another.

Note also that this condition is Σ^0_3 as a condition on $f_{\mathrm{I}}, f_{\mathrm{II}}$. Strictly speaking, the quantifiers over $\mathcal{M}_{\mathrm{I}}$, $\mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$, etc. should be regarded as natural number quantifiers ranging over the indices of defining formulas for members of the models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$. Clause (1) is then Σ^0_2, and (2) is Σ^0_3, since each bulleted item there is recursive in codes for the objects $\beta, a, \langle a_i, \sigma_i \rangle_{i \leq n}$ and the pair $\langle f_{\mathrm{I}}, f_{\mathrm{II}} \rangle$.

Denote the set of runs which I wins by F; so F is Σ^0_3.

Claim 2.3.4. *Player I has no winning strategy in $G(F)$.*

Proof. Suppose instead that I has some winning strategy in this game. By Shoenfield absoluteness (which holds in $\mathbf{\Pi}^1_1$-CA_0, see [Sim09]) there is such a winning strategy σ in L. Let α be the least admissible ordinal so that $\sigma \in L_\alpha$ (such exists since $\mathbf{\Pi}^1_1$-CA_0 implies the reals are closed under the hyperjump; see [Sac90]). Let f_{II} be the theory of L_α. Note that then L_α Σ_1-projects to ω, since it is the least admissible containing some real; in particular, it satisfies condition $(*)$. Let f_{I} be the theory that σ responds with.

Now σ is winning for I in $G(F)$; so $\mathcal{M}_{\mathrm{I}}$ is an ω-model. Since $\mathcal{M}_{\mathrm{II}}$ is wellfounded, (2) must fail, and since we assumed σ is winning for I, we have (1) holds; that is, either $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$ or $f_{\mathrm{I}} = f_{\mathrm{II}}$. If $f_{\mathrm{I}} = f_{\mathrm{II}}$, then II was simply copying I's play, so that $\sigma \in L_\alpha = \mathcal{M}_{\mathrm{I}}$, implying $f_{\mathrm{I}} \in \mathcal{M}_{\mathrm{I}}$, a contradiction to the fact that $\mathcal{M}_{\mathrm{I}}$ ω-projects to ω.

So $f_{\mathrm{II}} \in \mathcal{M}_{\mathrm{I}}$. The strategy σ is computable from f_{II}, so must also belong to $\mathcal{M}_{\mathrm{I}}$. But then, since $f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$, we again obtain the contradiction $f_{\mathrm{I}} \in \mathcal{M}_{\mathrm{I}}$. □

Claim 2.3.5. *If there is no model with an infinite depth Σ_2-nesting, then Player II has no winning strategy in $G(F)$.*

Proof. Towards a contradiction, let τ be a winning strategy for II; as in the previous claim, we may assume $\tau \in L$, and let α be the least admissible with $\tau \in L_\alpha$. Put $f_{\mathrm{I}} = \mathrm{Th}(L_\alpha)$; then f_{I} satisfies the condition $(*)$. Let $f_{\mathrm{II}} = \tau * f_{\mathrm{I}}$ be τ's response.

We claim that if $\mathcal{M}_{\mathrm{II}}$ is the model so determined, then $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$ (note $\mathbf{\Pi}^1_1$-CA_0 is enough to ensure (a real coding) the wellfounded ordinal of a model exists). Suppose otherwise; then $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) > \alpha$, and then $L_\alpha \in \mathcal{M}_{\mathrm{II}}$. Then $f_{\mathrm{I}} = \mathrm{Th}(L_\alpha)$ and τ belongs to $\mathcal{M}_{\mathrm{II}}$, so that $f_{\mathrm{II}} = \tau * f_{\mathrm{I}}$ does as well. As before, this contradicts the assumption that II wins the play; specifically, f_{II} fails to satisfy condition $(*)$.

So $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$. We claim $\mathcal{M}_{\mathrm{II}}$ is illfounded. Otherwise, either $o(\mathcal{M}_{\mathrm{II}}) = \alpha$, in which case we get $\mathcal{M}_{\mathrm{II}} = L_\alpha = \mathcal{M}_{\mathrm{I}}$, in which case (1) holds and I wins; or else $o(\mathcal{M}_{\mathrm{II}}) < \alpha$, so that $\mathcal{M}_{\mathrm{II}} = L_\gamma$ for some $\gamma < \alpha$, so that $f_{\mathrm{II}} = \mathrm{Th}(L_\gamma) \in L_\alpha = \mathcal{M}_{\mathrm{I}}$, and again (1) holds, contradicting that τ is winning for II.

So $\mathcal{M}_{\mathrm{II}}$ is illfounded with $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \alpha$. Set $\beta = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$. If there is no model bearing an infinite depth Σ_2-nesting, then by Lemma 2.3.1 there exists some nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal a, so that, for every nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal b with $b \leq^{\mathcal{M}_{\mathrm{II}}} a$, we have $T_2^b \not\subseteq T_2^\beta$. That is, for all such b, there is a Σ_2 sentence σ so that $L_\beta \not\models \sigma$, but $L_b^{\mathcal{M}_{\mathrm{II}}} \models \sigma$.

It is now straightforward to show β, a witness the winning condition (2). Set $a_0 = a$. Suppose inductively that a_i is a nonstandard $\mathcal{M}_{\mathrm{II}}$-ordinal with $a_i \leq^{\mathcal{M}_{\mathrm{II}}} a$. Then by choice of a, there is some Σ_2 formula σ so that $L_b^{\mathcal{M}_{\mathrm{II}}} \models \sigma$ and $L_\beta \not\models \sigma$; let σ_i be the least such under our fixed enumeration of formulae. If a_i is not limit in $\mathcal{M}_{\mathrm{II}}$, take a_{i+1} to be the greatest limit ordinal of $\mathcal{M}_{\mathrm{II}}$ below a_i; note then a_{i+1} is also nonstandard and below a.

Now if a_i is limit in $\mathcal{M}_{\mathrm{II}}$, we have that σ_i is of the form $(\exists u)(\forall v)\psi(u, v)$ for some Δ_0 formula ψ. Let a_{i+1} be least so that for some $x \in L_{a_i+1}^{\mathcal{M}_{\mathrm{II}}}$, we have $L_{a_i}^{\mathcal{M}_{\mathrm{II}}} \models (\forall v)\psi(x, v)$. Then $a_{i+1} <^{\mathcal{M}_{\mathrm{II}}} a_i$, and since $L_\beta \not\models \sigma_i$, we must have that a_{i+1} is nonstandard. Thus the construction proceeds, and we have that I wins the play $\langle f_{\mathrm{I}}, f_{\mathrm{II}}\rangle$ via condition (2). So τ cannot be a winning strategy. $\square$

These claims combine to show that if there is no model with an infinite depth Σ_2 nesting, then neither player has a winning strategy in the game $G(F)$. This completes the proof of the theorem. $\square$

We have thus shown that Σ^0_3 determinacy implies the existence of a model satisfying $\mathbf{\Pi}^1_2$-MI, and indeed, of some ordinal γ so that $L_\gamma \models \mathbf{\Pi}^1_2$-$\mathsf{MI}$. The meticulous reader will observe, however, that our proof of determinacy in section 2 really only made use of $(\Game\mathbf{\Sigma}^0_3)^\vee$ monotone inductive definitions. This may at first appear strange, in light of the fact that $(\Game\mathbf{\Sigma}^0_3)^\vee$ is a much smaller class than $\mathbf{\Pi}^1_2$. This situation is clarified somewhat by the following theorem, which shows that if γ is minimal with $L_\gamma \models \mathbf{\Pi}^1_2$-$\mathsf{MI}$, then the $\mathbf{\Pi}^1_2$ relations that are correctly computed in L_γ are precisely the $(\Game\mathbf{\Sigma}^0_3)^\vee$ relations.

Theorem 2.3.6. *Let γ be the least ordinal so that L_γ satisfies $\mathbf{\Pi}^1_2$-MI. Let z be a real in L_γ, and suppose $\Phi(u)$ is a $\mathbf{\Sigma}^1_2$ formula. Then there is a $\Game\mathbf{\Sigma}^0_3$ relation Ψ so that, for all reals x of L_γ, we have L_γ satisfies $\Phi(x)$ if and only if $\Psi(x)$ holds (in V, or equivalently, in L_γ).*

Proof. Fix such a formula $\Phi(x)$. Then there is a recursive tree T on ω^3 so that for all x, $\Phi(x)$ holds if and only if for some y, $T_{\langle x,y\rangle}$ is wellfounded. We define a version of the game from Theorem 2.3.2. This time, for a fixed real x, each player is required to produce their respective ω-models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ satisfying

$$V = L(x) + KP + \rho_\omega = \omega. \tag{$**$}$$

In addition, $\mathcal{M}_{\mathrm{I}}$ must satisfy the sentence "$(\exists y)T_{\langle x,y\rangle}$ is ranked"; whereas $\mathcal{M}_{\mathrm{II}}$ must satisfy its negation. If a winner has not been decided on the basis of one of these conditions being violated, then Player I wins if either of the conditions (1), (2) from the proof of Theorem 2.3.2 hold. Let F_x be the set of $f \in \omega^\omega$ so that Player I wins the play of the game on x, where $\langle f_{\mathrm{I}}, f_{\mathrm{II}}\rangle$, where $f_{\mathrm{I}}(n) = f(2n), f_{\mathrm{II}}(n) = f(2n+1)$ for all n. Let $F = \{\langle x, f\rangle \mid f \in F_x\}$. Then F is Σ^0_3; let $\Psi(x)$ be the statement "I has a winning strategy in the game $G(F_x)$".

Suppose $x \in L_\gamma$ is such that $L_\gamma \models \Phi(x)$. We claim $\Psi(x)$ holds; that is, I has a winning strategy in $G(F_x)$. Let y be a witness to truth of Φ, and let α be least $y \in L_\alpha(x)$ and $L_\alpha(x) \models \mathsf{KP}$. Then by admissibility, $L_\alpha(x)$ contains a ranking function for $T_{\langle x,y\rangle}$. Let σ be the strategy for I that always produces the theory of $L_\alpha(x)$. We claim σ is winning for Player I.

Suppose towards a contradiction that $\mathcal{M}_{\mathrm{II}}$ is the model produced by a winning play by II against σ; we can assume $\mathcal{M}_{\mathrm{II}} \in L_\gamma$. Then $\mathcal{M}_{\mathrm{II}}$ is an ω-model. It cannot be wellfounded, since then it would be of the form $L_\beta(x)$ for some β; but we can't have $\beta \geq \alpha$ (since $\mathcal{M}_{\mathrm{II}}$ cannot contain y, or else it would have a ranking function for $T_{\langle x,y\rangle}$), nor can $\beta < \alpha$ hold (since then (1) is satisfied, and I wins the play). So $\mathcal{M}_{\mathrm{II}}$ is illfounded, say with $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) = \beta$; by a similar argument, $\beta < \alpha$. Now since I does not win the play, the condition (2) fails, so there must be some infinite depth Σ_2-nesting based on $\mathcal{M}_{\mathrm{II}}$, by Lemma 2.3.1. But this contradicts the fact that the model $\mathcal{M}_{\mathrm{II}}$ belongs to L_γ, by minimality of γ and Theorem 2.2.11.

Conversely, suppose $\Phi(x)$ fails in L_γ. Suppose towards a contradiction that I wins the game $G(F_x)$; then by Theorem 2.1.1, there is such a strategy $\sigma \in L_\gamma$. Let $\mathcal{M}_{\mathrm{II}}$ be the least level of $L(x)$ containing σ. Note that $\mathcal{M}_{\mathrm{II}} \models (**)$ + "$(\forall y)T_{\langle x,y\rangle}$ is not ranked". By the argument in the proof of Theorem 2.3.2, we obtain failure of both (1) and (2), so that II wins the play, a contradiction to σ being a winning strategy. $\square$

CHAPTER 3

Determinacy in $\mathbf{\Sigma}^0_4$ and higher

In this chapter, we consider the next level, Σ^0_4, and more generally, levels of the form $\Sigma^0_{\alpha+3}$ for $1 < \alpha < \omega_1$. By an early result of Friedman [Fri71], full Borel determinacy requires ω_1 iterations of the Powerset axiom, and even Σ^0_5-DET is not provable in second order arithmetic nor indeed, full ZFC^- (ZFC minus the Powerset axiom). Martin later improved this to Σ^0_4-DET and proved the corresponding generalization for higher levels of the hyperarithmetical hierarchy; combining Montalbán-Shore's fine analysis [MS12] of $(n-\mathbf{\Pi}^0_3)$-DET with Martin's inductive proof [Mar85] of Borel determinacy, we may summarize the bounds for these levels known prior to our work as follows.

Theorem 3.0.7 (Martin, Friedman, Montalbán-Shore)**.** *For $\alpha < \omega_1$, $n < \omega$,*

$$\mathsf{Z}^- + \Sigma_1\text{-}\textit{Replacement} + \text{“}\mathcal{P}^\alpha(\omega)\ \textit{exists}\text{”} \vdash n\text{–}\mathbf{\Pi}^0_{1+\alpha+2}\text{-DET},\ \textit{but}$$
$$\mathsf{ZFC}^- + \text{“}\mathcal{P}^\alpha(\omega)\ \textit{exists}\text{”} \nvdash \Sigma^0_{1+\alpha+3}\text{-DET}.$$

Here Z is Zermelo Set Theory without Choice (including Comprehension, but excluding Replacement). Again the superscript “$-$” indicates removal of the Powerset axiom. Thus, $\alpha + 1$ iterations of the Powerset axiom are necessary to prove $\Sigma^0_{1+\alpha+3}$-DET. However, the question remained as to what additional ambient set theory is strictly necessary. More precisely, can one isolate a natural fragment of $\mathsf{Z}^- + \Sigma_1$-Replacement $+$“$\mathcal{P}^{\alpha+1}(\omega)$ exists” whose consistency strength is precisely that of $\Sigma^0_{1+\alpha+3}$-DET? Furthermore, can one characterize in a meaningful way the least level of L at which winning strategies in these games are constructed?

In this chapter, we show this is the case. We introduce a family of natural reflection principles, Π_1-RAP_α, and show in a weak base theory that the existence of a wellfounded model of Π_1-RAP_α is equivalent to $\Sigma^0_{1+\alpha+3}$-DET, for $\alpha < \omega_1^{\mathsf{CK}}$. In particular, we show that the least ordinal θ_α so that winning strategies in all $\Sigma^0_{1+\alpha+3}$ games belong to $L_{\theta_\alpha+1}$ is precisely the least so that $L_{\theta_\alpha} \models \Pi_1$-$\mathsf{RAP}_\alpha$. It turns out that in the $V = L$ context, the principles Π_1-RAP_α are equivalent to easily stated axioms concerning the existence of ranking functions for open games, so that the ordinals θ_α can be rather simply described. In particular, letting $\theta = \theta_0$, we have the following: L_θ is the least level of L satisfying "$\mathcal{P}(\omega)$ exists, and all wellfounded trees on $\mathcal{P}(\omega)$ are ranked."

The chapter is organized as follows. In Section 3.1, after introducing the abstract principles Π_1-$\mathsf{RAP}(U)$, we focus on Π_1-$\mathsf{RAP}(\omega)$, proving some basic consequences and obtaining useful equivalents in the $V = L$ context. In Section 3.2, we connect these principles to determinacy, in particular proving Σ^0_4-DET assuming the existence of a wellfounded model of Π_1-$\mathsf{RAP}(\omega)$. In Section 3.3, we prove our lower bound in the case of Σ^0_4-DET, making heavy use of the results of Section 3.1. Section 3.4 carries out the analogous arguments for levels of the hyperarithmetical hierarchy of the form $\Sigma^0_{\alpha+3}$, for $1 < \alpha < \omega_1^{\mathsf{CK}}$. We conclude in Section 3.5 with some remarks concerning the complexity of winning strategies.

3.1 The Π_1-Reflection to Admissibles Principle

We take as our background theory BST (Basic Set Theory), which consists of the axioms of Extensionality, Foundation, Pair, Union, and Δ_0-Comprehension. Unless otherwise stated, all of the models we consider satisfy at least BST (so "transitive model" really means "transitive model of BST"). Central to this chapter is Kripke-Platek Set Theory, KP, which is BST together with the axiom scheme of Δ_0-Collection; note that all axioms in the schema of Σ_1-Collection and Δ_1-Comprehension are then provable in KP. A transitive set M is called *admissible* if the structure $(M, \in)$ satisfies KP; KPI_0 is the theory asserting that every set x belongs to an admissible set, and has the same consequences for second order arithmetic as $\boldsymbol{\Pi}^1_1$-CA_0. The standard reference for admissible set theory is Barwise's [Bar75].

We now define the main theory of interest in this chapter.

Definition 3.1.1. Let U be a transitive set. The Π_1-*Reflection to Admissibles Principle for* U (denoted Π_1-$\mathsf{RAP}(U)$) is the assertion that $\mathcal{P}(U)$ exists, together with the following axiom scheme, for all Π_1 formulae $\phi(u)$ in the language of set theory: Suppose $Q \subseteq \mathcal{P}(U)$ is a set and $\phi(Q)$ holds. Then there is an admissible set M so that

- $U \in M$.
- $\bar{Q} = Q \cap M \in M$.
- $M \models \phi(\bar{Q})$.

We chose this particular formulation for its simplicity. The sets U we consider are sufficiently well-behaved that Π_1-$\mathsf{RAP}(U)$ gives a bit more.

Say U *admits power tuple coding* if there is a bijective map $c : \mathcal{P}(U)^{<\omega} \to \mathcal{P}(U)$ so that the relations $a \in c(s)$, $a \in c^{-1}(x)_i$, and $c(s) = x$ are all $\Delta_0(\{U\})$ (that is, definable from the parameter U with all quantifiers bounded). Note then that if M is transitive satisfying BST and $U \in M$, then any set $Q \subseteq \mathcal{P}(U)^{<\omega}$ in M can be coded by a set $Q \subseteq \mathcal{P}(U)$ in M.

Lemma 3.1.2. *Suppose U is a transitive set, admits power tuple coding, and Π_1-$\mathsf{RAP}(U)$ holds. Let $\phi(u_1, \ldots, u_m, v_1, \ldots, v_n)$ be a Π_1 formula and fix sets $p_i \subseteq U^{<\omega}$, $Q_j \subseteq \mathcal{P}(U)^{<\omega}$ for $1 \leq i \leq m$, $1 \leq j \leq n$ so that $\phi(p_1, \ldots, p_m, Q_1, \ldots, Q_n)$ holds. Then there is an admissible set M so that*

- *$U \in M$ and $M \models$ "$\mathcal{P}(U)$ exists".*
- *For $1 \leq i \leq m$, $p_i \in M$; for $1 \leq j \leq n$, $\bar{Q}_j = Q_j \cap M \in M$.*
- *$M \models \phi(p_1, \ldots, p_m, \bar{Q}_1, \ldots, \bar{Q}_n)$.*

Proof. First note that given $Q \subset \mathcal{P}(U)$, the relations $u = U$ and $v = Q$ are both Δ_0-definable from $Q' = Q \cup \{U\}$, and this allows us refer to the coding map $c : \mathcal{P}(U)^{<\omega} \to \mathcal{P}(U)$ in a

$\Delta_0(\{Q'\})$ fashion. So suppose $Q_1 \subseteq \mathcal{P}(U)^{<\omega}$ is given, and $(\forall x)\psi(Q_1, x)$ holds, where ψ is Δ_0. Let $Q = c[Q_1]$; the given Π_1 statement is equivalent to

$$(\forall u)(\forall x) u = c^{-1}[Q] \to \psi(u, x).$$

This can be phrased as a $\Pi_1(Q')$ statement and so can be reflected to an admissible set M where it holds of $\bar{Q}' = Q' \cap M$. Note then by absoluteness of the coding map c, we have $c^{-1}[\bar{Q}] = Q_1 \cap M$, so that $(\forall x)\psi(Q_1 \cap M, x)$ holds in M, as desired.

Similar uses of coding allow us to reflect statements involving finite lists of parameters $p_1, \ldots, p_m, Q_1, \ldots, Q_n$; that $p_i \cap M = p_i$ follows from transitivity of M and the assumption that $U \in M$. Finally, we can ensure $M \models$ "$\mathcal{P}(U)$ exists" by including $\mathcal{P}(U)$ as one of the Q_j; then $\bar{Q}_j = Q_j \cap M = \mathcal{P}(U)^M \in M$. $\square$

We will first be concerned mainly with Π_1-$\mathsf{RAP}(\omega)$, which we abbreviate simply as Π_1-RAP. Π_1-RAP does not imply Δ_0-Collection, so cannot prove KP. However, it does prove many Σ_1 consequences of admissibility. For example, from the following lemma, we have Σ_1-Recursion along wellfounded relations on $\mathcal{P}(\omega)$. Recall a relation R is *wellfounded* if every nonempty subset of its domain has an R-minimal element. For R wellfounded, let $\mathrm{tc}_R(a)$ denote the downwards R-closure of an element $a \in \mathrm{dom}(R)$,

$$\mathrm{tc}_R(a) := \{b \in \mathrm{dom}(R) \mid (\exists b_0, b_1, \ldots, b_n) b = b_0, b_n = a, b_i R b_{i+1} \text{ for all } i < n\}.$$

Lemma 3.1.3. *Work in Π_1-RAP and suppose R is a wellfounded binary relation on $\mathcal{P}(\omega)^{<\omega}$. Suppose further that $\phi(u, v, w)$ is a Σ_1 formula that provably in KP defines the graph of a binary class function G. Then for every $Q \subseteq \mathcal{P}(\omega)^{<\omega}$, there is a function $F : \mathrm{dom}(R) \to V$ so that for all $a \in \mathrm{dom}(R)$,*

$$F(a) = G(Q, F \upharpoonright \mathrm{tc}_R(a)).$$

Proof. Suppose towards a contradiction that R is wellfounded, but Q is such that no total function $F : \mathrm{dom}(R) \to V$ as in the lemma exists. This is a Π_1 statement in parameters R, Q, so by Π_1-RAP and Lemma 3.1.2, reflects to an admissible set M satisfying "$\mathcal{P}(\omega)$ exists".

Working in M, define the map F from $\bar{R}, \bar{Q}$ by its usual definition,

$$F(a) = Y \iff (\exists \bar{f} : \mathrm{tc}_{\bar{R}}(a) \to V)[Y = G(\bar{Q}, \bar{f}) \\ \wedge (\forall b \in \mathrm{dom}(\bar{f}))\bar{f}(b) = G(\bar{Q}, \bar{f} \restriction \mathrm{tc}_{\bar{R}}(b))].$$

The fact that $M \models \mathsf{KP}$ ensures this definition can be expressed in a Σ_1 fashion. Since we reflected a failure of this instance of Σ_1-Recursion to M, there must be some $y \in \mathrm{dom}(\bar{R})$ so that $F(y)$ does not exist. Consider the set

$$D = \{y \in \mathrm{dom}(\bar{R}) \mid M \models \text{“}F(y) \text{ does not exist”}\}.$$

We have D nonempty; note though, that we needn't have $D \in M$. By wellfoundedness of R, let $y_0 \in D$ be R-minimal. It follows that in M, $F(y)$ is defined whenever $y\bar{R}y_0$, so F is defined on $\mathrm{tc}_{\bar{R}}(y_0)$. By Σ_1-Replacement in M, we have that $\bar{f} = F \restriction \mathrm{tc}_{\bar{R}}(y_0)$ exists in M. But this $\bar{f}$ witnesses the fact that $F(y_0)$ exists, a contradiction. □

Corollary 3.1.4. *Assume Π_1-RAP and $\mathsf{DC}_{\mathbb{R}}$. Then whenever T is a tree on $\mathcal{P}(\omega)$, either T has an infinite branch, or T has a rank function, that is, a map $\rho : T \to \mathrm{ON}$ such that $\rho(s) < \rho(t)$ whenever $s \supsetneq t$.*

Proof. Suppose T is a tree on $\mathcal{P}(\omega)$ with no infinite branch. By Dependent Choice for reals, the relation $\supsetneq$ is wellfounded on T. Apply Σ_1-Recursion with the function $G(Q, F) = \sup\{F(s) \mid s \in \mathrm{dom}(F)\}$. □

It turns out that the statement that all wellfounded trees on $\mathcal{P}(\omega)$ are ranked is equivalent to Π_1-RAP in the $V = L$ context. Besides having intrinsic interest, this fact will be useful for our determinacy strength lower bounds. To prove it, we require a version of the truncation lemma for admissible structures (see [Bar75]) specialized to models of $V = L$.

For our purposes, “$V = L$” abbreviates BST plus the statement that every x belongs to J_α for some α, as well as the usual fine-structural consequences of this statement (acceptability, condensation, etc.). We will also make reference to Jensen's auxiliary S-hierarchy further stratifying the J-hierarchy: $S_0 = \emptyset$, $S_{\alpha+1}$ is the image of $S_\alpha \cup \{S_\alpha\}$ under a finite list of binary operations generating the rudimentary functions, and $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$; then $J_\alpha = S_{\omega\cdot\alpha}$. For details, see [SZ10].

Recall any model $\mathcal{M} = \langle M, \epsilon\rangle$ in the language of set theory has a unique largest downward ϵ-closed submodel on which ϵ is wellfounded, the *wellfounded part of* $\mathcal{M}$, denoted $\mathrm{wfp}(\mathcal{M})$. When ϵ is extensional on M, we identify $\mathrm{wfp}(\mathcal{M})$ with its transitive isomorph, and denote $\mathrm{wfo}(\mathcal{M}) = \mathrm{wfp}(\mathcal{M}) \cap \mathrm{ON}$.

Proposition 3.1.5. *Working in* KPI_0*, let* $\mathcal{M} = \langle M, \epsilon\rangle \models V = L$*, and suppose* $\mathcal{M}$ *is illfounded. Then* $L_{\mathrm{wfo}(\mathcal{M})}$ *is admissible.*

This differs from the usual truncation lemma both in that $\mathcal{M}$ is not itself assumed to be admissible, and in general $L_{\mathrm{wfo}(\mathcal{M})}$ needn't coincide with $\mathrm{wfp}(\mathcal{M})$ (even when $\mathcal{M} \models V = L$).

Proof. Given such $\mathcal{M}$, we know (working in KPI_0) that $\mathrm{wfp}(\mathcal{M})$ exists. Note $\mathrm{wfo}(\mathcal{M}) = \omega\cdot\alpha$ for some unique α. If $\alpha = 1$ then we're done. So suppose $\alpha > 1$. That $J_\alpha \models \mathsf{BST}$ is automatic. We only need to show $J_\alpha \models \Delta_0$-Collection (from which it follows that $L_\alpha = J_\alpha \models \mathsf{KP}$). One can show by induction on ξ that $S_\xi = S_\xi^{\mathcal{M}} \in \mathrm{wfp}(\mathcal{M})$ for all $\xi < \mathrm{wfo}(\mathcal{M})$; that is, $J_\alpha \subseteq \mathrm{wfp}(\mathcal{M})$.

Assume $a, p \in J_\alpha$ and

$$J_\alpha \models (\forall x \in a)(\exists y)\varphi(x, y, p),$$

where φ is Δ_0. Then in $\mathcal{M}$,

$$\mathcal{M} \models (\forall x \in a)(\exists y)\varphi(x, y, p).$$

Let σ be a nonstandard ordinal of $\mathcal{M}$. In $S_\sigma^{\mathcal{M}}$, define

$$\begin{aligned} F(x) = \xi \iff & x, p \in S_{\xi+1} \wedge (\exists y \in S_{\xi+1})\varphi(x, y, p) \\ & \wedge (\forall y \in S_\xi)(\neg\varphi(x, y, p) \vee x \notin S_\xi \vee p \notin S_\xi). \end{aligned}$$

Notice that $J_\alpha \subset S_\sigma^{\mathcal{M}}$, and by absoluteness, $F(x) < \omega\cdot\alpha$ for each $x \in a$. Since $\mathcal{M}$ satisfies BST, we have that the union of the $F(x)$,

$$\tau = \{\eta \in \sigma \mid (\exists x \in a)(\exists \xi \in \sigma)\eta \in \xi \wedge S_\sigma^{\mathcal{M}} \models F(x) = \xi\}$$

is an ordinal in $\mathcal{M}$, and must be contained in $\omega \cdot \alpha$. So $\tau \in \text{wfp}(\mathcal{M})$, hence $\tau < \omega \cdot \alpha$. We have $S_\tau \in J_\alpha$, and

$$(\forall x \in a)(\exists y \in S_\tau)\varphi(x, y, p).$$

This proves the needed instance of Δ_0-Collection, so $J_\alpha \models \mathsf{KP}$. $\square$

Theorem 3.1.6 (joint with Itay Neeman)**.** *Let $V = L$ and assume ω_1 exists, and that every tree on $\mathcal{P}(\omega)$ is either illfounded or ranked. Then Π_1-RAP holds; moreover, every instance of Π_1-RAP is witnessed by some L_α with α countable.*

Proof. We may assume ω_2 does not exist, since otherwise Π_1-RAP follows immediately. Suppose $Q \subseteq \mathcal{P}(\omega)$ and that $\phi(Q)$ holds for some Π_1 formula ϕ. Let $\tau > \omega_1$ be sufficiently large that $Q \in J_\tau$. Let T be the tree of attempts to build a complete, consistent theory of a model $\mathcal{M}$ so that

- $\mathcal{M}$ is illfounded,
- $\bar{Q} = \mathcal{M} \cap Q \in S_t$ for some $t \in \text{wfo}(\mathcal{M})$,
- $\mathcal{M} \models V = L + \phi(\bar{Q})$ + "ω_1 exists".

In slightly more detail: Let $\mathcal{L}^*$ be the language of set theory together with constants $\{d_n\}_{n\in\omega} \cup \{a_n\}_{n\in\omega} \cup \{t, q\}$. Fix some standard coding $\sigma \mapsto \#\sigma$ of sentences in the language of $\mathcal{L}^*$ so that $\#\sigma > k$ whenever d_k appears in σ. All nodes in T are pairs of the form $\langle f, g\rangle$, where $f : n \to \{0, 1\}$ and $g : n \to \tau \cup \mathcal{P}(\omega)$, and the set $\{\sigma \mid f(\#\sigma) = 1\}$ is a finite theory in $\mathcal{L}^*$, consistent with the following:

- "$d_{n+1} \in d_n$" for each $n \in \omega$.
- "t is an ordinal, q is a set of reals, and $q \in S_t$".
- $V = L + \phi(q)$ + "ω_1 exists and $\omega_1 \in t$".
- $\mu \to \psi(a_{\#\mu})$, for sentences μ of the form $(\exists x)(x \in t \lor x \subset \omega) \land \psi(x)$.

The point of the last clause is to have the a_i serve as Henkin constants witnessing statements asserting existence of a real or of an ordinal below t. Finally, the function g is required to assign values in $\mathcal{P}(\omega) \cup \tau$ to the Henkin constants in a way compatible with the theory; in particular, respecting the theory's order for elements of t (so that $f(\#(a_i \in a_j \in t)) = 1$ implies $g(i) < g(j) < \tau$), membership of reals in Q (so that $f(\#(a_i \in q)) = 1$ implies $g(i) \in Q$), and membership of naturals in reals ($f(\#(n \in a_i)) = 1$ implies $n \in g(i)$).

Suppose T is illfounded. A branch through T then yields f giving a complete and consistent theory in $\mathcal{L}^*$ together with assignment of constants g. Let $\mathcal{M}$ be the term model obtained from this theory. By construction, $\mathcal{M}$ is illfounded, $\mathcal{M} \models V = L$, and setting $\bar{Q} = \{g(i) \mid a_i^{\mathcal{M}} \in q^{\mathcal{M}}\}$, we have $\bar{Q} = Q \cap M \in S_t^{\mathcal{M}}$; moreover, by the assignment of elements of τ to terms below t, we have that $\mathcal{M}$ has wellfounded part containing $S_t^{\mathcal{M}}$. By Proposition 3.1.5, if $\alpha = \mathrm{wfo}(\mathcal{M})$, then $L_\alpha \models \mathsf{KP} + \phi(\bar{Q}) +$ "ω_1 exists", as needed.

Now suppose towards a contradiction that T is not illfounded. Since T is clearly coded by a tree on $\mathcal{P}(\omega)$, we have that T is ranked. Let $\rho : T \to \mathrm{ON}$ be the ranking function. We construct a branch through T using the function ρ; f will be the characteristic function of the complete theory of $J_{\rho(\emptyset)}$, interpreting t by τ, q by Q, and inductively choosing values $g(i)$ in $\tau \cup \mathcal{P}(\omega)$ to be $<_L$-least witnessing existential statements holding in $J_{\rho(\emptyset)}$. All that remains is to decide on interpretations for the constants d_n, corresponding to the descending sequence of ordinals. So let $x_0 = \rho(\emptyset)$, and having chosen the fragment s of the branch up to k, let $x_{k+1} = \rho(s)$. Then interpret d_i in the theory by x_i.

At each finite stage of the above construction, the theory chosen is satisfied under the appropriate interpretation in $J_{\rho(\emptyset)}$, so we may always extend the branch by one step. But then $\{x_k \mid k \in \omega\}$ is an infinite descending sequence of ordinals, a contradiction. □

We remark that with some extra work, the $V = L$ assumption can be replaced with more natural hypotheses. Namely, we have the converse of Lemma 3.1.3: If $\mathsf{DC}_{\mathbb{R}}$ holds and we have Σ_1-Recursion along wellfounded relations on $\mathbb{P}(\omega)$, then Π_1-RAP holds. The extra assumption of Σ_1-Recursion guarantees the existence of all levels of the hierarchy of sets constructible relative to the parameter set Q; the argument of Theorem 3.1.6 may then be carried out inside $L(Q)$ to give the desired instance of Π_1-RAP. ($\mathsf{DC}_{\mathbb{R}}$ is important in part to ensure the existence branches through illfounded trees.)

Proposition 3.1.7. *Suppose M is a transitive model of Π_1-RAP. Then Π_1-RAP holds in $L^M = \bigcup\{L_\alpha \mid \alpha \in M \wedge M \models$ "L_α exists"$\}$.*

Proof. Work in M. By Lemma 3.1.3, whenever $A \subseteq \mathcal{P}(\omega)$ codes a prewellorder of $\mathcal{P}(\omega)$, there is an ordinal α so that $\alpha = \mathrm{otp}(\mathcal{P}(\omega)/\equiv_A, <_A)$, and L_α exists. Since $\mathcal{P}(\omega)$ exists, there is a Δ_0 prewellorder of $\mathcal{P}(\omega)$ in order type ω_1, so L_{ω_1} exists. It follows that $L \models$ "ω_1 exists". So it's sufficient to show that every tree on $\mathcal{P}(\omega)$ in L is either ranked or illfounded in L.

So let T be a tree on $\mathcal{P}(\omega)$ with $T \in L^M$. If T is not ranked in L^M, reflect this fact to an admissible level L_α. Then $\bar{T} = T \cap L_\alpha \subseteq T$, and $L_\alpha \models$ "$\bar{T}$ is not ranked". Since L_α is admissible, there is a branch through $\bar{T}$ definable over L_α. In particular, $\bar{T} \subseteq T$ is illfounded in L. Hence by Theorem 3.1.6, Π_1-RAP holds in L $(= L^M)$, as needed. □

3.2 Proving determinacy

In this section, we prove a key lemma connecting the principles Π_1-$\mathsf{RAP}(U)$ to determinacy for certain infinite games, and use this lemma to give a proof of Σ^0_4-DET starting from a model of Π_1-RAP. We first recall some basic definitions and terminology.

Fix a set X. By a *tree on X* we mean a set $T \subseteq X^{<\omega}$ closed under initial segments. Let $[T]$ denote the set of infinite branches of T. For a set $A \subseteq [T]$, the *game on T with payoff A*, denoted $G(A;T)$, is played by two players, I and II, who alternate choosing elements of X,

$$\begin{array}{c|ccccccc} I & x_0 & & x_2 & \ldots & x_{2n} & & \ldots \\ \hline II & & x_1 & & \ldots & & x_{2n+1} & \ldots \end{array}$$

with the requirement that $\langle x_0, \ldots, x_n\rangle \in T$ for all n. If a terminal node is reached, the last player to make a move wins; otherwise, we obtain an infinite play $x \in [T]$, and we say I wins the play if $x \in A$; otherwise, II wins.

A *strategy for I* in a game on T is a partial function $\sigma : T \rightharpoonup X$ that assigns to an even-length position $s \in T$ a legal move x for I at s, that is, $x \in X$ so that $s^\frown\langle x \rangle \in T$. We require the domain of σ to be closed under legal moves by II as well as moves by σ; note then that due to the presence of terminal nodes in the tree, I needn't have a strategy at all. Strategies for II are defined analogously. We say a strategy σ is *winning* for I (II) in $G(A;T)$ if every play according to σ belongs to A ($[T] \setminus A$). A game $G(A;T)$ is *determined* if one of the players has a winning strategy. For a pointclass Γ, Γ-DET denotes the statement that $G(A;\omega^{<\omega})$ is determined for all $A \subseteq \omega^\omega$ in Γ.

Our proof of Σ^0_4-DET hinges on the use of the technical concept of an *unraveling*, introduced by Martin in his proof of Borel determinacy.

Definition 3.2.1. Let S be a tree. A *covering* of S is a triple $\langle T, \pi, \psi \rangle$ consisting of a tree T and continuous maps $\pi : T \to S$, $\psi : \mathrm{Strat}(T) \to \mathrm{Strat}(S)$ such that

(i) For all $\psi \in \mathrm{Strat}(T)$, $\psi(\sigma)$ is a strategy for the same player as σ;

(ii) For all $x \in [S]$, if x is compatible with $\psi(\sigma)$, then there is some $y \in [T]$ compatible with σ so that $\pi(y \restriction i) = x \restriction i$ for all $i \in \omega$.

We say a covering *unravels* a set $A \subseteq [S]$ if $\pi^{-1}(A)$ is clopen.

Martin showed that any countable collection of closed sets can be *simultaneously* unraveled. So if A is a $\mathbf{\Sigma}^0_{1+\alpha}$ subset of $[S]$ and $\langle T, \pi, \psi \rangle$ is the simultaneous unraveling of the countably many open sets involved in the construction of A, then $\pi^{-1}(A)$ is $\mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$. Thus the unraveling allows us to reduce the determinacy of $G(A;S)$ to that of the topologically simpler game $G(\pi^{-1}(A);T)$.

Our sights set on Σ^0_4-DET, the way forward is clear: apply the unraveling to all open sets involved in the construction of a $\mathbf{\Sigma}^0_4$ set A, and prove that the $\mathbf{\Sigma}^0_3$ game $G(\pi^{-1}(A);T)$ is determined, roughly imitating Davis's proof of $\mathbf{\Sigma}^0_3$-DET. Happily, the unraveling of countably many closed sets in $[S]$ can be carried out in rudimentarily closed models of "$H_{|S|^+}$ exists".

However, the tree T is on a set of higher type than S, and we will need to take some pains to prove the determinacy of the unraveled game from minimal assumptions.

In particular, it will be crucial for us that the unraveling tree T is one in which Player I's moves are from $S \times \mathcal{P}(S)$, whereas II's moves are in $S \times 2$.[1] Thus the unraveling tree is "one-sided," in the sense that only Player I's moves in the unraveled game are necessarily of higher type than those in the game on S. These one-sided trees are central to our arguments and, it seems, have not previously been isolated for uses in the literature, though they are produced in the constructions of [Hur93], [Mar]. Let us say a tree T in which I plays moves in X and II plays moves in Y is a *tree on* X, Y.

Theorem 3.2.2 (Martin [Mar], Hurkens [Hur93])**.** *Suppose S is a tree, and $\{K_n\}_{n\in\omega}$ is a collection of closed subsets in $[S]$. Then there is a covering $\langle T, \pi, \psi\rangle$ of S that simultaneously unravels the K_n, and so that T is a tree on $S \times \mathcal{P}(S), 2 \times S$.*

In the situation of this section, $S = \omega^{<\omega}$, so the unraveling tree is (isomorphic to) a tree on $\mathcal{P}(\omega), \omega$. Since II's moves are in ω, I's strategies are *countable* objects. The upshot is that as long as $H(\omega_1)$ exists, we can bound quantification over strategies for I by $H(\omega_1)$, thus keeping the complexity of formulae low.

We shall see that Π_1-RAP does not itself imply Σ^0_4-DET; rather, it only guarantees that for every Σ^0_4 set $A \subseteq \omega^\omega$, either I has a winning strategy in $G(A; \omega^{<\omega})$, or else there is a $\mathbf{\Delta}_1$-definable winning strategy for II. Since we aren't assuming Δ_1-Comprehension, it is possible that II's definable strategy will not be a set, and indeed, this scenario must occur in minimal models of Π_1-RAP. Thus the hypothesis that guarantees determinacy of all Σ^0_4 games is the existence of such a model. Note that this situation is in complete analogy with that of Σ^0_1-DET and models of KP (see [Bla72]).

The following is an abstract form of Davis's Lemma towards Σ^0_3-Determinacy, stated for

[1] Though this is true of Martin's original construction, it is not true of the first published version in [Mar85], where both players play quasistrategies for the starting tree S. The presentations of [Hur93], [Mos09], [Mar] incorporate an innovation due to Hurkens which avoids the use of quasistrategies, thus producing an unraveling tree with the "one-sidedness" described above. Our results may indicate that this avoidance of quasistrategies is essential to a fine calibration of strength.

trees on $\mathcal{P}(X), X$. It includes some technical assumptions on X since we do not in general assume the Axiom of Choice.

Lemma 3.2.3. *Let X be a transitive set so that $H(|X|^+)$ exists and Π_1-$\mathsf{RAP}(X)$ holds. Assume X can be wellordered and that $\mathsf{DC}_{\mathcal{P}(X)}$ holds; further assume that there is a uniform Δ_0-definable coding of $X^{<\omega}$ by elements of X, as well as elements of $\mathcal{P}(X)^\omega$ by $\mathcal{P}(X)$. Let T be a tree on $\mathcal{P}(X), X$ with $B \subseteq A \subseteq [T]$, where B is $\mathbf{\Pi}^0_2$ and A is Borel. Then for any $p \in T$, either*

1. *I has a winning strategy in $G(A; T_p)$, or*
2. *There is $W \subseteq T_p$ a quasistrategy for II so that*
   - *$[W] \cap B = \emptyset$;*
   - *I does not win $G(A; W)$.*

Proof. In the event (2) holds, we say p is *good* (relative to A, B, T). Notice that goodness of p is Σ_1 in parameters; for this, it is important that moves for II come from X, so that strategies for Player I, who plays in $\mathcal{P}(X)$, can be coded by subsets of X. We assume (2) fails for some p and show (1) must hold. So suppose $p \in T$ is not good; using our canonical coding we can reflect this Π_1 statement to an admissible set N with $X \in N$; since ω-sequences in $\mathcal{P}(X)$ are coded by elements of $\mathcal{P}(X)$, $\mathsf{DC}_{\mathcal{P}(X)}$ is Π_1, so we can further assume this choice principle holds in N. This move to an admissible set is our use of the main strength assumption of the lemma, namely Π_1-$\mathsf{RAP}(X)$.

So work in N. By an abuse of notation we prefer to refer to N's versions of the relevant objects A, B, T by the same names. Since B is $\mathbf{\Pi}^0_2$ in $[T]$, we have $B = \bigcap_{n\in\omega} D_n$, where each D_n is open in $[T]$; that is, there are sets $U_n \subseteq T$ so that for all n, $D_n = \{x \in T \mid (\exists i) x \restriction i \in U_n\}$. Adjusting the sets U_n as necessary, we may assume that each U_n contains only nodes with odd length $\geq n$.

We now define an operator $\Gamma : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows. Fix $Y \subseteq T$. For $n \in \omega$, let T^Y_n

be the tree $\{s \in T \mid (\forall i < |s|) s \upharpoonright i \notin U_n \setminus Y\}$. Then let

$$\Gamma(Y) = \{q \in T \mid (\exists n) \text{ I does not win } G(A; T_n^Y) \text{ from } q\}.$$

The reason we truncate the tree following minimal odd-length nodes in $U_n \setminus Y$ is that we will be considering an auxiliary game where Player I is trying to enter the open set D_n while avoiding Y; the auxiliary game on T_n^Y ends when a node in $U_n \setminus X$ is reached, and in this case, Player I is the winner, because the length of the node reached is odd by the way we defined U_n.

Note that the definition of Γ is such that "$x \in \Gamma(Y)$" is Δ_0 in parameters, by our remarks about quantification over strategies for I. Now define recursively sets $\mathcal{G}_\alpha \subseteq T$ for all $\alpha \in \mathrm{ON}$, by

- $\mathcal{G}_0 = \emptyset$;
- $\mathcal{G}_{\alpha+1} = \Gamma(\mathcal{G}_\alpha)$;
- $\mathcal{G}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{G}_\lambda$ for λ limit.

Note KP gives us enough transfinite recursion to ensure that the sets $\mathcal{G}_\alpha$ exist for all ordinals α; it is crucial for this that the operator Γ is Δ_0. Note also that Γ is a monotone operator (i.e., if $Y \subseteq Y'$ then $\Gamma(Y) \subseteq \Gamma(Y')$), and therefore the sets $\mathcal{G}_\alpha$ are increasing.

Claim 3.2.4. *If $q \in T$ belongs to $\mathcal{G}_\alpha$ for some α, then q is good.*

Proof. The proof is by induction on α. So suppose q, α are such that α is least with $q \in \mathcal{G}_{\alpha+1}$. Let n_0 be the least witness to this, so that I doesn't win $G(A; T_{n_0}^{\mathcal{G}_\alpha})$ from q. We describe a quasistrategy W for II at q as follows: play according to II's non-losing quasistrategy in $G(A; T_{n_0}^{\mathcal{G}_\alpha})$. If at any point we reach a position $r \in U_{n_0}$, then because we are inside II's non-losing quasistrategy, we must have $r \in \mathcal{G}_\alpha$. By inductive hypothesis, r is good; so switch to W^r witnessing goodness of r. Note that W exists by admissibility (the assertion "W^r witnesses goodness of r in T" is Δ_0 in parameters).

We claim this W witnesses goodness of q. For let $x \in [W]$; if $x \restriction i \notin U_{n_0}$ for all $i \in \omega$, then by openness of D_{n_0} we have $x \notin D_{n_0}$, so $x \notin B$. If $x \restriction i \in U_{n_0}$ for some i, then we must have switched to $W^{x \restriction i}$ at the least such i; since $[W^{x \restriction i}] \cap B = \emptyset$, we have $x \notin B$.

We need to show I doesn't win $G(A; W)$. So suppose σ is a strategy for I in W. If all positions r compatible with σ satisfy $r \notin U_{n_0}$, then σ must stay inside II's non-losing quasistrategy for $G(A; T_{n_0}^{\mathcal{G}_\alpha})$. This implies that σ can be extended to a winning strategy for I in $G(A; T_{n_0}^{\mathcal{G}_\alpha})$ from q: Play by σ so long as II stays inside W, and if II strays to r outside W switch to a winning strategy for I in $G(A; T_r)$, which exists by definition of W. (Note that the existence of this extension of σ requires a use of $\mathsf{DC}_{\mathcal{P}(X)}$, to choose the strategies Player I switches to.) But this contradicts the assumptions on q and n_0. If on the other hand $r \in U_{n_0}$ for some r compatible with σ, then $r \in \mathcal{G}_\alpha$, so by inductive hypothesis and definition of the quasistrategy W, W_r witnesses the goodness of r. But that means I doesn't have a winning strategy in $G(A; W_r)$; again, σ cannot be winning for I. $\square$

Since we have (in N) that p is not good, we get $p \notin \mathcal{G}_\infty := \bigcup_{\alpha \in \mathrm{ON}} \mathcal{G}_\alpha$. Note that since we do not have Σ_1-Comprehension in N, $\mathcal{G}_\infty$ may not be a set; indeed, if $\mathcal{G}_\alpha \neq \mathcal{G}_{\alpha+1}$ for all α, then $\mathcal{G}_\infty$ must be a proper class, by admissibility.

Claim 3.2.5. $\Gamma(\mathcal{G}_\infty) = \mathcal{G}_\infty$.

Proof. Since $\mathcal{G}_\infty$ needn't be a set, we should say what we mean by $\Gamma(\mathcal{G}_\infty)$: This is the class of $q \in [T]$ so that for some n, I doesn't win $G(A; T_n^{\mathcal{G}_\infty})$ from q. Expanding our definition, this is the same as

$$\begin{aligned}
&(\exists n \in \omega)(\forall \sigma \in H(|X|^+)) \text{ if } \sigma \text{ is a partial strategy for I in } T_q, \text{ then} \\
&\quad (\exists s \in T_q) s \text{ is compatible with } \sigma, \text{ and is a win for II or not in } \mathrm{dom}(\sigma), \text{ or} \\
&\quad (\exists x \in [T_q]) x \text{ is compatible with } \sigma, x \notin A, \text{ and } (\forall i)(x \restriction i \in \mathcal{G}_\infty \cup (T \setminus U_n))
\end{aligned}$$

This statement is Σ_1, because $x \restriction i \in \mathcal{G}_\infty$ is Σ_1 (it is the statement $(\exists \alpha) x \restriction i \in \mathcal{G}_\alpha$, and the relation $s \in \mathcal{G}_\alpha$ is $\boldsymbol{\Delta}_1$ as a relation on $T \times \mathrm{ON}$), and all other quantifiers are bounded.

Suppose $q \in \Gamma(\mathcal{G}_\infty)$. Then by Σ_1-Collection (recall we are working in the admissible set N), there is a bound α on ordinals witnessing the "$x \upharpoonright i \in \mathcal{G}_\infty$" clause in the above statement, for various σ, x. That is, $q \in \Gamma(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\infty$; so $\Gamma(\mathcal{G}_\infty) \subseteq \mathcal{G}_\infty$. The reverse inclusion is trivial. $\square$

Using this stabilization of the operator Γ we describe a winning strategy for I in $G(A; T_p)$. Since $p \notin \mathcal{G}_\infty = \Gamma(\mathcal{G}_\infty)$, we have for all n that I wins $G(A; T_n^{\mathcal{G}_\infty})$ at p. Let σ_0 witness this for $n_0 = 0$. Have I play according to σ_0 from the initial position. Now suppose strategies σ_i, n_i have been defined; if at any point in the strategy σ_i I reaches a position p_i with $p_i \in U_{n_i} \setminus \mathcal{G}_\infty$, then I wins $G(A; T_n^{\mathcal{G}_\infty})$ from p_i for every n, since $p_0 \notin \mathcal{G}_\infty$. Let n_{i+1} be least so that no initial segment of p_i is in $U_{n_{i+1}}$ (such exists because of the way we defined the U_n). Let σ_{i+1} be winning for I in $G(A; T_{n_{i+1}}^{\mathcal{G}_\infty})$, and have I continue according to this strategy. Any infinite play against the strategy σ so defined must either enter the sets $U_0, U_1, U_2, \ldots$ one by one, thus belonging to $B = \bigcap_{n\in\omega} D_n$, or else the play avoids $U_n \setminus \mathcal{G}_\infty$ for some least n; but then the play is compatible with σ_n, so must belong to A.

Some remarks regarding definability are in order. The inductive construction of Player II's goodness-witnessing quasistrategies was uniform, so no choice was required. The same cannot be said of Player I's winning strategy; at the very least, $\mathsf{DC}_{\mathcal{P}(X)}$ is needed to select the various σ_n, and even if N possesses a Δ_1-definable wellordering of $H(|X|^+)$, the definition of σ will (in general) be Σ_2, so the strategy needn't be a set in N. However, we are able to define the strategy over N from the point of view of our model of Π_1-$\mathsf{RAP}(X)$, using $\mathsf{DC}_{\mathcal{P}(X)}$.

We claim that the strategy σ we have described is winning (in V) in $G(A; T_p)$ (where now T is the true unraveling tree). Note first that σ really is a strategy in the true T, since II's moves are in X, and so N's version of T is closed under moves by II.

Suppose $x \in [T_p]$ is according to σ. Then $x \upharpoonright i \in N$ for all i. If for some n the play never enters $U_n \setminus \mathcal{G}_\infty$, then x must be according to some $\sigma_n \in N$. We have N is admissible, A is Borel, and σ_n is winning in $G(A; T_n^{\mathcal{G}_\infty})$ in N. So by absoluteness, $x \in A$.

So suppose x enters $U_n \setminus \mathcal{G}_\infty$ for every n. Then $x \in \bigcap_{n\in\omega} D_n = B \subseteq A$. Either way, we have any play x compatible with σ is in A, so we have that σ is winning in $G(A; T_p)$. That is, case (1) of the lemma holds. $\square$

Theorem 3.2.6. *Suppose M is a transitive model of $H(\omega_1)$ exists + Π_1-RAP, and that M has a wellordering of its reals. Let $A \subseteq \omega^\omega$ be Σ^0_4. Then either I wins $G(A; \omega^{<\omega})$ with a strategy in M, or II has a winning strategy in $G(A; \omega^{<\omega})$ that is $\boldsymbol{\Delta}_1$-definable over M.*

Proof. Working inside M, let $A \subseteq \omega^\omega$ be Σ^0_4, and using Theorem 3.2.2, let $\langle T, \pi, \psi \rangle$ be the simultaneous unraveling of all Π^0_1 sets; note the unraveling is definable over $H(\omega_1)$ with reference to some wellordering of $\mathcal{P}(\omega)$, so belongs to M. By a standard coding, T may be regarded as a tree on $\mathcal{P}(\omega), \omega$. Let $\bar{A} = \pi^{-1}(A)$. Then $\bar{A}$ is $\boldsymbol{\Sigma}^0_3$, and we have $\bar{A} = \bigcup_{k \in \omega} B_k$ for some family $\{B_k \mid k \in \omega\}$ of $\boldsymbol{\Pi}^0_2$ subsets of $[T]$.

If I wins $G(\bar{A}; T)$, say with σ, then $\psi(\sigma)$ is easily seen to be a winning strategy in M for I in $G(A; \omega^{<\omega})$ that continues to be winning in V. So suppose I does not win $G(\bar{A}; T)$. By Lemma 3.2.3, there is a quasistrategy $W_0^\emptyset$ for II witnessing the goodness of $\emptyset$ relative to $B_0, \bar{A}$ in T. We may assume this $W_0^\emptyset$ is obtained from the uniform construction of goodness-witnessing quasistrategies for II, as in the proof of Lemma 3.2.3. We may furthermore assume that $W_0^\emptyset$ is non-losing for II; that is, I doesn't win $G(\bar{A}; (W_0^\emptyset)_p)$ for any $p \in W_0^\emptyset$.

Suppose now that we have some fixed quasistrategy $W_k^p \subseteq T$ for II in T_p, with $p \in W_k^p$ a position of length $2k$, and that I doesn't win $G(\bar{A}; W_k^p)$. For any $q \in W_k^p$ of length $2k+2$, let W_{k+1}^q be the (uniformly constructed) goodness-witnessing quasistrategy at q guaranteed by applying Lemma 3.2.3 (to $\bar{A}, B_{k+1}, W_k^p$).

We then define a quasistrategy for II in $G(\bar{A}; T)$ by inductively taking the common refinement of the W_k^p. That is, at positions p in T of length $2k$, if $p^\frown\langle a \rangle \in W_k^p$, then those moves b for II at $p^\frown\langle a \rangle$ are permitted exactly when $q^\frown\langle a, b \rangle \in W_k^p$. Let W be the quasistrategy for II so obtained.

Notice that W is Δ_1-definable over M. Furthermore, if $x \in [W]$, then $x \in [W_k^{x \restriction 2k}]$ for all k; since each $W_k^{x \restriction k}$ witnesses goodness of $x \restriction k$ relative to B_k, we have $x \notin B_k$ for all k, hence $x \notin A$.

A strategy τ for II is easily obtained by refining W, choosing a single successor node at each position of odd length (recall II's moves are in ω). Now, τ need not be an element of M, so it's not clear that we can even apply the unraveling map to τ. However, *all of τ's finite parts belong to M*, in the sense that for any position p of even length reachable by τ, the set

$$\{\langle q, \tau(q)\rangle \mid q = p^\frown\langle a\rangle \in T \text{ for some } a \in \mathcal{P}(\omega)\}$$

belongs to M (because $W_k^p \in M$, and M satisfies Δ_0-Comprehension). Since the unraveling maps are continuous, these are precisely the fragments of the strategy τ that are required to define $\psi(\tau)$ and carry out the proof of the lifting property (ii) of the unraveling, and this property holds *even of plays $x \notin M$*. So, applying the unraveling map, we have that $\tau' = \psi(\tau)$ is a $\mathbf{\Delta}_1$-definable strategy for Player II (boldface because the wellordering of $\mathcal{P}(\omega)$ is a necessary parameter in defining the unraveling map ψ).

We claim τ' wins $G(A;\omega^{<\omega})$ for II in V. Suppose towards a contradiction that $x \in A$ is a play in ω^ω compatible with τ'. Then using the lifting property of the unraveling, we have a play $y \in [T]$ (though possibly $\notin M$) so that y is compatible with τ, and $\pi(y) = x$ (in particular, $y \in \pi^{-1}(A) = \bar{A}$). Then $y \in B_k$ for some k. Now σ on $T_{y\restriction 2k}$ is a refinement of $W_k^{y\restriction 2k}$, so we must have $y \in [W_k^{y\restriction 2k}]$. But inside M, we have $[W_k^{y\restriction 2k}] \cap B_k = \emptyset$; in particular, M thinks the tree of attempts to find a branch through $W_k^{y\restriction 2k}$ in B_k is wellfounded, hence ranked in M, by Claim 3.1.4. By absoluteness, this contradicts $y \in B_k$. □

The uses of Choice in the previous theorems can be removed in a number of ways. One is to discuss "determinacy mod choice" (see [Mos09]); more typical is to relativize to L and use absoluteness. By Proposition 3.1.7, this can be done. We obtain

Theorem 3.2.7. *If there is a transitive model of Π_1-RAP, then Σ^0_4-DET holds.*

3.3 The lower bound

Let (T) denote the sentence

$$\mathcal{P}(\omega) \text{ exists, and every tree } T \text{ on } \mathcal{P}(\omega) \text{ is either ranked or illfounded.}$$

Let θ be the least ordinal so that $L_\theta \models (\mathsf{T})$. Note that $L_\theta \models$"ω_1 is the largest cardinal," and $\rho_1^{L_\theta} = \omega$. The results of the previous two sections show

Theorem 3.3.1. *Let $A \subseteq \omega^\omega$ be Σ^0_4. Then either I wins $G(A; \omega^{<\omega})$ as witnessed by a strategy $\sigma \in L_\theta$, or else II has a winning strategy τ that is Δ_1-definable over L_θ.*

Corollary 3.3.2. *$L_{\theta+1} \models \Sigma^0_4$-DET.*

In particular, the existence of θ implies Σ^0_4-DET. In this section we prove the converse.

Theorem 3.3.3 (KPI_0)**.** *Σ^0_4-DET implies θ exists.*

Applying the Theorem in $L_{\omega_1^{L_\theta}}$, we have failure of Σ^0_4-DET in L_θ. Thus Theorem 3.3.1 is optimal:

Corollary 3.3.4. *$L_\theta \not\models \Sigma^0_4$-DET.*

Note that in the following sections, we make use of the fine structure for levels of the J-hierarchy. Of particular importance are the Σ_1-projectum of L_α, denoted $\rho_1^{L_\alpha}$; the ultimate projectum, $\rho_\omega^{L_\alpha}$; and the uniformly definable Σ_1-Skolem function, $h_1^{L_\alpha}$ (the reader unfamiliar with these notions should consult [SZ10]).

Proof of Theorem 3.3.3. Work in KPI_0, assume Σ^0_4-DET, and towards a contradiction, that θ does not exist; that is, there is no α so that $L_\alpha \models (\mathsf{T})$. Since KPI_0 holds in V, it holds also in L, and by the assumption that θ does not exist, all ordinals must be countable in L (since otherwise, the least admissible set containing ω_1^L as an element would satisfy (T)); in particular, there are unboundedly many α so that L_α is admissible and has ultimate projectum ω.

We define a game G with Π^0_4 winning condition, and argue that G cannot be determined. The game is essentially a Friedman game where the players compete to play models of $V = L + \mathsf{KP}$ with largest possible wellfounded part. It proceeds as follows: players I and II play reals $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively, that are the characteristic functions of consistent theories extending

$$V = L + \mathsf{KP} + \text{all sets are countable} + \rho_\omega = \omega + (\forall\alpha)L_\alpha \not\models (\mathsf{T})\,.$$

The first player to break this rule loses. Otherwise, the theories uniquely determine *term models* $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ minimal satisfying $f_{\mathrm{I}}, f_{\mathrm{II}}$, respectively. If one of the player's models has nonstandard ω, then that player loses; if both have nonstandard ω, then I loses.

The remainder of our winning condition will essentially assert that $\mathcal{M}_{\mathrm{II}}$ does not have wellfounded part strictly larger than that of $\mathcal{M}_{\mathrm{I}}$; failure of (T) in all initial segments of $\mathcal{M}_{\mathrm{II}}$ is what enables us to do this in a Π^0_4 manner.

In order to make the condition easier to parse, we will typically quantify over sets such as $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$ rather than ω, and will frequently compress the Π^0_1 condition "i codes a real $x \in \mathcal{M}_{\mathrm{I}}$ and j codes a real $y \in \mathcal{M}_{\mathrm{II}}$ so that $x = y$" as simply "$x = y$", with the hope that this will make the intended meaning clearer. For example, we write

$$(\forall x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}})x = y$$

regarding this as an abbreviation for

$$\begin{aligned}(\forall i)(\exists j)(\forall n)[f_{\mathrm{II}}(\text{``}\exists! u\varphi_i(u) \wedge u \subseteq \omega\text{''}) = 1 \rightarrow (f_{\mathrm{I}}(\text{``}\exists! u\varphi_j(u) \wedge u \subseteq \omega\text{''}) = 1 \wedge \\ (f_{\mathrm{II}}(\text{``}\forall u\varphi_i(u) \rightarrow n \in u\text{''}) = 1 \leftrightarrow f_{\mathrm{I}}(\text{``}\forall u\varphi_j(u) \rightarrow n \in u\text{''}) = 1))],\end{aligned}$$

where $\langle \varphi_i \mid i \in \omega\rangle$ is some fixed recursive enumeration of formulae. We condense this further as "$\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$", which we see is Π^0_3 in $f_{\mathrm{I}}, f_{\mathrm{II}}$. In what follows, we typically omit mentioning $f_{\mathrm{I}}, f_{\mathrm{II}}$, which are allowed parameters in all our complexity calculations, and simply say the relation is Π^0_3.

Claim 3.3.5. *Suppose $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models, and let $x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$. Then*

$$\text{“}(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(y \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \wedge (y <_L x)^{\mathcal{M}_{\mathrm{II}}})\text{”}$$

is Σ^0_3.

Proof. The statement that there exists a real in $\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}}$ is Σ^0_3, as the previous discussion shows. Once a code for such a y has been fixed, the question of whether $(y <_L x)^{\mathcal{M}_{\mathrm{II}}}$ holds is decided by the theory of $\mathcal{M}_{\mathrm{II}}$, hence recursive in f_{II}. □

This claim is enough to give a Π^0_4 condition that holds whenever there are reals in $\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}}$ and no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least such. To handle the remaining possibilities, we will need to look closely at the manner in which (T) fails in levels of $\mathcal{M}_{\mathrm{II}}$.

Definition 3.3.6. Recall for $x \in L$ that $\mathrm{rank}_L(x)$ is defined as the least ρ so that $x \in J_{\rho+1}$. Working in $\mathcal{M}_{\mathrm{II}}$, suppose $x \in \mathcal{P}(\omega)$, and that $L_{\mathrm{rank}_L(x)} \models \omega_1$ exists. Inductively define

$$\delta(0, x) = \mathrm{rank}_L(x)$$

$$\delta(k+1, x) = \begin{cases} \delta \text{ least s.t. } J_{\delta(k,x)} \models \text{“}\omega_1 \text{ exists and} & \\ (\exists T \in J_{\delta+1})\ T \text{ is a tree on } \mathcal{P}(\omega) \text{ that} & \\ \text{is neither ranked nor illfounded,”} & \text{if such exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We stress that this definition is internal to $\mathcal{M}_{\mathrm{II}}$. Thus if $\mathcal{M}_{\mathrm{II}}$ has standard ω and $\delta(0, x)$ exists, then $\langle \delta(k, x) \rangle$ is a strictly descending sequence of ordinals, so is finite. The fact that (T) fails in every level of $\mathcal{M}_{\mathrm{II}}$ implies that the smallest element of $\langle \delta(k, x) \rangle$ is $\omega_1^{J_{\delta(0,x)}}$.

Notice that if $\mathcal{M}_{\mathrm{II}}$ is illfounded with $\omega_1^{J_{\delta(0,x)}} < \mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \subseteq \delta(0, x)$, then there is some unique k so that $\delta(k+1, x)$ is wellfounded but $\delta(k, x)$ is nonstandard. By the defining property of $\delta(k+1, x)$, there is some tree $T \in L_{\delta(k+1,x)+1} \subset L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$ that is neither ranked nor illfounded in $L_{\delta(k,x)}$, hence neither ranked nor illfounded in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$. The latter set is admissible, so T is in fact illfounded, and a branch through T is definable over $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$. The main use of this fact is the following Lemma:

Lemma 3.3.7. *There is a Σ^0_3 relation $R(k,\gamma,x)$ such that if $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models obtained from a play of the game according to the rules of G given thus far, $\mathcal{M}_{\mathrm{I}}$ is wellfounded, and x is the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element of $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$, then we have the following:*

(A) $(\forall k \in \omega)(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}) R(k,\gamma,x) \to \delta(k+1,x)$ *is standard;*

(B) $(\forall k \in \omega)$ *if* $\delta(k,x)$ *is nonstandard and* $\delta(k+1,x)$ *is wellfounded, then* $(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})[R(k,\gamma,x) \leftrightarrow (\gamma < \delta(k,x))^{\mathcal{M}_{\mathrm{II}}} \wedge \gamma$ *is nonstandard*$]$.

Note that when we assert "R is a Σ^0_3 relation on $\omega \times {\mathcal{M}_{\mathrm{II}}}^2$" this should be understood to mean that the corresponding relation $\tilde{R} \subseteq \omega^3 \times (\omega^\omega)^2$ (on the codes) is Σ^0_3 (as a relation on $i, j, k, f_{\mathrm{I}}, f_{\mathrm{II}}$).

We shall give the proof of Lemma 3.3.7 shortly. For now, we use the Lemma to finish giving our winning condition. Suppose I, II play consistent theories $f_{\mathrm{I}}, f_{\mathrm{II}}$ that yield term models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively, both with wellfounded ω. I wins if

$$(\exists x \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}) x \text{ codes a wellorder of } \omega \text{ in } \mathcal{M}_{\mathrm{II}}, \text{ but not in } \mathcal{M}_{\mathrm{I}}.$$

(Notice this immediately implies illfoundedness of $\mathcal{M}_{\mathrm{II}}$.) Otherwise, I wins just in case the following holds:

1. $(\forall x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})$ if $x \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$, then
   (a) $(\exists y \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(y \notin \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \wedge (y <_L x)^{\mathcal{M}_{\mathrm{II}}})$, or
   (b) $(\exists k,\gamma) R(k,\gamma,x)$
   $\wedge (\forall k,\gamma)[R(k,\gamma,x) \to (\exists k',\gamma') R(k',\gamma',x) \wedge \langle k',\gamma' \rangle <_{\mathrm{Lex}} \langle k,\gamma \rangle]$,

   and
2. $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{M}_{\mathrm{I}}$ implies
   (a) $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, or
   (b) $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$.

Here $<_{\text{Lex}}$ is the lexicographic order on the product $(\omega, \in) \times (\text{ON}^{\mathcal{M}_{\text{II}}}, \in^{\mathcal{M}_{\text{II}}})$. Note if (1b) holds, we immediately have that $\mathcal{M}_{\text{II}}$ is illfounded.

Claim 3.3.8. *Assuming θ does not exist, I does not win G.*

Proof. Suppose towards a contradiction that σ is a winning strategy for I in G. Applying Shoenfield absoluteness (which is provable in $\mathbf{\Pi}^1_1$-CA_0 [Sim09], hence in KPI_0), we may assume $\sigma \in L$. Let α be the least admissible ordinal so that $\sigma \in L_\alpha$. Then L_α projects to ω and satisfies "all sets are countable". Since θ does not exist, we also have L_α satisfies $(\forall \xi) L_\xi \not\models (\mathsf{T})$. Let f_{II} be the theory of L_α, so that $\mathcal{M}_{\text{II}} = L_\alpha$. Let $\mathcal{M}_{\text{I}}$ be the model that σ replies with.

Since $\mathcal{M}_{\text{II}}$ is wellfounded, there cannot be any real in $\mathcal{M}_{\text{II}} \setminus \mathcal{M}_{\text{I}}$, since the $<_L$-least such would be a witness to failure of (1). So $\mathcal{P}(\omega)^{\mathcal{M}_{\text{II}}} \subseteq \mathcal{M}_{\text{I}}$. In particular, $\sigma \in \mathcal{M}_{\text{I}}$, and we can't have $\text{Th}(\mathcal{M}_{\text{II}}) \in \mathcal{M}_{\text{I}}$, since then $\text{Th}(\mathcal{M}_{\text{I}}) = \sigma * f_{\text{II}} \in \mathcal{M}_{\text{I}}$, contradicting the fact $\mathcal{M}_{\text{I}}$ projects to ω. So (2a) fails, and (2b) must hold; in particular, $\mathcal{P}(\omega)^{\mathcal{M}_{\text{I}}} = \mathcal{P}(\omega)^{\mathcal{M}_{\text{II}}}$. This implies $\mathcal{M}_{\text{I}} = \mathcal{M}_{\text{II}}$, since both models satisfy "all sets are countable", and we again have the contradiction $\text{Th}(\mathcal{M}_{\text{I}}) \in \mathcal{M}_{\text{I}}$, since in this case f_{II} is just copying the play by σ. We have that $(1) \wedge (2)$ must fail, so σ cannot be a winning strategy for I. $\square$

Claim 3.3.9. *Assuming θ does not exist, II does not win G.*

Proof. As before, assume for a contradiction that $\tau \in L$ is a winning strategy for II in G. Let α be admissible with $\tau \in L_\alpha$ and L_α projecting to ω; again, $L_\xi \not\models (\mathsf{T})$ for all $\xi \in \alpha$, since θ does not exist. Let f_{I} be $\text{Th}(L_\alpha)$, so $\mathcal{M}_{\text{I}} = L_\alpha$, and suppose τ replies with model $\mathcal{M}_{\text{II}}$.

We claim $\text{wfo}(\mathcal{M}_{\text{II}}) \leq \alpha$. For otherwise, we would have $\tau \in \mathcal{M}_{\text{I}} \in \mathcal{M}_{\text{II}}$, so that $\text{Th}(\mathcal{M}_{\text{II}}) = \tau * \text{Th}(\mathcal{M}_{\text{I}}) \in \mathcal{M}_{\text{II}}$, a contradiction to the fact that $\mathcal{M}_{\text{II}}$ projects to ω.

Suppose $\text{wfo}(\mathcal{M}_{\text{II}}) = \alpha$. If $\mathcal{M}_{\text{II}}$ is wellfounded, then $\mathcal{M}_{\text{I}} = \mathcal{M}_{\text{II}}$, so that (1) holds vacuously and (2) holds via (2b), a contradiction to τ being winning for II. So $\mathcal{M}_{\text{II}}$ is illfounded. By overspill, there are countable codes for nonstandard ordinals in $\mathcal{M}_{\text{II}}$, and there is no $<_L^{\mathcal{M}_{\mathrm{II}}}$ least such; since II wins the play, none of these codes can belong to $\mathcal{M}_{\mathrm{I}}$. But then (1) holds via (1a), and (2) holds vacuously, again a contradiction.

So we must have $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) < \alpha$. Again $\mathcal{M}_{\mathrm{II}}$ cannot be wellfounded, for then (1) holds vacuously and (2) holds via (2a). Since $\mathcal{M}_{\mathrm{II}}$ is illfounded, there is some $x \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \backslash \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$. Since (1a) fails, we can let x be the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least such. We must have $L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)} \models$ "ω_1 exists", and by minimality of x, this ω_1 is standard and contained in $\mathcal{M}_{\mathrm{I}}$. It follows that $\delta(0, x)$ exists, and there is a unique k so that $\delta(k, x)$ is nonstandard and $\delta(k+1)$ is wellfounded. In particular, $R(k, \gamma, x)$ holds for any nonstandard $\gamma < \delta(k, x)$ by (B) of Lemma 3.3.7. And by (A) of the same Lemma, $R(k', \gamma', x)$ cannot hold for any k', γ' with $k' < k$. But then (1b) holds, contradicting that II wins the play. $\square$

It is easy to check by computations similar to those we have given that G has a Π^0_4 winning condition. Since G is non-determined when θ doesn't exist, this completes the proof of Theorem 3.3.3, modulo the proof of Lemma 3.3.7. $\square$

Proof of Lemma 3.3.7. Recall that for any ω-model M of $V = L$, h_1^M is the uniformly definable Σ_1 Skolem function, $h_1^M : \omega \times (\mathrm{ON}^M)^{<\omega} \rightharpoonup M$. Notice that if M also satisfies "ω_1 exists" $\wedge (\forall \alpha) L_\alpha \not\models (\mathsf{T})$, then we have

$$M = h_1^M[(\omega_1^M + 1)^{<\omega}].$$

To see this, suppose for a contradiction that $H = h_1[(\omega_1^M + 1)^{<\omega}]$ is a proper subset of M. Failure of (T) in initial segments of M implies ω_1^M is the largest cardinal of M; since the $<_L$-least surjection of ω_1^M onto x is Σ_1-definable in the parameter x, we have that H contains such a surjection for each $x \in H$, and since $\omega_1^M \subseteq H$, we have that H is transitive in M; it follows that $H = J_\alpha^M$ for some $\alpha \in \mathrm{ON}^M$.

Let $T \in H$ be a tree. If T is ranked in M, then the same holds in H since $H \prec_1 M$. Otherwise $\{s \in T \mid T_s \text{ is not ranked in } H\}$ is a subtree of T with no terminal nodes, and this belongs to M by Δ_0-Comprehension. It follows that there is a branch through T in

M, so such must belong to H, again by Σ_1-elementarity. But then $H = J^M_\alpha$ is a model of $V = L + (\mathsf{T})$, contradicting our assumption.

Thus in the models we work with, we can talk about uncountable objects by taking images of countable ordinals by h_1.

We define the Σ^0_3 relation $R(k, \gamma, x)$ to be the conjunction of the following:

1. $\mathcal{M}_{\mathrm{II}} \models$ "$\delta(0, x)$ exists and $(\delta(k+1, x) < \gamma < \delta(k, x))$";
2. $(\exists \beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}})$
   (a) $(J_\beta \models \mathsf{KP} +$ "ω_1 exists"$)^{\mathcal{M}_{\mathrm{I}}}$
   (b) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})(z \in J_\gamma)^{\mathcal{M}_{\mathrm{II}}} \to (z \in J_\beta)^{\mathcal{M}_{\mathrm{I}}}$
   (c) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
   If $\mathcal{M}_{\mathrm{I}} \models$ "z codes $\vec{\xi}, \vec{\eta} \in (\omega_1^{J_\beta})^{<\omega}$ such that $h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta}), h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta})$ exist", then $\mathcal{M}_{\mathrm{II}} \models$ "z codes $\vec{\xi}', \vec{\eta}' \in (\omega_1^{J_\gamma})^{<\omega}$ such that $h_1^{J_\gamma}(\vec{\xi}', \omega_1^{J_\gamma}), h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma})$ exist", and $(h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta}) \in h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta}))^{\mathcal{M}_{\mathrm{I}}}$ iff $(h_1^{J_\gamma}(\vec{\xi}', \omega_1^{J_\gamma}) \in h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma}))^{\mathcal{M}_{\mathrm{II}}}$;
   (d) $(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
   If $\mathcal{M}_{\mathrm{I}} \models$ "z codes $\vec{\xi}, \vec{\eta} \in \omega_1^{J_\beta}$ such that $h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta})$ exists"
   and $\mathcal{M}_{\mathrm{II}} \models$ "z codes $\vec{\xi}', \vec{\eta}' \in \omega_1^{J_\gamma}$ such that $h_1^{J_{\rho+1}}(\vec{\xi}', \omega_1^{J_\gamma})$ exists, where $\rho = \max\{\omega_1^{J_\gamma}, \mathrm{rank}_L(h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma}))\}$", then $\mathcal{M}_{\mathrm{I}} \models$ "$h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta})$ exists";
   (e) $(\exists t \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
      i. $\mathcal{M}_{\mathrm{I}} \models$ "t codes $\vec{\tau} \in (\omega_1^{J_\beta})^{<\omega}$ with $h_1^{J_\beta}(\vec{\tau}, \omega_1^{J_\beta})$ a tree on $P(\omega)^{J_\beta}$, that is neither ranked nor illfounded in J_β";
      ii. $\mathcal{M}_{\mathrm{II}} \models$ "t codes $\vec{\tau}' \in (\omega_1^{J_\gamma})^{<\omega}$ with $h_1^{J_\gamma}(\vec{\tau}', \omega_1^{J_\gamma})$ a tree on $\mathcal{P}(\omega)^{J_\gamma}$ that witnesses the defining property of $\delta(k+1, x)$";
      iii. $(\forall s \in \mathcal{P}(\omega)^{<\omega} \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})$
      $(s \in h_1^{J_\beta}(\vec{\tau}, \omega_1^{J_\beta}))^{\mathcal{M}_{\mathrm{I}}} \leftrightarrow (s \in h_1^{J_\gamma}(\vec{\tau}', \omega_1^{J_\gamma}))^{\mathcal{M}_{\mathrm{II}}}$.

Before proceeding with the proof, we feel obligated to make a few remarks about the intended meaning of the relation $R(k, \gamma, x)$. Suppose x is $<_L^{\mathcal{M}_{\mathrm{II}}}$-least in $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$. $R(k, \gamma, x)$

is meant to hold just in case there is an admissible level J_β of $\mathcal{M}_{\mathrm{I}}$ in which ω_1 exists (2a) so that any real in $\mathcal{M}_{\mathrm{I}} \cap J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ belongs to J_β (2b), and such that the Σ_1-hull of $X \cup \{\omega_1^{J_\beta}\}$, where X is the set of ordinals of $\mathcal{M}_{\mathrm{I}}$ coded by reals in $\mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}$, is $\in$-embeddable (2c) onto an initial segment of $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ (2d); by (e), this initial segment contains $\delta(k+1, x)$ (thus if $\mathcal{M}_{\mathrm{I}}$ is wellfounded and $R(k, \gamma, x)$ holds, we must have that $\delta(k+1, x)$ is standard).

The hope is that β will be equal to wfo($\mathcal{M}_{\mathrm{II}}$), which then implies γ is illfounded; but because we can only refer to the hull of X (rather than all of J_β) embedding into $\mathcal{M}_{\mathrm{II}}$ (since the latter statement would be Π_3^0), it can happen that β is strictly larger than wfo($\mathcal{M}_{\mathrm{II}}$), in which case γ could be standard. This is where the trees come in: we ensure γ is nonstandard by taking (the unique) k so that $\delta(k, x)$ is nonstandard. Then $\delta(k+1, x)$ is standard by the above remarks, and if $\gamma \in \delta(k, x)$ is nonstandard, there will be an illfounded tree in $\mathcal{M}_{\mathrm{I}} \cap (J_\gamma)^{\mathcal{M}_{\mathrm{II}}}$ whose illfoundedness has not yet been witnessed inside $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$. Then wfo($\mathcal{M}_{\mathrm{II}}$) is uniquely identified as that level of $\mathcal{M}_{\mathrm{I}}$ at which a branch through this tree is first constructed.

Note first, though, that this relation is Σ_3^0. The main thing is, we repeatedly used expressions of the form

$$(\forall z \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}})(\text{Boolean comb. of statements internal to } \mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}});$$

c.f. (2b,c,d) and (iii) in (2e). These should be regarded as abbreviations for

$$(\forall z \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}})(\forall z' \in \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}})(z' = z \to (\text{Boolean comb...}))$$

which is clearly Π_2^0 (recall "$z' = z$" is Π_1^0 and internal statements are recursive in the codes). Re-envisioning the statement of $R(k, \gamma, x)$ appropriately, it is now easy to check that it is Σ_3^0.

We now prove that $R(k, \gamma, x)$ is as desired. So let $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ and $x \in \mathcal{M}_{\mathrm{II}}$ satisfy the hypotheses of Lemma 3.3.7, namely, that $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models projecting to ω and satisfying KP, $V = L$, "all sets are countable" and "θ does not exist"; that $\mathcal{M}_{\mathrm{I}}$ is wellfounded, and that x is minimal in $\mathcal{P}(\omega) \cap (\mathcal{M}_{\mathrm{II}} \setminus \mathcal{M}_{\mathrm{I}})$. To prove (A), suppose $k \in \omega$ and $\gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$ are such that $R(k, \gamma, x)$ holds. We need to show $\delta(k+1, x)$ is wellfounded.

Let $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$ witness (2) in the definition of R, with $t \in \mathcal{P}(\omega) \cap J_\beta^{\mathcal{M}_{\mathrm{I}}} \cap J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ a witness to (2e). Then let $T \in J_\beta^{\mathcal{M}_{\mathrm{I}}}$, $T' \in J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ be the trees whose existence is asserted in clauses (i),(ii) of (2e). There is a real y computable from t so that y codes tuples $\vec{\eta}$ in $\mathcal{M}_{\mathrm{I}}$ and $\vec{\eta}' \in \mathcal{M}_{\mathrm{II}}$ with $h_1^{J_\beta}(\vec{\eta}, \omega_1^{J_\beta}) = \mathrm{rank}_L(T)$ in $\mathcal{M}_{\mathrm{I}}$, and $h_1^{J_\gamma}(\vec{\eta}', \omega_1^{J_\gamma}) = \mathrm{rank}_L(T') = \delta(k+1, x)$ in $\mathcal{M}_{\mathrm{II}}$.

Suppose towards a contradiction that $\mathcal{M}_{\mathrm{II}}$ is illfounded below $\delta(k+1,x)$. Fix a sequence $\langle \alpha_n \mid n \in \omega \rangle$ of ordinals α_n of $\mathcal{M}_{\mathrm{II}}$ so that $\alpha_0 = \delta(k+1,x)$ and $(\alpha_{n+1} \in \alpha_n)^{\mathcal{M}_{\mathrm{II}}}$ for all n. Inductively, fix tuples $\vec{\xi}'_n \in \omega_1^{J_\gamma}$ in $\mathcal{M}_{\mathrm{II}}$, as follows: $\vec{\xi}'_0 = \vec{\eta}'$. If $\vec{\xi}'_n$ is fixed so that $h_1^{J_\gamma}(\vec{\xi}'_n, \omega_1^{J_\gamma}) = \alpha_n$ in $\mathcal{M}_{\mathrm{II}}$, let $\rho_n = \max\{\omega_1^{J_\gamma}, \alpha_n\}$ and fix $\vec{\xi}'_{n+1}$ so that $h_1^{J_{\rho_n+1}}(\vec{\xi}'_{n+1}, \omega_1^{J_\gamma}) = \alpha_{n+1}$. Such $\vec{\xi}'_{n+1}$ is guaranteed to exist by the fact that $J_{\rho_n+1}^{\mathcal{M}_{\mathrm{II}}}$ satisfies "ω_1 exists $\wedge (\forall \alpha) L_\alpha \not\models (\mathsf{T})$".

Now each $\vec{\xi}'_n$ is coded by some real $y_n \in J_\gamma^{\mathcal{M}_{\mathrm{II}}}$, and $\mathcal{P}(\omega)^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}} \subset \mathcal{M}_{\mathrm{I}}$ by the minimality assumption on x. So $y_0 \in \mathcal{M}_{\mathrm{I}}$, and we have in $\mathcal{M}_{\mathrm{I}}$ that y_0 codes $\vec{\xi}_0 \in J_\beta$ so that $h_1^{J_\beta}(\vec{\xi}_0, \omega_1^{J_\beta})$ exists. By inductively applying condition (2d), we can pull back the tuples $\vec{\xi}'_n$ of $\mathcal{M}_{\mathrm{II}}$ to tuples $\vec{\xi}_n$ of $\mathcal{M}_{\mathrm{I}}$ so that for all n, $h_1^{J_\beta}(\vec{\xi}_n, \omega_1^{J_\beta})$ exists. But then by (2c), $\langle h_1^{J_\beta}(\vec{\xi}_n, \omega_1^{J_\beta}) \mid n \in \omega \rangle$ is an infinite $\in^{\mathcal{M}_{\mathrm{I}}}$-descending sequence. This contradicts wellfoundedness of $\mathcal{M}_{\mathrm{I}}$.

Now let us prove (B) of the Lemma. For the rest of the proof, we let $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}, x$ be as above, and suppose further that k is (unique) such that $\delta(k,x)$ is nonstandard and $\delta(k+1,x)$ is wellfounded. Let $\gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$.

Suppose first that $(\gamma < \delta(k,x))^{\mathcal{M}_{\mathrm{II}}}$ and that γ is nonstandard. Clearly (1) in the definition of R holds. Let $\beta = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$. Then $\delta(k+1,x) < \beta \subset \delta(k,x)$, so $J_\beta \models \mathsf{KP} +$ "ω_1 exists." Our minimality assumption on x ensures $\mathcal{P}(\omega)^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}} \subset \mathcal{M}_{\mathrm{I}}$, so that in particular, the ω_1 of $J_\gamma^{\mathcal{M}_{\mathrm{II}}}$ is a subset of the ordinals of $\mathcal{M}_{\mathrm{I}}$. Indeed, it must be a proper subset, as $\mathcal{M}_{\mathrm{I}}$ projects to ω. It follows that $\beta \in \mathcal{M}_{\mathrm{I}}$, so is a witness to (2a); and the fact that $\mathcal{P}(\omega)^{J_\beta} = \mathcal{P}(\omega)^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}}$ implies (2b).

Now, J_β is an initial segment of $\mathcal{M}_{\mathrm{II}}$, with $\beta \subseteq \gamma$, and $\omega_1^{J_\beta} = (\omega_1^{J_\gamma})^{\mathcal{M}_{\mathrm{II}}}$. It follows that the map

$$h_1^{J_\beta}(\vec{\xi}, \omega_1^{J_\beta}) \mapsto (h_1^{J_\gamma}(\vec{\xi}, \omega_1^{J_\gamma}))^{\mathcal{M}_{\mathrm{II}}}$$

is $\in$-preserving (so (2c) holds) and is onto the initial segment of $\mathcal{M}_{\mathrm{II}}$ corresponding to J_β, by upward absoluteness of the Σ_1-Skolem function h_1 (so (2d) holds).

Finally, by definition there is some tree $T \in J^{\mathcal{M}_{\mathrm{II}}}_{\delta(k+1,x)+1}$ that is neither ranked nor illfounded in $(J_{\delta(k,x)})^{\mathcal{M}_{\mathrm{II}}}$. Since $\delta(k+1,x)$ is a true ordinal and $\delta(k+1,x) < \beta$, we have $T \in J_\beta$ and T is neither ranked nor illfounded in J_β. If we let $t \in J_\beta$ be any real coding $\vec{\tau}$ so that $h_1^{J_\beta}(\vec{\tau}, \omega_1^{J_\beta}) = T$, then t is a witness to (2e). Thus $R(k,\gamma,x)$ is satisfied as needed.

Conversely, suppose γ is such that $R(k,\gamma,x)$ holds. Let this be witnessed by $\beta \in \mathcal{M}_{\mathrm{I}}$ and $t \in \mathcal{P}(\omega) \cap \mathcal{M}_{\mathrm{I}} \cap \mathcal{M}_{\mathrm{II}}$. We immediately have $\gamma < \delta(k,x)$, by (1); all that's left is to show γ is nonstandard.

First consider the case that $\omega_1^{J_\beta} = \omega_1^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}}$. Then $P(\omega)^{J_\beta} \subseteq \mathcal{M}_{\mathrm{II}}$, so by (2c,d), J_β is isomorphic to an initial segment of J_γ. By (2a), J_β is admissible. If we had $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$, then by failure of (T) in $J_\beta^{\mathcal{M}_{\mathrm{II}}}$, we must have that $\omega_1^{J_\beta}$ is countable in $J_{\beta+1}^{\mathcal{M}_{\mathrm{II}}}$. But $\beta \leq \gamma < \delta(0,x)$, while $\omega_1^{J_\beta} = \omega_1^{J_\gamma} = \omega_1^{J_{\delta(0,x)}}$ (the latter computed in J_γ), a contradiction. So $\beta \notin \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}}$, even though $\beta \subseteq \gamma$. It follows that γ is nonstandard.

Now suppose $\omega_1^{J_\gamma^{\mathcal{M}_{\mathrm{II}}}} < \omega_1^{J_\beta}$ (the reverse inequality is impossible by our minimality assumption on x). Let $\alpha = \mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$ and let $T \in \mathcal{M}_{\mathrm{I}}$, $T' \in \mathcal{M}_{\mathrm{II}}$ be the trees witnessing (i) and (ii), respectively, in (2e). T' is a tree on $\mathcal{P}(\omega)^{J_\gamma} \subseteq \mathcal{M}_{\mathrm{I}}$, so by (iii) of (2e), we have $T \cap (\mathcal{P}(\omega)^{J_\gamma})^{<\omega} = T'$. Now, since $J^{\mathcal{M}_{\mathrm{II}}}_{\delta(k,x)} \models$ "T' is neither ranked nor illfounded" and $\alpha \subseteq \delta(k,x)$, we must have that $T' \in J_\alpha$ and $J_\alpha \models$ "T' is neither ranked nor illfounded" (since being either ranked or illfounded is Σ_1 and would reflect from J_α to $J^{\mathcal{M}_{\mathrm{II}}}_{\delta(k,x)}$). But J_α is admissible, so there is a branch through T', hence through T, definable over J_α. Since $\alpha < \omega_1^{J_\beta}$, we must have that T is illfounded in J_β. But this contradicts (i) of (2e). This contradiction completes the proof of Lemma 3.3.7. □

The proofs we have given all make reference to the lightface hierarchy, and it's easy to see these relativize to real parameters x. Letting $\theta(x)$ be the least ordinal α so that $L_\alpha[x] \models (\mathsf{T})$, we have

Theorem 3.3.10 (KPI_0). *For all reals x, $\Sigma^0_4(x)$-DET if and only if $\theta(x)$ exists.*

Since (boldface) $\mathbf{\Sigma}^0_1 \wedge \mathbf{\Pi}^0_1$-DET implies closure under the next admissible, the boldface result goes through in the weaker theory BST:

Theorem 3.3.11 (BST). *$\mathbf{\Sigma}^0_4$-DET if and only if $\theta(x)$ exists for every $x \subseteq \omega$.*

3.4 Generalizing to $\Sigma^0_{\alpha+3}$-DET, for $\alpha > 1$.

The generalization of the results about Σ^0_4 from the last two sections to all pointclasses of the form $\Sigma^0_{\alpha+3}$ is obtained in a manner similar to that in the inductive proof of Theorem 3.0.7 (see [Mar]). The most significant modification to those arguments involves the identification of the correct higher analogues of Π_1-RAP and (T).

Definition 3.4.1 (Π_1-RAP_α)**.** Let $\alpha < \omega_1^{\mathsf{CK}}$. Π_1-RAP_α denotes the theory consisting of "$\mathcal{P}^\alpha(\omega)$ exists" together with the axioms of the schema Π_1-$\mathsf{RAP}(\mathcal{P}^\alpha(\omega))$.

In particular, Π_1-RAP_α entails the existence of $\mathcal{P}^{\alpha+1}(\omega)$, and any Π_1 statement in parameters from $\mathcal{P}^{\alpha+2}(\omega)$ can be reflected to an admissible set M with $\mathcal{P}^\alpha(\omega) \subset M$. Note that since α is computable, we can make sense of this theory being satisfied in (not necessarily wellfounded) ω-models of KP, since α will be computed correctly in any such model. Note also Π_1-RAP is the same as Π_1-RAP_0.

The following is the general form of Theorem 3.2.6.

Theorem 3.4.2. *Suppose M is a transitive model of "$H(|\mathcal{P}^\alpha(\omega)|^+)$ exists" + Π_1-RAP_α, and that M has a wellordering of $\mathcal{P}^{\alpha+1}(\omega)$. Let $A \subseteq \omega^\omega$ be $\Sigma^0_{1+\alpha+3}$. Then either I wins $G(A;\omega^{<\omega})$ with a strategy in M, or II has a winning strategy in $G(A;\omega^{<\omega})$ that is $\mathbf{\Delta}_1$-definable over M.*

Proof. As Martin [Mar85] shows, the unraveling of closed sets can be iterated into the transfinite, taking inverse limits of the unraveling trees at limit stages. Precisely, assuming $\mathcal{P}^{\alpha+1}(\omega)$ exists and can be wellordered, there is a cover $\langle T, \pi, \psi\rangle$ that simultaneously unravels all $\Pi^0_{1+\alpha}$ sets, and so that T is a tree on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$. Furthermore, this cover is definable over $H(|\mathcal{P}^\alpha(\omega)|^+)$.

So, work in M as in the hypothesis of the Theorem. Let A be a $\Sigma^0_{1+\alpha+3}$ subset of ω^ω; and let $\langle T, \pi, \psi\rangle$ be the simultaneous unraveling of all $\Pi^0_{1+\alpha}$ sets. Then $\bar{A} = \pi^{-1}(A)$ is a $\mathbf{\Sigma}^0_3$ subset of $[T]$. Applying Lemma 3.2.3 with $X = \mathcal{P}^\alpha(\omega)$, we have that for any position $p \in T$, either I wins $G(\bar{A}; T)$ or, for any $\mathbf{\Pi}^0_2$ set $B \subseteq \bar{A} \subseteq [T]$, p is good for II relative to $\bar{A}, B, T$.

The remainder of the proof then is exactly like that of Theorem 3.2.6. If I doesn't win $G(\bar{A}; T)$, then we can take W for II to be the common refinement at stage k of goodness-witnessing quasistrategies relative to the various B_k (where $\bar{A} = \bigcup_{k\in\omega} B_k$). The desired strategy is $\psi(\tau)$, where τ is any strategy for II refining W; as before, M is sufficiently closed under local definitions involving τ to ensure $\psi(\tau)$ is truly winning for I in $G(A; T)$. $\square$

For the lower bound argument, it will again be helpful to have a natural principle involving trees on $\mathcal{P}^{\alpha+1}(\omega)$ which is equivalent in models of $V = L$ to Π_1-RAP_α. Consider a game tree T. The Gale-Stewart theorem applied to the game $G([T]; T)$ tells us that either I has a strategy in T, or the game tree T is *ranked for Player II*, in the sense that there is a partial map $\rho : T \rightharpoonup \mathrm{ON}$ so that for every s of even length in the domain of ρ and every a with $s^\frown\langle a\rangle \in T$, there is some b so that $\rho(s^\frown\langle a, b\rangle) < \rho(s)$. We let $(\mathsf{T})_\alpha$ denote the following special case of this fact, for $\alpha < \omega_1^{\mathsf{CK}}$.

Definition 3.4.3 ($(\mathsf{T})_\alpha$)**.** Suppose T is a tree on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$. Then either

- Player I has a strategy in T, or
- The game tree T is ranked for Player II.

Although $(\mathsf{T})_0$ clearly implies (T), and it follows from what we've shown that in L, (T) implies Π_1-RAP and hence $(\mathsf{T})_0$, it's less clear that $(\mathsf{T})_0$ and (T) are equivalent in general (under, say, the weak base theory BST). At this point, we conjecture they are but have not been able to show it.

Lemma 3.4.4. *Suppose $V = L$ and that $\mathcal{P}^{\alpha+1}(\omega)$ exists. Then Π_1-RAP_α holds if and only if $(\mathsf{T})_\alpha$ holds.*

Proof. Assume Π_1-RAP_α. Given a tree T on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$, suppose T is not ranked for II. Then this can be reflected to an admissible set M; in an admissible structure, every closed game is either won by Player II (the open Player), or there is a definable winning strategy for Player I (the closed Player). Since if II won $G([T];T)$ in M this would easily furnish a rank function ρ for II, we must have a strategy for I that is definable over M, hence belongs to V, and is winning for I in the restriction of the game tree to M, hence (since $\mathcal{P}^\alpha(\omega) \subset M$) winning for I in V.

Conversely, suppose $(\mathsf{T})_\alpha$ holds; clearly, using the uniform definable bijection in L of $\mathcal{P}^\alpha(\omega)$ with ω_α, it is sufficient to show the version of Π_1-RAP_α holds involving parameters $Q \subseteq \mathcal{P}(\omega_\alpha)$. So let $\phi(Q)$ be Π_1 and true in V, with $Q \subseteq \mathcal{P}(\omega_\alpha)$. Let $\tau > \omega_{\alpha+1}$ be large enough that $Q \in J_\tau$. Consider a game tree T defined as follows: Player II plays ordinals $\xi_n < \omega_\alpha$. The moves of Player I are fragments $\langle f, g\rangle$ much like the nodes of the tree T used in the proof of Theorem 3.1.6; f is the characteristic function of a consistent theory in the language of set theory plus constant symbols t, q and a_n, c_n, d_n for $n \in \omega$; g assigns elements of $\tau \cup \mathcal{P}(\omega_\alpha)$ to certain of the constants a_n.

The theory played by I is subject to the following rules: it must extend $V = L +$ "$\mathcal{P}^{\alpha+1}(\omega)$ exists", and assert that q is a subset of $\mathcal{P}(\omega_\alpha)$ belonging to its J_t (t an ordinal), and $\phi(q)$ must hold; the a_n must act as Henkin constants for statements asserting the existence of elements of $t \cup \mathcal{P}(\omega_\alpha)$; the $\in$-ordering of the constants c_n must agree with that of the ordinals ξ_n played by II; and $d_{n+1} < d_n$ for all n. Moreover, the assignment of the Henkin constants a_n must respect the order on t as asserted by the theory, as well as membership of the ξ_n in subsets of ω_α (so that $f(\#(c_i \in a_j)) = 1$ iff $\xi_i \in g(j)$) and of the elements of $\mathcal{P}(\omega_\alpha)$ in Q (so that $f(\#(a_j \in q)) = 1$ iff $g(j) \in Q$).

Since GCH holds, the tree T is evidently equivalent to one on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$. We claim Player I has a strategy in T. Otherwise, by $(\mathsf{T})_\alpha$, T is ranked for Player II. Let ρ be the rank function. Consider a play of the game where I plays the theory of $L_{\rho(\emptyset)+1}$; I interprets q by Q, the constants a_n are interpreted by witnesses to the appropriate existential statements, the c_n are interpreted as the ξ_n played by II, and the d_n are interpreted by ordinals furnished by the rank function (when the time comes to interpret the constant d_n, we must be at a position p of length at least $2n$, so interpret d_n by $\rho(p \upharpoonright 2n)$).

We have described how to obtain an infinite play for I; but this gives an infinite descending sequence of ordinals, a contradiction.

So Player I has a strategy σ in T; then $\sigma : \mathcal{P}^\alpha(\omega)^{<\omega} \rightharpoonup \mathcal{P}^{\alpha+1}(\omega)$ is an element of $H(|\mathcal{P}^{\alpha+1}(\omega)|) = L_{\omega_{\alpha+1}}$. Let G be a $L_{\omega_{\alpha+1}}$-generic filter for the poset $\mathrm{Col}(\omega, \omega_\alpha)$ to collapse ω_α to ω. (Note this makes sense since $L_{\omega_{\alpha+1}} \models \mathsf{ZFC}^-$.) Have II play against I's strategy σ with G, so that II plays an enumeration of ω_α in order-type ω.

Now in $L_{\omega_{\alpha+1}}[G]$, $\sigma * G$ yields a complete theory of an illfounded model $\mathcal{M}$ of $V = L$ + "$\mathcal{P}^{\alpha+1}(\omega)$ exists" + $\phi(\bar{Q})$; by the rules of the game, $\mathcal{M}$ is wellfounded up to $\tau > \omega^M_{\alpha+1}$; and since II plays all ordinals below ω_α, we have $\omega_\alpha^{\mathcal{M}} = \omega_\alpha^V$, so that $\mathcal{P}^\alpha(\omega) \subseteq \mathrm{wfp}(\mathcal{M})$. By Proposition 3.1.5, $L_{\mathrm{wfo}(\mathcal{M})}$ is admissible, and satisfies $\phi(\bar{Q})$, thus witnessing the desired instance of Π_1-RAP_α. □

By a similar argument to that given in Proposition 3.1.7, we can use the equivalence of Π_1-RAP_α with $(\mathsf{T})_\alpha$ to show that for transitive models M, Π_1-RAP_α reflects from M to L^M; thus we can eliminate the need for the Axiom of Choice in Theorem 3.4.2:

Theorem 3.4.5. *For all $\alpha < \omega_1^{\mathsf{CK}}$, if there is a transitive model of Π_1-RAP_α, then $\Sigma^0_{\alpha+3}$-DET holds.*

Let θ_α be least so that $L_{\theta_\alpha} \models$ "$\mathcal{P}^{\alpha+1}(\omega)$ exists" + $(\mathsf{T})_\alpha$.

Theorem 3.4.6 (KPI)**.** *$\Sigma^0_{1+\alpha+3}$-DET implies θ_α exists.*

Proof. As before, we define a Friedman-style game G with a $\Pi^0_{1+\alpha+3}$ winning condition. Player I and II play reals $f_{\mathrm{I}}, f_{\mathrm{II}}$ coding the characteristic functions of consistent theories extending

$$V = L + \mathsf{KP} + \text{all sets are countable} + \rho_\omega = \omega + (\forall \eta) L_\eta \not\models (\mathsf{T})_\alpha\,.$$

The rules dictate that the models obtained must both be ω-models; if not, the winner is decided appropriately.

We need a lemma concerning the complexity of comparing elements of $\mathcal{M}_{\mathrm{I}}$ to those of $\mathcal{M}_{\mathrm{II}}$. Essentially, it states that increasing the type of the elements by 1 increases the Borel rank of the equality relation by 1. The main complication is that sensibly comparing elements of $\mathcal{P}^{\beta+1}(\omega)$ requires equality of $\mathcal{P}^{\beta}(\omega)$ between the levels in $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ where these elements are constructed.

Lemma 3.4.7. *Let $\beta < \omega_1^{CK}$. Let $\mu, \nu \in \mathrm{ON} \cup \{\mathrm{ON}\}$ of $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively. Then*

- *The relation "$\mathcal{P}^{\beta}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\beta}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}}$" is $\Pi^0_{1+\beta+1}$;*
- *Suppose $x, y \in \mathcal{P}^{\beta+1}(\omega)$ of $L_\mu^{\mathcal{M}_{\mathrm{I}}}$ and $L_\nu^{\mathcal{M}_{\mathrm{II}}}$, respectively; and that the clause above holds. Then the relation "$x = y$" is $\Pi^0_{1+\beta}$.*

As usual, we mean that the relations in $f_{\mathrm{I}}, f_{\mathrm{II}}$ and the codes for μ, ν, x, y have the stated complexity.

Proof. By induction on β. For $\beta = 0$, we regard the statement that "$\omega^{\mathcal{M}_{\mathrm{I}}} = \omega^{\mathcal{M}_{\mathrm{II}}}$" as asserting that both models have standard ω, which is Π^0_2; and we have already seen that if this is the case, then equality of reals x, y is Π^0_1 in the codes.

If $\beta = \gamma + 1$, then the relation "$\mathcal{P}^{\beta}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\beta}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}}$" is captured by

$$\mathcal{P}^{\gamma}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}} = \mathcal{P}^{\gamma}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}},\ (\forall x \in \mathcal{P}^{\gamma+1}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}})(\exists y \in \mathcal{P}^{\gamma+1}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}})(x = y),$$
$$\text{and } (\forall x \in \mathcal{P}^{\gamma+1}(\omega)^{L_\nu^{\mathcal{M}_{\mathrm{II}}}})(\exists y \in \mathcal{P}^{\gamma+1}(\omega)^{L_\mu^{\mathcal{M}_{\mathrm{I}}}})(y = x).$$

By inductive hypothesis, the first clause is $\Pi^0_{1+\gamma+1}$, and "$x = y$" (and $y = x$) here has complexity $\Pi^0_{1+\gamma}$. So the whole expression is $\Pi^0_{1+\gamma+2}$, that is, $\Pi^0_{1+\beta+1}$.

For the second item, let $x, y \in \mathcal{P}^{\beta+1}(\omega)$ of $L_\mu^{\mathcal{M}_{\mathrm{I}}}, L_\nu^{\mathcal{M}_{\mathrm{II}}}$, respectively. Then $x = y$ iff

$$(\forall u \in \mathcal{P}^{\beta}(\mathcal{M}_{\mathrm{I}}))(\forall v \in \mathcal{P}^{\beta}(\mathcal{M}_{\mathrm{I}}))(u = v \to ((u \in x)^{\mathcal{M}_{\mathrm{I}}} \leftrightarrow (v \in y)^{\mathcal{M}_{\mathrm{II}}})).$$

By inductive hypothesis, "$u = v$" is $\Pi^0_{1+\gamma}$. So the displayed line is $\Pi^0_{1+\beta}$, as claimed.

The proof at limits is similar, and in fact, since equality of $\mathcal{P}^\lambda(\omega)$ between the models is equivalent for limit λ to equality of $\mathcal{P}^\xi(\omega)$ for all $\xi < \lambda$, both relations in this case turn out to be Π^0_λ. (Note the importance of the fact that λ is assumed to be recursive, and the relations above are uniform in the codes.) $\square$

We seek to describe the level of least disagreement of $\mathcal{M}_{\mathrm{II}}$ with $\mathcal{M}_{\mathrm{I}}$. Previously, this was witnessed by the least constructed real of $\mathcal{M}_{\mathrm{II}}$ not belonging to $\mathcal{M}_{\mathrm{I}}$; in the present situation, we look for sets witnessing least disagreement of type $\beta \leq \alpha$, in the following sense:

Definition 3.4.8. Suppose $\beta < \omega_1^{\mathsf{CK}}$ and that $x \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{II}}}$. We say x *witnesses disagreement at* β if for some $\mu \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$, $x \subseteq \mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)}} = \mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{I}}}_\mu}$ (in particular, both models believe $\mathcal{P}^\beta(\omega)$ exists), and for every $z \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{I}}}$ there is some u belonging to this common $\mathcal{P}^\beta(\omega)$ that is in the symmetric difference of x and z.

Arguing as in the proof of Lemma 3.4.7, the relation "x witnesses disagreement at β" is $\Sigma^0_{1+\beta+2}$ in the codes.

Just as before, we require a means of identifying the height of $\mathrm{wfp}(\mathcal{M}_{\mathrm{II}})$ in the event that $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})}$ satisfies "$\omega_{\alpha+1}$ exists". The device is again a function that steps down incrementally from an ordinal to its $\omega_{\alpha+1}$, using failures of $(\mathsf{T})_\alpha$.

Definition 3.4.9. Inside $\mathcal{M}_{\mathrm{II}}$, suppose $x \in \mathcal{P}^{\alpha+1}(\omega)$ and $L_{\mathrm{rank}_L(x)} \models$ "$\omega_{\alpha+1}$ exists". Then put

$$\delta_\alpha(0,x) = \mathrm{rank}_L(x)$$

$$\delta_\alpha(k+1,x) = \begin{cases} \delta \text{ least s.t.} J_{\delta_\alpha(k,x)} \models \text{“}\omega_{\alpha+1} \text{ exists and} & \\ (\exists T \in J_{\delta+1}) T \text{ is a tree on } \mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega) & \\ \text{ witnessing failure of } (\mathsf{T})_\alpha\text{”} & \text{if such exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We have the following analogue of Lemma 3.3.7.

Lemma 3.4.10. *There is a $\Sigma^0_{1+\alpha+2}$ relation $R_\alpha(k,\gamma,x)$ such that if $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$ are ω-models obtained from a play of the game according to the rules of G given thus far, $\mathcal{M}_{\mathrm{I}}$ is wellfounded, $L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)} \models$ "$\omega_{\alpha+1}$ exists", and x is the $<_L^{\mathcal{M}_{\mathrm{II}}}$-least element of $\mathcal{P}^{\alpha+1}(\omega)$ witnessing disagreement at α, then we have the following:*

(A) $(\forall k \in \omega)(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})[R_\alpha(k,\gamma,x) \to \delta_\alpha(k+1,x)$ *is standard*];

(B) $(\forall k \in \omega)$ *if* $\delta_\alpha(k,x)$ *is nonstandard and* $\delta_\alpha(k+1,x)$ *is wellfounded, then*
$(\forall \gamma \in \mathrm{ON}^{\mathcal{M}_{\mathrm{II}}})[R_\alpha(k,\gamma,x) \leftrightarrow (\gamma < \delta_\alpha(k,x))^{\mathcal{M}_{\mathrm{II}}} \wedge \gamma$ *is nonstandard*].

The definition of R_α and the proof of the Lemma closely resemble those in Lemma 3.3.7, so we omit them; note though that in addition to the obvious modifications, we require of any $\beta \in \mathrm{ON}^{\mathcal{M}_{\mathrm{I}}}$ witnessing $R_\alpha(k,\gamma,x)$ that $\mathcal{P}^\alpha(\omega)^{L^{\mathcal{M}_{\mathrm{I}}}_\beta} = \mathcal{P}^\alpha(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(x)}}$ (which is $\Pi^0_{1+\alpha+1}$) so that comparing codes for elements of $\omega_{\alpha+1}$ makes sense. Observe now the engine making the lemma go is the fact that if T is a game tree in an admissible structure which does not have a ranking function for II, then there is a strategy for I (the closed player) defined over the admissible set. The role before played by the newly defined branch now belongs to this strategy.

We may now give the winning condition. Suppose a play $f_{\mathrm{I}}, f_{\mathrm{II}}$ with term models $\mathcal{M}_{\mathrm{I}}, \mathcal{M}_{\mathrm{II}}$, respectively, is such that no rules have so far been broken. I wins the game if there are $\beta \leq \alpha$ and sets z, z' in $\mathcal{P}^{\beta+1}(\omega)$ of $L^{\mathcal{M}_{\mathrm{I}}}_{\mathrm{rank}_L(z)+1}, L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(z')+1}$, respectively, so that

- $\mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{I}}}_{\mathrm{rank}_L(z)}} = \mathcal{P}^\beta(\omega)^{L^{\mathcal{M}_{\mathrm{II}}}_{\mathrm{rank}_L(z')}}$,
- $z = z'$,
- z' codes an ordinal in $\mathcal{M}_{\mathrm{II}}$, but codes an illfounded linear order in $\mathcal{M}_{\mathrm{I}}$.

Call this condition $(*)$ (and notice $(*)$ is $\Sigma^0_{1+\beta+2}$). Otherwise, I wins just in case

1. $(\forall \beta \leq \alpha)(\forall x \in \mathcal{P}^{\beta+1}(\omega)^{\mathcal{M}_{\mathrm{II}}})$ if x witnesses disagreement at β, then

(a) $(\exists \beta' \leq \alpha)(\exists y \in \mathcal{P}^{\beta'+1}(\omega)^{\mathcal{M}_{\mathrm{II}}})$

y witnesses disagreement at β' and $(\mathrm{rank}_L(y) < \mathrm{rank}_L(x))^{\mathcal{M}_{\mathrm{II}}})$, or

(b) $(\exists k, \gamma) R_\alpha(k, \gamma, x)$

$\wedge\, (\forall k, \gamma)[R_\alpha(k, \gamma, x) \rightarrow (\exists k', \gamma') R_\alpha(k', \gamma', x) \wedge \langle k', \gamma' \rangle <_{\mathrm{Lex}} \langle k, \gamma \rangle]$,

and

2. $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \subseteq \mathcal{M}_{\mathrm{I}}$ implies

(a) $\mathrm{Th}(\mathcal{M}_{\mathrm{II}}) \in \mathcal{M}_{\mathrm{I}}$, or

(b) $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}} \subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}}$.

That this game is $\Pi^0_{1+\alpha+3}$ is by now a routine computation. We claim I has no winning strategy if θ_α does not exist. For suppose σ is such; we can assume by absoluteness that $\sigma \in L$, so let μ be the least admissible ordinal with $\sigma \in L_\mu$, and let f_{II} be the theory of L_μ. Let $\mathcal{M}_{\mathrm{I}}$ be the model given by $f_{\mathrm{I}} = \sigma * f_{\mathrm{II}}$. If $\mathcal{M}_{\mathrm{I}}$ is wellfounded, then it has ordinal height strictly less than μ, since $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \notin \mathcal{M}_{\mathrm{I}}$. But then $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \in \mathcal{M}_{\mathrm{II}}$ is a witness to failure of (1) (with $\beta = 0$).

So $\mathcal{M}_{\mathrm{I}}$ must be illfounded, and $\mathrm{Th}(\mathcal{M}_{\mathrm{I}}) \notin \mathcal{M}_{\mathrm{I}}$ again implies $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) \leq \mu$. It can't be the case that $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) = \mu$, for then (2) fails (since $\mathcal{M}_{\mathrm{I}}$ can be computed by applying σ to $\mathrm{Th}(\mathcal{M}_{\mathrm{II}})$, so this latter real cannot belong to $\mathcal{M}_{\mathrm{I}}$). So $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}}) < \mu$. By admissibility (and failure of $(\mathsf{T})_\alpha$), there is a largest cardinal in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$. So we must have $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})} \models$ "ω_β is the largest cardinal", for some $\beta \leq \alpha + 1$. If $\beta = \alpha + 1$, then failure of $(\mathsf{T})_\alpha$ in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$ implies this model projects to its ω_α; so there is a subset of $\mathcal{P}^\alpha(\omega)$ in $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})+1}$ that codes a wellorder isomorphic to $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})$, and this must be rank_L-minimal witnessing disagreement at α; but then (1) fails. Similarly, if $\beta < \alpha + 1$, then there is some least level above $\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})$ projecting to ω_β of $L_{\mathrm{wfo}(\mathcal{M}_{\mathrm{I}})}$, and an element x of $\mathcal{P}^{\beta+1}(\omega)$ can be found to witness failure of (1). But this contradicts our assumption that σ was winning for Player I.

All that's left is to show that II doesn't win if θ_α doesn't exist. So suppose τ is a winning strategy in L; have I play $\mathcal{M}_{\mathrm{I}} = L_\nu$, the least admissible level of L containing τ. As before, we must have that $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}}) \leq \nu$.

Since II wins, $\mathcal{M}_{\mathrm{II}}$ must be illfounded (if $\mathcal{M}_{\mathrm{II}}$ is wellfounded then (1) holds vacuously and II holds via (2b) if $\mathcal{M}_{\mathrm{I}} = \mathcal{M}_{\mathrm{II}}$ and (2a) otherwise). It follows that $\mathcal{M}_{\mathrm{II}}$ has countable codes for nonstandard ordinals; if these belong to M_I then I wins via condition $(*)$, a contradiction. So it must be that $\mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{II}}} \not\subseteq \mathcal{P}(\omega)^{\mathcal{M}_{\mathrm{I}}}$, hence (2) holds vacuously. Now $\mathrm{wfo}(\mathcal{M}_{\mathrm{II}})$ has a largest cardinal, say ω_β for some $\beta \leq \alpha + 1$. If $\beta \leq \alpha$, then by overspill, there are nonstandard ordinals of $\mathcal{M}_{\mathrm{II}}$ coded by subsets of $\mathcal{P}^\beta(\omega)$. Since II wins the game (so in particular $(*)$ fails), these cannot be coded by any element of $\mathcal{P}^{\beta+1}(\omega)$ in $\mathcal{M}_{\mathrm{I}}$. We thus obtain codes witnessing disagreement at β, and by overspill, there is no $<_L^{\mathcal{M}_{\mathrm{II}}}$-least such; this witnesses (1) via (1a), a contradiction. If $\beta = \alpha + 1$, on the other hand, then I wins the game via (1b) (here making use of Lemma 3.4.10). This contradiction completes the proof. $\square$

As before, we can relativize and obtain a boldface result.

Theorem 3.4.11 (KPI_0)**.** *For all reals x and ordinals $\alpha < \omega_1^x$, $\Sigma^0_{1+\alpha+3}(x)$-DET if and only if $\theta_\alpha(x)$, the least ordinal so that $L_{\theta_\alpha(x)}[x] \models \Pi_1$-$\mathsf{RAP}_\alpha$, exists.*

Theorem 3.4.12 (BST)**.** *$\boldsymbol{\Sigma}^0_{1+\alpha+3}$-DET if and only if $\theta_\alpha(x)$ exists for all $x \subseteq \omega$.*

It is interesting to note that game trees on $\mathcal{P}^{\alpha+1}(\omega), \mathcal{P}^\alpha(\omega)$ appear to be crucial on both sides of the argument, though they are used in very different ways. Though $(\mathsf{T})_\alpha$ and Π_1-RAP_α are equivalent in levels of L, it is not clear whether this equivalence is provable in a more general setting, say, that of $\mathsf{BST} + \mathsf{DC}$. We are led to wonder whether the (ostensibly weaker) axioms $(\mathsf{T})_\alpha$ could replace Π_1-RAP_α as the essential ingredient in the proof of Lemma 3.2.3.

3.5 Borel determinacy and inductive definitions

For a pointclass Γ, $o(\Gamma)$ is defined to be the supremum of lengths of inductive definitions obtained by iterating Γ operators; $o(\Gamma\text{-mon})$ is the supremum of lengths of *monotone* inductive definitions (see [Mar81] for full definitions).

The simplest winning strategies in games below Σ^0_3 can often be obtained by iterating an operator that gathers "sure winning positions", and this is reflected in the tight connection between the lengths of monotone inductive definitions and the location in L where winning strategies are first constructed. For example, $o(\Game\Sigma^0_1\text{-mon}) = o(\Pi^1_1\text{-mon}) = \omega_1^{\mathsf{CK}}$, and by the results of Solovay, winning strategies in Σ^0_2 games are constructed by $o(\Game\Sigma^0_2) = o(\Sigma^1_1\text{-mon})$ in L (for Player I) or in the next admissible (for Player II). Welch [Wel12] has conjectured that a similar result holds for $o(\Game\Pi^0_3\text{-mon})$ and Σ^0_3 determinacy.

It is natural to ask whether $o(\Game\Sigma^0_{1+\alpha+3}\text{-mon})$ is related to the ordinals θ_α in this way. We content ourselves with some coarse bounds that follow easily from arguments given above. For simplicity, we restrict to the case Σ^0_4; analogous bounds hold for the higher pointclasses.

Proposition 3.5.1. *Put $\kappa = {\omega_1}^{L_\theta}$. For $i \in \omega$, define α_i to be the least ordinal so that $L_{\alpha_i} \prec_{\Sigma_i} L_\kappa$. Then $\alpha_1 \leq o(\Game\Sigma^0_4) < o(\Game\Pi^0_4\text{-}mon) < \alpha_2$.*

Proof. If Player I wins a Σ^0_4 game, then there is a winning strategy for I in L_{α_1}. For a fixed parameter-free Σ_1-formula ψ, we define a modified version G_ψ of the game G of Section 4 by requiring Player II to play a minimal model of $V = L + \mathsf{KP} + \psi + (\forall\alpha)L_\alpha \not\models (\mathsf{T})$ (and putting no additional restrictions on Player I). Then Player II wins G_ψ if and only if $L_{\alpha_1} \models \psi$. So, the (set of codes for the) Σ_1-theory of L_θ is a $\Game\Sigma^0_4$ set of integers (indeed, it is a *complete* $\Game\Sigma^0_4$ set of integers; compare [Wel12]), and furnishes a $\Game\Sigma^0_4$ prewellordering of ω of order type α_1; this establishes the first inequalitity.

The second inequality is a consequence of Theorem A of [Mar81].

Next notice that $o(\Game\Pi^0_4\text{-mon}) < \kappa$, since L_κ is a model of ZFC^-, and for $x \in \mathbb{R} \cap L_\kappa$, the statement that Player II wins some $\Sigma^0_4(x)$ game $G(A, \omega^{<\omega})$ is Π_1 over L_κ in the parameter x (it is equivalent to the statement "there is no β so that L_β is a model of KPI in which I wins $G(A, \omega^{<\omega})$"). It can easily be verified that being the fixed point of a monotone Π_1-inductive operator is Δ_2 in models of ZFC^-. So the existence of a fixed point is Σ_2, and reflects to L_{α_2}.

This establishes $o(\Game\Pi^0_4\text{-mon}) \leq \alpha_2$. By the existence of a universal $\Game\Pi^0_4$-mon-monotone inductive definition (see Section 3 of [Mar81]), the inequality is strict. $\square$

Note that a winning strategy for Player I in a Σ^0_4 game can be computed from the Σ_1-theory of L_θ. So winning strategies for the Σ^0_4 player are at worst $\Game\Sigma^0_4$. As we have seen, winning strategies for the Π^0_4 player are rather more complicated, and needn't belong to $\Game\Pi^0_4$; at best, they are $\boldsymbol{\Delta}_1$-definable over L_θ in parameters.

CHAPTER 4

A connection with higher order reverse mathematics

Reverse mathematics, initiated and developed by Harvey Friedman, Stephen Simpson, and many others, is the project of classifying theorems of ordinary mathematics according to their intrinsic strength (a thorough account of the subject is given in [Sim09]). This project has been enormously fruitful in clarifying the underlying strength of theorems, classical and modern, formalizable in second order arithmetic. However, the second order setting precludes study of objects of higher type (e.g., arbitrary functions $f : \mathbb{R} \to \mathbb{R}$), and a number of frameworks have been proposed for reverse math in higher types. For example, Kohlenbach [Koh05] develops a language and base theory RCA_0^ω to accommodate all finite types, and shows it is conservative over the second order theory RCA_0; Schweber [Sch13] defines a theory RCA_0^3 for three types over which RCA_0^ω is conservative.

In this chapter, we are interested in higher types because of their necessary use in proofs of true statements of second order arithmetic, namely, in proofs of Borel determinacy. The reverse mathematical strength of determinacy for the first few levels of the Borel hierarchy has been well-investigated ([Ste77], [Tan91], [Wel11], [Wel12], [MS12]). However, as Montalbán and Shore [MS12] show, determinacy even for ω-length differences of Π^0_3 sets is not provable in Z_2, full second order arithmetic, and by the celebrated results of Friedman [Fri71] and Martin [Mar85], [Mar], determinacy for games with Σ^0_{n+4} payoff, for $n \in \omega$, requires the existence of $\mathcal{P}^{n+1}(\omega)$, the $n+1$-st iterated Power set of ω.

In light of this, the third order framework developed by Schweber [Sch13] seems a natural setting for investigating the strength of Σ^0_4 determinacy. In addition to defining the base theory RCA_0^3, Schweber introduces a number of natural versions of transfinite recursion principles in the third order context; he then proceeds to show that many of these are not equivalent over the base theory. In particular, he shows that Open determinacy for games played with *real-number* moves ($\Sigma_1^{\mathbb{R}}$-DET) is *strictly stronger* than Clopen determinacy ($\Delta_1^{\mathbb{R}}$-DET) . The argument given there is a technical forcing construction, and it is asked ([Sch13] Question 5.2) whether this separation is witnessed by some level of Gödel's L, say the least satisfying "$P(\omega)$ exists + $\Delta_1^{\mathbb{R}}$-DET".

Recalling the results of the previous chapter, this question should be connected to the minimal level of L at which winning strategies in Σ_4^0 games are constructed. In Chapter 3, we proved

Theorem 4.0.2. *Working over $\mathbf{\Pi}_1^1$-CA_0, the determinacy of all Σ_4^0 games is equivalent to the existence of a countable ordinal θ so that $L_\theta \models$ "$\mathcal{P}(\omega)$ exists, and for any tree T of height ω, either T has an infinite branch or there is a map $\rho : T \to \mathrm{ON}$ so that $\rho(x) < \rho(y)$ whenever $x \supsetneq y$."*

If θ is the least such ordinal, then it is also the least ordinal so that every Σ_4^0 game is determined as witnessed by a strategy in $L_{\theta+1}$.

We found L_θ is a model of $\mathsf{RCA}_0^3 + \Delta_1^{\mathbb{R}}$-DET $+\neg\Sigma_1^{\mathbb{R}}$-DET, answering Schweber's question in the affirmative.

In light of this result, it is plausible that the results of Chapter 3 could be elegantly reformulated in terms of higher-order arithmetic. Indeed, the defining property of L_θ bears a resemblance to that of β-model from reverse mathematics: a structure (ω, S) (where $S \subseteq \mathcal{P}(\omega)$) in the language of second order arithmetic is a *β-model* if it satisfies all true Σ_1^1 statements in parameters from S. Can the three-sorted structure $(\omega, (\mathbb{R})^{L_\theta}, (\omega^{\mathbb{R}})^{L_\theta})$ be characterized as a minimal β-model of some natural theory in third order arithmetic?

We here provide such a characterization. We describe a translation from β-models in third-order arithmetic to transitive models of set theory, much in the spirit of the second order translation given in [Sim09]. Combining these results with the theorem, we obtain: Σ_4^0-DET is equivalent over $\mathbf{\Pi}_1^1$-CA_0 to the existence of a countably-coded β-model of *projective* *transfinite recursion*, or Π^1_∞-$\mathsf{TR}_\mathbb{R}$; as we shall see, the latter theory is the natural analogue of ATR_0 in the third-order setting, and is equivalent (modulo the existence of selection functions for $\mathbb{R}$-indexed sets of reals) to $\Delta^\mathbb{R}_1$-DET.

4.1 Separating $\Sigma^\mathbb{R}_1$-DET and $\Delta^\mathbb{R}_1$-DET

We begin by showing that L_θ is a witness to the main separation result of Schweber [Sch13].

Theorem 4.1.1. *L_θ is a model of $\Delta^\mathbb{R}_1$-DET, but not of $\Sigma^\mathbb{R}_1$-DET.*

Proof. Working in L_θ, suppose $T \subseteq \mathcal{P}(\omega)^{<\omega}$ is a tree with no infinite branch. We will show that the game where Players I and II alternate choosing nodes of a branch through T is determined (here a player loses if he is the first to leave T).

Recall that in the previous chapter, it is shown that L_θ is a model of the following Π_1-*Reflection Principle* (Π_1-RAP): Whenever Q is a set of reals (that is, $Q \subseteq \mathcal{P}(\omega)$) and $\varphi(Q)$ is a true Π_1 formula, there is some admissible set M so that $Q \cap M \in M$, $M \models$ "$\mathcal{P}(\omega)$ exists", and $\varphi(Q \cap M)$ holds in M.

Suppose the game on T is undetermined. This is a Π_1 statement in parameters: it states that for any strategy σ for either player, there is a finite sequence $s \in \mathcal{P}(\omega)^{<\omega}$ against which this strategy loses the game on T (note that we may use $\mathcal{P}(\omega)$ as a parameter, so the existential quantifier is bounded). By Π_1-RAP, let M be an admissible set with $\bar{T} = T \cap M \in M$ so that $M \models$ "$\mathcal{P}(\omega)$ exists and neither player wins the game on $\bar{T}$". Note that $\bar{T}$ is a wellfounded tree, and by basic properties of admissible sets, we have a map $f : \bar{T} \to \mathrm{ON} \cap M$ in M so that $s \subsetneq t$ implies $f(s) > f(t)$. Working in M, we may therefore define by induction on the wellfounded relation $\supsetneq \cap (\bar{T} \times \bar{T})$ a partial function $\rho : \bar{T} \to \mathrm{ON}$ in M by

$$\rho(s) = \mu\alpha[(\forall x)(\exists y) s^\frown\langle x\rangle \in \bar{T} \to \rho(s^\frown\langle x, y\rangle) < \alpha].$$

Let us say an element in the domain of ρ is *ranked*. We claim for every $s \in \bar{T}$, either s is ranked or some real x exists with $s^\frown\langle x\rangle \in \bar{T}$ ranked. For suppose not, and let s be $\supsetneq$-minimal such. Then whenever x is such that $s^\frown\langle x\rangle \in \bar{T}$, there is some y so that $\rho(s^\frown\langle x, y\rangle)$ exists. By admissibility, we can find some α so that if $s^\frown\langle x\rangle \in \bar{T}$, then for some y, $\rho(s^\frown\langle x, y\rangle) < \alpha$.

So, either $\emptyset$ is ranked, or $\langle x\rangle$ is ranked for some x. It is easy to see that a winning strategy in the game on $\bar{T}$ (for II in the first case, I in the second) is definable from ρ. But this contradicts the fact that the game on $\bar{T}$ is not determined in M.

So $\Delta_1^{\mathbb{R}}$-DET holds in L_θ. It remains to show $\Sigma_1^{\mathbb{R}}$-DET fails. Note that if $T \in L_\theta$ is a tree on $\mathcal{P}(\omega)^{L_\theta}$, then if σ is a winning strategy (for either player) for the game on T in L_θ, σ is also winning in V (if σ is for the closed player, then being a winning strategy is simply the statement that no terminal nodes are reached by σ when it is σ's turn; if σ is for the open player, then the tree of plays in T compatible with σ is wellfounded, so is ranked in L_θ, hence wellfounded in V).

Note further that L_θ is not admissible, and Σ_1-projects to ω with parameter $\{\omega_1^{L_\theta}\}$; in particular, L_θ does not contain the real $\{k \mid L_\theta \models \phi_k(\omega_1^{L_\theta})\}$ (here $\langle\phi_k\rangle_{k\in\omega}$ is some standard fixed enumeration of Σ_1 formulae with one free variable). We will define an open game on $L_{\omega_1^{L_\theta}}$ so that Player II (the closed player) wins in V, but any winning strategy for II computes this theory; by what was just said, no winning strategy can belong to L_θ.

For the rest of the proof, we let ω_1 denote $\omega_1^{L_\theta}$. The game proceeds as follows: In round -1, Player I plays an integer k; Player II responds with 0 or 1, and a model M_0. In all subsequent rounds $n < \omega$, Player I plays a real x_n in $\mathcal{P}(\omega)^{L_\theta}$, and Player II responds with a pair π_n, M_{n+1}:

I	k		x_0		x_1		$\dots$	
II		$i \in \{0,1\}, M_0$		π_0, M_1		π_1, M_2		$\dots$

Player II must maintain the following conditions, for all $n \in \omega$:

- M_n is a countable transitive model of "$\mathcal{P}(\omega)$ exists";
- $\pi_n : M_n \to M_{n+1}$ is a Σ_0-elementary embedding with $\pi_n(\omega_1^{M_n}) = \omega_1^{M_{n+1}}$;
- $x_n \in M_{n+1}$, and for all trees $T \in M_n$, $\pi_n(T)$ is either ranked or illfounded in M_{n+1};

- For all $a \in M_n$, $M_{n+1} \models (\exists\alpha)\pi_n(a) \in L_\alpha$;
- $M_n \models \phi(\omega_1^{M_n})$ if and only if $i = 1$.

Note the second condition entails that π_n has critical point $\omega_1^{M_n}$. The first player to violate a rule loses; Player II wins all infinite plays where no rules are violated.

We first claim that Player II wins this game in V. We describe a winning strategy. If I plays k, have Player II respond with 1 if and only if $\phi(\omega_1)$ holds in L_θ. If 1 was played, let $\alpha_0 < \theta$ be sufficiently large that $L_{\alpha_0} \models \phi(\omega_1)$; otherwise let $\alpha_0 = \omega_1 + \omega$. Inductively, let $\alpha_{n+1} < \theta$ be the least limit ordinal so that every wellfounded tree in L_{α_n} is ranked in $L_{\alpha_{n+1}}$. (Note such exists: the direct sum of all wellfounded trees $T \in L_{\alpha_n}$ belongs to L_{α_n+1}, since L_{α_n} has Σ_1 projectum $\omega_1^{L_\theta}$. If β is large enough that this sum is ranked in L_β, then all wellfounded trees of L_{α_n} are also ranked in L_β.)

Now let H_0 be the Σ_0-Hull of $\{L_{\omega_1}\}$ in L_{α_0} (that is, H_0 is the closure in L_{α_0} of $\{L_{\omega_1}\}$ under taking $<_L$-least witnesses to bounded existential quantifiers). Let M_0 be its transitive collapse. Inductively, having defined $H_n \subset L_{\alpha_n}$ and given a real x_n played by I, let H_{n+1} be the Σ_0-Hull of $H_n \cup \{L_{\alpha_n}, x_n\} \cup \{f \in L_{\alpha_{n+1}} \mid f$ is the rank function of a wellfounded tree $T \in H_n\}$ inside $L_{\alpha_{n+1}}$. Let M_{n+1} be its transitive collapse, and $\pi_{n,n+1} : M_n \to M_{n+1}$ be the map induced by the inclusion embedding. Inductively, each H_n (hence M_n) is countable and belongs to L_θ (since θ is limit). The remaining rules are clearly satisfied by the π_n, M_n. So II wins in V, as desired.

All that's left is to show that any winning strategy for II responds to k with 1 if and only if $L_\theta \models \phi_k(\omega_1)$; it follows that no winning strategy (for either player) can belong to L_θ. So suppose σ is winning for II. Let $\langle x_n \rangle_{n\in\omega}$ be an enumeration of the reals of $\mathcal{P}(\omega)^{L_\theta}$. Then σ replies with a sequence $\langle \pi_n, M_n \rangle_{n\in\omega}$ of models and embeddings; these form a directed system. Let M_ω be the direct limit, with $\pi_{n,\omega} : M_n \to \omega$ the limit embedding. Since $\mathrm{crit}(\pi_n) = \omega_1^{M_n}$ for each n, the ω_1 of M_ω is wellfounded. Moreover, by the rules of the game, M_ω is a model of $V = L+$ "all wellfounded trees are ranked", and since all reals of L_θ were played, we have $\omega_1^{M_\omega} = \omega_1^{L_\theta}$.

Now suppose towards a contradiction that σ played a truth value of $\phi(\omega_1)$ that disagreed with the truth value of $\phi(\omega_1)$ in L_θ. By the previous paragraph, M_ω satisfies that it is the minimal model of $V = L+$ "all wellfounded trees are ranked," and since it satisfies a formula not true in L_θ, M_ω is illfounded; let $\mathrm{wfo}(M_\omega)$ be the supremum of its wellfounded ordinals. By the truncation lemma for models of $V = L$ (see Proposition 3.1.5), $L_{\mathrm{wfo}(M_\omega)}$ is admissible. But by minimality in the definition of θ, no α with $\omega_1 < \alpha \leq \theta$ can have L_α be admissible. So we must have $\mathrm{wfo}(M_\omega) > \theta$. But as remarked above, L_θ Σ_1-projects to ω, whereas $L_{\mathrm{wfo}(M_\omega)}$ is a proper end extension of L_θ with the same ω_1. This is a contradiction. $\square$

4.2 Third order β-models and transfinite recursion

We adopt the definitions of the language and structures of third-order arithmetic introduced in [Sch13]. We briefly recall some salient points. The language L^3 is a many-sorted first order language consisting of three sorts: s_0 corresponds to naturals, s_1 to functions $x : \omega \to \omega$, and s_2 to functionals $F : \omega^\omega \to \omega$. Non-logical symbols include the usual signature $\{+, \times, <, 0, 1\}$ for arithmetic on s_0, application operators $\cdot_0$ and $\cdot_1$, equality relations $=_0, =_1, =_2$ for their respective sorts, and binary operations $* : s_2 \times s_1 \to s_1$ and $\frown : s_0 \times s_1 \to s_1$. The latter operations are introduced to allow for coding. Namely, under the intended interpretation,

$$k^\frown x = \langle k, x(0), x(1), x(2), \dots \rangle$$
$$F * x = \langle F(0^\frown x), F(1^\frown x), F(2^\frown x), \dots \rangle.$$

Here of course we are denoting type 1 objects $x : \omega \to \omega$ as $\langle x(0), x(1), \dots \rangle$. Note that in what follows we will adopt the convention that the first time a fresh variable appears, its type will be denoted by a superscript (x^1, F^2, etc.).

Recall that a structure $\mathcal{M} = (M_0, M_1)$ in the language of second order arithmetic is a *β-model* if M_0 is isomorphic to the standard natural numbers, and whenever $x \in M_1$ and $\phi(u)$ is a Σ^1_1 formula, we have $\mathcal{M} \models \phi(x)$ if and only if $\phi(x)$ is true (understood as a statement about the unique real corresponding to x). Equivalently, $\mathcal{M}$ is a β-model if whenever $T \subseteq \omega^{<\omega}$ is a tree coded by a real in M_1 (under some standard coding of finite sequences of naturals), we have that $\mathcal{M} \models$ "T is illfounded" whenever T is illfounded. For simplicity's sake, we use the latter characterization to define a notion of β-model in the third-order context.

Fix a coding $\langle \cdot \rangle : \mathbb{R}^{\leq\omega} \to \mathbb{R}$ of length $\leq \omega$ sequences of reals by reals, in such a way that if x codes a sequence, then the length $\mathrm{lh}(x)$ of the sequence coded is uniquely determined, $(x)_i$ denotes the i-th element of the sequence, and x is the unique real so that $\langle (x)_i \rangle_{i<\mathrm{lh}(x)} = x$. By a tree on $\mathbb{R}$, we mean a functional $T^2 : \mathbb{R} \to 2$ so that T takes value 1 only on codes for finite sequences, so that $\{\langle x_0, \ldots, x_i \rangle \mid T(\langle x_0, \ldots, x_i \rangle) = 1\}$ is a tree in the usual sense.

Definition 4.2.1. Let $\mathcal{M} = (M_0, M_1, M_2)$ be an L^3 structure. We say $\mathcal{M}$ is a β-*model* if $M_0 = \omega$, $M_1 \subseteq \omega^\omega = \mathbb{R}$, and $M_2 \subseteq \omega^{M_1}$; and whenever T^2 is (a functional in M_2 coding) a tree on M_1, if T has an infinite branch, then $\mathcal{M}$ satisfies $(\exists x^1)(\forall k)(x)_k \subsetneq (x)_{k+1} \wedge T((x)_k) = 1$.

That is, trees on $\mathbb{R}^{\mathcal{M}}$ in $\mathcal{M}$ are wellfounded (in V) if and only if they are wellfounded in $\mathcal{M}$. We would have liked to define $\mathcal{M}$ to be a β-model if for any Σ^1_1 formula $\exists x^1 \phi(x, y, F)$ with parameters from $M_1 \cup M_2$, we have, for any $y \in M_1$, $F \in M_2$, that $\mathcal{M} \models \exists x \phi(x, y, F)$ if and only if $\exists x \phi(x, y, F)$ is true; but we must be careful about what we mean by "true". For, if x is a real not in M_1, then the value $F(x)$ is not defined. There are a number of ways to get around this, e.g., by appropriately altering the language L^3 and our base theory to accommodate a built-in coding of sequences of reals by reals. But it is more straightforward in our case to use the definition of β-model above.

We will be primarily interested in models of fragments of set theory, considered as β-models of third-order arithmetic. If $\mathcal{M} = (M, \in)$ is a transitive set with $\omega \in M$, we will refer to $\mathcal{M}$ as a model of third-order arithmetic, keeping in mind we are really referring to the structure $(\omega, M \cap \omega^\omega, M \cap \omega^{M \cap \mathbb{R}})$. It is immediate from our definitions that L_θ is a β-model. Indeed, whenever α is an ordinal with $\omega_1^{L_\theta} < \alpha \leq \theta$, then L_α is a β-model; this follows from the fact that branches through trees on $\mathbb{R}$ are themselves (coded by) reals. Consequently, taking collapses of Skolem hulls, we have many β-models L_γ with $\gamma < \omega_1^{L_\theta}$.

Our aim is to show that L_θ can be recovered from certain β-models, from which it will follow that L_θ is the minimal β-model of $\Delta^{\mathbb{R}}_1$-DET. Our starting point is a connection between $\Delta^{\mathbb{R}}_1$-DET and the third-order analogue of ATR_0.

Definition 4.2.2. Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$ is the theory in third-order arithmetic that asserts the following, for every Π^1_n formula $\phi(x^1, Y^2)$ with the displayed free variables. Suppose $W \subseteq \mathbb{R} \times \mathbb{R}$ is a regular relation. Then there is a functional $\theta : \mathbb{R} \times \mathbb{R} \to 3$ so that

$$(\forall a^1 \in \mathrm{dom}(W))(\forall x^1)\theta(a, x) = \begin{cases} 1 \text{ if } \phi(x, \theta \restriction \{b \mid \langle b, a\rangle \in W\}), \\ 0 \text{ otherwise.} \end{cases}$$

Here for $A \subseteq \mathbb{R}$, $\theta \restriction A$ denotes the functional θ' so that for all x, if $b \in A$, $\theta'(b, x) = \theta(b, x)$, and if $b \notin A$ then $\theta'(b, x) = 2$.

Note here we regard a functional $W : \mathbb{R} \to \omega$ as a binary relation if it determines the characteristic function of one; i.e., if there is a set $\mathrm{dom}(W) \subseteq \mathbb{R}$ so that $W(x) < 2$ whenever $x = \langle a, b\rangle$ for some $a, b \in \mathrm{dom}(W)$, and otherwise $W(x) = 2$. A binary relation is *regular* if whenever $A \subseteq \mathrm{dom}(W)$ is non-empty, there is some W-minimal $a \in A$. Be warned: we will routinely conflate the functionals of third-order arithmetic and the subsets of $\mathbb{R}, \mathbb{R}^{<\omega}, \mathbb{R}^\omega$, etc., which these functionals represent.

The idea of Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$ is that for each $a \in \mathrm{dom}(W)$, the map $x \mapsto \theta(a, x)$ is the characteristic function of the set of reals obtained by iterating the defining formula ϕ along the wellfounded relation W on $\mathbb{R}$ up to a. Note that strictly speaking, Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$ is projective *wellfounded recursion*, in that the relation W along which we iterate is not required to be a wellorder. This suits our purposes because we will iterate definitions along wellfounded trees on $\mathbb{R}$; taking the Kleene-Brouwer ordering of such a tree requires a wellordering of $\mathbb{R}$, but we would like to use as little choice as possible.

The following lemma makes reference to the theories $\mathsf{TR}_1(\mathbb{R})$ and $\mathsf{SF}(\mathbb{R})$, both introduced in [Sch13]. $\mathsf{TR}_1(\mathbb{R})$ is the restriction of Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$ to the case that ϕ is Σ^1_1 and W is a wellorder; $\mathsf{SF}(\mathbb{R})$ asserts the existence of selection functions for $\mathbb{R}$-indexed collections of sets of reals.

Lemma 4.2.3. *The following theories are equivalent over* RCA^3_0*:*

(1) $\Delta^{\mathbb{R}}_1$-DET*;*

(2) $\mathsf{TR}_1(\mathbb{R}) + \mathsf{SF}(\mathbb{R})$*;*

(3) Π^1_∞-$\mathsf{TR}_\mathbb{R} + \mathsf{SF}(\mathbb{R})$*.*

Proof. Clearly, (3) implies (2). The equivalence of (1) and (2) is proved in [Sch13]; and the proof that (1) implies the Σ^1_1 case in (3) is the essentially same proof given there for $\mathsf{TR}_1(\mathbb{R})$ with the appropriate adjustments. So all that is left to show is that Σ^1_1-wellfounded recursion implies Π^1_∞-$\mathsf{TR}_\mathbb{R}$.

So suppose inductively that we have Σ^1_n-wellfounded recursion, that W is a wellfounded relation on $\mathbb{R}$, and that $\phi(w^1, x^1, Y^2)$ is a Π^1_n formula. We wish to prove the instance of wellfounded recursion along W with formula $(\exists w)\phi$.

Without loss of generality, we may assume the real $\bar{0} = \langle 0, 0, 0 \ldots \rangle$ does not belong to $\mathrm{dom}(W)$. We define $\bar{W}$ to be a binary relation on $\omega \times \mathbb{R}$ so that $\bar{W}$ is isomorphic to the product $3 \times W$, with an additional minimal element $\bar{0}$ below points of the form $0^\frown x$; namely, when $i^\frown x, j^\frown y \neq \bar{0}$, set

$$\bar{W}(i^\frown x, j^\frown y) = \begin{cases} 1 & \text{if } i, j < 3 \text{ and } W(x,y) = 1 \text{ or } x = y \text{ and } i < j, \\ 0 & \text{if } i, j < 3 \text{ and } W(x,y) = 0 \text{ and } (x \neq y \text{ or } i \geq j) \\ 2 & \text{in all other cases.} \end{cases}$$

Furthermore set

$$\bar{W}(\bar{0}, i^\frown x) = \begin{cases} 1 & \text{if } i = 0 \text{ and } x \in \mathrm{dom}(W), \\ 0 & \text{if } i \in \{1,2\} \text{ and } x \in \mathrm{dom}(W), \\ 2 & \text{otherwise.} \end{cases}$$

Finally, $\bar{W}(x, \bar{0}) = 0$ for all $x \in \mathrm{dom}(\bar{W})$.

The idea is to iterate Σ^1_n-wellfounded recursion along $\bar{W}$, breaking up into the three stages of applying $\neg\phi$, taking complements, and taking projections. Let us define the formula

$\bar{\phi}(z,Y)$ by

$$\begin{aligned}\bar{\phi}(z,Y) \iff &(\exists i^0, a^1) a \in \mathrm{dom}(W), Y(i^\frown a, z) = 2 \text{ and}\\ &i = 0, Y(\bar{0}, z) < 2, \text{ and } z = \langle w^1, x^1\rangle, \neg\phi(w, x, [\langle b, y\rangle \mapsto Y(2^\frown b, y)]); \text{ or}\\ &i = 1, \text{ and } Y(0^\frown a, z) = 0; \text{ or}\\ &i = 2 \text{ and } (\exists w^1) Y(1^\frown a, \langle w, z\rangle) = 1.\end{aligned}$$

The point of introducing $\bar{0}$ as a $\bar{W}$-minimal element below points of the form $0^\frown x$ is that we would like to make sure the witness $i^\frown a$ to $\bar{\phi}$ are $\bar{W}$-minimal at which the fragment Y of θ has not yet been defined (that is, minimal so that $Y(i^\frown a, z) = 2$). This would require one quantifier too many in the $i = 0$ case. We get around this by exploiting the way $Y = \theta \restriction \{b \mid \langle b, i^\frown a\rangle \in \bar{W}\}$ is defined to use $\bar{0}$ as a trigger: Since we always have $Y(\bar{0}, z) = 2$ if $i \neq 0$, we know that the $i = 0$ case will only be satisfied when we are applying the recursion at $\theta \restriction \{b \mid \langle b, 0^\frown a\rangle\}$ for some a. (Note that the desired minimality holds for similar reasons in the cases $i = 1, 2$.)

To see $\bar{\phi}$ is Σ^1_n, it is enough to show the relation $\neg\phi(w, x, [\langle b, y\rangle \mapsto Y(2^\frown b, y)])$ is Σ^1_n (as a relation on w, x, Y). But this follows from the fact (checked to be provable in RCA^3_0) that if Y' is a functional Π^0_∞-definable from Y, then for any Σ^1_n formula π, there is, uniformly in π and the definition of Y' from Y, a Σ^1_n formula π', so that

$$(\forall x^1)\pi'(x, Y) \iff \pi(x, Y').$$

We obtain the result by applying Σ^1_n-wellfounded recursion to $\bar{W}$ with $\bar{\phi}$. From the θ obtained, the desired instance of Σ^1_{n+1} recursion is witnessed by the relation $\langle a, x\rangle \mapsto \theta(2^\frown a, x)$ (which exists by Δ^0_1-Comprehension). $\square$

We remark that the uniqueness of the functional θ is provable from the Σ^1_1-Comprehension scheme (which itself follows from $\mathsf{TR}_1(\mathbb{R})$), using regularity of the relation W applied to $\{a \in \mathrm{dom}(W) \mid (\exists x^1)\theta_1(a, x) \neq \theta_2(a, x)\}$.

4.3 From β-models to set models.

In this section we show that from any β-model $\mathcal{M}$ of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, one can define a transitive set model M^{set} with the same reals and functionals; and furthermore, any set model so obtained contains L_θ as a subset. By what we have shown, L_θ is a β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, so this proves that L_θ is the minimal β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$.

These results are essentially a recapitulation in the third-order context of the correspondence between β-models of ATR_0 and wellfounded models of $\mathsf{ATR}_0^{\text{set}}$ described in Chapter VII.3-4 of [Sim09]; therefore we omit most details, taking care mainly where the special circumstances of the third-order situation arise.

Let $\mathcal{M}$ be a L^3-structure modelling Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Working inside $\mathcal{M}$, we say $T : \mathbb{R} \to \omega$ is a *suitable tree* if

1. T codes a tree on $\mathbb{R}$,
2. T is non-empty, i.e. $T(\langle\rangle) = 1$, and
3. T is regular: if $A \subseteq T$, there is $a \in A$ with no proper extension in A.

The third item is understood to quantify over type-2 objects corresponding to characteristic functions of subsets of T. We take suitable trees to be regular because this is what's required by Π^1_∞-$\mathsf{TR}_\mathbb{R}$ and is possibly stronger than non-existence of a branch; of course the two are equivalent assuming $\mathsf{DC}_\mathbb{R}$, in particular, in β-models.

Now suppose $\mathcal{M}$ is a β-model. If T is a tree on $\mathbb{R}^\mathcal{M}$ coded by some functional in M_2, then T is suitable in $\mathcal{M}$ if and only if T is (non-empty and) wellfounded. We will define M^{set} to be the set of collapses of suitable trees in $\mathcal{M}$. Namely, given a wellfounded tree, define by recursion on the wellfounded relation $\supsetneq \cap (T \times T)$,

$$f(s) = \{f(s^\frown\langle a\rangle) \mid a \in \mathbb{R} \wedge s^\frown\langle a\rangle \in T\}.$$

Then put $|T| = f(\langle\rangle)$. Notice that $|T|$ need not be transitive, as we only take $f(s)$ to be the pointwise image of *one-step* extensions of s. We define

$$M^{\text{set}} = \{|T| \mid T \in M_2 \text{ is a suitable tree}\}.$$

Such M^{set} is transitive: If T is a suitable tree in $\mathcal{M}$ then any $x \in |T|$ is $|T_s|$ for some $s \in T$. But $T_s = \{t \mid s^\frown t \in T\}$ is evidently a suitable tree, and belongs to $\mathcal{M}$ by Δ^0_1-Comprehension.

Although we are interested primarily in β-models of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, it is worth making a definition of M^{set} that works for ω-models of Π^1_∞-$\mathsf{TR}_\mathbb{R}$, that is, models $\mathcal{M}$ with standard ω so that $M_1 \subseteq \mathbb{R}$ and $M_2 \subseteq \omega^{M_1}$. Working inside such an $\mathcal{M}$, say that $\mathrm{ISO}(T^2, X^2)$ holds, where T is a suitable tree, if $X \subseteq T \times T$ and for all $s, t \in T$, we have

$$\begin{aligned}\langle s,t\rangle \in X \iff (\forall x^1)[&s^\frown\langle x\rangle \in T \to (\exists y^1)(t^\frown\langle y\rangle \in T \wedge \langle s^\frown\langle x\rangle, t^\frown\langle y\rangle\rangle \in X)\\ &\wedge t^\frown\langle x\rangle \in T \to (\exists y^1)(s^\frown\langle y\rangle \in T \wedge \langle s^\frown\langle y\rangle, t^\frown\langle x\rangle\rangle \in X)].\end{aligned}$$

(The point is, $\langle s,t\rangle \in X$ if and only if $|T_s| = |T_t|$). The existence and uniqueness of an X so that $\mathrm{ISO}(T, X)$ holds is provable in Π^1_∞-$\mathsf{TR}_\mathbb{R}$, using the fact that T is suitable. Letting $\bar{n}$ denote the real $\langle n, n, n, \ldots\rangle$, we may define $S \oplus T$, for suitable trees S, T, as the set of sequences $\{\langle\bar{0}\rangle^\frown s \mid s \in S\} \cup \{\langle\bar{1}\rangle^\frown t \mid t \in T\}$. Then set $S =^* T$ iff for the unique X with $\mathrm{ISO}(S \oplus T, X)$, we have $\langle\langle\bar{0}\rangle, \langle\bar{1}\rangle\rangle \in X$; and set $S \,\epsilon\, T$ iff for the unique X with $\mathrm{ISO}(S \oplus T, X)$, there is some real x so that $\langle\langle\bar{0}\rangle, \langle\langle\bar{1}, x\rangle\rangle \in X$. Then provably in Π^1_∞-$\mathsf{TR}_\mathbb{R}$, $=^*$ is an equivalence relation on the class of suitable trees, and ϵ is well-defined and extensional relation on the equivalence classes $[T]_{=^*}$, so inducing a relation $\in^*$ on these. We define

$$M^{\text{set}} = \langle\{[T]_{=^*} \mid T \in M_2 \text{ is a suitable tree in } \mathcal{M}\}, \in^*\rangle.$$

For β-models $\mathcal{M}$, the M^{set} we obtain is a wellfounded structure, and is isomorphic to the transitive set M^{set} defined above, via the map $[T]_{=^*} \mapsto |T|$. For brevity, we will from now on refer to $[T]_{=^*}$ as $|T|$ (even for T in non β-models, so that T may be illfounded in V).

Recall now some basic axiom systems in the language of set theory. BST is the theory consisting of Extensionality, Foundation, Pair, Union, and Δ_0-Comprehension. Axiom Beta, which we denote $\mathrm{Ax}\,\beta$, states that every regular relation r has a collapse map; that is, a map $f : \mathrm{dom}(r) \to V$ so that for all $x \in \mathrm{dom}(r)$, $f(x) = \{f(y) \mid \langle y, x\rangle \in r\}$.

Proposition 4.3.1. *Let $\mathcal{M}$ be an ω-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Then*

1. *M^{set} is an ω-model of $\mathsf{BST} + \mathrm{Ax}\,\beta +$ "$\mathcal{P}(\omega)$ exists".*

2. *$\mathcal{M}$ and M^{set} have the same reals $x : \omega \to \omega$ and functionals $F : \mathbb{R} \to \omega$; that is, $M_1 = \mathbb{R} \cap M^{\text{set}}$ and $M_2 = (\omega^{\mathbb{R}\cap M^{\text{set}}}) \cap M^{\text{set}}$.*

3. *In M^{set}, every set is hereditarily of size at most 2^ω; that is, for all $x \in M^{\text{set}}$, there is an onto map $f : \mathcal{P}(\omega)^{M^{\text{set}}} \to \mathrm{tcl}(x)$ in M^{set}, where $\mathrm{tcl}(x)$ denotes the transitive closure of x.*

4. *If $\alpha \in \mathrm{ON}^{M^{\text{set}}}$, then $M^{\text{set}} \models$ "L_α exists"; furthermore, $L_\alpha^{M^{\text{set}}} = L_\alpha$ when α is in the wellfounded part of M^{set}.*

5. *M^{set} is wellfounded if and only if $\mathcal{M}$ is a β-model.*

Proof. (1) Since $\mathcal{M}$ is an ω-model of RCA_0^3, the tree

$$\{\langle \bar{n_0}, \bar{n_1}, \ldots, \bar{n_k}\rangle \mid (\forall i < k) n_{i+1} < n_i\}$$

belongs to M_2. Clearly it is a suitable tree in $\mathcal{M}$, and $|T| \in M^{\text{set}}$ is the ω of M^{set}. That $\mathcal{P}(\omega)$ exists in M^{set} is a similar exercise in coding: given any real x, there is a canonical tree $T(x)$ so that $|T(x)| = x$, membership in $T(x)$ being uniformly Π^1_∞-definable from x; and from any suitable tree collapsing to a real, one can define in Π^1_∞-$\mathsf{TR}_\mathbb{R}$ the x it collapses to. So $\mathcal{P}(\omega)^{M^{\text{set}}}$ is precisely $|T|$, where $T = \{\langle\rangle\} \cup \{\langle x\rangle^\frown s \mid s \in T(x)\}$.

For the axioms of BST, Extensionality follows from the fact that the relation $\in^*$ is extensional on M^{set}. Pair and Union are straightforward, only requiring Σ^1_1-Comprehension to show that from given suitable trees $S, T \in \mathcal{M}_2$, one can define trees corresponding to $\{|S|, |T|\}$ and $\bigcup |S|$.

Δ_0-Comprehension is similar. Notice here that although the relations $= *$ and $\in^*$ are in general Σ^2_1, when restricted to a given tree T with parameter X witnessing $\mathrm{ISO}(T, X)$, the relations $|T_s| =^* |T_t|$ and $|T_s| \in^* |T_t|$, regarded as binary relations T, are each Π^1_2 in the parameters T, X. From this, one shows by induction on formula complexity that for any Δ_0 formula $\phi(u_1, \ldots, u_k)$ in the language of set theory, the k-ary relation on T defined by

$$P(s_1, \ldots, s_k) \iff M^{\mathrm{set}} \models \phi(|T_{s_1}|, \ldots, |T_{s_k}|)$$

is Π^1_n for some n (again, in the parameter X). Δ_0-Comprehension is then straightforward to prove.

For Foundation, suppose towards a contradiction T is a suitable tree so that in M^{set}, $|T|$ is a non-empty set with no $\in^*$-minimal element. Let X witness $\mathrm{ISO}(T, X)$. Then

$$A = \{s \in T \mid (\exists x^1)\langle x\rangle \in T \wedge \langle\langle x\rangle, s\rangle \in X\}$$

is a set of nodes in T such that every element of A can be properly extended in A. This contradicts suitability of T.

$\mathrm{Ax}\,\beta$ is in a similar vein. Given a suitable tree R so that $|R| = r$ is a regular relation in M^{set}, verify that the relation $W = \{\langle s, t\rangle \in R \times R \mid \langle |R_s|, |R_t|\rangle \in r\}$ is a regular relation in $\mathcal{M}$. A tree F so that $|F| : \mathrm{dom}(r) \to \mathrm{ON}$ is precisely the collapse map is then defined by Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$ along the relation W.

(2) The inclusion $\subseteq$ is another coding exercise. The reverse follows from an application of Δ_0-Comprehension in M^{set}.

(3) Define a suitable F so that $f = |F| \supseteq \{\langle s, |T_s|\rangle \mid s \in T\}$.

(4) The construction is very nearly identical to that of Lemma VII.4.2 of [Sim09]. The only modifications are that we work in Π^1_∞-$\mathsf{TR}_{\mathbb{R}}$, and so do not induct along a wellorder; rather, we induct along the suitable tree A for which $\alpha = |A|$. The ramified language we define therefore makes use of variables v_i^a, where $i \in \omega$ and $a \in A$, intended to range over $L^{M^{\mathrm{set}}}_{|T_a|}$. The rest of the proof is unchanged.

(5) Evidently if $\mathcal{M}$ is a β-model, every suitable tree in $\mathcal{M}$ is in fact wellfounded, so that $\in^*$ is a wellfounded relation. Conversely, if $\mathcal{M}$ is not a β-model, there some tree T which $\mathcal{M}$ thinks is suitable, but is not wellfounded. Then if $\langle s_n\rangle_{n\in\omega}$ is a branch through T, the sequence $\langle |T_{s_n}|\rangle_{n\in\omega}$ witnesses illfoundedness of $\in^*$. $\square$

Theorem 4.3.2. *Let $\mathcal{M}$ be a β-model of Π^1_∞-$\mathsf{TR}_\mathbb{R}$. Then $L_\theta \subseteq M^{\text{set}}$.*

Proof. Work in M^{set}. Notice that ω_1 exists by an application of $\mathrm{Ax}\,\beta$ to the regular relation $\{\langle x, y\rangle \mid x, y$ are wellorders of ω with x isomorphic to an initial segment of $y\}$. Now if $\omega_1^L < \omega_1$, we're done, since L_{ω_1} is then a model of $\mathsf{ZF}^- +$ "$\mathcal{P}(\omega)$ exists", so θ must exist and be less than ω_1. So we can suppose $\omega_1^L = \omega_1$.

We have that every tree on $\mathcal{P}(\omega)$ is either ranked or illfounded by $\mathrm{Ax}\,\beta$; we claim the same is true in L. For suppose $T \in L$ is a tree on $\mathcal{P}(\omega) \cap L$. If T is ranked, then let $\rho : T \to \mathrm{ON}$ be the ranking function. Let α be large enough that $T \in L_\alpha$. Then it is easily checked that $\rho \in L_{\alpha+\omega\cdot\rho(\emptyset)}$; note the latter exists because (by $\mathrm{Ax}\,\beta$) the ordinals are closed under ordinal $+$ and $\cdot$.

Now suppose T is illfounded. Then let $x = \langle x_i \rangle_{i\in\omega}$ be a branch through T. Note that each $x_i \in L$, hence in L_{ω_1}. Let $\alpha_i < \omega_1^L$ be sufficiently large that $x_i \in L_{\alpha_i}$. Since $\omega_1 = \omega_1^L$, the map $i \mapsto \alpha_i$ is bounded in ω_1^L (note M^{set} models $\mathsf{DC}_\mathbb{R}$, so ω_1 is regular in M^{set}). So we have some admissible level L_γ with $\gamma < \omega_1$ so that $\alpha = \sup_{i\in\omega} \alpha_i < \gamma$; but then $T \cap L_\alpha$ is an illfounded tree, so has some branch definable over L_γ. So we have a branch through T in L. $\square$

4.4 Higher levels

For a transitive set U, let $\Delta_1(U)$-DET and $\Sigma_1(U)$-DET denote, respectively, clopen and open determinacy for game trees $T \subseteq U^{<\omega}$. We recall from Chapter 3 the principles Π_1-$\mathsf{RAP}(U)$:

Definition 4.4.1. Let U be a transitive set. The Π_1-*Reflection to Admissibles Principle for* U (denoted Π_1-$\mathsf{RAP}(U)$) is the assertion that $\mathcal{P}(U)$ exists, together with the following axiom scheme, for all Π_1 formulae $\phi(u)$ in the language of set theory: Suppose $Q \subseteq \mathcal{P}(U)$ is a set and $\phi(Q)$ holds. Then there is an admissible set M so that

- $U \in M$.
- $\bar{Q} = Q \cap M \in M$.

- $M \models \phi(\bar{Q})$.

For $n \in \omega$, let θ_n be the least ordinal so that L_{θ_n} is a model of "$\mathcal{P}^n(\omega)$ exists" plus Π_1-$\mathsf{RAP}(\mathcal{P}^n(\omega))$; note $\theta = \theta_0$, and by the definition of Π_1-$\mathsf{RAP}(U)$, $L_{\theta_n} \models$ "$\mathcal{P}^{n+1}(\omega)$ exists" + "ω_{n+1} is the largest cardinal". Furthermore, L_{θ_n} Σ_1-projects to ω with parameter $\{\omega_{n+1}\}$, and we have the following characterisation of the ordinals θ_n in terms of trees:

Proposition 4.4.2. *Say T is a tree on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$ if whenever $s \in T$, we have $s_{2n} \in \mathcal{P}^{n+1}(\omega)$ and $s_{2n+1} \in \mathcal{P}^n(\omega)$. Consider a closed game on such a tree, that is, a game where players cooperate to choose a branch through the tree, and player I wins precisely the infinite plays. Then θ_n is the least ordinal so that L_θ satisfies "for every tree T on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$, either I wins the closed game on T, or the game is ranked for player II".*

Note that a winning strategy for I in such a game is (coded by) an element of $\mathcal{P}^{n+1}(\omega)$; a ranking function for II (the open player) is a partial function $\rho : T \rightharpoonup \mathrm{ON}$ so that $\rho(\emptyset)$ exists, and whenever $s \in T$ has even length and $\rho(s)$ is defined, we have $(\forall x)(\exists y) s^\frown\langle x\rangle \in T \to \rho(s^\frown\langle x, y\rangle) < \rho(s)$.

We obtain a generalization of Schweber's separation result to higher types by looking at the models L_{θ_n}:

Theorem 4.4.3. *L_{θ_n} is a model of $\Delta_1(\mathcal{P}^{n+1}(\omega))$-DET, but not of $\Sigma_1(\mathcal{P}^{n+1}(\omega))$-DET, for each $n \in \omega$.*

Proof. The proof of $\Delta_1(\mathcal{P}^{n+1}(\omega))$-DET is exactly like that of $\Delta_1^{\mathbb{R}}$-DET in Theorem 4.1.1: given a parameter set Q coding a wellfounded tree T on $\mathcal{P}^{n+1}(\omega)$, if neither player wins the game on T, reflect this Π_1 statement to an admissible set M containing $\mathcal{P}^n(\omega)$. Use the fact that $T \cap M \in M$ is wellfounded to contradict admissibility.

To see that $\Sigma_1(\mathcal{P}^{n+1}(\omega))$-DET fails, again define a game where the open player proposes a Σ_1 formula $\phi(\omega_{n+1})$, and the closed player chooses a truth value and plays approximations to the model L_{θ_α} (now using the characterization of Proposition 4.4.2, closing under the operation sending a game tree on $\mathcal{P}^{n+1}(\omega), \mathcal{P}^n(\omega)$ to a winning strategy for I or ranking function for II, whichever exists), while player II lists elements of $\mathcal{P}^{n+1}(\omega)$ that must be included in the model. As before II has no winning strategy in V, so none in L_{θ_n}, and any winning strategy for I computes the $\Sigma_1(\{\omega_{n+1}\})$ theory of L_{θ_n}, so cannot belong to L_{θ_n}. $\square$

Note that we haven't attempted to give these results in the context of some standard base theory of n-th order arithmetic, but the models L_{θ_n}, being models of BST, should clearly be models of any reasonable such base theory.

4.5 Remarks: Provability versus β-consistency strength

We have shown that $\Sigma_1^{\mathbb{R}}$-DET, Σ_4^0-DET, and $\Delta_1^{\mathbb{R}}$-DET are strictly decreasing in consistency strength when we require the models under consideration to satisfy some mild absoluteness. For by the results of the last section, any β-model of $\Sigma_1^{\mathbb{R}}$-DET contains a copy of L_θ, and the argument of Theorem 4.1.1 then applies; it follows that any β-model of $\Sigma_1^{\mathbb{R}}$-DET must contain the Σ_1-theory of L_θ, from which winning strategies in Σ_4^0 games are computable. So a β-model of $\Sigma_1^{\mathbb{R}}$-DET always satisfies Σ_4^0-DET, in fact, (boldface) $\boldsymbol{\Sigma}_4^0$-DET.

This establishes the implication (1) $\Rightarrow$ (2) of Theorem 1.2.7. The implication (2) $\Rightarrow$ (3) is then made clear by absoluteness between β-models and V.

Now, we have (working in $\boldsymbol{\Pi}_1^1$-CA_0) that Σ_4^0-DET is equivalent to the existence of L_θ, which by Theorem 4.1.1 a β-model of $\Delta_1^{\mathbb{R}}$-DET, so Σ_4^0-DET is (consistency strength-wise) strictly stronger than $\Delta_1^{\mathbb{R}}$-DET. Theorem 4.1.1 also shows that any β-model of $\Sigma_1^{\mathbb{R}}$-DET must strictly extend the β-model obtained from L_θ, and hence satisfy that there exists a β-model of $\Delta_1^{\mathbb{R}}$-DET. But it is unclear whether $\Sigma_1^{\mathbb{R}}$-DET outright implies the existence of a β-model of $\Delta_1^{\mathbb{R}}$-DET, that is, whether Σ_4^0-DET is provable from the third order theory $\Sigma_1^{\mathbb{R}}$-DET.

Indeed, Schweber asks (Question 5.1 of [Sch13]) whether $\Sigma_1^{\mathbb{R}}$-DET and $\Delta_1^{\mathbb{R}}$-DET have the same second order consequences, and Σ_4^0-DET would be an interesting counterexample. However, the present study doesn't rule out the possibility that there are (necessarily non-β-)

models of $\Sigma_1^{\mathbb{R}}$-DET in which Σ_4^0-DET fails. One can show that there is no model of $\Sigma_1^{\mathbb{R}}$-DET whose reals are precisely those of L_θ, and so any such (set) model will be illfounded with wellfounded part well below θ. The problem of constructing such a model (if one exists) then seems a difficult one.

CHAPTER 5

Measurable cardinals and Borel-on-$\mathbf{\Pi}^1_1$ determinacy

This chapter continues the analysis of determinacy strength, but now for sets more complicated than $\mathbf{\Pi}^1_1$. At this level, our results will have the form of equivalences between determinacy and *inner models for large cardinals*; such results fall into the realm of *descriptive inner model theory.*

One early such result, due to Martin and Harrington [Mar70], [Har78] is the equivalence of analytic determinacy, $\mathbf{\Pi^1_1}$-DET, with the statement that for every real x, $x^{\#}$ *exists*: there is a nontrivial elementary embedding $j : L[x] \to L[x]$ from the class of sets constructible from x to itself. Note that $x^{\#}$ may be regarded as the unique minimal iterable model of the form $\langle L_\alpha[x], U\rangle$, where U is a normal ultrafilter on $\mathcal{P}^{L_\alpha}(\kappa)$ for some cardinal κ of $L_\alpha[x]$. This model is an example of a *mouse*. We are interested in results such as this which equate determinacy hypotheses with the existence of mice.

Recall $\mathbf{\Pi}^1_1$ is the pointclass of complements of projections of closed sets. Continuing our practice of restricting pointclasses to Baire space, ω^ω, we say that $A \subseteq \omega^\omega$ is in $\mathbf{\Pi}^1_1$ if and only if there is a tree T on $(\omega \times \omega)^{<\omega}$ such that

$$A = \neg p[T] = \{x \in \omega^\omega \mid \forall y \in \omega^\omega \langle x, y\rangle \notin [T]\}.$$

The pointclass $\mathbf{\Pi}^1_1$ properly contains the collection $\mathbf{\Delta}^1_1$ of Borel sets, and is a least such, in the sense that the Borel Wadge degrees are cofinal in $\mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$.

The pointclass $\mathbf{\Pi}^1_1$ is not a σ-algebra (it is not closed under complementation), so we define $\mathcal{B}(\mathbf{\Pi}^1_1)$, the class of *Borel-on-coanalytic sets* to be the smallest σ-algebra of subsets of ω^ω containing $\mathbf{\Pi}^1_1$. The results of this section concern sets in the pointclass $\mathcal{B}(\mathbf{\Pi}^1_1)$.

As in the case of $\mathbf{\Delta}^1_1$, we stratify $\mathcal{B}(\mathbf{\Pi}^1_1)$ into a hierarchy by closing $\mathbf{\Pi}^1_1$ under iteration of the operations of complementation and countable union. Set

$$\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1) = \left\{ \bigcup_{n\in\omega} A_n \mid \text{ each } A_n \text{ is a Boolean combination of } \mathbf{\Pi}^1_1 \text{ sets} \right\}$$

$$\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_1) = \left\{ \bigcup_{n\in\omega} \neg A_n \mid \text{ each } A_n \text{ is in } \mathbf{\Sigma}^0_{\eta_n}(\mathbf{\Pi}^1_1), \text{ for some } \eta_n < \alpha \right\}$$

Note that $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_1)$ is exactly what we would get if we had defined $\mathbf{\Sigma}^0_\alpha$ starting with the algebra generated by $\mathbf{\Pi}^1_1$ in place of the class of clopen sets. We have the same inclusion relations between the classes $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_1)$, their duals $\mathbf{\Pi}^0_\alpha(\mathbf{\Pi}^1_1)$, and the ambiguous pointclasses $\mathbf{\Delta}^0_\alpha(\mathbf{\Pi}^1_1)$ as in the Borel case; and $\mathcal{B}(\mathbf{\Pi}^1_1) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_1)$. Note also that $\mathcal{B}(\mathbf{\Pi}^1_1) \subsetneq \mathcal{A}\mathbf{\Pi}^1_1$, where the latter is the collection of sets obtained by applying Suslin's operation A to countable families of $\mathbf{\Pi}^1_1$ sets.

As in previous sections, we would like to obtain the sharpest results possible, and so we will work with a lightface (effective) version of this hierarchy. A development of the classes $\Sigma^0_\alpha(\Pi^1_1)$ using recursive Borel codes (in the fashion of [Mos09], Section 3H) produces the pointclasses we are interested in; the construction is verbatim identical, with an effective enumeration of Π^1_1 codes in place of that of Σ^0_1 codes. For the first ω levels of this hierarchy, we also have the following characterization, which will be useful for our lower bound arguments.

Proposition 5.0.1. *Let $z \in \omega^\omega$ and $n \geq 1$. Then $A \subseteq \omega^\omega$ is $\Sigma^0_n(\Pi^1_1(z))$ if and only if there is a Σ_{n+1} formula in the language of set theory, $\theta(u, v)$, so that for all $x \in \omega^\omega$, we have*

$$x \in A \iff L_{\omega_1^{\langle x,z\rangle}}[x, z] \models \theta[x, z].$$

We analyze the strength of hypotheses of the form: "all $\Sigma^0_\alpha(\Pi^1_1)$ are determined". For lower bounds on the strength of such assertions, we will require some of the technology of core model theory. Neeman [Nee00] has shown that if the reals are closed under the sharp function, then $\mathcal{A}\mathbf{\Pi}^1_1$-DET is equivalent to the existence of an iterable mouse satisfying $(\exists\kappa)V = L(V_{\kappa+2}) + o(\kappa) = \kappa^{++}$. Consequently, we will only require fine structural mice with *sequences of measures.* We therefore forego the state-of-the-art of inner models, and will get by just on the terminology and notation established in Mitchell [Mit10].

5.1 Unraveling $\boldsymbol{\Sigma}^0_1(\boldsymbol{\Pi}^1_1)$ sets and proving determinacy

In [Nee00], Neeman constructs an unraveling of $\Sigma^0_1(\Pi^1_1)$ sets by a tree on $V_{\kappa+2}$ from the assumption that for every $A \in V_{\kappa+2}$, there is a measure μ so that $A \in \mathrm{Ult}(V, \mu)$; in particular, the hypothesis is satisfied in premice satisfying $o(\kappa) = \kappa^{++}$. Combining this result with the methods of Chapter 3, we produce proofs of determinacy for sets in pointclasses of the form $\boldsymbol{\Sigma}^0_{1+\alpha+3}(\boldsymbol{\Pi}^1_1)$ from optimal hypotheses. We should be somewhat careful, however, as for our arguments to work, we need to unravel countably many $\Sigma^0_1(\Pi^1_1)$ sets with a tree on $V_{\kappa+2}, V_{\kappa+1}$, whereas the unraveling presented in [Nee00] is only a tree on $V_{\kappa+2}$.

Theorem 5.1.1. *Suppose M is a mouse satisfying "there exists a measurable cardinal κ with $o(\kappa) = \kappa^{++}$, and Π_1-$\mathsf{RAP}_{\kappa+1+\alpha}$ holds." Then $\Sigma^0_{1+\alpha+3}$-*DET *holds.*

Proof. Fix an ordinal α and measure sequence $\mathcal{U}$ so that $M = L_\alpha[\mathcal{U}]$; let κ be as in the statement of the theorem. For $A \in V^M_{\kappa+2}$, there exists, by acceptability, an ordinal $\eta < \kappa^{++M}$ so that $A \in L_\eta[\mathcal{U}]$. Since $o(\kappa) = \kappa^{++}$ in M, we have some γ with $\eta < \gamma < \kappa^{++}$ so that $U_\gamma \neq \emptyset$. By coherence of $\mathcal{U}$, $L_\gamma[\mathcal{U}] = L_\gamma[i_{\mathcal{U}_\gamma}(\mathcal{U})]$. In particular, A belongs to the ultrapower of M by $\mathcal{U}_\gamma$.

We have that $\kappa \in M$ satisfies the necessary hypothesis to carry out the unraveling of Π^1_1 sets in [Nee00]. In section 8 of that paper, Neeman shows that countably many $\Sigma^0_1(\Pi^1_1)$ sets may be simultaneously unraveled by a tree T on $V_{\kappa+2}$. The definition of the tree is given in Lemma 8.6 of that paper; as defined, the tree has Player II (who is there called S) playing sequences of elements of an unraveling tree on $V_{\kappa+2}$. The tree may be modified in the following way: in the terminology of [Nee00], when the player S is given the opportunity to accept or reject, S plays $r \in T_F$ of length $2j+1$ with $N_r = \emptyset$, then F chooses a (possibly different) $q \in T_F$ with the same length, and so that $h(q \restriction 2l+1) = h(r \restriction 2l+1), \ldots h(q \restriction 2j+1) = h(r \restriction 2j+1)$. The proof goes through with this change, and produces a full covering (rather than a demi-covering) by a tree on $V_{\kappa+2}, V_{\kappa+1}$, as needed; note further that the tree is definable over $V^M_{\kappa+2} \subseteq L_{\kappa^{++M}}[\mathcal{U}]$.

So let $T \in M$ be the tree on $V_{\kappa+2}, V_{\kappa+1}$ obtained by simultaneously unraveling all $\Sigma^0_1(\Pi^1_1)$ sets, with covering map $\pi : T \to \omega^{<\omega}$. Let $\langle A_n \rangle_{n\in\omega}$ be an effective enumeration of all $\Pi^0_{1+\alpha}(\Pi^1_1)$ sets. Then each set $\pi^{-1}(A_n)$ is $\mathbf{\Pi}^0_\alpha$ in $[T]$ if $\alpha > 0$, and $\mathbf{\Delta}^0_1$ if $\alpha = 0$. Note that our assumption that M satisfies $\Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$ implies that $V_{\kappa+1+\alpha+1}$ exists in M. By the argument of [Mar85], we can iterate the unraveling α many times ($\alpha > 0$), to obtain a cover T' on $V_{\kappa+1+\alpha+1}, V_{\kappa+1+\alpha}$ that simultaneously unravels any fixed sequence of $\mathbf{\Pi}^0_\alpha$ sets of T. Let T' be the simultaneous unraveling of the sets $\langle \pi^{-1}(A_n)\rangle_{n\in\omega}$, and let $\pi' : T' \to T$ be the covering map, if $\alpha > 0$; otherwise let $T' = T$ and π the identity map.

Suppose A is a fixed $\Sigma^0_{1+\alpha+3}(\Pi^1_1)$ set. Then in T', $(\pi' \circ \pi)^{-1}(A)$ is $\mathbf{\Sigma}^0_3$. Now M and $T' \in M$ satisfy the hypotheses of Lemma 3.2.3. We may therefore carry out the argument for $\mathbf{\Sigma}^0_3$ determinacy over this model, obtaining either a winning strategy σ for Player I in M, or a $\mathbf{\Delta}_1(M)$-definable winning strategy τ for II whose finite fragments belong to M. This is enough to apply the unraveling maps to the strategies obtained; therefore either $\psi(\sigma)$ is a winning strategy for I in the game $G(A; \omega^{<\omega})$ and σ belongs to M, or $\psi(\tau)$ is a winning strategy for II in $G(A; \omega^{<\omega})$, and is $\mathbf{\Delta}_1(M)$-definable. $\square$

5.2 Inner models with sequences of measures

We take for granted the basic notions and terminology for *mice with sequences of measures* as defined in [Mit10]. In particular, in this section, $\mathcal{U}$ always denotes a *coherent sequence of (partial) measures*, so that the structure $L_\alpha[\mathcal{U}]$ is acceptable. We make special note of the fact that whenever a measure $\mathcal{U}_\gamma$ on the sequence is non-empty, then it is indexed so that (if α is largest so that $\mathcal{U}_\gamma$ is a total measure in $L_\alpha[\mathcal{U}]$) we have $\gamma = \kappa^{++\mathrm{Ult}(L_\alpha[\mathcal{U}], \mathcal{U}_\gamma)}$.

Lemma 5.2.1. *Let M be a countable mouse, and suppose N is a countable premouse so that M and N disagree on their constructible wellorder of $\mathbb{R}$. Then the comparison of M and N must fail at some countable stage θ, so that either there are infinitely many drops on the N-side, or the model N_θ is illfounded with* $\mathrm{wfo}(N_\theta) \leq o(M_\theta)$.

Proof. Let $\mathcal{I} = \langle M_\nu, i_{\nu,\nu'}, U_\nu \rangle_{\nu \le \theta}$, $\mathcal{J} = \langle N_\nu, j_{\nu,\nu'}, V_\nu \rangle_{\nu \le \theta}$ be the iterations obtained by comparing M, N, with respective drop sets $D^{\mathcal{I}}, D^{\mathcal{J}}$. First we show the length θ of the comparison is countable; the argument is the standard one. If not, then $M_{\omega_1}, N_{\omega_1}$ are both defined, and must be wellfounded. Let H be a countable elementary substructure of H_{ω_2} so that the restrictions $\mathcal{I} \restriction \omega_1, \mathcal{J} \restriction \omega_1$ both belong to H, and so that $H \cap \omega_1$ is an ordinal, call it μ. Let $\pi : \bar{H} \to H$ be the anticollapse isomorphism. Then $\mathrm{crit}(\pi) = \mu$, and $\pi(\mu) = \omega_1$. Let $\langle \bar{M}_\nu, \bar{i}_{\nu,\nu'}, \bar{U}_\nu \rangle_{\nu' < \nu \le \mu}$ be the collapsed versions of the iteration in $\bar{H}$. By countability of the models M_ν for $\nu < \omega_1$ and since $\mu \subseteq H$, we have $\bar{M}_\nu = M_\nu$ for all $\nu < \mu$. Since we take direct limits at limit stages, we have $\bar{M}_\mu = M_\mu$ as well, and $\bar{i}_{\nu,\mu} = i_{\nu,\mu}$ for any $\nu < \mu$. Furthermore, $\pi \restriction M_\mu : M_\mu \to M_{\omega_1}$ is an elementary embedding; we claim it is equal to i_{μ,ω_1}.

For any $x \in M_\mu$, there is some $\xi < \mu$ and $\bar{x} \in M_\xi$ so that $i_{\xi,\mu}(\bar{x}) = x$. For such $x, \bar{x}$, we have

$$\pi(x) = \pi(i_{\xi,\mu}(\bar{x})) = \pi(\bar{i}_{\xi,\mu})(\pi(\bar{x})) = i_{\xi,\omega_1}(\bar{x}) = i_{\mu,\omega_1}(x),$$

by elementarity of π and the fact that $\pi(\mu) = \omega_1$, $\mathrm{crit}(\pi) = \mu$.

An identical argument shows $\pi \restriction N_\mu : N_\mu \to N_{\omega_1}$ is equal to j_{μ,ω_1}. Now for any $X \in M_\mu \cap N_\mu$ with $X \subseteq \mu$, we have

$$X \in U_\mu \iff \mu \in i_{\mu,\omega_1}(X) \iff \mu \in \pi(X) \iff \mu \in j_{\mu,\omega_1}(X) \iff X \in V_\mu.$$

So $U_\mu = V_\mu$. But this contradicts the fact that these measures were applied at the same stage of the comparison.

We have established that $\theta < \omega_1$. We continue by supposing that there are only finitely many drops on the N-side of the comparison, so that N_θ is defined, and showing $\mathrm{wfo}(N_\theta) \le o(M_\theta)$.

For premice P, let $<^P$ denote the canonical wellorder in P furnished by the fact that P is a premouse satisfying $V = L[\mathcal{U}]$ for some sequence of measures. Since M and N disagree on their constructible wellorder of $\mathbb{R}$, we fix reals x, y so that $x <^M y$ while $y <^N x$; here we extend the meaning of $x <^M y$ to include the case that $x \in M$ while $y \notin M$, and similarly for $<^N$.

Since M is a mouse, M_θ is wellfounded. Suppose towards a contradiction that $\mathrm{wfo}(N_\theta) > o(M_\theta)$, so that M_θ is a proper initial segment of N_θ. Then there must be a drop on the M-side, as otherwise we would have $x <^{M_\theta} y$, implying $x <^{N_\theta} y$, a contradiction to elementarity of the iteration maps and our assumption that $y <^N x$. Let ι be the largest element of $D^{\mathcal{I}}$. Since we drop at ι, all models M_ν for $\iota < \nu \leq \theta$ fail to be sound. In particular, M_θ is not sound; but this contradicts the fact that N_θ is a premouse.

So we must have $\mathrm{wfo}(N_\theta) \leq o(M_\theta)$. We just need to see that N_θ is illfounded. Suppose not; then $N_\theta \trianglelefteq M_\theta$. As before, we must have a drop on the N-side, and N_θ fails to be sound. Since M_θ is a premouse, we must have $N_\theta = M_\theta$, and we drop on the M-side as well. Let $\iota = \max D^{\mathcal{I}}, \eta = \max D^{\mathcal{J}}$ be the last drops. Now since M_ι, N_η are both sound, these must be the transitive collapses of the core of $M_\theta = N_\theta$, with $i_{\iota,\theta} = j_{\eta,\theta}$ the anticore embedding. But this implies $U_\iota = V_\eta$ as in the proof that θ is countable, so that the same measure was applied on both sides of the iteration, a contradiction. $\square$

Since we work with an ambient anti-large cardinal hypothesis, we have a sharper bound on the length of iterations. For the remainder of this thesis, we assume that the models we work with satisfy

$$(\forall\gamma) \text{ if } \mathcal{U}_\gamma \neq \emptyset, \text{ then } L_\gamma[\mathcal{U}] \not\models (\exists\kappa) o(\kappa) = \kappa^{++}. \tag{$\dagger$}$$

Note that by our indexing convention, we then have that whenever $M = L_\alpha[\mathcal{U}]$ satisfies ($\dagger$), is iterable, and $\mathcal{U}_\gamma$ is a nonempty measure with critical point κ, that $\sup\{\delta < \gamma \mid \mathcal{U}_\delta \neq \emptyset\}$ is an ordinal strictly less than γ with cardinality at most κ^+ in $L_\gamma[\mathcal{U} \restriction \gamma]$.

In the following lemma, ω_1^z for reals z is the least z-admissible ordinal, that is, the least α so that $L_\alpha[z] \models \mathsf{KP}$; we will let $\mathcal{A}_z$ denote $L_{\omega_1^z}[z]$, the least admissible set containing z. $\langle w, z\rangle$ denotes the real $\langle w(0), z(0), w(1), z(1), \ldots\rangle$.

Theorem 5.2.2. *Let w, z be codes for countable premice M, N, respectively, so that M is a mouse, and both satisfy ($\dagger$). Then the comparison of M, N has length at most $\omega_1^{\langle w,z\rangle}$; moreover, if $\omega_1^{\langle w,z\rangle} \in \mathrm{wfo}(N_{\omega_1^{\langle w,z\rangle}})$, then setting $\kappa = \omega_1^{\langle w,z\rangle}$, we have*

1. *In M_κ, κ^+ exists;*

2. *M_κ and N_κ have the same subsets of κ, and therefore $\kappa^{+M_\kappa} = \kappa^{+N_\kappa}$;*

3. *Let $\mathcal{U}$, $\mathcal{V}$ be the measure sequences in M_κ, N_κ; then for all $\alpha < \mathrm{wfo}(N_\kappa)$, $\mathcal{U}_\alpha = \mathcal{V}_\alpha$.*

4. *Let $\delta = \sup\{\gamma < \mathrm{wfo}(N_\kappa) \mid \mathcal{V}_\gamma \neq \emptyset\}$, and suppose $\delta < \mathrm{wfo}(N_\kappa)$. Then we have $M_\kappa | \mathrm{wfo}(N_\kappa) = N_\kappa | \mathrm{wfo}(N_\kappa)$, and in both models, $o(\kappa) = \kappa^{++}$ holds.*

Proof. Suppose the comparison lasts to stage $\omega_1^{\langle w,z\rangle}$ with iterations $\mathcal{I} = \langle M_\nu, i_{\nu,\nu'}\rangle$, $\mathcal{J} = \langle N_\nu, j_{\nu,\nu'}\rangle$. Working in $\mathcal{A}_{\langle w,z\rangle}$, note that by admissibility and Δ_1-definability of the iterations in the codes w, z, that there are finitely many drops on the N-side of the comparison. Put $\alpha_0 = \max \omega_1^{\langle w,z\rangle} \cap (D^{\mathcal{I}} \cup D^{\mathcal{J}})$. If $\mathrm{wfo}(N_{\omega_1^{\langle w,z\rangle}}) = \omega_1^{\langle w,z\rangle}$, we're done; we may assume therefore that $\omega_1^{\langle w,z\rangle} \in \mathrm{wfp}(N_{\omega_1^{\langle w,z\rangle}})$.

The following claim is Lemma 2 in [Ste82].

Claim 5.2.3. *Suppose $\omega_1^{\langle w,z\rangle} \in \mathrm{wfp}(N_{\omega_1^{\langle w,z\rangle}})$. For $\nu < \omega_1^{\langle w,z\rangle}$, let $\kappa_\nu = \mathrm{crit}(i_{\nu,\omega_1^{\langle w,z\rangle}})$ and $\lambda_\nu = \mathrm{crit}(j_{\nu,\omega_1^{\langle w,z\rangle}})$. Then there is a club $C \subseteq \omega_1^{\langle w,z\rangle}$, $\mathbf{\Delta}_1$-definable over $\mathcal{A}_{\langle w,z\rangle}$, so that for all $\nu, \nu' \in C$,*

$$i_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'} = \lambda_{\nu'} = j_{\nu,\nu'}(\lambda_\nu).$$

Proof of Claim 5.2.3. Work in $\mathcal{A}_{\langle w,z\rangle}$. Fix $\nu_0 > \alpha_0$ (so ν_0 is past the last drop of M) so that $i_{\nu_0,\omega_1^{\langle w,z\rangle}}(\alpha) = \omega_1^{\langle w,z\rangle}$ for some $\alpha \in M_{\nu_0}$, and set $\alpha_\nu = i_{\nu_0,\nu}(\alpha)$ for all $\nu > \nu_0$. Since $\beta < \omega_1^{\langle w,z\rangle}$ if and only if $\beta < \kappa_\nu$ for some ν, we have for any $\eta < \alpha_\nu$ that there exists $\nu' > \nu$ with $i_{\nu,\nu'}(\eta) < \kappa_{\nu'}$. For $\nu > \nu_0$, define

$$f(\nu) = \text{ least } \gamma \text{ such that for all } \eta < \alpha_\nu, \text{ we have } i_{\nu,\gamma}(\eta) < \kappa_\gamma.$$

Such exists by admissibility and $\Delta_1(w,z)$-definability of the iteration. The set C_0 of closure points of f is a $\mathbf{\Delta}_1$-definable club class.

We claim $\kappa_\nu = \alpha_\nu$ for all $\nu \in C_0$, from which it follows that $i_{\nu,\nu'}(\kappa_\nu) = \kappa_\nu'$ for all $\nu, \nu' \in C_0$. Clearly $\alpha_\nu \geq \kappa_\nu$ for all $\nu > \nu_0$. If $\eta < \alpha_\nu$, then $\eta = i_{\nu',\nu}(\bar\eta)$ for some $\bar\eta \in \mathrm{ON}^{M_{\nu'}}$. Since ν is a closure point of f, we have some γ, $\nu' < \gamma < \nu$, so that $i_{\nu',\gamma}(\bar\eta) < \kappa_\gamma$. Then since

$\mathrm{crit}(i_{\gamma,\nu}) = \kappa_\gamma$, $\eta = i_{\nu',\nu}(\bar{\eta}) < \kappa_\gamma$. We have shown $\sup_{\nu'<\nu} \kappa'_\nu \geq \alpha_\nu$ for $\nu \in C_0$, implying $\alpha_\nu \leq \kappa_\nu$ for such ν.

Repeating the argument with the iteration $\mathcal{J}$ and intersecting with C_0 gives the desired club. $\square$

From now on, then, we fix a club C as in the claim and let κ_ν for $\nu \in C$ denote the critical points on both sides of the iteration. Since $i_{\nu,\omega_1^{\langle w,z\rangle}}(\kappa_\nu) = \omega_1^{\langle w,z\rangle}$ for all such ν, we from now on let κ denote $\omega_1^{\langle w,z\rangle}$. By elementarity and the fact that $M_\nu \models$ "κ_ν^+ exists" (by our indexing convention for measures), we have that κ^+ exists in M_κ, so (1) of the theorem holds.

We isolate for later use a consequence of the last claim.

Claim 5.2.4. *There is a $\Sigma^0_2(\Pi^1_1)$ property of pairs w, z that holds if and only if w, z are reals coding premice M, N respectively so that the comparison of M, N lasts to stage $\omega_1^{\langle w,z\rangle}$ and $\omega_1^{\langle w,z\rangle} \in M_{\omega_1^{\langle w,z\rangle}}$.*

Proof of Claim 5.2.4. Denote the iterations of the comparison by $\langle M_\nu, i_{\nu,\nu'}\rangle, \langle N_\nu, j_{\nu,\nu'}\rangle$ as before. We claim the desired $\Sigma^0_2(\Pi^1_1)$ property is

$$\mathcal{A}_{\langle w,z\rangle} \models (\exists \nu)(\forall \mu > \nu)(\exists \nu' > \mu) i_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'}.$$

By Claim 5.2.3, if $\omega_1^{\langle w,z\rangle} \in M_{\omega_1^{\langle w,z\rangle}}$ then there exists a club C witnessing that this property holds; conversely, if ν is such that $i_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'}$ for arbitrarily large $\nu' < \omega_1^{\langle w,z\rangle}$, then $i_{\nu,\omega_1^{\langle w,z\rangle}}(\kappa_\nu) = \sup_{\nu'<\omega_1^{\langle w,z\rangle}} i_{\nu,\nu'}(\kappa_\nu) = \omega_1^{\langle w,z\rangle}$. $\square$

Claim 5.2.5. $\mathcal{P}(\kappa) \cap M_\kappa = \mathcal{P}(\kappa) \cap N_\kappa$.

Proof. For suppose $X \in \mathcal{P}(\kappa) \cap M_\kappa$. Let $\xi > \alpha_0$ be sufficiently large that $\bar{X} = X \cap \kappa_\xi \in M_\xi$ with $i_{\xi,\kappa}(\bar{X}) = X$. For every $\nu \in C$ above ξ, we must have $i_{\xi,\nu}(\bar{X}) = X \cap \kappa_\nu \in N_\nu$, since otherwise we would have truncated on the N-side, a contradiction to $\nu > \alpha_0$.

Suppose towards a contradiction that for every $\nu > \xi$ in C, we had $j_{\nu,\kappa}(X \cap \kappa_\nu) \neq X$. Then the map defined on C by

$$f(\nu) = \text{ the least } \mu \text{ such that } j_{\nu,\mu}(i_{\xi,\nu}(\bar{X})) \neq i_{\xi,\mu}(\bar{X})$$

is $\mathbf{\Delta}_1$-definable. By admissibility, let ν be a limit point of C with $f"\nu \subseteq \nu$. We have $X \cap \kappa_\nu \in N_\nu$, so that $X \cap \kappa_\nu = j_{\nu',\nu}(\bar{Y})$ for some $\nu' < \nu$ in C and $\bar{Y} \in N_{\nu'}$. But this is a contradiction to the fact that ν is a closure point of f, since necessarily $\bar{Y} = X \cap \kappa_{\nu'}$.

By symmetry, we have the desired equality. $\square$

Note the last claim is just an immediate consequence of the fact that we truncate in fine structural iterations. It follows that $\kappa^{+M_\kappa} = \kappa^{+N_\kappa}$, and we have (2) of the theorem; from now on, we just write 'κ^+' without risk of ambiguity.

Let us say m is a *code for* $X \in M_\kappa$ if in $\mathcal{A}_{\langle w,z\rangle}$, we have that

$$h_1(m) = \langle \nu, \bar{X}\rangle, \text{ where } \nu \in C, \bar{X} \in M_\nu, \text{ and } i_{\nu,\kappa}(\bar{X}) = X,$$

(where here h_1 is the canonical $\Sigma_1(w,z)$ Skolem function for $\mathcal{A}_{\langle w,z\rangle}$); similarly for elements of N_κ. Statements such as "m codes a subset of κ^+ in M_κ" then have the obvious meaning; note that the property of coding a subset of $\mathcal{P}(\kappa)$ in M_κ is captured by the statement

$$h_1(m) = \langle \nu, \bar{X}\rangle, \text{ where } \bar{X} \subseteq P(\kappa_\nu) \text{ and } (\forall \eta)(\exists \nu' > \nu) i_{\nu,\nu'}(\kappa_\nu) = \kappa'_\nu,$$

by Lemma 5.2.3. This is Π_2 over $\mathcal{A}_{\langle w,z\rangle}$, so by Proposition 5.0.1, is a $\Pi^0_1(\Pi^1_1)$ relation about m, w, z.

The following observation is crucial to our lower bounds. It is the analogue to the that fact that comparing reals in countably-coded ω-models is Π^0_1 in the codes.

Lemma 5.2.6. *Suppose $P \in M_\kappa, Q \in N_\kappa$ are subsets of $\mathcal{P}(\kappa)$. Then $P = Q$ if and only if there is a club (equivalently, unbounded) $D \subseteq C$ so that for all $\nu \in D$, $P_\nu = Q_\nu$ (where here $i_{\nu,\kappa}(P_\nu) = P$ and $j_{\nu,\kappa}(Q_\nu) = Q$ for all $\nu \in D$).*

If p, q are codes for P, Q, respectively, then this club is uniformly $\Delta_1(\mathcal{A}_{\langle w,z\rangle})$ in the codes; consequently, the relation

$$p, q \text{ are codes for } P \in M_\kappa \cap \mathcal{P}^2(\kappa),\ Q \in N_\kappa \cap P^2(\kappa),\ \text{respectively, and } P = Q$$

is $\Pi^0_1(\Pi^1_1)$ as a relation on $\langle w, z, p, q\rangle$ when $\omega_1 \in \mathrm{wfo}(M_{\omega_1}) \cap \mathrm{wfo}(N_{\omega_1})$.

Proof. Suppose such an unbounded D exists. If $X \subseteq \kappa$ witnesses disagreement between P and Q, then let $\nu \in D$ be sufficiently large that $i_{\nu,\kappa}(X \cap \kappa_\nu) = X = j_{\nu,\kappa}(X \cap \kappa_\nu)$. But then by elementarity, $X \cap \kappa_\nu$ is in the symmetric difference of P_ν, Q_ν, contradicting the assumption on D.

Conversely, supposing $P = Q$, let ν be sufficiently large that P_ν, Q_ν exist, and define f to be the map on C by setting, for $\gamma > \nu$ in C, $f(\gamma) = \delta$ iff δ is least in C such that

$$(\forall X \in \mathcal{P}(\kappa_\gamma) \cap M_\gamma)[X \in P_\gamma \triangle Q_\gamma \to (\exists \eta < \delta) i_{\gamma,\eta}(X) \neq j_{\gamma,\eta}(X)].$$

Note such an ordinal exists, by admissibility and the fact that any disagreement between P_γ, Q_γ is eventually iterated away. The map f is $\boldsymbol{\Delta}_1$ in parameters, so its set D of closure points is $\boldsymbol{\Delta}_1$, and clearly satisfies the condition of the lemma.

For the final claim, let $p, q \in \omega$ code $P \in M_\kappa \cap \mathcal{P}^2(\kappa)$ and $Q \in N_\kappa \cap \mathcal{P}^2(\kappa)$, respectively. By what was just shown, $P = Q$ if and only if $\mathcal{A}_{\langle w,z\rangle}$ satisfies, for all $\nu, \mu \in C$, $\bar{P} \in M_\nu \cap \mathcal{P}^2(\kappa_\nu)$, and $\bar{Q} \in N_\mu \cap \kappa_\mu$,

$$h_1(p) = \langle \nu, \bar{P}\rangle, h_1(q) = \langle \mu, \bar{Q}\rangle \text{ implies } (\forall \eta)(\exists \nu' > \eta) i_{\nu,\nu'}(\bar{P}) = j_{\mu,\nu'}(\bar{Q}).$$

By the remarks preceding the lemma, the desired statement is Π_2 over $\mathcal{A}_{\langle w,z\rangle}$, so again by Proposition 5.0.1, the relation is $\Pi^0_1(\Pi^1_1)$. □

The following claim is the final ingredient in showing the iteration terminates, and will be useful in comparing the final models besides.

Claim 5.2.7. *Let $\mathcal{U}$,$\mathcal{V}$ denote the measure sequences of M_κ, N_κ, respectively. Let $\beta =$ $\mathrm{wfo}(N_\kappa)$. Then $\mathcal{U} \restriction \beta = \mathcal{V} \restriction \beta$.*

Proof. Suppose towards a contradiction that $\alpha < \beta$ is least so that $\mathcal{U}_\alpha \neq \mathcal{V}_\alpha$. Note that then $\alpha \leq \kappa^{++}$ on both sides. Since M_κ, N_κ have the same subsets of κ, the measure indexed must be total. By our anti-large cardinal assumption, we have

$$\delta = \sup\{\gamma < \alpha \mid \mathcal{U}_\gamma \neq \emptyset\} = \sup\{\gamma < \alpha \mid \mathcal{V}_\gamma \neq \emptyset\} < \alpha.$$

Since $\kappa^{+M_\kappa|\alpha} = \kappa^{+N_\kappa|\alpha}$ is the largest cardinal of $M_\kappa|\alpha = N_\kappa|\alpha$, there exists a surjection of $\mathcal{P}(\kappa)$ onto $\delta + 1$ in both models. Furthermore, the constructibly-least wellorder $\prec$ of $\mathcal{P}(\kappa)$ with ordertype $\delta + 1$ is the same in M_κ and in N_κ.

Let therefore $P \in M_\kappa$ code $\mathcal{U} \restriction \delta+1$ as a subset of $\mathcal{P}(\kappa)$ in some standard fashion, via the wellorder $\prec$; similarly for $Q \in N_\kappa$. Then $P = Q$, and we obtain a club D as in Lemma 5.2.6. Now we have for $\nu \in D$ that there exists $\delta_\nu \in M_\nu$ so that $i_{\nu,\kappa}(\delta_\nu) = \delta$, by elementarity; since $P_\nu = Q_\nu$ we have $j_{\nu,\kappa}(\delta_\nu) = \delta$ as well, and furthermore, letting $\mathcal{U}^\nu, \mathcal{V}^\nu$ be the measure sequences in M_ν, N_ν, respectively, we have

$$\mathcal{U}^\nu \restriction \delta_\nu + 1 = \mathcal{V}^\nu \restriction \delta_\nu + 1.$$

Since $\mathrm{crit}(i_{\nu,\kappa}) = \mathrm{crit}(j_{\nu,\kappa}) = \kappa_\nu$ for $\nu \in D$, we must apply measures with critical point κ_ν on both sides of the iteration. If we let $\alpha_\nu^M, \alpha_\nu^N$ be the least indices of a measure above δ_ν, then

$$\alpha_\nu^M = \kappa_\nu^{++} = \kappa_\nu^{++L[\mathcal{U}^\nu \restriction \delta_\nu+1]} = \kappa_\nu^{++L[\mathcal{V}^\nu \restriction \delta_\nu+1]},$$

by agreement of the measure sequences and the way measures are indexed. Then by elementarity, $i_{\nu,\kappa}(\alpha_\nu^M) = \alpha = j_{\nu,\kappa}(\alpha_\nu^N)$. We have that $\mathcal{U}_\alpha$ and $\mathcal{V}_\alpha$ are both nonempty, and must then disagree. Suppose $X \subset \kappa$ witnesses this disagreement, and let $\nu \in D$ be sufficiently large that $i_{\nu,\kappa}(\bar{X}) = X = j_{\nu,\kappa}(\bar{X})$, where $\bar{X} = X \cap \kappa_\nu$. We have by elementarity that $\bar{X} \in \mathcal{U}^\nu_{\alpha_\nu^M} \triangle \mathcal{V}^\nu_{\alpha_\nu^N}$, whereas

$$\bar{X} \in \mathcal{U}^\nu_{\alpha_\nu^M} \iff \kappa_\nu \in i_{\nu,\kappa}(\bar{X}) \iff \kappa_\nu \in X \iff \kappa_\nu \in j_{\nu,\kappa}(\bar{X}) \iff \bar{X} \in \mathcal{V}^\nu_{\alpha_\nu^N},$$

since these measures were applied at stage ν, as witnessed by the disagreement on $X \cap \kappa_\nu$ and since $\nu \in D$; this is a contradiction. $\square$

The claim implies (3) of the theorem. For (4), set $\beta = \mathrm{wfo}(N_\kappa)$, and suppose $\delta = \sup\{\gamma < \beta \mid \mathcal{U}_\gamma \neq \emptyset\} < \beta$. If $\delta = \kappa^{++N_\kappa|\beta}$, then (4) follows from (3). Otherwise, we have equality of the measure sequences up to $\delta+1$, and M_κ, N_κ contain the same least constructible surjection of $\mathcal{P}(\kappa)$ onto δ. By the same argument as in (3), the iteration would have halted before $\omega_1^{\langle w,z\rangle}$ if there were no nonempty measure $\mathcal{V}_b$ indexed at some nonstandard ordinal b of N_κ; since $\delta < \beta$, there is a least such b, and we must have $\mathcal{V}_b = \mathcal{U}_\eta$ for some $\eta < o(\kappa)^{M_\kappa}$, again by the argument of (3). But this is impossible, since $\mathrm{wfo}(N_\kappa) = \beta < \eta$ is definable from the measure $\mathcal{U}_\eta = \mathcal{V}_b$ and $\mathcal{P}(\kappa)$.

We still haven't shown the iteration terminates at stage $\kappa = \omega_1^{\langle w,z\rangle}$. By the arguments just given, the only way this could be is if N_κ were wellfounded and some disagreement existed at a measure with critical point strictly above κ. But this would violate the assumption that (†) holds in N, since $o(\kappa) = \kappa^{++}$ in N_κ. □

We require some lemmas analogous to Lemmas 3.4.7 and 3.4.10 for identifying overspill of higher type.

Lemma 5.2.8. *Suppose M, N are premice coded by reals w, z so that, setting $\kappa = \omega_1^{\langle w,z\rangle}$, we have that the comparison of M, N lasts to stage $\omega_1^{\langle w,z\rangle}$, M_κ, N_κ are both defined, and κ belongs to the wellfounded parts of both models. Fix $\beta < \omega_1^{\mathsf{CK}}$, and let $\mu, \nu \in \mathrm{ON} \cup \{\mathrm{ON}\}$ of M_κ, N_κ (with codes m, n), respectively. Then*

- *The relation "$V_{\kappa+1+\beta}^{M_\kappa|\mu} = V_{\kappa+1+\beta}^{N_\kappa|\nu}$" is $\Pi^0_{1+\beta+1}(\Pi^1_1)$ (as a relation on $\langle w, z, m, n\rangle$);*
- *Suppose $a, b \in V_{\kappa+1+\beta+1}$ of $M_\kappa|\mu$ and $N_\kappa|\nu$, respectively; and that the clause above holds. Then the relation "$a = b$" is $\Pi^0_{1+\beta}(\Pi^1_1)$ (on w, z, m, n and codes for a, b).*

Proof. For $\beta = 0$, we automatically have $V_{\kappa+1}^M = V_{\kappa+1}^N$, by Lemma 5.2.2; and equality of subsets of $V_{\kappa+1}$ between the models is $\Pi^0_1(\Pi^1_1)$, by Lemma 5.2.6. For $\beta > 0$, the lemma is proved by induction in the same manner as the proof of Lemma 3.4.7. □

Definition 5.2.9. Let M_κ, N_κ be as in the previous lemma. Suppose $\beta < \omega_1^{\mathsf{CK}}$. We say *$a$ witnesses β-disagreement past $\kappa + 1$* if either

- $a \in V^{M_\kappa}_{\kappa+1+\beta+1}$, and for some $\mu \in \mathrm{ON}^{M_\kappa} \cup \{\mathrm{ON}^{M_\kappa}\}$, $a \subseteq V^{N_\kappa|\operatorname{rank}_{L[U]}(a)}_{\kappa+1+\beta} = V^{M_\kappa|\mu}_{\kappa+1+\beta}$ (in particular, both models believe $V_{\kappa+1+\beta}$ exists), and for every $b \in V^{N_\kappa}_{\kappa+1+\beta+1}$ there is some u belonging to this common $V_{\kappa+1+\beta}$ that is in the symmetric difference of a and b; or

- $a = \infty$, where we define ∞ to be a symbol so that $x <^{N_\kappa} \infty$ for all $x \in N_\kappa$; and in N_κ, $\kappa^{+1+\beta+1}$ is the largest cardinal, and $V^{N_\kappa}_{\kappa+1+\beta} \subseteq M_\kappa$.

In analogy with Chapter 3, the relation "a witnesses β-disagreement past $\kappa+1$" is $\Sigma^0_{1+\beta+2}(\Pi^1_1)$ as a relation on the reals w, z and code i for M, N, a. The point is again that witnessing β-disagreement is supposed to isolate a point where $V^{N_\kappa}_{\kappa+1+\beta}$ of N_κ is an element of M_κ, and an element of $V^{N_\kappa}_{\kappa+1+\beta+1}$ is constructed that does not appear anywhere in M_κ. We need the new, second clause because the model N_κ may not project to $\kappa^{+1+\beta}$ unboundedly often when $\kappa^{+1+\beta+1}$ is the largest cardinal of N_κ. If this happens, we would still like to say that a β-disagreement exists, and let the symbol ∞ serve as the witness; we let $x <^{N_\kappa} \infty$ for all $x \in N_\kappa$.

Suppose N is a premouse satisfying: for all ordinals α, whenever $o(\kappa) = \kappa^{++}$ in $L_\alpha[\mathcal{U}]$, then $L_\alpha[\mathcal{U}] \not\models \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$. Suppose a of $V^{N_\kappa}_{\kappa+1+\alpha+1} \cup \{\infty\}$ is constructed at a stage η where $o(\kappa) = \kappa^{++}$ in $N_\kappa|\eta$ (here $\eta = \mathrm{ON}^{N_\kappa}$ if $a = \infty$). We define a finite descending sequence of ordinals with final element $\kappa^{+1+\alpha}$, as follows.

$$\delta^{\mathcal{U}}_\alpha(0,a) = \begin{cases} \delta \text{ least s.t. } a \in N|\delta+1 & \text{if } N_\kappa|\delta \models \text{“}V_{\kappa+1+\alpha+1} \text{ exists”}, \\ \mathrm{ON}^{N_\kappa} & \text{if } a = \infty \text{ and } N_\kappa \models \text{“}V_{\kappa+1+\alpha+1} \text{ exists”}; \end{cases}$$

$$\delta^{\mathcal{U}}_\alpha(k+1,a) = \begin{cases} \delta \text{ least s.t. } N|\delta(k,a) \models \text{“}V_{\kappa+1+\alpha+1} \text{ exists and} & \\ (\exists Q \in N|\delta+1)\ Q \subseteq V_{\kappa+1+\alpha+1} \text{ witnesses} & \\ \text{failure of } \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha} \text{ in } N|\delta^{\mathcal{U}}_\alpha(k,a),\text{”} & \text{if such exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This is the analogue to the sequence $\delta_\alpha(k,x)$ from Definition 3.4.9. Furthermore, we obtain

Lemma 5.2.10. *There is a $\Sigma^0_{1+\alpha+2}(\Pi^1_1)$ relation $R^{\mathcal{U}}_\alpha(k,\gamma,a)$ such that the following holds, whenever w,z produce models M,N satisfying "for all α, if $L_\alpha[\mathcal{U}] \models o(\kappa) = \kappa^{++}$, then $L_\alpha[\mathcal{U}] \not\models \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$". Suppose the comparison of M,N lasts to stage $\kappa = \omega_1^{\langle w,z\rangle}$, M_κ is wellfounded, $N_\kappa | \operatorname{rank}_{L[\mathcal{U}]}(a) \models$ "$V_{\kappa+1+\alpha+1}$ exists", and a is the $<^{N_\kappa}$-least element of $V^{N_\kappa}_{\kappa+1+\alpha+1} \cup \{\infty\}$ witnessing α-disagreement past $\kappa+1$. Then:*

(A) $(\forall k \in \omega)(\forall \gamma \in \mathrm{ON}^{N_\kappa}) R^{\mathcal{U}}_\alpha(k,\gamma,a) \to \delta^{\mathcal{U}}_\alpha(k+1,a)$ *is standard;*

(B) $(\forall k \in \omega)$ *if* $\delta^{\mathcal{U}}_\alpha(k,a)$ *is nonstandard and* $\delta^{\mathcal{U}}_\alpha(k+1,a)$ *is wellfounded, then* $(\forall \gamma \in \mathrm{ON}^{N_\kappa})[R^{\mathcal{U}}_\alpha(k,\gamma,a) \leftrightarrow (\gamma < \delta^{\mathcal{U}}_\alpha(k,a))^{N_\kappa} \wedge \gamma$ *is nonstandard*$]$.

5.3 The lower bound

We assume in this section that $\mathbb{R}$ is closed under the sharp function. Our lower bound argument, like those in [Ste82], relies on the following result of Kechris [Kec78a].

Theorem 5.3.1. *Let Γ be a pointclass closed under recursive substitutions. Suppose Γ-DET holds. Then every $\Game\Gamma$ set has the Baire property; in particular, there does not exist a $\Game\Gamma$ wellordering of the reals $\mathbb{R}$.*

We will show that if there is no mouse satisfying $(\exists\kappa) o(\kappa) = \kappa^{++} + \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$, then the canonical wellorder of the reals in K is $\Game\Pi^0_{1+\alpha+3}(\Pi^1_1)$. By the theorem, we have a failure of $\Sigma^0_{1+\alpha+3}$-DET in K, which reflects to V by $\mathbf{\Sigma}^1_3$-absoluteness of K [SW98].

Theorem 5.3.2. *Assume $x^\#$ exists for all reals x. Suppose $\Sigma^0_{1+\alpha+3}(\Pi^1_1)$-DET holds, where $\alpha < \omega_1^{\mathsf{CK}}$. Then there is a mouse satisfying $o(\kappa) = \kappa^{++} + \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$.*

Proof. We prove the contrapositive. Assume there exists no mouse M so that $o(\kappa) = \kappa^{++} + \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}$ holds in M. Then the same is true in K, the core model for sequences of measures (see e.g. Section 4.1 of [Mit10] for a presentation of the core model in this setting). Working now in K, we define a game $G_{x,y}$, for all pairs of reals x,y. The players are required to produce reals w, z coding countable premice M, N, respectively, both satisfying

$$V = L[\mathcal{U}], \text{ and } (\forall\alpha) L_\alpha[\mathcal{U} \restriction \alpha] \not\models o(\kappa) = \kappa^{++} \wedge \Pi_1\text{-}\mathsf{RAP}_{\kappa+1+\alpha}.$$

We furthermore require Player I to play w so that $x <^M y$, and Player II to play z so that $y \leq^N x$, where $<^M, <^N$ are the respective constructibility orders of the premice. Note that being a countable premouse is a first-order property, so the winning conditions so far expressed are arithmetical in x, y.

Now, suppose the players have produced premice M, N so that no rules have been broken. We take the comparison of these premice, obtaining a sequence of models $\langle M_\nu, N_\nu \rangle_{\nu \leq \theta}$, for some $\theta \leq \omega_1^{\langle w,z \rangle}$. If the iteration terminates before $\omega_1^{\langle w,z \rangle}$, then let μ be least such that one or both of M_μ, N_μ is undefined or illfounded. If M_μ is the offending model, Player I loses; otherwise, Player II loses. If $\theta = \omega_1^{\langle w,z \rangle}$, and if $\omega_1^{\langle w,z \rangle} \notin \mathrm{wfp}(M_{\omega_1^{\langle w,z \rangle}})$, then Player I loses. Otherwise, Player II loses if $\omega_1^{\langle w,z \rangle} \notin \mathrm{wfp}(N_{\omega_1^{\langle w,z \rangle}})$. Note by Claim 5.2.4 that the winning condition so far is a difference of $\Sigma^0_2(\Pi^1_1(x, y))$ conditions on w, z.

So suppose now that the iteration reaches stage $\omega_1^{\langle w,z \rangle}$, and that $\omega_1^{\langle w,z \rangle}$ belongs to the wellfounded parts of both models. Let $\kappa = \omega_1^{\langle w,z \rangle}$. I wins if

- $(\forall\beta \leq \alpha)(\forall a \in V^{N_\kappa}_{\kappa+1+\beta+1} \cup \{\infty\})$ if a witnesses β-disagreement past $\kappa + 1$, then either
    1. $(\exists\beta' \leq \alpha)(\exists b \in V^{N_\kappa}_{\kappa+1+\beta+1})$ b witnesses β'-disagreement past $\kappa + 1$ and $b <^{N_\kappa} a$, or
    2. $(\exists\kappa, \gamma) R^{\mathcal{U}}_\alpha(k, \gamma, a) \wedge (\forall k, \gamma)[R^{\mathcal{U}}_\alpha(k, \gamma, a) \to (\exists k', \gamma') R^{\mathcal{U}}_\alpha(k', \gamma', a) \wedge \langle k', \gamma' \rangle <_{\mathrm{Lex}} \langle k, \gamma \rangle]$.

The winning condition should be considered to be the obvious analogue of the condition in the proof of Theorem 3.4.6; because comparing subsets of κ^+ is $\Pi^0_1(\Pi^1_1(x, y))$ in the codes w, z, we essentially may substitute each appearance of ω in that argument with $V_{\kappa+1}$. By Lemmas 5.2.8 and 5.2.10 and remarks preceding it, the entire winning condition is $\Pi^0_{1+\alpha+3}(\Pi^1_1(x, y))$.

We claim that I has a winning strategy in the game $G_{x,y}$ if and only if $x <^K y$. For suppose $x <^K y$. Let η be a countable ordinal so that $x, y \in K|\eta$. We claim I wins the game by playing a real w coding $K|\eta$. For suppose Player II plays a premouse N in response.

Since w codes an iterable mouse, if Player I has not already won by stage $\omega_1^{\langle w,z\rangle}$, then we have $M_{\omega_1^{\langle w,z\rangle}}, N_{\omega_1^{\langle w,z\rangle}}$ are both defined, $\omega_1^{\langle w,z\rangle} \in N_{\omega_1^{\langle w,z\rangle}}$, and $N_{\omega_1^{\langle w,z\rangle}}$ is illfounded.

Put $\kappa = \omega_1^{\langle w,z\rangle}$, and denote the measure sequences in M_κ, N_κ by $\mathcal{U}, \mathcal{V}$ respectively. Suppose $\delta = \sup\{\gamma < \text{wfo}(N_\kappa) | \mathcal{V}_\gamma \neq \emptyset\} = \text{wfo}(N_\kappa)$. Then each measure indexed in the illfounded part of N_κ is a witness to 0-disagreement past $\kappa + 1$, and by overspill, there is no least such; so the winning condition holds via (1).

So suppose $\delta < \text{wfo}(N_\kappa)$. Then by Theorem 5.2.2, we have $N_\kappa | \text{wfo}(N_\kappa) \models o(\kappa) = \kappa^{++}$. By failure of Π_1-$\mathsf{RAP}_{\kappa+1+\alpha}$ in initial segments of N_κ, there is some β with $1 < \beta \leq \alpha + 1$ so that $\kappa^{+\beta}$ is the largest cardinal of $N_\kappa | \text{wfo}(N_\kappa)$.

Note that the witnesses for α-disagreement past $\kappa + 1$ are exactly the subsets of $\kappa^{+\alpha}$ constructed at nonstandard levels of N_κ, and ∞. If $\beta < \alpha + 1$, then $N_\kappa | \text{wfo}(N_\kappa)$ projects unboundedly often to $\kappa^{+\beta}$. By overspill there are nonstandard levels of N_κ constructing subsets of κ^α, and no least such. Hence there is an infinite $<^{N_\kappa}$-descending sequence of elements of $V^{N_\kappa}_{\kappa+1+\beta+1}$ witnessing β-disagreement past $\kappa + 1$, so that I wins via (1).

If $\beta = \alpha + 1$, then there is some $<^{N_\kappa}$-least set a witnessing α-disagreement past $\kappa + 1$; namely, the least nonstandard level projecting to $\kappa^{+\alpha}$ if there is one, and ∞ otherwise. Then by Lemma 5.2.10, every nonstandard γ below the level at which a is constructed is a witness to $R^{\mathcal{U}}_\alpha(k, \gamma, a)$. So Player I wins via (2).

This shows I has a winning strategy when $x <^K y$. So suppose $y \leq^K x$; we claim II wins by playing z coding $K|\eta$ with η sufficiently large that $x, y \in K|\eta$. For suppose Player I responds with a play w coding a premouse M; M is not iterable, so if the comparison lasts to stage $\kappa = \omega_1^{\langle w,z\rangle}$, then we know M_κ is illfounded with $M_\kappa | \text{wfo}(M_\kappa) \trianglelefteq N_\kappa$. If $\kappa \notin \mathcal{M}_\kappa$, then II wins; so assume $\kappa \in M_\kappa$.

Now either $N_\kappa \trianglelefteq M_\kappa$, in which case ∞ is a witness to β-disagreement past $\kappa + 1$, where β is largest such that $\kappa^{+\beta+1}$ exists in M_κ; or else there is some $a \in V^{N_\kappa}_{\kappa+1+\beta+1}$ witnessing β-disagreement past $\kappa + 1$, with $\beta \leq \alpha$. Then the $<^{N_\kappa}$-least such is a witness to the failure of the winning condition, since N_κ is wellfounded.

We have shown that $<^K$ is a $\Game\Pi^0_{1+\alpha+3}(\Pi^1_1)$ wellorder of the reals of K. It follows from Theorem 5.3.1 that $\Pi^0_{1+\alpha+3}(\Pi^1_1)$-DET fails in K; by $\boldsymbol{\Sigma}^1_3$-absoluteness of the core model K [SW98], this failure reflects to V, completing the proof of the theorem. $\Box$

References

[Bar75] Jon Barwise. *Admissible sets and structures*. Springer-Verlag, Berlin-New York, 1975. An approach to definability theory, Perspectives in Mathematical Logic.

[Bla72] Andreas Blass. “Complexity of winning strategies.” *Discrete Math.*, **3**:295–300, 1972.

[Bla75] Andreas Blass. “Equivalence of two strong forms of determinacy.” *Proc. Amer. Math. Soc.*, **52**:373–376, 1975.

[Dav64] Morton Davis. “Infinite games of perfect information.” In *Advances in game theory*, pp. 85–101. Princeton Univ. Press, Princeton, N.J., 1964.

[Fri71] Harvey M. Friedman. “Higher set theory and mathematical practice.” *Ann. Math. Logic*, **2**(3):325–357, 1970/1971.

[Fri08] Sy-David Friedman. “Parameter-free uniformisation.” *Proc. Amer. Math. Soc.*, **136**(9):3327–3330, 2008.

[GS53] David Gale and F. M. Stewart. “Infinite games with perfect information.” In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pp. 245–266. Princeton University Press, Princeton, N. J., 1953.

[Har78] Leo Harrington. “Analytic determinacy and $0^\sharp$.” *J. Symbolic Logic*, **43**(4):685–693, 1978.

[Hur93] A.J.C. Hurkens. *Borel Determinacy without the Axiom of Choice*. Katholieke Universiteit Nijmegen, Nijmegen, The Netherlands, 1993. Thesis (Ph.D.)–Katholieke Universiteit Nijmegen.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[Kec78a] Alexander S. Kechris. “Forcing in analysis.” In *Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977)*, volume 669 of *Lecture Notes in Math.*, pp. 277–302. Springer, Berlin, 1978.

[Kec78b] Alexander S. Kechris. “On Spector classes.” In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pp. 245–277. Springer, Berlin, 1978.

[KM78] Alexander S. Kechris and Yiannis N. Moschovakis, editors. *Cabal Seminar 76–77*, volume 689 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.

[KMM81] Alexander S. Kechris, Donald A. Martin, and Yiannis N. Moschovakis, editors. *Cabal Seminar 77–79*, volume 839 of *Lecture Notes in Mathematics*. Springer, Berlin, 1981.

[KMM83] A. S. Kechris, D. A. Martin, and Y. N. Moschovakis, editors. *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[KMS88] A. S. Kechris, D. A. Martin, and J. R. Steel, editors. *Cabal Seminar 81–85*, volume 1333 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[Koh05] Ulrich Kohlenbach. "Higher order reverse mathematics." In *Reverse mathematics 2001*, volume 21 of *Lect. Notes Log.*, pp. 281–295. Assoc. Symbol. Logic, La Jolla, CA, 2005.

[Kon37] Motokiti Kondô. "L'uniformisation des complémentaires analytiques." *Proc. Imp. Acad.*, **13**(8):287–291, 1937.

[Mar] Donald A. Martin. *Determinacy.* Unpublished Manuscript.

[Mar70] Donald A. Martin. "Measurable cardinals and analytic games." *Fund. Math.*, **66**:287–291, 1969/1970.

[Mar74] Donald A. Martin. "Analysis $\vdash \Delta^0_4$-Determinacy." 1974. circulated handwritten notes.

[Mar80] Donald A. Martin. "Infinite games." In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pp. 269–273. Acad. Sci. Fennica, Helsinki, 1980.

[Mar81] Donald A. Martin. "Π^1_2 monotone inductive definitions." In *Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79)*, volume 839 of *Lecture Notes in Math.*, pp. 215–233. Springer, Berlin-New York, 1981.

[Mar85] Donald A. Martin. "A purely inductive proof of Borel determinacy." In *Recursion theory (Ithaca, N.Y., 1982)*, volume 42 of *Proc. Sympos. Pure Math.*, pp. 303–308. Amer. Math. Soc., Providence, RI, 1985.

[Mau81] R. Daniel Mauldin, editor. *The Scottish Book.* Birkhäuser, Boston, Mass., 1981. Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

[Mit74] William J. Mitchell. "Sets constructible from sequences of ultrafilters." *J. Symbolic Logic*, **39**:57–66, 1974.

[Mit10] William J. Mitchell. "The covering lemma." In *Handbook of set theory. Vols. 1, 2, 3*, pp. 1497–1594. Springer, Dordrecht, 2010.

[Mos71] Yiannis N. Moschovakis. "Uniformization in a playful universe." *Bull. Amer. Math. Soc.*, **77**:731–736, 1971.

[Mos09] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[MS62] Jan Mycielski and H. Steinhaus. "A mathematical axiom contradicting the axiom of choice." *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **10**:1–3, 1962.

[MS64] Jan Mycielski and S. Świerczkowski. "On the Lebesgue measurability and the axiom of determinateness." *Fund. Math.*, **54**:67–71, 1964.

[MS88] Donald A. Martin and John R. Steel. "Projective determinacy." *Proc. Nat. Acad. Sci. U.S.A.*, **85**(18):6582–6586, 1988.

[MS89] Donald A. Martin and John R. Steel. "A proof of projective determinacy." *J. Amer. Math. Soc.*, **2**(1):71–125, 1989.

[MS94] D. A. Martin and J. R. Steel. "Iteration trees." *J. Amer. Math. Soc.*, **7**(1):1–73, 1994.

[MS12] Antonio Montalbán and Richard A. Shore. "The limits of determinacy in second-order arithmetic." *Proc. Lond. Math. Soc. (3)*, **104**(2):223–252, 2012.

[MS13] Antonio Montalbán and Richard A. Shore. "The limits of determinacy in second-order arithmetic: consistency and complexity strength." *Unpublished*, 2013. to appear.

[MT08] MedYahya Ould MedSalem and Kazuyuki Tanaka. "Weak determinacy and iterations of inductive definitions." In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pp. 333–353. World Sci. Publ., Hackensack, NJ, 2008.

[Nee00] Itay Neeman. "Unraveling $\boldsymbol{\Pi}^1_1$ sets." *Ann. Pure Appl. Logic*, **106**(1-3):151–205, 2000.

[Nee06] Itay Neeman. "Unraveling Π^1_1 sets, revisited." *Israel J. Math.*, **152**:181–203, 2006.

[NM44] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey, 1944.

[Oxt57] John C. Oxtoby. "The Banach-Mazur game and Banach category theorem." In *Contributions to the theory of games, vol. 3*, Annals of Mathematics Studies, no. 39, pp. 159–163. Princeton University Press, Princeton, N. J., 1957.

[Par72] J. B. Paris. "$\mathsf{ZF} \vdash \Sigma^0_4$ determinateness." *J. Symbolic Logic*, **37**:661–667, 1972.

[Sac90] Gerald E. Sacks. *Higher recursion theory.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

[Sch13] N. Schweber. "Transfinite Recursion in Higher Reverse Mathematics." *ArXiv e-prints*, October 2013.

[Sho61] J. R. Shoenfield. "The problem of predicativity." In *Essays on the foundations of mathematics*, pp. 132–139. Magnes Press, Hebrew Univ., Jerusalem, 1961.

[Sim79] John Carson Simms. *Semihypermeasurables and* $\Pi^0_1(\Pi^1_1)$ *Games.* ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)–The Rockefeller University.

[Sim09] Stephen G. Simpson. *Subsystems of second order arithmetic.* Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition, 2009.

[Ste77] John Robert Steel. *Determinateness and Subsystems of Analysis.* ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)–University of California, Berkeley.

[Ste82] John R. Steel. "Determinacy in the Mitchell models." *Ann. Math. Logic*, **22**(2):109–125, 1982.

[SW98] J. R. Steel and P. D. Welch. "Σ^1_3 absoluteness and the second uniform indiscernible." *Israel J. Math.*, **104**:157–190, 1998.

[SZ10] Ralf Schindler and Martin Zeman. "Fine structure." In *Handbook of set theory. Vols. 1, 2, 3*, pp. 605–656. Springer, Dordrecht, 2010.

[Tan90] Kazuyuki Tanaka. "Weak axioms of determinacy and subsystems of analysis. I. Δ^0_2 games." *Z. Math. Logik Grundlag. Math.*, **36**(6):481–491, 1990.

[Tan91] Kazuyuki Tanaka. "Weak axioms of determinacy and subsystems of analysis. II. Σ^0_2 games." *Ann. Pure Appl. Logic*, **52**(1-2):181–193, 1991. International Symposium on Mathematical Logic and its Applications (Nagoya, 1988).

[Wel11] P. D. Welch. "Weak systems of determinacy and arithmetical quasi-inductive definitions." *J. Symbolic Logic*, **76**(2):418–436, 2011.

[Wel12] P. D. Welch. "$G_{\delta\sigma}$ games." *Isaac Newton Institute Preprint Series*, pp. 1–10, 2012. Available online at http://www.maths.bris.ac.uk/∼mapdw/Sigma0-3-5-INI.pdf.

[Wol55] Philip Wolfe. "The strict determinateness of certain infinite games." *Pacific J. Math.*, **5**:841–847, 1955.

[YT12] Keisuke Yoshii and Kazuyuki Tanaka. "Infinite games and transfinite recursion of multiple inductive definitions." In *How the world computes*, volume 7318 of *Lecture Notes in Comput. Sci.*, pp. 374–383. Springer, Heidelberg, 2012.