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On Combinatorial Problems of Extremal Nature and Games

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Abstract of the Dissertation

On Combinatorial Problems of Extremal Nature and Games

by

Humberto Silva Naves

Doctor of Philosophy in Mathematics

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Professor Benjamin Sudakov, Chair

Extremal graph theory is a branch of discrete mathematics and also the central theme of extremal combinatorics. It studies graphs which are extremal with respect to some parameter under certain restrictions. A typical result in extremal graph theory is Mantel's theorem. It states that the complete bipartite graph with equitable parts is the graph the maximizes the number of edges among all triangle-free graphs. One can say that extremal graph theory studies how local properties of a graph influence its global structure.

Another fundamental topic in the field of combinatorics is the probabilistic method, which is a nonconstructive method pioneered by Paul Erdős for proving the existence of a prescribed kind of mathematical object. One particular application of the probabilistic method lies in the field of *positional games*, more specifically *Maker-Breaker* games.

My dissertation focus mainly on various Turán-type questions and their applications to other related areas as well as the employment of the probabilistic method to study extremal problems and positional games.

The dissertation of Humberto Silva Naves is approved.

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Benjamin Sudakov, Committee Chair

University of California, Los Angeles

2014

To my beloved family . . .

I could never have done this without your love,

support, and endless encouragement.

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Chapter 2 is a version of the paper *Cliques and independent sets* coauthored with Hao Huang, Nati Linial, Yuval Peled, and Benny Sudakov, which has been accepted to Combinatorica. Chapter 3 is a version of the paper *Discrepancy of random graphs and hypergraphs*, coauthored with Jie Ma and Benny Sudakov, which has been accepted to Random Structures & Algorithms with **DOI: 10.1002/rsa.20522**. Chapter 4 is a version of the paper *Generating random graphs in biased Maker-Breaker games*, coauthored with Asaf Ferber and Michael Krivelevich, which has been submitted, and whose preprint is posted on arXiv with identification number **arXiv:1310.4096**. All these papers were written under the supervision of Benny Sudakov.

Vita

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Publications

S. Das, H. Huang, J. Ma, H. Naves and B. Sudakov. *A problem of Erdős on the minimum number of k-cliques*, J. Combinatorial Theory Ser. B **103** (2013) 344–373.

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J. Ma, H. Naves and B. Sudakov. *Discrepancy of random graphs and hypergraphs*, Random Structures and Algorithms, DOI: 10.1002/rsa.20522.

H. Huang, N. Linial, H. Naves, Y. Peled and B. Sudakov. *On the* 3*-local profiles of graphs*, submitted, 2012.

A. Ferber, M. Krivelevich and H. Naves. *Generating random graphs in biased Maker-Breaker games*, submitted, 2013.

H. Naves, and O. Pikhurko, *How unproportional must a graph be?*, submitted, 2014.

CHAPTER 1

Introduction

Extremal graph theory is one of the central areas of extremal combinatorics. It studies graphs which are extremal with respect to some parameter (e.g., number of edges, triangle density, number of proper colorings) under certain restrictions. A typical result in extremal graph theory is Mantel's theorem. It states that the complete bipartite graph with equal parts maximizes the number of edges among all triangle-free graphs. The problems in this field are often related to other areas including number theory, analysis, geometry, computer science and information theory.

Another fundamental topic is the probabilistic method, which is a nonconstructive method pioneered by Paul Erdős for proving the existence of a prescribed kind of mathematical object. This branch of mathematics has developed spectacularly in the past decades. Several hundred papers employ probabilistic ideas, and many interesting open problems arose from what has become one of the most indispensable tools in modern combinatorics.

One particular application of the probabilistic method lies in the field of *positional games*. These are combinatorial games described by a finite set of positions (the board) and by a family of subsets of the board (the winning sets). Two players alternatively claim previously unclaimed positions until they fully occupy the board. The variety of games that falls in this category ranges from recreationally popular games such as Tic-Tac-Toe and Hex to abstract games played on graphs and hypergraphs. The different types of positional games are characterized by the rules used to select the winner. For instance, in a *Maker-Breaker* game, the first player (*Maker*) has to occupy a winning set to win, while the second player (*Breaker*) has to stop Maker from doing so. If Breaker successfully prevents Maker from occupying a winning set to the end of the game, then Breaker wins.

My thesis focuses on various aspects of extremal combinatorics, including Turán-type questions and their applications to related areas, as well as the employment of the probabilistic method to study extremal graph problems and positional games. The following is a brief list of the topics that will be covered in this thesis.

On the densities of cliques and independent sets in graphs

A variety of problems in extremal combinatorics can be stated as the following general question: For two given graphs H_1 and H_2, if the number of induced copies of H_1 in a n-vertex graph G is known, what is the maximum or minimum number of induced copies of H_2 in G? In its full generality, this question seems currently out of reach, but several special cases already have important implications in combinatorics, as well as other branches of mathematics and computer science. For instance, Turán proved that the maximal edge density in any graph with no cliques of size r is attained by an $r-1$ partite graph. Kruskal and Katona found that cliques, among all graphs, maximize the number of induced copies of K_s when $r < s$ and the number of induced copies of K_r is fixed.

In Chapter 2, we study the following analogue of the Kruskal-Katona theorem: Suppose that the number of blue r-cliques in a red/blue coloring of the edges of the complete graph K_n is known and fixed. What is the largest possible number of red s-cliques under this assumption? Using the shifting technique from extremal set theory together with some powerful analytical methods, we resolve this problem in general and prove that in the extremal coloring either the blue edges or the red edges form a clique. As a corollary, our result also gives a simple description of the extremal red/blue coloring of K_n that maximizes the minimum number of red r-cliques and the number blue r-cliques. This fact is quite surprising, since little is known about the extremal red/blue coloring of K_n that minimizes the number of monochromatic r-cliques (blue and red altogether).

Discrepancy of graphs and hypergraphs

The *discrepancy* of an n-vertex k-uniform hypergraph H with edge density $\rho_H = e(H)/\binom{n}{k}$ is $\mathrm{disc}(H) = \max_{S \subseteq V(H)} \left| e(S) - \rho_H \binom{|S|}{k} \right|$, where $e(S) = e(H[S])$ is the number of edges in the sub-hypergraph induced by S. This important concept appears naturally in various branches of combinatorics and was studied by many researchers in recent years. The discrepancy can be viewed as a measure of how uniformly the edges of H are distributed among the vertices, and it is closely related to the theory of quasi-random graphs, as the property $\mathrm{disc}(G) = o(|V(G)|^2)$ implies the quasi-randomness of the graph G. A natural generalization is the relative discrepancy of two hypergraphs. Let G and H be two k-uniform hypergraphs over the same vertex set V, with $|V| = n$. The *relative discrepancy of G with respect to H* is $\mathrm{disc}(G, H) = \max_\pi \left| e(G_\pi \cap H) - \rho_G \rho_H \binom{n}{k} \right|$, over all bijections $\pi : V \to V$. Thus $\mathrm{disc}(G, H)$ measures by how much the overlap $e(G_\pi \cap H)$ can deviate from its average. Bollobás and Scott introduced and studied this notion, and showed that for any two n-vertex graphs G and H, if $\frac{16}{n} \leq \rho_G, \rho_H \leq 1 - \frac{16}{n}$, then $\mathrm{disc}(G, H) \geq c \cdot f(\rho_G, \rho_H) \cdot n^{\frac{3}{2}}$, where c is an absolute constant and $f(x, y) = x^2(1 - x)^2 y^2 (1 - y)^2$. In addition, they asked the following question. For two random n-vertex graphs G, H with constant edge probability p, what is the expected value of $\mathrm{disc}(G, H)$?

Answering this question in a strong form, we determine in Chapter 3 the discrepancy between two random k-uniform hypergraphs, up to a constant factor depending solely on k. In particular, we show that Bollobás and Scott's lower bound on the relative discrepancy is not tight if G and H are both random with constant edge probability.

Generating random graphs in biased Maker-Breaker games

There is a striking relation between the theory of biased Maker-Breaker games and the theory of random graphs, frequently referred to as the *random graph intuition*. Roughly speaking, it suggests that on a Maker-Breaker game played on the edges of a complete graph, claiming edges at random often yields a good strategy for Maker. For instance, Bednarska and Łuczak proved that this "random strategy" guarantees Maker's victory in the game whose winning sets consist of all copies of a prescribed target graph H. In their proof, the graph obtained by Maker at the end of the game is not exactly a random graph, since Maker occasionally attempts to pick an edge which already belongs to Breaker. Thus, in order to prove their result, Bednarska and Łuczak showed that with a positive probability, even after removing a small fraction of the total number of edges, a random graph still contains many copies of the target graph H. This particular statement relates to the *global resilience* of random graphs with respect to the property "containing a copy of H". The systematic study of resilience of random and pseudorandom graphs was initiated by Sudakov and Vu, and since then, this field has attracted substantial research interest.

In Chapter 4 we develop a refined version of Bednarska and Łuczak's approach, connecting biased Maker-Breaker games and problems about *local resilience* in random graphs. For a monotone increasing graph property $\mathcal{P}$, the *local resilience* of G with respect to $\mathcal{P}$ is the minimum number r such that by deleting an r proportion of the edges incident to v, for each vertex v in G, one can obtain a subgraph of G not satisfying the property $\mathcal{P}$. Since one can destroy many natural properties by making local changes (for example, by isolating a vertex), it is natural to limit the number of edges touching any vertex that Breaker is allowed to claim. By studying the local resilience of some relevant graph properties, we show that if Breaker's bias is $b = \Theta(\frac{n}{\ln n})$, Maker can build a graph that contains copies of all spanning trees having maximum degree $\Delta = O(1)$ with a bare path of linear length (a bare path in a tree T is a path with all interior vertices of degree exactly two in T).

Each subsequent chapter will contain their own introduction where backgrounds and motivations of the problems are discussed in more details.

CHAPTER 2

Cliques and independent sets

2.1 Introduction

As usual we denote by K_s the complete graph on s vertices and by $\overline{K}_s$ its complement, the edgeless graph on s vertices. By the celebrated Ramsey's theorem, for every two integers r, s every sufficiently large graph must contain K_r or $\overline{K}_s$. Turán's theorem can be viewed as a quantitative version of the case $s = 2$. Namely, it shows that among all $\overline{K}_r$-free n-vertex graphs, the graph with the least number of K_2 (edges) is a disjoint union of $r - 1$ cliques of nearly equal size. More generally, one can ask the following question. Fix two graphs H_1 and H_2, and suppose that we know the number of induced copies of H_1 in an n-vertex graph G. What is the maximum (or minimum) number of induced copies of H_2 in G? In its full generality, this problem seems currently out of reach, but some special cases already have important implications in combinatorics, as well as other branches of mathematics and computer science.

To state these classic results, we introduce some notation. Adjacency between vertices u and v is denoted by $u \sim v$, and the neighbor set of v is denoted by $N(v)$. If necessary, we add a subscript G to indicate the relevant graph. The collection of induced copies of a k-vertex graph H in an n-vertex graph G is denoted by $\mathrm{Ind}(H; G)$, i.e.

$$\mathrm{Ind}(H; G) := \{X \subseteq V(G) : G[X] \simeq H\}$$

and the *induced H-density* is defined as

$$d(H; G) := \frac{|\mathrm{Ind}(H; G)|}{\binom{n}{k}}.$$

In this language, Turán's theorem says that if $d(K_r;G)=0$ then $d(K_2;G)\le 1-\frac{1}{r-1}$ and this bound is tight. For a general graph H, Erdős and Stone [ES46] determined $\max d(K_2;G)$ when $d(H;G)=0$ and showed that the answer depends only on the chromatic number of H. Zykov [Zyk49] extended Turán's theorem in a different direction. Given integers $2\le r<s$, he proved that if $d(K_s;G)=0$ then $d(K_r;G)\le \frac{(s-1)\cdots(s-r)}{(s-1)^r}$. The balanced complete $(s-1)$-partite graphs shows that this bound is also sharp.

For fixed integers $r<s$, the Kruskal-Katona theorem [Kat87, Kru63] states that if $d(K_r;G)=\alpha$ then $d(K_s;G)\le \alpha^{s/r}$. Again, the bound is tight and is attained when G is a clique on some subset of the vertices. On the other hand, the problem of *minimizing* $d(K_s;G)$ under the same conditions is much more difficult. Even the case $r=2$ and $s=3$ has remained unsolved for many years until it was recently answered by Razborov [Raz08] using his newly-developed flag algebra method. Subsequently, Nikiforov [Nik11] and Reiher [Rei12] applied complicated analytical techniques to solve the cases $(r,s)=(2,4)$, and ($r=2$, arbitrary s), respectively.

In this chapter, we study the following natural analogue of the Kruskal-Katona theorem. Given $d(\overline{K}_r;G)$, how large can $d(K_s;G)$ be? For integers $a\ge b>0$ we let $Q_{a,b}$ be the a-vertex graph whose edge set is a clique on some b vertices. The complement of this graph is denoted by $\overline{Q}_{a,b}$. Let $\mathcal{Q}_a$ denote the family of all graphs $Q_{a,b}$ and its complement $\overline{Q}_{a,b}$ for $0<b\le a$. Note that for $r=2$ or $s=2$, the Kruskal-Katona theorem implies that the extremal graph comes from $\mathcal{Q}_n$. Our first theorem shows that a similar statement holds for all r and s.

Theorem 2.1.1. *Let $r,s\ge 2$ be integers and suppose that $d(\overline{K}_r;G)\ge p$ where G is an n-vertex graph and $0\le p\le 1$. Let q be the unique root of $q^r+rq^{r-1}(1-q)=p$ in $[0,1]$. Then $d(K_s;G)\le M_{r,s,p}+o(1)$, where*

$$M_{r,s,p}:=\max\left\{(1-p^{1/r})^s+sp^{1/r}(1-p^{1/r})^{s-1},(1-q)^s\right\}.$$

Namely, given $d(\overline{K}_r;G)$, the maximum of $d(K_s;G)$ (up to $\pm o_n(1)$) is attained in one of two graphs, (or both), one of the form $Q_{n,t}$ and another $\overline{Q}_{n,t'}$.

We obtain as well a *stability version* of Theorem 2.1.1. Two n-vertex graphs H and G are ε-*close* if it is possible to obtain H from G by adding or deleting at most εn^2 edges. As the next theorem shows, every near-extremal graph G for Theorem 2.1.1 is ε-close to a specific member of $\mathcal{Q}_n$.

Theorem 2.1.2. *Let $r, s \geq 2$ be integers and let $p \in [0,1]$. For every $\varepsilon > 0$, there exists $\delta > 0$ and an integer N such that every n-vertex graph G with $n > N$ satisfying $d(\overline{K}_r; G) \geq p$ and $|d(K_s;G) - M_{r,s,p}| \leq \delta$, is ε-close to some graph in $\mathcal{Q}_n$.*

For instance, the green curve in Figure 2.1 is $(d(\overline{K}_3; Q_{n,\theta n}), d(K_3; Q_{n,\theta n}))$ for $\theta \in [0,1]$, and the red curve defined the same with $\overline{Q}_{n,\theta n}$. The maximum between the curves is the extremal function in Theorem 2.1.1. The intersection of the curves represents the solution of the max-min problem in Theorem 2.1.3.

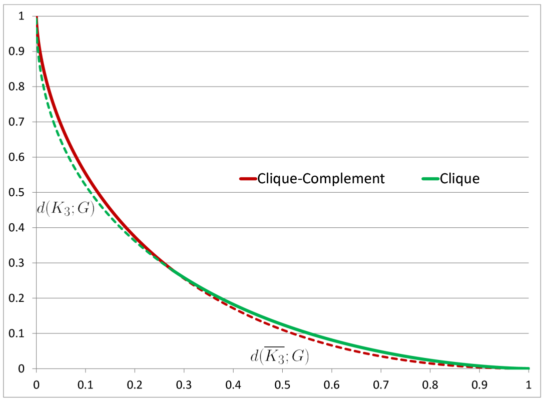

Figure 2.1: Illustration for the case $r = s = 3$.

Rather than talking about an n-vertex graph and its complement, we can consider a two-edge-coloring of K_n. A quantitative version of Ramsey Theorem asks for the minimum number of monochromatic s-cliques over all such colorings. Goodman [Goo59] showed that for $r = s = 3$, the optimal answer is essentially given by a random two-coloring of $E(K_n)$. In other words, $\min_G d(K_3;G) + d(\overline{K}_3;G) = 1/4 - o(1)$. Erdős [Erd62] conjectured that the same random coloring also minimizes $d(K_r;G) + d(\overline{K}_r;G)$ for all r, but this was refuted by Thomason [Tho89] for all $r \geq 4$. A simple consequence of Goodman's inequality is that $\min_G \max\{d(K_3; G), d(\overline{K}_3; G)\} = 1/8$. The following construction by Franek and Rödl [FR93] shows that the analogous statement for $r \geq 4$ is again false. Let H be a graph with vertex set $[2]^{13}$, the collection of all 8192 binary vectors of length 13. Two vertices are adjacent if the Hamming distance between the corresponding binary vectors is a number in $\{1, 4, 5, 8, 9, 11\}$. Let G be obtained from H by replacing each vertex with a clique of size n, and every edge with a complete bipartite graph. The number of K_4 and $\overline{K}_4$ in G can be easily expressed in terms of the parameters of H (see [FR93]), for large enough n one can show that $d(K_4; G) < 0.99 \cdot \frac{1}{64}$ and $d(\overline{K}_4; G) < 0.993 \cdot \frac{1}{64}$.

While the min-max question remains at present very poorly understood, we succeeded to completely answer the max-min version of this problem.

Theorem 2.1.3.

$$\max_G \min\{d(K_r; G), d(\overline{K}_r; G)\} = \rho^r + o(1),$$

where ρ is the unique root in $[0, 1]$ of the equation $\rho^r = (1-\rho)^r + r\rho(1-\rho)^{r-1}$.

This theorem follows easily from Theorem 2.1.1. Moreover, using Theorem 2.1.2, we can also show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that every n-vertex graph G with $\min\{d(K_r; G), d(\overline{K}_r; G)\} > \rho^r - \delta$ is ε-close to a clique of size ρn or to the complement of this graph.

Here we prove these theorems using the method of shifting. In the next section we describe this well-known and useful technique in extremal set theory. Using shifting, we show how to reduce the problem to *threshold graphs*. Section 2.3 contains the proof of our main result for threshold graphs and Section 2.4 contains the proof of the stability result. In Section 2.5 we sketch a second proof for the case $r = s$, based on a different representation of threshold graphs. We make a number of comments on the analogous problems for hypergraphs in Section 2.6. In the final section of this chapter we discuss some open problems and further research directions.

2.2 Shifting

Shifting is one of the most important and widely-used tools in extremal set theory. This method allows one to reduce many problems to more structured instances which are usually easier to analyze. Our treatment is rather shallow and we refer the reader to Frankl's survey article [Fra87] for a fuller account.

Let $\mathcal{F}$ be a family of subsets of a finite set V, and let u, v be two distinct elements of V. We define the (u,v)-*shift map* $S_{u\to v}$ as follows: for every $F \in \mathcal{F}$, let

$$S_{u\to v}(F,\mathcal{F}) := \begin{cases} (F\cup\{v\})\setminus\{u\} & \text{if } u\in F, v\notin F \text{ and } (F\cup\{v\})\setminus\{u\}\notin\mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

We define the (u,v)-shift of $\mathcal{F}$, to be the following family of subsets of V: $S_{u\to v}(\mathcal{F}) := \{S_{u\to v}(F,\mathcal{F}) : F\in\mathcal{F}\}$. We observe that $|S_{u\to v}(\mathcal{F})| = |\mathcal{F}|$. In this context, one may think of $\mathcal{F}$ as a hypergraph over V. When all sets in $\mathcal{F}$ have cardinality 2, this is a graph with vertex set V. Figure 2.2 shows what happens to the neighborhoods of u and v when the shifting is applied to a graph.

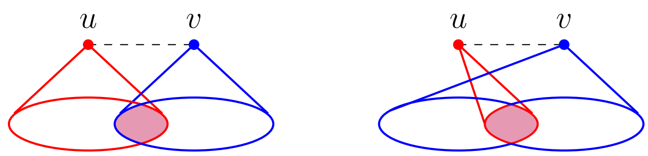

Figure 2.2: Graph shifting

As the next lemma shows, shifting of graph does not reduce the number of l-cliques in it, for every l. Recall that $\mathrm{Ind}(K_l;G)$ denotes the collection of all cliques of size l in G.

Lemma 2.2.1. *For every integer $l>0$, every graph G, and every $u\neq v\in V(G)$ there holds*

$$S_{u\to v}(\mathrm{Ind}(K_l;G)) \subseteq \mathrm{Ind}(K_l;S_{u\to v}(G)).$$

Proof. Let $A = S_{u\to v}(B,G)$, where B is an l-clique in G. First, consider the cases when $u\notin B$ or both $u,v\in B$ or $B\setminus\{u\}\cup\{v\}$ is also a clique in G. Then $A=B$ and we need to show that B remains a clique after shifting. Which edge in B can be lost by shifting? It must be some edge uw in B that gets replaced by the non-edge vw (otherwise we can not shift uw). Note that vw is not in B, since B is a clique. Hence $u, w \in B$ and $v \notin B$. But then $B \setminus \{u\} \cup \{v\}$ is not a clique, contrary to our assumption.

In the remaining case when $u \in B$, $v \notin B$ and $B \setminus \{u\} \cup \{v\}$ is not a clique in G, we need to show that $A = B \setminus \{u\} \cup \{v\}$ is a clique after shifting $S_{u \to v}(G)$. Every pair of vertices in $A \setminus \{v\}$ belongs to B and the edge they span is not affected by the shifting. So consider $v \neq w \in A$. If $vw \in E(G)$, this edge remains after shifting. If, however, $vw \notin E(G)$, note that $uw \in E(G)$ since both vertices belong to the clique B. In this case $vw = S_{u \to v}(uw, G)$ and the claim is proved. $\square$

Since shifting edges from u to v is equivalent to shifting non-edges from v to u, it is immediate that $S_{u \to v}(\mathrm{Ind}(\overline{K_l}; G)) \subseteq \mathrm{Ind}(\overline{K_l}; S_{u \to v}(G))$. Therefore we obtain the following corollary.

Corollary 2.2.2. *Let G be a graph, let $H = S_{u \to v}(G)$ and let l be a positive integer. Then*

$$d(K_l; H) \geq d(K_l; G) \quad \textit{and} \quad d(\overline{K}_l; H) \geq d(\overline{K}_l; G).$$

We say that vertex u *dominates* vertex v if $S_{v \to u}(\mathcal{F}) = \mathcal{F}$. In the case when $\mathcal{F}$ is a set of edges of G, this implies that every $w \neq u$ which is adjacent to v is also adjacent to u. If $V = [n]$, we say that a family $\mathcal{F}$ is *shifted* if i dominates j for every $i < j$. Every family can be made shifted by repeated applications of shifting operations $S_{j \to i}$ with $i < j$. To see this note that a shifting operation that changes $\mathcal{F}$ reduces the following non-negative potential function $\sum_{A \in \mathcal{F}} \sum_{i \in A} i$. As Corollary 2.2.2 shows, it suffices to prove Theorem 2.1.1 for shifted graphs.

In Section 2.3 we use the notion of *threshold graphs*. There are several equivalent ways to define threshold graph (see, e.g., [CH77]), and we adopt the following definition.

Definition 2.2.3. *We say that $G = (V, E)$ is a threshold graph if there is an ordering of V so that every vertex is adjacent to either all or none of the preceding vertices.*

Lemma 2.2.4. *A graph is shifted if and only if it is a threshold graph.*

Proof. Let G be a shifted graph. We may assume that $V = [n]$, and i dominates j in G for every $i < j$. Consider the following order of vertices,

$$\ldots,\ 3,\ N_G(2) \setminus N_G(3),\ 2,\ N_G(1) \setminus N_G(2),\ 1,\ V \setminus N_G(1)\ ,$$

where the vertices inside the sets that appear here are ordered arbitrarily. We claim that this order satisfies Definition 2.2.3. First, every vertex $v \notin N_G(1)$ is isolated. Indeed, if $u \sim v$, then necessarily $v \sim 1$, since 1 dominates u. Therefore, vertex 1 and its non-neighbors satisfy the condition in the definition. The proof that G is threshold proceeds by induction applied to $G[N_G(1)]$.

Conversely, let G be a threshold graph. Let $v_1, v_2, \ldots, v_n$ be an ordering of V as in Definition 2.2.3. We say that a vertex is good (resp. bad) if it is adjacent to all (none) of its preceding vertices. Consider two vertices v_i and v_j. It is straightforward to show that v_i dominates v_j if either (1) v_i is good and v_j is bad, (2) they are both good and $i > j$ or (3) they are both bad and $i < j$. Therefore we can reorder the vertices by first placing the good vertices in reverse order followed by the bad vertices in the regular order. This new ordering demonstrates that G is shifted. □

2.3 Main result

In this section, we prove Theorem 2.1.1. It will be convenient to reformulate the theorem, in a way that is analogous to the Kruskal-Katona theorem.

Theorem 2.3.1. *Let $r, s \geq 3$ be integers and let $a, b > 0$ be real numbers. The maximum (up to lower order terms) of the function*

$$f(G) := \min\{a \cdot d(K_s; G), b \cdot d(\overline{K}_r; G)\}$$

over all n-vertex graphs is attained in one of two graphs, (or both), one of the form $Q_{n,t}$ and another $\overline{Q}_{n,t'}$. In particular, $f(G) \leq \max\{a \cdot \alpha^s, b \cdot \beta^r\} + o(1)$, where α is the unique root in $[0,1]$ of $a \cdot \alpha^s = b \cdot [(1-\alpha)^r + r\alpha(1-\alpha)^{r-1}]$ and β is the unique root in $[0,1]$ of $b \cdot \beta^r = a \cdot [(1-\beta)^s + s\beta(1-\beta)^{s-1}]$.

We turn to show how to deduce Theorem 2.1.1 from Theorem 2.3.1. We assume that $r, s \geq 3$, since the other cases follow from Kruskal-Katona theorem.

Proof of Theorem 2.1.1. Let M be the maximum of $d(K_s; G)$ over all graphs G on n vertices with $d(\overline{K}_r; G) \geq p$. Fix such an extremal G with $d(\overline{K}_r; G) = p' \geq p$ and $d(K_s; G) = M$. Now apply Theorem 2.3.1 with $a = p$ and $b = M$ and the same n, r and s. The extremal graph G' that Theorem 2.3.1 yields, satisfies

$$f(G') \geq f(G) = \min\{a \cdot d(K_s; G), b \cdot d(\overline{K}_r; G)\} = p \cdot M,$$

hence $d(K_s; G') \geq M$ and $d(\overline{K}_r; G') \geq p$. Therefore, the same G' is extremal for Theorem 2.1.1 as well and we know that the maximum in this theorem is achieved asymptotically by a graph of $\mathcal{Q}_n$.

Note that we can always assume that in the extremal graph $d(\overline{K}_r; G') = p$ since otherwise we can add edges to G' without decreasing $d(K_s; G')$ until $d(\overline{K}_r; G') = p$ is obtained. Therefore the maximum is attained either by a graph of the form $\overline{Q}_{n,p^{1/r}n}$ or by $Q_{n,(1-q)n}$, where $q^r + rq^{r-1}(1-q) = p$. This implies that asymptotically the maximum in Theorem 2.1.1 is indeed

$$M_{r,s,p} = \max\{(1-p^{1/r})^s + sp^{1/r}(1-p^{1/r})^{s-1}, (1-q)^s\}.$$

□

By Corollary 2.2.2 and Lemma 2.2.4, $f(G)$ is maximized by a threshold graph. We turn to prove Theorem 2.3.1 for threshold graphs. Let G be a threshold graph on an ordered vertex set V, as in Definition 2.2.3. There exists an integer $k > 0$, and a partition $A_1, \ldots, A_{2k}$ of V such that

1. If $v \in A_i$ and $u \in A_j$ for $i < j$, then $v < u$.

2. Every vertex in A_{2i-1} (respectively A_{2i}) is adjacent to all (none) of its preceding vertices.

Let $x_i = \frac{|A_{2i-1}|}{|V|}$ and $y_i = \frac{|A_{2i}|}{|V|}$. Clearly $\sum_{i=1}^{k}(x_i + y_i) = 1$. Up to a negligible error-term,

$$d(K_s;G) = p(\mathbf{x},\mathbf{y}) := \left(\sum_{i=1}^{k} x_i\right)^s + s \cdot \sum_{i=1}^{k-1}\left[y_i \cdot \left(\sum_{j=i+1}^{k} x_j\right)^{s-1}\right],$$

$$d(\overline{K}_r;G) = q(\mathbf{x},\mathbf{y}) := \left(\sum_{i=1}^{k} y_i\right)^r + r \cdot \sum_{i=1}^{k}\left[x_i \cdot \left(\sum_{j=i}^{k} y_j\right)^{r-1}\right].$$

Where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$. Occasionally, p will be denoted by p_s and q by q_r to specify the parameter of these functions.

Our problem can therefore be reformulated as follows. For given integers $k \geq 2$, $r, s \geq 3$ and real $a, b > 0$, let $W_k \subseteq \mathbb{R}^{2k}$ be the set

$$W_k := \left\{(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k) \in \mathbb{R}^{2k} : x_i, y_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{k}(x_i + y_i) = 1\right\}.$$

Let $p, q : W_k \to \mathbb{R}$ be the two homogeneous polynomials defined above, We are interested in maximizing the real function

$$\varphi(\mathbf{x},\mathbf{y}) := \min\{a \cdot p(\mathbf{x},\mathbf{y}), b \cdot q(\mathbf{x},\mathbf{y})\}.$$

This problem is well defined since W_k is compact and φ is continuous.

We say that $(\mathbf{x},\mathbf{y}) \in W_k$ is *non-degenerate* if the set of zeros in the sequence $(y_1, x_2, y_2, \ldots, x_k, y_k)$, with x_1 omitted, forms a suffix. If $(\mathbf{x},\mathbf{y}) \in W_k$ is degenerate, then there is a non-degenerate $(\mathbf{x}',\mathbf{y}') \in W_k$ with $\varphi(\mathbf{x},\mathbf{y}) = \varphi(\mathbf{x}',\mathbf{y}')$. Indeed, if $y_i = 0$ and $x_{i+1} \neq 0$ for some $1 \leq i < k$, let $(\mathbf{x}',\mathbf{y}') \in W_{k-1}$ be defined by

$$\mathbf{x}' = (x_1, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_k)$$

$$\mathbf{y}' = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k)$$

It is easy to verify that $p(\mathbf{x},\mathbf{y}) = p(\mathbf{x}',\mathbf{y}')$ and $q(\mathbf{x},\mathbf{y}) = q(\mathbf{x}',\mathbf{y}')$. By induction on k, we assume that $(\mathbf{x}',\mathbf{y}')$ is non-degenerate, and by padding $\mathbf{x}'$ and $\mathbf{y}'$ with a zero, the claim is proved. The case $x_i = 0$ and $y_i \neq 0$ is proved similarly. In particular, φ has a non-degenerate maximum in W_k.

Our purpose is to show that the original problem is optimized by graphs from $\mathcal{Q}_n$. This translates to the claim that a non-degenerate $(\mathbf{x}, \mathbf{y})$ that maximizes φ is supported only on either x_1, y_1 or y_1, x_2, which corresponds to either a clique $Q_{n,t}$ or a complement of a clique $\overline{Q}_{n,t}$, respectively.

Lemma 2.3.2. *Let $(\mathbf{x}, \mathbf{y}) \in W_k$ be a non-degenerate maximum of φ. If $x_1 > 0$, then for every $i \geq 2$, $x_i = y_i = 0$. On the other hand, if $x_1 = 0$ then $y_i = 0$ for every $i \geq 2$, and $x_i = 0$ for every $i \geq 3$.*

Proof. We note first that the second part of the lemma is implied by the first part. Define $\mathbf{x}'$ by

$$x'_i := \begin{cases} x_{i+1} & \text{if } i < k, \\ 0 & \text{if } i = k. \end{cases}$$

Clearly, if $x_1 = 0$, then $p_s(\mathbf{x}, y) = q_s(\mathbf{y}, x')$, $q_r(\mathbf{x}, y) = p_r(\mathbf{y}, x')$, and

$$\varphi'(\mathbf{y}, \mathbf{x}') := \min\{b \cdot p_r(\mathbf{y}, \mathbf{x}'), a \cdot q_s(\mathbf{y}, \mathbf{x}')\} = \varphi(\mathbf{x}, \mathbf{y}).$$

Since φ attains its maximum when $x_1 = 0$, maximizing it is equivalent to maximizing $\varphi'(\mathbf{y}, x')$. Since $(\mathbf{x}, \mathbf{y})$ is non-degenerate, $y_1 > 0$, and applying the first part of Lemma 2.3.2 for $\varphi'(\mathbf{y}, \mathbf{x}')$ finishes the proof, by obtaining that for every $i \geq 2$, $y_i = x'_i = 0$.

The first part of Lemma 2.3.2 is proved in the following lemmas. We successively show that $x_3 = 0$, then $y_2 = 0$ and finally $x_2 = 0$. □

Here is a local condition that maximum points of φ satisfy.

Lemma 2.3.3. *If φ takes its maximum at a non-degenerate $(\mathbf{x}, \mathbf{y}) \in W_k$, then $a \cdot p(\mathbf{x}, \mathbf{y}) = b \cdot q(\mathbf{x}, \mathbf{y})$.*

Proof. Note that $0 < y_1 < 1$, since $(\mathbf{x}, \mathbf{y}) \in W$ is non-degenerate. We consider two perturbations of the input, one of which increases $p(\mathbf{x}, \mathbf{y})$, and the other increases $q(\mathbf{x}, \mathbf{y})$. Consequently, if $a \cdot p(\mathbf{x}, \mathbf{y}) \neq b \cdot q(\mathbf{x}, \mathbf{y})$, by applying the appropriate perturbation, we increase the smallest between $a \cdot p(\mathbf{x}, \mathbf{y})$ and $b \cdot q(\mathbf{x}, \mathbf{y})$, thus increasing $\min\{a \cdot p(\mathbf{x}, \mathbf{y}), b \cdot q(\mathbf{x}, \mathbf{y})\}$, contrary to the maximality assumption.

To define the perturbation that increases p, let $\mathbf{x}' = \mathbf{x} + t\mathbf{e}_1$ and $\mathbf{y}' = \mathbf{y} - t\mathbf{e}_1$, where $0 < t < y_1$, and $\mathbf{e}_1$ is the first unit vector in $\mathbb{R}^k$. Then, $(\mathbf{x}', \mathbf{y}') \in W$ and

$$\frac{\partial p(\mathbf{x}', \mathbf{y}')}{\partial t} = s\left(t + \sum_{i=1}^{k} x_i\right)^{s-1} - s \cdot \left(\sum_{j=2}^{k} x_j\right)^{s-1} > 0$$

as claimed.

In order to increase q, consider two cases. If $x_1 = 0$, let $\mathbf{x}' = \mathbf{x} - t\mathbf{e}_2$ and $\mathbf{y}' = \mathbf{y} + t\mathbf{e}_1$, where $0 < t < x_2$. Then, $(\mathbf{x}', \mathbf{y}') \in W$ and

$$\frac{\partial q(\mathbf{x}', \mathbf{y}')}{\partial t} = r\left(t + \sum_{i=1}^{k} y_i\right)^{r-1} - r \cdot \left(\sum_{j=k}^{n} y_j\right)^{r-1} > 0.$$

If $x_1 > 0$, we let $\mathbf{x}' = \mathbf{x} - t\mathbf{e}_1$ and $\mathbf{y}' = \mathbf{y} + t\mathbf{e}_1$, where $0 < t < x_1$. Then,

$$\frac{\partial q(\mathbf{x}', \mathbf{y}')}{\partial t} = r(x_1 - t)(r-1)\left(t + \sum_{i=1}^{k} y_i\right)^{r-2} > 0.$$

□

Lemma 2.3.4. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $x_3 = 0$.*

Proof. Suppose, that $x_3 > 0$ and let $1 \leq l \leq m \leq k$. Then

$$\frac{\partial p}{\partial x_l} = s \cdot \left(\sum_{i=1}^{k} x_i\right)^{s-1} + s(s-1) \cdot \sum_{i=1}^{l-1}\left[y_i \cdot \left(\sum_{j=i+1}^{k} x_j\right)^{s-2}\right],$$

$$\frac{\partial q}{\partial x_l} = r \cdot \left(\sum_{j=l}^{k} y_j\right)^{r-1},$$

and

$$\frac{\partial^2 p}{\partial x_l \partial x_m} = s(s-1) \cdot \left(\sum_{i=1}^{k} x_i\right)^{s-2} + s(s-1)(s-2) \cdot \sum_{i=1}^{l-1}\left[y_i \cdot \left(\sum_{j=i+1}^{k} x_j\right)^{s-3}\right],$$

$$\frac{\partial^2 q}{\partial x_l \partial x_m} \equiv 0.$$

Clearly $\frac{\partial^2 p}{\partial x_l \partial x_m} = \frac{\partial^2 p}{\partial x_l^2}$, for $l \leq m$. We define two matrices $\mathbf{A}$ and $\mathbf{B}$ as following.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\partial p}{\partial x_1} & frac \partial p \partial x_2 & \frac{\partial p}{\partial x_3} \\ \frac{\partial q}{\partial x_1} & \frac{\partial q}{\partial x_2} & \frac{\partial q}{\partial x_3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1 \partial x_2} & \frac{\partial^2 p}{\partial x_1 \partial x_3} \\ \frac{\partial^2 p}{\partial x_1 \partial x_2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_2 \partial x_3} \\ \frac{\partial^2 p}{\partial x_1 \partial x_3} & \frac{\partial^2 p}{\partial x_2 \partial x_3} & \frac{\partial^2 p}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1^2} \\ \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_2^2} \\ \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_3^2} \end{bmatrix}.$$

It is easy to see that if $(\mathbf{x}, \mathbf{y})$ is non-degenerate with $x_3 > 0$, then $\frac{\partial^2 p}{\partial x_3^2} > \frac{\partial^2 p}{\partial x_2^2} > \frac{\partial^2 p}{\partial x_1^2} > 0$. This implies that $\mathbf{B}$ is positive definite.

For a vector $\mathbf{v} \in \mathbb{R}^3$ and $\varepsilon > 0$, we define $\mathbf{x}'$ by

$$x_i' = \begin{cases} x_i + \varepsilon v_i & \text{if } i \leq 3, \\ x_i & \text{if } i > 3, \end{cases}$$

If $\mathbf{A}$ is invertible, let $\mathbf{v}$ be the (unique) vector for which

$$\mathbf{A} \cdot \mathbf{v}^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

In particular $\sum_i x_i' = \sum_i x_i$. For ε sufficiently small,

$$p(\mathbf{x}', \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) + \varepsilon + O(\varepsilon^2) > p(\mathbf{x}, \mathbf{y})$$

$$q(\mathbf{x}', \mathbf{y}) = q(\mathbf{x}, \mathbf{y}) + \varepsilon > q(\mathbf{x}, \mathbf{y})$$

contrary to the maximality of $(\mathbf{x}, \mathbf{y})$.

If $\mathbf{A}$ is singular, pick some $\mathbf{v} \neq 0$ with $\mathbf{A} \cdot \mathbf{v}^T = \mathbf{0}$. Again $\sum_i x_i' = \sum_i x_i$. Since $\mathbf{B}$ is positive definite, for a sufficiently small ε,

$$p(\mathbf{x}', \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) + \frac{\varepsilon^2}{2} \cdot \mathbf{v} \cdot \mathbf{B} \cdot \mathbf{v}^T + O(\varepsilon^3) > p(\mathbf{x}, \mathbf{y})$$

$$q(\mathbf{x}', \mathbf{y}) = q(\mathbf{x}, \mathbf{y}),$$

Contradicting Lemma 2.3.3. $\square$

Lemma 2.3.5. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $y_2 = 0$.*

Proof. Suppose, towards contradiction, that $y_2 \neq 0$. Let

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

where

$$a_1 = \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_2} = -s(s-1) \cdot y_1 \cdot x_2^{s-2}, \qquad b_1 = \frac{\partial q}{\partial x_1} - \frac{\partial q}{\partial x_2} = r \cdot ((y_1+y_2)^{r-1} - y_2^{r-1}),$$
$$a_2 = \frac{\partial p}{\partial y_1} - \frac{\partial p}{\partial y_2} = s \cdot x_2^{s-1}, \qquad b_2 = \frac{\partial q}{\partial y_1} - \frac{\partial q}{\partial y_2} = -r(r-1) \cdot x_2 \cdot y_2^{r-2},$$

If $\operatorname{rank}(\mathbf{M}) = 2$, then there is a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that $\mathbf{M} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Define $x_1' = x_1 + \varepsilon v_1, x_2' = x_2 - \varepsilon v_1$ and $y_1' = y_1 + \varepsilon v_2, y_2' = y_2 - \varepsilon v_2$. Then $x_1' + x_2' + y_1' + y_2' = 1$ and for sufficiently small $\varepsilon > 0$

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}') &= p(\mathbf{x}, \mathbf{y}) + \varepsilon \left(\frac{\partial p}{\partial x_1} v_1 - \frac{\partial p}{\partial x_2} v_1 + \frac{\partial p}{\partial y_1} v_2 - \frac{\partial p}{\partial y_2} v_2 \right) + O(\varepsilon^2) \\ &= p(\mathbf{x}, \mathbf{y}) + \varepsilon (a_1 v_1 + a_2 v_2) + O(\varepsilon^2) = p(\mathbf{x}, \mathbf{y}) + \varepsilon + O(\varepsilon^2) > p(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Similarly $q(\mathbf{x}', \mathbf{y}') = q(\mathbf{x}, \mathbf{y}) + \varepsilon + O(\varepsilon^2) > q(\mathbf{x}, \mathbf{y})$. Thus $(\mathbf{x}, \mathbf{y})$ cannot be a maximum of φ. Hence, $\operatorname{rank}(\mathbf{M}) \leq 1$, and in particular

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0,$$

which implies that

$$0 = x_2^{s-1} y_2^{r-1} \left((r-1)(s-1) \frac{y_1}{y_2} - \left(\frac{y_1}{y_2} + 1 \right)^{r-1} + 1 \right),$$

The function

$$g(\alpha) = (r-1)(s-1)\alpha - (\alpha+1)^{r-1} + 1$$

is strictly concave for $\alpha > 0$ and vanishes at 0. Since $\alpha = 0$ is not a maximum of g, the equation $g\left(\frac{y_1}{y_2}\right) = 0$ determines $\frac{y_1}{y_2}$ uniquely.

Denote $\alpha = \frac{y_1}{y_2}$, and consider the following change of variables.

$$x_1' = x_1 + \frac{1}{1 + (r-1)(s-1)\alpha} \cdot x_2, \qquad x_2' = \frac{(r-1)(s-1)\alpha}{1 + (r-1)(s-1)\alpha} \cdot x_2$$
$$y_1' = y_1 + y_2 = (\alpha+1) y_2, \qquad y_2' = 0$$

Clearly, $x_1' + x_2' = x_1 + x_2$ and $y_1' + y_2' = y_1 + y_2$. Moreover,

$$\begin{aligned}
q(\mathbf{x}', \mathbf{y}') &= (y_1')^r + r \cdot x_1' \cdot (y_1')^{r-1} \\
&= (y_1 + y_2)^r + r \cdot x_1 \cdot (y_1 + y_2)^{r-1} + \frac{r \cdot x_2 \cdot (y_1 + y_2)^{r-1}}{1 + (r-1)(s-1)\alpha} \\
&= (y_1 + y_2)^r + r \cdot x_1 \cdot (y_1 + y_2)^{r-1} + \frac{r \cdot (1+\alpha)^{r-1} \cdot x_2 \cdot y_2^{r-1}}{(1+\alpha)^{r-1}} = q(\mathbf{x}, \mathbf{y}) \\
p(\mathbf{x}', \mathbf{y}') &= (x_1' + x_2')^s + s \cdot y_1' \cdot (x_2')^{s-1} \\
&= (x_1 + x_2)^s + s \cdot (\alpha + 1) \cdot \left(\frac{(r-1)(s-1)\alpha}{1 + (r-1)(s-1)\alpha} \right)^{s-1} \cdot y_2 \cdot x_2^{s-1} \\
&> (x_1 + x_2)^s + s \cdot \alpha \cdot y_2 \cdot x_2^{s-1} = p(\mathbf{x}, \mathbf{y}),
\end{aligned}$$

Where the last inequality is a consequence of Lemma 2.3.6 below. This contradicts Lemma 2.3.3.

□

Lemma 2.3.6. *Let $r, s \geq 3$ be integers. Let $\alpha > 0$ be the unique positive root of*

$$(\alpha + 1)^{r-1} - 1 = (r-1)(s-1)\alpha.$$

Then

$$\left(1 + \frac{1}{(r-1)(s-1)\alpha} \right)^{s-1} < 1 + \frac{1}{\alpha}.$$

Proof. First, we show that $(r-1)\alpha > 1$. Let $t = (r-1)\alpha$ and assume, by contradiction, that $t \leq 1$. For $0 < t \leq 1$, we have $e^t < 1 + 2t$. On the other hand, $e \geq (1+\alpha)^{1/\alpha}$, implying $e^t \geq (1+\alpha)^{t/\alpha} = (1+\alpha)^{r-1}$. Thus we have $2t > (1+\alpha)^{r-1} - 1 = (r-1)(s-1)\alpha = (s-1)t$, which implies $2 > s-1$, a contradiction. Therefore $(r-1)\alpha > 1$. Also, since $1+x < e^x$ for all $x > 0$, we have that $\left(1 + \frac{1}{(r-1)(s-1)\alpha}\right)^{s-1} < e^{\frac{1}{(r-1)\alpha}}$. So it suffices to show that $e^{\frac{1}{(r-1)\alpha}} \leq 1 + \frac{1}{\alpha}$. But since $(r-1)\alpha > 1$, we have

$$\left(1 + \frac{1}{\alpha} \right)^{(r-1)\alpha} > 1 + \frac{(r-1)\alpha}{\alpha} = r \geq 3 > e,$$

which finishes the proof of the lemma. □

Lemma 2.3.7. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $x_2 = 0$.*

Proof. This proof is very similar to the proof of Lemma 2.3.5. Now $x_1, x_2, y_1 > 0$ and $x_1 + x_2 + y_1 = 1$. Also

$$\begin{aligned} p(\mathbf{x},\mathbf{y}) &= (x_1 + x_2)^s + s \cdot y_1 \cdot x_2^{s-1}, \\ q(\mathbf{x},\mathbf{y}) &= y_1^r + r \cdot x_1 \cdot y_1^{r-1}. \end{aligned}$$

Let

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_2} = -s(s-1) \cdot y_1 \cdot x_2^{s-2}, & b_1 &= \frac{\partial q}{\partial x_1} - \frac{\partial q}{\partial x_2} = r \cdot y_1^{r-1}, \\ a_2 &= \frac{\partial p}{\partial y_1} - \frac{\partial p}{\partial x_1} = -s \cdot ((x_1 + x_2)^{s-1} - x_2^{s-1}), & b_2 &= \frac{\partial q}{\partial y_1} - \frac{\partial q}{\partial x_1} = r(r-1) \cdot x_1 \cdot y_1^{r-2}, \end{aligned}$$

If $\mathbf{M}$ is nonsingular, then there is a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that $\mathbf{M} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Define $x_1' = x_1 + \varepsilon(v_1 - v_2), x_2' = x_2 - \varepsilon v_1$ and $y_1' = y_1 + \varepsilon v_2$. Then $x_1' + x_2' + y_1' = 1$ and for sufficiently small $\varepsilon > 0$

$$\begin{aligned} p(\mathbf{x}',\mathbf{y}') &= p(\mathbf{x},\mathbf{y}) + \varepsilon \left(\frac{\partial p}{\partial x_1}(v_1 - v_2) - \frac{\partial p}{\partial x_2} v_1 + \frac{\partial p}{\partial y_1} v_2 \right) + O(\varepsilon^2) \\ &= p(\mathbf{x},\mathbf{y}) + \varepsilon \left(a_1 v_1 + a_2 v_2 \right) + O(\varepsilon^2) = p(\mathbf{x},\mathbf{y}) + \varepsilon + O(\varepsilon^2) > p(\mathbf{x},\mathbf{y}). \end{aligned}$$

Similarly $q(\mathbf{x}',\mathbf{y}') = q(\mathbf{x},\mathbf{y}) + \varepsilon + O(\varepsilon^2) > q(\mathbf{x},\mathbf{y})$ and therefore $(\mathbf{x},\mathbf{y})$ cannot be a maximum of φ. Hence,

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0,$$

which implies

$$0 = y_1^{r-1} x_2^{s-1} \left((r-1) \cdot (s-1) \cdot \frac{x_1}{x_2} - \left(\frac{x_1}{x_2} + 1 \right)^{s-1} + 1 \right).$$

Let $\gamma = \frac{x_1}{x_2} > 0$. Then $1 + (r-1)(s-1)\gamma - (1+\gamma)^{s-1} = 0$ and concavity of the left hand side shows that γ is determined uniquely by this equation. Now make the following substitution:

$$\begin{aligned} x_1' &= 0 \\ x_2' &= x_1 + x_2 = (1+\gamma) \cdot x_2 \\ y_1' &= \frac{1}{1 + (r-1)(s-1)\gamma} \cdot y_1 \\ y_2' &= \frac{(r-1)(s-1)\gamma}{1 + (r-1)(s-1)\gamma} \cdot y_1 \end{aligned}$$

Clearly $x_1' + x_2' = x_1 + x_2$ and $y_1' + y_2' = y_1$. Moreover

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}') &= (x_2')^s + s \cdot y_1' \cdot (x_2')^{s-1} \\ &= (x_1 + x_2)^s + s \cdot y_1 \cdot x_2^{s-1} = p(\mathbf{x}, \mathbf{y}) \\ q(\mathbf{x}', \mathbf{y}') &= (y_1' + y_2')^r + r \cdot x_2' \cdot (y_2')^{r-1} \\ &= y_1^r + r \cdot \frac{(1+\gamma)}{\gamma} \cdot \left(\frac{(r-1)(s-1)\gamma}{1 + (r-1)(s-1)\gamma} \right)^{r-1} \cdot x_1 \cdot y_1^{r-1} \\ &> y_1^r + r \cdot x_1 \cdot y_1^{r-1} = q(\mathbf{x}, \mathbf{y}), \end{aligned}$$

Where the last inequality follows from Lemma 2.3.6, with r and s switched. Again, this contradicts Lemma 2.3.3. □

By combining Lemma 2.3.3 and Lemma 2.3.7, we obtain a proof of Lemma 2.3.2, which states that the maximum of φ is attained by a non-degenerate $(\mathbf{x}, \mathbf{y})$ supported only on either x_1, y_1 or y_1, x_2. In the first case, let $x_1 = \alpha$ and $y_1 = 1 - \alpha$. Then by Lemma 2.3.3, $a \cdot p(\mathbf{x}, \mathbf{y}) = a \cdot \alpha^s = b \cdot q(\mathbf{x}, \mathbf{y}) = b\big[(1-\alpha)^r + r\alpha(1-\alpha)^{r-1}\big]$ and $\varphi(\mathbf{x}, \mathbf{y}) = a \cdot \alpha^s$. In the second case, let $y_1 = \beta$ and $x_2 = 1-\beta$. Then $b \cdot q(\mathbf{x}, \mathbf{y}) = b \cdot \beta^r = a \cdot p(\mathbf{x}, \mathbf{y}) = a\big[(1-\beta)^s + s(1-\beta)^{s-1}\big]$ and $\varphi(\mathbf{x}, \mathbf{y}) = b \cdot \beta^r$. This shows that the maximum of φ is $\max\{a \cdot \alpha^s, b \cdot \beta^r\}$ with α, β satisfying the above equations. In terms of the original graph, this proves that φ is maximized by a graph of the form $Q_{n,t}$ or $\overline{Q}_{n,t}$, respectively. In particular, our problem has at most two extremal configurations (in some cases a clique and the complement of a clique can give the same value of φ).

2.4 Stability analysis

In this section we discuss the proof of Theorem 2.1.2. In essentially the same way that Theorem 2.3.1 implies Theorem 2.1.1, this theorem follows from a stability version of Theorem 2.3.1:

Theorem 2.4.1. *Let $r, s \geq 3$ be integers and let $a, b > 0$ be real. For every $\varepsilon > 0$, there exists $\delta > 0$ and an integer N such that every n-vertex G with $n > N$ for which*

$$f(G) \geq \max\{a \cdot \alpha^s, b \cdot \beta^r\} - \delta$$

is ε-close to some graph in $\mathcal{Q}_n$. Here f, α and β are as in Theorem 2.3.1.

Proof. If G is a threshold graph, the claim follows easily from Lemma 2.3.2. Since G is a threshold graph, $f(G) = \varphi(\mathbf{x}, \mathbf{y}) + o(1)$ for some $(\mathbf{x}, \mathbf{y}) \in W_k$ and some integer k. As this lemma shows, the continuous function φ attains its maximum on the compact set W_k at most twice, and this in points that correspond to graphs from $\mathcal{Q}_n$. Since $f(G)$ is δ-close to the maximum, it follows that $(\mathbf{x}, \mathbf{y})$ must be ε'-close to at least one of the two optimal points in W_k. This, in turn implies ε-proximity of the corresponding graphs.

For the general case, we use the stability version of the Kruskal-Katona theorem due to Keevash [Kee08]. Suppose G is a large graph such that $f(G) \geq \max\{a \cdot \alpha^s, b \cdot \beta^r\} - \delta$. Let G_1 be the shifted graph obtained from G. Thus G_1 is a threshold graph with the same edge density as G, and $f(G_1) \geq f(G)$ by Corollary 2.2.2. Pick a small $\varepsilon' > 0$. We just saw that for δ sufficiently small, G_1 is ε'-close to $G_{max} \in \mathcal{Q}_n$. As we know, either $G_{max} = Q_{n,t}$ or $G_{max} = \bar{Q}_{n,t}$ for some $0 < t \leq n$. We deal with the former case, and the second case can be done similarly. Now $|d(K_2; G) - d(K_2; G_{max})| \leq \varepsilon'$, since G and G_1 have the same edge density. Moreover, $d(K_s; G) \geq d(K_s; G_{max}) - \delta/a$, because $f(G) \geq f(G_{max}) - \delta$. Since G_{max} is a clique, it satisfies the Kruskal-Katona inequality with equality. Consequently G has nearly the maximum possible K_s-density for a given number of edges. By choosing ε' and δ small enough and applying Keevash's stability version of Kruskal-Katona inequality, we conclude that G and G_{max} are ε-close. $\square$

2.5 Second proof

In this section we briefly present the main ingredients for an alternative approach to Theorem 2.1.1. We restrict ourselves to the case $r = s$. This proof reduces the problem to a question in the calculus of variations. Such calculations occur often in the context of shifted graphs.

Let G be a shifted graph with vertex set $[n]$ with the standard order. Then, there is some $n \geq i \geq 1$ such that $A = \{1, ..., i\}$ spans a clique, whereas $B = \{i+1, ..., n\}$ spans an independent set. In addition, there is some non-increasing function $F : A \to B$ such that for every $j \in A$ the highest index neighbor of j in B is $F(j)$. Let x be a relative size of A and $1-x$ relative size of B. In this case we can express (up to a negligible error term)

$$d(\overline{K}_k; G) = \binom{n}{k}^{-1}\left[\binom{(1-x)n}{k} + \sum_{1\leq j\leq xn}\binom{n-F(j)}{k-1}\right] = (1-x)^k + \frac{k}{n}\sum_{1\leq j\leq xn}\left(\frac{n-F(j)}{n}\right)^{k-1}$$
$$= (1-x)^k + kx(1-x)^{k-1}\sum_{1\leq j\leq xn}\frac{1}{nx}\left(1-\frac{F(j)-xn}{(1-x)n}\right)^{k-1}.$$

Let f be a non-increasing function $f : [0,1] \to [0,1]$ such that $f(t) = \frac{F(j)-xn}{(1-x)n}$ for every $\frac{j-1}{xn} \leq t \leq \frac{j}{xn}$ (Think of f as a relative version of F both on its domain with respect to A and its codomain with respect to B). Then we can express $d(\overline{K}_k; G)$ in terms of x and f

$$d(\overline{K}_k; G) = (1-x)^k + kx(1-x)^{k-1}\int_0^1 (1-f(t))^{k-1}dt = d(\overline{K_k}; G_{x,f}).$$

Similarly one can show that

$$d(K_k; G) = x^k + kx^{k-1}(1-x)\int_0^1 (k-1)t^{k-2}f(t)dt = d(K_k; G_{x,f}).$$

Note that in this notation, $x = \theta$, $f = 0$ (resp. $x = 1-\theta$, $f = 1$) corresponds to $Q_{n,\theta\cdot n}$, (resp. $\overline{Q}_{n,\theta\cdot n}$).

To prove Theorem 2.1.1 for the case $r = s = k$, we show that assuming $d(K_k; G_{x,f}) \geq \alpha$, the maximum of $d(\overline{K_k}; G_{x,f})$ is attained for either $f = 0$ or $f = 1$. For this purpose, we prove upper bounds on the integrals.

Lemma 2.5.1. *If $f : [0,1] \to [0,1]$ is a non-increasing function, then*

$$\int_0^1 (1-f(t))^{k-1} dt \leq \max\left\{1-\left(\int_0^1 (k-1)t^{k-2}f(t)dt\right)^{\frac{1}{k-1}}, \left(1-\int_0^1 (k-1)t^{k-2}f(t)dt\right)^{k-1}\right\}.$$

The bounds in Lemma 2.5.1 are tight. Equality with the first term holds for f that takes only the values 1 and 0, and equality with the second term occurs for f a constant function. Proving Theorem 2.1.1 for such functions is done using rather standard (if somehow tedious) calculations. Lemma 2.5.1 itself is reduced to the following lemma through a simple affine transformation and normalization.

What non-decreasing function in $[0,1]$ minimizes the inner product with a given monomial?

Lemma 2.5.2. *Let $g : [0,1] \to [0,B]$ be a non-decreasing function with $B \geq 1$ and $\|g\|_{k-1} = 1$. Then*

$$\langle (k-1)t^{k-2}, g\rangle = \int_0^1 (k-1)t^{k-2}g(t)dt \geq \min\left\{B\left(1-\left(1-\frac{1}{B^{k-1}}\right)^{k-1}\right), 1\right\}.$$

Equality with the first term holds for

$$g(t) = \begin{cases} 0 & \text{if} \quad t < 1-\frac{1}{B^{k-1}} \\ B & \text{if} \quad t \geq 1-\frac{1}{B^{k-1}} \end{cases}$$

The second equality holds for $g = 1$.

We omit the proof which is based on standard calculations and convexity arguments.

2.6 Shifting in hypergraphs

In this section, we will discuss a possible extension of Lemma 2.2.1 to hypergraphs. Consider two set systems $\mathcal{F}_1$ and $\mathcal{F}_2$ with vertex sets V_1 and V_2 respectively. A (not necessarily induced) *labeled copy of $\mathcal{F}_1$ in $\mathcal{F}_2$* is an injection $I : V_1 \to V_2$ such that $I(F) \in \mathcal{F}_2$ for every $F \in \mathcal{F}_1$. We denote by $\mathrm{Cop}(\mathcal{F}_1; \mathcal{F}_2)$ the set of all labeled copies of $\mathcal{F}_1$ in $\mathcal{F}_2$ and let

$$t(\mathcal{F}_1; \mathcal{F}_2) := |\mathrm{Cop}(\mathcal{F}_1; \mathcal{F}_2)|.$$

Recall that a vertex u *dominates* vertex v if $S_{v\to u}(\mathcal{F}) = \mathcal{F}$. If either u dominates v or v dominates u in a family $\mathcal{F}$, we call the pair $\{u, v\}$ *stable* in $\mathcal{F}$. If every pair is stable in $\mathcal{F}$, then we call $\mathcal{F}$ a *stable set system.*

Theorem 2.6.1. *Let $\mathcal{H}$ be a stable set system and let $\mathcal{F}$ be a set system. For every two vertices u, v of $\mathcal{F}$ there holds*

$$t(\mathcal{H}; S_{u\to v}(\mathcal{F})) \geq t(\mathcal{H}; \mathcal{F}).$$

Corollary 2.6.2. *Let G be an arbitrary graph and let H be a threshold graph H. Then*

$$t(H; S_{u\to v}(G)) \geq t(H; G),$$

for every two vertices u, v of G.

Proof of Theorem 2.6.1 (sketch). We define a new shifting operator $\tilde{S}_{u\to v}$ for sets of labeled copies. First, for every $u, v \in V$, and a labeled copy $I : U \to V$, define $I_{u\leftrightarrow v} : U \to V$ by

$$I_{u\leftrightarrow v}(w) = \begin{cases} I(w) & \text{if } I(w) \neq u, v, \\ v & \text{if } I(w) = u, \\ u & \text{if } I(w) = v \end{cases}$$

For $\mathcal{I}$ a set of labeled copies, $I \in \mathcal{I}$, we let

$$\tilde{S}_{u\to v}(I, \mathcal{I}) = \begin{cases} I_{u\leftrightarrow v} & \text{if } I_{u\leftrightarrow v} \notin \mathcal{I} \text{ and } \mathrm{Im}(I) \cap \{u, v\} = \{u\}, \\ I_{u\leftrightarrow v} & \text{if } I_{u\leftrightarrow v} \notin \mathcal{I}, \{u, v\} \subset \mathrm{Im}(I), \text{ and } I^{-1}(u) \text{ dominates } I^{-1}(v) \text{ in } \mathcal{H}, \\ I & \text{otherwise.} \end{cases}$$

Finally, let $\tilde{S}_{u\to v}(\mathcal{I}) := \{\tilde{S}_{u\to v}(I, \mathcal{I}) : I \in \mathcal{I}\}$. Clearly, $|\tilde{S}_{u\to v}(\mathcal{I})| = |\mathcal{I}|$, and we prove that

$$\tilde{S}_{u\to v}(\mathrm{Cop}(\mathcal{H}; \mathcal{F})) \subseteq \mathrm{Cop}(\mathcal{H}; S_{u\to v}(\mathcal{F}))$$

thereby proving that $t(\mathcal{H}; S_{u\to v}(\mathcal{F})) \geq t(\mathcal{H}; \mathcal{F})$. As often in shifting, the proof is done by careful case analysis which is omitted. $\square$

2.7 Final remarks

In this chapter, we studied the relation between the densities of cliques and independent sets in a graph. We showed that if the density of independent sets of size r is fixed, the maximum density of s-cliques is achieved when the graph itself is either a clique on a subset of the vertices, or a complement of a clique. On the other hand, the problem of minimizing the clique density seems much harder and has quite different extremal graphs for various values of r and s (at least when $\alpha = 0$, see [DHM13, PV13]).

Question 2.7.1. *Given that* $d(\overline{K}_r; G) = \alpha$ *for some integer* $r \geq 2$ *and real* $\alpha \in [0, 1]$*, which graphs minimize* $d(K_s; G)$*?*

In particular, when $\alpha = 0$ we ask for the least possible density of s-cliques in graphs with independence number $r - 1$. This is a fifty years old question of Erdős, which is still widely open. Das et al [DHM13], and independently Pikhurko [PV13], solved this problem for certain values of r and s. It would be interesting if one can describe how the extremal graph changes as α goes from 0 to 1 in these cases. As mentioned in Section 2.1, the problem of minimizing $d(K_s; G)$ in graphs with fixed density of r-cliques for $r < s$ is also open and so far solved only when $r = 2$.

CHAPTER 3

Discrepancy of random graphs and hypergraphs

3.1 Introduction

A *hypergraph* H is an ordered pair $H = (V, E)$, where V is a finite set (the *vertex set*), and E is a family of distinct subsets of V (*the edge set*). The hypergraph H is k-uniform if all its edges are of size k. In this chapter we consider only k-uniform hypergraphs. The *edge density* of a k-uniform hypergraph H with n vertices is $\rho_H = e(H)/\binom{n}{k}$. We define the *discrepancy* of H to be

$$\operatorname{disc}(H) = \max_{S \subseteq V(H)} \left| e(S) - \rho_H \binom{|S|}{k} \right|, \tag{3.1.1}$$

where $e(S) = e(H[S])$ is the number of edges in the sub-hypergraph induced by S. The discrepancy can be viewed as a measure of how uniformly the edges of H are distributed among the vertices. This important concept appears naturally in various branches of Combinatorics and has been studied by many researchers in recent years. The discrepancy is closely related to the theory of quasi-random graphs (see [CGW89]), as the property $\operatorname{disc}(G) = o(|V(G)|^2)$ implies the quasi-randomness of the graph G.

Erdős and Spencer [ES71] proved that for $k \geq 2$, any k-uniform hypergraph H with n vertices has a subset S satisfying $\left| e(S) - \frac{1}{2}\binom{|S|}{k} \right| \geq cn^{\frac{k+1}{2}}$, which implies the bound $\operatorname{disc}(H) \geq cn^{\frac{k+1}{2}}$ for k-uniform hypergraphs H of edge density $\frac{1}{2}$. Erdős, Goldberg, Pach and Spencer [EGP88] obtained a similar lower bound for graphs of edge density smaller than $\frac{1}{2}$. These results were later generalized by Bollobás and Scott in [BS06], who proved the inequality $\operatorname{disc}(H) \geq c_k \sqrt{r} n^{\frac{k+1}{2}}$ for k-uniform hypergraphs H, whenever $r = \rho_H(1 - \rho_H) \geq 1/n$. The random hypergraphs show that all the aforementioned lower bounds are optimal up to constant factors. For more discussion and general accounts of discrepancy, we refer the interested reader to Beck and Sós [BS96], Bollobás and Scott [BS06], Chazelle [Cha00], Matoušek [Mat99] and Sós [S83].

A similar notion is the relative discrepancy of two hypergraphs. Let G and H be two k-uniform hypergraphs over the same vertex set V, with $|V| = n$. For a bijection $\pi : V \to V$, let G_π be obtained from G by permuting all edges according to π, i.e., $E(G_\pi) = \pi(E(G))$. The *overlap* of G and H with respect to π, denoted by $G_\pi \cap H$, is a hypergraph with the same vertex set V and with edge set $E(G_\pi) \cap E(H)$. The *discrepancy of G with respect to H* is

$$\mathrm{disc}(G,H) = \max_{\pi} \left| e(G_\pi \cap H) - \rho_G \rho_H \binom{n}{k} \right|, \tag{3.1.2}$$

where the maximum is taken over all bijections $\pi : V \to V$. For random bijections π, the expected size of $E(G_\pi) \cap E(H)$ is $\rho_G \rho_H \binom{n}{k}$; thus $\mathrm{disc}(G,H)$ measures how much the overlap can deviate from its average. In a certain sense, the definition (3.1.2) is more general than (3.1.1), because one can write $\mathrm{disc}(H) = \max_{1 \le i \le n} \mathrm{disc}(G_i, H)$, where G_i is obtained from the complete i-vertex k-uniform hypergraph by adding $n - i$ isolated vertices.

Bollobás and Scott introduced the notion of relative discrepancy in [BS11] and showed that for any two n-vertex graphs G and H, if $\frac{16}{n} \le \rho_G, \rho_H \le 1 - \frac{16}{n}$, then $\mathrm{disc}(G,H) \ge c \cdot f(\rho_G, \rho_H) \cdot n^{\frac{3}{2}}$, where c is an absolute constant and $f(x,y) = x^2(1-x)^2 y^2 (1-y)^2$. As a corollary, they proved a conjecture in [EGP88] regarding the *bipartite discrepancy* $\mathrm{disc}(G, K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$. Moreover, they also conjectured that a similar bound holds for k-uniform hypergraphs, namely, there exists $c = c(k, \rho_G, \rho_H)$ for which $\mathrm{disc}(G,H) \ge cn^{\frac{k+1}{2}}$ holds for any k-uniform hypergraphs G and H satisfying $\frac{1}{n} \le \rho_G, \rho_H \le 1 - \frac{1}{n}$.

In their paper, Bollobás and Scott also asked the following question (see Problem 12 in [BS11]). Given two random n-vertex graphs G and H with constant edge probability p, what is the expected value of $\mathrm{disc}(G,H)$? In this chapter, we solve this question completely for general k-uniform hypergraphs. Let $\mathcal{H}_k(n,p)$ denote the random k-uniform hypergraph on n vertices, in which every edge is included independently with probability p. We say that an event happens *with high probability*, or w.h.p. for brevity, if it happens with probability at least $1 - n^{-\omega(1)}$, where here and later $\omega(1)$ denotes an arbitrary function tending to infinity together with n.

Theorem 3.1.1. *For positive integers n and k, let $N = \binom{n-\frac{n}{k}}{k-1}$. Let G and H be two random hypergraphs distributed according to $\mathcal{H}_k(n,p)$ and $\mathcal{H}_k(n,q)$ respectively, where $\frac{\omega(1)}{N} \le p \le q \le \frac{1}{2}$.*

(1) DENSE CASE *– If $pqN > \frac{1}{30}\log n$, then w.h.p. $\mathrm{disc}(G,H) = \Theta_k\left(\sqrt{pq\binom{n}{k}n\log n}\right)$;*

(2) SPARSE CASE *– If $pqN \le \frac{1}{30}\log n$, let $\gamma = \frac{\log n}{pqN}$; then*

(2.1) if $pN \ge \frac{\log n}{5\log\gamma}$, then w.h.p. $\mathrm{disc}(G,H) = \Theta_k\left(\frac{n\log n}{\log\gamma}\right)$.

(2.2) if $pN < \frac{\log n}{5\log\gamma}$, then w.h.p. $\mathrm{disc}(G,H) = \Theta_k\left(p\binom{n}{k}\right)$.

The previous theorem also provides tight bounds when p and/or $q \ge \frac{1}{2}$, as we shall see in the concluding remarks. The result of Theorem 3.1.1 in the sparse range is closely related to the recent work of the third author with Lee and Loh [LLS13]. Among other results, the authors of [LLS13] show that two independent copies G,H of the random graph $G(n,p)$ with $p \ll \sqrt{\log n/n}$ w.h.p. have overlap of order $\Theta\left(n\frac{\log n}{\log\gamma}\right)$, where $\gamma = \frac{\log n}{p^2 n}$. Hence $\mathrm{disc}(G,H) = \Theta\left(n\frac{\log n}{\log\gamma}\right)$ holds, since in this range of edge probability, $n\frac{\log n}{\log\gamma}$ is larger than the average overlap $p^2\binom{n}{2}$. Our proof in the sparse case borrows some ideas from [LLS13]. On the other hand, one can not use their approach for all cases; hence some new ideas were needed to prove Theorem 3.1.1.

We would like to remark that Bollobás and Scott [BS] independently obtained similar results as in Theorem 3.1.1.

It will become evident from our proof that the problem of determining the discrepancy can be essentially reduced to the following question. Let $K > 0$, and let X be a binomial random variable with parameters m and ρ. What is the maximum value of $\Lambda = \Lambda(m,\rho,K)$ satisfying $\mathbb{P}\big[X - m\rho > \Lambda\big] \ge e^{-K}$? This question is related to the rate function of binomial distribution. In all cases, the discrepancy in the statement of Theorem 3.1.1 is w.h.p.

$$\mathrm{disc}(G,H) = \Theta_k\left(n \cdot \Lambda\Big(p\binom{n-1}{k-1}, q, \log n\Big)\right). \tag{3.1.3}$$

Note that $p\binom{n-1}{k-1}$ is roughly the size of the neighborhood of a vertex in the hypergraph G.

The rest of this chapter is organized as follows. Section 3.2 contains a high level outline of our proof. It also includes the definition of the *probabilistic discrepancy* $\mathrm{disc}_P(G,H)$. Section 3.3 contains a list of inequalities and technical lemmas used throughout the chapter. In particular, we demonstrate that $\mathrm{disc}_P(G,H)$ w.h.p. does not deviate too much from $\mathrm{disc}(G,H)$. In Section 3.4, we establish the upper bound for $\mathrm{disc}(G,H)$ based on a similar bound for $\mathrm{disc}_P(G,H)$. In Section 3.5, we give a detailed proof of the lower bound for $\mathrm{disc}(G,H)$. The final section contains some concluding remarks. In this chapter, the function log refers to the natural logarithm and all asymptotic notation symbols (Ω, O, o and Θ) are with respect to the variable n. Furthermore, the k-subscripts in these symbols indicate the dependence on k in the relevant constants.

3.2 Outline of the proof

In this section, we describe the main ideas in the proof of Theorem 3.1.1. In order to determine $\mathrm{disc}(G,H)$, we introduce a related quantity, the *probabilistic discrepancy* $\mathrm{disc}_P(G,H)$. Let G and H be two random hypergraphs over the same vertex set V, distributed according to $\mathcal{H}_k(n,p)$ and $\mathcal{H}_k(n,q)$, respectively. The *probabilistic discrepancy* of G with respect to H is defined by

$$\mathrm{disc}_P(G,H) = \max_{\pi} \left| e(G_\pi \cap H) - pq\binom{n}{k} \right|,$$

where the maximum is taken over all bijections $\pi : V \to V$. In Section 3.4, we show that $\mathrm{disc}_P(G,H)$ is w.h.p. very close to $\mathrm{disc}(G,H)$, hence, to bound $\mathrm{disc}(G,H)$, it suffices to show corresponding bounds for $\mathrm{disc}_P(G,H)$.

The proof of the upper bound for $\mathrm{disc}_P(G,H)$ is fairly standard. In case (2.2) of the main theorem, the proof is trivial, as w.h.p. $e(G) < 2p\binom{n}{k}$. For the remaining cases, we remark that for any fixed permutation $\pi : V \to V$, the overlap $G_\pi \cap H$ is a random hypergraph distributed according to $\mathcal{H}_k(n,pq)$. The upper bound then follows from a straightforward union bound argument over all possible permutations π, together with the application of concentration inequalities for the binomial distribution. The remaining details of this particular argument are presented in Section 3.4.

In Section 3.5, we show that w.h.p. there exists a permutation π such that the corresponding overlap $e(G_\pi \cap H)$ is much bigger than $pq\binom{n}{k}$. Note that $e(G_\pi \cap H) > pq\binom{n}{k}$, so the discrepancy is "positive" here. In the proof, we fix an arbitrary set $L \subseteq V$ of size $|L| = \frac{n}{k}$, and restrict the set of possible permutations to bijections permuting only the elements of L. Then, we gradually expose the edges (belonging to both G and H) in two rounds. In the first round, we expose the edges having exactly one vertex in L, while keeping unexposed the edges having zero or at least two vertices in L. This way, the overall contribution to the discrepancy from the edges exposed in the first round is exactly the sum of the contributions from each individual choice of $\pi(x)$. To be more precise, let R be the set of all $(k-1)$-subsets of $V \setminus L$; for each $u \in L$, let $N_G(u)$ be the collection of all $(k-1)$-sets $T \in R$ such that $\{u\} \cup T$ is an edge of G, and let $N_H(u)$ be defined similarly; finally, for each pair $u, v \in L$, let $\mathrm{codeg}(u, v)$ denote the size of $N_G(u) \cap N_H(v)$. The total number of edges in the overlap $G_\pi \cap H$ having exactly one vertex in L is precisely the sum

$$\sum_{x \in L} \mathrm{codeg}(x, \pi(x)). \tag{3.2.1}$$

See Figure 3.1 for more details. The size $|L| = \frac{n}{k}$ was appropriately chosen to maximize the number of edges having precisely one vertex in L. Additionally, we remark that $|R| = \binom{n-\frac{n}{k}}{k-1}$, which is exactly the value of N in the statement of Theorem 3.1.1. The following inequality will be used extensively later in the chapter. It relates N and the binomial coefficient $\binom{n}{k}$ for large enough n, as

$$\frac{1}{3}\binom{n}{k} \le N\frac{n}{k} \le \frac{1}{2}\binom{n}{k}.$$

Having found the bijection π with big overlap in the exposed edges (we have not yet explained how to obtain such bijection), the final step would be to expose the remaining edges of both hypergraphs (second round exposure) and compute the overall discrepancy. The potential "loss" in this final step will be w.h.p. much smaller than the "gain" we already obtained in the previous steps.

It remains to explain how to obtain the bijection π. We define the *connection* graph $\Gamma = \Gamma(G, H)$ as follows. The set of vertices of Γ is the union of two disjoint copies of L,

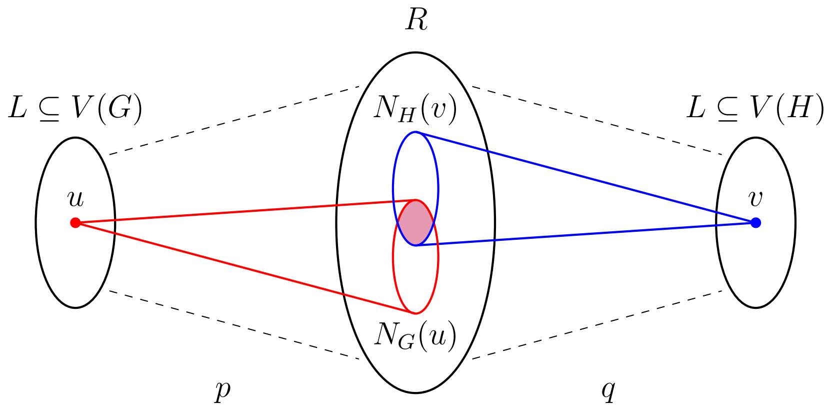

Figure 3.1: Edges of G and H having one vertex in L.

which we will refer to as L_G and L_H, respectively. We will add an edge between $u \in L_G$ and $v \in L_H$ in Γ when $\text{codeg}(u, v)$ is sufficiently large. The notion of large here will vary, depending on which case (dense or sparse) we are trying to prove. Because of (3.2.1), in order to maximize the overlap, it will suffice to show the existence of a large matching in auxiliary graph Γ.

In the dense case, we prove that we can find a nearly regular subgraph of Γ (i.e., all the degrees are roughly the same) and thus the existence of the desired bijection π easily follows from well-known theorem of Vizing. For more details, see Section 3.5.1. In the sparse case, the proof is slightly different. To find the matching in Γ, we divide L_G into chunks, each having size $n^{2/5}$. Then, for each chunk in L_G, we expose the neighborhoods of its vertices to R and w.h.p. we show that these neighborhoods can be made disjoint by removing very few edges. Finally, we start matching the vertices in L_H with the vertices in L_G. This is done by exposing the neighborhood of a vertex in L_H (one by one, according to an arbitrary predetermined order), and matching it with a high codegree vertex in L_G. The details of this construction are contained in Section 3.5.2.

3.3 Auxiliary results

In this section we list and prove some useful concentration inequalities about the binomial and hypergeometric distributions. In addition, we prove that $\text{disc}_P(G, H)$ (defined in the previous section) is w.h.p. very close to $\mathrm{disc}(G,H)$. Lastly, we prove a corollary from the well-known Vizing's Theorem which asserts the existence of a linear-size matching in *nearly regular* graphs (i.e., the maximum degree is close to the average degree). We will not attempt to optimize our constants, preferring rather to choose values which provide a simpler presentation. Let us start with classical Chernoff-type estimates for the tail of the binomial distribution (see, e.g., [AS08]).

Lemma 3.3.1. *Let $X = \sum_{i=1}^{l} X_i$ be the sum of independent zero-one random variables with average $\mu = \mathbb{E}[X]$. Then for all non-negative $\lambda \le \mu$, we have $\mathbb{P}[|X-\mu| > \lambda] \le 2e^{-\frac{\lambda^2}{4\mu}}$.*

The following lower tail inequality (see [AS08]) is due to Janson.

Lemma 3.3.2. *Let $A_1, A_2, \dots, A_l$ be subsets of a finite set Ω, and let R be a random subset of Ω for which the events $r \in R$ are mutually independent over $r \in \Omega$. Define X_j to be the indicator random variable of $A_j \subset R$. Let $X = \sum_{j=1}^{l} X_j$, $\mu = \mathbb{E}[X]$, and $\Delta = \sum_{i\sim j} \mathbb{E}[X_i \cdot X_j]$, where $i \sim j$ means that X_i and X_j are dependent (i.e., A_i intersects A_j). Then for any $\lambda > 0$,*

$$\mathbb{P}[X \le \mu - \lambda] < e^{-\frac{\lambda^2}{2\mu+\Delta}}.$$

Next, we establish that the difference between $\mathrm{disc}(G,H)$ and $\mathrm{disc}_P(G,H)$ is w.h.p. very small. This difference is, in fact, much smaller than any bound stated in Theorem 3.1.1. Thus, to prove bounds for $\mathrm{disc}(G,H)$, it suffices to show corresponding bounds for $\mathrm{disc}_P(G,H)$.

Lemma 3.3.3. *Let G and H be two random hypergraphs over the same vertex set V, distributed according to $\mathcal{H}_k(n,p)$ and $\mathcal{H}_k(n,q)$, respectively. With probability at least $1-4e^{-\sqrt{n}}$, the inequality $|\mathrm{disc}(G,H) - \mathrm{disc}_P(G,H)| \le 2\varepsilon$ holds, where $\varepsilon = 4n^{\frac{1}{4}}\sqrt{pq\binom{n}{k}}$.*

Proof. Since $p\binom{n}{k} = \Omega(n)$, applying Lemma 3.3.1 to the random variable $e(G)$ for $\lambda = 2n^{\frac{1}{4}}\sqrt{p\binom{n}{k}} \le p\binom{n}{k}$ yields

$$\mathbb{P}\left[\left|e(G) - p\binom{n}{k}\right| \le 2n^{\frac{1}{4}}\sqrt{p\binom{n}{k}}\right] \ge 1 - 2e^{-\sqrt{n}}.$$

Similarly, we have $\mathbb{P}\left[|e(H) - q\binom{n}{k}| \leq 2n^{\frac{1}{4}}\sqrt{q\binom{n}{k}}\ \right] \geq 1 - 2e^{-\sqrt{n}}$. Therefore, with probability at least $1 - 4e^{-\sqrt{n}}$, $|\rho_G - p| \leq 2n^{\frac{1}{4}}\left(p/\binom{n}{k}\right)^{1/2}$ and $|\rho_H - q| \leq 2n^{\frac{1}{4}}\left(q/\binom{n}{k}\right)^{1/2}$. But if $|AB - A_0B_0| \geq \epsilon_1\epsilon_2 + |A_0|\epsilon_2 + |B_0|\epsilon_1$, then either $|A - A_0| \geq \epsilon_1$ or $|B - B_0| \geq \epsilon_2$. Together, these inequalities imply

$$\left|\rho_G\rho_H\binom{n}{k} - pq\binom{n}{k}\right| \leq 4\sqrt{pqn} + 2pn^{\frac{1}{4}}\sqrt{q\binom{n}{k}} + 2qn^{\frac{1}{4}}\sqrt{p\binom{n}{k}} \leq 2\varepsilon,$$

completing the proof of the lemma. □

In the proof of the dense case of the main theorem we will need a lower bound for the tail of the hypergeometric distribution. To prove it we use the following well-known estimates for the binomial coefficient.

Proposition 3.3.4. *Let $H(p) = -p\log p - (1-p)\log(1-p)$ (the binary entropy), then for any integer $m > 0$ and real $p \in (0,1)$ satisfying $pm \in \mathbb{Z}$ we have*

$$\frac{\sqrt{2\pi}}{e^2} \leq \binom{m}{pm}\sqrt{mp(1-p)}e^{-mH(p)} \leq \frac{e}{2\pi}.$$

Proof. This can be derived from Stirling's formula $\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \leq m! \leq e\sqrt{m}\left(\frac{m}{e}\right)^m$. □

Lemma 3.3.5. *Let d_1, d_2, Δ and N be integers and K be a real parameter such that $1 \leq d_1, d_2 \leq \frac{2N}{3}$, $1 \leq K \leq \frac{d_1d_2}{100N}$ and $\Delta = \sqrt{\frac{d_1d_2K}{N}}$. Then*

$$\sum_{t \geq \frac{d_1d_2}{N} + \Delta} \frac{\binom{d_1}{t}\binom{N-d_1}{d_2-t}}{\binom{N}{d_2}} \geq e^{-40K}.$$

Proof. For convenience, we write $f(t) = \binom{d_1}{t}\binom{N-d_1}{d_2-t}/\binom{N}{d_2}$. In order to show the desired lower bound of the hypergeometric sum, it suffices to prove that

$$f(t) \geq \frac{4e^{-40K}}{\sqrt{\frac{d_1d_2}{N} + \Delta}},$$

for every integer $t = \frac{d_1d_2}{N} + \theta\Delta$ with $1 \leq \theta \leq 2$. Indeed, to see this, note that there are at least $\lfloor\Delta\rfloor \geq \frac{\Delta}{2}$ integers between $\frac{d_1d_2}{N} + \Delta$ and $\frac{d_1d_2}{N} + 2\Delta$ and

$$\Delta > \frac{1}{2}\sqrt{\Delta^2 + \Delta} \geq \frac{1}{2}\sqrt{\frac{d_1d_2}{N} + \Delta}.$$

Next we prove the bound for $f(t)$. For our choice of Δ, the inequality $\Delta \leq \frac{d_1}{15}$ is true since

$$\Delta = \sqrt{\frac{d_1 d_2 K}{N}} = d_1 \sqrt{\frac{d_2}{N} \cdot \frac{K}{d_1}} \leq d_1 \sqrt{\frac{d_2}{N} \cdot \frac{d_2}{100N}} = \frac{d_1}{10} \cdot \frac{d_2}{N} \leq \frac{d_1}{15}.$$

Similarly $\Delta \leq \frac{d_2}{15}$. Let $x = \frac{d_2}{N}$, $y = \frac{\theta\Delta}{d_1}$ and $z = \frac{\theta\Delta}{N-d_1}$. Then $t = (x+y)d_1$ and $d_2 - t = (x-z)(N-d_1)$. But $0 < x+y < 1$, because $0 < x \leq \frac{2}{3}$ and $0 < y \leq \frac{2\Delta}{d_1} < \frac{1}{3}$. Furthermore, $0 < x - z < 1$, because $\frac{z}{x} = \frac{\theta\Delta N}{d_2(N-d_1)} \leq \frac{3\theta\Delta}{d_2} \leq \frac{2}{5}$ and $x \leq \frac{2}{3}$. By Proposition 3.3.4, we have

$$f(t) = \frac{\binom{d_1}{(x+y)d_1}\binom{N-d_1}{(x-z)(N-d_1)}}{\binom{N}{xN}} \geq \frac{4\pi^2}{e^5}\sqrt{R}e^{-L},$$

where $L = -d_1 \cdot H(x+y) - (N-d_1) \cdot H(x-z) + N \cdot H(x)$ and

$$R = \frac{x(1-x)N}{(x-z)(1-x+z)(x+y)(1-x-y)d_1(N-d_1)} \geq \frac{1}{(x+y)d_1} \geq \frac{1}{2} \cdot \frac{1}{\frac{d_1 d_2}{N} + \Delta}.$$

Here we used $z \leq x$ for the first the inequality; and we used $\theta \leq 2$ and the identity $(x+y)d_1 = t = \frac{d_1 d_2}{N} + \theta\Delta$ for the second inequality. Because $d_1 y = (N-d_1)z = \theta\Delta$ and $\log(1+s) \leq s$, we obtain

$$\begin{aligned}
L &= d_1 \left[(x+y)\log\left(1+\frac{y}{x}\right) + (1-x-y)\log\left(1-\frac{y}{1-x}\right)\right] \\
&\quad + (N-d_1)\left[(x-z)\log\left(1-\frac{z}{x}\right) + (1-x+z)\log\left(1+\frac{z}{1-x}\right)\right] \\
&\leq d_1\left[\frac{(x+y)y}{x} - \frac{(1-x-y)y}{1-x}\right] + (N-d_1)\left[-\frac{(x-z)z}{x} + \frac{(1-x+z)z}{1-x}\right] \\
&= \theta\Delta \cdot (y+z) \cdot \left(\frac{1}{x} + \frac{1}{1-x}\right) = \frac{\theta^2\Delta^2 N^3}{d_1(N-d_1)d_2(N-d_2)} \leq 36K.
\end{aligned}$$

Thus we always have $f(t) \geq \frac{4\pi^2}{\sqrt{2}e^5} \cdot \frac{e^{-36K}}{\sqrt{\frac{d_1 d_2}{N}+\Delta}} \geq \frac{4e^{-40K}}{\sqrt{\frac{d_1 d_2}{N}+\Delta}}$, completing the proof. □

The next lemma will be used to prove the lower bound in the sparse case of Theorem 3.1.1 and was inspired by an analogous result in [LLS13].

Lemma 3.3.6. *For positive integers n and k, let $N = \binom{n-\frac{n}{k}}{k-1}$, $\frac{\omega(1)}{N} \leq p \leq q \leq \frac{1}{2}$ and suppose that $pqN \leq \frac{1}{30}\log n$. Define $\gamma = \frac{\log n}{pqN}$. Let $N_1, \ldots, N_s \subseteq B$ be $s \geq n^{1/3}$ disjoint sets of size $(1+o(1))Np$, and consider the random set B_q, obtained by taking each element of B independently with probability q. Then w.h.p., there is an index i for which*

(1) $|B_q \cap N_i| \geq \frac{\log n}{6 \log \gamma}$ *if* $pN \geq \frac{\log n}{5 \log \gamma}$*;*

(2) $N_i \subseteq B_q$ *if* $pN < \frac{\log n}{5 \log \gamma}$.

Proof. If $pN \geq \frac{\log n}{5 \log \gamma}$, let $t = \frac{\log n}{6 \log \gamma}$. Clearly $1 - q \geq e^{-3q/2}$ when $q \leq 1/2$. For a fixed index i, the probability that $|B_q \cap N_i| \geq t$ is at least $\binom{|N_i|}{t} q^t (1-q)^{|N_i|-t}$. Using the bounds $\binom{a}{n} \geq (\frac{a}{b})^b$ for $a \geq b$, and $\frac{1}{30} \log n \geq Npq = \frac{\log n}{\gamma}$, we obtain

$$\binom{|N_i|}{t} q^t (1-q)^{|N_i|-t} \geq \left(\frac{(1+o(1))Npq}{t}\right)^t e^{-2pqN} \geq \left(\frac{5 \log \gamma}{\gamma}\right)^{\frac{\log n}{6 \log \gamma}} n^{-1/15}$$
$$\geq n^{-1/6} \cdot n^{-1/15} \geq n^{-0.3}.$$

Hence the expected number of indices i such that $|B_q \cap N_i| \geq t$ is at least $sn^{-0.3} \geq n^{1/30}$. Since the sets N_i are disjoint, these events are independent for different choices of i. Therefore by Lemma 3.3.1 w.h.p. we can find such an index (actually many).

If $pN < \frac{\log n}{5 \log \gamma}$, then $q = \frac{\log n}{\gamma p N} > \frac{5 \log \gamma}{\gamma} \geq \gamma^{-1}$. Therefore the probability that some $N_i \subseteq B_q$ is

$$q^{|N_i|} \geq \gamma^{-(1+o(1))Np} \geq \gamma^{-\frac{\log n}{4 \log \gamma}} = n^{-1/4},$$

and we can complete the proof as in the first case. $\square$

The last lemma in this section, which can be easily derived from Vizing's Theorem, will be used to find a linear-size matching in nearly regular graphs.

Lemma 3.3.7. *Every graph G with maximum degree $\Delta(G)$, contains a matching of size at least $\frac{e(G)}{\Delta(G)+1}$.*

Proof. By Vizing's Theorem, the graph G has a proper edge coloring $f : E(G) \to \{1, 2, \ldots, 1 + \Delta(G)\}$. For each color $1 \leq c \leq \Delta(G) + 1$, the edges $f^{-1}(c)$ form a matching in G. By the pigeonhole principle, there is a color c such that $f^{-1}(c)$ has at least $\frac{e(G)}{\Delta(G)+1}$ edges. $\square$

3.4 Upper bounds

In this section we prove the upper bound for the discrepancy in Theorem 3.1.1. By Lemma 3.3.3, it suffices to prove the corresponding bounds for $\mathrm{disc}_P(G, H)$ instead.

Lemma 3.4.1. *Let G and H be as in Theorem 3.1.1. Then w.h.p.* $\mathrm{disc}_P(G,H)$ *satisfies the stated upper bounds of Theorem 3.1.1.*

Proof. Since the number of edges of G is distributed binomially and $p\binom{n}{k} = \Omega(n)$, by Lemma 3.3.1, we have $e(G) < 2p\binom{n}{k}$ with probability at least $1-e^{-\Theta(n)}$. Since $\mathrm{disc}_P(G,H)$ is bounded by $\max\left\{e(G), pq\binom{n}{k}\right\}$, this implies the assertion in the case (2.2) of Theorem 3.1.1.

For any fixed bijection $\pi : V \to V$, the number of edges in $G_\pi \cap H$ is distributed binomially with parameters $\binom{n}{k}$ and pq. If $pq\binom{n}{k} > 4n\log n$ let $\lambda = 2\sqrt{pq\binom{n}{k}n\log n} \le pq\binom{n}{k}$. Then by Lemma 3.3.1, the probability that $\left|e(G_\pi \cap H) - pq\binom{n}{k}\right| > \lambda$ is at most $2e^{-n\log n}$. On the other hand, if $pq\binom{n}{k} \le 4n\log n$, let $\gamma' = 4e\frac{n\log n}{pq\binom{n}{k}} \ge e > 1$ and $\lambda = \frac{4e^2 n\log n}{\log\gamma'} \ge \frac{4e^2 n\log n}{\gamma'} = epq\binom{n}{k}$. Since $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$, the probability that $e(G_\pi \cap H) > \lambda$ is at most

$$\binom{\binom{n}{k}}{\lambda}(pq)^\lambda \le \left(\frac{e\binom{n}{k}pq}{\lambda}\right)^\lambda = \left(\frac{4e^2 n\log n}{\gamma'\lambda}\right)^\lambda = \left(\frac{\gamma'}{\log\gamma'}\right)^{-\frac{4e^2 n\log n}{\log\gamma'}} < e^{-n\log n}.$$

In either case, since there are $n!$ possible bijections $\pi : V \to V$, by the union bound

$$\mathbb{P}\left[\mathrm{disc}_P(G,H) > \lambda\ \right] \le n! \cdot 2e^{-n\log n} \le e^{-n/2},$$

which finishes the proof of the upper bound in case (1). Since γ (defined in Theorem 3.1.1) satisfies $\gamma = \Theta_k(\gamma')$, this implies upper bound in case (2.1) as well. Finally, observe that we divided the dense and sparse cases in this proof, according to whether $pq\binom{n}{k}$ is bigger (or smaller) than $4n\log n$, a threshold slightly different than the one used in Theorem 3.1.1. This difference is not essential though, as for p, q satisfying both $pq\binom{n}{k} \le 4n\log n$ and $pqN \ge \frac{1}{30}\log n$, we have $\sqrt{pq\binom{n}{k}n\log n} = \Theta_k\left(\frac{4e^2 n\log n}{\log\gamma'}\right)$. $\square$

3.5 Lower bounds

In this section we prove the lower bounds in Theorem 3.1.1. As we previously explained, it is enough to obtain these bounds for $\mathrm{disc}_P(G,H)$. We divide the proof into two cases. The first (*dense case*) will be discussed in the next subsection. The second (*sparse case*) will be discussed in subsection 3.5.2. Throughout the proofs, we assume that k is fixed and n is tending to infinity.

3.5.1 Dense Case

Let $N = \binom{n-\frac{n}{k}}{k-1}$ and let p, q be such that $pqN > \frac{1}{30}\log n$. Select an arbitrary set $L \subseteq V$ of size $|L| = \frac{n}{k}$. We prove that w.h.p. there exists an *L-bijection* $\pi : V \to V$ with overlap

$$e(G_\pi \cap H) \geq pq\binom{n}{k} + \Theta_k\left(n \cdot \sqrt{pqN\log n}\right) = pq\binom{n}{k} + \Theta_k\left(\sqrt{pq\binom{n}{k}n\log n}\right), \quad (3.5.1)$$

where an L-bijection $\pi : V \to V$ is a bijection from V to V which only permutes the elements of L, i.e., $\pi(x) = x$ for all $x \notin L$.

We start by describing the construction outlined in Section 3.2 in more details. From the random hypergraph G we construct a random bipartite graph $\widetilde{G}$ with vertex set $L_G \cup R$, where $L_G = L$ and R is the set of all $(k-1)$-tuples in $V \setminus L$. Note that $|R| = N$. The vertices $v_1 \in L_G$ and $\{v_2, v_3, \ldots, v_k\} \in R$ are adjacent if $\{v_1, v_2, \ldots, v_k\}$ forms an edge in the hypergraph G. With slight abuse of notation, we view $\widetilde{G}$ as a sub-hypergraph of G, containing all edges e having exactly one vertex in L, i.e. $|e \cap L| = 1$. Similarly, from the random hypergraph H we construct a random bipartite graph $\widetilde{H}$ with vertex set $L_H \cup R$. Figure 3.1 shows the resulting bipartite graphs.

Given an L-bijection $\pi : V \to V$, we divide the edge set of $G_\pi \cap H$ into two subsets: the edge set of $\widetilde{G}_\pi \cap \widetilde{H}$ and its complement. To prove our result we first expose the random edges in $\widetilde{G}$ and $\widetilde{H}$, and show how to find an L-bijection π having overlap at least $\Theta_k\left(n \cdot \sqrt{pqN\log n}\right)$ more than the expectation. Then we fix such π and expose all the remaining edges in G and H showing that the contribution of these edges to $G_\pi \cap H$ does not deviate much from the expected contribution. More precisely, let $e_\pi = |E((G - \widetilde{G})_\pi) \cap E(H - \widetilde{H})|$, then $e(G_\pi \cap H) = e(\widetilde{G}_\pi \cap \widetilde{H}) + e_\pi$. Moreover, e_π is distributed according to $\mathrm{Bin}(m, pq)$, where $\frac{1}{2}\binom{n}{k} \leq m = \binom{n}{k} - N\frac{n}{k} \leq \binom{n}{k}$. Thus w.h.p. $|e_\pi - pqm| < \sqrt{pqm} \cdot \log n$, as Lemma 3.3.1 shows. Also, $\sqrt{pqm} \cdot \log n \ll \sqrt{pq\binom{n}{k}n\log n} = \Theta_k\left(n\sqrt{pqN\log n}\right)$. To obtain (3.5.1), it is therefore enough to show that w.h.p. there exists an L-bijection π such that

$$e(\widetilde{G}_\pi \cap \widetilde{H}) \geq \frac{n}{k} \cdot \left(pqN + \Theta_k\left(\sqrt{pqN\log n}\right)\right). \quad (3.5.2)$$

since then w.h.p.,

$$\begin{aligned}
e(G_\pi \cap H) &= e(\widetilde{G}_\pi \cap \widetilde{H}) + e_\pi \\
&\geq \frac{n}{k}(pqN + \Theta_k(\sqrt{pqN\log n})) + pqm - \sqrt{pqm}\log n \\
&= \frac{n}{k}\Theta_k(\sqrt{pqN\log n}) + pq\binom{n}{k} - \sqrt{pqm}\log n \\
&= pq\binom{n}{k} + \Theta_k\left(\sqrt{pq\binom{n}{k}n\log n}\right).
\end{aligned}$$

We define an auxiliary bipartite graph $\Gamma = \Gamma(\widetilde{G}, \widetilde{H})$ as follows. A vertex $u \in L_G$ *survives* if $|\deg_{\widetilde{G}}(u) - pN| \leq 2\sqrt{2pN}$ and similarly, a vertex $v \in L_H$ *survives* if $|\deg_{\widetilde{H}}(v) - qN| \leq 2\sqrt{2qN}$. Let S_G and S_H be the sets of all surviving vertices of $\widetilde{G}$ and $\widetilde{H}$, respectively. Let $s_G = |S_G|$ and $s_H = |S_H|$. The set of vertices of Γ is the union of S_G and S_H. The edges of Γ are defined by the property

$$u \sim_\Gamma v \iff \operatorname{codeg}(u,v) \geq \frac{\deg_{\widetilde{G}}(u)\deg_{\widetilde{H}}(v)}{N} + 10^{-2}\sqrt{pqN\log n},$$

where $\operatorname{codeg}(u,v)$ denotes the *codegree* of $u \in L_G$ and $v \in L_H$, i.e. $\operatorname{codeg}(u,v) = |N_{\widetilde{G}}(u) \cap N_{\widetilde{H}}(v)|$. The graph Γ has many vertices in both parts, as the following simple lemma demonstrates

Lemma 3.5.1. *W.h.p. each part of Γ has size at least $\frac{n}{4k}$.*

Proof. Let α be the probability that some vertex u survives in L_G. Since $pN \geq 8$, we have that $2\sqrt{2pN} \leq pN$. Thus Lemma 3.3.1 applied to $\deg_{\widetilde{G}}(u)$ implies $\alpha \geq 1 - 2e^{-2} \geq 1/2$. Since the events that vertices survive are independent, s_G stochastically dominates the binomial distribution with parameters n/k and $1/2$. Thus, again by Lemma 3.3.1, w.h.p. $s_G \geq n/(4k)$ and a similar estimate holds for s_H. □

To prove (3.5.2), we will show that the following two statements hold w.h.p.

(a) Γ has a matching $M = \{(u_1, v_1), \ldots, (u_l, v_l)\}$ of size $l = \frac{n}{50k}$;

(b) there exists an L-bijection π such that $\pi(u_i) = v_i$ for all $i = 1, 2, \ldots, l$, and,

$$\sum_{u \in L_G \setminus \{u_1, u_2, \ldots, u_l\}} \operatorname{codeg}(u, \pi(u)) \geq \left(\frac{n}{k} - l\right)pqN - 2\frac{n}{k}\sqrt{pqN}.$$

Indeed, for any two adjacent vertices u, v in Γ, we have

$$\frac{\deg_{\widetilde{G}}(u)\deg_{\widetilde{H}}(v)}{N} \geq \frac{(pN-\sqrt{8pN})(qN-\sqrt{8qN})}{N} \geq pqN - 6\sqrt{pqN}.$$

Thus using (a), (b) and $l = \frac{n}{50k}$ we obtain

$$\begin{aligned}
e(\widetilde{G}_\pi \cap \widetilde{H}) &= \sum_{u \in L_G} \text{codeg}(u, \pi(u)) \geq \sum_{i=1}^{l} \text{codeg}(u_i, v_i) + \left(\frac{n}{k} - l\right) pqN - 2\frac{n}{k}\sqrt{pqN} \\
&\geq \sum_{i=1}^{l} \left[\frac{\deg_{\widetilde{G}}(u_i)\deg_{\widetilde{H}}(v_i)}{N} + 10^{-2}\sqrt{pqN\log n}\right] + \left(\frac{n}{k} - l\right) pqN - 2\frac{n}{k}\sqrt{pqN} \\
&\geq \sum_{i=1}^{l} \left[pqN - 6\sqrt{pqN}\right] + \frac{n}{50k}10^{-2}\sqrt{pqN\log n} + \left(\frac{n}{k} - l\right) pqN - 2\frac{n}{k} \cdot \sqrt{pqN} \\
&\geq \frac{n}{k}\left(pqN + 10^{-4}\sqrt{pqN\log n}\right)
\end{aligned}$$

We need the following lemma in order to prove that (b) holds.

Lemma 3.5.2. *Let $0 < \alpha < 1$ be any absolute constant. Then with probability at least $1 - e^{-\frac{n}{k}}$, any two subsets $A \subseteq L_G$ and $B \subseteq L_H$ with $|A| = |B| = \frac{\alpha n}{k}$ satisfy*

$$X_{A,B} := \sum_{u \in A, v \in B} \text{codeg}(u, v) \geq \left(\frac{\alpha n}{k}\right)^2 pqN - 2\alpha\left(\frac{n}{k}\right)^2 \sqrt{pqN}.$$

Proof. Let $X_{w,u,v}$ be the indicator of $wu \in E(\widetilde{G})$ and $wv \in E(\widetilde{H})$ for $w \in R, u \in A, v \in B$. So $X_{A,B} = \sum_{w \in R, u \in A, v \in B} X_{w,u,v}$ and $\mathbb{E}[X_{w,u,v}] = pq$. Moreover, $X_{w,u,v}$ and $X_{w',u',v'}$ are dependent if and only if $wu = w'u'$ or $wv = w'v'$. Thus, $\mu = \mathbb{E}[X_{A,B}] = \left(\frac{\alpha n}{k}\right)^2 Npq$ and

$$\Delta = \sum_{w \in R, u \in A} \sum_{v, v' \in B} \mathbb{E}[X_{w,u,v} \cdot X_{w,u,v'}] + \sum_{w \in R, v \in B} \sum_{u, u' \in A} \mathbb{E}[X_{w,u,v} \cdot X_{w,u',v}] = \frac{\alpha n}{k}\binom{\frac{\alpha n}{k}}{2} Npq\,(p+q),$$

where μ and Δ are defined as in Lemma 3.3.2. Let F be the event that there exists at least one pair of subsets $A \subseteq L_G, B \subseteq L_H$ with $|A| = |B| = \frac{\alpha n}{k}$ satisfying $X_{A,B} < (\frac{\alpha n}{k})^2 Npq - 2\alpha(\frac{n}{k})^2\sqrt{Npq}$. By the union bound and by Lemma 3.3.2, we have

$$\begin{aligned}
\mathbb{P}[F] &\leq \sum_{A \in \binom{L_G}{\alpha n}, B \in \binom{L_H}{\alpha n}} \mathbb{P}\left[X_{A,B} < \mu - 2\alpha\left(\frac{n}{k}\right)^2\sqrt{Npq}\right] \leq \binom{\frac{n}{k}}{\frac{\alpha n}{k}}^2 e^{-\frac{\left(2\alpha(\frac{n}{k})^2\sqrt{Npq}\right)^2}{2\mu+\Delta}} \\
&\leq \left(\frac{e}{\alpha}\right)^{\frac{2\alpha n}{k}} e^{-3\frac{n}{k}} \leq e^{-\frac{n}{k}},
\end{aligned}$$

since $2\mu + \Delta \leq \frac{4}{3}\left(\frac{\alpha n}{k}\right)^3 Npq$, $\alpha < 1$ and $\alpha\log(e/\alpha) \leq 1$ for all such α. $\square$

Let $M = \{(u_1, v_1), \ldots, (u_l, v_l)\}$ be a matching satisfying (a) and let $A = L_G \backslash \{u_1, u_2, \ldots, u_l\}$ and $B = L_H \setminus \{v_1, v_2, \ldots, v_l\}$. Write $|A| = |B| = \frac{n}{k} - l = \frac{\alpha n}{k}$, where $\alpha = \frac{49}{50}$. Consider $X_{A,B} = \sum_{u \in A, v \in B} \operatorname{codeg}(u, v)$. Then, by Lemma 3.5.2, with probability at least $1 - e^{-\frac{n}{k}}$, we have

$$\sum_{u \in A, v \in B} \operatorname{codeg}(u, v) \geq \left(\frac{n}{k} - l\right)^2 pqN - 2\frac{n}{k}\left(\frac{n}{k} - l\right)\sqrt{pqN}.$$

Since the complete bipartite graph with parts A, B is a disjoint union of $\frac{n}{k} - l$ perfect matchings, by the pigeonhole principle, there exists a matching M' between A and B such that

$$\sum_{(u,v) \in M'} \operatorname{codeg}(u, v) \geq \frac{\sum_{u \in A, v \in B} \operatorname{codeg}(u, v)}{\frac{n}{k} - l} \geq \left(\frac{n}{k} - l\right) pqN - \frac{2n}{k}\sqrt{pqN}.$$

Then the matching $M \cup M'$ between L_G and L_H gives the desired L-bijection π and proves (b).

To finish the proof we need to establish (a). If Γ is nearly regular, then by Lemma 3.3.7, Γ would contain a linear-size matching. Unfortunately, it is not clear that this is the case. However, we will show that it is possible to delete some edges of Γ at random and obtain a *pruned graph* Γ', which is nearly regular. Let

$$f(d_1, d_2) := \mathbb{P}\left[u \sim_\Gamma v | \deg_{\widetilde{G}}(u) = d_1, \deg_{\widetilde{H}}(v) = d_2\right],$$

where $|d_1 - pN| \leq 2\sqrt{2pN}$ and $|d_2 - qN| \leq 2\sqrt{2qN}$. Let f_0 be the minimum of $f(d_1, d_2)$ over all pairs (d_1, d_2) in the domain of f. Suppose that $f_0 \geq n^{-\frac{1}{2}}$, which we shall prove later. We keep each edge uv of Γ in Γ' independently with probability $\frac{f_0}{f(d_1,d_2)}$, where $d_1 = \deg_{\widetilde{G}}(u)$ and $d_2 = \deg_{\widetilde{H}}(v)$. Then, we claim that for any vertex $u \in S_G$, $\deg_{\Gamma'}(u)$ is binomially distributed with parameters s_H and f_0. Indeed, by definition, $\mathbb{P}\left[u \sim_{\Gamma'} v | \deg_{\widetilde{G}}(u) = d_1, \deg_{\widetilde{H}}(v) = d_2\right] = f_0$ for all possible d_1, d_2. Moreover, conditioning on the neighbors of u in $\widetilde{G}$ and on the values of the degrees $\deg_{\widetilde{H}}(v_1)$, $\deg_{\widetilde{H}}(v_2)$, $\ldots$, $\deg_{\widetilde{H}}(v_m)$, the events $u \sim_\Gamma v_1, u \sim_\Gamma v_2, \ldots,$ and $u \sim_\Gamma v_m$ are all independent. Therefore, by definition of Γ', it is easy to see that $u \sim_{\Gamma'} v_1$, $u \sim_{\Gamma'} v_2$, $\ldots$, and $u \sim_{\Gamma'} v_m$ are independent as well. Thus for any $u \in S_G$, $\deg_{\Gamma'}(u) \sim \operatorname{Bin}(s_H, f_0)$ and similarly, $\deg_{\Gamma'}(v) \sim \operatorname{Bin}(s_G, f_0)$ for all $v \in S_H$.

Conditioning on the degrees of all vertices in $\widetilde{G}, \widetilde{H}$, we obtain sets S_G and S_H, which w.h.p. satisfy the assertion of Lemma 3.5.1, i.e., $|S_G| = s_G \geq \frac{n}{4k}$ and $|S_H| = s_H \geq \frac{n}{4k}$. Thus both $s_G f_0$ and $s_H f_0$ are $\Omega_k(\sqrt{n})$. Since all degrees in Γ' are binomially distributed, Lemma 3.5.1 together with the union bound imply that w.h.p. all vertices $u \in S_G, v \in S_H$ satisfy

$$\frac{s_H f_0}{2} \leq \deg_{\Gamma'}(u) \leq \frac{3 s_H f_0}{2} \quad \text{and} \quad \frac{s_G f_0}{2} \leq \deg_{\Gamma'}(v) \leq \frac{3 s_G f_0}{2}.$$

Therefore, the max-degree $\Delta(\Gamma') \leq \max\left\{\frac{3 s_H f_0}{2}, \frac{3 s_G f_0}{2}\right\} \leq \frac{3 n f_0}{2k}$ and $e(\Gamma') \geq \frac{s_G s_H f_0}{2} \geq \frac{n^2 f_0}{32 k^2}$. Thus by Lemma 3.3.7, Γ' has a matching of size at least $\frac{e(\Gamma')}{\Delta(\Gamma')+1} \geq \frac{n}{50k}$, completing the proof of (a).

It remains to prove the bound $f_0 \geq n^{-\frac{1}{2}}$. Let $K = \frac{\log n}{5000} \geq 1$. Since pN tends to infinity, $p \leq q \leq 1/2$ and $|d_1 - pN| \leq 2\sqrt{2pN}$, we have $1 \leq d_1 = (1+o(1))pN \leq \frac{2N}{3}$. Similarly $1 \leq d_2 = (1+o(1))qN \leq \frac{2N}{3}$. Also recall that $pqN \geq \frac{1}{30}\log n$, which implies

$$\frac{d_1 d_2}{100N} = (1+o(1))\frac{pqN}{100} \geq (1+o(1))\frac{\log n}{3000} > K.$$

Therefore we can apply Lemma 3.3.5 with $\Delta = \sqrt{\frac{d_1 d_2 K}{N}} > \frac{\sqrt{pqN \log n}}{100}$. By the definition of $f(d_1, d_2)$, we have

$$f(d_1, d_2) = \sum_{t \geq \frac{d_1 d_2}{N} + \frac{\sqrt{pqN\log n}}{100}} \frac{\binom{d_1}{t}\binom{N-d_1}{d_2-t}}{\binom{N}{d_2}} \geq \sum_{t \geq \frac{d_1 d_2}{N} + \Delta} \frac{\binom{d_1}{t}\binom{N-d_1}{d_2-t}}{\binom{N}{d_2}} \geq e^{-40K} > n^{-\frac{1}{2}}.$$

This completes the proof. $\square$

3.5.2 Sparse case

In this subsection, we prove the lower bound in the sparse case $pqN \leq \frac{1}{30}\log n$. Note that, since $p \leq q$ and $\binom{n}{k} \leq 3N\frac{n}{k}$ in this case, we have $p \leq N^{-1/2+o(1)}$ and $pq\binom{n}{k} < n \log n$. The proof runs along the same lines as that of the dense case differing only in the application of Lemma 3.3.6 to obtain an L-bijection $\pi : V \to V$ whose sum of codegrees $\sum_{u \in L_G} \operatorname{codeg}(u, \pi(u))$ is large. Suppose first that $pN \geq \frac{\log n}{5 \log \gamma}$. Recall that $\gamma = \frac{\log n}{pqN} \geq 30$ and thus $\frac{\log n}{6 \log \gamma} \geq \frac{\log n}{42 \log \gamma} + \frac{\log n}{\gamma} = \frac{\log n}{42 \log \gamma} + pqN$. Also, $\sqrt{pqm}\log n \leq \sqrt{pq\binom{n}{k}}\log n \ll \frac{\log n}{42 \log \gamma}\frac{n}{k}$. Therefore it is enough to find a bijection π between L_G and

L_H such that $\sum_{u \in L_G} \mathrm{codeg}(u, \pi(u)) \geq (1+o(1))\frac{n}{k} \cdot \frac{\log n}{6 \log \gamma}$. Using such bijection, together with above inequalities and $m + N\frac{n}{k} = \binom{n}{k}$, we obtain that

$$\begin{aligned} e(G_\pi \cap H) &= \sum \mathrm{codeg}(u, \pi(u)) + e_\pi \\ &\geq (1+o(1))\frac{n}{k}\frac{\log n}{6 \log \gamma} + pqm - \sqrt{pqm}\log n \\ &\geq (1+o(1))\frac{\log n}{42 \log \gamma}\frac{n}{k} + pq\binom{n}{k}. \end{aligned}$$

Analogous to the dense case, we define the connection graph $\Gamma = \Gamma(\widetilde{G}, \widetilde{H})$ for the sparse case. But the criterion to add edges to Γ is different – u and v are joined if and only if $\mathrm{codeg}(u, v) \geq \frac{\log n}{6 \log \gamma}$. Again, our goal is to find a large matching in Γ, but the strategy will be slightly different this time.

Partition the vertices of L_G into $r = \frac{n}{ks}$ disjoint sets $S_1, \ldots, S_r$ each of size $s = n^{2/5}$. We will construct π by applying the following greedy algorithm to each set. Let us start with S_1. The algorithm will reveal the edges emanating from S_1 to R in $\widetilde{G}$ by repeatedly exposing the neighborhood of a vertex in S_1, one at a time. Throughout this process, we construct a subset $S_1' \subseteq S_1$ of size $(1 + o(1))|S_1|$ and a family of disjoint sets $N_u \subseteq R$, such that each N_u has size $(1 + o(1))Np$ and is contained in the neighborhood of u, for all $u \in S_1'$. At each step, we pick a fresh vertex u in S_1 and expose its neighborhood. If u has a set of $(1 + o(1))Np$ neighbors which is disjoint from N_w for all w in the current S_1', denote this particular set by N_u and put u in the set S_1'; otherwise move to the next fresh vertex in S_1, until there are none left. The union $X = \cup_{w \in S_1'} N_w$ always has size at most $O(pN \cdot s) \leq N^{0.9+o(1)}$. Moreover, every vertex in $R \setminus X$ is adjacent to u independently with probability p. Since $pN \geq \omega(1)$ tends to infinity with n, the set of neighbors of u outside X has size $(1 + o(1))|R \setminus X|p = (1 + o(1))Np$ with probability $1 + o(1)$. Thus, there exists an absolute lower bound $p_0 = 1 + o(1)$ such that the event "S_1' contains u" occurs with probability at least p_0, for all u. Furthermore, conditioned on the sizes of $R \setminus X$, these events are independent for different vertices u. A straightforward coupling argument shows that the number of elements in S_1' can be bounded below by a binomial random variable with s trials and probability p_0. Therefore, by Lemma 3.3.1, w.h.p. $|S_1'| = (1 + o(1))|S_1|$. Next, we construct the partial matching for S_1. Consider the disjoint sets N_u, for $u \in S_1'$, each of size $(1+o(1))Np$. Pick an arbitrary vertex v in L_H and expose its neighbors in $\widetilde{H}$. This is a random subset N_v of R, obtained by taking each element independently with probability q. Therefore by case (1) of Lemma 3.3.6, w.h.p there is a vertex $u \in S_1'$ such that $\mathrm{codeg}(u,v) \geq |N_u \cap N_v| \geq \frac{\log n}{6\log\gamma}$. Define $\pi(u)=v$, remove u from S_1', remove v from L_H and continue. Note that, as long as there are at least $n^{1/3}$ vertices remaining in S_1', we can match one of them with a newly exposed vertex from L_H such that the codegree of this pair is at least $\frac{\log n}{6\log\gamma}$. Once the number of vertices in S_1' drops below $n^{1/3}$, leave the remaining vertices unmatched. W.h.p. we can match a $1+o(1)$ fraction of the vertices in S_1.

Continue the above procedure for $S_2,\dots,S_r$ as well. At the end of the process, we will have matched a $1+o(1)$ fraction of all the vertices in L_G with distinct vertices in L_H such that codegree of every matched pair is at least $\frac{\log n}{6\log\gamma}$. Therefore the sum of the codegrees of this partial matching is at least $(1+o(1))\frac{n}{k}\cdot\frac{\log n}{6\log\gamma}$. To obtain the bijection π, one can match the remaining vertices in L_G and L_H arbitrarily.

When $pN < \frac{\log n}{5\log\gamma}$ the same proof as above together with case (2) of Lemma 3.3.6 yields a bijection π such that $\sum_{u\in L_G}\mathrm{codeg}(u,\pi(u)) \geq (1+o(1))\frac{n}{k}\cdot pN$. Since $q \leq \frac{1}{2}$, $p \geq \frac{\omega(1)}{N}$ and $m=\binom{n}{k}-N\frac{n}{k}$, this implies

$$\begin{aligned} e(G_\pi \cap H) &\geq (1+o(1))\frac{n}{k}pN + pqm - \sqrt{pqm}\log n \\ &= \Theta_k\left(p\binom{n}{k}\right) + pq\binom{n}{k}. \end{aligned}$$

finishing the analysis of the sparse case. □

3.6 Concluding remarks

As we stated in Section 3.1, Theorem 3.1.1 also yields tight bounds when p and/or $q > \frac{1}{2}$. For any G and H, one can check that $\mathrm{disc}(G,\overline{H}) = \mathrm{disc}(G,H)$, where $\overline{H}$ is the complement of H. Moreover, $\overline{H}$ is distributed according to $\mathcal{H}_k(n,1-q)$, hence we can reduce the case $q > \frac{1}{2}$ to the case $q' = 1-q \leq \frac{1}{2}$; the same holds when we take the complement of G instead. We remark that one can determine the discrepancy when p is smaller than $\frac{\omega(1)}{N}$, but we chose not to discuss this range here, since the proof is similar to the sparse case and it wouldn't provide any new insight.

The definition of discrepancy can be rephrased as

$$\operatorname{disc}(G,H) = \max\{\operatorname{disc}^+(G,H), \operatorname{disc}^-(G,H)\},$$

where $\operatorname{disc}^+(G,H) = \max_\pi e(G_\pi \cap H) - \rho_G\rho_H\binom{n}{k}$ and $\operatorname{disc}^-(G,H) = \rho_G\rho_H\binom{n}{k} - \min_\pi e(G_\pi \cap H)$ are the *one-sided relative discrepancies.* In fact, all the lower bounds we obtained are for $\operatorname{disc}^+(G,H)$, and some of them are not true for $\operatorname{disc}^-(G,H)$. This is because $\operatorname{disc}^-(G,H) \leq \rho_G\rho_H\binom{n}{k} \simeq pq\binom{n}{k}$ and in the sparse case, $pq\binom{n}{k}$ could be much smaller than $\operatorname{disc}(G,H)$. Under the same hypothesis and using similar ideas as in Theorem 3.1.1, one can show that

$$\operatorname{disc}^-(G,H) = \begin{cases} \Theta_k\left(\sqrt{pq\binom{n}{k}n\log n}\right) & \text{if } pqN > \frac{1}{30}\log n; \\ \Theta_k\left(pq\binom{n}{k}\right) & \text{otherwise.} \end{cases}$$

The last equation is related to the lower tail of the binomial distribution.

It would be interesting to determine the exact dependence on k of the relative discrepancy. It also worth mentioning that there are a substantial number of open problems about $\operatorname{disc}(G,H)$ and its related topics in [BS11].

CHAPTER 4

Generating random graphs in biased Maker-Breaker games

4.1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the $(a : b)$ Maker-Breaker game $\mathcal{F}$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first. The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker claims a board elements per turn, whereas Breaker claims b elements. The parameters a and b are called the *bias* of Maker and of Breaker, respectively. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by either of the players, then Breaker wins the game. We say that the $(a : b)$ game $\mathcal{F}$ is *Maker's win* if Maker has a strategy that ensures his victory against any strategy of Breaker, otherwise the game is *Breaker's win*. The most basic case is $a = b = 1$, the so-called *unbiased* game, while for all other choices of a and b the game is called a *biased* game. Note that being the first player is never a disadvantage in a Maker-Breaker game. Therefore, in order to prove that Maker can win some Maker-Breaker game as the first or the second player it is enough to prove that he can win this game as a second player. Hence, we will always assume that Maker is the second player to move.

It is natural to play Maker-Breaker games on the edge set of a graph $G = (V, E)$. In this case, $X = E$ and the winning sets are all the edge sets of subgraphs of G which possess some given monotone increasing graph property $\mathcal{P}$. In this case, we refer to this game as the $(a : b)$ game $\mathcal{P}(G)$. In the special case where $G = K_n$ we denote $\mathcal{P}_n := \mathcal{P}(K_n)$. In the *connectivity game*, Maker wins if and only if his edges contain a spanning tree. In the *perfect matching* game $\mathcal{M}(G)$ the winning sets are all sets of $\lfloor |V(G)|/2 \rfloor$ independent edges of G. Note that if $|V(G)|$ is odd, then such a matching covers all vertices of G but one. In the *Hamiltonicity game* $\mathcal{H}(G)$ the winning sets are all edge sets of Hamilton cycles of G. Given a positive integer k, in the *k-connectivity game* $\mathcal{C}^k(G)$ the winning sets are all edge sets of k-vertex-connected spanning subgraphs of G. Given a graph H, in the *H-game* played on G, the winning sets are all the edge sets of copies of H in G.

Playing unbiased Maker-Breaker games on the edge set of K_n is frequently in favor of Maker. For example, it is easy to see (and also follows from [Leh64]) that for every $n \geq 4$, Maker can win the unbiased connectivity game in $n-1$ moves (which is clearly also the fastest possible strategy). Other unbiased games played on $E(K_n)$ like the perfect matching game, the Hamiltonicity game, the k-vertex-connectivity game and the T-game where T is a spanning tree with bounded maximum degree, are also known to be easy win for Maker (see e.g, [HKS09], [FH14], [CFG13]). It thus natural to give Breaker more power by allowing him to claim $b > 1$ elements in each turn.

Note that Maker-Breaker games are known to be *bias monotone*. That means that none of the players can be harmed by claiming more elements. Therefore, it makes sense to study $(1 : b)$ games and the parameter b^* which is the *critical bias* of the game, that is, b^* is the maximal bias b for which Maker wins the corresponding $(1 : b)$ game $\mathcal{F}$.

There is a striking relation between the theory of biased Maker-Breaker games and the theory of random graphs, frequently referred to as the *Erdős paradigm*. Roughly speaking, it suggests that the critical bias for the game played by two "clever players" and the appropriately defined critical bias for the game played by two "random players" are asymptotically the same. In this "random players" version of the game, both players use the *random strategy*, i.e., Maker claims one random unclaimed element, while Breaker claims b random unclaimed elements from the board $E(K_n)$, per move. Note that the resulting graph occupied by Maker at the end of the game is the random graph $\mathbb{G}(n, m)$ with n vertices and $m = \lfloor \frac{1}{1+b}\binom{n}{2} \rfloor$ edges. Therefore, if the winning sets consist of all the edge sets of subgraphs of K_n which possess some monotone graph property $\mathcal{P}$, a natural guess for the critical bias is b^* for which $m^* = \frac{1}{1+b^*}$ is the threshold for the property that $\mathbb{G}(n,m)$ typically possesses $\mathcal{P}$. For this reason, the Erdős paradigm is also known as the *random graph intuition.*

Chvátal and Erdős were the first to indicate this phenomenon in their seminal paper [CE78]. They showed that Breaker, playing with bias $b = \frac{(1+\varepsilon)n}{\log n}$, can isolate a vertex in Maker's graph while playing on the board $E(K_n)$. It thus follows that Breaker wins every game for which the winning sets consist of subgraphs of K_n with positive minimum degree. What is most surprising about their result is that at the end of the game, Maker's graph consists of roughly $m = \frac{1}{2}n \log n$ edges which is (asymptotically) the threshold for a random graph $\mathbb{G}(n,m)$ to stop "having isolated vertices" (for more details on properties' thresholds for random graphs, the reader is referred to [Bol98] and [JLR11]). In this spirit, the results of Chvátal and Erdős in [CE78] hint that $b^* = \frac{n}{\log n}$ is actually the critical bias for many games whose target sets consist of graphs having some property $\mathcal{P}$, for which the threshold probability is $p = \frac{\log n}{n}$ (such as the connectivity game, the perfect matching game and the Hamiltonicity game). Gebauer and Szabó showed in [GS09] that the critical bias for the connectivity game played on $E(K_n)$ is asymptotically equal to $n/\log n$. In a relevant development, Krivelevich proved in [Kri11] that the critical bias to build a Hamilton cycle is indeed $(1 + o(1))n/\log n$.

Another striking result exploring the relation between results in Maker-Breaker games played on graphs and threshold probabilities for properties of random graphs is due to Bednarska and Łuczak in [BL00]. Given a graph G on at least three vertices we define

$$m(G) = \max\left\{\frac{|E(H)| - 1}{|V(H)| - 2} : H \subseteq G and |V(H)| \geq 3\right\}.$$

Bednarska and Łuczak proved that the threshold bias for the H-game is of order $\Theta\left(n^{1/m(H)}\right)$. The most surprising part in their proof is the side of Maker, where for this part they proved the following:

Theorem 4.1.1 (Theorem 2 in [BL00]). *For every graph H which contains a cycle there exists a constant c_0 such that for every sufficiently large integer n and $q \leq c_0 n^{1/m(H)}$ Maker has a random strategy for the H-game that succeeds with probability $1 - o(1)$ against any strategy of Breaker.*

Stating it intuitively, they proved that an "optimal" strategy for Maker is just to claim edges at random without caring about Breaker's moves! Note that since a Maker-Breaker game is a deterministic game, it follows that if Maker has a random strategy that works with non-zero probability against any given strategy of Breaker, then the game is Maker's win (otherwise Maker's strategy should work with probability zero against Breaker's winning strategy).

In the proof of Theorem 4.1.1, the graph obtained by Maker at the end of the game is not exactly a random graph, since some *failure* edges might exist (that is, it might happen that by choosing random edges, Maker attempts occasionally to pick an edge e which already belongs to Breaker). Thus, in order to prove their result, Bednarska and Łuczak not only proved that random graphs typically contain copies of the target graph H, but they also showed that with a positive probability, even after removing a small fraction of the total number of edges, these graphs still contain many copies of H. This particular statement relates to the *resilience* of random graphs with respect to the property "containing a copy of H".

Given a monotone increasing graph property $\mathcal{P}$ and a graph G which satisfies $\mathcal{P}$, the *resilience of G with respect to $\mathcal{P}$* measures how much one should change G in order to destroy $\mathcal{P}$. There are two natural ways to define it quantitatively. The first one is the following:

Definition 4.1.2. *For a monotone increasing graph property $\mathcal{P}$, the* global resilience *of G with respect to $\mathcal{P}$ is the minimum number r such that by deleting $r \cdot e(G)$ edges from G one can obtain a graph G' not having $\mathcal{P}$.*

Since one can destroy many natural properties by small changes (for example, by isolating a vertex), it is natural to limit the number of edges touching any vertex that one is allowed to delete. This leads to the following definition of *local resilience*.

Definition 4.1.3. *For a monotone increasing graph property $\mathcal{P}$, the* local resilience *of G with respect to $\mathcal{P}$ is the minimum number r such that by deleting at each vertex v at most $r \cdot d_G(v)$ edges one can obtain a graph not having $\mathcal{P}$.*

Sudakov and Vu initiated the systematic study of resilience of random and pseudorandom graphs in [SV08]. Since then, this field has attracted substantial research interest (see, e.g. [BCS11, BKS11a, BKS11b, BKT09, FK08, KLS10, LS12]).

Going back to Theorem 4.1.1, Bednarska and Łuczak actually proved that playing according the random strategy, Maker can typically build a graph $G \sim G(n, m)$ minus some ε-fraction of its edges. They then showed that for a given graph H and an appropriate m, the global resilience of a typical $G \sim G(n, m)$ with respect to the property "containing a copy of H" is at least ε. It thus natural to seek an alternative theorem which provides the analogous local resilience argument.

The main result in this chapter uses a sophisticated version of the argument in [BL00]. Let G be a graph and let $0 < p < 1$. The model $\mathcal{G}(G, p)$ is a random subgraph G' of G, obtained by retaining each edge of G in G' independently at random with probability p. For the special case where $G = K_n$, we denote $\mathcal{G}(n, p) = \mathcal{G}(K_n, p)$, which is the well-known Erdős-Rényi model of random graphs. In the same manner we define $\mathcal{D}(D, p)$ in case that D is a *directed* graph. We also denote by $\mathcal{D}(n, p)$ the special case where each pair of the $n(n-1)$ oriented pairs of $\{1, \ldots, n\}$ is being chosen with probability p independently at random. Our main result is the following.

Theorem 4.1.4. *For every positive constant* $0 < \varepsilon < 1/2$ *and a sufficiently large integer* n *the following holds. Suppose that*

(i) $0 \leq p = p(n) \leq 1$, *and*

(ii) G *is a graph with* $|V(G)| = n$, *and*

(iii) $\delta(G) \geq \frac{11 \log n}{\varepsilon p}$,

then in the $(1 : \frac{\varepsilon}{40p})$ *game played on* $E(G)$, *Maker a.a.s can build a graph* $M = G' \setminus F$, *where* $G' \sim \mathcal{G}(G, p)$ *and* F *is a graph which satisfies* $d_F(v) \leq \varepsilon d_G(v) p$, *for each* $v \in V(G)$.

As an easy corollary to Theorem 4.1.4 we establish the following directed analog.

Theorem 4.1.5. *For every positive constant $0 < \varepsilon < 1/2$ and a sufficiently large integer n the following holds. Suppose that*

(i) $0 \le p = p(n) \le 1$, *and*

(ii) D *is a directed graph with* $|V(D)| = n$, *and*

(iii) $\delta^0(D) \ge \frac{11 \log n}{\varepsilon p}$,

then in the $(1 : \frac{\varepsilon}{40p})$ *game played on* $E(D)$, *Maker a.a.s can build a graph* $M = D' \setminus F$, *where* $D' \sim \mathcal{D}(D, p)$ *and* F *is a graph which satisfies* $d_F^+(v) \le \varepsilon d_D^+(v)p$ *and* $d_F^-(v) \le \varepsilon d_D^-(v)p$, *for each* $v \in V(D)$, *where* $d_F^+(v)$ *and* $d_F^-(v)$ *denote the out- and in-degrees of* v *in* F, *respectively, and* $\delta^0(D)$ *is the minimum of all out- and in-degrees in* D.

Proof. For a directed graph D one can define the following bipartite graph G_D: the parts of G_D are two disjoint copies of $V(D)$, denoted by A and B. For any $a \in A$ and $b \in B$, the (undirected) edge ab belongs to $E(G_D)$ if and only if the directed edge ab belongs to $E(D)$. Note that the mapping $D \to G_D$ is a bijection between all the directed graphs on n vertices (self loops and double edges are allowed!) to the set of bipartite graphs with two parts of size n each, and apply Theorem 4.1.4 to G_D in the obvious way. $\square$

Theorems 4.1.4 and 4.1.5 connect between Maker's side in biased Maker-Breaker games on graphs or directed graphs and local resilience; it thus allows to use (known) results about local resilience to give a lower estimate for the critical bias in biased Maker-Breaker games. We now present our concrete results for biased games, all of them are applications of Theorems 4.1.4 and 4.1.5 and corresponding local resilience results for random graphs.

First, as a warm up we prove the following theorem which shows that the critical bias for the Hamiltonicity game played on $E(K_n)$ is $\Theta(\frac{n}{\log n})$.

Theorem 4.1.6. *There exists a constant $\delta > 0$ for which for every sufficiently large integer n the following holds. Suppose that $b \le \delta n / \log n$, then Maker has a winning strategy in the $(1 : b)$ Hamiltonicity game played on $E(K_n)$.*

The proof of Theorem 4.1.6 is just a warm up since the critical bias for the Hamiltonicity game played on $E(K_n)$ is known to be $\frac{(1+o(1))n}{\log n}$ (for details, see [Kri11]).

As a second application, by obtaining a directed version of Theorem 4.1.4 we prove the following theorem which shows that playing against a bias $b = o(\sqrt{n})$, Maker can build an oriented Hamilton cycle while playing on the complete directed graph on n vertices (that is, between every pair of vertices there is an edge in either direction).

Theorem 4.1.7. *Let $b = o(\sqrt{n})$. Then in the $(1 : b)$ game played on the edge set of the complete directed graph on n vertices, Maker has strategy to build an oriented Hamilton cycle.*

Asaf, Hefetz, and Krivelevich showed in [HFK11] that if T is a tree on n vertices and $\Delta(T) \leq n^{0.05}$, then in the $(1 : b)$ game, Maker has a strategy to win the T-game, for every $b \leq n^{0.005}$, in $n + o(n)$ moves. They also asked for improvements of the parameter b. In this chapter, as a third application of our main result, we show how to obtain such an improvement for a large family of trees. Those are trees T with $\Delta(T) = O(1)$ containing a *bare path* of length $\Theta(n)$, where a bare path is a path for which all the interior vertices are of degree exactly two in T. In fact we prove the following much stronger result:

Theorem 4.1.8. *For every $\alpha > 0$ and $D > 0$ there exists a $\delta := \delta(\alpha, D) > 0$ such that for every sufficiently large integer n the following holds. For $b \leq \frac{\delta n}{\log n}$, in the $(1 : b)$ Maker-Breaker game played on $E(K_n)$, Maker has a strategy to build a graph which contains copies of all the spanning trees T such that:*

(i) $\Delta(T) \leq D$, *and*

(ii) T *admits a bare path of length* αn.

Remark. Note that the bias b in Theorem 4.1.8 is best possible up to a constant factor, as Chvátal and Erdős showed [CE78] that for $b = \frac{(1+\varepsilon)n}{\log n}$ Breaker can isolate a vertex in Maker's graph.

The rest of the chapter is organized as follows: In Section 4.2 we present some auxiliary results. In Section 4.3 we prove Theorem 4.1.4, and in Section 4.4 we show how to apply

Theorem 4.1.4 combined with local resilience statements (introduced in Subsection 4.2.3) to various games. In this chapter, the function log refers to the natural logarithm and all asymptotic notation symbols (Ω, O, o and Θ) are with respect to the variable n.

4.2 Auxiliary results

In this section we present some auxiliary results that will be used throughout the chapter.

4.2.1 Binomial distribution bounds

We use extensively the following well-known bounds on the lower and upper tails of the Binomial distribution due to Chernoff (see, e.g., [AS08, Theorems A.1.11, A.1.13, and A.1.12]).

Lemma 4.2.1. *If* $X \sim \mathrm{Bin}(n,p)$, *then*

- $\mathbb{P}\left[X < (1-a)np\right] < \exp\left(-\frac{a^2np}{2}\right)$ *for every* $a > 0$.
- $\mathbb{P}\left[X > (1+a)np\right] < \exp\left(-\frac{a^2np}{3}\right)$ *for every* $0 < a < 1$.

Lemma 4.2.2. *Let* $X \sim \mathrm{Bin}(n,p)$ *and* $k \in \mathbb{N}$. *Then*

$$\mathbb{P}[X \geq k] \leq \left(\frac{enp}{k}\right)^k.$$

4.2.2 The MinBox game

Consider the following variant of the classical *Box Game* introduced by Chvátal and Erdős in [CE78], which we refer to as the *MinBox game*. The game $MinBox(n, D, \alpha, b)$ is a $(1 : b)$ Maker-Breaker game played on a family of n disjoint sets (*boxes*), each having size at least D. Maker's goal is to claim at least $\alpha|F|$ elements from each box F. In the proof of our main result, we make use of a specific strategy S for Maker in the MinBox game. This strategy not only ensures his victory, but also allows Maker to maintain a reasonable proportion of elements in all boxes throughout the game.

Before describing the strategy, we need to introduce some notation. Assume that a MinBox game is in progress, let $w_M(F)$ and $w_B(F)$ denote the number of Maker's and

Breaker's current elements in box F, respectively. Furthermore, let $\mathsf{dang}(F) := w_B(F) - b \cdot w_M(F)$ be the *danger value* of F. Finally, we say that a box F is *free* if it contains an element not yet claimed by either player, and it is *active* if $w_M(F) < \alpha|F|$. Maker's strategy is as follows:

Strategy S: In any move of the game, Maker identifies one free active box having maximal danger value (breaking ties arbitrarily), and claims one arbitrary free element from it.

We are ready to state the following theorem.

Theorem 4.2.3. *Let n, b, and D be positive integers, and $0 < \alpha < 1$. Assume that Maker plays the game $MinBox(n, D, \alpha, b)$ according to the strategy S described above. Then he ensures that, throughout the game, every active box F satisfies*

$$\mathsf{dang}(F) \leq b(\log n + 1).$$

In particular, if $\alpha < \frac{1}{1+b}$ and $D \geq \frac{b(\log n+1)}{1-\alpha(b+1)}$, then S is a winning strategy for Maker in this game.

The proof of this result can be found in the Appendix. We remark that it is very similar to the proof of Theorem 1.2 in [GS09].

4.2.3 Local resilience

In this subsection we describe several results related to local resilience of monotone graph properties. The main result of this chapter (Theorem 4.1.4) shows a connection between local resilience of graphs and Maker-Breaker games, therefore, in order to be able to apply it, we first need to present some results related to local resilience of various properties of random graphs.

The first statement of this section is a theorem from [LS12] providing a good bound on the local resilience of a random graph with respect to the property "being Hamiltonian". This result will be used in the proof of Theorem 4.1.6 for the Hamiltonicity game. We remark that for our purposes, prior (and weaker) results on the local resilience of a random graph with respect to Hamiltonicity (for example those in [FK08]) would suffice.

Theorem 4.2.4 (Theorem 1.1, [LS12])**.** *For every positive $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for $p \geq \frac{C\log n}{n}$, a graph $G \sim \mathbb{G}(n,p)$ is a.a.s such that the following holds. Suppose that H is a subgraph of G for which $G' = G \setminus H$ has minimum degree at least $(1/2+\varepsilon)np$, then G' is Hamiltonian.*

The following result from [FNN14] is related to the local resilience of a typical $D \sim \mathcal{D}(n,p)$ with edge probability $p = \omega\left(\frac{1}{\sqrt{n}}\right)$, with respect to the property "being Hamiltonian" (where a Hamilton cycle in a directed graph is an oriented cycle passing through all the vertices).

Theorem 4.2.5 (Theorem 1.4, [FNN14])**.** *Let n be a sufficiently large integer and let $p = \omega(1/\sqrt{n})$. Then $D \sim \mathcal{D}(n,p)$ is a.a.s such that the following holds. Suppose that $H \subseteq D$ is any subgraph of D, and $d^+_H(v), d^-_H(v) \leq np/16$ for every $v \in V(D)$, then $D' := D \setminus H$ is Hamiltonian.*

The following theorem shows that a sparse random graph $G \sim \mathbb{G}(n,p)$ typically contains a copy of every tree T having a bare path of linear length and having bounded maximum degree, even if one delete a small fixed fraction of edges from each vertex $v \in V(G)$. This result relates to the local resilience of the property of being universal for this particular class of trees, and it is an essential component in the proof of Theorem 4.1.8.

Theorem 4.2.6. *For every $\alpha > 0$ and $D > 0$, there exist $\varepsilon > 0$ and C_0 such that for every $p \geq C_0 \log n/n$, $G \sim \mathbb{G}(n,p)$ is a.a.s such that the following holds. For every subgraph $H \subseteq G$ with $\Delta(H) \leq \varepsilon np$, the graph $G' = G \setminus H$ contains copies of all spanning trees T such that:*

(i) $\Delta(T) \leq D$*, and*

(ii) *T contains a bare path of length at least αn.*

In order to prove Theorem 4.2.6 we need the following theorem due to Balogh, Csaba and Samotij [BCS11] about the local resilience of random graphs with respect to the property "containing all the almost spanning trees with bounded degree".

Theorem 4.2.7 (Theorem 2, [BCS11]). *Let β and γ be positive constants, and assume that $D \geq 2$. There exists a constant $C_0 = C_0(\beta, \gamma, D)$ such that for every $p \geq C_0/n$, a graph $G \sim \mathbb{G}(n,p)$ is a.a.s such that the following holds. For every subgraph H of G for which $d_H(v) \leq (1/2 - \gamma)d_G(v)$ for every $v \in V(G)$, the graph $G' = G \setminus H$ contains all trees of order at most $(1-\beta)n$ and maximum degree at most D.*

Proof of Theorem 4.2.6. Let $\alpha > 0$ and $D > 0$ be two positive constants. Let $\varepsilon \ll \alpha$ and let $C_0 = C_0(\varepsilon) > 0$ be a sufficiently large constant. Let $G \sim \mathbb{G}(n,p)$ be a typical random graph, $H \subseteq G$ be any subgraph with $\Delta(H) \leq \varepsilon np$ and denote $G' = G \setminus H$. We wish to show that G' contains a copy of every spanning tree T which satisfies (i) and (ii). This can be done as follows:

(1) Assume that G has been generated by a two-round-exposure and is presented as $G = G_1 \cup G_2$, where $G_1, G_2 \sim \mathcal{G}(n,q)$, with $q > p/2$ (because of the monotonicity of all the properties we mention, we treat q as $p/2$).

(2) Let V_0 be a random subset of $V(G)$ of size $|V_0| = \alpha n/2$ and denote $G_1' = G_1[V(G) \setminus V_0]$. Note that $G_1' \sim \mathcal{G}((1-\alpha/2)n, p/2)$ and that a.a.s $d_{G_1'}(v) \geq (1 - \alpha/2 - \varepsilon)np/2$ for every $v \in V(G_1')$ (this can be easily shown using Lemma 4.2.1 and choosing C_0 appropriately).

(3) Let T be a tree which satisfies (i) and (ii), and let $P = v_0 v_1 \ldots v_t$ be a bare path of T with $t = \alpha n$. Let T' be the tree obtained from T by deleting $v_1, \ldots, v_{t-1}$ and adding the edge $v_0 v_t$. Note that $|V(T')| = (1-\alpha)n + 1$.

(4) Applying Theorem 4.2.7 to G_1', using the fact that $\varepsilon \ll \alpha$ we conclude that there exists a copy T'' of T' in $G_1' \setminus H$. Let x and y denote the images (in T'') of v_0 and v_t (from T'), respectively.

(5) Let $V' = (V(G) \setminus V(T')) \cup \{x, y\}$. In order to complete the proof, we should be able to show that $(G \setminus H)[V']$ contains a Hamilton path with x and y as its endpoints. Note that $V_0 \subseteq V'$ and that $V' \setminus V_0$ and the two designated vertices x and y heavily depend on the tree T which we are trying to embed. Therefore, we wish to show that G is a.a.s such that for every possible option for V' (with two designated vertices x and y), $(G \setminus H)[V']$

contains a Hamilton path with x and y as its endpoints. For this, first note that since V_0 is a random subset of vertices, $V_0 \subseteq V'$ and $G_1 \sim \mathcal{G}(n, p/2)$, using Lemma 4.2.1 we conclude that $\delta\left(\left(G_1 \setminus H\right)[V']\right) \geq (\alpha/4-\varepsilon)np-\varepsilon np = (\alpha/4-2\varepsilon)np$ (here we assume that C_0 is large enough). Next, we show that a graph $G_1 \sim \mathcal{G}(n, p/2)$ is a.a.s such that any subgraph $D \subseteq G_1$ on $\alpha n+1$ vertices with $\delta(D) \geq (\alpha/4-2\varepsilon)np$ has "good" expansion properties (our candidate for D will be $\left(G_1 \setminus H\right)[V']$).

Claim 4.2.8. *A graph $G_1 \sim \mathcal{G}(n, p/2)$ is a.a.s such that for any subgraph $D \subset G_1$ with $|V(D)| = \alpha n + 1$ and with $\delta(D) \geq (\alpha/4 - 2\varepsilon)np$, the following holds:*

$$|N_D(X) \setminus X| \geq 2|X| + 2$$

for every $X \subseteq V(D)$ with $|X| \leq |V(D)|/5$.

Proof. Using Lemma 4.2.2, it easy to show that for every subset $S \subseteq V(G)$ of size at most (say) $\frac{\alpha n}{\sqrt{\log n}}$, the number of edges of G with both endpoints in S is at most $|E_G(S)| \leq |S|np/\log\log n$. Therefore, using the fact that $\delta(D) = \Theta(np)$ we conclude that $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every subset $X \subseteq V(D)$ of size at most $\frac{\alpha n}{3\sqrt{\log n}}$ (otherwise, the average degree of $X \cup N(X)$ is $\Theta(np)$, contradiction). In addition, since $\varepsilon \ll \alpha$ and since $\delta(D) \geq (\alpha/4 - 2\varepsilon)np$, it follows (again, using Lemma 4.2.1) that $|E_D(X, Y)| \neq 0$ for every two disjoint subsets of vertices X and Y of sizes $|X| = \frac{\alpha n}{3\sqrt{\log n}}$ and $|Y| = \alpha n/10$ (for example by showing that $|E_G(X, Y)| \geq \alpha^2 n^2 p/31\sqrt{\log n}$ for each such X, Y and therefore the average degree of the vertices of X into Y is at least $30/31 \cdot \alpha np$ and it cannot be that in D all of them are gone). Therefore, we conclude that $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every $|X| \leq |V(D)|/5$ as well (otherwise, there exist a subset $X' \subseteq X$ of size exactly $\frac{n}{3}\sqrt{\log n}$ and a subset $Y \subseteq V(D) \setminus (X \cup N_D(X))$ of size exactly $\alpha n/10$ with $E_D(X, Y) = \emptyset$).

□

A routine way to turn a non-Hamiltonian graph D that satisfies some expansion properties (as in Claim 4.2.8) into a Hamiltonian graph is by using *boosters*. Roughly speaking, a booster is a non-edge e of D such that the addition of e to D creates a path which is longer than a longest path of D, or turns D into a Hamiltonian graph. In order to turn D into a Hamiltonian graph, we start by adding a booster e of D. If the new graph $D \cup \{e\}$ is not Hamiltonian then one can continue by adding a booster of the new graph. Note that after at most $|V(D)|$ successive steps the process must terminate and we end up with a Hamiltonian graph. The main point using this method is that it is well-known (for example, see [Bol98]) that a non-Hamiltonian graph D with "good" expansion properties has many boosters. However, our goal is a bit different. We wish to turn D into a graph that contains a Hamilton path with x and y as its endpoints. In order to do so, we add one (possibly) fake edge xy to D and try to find a Hamilton cycle that contains the edge xy. Then, the path obtained by deleting this edge from the Hamilton cycle will be the desired path. For that we need to define the notion of *e-boosters*.

Given a graph D and any edge $e \in \binom{V(D)}{2}$ (e might be a non-edge of D), consider a path P of $D \cup \{e\}$ of maximal length which contains the edge e. A non-edge e' of D is called an e-booster if $D \cup \{e, e'\}$ contains a path P' which passes through e and which is longer than P, or that $D \cup \{e, e'\}$ contains a Hamilton cycle that uses e. The following lemma shows that every connected and non-Hamiltonian graph D with "good" expansion properties has many e-boosters for every possible e.

Lemma 4.2.9. *Let D be a connected graph for which $|N_D(X) \setminus X| \geq 2|X| + 2$ holds for every subset $X \subseteq V(D)$ of size $|X| \leq k$. Then, for every pair $e \in \binom{V(D)}{2}$ such that $D \cup \{e\}$ does not contains a Hamilton cycle which uses the edge e, the number of e-boosters for D is at least $(k+1)^2/2$.*

The proof of the previous lemma is very similar to the proof of the well-known Pósa's lemma using the ordinary boosters and hence we omit it (it can be found for example in [FK08], Lemma 4). The only difference is that in the proof of Lemma 4.2.9 we forbid rotations that destroy the edge e; and so the number of possible rotations with a given fixed endpoint drops by at most two.

Lastly, we complete the proof of Theorem 4.2.6 by showing that $G_2 \sim \mathcal{G}(n, p/2)$ is a.a.s such that for every subgraph $H \subset G$ with $\Delta(H) \leq \varepsilon np$, for every subset $V' \subseteq V(G)$ of size $|V'| = \alpha n+1$ for which $\delta((G_1 \setminus H)[V']) \geq (\alpha/4-2\varepsilon)np$, for every pair $e = xy \in \binom{V'}{2}$ and for every subset E of at most αn pairs of $V(D)$, G_2 contains at least $\alpha^2 n^2/100$ e-boosters for $(G_1 \setminus H)[V'] \cup E \cup \{e\}$.

Lemma 4.2.10. *$G_2 \sim \mathcal{G}(n, p/2)$ is a.a.s such that for every subgraph $H \subset G$ with $\Delta(H) \leq \varepsilon np$, for every subset $V' \subseteq V(G)$ of size $|V'| = \alpha n+1$ for which $\delta((G_1 \backslash H)[V']) \geq (\alpha/4 - 2\varepsilon)np$, for every pair $e = xy \in \binom{V'}{2}$ and for every subset E_0 of at most αn pairs of V', G_2 contains at least $\alpha^2 n^2/100$ e-boosters for $(G_1 \setminus H)[V'] \cup E_0 \cup \{e\}$.*

Proof. Combining Claim 4.2.8 with Lemma 4.2.9 we conclude that for every such V', $e \in \binom{V'}{2}$, and for every subset E_0 of at most αn pairs of V', $G_{E_0,e,V'} = (G_1 \setminus H)[V'] \cup E_0 \cup \{e\}$ has at least $\alpha^2 n^2/3$ e-boosters. For a fixed choice of such V', e and E_0, using Lemma 4.2.1 it follows that the probability that G_2 will have at most $\alpha^2 n^2 p/100$ e-boosters for $G_{E_0,e,V'}$ is at most $\exp(-Cn^2 p)$, where C is a constant which depends only in α. Applying the union bound, running over all the options for choosing H, e and E_0 we obtain that the probability that there exist such V', e, and E_0 such that G_2 contains at most $\alpha^2 n^2 p/100$ e-boosters is at most

$$\sum_{t=1}^{\varepsilon n^2 p} \binom{e(G_1)}{t} n^2 \binom{\alpha^2 n^2}{\alpha n} \exp(-Cn^2 p) \leq$$

$$\varepsilon n^2 p \binom{e(G_1)}{\varepsilon n^2 p} n^2 \binom{\alpha^2 n^2}{\alpha n} \exp(-Cn^2 p) \leq$$

$$\varepsilon n^4 p \left(\frac{e\alpha^2 n^2 p}{\varepsilon n^2 p}\right)^{\varepsilon n^2 p} (e\alpha n)^{\alpha n} \exp(-Cn^2 p) \leq$$

$$\varepsilon n^4 p \left(\frac{e\alpha^2}{\varepsilon}\right)^{\varepsilon n^2 p} (e\alpha n)^{\alpha n} \exp(-C \cdot C_0 n \log n) = o(1),$$

where the last inequality holds since $\varepsilon \ll \alpha$ and since C_0 is large enough. $\square$

This completes the proof of Theorem 4.2.6. $\square$

4.3 Proof of the main result

Proof of Theorem 4.1.4. We first present a strategy for Maker, and then prove that by playing according to this strategy, Maker a.a.s achieves his goal. In this strategy, Maker will gradually generate a random graph $G' \sim \mathcal{G}(G, p)$, by tossing a biased coin on each edge of G, and declaring that it belongs to G' independently with probability p. Each edge which Maker has tossed a coin for is called *exposed*, and we say that Maker is *exposing* an edge $e \in E(G)$ whenever he tosses a coin to decide about the appearance of e in G'. To keep track of the unexposed edges, Maker maintains a set $U_v \subseteq N_G(v)$ of the *unexposed neighbors* of v, for each vertex v in G; i.e. $u \in U_v$ if and only if the edge vu remains to be exposed. Initially, $U_v = N_G(v)$ for all $v \in V(G)$. We remark that Maker will expose all edges of G, even those that belong to Breaker.

In every turn, Maker chooses an *exposure vertex* v (we will later discuss the choice of the exposure vertex) and starts to expose edges connecting v to vertices in U_v, one by one in an arbitrary order, until one edge in G' is found (that is, until he has a first success). If this exposure happens to reveal an edge $vu \in E(G')$ not yet claimed by Breaker, Maker claims it and completes his move. Otherwise, either the exposure failed to reveal a new edge in G' (failure of *type I*), or the newly found edge already belongs to Breaker (failure of *type II*). In either case, Maker skips his move. Let $f_I(v)$ and $f_{II}(v)$ denote the number of failures of type I and II, respectively, for the exposure vertex v. To complete the proof, it suffices to show that a.a.s Maker can ensure that $f_{II}(v) \leq \varepsilon d_G(v)p$ for all vertices $v \in V(G)$ at the end of the game (note that if a failure of type I occurs, it does harm Maker in claiming edges of the generated random graph G').

To keep the failures of type II under control, concurrently to the game played on G, we simulate a game $MinBox(n, 4\delta(G), p/2, 2b)$. In this simulated game, there is one box F_v for each $v \in V(G)$ which helps us to keep track on the exposure of edges touching v. Initially, we set the sizes of the boxes as $|F_v| = 4d_G(v)$. Now, we describe Maker's strategy.

Maker's strategy S_M: Maker's strategy is divided into the following two stages.

Stage 1: Before Maker's move, he updates the status of the simulated game by pretending that Breaker claimed one free element from both F_v and F_u, for each edge vu occupied in Breaker's last move. Maker then identifies a free active box F_v having highest danger value in the simulated game (breaking ties arbitrarily). If there is no such box, Maker proceeds to the second stage of the strategy. Otherwise, let F_v be such a box. Maker claims one free element from F_v, and selects v as the exposure vertex. Let $\sigma : [m] \to U_v$ be an arbitrary permutation on U_v, where $m := |U_v|$. Maker starts tossing a biased coin for vertices in U_v, independently at random, according to the ordering of σ.

(a) If there were no successes, then Maker declares this turn as a *failure of type I*, thereby incrementing $f_I(v)$, and skips his move in the original game. Maker then claims $\frac{p}{2}\cdot|F_v|-1$ additional free elements from F_v in the simulated game, and updates $U_v := \emptyset$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq m$.

(b) Assume that Maker's first success has happened at the kth coin tossing. If the edge $v\sigma(k)$ is not free, then Maker declares $v\sigma(k)$ as a *failure of type II*, increments $f_{II}(v)$ by one, and skips his move in the original game. Maker then updates $U_v := U_v \setminus \{\sigma(i) : i \leq k\}$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq k$.

(c) Otherwise, Maker claims the edge $v\sigma(k)$. In this case Maker also claims a free element from box $F_{\sigma(k)}$ and then updates $U_v := U_v \setminus \{\sigma(i) : i \leq k\}$, and $U_{\sigma(i)} := U_{\sigma(i)} \setminus \{v\}$ for each $i \leq k$.

Stage 2: In this stage, there are no free active boxes. Let $U := \{vu : v \in V(G), u \in U_v\}$. For each $e = vu \in U$, Maker declares a failure of type II on both u and v (i.e., increments both $f_{II}(u)$ and $f_{II}(v)$ by one) with probability p, independently at random. After the end of this stage, Maker stops playing the game altogether, and skips all his subsequent moves.

We now prove that by following S_M, Maker typically achieves his goal. For the sake of notation, at any point during the game, we denote by $d_M(v)$ and $d_B(v)$ the degrees of v in the subgraphs currently occupied by Maker and Breaker, respectively. The proof will follow from the next four claims.

Claim 4.3.1. *At any point during the first stage, we have $w_M(F_v) \leq (1 + 2p)d_G(v)$ and*

$w_B(F_v) \leq d_G(v)$ for every box F_v in the simulated game. In particular, no box is ever exhausted of free elements.

Proof. Clearly $w_B(F_v) = d_B(v) \leq d_G(v)$. Moreover, $w_M(F_v) = d_M(v) + \frac{p}{2}|F_v|f_I(v) + f_{II}(v)$. Since $d_M(v) + f_{II}(v) \leq d_G(v)$ and $f_I(v) \leq 1$, as F_v becomes inactive after a failure of type I on v, we have $w_M(F_v) \leq d_G(v) + \frac{p}{2} \cdot |F_v| \leq (1+2p)d_G(v)$, as required. □

Claim 4.3.2. *For every $v \in V(G)$, F_v becomes inactive before $d_B(v) \geq \varepsilon d_G(v)/5$.*

Proof. Let $v \in V(G)$ be any vertex of $V(G)$. Since in the simulated game Maker follows the strategy described in Theorem 4.2.3, we must have

$$\mathsf{dang}(F_v) = w_B(F_v) - 2b \cdot w_M(F_v) \leq 2b(\log n + 1) \tag{4.3.1}$$

for every active box F_v. Assume that there exists a vertex $v \in V(G)$ for which F_v is still active and $w_B(F_v) = d_B(v) \geq \varepsilon d_G(v)/5$. Recall that $b = \frac{\varepsilon}{40p}$, and by (4.3.1) it follows that

$$\begin{aligned} 2\frac{\varepsilon}{40p}(\log n + 1) &\geq w_B(F_v) - 2\frac{\varepsilon}{40p} \cdot w_M(F_v) \\ &\geq \varepsilon d_G(v)/5 - \frac{\varepsilon}{20p} \cdot w_M(F_v). \end{aligned}$$

Therefore, we obtain that $w_M(F_v) \geq 4d_G(v)p - (\log n + 1)$, and by the assumption that $\delta(G) \geq \frac{11 \log n}{\varepsilon p}$, we conclude that $w_M(F_v) > 3.8d_G(v)p \geq \frac{p}{2}|F_v| = 2d_G(v)p$, a contradiction. □

Claim 4.3.3. *All edges of G' are a.a.s exposed before the beginning of Stage 2.*

Proof. Suppose there exists a vertex v at the beginning of the second stage, such that $U_v \neq \emptyset$. Since $U_v \neq \emptyset$, we must have $f_I(v) = 0$. Moreover, because F_v is not active, we must also have $w_M(F_v) = d_M(v) + f_{II}(v) \geq \frac{p}{2}|F_v| = 2d_G(v)p$. This implies that $d_{G'}(v) \geq d_M(v) + f_{II}(v) \geq 2d_G(v)p$. Now, since $d_{G'}(v) \sim \text{Bin}(d_G(v), p)$, using Lemma 4.2.1, it follows that

$$\mathbb{P}\left[\text{Bin}(d_G(v), p) \geq 2d_G(v)p\right] < e^{-d_G(v)p/3} = o\left(\frac{1}{n}\right).$$

Applying the union bound, it thus follows that with probability $1 - o(1)$, there exists no such vertex, proving the claim. □

Claim 4.3.4. *For every $v \in V(G)$ a.a.s $f_{II}(v) \leq \varepsilon d_G(v)p$.*

Proof. Let $v \in V(G)$ be any vertex. By Claim 4.3.2, during Stage 1 Breaker can touch v at most $\varepsilon d_G(v)/5$ times before F_v becomes inactive. Since a failure of type II occurs if Maker has a success on one of Breaker's edges, it follows that $f_{II}(v) \sim \mathrm{Bin}(m, p)$, where $m \leq \varepsilon d_G(v)/5$. Applying Lemma 4.2.2 to $f_{II}(v)$ we conclude that the probability for having more than $\varepsilon d_G(v)p$ edges vu which are failures of type II is at most

$$\mathbb{P}\left[\mathrm{Bin}(\varepsilon d_G(v)/5, p) \geq \varepsilon d_G(v)p\right] \leq \left(\frac{e\varepsilon d_G(v)p/5}{\varepsilon d_G(v)p}\right)^{\varepsilon d_G(v)p} = o\left(\frac{1}{n}\right).$$

Applying the union bound we obtain that the probability that there is such a vertex is $o(1)$. Moreover, by Claim 4.3.3, with probability $1 - o(1)$ all the edges of G' were exposed before the beginning of Stage 2. Therefore, a.a.s. $f_{II}(v) \leq \varepsilon d_G(v)p$ for all $v \in V(G)$. $\square$

This completes the proof of Theorem 4.1.4. $\square$

4.4 Applications

In this section we show how to apply Theorems 4.1.4 and 4.1.5 in order to prove Theorems 4.1.6, 4.1.7 and 4.1.8. We start with proving Theorem 4.1.6, which states that Maker can win the Hamiltonicity game played on $E(K_n)$ against an asymptotically optimal (up to a constant factor) bias of Breaker.

Proof of Theorem 4.1.6. Fix $\varepsilon = \frac{1}{6}$. Let $C_1 = C(\frac{1}{6})$ be as in Theorem 4.2.4, and let C_2 be a constant for which $G \sim \mathbb{G}(n, p)$ is a.a.s. such $\delta(G) \geq \frac{5}{6}np$ for every $p \geq \frac{C \log n}{n}$. Denote by $C := \max\{C_1, C_2\}$. Now, let $\delta = \frac{\varepsilon}{40C}$ and note that $\frac{\delta n}{\log n} = \frac{\varepsilon}{40p}$, where $p = \frac{C \log n}{n}$. Therefore, applying Theorem 4.1.4 we obtain that for sufficiently large integer n, in the $(1 : \frac{\delta n}{\log n})$ game played on $E(K_n)$ Maker can build a subgraph $G' = G \setminus H$ where $G \sim \mathbb{G}(n, p)$ and $\Delta(H) \leq \varepsilon np$. Now, since a.a.s. we have that $\delta(G) \geq \frac{5}{6}np$, it follows that $\delta(G') \geq (\frac{1}{2} + \frac{1}{6})np$. Applying Theorem 4.2.4 we conclude that G' is a.a.s. Hamiltonian. This completes the proof. $\square$

Next, we prove Theorem 4.1.7

Proof of Theorem 4.1.7. Let $b = o(\sqrt{n})$ and $0 < \varepsilon < 1/20$. Let $0 < p < 1$ be such that the following two properties hold: $b \leq \frac{\varepsilon}{40p}$ and $p = \omega\left(1/\sqrt{n}\right)$. Applying Theorem 4.1.5 we conclude that for sufficiently large integer n, in the $(1 : b)$ game played on the complete directed graph on n vertices, Maker can build a subgraph $D' = D \setminus H$ for which $D \sim \mathcal{D}(n, p)$ and $\Delta^{\pm}(H) \leq \varepsilon np$, where $\Delta^{\pm}(H) = \max\{d_H^+(v), d_H^-(v)\}$ and the maximum runs over all $v \in V(H)$. Since $p \gg 1/\sqrt{n}$, an easy application of the Chernoff bounds (Lemma 4.2.1) shows that $d_D^+(v), d_D^-(v) = (1 + o(1))np$ for each $v \in V(D)$. Combining it with the fact that $\varepsilon < 1/20$, we obtain that $\Delta^{\pm}(H) \leq np/16$. Therefore, applying Theorem 4.2.5 to D we conclude that D' contains an oriented Hamilton cycle. This completes the proof. $\square$

Finally, we prove Theorem 4.1.8.

Proof of Theorem 4.1.8. Let $\alpha > 0$ and $D > 0$ be two positive constants. Let $\varepsilon > 0$ and C be as in Theorem 4.2.6 (applied to α and D). Now, let $\delta = \frac{\varepsilon}{40C}$ and note that $\frac{\delta n}{\log n} = \frac{\varepsilon}{40p}$, where $p = \frac{C \log n}{n}$. Therefore, by applying Theorem 4.1.4 we obtain that for sufficiently large integer n, in the $(1 : \frac{\delta n}{\log n})$ game played on $E(K_n)$ Maker can build a subgraph $G' = G \setminus H$ where $G \sim \mathbb{G}(n, p)$ and $\Delta(H) \leq \varepsilon np$. Now, using Theorem 4.2.4 we conclude that G' a.a.s. contains a copy of every tree T on n vertices with $\Delta(T) \leq D$ and with a bare path of length at least αn. This completes the proof. $\square$

APPENDIX A

Detailed proofs

A.1 Second proof in details

We turn to desribe in details the second proof of Theorem 2.1.1 for k-cliques and k-anticliques, which was sketched in Section 2.5. Some of the purely technical parts of this proof are collected together in Lemma A.1.2. First, we derive the proof of the theorem from Lemma 2.5.1.

Proof. Let $p \in [0,1]$, and q be the unique root of $q^k + kq^{k-1}(1-q) = p$ in $[0,1]$. We need to show that every $x \in [0,1]$ and non-increasing $f : [0,1] \to [0,1]$ with $d(\overline{K}_k; G_{x,f}) \geq p$ satisfy that $d(K_k; G_{x,f}) \leq \Phi_k(p)$, where

$$\Phi_k(p) := M_{k,k,p} = \max\{(1-p^{1/k})^k + kp^{1/k}(1-p^{1/k})^{k-1}, (1-q)^k\}.$$

Namely, that $d(K_k; G_{x,f})$ is maximized when either $f = 0$ or $f = 1$.

By Lemma 2.5.1, we can assume that f is either constant or that it only takes the values 1 and 0. Consider the latter, and for some $y \in [0,1]$, let

$$f(t) = \begin{cases} 1 & \text{if} \quad t \leq 1-y \\ 0 & \text{if} \quad t > 1-y \end{cases}$$

For convenience, we prove the statement for $G_{1-x,f}$, thus we need to prove that

$$\max\ (1-x)^k + kx(1-x)^{k-1}(1-y)^{k-1} \text{ s.t.}$$

$$x^k + kx^{k-1}(1-x)y \geq p$$

is attained when either $x = q$, $y = 1$ or $x = p^{1/k}$, $y = 0$. By monotonicity of $d(K_k; G_{x,f})$ and $d(\overline{K}_k; G_{x,f})$, we assume that $x^k + kx^{k-1}(1-x)y = p$, hence $x \in [q, p^{1/k}]$ and

$$y = \frac{p - x^k}{kx^{k-1}(1-x)}.$$

We rewrite the objective function as

$$K(x) = (1-x)^k + kx(1-x)^{k-1}\left(1 - \frac{p - x^k}{kx^{k-1}(1-x)}\right)^{k-1}.$$

This part of the proof is completed in Lemma A.1.2 below, where it is shown (among other things) that the maximum of this function in the interval $[q, p^{1/k}]$ occurs at an endpoint.

We can now deal with the case of a constant f. First, note that $\Phi_k(\Phi_k(p)) = p$ for every $p \in [0,1]$. Indeed, the curves $(p, (1-p^{1/k})^k + kp^{1/k}(1-p^{1/k})^{k-1})$ and $(p, (1-q)^k)$ are reflections of each other with respect to the line $y = x$. Therefore Φ_k is symmertic with respect to this line. Let $f = y$ be a constant function, and suppose that $d(K_k; G_{x,f}) = x^k + kx^{k-1}(1-x)y > \Phi_k(p)$. Then, as we have just shown,

$$d(\overline{K}_k; G_{x,f}) = (1-x)^k + kx(1-x)^{k-1}(1-y)^{k-1} < \Phi_k(\Phi_k(p)) = p$$

and the proof is completed. □

We now turn to deduce Lemma 2.5.1 from Lemma 2.5.2

Proof of Lemma 2.5.1. Let $f : [0,1] \to [0,1]$ be non-increasing and let $h(t) = 1 - f(t)$. Then $g(t) := h(t)/\|h\|_{k-1}$ is l_{k-1}-normalized and non-decreasing. We apply Lemma 2.5.2 with $B = 1/\|h\|_{k-1}$ to conclude that

$$\frac{\langle (k-1)t^{k-2}, h\rangle}{\|h\|_{k-1}} \geq \min\left\{\frac{1}{\|h\|_{k-1}}\left(1 - (1 - \|h\|_{k-1}^{k-1})^{k-1}\right), 1\right\}.$$

which we rewrite as

$$\langle (k-1)t^{k-2}, h\rangle \geq \min\left\{1 - (1 - \|h\|_{k-1}^{k-1})^{k-1}, \|h\|_{k-1}\right\}.$$

In the first case,

$$\left(1 - \|h\|_{k-1}^{k-1}\right)^{k-1} \geq 1 - \langle (k-1)t^{k-2}, h\rangle$$

Since $h = 1 - f$ this becomes

$$\left(1 - \int_0^1 (1 - f(t))^{k-1} dt\right)^{k-1} \geq 1 - \int_0^1 (k-1)t^{k-2}(1 - f(t))dt = \int_0^1 (k-1)t^{k-2} f(t) dt$$

which implies,

$$\int_0^1 (1-f(t))^{k-1}dt \le 1 - \left(\int_0^1 (k-1)t^{k-2}f(t)dt\right)^{\frac{1}{k-1}}.$$

Otherwise,

$$\langle (k-1)t^{k-2}, h\rangle^{k-1} \ge \|h\|_{k-1}^{k-1},$$

which implies,

$$\left(1 - \int_0^1 (k-1)t^{k-2}f(t)dt\right)^{k-1} \ge \int_0^1 (1-f(t))^{k-1}dt.$$

□

It only remains to prove Lemma 2.5.2. By a standard density argument it suffices to prove it for step functions, which we do in the following claim by induction on the number of steps.

Claim A.1.1. *Let $g : [0,1] \to [0,B]$ be an non-decreasing step function with $n \ge 2$ steps. Namely, there is a partition $X = (x_0 = 0 < x_1 < \ldots < x_{n-1} < x_n = 1)$ and real numbers $0 = T_0 \le T_1 \le \ldots \le T_n \le T_{n+1} = B$ such that*

$$g\mid_{[x_{i-1},x_i]} = T_i \quad \textit{for all} \quad 1 \le i \le n.$$

Suppose further that $\|g\|_{k-1} = 1$. Let $1 \le i \le n-1$. Fix the partition X and all the T_j, except possibly T_i, T_{i+1} subject to the condition that the modified function is non-decreasing and has l_{k-1} norm 1. Then $\langle g, (k-1)t^{k-2}\rangle$ is minimized when either $T_{i-1} = T_i$, $T_i = T_{i+1}$ or $T_{i+1} = T_{i+2}$.

Proof. We need to solve the following optimization problem.

$$\text{Minimize} \quad (x_i^{k-1} - x_{i-1}^{k-1})t_i + (x_{i+1}^{k-1} - x_i^{k-1})t_{i+1}, \quad \text{subject to} \tag{A.1.1}$$

$$T_{i-1} \le t_i \le t_{i+1} \le T_{i+2} \tag{A.1.2}$$

$$\mu_i t_i^{k-1} + \mu_{i+1} t_{i+1}^{k-1} = \mu_i T_i^{k-1} + \mu_{i+1} T_{i+1}^{k-1}, \tag{A.1.3}$$

where $\mu_i = x_i - x_{i-1}$. By Equation (A.1.3)

$$t_{i+1} = \left(T_{i+1}^{k-1} + \frac{\mu_i}{\mu_{i+1}}(T_i^{k-1} - t_i^{k-1})\right)^{\frac{1}{k-1}}. \tag{A.1.4}$$

The inequalities $t_i \leq t_{i+1} \leq T_{i+2}$ in (A.1.2) yield

$$t_i \leq \left(\frac{\mu_i T_i^{k-1} + \mu_{i+1} T_{i+1}^{k-1}}{\mu_i + \mu_{i+1}} \right)^{\frac{1}{k-1}} . \tag{A.1.5}$$

and

$$t_i \geq \left(\frac{\max\{0, \mu_i T_i^{k-1} + \mu_{i+1} T_{i+1}^{k-1} - \mu_{i+1} T_{i+2}^{k-1}\}}{\mu_i} \right)^{\frac{1}{k-1}} . \tag{A.1.6}$$

Therefore, we can restate our problem as the following optimization problem

$$\text{Minimize} \quad (x_i^{k-1} - x_{i-1}^{k-1}) t_i + (x_{i+1}^{k-1} - x_i^{k-1}) \left(T_{i+1}^{k-1} + \frac{\mu_i}{\mu_{i+1}} (T_i^{k-1} - t_i^{k-1}) \right)^{\frac{1}{k-1}}$$

$$\text{subject to (A.1.5), (A.1.6) and } t_i \geq T_{i-1}$$

This problem is feasible since $t_i = T_i$ is a valid solution, and therefore the domain is an interval. The objective function is well defined in this segment due to (A.1.5) and is concave. This is because it is a positive linear combination of a linear function and a concave function of the form $(a - bx^n)^{\frac{1}{n}}$. Therefore, it is minimized at an endpoint of the interval, which corresponds precisely to $t_i = T_{i-1}$, $t_i = t_{i+1}$ or $t_{i+1} = T_{i+2}$. $\square$

We now turn to the proof of Lemma 2.5.2.

Proof of Lemma 2.5.2. Let $g : [0, 1] \to [0, B]$, with l_{k-1}-norm 1, be a non-decreasing n-step function.

Case 1. If $n = 1$ then g is constant, therefore $g = 1$ and $\langle g, (k-1)t^{k-2} \rangle = 1$.

Case 2. If $n = 2$, we use Claim A.1.1 for $i = 1$ to conclude that $T_1 = 0$ or $T_2 = B$.

Case 2.1. If $T_1 = 0$, then

$$T_2^{k-1} \cdot (1 - x_1) = 1 \implies x_1 = 1 - \frac{1}{T_2^{k-1}},$$

and therefore

$$\begin{aligned}\langle g, (k-1)t^{k-2} \rangle &= T_2 \left(1 - \left(1 - \frac{1}{T_2^{k-1}} \right)^{k-1} \right) \\ &\geq \min \left\{ B \left(1 - \left(1 - \frac{1}{B^{k-1}} \right)^{k-1} \right), 1 \right\}.\end{aligned}$$

The last concavity inequality is shown in Lemma A.1.2.

Case 2.2. If $T_2 = B$, then

$$x_1 T_1^{k-1} + (1-x_1)B^{k-1} = 1 \implies T_1 = \left(\frac{B^{k-1}x_1 - B^{k-1} + 1}{x_1}\right)^{\frac{1}{k-1}}.$$

hence $1 - \frac{1}{B^{k-1}} \leq x_1 \leq 1$, and

$$\begin{aligned}\langle g, (k-1)t^{k-2}\rangle &= x_1^{k-1}T_1 + (1 - x_1^{k-1})B \\ &= x_1^{k-1-\frac{1}{k-1}}\left(B^{k-1}x_1 - B^{k-1} + 1\right)^{\frac{1}{k-1}} - Bx_1^{k-1} + B.\end{aligned}$$

By Lemma A.1.2, this is a concave function of x_1, and is therefore minimized either for $x_1 = 1$, where g is a 1-step function, or for $x_1 = 1 - \frac{1}{B^{k-1}}$, where $T_1 = 0$ which returns us to the Case 2.1.

Case 3. If $n = 3$, we use Claim A.1.1 for $i = 1, 2$, and obtain a 3-step function with $T_1 = 0$ and $T_3 = B$. Therefore,

$$(x_2 - x_1)T_1^{k-1} + (1 - x_2)B^{k-1} = 1 \implies T_2 = \left(\frac{B^{k-1}x_2 - B^{k-1} + 1}{x_2 - x_1}\right)^{\frac{1}{k-1}}$$

Since $0 \leq T_2 \leq B$ we conclude that $0 \leq x_1 \leq 1 - \frac{1}{B^{k-1}}$ and $1 - \frac{1}{B^{k-1}} \leq x_2 \leq 1$. Furthermore,

$$\begin{aligned}\langle g, (k-1)t^{k-2}\rangle &= (x_2^{k-1} - x_1^{k-1})T_2 + (1 - x_2^{k-1})B \\ &= \frac{x_2^{k-1} - x_1^{k-1}}{(x_2 - x_1)^{\frac{1}{k-1}}}\left(B^{k-1}x_2 - B^{k-1} + 1\right)^{\frac{1}{k-1}} + B - Bx_2^{k-1}\end{aligned}$$

By Lemma A.1.2, for fixed $x_2 \geq 1 - \frac{1}{B^{k-1}}$, this is minimized when either $x_1 = 0$, which is precisely Case (2.2), or when $x_1 = 1 - \frac{1}{B^{k-1}}$, implying that $T_2 = T_3 = B$, which brings us back to Case 2.

Case 4. If $n \geq 4$, we apply Claim A.1.1 for $i = 2$ and reduce the number of steps without increasing $\langle g, (k-1)t^{k-2}\rangle$. This completes the proof by induction. □

We finally provide a proof for the following technical lemma.

Lemma A.1.2. *The following statements hold.*

1. *Let $\alpha \in [0,1]$ and $c_k(\alpha)$ the root of $x^k + kx^{k-1}(1-x) = \alpha$ in $[0,1]$. The function*

$$K(x) = (1-x)^k + k(1-x)^{k-1}x\left(1 - \frac{\alpha^k - x^k}{k(1-x)x^{k-1}}\right)^{k-1}$$

has no local maximum in $(c_k(\alpha), \alpha)$.

2. Let $f(x) = x(1-(1-\frac{1}{x^r})^r)$*,* $r \geq 2$ *an integer,* $B \geq 1$ *and* $x \in [1, B]$*, then*

$$f(x) \geq \min\{f(1), f(B)\}.$$

3. The function

$$g(x) = x^{r-\frac{1}{r}}\left(B^r x - B^r + 1\right)^{\frac{1}{r}} - Bx^r + B$$

is concave in $(1 - \frac{1}{B^r}, 1)$*, where* $r \geq 2$ *an integer.*

4. The function $h(x) = \frac{a^r - x^r}{(a-x)^{\frac{1}{r}}}$ *has no local minimum in* $(0, a)$*. Hence, for fixed* $x_2 \geq 1 - \frac{1}{B^{k-1}}$

$$\frac{x_2^{k-1} - x_1^{k-1}}{(x_2 - x_1)^{\frac{1}{k-1}}}\left(B^{k-1}x_2 - B^{k-1} + 1\right)^{\frac{1}{k-1}} + B - Bx_2^{k-1}$$

is minimized at an endpoint of the interval $0 \leq x_1 \leq 1 - \frac{1}{B^{k-1}}$*.*

Proof. 1. We define

$$Z = Z(x) = 1 - \frac{\alpha^k - x^k}{kx^{k-1}(1-x)} = \frac{x^k + kx^{k-1}(1-x) - \alpha^k}{kx^{k-1}(1-x)}.$$

Note that Z varies between 0 and 1 as x ranges over the interval $[c_k(\alpha), \alpha]$ and in particular $Z(c_k(\alpha)) = 0$ and $Z(\alpha) = 1$. Also,

$$\log Z = \log(x^k + kx^{k-1}(1-x) - \alpha^k) - \log k - (k-1)\log x - \log(1-x).$$

Therefore,

$$\frac{Z'}{Z} = \frac{k(k-1)x^{k-2}(1-x)}{x^k + kx^{k-1}(1-x) - \alpha^k} - \frac{k-1}{x} + \frac{1}{1-x} = \frac{k-1}{xZ} - \frac{k-1}{x} + \frac{1}{1-x}$$

and

$$Z' = \frac{k-1}{x}(1-Z) + \frac{Z}{1-x} = \frac{(kx-k+1)Z + (k-1)(1-x)}{x(1-x)}.$$

We need to show that $K(x) = (1-x)^k + kx(1-x)^{k-1}Z^{k-1}$ has no local maximum in the interval $[c(\alpha), \alpha]$.

$$K'(x) = -k(1-x)^{k-1} + k(1-x)^{k-1}Z^{k-1} - k(k-1)x(1-x)^{k-2}Z^{k-1} +$$

$$k(k-1)x(1-x)^{k-1}Z^{k-2}Z' =$$

$$-k(1-x)^{k-2}\left(1-x+(kx-1)Z^{k-1}-(k-1)Z^{k-2}((kx-k+1)Z+(k-1)(1-x))\right)=$$

$$-k(1-x)^{k-1}\left(k(k-2)Z^{k-1}-(k-1)^2Z^{k-2}+1\right).$$

Suppose $x_0 \in (c(\alpha),\alpha)$ is a critical point, we claim that $Z(x_0) < 1-\frac{1}{k}$. Indeed, $Z(x_0)$ is a root of the polynomial $q(z) = k(k-2)z^{k-1} - (k-1)^2z^{k-2} + 1$. Since

$$q'(z) = (k-2)(k-1)z^{k-3}(kz-k+1),$$

q is increasing in $[1-\frac{1}{k},1]$ and therefore for every $z \in [1-\frac{1}{k},1)$, $q(z) < q(1) = 0$. Hence q has no roots in this interval, which therefore does not contain $Z(x_0)$.

Now, we compute the second derivative of K_2 in x_0,

$$K_2''(x_0) = -k(1-x_0)^{k-1}\cdot(k-1)(k-2)Z^{k-3}(x_0)Z'(x_0)(kZ(x_0)-k+1) \geq 0$$

since $Z \geq 0$, $Z' \geq 0$, and $(kZ(x_0)-k+1) < 0$. This proves that x_0 is not a local maximum.

2. It is sufficient to see that $f_1(x) = f(\frac{1}{x})$ has no local minimum in $(0,1)$. Indeed,

$$f_1(x) = \frac{1-(1-x^r)^r}{x}.$$

Therefore,

$$f_1'(x) = \frac{r^2x^r(1-x^r)^{r-1}+(1-x^r)^r-1}{x^2}$$

Denote $y(x) = 1-x^r$ and $q(y) = r^2(1-y)y^{r-1}+y^r-1$. Note that

$$q'(y) = -r(r-1)y^{r-2}((1+r)y-r),$$

which implies that q is decreasing if $y \in [\frac{r}{r+1},1]$, therefore q has no roots in this interval as $q(y) > q(1) = 0$. Consequently, if $x_0 < 1$ is a critical point of f_1, then $y(x_0) < \frac{r}{r+1}$. The numerator of the second derivative of f_1 in x_0 is

$$q'(y(x_0))\cdot y'(x_0) = r^2(r-1)x_0^{r-1}y(x_0)^{r-2}((1+r)y-r) < 0,$$

and f_1 does not have a local minimum.

3. Consider the change of variables $y = B^r x - B^r + 1$ and denote $a = B^r - 1$. Then,

$$g(x) = \frac{(y+a)^{\frac{r^2-1}{r}} y^{\frac{1}{r}}}{B^{r^2-1}} - \frac{(y+a)^r}{B^{r^2-1}} + B, \quad y \in (0,1), \quad a \geq 0$$

We need to show that for every $a \geq 0$, the function

$$G(y) = (y+a)^{\frac{r^2-1}{r}} y^{\frac{1}{r}} - (y+a)^r$$

is concave in $(0,1)$. Indeed,

$$\begin{aligned} G''(y) = &\frac{(r^2-1)(r^2-r-1)}{r^2}(y+a)^{\frac{r^2-2r-1}{r}} y^{\frac{1}{r}} \\ &+ 2\frac{r^2-1}{r^2}(y+a)^{\frac{r^2-r-1}{r}} y^{\frac{1-r}{r}} + \frac{1-r}{r^2}(y+a)^{\frac{r^2-1}{r}} y^{\frac{1-2r}{r}} - r(r-1)(y+a)^{r-2}. \end{aligned}$$

We need to prove that $G'' \leq 0$. At $a = 0$,

$$G''(y) = \frac{y^{r-2}}{r^2}\left((r^2-1)(r^2-r-1) + 2(r^2-1) + 1 - r - r^3(r-1)\right) = 0.$$

If $a > 0$, let $c = \frac{y}{a} > 0$. $G''(y)$ can be written as

$$\frac{a^{r-2}}{r^2}(c+1)^{\frac{r^2-2r-1}{r}} c^{\frac{1-2r}{r}} (r-1)\left(r^3c^2 + 2rc - 1 - r^3(1+c)^{\frac{1}{r}} c^{\frac{2r-1}{r}}\right).$$

We need to prove that

$$r^3c^2 + 2rc - 1 \leq r^3c^2\left(1 + \frac{1}{c}\right)^{\frac{1}{r}}$$

We multiply by $\frac{1}{c^2}$ and let $z = \frac{1}{c}$ to rewrite this as

$$r^3 + 2rz - z^2 \leq r^3(1+z)^{\frac{1}{r}}$$

For $z = 0$ this holds as equality, so it is sufficient to prove the inequality for the derivatives,

$$2r - 2z \leq r^2(1+z)^{\frac{1-r}{r}}.$$

For $z \geq r$, this inequality holds as $2r - 2z \leq 0 \leq r^2(1+z)^{\frac{1-r}{r}}$. For $0 \leq z \leq r$ we can raise both sides to the r-th power,

$$2^r(r-z)^r(1+z)^{r-1} \leq r^{2r}$$

This holds for $z = 0$ and $z = r$, so we only need to check that it holds for critical points. If $z \in (0, r)$, then,

$$\frac{d\left((r-z)^r (1+z)^{r-1}\right)}{dz} = 0 \implies z = \frac{r^2 - 2r}{2r-1},$$

hence, for $0 \le z \le r$,

$$2^r (r-z)^r (1+z)^{r-1} \le 2^r \left(\frac{r^2+r}{2r-1}\right)^r \left(\frac{r^2-1}{2r-1}\right)^{r-1}.$$

Consequently, it is enough prove that

$$2^r \left(\frac{r^2+r}{2r-1}\right)^r \left(\frac{r^2-1}{2r-1}\right)^{r-1} \le r^{2r}$$

which we write as

$$1 \le \left(\frac{2r-1}{2r-2}\right)^{r-1} \left(\frac{2r^2-r}{(r+1)^2}\right)^{r-\frac{1}{2}} \left(\frac{\sqrt{2r^2-r}}{2}\right)$$

For $r \ge 4$, each of these three terms is greater than 1, and for $r = 2, 3$ it can be verified by assignment.

4. Let

$$h(x) = \frac{a^r - x^r}{(a-x)^{\frac{1}{r}}}.$$

Then,

$$h'(x) = \frac{(r^2-1)x^r - r^2 a x^{r-1} + a^r}{r(a-x)^{\frac{r+2}{r}}}.$$

If $0 \le x_0 < a$ is a critical point, then it is a root of the polynomial $q(x) = (r^2-1)x^r - r^2 a x^{r-1} + a^r$. Note that $q'(x) = r(r-1)x^{r-2}((r+1)x - ra)$, and therefore positive for $x \in [\frac{r}{r+1}a, a]$. Hence, for such x, $q(x) > q(a) = 0$, and therefore $x_0 < \frac{r}{r+1}a$. The numerator of the second derivative of h in x_0 equals to $q'(x_0)$ which is thereby negative. Hence, x_0 is not a local minimum. □

A.2 Hypergraph theorem in details

In this section we describe the detailed proof of Theorem 2.6.1 which was sketched in Section 2.6. Theorem 2.6.1 states that the number of labeled copies of a stable set system $\mathcal{H}$ in an arbitrary set system $\mathcal{F}$ does not decrease after the shifting, i.e.,

$$t(\mathcal{H}; S_{u\to v}(\mathcal{F})) \geq t(\mathcal{H}; \mathcal{F}).$$

Proof of Theorem 2.6.1. Let us recall the definition of the shifting operator $\tilde{S}_{u\to v}$ for sets of labeled copies. For $\mathcal{I}$ a set of labeled copies, we defined $\tilde{S}_{u\to v}(\mathcal{I}) = \{\tilde{S}_{u\to v}(I, \mathcal{I}) : I \in \mathcal{I}\}$, where

$$\tilde{S}_{u\to v}(I, \mathcal{I}) = \begin{cases} I_{u\leftrightarrow v} & \text{if } I_{u\leftrightarrow v} \notin \mathcal{I} \text{ and } \mathrm{Im}(I) \cap \{u, v\} = \{u\}, \\ I_{u\leftrightarrow v} & \text{if } I_{u\leftrightarrow v} \notin \mathcal{I}, \{u, v\} \subset \mathrm{Im}(I), \text{ and } I^{-1}(u) \text{ dominates } I^{-1}(v) \text{ in } \mathcal{H}, \\ I & \text{otherwise,} \end{cases}$$

and $I_{u\leftrightarrow v} : U \to V$ was defined by

$$I_{u\leftrightarrow v}(w) = \begin{cases} I(w) & \text{if } I(w) \neq u, v, \\ v & \text{if } I(w) = u, \\ u & \text{if } I(w) = v. \end{cases}$$

Henceforth, let $\mathcal{I} := \mathrm{Cop}(\mathcal{H}; \mathcal{F})$ be the family of all labeled copies of $\mathcal{H}$ in $\mathcal{F}$, let $\mathcal{F}' = S_{u\to v}(\mathcal{F})$ be the shifted set system and let $\mathcal{I}' = \tilde{S}_{u\to v}(\mathcal{I}) := \{\tilde{S}_{u\to v}(I, \mathcal{I}) : I \in \mathcal{I}\}$ be the set of all labeled copies after the shifting. Clearly $|\mathcal{I}'| = |\mathcal{I}|$, thus in order to show $t(\mathcal{H}; \mathcal{F}') \geq t(\mathcal{H}; \mathcal{F})$, it suffices to prove that

$$\tilde{S}_{u\to v}(\mathcal{I}) \subseteq \mathrm{Cop}(\mathcal{H}; \mathcal{F}').$$

Let $I \in \mathcal{I}$ be an arbitrary labeled copy of $\mathcal{H}$, and let $I' = \tilde{S}_{u\to v}(I, \mathcal{I})$. For any $H \in \mathcal{H}$, it is true that $I(H) \in \mathcal{F}$. Let us now show, by careful case analysis, that $I'(H) \in \mathcal{F}'$ for all $H \in \mathcal{H}$, thereby proving that $I' \in \mathrm{Cop}(\mathcal{H}; \mathcal{F}')$. One of the following is true

Case (i): $\mathrm{Im}(I') \cap \{u, v\} = \emptyset$. Clearly $I = I'$. On the other hand, since $I(H) \cap \{u, v\} = \emptyset$, it is true that $S_{u\to v}(I(H); \mathcal{F}) = I(H)$, hence $I(H) \in \mathcal{F}'$. Thus $I'(H) \in \mathcal{F}'$.

Case (ii): $\mathrm{Im}(I') \cap \{u, v\} = \{u\}$. In this case, the equality $I' = I$ holds. But $\tilde{S}_{u\to v}(I, \mathcal{I}) = I$ implies that $I_{u\leftrightarrow v} \in \mathcal{I}$, and thus it is true that $I_{u\leftrightarrow v}(H) \in \mathcal{F}$. It becomes clear that $S_{u\to v}(I(H); \mathcal{F}) = I(H)$ since $I_{u\leftrightarrow v}(H) \in \mathcal{F}$, hence $I(H) \in \mathcal{F}'$. Thus $I'(H) \in \mathcal{F}'$.

Case (iii) $\mathrm{Im}(I') \cap \{u, v\} = \{v\}$: Let us now analyze two subcases:

1. $I = I'$: This is the straightforward subcase. Since $u \notin I(H)$, it is true that $S_{u\to v}(I(H); \mathcal{F}) = I(H)$, hence $I(H) \in \mathcal{F}'$. We conclude that $I'(H) \in \mathcal{F}'$.

2. $I = I_{u\leftrightarrow v}$: Let $X = S_{u\to v}(I(H), \mathcal{F})$. If $X = I_{u\leftrightarrow v}(H)$ then clearly $I'(H) = I_{u\leftrightarrow v}(H) \in \mathcal{F}'$. If $X \neq I_{u\leftrightarrow v}(H)$, it must be true that $X = I(H)$ and $I(H) \cap \{u, v\} = \{u\}$. But $S_{u\to v}(I(H), \mathcal{F}) = I(H)$ only if $I_{u\leftrightarrow v}(H) \in \mathcal{F}$. Hence $I_{u\leftrightarrow v}(H) = S_{u\to v}(I_{u\leftrightarrow v}(H), \mathcal{F}) \in \mathcal{F}'$, because $v \in I_{u\leftrightarrow v}(H)$.

Case (iv): $\mathrm{Im}(I') \cap \{u, v\} = \{u, v\}$. Dividing into three more subcases:

1. $I_{u\leftrightarrow v} \in \mathcal{I}$: We have $I' = I$. Clearly $S_{u\to v}(I(H), \mathcal{F}) = I(H)$ because $I_{u\leftrightarrow v}(H) \in \mathcal{F}$, hence $I'(H) = I(H) \in \mathcal{F}'$.

2. $I_{u\leftrightarrow v} \notin \mathcal{I}$ but $I^{-1}(u)$ does not dominate $I^{-1}(v)$ in $\mathcal{H}$: Again, it is true that $I' = I$. Moreover $I^{-1}(v)$ must dominate $I^{-1}(u)$ in $\mathcal{H}$, because $\mathcal{H}$ is stable. We will now show that $S_{u\to v}(I(H), \mathcal{F}) = I(H)$. Suppose, towards contradiction, that $S_{u\to v}(I(H), \mathcal{F}) \neq I(H)$. This can only happen when $I(H) \cap \{u, v\} = \{u\}$ and $I_{u\leftrightarrow v}(H) \notin \mathcal{F}$. Let $H' = (H \cup \{I^{-1}(v)\}) \setminus \{I^{-1}(u)\}$. Because $I^{-1}(v)$ dominates $I^{-1}(u)$ in $\mathcal{H}$, we must have $S_{I^{-1}(u)\to I^{-1}(v)}(H, \mathcal{H}) = H$, hence $H' \in \mathcal{H}$. But this is a contradiction because $I_{u\leftrightarrow v}(H) = I(H')$ and $I(H') \in \mathcal{F}$ since $H' \in \mathcal{H}$. Therefore $I(H) = S_{u\to v}(I(H), \mathcal{F}) \in \mathcal{F}'$.

3. $I_{u\leftrightarrow v} \notin \mathcal{I}$ and $I^{-1}(u)$ dominates $I^{-1}(v)$ in $\mathcal{H}$: The identity $I' = I_{u\leftrightarrow v}$ holds. If $S_{u\to v}(I(H), \mathcal{F}) = I_{u\leftrightarrow v}(H)$, then clearly $I'(H) \in \mathcal{F}$. If $S_{u\to v}(I(H), \mathcal{F}) \neq I_{u\leftrightarrow v}(H)$, then it is either because

   (a) $I(H) \cap \{u, v\} = \{u\}$ and $I_{u\leftrightarrow v}(H) \in \mathcal{F}$: It is true that $I_{u\leftrightarrow v}(H) = S_{u\to v}(I_{u\leftrightarrow v}(H), \mathcal{F}) \in \mathcal{F}'$; or because

   (b) $I(H) \cap \{u, v\} = \{v\}$: Let $H' = (H \cup \{I^{-1}(u)\}) \setminus \{I^{-1}(v)\}$. Since $I^{-1}(u)$ dominates $I^{-1}(v)$ in $\mathcal{H}$, clearly $H' \in \mathcal{H}$. Thus $I(H') \in \mathcal{F}$, hence $I_{u\leftrightarrow v}(H) = I(H') \in \mathcal{F}$. Therefore $I_{u\leftrightarrow v}(H) = S_{u\to v}(I_{u\leftrightarrow v}(H), \mathcal{F}) \in \mathcal{F}'$;

In all cases we have $I'(H) \in \mathcal{F}'$, therefore $I' \in \mathrm{Cop}(\mathcal{H};\mathcal{F}')$, which implies $t(\mathcal{H};\mathcal{F}') \geq t(\mathcal{H};\mathcal{F})$. $\square$

A.3 Proof of the MinBox game

Proof of Theorem 4.2.3. The proof of this theorem is very similar to the proof of Theorem 1.2 in [GS09]. Since claiming an extra element is never a disadvantage for any of the players, we can assume that Breaker is the first player to move. For a subset $X \subseteq E(\mathcal{H})$, let $\overline{\mathsf{dang}}(X) = \frac{\sum_{F \in X} \mathsf{dang}(F)}{|X|}$ denote the average danger of the boxes in X. The game ends when there are no more free elements left.

Suppose, towards a contradiction, that there exists a strategy for Breaker that ensures the existence of an active box F satisfying $\mathsf{dang}(F) > b(\log n + 1)$ at some point during the game. Denote the first time when this happens by g. Let $I = \{F_1, \ldots, F_g\}$ be the set which defines Maker's game, i.e, in his i^{th} move, Maker plays at F_i for $1 \leq i \leq g-1$ and F_g is the first active box satisfying $\mathsf{dang}(F_g) > b(\log n + 1)$. For every $0 \leq i \leq g-1$, let $I_i = \{F_{g-i}, \ldots, F_g\}$. Following the notation of [GS09], let $\mathsf{dang}_{B_i}(F)$ and $\mathsf{dang}_{M_i}(F)$ denote the danger value of a box F, *directly before* Breaker's and Maker's i^{th} move, respectively. Notice that in his g^{th} move, Breaker increases the danger value of F_g to at least $b(\log n + 1)$. This is only possible if $\mathsf{dang}_{B_g}(F_g) > b(\log n + 1) - b = b \log n$.

Analogously to the proof of Theorem 1.2 in [GS09], we state the following lemmas which estimate the change of the average danger after a particular move (by either player). In the first lemma we estimate the changes after Maker's moves.

Lemma A.3.1. *Let i, $1 \leq i \leq g-1$,*

(i) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$.

(ii) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq \frac{b}{|I_i|}$.

In the second lemma we estimate the changes after Breaker's moves.

Lemma A.3.2. *Let i be an integer, $1 \leq i \leq g-1$. Then,*

$$\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \le \frac{b}{|I_i|}.$$

Combining Lemmas A.3.1 and A.3.2, we get the following corollary which estimates the change of the average danger after a full round.

Corollary A.3.3. *Let i be an integer, $1 \le i \le g-1$.*

(i) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge 0$.

(ii) if $I_i \ne I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge -\frac{b}{|I_i|}$

In order to complete the proof, we prove that before Breaker's first move, $\overline{\mathsf{dang}}_{B_1}(I_{g-1}) > 0$, thus obtaining a contradiction. To that end, let $|I_g| = r$ and let $i_1 < \ldots < i_{r-1}$ be those indices for which $I_{i_j} \ne I_{i_j-1}$. Note that $|I_{i_j}| = j+1$. Recall that $\mathsf{dang}_{B_g}(F_g) > b \log n$, therefore

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \left(\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \right) \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \left(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1}) \right) \quad \text{[by Corollary A.3.3 } (i)\text{]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{b}{j+1} \quad \text{[by Corollary A.3.3 } (ii)\text{]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - b \log n > 0,
\end{aligned}$$

finishing the proof. $\square$

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