On extremizers for certain inequalities of the k-plane transform and related topics

by

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Abstract

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This dissertation is concerned with determining optimal constants and extremizers, functions which achieve them, for certain inequalities arising in harmonic analysis.

The main inequality considered is the L^p-L^q inequality for the k-plane transform. It was shown in [11] that the k-plane transform is a bounded operator from L^p of Euclidean space to L^q of the Grassmann manifold of all affine k-planes in $\mathbb{R}^d$ for certain exponents depending on k and d. Specifically, for $1 \leq q \leq d+1$ and $p = \frac{dq}{n-d+dq}$ there exists a finite positive constant $A_0 > 0$ such that

$$\|T_{k,d} f\|_{L^q(\mathcal{M}_{k,d})} \leq A_0 \|f\|_{L^p(\mathbb{R}^d)}.$$

Extremizers of the inequality have previously been shown to exist when $q = 2$ by Baernstein and Loss [3], when $k = 2$ and q is an integer, also in [3], when $k = d-1$ and $q = d+1$ by Christ [12], and when $q = d+1$ for general k by Drouot [17]. In each of these cases, $f_0(x) = (1 = |x|^2)^{\frac{-(d-k)}{2(p-1)}}$ is an extremizer. When $q = 2$ [3] or $k = n-1$ and $q = d+1$ [12] this extremizer has been shown to be unique up to composition with certain explicit symmetries of the inequality.

Chapter 3 contains two proofs that when q is an integer, there exist extremizers, functions which achieve equality in the inequality with the sharp constant.

Chapter 4 extends Christ's uniqueness result for the endpoint case from $k = n-1$ to general k. In particular, we show that for $q = d+1$ for $k \in [1, d-1]$, the extremizing function is unique up to composition with affine maps. This is achieved by modifying the methods of [12] to apply to functions which are only assumed to be measurable L^p functions (rather than smooth L^p functions).

Chapter 6 shows that when q and $\frac{1}{p-1}$ are both integers, all extremizers are infinitely differentiable. This involves a family of weighted inequalities for the k-plane transform and the analysis of a nonlinear Euler-Lagrange equation.

Chapter 7, considers the related question of extremizing n-tuples of characteristic functions for certain multilinear inequalities of Hardy-Riesz-Brascamp-Lieb-Luttinger-Rogers type. Extremizing n-tuples are characterized in a special case. This chapter is joint work with Christ.

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Contents

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Chapter 1

Introduction

1.1 Optimal constants and extremizers

Inequalities abound in harmonic analysis. There are bounds for linear operators, bounds for multilinear forms, rearrangement inequalities, and many others. Here are a few examples:

- *The Hausdorff-Young inequality*: Let $p \in [1,2]$ and $q = \frac{p}{p-1}$. Denote by $\widehat{f}$ the Fourier transform of f. Then there exists a constant A such that for all $f \in L^p$,

$$\|\hat{f}\|_{L^q} \leq A\|f\|_{L^p}.$$

- *The Hardy-Littlewood-Sobolev inequality*: Let $p, r > 1$ and $0 < \lambda < n$ such that $\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{r} = 2$. Then there exists a constant A such that for all $f \in L^p$ and $h \in L^r$,

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda} h(y) dx dy \right| \leq A\|f\|_{L^p}\|h\|_{L^r}.$$

- *The Riesz rearrangement inequality*: Let f, g, and h be nonnegative measurable functions and let f^*, g^*, and h^* be their radially symmetric decreasing rearrangements. Then

$$\int f(x)g(x-y)h(y)\ dxdy \leq \int f^*(x)g^*(x-y)h^*(y)\ dxdy.$$

- *The Sobolev inequality*: Fix $d > 1$. Let $1 < p < d$ and take $q = \frac{pd}{d-p}$. Let Du denote the gradient of a function u. Then there exists a constant A such that for all $f \in L^p$,

$$\|u\|_{L^q} \leq A\|Du\|_{L^p}.$$

In each case, a natural question to ask is, “What is the optimal value for the constant A?” In cases such as the Riesz rearrangement inequality, where the constant A is not explicitly mentioned, we ask, “Is 1 the optimal value for the constant A?”

For those inequalities involving only one function, when the right hand side is nonzero, dividing yields

$$R(f) \leq A$$

where $R(f)$ is a ratio determined by the particular inequality. [1]. Now, let $\mathcal{A}$ to denote the class of functions for which the right hand side of the inequality is finite and nonzero. The optimal constant satisfies

$$A = sup_{f\in\mathcal{A}} R(f).$$

This leads to another natural question. Is this supremum actually a maximum? Are there functions which achieve the optimal ratio? If so, what can we say about these functions? Such functions are called extremizers.

When the inequality involves multiple functions, the same considerations apply, but now $\mathcal{A}$ will consist of n-tuples of functions for which the right hand side is finite and nonzero. If A is actually a maximum, then there will be a n-tuple of functions which achieve the optimal ratio and such a n-tuple of functions will be called an extremizing n-tuple.

Extremizers and optimal constants have been determined for many cases of these inequalities and for many others. While concentration compactness [34] and related ideas often provide a method for demonstrating existence of extremizers, identification of optimal constants and/or extremizers typically requires exploitation of specific symmetries or geometric properties of the inequality in question and thus no truly general method exists. A sample of such results is given in the next section.

1.2 A few key results from the literature

In this section we review several important results from the literature regarding optimal constants and extremizers. This is by no means an exhaustive survey, but simply covers a few famous examples (some of which closely relate to the work that follows).

For the Hausdorff-Young inequality the optimal constant is $A_H = \frac{(1/p)^{1/p}}{(1/q)^{1/q}}^{d/2}$ and Gaussian functions are extremizers. When $p = q = 2$ this is obvious. Plancherel's theorem tells us that equality is obtained with $A = 1$ for all functions. For pairs (p, q) where $p \in [1, 2)$ and $q = \frac{p}{p-1}$ is an even integer, this result was obtained by Babenko [2] who used methods of entire functions. Beckner [4] generalized the result to all pairs $p \in [1, 2)$ and $q = \frac{p}{p-1}$. Beckner's landmark proof begins by relating the Hausdorff-Young inequality to a multiplier inequality on the Hermite semigroup. In this setting, he is able to harness surprising additional structure relating the problem to both the central limit theorem and an inequality for the group with two elements. That Gaussian functions are extremizers follows direct calculation once the optimal constants are known. The sharp version of this inequality is sometimes called the Babenko-Beckner inequality.

Further, Lieb [33] shows that Gaussian functions are the unique extremizers. He proves the more general result that Gaussian functions are the unique extremizers for all L^p-L^q inequalities

[1]For example, for the Hausdorff-Young inequality $R(f) = \frac{\|\hat{f}\|_{L^q}}{\|f\|_{L^p}}$

for operators associated to Gaussian kernels whenever $1 < p \leq q < \infty$, and for $p > q$ in some special cases. His methods rely on the use of Minkowski's integral inequality and conditions for equality therein.

For the Riesz rearrangement inequality, 1 must be the optimal constant. Lieb in [32] proves that when $g = g^*$ and g^* is strictly symmetrically decreasing, a pair of nonnegative functions f and g achieve equality if and only if for some $v \in \mathbb{R}^d$, $f(x-v) = f^*(x)$ and $g(x-v) = g^*(x)$. Burchard in [7],[8] shows that if f, g, and h are nonnegative and two of f, g, and h have strictly symmetric decreasing rearrangements then there exists an affine map ϕ such that $f(\phi(x)) = f^*(x)$, $f(\phi(x)) = f^*(x)$, $g(\phi(x)) = g^*(x)$, and $h(\phi(x)) = h^*(x)$. Both results are proved by decomposing each function using the layer cake decomposition to reduce to the case of indicator functions of sets. Burchard identifies all cases equality in this situation. Burchard's result in fact applies in the more general Brascamp-Lieb-Luttinger-Rogers setting and is discussed in more depth in Section 1.5.

For the Hardy-Littlewood-Sobolev inequality, existence of extremizers was shown by Lieb in [30]. When $p = r$, the unique extremizer is $(1 + |x|^2)^{-d/p}$ up to dilation, translation, and multiplication by a complex constant. The sharp constant is given by

$$A_{p,\lambda,d} = \pi^{\lambda/2} \frac{\Gamma(d/2 - \lambda/2)}{\Gamma(d - \lambda/2)} \left(\frac{\Gamma(d/2)}{\Gamma(d)} \right)^{\lambda/n-1}.$$

As a corollary of this result Lieb [30] also determines the extremizers and sharp constant when $p = 2$ or $q = 2$.

Lieb proves the existence of extremizers by using Riesz's rearrangement inequality to reduce to the case of radially symmetric decreasing functions, where there is sufficient compactness to prove that after an appropriate dilation any extremizing sequence has a convergent subsequence which converges to a nonzero function. When $p = r$ there is an additional inversion symmetry which Lieb exploits to determine the form of the extremizer. He also uses the cases of equality in the Riesz rearrangement inequality, to show that all extremizers are radial up to translation.

For the Sobolev inequality, the optimal constant is

$$A_{d,p} = \pi^{-1} d^{-1/p} \left(\frac{p-1}{d-p} \right)^{1-1/p} \left(\frac{\Gamma(1 + d/2)\Gamma(m)}{\Gamma(d/p)\Gamma(1 + m - m/p)} \right)^{1/m}$$

and the extremizers are the functions $\left(a + b|x|^{p/(p-1)}\right)^{1-d/p}$ for $a, b > 0$. This was proved by Talenti [45] using methods from rearrangement inequalities and the calculus of variations in one dimension. Unlike Lieb's in work on the Hardy-Littlewood-Sobolev inequality, existence of extremizers is shown by exhibiting the extremizer, rather than by a convergence argument.

1.3 The k-plane transform

The bulk of this thesis focuses on identifying the optimal constants and extremizers for L^p-L^q inequalities of the k-plane transform. Here the optimal constant is the operator norm of the transform.

The k-plane transform takes complex-valued functions on $\mathbb{R}^d$ to complex-valued functions on the Grassmannian manifold of all affine k-dimensional planes in $\mathbb{R}^d$ by mapping a function f to its integrals over all k-planes. The $(d-1)$-plane transform is better known as the Radon transform and the 1-plane transform, as the X-ray transform. These transforms have applications in partial differential equations, radio astronomy, and image processing, among others.

Discussing functions defined on the manifold of all k-planes requires a certain amount of notation. Let's begin with $\mathcal{G}_{k,d}$, the Grassmannian manifold of all k-planes in $\mathbb{R}^d$ passing through the origin (or equivalently, the Grassmannian manifold of all k-dimensional linear subspaces of $\mathbb{R}^d$). We use θ to denote elements of $\mathcal{G}_{k,d}$. Both the manifold of all lines containing the origin, $\mathcal{G}_{1,d}$, and the manifold of all hyperplanes containing the origin, $\mathcal{G}_{d-1,d}$, are double-covered by the sphere, as both lines and hyperplanes may be specified by unit vectors with the identification $\theta = -\theta$. In general, $\mathcal{G}_{k,d}$ is a compact manifold of dimension $k(d-k)$. Further, the orthogonal group group acts transitively on $\mathcal{G}_{k,d}$. This action gives rise to a natural probablity measure on $\mathcal{G}_{k,d}$. Let $d\gamma_O$ denote unit Haar measure on the orthogonal group, and fix $\theta_0 \in \mathcal{G}_{k,d}$. Define the measure of a set $E \in \mathcal{G}_{k,d}$ by

$$\gamma(E) = \gamma_O(\{g \in O(d) : g(\theta_0) \in E\}).$$

Next, denote by $\mathcal{M}_{k,d}$ the Grassmannian manifold of all affine k-planes in $\mathbb{R}^d$. Each affine k-plane may be specified by a k-dimensional linear subspace, θ, and an orthogonal translation $y \in \theta^\perp$, where $\theta^\perp$ denotes the $(d-k)$-dimensional subspace orthogonal to θ. Thus the dimension of $\mathcal{M}_{k,d}$ is $(d-k)(k+1)$. When $k = d-1$, the dimension of $\mathcal{M}_{d-1,d}$ is d, and in general the dimension of $\mathcal{M}_{k,d}$ is at least d. For a measure on $\mathcal{M}_{k,d}$, we take the measure formed by pairing $d\gamma(\theta)$ on $\mathcal{G}_{k,d}$, and Lebesgue measure on the $(d-k)$-dimensional subspace orthogonal to θ, denoted $d\lambda_{\theta^\perp}(y)$.

Thus,

$$\|F\|_{L^q(\mathcal{M}_{k,d})} = \left(\int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} |F(\theta, y)|^q d\lambda_{\theta^\perp}(y) d\gamma(\theta) \right)^{1/q}.$$

Using this notation, the k-plane transform in $\mathbb{R}^d$ is given by

$$T_{k,d} f(\theta, y) = \int_{x \in \theta} f(x+y) \, d\lambda_\theta(x).$$

For example, let $\mathbb{1}_E$ be the indicator function of some measurable set $E \subset \mathbb{R}^d$. Then

$$T_{k,d} \mathbb{1}_E(\theta, 0) = \int_\theta \mathbb{1}_E(x) d\lambda_\theta(x) = k\text{-dim volume of } |E \cap \theta|.$$

$L^p(\mathbb{R}^d)$-$L^q(\mathcal{M}_{k,d})$ boundedness of the k-plane transform has been studied by several authors: Strichartz [44], Oberlin and Stein, [35], Calderón [9], Drury [21], [19], and Christ [11]. The simplest such result is the following inequality.

Claim 1. *For all* $f \in L^1(\mathbb{R}^d)$

$$\|T_{k,d} f\|_{L^1(\mathcal{M}_{k,d})} \leq \|f\|_{L^1(\mathbb{R}^d)}. \tag{1.1}$$

Further, equality holds if and only if there exists $c \in \mathbb{C}$ such that $f = c|f|$ almost everywhere.

Proof. By definition,

$$\begin{aligned}\int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} |T_{k,d} f(\theta, y)| d\lambda_{\theta^\perp}(y) d\gamma(\theta) &= \int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} \left| \int_{x \in \theta} f(x+y)\, d\lambda_\theta(x) \right| d\lambda_{\theta^\perp}(y)\, d\gamma(\theta) \\ &\leq \int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} \int_{x \in \theta} |f(x+y)|\, d\lambda_\theta(x) d\lambda_{\theta^\perp}(y)\, d\gamma(\theta)\end{aligned}$$

Changing variables,

$$\int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} |T_{k,d} f(\theta, y)| d\lambda_{\theta^\perp}(y) d\gamma(\theta) \leq \int_{\mathcal{G}_{k,d}} \int_{\mathbb{R}^d} |f(x)|\, d\lambda_{\mathbb{R}^d}(x)\, d\gamma(\theta)$$

As the measure on $\mathcal{G}_{k,d}$ is a probability measure, this yields (1.1).

Equality occurs in the inequality $|\int f dx| \leq \int |f| dx$ if and only if there exists $c \in \mathbb{C}$ such that $f = c|f|$. Thus such functions are the unique extremizers. □

Christ [11] showed that for $q = d+1$ and $p = \frac{d+1}{k+1}$ there exists a finite positive constant A such that for all $f \in L^p(\mathbb{R}^d)$,

$$\|T_{k,d} f\|_{L^{d+1}(\mathcal{M}_{k,d})} \leq A \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)}. \tag{1.2}$$

Thus by interpolation [11], for $1 \leq q \leq d+1$ and $p = \frac{dq}{d-k+kq}$ there exists a finite positive constant A such that for all $f \in L^p(\mathbb{R}^d)$,

$$\|T_{k,d} f\|_{L^q(\mathcal{M}_{k,d})} \leq A \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.3}$$

Simple examples (families of characteristic functions of balls and of appropriately chosen boxes) show that these are the only possible $L^p(\mathbb{R}^d)$-$L^q(\mathcal{M}_{k,d})$ inequalities.

Mixed norm inequalities of the k-plane transform are also of interest, though they are not studied here. A useful survey of such results is [18].

1.4 Extremizers of the k-plane transform

For the k-plane transform the question of optimal constants and extremizers was first considered by Baernstein and Loss in [3]. They conjectured:

Conjecture 1. *For all $a, b > 0$, $f_0(x) = (a + b|x|^2)^{\frac{-(d-k)}{2(p-1)}}$ is an extremizer of* (1.3).

As f_0 is nonnegative and nonzero, the conjecture is true when $q = 1 = p$. In this case, f_0 is far from unique– all nonnegative functions which are not identically zero extremize. More interestingly, the conjecture is also true when $q = 2$ and, thus, $p = \frac{2d}{d+k}$. Following Baernstein and Loss [3], specializing Drury's identity (here Lemma 10) to the case $q = 2$ gives that when

$q = 2$, (1.3) is equivalent to the Hardy-Littlewood-Sobolev inequality with $\lambda = d - k$, $q = 2$, and $p = \frac{2d}{d+k}$. Thus by the work of Lieb [30], f is an extremizer if and only if

$$f(x) = c(\gamma + |x - a|^2)^{-(d+k)/2}$$

for some $c \in \mathbb{C}$, $\gamma > 0$ and $a \in \mathbb{R}^d$.

Baernstein and Loss [3] also prove the conjecture for the 2-plane transform in the case when $q \in (1, d+1] \cap \mathbb{Z}$, by using symmetrization to reduce to the case of radial functions. In this setting, the k-plane transform becomes a one-dimensional integral operator for which (up to a change of variables) extremizers were found by Bliss [5] using methods from the calculus of variations. In this case, a radial function is an extremizer if and only if

$$f(|x|) = c(1 + \gamma|x|^2)^{\frac{-(d-2)}{2(p-1)}}$$

for some $c \in \mathbb{C}$ and $\gamma > 0$.

For $k = 2$, the characterization of non-radial extremizers remains open unless $q = 1, 2$, or $d + 1$.

The distinction between integer and non-integer q arises because the tool used to reduce to the radial case is the rearrangement inequality $\|T_{k,d}f\|_q \leq \|T_{k,d}f^*\|_q$ where f^* is the radial non-increasing rearrangement of f, which is proved [11],[3] by rewriting the L^q norm in terms of a multilinear form and thus only known for integer q.

Recently the conjecture has also been proved for the Radon transform when $q = d+1$ by Christ [12] and for the k-plane case when $q = d + 1$ by Drouot [17].

Christ additionally proves a uniqueness result. When $q = d + 1$ and $k = d - 1$, f is an extremizer of (1.3) if and only if

$$f(x) = c(1 + |\phi(x)|^2)^{\frac{-d}{2}}$$

where $c \in \mathbb{C}$ and ϕ is an invertible affine endomorphism of $\mathbb{R}^d$.

In Chapter 4 we extend this endpoint uniqueness result to the k-plane transform case for $1 \leq k \leq d - 2$.

While a complete characterization of extremizers in all cases remains unavailable, we also show that extremizers are smooth in the case that $q - 1$ and $\frac{1}{p-1}$ are integers larger than 1. This is the content of Chapter 6.

1.5 Extremizing n-tuples of characteristic functions for certain multilinear inequalities of Hardy-Riesz-Brascamp-Lieb-Luttinger-Rogers type

In the characterization of extremizers in both [12] and Chapter 4 a key step is the application of a theorem by Burchard [7],[8] on extremizers for Riesz's rearrangement inequality. In Chapter 7, which is joint work with Christ, we prove a limited extension of her result.

The (very) general setting incorporates mutlilinear rearrangement inequalities of the type studied by Hardy, Riesz [36], Brascamp, Lieb and Luttinger [6] and Rogers [37]. We consider integral expressions depending on three key parameters: $m > k$ and n all positive integers. Let $\{L_j\}_{j=1}^n$ be linear surjective maps $\mathbb{R}^m \to \mathbb{R}^k$. Let $E_j \subset \mathbb{R}^k$ be Lebesgue measurable sets with positive, finite measures, and consider

$$I(E_1, \cdots, E_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{d} \mathbb{1}_{E_j}(L_j(x))\, dx.$$

Definition 1. *Extremizing n–tuples $(E_1, \cdots, E_n)$ of measurable sets are those n-tuples that maximize I among all n–tuples with specified measures $|E_j|$.*

The generalization of the Riesz rearrangement inequality, proved by Brascamp, Lieb, and Luttinger [6] and, independently, Rogers [37], holds in this setting. Thus, among sets with specified measures, the functional I attains its maximum value when, up to sets of measure zero, each E_j equals E_j^*, the ball centered at the origin with the same measure as E_j. Specifically,

$$I(E_1, \cdots, E_n) \leq I(E_1^*, \cdots, E_n^*).$$

Thus, the main issue is the question of uniqueness.

Burchard characterized all extremizing n-tuples of measurable sets in the case where k is arbitrary, $m/k = r$ is an integer and $n = r + 1$ [7],[8], and I has the form

$$\mathcal{I}_{\mathcal{B}}(E_1, \ldots, E_n) = \int_{\mathbb{R}^{k(n-1)}} \left(\prod_{i=1}^{n-1} \mathbb{1}_{E_i}(x_i) \right) \mathbb{1}_{E_n} \left(\sum_{i=1}^{n-1} x_i \right) \, dx_1 \ldots dx_{n-1}.$$

Definition 2. *A set of positive numbers $\{\rho_i\}_{i=1}^n$ is admissible if they satisfy this generalization of the triangle inequality:*

$$\sum_{\substack{j=1 \\ j \neq i}}^{d} \rho_j \geq \rho_i \textit{ for all } i \in [1, n].$$

The set $\{\rho_i\}_{i=1}^n$ is strictly admissible if $\geq$ can be replaced by $>$.

Theorem 1. *[Burchard's theorem for indicator functions, [8], [7]] Let $n \geq 3$. Let E_i for $i \in [1, n]$ be sets of finite positive measure in $\mathbb{R}^k$. Denote by ρ_i the radii of the E_i^*. If the family $\{\rho_i\}_{i=1}^n$ is strictly admissible, then $(E_1, \ldots, E_n)$ is an extremizing n-tuple for $\mathcal{I}_{\mathcal{B}}$ if and only if there exist vectors $c_i \in \mathbb{R}^k$ for $i \in [1, n]$ and numbers $\alpha_i \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} c_i = 0$, and an ellipsoid $\mathcal{E} \subset \mathbb{R}^k$ centered at the origin, such that up to sets of measure zero*

$$E_i = c_i + \alpha_i \mathcal{E}.$$

If the family $\{\rho_i\}_{i=1}^n$ *is admissible but not strictly admissible, then* $(E_1, \dots, E_n)$ *is an extremizing* n*-tuple for* I_B *if and only if for each* $i \in [1, n]$ *there exist vectors* $c_i \in \mathbb{R}^k$ *and numbers* $\alpha_i \in \mathbb{R}_+$ *such that* $\sum_{i=1}^{d} c_i = 0$*, and there exists a convex set* $\mathcal{M} \subset \mathbb{R}^k$ *centered at the origin, such that up to sets of measure zero*

$$E_i = c_i + \alpha_i \mathcal{M}.$$

Note that in the case $k = 1$ the results for strict admissibility and admissibility are equivalent.

In Chapter 7, which is joint work with Christ, we characterize all extremizing n-tuples in the case where $k = 1$ and $m = 2$ but $n \geq m + 1$ under an appropriate nondegeneracy condition on the maps L_j.

Chapter 2

Preliminary results regarding the k-plane transform

We here record several basic results regarding the k-plane transform for later use.

2.1 Simple symmetries

A symmetry of (1.3) is a "nice"[1] operation on L^p functions such that

- $$\|\mathcal{J}(f)\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$$
- $$\|T_{k,n}\mathcal{J}(f)\|_{L^q(\mathcal{M}_{k,d})} = \|T_{k,n}f\|_{L^q(\mathcal{M}_{k,d})}.$$

Clearly, if $\mathcal{J}$ is a symmetry of (1.3) and f is an extremizer of (1.3) then $\mathcal{J}(f)$ is also an extremizer of (1.3).

We next show that translation and dilation (with the appropriate Jacobian factor) are symmetries of the k-plane transform.

Claim 2. *For any $v \in R^d$, let $\tau_v(f) = f(x+v)$. For all $v \in R^d$,*

$$T_{k,d}\tau_v(f) = \tau_{P_{\theta_\perp}(v)} T_{k,d} f(\theta, y)$$

where the translation on $\mathcal{M}_{k,d}$ acts only in the y variable. Thus,

$$\|T_{k,d}\tau_v(f)\|_{L^q(\mathcal{M}_{k,d})} = \|T_{k,d}(f)\|_{L^q(\mathcal{M}_{k,d})}$$

[1] For the purposes of the thesis we will Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a function with a well defined Jacobian determinant. Define $\mathcal{J} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ by $\mathcal{J}f = |J_\varphi|^{1/p}(f \circ \varphi)$ where $|J_\varphi|$ is the Jacobian determinant of φ.

Proof. Fix θ and $y \in \theta^{\perp}$. By definition,

$$T_{k,d}\tau_v f(\theta, y) = \int_{x\in\theta} f(x + y + v)\, d\lambda_\theta(x).$$

Let P_θ denote projection on the k-plane θ and $P_{\theta^\perp}$ denote projection on to the $(d-k)$-plane perpendicular to θ. Then $v = P_\theta(v) + P_{\theta_\perp}(v)$ and

$$T_{k,d}\tau_v f(\theta, y) = \int_{x\in\theta} f(x + y + P_\theta(v) + P_{\theta_\perp}(v))\, d\lambda_\theta(x).$$

As $x + P_\theta(v) \in \theta$, making the change variables $z = x + P_\theta(v)$ gives

$$T_{k,d}\tau_v f(\theta, y) = \int_{z\in\theta} f(z + y + P_{\theta_\perp}(v))\, d\lambda_\theta(z).$$

The second statement follows directly from the first by a similar change of variables. □

Claim 3. *For all $r \neq 0$, define $\mathcal{J}_r(f) = r^{-d/p} f(r^{-1}x)$. If $1 \leq q, p$ are related by $p = \frac{dq}{d-k+kq}$, then for all $r \neq 0$*

$$\|T_{k,d}[\mathcal{J}_r(f)]\|_{L^q(\mathcal{M}_{k,d})} = \|T_{k,d}(f)\|_{L^q(\mathcal{M}_{k,d})}$$

Proof. It is enough to prove $T_{k,d}[\mathcal{J}_r(f)] = r^{(k-d)/q} T_{k,d} f(\theta, r^{-1}y)$. First,

$$T_{k,d}[r^{-d/p} f(r^{-1}x)] = \int_{x\in\theta} r^{-d/p} f(r^{-1}(x) + r^{-1}(y))\, d\lambda_\theta(x).$$

Changing variables from $z = r^{-1}x$,

$$\int_{x\in\theta} r^{-d/p} f(r^{-1}(x) + r^{-1}(y))\, d\lambda_\theta(x) = \int_{z\in\theta} r^{d/p} f(z + r^{-1}(y))\, r^k d\lambda_\theta(z).$$

Lastly, $-d/p + k = \frac{-d+k-kq}{q} + k = \frac{-d+k-kq+kq}{q} = \frac{k-d}{q}$. □

2.2 The dual k-plane transform

Define

$$T^*_{k,d} f(x) = \int_{\mathcal{G}_{k,d}} f(\theta, P_{\theta^\perp}(x))\, d\theta$$

where $P_{\theta^\perp}(x)$ is the projection of x on to the $(d-k)$-dimensional plane perpendicular to θ. This is the dual transform in the sense that if f and g are Schwartz functions then $\int_{\mathcal{M}_{k,d}} (T_{k,d} f)\bar{g} d\lambda_{\theta^\perp}(y) d\theta = \int_{\mathbb{R}^d} f\overline{(T^*_{k,n} g)} dx$.

The dual k-plane transform takes a function defined on the space of all k-planes and returns a function defined on $\mathbb{R}^d$ by assigning to x the average value of the function on k-planes passing through the point x.

That this transform is dual to the k-plane transform is seen as follows,

$$\begin{aligned}\int_{\mathcal{M}_{k,d}} (T_{k,d}f)\bar{g}d\lambda_\theta(y)d\theta &= \int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} (T_{k,d}f)\bar{g}d\lambda_{\theta^\perp}(y)d\theta \\ &= \int_{\mathcal{G}_{k,d}} \int_{\theta^\perp} \int_\theta f(x+y)\bar{g}d\lambda_\theta(x)d\lambda_{\theta^\perp}(y)d\theta\end{aligned}$$

Make the change of variables $z = x + y$. Note that as x ranges over θ and y ranges over $\theta^\perp$, z ranges over $\mathbb{R}^d$, and also that $y = P_{\theta^\perp}(z)$. Thus,

$$\begin{aligned}\int_{\mathcal{M}_{k,d}} (T_{k,d}f)\bar{g}d\lambda_\theta(y)d\theta &= \int_{\mathcal{G}_{k,d}} \int_{\mathbb{R}^d} f(z)\bar{g}(\theta, P_{\theta^\perp}(z))dzd\theta \\ &= \int_{\mathbb{R}^d} f(z) \int_{\mathcal{G}_{k,d}} \bar{g}(\theta, P_{\theta^\perp}(z))d\theta dz.\end{aligned}$$

2.3 Basic properties of $T_{k,d}$ and $T^*_{k,d}$

Let $\mathscr{S}$ denote the Schwartz class of functions on $\mathbb{R}^d$. Let $\mathscr{S}(\mathcal{M}_{k,d})$ denote the class of functions on $\mathcal{M}_{k,d}$, satisfying "for all θ, $f(\theta, y) \in \mathscr{S}(\theta^\perp) \sim \mathscr{S}(\mathbb{R}^{d-k})$". We call this class the Schwartz class of functions on $\mathcal{M}_{k,d}$. Note that for this class of functions $T_{k,d}f$ is always well defined.

Lemma 1. *Let $f \in \mathscr{S}$ be a radial function, Then the function g defined on on $[0, \infty)$ by*

$$g(r) = \int_0^\infty f((s^2 + r^2)^{1/2})s^{k-1}ds$$

satisfies $T_{k,d}f(\theta, y) = g(|y|)$. Moreover, if f is symmetric decreasing, then g is decreasing as well.

Proof. Fix θ and $y \in \theta^\perp$. By definition,

$$T_{k,d}f(\theta, y) = \int_{x\in\theta} f(x + y)\ d\lambda_\theta(x).$$

As f is radial, and x and y are orthogonal,

$$T_{k,d}f(\theta, y) = \int_{x\in\theta} f((|x|^2 + |y|^2)^{1/2})\ d\lambda_\theta(x).$$

Let R_θ be the rotation of $\mathbb{R}^n$ such that $R_\theta(\theta) = \mathbb{R}^k$. As f is radial, it is invariant under rotation, thus

$$T_{k,d}f(\theta, y) = \int_{x\in\mathbb{R}^k} f((|x|^2 + |y|^2)^{1/2})\ d\lambda_\theta(x).$$

Changing variables we have

$$T_{k,d}f(\theta, y) = \int_0^\infty f((r^2 + |y|^2)^{1/2})r^{k-1}\ d\lambda_\theta(x).$$

□

Lemma 2. *For all $f \in \mathscr{S}(\mathbb{R}^d)$, for all $g \in \mathscr{S}(\mathcal{M}_{k,d})$, for all $\theta \in \mathcal{G}_{k,d}$, for all $\xi \in \theta^\perp$*

$$\widehat{T_{k,d}f(\theta,}\xi) = \hat{f}(\xi)$$

and

$$\widehat{T^*_{k,d}g}(\xi) = \int_{\{\theta:\theta\perp\xi\}} \widehat{g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta)$$

where for functions on $\mathcal{M}_{k,d}$ the Fourier transform is taken only in the y-variable and $d\gamma_{\xi^\perp}$ represent the restriction of the measure $d\gamma$ to the subset of k-planes which are perpendicular to ξ.

Proof. Fix $\theta \in \mathcal{M}_{k,d}$.

$$\begin{aligned}\widehat{T_{k,d}f(\theta,}\xi) &= \int T_{k,d}f(\theta,y)e^{-2\pi i\xi\cdot y}d\lambda_{\theta^\perp}(y)\\ &= \int_{\theta^\perp}\int_\theta f(x+y)e^{-2\pi i\xi\cdot y}\ d\lambda_\theta(x)d\lambda_{\theta^\perp}(y)\end{aligned}$$

Make the change of variable $z = x + y$. Note that for $\xi \in \theta^\perp$ $\xi \cdot z = \xi \cdot y$. Thus,

$$\begin{aligned}\widehat{T_{k,d}f(\theta,}\xi) &= \int_{\mathbb{R}^d} f(z)e^{-2\pi i\xi\cdot z}d\lambda_{\mathbb{R}^d}(z)\\ &= \widehat{f}(\xi)\end{aligned}$$

Now,

$$\begin{aligned}\widehat{T^*_{k,d}g}(\xi) &= \int_{\mathbb{R}^d} T^*_{k,d}g(x)e^{-2\pi ix\cdot\xi}dx\\ &= \int_{\mathbb{R}^d}\int_{\mathcal{G}_{k,d}} g(\theta,P_{\theta^\perp}(x))e^{-2\pi ix\cdot\xi}d\gamma(\theta)dx\\ &= \int_{\mathcal{G}_{k,d}}\int_\theta\int_{\theta^\perp} g(\theta,x_1)e^{-2\pi i(x_1\cdot\xi_1+x_2\cdot\xi_2)}d\lambda_{\theta^\perp}(x_1)d\lambda_\theta(x_2)d\gamma(\theta)\end{aligned}$$

where $x_1 = P_{\theta^\perp}(x)$ and $x_2 = P_\theta(x)$ and similarly for ξ.

$$\begin{aligned}\widehat{T^*_{k,d}g}(\xi) &= \int_{\mathcal{G}_{k,d}}\int_\theta\left(\int_{\theta^\perp} g(\theta,x_1)e^{-2\pi i(x_1\cdot\xi_1)}d\lambda_{\theta^\perp}(x_1)\right)e^{-2\pi i(x_2\cdot\xi_2)}d\lambda_\theta(x_2)d\gamma(\theta)\\ &= \int_{\{\theta:\theta\perp\xi\}}\widehat{g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta)\end{aligned}$$

□

2.4 Rearrangement inequalities

Rearrangment inequalities play a key role in many results regarding optimal constants and extremizers. An excellent introduction is provided in [31] but we record the main definitions and properties here for easy reference.

Definition 3. *For any measurable set E, $|E| < \infty$, let E^* be the open ball with the same measure*

Definition 4. *The radial symmetric decreasing (nonincreasing) rearrangement of f is given by*

$$f^*(x) = \int_0^\infty \mathbb{1}^*_{|f|>t}(x)dt$$

Thus f^* is radial and nonincreasing. Further $\|f^*\|_{L^p} = \|f\|_{L^p}$, which follows immediately from the formula

$$\|f\|_{L^p} = p\int_0^\infty t^{p-1}\mu\{|f| > t\})dt.$$

2.5 Euler-Lagrange equation

Recall that our goal is to find the optimal constant A_0 in (1.3). We know that

$$A_0 = \sup_{\{f:\|f\|_{L^p}\neq 0\}} \frac{\|T_{k,d}f\|_{L^q}}{\|f\|_{L^p}}.$$

For $q \in (1, d+1]$, the nonnegative critical points of this functional satisfy the Euler-Lagrange equation

$$f = \lambda(T^*_{k,d}[(T_{k,d}f)^{q_{el}}])^{p_{el}} \tag{2.1}$$

where $q_{el} = q-1$, $p_{el} = \frac{1}{p-1}$, and $\lambda = \left(\|f\|^p_{L^p(\mathbb{R}^d)}\|T_{k,d}f\|^{-q}_{L^q(\mathcal{M}_{k,d})}\right)^{p_{el}}$.

Here we present a purely formal computation of the Euler-Lagrange equation and second variation. For an example of this computation done rigorously see [15].

$$\begin{aligned}\|F+zG\|^q_{L^q} &= \int (F+zG)^q dx \\ &= \int F^q + zqF^{q-1}G + z^2\binom{q}{2}F^{q-2}G^2 + O(z^3)dx\end{aligned}$$

$$\|T_{k,d}(f+zg)\|^q_{L^q} = \int\int (T_{k,d}f)^q + zq(T_{k,d}f)^{q-1}T_{k,d}g + z^2\binom{q}{2}(T_{k,d}f)^{q-2}(T_{k,d}g)^2 + O(z^3)dyd\theta$$

$$
\begin{aligned}
\|f+zg\|_{L^p}^q &= \left(\int (f+zg)^p dx\right)^{q/p} \\
&= \left(\int f^p + zpf^{p-1}g + z^2\binom{p}{2}(f)^{p-2}g^2 + O(z^3)dx\right)^{q/p} \\
&= \left(\int f^p dx + \int zpf^{p-1}g + z^2\binom{p}{2}(f)^{p-2}g^2 + O(z^3)dx\right)^{q/p} \\
&= \left(\int f^p dx\right)^{q/p} + \frac{q}{p}\left(\int f^p dx\right)^{q/p-1}\int zpf^{p-1}g + z^2\binom{p}{2}(f)^{p-2}g^2 dx \\
&\quad + \binom{q/p}{2}\left(\int f^p dx\right)^{q/p-2}\left(\int zpf^{p-1}g + z^2\binom{p}{2}(f)^{p-2}g^2 dx\right)^2 + O(z^3) \\
&= \|f\|_p^q + zq\|f\|_p^{q-p}\int f^{p-1}gdx + z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{q-p}\int f^{p-2}g^2 dx \\
&\quad + (zp)^2\binom{q/p}{2}\|f\|_p^{q-2p}\left(\int f^{p-1}gdx\right)^2 + O(z^3) \\
&= \|f\|_p^q\left(1 + zq\|f\|_p^{-p}\int f^{p-1}gdx + z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2 dx\right. \\
&\quad \left. + (zp)^2\binom{q/p}{2}\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2\right) + O(z^3)
\end{aligned}
$$

$$
\begin{aligned}
\|f+zg\|_{L^p}^{-q} &= \|f\|_p^{-q}\sum_{n=0}^{\infty}\left(-zq\|f\|_p^{-p}\int f^{p-1}gdx - z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2 dx\right. \\
&\quad \left. - (zp)^2\binom{q/p}{2}\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2\right)^n + O(z^3) \\
&= \|f\|_p^{-q}\left(1 - zq\|f\|_p^{-p}\int f^{p-1}gdx - z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2 dx\right. \\
&\quad \left. - z^2p^2\binom{q/p}{2}\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2 + z^2q^2\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2\right) + O(z^3) \\
&= \|f\|_p^{-q}\left(1 - zq\|f\|_p^{-p}\int f^{p-1}gdx - z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2 dx\right. \\
&\quad \left. + z^2\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2\left(q^2 - p^2\binom{q/p}{2}\right)\right) + O(z^3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\|T_{k,d}(f+zg)\|_{L^q}^q}{\|f+zg\|_{L^p}^q} &= \left(\int\int (T_{k,d}f)^q + zq(T_{k,d}f)^{q-1}T_{k,d}g + z^2\binom{q}{2}(T_{k,d}f)^{q-2}(T_{k,d}g)^2\ dyd\theta\right) \\
&\quad + \left(\|f\|_p^{-q}\left(1 - zq\|f\|_p^{-p}\int f^{p-1}gdx - z^2\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2dx\right.\right. \\
&\quad \left.\left. + z^2\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2\left(q^2 - p^2\binom{q/p}{2}\right)\right)\right) + O(z^3) \\
&= \|T_{k,d}f\|_{L^q}^q\|f\|_p^{-q} \\
&\quad + zq\|f\|_p^{-q}\left(\int\int (T_{k,d}f)^{q-1}T_{k,d}g\ dyd\theta - \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q\int f^{p-1}gdx\right) \\
&\quad - z^2q^2\|f\|_p^{-q-p}\int\int (T_{k,d}f)^{q-1}T_{k,d}g\ dyd\theta\int f^{p-1}gdx \\
&\quad + z^2\|T_{k,d}f\|_{L^q}^q\|f\|_p^{-q}\left(\|f\|_p^{-2p}\left(q^2 - p^2\binom{q/p}{2}\right)\left(\int f^{p-1}gdx\right)^2\right. \\
&\quad \left. - \frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2dx\right) \\
&\quad + z^2\|f\|_p^{-q}\binom{q}{2}\int\int (T_{k,d}f)^{q-2}(T_{k,d}g)^2\ dyd\theta + O(z^3)
\end{aligned}
$$

If f is critical point, then for all g nice test functions:

$$
\begin{aligned}
0 &= \int\int (T_{k,d}f)^{q-1}T_{k,d}g\ dyd\theta - \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q\int f^{p-1}gdx \\
&= \int T_{k,d}^*\left([T_{k,d}f]^{q-1}\right)gdx - \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q\int f^{p-1}gdx \\
&= \int\left(T_{k,d}^*[(T_{k,d}f)^{q-1}] - \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q f^{p-1}\right)gdx
\end{aligned}
$$

Thus,

$$T_{k,d}^*[(T_{k,d}f)^{q-1}] - \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q f^{p-1} = 0$$

$$f = \left(\frac{\|f\|_p^p}{\|T_{k,d}f\|_{L^q}^q}T_{k,d}^*[(T_{k,d}f)^{q-1}]\right)^{\frac{1}{p-1}}$$

Yielding the generalized Euler-Lagrange Equation,

$$f = \lambda\left(T_{k,d}^*[(T_{k,d}f)^{q-1}]\right)^{\frac{1}{p-1}}$$

Considering the second variation we have, if f is an extremizer,

$$
\begin{aligned}
&- q^2\|f\|_p^{-q-p}\int\int(T_{k,d}f)^{q-1}T_{k,d}g\ dyd\theta\int f^{p-1}gdx \\
&+ \|T_{k,d}f\|_{L^q}^q\|f\|_p^{-q}\left(\|f\|_p^{-2p}\left(q^2-p^2\binom{q/p}{2}\right)\left(\int f^{p-1}gdx\right)^2-\frac{q}{p}\binom{p}{2}\|f\|_p^{-p}\int f^{p-2}g^2dx\right) \\
&\qquad + \|f\|_p^{-q}\binom{q}{2}\int\int(T_{k,d}f)^{q-2}(T_{k,d}g)^2\ dyd\theta \le 0
\end{aligned}
$$

If f is an extremizer then it satisfies the equation:

$$\int\int(T_{k,d}f)^{q-1}T_{k,d}g\ dyd\theta = \|f\|_p^{-p}\|T_{k,d}f\|_{L^q}^q\int f^{p-1}gdx.$$

Thus,

$$
\begin{aligned}
&(q-1)\int\int(T_{k,d}f)^{q-2}(T_{k,d}g)^2\ dyd\theta \\
&\qquad -(q-p)\|T_{k,d}f\|_{L^q}^q\|f\|_p^{-2p}\left(\int f^{p-1}gdx\right)^2 \\
&\qquad\qquad -(p-1)\|T_{k,d}f\|_{L^q}^q\|f\|_p^{-p}\int f^{p-2}g^2dx \le 0
\end{aligned}
$$

2.6 Nonnegative extremizers are strictly positive

In order to show that nonnegative extremizers of (1.3) are positive almost everywhere, we instead prove a slightly more general statement.

Note that all nonnegative extremizers of (1.3) satisfy the Euler-Lagrange equation

$$f(x) = \lambda(T_{k,n}^*[(T_{k,n}f)^{q_0}])^{p_0}(x) \tag{2.2}$$

where $q_0 = q-1$, $p_0 = \frac{1}{p-1}$, λ depends on p, q, n, k and f, and $T_{k,n}^*$ is the dual of the k-plane transform.

Proposition 1. *If $f(x) \in L^p(\mathbb{R}^n)$ is a nonnegative solution of (2.1) with $q \geq 2$, then either $f(x) > 0$ for almost every $x \in \mathbb{R}^n$ or $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.*

The proof relies on the following lemma.

Lemma 3. *For any nonnegative solution $f(x) \in L^p(\mathbb{R}^n)$ of (2.1) with $q \geq 2$,*
$f(x) \geq C(\lambda)(T_{k,n}^*T_{k,n}f(x))^{p_0q_0}$ almost everywhere.

Proof. Let $d\theta$ be the unique Haar probability measure on $\mathcal{G}_{k,n}$ and $P(x,\theta^{\perp})$ be the projection of x onto $\theta^{\perp}$, the orthogonal complement of θ. Then, writing out $T^*_{k,n}$ explicitly,

$$f(x) = \lambda \left(\int_{\mathcal{G}_{k,n}} [T_{k,n} f(\theta, P(x,\theta^{\perp}))]^{q_0} d\theta \right)^{p_0}.$$

As $q_0 = q - 1 = n \geq 1$, Hölder's inequality applies.

$$\int_{\mathcal{G}_{k,n}} g(\theta) d\theta \leq \left(\int_{\mathcal{G}_{k,n}} g(\theta)^{q_0} d\theta \right)^{1/q_0} \left(\int_{\mathcal{G}_{k,n}} 1 d\theta \right)^{1/q_0'} = C \left(\int_{\mathcal{G}_{k,n}} g(\theta)^{q_0} d\theta \right)^{1/q_0}.$$

Thus, $\int_{\mathcal{G}_{k,n}} [T_{k,n} f(\theta, P(x,\theta^{\perp}))]^{q_0} d\theta \geq \left(\int_{\mathcal{G}_{k,n}} (T_{k,n} f(\theta, P(x,\theta^{\perp})) d\theta \right)^{q_0}$. As $p = \frac{n+1}{k+1} \geq 1$, $p_0 = \frac{1}{p-1} > 0$ and therefore

$$f(x) \geq C\lambda \left(\int_{\mathcal{G}_{k,n}} [(T_{k,n} f(\theta, P(x,\theta^{\perp}))]\, d\theta \right)^{p_0 q_0}.$$

Again applying the definition of $T^*_{k,n}$ proves the statement, with the qualification that as our function satisfies (2.1) with equality in L^p, the statement holds only almost everywhere. □

Proof of Proposition. Writing out $T^*_{k,n} T_{k,n}$ using Fuglede's formula [23]

$$f(x) \geq C\lambda \left(\int f(y) |y - x|^{k-n} dx \right)^{p_0 q_0}.$$

If there is a set of positive measure on which $f(x) = 0$ then for some x_0 ,

$$C\lambda (\int f(y) |y - x_0|^{k-n} dx)^{p_0 q_0} = 0.$$

$$\int f(y) |y - x_0|^{k-n} dx = 0.$$

As $|y - x_0|^{k-n}$ is positive except at $y = x_0$, $f(y) = 0$ almost everywhere. □

Chapter 3

Existence of extremizers when q is an integer

3.1 Main result and methods

In this chapter we give two proofs of the fact that when $q \in [1, d+1]$ is an integer, extremizers of the k-plane transform inequality (1.3) exist. The idea of the proof is simple: show that some extremizing sequence converges. Given this, it will be advantageous to work in a space of functions with better compactness properties than L^p. Specifically, we use a rearrangement inequality to reduce to the case of radial symmetric decreasing functions. It is this step that imposes the restriction that q be an integer, as the rearrangement inequality is only known in this case. It is an open question whether the rearrangement inequality holds in the noninteger case. If it does, then either proof would give existence of extremizers in all cases.

Additionally, the symmetries of our inequality pose an obstacle to convergence. Suppose an extremizer exists, say f, and let v be a non-zero vector. Then each of the sequences $f_n(x) = f(x+nv)$ and $f = n^{d/p} f(nx)$ are extremizing but converge pointwise to zero as $n \to \infty$. Similar problems may arise with a generic extremizing sequence. Restricting to radial symmetric decreasing functions removes the translation symmetry. We show that some subsequence of any extremizing sequence of radially symmetric nonincreasing functions $\{f_n\}$ converges after an n-dependent dilation.

We give two proofs, one following Lieb's proof for existence of extremizers for the Hardy-Littlewood-Sobolev inequality [30], and one following Drouot's proof for existence of extremizers in the endpoint case [17]. Both proofs follow the same general outline:

Reduce to radially symmetric functions using the rearrangement inequality for the k-plane transform.

Lemma 4 (k-plane rearrangement, [11])**.** *When $q \in [1, d+1]$ is an integer, for all $f \in L^p(\mathbb{R}^d)$,*

$$\|T_{k,d} f\|_{L^q} \leq \|T_{k,d} f^*\|_{L^q}.$$

Christ gives this result for $q = d + 1$ in [11], in a multilinear form from which the inequality for smaller q can be derived. Another proof is given in [3].

Given Lemma 4 and invariance of the ratio of norms under multiplication by a constant, the sharp constant A_0 in is given by

$$A_0 = \sup_{\{f:\|f\|_{L^p}=1, f=f^*\}} \frac{\|T_{k,d}f\|_{L^q}}{\|f\|_{L^p}}. \tag{3.1}$$

With this notation our main result is:

Theorem 2. *Let $q \in (1, d+1]$ be an integer. Let f_j be an extremizing sequence for* (3.1). *Then there exist a subsequence, which we will continue to denote f_j, and a sequence $\sigma_j \in (0,\infty)$ such that the new extremizing sequence $|\sigma_j|^{-d/p} f_j(\sigma_j^{-1}x)$ is relatively compact in $L^p(\mathbb{R}^d)$. In particular, the supremum in* (3.1) *is attained.*

Again, Drouot in [17] has already proved this in the $q = d+1$ case. A more general version of this result has been proved by Christ in [12] in the case that $q = n+1$ and $k = n-1$.

Restricting to sequences of radially symmetric decreasing functions allows us to extract a sequence which converges pointwise (except perhaps at zero) by Helly's selection principle.

Lemma 5 (Helly's selection principle (see for instance [31])). *For any sequence, $\{f_j\}$, of functions $[0,\infty) \to [0,\infty)$ (resp. $\mathbb{R}^d \to [0,\infty)$) which are decreasing (resp. radial symmetric decreasing), and for which there exists a finite positive constant B, such that $\|f_j\|_{L^p} \leq B$ for all j, there exists a function $f \in L^p$ and a subsequence, which we will continue to denote f_j, such that $f_j \to f$ pointwise except perhaps at the origin.*

Given pointwise convergence, we now wish to apply a powerful theorem of Lieb's from [30].

Lemma 6 (Lieb [30] Lemma 2.7). *Let (M, Σ, μ) and (M', Σ', μ') be measure spaces and let X (resp. Y) be $L^p(M, \Sigma, \mu)$ (resp. $L^q(M', \Sigma', \mu')$) with $1 \leq p \leq q < \infty$. Let A be a bounded linear operator from X to Y. For $f \in X$, $f \neq 0$, let*

$$R(f) = \|Af\|_Y / \|f\|_X \text{ and } N = sup\{R(f) | f \neq 0\}$$

Let $\{f_j\}$ be a uniformly norm-bounded maximizing sequence for N and suppose that $f_j \to f \neq 0$ and that $Af_j \to Af$ pointwise almost everywhere. Then f maximizes R, ie, $R(f) = N$. Moreover, if $p < q$ and if $lim\|f_j\|_X = C$ exists, then $\|f\|_X = C$ and hence $\|Af_j\|_Y \to \|Af\|_Y$.

Thus we are required to prove that there exists a sequence $\sigma_j \in (0,\infty)$ such that the new extremizing sequence $|\sigma_j|^{-d/p} f_j(\sigma_j^{-1}x)$ converges pointwise to a nonzero function and that $T_{k,d}f_j \to T_{k,d}f$ pointwise.

3.2 Conclusion of the proof of Theorem 2

Lemma 7. *Let f_j be an extremizing sequence for* (3.1) *such that for each j, $\|T_{k,d}f_j\|_{L^q} \geq A_0/2$. Then there exists a sequence $\sigma_j \in (0,\infty)$ and a ball $\mathcal{B}$ such that for each j, $|\sigma_j|^{-d/p} f_j(\sigma_j^{-1}x) \geq \mathbb{1}_{\mathcal{B}}(x)$.*

This follows from Lorentz norm estimates for the k-plane transform.

Lemma 8. *[11] There exists a constant C such that for all $f \in L^{\frac{d+1}{k+1},d+1}$,*

$$\|T_{k,d}f\|_{L^{d+1}(\mathcal{M}_{k,d})} \leq C\|f\|_{L^{\frac{d+1}{k+1},d+1}}.$$

By interpolation, this yields

Lemma 9. *For $q \in [1, d+1]$ and $p = \frac{nq}{n-k+kq}$, there exists a constant C such that for all $f \in L^{p,q}$,*

$$\|T_{k,d}f\|_{L^q(\mathcal{M}_{k,d})} \leq C\|f\|_{L^{p,q}}.$$

Note that for $q > 1$ as $p = \frac{n}{n+k(q-1)}q < q$, this estimate is stronger then that of (1.3).

Proof. The $q = d+1$ endpoint is exactly Christ's estimate. The $q = 1$ endpoint is the inequality $\|T_{k,d}f\|_{L^1(\mathcal{M}_{k,d})} \leq \|f\|_{L^1}$. These estimates certainly imply the corresponding estimates for weak L^q. Thus, using the off-diagonal Marcinkiewicz interpolation theorem [43], for $q \in (1, d+1)$ and $p = \frac{nq}{n-k+kq}$, for all $r \in (0,\infty]$, there exists a constant C such that for all $f \in L^{p,r}$,

$$\|T_{k,d}f\|_{L^{q,r}(\mathcal{M}_{k,d})} \leq C\|f\|_{L^{p,r}}.$$

Taking $r = q$ gives the theorem. □

This proof is essentially the same as that given in [17].

Proof of Lemma 7. First note that for $1 \leq p < q < \infty$,

$$\|f\|_{L^{p,q}}^q \leq \|f\|_{L^{p,\infty}}^{q-p}\|f\|_p^p \tag{3.2}$$

which can be seen as follows.

$$\begin{aligned}
\|f\|_{L^{p,q}}^q &= \int_0^\infty t^q \left(\mu\{x : |f(x)| > t\}\right)^{q/p} \frac{dt}{t} \\
&= \int_0^\infty \left(t^p \mu\{x : |f(x)| > t\}\right) \left(t^{q-p} \left(\mu\{x : |f(x)| > t\}\right)^{(q-p)/p} \frac{dt}{t}\right. \\
&\leq \sup_{t>0} (t^{q-p} \left(\mu\{x : |f(x)| > t\}\right)^{(q-p)/p} \int_0^\infty \left(t^p \mu\{x : |f(x)| > t\}\right) \frac{dt}{t} \\
&\leq \|f\|_{L^{p,\infty}}^{q-p}\|f\|_{L^p}^p.
\end{aligned}$$

Therefore as for each j, $\|T_{k,d}f_j\|_{L^q} \geq A_0/2$ and $\|f_j\|_{L^p} = 1$,

$$\begin{aligned}(A_0/2)^q &\leq \|T_{k,d}f_j\|^q_{L^q}\\ &\leq C\|f_j\|^q_{L^{p,q}}\\ &\leq C\|f_j\|^{q-p}_{L^{p,\infty}}\\ &\leq C\sup_{t>0}(t^{q-p}\left(\mu\{x : |f_j(x)| > t\}\right)^{(q-p)/p}\end{aligned}$$

Thus, there exists t_j such that

$$t_j^p\left(\mu\{x : |f_j(x)| > t_j\}\right) \geq \left(\frac{A_0^q}{2^{q+1}C}\right)^{\frac{p}{q-p}}.$$

Set $\sigma_j = t_j^{-p/n}$. Then

$$\mu\{x : \sigma_j^{d/p}|f_j(\sigma_j x)| > 1\} = t_j^p\left(\mu\{x : |f_j(x)| > t_j\}\right).$$

Let $\mathcal{B}$ be the ball centered at origin with volume $\left(\frac{A_0^q}{2^{q+1}C}\right)^{\frac{p}{q-p}}$. As, f_j is symmetric decreasing, it follow from the above estimates that

$$\sigma_j^{d/p}f_j(\sigma_j x) \geq \mathbb{1}_{\mathcal{B}}(x).$$

□

Proof of Theorem 2 . Let f_j be an extremizing sequence for (3.1). Thus $\|f_j\|_{L^p} = 1$, f_j is radial symmetric decreasing, and there exists a subsequence, which we will continue to denote f_j, such that $\|T_{k,d}f\|_{L^q} \geq A_0/2$. Take σ_j as guaranteed by Lemma 7. Applying Helly's selection principle (Lemma 5) to $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x)$, there exist a function $f \in L^p$ and a subsequence, which we will continue to denote f_j, such that $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x) \to f$ pointwise except at the origin. By Lemma 7 for each j, $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x) \geq \mathbb{1}_{\mathcal{B}}$. Thus $f \geq \mathbb{1}_{\mathcal{B}}$, and in particular, $f \neq 0$.

As f_j is radial symmetric decreasing, $T_{k,d}f_j(\theta, y)$ is as well. Let $g_j = T_{k,d}f_j(\theta, y)$. Applying Helly's selection principle there exists g such that some subsequence still denoted g_j satisfies $g_j \to g$ pointwise except at the origin. As $q \in (1, d+1]$, $p \in (1, \infty)$. Also $\|f_j\|_{L^p(\mathbb{R}^d)} = 1$, and so by (1.3), $\|g_j\|_{L^q(\mathcal{M}_{k,d})} = \|T_{k,d}f_j(\theta, y)\|_{L^q(\mathcal{M}_{k,d})} \leq A_0$. Thus each of these sequences converges weakly. Finally, T is a bounded linear operator for L^p to $L^q(\mathcal{M}_{k,d})$ and thus also is a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathcal{M}_{k,d})$ when each is endowed with the weak topology. Hence $g = T_{k,d}f$, and the conditions of Lemma 6 are satisfied.

□

3.3 Conclusion of the proof of Theorem 2 by Lieb's method

Proof. Recall that by Lieb's lemma (stated above as Lemma 6), it is enough to show that given an extremizing sequence f_j, after extracting a subsequence which we will continue to denote f_j, there exists a sequence $\sigma_j \in (0, \infty)$ such that the new extremizing sequence $|\sigma_j|^{-d/p} f_j(\sigma_j^{-1}x)$ converges pointwise to a nonzero function and that $T_{k,d} f_j \to T_{k,d} f$ pointwise.

Define $F(u) = e^{ud/p} f(e^u)$. Let c_d be the volume of the unit sphere in $\mathbb{R}^d$. Firstly, $\|F\|_{L^p(\mathbb{R})} = c_d^{1/p} \|f\|_{L^p(\mathbb{R}^d)}$. Secondly, $\|f\|_{L^{p,\infty}} = \sup_{t>0} c_d^{1/p} t^{d/p} f^*(t)$, and hence, $\|F\|_{L^\infty(\mathbb{R})} = c_d^{1/p} \|f\|_{L^{p,\infty}}$.

By Lemma 1, regarding the action of the k-plane on radial functions, there exists a decreasing function h defined on on $[0, \infty)$ such that $T_{k,d} f(\theta, y) = h(|y|)$. Set $H(v) = e^{v(d-k)/q} h(e^v)$. First, $\|H\|_{L^q(\mathbb{R})} = c_{n-k}^{1/q} \|T_{k,d} f(\theta, y)\|_{L^q(\mathcal{M}_{k,d})}$

By Lemma 1,

$$h(r) = \int_0^\infty f((s^2 + r^2)^{1/2}) s^{k-1} ds.$$

Thus,

$$H(v) = e^{v(d-k)/q} \int_0^\infty f((e^{2v} + s^2)^{1/2}) s^{k-1} ds.$$

Changing variables so that $e^u = (e^{2v} + s^2)^{1/2}$, yields

$$H(v) = e^{v(d-k)/q} \int_v^\infty f(e^u)(e^{2u} - e^{2v})^{\frac{k-2}{2}} e^{2u} du.$$

Using that $F(u) = e^{ud/p} f(e^u)$

$$H(v) = \int_v^\infty F(u)(e^{2u} - e^{2v})^{\frac{k-2}{2}} e^{(2-\frac{n}{p})u} e^{v(d-k)/q} du.$$

Now $p = \frac{nq}{n-q+kq}$, thus $2 - \frac{n}{p} = 2 - k - \frac{n-k}{q}$ and

$$H(v) = \int_v^\infty F(u)(1 - e^{2(v-u)})^{\frac{k-2}{2}} e^{(v-u)(d-k)/q} du.$$

Define $L_{d,k} = (1 - e^{2u})^{\frac{k-2}{2}} e^{u(d-k)/q} \mathbb{1}_{(-\infty,0]}$. Then,

$$H = F * L_{d,k}.$$

Now, $L_{d,k}$ is an integrable function provided that $(k-2)/2 > -1$ and $(d-k)/q > 0$ which hold in all cases that we consider. Thus by Young's inequality,

$$\|H\|_{L^\infty(\mathbb{R})} \leq \|F\|_{L^\infty(\mathbb{R})} \|L_{d,k}\|_{L^1(\mathbb{R})} \tag{3.3}$$

Let f_j be an extremizing sequence for (3.1). Thus $\|f_j\|_{L^p} = 1$ and f_j is radial symmetric decreasing.

Define $a_j = \sup_{r \in [0,\infty)} r^{d/p} f_j(r) = c{-}1/d \|f_j\|_{L^{p,\infty}}$. Note that a_j is invariant under dilations which preserve the L^p norm and $a_j \leq c_d$ where c_d is the volume of the unit sphere in $\mathbb{R}^d$, because

$$1 = c_d \int_0^\infty r^{n-1} f_j(r)^p \, dr \geq c_d \int_0^R r^{n-1} f_j(r)^p \geq c_d R^n f_j(R)^p.$$

By Lemma 9 and equation (3.2), $\|T_{k,d}f_j\|_{L^q(\mathcal{M}_{k,d})} \leq C\|f_j\|_{L^p}^{p/q}\|f_j\|_{L^{p,\infty}}^{1-p/q}$. Thus, if f_j is an extremizing sequence, and $\|f_j\|_{L^p} = 1$ it cannot be that the a_j tend to zero.

Choose $\alpha > 0$, such that for all j, $a_j > 2\alpha$. Next choose σ_j such that $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}) > \alpha$. For all $|x| \leq 1$, $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x) > \alpha$ as f_j is a radial symmetric decreasing function.

Applying Helly's selection principle, after extracting a subsequence if necessary, the sequence $|\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x)$ converges pointwise except perhaps at zero to a function f. By pointwise convergence, for all $x \neq 0$, $|x| \leq 1$, $f(x) > \alpha$, and in particular f is nonzero.

Let $g_j(x) = |\sigma_j|^{-d/p}f_j(\sigma_j^{-1}x)$ and $G_j(u) = e^{ud/p}g_j(e^u)$. Now $\|G_j\|_\infty \leq c_d$ and the G_j converge pointwise to $G = e^{ud/p}f(e^u)$. Thus, by dominated convergence $L_{d,k} * (G_j)$ converges pointwise to $L_{d,k} * G$, and hence $T_{k,d}g_j$ converges pointwise to $T_{k,d}f$ as required. $\square$

Chapter 4

Uniqueness of extremizers in the endpoint case

4.1 Main result and methods

The main result of this chapter is:

Theorem 3. *$f \in L^{(d+1)/(k+1)}(\mathbb{R}^d)$ is an extremizer of the inequality* (1.2) *if and only if*

$$f(x) = c(1 + |\phi(x)|^2)^{-(k+1)/2}$$

for some $c \in \mathbb{C} - \{0\}$ and some invertible affine endomorphism, ϕ, of $\mathbb{R}^d$.

The proof of Theorem 3 has two main steps. The first, done by Drouot in [17], is to show that extremizers exist and that $f = c(1+|x|^2)^{-(k+1)/2}$ is a radial nonincreasing extremizer. Drouot further proved the conditional result that if every extremizer of (1.2) has the form $f \circ \phi$ for f a radial nonincreasing extremizer and ϕ an affine map, then all extremizers have the the form required in Theorem 3. This paper concerns the second step, showing that the conditional step holds.

Proposition 2. *For any nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of* (1.2) *there exists ϕ an invertible affine transformation of $\mathbb{R}^d$, such that $f = F \circ \phi$ for F some radial nonincreasing function $F : \mathbb{R}^d \to [0, \infty)$.*

The proof of Proposition 2 is modeled on that of Christ in [12], which treats the Radon transform case. The basic idea is to show that the superlevel sets of any extremizer are homothetic ellipsoids. As radial function are precisely those functions whose superlevel sets are spheres, functions which are radial after composition with an affine map are those functions whose superlevel sets are homothetic ellipsoids. This is done by applying a theorem of Burchard [7],[8] characterizing m-tuples of characteristic functions which extremize a certain class of multilinear rearrangement inequalities as homothetic ellipsoids.

The change in dimension from that treated by [12] presents two difficulties. First, the result in [8],[7] applies directly in the case of the Radon transform, but this result must be adapted before it applies for the k-plane transform case. This is dealt with in Section 4.4.

More crucially, while in the case of the Radon transform it was known before [12] that extremizers of the endpoint inequality are smooth, in the general case they are not yet even known to be continuous. When setting ourselves up to use Burchard's theorem we are forced to work with cross-sections of the superlevel sets, which are, firstly, measure zero sets, and, secondly, have to ”line up correctly” so that they ”stack back up” into superlevel sets which are themselves homothetic ellipses. We adapt the methods of [12] handle extremizers which are only assumed to be measurable L^p functions.

4.2 Proof of Theorem 3 from Proposition 2

In [17], Drouot proves the following conditional result which is the starting place for our argument.

Theorem 4 (Drouot, [17]). *Let $1 \leq k \leq d-1$. Assume that any extremizer $f \in L^p(\mathbb{R}^d)$ for the k-plane transform inequality* (1.2) *can be written $f \circ \phi$ with f a radial nonincreasing extremizer and ϕ an invertible affine map. Then any extremizer can be written*

$$f = c(1 + |\phi(x)|^2)^{-(k+1)/2}$$

with $c \in \mathbb{C}$ and ϕ an invertible affine map.

Proof of Theorem 3 from Proposition 2. It is easy to see that if $f \in L^p(\mathbb{R}^d)$ is an extremizer of (1.2) then $f = c|f|$ for some $c \in \mathbb{C} - \{0\}$, thus it suffices to consider nonnegative extremizers. By Proposition 1, the conditions of Drouot's theorem are satisfied for all nonnegative functions, and thus any extremizer can be written $f = c(1 + |\phi(x)|^2)^{-(k+1)/2}$ for some $c \in \mathbb{C}$ and ϕ an invertible affine map. That any such function is an extremizer follows as $f = c(1 + |x|^2)^{-(k+1)/2}$ is an extremizer, and invertible affine maps are symmetries of (1.2) ([17]). □

4.3 Direct Symmetrization

Following Christ's proof in [12], we begin by rewriting $\|T_{k,d}f\|_{L^q}$ as a multilinear form to which we may apply symmetrization results.

Lemma 10 (Drury's Identity, [21]). *Let $f \in L^p(\mathbb{R}^d)$ be a nonnegative function. There exists $C \in \mathbb{R}_+$ depending only on d and k such that*

$$\|T_{k,d}f\|^q_{L^q(\mathcal{M}_{k,d})}$$
$$= C \int \prod_{i=0}^{k} f(x_i) \left(\prod_{i=k+1}^{d} \int_{\pi(x_0,\ldots,x_k)} f(x_i)\, d\sigma \right) det^{(k-d)}(x_0, \ldots, x_k)\, dx_0 \ldots dx_k$$

where det$^{(k-d)}(x_0,\dots,x_k)$ is the k-dimensional volume of the simplex determined by $x_0,\dots,x_k$ in $\mathbb{R}^d$ raised to the power $(k-d)$ and $d\sigma$ is the surface measure on $\pi(x_0,\dots,x_k)$.

Next reorganize Drury's identity, separating $\mathbb{R}^d$ into $\mathbb{R}^k\times\mathbb{R}^{d-k}$ with coordinates $x'\in\mathbb{R}^k$ and $v\in\mathbb{R}^{d-k}$.

Lemma 11. *Let $f\in L^p(\mathbb{R}^d)$ be a nonnegative function. There exists $C\in\mathbb{R}_+$ depending only on d and k such that*

$$\|T_{k,d}f\|^q_{L^q(\mathcal{M}_{k,d})} = C\int_{\mathbb{R}^{(d-k)(k+1)}}\Delta^{(k-d)}(x'_0,\dots,x'_k)\int_{\mathbb{R}^{k(n+1)}}\prod_{i=0}^k f(x'_i,v_i)\prod_{i=k+1}^d f(x'_i,\sum_{j=0}^k b_{i,j}v_j)\ dv_0\dots dv_k dx'_0\dots dx'_d$$

where $b_{i,j}$ are certain measurable real-valued functions of $x'_0,\dots,x'_k$, x'_i, i and j.

Proof. This is essentially a change of coordinates. Let $x_i=(x'_i,v_i)$ for $i\in[0,d]$. Take x'_i to be an independent variable in $\mathbb{R}^k$ for each $i\in[0,d]$, and take v_i to be an independent variable in $\mathbb{R}^{(d-k)}$ for $i\in[0,k]$. Then for $i\in[k+1,d]$, v_i will be determined by $x'_0,\dots,x'_k$, x'_i, and $v_0,\dots,v_k$ so that for $i\in[k+1,d]$, each (x'_i,v_i) lies in the k-plane spanned by $\{(x'_i,v_i)\}_{i=0}^k$. Specifically, let $A:\mathbb{R}^k\to\mathbb{R}^{d-k}$ be the unique affine map determined by $(k+1)$-tuple of equations $\{A(x'_i)=v_i\}_{i=0}^k$. Then for $i\in[k+1,d]$, set $v_i=A(x'_i)$.

Our goal is to express $d\sigma$ in terms of dx'_i for $i\in[k+1,d]$. The parameterization above of $\pi(x_0,\dots,x_k)$ takes the k-simplex in $\mathbb{R}^k$ spanned by $(x'_0,\dots,x'_k)$ which has volume $\Delta(x'_0,\dots,x'_k)$ to the k-simplex in $\mathbb{R}^d$ spanned by $(x_0,\dots,x_k)$ which has volume $\det(x_0,\dots,x_k)$. Therefore, for each x_i with $i\in[k+1,d]$, $d\sigma(x_i)=\frac{\det(x_0,\dots,x_k)}{\Delta(x'_0,\dots,x'_k)}dx'_i$. As $n-k$ terms of this type appear in Drury's identity, the $\det(x_0,\dots,x_k)$ terms cancel leaving

$$\|T_{k,d}f\|^{d+1}_{L^{n+1}(\mathcal{M}_{k,d})} = C\iint\prod_{i=0}^k f(x'_i,v_i)\prod_{i=k+1}^d f(x'_i,A(x'_i))\Delta^{(k-d)}(x'_0,\dots,x'_k)\ dv_1\dots dv_k dx'_0\dots dx'_d.$$

Finally, a computation by Cramer's rule shows that for $i\in[k+1,d]$, $A(x'_i)=\sum_{j=0}^k b_{i,j}v_j$ for coefficients $b_{i,j}$ given by

$$b_{i,j}=\frac{\Delta(x'_0,\dots,x'_{j-1},x'_i,x'_{j+1},\dots,x'_k)}{\Delta(x'_0,\dots,x'_k)}. \tag{4.1}$$

The formula (4.1) gives $b_{i,j}=\delta_{i,j}$ if $0\le i\le k$. Define $b_{i,j}=\delta_{i,j}$ for all $0\le i\le k$. □

The inner integral in Lemma 11 becomes

$$\int\prod_{i=0}^k f(x'_i,v_i)\prod_{i=k+1}^d f(x'_i,\sum_{j=0}^k b_{i,j}v_j)\ dv_0\dots dv_k=\int\prod_{i=0}^d f_{x'_i}(\sum_{j=0}^k b_{i,j}v_j)dv_0\dots dv_k.$$

Definition 5. *For $b_{i,j}$ with $i \in [0,d]$ and $j \in [0,k]$ depending on $(x'_0,\ldots,x'_d)$, given by* (4.1)*, and $F_i : \mathbb{R}^{d-k} \to \mathbb{R}$ for all $i \in [0,d]$, let $\mathcal{T}_{x'_0,\ldots,x'_d}$ denote the operator given by*

$$\mathcal{T}_{x'_0,\ldots,x'_d}(F_0,\ldots,F_d) = \int \prod_{i=0}^{d} F_i(\sum_{j=0}^{k} b_{i,j}v_j)dv_0\ldots dv_k.$$

As the $b_{i,j}$ are real valued $\mathcal{T}_{x'_0,\ldots,x'_d}$ is precisely the type of multilinear form addressed by Brascamp, Lieb, and Luttinger's rearrangement inequality:

Theorem 5 (Brascamp, Lieb, and Luttinger's rearrangement inequality, [6])**.** *Let $f_i(x)$ for $1 \le i \le m$ be nonnegative measurable functions on $\mathbb{R}^d$, and let $a_{i,j}$ for $1 \le i \le m$ and $1 \le j \le k$ be real numbers. Then*

$$\int_{\mathbb{R}^{nk}} \prod_{i=1}^{m} f_i(\sum_{j=1}^{k} a_{i,j}x_j)\ dx_1\ldots dx_k \le \int_{\mathbb{R}^{nk}} \prod_{i=1}^{m} f_i^*(\sum_{j=1}^{k} a_{i,j}x_j)\ dx_1\ldots dx_k.$$

Thus,

$$\mathcal{T}_{x'_0,\ldots,x'_d}(F_0,\ldots,F_d) \le \mathcal{T}_{x'_0,\ldots,x'_d}(F_0^*,\ldots,F_d^*). \tag{4.2}$$

Moreover,

Lemma 12. *For every nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of* (1.2) *and every symmetry $\mathcal{J}$ of* (1.2)*, for almost every $x'_0,\ldots,x'_d$*

$$\mathcal{T}_{x'_0,\ldots,x'_d}(\mathcal{J}(f)_{x'_0},\ldots,\mathcal{J}(f)_{x'_d}) = \mathcal{T}_{x'_0,\ldots,x'_d}(\mathcal{J}(f)^*_{x'_0},\ldots,\mathcal{J}(f)^*_{x'_d}).$$

Proof. As $\mathcal{J}$ is a symmetry of (1.2) , $\mathcal{J}(f)$ is an extremizer of (1.2) , hence it suffices to consider $\mathcal{J}$ the identity transformation on $L^p(\mathbb{R}^d)$. Multiplying both sides of (4.2) by $\Delta(x'_0,\ldots,x'_k)^{(k-d)}$ gives

$$\Delta(x'_0,\ldots,x'_k)^{(k-d)}\mathcal{T}_{x'_0,\ldots,x'_d}(f_{x'_0},\ldots,f_{x'_d}) \le \Delta(x'_0,\ldots,x'_k)^{(k-d)}\mathcal{T}_{x'_0,\ldots,x'_d}(f^*_{x'_0},\ldots,f^*_{x'_d}). \tag{4.3}$$

Let $f^\sharp(x,v) = f^*_x(v)$. Then integrating in each x'_i shows

$$\|T_{k,d}f\|^q_{L^q(\mathcal{M}_{k,d})} \le \|T_{k,d}f^\sharp\|^q_{L^q(\mathcal{M}_{k,d})}. \tag{4.4}$$

Since f is an extremizer, there is equality in (4.4). Hence, there is equality in (4.3) for almost every $x'_0,\ldots,x'_k$. Multiplying by $\Delta(x'_0,\ldots,x'_k)^{(d-k)}$, which is nonzero for almost every $x'_0,\ldots,x'_k$, proves the proposition. □

We further reduce to the case where $\mathcal{T}_{x'_0,\ldots,x'_d}$ is applied to characteristic functions of superlevel sets of extremizers. This requires the layer cake decomposition of a function.

Proposition 3 (Layer cake decomposition (see for instance [31]))**.** *If f is a nonnegative measurable function, then*

$$f(x) = \int_0^\infty \mathbb{1}_{\{f(x)>t\}}(x)dt.$$

To implement this reduction we will need a proposition parallel to Lemma 12 for superlevel sets.

Proposition 4. *For every nonnegative extremizer f of* (1.2) *, for almost every $x_0', \dots, x_d'$ and almost every $s_0, \dots, s_d$,*

$$\mathcal{T}_{x_0',\dots,x_d'}(E(x_0', s_0), \dots, E(x_d', s_d)) = \mathcal{T}_{x_0',\dots,x_d'}(E(x_0', s_0)^*, \dots, E(x_d', s_d)^*) \tag{4.5}$$

where $E(x_i', s_i)$ is shorthand for $\mathbb{1}_{E(x_i', s_i)}$.

Proof. Applying the layer cake decomposition to each $F_{x_i'}$,

$$\mathcal{T}_{x_0',\dots,x_d'}(F_{x_0'}, \dots, F_{x_d'}) = \int_{(0,\infty)^{n+1}} \int_{(\mathbb{R}^{d-k})^{k+1}} \prod_{i=0}^{d} \mathbb{1}_{E(x_i',s_i)}\left(\sum_{j=0}^{k} b_{i,j} v_j\right) \prod_{l=0}^{k} dv_l \prod_{m=0}^{d} ds_m$$

Similarly,

$$\mathcal{T}_{x_0',\dots,x_d'}(F^*_{x_0'}, \dots, F^*_{x_d'}) = \int_{(0,\infty)^{n+1}} \int_{(\mathbb{R}^{d-k})^{k+1}} \prod_{i=0}^{d} \mathbb{1}_{E^*(x_i',s_i)}\left(\sum_{j=0}^{k} b_{i,j} v_j\right) \prod_{l=0}^{k} dv_l \prod_{m=0}^{d} ds_m.$$

Again by the result of Brascamp, Lieb, and Luttinger in [6],

$$\int_{(\mathbb{R}^{d-k})^{k+1}} \prod_{i=0}^{d} \mathbb{1}_{E(x_i',s_i)}\left(\sum_{j=0}^{k} b_{i,j} v_j\right) \prod_{l=0}^{k} dv_l \le \int_{(\mathbb{R}^{d-k})^{k+1}} \prod_{i=0}^{d} \mathbb{1}_{E^*(x_i',s_i)}\left(\sum_{j=0}^{k} b_{i,j} v_j\right) \prod_{l=0}^{k} dv_l$$

Integrating in s_i gives

$$\mathcal{T}_{x_0',\dots,x_d'}(f_{x_0'}, \dots, f_{x_d'}) \le \mathcal{T}_{x_0',\dots,x_d'}(f^*_{x_0'}, \dots, f^*_{x_d'}).$$

As equality holds here for almost every $x_0', \dots, x_d'$ and the product of characteristic functions is nonnegative, equality must hold in (4.5) for almost every $x_0', \dots, x_d'$, for almost every $s_0, \dots, s_d$. □

4.4 Inverse symmetrization for superlevel sets

In [12], Christ performs a change of variables and applies Burchard's Theorem ([8],[7]) to conclude that the superlevel sets of the f_{x_i} are intervals. Here, because of the change in the relationship between the dimension and the number of functions, the result does not apply directly. Before applying Burchard's Theorem ([8],[7]), we must first show that the extra $n - k$ functions are redundant given a modified admissibility condition and then apply a change of variables so that the functions, rather than the functional, depend on $b_{i,j}$.

Definition 6. *A set of positive numbers* $\{\rho_i\}_{i=0}^d$ *is permissible with respect to* $(x'_0, \ldots, x'_d)$ *if:*

$$\sum_{\substack{j=0 \\ j\neq i}}^{k+1} |b_{(k+1),j}|\rho_j > |b_{(k+1),i}|\rho_i \quad \textit{for all } i \in [0, k+1] \tag{4.6}$$

$$\sum_{j=0}^{k} |b_{i,j}|\rho_j < \rho_i \quad \textit{for all } i \in [k+2, d] \tag{4.7}$$

where the $b_{i,j}$ *are determined for* $i \in [0, d]$ *and* $j \in [0, k]$ *by* $x'_0, \ldots, x'_d$ *according to* (4.1) *and* $b_{(k+1),(k+1)} = 1$.

Lemma 13. *For* $i \in [0, d]$ *let* $E_i \subset \mathbb{R}^{d-k}$ *be a set of finite positive measure. Let* ρ_i *be the radius of* E_i^**. If the set* $\{\rho_i\}_{i=0}^d$ *is permissible with respect to* $(x'_0, \ldots, x'_d)$ *and* $\mathcal{T}_{x'_0,\ldots,x'_d}(E_0, \ldots, E_d) = \mathcal{T}_{x'_0,\ldots,x'_d}(E_0^*, \ldots, E_d^*)$ *then*

$$\mathcal{T}_{x'_0,\ldots,x'_d}(E_0, \ldots, E_d) = \mathcal{T}_{x'_0,\ldots,x'_d}(E_0, \ldots, E_{k+1}, \mathbb{R}, \ldots, \mathbb{R})$$

and

$$\mathcal{T}_{x'_0,\ldots,x'_d}(E_0, \ldots, E_{k+1}, \mathbb{R}, \ldots, \mathbb{R}) = \mathcal{T}_{x'_0,\ldots,x'_d}(E_0^*, \ldots, E_{k+1}^*, \mathbb{R}, \ldots, \mathbb{R})$$

Proof. By definition $\mathcal{T}_{x'_0,\ldots,x'_d}(E_0^*, \ldots, E_d^*) = \int \prod_{i=0}^d \mathbb{1}_{E_i^*}(\sum_{j=0}^k b_{i,j}v_j)dv_0 \ldots dv_k$. Recall that $b_{i,j} = \delta_{i,j}$ if $i, j \in [0, k]$. Consider

$$\prod_{i=0}^{k} \mathbb{1}_{E_i^*}(v_i)\mathbb{1}_{E_l^*}(\sum_{j=0}^{k} b_{l,j}v_j).$$

For $l \in [k+2, d]$, from the definition of permissibility (4.7),

$$\rho_l > \sum_{j=0}^{k} |b_{l,j}|\rho_j.$$

As ρ_j is the radius of the open ball E_j^* which is centered at the origin, it follows that for any choice of vectors $v_j \in E_j^*$,

$$\sum_{j=0}^{k} |b_{l,j}|v_j \in E_l^*.$$

Therefore,

$$\prod_{i=0}^{k} \mathbb{1}_{E_i^*}(v_i)\mathbb{1}_{E_l^*}(\sum_{j=0}^{k} b_{l,j}v_j) = \prod_{i=0}^{k} \mathbb{1}_{E_i^*}(v_i).$$

Because this holds for every $l \in [k+2, d]$,

$$\prod_{i=0}^{k} \mathbb{1}_{E_i^*}(v_i) \prod_{l=k+2}^{d} \mathbb{1}_{E_l^*}(\sum_{j=0}^{k} b_{l,j}v_j) = \prod_{i=0}^{k} \mathbb{1}_{E_i^*}(v_i).$$

Multiplying by $\mathbb{1}_{E^*_{k+1}}(\sum_{j=0}^k b_{l,j}v_j)$ yields,

$$\prod_{l=1}^{d} \mathbb{1}_{E_l^*}(\sum_{j=0}^{k} b_{l,j}v_j) = \prod_{i=0}^{k+1} \mathbb{1}_{E_i^*}(v_i).$$

Multiply the right hand side by one in the form $\prod_{l=k+2}^d \mathbb{1}_{\mathbb{R}}(\sum_{j=0}^k b_{l,j}v_j)$ and integrate in v_j for $j \in [0,k]$ to obtain

$$\mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\dots,E_d^*) = \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\dots,E_{k+1}^*,\mathbb{R},\dots,\mathbb{R}). \tag{4.8}$$

Now

$$\prod_{i=0}^{d} \mathbb{1}_{E_i}(\sum_{j=0}^{k} b_{i,j}v_j) \le \prod_{i=0}^{k+1} \mathbb{1}_{E_i}(\sum_{j=0}^{k} b_{i,j}v_j)$$

because each term in the product is a characteristic function. Hence

$$\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_d) \le \mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_{k+1},\mathbb{R},\dots,\mathbb{R}).$$

Combining this with (4.8) and the fact that $\mathcal{T}_{x'_0,\dots,x'_d}$ satisfies rearrangement inequalities yields

$$\begin{aligned}\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_d) &\le \mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_{k+1},\mathbb{R},\dots,\mathbb{R})\\ &\le \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\dots,E_{k+1}^*,\mathbb{R},\dots,\mathbb{R}) = \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\dots,E_d^*)\end{aligned}$$

Since by assumption $\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_d) = \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\dots,E_d^*)$ equality must hold at every step. □

Theorem 6 (An adaption of Burchard's theorem for indicator functions)**.** *Let E_i be sets of finite positive measure in $\mathbb{R}^{d-k}$ for $i \in [0,d]$. Denote by ρ_i the radius of E_i^*. If the family ρ_i is permissible with respect to $(x'_0,\dots,x'_d)$ and*

$$\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\cdots,E_d) = \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\cdots,E_d^*)$$

then for each $i \in [0,k+1]$ there exist vectors $\beta_i \in \mathbb{R}^{d-k}$ and numbers $\alpha_i \in \mathbb{R}_+$ such that $\sum_{i=0}^k \beta_i = \beta_{k+1}$, and there exists an ellipsoid $\mathcal{E}$ which is centered at the origin and independent of i such that, up to null sets,

$$b_{(k+1),i}E_i = \beta_i + \alpha_i\mathcal{E}$$

where the $b_{i,j}$ are determined for $i \in [0,d]$ and $j \in [0,k]$ by $x'_0,\dots,x'_d$ according to (4.1) *and* $b_{(k+1),(k+1)} = 1$.

Proof. By Lemma 13, $\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\cdots,E_{k+1},\mathbb{R},\dots,\mathbb{R}) = \mathcal{T}_{x'_0,\dots,x'_d}(E_0^*,\cdots,E_{k+1}^*,\mathbb{R},\dots,\mathbb{R})$. Set $y_0 = b_{(k+1),0}v_0$ and $y_i = -b_{(k+1),i}v_i$ for $i \in [1,k]$. Recall that $b_{i,j} = \delta_{i,j}$ if $i,j \in [0,k]$.

$$\begin{aligned}\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_{k+1},\mathbb{R},\dots,\mathbb{R}) &= \int \prod_{i=0}^{k} \mathbb{1}_{E_i}(v_i)\mathbb{1}_{E_{k+1}}(\sum_{j=0}^{k} b_{(k+1),j}v_j)dv_0\dots dv_k\\ &= c\int \prod_{i=0}^{k} \mathbb{1}_{E_i}(b_{(k+1),i}^{-1}y_i)\mathbb{1}_{E_{k+1}}(y_0 - \sum_{j=1}^{k} y_j)dy_0\dots dy_k.\end{aligned}$$

Therefore,

$$\mathcal{T}_{x'_0,\dots,x'_d}(E_0,\dots,E_{k+1},\mathbb{R},\dots,\mathbb{R}) = c\mathcal{I}(E_{k+1}, b_{(k+1),0}E_0,\dots,b_{(k+1),k}E_k).$$

The permissibility condition (4.6) is precisely the requirement that the radii of $\{b_{(k+1),i}E_i^*\}_{i=0}^k \bigcup E_{k+1}^*$ are strictly admissible. Thus as the family ρ_i is permissible with respect to $(x'_0,\dots,x'_d)$, Burchard's Theorem applied to $\{b_{(k+1),i}E_i^*\}_{i=0}^k \bigcup E_{k+1}^*$ gives the result. □

4.5 Identifying $(d-k)$-cross sections of superlevel sets

Definition 7. *To each nonnegative extremizer f of* (1.2) *, associate a function $\rho(x', s)$ which is the radius of the ball $E^*(x', s)$.*

In this section we show that almost every $(d-k)$-cross section of almost every superlevel set is, up to a null set, an ellipsoid. The main step is to show that each such set of positive measure can be associated to an $(d+1)$-tuple of sets to which Burchard's theorem in the form of Theorem 6 may be applied. We construct such $(d+1)$-tuples predominantly following the proof of Lemma 5.4 in [12]. Our proof differs in that it is not yet known that extremizers are continuous, so we will rely on Lebesgue points of the function $\rho(x', s)$. The goal is:

Proposition 5. *Let f be any nonnegative extremizer of* (1.2) *. For almost every $x' \in \mathbb{R}^k$, for almost every $s \in \mathbb{R}_+$ the set $E(x', s)$ differs from an ellipsoid by a null set.*

Before we wade into the proof we need a few technical lemmas.

Lemma 14. *For every nonnegative extremizer f of* (1.2) *the associated function $\rho(x', s)$ is in $L^1_{loc}(\mathbb{R}^k \times \mathbb{R}_+)$. In particular, almost every $(x', s) \in \mathbb{R}^k \times \mathbb{R}_+$ is a Lebesgue point of the function $(x', s) \to \rho(x', s)$.*

Proof. This is a direct consequence of the definition, the observation that $f \in L^p(\mathbb{R}^d)$, and Fubini's theorem. □

Lemma 15. *Any nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of* (1.2) *satisfies $f(x) > 0$ for almost every $x \in \mathbb{R}^d$.*

The proof is deferred to the last section of the paper.

Lemma 16. *Let $\{u_i\}_{i=0}^k$ be a set of pairwise-distinct unit vectors such that the volume of the simplex with vertices $0, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k$ is independent of the choice of j. Let $\tau > 0$. If $x'_i = x'_{k+1} + \tau u_i$ for $i \in [0, k]$, then for all $j \in [0, k]$ $b_{(k+1),j} = \frac{1}{k+1}$.*

Proof. Note that such u_i exist in every dimension, take the vertices of a regular triangle, tetrahedron, etc. Take u_i and τ as in the statement of the lemma. Set $x'_i = x'_{k+1} + \tau u_i$ for $i \in [0, k]$. By choice of u_i each of volumes $\Delta(x'_0, \dots, x'_{j-1}, x'_{k+1}, x'_{j+1}, \dots, x'_k)$ are equal. Plugging this into (4.1), the definition of $b_{i,j}$, produces $b_{(k+1),j} = \frac{1}{k+1}$ for $j \in [0, k]$. □

Proof of Proposition 5 . Fix any $x'_{k+1} \in \mathbb{R}^k$ such that $f_{x'_{k+1}}$ is in $L^p(\mathbb{R}^{d-k})$, $f_{x'_{k+1}}$ is positive almost everywhere and for almost every $s \in \mathbb{R}_+$, (x'_{k+1}, s) is a Lebesgue point of the function $(x', s) \to \rho(x', s)$. Almost every $x' \in \mathbb{R}^k$ satisfies all of these conditions: the first because $f \in L^p(\mathbb{R}^d)$, the second by Lemma 15, and the third by Lemma 14. Consider $s_{k+1} \in \mathbb{R}_+$. Either $\rho(x'_{k+1}, s_{k+1}) > 0$ or $\rho(x'_{k+1}, s_{k+1}) = 0$. In the latter case, $|E(x'_{k+1}, s_{k+1})| = 0$ and the conclusion of Proposition 5 is vacuously true. Hence it suffices to consider s_{k+1} such that $\rho(x'_{k+1}, s_{k+1}) > 0$. Fix some such $s_{k+1} \in \mathbb{R}_+$.

Our goal is to construct a family of sets $\{S_i \subset (\mathbb{R}^k, \mathbb{R}_+) : k+1 \neq i \in [0, d]\}$ depending on (x'_{k+1}, s_{k+1}) satisfying two conditions: first $|S_i| > 0$ for $i \in [0, d] \setminus \{k+1\}$ and second, if for each $i \in [0, d] \setminus \{k+1\}$, $(x'_i, s_i) \in S_i$ then $\{\rho(x'_i, s_i)\}_{i=0}^d$ is permissible with respect to $(x'_0, \ldots, x'_d)$. Proposition 4 guarantees that for almost every $(x_{k+1}, s_{k+1}) \in \mathbb{R}^k \times \mathbb{R}_+$ for which such a family exists, for almost every $(x'_0, s_0), \ldots, (x'_k, s_k), (x'_{k+2}, s_{k+2}) \ldots, (x'_d, s_d)$ equality in (4.5) holds in addition to permissibility. Applying Burchard's Theorem 6 for superlevel to the sets $E(x'_0, s_0), \ldots, E(x'_d, s_d)$ for which both equality and permissibility hold, produces the desired conclusion.

The first permissibility condition (4.6) doesn't depend on ρ_i for $i \in [k+2, d]$. Thus we begin by constructing $\{S_i\}_{i=0}^k$ such that if $(x'_i, s_i) \in S_i$ then $\{\rho(x'_i, s_i)\}_{i=0}^{k+1}$ satisfies (4.6).

Choose $r_{k+1} \in (0, s_{k+1})$ such that (x'_{k+1}, r_{k+1}) is a Lebesgue point of the function $(x', s) \to \rho(x', s)$ and $\rho(x'_{k+1}, r_{k+1}) > \rho(x'_{k+1}, s_{k+1})$. The first condition holds for almost every $r_{k+1} > 0$ by choice of x'_{k+1}. The second must be satisfied by some r_{k+1} as $f_{x'_{k+1}}(v) \in L^p(\mathbb{R}^{d-k})$ is almost everywhere positive, and thus larger superlevel sets always exist.

Our strategy for constructing the family $\{S_i\}_{i=0}^k$ will be to find sets with positive measure of $(x', s) \in \mathbb{R}^k \times \mathbb{R}_+$ such that each $\rho(x', s)$ is approximately $\rho(x'_{k+1}, r_{k+1})$ and the $b_{(k+1),j}(x'_0, \ldots, x'_{k+1})$ for $j \in [0, k]$ are approximately equal to one another.

Fix $\epsilon_\rho > 0$ such that $4\epsilon_\rho < \min\left(\rho(x'_{k+1}, r_{k+1}) - \rho(x'_{k+1}, s_{k+1}), \rho(x'_{k+1}, s_{k+1})\right)$. This will be the tolerance in the size of superlevel sets.

Let $\mathcal{B}(\delta_\rho) = B(x'_{k+1}, \delta_\rho) \times B(r_{k+1}, \delta_\rho)$. Since (x'_{k+1}, r_{k+1}) is a Lebesgue point of the function $(x', s) \to \rho(x', s)$, there is a $\delta_\rho > 0$ such that

$$\frac{1}{|\mathcal{B}(\delta_\rho)|} \int_{\mathcal{B}(\delta_\rho)} |\rho(x', s) - \rho(x'_{k+1}, r_{k+1})| dx' ds < \frac{\epsilon_\rho}{2(k+1)}.$$

Hence,

$$\left| \left\{ (x', s) : |\rho(x', s) - \rho(x'_{k+1}, r_{k+1})| > \frac{\epsilon_\rho}{2} \right\} \bigcap \mathcal{B}(\delta_\rho) \right| < \frac{|\mathcal{B}(\delta_\rho)|}{k+1}.$$

Therefore, by the pigeonhole principle, it is possible to choose $\{u_i\}_{i=0}^k$ as in the statement of Lemma 16, $\tau \in (0, \delta_\rho)$, and $r_i \in B(r_{k+1}, \delta_\rho)$ for $i \in [0, k]$ such that $z'_i = x'_{k+1} + \tau u_i$ satisfy $|\rho(z'_i, s_i) - \rho(z'_{k+1}, r_{k+1})| < \epsilon_\rho/2$ and each of the (z'_i, r_i) are in turn Lebesgue points of $(x', s) \to \rho(x', s)$. Note that as computed in Lemma 16 for $j \in [0, k]$, $b_{(k+1),j}(z'_0, \ldots, z'_k, x'_{k+1}) = \frac{1}{k+1}$. Direct computation verifies that $\{\rho(z'_i, r_i)\}_{i=0}^k \cup \{\rho(x'_{k+1}, s_{k+1})$ satisfy (4.6).

Fix $\epsilon_b < \epsilon_\rho \left((k+1)\rho(x'_{k+1}, r_{k+1})\right)^{-1}$. This will be the tolerance in the variation of the coefficients $b_{(k+1),j}$. For each $j \in [0, k]$ the function $b_{(k+1),j} : \mathbb{R}^{(d-k)(k+1)} \to \mathbb{R}$ is continuous, as it is a multilinear function of $\{x'_i\}_{i=0}^k$. Therefore, there exists $\delta_b > 0$ such that if $x'_i \in B(z'_i, \delta_b)$ for $i \in [0,k]$ and $y'_{k+1} \in B(x'_{k+1}, \delta_b)$, then $|b_{(k+1),j}(z'_0, \ldots, z'_k, x'_{k+1}) - b_{(k+1),j}(x'_0, \ldots, x'_k, y'_{k+1})| < \epsilon_b$.

Set $S_i = \{(x_i, s_i) : |\rho(x'_i, s_i) - \rho(x'_{k+1}, r_{k+1})| < \epsilon_\rho, x'_i \in B(z'_i, \delta_b)\}$, for $i \in [0,k]$. To see that for $i \in [0,k]$, $|S_i| > 0$ recall that each (z'_i, r_i) is a Lebesgue point of the function $(x', s) \to \rho(x', s)$. Thus, there exists a small radius $\delta \in (0, \delta_b)$ such that for all $i \in [0,k]$ the condition $|\rho(z'_i, r_i) - \rho(x'_i, s_i)| < \epsilon_\rho/2$ is satisfied by at least half of the (x'_i, s_i) such that $x'_i \in B(z'_i, \delta)$ and $|r_i - s_i| \leq \delta$. By the triangle inequality, such (x'_i, s_i) also satisfy $|\rho(x'_i, s_i) - \rho(x'_{k+1}, r_{k+1})| < \epsilon_\rho$.

We now verify that any $(k+1)$-tuple $(x'_i, s_i)_{i=0}^k$ such $(x'_i, s_i) \in S_i$ fulfills the permissibility condition.

By Lemma 16, $b_{(k+1),j}(z'_0, \ldots, z'_k, x'_{k+1}) = \frac{1}{k+1}$, therefore $b_{(k+1),j} = b_{(k+1),j}(x'_0, \ldots, x'_{k+1}) \in (\frac{1}{k+1} - \epsilon_b, \frac{1}{k+1} + \epsilon_b)$.

$$\sum_{j=0}^{k} |b_{(k+1),j}| \rho(x'_j, s_j) \geq (1 - (k+1)\epsilon_b)(\rho(x'_{k+1}, r_{k+1}) - \epsilon_\rho)$$
$$\geq \rho(x'_{k+1}, r_{k+1}) - (k+1)\epsilon_b \rho(x'_{k+1}, r_{k+1}) - \epsilon_\rho$$

As $\epsilon_b < \frac{\epsilon_\rho}{(k+1)\rho(x'_{k+1}, r_{k+1})}$ and $2\epsilon_\rho < \rho(x'_{k+1}, r_{k+1}) - \rho(x'_{k+1}, s_{k+1})$,

$$\sum_{j=0}^{k} |b_{(k+1),j}| \rho(x'_j, s_j) > \rho(x'_{k+1}, s_{k+1}).$$

Fix any $i \in [0,k]$, then

$$\sum_{\substack{j=0 \\ j \neq i}}^{k+1} |b_{(k+1),j}| \rho(x'_j, s_j) \geq \frac{k - k\epsilon_b}{k+1} \left(\rho(x'_{k+1}, r_{k+1}) - \epsilon_\rho \right) + \rho(x'_{k+1}, s_{k+1})$$
$$\geq \frac{k\rho(x'_{k+1}, r_{k+1})}{k+1} + \rho(x'_{k+1}, s_{k+1}) - \frac{k}{(k+1)}(\epsilon_\rho + \epsilon_b \rho(x'_{k+1}, r_{k+1})).$$

As $\epsilon_b < \frac{\epsilon_\rho}{(k+1)\rho(x'_{k+1}, r_{k+1})}$ and $\frac{k(k+2)}{(k+1)^2} < 1$,

$$\sum_{\substack{j=0 \\ j \neq i}}^{k+1} |b_{(k+1),j}| \rho(x'_j, s_j) \geq \frac{k\rho(x'_{k+1}, r_{k+1})}{k+1} + \rho(x'_{k+1}, s_{k+1}) - \epsilon_\rho \geq \frac{k\rho(x'_{k+1}, r_{k+1})}{k+1} + 3\epsilon_\rho$$

Additionally as $\epsilon_b \leq \frac{\epsilon_\rho}{(k+1)\rho(x'_{k+1}, r_{k+1})} < 1$, and $k \geq 1$,

$$|b_{(k+1),j}| \rho(x'_j, s_j) < (\frac{1}{k+1} + \epsilon_b)(\rho(x'_{k+1}, r_{k+1}) + \epsilon_\rho)$$
$$\leq \frac{\rho(x'_{k+1}, r_{k+1})}{k+1} + \frac{3}{k+1}\epsilon_\rho \leq \frac{\rho(x'_{k+1}, r_{k+1})}{k+1} + 2\epsilon_\rho.$$

Therefore

$$\sum_{\substack{j=0\\ j\neq i}}^{k+1} |b_{(k+1),j}|\rho(x'_j, s_j) > |b_{(k+1),i}|\rho(x'_j, s_j).$$

To prove the proposition, it remains to find a family $\{S_i\}_{i=k+2}^{d}$. Given the construction above, for $i \in [0,k]$ if $(x'_i, s_i) \in S_i$, $\rho(x'_i, s_i) < \rho(x'_{k+1}, r_{k+1}) + \epsilon_\rho$. Moreover, if $i \geq k+1$ then $b_{i,j}(x'_0, \dots, x'_d) = b_{(k+1),j}(x'_0, \dots, x'_k, x'_i)$. Therefore $|b_{i,j}(x'_0, \dots, x'_d) - b_{(k+1),j}(z'_0, \dots, z'_k, x_{k+1})| < \frac{1}{k+1} + \epsilon_b$. Hence, there exists $C' \in \mathbb{R}$ such that if $(x'_j, s_j) \in S_j$ for $j \in [0,k]$

$$\sum_{j=0}^{k} |b_{i,j}|\rho(x'_j, s_j) \leq C'.$$

For each $i \in [k+2, d]$, set $S_i = \{(x'_i, s_i) : x'_i \in B(x'_{k+1}, \delta) \text{ and } \rho(x'_i, s_i) > C'\}$. S_i for $i \in [k+2, d]$ has positive measure by positivity of the nonnegative extremizer f (see Lemma 15). Moreover, if $(x'_i, s_i) \in S_i$ for $i \in [0,d] \setminus \{k+1\}$, then

$$\sum_{j=0}^{k} |b_{i,j}|\rho(x'_j, s_j) \leq C' < \rho(x'_i, s_i)$$

and hence (4.7) is satisfied. □

4.6 Identifying $(d-k)$-cross sections part II: shared geometry

Thus far we have shown that almost all $(d-k)$-dimensional cross sections of the superlevel sets of extremizers are ellipsoids up to null sets. The next step is to show that these elliptical cross sections almost always have the same geometry, i.e., they are translations and dilations of a single ellipsoid in $\mathbb{R}^{d-k}$. Further, we show that the translations are given by an affine function.

We have not yet used the full strength of Burchard's theorem. Applying Theorem 6:

Lemma 17. *For every nonnegative extremizer f of* (1.2) *, for almost every $x' \in \mathbb{R}^k$, for almost every $s \in \mathbb{R}_+$, there exist an ellipsoid $\mathcal{E}(x') \subset \mathbb{R}^{d-k}$ centered at the origin, a vector $\gamma(x') \in \mathbb{R}^{d-k}$ and a number $\alpha(x', s) \in \mathbb{R}$ such that, up to a null set,*

$$E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}(x').$$

Proof. It is enough to prove the lemma for almost every $x' \in \mathbb{R}^{d-k}$, for almost any pair s and $\tilde{s}$ such that both $\rho(x'_{k+1}, s)$ and $\rho(x_{k+1}, \tilde{s})$ are nonzero.

Take $x'_{k+1} \in \mathbb{R}^{d-k}$ satisfying the conditions of the construction of Proposition 5. Apply the construction, with r_{k+1} chosen so that $\rho(x'_{k+1}, r_{k+1})$ is greater than both $\rho(x'_{k+1}, s)$ and $\rho(x'_{k+1}, \tilde{s})$. This produces a family of measurable sets $\{S_i \subset \mathbb{R}^{d-k} \times \mathbb{R}_+ : k+1 \neq i \in [0,d]\}$, each with positive measure, such that if $(x'_i, s_i) \in S_i$ then $\{\rho(x'_i, s_i)\}_{i=0}^{d}$ is permissible with respect to $(x'_0, \dots, x'_d)$

both for $s_{k+1} = s$ and for $s_{k+1} = \tilde{s}$. By Proposition 4 , for almost every $x'_{k+1} \in \mathbb{R}^k$ and almost every pair $(s, \tilde{s}) \in \mathbb{R}^2_+$, for almost every family $\{(x'_i, s_i) : k+1 \neq i \in [0, d]\}$ with $(x'_i, s_i) \in S_i$, the $(d+1)$-tuple of sets $\{E(x'_i, s_i)\}_{i=0}^d$ produces equality in equation (4.5), both for $s_{k+1} = s$ and for $s_{k+1} = \tilde{s}$.

For any $(d+1)$-tuple of sets $\{E(x'_i, s_i)\}_{i=0}^d$ which produces equality in equation (4.5) and is such that the set $\{\rho(x'_i, s_i)\}_{i=0}^d$ is permissible, Burchard's Theorem (Theorem 6) gives that for $i \in [0, k+1]$ there exist numbers $\alpha(x'_i, s_i) \in \mathbb{R}_+$, vectors $\beta(x'_i, s_i) \in \mathbb{R}^{d-k}$ satisfying $\sum_{i=0}^k \beta(x'_i, s_i) = \beta(x'_{k+1}, s_{k+1})$, and a fixed ellipsoid $\mathcal{E}(x'_i, s_i)$ which is centered at the origin and independent of i such that, up to null sets,

$$b_{(k+1),i} E(x'_i, s_i) = \beta(x'_i, s_i) + \alpha(x'_i, s_i)\mathcal{E}(x'_i, s_i).$$

Recall that $b_{(k+1),i}$ is given by (4.1) for $i \in [0, k]$ and $b_{(k+1),(k+1)} = 1$. As $\mathcal{E}(x'_i, s_i)$ is determined by $\{(x'_i, s_i)\}_{i=0}^k$, $\mathcal{E}(x'_{k+1}, s) = \mathcal{E}(x'_{k+1}, \tilde{s})$. Set $\mathcal{E}(x'_{k+1}) = \mathcal{E}(x'_{k+1}, \tilde{s})$. Similarly, $\beta(x'_{k+1}, s_{k+1}) = \sum_{i=0}^k \beta(x'_i, s_i)$, thus $\beta(x'_{k+1}, s) = \beta(x'_{k+1}, \tilde{s})$. Set $\gamma(x'_{k+1}) = \beta(x'_{k+1}, s)$.

With this terminology, for almost every $x'_{k+1} \in \mathbb{R}^{d-k}$, for almost every pair $s, \tilde{s} \in \mathbb{R}_+ \times \mathbb{R}_+$, both for $s_{k+1} = s$ and for $s_{k+1} = \tilde{s}$, up to a null set,

$$E(x'_{k+1}, s_{k+1}) = \gamma(x'_{k+1}) + \alpha(x'_{k+1}, s_{k+1})\mathcal{E}(x'_{k+1}). \qquad \square$$

Because superlevel sets are nested, this result extends to:

Proposition 6. *For every nonnegative extremizer f of* (1.2) *, for all $s \in \mathbb{R}_+$, for almost every $x' \in \mathbb{R}^k$, there exist an ellipsoid centered at the origin $\mathcal{E}(x') \subset \mathbb{R}^{d-k}$, a vector $\gamma(x') \in \mathbb{R}^{d-k}$, and a number $\alpha(x', s) \in \mathbb{R}$ such that $E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}(x')$ up to a null set.*

Proof. Fix any $\tilde{s} \in \mathbb{R}_+$. Fix any $x' \in \mathbb{R}^k$ such that for almost every $s \in \mathbb{R}_+$, $E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}(x')$ up to a null set. By Lemma 17, this condition is satisfied by almost every $x' \in \mathbb{R}^k$. Because superlevel sets are nested, for any sequence s_d approaching $\tilde{s}$ from above, $E(x', \tilde{s}) = \bigcup_{s_d} E(x', s_d)$. By our choice of $x' \in \mathbb{R}^k$, this sequence s_d can be chosen such that for each $n \in$, $E(x', s_d) = \gamma(x') + \alpha(x', s_d)\mathcal{E}(x')$ up to a null set. As the union of a countable collection of null sets is a null set,

$$E(x', \tilde{s}) = \bigcup_{s_d} \gamma(x') + \alpha(x', s_d)\mathcal{E}(x')$$

up to a null set.

Set $\alpha(x', \tilde{s}) = \lim_{n\to\infty} \alpha(x', s_d)$. This limit exists because $\alpha(x', s_d)$ is nondecreasing and bounded as $n \to \infty$. The first condition holds because superlevel sets are nested. The second because x' was chosen to satisfy the conditions of the construction in Proposition 5 which require that $f_{x'_{k+1}}$ is in $L^p(\mathbb{R}^{d-k})$ and thus that each superlevel set of $f_{x'_{k+1}}(v)$ has finite measure.

Therefore, up to a null set,

$$E(x', \tilde{s}) = \gamma(x') + \alpha(x', \tilde{s})\mathcal{E}(x'). \qquad \square$$

Our next goal is to show that there exists an ellipsoid centered at the origin $\mathcal{E} \subset \mathbb{R}^{d-k}$ such that for every $x' \in \mathbb{R}^k$, $\mathcal{E}(x') = \mathcal{E}$ and further that $\gamma(x')$ is an affine function. A proof similar to that given for Lemma 17 holds if the extremizers are known to be continuous. However, for extremizers that are only known to be measurable, there is an extra step. We show that the results proved so far imply that any superlevel set of an extremizer is convex up a null set and thus there exists a representative of $f \in L^p(\mathbb{R}^d)$ whose superlevel sets are convex. This function will have the properties of continuous functions that are relevant to the proof.

Definition 8. *A set E is almost Lebesgue convex if for almost every pair $(x, y) \in E \times E$ the line segment $\overline{xy} \subset E$ up to a one-dimensional null set.*

In Section 4.8 we prove Lemma 20: A set E is almost Lebesgue convex if and only if there exists an open convex set $\mathcal{C}$ such that $|E \Delta \mathcal{C}| = 0$ and in this case, $\mathcal{C}$ is the convex hull of the Lebesgue points of E.

Proposition 7. *For every nonnegative extremizer f of* (1.2) *, for every $s \in \mathbb{R}_+$ the set $E_s = \{x \in \mathbb{R}^d : f(x) > s\}$ is an almost Lebesgue convex set.*

We will first show:

Lemma 18. *For every nonnegative extremizer f of* (1.2) *, for every $s \in \mathbb{R}_+$, for every k-plane $\theta \in \mathcal{M}_{k,n}$, for almost every $x' \in \theta$, and for almost every pair $(v_1, v_2) \in \theta^\perp \times \theta^\perp$ such that $x' + v_1 \in E_s$ and $x' + v_2 \in E_s$, the line segment connecting $x' + v_1$ and $x' + v_2$ is contained in E_s up to a one-dimensional null set.*

Note that unlike most claims in this paper, which are of the almost everywhere variety, this result holds for every superlevel set and every k-plane.

Proof. For any k-plane θ there is an affine map A taking θ to $\mathbb{R}^k$. As A is affine, the mapping $f \mapsto f \circ A$ is a symmetry of (1.2) . Therefore $f \circ A$ is also an nonnegative extremizer of (1.2) and it suffices to consider the case where $\theta = \mathbb{R}^k \subset \mathbb{R}^d$. By Proposition 6, for all $s \in \mathbb{R}_+$, for almost every $x' \in \mathbb{R}^k$, $E(x', s)$ is an ellipsoid, and hence convex, up to an $(d-k)$-dimensional null set, so the claim follows from the only if direction of Lemma 20. □

Proof of Proposition 7. Factor $\mathbb{R}^d \times \mathbb{R}^d$ as the product $\mathcal{G}_{k,k+1} \times \mathbb{R}^k \times \mathbb{R}^{d-k} \times \mathbb{R}^{d-k}$, losing a null set, as follows. For $x = (x_1, \cdots, x_d)$ write $x'' = (x_1, \ldots, x_{k+1})$. Almost every pair (x'', y'') determines a line ℓ in $\mathbb{R}^{k+1}$. There is a unique k-plane, θ, in $\mathbb{R}^{k+1}$ that passes through the origin and is perpendicular to ℓ . Let $x' \in \mathbb{R}^k$ denote the projection of x onto θ. As θ is perpendicular to ℓ, the projection of y onto θ is also x'. Let v_x be the projection of x onto $\theta^\perp$, the $(d-k)$-dimensional subspace perpendicular to θ, and similarly for v_y. The 4-tuple (θ, x', v_x, v_y) completely specifies the pair (x, y).

By Lemma 18 the set of 4-tuples (θ, x', v_x, v_y) such that the line segment connecting $x' + v_x$ and $x' + v_y$ is contained in E_s up to a null set has full measure. Thus the set of (x, y) such that $\overline{xy} \subset E_s$ up to a one-dimensional null set has full measure as well. □

Proposition 8. *For every nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of* (1.2) *, there exists $\tilde{f} \in L^p(\mathbb{R}^d)$ such that $\tilde{f} = f$ almost everywhere and every superlevel set of $\tilde{f}$ is open and convex.*

Proof. Let f be any nonnegative extremizer of (1.2) . Let $E_s = \{x : f(x) > s\}$. By Proposition 7 for every $s \in \mathbb{R}_+$, the convex hull of the Lebesgue points of E_s, $\mathcal{C}_s$, is open and satisfies $|E_s \Delta \mathcal{C}_s| = 0$. Define

$$\tilde{f}(x) = \int_0^\infty \mathbb{1}_{\mathcal{C}_s}(x) ds. \tag{4.9}$$

Because $|E_s \Delta \mathcal{C}_s| = 0$, $\tilde{f}(x) = f(x)$ almost everywhere .

Observe that the sets $\mathcal{C}_s$ are nested. Take $r > t > 0$. $E_r \subset E_t$, thus the set of Lebesgue points of E_r is contained in the set of Lebesgue points of E_t. As $\mathcal{C}_r$ and $\mathcal{C}_t$ are the convex hulls of the Lebesgue points of E_r and E_t respectively, $\mathcal{C}_r \subset \mathcal{C}_t$.

For each $s \in \mathbb{R}_+$, define $\tilde{E}_s = \{x : \tilde{f}(x) > s\}$. Using (4.9) and that the sets $\mathcal{C}_s$ are nested, $\tilde{E}_s = \bigcup_{t>s} \mathcal{C}_t$. As the union of open sets is open, $\tilde{E}_s$ is open. Further, as the union of nested convex sets is convex, $\tilde{E}_s$ is also convex. □

Corollary 1. *Any nonnegative extremizer f of* (1.2) *agrees almost everywhere with a lower semi-continuous function.*

Corollary 2. *Let f be a nonnegative extremizer of* (1.2) *whose superlevel sets are open and convex. For every $s \in \mathbb{R}_+$, the function $x' \to \rho(x', s)$ is continuous on the interior of $\{x' : \rho(x', s) \neq 0\}$.*

Proof. Fix any $x' \in \mathbb{R}^k$ and $y' \in \mathbb{R}^k$ such that $|E(x', s)| \neq 0$ and $|E(y', s)| \neq 0$. By the Brunn-Minkowski inequality,

$$|tE(x', s) + (1-t)E(y', s))|^{1/n} \geq t|E(x', s)|^{1/n} + (1-t)|E(y', s)|^{1/n}.$$

By convexity of the superlevel set E_s,

$$tE(x', s) + (1-t)E(y', s)) \subset E(tx' + (1-t)y', s).$$

Thus,

$$\rho(tx' + (1-t)y', s) \geq t\rho(x', s) + (1-t)\rho(y', s).$$

Hence $x' \to \rho(x', s)$ is concave on $\{x' : \rho(x', s) \neq 0\}$. Using that a concave function on an open set is continuous, $x' \to \rho(x', s)$ is continuous on the interior of the set $\{x' : \rho(x', s) \neq 0\}$. □

Proposition 9. *Let f be a nonnegative extremizer of* (1.2) *whose superlevel sets are open and convex. There exist an ellipsoid centered at the origin $\mathcal{E} \subset \mathbb{R}^{d-k}$, an affine function $\gamma(x')$, and numbers $\alpha(x', s) \in [0, \infty)$ such that for every $(x', s) \in \mathbb{R}^k \times \mathbb{R}_+$ satisfying $|E(x', s)| > 0$*

$$E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}.$$

Proof. By Proposition 6, for all $s \in \mathbb{R}_+$, for almost every $x' \in \mathbb{R}^k$, there exist an ellipsoid centered at the origin $\mathcal{E}(x') \subset \mathbb{R}^{d-k}$, a vector $\gamma(x') \in \mathbb{R}^{d-k}$, and a number $\alpha(x', s) \in \mathbb{R}$ such that up to a null set,

$$E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}(x'). \tag{4.10}$$

As $E(x', s)$ is open and convex, when $|E(x', s)| > 0$ there is true equality in (4.10), not just equality up to a null set. It remains to see that $\mathcal{E}(x')$ is independent of x' and $\gamma(x')$ is an affine function.

By the convexity established in Proposition 8, it suffices to show that for almost every $z' \in \mathbb{R}^k$ there exists some $\delta > 0$ such that for almost every $x' \in B(z', \delta)$, $\mathcal{E}(z') = \mathcal{E}(x')$ and $\gamma(x')$ is almost everywhere equal to an affine function on $B(z', \delta)$.

Fix $z'_{k+1} \in \mathbb{R}^k$ satisfying the conditions of the construction in Proposition 5 and take $s_{k+1} \in \mathbb{R}_+$ such that z'_{k+1} is in the interior of $\{x' : \rho(x', s_{k+1}) \neq 0\}$. Such an s_{k+1} always exists by positivity of nonnegative extremizers and convexity of each superlevel set.

By essentially the same argument used for the construction in Proposition 5, there exist $\delta_b > 0$, $\epsilon_\rho > 0$, and $\{S_i : k + 1 \neq i \in [0, d]\}$ such that if $|\rho(x'_{k+1}, s_{k+1}) - \rho(z'_{k+1}, s_{k+1})| < \epsilon_\rho$ and $x'_{k+1} \in B(z'_{k+1}, \delta_b)$ and $(x'_i, s_i) \in S_i$ for $i \in [0, d] \setminus \{k+1\}$, then $\{\rho(x'_i, s_i)\}_{i=0}^d$ is permissible with respect to $(x'_0, \ldots, x'_{d+1})$.

The only change required is in the definition of δ_b. Whereas in the original proof $b_{(k+1),j}$ is viewed as a function of the $(k + 1)$ variables $\{x'_i\}_{i=0}^k$ with x'_{k+1} fixed, here x'_{k+1} varies as well. Thus $b_{(k+1),j}$ is a function of the $(k + 2)$ variables $\{x'_i\}_{i=0}^{k+1}$. As this function is continuous, there exists $\delta_b > 0$ such that if $x'_i \in B(z'_i, \delta_b)$ for $i \in [0, k + 1]$, then $|b_{(k+1),j}(x'_0, \ldots, x'_{k+1}) - b_{(k+1),j}(z'_0, \ldots, z'_k, z'_{k+1})| < \epsilon_b$, where for $i \in [0, k]$, z'_i is fixed as in Proposition 5. As there is an extra ϵ_ρ in the computation of permissibility in Proposition 5, the same computation gives permissibility here.

By Corollary 2, there exists $\delta_1 > 0$ such that for all $x' \in B(z'_{k+1}, \delta_1)$,

$$|\rho(x', s_{k+1}) - \rho(z'_{k+1}, s_{k+1})| < \epsilon_\rho.$$

Set $\delta_{z'_{k+1}} = \min(\delta_1, \delta_b)$. Then, for every $x'_{k+1} \in B(z'_{k+1}, \delta_{z'_{k+1}})$ if $(x'_i, s_i) \in S_i$ for $i \in [0, d] \setminus \{k + 1\}$, $\{\rho(x'_i, s_i)\}_{i=0}^d$ is permissible with respect to $(x'_0, \ldots, x'_{d+1})$.

By Proposition 4, for almost every $(z'_{k+1}, s_{k+1}) \in \mathbb{R}^k \times \mathbb{R}_+$ satisfying the conditions above, for almost every for almost every family $\{(x'_i, s_i) : k + 1 \neq i \in [0, d]\}$ with $(x'_i, s_i) \in S_i$, for almost every $x'_{k+1} \in B(z'_{k+1}, \delta_{z'_{k+1}})$, the family $\{E(x'_i, s_i)\}_{i=0}^d$ produces equality in (4.5) and the family $\{\rho(x'_i, s_i)\}_{i=0}^d$ is permissible. Thus for almost every $z'_{k+1} \in \mathbb{R}^k$, there exist an $s_{k+1} \in \mathbb{R}^k$ and a family $\{(x'_i, s_i) : k + 1 \neq i \in [0, d]\}$ such that for almost every $x'_{k+1} \in B(z'_{k+1}, \delta_{z'_{k+1}})$, the $(d + 1)$-tuple of sets $\{E(x'_i, s_i)\}_{i=0}^d$ satisfies the conditions of Burchard's theorem.

Applying Burchard's theorem and Lemma 17 gives that there exist vectors $\beta(x'_i) \in \mathbb{R}^{d-k}$ satisfying $\sum_{i=0}^k \beta(x'_i) = \beta(x'_{k+1})$, numbers $\alpha(x'_i, s_i) \in \mathbb{R}_+$ and a fixed ellipsoid $\mathcal{E}(x'_i)$ which is centered at the origin and independent of i, such that up to null sets,

$$b_{(k+1),i}(x'_0, \ldots, x'_{k+1})E(x'_i, s_i) = \beta(x'_i) + \alpha(x'_i, s_i)\mathcal{E}(x'_i).$$

Therefore $\mathcal{E}(x'_{k+1})$ is determined by $\{x'_i\}_{i=0}^k$ and must be the same for almost every $x'_{k+1} \in B(z'_{k+1}, \delta_{b,s_{k+1}})$.

Set[1] $\gamma(x'_i) = \beta(x'_i)/b_{(k+1),i}(x'_0, \ldots, x'_{k+1})$. Therefore for almost every $x'_{k+1} \in B(z'_{k+1}, \delta_{b,s_{k+1}})$,

$$\gamma(x'_{k+1}) = \sum_{i=0}^{k} b_{(k+1),i}(x'_0, \ldots, x'_{k+1})\gamma(x'_i).$$

For $i \in [0, k]$, $b_{(k+1),i}(x'_0, \ldots, x'_{k+1})$ defined by (4.1)) is an affine function of x_{k+1}, thus $\gamma(x'_{k+1})$ is as well. □

4.7 Proof of Proposition 2

Proposition 10. *Let f be a nonnegative extremizer of* (1.2) *whose superlevel sets are open and convex. Let $v \to f^\sharp(x', v)$ be the symmetric nonincreasing rearrangement of $v \to f(x', v)$ for each $x' \in \mathbb{R}^{(d-k)}$. Then there exist $\gamma(x') : \mathbb{R}^k \to \mathbb{R}^{d-k}$ an affine function and $L : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ an invertible linear map such that $f(x', L(v) + \gamma(x')) = f^\sharp(x', v)$.s*

Proof. Let $f \in L^p(\mathbb{R}^d)$ be any nonnegative extremizer of (1.2) whose superlevel sets are open and convex. By Proposition 9, there exist an ellipsoid centered at the origin $\mathcal{E} \subset \mathbb{R}^{d-k}$, an affine function $\gamma(x')$, and numbers $\alpha(x', s) \in [0, \infty)$ such that for every $(x', s) \in \mathbb{R}^k \times \mathbb{R}_+$ satisfying $|E(x', s)| > 0$

$$E(x', s) = \gamma(x') + \alpha(x', s)\mathcal{E}.$$

Let $L : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ be the linear map taking the unit ball to $\mathcal{E}$. Thus for each $x' \in \mathbb{R}^k$, each superlevel set of the function $v \to f(x', L(v) + \gamma(x'))$ is a ball centered at the origin or the empty set. □

To prove Proposition 2, we follow the proof in [12] for the Radon transform with modifications to allow for the change in dimension. This proof requires some notation from group theory. Let $\mathfrak{A}(d)$ denote the affine group and $O(d)$ denote the orthogonal group, each in $\mathbb{R}^d$. Similarly, let $O(d-k)$ denote the orthogonal group in $\mathbb{R}^{d-k}$.

Definition 9. *Fix $k \in [1, d-1]$. For $\varphi \in O(d)$ define a scaled skew reflection associated to φ to be any element of $\mathfrak{A}(d)$ with the form*

$$\Phi_\varphi = \varphi^{-1}\psi^{-1}\mathcal{L}^{-1}R\mathcal{L}\psi\varphi$$

where $\psi(x', v) = (x', v + \gamma(x'))$ for $\gamma(x') : \mathbb{R}^k \to \mathbb{R}^{d-k}$ an affine mapping, $\mathcal{L}(x', v) = (x', L(v))$ for $L : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ an invertible linear map, and
$R(x', v_1, \ldots, v_{d-k}) = (x', v_1, \ldots, v_{n-k-1}, -v_{d-k})$.

[1] Note that $\gamma(x'_{k+1}) = \beta(x'_{k+1})$ so this definition agrees with the definition of $\gamma(x'_{k+1})$ given in Lemma 17.

We use the term scaled skew reflection, to distinguish these functions the the skew reflections used in [12], where the linear map $\mathcal{L}$ is unnecessary as ellipsoids and spheres are both intervals in 1 dimension. The following results are proved in [12] where scaled skew reflections are replaced by skew reflections.

Lemma 19. *For every nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of (1.2), for each $\varphi \in O(d)$ there exists a scaled skew reflection associated to φ, Φ_φ, such that$f \circ \Phi_\varphi = f$ almost everywhere.*

Proof. Given a nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of (1.2) and an orthogonal transformation $\varphi \in O(d)$, take $\gamma(x') : \mathbb{R}^k \to \mathbb{R}^{d-k}$ and $L : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ to be the affine function and invertible linear map guaranteed by Proposition 10 applied to the extremizer that agrees almost everywhere whose level sets are open and convex with $f \circ \varphi$. Set $\mathcal{L} = (x', L^{-1}(v))$ and $\psi(x', v) = (x', v - \gamma(x'))$. Then by Proposition 10, $f \circ \Phi_\varphi = f$ almost everywhere. □

Proposition 11. *Let $f : \mathbb{R}^d \to [0, \infty)$ be a measurable function such that each superlevel set is convex and bounded. Suppose $\{x : f(x) > 0\}$ has positive Lebesgue measure and for each $\varphi \in O(d)$ there exists a scaled skew reflection associated to φ, Φ_φ, such that $f \circ \Phi_\varphi = f$ almost everywhere, then there exists $\phi \in \mathfrak{A}(d)$ such that $f \circ \phi = (f \circ \phi)^*$ almost everywhere.*

Proof. For each $s \in \mathbb{R}_+$ set $E_s = \{x : f(x) > s\}$. Let $G \subset \mathfrak{A}(d)$ be the subgroup of all $g \in \mathfrak{A}(d)$ such that $g(E_s) = E_s$ up to a null set for each $s \in \mathbb{R}_+$. As for some $s \in \mathbb{R}_+$ the set E_s has positive measure and for each $s \in \mathbb{R}_+$ the set E_s is bounded, G is compact. For each $\varphi \in O(d)$ there exists a scaled skew reflection associated to φ, Φ_φ, such that$f \circ \Phi_\varphi = f$ and hence $\Phi_\varphi \in G$. Any compact subgroup of $\mathfrak{A}(d)$ is conjugate by an element of $\mathfrak{A}(d)$ to a subgroup of $O(d)$ (see [26] pg 256). Thus, there exists $\phi \in \mathfrak{A}(d)$ such that for all $\varphi \in O(d)$, $\phi^{-1}\Phi_\varphi\phi \in O(d)$. Set $\tilde{\Phi}_\varphi = \phi^{-1}\Phi_\varphi\phi$.

Express $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}$. The transformation $\psi^{-1}\mathcal{L}^{-1}R\mathcal{L}\psi$ acts as the identity on $\mathbb{R}^{d-1}$, so $\tilde{\Phi}_\varphi$ acts as the identity on $\phi^{-1}\varphi^{-1}(R^{d-1})$. For a scaled skew reflection Φ_φ, $\tilde{\Phi}_\varphi$ is an orthogonal reflection. Thus $\tilde{\Phi}_\varphi$ must be reflection about the hyperplane parallel to $\phi^{-1}\varphi^{-1}(R^{d-1})$ passing through origin. As φ ranges over $O(d)$, the hyperplane parallel to $\phi^{-1}\varphi^{-1}(R^{d-1})$ passing through origin ranges over $\mathcal{G}_{(d-1),n}$. Thus the conjugated subgroup $\phi^{-1}G\phi$ contains a reflection about each $(d-1)$-dimensional subspace of $\mathbb{R}^d$. These transformations generate the orthogonal group, so for each $s \in \mathbb{R}_+$, $\phi(E_s)$ is a convex set fixed under every orthogonal transformation. Therefore, for each $s \in \mathbb{R}_+$, $\phi(E_s)$ must be a ball. □

Proof of Proposition 2. For every nonnegative extremizer $f \in L^p(\mathbb{R}^d)$ of (1.2) , each superlevel set E_s of f is convex. As $f \in L^p(\mathbb{R}^d)$, each E_s has finite measure. As a convex set with positive finite measure is bounded, for every $s \in \mathbb{R}_+$, E_s is bounded. Given this and Lemma 19, $\tilde{f}$ satisfies the conditions of Proposition 11. Hence $\tilde{f} = F \circ \phi$ for some radial function F and ϕ some affine transformation of $\mathbb{R}^d$. As $\tilde{f}$ and f are equal in L^p, this suffices. □

4.8 Almost Lebesgue convexity

Recall Definition 8: A set E is almost Lebesgue convex if for almost every pair $(x, y) \in E \times E$ the line segment $\overline{xy} \subset E$ up to a one-dimensional null set. Throughout this section E_L will denote the set of Lebesgue points of a set E, and for any set A, $ch(A)$ will be the convex hull of A.

Lemma 20. *A set E is almost Lebesgue convex if and only if there exists an open convex set $\mathcal{C}$ such that $|E\Delta\mathcal{C}| = 0$. In this case, $\mathcal{C}$ is the convex hull of the Lebesgue points of E.*

We start with two lemmas that together prove the "only if" direction when $|E| > 0$.

Lemma 21. *For any set E with positive measure, if for almost every $(d+1)$-tuple $(x_1, \ldots, x_{d+1}) \in E^{d+1}$, the convex hull $ch(x_1, \ldots, x_{d+1}) \subset E$ up to an n-dimensional null set, then the convex hull of the Lebesgue points of E, $ch(E_L)$, is an open convex set, and $|ch(E_L)\Delta E| = 0$.*

Proof. As $E_L \subset ch(E_L)$, $|E \setminus ch(E_L)| < |E \setminus E_L| = 0$. Thus $|E \setminus ch(E_L)| = 0$.

It remains to show that $ch(E_L)$ is open and $|ch(E_L) \setminus E| = 0$. The main step is to show that for each $(d + 1)$-tuple of points $\{x_1, \ldots, x_{d+1}\} \in E_L^{d+1}$, there exists an open set $O_{\{x_1,\ldots,x_{d+1}\}}$, such that $ch(x_1, \ldots, x_{d+1}) \subset O_{\{x_1,\ldots,x_{d+1}\}} \subset ch(E_L)$ and $O_{\{x_1,\ldots,x_{d+1}\}} \setminus E$ is a null set.

This claim implies the lemma as follows: By definition,

$$ch(E_L) = \bigcup_{\{x_1,\ldots,x_{d+1}\}\in E_L^{d+1}} ch(x_1, \ldots, x_{d+1}).$$

As $ch(x_1, \ldots, x_{d+1}) \subset O_{\{x_1,\ldots,x_{d+1}\}}$,

$$ch(E_L) \subset \bigcup_{\{x_1,\ldots,x_{d+1}\}\in E_L^{d+1}} O_{\{x_1,\ldots,x_{d+1}\}}.$$

Similarly, because each $O_{\{x_1,\ldots,x_{d+1}\}} \subset ch(E_L)$,

$$ch(E_L) \supset \bigcup_{\{x_1,\ldots,x_{d+1}\}\in E_L^{d+1}} O_{\{x_1,\ldots,x_{d+1}\}}.$$

Therefore,

$$ch(E_L) = \bigcup_{\{x_1,\ldots,x_{d+1}\}\in E_L^{d+1}} O_{\{x_1,\ldots,x_{d+1}\}}.$$

As $ch(E_L)$ is a union of open sets, it is open. Moreover, by the second countability of $\mathbb{R}^d$, there exists $\{O_i\}$ a countable collection of $O_{\{x_1,\ldots,x_{d+1}\}}$, such that $ch(E_L) = \bigcup_{i=1}^{\infty} O_i$. Thus $ch(E_L)\setminus E \subset \bigcup_{i=1}^{\infty}(O_i \setminus E)$, which is a null set by countable additivity.

It remains to construct these $O_{\{x_1,\ldots,x_{d+1}\}}$. Begin by observing that given the conditions of the lemma, if $x \in E_L$ then there exists $\delta > 0$ such that $B(x, \delta) \subset E$ up to a d-dimensional null set. Since $x \in E_L$, there exists $\delta' > 0$ such that $|B(x, \delta') \cap E| \geq \frac{1}{d+1}|B(x, \delta')|$. Applying the pigeonhole principle, there exists an n-tuple $\{x_i\}_{i=1}^{d+1}$ such that x is in the interior of $ch(x_1, \ldots, x_{d+1})$ and $ch(x_1, \ldots, x_{d+1}) \subset E$ up to a d-dimensional null set. Therefore, there exists $\delta > 0$ such that $B(x, \delta) \subset ch(x_1, \ldots, x_{d+1})$, $B(x, \delta) \subset E$ up to a d-dimensional null set.

For any $(d+1)$-tuple of points $(x_1, \ldots, x_{d+1}) \in E_L^{d+1}$, using the observation above, there exists a set of positive measure in E_L^{d+1} of $y_1, \ldots, y_{d+1}$ such that $ch(x_1, \ldots, x_{d+1}) \subset ch(y_1, \ldots, y_{d+1})$.

By the hypothesis of the lemma, for almost every such $(d+1)$-tuple, $ch(y_1, \dots, y_{d+1}) \subset E$ up to a null set. Pick one of these $(d+1)$-tuples and take $O_{(x_1,\dots,x_{d+1})}$ to be the interior of $ch(y_1, \dots, y_{d+1})$. □

Lemma 22. *If $E \subset \mathbb{R}^d$ is an almost Lebesgue convex set with positive measure and $m \in [2, d+1]$, then for almost every m-tuple $(x_1, \dots, x_m) \in E^m$, the convex hull $ch(x_1, \dots, x_m) \subset E$ up to an $(m-1)$-dimensional null set.*

Proof. The proof proceeds by induction on m. If $E \subset \mathbb{R}^d$ is almost Lebesgue convex, then by definition the base case $m = 2$ holds. Assume $m \in [2, d]$ and the statement is true for m. We seek to prove that for almost every x_0, for almost every $x_1, \dots, x_m$, $ch(x_0, \dots, x_m) \subset E$ up to an m-dimensional null set.

Fix $x_0 \in E$ such that for almost every y, $|\overline{x_0 y} \setminus E| = 0$. By almost Lebesgue convexity, it is enough to prove the statement for every such x_0. Working in polar coordinates centered at x, define $r_\theta = \sup\{r : |\overline{x_0(\theta, r)} \setminus E| = 0\}$. Set

$$S_{x_0} = \bigcup_{\theta \in \mathbb{S}^{n-1}} \overline{x_0(\theta, r_\theta)}.$$

By the definition of r_θ, $|S_{x_0} \setminus E| = 0$. Moreover, because $|\overline{x_0 y} \setminus E| = 0$ for almost every y, $|E \setminus S_{x_0}| = 0$. Therefore, $|S_{x_0} \Delta E| = 0$.

Parameterize m-tuples in $\mathbb{R}^d$, losing a null set, by $(\pi, y, v_1, \dots, v_m)$ where $\pi \in \mathcal{G}_{m-1,n}$, $y \in \pi^\perp$, $v_i \pi$ for $i \in [1, m]$. Let (π, y) denote the $(m-1)$-plane π translated by y. By the induction hypothesis, for almost every π, for almost every y, for almost every m-tuple, $v_1, \dots, v_m \in \pi^m$ such that $v_1 + y, \dots, v_m + y \in E^m$, $ch(v_1 + y, \dots, v_m + y) \subset E$ up to an $(m-1)$-dimensional null set.

Fix $\pi \in \mathcal{G}_{m-1,n}$ such that this condition holds. For almost every $y \in \pi^\perp$, $(\pi, y) \cap E$ satisfies the conditions of Lemma 21, hence there is a convex set $\mathcal{C}_{(\pi,y)}$ such that $|((\pi, y) \cap E)\Delta\mathcal{C}_{(\pi,y)}| = 0$. For the null set of $y \in \pi^\perp$ for which such a set does not exist, let $\mathcal{C}_{(\pi,y)}$ be the empty set. Set

$$\mathcal{C}_\pi = \bigcup_{y \in \pi^\perp} \mathcal{C}_{(\pi,y)}.$$

Then $|\mathcal{C}_\pi \Delta E| = 0$, and moreover, $|\mathcal{C}_\pi \Delta S_{x_0}| = 0$. Thus for almost every $y \in \pi^\perp$, $|\mathcal{C}_{(\pi,y)} \Delta (S_{x_0} \cap (\pi, y))| = 0$. Using that $|\mathcal{C}_{(\pi,y)} \setminus S_{x_0}| = 0$ and S_{x_0} is star-shaped about x_0, for almost every $y \in \pi^\perp$, for almost every m-tuple $v_1, \dots, v_m \in \pi^m$ such that $v_1 + y, \dots, v_m + y \in E^m$, $|ch(x_0, v_1 + y, \dots, v_m + y) \setminus S_{x_0}| = 0$. As $|S_{x_0} \Delta E| = 0$, it follows that for almost every m-tuple, $v_1, \dots, v_m \in \pi^m$ such that $v_1 + y, \dots, v_m + y \in E^m$, $|ch(x_0, v_1 + y, \dots, v_m + y) \setminus E| = 0$. □

Proof of Lemma 20 . First consider the case that $|E| = 0$. Any null set is almost Lebesgue convex. The set of Lebesgue points for any null set is the empty set which is an open convex set equal to E up to a null set. Hence the theorem holds when $|E| = 0$.

The "only if" direction when $|E| > 0$ is addressed by Lemmas 21 and 22.

To see the "if" direction, assume there exists an open convex set $\mathcal{C}$ such that $|E\Delta\mathcal{C}| = 0$. As $|E| > 0$, $|E \cap \mathcal{C}| > 0$. Fix any $x \in E \cap \mathcal{C}$. Take polar coordinates centered at x. For every $\theta \in \mathbb{S}^d$, define $r_\theta = \inf\{r : (\theta, r) \notin \mathcal{C}\}$. $r_\theta > 0$ as $\mathcal{C}$ is open. Further as $|\mathcal{C} \setminus E| = 0$, for almost every θ, for every $0 < r < r_\theta$ such that $(\theta, r) \in E$ the line segment in the direction θ up to distance r is contained in E up to a one-dimensional null set. As almost every point of $\mathcal{C}$ will be some (θ, r) such that this condition holds, it will hold for almost every point of E as well. As almost every $x \in E$ is in $E \cap \mathcal{C}$, this suffices. $\square$

Chapter 5

Alternate perspectives on the endpoint inequality

5.1 Main results and methods

Central to the uniqueness argument is a multilinear form (Drury's identity) that gives the L^q norm of the k-plane transform. A related multilinear form has been studied by Valdimarsson using similar methods in [46]. As in Valdimarsson's case there is a certain amount of geometric invariance that allows us to immediately extend our result for the k-plane transform Euclidean space to the k-plane transform in elliptic space. This transform was originally introduced by Funk [24]. See Helgason (for instance [27]) for the modern perspective. The question of L^p-L^q inequalities for the k-plane transform in elliptic space has been considered by Strichartz [44], Christ [11], and Drury [20].

The k-plane transform in elliptic space is defined as follows. Let F be a function defined on $\mathcal{G}_{1,d}$, the set of lines through the origin in $\mathbb{R}^d$. Let $\pi \in \mathcal{G}_{k,d}$ be a k-plane passing through the origin in $\mathbb{R}^d$. There is a unique probability measure invariant under the action of the orthogonal group on the space of lines through the origin contained in π analogous to that for $\mathcal{G}_{1,k}$. This measure will be denoted by $d\gamma_\pi$. The k-plane transform in elliptic space is given by

$$T^E_{k,d}F(\pi) = \int_{\theta \subset \pi} F(\theta)\; d\gamma_\pi(\theta).$$

Christ [11] proves that there exists a finite indeterminate constant A_E such that for all $f \in L^p(\mathbb{R}^d)$,

$$\left(\int_{\mathcal{G}_{k,d}} |T^E_{k,d}F(\pi)|^d d\gamma(\pi)\right)^{1/d} \leq A_E \left(\int_{\mathcal{G}_{1,d}} |F(\theta)|^{\frac{d}{k}} d\gamma(\theta)\right)^{\frac{k}{d}}. \tag{5.1}$$

Assign coordinates on $\mathcal{G}_{1,d}$, losing a null set, by identifying each unit vector θ in the northern hemisphere with the line it spans. For a linear map L, $L(\theta)$ is the image of the unit vector θ under the map L. The main result of Section 3 is:

Theorem 7. *$F \in L^{\frac{d}{k}}(\mathcal{G}_{1,d})$ is an extremizer of the inequality* (5.1) *if and only if*

$$F(\theta) = c\,|L(\theta)|^{-k}$$

for some $c \in \mathbb{C} - \{0\}$ and some invertible linear endomorphism L of $\mathbb{R}^d$.

We also consider a third variant of the k-plane transform, $T^{\sharp}_{k,d}$. Denote the space of $k \times (d-k)$ matrices by $Mat(k, d-k)$. Let $f : \mathbb{R}^d \to \mathbb{C}$, $A \in Mat(k, d-k)$ and $b \in \mathbb{R}^{(d-k)}$. Then $T^{\sharp}_{k,d}f$ is given by:

$$T^{\sharp}_{k,d}f(A, b) = \int_{\mathbb{R}^k} f(x', A(x') + b)dx'.$$

We view $T^{\sharp}_{k,d}f(A, b)$ as a function on $\mathbb{R}^{(k+1)(d-k)}$ by identifying $Mat(k, d-k) \times \mathbb{R}^{(d-k)}$ with $\mathbb{R}^{(k+1)(d-k)}$ by first identifying $Mat(k, d-k)$ with $\mathbb{R}^{d-k} \times \ldots \times \mathbb{R}^{d-k}$. As usual, equip $\mathbb{R}^{(k+1)(d-k)}$ with Lebesgue measure. The main result of section 4 is:

Theorem 8. *There exists a finite constant $A^{\sharp} \in \mathbb{R}_+$ such that for all $f \in L^p(\mathbb{R}^d)$*

$$\left(\int_{\mathbb{R}^{k(d-k)}} \int_{\mathbb{R}^{d-k}} |T^{\sharp}_{k,d}f(A, b)|^q dA db\right)^{1/q} \leq A^{\sharp}\|f\|_{L^p(\mathbb{R}^d)}. \tag{5.2}$$

Further, $f \in L^p(\mathbb{R}^d)$ is an extremizer of (5.2) *if and only if it is an extremizer of* (1.2).

Again, this is an extension of a result in [12] where Theorem 8 is proved in the case that $k = n - 1$.

5.2 k-plane transform in elliptic space

At the heart of this section is a correspondence between the k-plane transform in Euclidean space and the $(k+1)$-plane transform in elliptic space when $q = d + 1$. This correspondence was originally observed by Drury [20] for the l-to-k plane transform and its elliptic analog. Valdimarsson [46] uses a similar correspondence to extend his results on extremizers in $L^p(\mathbb{R}^d)$ for a multilinear form similar to the form which appears in Drury's identity to extremizers in $L^p(\mathbb{S}^{d+1} \cap \{x_{d+1} > 0\})$ for a corresponding version of the multilinear form.

Recall that the $(k+1)$-plane transform in elliptic space, defined in the introduction, is a bounded operator from $L^{\frac{d+1}{k+1}}(\mathcal{G}_{1,d+1})$ to $L^{d+1}(\mathcal{G}_{k+1,d+1})$. Define a map from $\mathbb{R}^d$ to $\mathcal{G}_{1,d+1}$ by embedding $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ as $\{x_{d+1} = 1\}$ and associating to each point $(x, 1)$ the line it spans. Parameterize $\mathcal{G}_{1,d+1}$ by $\theta \in \mathbb{S}^{d+1} \bigcap \{x_{d+1} > 0\}$, losing a null set, by associating unit vectors in the northern hemisphere with the lines they span. In these coordinates, the map described is a nonlinear projection onto the northern hemisphere:

$$\mathcal{S}(x) = \frac{1}{(1 + |x|^2)^{1/2}}(x_1, \ldots, x_n, 1).$$

Let $d\sigma$ denote surface measure on the northern hemisphere and set $c_n = \int_{\mathbb{S}^{d+1}} \mathbb{1}_{\{\theta_{d+1}>0\}}(\theta)d\sigma$. For this parametrization of $\mathcal{G}_{1,d+1}$ the natural probability measure is $c_n^{-1}\mathbb{1}_{\{\theta_{d+1}>0\}}(\theta)d\sigma$.

To a function $f \in L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$, associate the function $F \in L^{\frac{d+1}{k+1}}(\mathcal{G}_{1,d+1})$ defined by

$$F(\theta) = (\theta_{d+1})^{-(k+1)} f(\mathcal{S}^{-1}(\theta)).$$

Observe that $c_n^{1/p}\|F\|_{L^p(\mathcal{G}_{1,d+1})} = \|f\|_{L^p(\mathbb{R}^d)}$ when $p = \frac{d+1}{k+1}$.

Lemma 23. *There exists $C \in \mathbb{R}_+$ depending only on n and k such that for every $f \in L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$ and its associated function $F \in L^{\frac{d+1}{k+1}}(\mathcal{G}_{1,d+1})$*

$$||T^E_{k+1,d+1}F(\theta)||_{L^{d+1}(\mathcal{G}_{k+1,d+1})} = C||T_{k,d}f||_{L^{n+1}(\mathcal{M}_{k,d})}. \tag{5.3}$$

Proof. The nonlinear projection above also gives us a map from $\mathcal{G}_{k+1,d+1}$ to $\mathcal{M}_{k,d}$. For any $\pi \in \mathcal{G}_{k+1,d+1}$, let $\Pi \in \mathcal{M}_{k,d}$ be $\pi \cap \{x_{d+1} = 1\}$ thought of as a k-plane in $\mathbb{R}^d$. Note that each line $\theta \in \pi$ corresponds to a point $\mathcal{S}^{-1}(\theta) \in \Pi$. Let $b(\Pi)$ denote the distance from Π to the origin in $\mathbb{R}^{d+1}$. In [20], Drury showed that there exists $c \in \mathbb{R}_+$ depending only on k and d such that the natural measure on $\mathcal{G}_{1,d+1}$, denoted $d\gamma$, is related to the natural product measure, denoted $d\mu$, on $\mathcal{M}_{k,d}$ by

$$d\gamma(\pi) = c(b(\Pi))^{-(d+1)}d\mu(\Pi).$$

The next step is to relate the natural measure on the set of linear subspaces contained in π, denoted $d\gamma_\pi$, to the natural product measure on the set of lines contained in Π, denoted $d\lambda_\Pi$. As each of the measures in question is invariant under rotation[1], it is enough to consider π passing through the north pole of $\mathbb{S}^{d+1}$ and Π passing through $(0, \dots, 0, b(\Pi))$. In this case our map corresponds to division by $b(\Pi)$ followed by our original projection. Thus,

$$\theta_{d+1}^{-(k+1)} d\gamma_\pi(\theta) = c_n b(\Pi) d\lambda_\Pi(x).$$

Therefore,

$$\begin{aligned} T^E_{k+1,d+1}\left((\theta_{d+1})^{-(k+1)}f(\mathcal{S}^{-1}(\theta))\right)(\pi) &= \int_{\theta\subset\pi} (\theta_{d+1})^{-(k+1)}f(\mathcal{S}^{-1}(\theta))d\gamma_\pi(\theta) \\ &= c_n \int_{x\in\Pi} f(x)(b(\Pi))\, d\lambda_\Pi(x). \end{aligned}$$

Now,

$$\begin{aligned} ||T^E_{k+1,d+1}F||^{d+1}_{L^{d+1}(\mathcal{G}_{k+1,d+1})} &= \int_{\mathcal{G}_{1,d+1}} \left[T^E_{k,d}\left((\theta_{d+1})^{-(k+1)}f(\mathcal{S}^{-1}(\theta))\right)\right]^{d+1} d\gamma(\pi) \\ &= C\int_{\mathcal{M}_{k,d}} \left[\int_{x\in\Pi} f(x)(b(\Pi))\, d\lambda_\Pi(x)\right]^{d+1} (b(\Pi))^{-(d+1)}d\mu(\Pi) \\ &= C||T_{k,d}f||^{d+1}_{L^{d+1}(\mathcal{M}_{k,d})} \end{aligned}$$

□

[1]To rotate the northern hemisphere, rotate the sphere and send any points of the northern hemisphere mapped into the southern hemisphere to their antipodal points.

Proof of theorem 7. By Lemma 23, there exists $C \in \mathbb{R}_+$ depending only on n and k such that for any $f \in L^p(\mathbb{R}^d)$ with $p = \frac{d+1}{k+1}$,

$$\frac{\|T_{k,d}f\|_{L^{d+1}(\mathcal{M}_{k,d})}}{\|f\|_{L^p(\mathbb{R}^d)}} = C\frac{\|T^E_{k,d}F\|_{L^{d+1}(\mathcal{G}_{k+1,d+1})}}{\|F\|_{L^p(\mathcal{G}_{1,d+1})}}.$$

It follows immediately that $f \in L^p(\mathbb{R}^d)$ is an extremizer of (1.2) if and only if F is an extremizer of (5.1).

By Theorem 3 any extremizer of (1.2) has the form $f(x) = c(1+|\phi(x)|^2)^{-(k+1)/2}$ where ϕ is an affine endomorphism of $\mathbb{R}^d$. It remains to compute the associated F. Observe for any such ϕ there exists L, an invertible transformation of $\mathbb{R}^{d+1}$, such that $(1+|\phi(x)|^2) = |L(x,1)|^2$. Therefore,

$$\begin{aligned} F(\theta) &= (\theta_{d+1})^{-(k+1)} f(\mathcal{S}^{-1}(\theta)) \\ &= c(\theta_{d+1})^{-(k+1)}(|L(\mathcal{S}^{-1}(\theta),1)|^2)^{-(k+1)/2} \\ &= c|L(\theta_1,\ldots,\theta_{d+1})|^{-(k+1)}. \end{aligned}$$ □

This perspective gives insight into the additional symmetry J used in Christ [12] and Drouot's [17] work. Define $\mathcal{S}^* : L^p(\mathbb{R}^d) \to L^p(\mathcal{G}_{1,d+1})$ by $\mathcal{S}^*(f) = F$. Denote by sgn the standard sign function. Set $Jf(s,y) = |s|^{-k-1}f(s^{-1}, s^{-1}y)$ and

$$RF(\theta) = F(\text{sgn}(\theta_1)\theta_{d+1}, \text{sgn}(\theta_1)\theta_2, \ldots, \text{sgn}(\theta_1)\theta_n, |\theta_1|).$$

Lemma 24. *For every $f \in L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$,*

$$\mathcal{S}^*Jf(\theta) = R\mathcal{S}^*f(\theta).$$

Proof.

$$R\mathcal{S}^*f(\theta) = |\theta_1|^{-k-1} f\left(\frac{\text{sgn}(\theta_1)\theta_{d+1}}{|\theta_1|}, \frac{\text{sgn}(\theta_1)\theta_2}{|\theta_1|}, \ldots, \frac{\text{sgn}(\theta_1)\theta_n}{|\theta_1|}\right).$$

Similarly,

$$\mathcal{S}^*Jf(\theta) = |\frac{\theta_1}{\theta_{d+1}}|^{-(k+1)}(\theta_{d+1})^{-(k+1)} f\left(\frac{\theta_{d+1}}{\theta_1}, \frac{\theta_2}{\theta_1}, \ldots, \frac{\theta_n}{\theta_1}\right).$$

As $\theta_{d+1} > 0$, $\mathcal{S}^*Jf(\theta) = R\mathcal{S}^*f(\theta)$ as claimed. □

As the reflection R is clearly a symmetry of (5.1), J must be a symmetry of (1.2) by Lemma 23.

5.3 Another related family of operators

In this section we present yet another realization of the inequality (1.2), this time for the operator $T^\sharp_{k,d}$ which was defined in the introduction. Recall that $T^\sharp_{k,d}$ takes functions on $\mathbb{R}^d$ to functions on $\mathbb{R}^{(k+1)(d-k)}$.

Lemma 25. *Let $f \in L^p(\mathbb{R}^d)$ be a nonnegative continuous function. Then there exists $C \in \mathbb{R}_+$ depending only on n and k such that*

$$\|T_{k,d}f\|_{L^q(\mathcal{M}_{k,d})} = C\|T^\sharp_{k,d}f\|_{L^q(\mathbb{R}^{(k+1)(d-k)})}.$$

The proof is a generalization of that used in [12] in the case $k = n - 1$.

Proof. By Lemma 11, it suffices to show that for any nonnegative continuous function f,

$$\|T^\sharp_{k,d}f\|^q_{L^q(\mathbb{R}^{(k+1)(d-k)})} = \int \Delta^{(k-n)}(x'_0, \ldots, x'_k) \int \prod_{i=0}^{k} f(x'_i, v_i) \prod_{i=k+1}^{d} f(x'_i, \sum_{j=0}^{k} b_{i,j} v_j)\ dv_0 \ldots dv_k dx'_0 \ldots dx'_d. \quad (5.4)$$

Let c_{d-k} be the volume of the unit sphere in $(d-k)$ dimensions. Observe that

$$T^\sharp_{k,d}f(A, b) = \int_{\mathbb{R}^k} f(x', A(x') + b) dx' = \lim_{\epsilon \to 0} \left(c_{d-k}\epsilon^{d-k}\right)^{-1} \int_{\mathbb{R}^k}\int_{\mathbb{R}^{d-k}} f(x', A(x') + b + t)\mathbb{1}_{|t|<\epsilon}\ dt dx'.$$

Taking dA to be Lebesgue measure on the entries of A, and db to be Lebesgue measure on $\mathbb{R}^{d-k}$,

$$\int \left(T^\sharp_{k,d}f(A, b)\right)^{d+1} dA db = \int \prod_{j=0}^{k} \left(\lim_{\epsilon \to 0} \left(c_{d-k}\epsilon^{d-k}\right)^{-1} \int f(x'_j, A(x'_j) + b + t_j)\mathbb{1}_{|t_j|<\epsilon}\ dt_j dx'_j\right) \prod_{j=k+1}^{n} \left(\int_{\mathbb{R}^k} f(x'_j, A(x'_j) + b) dx'_j\right) dA db.$$

Apply the change of variables $s_j = Ax_j + b + t_j$ for $j \in [0, k]$ and Tonelli's theorem to obtain

$$\int \left(T^\sharp_{k,d}f(A, b)\right)^{d+1} dA db = \int \prod_{j=0}^{k} f(x'_j, s_j) \int \prod_{j=k+1}^{n} f(x'_j, A(x'_j) + b) \prod_{j=0}^{k} \left(\lim_{\epsilon \to 0} \left(c_{d-k}\epsilon^{d-k}\right)^{-1} \mathbb{1}_{|s_j - Ax_j + b|<\epsilon}\right) dA db \prod_{j=0}^{k} ds_j dx'_j \prod_{j=k+1}^{n} dx'_j. \quad (5.5)$$

Consider the inner integral, now viewing $\mathbb{1}_{|s_j - Ax_j + b|<\epsilon}$ as a cutoff function in A and b. Let a_i be the i-th row of A and b_i be the i-th entry of b. Let L be the linear map such that $L(a_i, b_i) = (a_i \cdot x_j + b_i)_{j=0}^{k}$. Then L has a Jacobian $\mathcal{J}_L$ given by

$$\mathcal{J}_L = \begin{pmatrix} x'_{0,1} & \cdots & x'_{0,k} & 1 \\ \vdots & & \vdots & \vdots \\ x'_{k,1} & \cdots & x'_{k,k} & 1 \end{pmatrix} = \Delta(x'_0, \ldots, x'_k).$$

Let A_0, b_0 such that $A_0x'_j + b_0 = s_j$. As f is assumed to be continuous,

$$\lim_{\epsilon\to 0} \left(c_{d-k}\epsilon^{d-k}\right)^{-1} \int \prod_{j=k+1}^{n} f(x'_j, A(x'_j)+b) \prod_{j=0}^{k} \left(\mathbb{1}_{|s_j - Ax_j + b| < \epsilon}\right) dAdb =$$

$$\Delta(x'_0, \ldots, x'_k)^{k-n} \delta_{(A,b)(A_0,b_0)} \prod_{j=k+1}^{n} f(x'_j, A(x'_j)+b).$$

Substituting this into (5.5) gives the result. □

Proof of theorem 8. Using Lemma 25 and standard approximation arguments, it follows that for any nonnegative function $f \in L^p(\mathbb{R}^d)$, $\|T_{k,d}f\|_{L^q(\mathcal{M}_{k,d})} = C\|T^\sharp_{k,d}f\|_{L^q(\mathbb{R}^{(k+1)(d-k)})}$. As

$$\|T^\sharp_{k,d}f\|_{L^q(\mathbb{R}^{(k+1)(d+1)})} \leq \|T^\sharp_{k,d}|f|\|_{L^q(\mathbb{R}^{(k+1)(d+1)})},$$

it follows directly from Lemma 25 and Theorem (3) that $T^\sharp_{k,d}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^{(k+1)(d+1)})$. Moreover, as $\|T_{k,d}f\|_{L^q(\mathcal{M}_{k,d})} \leq \|T_{k,d}|f|\|_{L^q(\mathcal{M}_{k,d})}$ as well,

$$\sup_{\{g:\|g\|_{L^p(\mathbb{R}^d)}\neq 0\}} \frac{\|T_{k,d}g\|_{L^q(\mathcal{M}_{k,d})}}{\|g\|_{L^p(\mathbb{R}^d)}} = \sup_{\{g:\|g\|_{L^p(\mathbb{R}^d)}\neq 0, g>0\}} \frac{\|T_{k,d}g\|_{L^q(\mathcal{M}_{k,d})}}{\|g\|_{L^p(\mathbb{R}^d)}}.$$

By Lemma 25 there exists $C \in \mathbb{R}_+$ depending only on n and k such that

$$\sup_{\{g:\|g\|_{L^p(\mathbb{R}^d)}\neq 0, g>0\}} \frac{\|T_{k,d}g\|_{L^q(\mathcal{M}_{k,d})}}{\|g\|_{L^p(\mathbb{R}^d)}} = \sup_{\{g:\|g\|_{L^p(\mathbb{R}^d)}\neq 0, g>0\}} \frac{C\|T^\sharp_{k,d}g\|_{L^q(\mathcal{M}_{k,d})}}{\|g\|_{L^p(\mathbb{R}^d)}}.$$

Therefore, a nonnegative function $f \in L^p(\mathbb{R}^d)$ is an extremizer of (1.2) if and only if it is a nonnegative extremizer of (5.2). As any extremizer has the form $f = c|f|$ for some complex number c, this suffices. □

Again, the pseudo-conformal symmetry J used to execute the method of competing symmetries in [17] is a natural symmetry of (5.2). Here, J intertwines with changing the identification of $\mathbb{R}^{(k+1)(d-k)} = \mathbb{R}^{k(d-k)} \times \mathbb{R}^{(d-k)} \simeq Mat(k, d-k) \times \mathbb{R}^{(d-k)} = \{(A, b)\}$ by interchanging b and the first row of A. Recall that $Jf = |s|^{-k-1}f(s^{-1}, s^{-1}y)$. Let A_b be the matrix A with the first row replaced by b and a_1 be the first row of A. Then define $R^\sharp F(A, b) = F(A_b, a_1)$.

Lemma 26. *For every $f \in L^p(\mathbb{R}^d)$,*

$$T^\sharp_{k,d}Jf = R^\sharp T^\sharp_{k,d}f.$$

Proof.

$$\begin{aligned} T^{\sharp}_{k,d}Jf(A,b) &= \int_{\mathbb{R}^k} (Jf(x', A(x')+b))dx' \\ &= \int_{\mathbb{R}^k} |s|^{-(k+1)} f(s^{-1}, s^{-1}x', s^{-1}(A(s,x')+b)))dx' \\ &= \int_{\mathbb{R}^k} (|s|^{-(k+1)} f(s^{-1}, s^{-1}x', (A(1, s^{-1}x') + s^{-1}b)))dx' \end{aligned}$$

.

Change variables so that $t = s^{-1}$ and $w = s^{-1}x'$ to obtain

$$\begin{aligned} T^{\sharp}_{k,d}Jf(A,b) &= \int_{\mathbb{R}^k} (|f(t, w', (A(1,w')+tb)))dx' \\ &= \int_{\mathbb{R}^k} (|f(t, w', (A_b(t,w')+a_1)))^{d+1}dx' \\ &= R^{\sharp}T^{\sharp}_{k,d}f. \end{aligned}$$

□

Chapter 6

Smoothness of extremizers when q and $\frac{1}{p-1}$ are integers

6.1 Main results and methods

While a characterization of extremizers of (1.3) for all q, or even all integer q, remains beyond us, in this chapter we prove that when q and $\frac{1}{p-1}$ are integers all extremizers are infinitely differentiable.

Recall that extremizers are functions which maximize the functional

$$\Phi(f) = \frac{\|T_{k,d}f\|_{L^q(\mathcal{M}_{k,d})}}{\|f\|_{L^p(\mathbb{R}^d)}}.$$

For $q \in (1, d+1]$, as discussed in Chapter 2, the nonnegative critical points of this functional satisfy the Euler-Lagrange equation (2.1):

$$f = \lambda (T^*_{k,d}[(T_{k,d}f)^{q_{el}}])^{p_{el}}$$

where $q_{el} = q - 1$, $p_{el} = \frac{1}{p-1}$, and $\lambda = \left(\|f\|^p_{L^p(\mathbb{R}^d)} \|T_{k,d}f\|^{-q}_{L^q(\mathcal{M}_{k,d})}\right)^{p_{el}}$.

We show

Theorem 9. *Let $d \geq 2$ and $\lambda \in \mathbb{R}$. Take $q_0 \in (1, d+1]$, $p_0 = \frac{d}{d-k+kq_0}$, such that both $q_0 - 1$ and $\frac{1}{p_0-1}$ are integers. Let $f \in L^{p_0}(\mathbb{R}^d)$ be any real-valued solution of the generalized Euler-Lagrange equation* (2.1)*. Then $f \in C^\infty$, all partial derivatives of f are bounded, and there exists $\delta > 0$ such that $(1 + |x|^2)^\delta D^s f(x) \in L^{p_0}(\mathbb{R}^d)$ for all $s \geq 0$.*

Corollary 3. *When $q_0 - 1$ and $\frac{1}{p_0-1}$ are integers, all extremizers of the corresponding L^p-L^q inequality are smooth.*

Proof. $T_{k,n}f \leq T_{k,n}|f|$. Thus, if f is an extremizer $|f|$ is a well, and moreover, $T_{k,n}f = T_{k,n}|f|$. From which it follows that $f = c|f|$. □

Complex-valued critical points also satisfy the generalized Euler-Lagrange equation (2.1) with λ as above, when the powers of the complex numbers on the right hand side of (2.1) are interpreted appropriately. Specifically, if $z \in \mathbb{C}$ and $0 \neq s \in \mathbb{R}$, z^s is interpreted as $z|z|^{s-1}$. When s is an odd integer, $s-1$ is even, and $|z|^{s-1}$ can be written as a product of positive integer powers of z and $\bar{z}$. This allows the argument for Theorem 9 to be carried out for complex-valued functions with straightforward modifications to the formulas to account for various complex conjugations, but only when both q_{el} and p_{el} are odd integers.

Remark 1. *Theorem 1 holds for complex valued solutions of the generalized Euler-Lagrange equation* (2.1) *when both $q_0 - 1$ and $\frac{1}{p_0-1}$ are odd integers.*

The condition that $\frac{1}{p-1}$ and $q-1$ are both integers is satisfied infinitely often. For any pair of integers q_{el},p_{el} not both 1, for any $s \in \mathbb{N}$ taking $q = q_{el}+1$, $n = q_{el}(p_{el}+1)s$, and $k = (p_{el}q_{el}-1)s$ gives $\frac{1}{p-1} = p_{el}$ and $q-1 = q_{el}$.

The methods come from Christ and Xue [16], which concerns extremizers of an L^p-L^q inequality for a convolution operator related to the Radon transform. To apply their technique requires a limited theory of weighted inequalities for the k-plane transform.

Such inequalities have been studied previously by Solmon [40] and Rubin [39]. Weighted inequalities corresponding to known mixed norm estimates for the k-plane transform of radial functions have been studied by Kumar and Ray [29]. Solmon [40] considers weights $v_\alpha(x) = \langle x\rangle^{\alpha-d}$ where $0 < \alpha \leq k < d$. He shows that for $1 \leq p < n/k$ and $0 < \alpha < k$, $T_{k,d}$ is a bounded operator from L^p to $L^1(v_\alpha)$. Rubin [39] considers weights of the form $|x|^\mu$ where $\mu > k - n/p$. He shows that for $1 \leq p \leq \infty$, $1/p + 1/p' = 1$, $\nu = \mu - k/p'$ and $\mu > k - n/p$, $T_{k,d}$ is a bounded operator from $L^p(|x|^\mu)$ to $L^p(|y|^\nu)$.

We consider weights $\langle x\rangle^{\frac{d-k}{p-1}}$. Note that unlike in Solmon's case the power is positive. We prove that $T_{k,d}$ is a bounded operator from $L^p(\langle x\rangle^{\frac{d-k}{p-1}})$ to $L^q(\langle y\rangle^d)$ where q and p are paired as in (1.3) (rather than from one weighted L^p another as in Rubin's work).

Throughout the chapter, D^s will denote the Fourier-multiplier operator defined by $D^s f(x) = (|\xi|^s \hat{f}(\xi))^\vee$ and analogously, $\mathscr{D}^s f(\theta, y) = (|\xi|^s \hat{f}(\theta,\xi))^\vee$ where the Fourier transform is taken only in the y variable.

6.2 Weighted inequalities

Define the following weights:

$$w(x) = (1+|x|^2)^{\frac{d-k}{2(p_0-1)}} = \langle x\rangle^{p_{el}(d-k)}$$
$$w_*(\theta, y) = (1+|y|^2)^{n/2} = \langle y\rangle^d.$$

Lemma 27. *There exist positive finite constants C_1 and C_0 depending only on q_0, n and k, such that*

$$T_{k,d}(w^{-1}) = C_0 w_*^{-1/q_{el}}$$
$$T^*_{k,d}(w_*^{-1}) = C_1 w^{-1/p_{el}}.$$

The lemma follows from direct computation and is a special case of the formulas given in ([38], Example 2.2).

Lemma 28. *There exists a constant C such that for all $t \in [0,1]$, for all nonnegative functions $f : \mathbb{R}^d \to [0,\infty)$ and $g : \mathcal{M}_{k,d} \to [0,\infty)$,*

$$\left(\int_{\mathcal{M}_{k,d}} (T_{k,d}f)^{q_t} w_*^{tq_t/q_{el}} d\lambda_{\theta^\perp}(y)d\theta\right)^{1/q_t} \leq C\left(\int_{\mathbb{R}^d} f^{p_t} w^{tp_t} dx\right)^{1/p_t}$$

$$\left(\int_{\mathbb{R}^d} (T^*_{k,d}g)^{p'_t} w^{tp'_t/p_{el}} dx\right)^{1/p'_t} \leq C\left(\int_{\mathcal{M}_{k,d}} g^{q'_t} w_*^{tq'_t} d\lambda_{\theta^\perp}(y)d\theta\right)^{1/q'_t}$$

where $q_t = q_0/(1-t)$, $p_t = p_0/(1-t)$, p'_0 and q'_0 are the Hölder conjugates of p_0 and q_0, $p'_t = p'_0/(1-t)$ and $q'_t = q'_0/(1-t)$.

Proof. The proof relies on complex interpolation. Consider the analytic family of operators $T_z f = w_*^{z/q_{el}} T_{k,d}(w^{-z}f)$ on the strip $\{z : 0 \leq Re(z) \leq 1\}$. If $Re(z) = 0$ then $T_z f$ is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{q_0}(\mathcal{M}_{k,d})$. If $Re(z) = 1$ then $T_z f$ is bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathcal{M}_{k,d})$, by Lemma 27. Both bounds are uniform in $Im(z)$. Therefore, the first conclusion follows by complex interpolation.

The proof of the second inequality is similar. □

Definition 10. *For all complex valued functions $f : \mathbb{R}^d \to \mathbb{C}$ and $g : \mathcal{M}_{k,d} \to \mathbb{C}$, set*

$$\mathcal{T}(f) = (T_{k,d}f)^{q_{el}}$$
$$\mathcal{T}_*(g) = (T^*_{k,d}g)^{p_{el}}$$
$$\mathcal{S}(f) = \mathcal{T}_*(\mathcal{T}(f)).$$

With this notation, the generalized Euler-Lagrange equation (2.1) becomes

$$f = \lambda \mathcal{S}(f).$$

We now define weighted spaces tailored to $\mathcal{T}$, $\mathcal{T}_*$, and $\mathcal{S}$.

Definition 11. *Define the spaces $X_t, X_{*,t}$, and $Y_{*,t}$ to be the sets of all equivalence classes of measurable functions on $\mathbb{R}^d$, $\mathcal{M}_{k,d}$, and $\mathcal{M}_{k,d}$ respectively, for which the following weighted norms are finite:*

$$||f||_{X_t}^{p_t} = \int_{\mathbb{R}^d} |f|^{p_t} w^{tp_t} dx$$

$$||g||_{X_{*,t}}^{q'_t} = \int_{\mathcal{M}_{k,d}} |g|^{q'_t} w_*^{tq'_t} d\lambda_{\theta^\perp} d\mu$$

$$||g||_{Y_{*,t}}^{q_t} = \int_{\mathcal{M}_{k,d}} |g|^{q_t} w_*^{tq_t/q_{el}} d\lambda_{\theta^\perp} d\mu.$$

Lemma 29. *There exists a constant C such that for any $t \in [0,1]$, for any functions $f \in X_t$ and $g \in X_{*,t}$*

$$\begin{aligned} ||T_{k,d}f||_{Y_{*,t}} &\leq C||f||_{X_t} \\ ||\mathcal{T}f||_{X_{*,t}} &\leq C||f||_{X_t}^{q_{el}} \\ ||\mathcal{T}_*g||_{X_t} &\leq C||g||_{X_{*,t}}^{p_{el}} \end{aligned}$$

Proof. Let $f \in X_t$. As $|T_{k,d}f| \leq T_{k,d}|f|$,

$$||T_{k,d}f||_{Y_{*,t}} = \left(\int_{\mathcal{M}_{k,d}} |T_{k,d}f|^{q_t} w_*^{tq_t/q_{el}}\right)^{1/q_t} \leq \left(\int_{\mathcal{M}_{k,d}} (T_{k,d}|f|)^{q_t} w_*^{tq_t/q_{el}}\right)^{1/q_t}.$$

By the weighted inequalities of Lemma 28,

$$||T_{k,d}f||_{Y_{*,t}} \leq C\left(\int_{\mathbb{R}^d} |f|^{p_t} w^{tp_t}\right)^{1/p_t} \leq C||f||_{X_t}.$$

Next,

$$||\mathcal{T}f||_{X_{*,t}} = \left(\int_{\mathcal{M}_{k,d}} |\mathcal{T}f|^{q_t'} w_*^{tq_t'}\right)^{1/q_t'} = \left(\int_{\mathcal{M}_{k,d}} |T_{k,d}f|^{q_{el}q_t'} w_*^{tq_t'}\right)^{1/q_t'}.$$

Because $q_{el} = q_0 - 1$, it so happens that $q_0' = \frac{q_0}{q_{el}}$ and hence $q_{el}q_t' = q_t$. Again using that $|T_{k,d}f| \leq T_{k,d}|f|$ and Lemma 28,

$$||\mathcal{T}f||_{X_{*,t}} \leq \left(\int_{\mathcal{M}_{k,d}} (T_{k,d}|f|)^{q_t} w_*^{t\frac{q_t}{q_{el}}}\right)^{q_{el}/q_t} \leq C\left(\int_{\mathbb{R}^d} |f|^{p_t} w^{tp_t}\right)^{q_{el}/p_t} \leq C||f||_{X_t}^{q_{el}}.$$

Lastly for $g \in X_{*,t}$

$$||\mathcal{T}_*g||_{X_t} = \left(\int_{\mathbb{R}^d} |\mathcal{T}_*g|^{p_t} w^{tp_t}\right)^{1/p_t} = \left(\int_{\mathbb{R}^d} |T_{k,d}^*g|^{p_{el}p_t} w^{tp_t}\right)^{1/p_t}.$$

Similarly, $p_{el} = \frac{1}{p_0 - 1}$, so it happens that $p_0' = p_0 p_{el}$ and hence $p_{el}p_t = p_t'$. Here using that $|T_{k,d}^*g| \leq T_{k,d}|g|$ and Lemma 28,

$$\begin{aligned} ||\mathcal{T}_*g||_{X_t} &\leq \left(\int_{\mathbb{R}^d} (T_{k,d}^*|g|)^{p_t'} w^{tp_t'/p_{el}}\right)^{p_{el}/p_t'} \\ &\leq C\left(\int_{\mathcal{M}_{k,d}} |g|^{q_t'} v^{-tq_t'}\right)^{p_{el}/q_t'} = C||g||_{X_{*,t}}^{p_{el}}. \quad \square \end{aligned}$$

Corollary 4. *There exists a constant C such that for all $t \in [0,1]$, for all $f \in X_t$,*

$$||\mathcal{S}f||_{X_t} \leq C||f||_{X_t}^{q_{el}p_{el}}.$$

We will need the following properties of these spaces, which are direct consequences of the definitions and Hölder's inequality.

Lemma 30. *The following statements hold for the spaces X_t. They hold as well when X_t is replaced by either $X_{*,t}$ or $Y_{*,t}$.*

1. *If $\alpha < \beta$ then $X_\beta \subset X_\alpha$. In particular, there exists a constant C such that for all $0 \leq \alpha \leq \beta \leq 1$, for all $f \in X_\beta$,*
$$\|f\|_{X_\alpha} \leq C\|f\|_{X_\beta}.$$
2. *Let $0 \leq \alpha \leq \gamma \leq \beta < 1$. Let $\gamma = \theta\alpha + (1-\theta)\beta$. Then,*
$$\|f\|_{X_\gamma} \leq \|f\|_{X_\alpha}^{\theta}\|f\|_{X_\beta}^{1-\theta}$$

Let A_p denote the standard Muckenhoupt classes of weights. Whenever $w^{tp_t} \in A_{p_t}$ operators of Calderón-Zygmund type are bounded on $L^{p_t}(w^{tp_t}) = X_t$ (see for instance [41], pg 205).

Lemma 31. *For all sufficiently small t, the weight $w = (1+|x|^2)^{\frac{p_{el}(d-k)}{2}}$ satisfies $w^{tp_t} \in A_{p_t}(\mathbb{R}^d)$.*

Proof. For any polynomial u, $|u|^s \in A_p$ if $-1 < s(deg(u)) < p-1$ ([41], pg. 219). Thus for $t \geq 0$ $w^{tp_t} \in A_{p_t}$ if

$$p_{el}(d-k)tp_t < p_t - 1.$$

Using that $p_t = \frac{p_0}{1-t}$, this holds whenever

$$t < \frac{(p_0-1)^2}{p_0(d-k-1)+1}.$$

As $p_0 > 1$ and $1 \leq k \leq n-1$ this bound is strictly positive.

□

We will also need similar results for the weights w_*, adapted to the spaces $Y_{*,t}$ and $X_{*,t}$.

Lemma 32. *Let $w_*(\theta,y) = \langle y\rangle^d$. For all sufficiently small t, for every $\theta \in \mathcal{G}_{k,d}$, $w_*^{tq_t'}(\theta,y) \in A_{q_t'}(\theta^\perp)$ and $w_*^{tq_t/q_{el}}(\theta,y) \in A_{q_t}(\theta^\perp)$.*

Proof. Using again that for any polynomial u, $|u|^s \in A_p$ if $-1 < s(deg(u)) < p-1$, for $t \geq 0$ $(w_*)^{tq_t'} \in A_{q_t'}$ if $ntq_t' < q_t' - 1$. Using that $q_t' = \frac{q_0}{(q_0-1)(1-t)}$, gives that $(w_*)^{tq_t'} \in A_{q_t'}$ if

$$t < \frac{1}{q_0 n - q_0 + 1}.$$

Secondarily, $(w_*)^{tq_t/q_{el}} \in A_{q_t}$ if $nt\frac{q_t}{q_{el}} < q_t - 1$. Using that $q_t = \frac{q_0}{1-t}$ and $q_{el} = q_0 - 1$ gives that $(w_*)^{tq_t/q_{el}} \in A_{q_t}$ if

$$t < \frac{(q_0-1)^2}{q_0(n-1)+1}.$$

Again note that each of these bounds is strictly positive.

□

6.3 Smoothing

Lemma 33 (Strichartz [44][1])**.** *For $p \in (1,2]$, there exists a constant C such that*

$$\|D_y^{k/p'} T_{k,d} f\|_{L^p(\mathcal{M}_{k,d})} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

Lemma 34. *Let $q_0 \in (1, d+1]$ such that $p_0 = \frac{dq_0}{d-k+kq_0}$ satisfies $p_0 \in (1,2]$. Let $t > 0$. There exists $\gamma = \gamma(t) > 0$ such that for any $f \in X_t$, $D_y^\gamma(Tf) \in L^{q_0}(\mathcal{M}_{k,d})$. In particular, there exists a positive finite constant C such that for any $f \in X_t$,*

$$\|D_y^\gamma(T_{k,d}f)\|_{L^{q_0}(\mathcal{M}_{k,d})} \leq C\|f\|_{X_t}.$$

Proof. Let $t > 0$. Let $f \in X_t$. From Lemma 29, $\|T_{k,d}f\|_{Y_{*,t}} \leq C\|f\|_{X_t}$. There exists $r > q_0$ such that the space $Y_{*,t}$ embeds continuously into $L^r(\mathcal{M}_{k,d})$;

$$\|T_{k,d}f\|_{L^r(\mathcal{M}_{k,d})} \leq C\|f\|_{X_t}.$$

As $p_0 \in (1,2]$, Lemma 33 yields

$$\|D_y^{k/p'} T_{k,d} f\|_{L^{p_0}(\mathcal{M}_{k,d})} \leq C\|f\|_{L^{p_0}(\mathbb{R}^d)} \leq C\|f\|_{X_t}.$$

Using the analytic family of operators $D_y^{zk/p'} T_{k,d}$ to interpolate between these two estimates gives that for $\theta \in [0,1]$,

$$\|D_y^{\theta k/p'} T_{k,d} f\|_{L^{Q(\theta)}(\mathcal{M}_{k,d})} \leq C\|f\|_{X_t}$$

where $Q(\theta)^{-1} = \frac{1}{p_0}\theta + \frac{1}{r}(1-\theta)$. As $q_0 > 1$, $p_0 = \frac{d}{d-k+kq_0}(q_0) < q_0$. Therefore there exists $\theta \in (0,1)$, such that $Q(\theta) = q_0$.

□

6.4 Multilinear Bounds

For the rest of the chapter we require that p_0 and $q_0$have the property that $p_{el}, q_{el} \in \mathbb{Z}$.

[1]Strichartz actually proves a stronger mixed norm estimate.

Definition 12. *Let $\vec{f} = \{f_{i,j}\}$ for $i \in [1, p_{el}], j \in [1, q_{el}]$. Define the multilinear operator $\vec{S}$ by:*

$$\vec{S}(\vec{f}) = \prod_{i=1}^{p_{el}} T^*_{k,d}\left(\prod_{j=1}^{q_{el}} T_{k,d}(f_{i,j})\right).$$

Thus $\mathcal{S}(f) = \vec{S}(f, \ldots, f)$.

Lemma 35. *For each $\vec{f} = \{f_{i,j}\}$ for $i \in [1, p_{el}], j \in [1, q_{el}]$,*

$$|\vec{S}(\vec{f})| \leq \prod_{i=1}^{p_{el}} \prod_{j=1}^{q_{el}} \mathcal{S}(|f_{i,j}|)^{\frac{1}{p_{el}q_{el}}}.$$

Proof. As both $|T^*_{k,d}(g)| \leq T^*_{k,d}(|g|)$ and $|T_{k,d}(f)| \leq T_{k,d}(|f|)$,

$$|\vec{S}(\vec{f})| \leq \prod_{i=1}^{p_{el}} T^*_{k,d}\left(\prod_{j=1}^{q_{el}} T_{k,d}(|f_{i,j}|)\right).$$

Also,

$$T^*_{k,d}\left(\prod_{j=1}^{q_{el}} g_j\right) = \int_{\mathcal{G}_{k,d}} \prod_{j=1}^{q_{el}} g_j(\theta, P(x, \theta^\perp))d\theta$$

By repeated applications of Hölder's inequality,

$$T^*_{k,d}\left(\prod_{j=1}^{q_{el}} g_j\right) \leq \prod_{j=1}^{q_{el}}\left(\int_{\mathcal{G}_{k,d}} g_j(\theta, P(x, \theta^\perp))^{q_{el}}d\theta\right)^{1/q_{el}} \leq \prod_{j=1}^{q_{el}} (T^*_{k,d}(g_j^{q_{el}}))^{\frac{1}{q_{el}}}.$$

Applying this for each i with $g_j = T_{k,d}(|f_{i,j}|)$ gives

$$|\vec{S}(\vec{f})| \leq \prod_{i=1}^{p_{el}} \prod_{j=1}^{q_{el}} T^*_{k,d}\left([T_{k,d}(|f_{i,j}|)]^{q_{el}}\right)^{\frac{1}{q_{el}}} = \prod_{i=1}^{p_{el}} \prod_{j=1}^{q_{el}} \mathcal{S}(|f_{i,j}|)^{\frac{1}{p_{el}q_{el}}}.$$

□

Lemma 35 and a weighted multilinear version of Hölder's inequality (Christ and Xue's Lemma 3.1, [16]) combine to give multilinear estimates for $\mathcal{S}$ which will be used in the following sections.

Lemma 36 (Christ and Xue, [16]). *Let $p_0 \in [1, \infty)$, $t > 0$, and $p_t = \frac{p_0}{1-t}$. Let $X_t \subset L^p(\mathbb{R}^d)$ with norm given by $||f||_{X_t}^{p_t} = \int_{\mathbb{R}^d} |f|^{p_t} w^{tp_t}$ for some measurable function $w \geq 1$. Let A be any finite index set. Let $\theta_\alpha, t_\alpha \in [0, 1]$ for each $\alpha \in A$. Suppose that $\sum_{\alpha \in A} \theta_\alpha = 1$ and $1 - t = \sum_{\alpha \in A} \theta_\alpha(1 - t_\alpha)$. Then for any nonnegative functions $\{f_\alpha\}_{\alpha \in A}$,*

$$|| \prod_{\alpha \in A} f_\alpha^{\theta_\alpha} ||_{X_t} \leq \prod_{\alpha \in A} ||f_\alpha||_{X_{t_\alpha}}^{\theta_\alpha}.$$

Lemma 37. *Let $t \in [0,1]$. Let $A = \{(i,j) : 0 \le i \le q_{el}, 0 \le j \le p_{el}\}$. Let $\{f_\alpha : \alpha \in A\}$ satisfy $f_\alpha \in X_t$ for all $\alpha \in A$. Then $\vec{S}(\vec{f}) \in X_t$ and*

$$\|\vec{S}(\vec{f})\|_{X_t} \le C \prod_{\alpha \in A} \|f_\alpha\|_{X_t}.$$

Proof. Estimating $\vec{S}$ in terms of $\mathcal{S}$ by Lemma 35,

$$\||\vec{S}(\vec{f})\|_{X_t} \le \| \prod_{\alpha \in A} (\mathcal{S}(|f_\alpha|))^{\frac{1}{p_{el}q_{el}}} \|_{X_t}.$$

Applying Lemma 36 with $\theta_\alpha = \frac{1}{p_{el}q_{el}}$ and $t_\alpha = t$ for each $\alpha \in A$,

$$\| \prod_{\alpha \in A} (\mathcal{S}(f_\alpha))^{\frac{1}{p_{el}q_{el}}} \|_{X_t} \le \prod_{\alpha \in A} ||\mathcal{S}(f_\alpha)||_{X_t}^{\frac{1}{p_{el}q_{el}}}.$$

The lemma then follows from the bound on $\mathcal{S}$ from Corollary 4.

□

Lemma 38. *Let $t \in [0, \frac{1}{p_{el}q_{el}}]$. Let $A = \{(i,j) : 0 \le i \le q_{el}, 0 \le j \le p_{el}\}$. Let $\{f_\alpha : \alpha \in A\}$ such that $f_\alpha \in X_0$ for all $\alpha \in A$ and suppose further that there exists $\beta \in A$ such that $f_\beta \in X_{tp_{el}q_{el}}$. Then*

$$\|\vec{S}(\vec{f})\|_{X_t} \le C \left(\|f_\beta\|_{X_{tp_{el}q_{el}}} \prod_{\alpha \ne \beta} \|f_\alpha\|_{X_0} \right).$$

Proof. Again, estimating $\vec{S}$ in terms of $\mathcal{S}$ by Lemma 35,

$$\||\vec{S}(\vec{f})\|_{X_t} \le \| \prod_{\alpha \in A} (\mathcal{S}(|f_\alpha|))^{\frac{1}{p_{el}q_{el}}} \|_{X_t}.$$

Apply Lemma 36 with $\theta_\alpha = \frac{1}{p_{el}q_{el}}$, $t_\alpha = 0$ for $\alpha \ne \beta$, and $t_\beta = p_{el}q_{el}t$ to obtain

$$\| \prod_{\alpha \in A} (\mathcal{S}(f_\alpha))^{\frac{1}{p_{el}q_{el}}} \|_{X_t} \le \prod_{\alpha \in A} ||\mathcal{S}(f_\alpha)||_{X_{t_\alpha}}^{\frac{1}{p_{el}q_{el}}}.$$

Using the bound on $\mathcal{S}$ from Corollary 4 and that $t_\alpha = 0$ for $\alpha \ne \beta$ and $t_\beta = p_{el}q_{el}t$,

$$\prod_{\alpha \in A} ||\mathcal{S}(f_\alpha)||_{X_{t_\alpha}}^{\frac{1}{p_{el}q_{el}}} \le C \prod_{\alpha \in A} ||f_\alpha||_{X_{t_\alpha}} \le C ||f_\alpha||_{X_{p_{el}q_{el}t}} \prod_{\alpha \ne \beta} ||f_\alpha||_{X_0}.$$ □

6.5 Extra Decay

Proposition 12. *Let $q_0 \in (1, d+1]$ such that for $p_0 = \frac{dq_0}{d-k+kq_0}$, $q_{el} = q_0 - 1$ and $p_{el} = \frac{1}{p_0-1}$ are both integers. Let $d \geq 2$ and $\lambda \in \mathbb{R}$. Let $f \in L^{p_0}(\mathbb{R}^d)$ be a real- valued solution of the generalized Euler-Lagrange equation $f = \lambda \mathcal{S} f$. Then there exists $t > 0$ such that $f \in X_t$.*

The proof given here is essentially the same as that of Proposition 5.1 in [16].
For any pair of functions $\varphi, g \in X_0$ such that, additionally, $\varphi \in L^\infty(\mathbb{R}^d)$, set

$$\mathcal{L}(\varphi, g) = \lambda \mathcal{S}(\varphi + g) - \lambda \mathcal{S}(g) - \varphi.$$

For each $\epsilon > 0$, chose a decomposition of f, $f = \varphi_\epsilon + g_\epsilon$, such that $\|g_\epsilon\|_{X_0} < \epsilon$ and $\varphi_\epsilon \in L^\infty$ has bounded support. Define

$$A_\epsilon(h) = \lambda \mathcal{S}(h) + \mathcal{L}(\varphi_\epsilon, g_\epsilon).$$

As f is a solution of the generalized Euler-Lagrange equation, g_ϵ is a solution of $A_\epsilon(h) = h$ in the space X_0. Proposition 12 follows from showing that g_ϵ in fact has better decay. The main step is the following lemma.

Lemma 39. *Let $q_0 \in (1, d+1]$ such that for $p_0 = \frac{dq_0}{d-k+kq_0}$, $q_{el} = q_0 - 1$ and $p_{el} = \frac{1}{p_0-1}$ are both integers. Let $d \geq 2$ and $\lambda \in \mathbb{R}$. Let $f \in L^{p_0}(\mathbb{R}^d)$ be a real- valued solution of $f = \lambda \mathcal{S}(f)$. For each $\epsilon > 0$, let $f = \varphi_\epsilon + g_\epsilon$ be any decomposition such that $\|g_\epsilon\|_{X_0} < \epsilon$ and $\varphi_\epsilon \in L^\infty$ has bounded support. Then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$, there exists $t_\epsilon > 0$ such that for all $t \in [0, t_\epsilon]$, the fixed point equation $A_\epsilon(h) = h$ has a unique solution $h \in X_t$ satisfying $\|h\|_{X_t} \leq \epsilon^{1/2}$.*

Proof. We begin by estimating $\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_t}$ for small t. First, as $g_\epsilon + \varphi_\epsilon$ is a solution to the generalized Euler-Lagrange equation (2.1), there exists $\lambda \in \mathbb{C}$ such that $g_\epsilon + \varphi_\epsilon = \lambda \mathcal{S}(g_\epsilon + \varphi_\epsilon)$. Therefore,

$$\mathcal{L}(\varphi_\epsilon, g_\epsilon) = g_\epsilon - \lambda \mathcal{S} g_\epsilon.$$

By the bound on $\mathcal{S}$ from Corollary 4 and the triangle inequality for $\epsilon \leq 1$,

$$\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_0} \leq \|g_\epsilon\|_{X_0} + C\|g_\epsilon\|_{X_0}^{p_{el}q_{el}} \leq C\epsilon.$$

Next consider $t = \frac{1}{p_{el}q_{el}}$ the largest value of t for which the multilinear estimate Lemma 38 applies. Working directly from the definition of $\mathcal{L}$,

$$\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_{1/p_{el}q_{el}}} \leq |\lambda| \|\mathcal{S}(\varphi_\epsilon + g_\epsilon) - \mathcal{S}(g_\epsilon))\|_{X_{1/p_{el}q_{el}}} + \|\varphi_\epsilon\|_{X_{1/p_{el}q_{el}}}.$$

Expand $\mathcal{S}(\varphi_\epsilon + g) - \mathcal{S}(g_\epsilon)$ as a sum of $p_{el}q_{el} - 1$ terms each of the general form $\vec{S}(\vec{f_i})$ where $\vec{f_i} = (f_{i,\alpha} : \alpha \in A)$ and $f_{i,\alpha} \in \{\varphi_\epsilon, g_\epsilon\}$ and for each term there is at least one index β such that $f_{i,\beta} = \varphi_\epsilon$. Applying Lemma 38 to each such term gives,

$$\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_{\frac{1}{p_{el}q_{el}}}} \leq C \sum_{i=1}^{p_{el}q_{el}} \left(\|\varphi_\epsilon\|_{X_1} \prod_{\alpha \neq \beta} \|f_{i,\alpha}\|_{X_0} \right) + \|\varphi_\epsilon\|_{X_{\frac{1}{p_{el}q_{el}}}}.$$

As $\|g_\epsilon\|_{X_0} < \epsilon$, and for each $s \in [0,1]$, $\|\varphi_\epsilon\|_{X_s} < \|\varphi_\epsilon\|_{X_1}$, there exists a finite constant depending on φ_ϵ such that

$$\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_{1/p_{el}q_{el}}} \leq C_{\varphi_\epsilon}.$$

By Lemma 30, in particular, by convexity of the X_t norms, for sufficiently small $\epsilon > 0$, there exists $t_\epsilon > 0$, such that for each $t \in (0, t_\epsilon]$,

$$\|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_t} \leq \epsilon^{3/4}.$$

Consider now bounds for A_ϵ. By the triangle inequality,

$$\|A_\epsilon(h)\|_{X_t} \leq |\lambda|\|\mathcal{S}h\|_{X_t} + \|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_t}.$$

Using the bound on $\mathcal{S}$ from Corollary 4,

$$\|A_\epsilon(h)\|_{X_t} \leq C\|h\|_{X_t}^{p_{el}q_{el}} + \|\mathcal{L}(\varphi_\epsilon, g_\epsilon)\|_{X_t}.$$

Let $B_t(0, \epsilon^{1/2})$ be the open ball of radius $\epsilon^{1/2}$ centered at 0 in X_t. If $t \in (0, t_\epsilon]$, for $h \in B_t(0, \epsilon^{1/2})$,

$$\|A_\epsilon(h)\|_{X_t} \leq C\epsilon^{p_{el}q_{el}/2} + \epsilon^{3/4}. \tag{6.1}$$

For sufficiently small ϵ it follows that $\|A_\epsilon(h)\|_{X_t} < \epsilon^{1/2}$. Thus for sufficiently small ϵ, for every $t \in (0, t_\epsilon]$,

$$A_\epsilon(B_t(0, \epsilon^{1/2})) \subset B_t(0, \epsilon^{1/2}).$$

Consider $\tilde{h}, h \in B_t(0, \epsilon^{1/2})$.

$$\|A_\epsilon(h) - A_\epsilon(\tilde{h})\|_{X_t} = C\|\mathcal{S}h - \mathcal{S}\tilde{h}\|_{X_t}.$$

Write out $\mathcal{S}(h)$ and $\mathcal{S}(\tilde{h})$ in terms of the multilinear operator $\vec{S}$. Adding and subtracting terms of the form $\vec{\mathcal{S}}(h, \ldots, h, \tilde{h}, \tilde{h}, \ldots, \tilde{h})$, allows one to write $\mathcal{S}(h) - \mathcal{S}(\tilde{h})$ as a sum of $p_{el}q_{el}$ terms, where each term is of the general form $\vec{S}(\vec{f})$ with $\vec{f} = (f_\alpha : \alpha \in A)$ such that there is one index β such that $f_\beta = h - \tilde{h}$, and for $\alpha \neq \beta$, f_α is either h or $\tilde{h}$. Applying the multilinear estimate from Lemma 37 to each such term gives that there exists C independent of ϵ such that

$$\|A_\epsilon(h) - A_\epsilon(\tilde{h})\|_{X_t} \leq C\epsilon^{(p_{el}q_{el}-1)/2}\|h - \tilde{h}\|_{X_t}.$$

Thus when ϵ is sufficiently small $A_\epsilon : B_t(0, \epsilon^{1/2}) \to B_t(0, \epsilon^{1/2})$ is a strict contraction. Therefore, there exists a unique $h_\epsilon \in X_t$ such that $\|h_\epsilon\|_{X_t} \leq \epsilon^{1/2}$ and $A_\epsilon(h_\epsilon) = h_\epsilon$.

□

Proof of Proposition 12. Let ϵ_0 be the small quantity guaranteed in Lemma 39. Fix $\epsilon \in (0, \epsilon_0]$ and let $0 \leq s \leq t \leq t_\epsilon$. Let $h \in X_s$, $\|h\|_{X_s} \leq \epsilon^{1/2}$ and $A_\epsilon(h) = h$. Similarly, let $\tilde{h} \in X_t$, $\|\tilde{h}\|_{X_t} \leq \epsilon^{1/2}$ and $A_\epsilon(\tilde{h}) = \tilde{h}$. As $\tilde{h} \in X_t$ and $s \leq t$, $\tilde{h} \in X_s$ and further,

$$\|\tilde{h}\|_{X_s} \leq \|\tilde{h}\|_{X_t} \leq \epsilon^{1/2}.$$

Hence by the uniqueness result in Lemma 39, $h = \tilde{h}$.

Taking $s = 0$ and $h = g_\epsilon$, this uniqueness implies that $g_\epsilon \in X_t$ for all $t \in [0, t_\epsilon]$ for all sufficiently small ϵ.

□

6.6 Mollified derivatives

The next goal is to show that if $f \in X_\rho$ for some $\rho > 0$, then its derivatives exist and behave well. Following Christ and Xue [16] we use mollified derivatives to prove the smoothness result. We correct a small technical error from [16] in the definition of these mollified derivatives.

Recall that $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz class of functions on $\mathcal{M}_{k,d}$. Let $\mathscr{S}(\mathcal{M}_{k,d})$ denote the class of functions on $\mathcal{M}_{k,d}$, satisfying "$f(\theta, y) \in \mathscr{S}(\theta^\perp) \sim \mathscr{S}(\mathbb{R}^{d-k})$, uniformly in θ". We call this class the Schwartz class of functions on $\mathcal{M}_{k,d}$.

Definition 13. *For all $f \in \mathscr{S}(\mathbb{R}^d)$, for each $s \geq 0$ and $\Lambda \geq 1$ define the operator D^s_Λ by*

$$\widehat{D^s_\Lambda f}(\xi) = \frac{\langle \xi\rangle^s}{\langle \Lambda^{-1}\xi\rangle^s}\hat{f}(\xi).$$

Similarly, for all $g(\theta, y) \in \mathscr{S}(\mathcal{M}_{k,d})$, for each $s \geq 0$ and $\Lambda \geq 1$ define $\mathscr{D}^s_\Lambda$ by

$$\widehat{\mathscr{D}^s_\Lambda g_\theta}(\xi) = \frac{\langle \xi\rangle^s}{\langle \Lambda^{-1}\xi\rangle^s}\widehat{g_\theta}(\xi).$$

where the Fourier transform is taken only in the y variable.

Lemma 40. *For all sufficiently small $\rho \geq 0$, for each $\Lambda > 0$, there exists a constant C_Λ, such that for all $f \in X_\rho$,*

$$\|D^s_\Lambda f\|_{X_\rho} \leq C_\Lambda \|f\|_{X_\rho}.$$

Proof. For sufficiently small ρ, $w^{\rho p} \in A_p$ by Lemma 31.By the Hörmander-Mihlin multiplier theorem (for the simple case used here see [41], pg. 26) it is enough to check that for each multi-index α, $\left|\partial^\alpha_\xi\left(\frac{\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\right)\right| \leq \frac{C_\alpha}{|\xi|^{|\alpha|}}$. Direct computation shows,

$$\left|\partial^\alpha_\xi\left(\frac{\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\right)\right| = \Lambda^s\left|\partial^\alpha_\xi\left(\frac{1+|\xi|^2}{\Lambda^2+|\xi|^2}\right)^{s/2}\right| \leq \frac{C_\alpha\Lambda^{s+2}}{|\xi|^{|\alpha|+2}} \leq \frac{C_\alpha\Lambda^s}{|\xi|^{|\alpha|}} \tag{6.2}$$

□

Moreover, these operators interact nicely with the k-plane and dual k-plane transforms. Let $\mathscr{S}(\mathcal{M}_{k,d})$ denote the class of functions on $\mathcal{M}_{k,d}$, satisfying "for all θ, $f(\theta, y) \in \mathscr{S}(\theta^\perp) \sim \mathscr{S}(\mathbb{R}^{d-k})$". We call this class the Schwartz class of functions on $\mathcal{M}_{k,d}$.

Lemma 41. *For all $f \in \mathscr{S}(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathcal{M}_{k,d})$ the following formulas hold:*

$$\mathscr{D}^s_\Lambda T_{k,d} f = T_{k,d} D^s_\Lambda f$$

$$D^s_\Lambda T^*_{k,d} g = T^*_{k,d}\mathscr{D}^s_\Lambda g. \tag{6.3}$$

Proof. The key fact is that $T_{k,d}$ and $T^*_{k,d}$ interact nicely with the Fourier transform. Recall the result of Lemma 2: For all $f \in \mathscr{S}(\mathbb{R}^d)$, for all $g \in \mathscr{S}(\mathcal{M}_{k,d})$, for all $\theta \in \mathcal{G}_{k,d}$, for all $\xi \in \theta^\perp$

$$\widehat{T_{k,d}f_\theta}(\xi) = \hat{f}(\xi)$$

and

$$\widehat{T^*_{k,d}g}(\xi) = \int_{\{\theta:\theta\perp\xi\}} \widehat{g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta) \tag{6.4}$$

where for functions on $\mathcal{M}_{k,d}$ the Fourier transform is taken only in the y-variable and $d\gamma_{\xi^\perp}$ represents the restriction of the measure $d\gamma$ to the subset of k-planes which are perpendicular to ξ. The notation $g(\theta,y) = g_\theta(y)$ is used to emphasize that on $\mathcal{M}_{k,d}$ the Fourier transform is taken only in the y-variable.

We prove (6.3). The proof of the other equation is similar.

$$D^s_\Lambda T^*_{k,d}g(x) = \left(\widehat{D^s_\Lambda T^*_{k,d}}g(\xi)\right)^\vee = \left(\langle\xi\rangle^s\langle\Lambda^{-1}\xi\rangle^{-s}\widehat{T^*_{k,d}g}(\xi)\right)^\vee$$
$$= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\langle\xi\rangle^s\langle\Lambda^{-1}\xi\rangle^{-s}\widehat{T^*_{k,d}g}(\xi)d\xi.$$

By (6.4),

$$= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\langle\xi\rangle^s\langle\Lambda^{-1}\xi\rangle^{-s}\int_{\{\theta:\theta\perp\xi\}}\widehat{g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta)d\xi.$$
$$= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\int_{\{\theta:\theta\perp\xi\}}\langle\xi\rangle^s\langle\Lambda^{-1}\xi\rangle^{-s}\widehat{g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta)d\xi.$$

By the definition of $\mathscr{D}^s_\Lambda$,

$$= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\int_{\{\theta:\theta\perp\xi\}}\widehat{\mathscr{D}^s_\Lambda g}(\theta,\xi)d\gamma_{\xi^\perp}(\theta)d\xi.$$

Using (6.4) again,

$$= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\widehat{T^*[\mathscr{D}^s_\Lambda g]}(\xi)d\xi.$$

□

Mollified derivatives, like fractional derivatives, fail to satisfy Leibniz's rule for derivatives of products. The following lemmas provide an adequate substitute in our situation.

Lemma 42. *Let $d \geq 2$. Let $s \in (0,\infty)$ and $\Lambda \in (4,\infty)$. Suppose that $r^{-1} = p_j^{-1} + q_j^{-1}$ for $j = 1, 2$ and that all exponents r, p_j, q_j belong to the open interval $(1,\infty)$. Let $u \geq 0$ be a locally integrable function on $\mathbb{R}^d$. Suppose further that the weight u belongs to A_r and that $u = u_1v_1 = u_2v_2$ where $u_j^{p_j/r} \in A_{p_j}$ and $v_j^{q_j/r} \in A_{q_j}$. Then there exists a constant C such that for all $\Lambda > 4$ the following inequality holds, whenever the right hand side is finite:*

$$\|D^s_\Lambda(fg)\|_{L^r(u)} \leq C\|D^s_\Lambda(f)\|_{L^{p_1}(u^{p_1/r})}\|g\|_{L^{q_1}(v^{q_1/r})} + C\|f\|_{L^{p_2}(u^{p_2/r})}\|D^s_\Lambda(g)\|_{L^{q_2}(v^{q_2/r})}.$$

This lemma is a modification of the standard fractional Leibniz rule or Kato-Ponce inequality [28] to the setting of mollified derivatives. The proof follows the methods of Christ and Weinstein in [13] and is similar to that in Christ and Xue, [16]. The proof is a routine application of ideas from weighted Calderón-Zygmund theory, but is rather long and consequently is deferred to the next section.

We also require a modified version for the space $\mathcal{M}_{k,d}$.

Lemma 43. *Let $s \in (0,\infty)$and $\Lambda \in (4,\infty)$. Suppose that $r^{-1} = p_j^{-1} + q_j^{-1}$ for $j = 1,2$ and that all exponents r, p_j, q_j belong to the open interval $(1,\infty)$. Let $u \geq 0$ be a locally integrable function on $\mathcal{M}_{k,d}$. Suppose the weight u_θ belongs to $A_r(\theta^\perp)$, uniformly in θ, and that for each $\theta \in \mathcal{G}_{k,d}$ $u_\theta = u_{\theta,1}v_{\theta,1} = u_{\theta,2}v_{\theta,2}$ where $u_{\theta,j}^{p_j/r} \in A_{p_j}(\theta^\perp)$ and $v_{\theta,j}^{q_j/r} \in A_{q_j}(\theta^\perp)$. Then there exists a constant C such that $\mathscr{D}^s_\Lambda(fg) \in L^r(\mathcal{M}_{k,d}, u)$ and the following inequality holds, whenever the right hand side is finite:*

$$\|\mathscr{D}^s_\Lambda(fg)\|_{L^r(\mathcal{M}_{k,d},u)} \leq C\|\mathscr{D}^s_\Lambda(f)\|_{L^{p_1}(\mathcal{M}_{k,d},u^{p_1/r})}\|g\|_{L^{q_1}(\mathcal{M}_{k,d},v^{q_1/r})} + C\|f\|_{L^{p_2}(\mathcal{M}_{k,d},u^{p_2/r})}\|\mathscr{D}^s_\Lambda(g)\|_{L^{q_2}(\mathcal{M}_{k,d},v^{q_2/r})}.$$

Proof.

$$\|\mathscr{D}^s_\Lambda(fg)\|^r_{L^r(\mathcal{M}_{k,d},u)} = \int_{\mathcal{G}_{k,d}} \left(\int_{\theta^\perp} |\mathscr{D}^s_\Lambda(fg)|^r u_\theta d\lambda_{\theta^\perp} y\right) d\theta.$$

By Lemma 42 applied to the inner integral for each θ,

$$\|\|\mathscr{D}^s_\Lambda(fg)\|^r_{L^r(\mathcal{M}_{k,d},u)} \leq C\int_{\mathcal{G}_{k,d}} \Big(\|D^s_\Lambda(f)\|_{L^{p_1}(\theta^\perp,u_{\theta,1}^{p_1/r})}\|g\|_{L^{q_1}(\theta^\perp,v_{\theta,1}^{q_1/r})} + \|f\|_{L^{p_2}(\theta^\perp,u_{\theta,2}^{p_2/r})}\|D^s_\Lambda(g)\|_{L^{q_2}(\theta^\perp,v_{\theta,2}^{q_2/r})}\Big)^r d\theta$$

Using Minowski's integral inequality,

$$\|\|\mathscr{D}^s_\Lambda(fg)\|^r_{L^r(\mathcal{M}_{k,d},u)} \leq C\int_{\mathcal{G}_{k,d}} \Big(\|D^s_\Lambda(f)\|_{L^{p_1}(\theta^\perp,u_{\theta,1}^{p_1/r})}\|g\|_{L^{q_1}(\theta^\perp,v_{\theta,1}^{q_1/r})}\Big)^r d\theta + C\int_{\mathcal{G}_{k,d}} \Big(\|f\|_{L^{p_2}(\theta^\perp,u_{\theta,2}^{p_2/r})}\|D^s_\Lambda(g)\|_{L^{q_2}(\theta^\perp,v_{\theta,2}^{q_2/r})}\Big)^r d\theta.$$

The lemma then follows by Hölder's inequality. □

Applying Lemmas 42 and 43 to $\|D^s_\Lambda \mathcal{S} f\|_{X_\varrho}$ using the definition of $\mathcal{S}$ gives the following result. The conditions of these lemmas are met because Lemmas 32 and 31, guarantee that for sufficiently small t, w^{tp_t} and $w_*^{tp_t}$ are in the appropriate A_p space.

Corollary 5. *For all $\rho \geq 0$ sufficiently small, there exist $\varrho' \leq \varrho$ both in $(0,\rho)$ and a constant C such that for all $s > 0$ and $\Lambda \geq 4$, for all $f \in X_\rho$ satisfying $\|f\|_{X_\rho} = 1$,*

$$\|D^s_\Lambda \mathcal{S} f\|_{X_\varrho} \leq C\|f\|^{p_{el}q_{el}-1}_{X_\rho}\|\mathscr{D}^s_\Lambda T_{k,d} f\|_{Y_{*,\varrho'}}.$$

We require one last technical lemma. Recall that $D^s f$ is defined such that $\widehat{D^s f}(\xi) = \langle \xi \rangle^s \hat{f}(\xi)$.

Lemma 44. *For all $\gamma > 0$, for all $s > 0$, there exists a constant C such that for all sufficiently small $t \geq 0$, for all $\Lambda \geq 1$, for all $h \in X_t$,*

$$\|D^s_\Lambda D^{-\gamma} f\|_{X_t} \leq C \|D^s_\Lambda f\|_{X_t}^{1-\gamma/s} \|f\|_{X_t}^{\gamma/s}.$$

Proof. First, the operator $D^s_\Lambda D^{-s}$ is a Fourier multiplier operator with

$$m(\xi) = \langle \xi \rangle^{-s} \frac{\langle \xi \rangle^s}{\langle \Lambda^{-1} \xi \rangle^s} = \frac{1}{\langle \Lambda^{-1} \xi \rangle^s}.$$

For each multi-index α,

$$\left| \partial^\alpha_\xi (m) \right| \leq C_\alpha |\xi|^{-|\alpha|}.$$

Therefore ([41], pg 26,205),

$$\|D^s_\Lambda D^{-s} f\|_{X_t} \leq C \|f\|_{X_t}.$$

Additionally, $D^{-i\sigma}$ is bounded on X_t with a norm $\lesssim \langle \sigma \rangle^c$ uniformly for all σ [16], whence

$$\|D^s_\Lambda D^{-s+i\sigma} f\|_{X_t} \leq C \langle \sigma \rangle^c \|f\|_{X_t}.$$

Trivially, $\|D^s_\Lambda f\|_{X_t} \leq C \|D^s_\Lambda f\|_{X_t}$.

The lemma follows from these two estimates by complex interpolation applied to the the analytic family of operators $D^s_\Lambda D^{-z}$. □

6.7 Proof of Lemma 42 (the fractional derivative inequality for mollified derivatives)

Proof. Fix $\eta \in \mathscr{S}(\mathbb{R}^n)$, a radial function, such that $\eta(\xi) = 1$ if $|\xi| \leq 1$ and $\eta(\xi) = 0$ if $|\xi| \geq 2$. For each $j \in \{0, 1, 2, \ldots\}$, define P_j by $\widehat{P_j f}(\xi) = \hat{f}(\xi)\eta(2^{-j}\xi)$. For $j \geq 1$, define Q_j by $Q_j = P_j - P_{j-1}$. Fix κ such that $2^{\kappa-1} \leq \Lambda \leq 2^\kappa$. As $\Lambda \geq 4$, $\kappa \geq 3$. Define $R_\kappa f = f - P_\kappa f$. Note that $\widehat{P_j f}$ is supported in $\{\xi : |\xi| \leq 2^{j+1}\}$, $\widehat{Q_j f}$ is supported in $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, and $\widehat{R_\kappa f}(\xi)$ is supported in $\{\xi : |\xi| \geq 2^\kappa\}$.

Decompose

$$f = P_\kappa f + R_\kappa f = P_2 f + \sum_{i=3}^{\kappa} Q_i f + R_\kappa$$

and decompose g similarly. Using this decomposition,

$$fg = \sum_{j=3}^{\kappa} Q_j f \cdot P_j g + \sum_{j=3}^{\kappa} Q_j g \cdot P_{j-1} f \tag{6.5}$$

$$+ P_2 f P_2 g \tag{6.6}$$

$$+ R_\kappa f P_\kappa g + R_\kappa g P_\kappa f + R_\kappa f \cdot R_\kappa g \tag{6.7}$$

Let us first consider the term $\|D^s_\Lambda P_2 f P_2 g\|_{L^r(u)}$.

As $\widehat{P_2 f P_2 g}$ is supported in $|\xi| < 2^4$,

$$(D^s_\Lambda P_2 f P_2 g)\hat{} = \frac{\langle \xi \rangle^s}{\langle \Lambda^{-1}\xi \rangle^s} \eta(2^{-4}\xi) \left(\widehat{P_2 f} * \widehat{P_2 g} \right).$$

The function $\frac{\langle \xi \rangle^s}{\langle \Lambda^{-1}\xi \rangle^s} \eta(2^{-4}\xi) \in C_0^\infty(\mathbb{R}^d)$ uniformly in Λ. Thus, Therefore,

$$\|D^s_\Lambda P_2 f P_2 g\|_{L^r(u)} \leq C \|P_2 f P_2 g\|_{L^r(u)}.$$

Applying Hölder's inequality,

$$\|P_2 f P_2 g\|_{L^r(u)} \leq C \|P_2 f\|_{L^{p_1}(u^{p_1/r})} \|P_2 g\|_{L^{q_1}(v^{q_1/r})}.$$

The operator P_2 is given by convolution with a Schwartz function independent of Λ so

$$\|P_2 g\|_{L^{q_1}(v^{q_1/r})} \leq C \|g\|_{L^{q_1}(v^{q_1/r})}.$$

Lastly,

$$\widehat{P_2 f} = \eta(2^{-2}\xi) \left(\frac{\langle \Lambda^{-1}\xi \rangle^s}{\langle \xi \rangle^s} \right) \frac{\langle \xi \rangle^s}{\langle \Lambda^{-1}\xi \rangle^s} \hat{f}(\xi).$$

Direct computation and the observation that as $\Lambda \geq 4$, $|\xi| \leq 2\Lambda$ on the support of $\eta(2^{-2}\xi)$ yields,

$$\left| \partial^\alpha_\xi \left(\frac{\eta(2^{-2}\xi)(1 + |\Lambda^{-1}\xi|^2)^{s/2}}{(1 + |\xi|^2)^{s/2}} \right) \right| \leq C_\alpha.$$

As, additionally, this multiplier is supported on $\{\xi : |\xi| < 8\}$ and $u_1^{p_1/r} \in A_{p_1}$, $\|P_2(f)\|_{L^{p_1}(u^{p_1/r})} \leq \|D^s_\Lambda(f)\|_{L^{p_1}(u^{p_1/r})}$. Therefore,

$$\|D^s_\Lambda(P_2 f P_2 g)\|_{L^r(u)} \leq C_{s,n} \|D^s_\Lambda(f)\|_{L^{p_1}(u^{p_1/r})} \|g\|_{L^{q_1}(v^{q_1/r})}.$$

Moving on to the terms in (6.5) , consider the contribution from $D^s_\Lambda(\sum_{j=3}^\kappa Q_j f P_j g)$. Further decompose this sum as

$$D^s_\Lambda \left(\sum_{j=3}^\kappa Q_j f P_j g \right) = D^s_\Lambda P_0 (\sum_{j=3}^\kappa Q_j f P_j g) + \sum_{l=1}^\infty D^s_\Lambda Q_l (\sum_{j=3}^\kappa Q_j f P_j g). \tag{6.8}$$

As above $\|D^s_\Lambda P_0(\sum_{j=3}^\kappa Q_j f P_j g)\|_{L^r(u)}$, is handled by Hölder's inequality and weighted Calderón-Zygmund theory. For any h, $\widehat{P_0(h)}$ is supported in $|\xi| < 2$,

$$\widehat{D^s_\Lambda P_0(h)} = \frac{\langle \xi \rangle^s}{\langle \Lambda^{-1}\xi \rangle^s} \eta(2^{-1}\xi) \widehat{h}.$$

Arguing as above,

$$\|D_\Lambda^s P_0(\sum_{j=3}^{\kappa} Q_j f P_j g)\|_{L^r(u)} \le C\|\sum_{j=3}^{\kappa} Q_j f P_j g\|_{L^r(u)}.$$

Further for any j, $|P_j g(x)| = |2^{-nj}\widehat{\eta}(2^j x) * g(x)| \le \|\psi'\|_{L^1}\mathcal{M}|g|$ where $\psi' \in L^1$ is a radial decreasing majorant for $\widehat{\eta}$. Thus,

$$|P_j g(x)| \le C\mathcal{M}|g|, \tag{6.9}$$

and

$$\|\sum_{j=3}^{\kappa} Q_j f P_j g\|_{L^r(u)} \le C\|\mathcal{M}(g)P_\kappa f\|_{L^r(u)}.$$

Applying Hölder's inequality,

$$\|\mathcal{M}(g)P_\kappa f\|_{L^r(u)} \le C\|P_\kappa(f)\|_{L^{p_1}(u^{p_1/r})}\|\mathcal{M}g\|_{L^{q_1}(v^{q_1/r})}.$$

As, $v^{q_1/r} \in A_{q_1}$, $\|\mathcal{M}g\|_{L^{q_1}(v^{q_1/r})} \le \|g\|_{L^{q_1}(v^{q_1/r})}$. Lastly,

$$\widehat{P_\kappa f} = \eta(2^{-\kappa}\xi)\left(\frac{\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\frac{\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\hat{f}(\xi).$$

Direct computation yields,

$$\left|\partial_\xi^\alpha\left(\frac{\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\right| \le \frac{C_\alpha\langle\Lambda^{-1}\xi\rangle^{s-|\alpha|}}{\langle\xi\rangle^{s+|\alpha|}} \tag{6.10}$$

Now, $\eta(\xi)$ is smooth and supported on $|\xi| \le 4$, whence for any $j > 0$,

$$\left|\partial_\xi^\alpha\left(\eta(2^{-j}\xi)\right)\right| \le C_\alpha\langle\xi\rangle^{-|\alpha|} \tag{6.11}$$

Combining (6.10) and (6.11) and using that $|\xi| \le 2^{\kappa+1} \le 4\Lambda$ on the support of $\eta(2^{-\kappa}\xi)$,

$$\left|\partial_\xi^\alpha\left(\frac{\langle\xi\rangle^s\eta(2^{-\kappa}\xi)}{\langle\Lambda^{-1}\xi\rangle^s}\right)\right| \le C_\alpha\langle\xi\rangle^{-|\alpha|}$$

Therefore, as $u_1^{p_1/r} \in A_{p_1}$,

$$\|P_\kappa(f)\|_{L^{p_1}(u^{p_1/r})} \le \|D_\Lambda^s(f)\|_{L^{p_1}(u^{p_1/r})}.$$

Finally,

$$\|D_\Lambda^s P_0(\sum_{j=3}^{\kappa} Q_j f P_j g)\|_{L^r(u)} \le C\|D_\Lambda^s(f)\|_{L^{p_1}(u^{p_1/r})}\|g\|_{L^{q_1}(v^{q_1/r})}.$$

Returning to the second term in (6.8), using that Fourier multiplier operators commute and weighted Littlewood-Paley theory

$$\left\| D^s_\Lambda \left(\sum_{l=1}^{\infty} Q_l (\sum_{j=3}^{\kappa} Q_j f P_j g) \right) \right\|_{L^r(u)} \leq C \left\| \left\{ \tilde{Q}_l D^s_\Lambda \left(\sum_{j=3}^{\kappa} Q_j f P_j g \right) \right\}_{l=-1}^{\infty} \right\|_{L^r(\ell^2,u)}$$

The first step is to prove

$$\left\| \left\{ \tilde{Q}_l D^s_\Lambda \left(\sum_{j=3}^{\kappa} Q_j f P_j g \right) \right\}_{l=1}^{\kappa+4} \right\|_{L^r(\ell^2,u)} \leq C \left\| \left\{ 2^{ls} \tilde{Q}_l \left(\sum_{j=3}^{\kappa} Q_j f P_j g \right) \right\}_{l=1}^{\kappa+4} \right\|_{L^r(\ell^2,u)} \tag{6.12}$$

For any function h_l supported in $\{\xi : |\xi| \leq 2^{l+1}\}$,

$$\widehat{D^s_\Lambda(h_l)} = \frac{2^{-ls}\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s} \eta(2^{-l-2}\xi) 2^{ls} \widehat{h_l}(\xi).$$

Define $M_l(h)$ by $\widehat{M_l h} = m_l \widehat{h}$ where $m_l = \frac{2^{-ls}\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\eta(2^{-l-2}\xi)$. Let $\vec{M}$ be the vector valued operator $\vec{M}(\{h_l\}) = \{M_l h_l\}$. Thus,

$$\left\| \left\{ \tilde{Q}_l D^s_\Lambda \left(\sum_{j=3}^{\infty} Q_j f P_j g \right) \right\}_{l=1}^{\kappa} \right\|_{L^r(\ell^2,u)} = \left\| \vec{M} \left\{ 2^{ls} \tilde{Q}_l \left(\sum_{j=3}^{\infty} Q_j f P_j g \right) \right\}_{l=1}^{\kappa} \right\|_{L^r(\ell^2,u)}.$$

First,

$$\left| \partial^\alpha_\xi \left(\frac{\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s} \right) \right| = \left| \partial^\alpha_\xi \left(\frac{\Lambda^s}{(\Lambda^2 + |\xi|^2)^{s/2}} (1 + |\xi|^2)^{s/2} \right) \right| \leq C_\alpha \langle\xi\rangle^{s-|\alpha|}.$$

Whence,

$$\left| \partial^\alpha_\xi \left(\frac{2^{-ls}\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s} \right) \right| \leq C_\alpha 2^{-ls} \langle\xi\rangle^{s-|\alpha|}. \tag{6.13}$$

Combining (6.13) and (6.11) with $j = l + 2$ and using that $\eta(2^{-l-2}\xi)$ is supported on $|\xi| \leq 2^{l+3}$,

$$\left| \partial^\alpha_\xi (m_l(\xi)) \right| \leq \left| \partial^\alpha_\xi \left(\frac{2^{-ls}\langle\xi\rangle^s \eta(2^{-l-2}\xi)}{\langle\Lambda^{-1}\xi\rangle^s} \right) \right| \leq C_\alpha \langle\xi\rangle^{-|\alpha|}$$

As this bound is independent of l, $\vec{M}$ is a vector valued Calderón-Zygmund operator [25] and, thus, is bounded from $L^r(\ell^2, u)$ to $L^r(\ell^2, u)$ for any $u \in A_r$ which proves (6.12).

Next, noting that $Q_l(Q_j f P_j g) = 0$ if $l > j + 4$ and using Minkowski's integral inequality,

$$\left\| \left\{ 2^{ls} \tilde{Q}_l \left(\sum_{j=3}^{\kappa} Q_j f P_j g \right) \right\}_{l=1}^{\infty} \right\|_{L^r(\ell^2,u)} \leq C \left\| \sum_{t=-4}^{\infty} \left(\sum_{j=3}^{\kappa} \left| 2^{(j-t)s} \tilde{Q}_{j-t} (Q_j f P_j g) \right|^2 \right)^{1/2} \right\|_{L^r(u)}$$

As the sum $\sum_{t=-4}^{\infty} 2^{-ts}$ is a finite constant depending only on s,

$$\left\| \sum_{t=-4}^{\infty} \left(\sum_{j=3}^{\kappa} \left| 2^{(j-t)s} \tilde{Q}_{j-t} \left(Q_j f P_j g \right) \right|^2 \right)^{1/2} \right\|_{L^r(u)} \leq C \left\| \left\{ 2^{js} \tilde{Q}_{j-t} \left(Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)}$$

Further for any l, $|Q_l g(x)| = |P_l g(x) - P_{l-1} g(x)| \leq C \mathcal{M}|g|$ by (6.9).

$$\left\| \left\{ 2^{js} \tilde{Q}_{j-t} \left(Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)} \leq C \left\| \left\{ \mathcal{M} \left(2^{js} Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)}.$$

This last quantity is the $L^r(u)$ norm of Fefferman and Stein's ([22]) vector-valued maximal function (with $q = 2$), which is well known to be bounded on $L^r(\ell^2, u)$ for $u \in A_r$ (see [1]). Thus,

$$\left\| \left\{ \mathcal{M} \left(2^{js} Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)} \leq C \left\| \left\{ \left(2^{js} Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)}.$$

Again by (6.9),

$$\left\| \left\{ \left(2^{js} Q_j f P_j g \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)} \leq C \left\| \mathcal{M}(g) \left\{ \left(2^{js} Q_j f \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)}$$

Therefore, by Hölder's inequality and the factorization $u = u_1 v_1$,

$$\left\| \mathcal{M}(g) \left\{ \left(2^{js} Q_j f \right) \right\}_{j=3}^{\kappa} \right\|_{L^r(\ell^2,u)} \leq C \left\| \mathcal{M} g \right\|_{L^{q_1}(v_1^{q_1/r})} \left\| \left\{ \left(2^{js} Q_j f \right) \right\}_{j=3}^{\kappa} \right\|_{L_1^p(\ell^2, u_1^{p_1/r})}$$

Moreover, $\left\| \mathcal{M} g \right\|_{L^{q_1}(v_1^{q_1/r})} \leq C \left\| g \right\|_{L^{q_1}(v_1^{q_1/r})}$ as $v_1^{q_1/r} \in A_{q_1}$.

Finally, recall that Q_j is supported on $\{\xi : 2^{j-1}|\xi| \leq 2^{j+1}\}$. Let $\zeta(\xi)$ a Schwartz cut-off function such that $\zeta(\xi) = 1$ for $1/2 \leq |\xi| < 2$ and $\zeta(\xi) = 0$ for $|\xi| < 1/4$ and $|\xi| > 4$. Then,

$$\widehat{2^{js} Q_j f} = (\eta(2^{-j}\xi) - \eta(2^{-j+1}\xi))\zeta(2^{-j}\xi) \left(\frac{2^{js} \langle \Lambda^{-1}\xi \rangle^s}{\langle \xi \rangle^s} \right) \frac{\langle \xi \rangle^s}{\langle \Lambda^{-1}\xi \rangle^s} \hat{f}(\xi).$$

Direct computation shows that

$$\left| \partial_\xi^\alpha \left(\frac{2^{js} \langle \Lambda^{-1}\xi \rangle^s}{\langle \xi \rangle^s} \right) \right| \leq \frac{C_\alpha 2^{js} \langle \Lambda^{-1}\xi \rangle^{s-|\alpha|}}{\langle \xi \rangle^{s+|\alpha|}}.$$

Now, $\zeta(\xi)$ also satisfies the estimate $|\partial^\alpha (\zeta(\xi))| \leq C_{\alpha,\eta} \langle \xi \rangle^{-|\alpha|}$. Further, $\zeta(2^{-j}\xi)$ is supported on $\{\xi : 2^{j-2} \leq |\xi| \leq 2^{j+2}\}$ and $2^{j+2} < 8\Lambda$ for $j \leq \kappa$. This yields,

$$\left| \partial_\xi^\alpha \left(\zeta(2^{-j}\xi) \frac{2^{js} \langle \Lambda^{-1}\xi \rangle^s}{\langle \xi \rangle^s} \right) \right| \leq C_\alpha \langle \xi \rangle^{-|\alpha|}.$$

As this bound is independent of Λ and j ([25],Theorem 3.16),

$$\left\| \{ 2^{js} Q_j f \}_{j=3}^{\kappa} \right\|_{L^{p_1}(\ell^2, u_1^{p_1/r})} \leq C \left\| \{ Q_j (D_\Lambda^s f) \}_{j=3}^{\kappa} \right\|_{L^{p_1}(\ell^2, u_1^{p_1/r})}.$$

Applying weighted Littlewood-Paley theory,

$$\left\|\{Q_j(D^s_\Lambda f)\}_{j=3}^{\kappa}\right\|_{L^{p_1}(\ell^2,u_1^{p_1/r})} \leq C \left\|D^s_\Lambda f\right\|_{L^{p_1}(u_1^{p_1/r})}.$$

Therefore,

$$\left\|D^s_\Lambda\left(\sum_{l=1}^{\infty} Q_l(\sum_{j=3}^{\kappa} Q_j f P_j g)\right)\right\|_{L^r(u)} \leq C \left\|D^s_\Lambda f\right\|_{L^{p_1}(u_1^{p_1/r})} \left\|g\right\|_{L^{q_1}(v_1^{q_1/r})}.$$

Finally,

$$\|D^s_\Lambda(\sum_{j=3}^{\kappa} Q_j f \cdot P_{j-3} g)\|_{L^r(u)} \leq C \left\|D^s_\Lambda f\right\|_{L^{p_1}(u_1^{p_1/r})} \left\|g\right\|_{L^{q_1}(v_1^{q_1/r})}.$$

The same argument with the roles of f and g reversed shows that

$$\|D^s_\Lambda(\sum_{j=3}^{\kappa} Q_j g \cdot P_{j-3} f)\|_{L^r(u)} \leq C \left\|f\right\|_{L^{p_2}(u_2^{p_2/r})} \left\|D^s_\Lambda g\right\|_{L^{q_2}(v_2^{q_2/r})}.$$

It remains to bound the terms in (6.7). Consider $\|D^s_\Lambda(R_\kappa f \cdot P_\kappa g)\|_{L^r(u)}$. Write,

$$D^s_\Lambda\widehat{(R_\kappa f P_\kappa g)} = \frac{2^{-s\kappa}\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\left(2^{s\kappa}\widehat{R_\kappa f} * \widehat{P_\kappa g}\right).$$

Recall that $2^{\kappa-1} \leq \Lambda \leq 2^\kappa$. Making use of the estimate (6.2),

$$\left|\partial^\alpha_\xi\left(\frac{2^{-s\kappa}\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\right)\right| \leq \frac{C_\alpha \Lambda^s 2^{-s\kappa}}{|\xi|^{|\alpha|}} \leq \frac{C_\alpha}{|\xi|^{|\alpha|}}.$$

Therefore, as $u \in A_r$ ([41], pg. 26, 205),

$$\|D^s_\Lambda(R_\kappa f \cdot P_\kappa g)\|_{L^r(u)} \leq C\|2^{s\kappa} R_\kappa f \cdot P_\kappa g\|_{L^r(u)}.$$

Using Hölder's inequality and the factorization $u = u_1 v_1$,

$$\|2^{s\kappa} R_\kappa f \cdot P_\kappa g\|_{L^r(u)} \leq C \left\|2^{s\kappa} R_\kappa f\right\|_{L^{p_1}(u_1^{p_1/r})} \left\|P_\kappa g\right\|_{L^{q_1}(v_1^{q_1/r})}.$$

Using (6.9),

$$\left\|P_\kappa g\right\|_{L^{q_1}(v_1^{q_1/r})} \leq C \left\|\mathcal{M} g\right\|_{L^{q_1}(v_1^{q_1/r})} \leq C \left\|g\right\|_{L^{q_1}(v_1^{q_1/r})}.$$

Finally, consider $\left\|2^{s\kappa} R_\kappa f\right\|_{L^{q_1}(v_1^{q_1/r})}$. Take a function ψ such that $\psi(\xi) = 1$ on $|\xi| \geq \kappa$, $\psi(\xi) = 0$ on $|\xi| \leq \kappa - 1$, and $\left|\partial^\alpha_\xi\left(\psi(\xi)\right)\right| \leq C_\alpha \langle\xi\rangle^{-|\alpha|}$. Then,

$$\widehat{2^{s\kappa} R_\kappa f} = \left(\frac{2^{s\kappa}\psi(\xi)\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\frac{\langle\xi\rangle^s}{\langle\Lambda^{-1}\xi\rangle^s}\hat{f}(\xi).$$

We again have the estimate,

$$\left|\partial_\xi^\alpha\left(\frac{2^{s\kappa}\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\right| \leq \frac{C_\alpha 2^{s\kappa}\langle\Lambda^{-1}\xi\rangle^{s-|\alpha|}}{\langle\xi\rangle^{s+|\alpha|}}.$$

Now, $2^\kappa \leq 2\Lambda$. Hence, when $|\xi| > \Lambda$,

$$\left|\partial_\xi^\alpha\left(\frac{2^{s\kappa}\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\right| \leq \frac{C_\alpha(\Lambda^2+|\xi|^2)^s}{\langle\xi\rangle^{s+|\alpha|}\langle\Lambda^{-1}\xi\rangle^{|\alpha|}} \leq C_\alpha\langle\xi\rangle^{-|\alpha|}.$$

Whence,

$$\left|\partial_\xi^\alpha\left(\frac{2^{s\kappa}\psi(\xi)\langle\Lambda^{-1}\xi\rangle^s}{\langle\xi\rangle^s}\right)\right| \leq C_\alpha\langle\xi\rangle^{-|\alpha|}.$$

The other remainder terms are addressed analogously with only the additional observation that $\|R_\kappa g\|_{L^{q_1}(v_1^{q_1/r})} = \|g - P_\kappa(g)\|_{L^{q_1}(v_1^{q_1/r})}$ and, thus, by (6.9)

$$\|R_\kappa g\|_{L^{q_1}(v_1^{q_1/r})} \leq C\,\|\mathcal{M}g\|_{L^{q_1}(v_1^{q_1/r})} \leq C\,\|g\|_{L^{q_1}(v_1^{q_1/r})}.$$

□

6.8 Conclusion of Proof

This section is devoted to the proof of the following lemma from which Theorem 9 follows.

Proposition 13. *Let $q_0 \in (1, d+1]$ such that for $p_0 = \frac{dq_0}{d-k+kq_0}$, $q_{el} = q_0 - 1$ and $p_{el} = \frac{1}{p_0-1}$ are both integers. Let $d \geq 2$, $\lambda \in \mathbb{R}$, and $\rho > 0$. Let $f \in X_\rho$ be any real- valued solution of the generalized Euler-Lagrange equation $f = \lambda\mathcal{S}f$. Let $\lambda \in \mathbb{C}$. Then there exists $\varrho > 0$ such that for all $s \geq 0$, $D^s f \in X_\varrho$.*

Proof of Theorem 9 using Proposition 13. By Proposition 12, if $f \in L^{p_0}(\mathbb{R}^d)$ is a solution of the generalized Euler-Lagrange equation $f = \lambda\mathcal{S}f$, there exists $t > 0$ such that $f \in X_t$. Thus the conditions of Proposition 13 are met, and there exists $\varrho > 0$ such that for all $s \geq 0$, $D^s f \in X_\varrho$. The theorem then follows by Sobolev embedding (see for instance, [42]). □

Proof of Proposition 13 . Fix $\lambda \in \mathbb{R}$. Let $f \in X_\rho$ for some $\rho > 0$ be any solution of the generalized Euler-Lagrange equation $f = \lambda\mathcal{S}f$. It suffices to consider $\|f\|_{X_\rho} = 1$, as $F = f/\|f\|_{X_\rho}$ will satisfy $f = \lambda\|f\|_{X_\rho}^{p_{el}q_{el}}\mathcal{S}f$.

Fix $s > 0$. It is enough to prove that there exists a finite constant C independent of Λ such that for all $\Lambda \geq 4$, $\|D_\Lambda^s f\|_{X_\varrho} < C$.

Using Corollary[2] 5 there exist $\varrho' \leq \varrho$ both in $(0, \rho)$,

$$\|D_\Lambda^s f\|_{X_\varrho} \leq C\|f\|_{X_\rho}^{p_{el}q_{el}-1}\|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho'}} = C\|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho'}}. \tag{6.14}$$

[2] This step assumes that p_{el}, q_{el} are integers so that $D_\Lambda^s\mathcal{S}(f)$ can be written as the derivative of a product.

As $0 \le \varrho' \le \varrho$, by Lemma 30, in particular convexity of the $Y_{*,t}$ norms, there exists $\theta \in (0,1)$ such that

$$\|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho'}} \le \|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho}}^{\theta} \|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,0}}^{1-\theta}.$$

By Lemma 41, on how mollified derivatives and the k-plane transform commute,

$$\|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho'}} \le \|T_{k,d} D_\Lambda^s f\|_{Y_{*,\varrho}}^{\theta} \|T_{k,d} D_\Lambda^s f\|_{Y_{*,0}}^{1-\theta}.$$

Using the estimate for $T_{k,d}$ given in Lemma 29,

$$\|\mathscr{D}_\Lambda^s T_{k,d} f\|_{Y_{*,\varrho'}} \le \|D_\Lambda^s f\|_{X_\varrho}^{\theta} \|T_{k,d} D_\Lambda^s f\|_{Y_{*,0}}^{1-\theta}.$$

From this and (6.14), we conclude that

$$\|D_\Lambda^s f\|_{X_\varrho} \le C \|T_{k,d} D_\Lambda^s f\|_{L^{q_0}(\mathcal{M}_{k,d})}. \tag{6.15}$$

Let $\gamma = \gamma(\varrho)$ as guaranteed by Lemma 34 which applies because if $p_{el} = \frac{1}{p_0 - 1}$ is an integer then $p_0 \in (1,2]$. Write $T_{k,d} D_\Lambda^s f = \mathscr{D}^\gamma T_{k,d}(D_\Lambda^s D^{-\gamma} f)$. Applying the estimate from Lemma 34,

$$\|T_{k,d} D_\Lambda^s f\|_{L^{q_0}(\mathcal{M}_{k,d})} = \|\mathscr{D}^\gamma T_{k,d}(D_\Lambda^s D^{-\gamma} f)\|_{L^{q_0}(\mathcal{M}_{k,d})} \le C \|D_\Lambda^s D^{-\gamma} f\|_{X_\varrho}.$$

Applying Lemma 44,

$$\|D_\Lambda^s D^{-\gamma} f\|_{X_\varrho} \le C \|D_\Lambda^s f\|_{X_\varrho}^{1-\gamma} \|f\|_{X_\varrho}^{\gamma}.$$

Combining this estimate and (6.15),

$$\|D_\Lambda^s f\|_{X_\varrho} \le C \|D_\Lambda^s f\|_{X_\varrho}^{1-\gamma}.$$

By Lemma 40, $\|D_\Lambda^s f\|_{X_\varrho} \le C_\Lambda \|f\|_{X_\varrho}$ and hence is finite. Therefore $\|D_\Lambda^s f\|_{X_\varrho} \le C$ where C is independent of Λ.

For $s = 0$ the estimate holds as $0 < \varrho < \rho$. □

Chapter 7

Cases of equality in certain multilinear inequalities of Hardy-Riesz-Brascamp-Lieb-Luttinger-Rogers type

This chapter represents the joint work of myself and Michael Christ.

7.1 Main result and methods

In this chapter we characterize cases of equality in certain Hardy-Riesz-Brascamp-Lieb-Luttinger-Rogers rearrangement inequalities.

Let $m \geq 2$ and $n \geq m+1$ be positive integers. For $j \in \{1, 2, \cdots, n\}$ let $E_j \subset \mathbb{R}$ be Lebesgue measurable sets with positive, finite measures, and let L_j be surjective linear maps $\mathbb{R}^m \to \mathbb{R}$. This paper is concerned with the nature of those n–tuples $(E_1, \cdots, E_n)$ of measurable sets that maximize expressions

$$I(E_1, \cdots, E_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{n} \mathbb{1}_{E_j}(L_j(x))\, dx,$$

among all n–tuples with specified Lebesgue measures $|E_j|$. Our results apply only in the lowest-dimensional nontrivial case, $m = 2$, but apply for arbitrarily large n.

Definition 14. *A family $\{L_j\}$ of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^1$ is nondegenerate if for every set $S \subset \{1, 2, \cdots, n\}$ of cardinality m, the map $x \mapsto (L_j(x) : j \in S)$ from $\mathbb{R}^m$ to $\mathbb{R}^S$ is a bijection.*

For any Lebesgue measurable set $E \subset \mathbb{R}^1$ with finite Lebesgue measure, E^* denotes the nonempty closed[1] interval centered at the origin satisfying $|E| = |E^*|$. Brascamp, Lieb, and Luttinger [6] proved that among sets with specified measures, the functional I attains its maximum value when each E_j equals E_j^*, that is,

$$I(E_1, \cdots, E_n) \le I(E_1^*, \cdots, E_n^*). \tag{7.1}$$

In this paper we study the uniqueness question and show that these are the only maximizing n–tuples, up to certain explicit symmetries of the functional, in those situations in which a satisfactory characterization of maximizers can exist.

Inequalities of this type can be traced back at least to Hardy and to Riesz [36]. In the 1930s, Riesz and Sobolev independently showed that

$$\iint_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(y)\mathbb{1}_{E_3}(x+y)\,dx\,dy \le \iint_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1^*}(x)\mathbb{1}_{E_2^*}(y)\mathbb{1}_{E_3^*}(x+y)\,dx\,dy$$

for arbitrary measurable sets E_j with finite Lebesgue measures. Brascamp, Lieb, and Luttinger [6] later proved the more general result indicated above, and in a yet more general form in which the target spaces $\mathbb{R}^1$ are replaced by $\mathbb{R}^k$ for arbitrary $k \ge 1$, satisfying an appropriate equivariance hypothesis.

The first inverse theorem in this context, characterizing cases of equality, was established by Burchard [8], [7]. The cases $n \le m$ are uninteresting, since $I(E_1, \cdots, E_n) = \infty$ for all $(E_1, \cdots, E_n)$ when $n < m$, and equality holds for all sets when $n = m$. The results of Burchard [7] apply to the smallest nontrivial value of n for given m, that is to $n = m+1$, but not to larger n. We are aware of no further progress in this direction since that time. This paper treats a situation at the opposite extreme of the spectrum of possibilities, in which $m = 2$ is the smallest dimension of interest, but the number $n \ge 3$ of factors can be arbitrarily large.

Burchard's inverse theorem has more recently been applied to characterizations of cases of equality in certain inequalities for the Radon transform and its generalizations the k–plane transforms [12],[**Me**]. Cases of near but not exact equality for the Riesz-Sobolev inequality have been characterized still more recently [10],[14].

As was pointed out by Burchard [8], a satisfactory characterization of cases of equality is possible only if no set E_i is too large relative to the others. This is already apparent for the trilinear expression associated to convolution,

$$I(E_1, E_2, E_3) = \iint \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(y)\mathbb{1}_{E_3}(x+y)\,dx\,dy;$$

if $|E_3| > |E_1| + |E_2|$ and if E_1, E_2 are intervals, then equality holds whenever E_3 is the union of an arbitrary measurable set with the algebraic sum of those two intervals.

Consider any expression $I(E_1, \cdots, E_n)$ where the integral is taken over $\mathbb{R}^m$, $E_j \subset \mathbb{R}^1$, and $L_j : \mathbb{R}^m \to \mathbb{R}^1$ are linear and surjective. Set $S_j = \{x \in \mathbb{R}^m : L_j(x) \in E_j\}$. Then $I(E_1, \cdots, E_n)$ is equal to the m–dimensional Lebesgue measure of $\cap_j S_j$. Define also

$$S_j^\star = \{x \in \mathbb{R}^m : L_j(x) \in E_j^*\}. \tag{7.2}$$

[1]A more common convention is that E^* should be open, but this convention will be convenient in our proofs. If $E = \emptyset$ then $E^* = \{0\}$, rather than the empty set, under our convention.

Definition 15. *Let $(L_j : 1 \le j \le n)$ be an n-tuple of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}$. An n–tuple $(E_j : 1 \le j \le n)$ of subsets of $\mathbb{R}^1$ is admissible relative to (L_j) if each E_j is Lebesgue measurable and satisfies $0 < |E_j| < \infty$, and if there exists no index k such that $S_k^\star$ contains an open neighborhood of $\bigcap_{j\ne k} S_j^\star$.*

(E_j) is strictly admissible relative to (L_j) if each set E_j is Lebesgue measurable, $0 < |E_j| < \infty$ for all j, and there exists no index k such that $S_k^\star$ contains $\bigcap_{j\ne k} S_j^\star$.

Once the maps L_j are specified, admissibility of $(E_1, \dots, E_n)$ is a property only of the n–tuple of measures $(|E_1|, \dots, |E_n|)$. Its significance is easily explained. Suppose that $(e_1, \cdots, e_n)$ is a sequence of positive numbers such that an n-tuple of sets with these measures is not admissible. The sets $E_j^*, S_j^\star$ are determined by e_j. Choose an index k such that $S_k^\star \supset \cap_{j\ne k} S_j^\star$. For $j \ne k$ set $E_j = E_j^*$. Choose the unique closed interval I centered at 0 such that the strip $S = \{x : L_k(x) \in I\}$ contains $\cap_{j\ne k} S_j^\star$, but $|I|$ is as small as possible among all such intervals. Choose E_k to be the disjoint union of I with an arbitrary set of measure $|E_k| - |I|$. Then $I(E_1, \cdots, E_n) = I(E_1^*, \cdots, E_n^*)$, yet $E_k \setminus I$ is an artibrary set of the specified measure. Thus without admissibility, extremizing n-tuples are highly nonunique.

Admissibility and strict admissibility manifestly enjoy the following invariance property. Let Φ be an affine automorphism of $\mathbb{R}^m$, and for $j \in \{1, 2, \cdots, n\}$ let Ψ_j be affine automorphisms of $\mathbb{R}^1$. Each composition $\Psi_j \circ L_j \circ \Phi$ is an affine mapping from $\mathbb{R}^m$ to $\mathbb{R}^1$. Write $\Psi_j \circ L_j \circ \Phi(x) = \tilde L_j(x) + a_j$ where $\tilde L_j : \mathbb{R}^m \to \mathbb{R}^1$ is linear. Define $\tilde E_j = \Psi_j(E_j)$ for all j. Then $(E_j : 1 \le j \le n)$ is admissible relative to $(L_j : 1 \le j \le n)$ if and only if $(\tilde E_j : 1 \le j \le n)$ is admissible relative to $(\tilde L_j : 1 \le j \le n)$. Strict admissibility is invariant in the same sense.

$A \triangle B$ will denote the symmetric difference of two sets. $|E|$ will denote the Lebesgue measure of a subset of either $\mathbb{R}^1$ or $\mathbb{R}^2$. We say that sets A, B differ by a null set if $|A \triangle B| = 0$.

The following theorem, our main result, characterizes cases of equality, in the situation in which $I(E_1, \cdots, E_n)$ is defined by integration over $\mathbb{R}^2$ and $E_j \subset \mathbb{R}^1$.

Theorem 10. *Let $n \ge 3$. Let $(L_i : 1 \le i \le n)$ be a nondegenerate n-tuple of surjective linear maps $L_i : \mathbb{R}^2 \to \mathbb{R}^1$. Let $(E_i : 1 \le i \le n)$ be an admissible n–tuple of Lebesgue measurable subsets of $\mathbb{R}^1$. If $I(E_1, \cdots, E_n) = I(E_1^*, \cdots, E_n^*)$ then there exist a point $z \in \mathbb{R}^2$, and for each index i an interval $J_i \subset \mathbb{R}$, such that $|E_i \triangle J_i| = 0$ and the center point of J_i equals $L_i(z)$. Conversely, $I(E_1, \cdots, E_n) = I(E_1^*, \cdots, E_n^*)$ in all such cases.*

We conjecture that Theorem 10 extends to arbitrary $m \ge 2$.

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7.2 On admissibility conditions

For maps L_j from $\mathbb{R}^m$ to the simplest target space $\mathbb{R}^1$, which is the subject of this paper, the most general case treated by Burchard [7] concerns

$$\int_{\mathbb{R}^m} \mathbb{1}_{E_0}(x_1 + x_2 + \cdots + x_m) \prod_{j=1}^{m} \mathbb{1}_{E_j}(x_j)\, dx_1 \cdots dx_m, \tag{7.3}$$

where m is any integer greater than or equal to 2. Cases of equality are characterized under the admissibility condition

$$|E_i| \le \sum_{j \ne i} |E_j| \ \text{ for all } i \in \{0, 1, 2, \cdots, m\}. \tag{7.4}$$

Strict admissibility is the same condition, with inequality replaced by strict inequality for all i. This single case subsumes many cases, in light of the invariance property discussed above.

Lemma 45. *For the expression* (7.3)*, admissibility in the sense* (7.4) *is equivalent to admissibility in the sense of Definition 15. Likewise, the two definitions of strict admissibility are mutually equivalent.*

Proof. $S_0^\star = \{x : |\sum_{j=1}^n x_j| \le \frac{1}{2}|E_0|\}$, while for $j \ge 1$, $S_j^\star = \left\{x : |x_j| \le \frac{1}{2}|E_j|\right\}$. Thus $|E_0| \ge \sum_{j=1}^n |E_j|$ if and only if

$$S_0^\star \supset \left\{x : |x_j| \le \tfrac{1}{2}|E_j| \ \text{ for all } 1 \le j \le n\right\} = \cap_{j=1}^n S_j^\star.$$

Likewise, strict inequality is equivalent to inclusion of $\cap_{j=1}^n S_j^\star$ in the interior of $S_0^\star$.

For any $i \in \{1, \cdots, n\}$,

$$\cap_{j \ne i} S_j^\star = \left\{x : |x_k| \le \tfrac{1}{2}|E_k| \ \text{ for all } \ k \ne i \in \{1, 2, \cdots, n\}\right\} \bigcap \left\{x : |\sum_{l=1}^n x_j| \le \tfrac{1}{2}|E_0|\right\}$$

while

$$S_i^\star = \left\{x : |x_i| \le \tfrac{1}{2}|E_i|\right\}.$$

Therefore $|E_i| \ge \sum_{0 \le j \ne i} |E_j|$ if and only if $S_i^\star \supset \cap_{0 \le j \ne i} S_j^\star$, and strict inequality is equivalent to inclusion of $\cap_{0 \le j \ne i} S_j^\star$ in the interior of $S_i^\star$. □

The case $m = 2$, $n = 3$ of Theorem 10 says nothing new. Indeed, let $(L_j : 1 \le j \le 3)$ be a nondegenerate family of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^1$. By making a linear change of coordinates in $\mathbb{R}^2$ we can make $L_1(x, y) \equiv x$ and $L_2(x, y) \equiv y$, so that

$$I(E_1, E_2, E_3) = c \int_{\mathbb{R}^2} \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(y) \mathbb{1}_{E_3}(ax + by)\, dx\, dy$$

where a, b are both nonzero. This equals

$$c' \int_{\mathbb{R}^2} \mathbb{1}_{E_1}(x/a)\mathbb{1}_{E_2}(y/b)\mathbb{1}_{E_3}(x+y)\,dx\,dy = c' \int_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}_1}(x)\mathbb{1}_{\tilde{E}_2}(y)\mathbb{1}_{E_3}(x+y)\,dx\,dy$$

where $\tilde{E}_j$ are appropriate dilates and reflections of E_j.

We will need the following simple result concerning the stability of strict admissibility.

Lemma 46. *Let $(L_j : 1 \le j \le n)$ be a nondegenerate family of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^1$. Let $(E_1, \cdots, E_n)$ be a strictly admissible n-tuple of Lebesgue measurable subsets of $\mathbb{R}^1$. There exists $\varepsilon > 0$ such that any n-tuple $(E_1, \cdots, E_n)$ of Lebesgue measurable subsets of $\mathbb{R}^1$ satisfying $\big| |E_j| - |F_j| \big| < \varepsilon$ for all $j \in \{1, 2, \cdots, n\}$ is strictly admissible.*

Proof. Suppose that no ε satisfying the conclusion exists. Then there exists a sequence of n-tuples $((E_{j,\nu}) : \nu \in \mathbb{N})$ such that $|E_{j,\nu}| \to |E_j|$ as $\nu \to \infty$, for each $j \in \{1, 2, \cdots, n\}$, and such that for each $\nu \in \mathbb{N}$, $(E_{n,\nu} : 1 \le j \le n)$ is not admissible.

Let $E^*_{j,\nu} \subset \mathbb{R}^1$ be the associated closed intervals centered at 0. Let

$$S^\star_{j,\nu} = \{x \in \mathbb{R}^m : L_j(x) \in E^*_{j,\nu}\}$$

be the associated closed strips. The failure of strict admissibility means that for each ν there exists $J(\nu)$ such that $S^\star_{J(\nu),\nu} \supset \cap_{j \ne J(\nu)} S^\star_{j,\nu}$. By passing to a subsequence we may assume that $J(\nu) \equiv J$ is independent of ν.

Since $|E_{j,\nu}| \to |E_j|$, the closed strips $S^\star_{j,\nu}$ converge to the closed strips $S^\star_j$ as $\nu \to \infty$, in such a way that it follows immediately that $S^\star_J \supset \cap_{j \ne J} S^\star_j$. Therefore $(E_1, \cdots, E_n)$ is not strictly admissible. □

7.3 Truncation

Definition 16. *Let $E \subset \mathbb{R}^1$ have finite measure. Let $\alpha, \beta > 0$. If $\alpha + \beta \le |E|$ then the truncation $E(\alpha, \beta)$ of E is*

$$E(\alpha, \beta) = E \cap [a, b] \tag{7.5}$$

where $a, b \in \mathbb{R}$ are respectively the minimum and the maximum real numbers that satisfy

$$|E \cap (-\infty, a]| = \alpha \text{ and } |E \cap [b, \infty)| = \beta.$$

In the degenerate case in which $\alpha + \beta = |E|$, $E(\alpha, \beta)$ has Lebesgue measure equal to zero, and may be empty or nonempty. According to our conventions, $E(\alpha, \beta)^* = \{0\}$ in this circumstance, in either case. This convention will be convenient below.

Lemma 47. *Let $k \ge 1$. Let $\{E_i : i \in \{1, 2, \cdots, k\}\}$ be a finite collection of Lebesgue measurable subsets of $\mathbb{R}^1$ with positive, finite Lebesgue measures. Let $\alpha, \beta > 0$, and suppose that $|E_i| \ge \alpha + \beta$ for each index i. If $\cap_{i=1}^k E_i(\alpha, \beta) \ne \emptyset$ then*

$$\int_{\mathbb{R}} \prod_{i=1}^k \mathbb{1}_{E_i}(y)\,dy \le \alpha + \beta + \int_{\mathbb{R}} \prod_{i=1}^k \mathbb{1}_{E_i(\alpha,\beta)}(y)\,dy. \tag{7.6}$$

If E_i are closed intervals and if $\cap_{i=1}^k E_i(\alpha,\beta) \neq \emptyset$ then equality holds in inequality (7.6).

This generalizes a key element underpinning the work of Burchard [8], which in turn is related, but not identical, to the construction employed by Riesz [36].[2]

Proof. For each index i, let $a_i, b_i \in \mathbb{R}$ respectively be the smallest and the largest real numbers satisfying $|E_i \cap (-\infty, a_i]| = \alpha$ and $|E_i \cap [b_i, \infty)| = \beta$. Thus $E_i = [a_i, b_i]$. Let $a = \max_i a_i$ and $b = \min_i b_i$. Then $\bigcap_i E_i(\alpha,\beta) = (\cap_i E_i) \cap [a,b]$. It is given that $\bigcap_i E_i(\alpha,\beta)$ is nonempty, so $a \le b$.

Thus

$$\int_{\mathbb{R}} \prod_{i=1}^k \mathbb{1}_{E_i(\alpha,\beta)}(y)\,dy = |\cap_i E_i(\alpha,\beta)| = |(\cap_i E_i) \cap [a,b]|.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}} \prod_{i=1}^k \mathbb{1}_{E_i}(y)\,dy - \int_{\mathbb{R}} \prod_{i=1}^k \mathbb{1}_{E_i(\alpha,\beta)}(y)\,dy &= |(\cap_i E_i) \setminus [a,b]| \\ &= |(\cap_i E_i) \cap (-\infty, a)| + |(\cap_i E_i) \cap (b,\infty)|.\end{aligned}$$

Choose l such that $a_l = a$. Then $(\cap_i E_i) \cap (-\infty,a) \subset E_l \cap (-\infty, a)$ and hence

$$|(\cap_i E_i) \cap (-\infty,a)| \le |E_l \cap (-\infty,a)| = \alpha.$$

Similarly $|(\cap_i E_i) \cap (b,\infty)| \le \beta$.

For the converse, suppose that the E_i are closed intervals, and that $\cap_i E_i(\alpha,\beta) \neq \emptyset$. Then $\cap_i E_i(\alpha,\beta) = [a,b]$ where $a \le b$, as above. In the same way, $\cap_i E_i = [a^\star, b^\star]$ where $a^\star$ is the maximum of the left endpoints of the intervals E_i, and $b^\star$ is the minimum of their right endpoints. Obviously $a^\star = a - \alpha$ and $b^\star = b + \beta$. □

The next lemma is evident.

Lemma 48. *Let $0 \le \alpha, \beta < \infty$. Let $\{I_k\}$ be a collection of closed bounded subintervals of $\mathbb{R}$ satisfying $|I_k| \ge \alpha + \beta$. Suppose that $\cap_k I_k(\alpha,\beta) \neq \emptyset$, and that J is a closed subinterval of $\mathbb{R}$ satisfying $J(\alpha,\beta) \supset \cap_k I_k(\alpha,\beta)$. Then $J \supset \cap_k I_k$.*

7.4 Deformation

We change notation: The number of sets E_j will be $n+1$, and the index j will run through $\{0, 1, \cdots, n\}$. The index $j = 0$ will have a privileged role.

Consider a functional

$$I(E_0, \cdots, E_n) = \int_{\mathbb{R}^2} \prod_{j=0}^n \mathbb{1}_{E_j}(L_j(x))\,dx,$$

[2] Riesz considers only the case of three sets, truncates all three in this fashion, uses only the case $\alpha = \beta$, and works directly with the integral over $\mathbb{R}^2$ which defines $I(E_1, \cdots, E_n)$, rather than with one-dimensional integrals.

with $\{L_j : 0 \le j \le n\}$ nondegenerate. The invariance under changes of variables noted above, together with this nondegeneracy, make it possible to bring this functional into the form

$$I(E_0, \cdots, E_n) = c \int_{\mathbb{R}} \mathbb{1}_{E_0}(x) \int_{\mathbb{R}} \prod_{j=1}^{n} \mathbb{1}_{E_j}(y + t_j x)\, dy\, dx$$

where c is a positive constant, and the t_j are pairwise distinct. This is accomplished by means of a linear change of variables in $\mathbb{R}^2$ together with linear changes of variables in each of the spaces $\mathbb{R}^1_j$ in which the sets E_j lie. The sets E_j which appear here are images of the original sets E_j under invertible linear mappings of $\mathbb{R}^1_j$, but equality holds in the inequality (7.1) for this rewritten expression $I(E_0, \cdots, E_n)$ if and only if it holds for the original expression, and the property of admissibility is preserved.

With $I(E_0, \cdots, E_n)$ written in this form,

$$\begin{aligned} S_0^\star &= \{(x,y) \in \mathbb{R}^2 : |x| \le \tfrac{1}{2}|E_0|\} \\ S_j^\star &= \{(x,y) \in \mathbb{R}^2 : |y + t_j x| \le \tfrac{1}{2}|E_j|\} \text{ for } 1 \le j \le n. \end{aligned}$$

Let $\pi : \mathbb{R}^2 \to \mathbb{R}^1$ be the projection $\pi(x,y) = x$. Define

$$\begin{aligned} &E_j(r) = E_j(\tfrac{1}{2}r, \tfrac{1}{2}r) \ \text{ for } j \ge 1 \text{ and } 0 < r \le |E_j| \\ &E_j(0) = E_j \\ &E_0(r) \equiv E_0. \end{aligned}$$

Thus $|E_j(r)| = |E_j| - r$ for $j \ge 1$. Let $S_j^\star(r)$ be the associated strips; $S_0^\star(r) = S_0^\star$ while for $j \ge 1$,

$$S_j^\star(r) = \left\{(x,y) \in \mathbb{R}^2 : |y + t_j x| \le \tfrac{1}{2}|E_j| - \tfrac{1}{2}r\right\}$$

for $0 \le r \le \min_j |E_j|$. Thus if $j \ge 1$ and $r = |E_j|$ then $S_j^\star(r)$ is a line in $\mathbb{R}^2$.

The cases $n \ge 3$ of the next lemma will later be used to prove Theorem 10 by induction on n.

Lemma 49. *Let $n \ge 2$. Let $\{E_j : 0 \le j \le n\}$ be a strictly admissible family of $n+1$ Lebesgue measurable subsets of $\mathbb{R}^1$. Then there exists $\bar r \in (0, \min_{1 \le j \le n} |E_j|)$ such that*

$$\begin{aligned} &(E_j(\bar r) : 0 \le j \le n) \ \text{ is admissible} \\ &S_0^\star \supset \cap_{j \ge 1} S_j^\star(\bar r). \end{aligned}$$

The second conclusion says in particular that $(E_j(\bar r) : 0 \le j \le n)$ fails to be strictly admissible. Because admissibility is a property of the measures of sets only with no reference to their geometry, Lemma 49 concerns deformations of intervals centered at 0 and of associated strips, not of more general sets.

Proof. Define $\bar r$ to be the infimum of the set of all $r \in [0, \min_{k \ge 1} |E_k|]$ for which $(E_j(r) : 0 \le j \le n)$ fails to be strictly admissible. If $r = \min_{k \ge 1 k \ge 1} |E_k| = |E_i|$ then $|E_i(r)| = 0$ and therefore

$(E_j(r) : 0 \le j \le n)$ is not strictly admissible. Thus $\bar{r}$ is defined as the infinmum of a nonempty set, and $0 \le \bar{r} \le \min_{k\ge 1} |E_k|$.

Since $(E_0, \cdots, E_n) = (E_0(0), \cdots, E_n(0))$ is strictly admissible, and since strict admissibility is stable under small perturbations of the type under consideration, the $n+1$-tuple $(E_0(r), \cdots, E_n(r))$ is strictly admissible for all sufficiently small $r \ge 0$. Therefore $\bar{r} > 0$.

Consequently the definition of $\bar{r}$ implies one of two types of degeneracy: Either $|E_l^*(\bar{r})| = 0$ for some $l \ge 1$, or there exists $i \in \{0, 1, \cdots, n\}$ such that

$$S_i^\star(\bar{r}) \supset \cap_{j \ne i} S_j^\star(\bar{r}). \tag{7.7}$$

Claim 4. *The inclusion* (7.7) *must hold for at least one index* $i \in \{0, 1, \cdots, n\}$.

Proof. If not, then the other alternative must hold; there exists an index l such that $|E_l^*(\bar{r})| = 0$. In that case, $S_l^\star(\bar{r})$ is by definition equal to the line $\{(x, y) : y + t_l x = 0\}$, which contains 0. For each index $j \ne l$, the intersection of $S_j^\star(\bar{r})$ with $\mathcal{L}$ is a nonempty closed interval of finite nonnegative length, centered at 0. Choose $i \ne l$ for which the length of $S_i^\star(\bar{r}) \cap \mathcal{L}$ is maximal. Then $S_i^\star(\bar{r})$ contains $S_i^\star(\bar{r}) \cap \mathcal{L}$, which in turn contains $S_j^\star(\bar{r}) \cap \mathcal{L}$ for every $j \notin \{i, l\}$. Therefore (7.7) holds for this index i. □

Let

$$K = \cap_{j=1}^n S_j^\star(\bar{r}),$$

which is a nonempty balanced convex subset of $\mathbb{R}^2$. K is compact, by the nondegeneracy hypothesis, since E_j^* are compact intervals.

$\pi(K) \subset \mathbb{R}$ is a compact interval centered at 0, as is E_0^*. Therefore[3] $\pi(K) \subset E_0^*$, or $E_0^* \subset \pi(K)$.

Claim 5. *If* $\pi(K) \supset E_0^*$ *and if an index* i *satisfies* (7.7)*, then* $i = 0$.

Proof. Suppose that $\pi(K) \supset E_0^*$ and that $i \ne 0$ satisfies (7.7). For $1 \le j \le n$ define the closed intervals

$$J(x, j, r) = \left\{ y \in \mathbb{R}^1 : (x, y) \in S_j^\star(r) \right\} \subset \mathbb{R}^1. \tag{7.8}$$

For any $x \in \pi(K)$, these intervals have at least one point in common. Since $S_i^\star(\bar{r}) \supset \cap_{j\ne i} S_j^\star(\bar{r})$,

$$J(x, i, \bar{r}) \supset \cap_{j \ne i} J(x, j, \bar{r}) \text{ for any } x \in E_0^*.$$

Therefore by Lemma 48,

$$J(x, i, 0) \supset \cap_{1 \le j \ne i} J(x, j, 0) \text{ for all } x \in E_0^*. \tag{7.9}$$

Since $S_0^\star = \pi^{-1}(E_0^*)$ it then follows that

$$S_i^\star \supset S_i^\star \cap \pi^{-1}(E_0^*) \supset \cap_{1\le j\ne i} S_j^\star \cap \pi^{-1}(E_0^*) = \cap_{0 \le j \ne i} S_j^\star,$$

contradicting the hypothesis that $(E_0, \cdots, E_n)$ is strictly admissible. □

[3]This apparently innocuous step is responsible for the restriction $m = 2$ in our main theorem.

Claim 6. $\pi(K)$ *cannot properly contain* E_0^*.

Proof. Suppose that $\pi(K)$ properly contains E_0^*. By the preceding Claim, (7.7) holds for $i = 0$. Let $x \in \pi(K) \setminus E_0^*$. There exists $y \in \mathbb{R}$ such that $(x, y) \in K$. Since $x \notin E_0^*$, $(x, y) \notin S_0^\star = \pi^{-1}(E_0^*)$. Therefore $K = \cap_{j\geq 1} S_j^\star(\bar{r})$ is not contained in $S_0^\star = S_0^\star(\bar{r})$, contradicting (7.7). □

Claim 7. $\pi(K)$ *is not properly contained in* E_0^*.

Proof. If $\pi(K)$ is properly contained in E_0^*, then it is contained in the interior of E_0^*, since each of these sets is a closed interval centered at 0. Consequently K is contained in the interior of $\pi^{-1}(E_0^*) = S_0^\star = S_0^\star(\bar{r})$; that is, $\cap_{j\geq 1} S_j^\star(\bar{r})$ is contained in the interior of $S_0^\star$. Therefore for every $r' < \bar{r}$ sufficiently close to $\bar{r}$, $\cap_{j\geq 1} S_j^\star(r')$ is contained in $S_0^\star$. Thus $(E_0(r'), \cdots, E_n(r'))$ fails to be strictly admissible. This contradicts the definition of $\bar{r}$ as the infimum of the set of all r for which $(E_0(r), \cdots, E_n(r))$ fails to be strictly admissible. □

Combining the above four claims, we conclude that (7.7) holds for $i = 0$ and for no other index, and that $\pi(K) = E_0^*$.

Claim 8. $|E_j(\bar{r})| > 0$ *for every index* $j \in \{0, 1, \cdots, n\}$.

Proof. If $|E_l(\bar{r})| = 0$ then since $E_0(\bar{r}) = E_0$, the index l cannot equal 0. $S_l^\star(\bar{r})$ is the line $\mathcal{L} = \{(x, y) : y + t_l x = 0\}$. For each $j \neq l$, $S_j^\star(\bar{r}) \cap \mathcal{L}$ is a closed subinterval of $\mathcal{L}$ centered at 0. Therefore K is equal to the smallest of these subintervals.

Since $\pi(K) = E_0^*$, and since $\pi : \mathcal{L} \to \mathbb{R}$ is injective, K must equal $\mathcal{L} \cap S_0^\star = S_l^\star(\bar{r}) \cap S_0^\star$. Therefore $S_j^\star(\bar{r}) \cap \mathcal{L} \supset S_0^\star(\bar{r}) \cap \mathcal{L}$. Therefore every $i \notin \{0, l\}$ satisfies (7.7). Since $n \geq 2$ there are at least three indices $0 \leq i \leq n$, so there exists at least one index $i \notin \{0, l\}$. But we have shown that the only such index is $i = 0$, so this is a contradiction. □

To conclude the proof of Lemma 49, it remains to show that $(E_0(\bar{r}), \cdots, E_n(\bar{r}))$ must be admissible. We have shown that $|E_j(\bar{r})| > 0$ for all j. The failure of admissibility is a stable property for sets with positive measures, so if $(E_0(\bar{r}), \cdots, E_n(\bar{r}))$ were not admissible then there would exist $0 < r < \bar{r}$ for which $(E_0(r), \cdots, E_n(r))$ was not admissible, contradicting the minimality of $\bar{r}$. □

7.5 Conclusion of the Proof

The proof of Theorem 10 proceeds by induction on the degree of multilinearity of the form I, that is, on the number of sets appearing in $I(E_1, \cdots, E_n)$. The base case $n = 3$ is a restatement of the one-dimensional case of Burchard's theorem, in its invariant form, since the two definitions of admissibility are equivalent.

Assuming that the result holds for expressions involving n sets E_j, we will prove it for expressions involving $n + 1$ sets. Let $(E_0, \cdots, E_n)$ be any admissible $n + 1$–tuple of sets satisfying $I(E_0, \cdots, E_n) = I(E_0^*, \cdots, E_n^*)$.

Consider first the case in which $(E_j : 0 \le j \le n)$ is not strictly admissible. Then there exists i such that $S_i^\star \supset \cap_{j\ne i} S_j^\star$. By permuting the indices, we may assume without loss of generality that $i = 0$. Then

$$I(E_0, \cdots, E_n) \le I(\mathbb{R}, E_1, \cdots, E_n) \le I(\mathbb{R}, E_1^*, \cdots, E_n^*) = I(E_0^*, \cdots, E_n^*),$$

so $I(\mathbb{R}, E_1, \cdots, E_n) = I(\mathbb{R}, E_1^*, \cdots, E_n^*)$.

Defining

$$J(E_1, \cdots, E_n) = I(\mathbb{R}, E_1, \cdots, E_n),$$

we have $J(E_1, \cdots, E_n) = J(E_1^*, \cdots, E_n^*)$. Now $(E_1, \cdots, E_n)$ is admissible relative to $\{L_j : 1 \le j \le n\}$. For if not, then there would exist $k \in \{1, 2, \cdots, n\}$ for which $S_k^\star$ properly contained $\cap_{1\le j\ne k} S_j^\star$. Since $S_0^\star \supset \cap_{j\ge 1} S_j^\star$,

$$\cap_{1\le j\ne k} S_j^\star = S_0^\star \cap (\cap_{1\le j\ne k} S_j^\star).$$

so $S_k^\star$ would properly contain $\cap_{0\le j\ne k} S_j^\star$, contradicting the hypothesis that $(E_0, \cdots, E_n)$ is admissible.

By the induction hypothesis, equality in the rearrangement inequality for J can occur only if E_j differs from an interval by a null set, for each $j \ge 1$. Moreover, there must exist a point $z \in \mathbb{R}^2$ such that for every $j \in \{1, 2, \cdots, n\}$, $L_j(z)$ equals the center of the interval corresponding to E_j.

For $j \ge 1$, replace E_j by the unique closed interval which differs from E_j by a null set. By an affine change of variables in $\mathbb{R}^2$, we can write $I(E_0, \cdots, E_n)$ in the form

$$c \int \mathbb{1}_{E_0}(x) \int \prod_{j=1}^{n} \mathbb{1}_{E_j}(y + t_j x)\, dy\, dx \tag{7.10}$$

where $c \in (0, \infty)$ and $t_j \in \mathbb{R}$, and now for each $j \ge 1$, E_j is an interval centered at 0. The inner integral defines a nonnegative function F of $x \in \mathbb{R}$ which is continuous, nonincreasing on $[0, \infty)$, even, and has support equal to a certain closed bounded interval centered at 0. The condition that $(E_0, \cdots, E_n)$ is admissible but $S_0^\star \supset \cap_{j=1}^n S_j^\star$ means that this support is equal to the closed interval E_0^*. Among sets E satisfying $|E| = |E_0|$, $\int_E F < \int_{\mathbb{R}} F$ unless E differs from E_0^* by a null set. We have thus shown that in any case of nonstrict admissibility, all the sets E_j differ from intervals by null sets, and the centers c_j of these intervals are coherently situated, in the sense that $c_j = L_j(z)$ for a common point $z \in \mathbb{R}^2$.

Next consider the case in which $(E_0, \cdots, E_n)$ is strictly admissible. Change variables to put $I(E_0, \cdots, E_n)$ into the form (7.10). This replaces the sets E_j by their images under certain invertible linear transformations, but does not affect the validity of the two conclusions of the theorem.

Let $\bar r$ be as specified in Lemma 49. Set $\tilde E_j = E_j(\bar r)$, and recall that $\tilde E_0 = E_0$. Let $\tilde S_j^\star$ be the strips in $\mathbb{R}^2$ associated to the rearrangements $\tilde E_j^*$. By Lemma 47,

$$\int_{\mathbb{R}} \prod_{j=1}^{n} \mathbb{1}_{E_j}(y + t_j x) dy \le \bar r + \int_{\mathbb{R}} \prod_{j=1}^{n} \mathbb{1}_{\tilde E_j}(y + t_j x)\, dy$$

for each $x \in E_0$. Multiplying both sides by $\mathbb{1}_{E_0}(x)$ and integrating with respect to x gives

$$\int_{\mathbb{R}} \mathbb{1}_{E_0}(x) \int_{\mathbb{R}} \prod_{i=1}^{n} \mathbb{1}_{E_j}(y + t_j x)\, dy\, dx \le \bar{r}|E_0| + \int_{\mathbb{R}} \mathbb{1}_{E_0}(x) \int_{\mathbb{R}} \prod_{i=1}^{n} \mathbb{1}_{\tilde{E}_j}(y + t_j x)\, dy\, dx.$$

Thus

$$I(E_0, \dots, E_n) \le \bar{r}|E_0| + I(E_0, \tilde{E}_1, \dots, \tilde{E}_n). \tag{7.11}$$

By the general rearrangement inequality applied to the $n+1$–tuple $(E_0, E_1, \dots, E_n)$,

$$\bar{r}|E_0| + I(E_0, \tilde{E}_1, \dots, \tilde{E}_n) \le \bar{r}|E_0| + I(E_0^*, \tilde{E}_1^*, \dots, \tilde{E}_n^*). \tag{7.12}$$

Since $(\tilde{E}_j : 0 \le j \le n)$ is admissible, for each $x \in E_0$ there exists y such that $(x, y) \in \cap_{j \ge 1} \tilde{S}_j^\star$. Therefore by the second conclusion of Lemma 47,

$$\int_{\mathbb{R}} \prod_{i=1}^{n} \mathbb{1}_{E_j^*}(y + t_j x)\, dy = \bar{r} + \int_{\mathbb{R}} \prod_{i=1}^{n} \mathbb{1}_{\tilde{E}_j^*}(y + t_j x)\, dy.$$

Integrating both sides of this inequality with respect to $x \in E_0^*$ gives

$$I(E_0^*, E_1^*, \dots, E_n^*) = \bar{r}|E_0^*| + I(E_0^*, \tilde{E}_1^*, \dots, \tilde{E}_n^*). \tag{7.13}$$

Combining (7.11), (7.12), and (7.13) yields

$$I(E_0, \dots, E_n) \le \bar{r}|E_0| + I(E_0, \tilde{E}_1, \dots, \tilde{E}_n) \le \\ \bar{r}|E_0| + I(E_0^*, \tilde{E}_1^*, \dots, \tilde{E}_n^*) = I(E_0^*, E_1^*, \dots, E_n^*)$$

We are assuming that $I(E_0, E_1, \dots, E_n) = I(E_0^*, \tilde{E}_1^*, \dots, \tilde{E}_n^*)$, so equality holds in each inequality in this chain. Hence

$$I(E_0, \tilde{E}_1, \dots, \tilde{E}_n) = I(E_0^*, \tilde{E}_1^*, \dots, \tilde{E}_n^*).$$

Thus the $n+1$–tuple $(E_0, \tilde{E}_1, \cdots, \tilde{E}_n)$ is admissible but not strictly admissible, and achieves equality in the inequality (7.1). This situation was analyzed above. Therefore we conclude that E_0 coincides with an interval, up to a null set.

The same reasoning can be applied to E_j for all j, by permuting the indices, so each of the sets E_j is an interval up to a null set. In this case (returning to the above discussion in which the index $j = 0$ is singled out), each interval E_j has the same center as $E_j(\bar{r})$. The discussion above has established that the centers of the intervals $E_j(\bar{r})$ are coherently situated.

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