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The Economics of Collaborative Production and Consumption with Applications in Digital
Technologies

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Economics

by

Michael Sacks

Dissertation Committee:
Professor Linda Cohen, Chair
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2016

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT OF THE DISSERTATION

The Economics of Collaborative Production and Consumption with Applications in Digital Technologies

By

Michael Sacks

Doctor of Philosophy in Economics

University of California, Irvine, 2016

Professor Linda Cohen, Chair

The first chapter models the general mechanisms/tradeoffs underpinning the dynamics of collaborative production using a club theoretic framework and drawing on tools from differential game theory. Individuals with preferences that are far from the objective of the club may not immediately split and form a new club. Instead they may take advantage of the increasing returns from club membership and incubate their new club within an existing one. In equilibrium, clubs may not be able to prevent this type of behavior even if it is undesired. Moreover, there are a range of conditions under which clubs may *encourage* incubation of future competitors to take advantage of increasing returns themselves and build up their own capital base.

The second chapter abstracts away from the dynamics and focuses on the static trade-offs. Existing public and club good models assume monotonicity in the utility of both consumption and provision. A wide range of public and club goods violate these assumptions. Accounting for appropriate non-monotonicities dramatically alters the equilibrium structure and welfare. When the utility from consumption is no longer monotonic, increasing the number of contributors mitigates the free-rider problem, rather than exacerbating it. When both the consumption value and provision cost are non-monotonic, increasing the number of

contributors not only mitigates the free-rider problem, but leads to an over-provision problem in which both the number of contributors and the intensity of contributions are inefficiently high. When the population is large, every equilibrium yields over-provision. Lastly welfare-maximizing policies involve transferring surpluses from consumers to producers.

The third chapter illustrates the competitive aspects of collaborative production in the context of the software industry. I address whether both proprietary and open source software will survive and how producers of proprietary software differentiate themselves from open source competition. I analyze competition between a firm producing proprietary software and a community producing open source software. If the firm faces no competition, then the software caters to less technologically savvy individuals. When facing competition, the open source software caters to the most technologically savvy individuals, leading the firm to target even less savvy individuals than it would when acting as a monopolist.

Chapter 1

In the Club: Strategic Admission, Membership, and Endogenous Splits

1.1 Introduction

Communities and clubs play a fundamental role across many branches of economics, including industrial organization, the economics of innovation, political economy, and religion, to name a few. Thus it is not surprising that, over the past sixty years, researchers have labored to understand the complex processes underlying individuals' participation decisions in clubs, the clubs' membership rules, and the subsequent production of club goods.[1] From this long body of literature, three results stand out (besides the obvious free-rider problem): firstly, increased heterogeneity among members decreases the level of contributions (Alesina et al., 1999; Alesina and La Ferrara, 2000), which in turn incentivizes the club to impose membership rules (Iannaccone, 1992; McBride, 2007a; Aimone et al., 2013). Secondly, the time horizon matters with respect to the level of contributions (Fershtman and Nitzan, 1991;

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[1]Buchanan (1965) defines a club as an ownership-membership agreement. I follow this definition throughout the remainder of the paper.

Wirl, 1996; Fujiwara and Matsueda, 2009), and the formation of new, competing clubs. Finally, individuals are strategic in club selection and location (Austin (1995); Alesina and Spolaore (1997); Casella (2001); Haimanko et al. (2004), among others). While models looking at heterogeneous agents, club formation, or time dynamics separately reveal many significant insights, this reductionist approach misses important features stemming from their interaction.

In this paper, I develop a dynamic model analyzing contributions to non-congestible club goods incorporating heterogeneous individuals, capital accumulation within clubs, and competition among clubs. Heterogeneity is modeled horizontally, where each individual is endowed with an ideal location and each club is assigned a location by its members. At any point in time, an individual or coalition of individuals can non-cooperatively orchestrate a schism, forming a child club that splits from the parent club. Since capital accumulates over time, those individuals that split need not leave empty-handed; rather, the amount of capital the coalition leaves with is dependent upon the degree of rivalry and excludability of the contributions, plus the distance between the parent club and the child club. As the degree of heterogeneity between the two clubs increases, the capital becomes less useful, which implies that capital depreciates according to distance.

Individuals strategically choose their club affiliation and contribution schedule, where affiliations may change over time as new clubs emerge. I characterize the existence of three types of equilibria: the *cohesive equilibrium* in which a single group persists in the long run, the *splitting equilibrium* in which clubs split at the outset, and the *incubation equilibrium* in which a delayed split occurs. I refer to the underlying strategy played in the incubation equilibrium as *strategic membership*. There exist conditions under which all three equilibria coexist.

In general, the existence and subsequent payoff ranking of the equilibria are determined by two interdependent tensions. The *contribution-scale tension* illustrates the underlying

2

tradeoff between economies of scale (club size) and heterogeneity. As individuals are heterogeneous, the introduction of more contributors need not increase the utility of a given member, even though the club good is non-congestible. To illustrate this point, consider the development of open source software (OSS). The development team must choose where to locate the project within the characteristic space (e.g. features and usability), à la Lancaster (1966). Adding another individual to the development team may bias the location decision, which in turn could make a subset of the development team worse off than if the individual was not a member, even with the additional contributions. The *mission-scope tension* illustrates the underlying tradeoff between depreciation, the chosen location of the child club, and the members' ideal location. Forming a child club located far away from the parent club is costly, as the accumulated capital becomes less valuable. Reconsider the development of OSS and suppose that an OSS project is forked (the technical term for the splitting of a project). The coalition that forks the project must take into account the relevance of the accumulated source code that the group is leaving with when choosing where to locate the new OSS project on the spectrum of possible projects. Locating too close to the original project will reduce the value of the split because fewer individuals will be developing a similar project. Locating too far from the original project will also reduce the value of a split because the usefulness of the accumulated source code is decreasing in the distance between the two projects, so the development team must essentially start from scratch.

One standard approach to discriminating among equilibria in dynamic settings is to compare steady-state payoffs by assuming that the discount rate is sufficiently small so that individuals are sufficiently patient. An analysis of this kind is rather misleading in my setting. The steady-state payoff in the splitting equilibrium is higher than the steady-state payoff in the incubation equilibrium for all individuals, while the payoff ranking of the cohesive equilibrium can vary. However, the incubation equilibrium can yield higher payoffs prior to a split occurring, along the path to the steady state, because incubation allows more rapid capital accumulation by both the parent and child clubs. Hence taking into account the full dynamic

path, I identify conditions under which the incubation equilibrium payoff dominates both the cohesive and splitting equilibria.

I also identify instruments that the parent club can use to shape decisions at both the extensive and intensive margins. When there is a misalignment of preferences between individuals who split and those who remain in the parent club, or preferences are aligned and incubation is preferred, the parent club can utilize one of two instruments to its advantage. The first involves taxation and subsidization. If the parent club prefers that incubation does (does not) occur, then it can institute a subsidy (tax) to encourage (discourage) such behavior. The club acts strategically to determine who is admitted. Furthermore, if its own members are sufficiently patient *and* preferences are aligned, then any rents from incubation can be captured fully by the parent club by instituting a delayed benefits scheme, where all members are taxed at the outset, only to have the payments returned post-split. Each individual who remains receives her original payment plus a share of the payments made by those who split. The second instrument involves the use of intellectual property restrictions, which the parent club can use to limit incubation behavior. By controlling the degree of excludability and depreciation through protections such as patents, copyrights, non-disclosure agreements, and non-compete clauses, the parent club can eliminate all benefits from incubation. Taxes and subsidies can then be used to select between the cohesive and splitting equilibria.

The insights of the model are applied to both the internal development of proprietary software (PS) by a for-profit software vendor and the development of OSS by a community. PS vendors developing product lines can benefit by implementing an internal form of incubation, where the development teams work together by first developing shared features as a cohesive unit, then separating to simultaneously and independently develop each product in the line. With respect to OSS, I restrict attention to the history of the derivatives of Unix, namely Linux and the BSD (Berkeley Software Distribution) family of operating systems. Much of the evolution of the Linux kernel, the various Linux distributions, and the BSD operating

systems can be better understood in the context of the model developed in this paper. Traditionally, authors argued that open source software can be best understood by treating the development as the private provision of a public good (Johnson, 2002; Polanski, 2007).[2] I argue that the development process, illustrated by the development of the Linux kernel, more closely resembles that of a club good.

This paper contributes to three strands of literature in club theory: the effects of heterogeneity, club dynamics, and competition between clubs. These strands of literature have, for the most part, evolved independently over time. This paper analyzes heterogeneity, club dynamics, and competition in a unified framework, providing a link between the strands. The literature on the theory of club goods can be traced back to the works of Buchanan (1965) and Olson (1965). The first models were nonstrategic, in that the analysis was typically conducted considering welfare rather than what are now considered standard game theoretic tools and concepts (e.g. Nash equilibrium). Buchanan formally introduced the idea of a club good - a good located in an intermediate position on the spectrum from purely private goods to purely public goods while Olson's book provided a treatment of many of the aspects of clubs themselves, rather than the goods they produce. Since those early works, much has been done to advance various branches of the literature. For more details, see Cornes and Sandler (1996); Glazer et al. (1997); Sandler and Tschirhart (1997); Scotchmer (2002), and the references therein.

Fershtman and Nitzan (1991) and Wirl (1996) were among the first to analyze dynamic models of contribution to public goods with capital accumulation and depreciation. They find that the free-rider problem illustrated in static settings can (Fershtman and Nitzan, 1991) but need not (Wirl, 1996) persist in dynamic settings, depending on the equilibrium strategy chosen.[3] The authors model a single club to which all agents belong, thus abstracting from

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[2]See Bergstrom et al. (1986) and Bernheim (1986) for details on the private provision of public goods

[3]The free-rider problem persists and becomes more severe when the agents play a Markov perfect equilibrium in linear strategies. Wirl (1996) shows that nonlinear Markov perfect equilibrium strategies exist in which there is no freeriding.

club issues apart from their public goods component. These findings were later generalized in Fujiwara and Matsueda (2009). The same intuition can be applied to club goods when all membership decisions must be made at the outset, as the setting is isomorphic to analyzing multiple public goods with independent populations of interest. A two-club Tiebout model is considered in Glomm and Lagunoff (1999), where the clubs are differentiated in the contribution mechanism: voluntary versus involuntary contributions. The authors find that the involuntary mechanism is more likely to prevail over time when individuals are sufficiently patient. The present paper differs from Glomm and Lagunoff (1999) by making the club formation decision endogenous while holding the contribution mechanism fixed (as voluntary).

Marx and Matthews (2000) take a similar dynamic approach to modeling a club, but focus on a single club with voluntary contributions. The authors model an environment where there is a threshold of contributions that, once met, allows for provision of the good. Their focus is analyzing the time to completion in all possible Nash equilibria, including Bayesian strategies and perfect Bayesian equilibria. Georgiadis (2015) considers a dynamic structure, akin to Marx and Matthews (2000), and finds that when individuals only receive the benefit when the project is complete and the project is sufficiently far from completion, members of a larger club contribute more than members of a smaller club. In my model, whenever incubation is payoff dominant, this finding holds even when individuals receive a benefit at every stage of the project.

Two works, Arnold and Wooders (2005, 2009), are close to this paper. The first paper examines dynamic club formation with myopic homogeneous agents, which places the focus on the equilibrium club size. The second again considers myopic agents, but in a hedonic game setting. In contrast, this paper develops a dynamic model with forward looking agents where the focus is on the members' decision to split along with the parent club's reaction to the split.

The literature on competition between clubs has developed along side the literature on club dynamics. Austin (1995) studies coordinated secession from public good jurisdictions in a club goods framework. Alesina and Spolaore (1997), Alesina et al. (1999), and Alesina and La Ferrara (2000) develop models of group selection participation in the context of nation building and local public good provision under (horizontally) heterogeneous agents. The first paper focuses on the optimal size of nations while the latter two develop models which address the membership decisions of individuals. Casella (2001) analyzes the relationship between the number and size of public jurisdictions as a function of the size of the underlying markets when there is heterogeneity among the underlying individuals. Jaramillo et al. (2003) also develop a heterogeneous model, but look at vertical heterogeneity in terms of income inequality. Haimanko et al. (2004) create a generalized model of club formation that analyzes the differences between competing clubs. Ahn et al. (2008), Polborn (2008), and Aimone et al. (2013) further this line of work by studying models of endogenous group formation under varying contexts, and all find multiple clubs as an equilibrium outcome. McBride (2008) and Carvalho and Koyama (2015) model club competition in the context of religious schism. However, none of the works mentioned above characterize the timing of competition.

The remainder of the paper is structured as follows. Section 1.2 outlines a static version of the model and provides a comprehensive analysis of the underlying tradeoffs. Dynamics are introduced and analyzed in Section 1.3. In Section 1.4, I describe two instruments available to the parent club which can be used to alter the payoff ranking of the various equilibria: strategic admission and strategic loss. These insights are then applied in Section 1.5 to software development, both open source and proprietary. Section 1.6 discusses the implications of the paper and concludes.

1.2   Static Model

There is a community $\mathcal{N}$ consisting of a finite number of individuals N.[4] Each individual $i \in \mathcal{N}$ has a "preferred mission" $A_i \in \mathcal{A}$, where $\mathcal{A}$ is the set of missions, or *mission space*, henceforth referred to as a mission for short.

Assumption 1.1. *There exists a continuous function* $d : \mathcal{A} \times \mathcal{A} \to (0,1]$, *such that for any* $A, A', A'' \in \mathcal{A}$,

$$1 - d(A, A') = 0 \Leftrightarrow A = A' \tag{A1}$$

$$1 - d(A, A') = 1 - d(A', A) \tag{A2}$$

$$1 - d(A, A'') \leq 1 - d(A, A') + 1 - d(A', A'') \tag{A3}$$

$$\text{image}\left(d\left(\mathcal{A}^2\right)\right) = (0, 1]. \tag{A4}$$

Under assumption 1.1, specifically (A1) - (A3), the pair $(\mathcal{A}, 1 - d)$ forms a metric space, where (A4) guarantees that the image of d spans the entire codomain.[5] The individuals are divided into two types, type 1 and type 2, endowed with missions A_m, $m = 1, 2$. Suppose that $N_1 > N_2$. I refer to type 1 individuals as majority individuals and type 2 individuals as minority individuals. Let $\mu \equiv \frac{N_1}{N_1 + N_2}$ denote the distribution of types.

There is a set of clubs $\mathcal{K}$. Each club $k \in \mathcal{K}$ has its own mission A^k and produces a single, non-congestible club good.[6] The mission of each club is selected by the club's members through some exogenously given cooperative or noncooperative bargaining process, such as

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[4]I use individual as a blanket term to refer entities such as people or organizations.

[5]I also normalize the codomain such that the metric spans the interval $(0, 1]$, rather than $\mathbb{R}_+ \cup \{0\}$. For example, if $\mathcal{A} \subseteq \mathbb{R}_+^\ell$, $\ell \geq 1$, then a suitable metric $1 - d$ is $1 - \exp(-\|A - A'\|)$, where $\| \cdot \|$ is the vector norm.

[6]Throughout the paper, superscripts index clubs and subscripts index individuals.

Nash, Jr. (1950), Kalai and Smorodinsky (1975), Rubinstein (1982), Cho and Duggan (2009), or Abreu and Pearce (2015).[7]

Definition 1.1. $d\left(A'', A'\right)$ *is the match value of an individual/club with mission A' to a club/individual with mission A''. A match value of one is referred to as a perfect match.*

The cardinality of $\mathcal{K}$ is endogenously determined in a manner to be specified. Each individual i chooses her affiliation to a single club k and her contribution level $x_i^k \geq 0$. The size of the club is characterized by two measures: the number of members N^k and new a measure I call the effective number of members.

Definition 1.2. *Let $\hat{N}^k \equiv \sum_{j \in k} d(A^k, A_j) \leq N^k$ denote the effective number of members of club k.*

The effective number of members measures club size by aggregating the affiliates and weighting each by her match value.

Each member i of club k receives utility from both the aggregate level of contributions to the club $\sum_{j \in k} x_j^k$ and her own individual contributions to the club. Contributing to a club is costly. Denote the marginal cost of contributing by ψx_i^k, $\psi > 0$. Let $d(A^k, A_i)b > 0$ denote the marginal benefit of consumption and let $d(A^k, A_i)\sigma \geq 0$ denote the marginal benefit of contributing. Generically, each individual $i \in k$ receives utility

$$u\left(x_i^k, x_{-i}^k\right) = \mathbf{1}\left\{x_i^k > 0\right\} d(A^k, A_i) \left(b \sum_{j \in k} x_j^k + \sigma x_i^k\right) - \frac{\psi}{2}\left(x_i^k\right)^2, \tag{1.1}$$

---

[7]The model developed in Abreu and Pearce (2015) is especially useful, as it incorporates an endogenous threat point into a Nash bargaining framework. The endogenous threat point can be treated as the game developed in this paper.

where $\mathbf{1}\{\cdot\}$ is a binary indicator function equaling unity if the argument is true and zero otherwise. Contributing $x_i^k = 0$ is akin to nonmembership and generates zero utility. Without loss of generality, I normalize $b = 1$ so that $d(A^k, A_i)\sigma$ is reinterpreted as the relative marginal benefit of contributing. Hence $\sigma = 0$, the typical case studied, corresponds to the marginal benefit of consumption being infinitely more valuable than the marginal benefit of contributing, and $\sigma \to \infty$ corresponds to the opposite. In addition, when $b = 1$, ψx_i^k is the relative marginal cost of contributing.

A strategy consists of two components for each individual i: a membership decision $i \in k$, and a contribution decision x_i^k. In the static analysis, I will be restricting attention to two Nash equilibria. The first is the cohesive equilibrium: a Nash equilibrium in which all individuals belong to a single cohesive club. The second is what I call a splitting (or separating) equilibrium: a Nash equilibrium in which the individuals sort by type into two distinct clubs, denoted by p for parent and c for child. In the splitting equilibrium, I assume that, without loss of generality, all majority individuals join club p while all minority individuals join club c.

1.2.1   Static Analysis

Suppose that there is a single cohesive club, endowed with mission $A(\mu)$, which satisfies $d(A(\mu), A_1) > d(A(\mu), A_2)$. That is, the match value is higher for the majority type. This relationship follows from many bargaining models, including those mentioned earlier. For expositional convenience, I suppress the affiliation script k as $|\mathcal{K}| = 1$. When the community is cohesive, each individual i's objective is given by

$$\max_{x_i} d(A(\mu), A_i) \left(\sum_{j=1}^{N} x_j + \sigma x_i \right) - \frac{\psi}{2} x_i^2,$$

10

It follows that for every i, the chosen contribution level is proportional to that individual's match value:

$$x_i^* = \left(\frac{1+\sigma}{\psi}\right) d(A(\mu), A_i), \tag{1.2}$$

and aggregate contributions are proportional to the effective number of individuals:

$$\sum_{j=1}^{N} x_j^* = \left(\frac{1+\sigma}{\psi}\right) \hat{N}.$$

Thus the indirect utility function for individual i is given by

$$u(x_i^*, x_{-i}^*) = \left(\frac{1+\sigma}{2\psi}\right) \left[2\hat{N} - (1-\sigma)d(A(\mu), A_i)\right] d(A(\mu), A_i). \tag{1.3}$$

Note that N does not enter the indirect utility function directly, but through the effective number of individuals $\hat{N} = \sum_j d(A(\mu), A_j) = \sum_m d(A(\mu), A_m)N_m$. Furthermore the indirect utility function is increasing and convex in $d(A(\mu), A_i)$.

Taken together, these two findings suggest that increasing the number of members need not increase an individual member's utility, unless a change in the composition of members does not affect $A(\mu)$. *If a change in the composition of members alters $A(\mu)$, then the utility of an individual can actually decrease, even when N increases.* For example, suppose that there are 80 majority individuals and 20 minority individuals and that $A(\mu)$ is defined such that $d(A(\mu), A_1) = \mu$ and $d(A(\mu), A_2) = 1 - \mu$. These assumptions imply that $d(A(\mu), A_1) = 0.8$ and $d(A(\mu), A_2) = 0.2$, which yields $\hat{N} = 68$. Now, suppose there are 80 majority individuals and 21 minority individuals, which alters the distribution slightly to μ', so $d(A(\mu'), A_1) = 0.792$ and $d(A(\mu'), A_2) = 0.208$, which yields $\hat{N} = 67.728$. Therefore $\hat{N}$ decreases and for majority individuals and $d(A(\mu'), A_1) < d(A(\mu), A_1)$, so majority individuals are strictly worse off. Thus when stating that a club benefits from economies of scale, I refer to an increase in utility from a rise in $\hat{N}$, rather than a rise in N_1 or N_2.

Existence of the cohesive equilibrium can be characterized under many conditions. I focus on two equivalent conditions, one relating to the match value $d(A(\mu), A_i)$ and the other relating to the coefficient σ in the relative marginal benefit of contributing. First consider a notion of disagreement. An individual i disagrees with a club k if the individual and club have different missions. Define a critical level of disagreement by

$$\bar{d}_i = \frac{-d(A(\mu), A_{-i})N_{-i} + \sqrt{[d(A(\mu), A_{-i})N_{-i}]^2 + (1+\sigma)[2N_i - (1-\sigma)]}}{2N_i - (1-\sigma)}. \tag{1.4}$$

I refer to the level of disagreement as having crossed the *club-disagreement threshold* if $d(A(\mu), A_i) \leq \bar{d}_i$. Next consider the coefficient σ. As σ increases, the relative marginal benefit of contributing increases, which implies that individuals are willing to give up economies of scale in exchange for a higher match value, even to the point of being the only member of a club. Denote the cutoff value where the coefficient in the marginal benefit from contributing outweighs economies of scale by

$$\bar{\sigma}_i = \frac{2d(A(\mu), A_i)d(A(\mu), A_{-i})N_{-i} + (2N_i - 1)d(A(\mu), A_i)^2 - 1}{1 - d(A(\mu), A_i)^2}. \tag{1.5}$$

Since $N_1 > N_2$, it is easier for the minority individuals to cross the thresholds $\bar{d}_i$ and $\bar{\sigma}_i$, as they have a lower match value and thus benefit less from economies of scale than majority individuals. Therefore attention can be restricted to minority individuals.

Proposition 1.1. *There exists a cohesive equilibrium if and only if the following two equivalent statements are true: (i) $d(A(\mu), A_2) \geq \bar{d}_2$, i.e., the match value is above the club-disagreement threshold; (ii) $\sigma \leq \bar{\sigma}_2$, i.e., the relative marginal benefit from contributing is below the cutoff value.*

The proof and all subsequent proofs can be found in Appendix A.2. Given that the club exhibits economies of scale, if no cohesive equilibrium exists, then a splitting equilibrium

must exist. When an individual has the incentive to form a club by herself, she can only be made better off by including more individuals who share her identity (are of the same type). However, this line of reasoning does not complete the analysis. The existence of a cohesive equilibrium need not imply non-existence of a splitting equilibrium. Rather, there may be multiple equilibria, where both the cohesive and splitting equilibria exist under the same set of parameters. For the remainder of the section, I drop the supposition that $|\mathcal{K}| = 1$.

If the members of the community select the splitting equilibrium, then each type m individual belongs to a club consisting of only other type m individuals. Since all members of a given club are homogeneous, the bargaining problem is trivial and the club's mission perfectly matches the mission of its members. The objective of each individual $i \in k$ is

$$
\max_{x_i^k} \mathbf{1} \left\{ x_i^k > 0 \right\} \left(\sum_{j \in k} x_j^k + \sigma x_i^k \right) - \frac{\psi}{2} \left(x_i^k \right)^2
$$

by choice of x_i^k. For every i, the chosen contribution level is given by

$$
x_i^{k*} = \frac{1 + \sigma}{\psi} \tag{1.6}
$$

if $i \in k$ and zero otherwise, generating indirect utility of

$$
u \left(x_i^{k*}, x_{-i}^{k*} \right) = \left(\frac{1 + \sigma}{2\psi} \right) \left[2N^k - 1 + \sigma \right].
$$

If a type m individual were to unilaterally deviate from the splitting equilibrium, then she does so by joining the competing club. A second notion of disagreement, analogous to club-disagreement, can be introduced: individual-disagreement. Two individuals i and j disagree if they have different missions. When this difference reaches a critical limit, I say the disagreement has crossed the *individual-disagreement threshold*. Denote this threshold

by

$$\hat{d}_{i=m} = \frac{-N_m + \sqrt{N_{-m}^2 + (1 + \sigma)[2N_m - (1 - \sigma)]}}{1 + \sigma}. \tag{1.7}$$

A second cutoff value of σ can be defined accordingly. When σ is large enough, the losses of economies of scale due to a lack of cohesion are outweighed by the gains in the relative marginal benefit of contributing. Denote the threshold where these effects counterbalance each other by

$$\hat{\sigma}(i, j) = \frac{[2N_j + d(A_i, A_j)]d(A_i, A_j) - 2N_i + 1}{1 - d(A_i, A_j)^2}. \tag{1.8}$$

If an individual has the incentive to deviate, then she must be leaving the smaller club to join the larger club. This fact implies that attention can again be restricted to a minority individual unilaterally deviating to join the club consisting of majority individuals.

Proposition 1.2. *There exists a splitting equilibrium if and only if the following two equivalent statements are true: (i) $d(A_1, A_2) \leq \hat{d}_2$, i.e., the degree of individual-disagreement crosses the individual-disagreement threshold; (ii) $\sigma \geq \hat{\sigma}(2, 1)$, i.e., the relative marginal benefit of contributing is above the cutoff value.*

Careful inspection of Propositions 1.1 and 1.2 support the claim of multiple equilibria. Multiple equilibria exist if $\sigma \in [\hat{\sigma}(2, 1), \overline{\sigma}(2, 1)]$, or equivalently, if $d(A(\mu), A_2)$ is not too small while $d(A_1, A_2)$ is not too large. For example, set $\sigma = 0$ and note that $\hat{\sigma}(2, 1) \leq 0$ if

$$d(A_1, A_2)N_1 - N_2 \leq -\frac{1}{2}\left(1 + d(A_1, A_2)^2\right)$$

The right-hand side is bounded between -1 and $-\frac{1}{2}$, so as $d(A_1, A_2) \to 0$, the inequality is

14

necessarily satisfied. $\overline{\sigma}(2,1) \geq 0$ if

$$\hat{N} \geq \frac{1}{2}(1 + d(A(\mu), A_2)).$$

The right-hand side is bounded between $\frac{1}{2}$ and 1, so for $d(A(\mu), A_1)$ or $d(A(\mu), A_2)$ large enough, the left-hand side is weakly larger than the right-hand side as $N_1 + N_2 > 1$.

The interesting aspect of multiple equilibria in this context is how the ranking of the payoffs in the two equilibria depend on the match values as well as the distribution of majority and minority individuals. There are conditions under which a minority individual "prefers" the splitting equilibrium over the cohesive equilibrium in terms of utility, and *vice versa*. Similarly, there exist conditions under which a majority individual prefers one equilibrium over the other. The conditions for each type of individual to prefer a specific equilibrium need not coincide, and depend explicitly on the members' match values. A third cutoff value can be defined representing the point at which a minority/majority individual's match value is such that the individual is indifferent between the two equilibria. Denote this value for a type m individual by

$$\tilde{d}_m(\mu) = \frac{-d(A(\mu), A_{-m})N_{-m} + \sqrt{[d(A(\mu), A_{-m})N_{-m}]^2 + [2N_m - (1-\sigma)]^2}}{2N_m - (1-\sigma)}. \qquad (1.9)$$

Proposition 1.3. *The splitting equilibrium payoff dominates the cohesive equilibrium for a type m individual if and only if $d(A(\mu), A_m) \leq \tilde{d}_m(\mu)$, or equivalently, the match value of a type m individual to the cohesive club is sufficiently small.*

If $d(A(\mu), A_2) \leq \tilde{d}_2(\mu)$ and $d(A(\mu), A_1) \geq \tilde{d}_1(\mu)$ (or the reverse), then majority preferences and minority preferences are *misaligned*. To illustrate that preference misalignment is in fact a possibility, note that $\frac{\partial \tilde{d}_m(\mu)}{\partial \sigma} > 0$ and $\tilde{d}_1(\mu) > \tilde{d}_2(\mu)$. Therefore up to three regions of σ

(a) Misalignment: majority prefers cohesion, minority prefers splitting.

(b) Misalignment: majority prefers splitting, minority prefers cohesion.
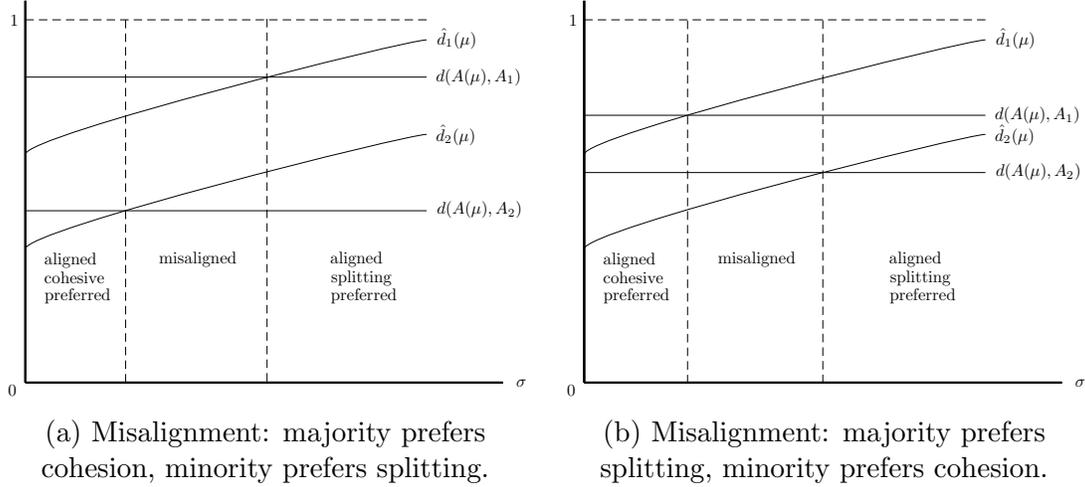
Figure 1.1: Preference (mis)alignment.

exist: one in which preferences are aligned and all individuals prefer the cohesive equilibrium, one in which preferences are aligned and all individuals prefer the splitting equilibrium, and one in which preferences are misaligned. The result in which preferences are misaligned and majority individuals prefer the cohesive equilibrium while minority individuals prefer the splitting equilibrium is illustrated in Figure 1.1(a). Note that majority individuals preferring the splitting equilibrium and minority individuals preferring the cohesive equilibrium is also a possibility.[8] This result is illustrated in Figure 1.1(b).

Propositions 1.1-1.3 illustrate a fundamental tension: the underlying tradeoff between match value and economies of scale. A cohesive club will often possess the greatest economies of scale since all individuals are members and total provision is strictly increasing in the number of members, *ceteris paribus*; however, the value of economies of scale is contingent upon the match value. A cohesive equilibrium implies heterogeneity among members.

Imagine that the feasible mission space is $\mathcal{A} \equiv [0, 1] \times [0, 1]$, and suppose that $A_1 = (1, 0)$ and $A_2 = (0, 1)$. For each distribution μ, any efficient bargaining process should lead to $A(\mu) = (y, 1 - y)$, for some $y \in [0, 1]$. Therefore the match value is strictly less than one for

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[8]For instance, when $N_1 \gg N_2$, so the benefits from an increase in club size are relatively small compared to the losses from a decrease in the match value.

at least one of the types. By segmenting the community according to type, those individuals with the lower match value necessarily find that the segmentation increases the value of contributions on both the extensive and intensive margins. Therefore total utility may in fact increase by sacrificing economies of scale in exchange for a higher match value. I call this tradeoff the contribution-scale tension.

Tension 1.1. (The Contribution-Scale Tension) *Under heterogeneity, there exists a tradeoff between economies of scale and match value. The economies of scale granted by a cohesive community necessarily decreases the match value for a non-degenerate subset of the community.*

In summary, there are two interesting equilibria in the static case: the cohesive equilibrium and the splitting equilibrium. There are conditions under which: (i) only a cohesive equilibrium exists; (ii) only a splitting equilibrium exists; (iii) both equilibria coexist. Conditional on both equilibria coexisting, there are conditions under which: (a) all individuals prefer the cohesive equilibrium; (b) all individuals prefer the splitting equilibrium; (c) There is preference misalignment across types, where either majority individuals receive a greater payoff in the cohesive equilibrium than in the splitting equilibrium, while minority individuals receive a greater payoff in the splitting equilibrium than in the cohesive equilibrium, or *vice versa*. In the dynamic extension of the model, to be described shortly, the tradeoffs described in the static setting all carry over. In addition, a second tension is introduced, which gives rise to a new equilibrium outcome of incubation/strategic membership.

1.3 Club Dynamics

In this section, I introduce a dynamic extension that relates directly to the static framework presented in the previous section and illustrates the underlying incentives for incubation and

strategic membership. It is not sufficient to focus on the steady state behavior. Rather, as I shall show, important insights are lost unless one analyzes the entire dynamic path. It turns out that analytical methods are not able to uncover all of the results with respect to incubation. When analytical results are not directly available, numerical methods are employed.

In Appendix A.1, I consider a second dynamic extension that restricts contributions from $\mathbb{R}_+$ to the binary space $\{0, 1\}$ and assumes complete myopia. These two restrictions allow for analytical solutions and more importantly lead to new results. A second form of incubation/strategic membership can be practiced by individuals called *passive strategic membership*, in which minority individuals do not contribute to any club at the outset. Instead, at some later time, they take the accumulated contributions of the parent club, consisting solely of majority individuals, and uses it to form a new competing child club. It is the discreteness of contributions that drives this new result.

1.3.1   Dynamic Model

Suppose that time, indexed by t, flows continuously with common discount rate $\delta > 0$. Contributions accumulate over time, but depreciate at the rate $\gamma \in (0, 1)$. At any time t, the change in accumulated contributions $C^k(t)$ to club k, henceforth referred to as installed base, is given by the dynamic

$$\dot{C}^k(t) = \sum_{j(t) \in k} x_j^k(t) - \gamma C^k(t), \tag{1.10}$$

where $x_i^k(t) \geq 0$ is now interpreted as individual i's contribution to club k at time t and $\dot{C}^k(t) = \frac{dC^k(t)}{dt}$. Individual i contributing to club k at time t is denoted by $i(t) \in k$. The

instantaneous utility function is analogous to (1.1):

$$u\left(x_i^k(t); C^k(t)\right) = \mathbf{1}\left\{x_i^k(t) > 0\right\} d(A^k, A_i)\left(C^k(t) + \sigma x_i^k(t)\right) - \frac{\psi}{2}x_i^k(t)^2.$$

The cohesive and splitting equilibria can still be characterized, as in the static case. When time and capital accumulation are considered, a third interesting equilibrium emerges. Rather than either remaining a cohesive community indefinitely or separating by type at the outset, individuals can incubate a child club within the parent club, leading to what I call an incubation equilibrium. In the incubation equilibrium, the community begins as a cohesive unit and remains so for a finite period of time, then splitting into multiple clubs. I refer to the period of time in which the community is cohesive as the *incubation period*, and denote the underlying strategy as active strategic membership, where a coalition of individuals temporarily joins a club in order to take advantage of economies of scale, only to then split once the installed base has grown to a sufficiently large level.

In order for active strategic membership to be a viable strategy, individuals must not leave empty-handed. There exists a wide range of interactions under which the installed base is non-rival across clubs. Take software development, where contributions are an amalgamation of source code and expertise. While an individual's expertise may be rival across clubs, the source code is non-rival.[9] One's use of the source code does not preclude another's use. Let $r \in [0, 1]$ denote the degree of rivalry of the installed base, where $r = 1$ signifies completely rival installed base and $r = 0$ signifies completely non-rival installed base. I assume that r is exogenous and fixed. There is also a potentially endogenous measure concerning the sharing of installed base: excludability.

Installed base may be excludable across clubs through the use of intellectual property restrictions such as patents, trademarks, licenses, and contracts (e.g., non-compete clauses). For

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[9]If individual i is an expert at some process z, then if that individual leaves the club, the club no longer has access to the individual with expertise in z.

instance, a software development team could choose a typical copyright license, preventing any who leave from retaining any of the source code, or the team could select what is known as a copyleft license, such as the GNU General Purpose License, the Apache License, or the various BSD licenses, which allow individuals who leave to take the source code and alter it to suit their own purposes (Open Source Initiative, n.d.b, 2015, n.d.a). Related to excludability is distance-based depreciation. As the child club moves further away from the parent club, the installed base may not be as relevant, as the match value of the clubs' missions is poor. Both excludability and distance-based depreciation can be captured by the same measure. Let $1 - \rho(0)$, $\rho(0) \in [0,1]$, denote the degree of excludability. The proportion of installed base retained by the child club depends on the match value of the two clubs $d(A^p, A^c)$ and $\rho(0)$. The retained proportion is denoted by the function $\rho\left(1 - d(A^p, A^c)\right) \in [0,1]$. As the distance between the two clubs $1 - d(A^p, A^c)$ increases, the retained proportion of installed base $\rho\left(1 - d(A^p, A^c)\right)$ decreases. For now, suppose that $\rho(0)$ is exogenous. Thus when a split occurs at time T^*, those who leave to form the child club take $\rho\left(1 - d(A_p, A_c)\right)(1-\gamma)C(T^*)$, while the parent club retains $[1 - r\rho(0)](1-\gamma)C(T^*)$, where $C(T^*)$ is the total installed base accumulated during the incubation period.[10]

1.3.2   Dynamic Analysis

The model described in section 1.3.1 allows the interactions to be represented as a linear-quadratic differential game. In the analysis to follow, I will be considering Markov perfect

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[10]A second cost of splitting may arise along the extensive margin, representing the social cost of splitting. The social cost can be thought of as the price to pay if splitting is considered "taboo," e.g. a stigma, or alternatively as the price of either forming a new mission or altering an existing mission. For example, religious schisms are socially costly, as the new mission must be justified, while the old mission must be criticized. Furthermore, social ostracization may occur. In OSS development, the act of forking is considered taboo and must be strongly justified (Stewart and Gosain, 2006). This cost is decreasing in the number of splits, à la Thomas Schelling's tipping point model (Schelling, 1978), or alternatively, Timur Kuran's preference falsification model Kuran (1987, 1995). Once the first split is justified, future splits become easier to justify. Let S represent the number of splits that have already occurred. The social cost of splitting can be written as $f(S)$, where $f(0)$ is the social cost of the first split, and $f(0) \geq f(1) \geq \ldots$. Since only a single split is considered in this paper, $f(0)$ can be normalized to zero without any loss of generality; however, future work involving multiple splits should explicitly consider this cost.

equilibria. In general, there are many, if not infinite Markov perfect equilibria in differential games. Therefore I restrict attention to the solutions found using Pontryagin's maximum principle and show that these solutions satisfy both the Markov property and subgame perfection. Each individual i's strategy is a pair consisting of a membership decision for every time t, and a contribution schedule $x_i^k(t)$ for every time t. The equilibrium requires that at almost every $t \in [0, \infty)$ and every possible state $C^k(t)$, each individual's contribution and affiliation decision satisfies the Nash property and that contributions satisfy the Markov property.[11] Derivations of each individual's equilibrium contribution schedule and the equilibrium state $C^k(t)$ can be found in Appendix A.3.

Suppose that there is a single club with mission $A(\mu)$, which is defined such that $d(A(\mu), A_1) > d(A(\mu), A_2)$. Each individual's objective is

$$\max_{(x_i(t))_{t=s}^{\infty}} \int_s^{\infty} e^{-\delta t} \left(d(A(\mu), A_i)[C(t) + \sigma x_i(t)] - \frac{\psi}{2} x_i(t)^2 \right) dt, \text{[12]}$$

for all $s \in [0, \infty)$, subject to the dynamic

$$\dot{C}(t) = \sum_j x_j(t) - \gamma C(t).$$

When the community is cohesive, each individual i chooses contribution level

$$x_i(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A(\mu), A_i) \tag{1.11}$$

at every time t, which as in the static case, is proportional to the match value. The installed

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[11] "[A]t almost every t[...]," means that the contributions must maximize the continuation value at all t except possibly at a finite, measure zero set of times t.

[12] Technically, the objective function is $\int_s^{\infty} e^{-\delta(t-s)} \left(d(A(\mu), A_i)[C(t) + \sigma x_i(t)] - \frac{\psi}{2} x_i(t)^2 \right) dt$. However, since $e^{-\delta(t-s)} = e^{-\delta t} e^{\delta s} \propto e^{-\delta t}$, $e^{\delta s}$ can be omitted without affecting the optimization. I utilize this expositional simplification for all dynamic optimizations in the text

base evolves along the continuous, strictly concave path

$$C(t) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} \hat{N}\left(1 - e^{-\gamma t}\right).$$

Since $\gamma > 0$, $C(t)$ converges to a steady-state value of $\frac{1+\sigma(\delta+\gamma)}{\gamma\psi(\delta+\gamma)}\hat{N}$ in the long run.

Substituting the individual contribution schedule and the installed base path into the utility function at $t = 0$ yields indirect utility of

$$V(d(A(\mu), A_i)) = \frac{(1 + \sigma(\delta + \gamma))\left(2\hat{N} + d(A(\mu), A_i)(\sigma(\delta + \gamma) - 1)\right)}{2\delta\psi(\delta + \gamma)^2} d(A(\mu), A_i). \qquad (1.12)$$

Observe the similarity between (1.12) and (1.3). $\hat{N}$ enters each indirect utility function in an identical manner, as does the match value $d(A(\mu), A_i)$. Consequently, cutoff values exist akin to those discussed in section 1.2.1, which characterize existence isomorphically to the static model. A formal dynamic extension of this result is omitted, as no new insights are uncovered.[13]

Next suppose that a split is orchestrated at time $T^* = 0$, where all majority individuals join club p and all minority individuals join club c. Each club sets its mission to match that of its members, so $d(A^k, A_i) = 1$ for all $i(t) \in k$, $k = p, c$. For every $i \in \mathcal{N}$,

$$x_i^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)}, \qquad (1.13)$$

for $i(t) \in k$ and zero otherwise. Club k's installed base evolves along the path

$$C^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} N^k\left(1 - e^{-\gamma t}\right).$$

As in the cohesive case, there is long run convergence to a steady-state level of $\frac{1+\sigma(\delta+\gamma)}{\gamma\psi(\delta+\gamma)}N^k$. Conditions isomorphic to those discussed in the static analysis can be described to char-

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[13]This result is available upon request.

acterize the existence of a splitting equilibrium, the occurrence of multiple equilibria, and the payoff ranking of the two equilibria. The same tradeoff between economies of scale and match value exists in both the static and dynamic structures. In the cohesive setting, each individual contributes less, but there are more individuals to make up for the smaller contribution level. Total contributions in the cohesive equilibrium are greater than total contributions in the splitting equilibrium at every time t if $\hat{N} > N^k$. Recall that $N_1 > N_2$, $N^p = N_1$, and $N^c = N_2$, so there exist conditions on $d(A(\mu), A_1)$ and $d(A(\mu), A_2)$ such that either: 1. $\hat{N} < N_2 < N_1$; 2. $N_2 < \hat{N} < N_1$; or 3. $N_2 < N_1 < \hat{N}$.

These three conditions illustrate why a minority individual may have the incentive to practice active strategic membership. When $N_2 < \hat{N}$, the minority individuals benefit from economies of scale. Figure 1.2 illustrates two possible dynamics under active strategic membership. Line a represents the installed base path under the splitting equilibrium. When assessing the decision to incubate, the distance-based depreciation becomes relevant. As the distance between the parent club and child club $1 - d(A(\mu), A^c)$ increases, $\rho(1 - d(A(\mu), A^c))$ decreases, which implies that the child club may not set A^c to perfectly match its members' missions. The minority individuals may be better off locating in an intermediate position, i.e. $d(A(\mu), A_2) < d(A^c, A_2) \leq d(A_2, A_2) = 1$. This relationship illustrates a second tension, which I call the mission-scope tension.

Tension 1.2. (The Mission-Scope Tension) *When splitting, locating the child club's mission closer to the members' missions than the parent club's mission increases the value of the child club to its members, but comes at a cost of increasing the losses due to splitting.*

Line b_1 illustrates one possible installed base path under the incubation equilibrium, which then falls to b_2 post-split due to the distance-based depreciation. Lines c_1 and c_2 illustrate a second possible incubation equilibrium path. $\overline{C}^k$ represent the steady-state installed base levels for $k = p, c$, conditional on incubation. Line a necessarily converges to a level above
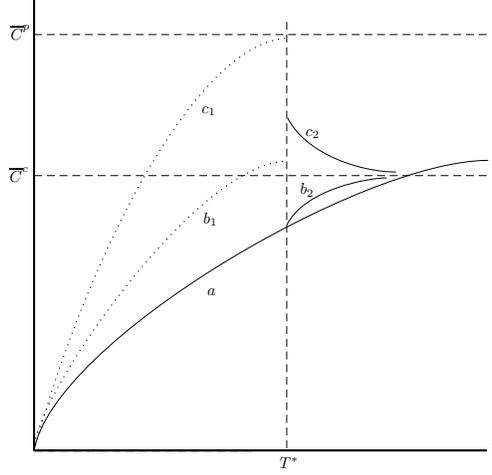
Figure 1.2: Incentives for Active Strategic Membership.

$\overline{C}^c$.

1.3.2.1 Active Strategic Membership

Suppose that a split occurs at some finite time $T^* > 0$, so the incubation period is given by the non-degenerate interval $[0, T^*]$. $C(T^*)$ is the installed base of the parent club at time T^*. Consider the subgames initiated post-split at time T^*, where all majority individuals are in club p and all minority individuals are in club c. There are an infinite number of subgames, one for each pair $(C(T^*), T^*)$. The minority individuals assign mission A^c to the child club, while the parent club keeps the original mission $A(\mu)$.[14] The child club retains $(1 - \gamma)\rho(1 - d(A(\mu), A^c))C(T^*)$ from the parent club, while the parent club retains $[1 - r\rho(0)]C(T^*)$ post-split. For expositional convenience, let

$$
\alpha^k = \begin{cases} 1 - r\rho(0) & \text{if } k = p \\ \rho(1 - d(A(\mu), A^c)) & \text{if } k = c \end{cases}
$$

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[14]If the parent club were to change its mission, then the utility from incubation must be greater than it would be had the parent club retained the original mission $A(\mu)$. Therefore assuming that the parent club keeps the original mission represents the most conservative case. Alternatively, one can think of inertia as the reason the mission remains the same. For instance, after a religious schism, it is rare that the original religion changes its practices.

represent the proportion of installed base retained by club k post-split. Each individual $i(t) \in k$ contributes an amount proportional to her match value

$$x_i^k(t > T^*) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A^k, A_i),\tag{1.14}$$

where the installed base accumulates according to

$$C^k(t > T^*) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}\hat{N}^k + \left[(1 - \gamma)\alpha^k C(T^*) - \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}\hat{N}^k\right]e^{-\gamma(t-T^*)}.$$

Study of the post-split problem is sufficient in determining the long run implications.

Post-split, each minority individual contributes more than she would in the cohesive equilibrium, while each majority individual's contributions are unaffected (compare equations (1.14) and (1.11)). However, each individual contributes no more than she would in a splitting equilibrium. When incubating, the contributions made by minority individuals are proportional to the contributions in the cohesive equilibrium. That is,

$$x_2^c(t > T^*) = \frac{d(A^c, A_2)}{d(A(\mu), A_2)}x_2(t > T^*),$$

so the difference is determined by the relative match value, which is greater than one. Depending on the relationship between $\hat{N}$ and $\hat{N}^c$, the steady-state level of installed base can be either higher or lower under incubation relative to cohesion. If $\hat{N}^c = d(A^c, A_2)N_2 > \hat{N} = d(A(\mu), A_1)N_1 + d(A(\mu), A_2)N_2$, then the steady-state installed base is greater under incubation. Even if the steady-state level of installed base is lower when incubating, minority individuals can still benefit since $d(A^c, A_2) > d(A(\mu), A_2)$, so both the installed base and individual contributions are more valuable at any given level. The decrease in contributions for minority individuals between the incubation equilibrium and the splitting equilibrium follows from the fact that in the splitting equilibrium, the minority individuals set the child club's mission such that the match value is one. When incubating, the members of the child

25

club may set the child club's mission such that the match value is less than one due to distance-based depreciation, as illustrated by the mission-scope tension. Thus in the long run, the splitting equilibrium induces a greater marginal benefit of both consumption and contribution, higher contributions from members, and a greater steady-state level of installed base. Since future utility is discounted, utility today is more valuable then utility in the future, so individuals are willing to choose actions which increase current utility, but decrease future utility. The benefits of incubation operate under this principle.

Proposition 1.4. *Suppose individuals become infinitely patient ($\delta \to 0$) and the cohesive equilibrium, splitting equilibrium, and incubation equilibrium all coexist. The splitting equilibrium always payoff dominates the incubation equilibrium. In addition for each type m individual, $m = 1, 2$: (i) the incubation equilibrium payoff dominates the cohesive equilibrium if and only if $d(A(\mu), A_m)$ is sufficiently small.; (ii) the cohesive equilibrium payoff dominates the splitting equilibrium if and only if $d(A(\mu), A_m)$ is sufficiently large.*

In the long run, the incubation equilibrium is always payoff dominated by the splitting equilibrium. This fact does not preclude an incubation equilibrium from existing. While practicing active strategic membership may not be the best option when individuals are sufficiently patient, it is still an option nonetheless.

Proposition 1.5. (Active Strategic Membership) *Suppose individuals become infinitely patient. If $d(A^c, A_2)$ and $d(A(\mu), A_1)$ are sufficiently large and $d(A(\mu), A_2)$ is sufficiently small, then there exists an incubation equilibrium in which minority individuals practice active strategic membership.*

If the match value of a minority individual to the child club is sufficiently large, then the difference between the indirect utility under incubation equilibrium post-split and the indirect utility under the splitting equilibrium is near zero, so the benefits of economies of

scale coupled with the relatively high value of the incubation equilibrium implies, by the contribution-scale tension, that there is no incentive for a minority individual to unilaterally deviate and form a club k with mission $A^k = A_2$. If the match value between a minority individual and the parent club is sufficiently small, then the value of the cohesive equilibrium to the minority individual is low, so no minority individual has the incentive to unilaterally deviate by not splitting at time T^* and instead remaining with the parent club. If the match value between a majority individual and the parent club is sufficiently large, then the benefits from economies of scale for majority individuals are large enough such that no majority individual wishes to deviate and form a new club k' with mission $A^{k'} = A_1$.

The implications of Proposition 1.4 only hold in the long run, as the results are valid only if individuals are sufficiently patient. However, the true benefits of incubation are of a short run nature, and are most prevalent when individuals are not too patient. When practicing active strategic membership, individuals are transferring utility from the future to the present. The splitting equilibrium always yields a greater payoff in the steady-state than the incubation equilibrium, but the incubation equilibrium can be more profitable in the short run. The short run incentives can be strong enough that the incubation equilibrium can payoff dominate the splitting equilibrium.

Consider the pre-split contributions by each individual. Each individual i chooses her pre-split contribution level, knowing that a split is imminent and given a split, she will contribute according to (1.14) post-split. Thus the pre-split problem for individual i is given by the family of finite-horizon problems

$$\max_{(x_i(t))_{t=s}^{T^*}} \int_s^{T^*} \left\{ e^{-\delta t} \left[d(A(\mu), A_i) \left(C(t) + \sigma x_i(t) \right) - \frac{\psi}{2} x_i(t)^2 \right] \right\} dt + B_i(C(T^*)),$$

for all $s \in [0, T^*]$, where

27

$$B_i(C(T^*)) = \int_{T^*}^{\infty} \left\{ e^{-\delta t} \left[d(A^k, A_i) \right. \right.$$

$$\times \left(\frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} \hat{N}^k + \left[(1 - \gamma)\alpha^k C(T^*) - \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} \hat{N}^k \right] e^{-\gamma(t - T^*)} \right.$$

$$\left. \left. + \sigma \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A^k, A_i) \right) - \frac{\psi}{2} \left(\frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A^k, A_i) \right)^2 \right] \right\} dt.$$

From this system, each individual i contributes

$$x_i(t \leq T^*) = \overline{x}_i + \left[\frac{(1 - \gamma)\alpha^k d(A^k, A_i) - d(A(\mu), A_i)}{\psi(\delta + \gamma)} \right] e^{-(\delta + \gamma)(T^* - t)}, \qquad (1.15)$$

where

$$\overline{x}_i = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A(\mu), A_i).$$

The pre-split installed base path is given by

$$C(t \leq T^*) = \omega_1 \left(1 - e^{-\gamma t} \right) + \omega_2 e^{-(\delta + \gamma)T^*} \left(e^{(\delta + \gamma)t} - e^{-\gamma t} \right),$$

where

$$\omega_1 = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} \hat{N}, \quad \omega_2 = \left(\frac{\gamma}{\delta + 2\gamma} \right) \left[\frac{(1 - \gamma)\sum_k \alpha^k \hat{N}^k - \hat{N}}{\gamma\psi(\delta + \gamma)} \right].$$

Note that $\omega_1(1 - e^{-\gamma t})$ is the installed base path under the cohesive equilibrium. While the pre-split contributions depend explicitly on T^*, the equilibrium choice of T^* is determined only by the post-split utility and the state at time T^*, $C(T^*)$. At $t = 0$, a forward looking individual chooses T^* to maximize the total stream of utility. However, at each instant, each individual reevaluates her decision up until time T^*, the point at which the split occurs. Anticipating this reaction, each individual contributes knowing that the split will occur at

time T^*, where T^* maximizes the discounted post-split stream of utility:

$$T^* \equiv \arg\max \left\{ B_{i=m=2} \left(\omega_1 \left(1 - e^{-\gamma T^*} \right) + \omega_2 \left(1 - e^{(\delta + 2\gamma)T^*} \right) \right) \right\}.$$

This value can only be computed numerically.

Careful analysis of the pre-split contributions yields insight into how the short run benefits of incubation can lead to payoff dominance of the incubation equilibrium. First compare (1.15) to (1.11). The pre-split contributions in the incubation equilibrium are equal to the pre-split contributions in the cohesive equilibrium, plus an extra term

$$\left[\frac{(1-\gamma)\alpha^k d(A^k, A_i) - d(A(\mu), A_i)}{\psi(\delta + \gamma)} \right] e^{-(\delta+\gamma)(T^*-t)}.$$

When the above term is positive, each individual contributes more under the incubation equilibrium than under the cohesive equilibrium. The term is positive if

$$\alpha^k > \frac{d(A(\mu), A_i)}{(1-\gamma)d(A^k, A_i)}. \tag{1.16}$$

recall that α^k represents the proportion of capital that each club k retains. If this value is sufficiently large, then anticipating the increased future benefits post-split, each individual contributes more, pre-split. If α^k is large relative to the value given in (1.16), then contributions under the incubation equilibrium are greater than contributions under the splitting equilibrium. This value may or may not be attainable, depending on other values, such as N_1 and N_2.

A similar comparison can be made with respect to the installed base path. The path under the incubation equilibrium is equal to the path under the cohesive equilibrium, plus an extra term, and is greater under the incubation equilibrium if α^k is sufficiently large for $k = 1, 2$.

The extra term is positive if

$$\sum_k \alpha^k \hat{N}^k > \frac{\hat{N}}{1-\gamma}. \tag{1.17}$$

It is clear that (1.16) and (1.17) are positively related. Furthermore, this increase is approximately hyperbolic for small δ, as it includes the term $e^{(\delta+\gamma)t} - e^{-\gamma t}$, so when the term is positive and δ is small, the increase can be quite large. Similar conditions exist for the same relationship to hold between the incubation equilibrium and the splitting equilibrium. The above argument illustrates the intuition behind the notion that the incubation equilibrium can payoff dominate both the cohesive equilibrium and the splitting equilibrium.

Unfortunately, the indirect utility function is complicated to the point that an analytical analysis of payoff dominance is infeasible. In order to illustrate the results more formally, I employ numerical methods. In the numerical analysis, I manipulate between one and three parameters at a time. In each round of the analysis, the discount rate δ changes. Along with δ, I manipulate either α^p and α^c, γ, $d(A(\mu), A^c)$ and $d(A^c, A_2)$, or σ. The rounds with manipulations to $(\delta, \alpha^p, \alpha^c)$ and $(\delta, d(A(\mu), A^c), d(A^c, A_2))$ have three parameters changing simultaneously while the remaining rounds have two parameters changing simultaneously. I normalize $\psi = 10$ and keep this parameter fixed throughout. Note that ψ enters the individual contributions in each of the three equilibria, given by (1.11), (1.13), (1.14), and (1.15), proportionally, so without loss of generality, ψ can be kept fixed without affecting the rankings of the equilibrium payoffs.

Result *There exists an open set of conditions such that the incubation equilibrium payoff dominates both the splitting equilibrium and the cohesive equilibrium.*

Table 1.1 summarizes the conditions under which the incubation equilibrium payoff dominates both the cohesive equilibrium and the splitting equilibrium for the minority individu-

Parameters	Intuition
$\delta \gg 0$	Individuals exhibit a degree of impatience, so they are willing to trade future benefits for current benefits.
$\alpha^p, \alpha^c \gg 0$	When both α^p, α^c are sufficiently large, both clubs are able to accumulate installed base via incubation.
$\gamma \ll 1$	Installed base can only accumulate rapidly when the depreciation rate is not too large.
$d(A(\mu), A^c) \ll d(A^c, A_2)$	If the match values are sufficiently large, then the post-split value of incubation is similar to the value of splitting.
$g(\delta) \leq \sigma \ll \infty$	If the relative benefit of contributing is too large, then the value of consuming the club good is irrelevant.

Table 1.1: Sufficient conditions for the incubation equilibrium to exhibit payoff dominance for minority individuals.

als. Every condition outlined must be satisfied simultaneously. The exact minimal/maximal value is dependent on the other parameters. Detailing the exact conditions is infeasible since there are an uncountably infinite number of values; for example, there is a unique δ for each σ such that incubation payoff dominates both of the other equilibria, *ceteris paribus*. To illustrate the result, I present some numerical results and comparative statics for a subset of the parameters. Details on the numerical analysis can be found in Appendix A.3.

As shown in Proposition 1.4, when δ is sufficiently small, the incubation equilibrium is always payoff dominated, but if individuals are not too patient, then they are willing to give up utility in the future in exchange for utility in the present. It only takes a small degree of impatience to generate payoff dominance for the incubation equilibrium. As there is an approximately hyperbolic component to contributions for a small δ, the utility from incubating increases significantly. The effect is non-monotonic and diminishing as δ increases (Figure 1.3a). Post-split individual contributions are decreasing in the discount rate, which in turn diminishes the incentive to incubate. Thus as the discount rate increases, the equilibrium incubation period T^* decreases, as illustrated in Figure 1.3b.

For the remaining numerical results, I restrict $\delta \in \{0.004, 0.05\}$, which corresponds to

(a) Payoff Dominance.

(b) Comparative Static.
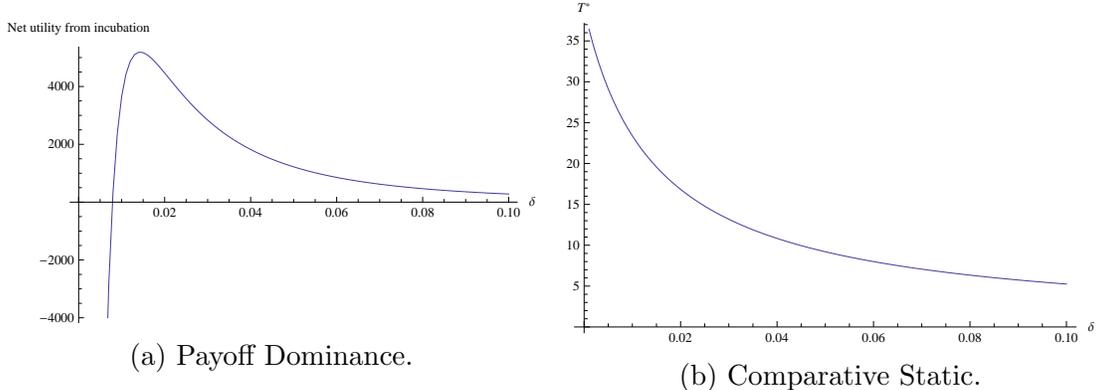
Figure 1.3: Properties of the incubation equilibrium as a function of δ.

(a) $\delta = 0.004$.

(b) $\delta = 0.05$.

Figure 1.4: Incubation equilibrium utility $-$ max{cohesive equilibrium utility, splitting equilibrium utility}, with respect to α^p and α^c.

discount factors of approximately 0.996 and 0.95, respectively.[15]   When $1 - r\rho(0)$ and $\rho(1 - d(A(\mu), A^c))$ are both sufficiently large, both majority and minority individuals are willing to contribute enough to make incubation worthwhile since both the child club and parent club retain a large proportion of the installed base. Figure 1.4 depicts the region in (α^p, α^c) space under which the incubation equilibrium payoff dominates both the cohesive equilibrium and the splitting equilibrium. The dark-gray region corresponds to the set of parameters under which the incubation equilibrium is payoff dominated while the light-gray region corresponds to the set of parameters in which the incubation equilibrium is payoff dominant. Note that in order for such a set of parameters to exist, δ must be sufficiently large, as illustrated by the relationship between Figures 1.4a and 1.4b. Conditional

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[15]The discount factor is given by $e^{-\delta} \approx 1 - \delta$ for small δ.

(a) $\delta = 0.004$.   (b) $\delta = 0.05$.
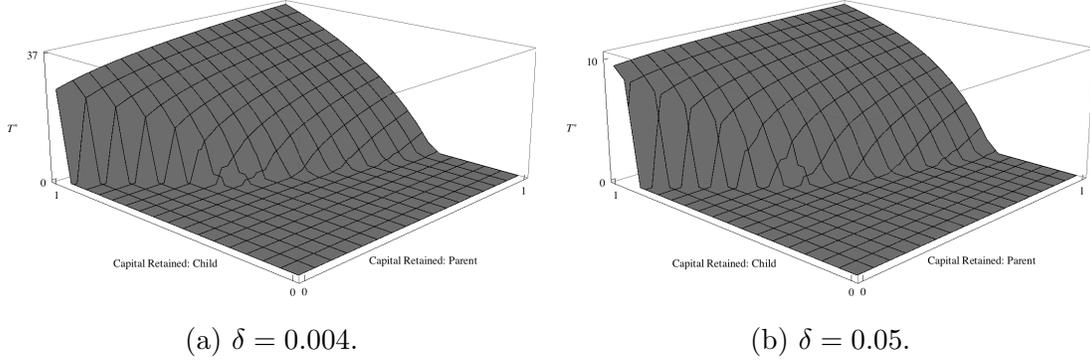
Figure 1.5: Comparative statics with respect to $1 - r\rho(0)$ and $\rho(1 - d(A(\mu), A^c))$.

(a) $\delta = 0.004$.   (b) $\delta = 0.05$.

Figure 1.6: Incubation equilibrium utility $-\max\{$cohesive equilibrium utility, splitting equilibrium utility$\}$ with respect to γ.

on an incubation equilibrium existing, the length of the incubation period is increasing in both $\rho(1 - d(A(\mu), A^c))$ and $1 - r\rho(0)$ (Figure 1.5). As $1 - r\rho(0)$ increases, the amount a majority individual is willing to contribute pre-split increases. At the same time, when $\rho(1 - d(A(\mu), A_c))$ increases, the value of incubation increases as minority individuals have the incentive to contribute more. Once $1 - \rho(0)$ and $\rho(1 - d(A(\mu), A_c))$ are large enough, the incubation period is a positive measure of time. As these two values increase, pre-split contributions by both type of individuals increase, which makes incubation more profitable, increasing the equilibrium incubation period.

With γ, there are no surprises. When there is very little depreciation, installed base is able to accumulate at a rapid rate, increasing the value of economies of scale to the members making incubation a valuable strategy. Figures 1.6a and 1.6b show that the more patient

(a) $\delta = 0.004$.

(b) $\delta = 0.05$.
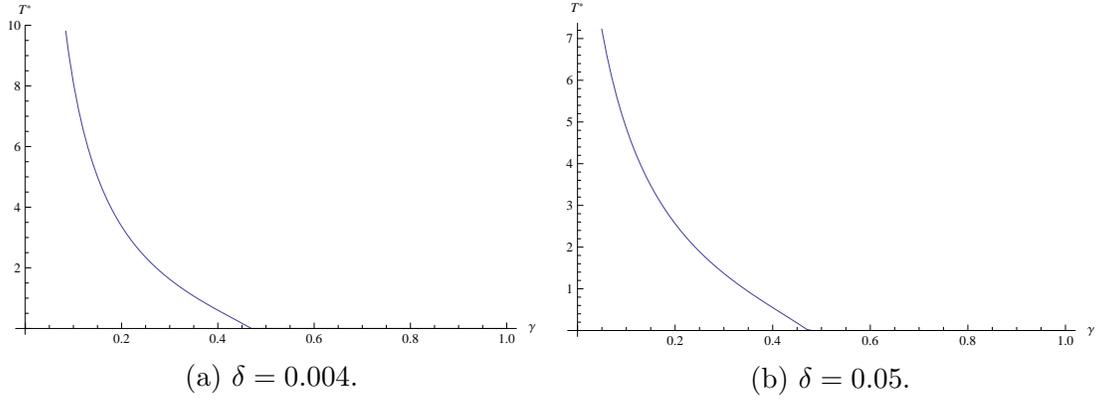
Figure 1.7: Comparative statics with respect to γ.

individuals are, the smaller γ must be to compensate and make incubation payoff dominant. Analysis of Figures 1.7a and 1.7b show that the equilibrium incubation period decreases as depreciation increases, since larger incubation periods become less valuable when installed base is unable to accumulate at a rapid pace.

The analysis of the match values $d(A(\mu), A_2)$ and $d(A^c, A_2)$ is conducted over a restricted parameter space. There are two conditions which must be satisfied. Firstly, the match value of a minority individual to the parent club must always be weakly lower than the match value of a majority individual to the parent club. Secondly, the match value of a minority individual to the child club must be strictly greater than the match value of a minority individual to the parent club, otherwise by the contribution-scale tension, the benefit from economies of scale dominates the benefit from a higher match value, since there is none.

As $d(A^c, A_2)$ increases, incubating becomes more valuable. However, when holding $d(A^c, A_2)$ fixed, the value of incubation decreases when $d(A(\mu), A_2)$ increases by the contribution-scale tension. For every $d(A(\mu), A_2)$, there is a unique $d(A^c, A_2)$ such that the incubation equilibrium is payoff dominant for the minority individuals, and this value is increasing in $d(A(\mu), A_2)$, as shown in Figures 1.8a and 1.8b. The equilibrium incubation period is decreasing in $d(A^c, A_2)$ since the amount minority individuals are willing to contribute both pre-split and post split is increasing in $d(A^c, A_2)$. The opposite is true when $d(A(\mu), A_2)$
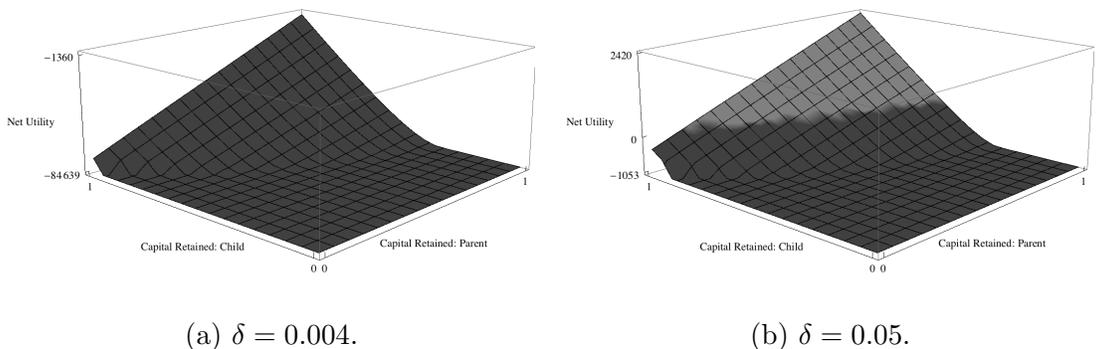
34
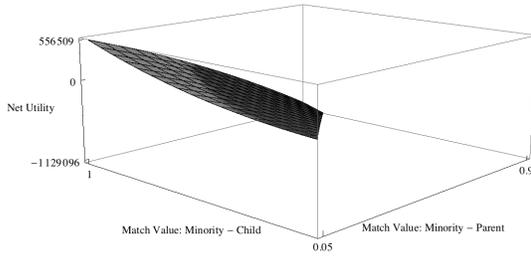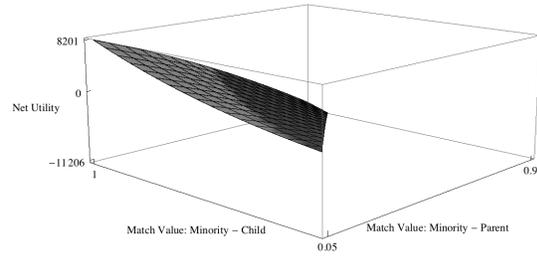
(a) $\delta = 0.004$.   (b) $\delta = 0.05$.

Figure 1.8: Incubation equilibrium utility $-\max\{$cohesive equilibrium utility, splitting equilibrium utility$\}$ with respect to the minority individuals' match values.
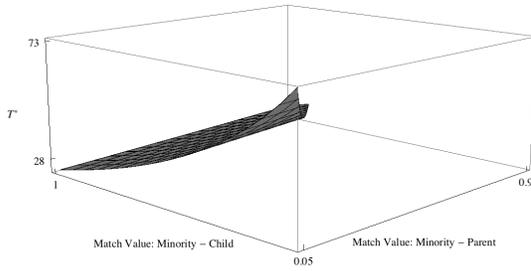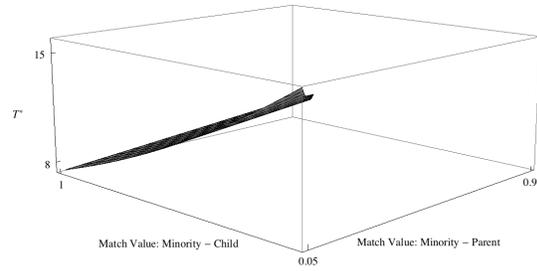
(a) $\delta = 0.004$.   (b) $\delta = 0.05$.

Figure 1.9: Comparative statics with respect to the minority individuals' match values.

increases, as post-split contributions are constant, but pre-split contributions are increasing (see Figures 1.9a and 1.9b).

The relationship between σ and payoff dominance is non-monotonic. When δ and σ are both small, incubation is payoff dominated solely due to the patience of minority individuals. If the marginal benefit of contributing increases while δ remains fixed, then the relative payoff when incubating actually decreases due to over-reliance on the installed base relative to individual contributions, so the cohesive equilibrium is the most valuable. If σ continues to increase, then the balance shifts and while there is still reliance on the total contributions, the greater future match value under the incubation equilibrium relative to the cohesive equilibrium leads to an increase in the relative payoff under the incubation equilibrium. As σ increases, not only does the incubation equilibrium payoff, but the difference between the incubation

(a) $\delta = 0.004$.

(b) $\delta = 0.05$.
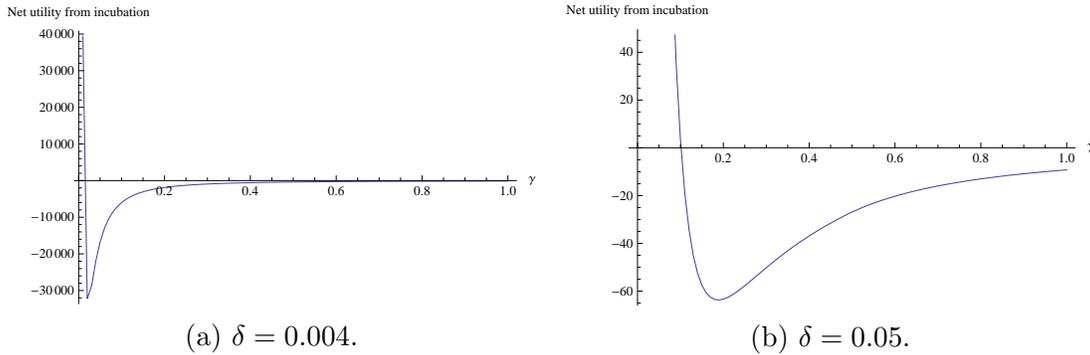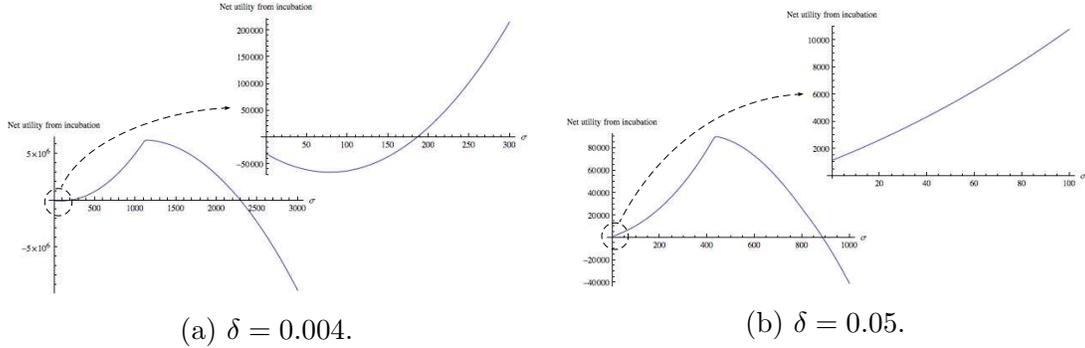
Figure 1.10: Incubation equilibrium utility $-\max\{$cohesive equilibrium utility, splitting equilibrium utility$\}$ with respect to the relative marginal benefit of contributing.

(b) $\delta = 0.05$.

(a) $\delta = 0.004$.
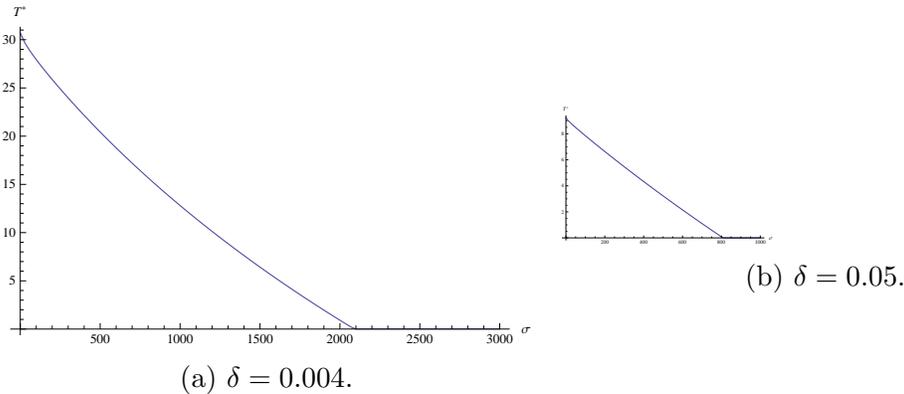
Figure 1.11: Comparative statics with respect to the relative marginal benefit of contributing.

equilibrium payoff and the payoffs from both the cohesive equilibrium and the splitting equilibrium. However, there exists a cutoff point where eventually, by the contribution-scale tension, the match value effects take over and the relative payoff from incubating is decreasing, to the point where it is no longer payoff dominant (Figure 1.10a). The splitting equilibrium becomes the payoff dominant equilibrium for σ sufficiently large. A similar pattern holds when δ increases, except the lower range of σ under which the incubation equilibrium is payoff dominated collapses to zero. The remaining pattern matches that of the case of δ small (Figure 1.11b). The equilibrium incubation period is weakly decreasing in σ, with the equality strict whenever $T^* > 0$ (Figures 1.11a and 1.11b).

36

1.4   (Dis)incentivizing Incubation

This section focuses on ways the parent club (as a centralized unit) can react to strategic membership by introducing two instruments the majority individuals can use to exert a degree of control over and profit from strategic membership. The first works through taxation/subsidization and the second through the use of exclusions such as intellectual property restrictions. These instruments can be thought of as policy parameters. The timing of their implementation is as follows. Prior to the bargaining process leading to the setting of the mission(s), the parent club announces its decisions regarding taxation/subsidization and intellectual property restrictions. The game then proceeds as described in Section 1.3.1. The results from the previous sections can be reinterpreted as Markov perfect equilibria of the subgame induced by the parent club's choice of instruments.

1.4.1   Membership Inclusion and Exclusion: Strategic Admission

Suppose that the parent club can control who may join and who may not. This measure of control leads to the concept of strategic admission, where individuals are either permitted to or prevented from joining the club given that other club members know that this individual will eventually split from the club. Before stating the results, I define some simplifying notation. Let $[0, \hat{T}]$ denote the optimal incubation period from the perspective of the parent club.[16]   Denote $T' \equiv \max\{\hat{T}, T^*\}$ and denote $T'' \equiv \min\{\hat{T}, T^*\}$. Let $V_{i(T')T''}$ and $V_{i(T')T'}$ represent the indirect utility functions of a member i of the parent club, given that a split occurs at times T'' and T', respectively. That is, $V_{i(T')T''}$ and $V_{i(T')T'}$ represent the indirect utilities of those individuals who retain their affiliation to the parent club post-split. Let $W_{i(T')T''}$ and $W_{i(T')T'}$ represent the indirect utility functions of an individual i who splits at

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[16]If $\hat{T} = \infty$, then analysis is conducted over the extended nonnegative real line $\mathbb{R}_+ \cup \{\infty\}$, so $[0, \infty]$ is well defined.

times T'' and T', respectively.[17] Members of the parent club are indexed by $j(t) \in p$ and the child club by $j(t) \in c$.

The timing of a split impacts both minority individuals and majority individuals. Formally, let the impact of delaying a split be the absolute value of the difference between the sum of indirect utilities when a split occurs at T' and T''. Thus the impact on a club, either parent or child, can be written as

$$\left| \sum_{j(T') \in k} \left(z_{j(T')T'} - z_{j(T')T''} \right) \right|, \tag{1.18}$$

for $(z, k) \in \{(V, p), (W, c)\}$. By removing the absolute value restrictions from (1.18), it can be determined whether preferences are aligned or misaligned. Preferences are misaligned if

$$\text{sign} \left\{ \sum_{j(T') \in p} \left(V_{j(T')T'} - V_{j(T')T''} \right) \right\} \neq \text{sign} \left\{ \sum_{j(T') \in c} \left(W_{j(T')T'} - W_{j(T')T''} \right) \right\}. \tag{1.19}$$

Preferences are misaligned if the time at which splitters prefer to split does not coincide with the time at which non-splitters prefer the split occur. Delay is beneficial for a group if

$$\sum_{j(T') \in k} \left(z_{j(T')T'} - z_{j(T')T''} \right) > 0, \tag{1.20}$$

and harmful if (1.20) is not satisfied.

Result (Strategic Admission) *If the impact of delaying the split on non-splitters is larger than the impact of delaying the split on splitters, and the preferences of the two groups are misaligned, then there exists a price ϕ, located in the convex hull of the impacts, such that the parent club offers those who split ϕ to: (i) increase the incubation period from $[0, T'']$ to $[0, T']$ if those who do not split benefit from the delay while those who split are harmed by the*

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[17]For the remainder of this section, the analysis is contingent upon the value of the indirect utility function. Thus the specified assumptions and function forms from Sections 1.2 and 1.3 are unnecessary.

delay; (ii) decrease the incubation period from $[0, T']$ to $[0, T'']$ if those who do not split are harmed by the delay while those who split benefit from the delay.

If there are economies of scale, it may be profitable for the club to keep those members for an extended period of time, i.e. an incubation period. If the benefits from allowing the splitting members to incubate are greater than the losses to the splitters when delaying the split by $T' - T''$, then the club can offer a fixed payment, divided amongst the splitters, to delay their decision by $T' - T''$: incubation is subsidized. Similarly, if the splitters would rather wait while the club is worse off with strategic membership, then the club can charge a fee for membership that those who plan on splitting are not willing to pay. Furthermore when the impact of delaying the split on the parent club members is smaller than the impact on the child club members, the parent club may still have an option at its disposal. While the parent club cannot shift the length of the incubation period to its preferred length, it can pay a price ϕ to adjust the incubation period so that it is *closer* to the preferred length $[0, \hat{T}]$. For example, if $\hat{T} < T^*$, but the impacts are such that the parent club is not willing to pay the minimum price ϕ to alter the child club's behavior, the parent club may be able to offer a price ϕ to adjust the incubation period from $[0, T^*]$ to $[0, \theta T^* + (1-\theta)\hat{T}]$, $\theta \in (0, 1)$.

There are two possibilities. Either the club can pay to bring individuals in that otherwise would not join/keep them for an extended period of time, or the club can charge a membership fee to keep undesired members out. Under decentralized clubs, where utility is increasing in the number of effective individuals, it is likely that incubation will occur, where the club temporarily subsidizes the membership of those who would prefer to form their own club.[18] When clubs are centralized and total contributions may decrease when individuals who are not a good fit join, the club can charge a fee large enough to keep those individuals from joining. This result is a generalization of both Iannaccone (1992, 1994) and McBride (2007a), where the fee can be interpreted as being equivalent to the costly sacrifice in reli-

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[18]Or equivalently, make joining difficult.

gion (Iannaccone, 1992), or the Mormon church requiring individuals to donate of 10% of their income (McBride, 2007a). Alternatively, the subsidy is akin to inviting free riders, à la McBride (2007b). This distinction between centralized and decentralized clubs is important, as Glomm and Lagunoff (1999) show that centralized clubs are more likely to persist in the long run than decentralized clubs.[19]

There is a corollary to the strategic admission result. Suppose that preferences are aligned, and specifically, both groups prefer strategic membership. Then the parent club can tax those interested in practicing strategic membership and capture the rents from incubating. To illustrate this instrument, suppose that preferences are aligned and minority individuals want to practice strategic membership. Those individuals are willing to pay any tax $\phi \in \left[0, W_{j=m=2(T^*)T^*} - W_{j=m=2(T^*)0}\right]$ for the ability to do so. If types are observable, then this result is bad news for those practicing strategic membership. Any rents from strategic membership are captured by the parent club; however, since this tax can be treated as a fixed cost, it has no impact on the marginal contribution of any individual, and thus no efficiency implications.

It is worth noting that if individuals are sufficiently patient, then the parent club can still extract all incubation rents regardless of whether or not types are observable. In particular, if $\delta \leq -\frac{1}{T^*} \ln\left(\frac{N_1}{N}\right)$, then all individuals will accept a taxation scheme in which: (i) each individual pays a value of ϕ at the outset; (ii) immediately following the split, each individual who remains in the parent club receives their payment ϕ; (iii) the payments made by those who split are divided evenly amongst those who remain. At time 0, each agent loses ϕ. At time T^*, those who remain receive the payment back, though the value is discounted by $e^{-\delta T^*}$. This tax scheme is preferable (and thus implementable) if $-\left(1 - e^{-\delta T^*}\right)\phi + \left(\frac{N_2}{N_1}\right)e^{-\delta T^*}\phi \geq 0$, which simplifies to $\delta \leq -\frac{1}{T^*} \ln\left(\frac{N_1}{N}\right)$.

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[19]Note that the result is stated over the indirect utility functions and is thus independent of the mechanism used to choose the contribution levels.

1.4.2 Excludability of Inputs: Strategic Loss

There are two channels through which strategic loss can be practiced. One is manipulating the degree of excludability through $\rho(0)$ and the second is through forcing the child club to locate a minimum distance from the parent club, affecting the depreciation through $d(A(\mu), A^c)$. The first is accomplished through intellectual property restrictions including copyrights, trademarks, and patents and the second is accomplished through the use of non-compete clauses.

With respect to excludability, if incubation is preferred, then $\rho(0)$ must be set sufficiently high such that there is a large benefit to incubating. Alternatively, if these members are undesirable, then $\rho(0)$ should be set sufficiently low such that there is no benefit to incubation and those individuals with the desire to split either do so immediately or not at all. $\rho(0)$ can be affected though the use of intellectual property. For example, assigning a copyright license software prevents other software developers from freely taking, manipulating, and repackaging the source code and branding it as their own. If a coalition of software developers orchestrates a split of copyrighted software, then all the coalition retains when splitting is experience and reputation. The source code in its entirety must remain with the parent organization. If the parent club desires incubation, then the parent club can implement a copyleft license, such as the BSD 2-Clause License, the GNU General Purpose License, and the Apache License. Under these licenses, individuals are free to take, modify, and repurpose the source code, each to a differing extent.[20]

It is impossible to analyze $\rho(0)$ on its own when studying the parent club's preference for incubation. The degree of rivalry r must be considered simultaneously. Recall that $1 - r\rho(0)$

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[20]While each license mentioned requires that contributions made by earlier contributors must still be attributed to the contributors, the GPL license further requires that any software developed using the source code must also be branded with the GPL license. The Apache and BSD licenses allow the source code to be branded under a new license, even a proprietary (copyright) license. For example, Apple utilized a fork of FreeBSD in the development of its Darwin system, which is a component of the proprietary OS X operating system. Patents and trademarks can be utilized in a similar fashion.

is the proportion of installed base the parent club retains following a split. If $r = 1$, so contributions are completely rival, then the parent club never wants to set $\rho(0) = 1$, as that implies that post-split, it loses everything. If $r = 0$, then the amount the parent club retains is independent of $\rho(0)$; however, the parent club will not be indifferent between $\rho(0) \in [0, 1]$ because both contributions and the decision on whether or not to the child club has the incentive to incubate are affected. Recall that the child club retains $\rho(1 - d(A(\mu), A^c))$ post-split, which can be rewritten as $\rho(0 + 1 - d(A(\mu), A^c))$. As $\rho(0)$ increases, $\rho(0 + 1 - d(A(\mu), A^c))$ necessarily decreases and as $\rho(0)$ decreases, $\rho(0 + 1 - d(A(\mu), A^c))$ is weakly increasing, strictly so if $\rho(0 + 1 - d(A(\mu), A^c)) > 0$.

By implementing restrictions such as non-compete clauses in membership contracts, the parent club can limit the options for the child club. In non-compete clauses, one party often agrees that it will not form an entity that competes too closely against the other party, where closeness can refer to either the geographic distance or the distance in the product characteristic space. Requiring a noncompete clause is equivalent to imposing a restriction on A^c such that if the child club incubates within the parent club, then the distance between the two clubs must be sufficiently large: $1 - d(A(\mu), A^c) \geq \zeta$, for some $\zeta \in [0, 1]$. Restricting this distance to be sufficiently large places an upper bound on $\rho(1 - d(A(\mu), A^c))$, similar to the impact of copyrights and patents on $\rho(0)$. Requiring this distance to be great enough, plus limiting competition through copyrights can lead to $\rho(1 - d(A(\mu), A^c)) = 0$ for all feasible A^c under the constraint $1 - d(A(\mu), A^c) \geq \zeta$.

The parent club has the incentive to practice strategic loss when either the cohesive equilibrium or splitting equilibrium payoff dominates the incubation equilibrium for majority individuals or alternatively, whenever preferences are misaligned. When the match value of majority individuals to the parent club is sufficiently large, the utility under the cohesive equilibrium will be greater than the equilibrium under the splitting equilibrium. The negative difference in utility caused by a small difference in match values between the cohesive

and splitting equilibria is outweighed by the positive difference due to economies of scale given the presence of minority individuals (the contribution-scale tension). If the match value of the minority individuals to the parent club is not too small, then they too will prefer the cohesive equilibrium to the splitting equilibrium, as the benefits from economies of scale outweigh the costs of a poor match value, relative to the splitting equilibrium. If this is the case, then the parent club utilizes strategic loss to ensure that the cohesive equilibrium is payoff dominant for the minority individuals.

The numerical analysis conducted in Section 1.3.2.1 is useful in illustrating the effectiveness of the strategies under strategic loss. Figure 1.4 shows that when $\rho(1 - d(A(\mu), A^c))$ is sufficiently small, the incubation equilibrium is payoff dominated for the minority individuals. Therefore utilizing restrictions to place an upper bound on $\rho(1 - d(A(\mu), A^c))$ is sufficient to limit the profitability of strategic membership. Furthermore by Proposition 1.5, the parent club can utilize its control over depreciation to keep $d(A^c, A_2)$ small enough that an incubation equilibrium no longer exists.

1.5 Applications

Software development is often characterized by one of two development methodologies. The first is the PS model and the second is the OSS model.[21] I shall now apply the analysis to both methodologies of software development. One feature that both methodologies share is that software development is often a collaborative process, requiring contributions from multiple individuals. PS development is more centralized, often coordinated and enclosed within a firm that takes advantage of the available intellectual property restrictions (often copyrights) to keep its source code proprietary. Nonetheless, the theory developed in this paper is still applicable to help better understand the firm's options with respect to internal structure

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[21]A third methodology has also been introduced recently, which is a hybrid of the two methodologies, referred to as the mixed-source model (Casadesus-Masanell and Llanes, 2011).

through the allocation of the software developers within the firm. The development team can be viewed as a club within the firm, where all clubs (parent and child) fall under the umbrella of the firm. The OSS development methodology corresponds to a decentralized venture, where programmers from all walks of life coordinate in a bazaar-like fashion (Raymond, 1999), and the source code is kept openly available for all to view, edit, and distribute (Open Source Initiative, n.d.*c*). An empirical feature of OSS is that projects have been known to fork over time, with development continuing on both the new child project, as well as the original parent project. This section examines each methodology and shows how this theory can be applied to understanding the underlying processes.

With respect to OSS, I use the model to explain the evolution of two open source derivatives of Unix, the BSD family of operating systems and Linux, considering both the Linux kernel and the vast array of Linux distributions. I then apply the model to proprietary software, showing how a firm developing multiple products can choose between three development structures, each corresponding to one of the equilibria described in this paper, to maximize profits.

1.5.1 Open Source Software Forks

Before considering specific forks, I first provide a brief overview of the process of forking. Forking occurs when an OSS project splits into multiple projects and is typically utilized in two contexts. An individual will often fork a project to provide themselves with a separate code base for developing and testing new features, without affecting the mainline project. When the added developments are ready, the individual makes a request to reintegrate the fork, adding the new features to the original project. Development of the fork then ceases. The second context is the one this paper is concerned with, which are permanent forks. Individuals fork a project with the intent to use the installed base to take the new

project in a different direction than the original. The development of both projects continues indefinitely.[22]

While there is only one official Linux kernel, there are many distributions that are built upon the Linux kernel. The Linux kernel represents the operating system, which can be viewed as the central nervous system of a computer. It relays information between the software and hardware. End-users often interact with the distribution, which includes the operating system, plus what are known as "userland" utilities, such as a graphical user interface, a file-system manager, a document typesetter, and an internet browser. In 1993, there were three major Linux distributions: Debian, Red Hat, and Slackware Linux. Presently there are currently over 250 Linux distributions listed on Distrowatch, a website dedicated to providing information on OSS distributions. Many of these distributions can be traced back to these three parent distributions and are the product of the splitting of OSS development communities.[23] One such distribution, Ubuntu, is one of the most popular Linux distributions, ranked at #3 by Distrowatch.[24] Ubuntu was forked from Debian in 2004 by Mark Shuttleworh, along with a small team of Debian developers.[25] Shuttleworth and his team felt that Debian was not accessible enough, in terms of ease-of-use, so set out to develop an easy to use distribution based on Debian. Thus the fork can be attributed to differences in missions, on the dimension of usability.

Ubuntu is among the most forked Linux distributions, Debian notwithstanding, as every fork of Ubuntu can also be interpreted as a fork of Debian as well. Ubuntu formally recognizes nine of the forks, which it refers to as "flavors."[26] However, there are over seventy forks of Ubuntu, and some of those forks, such as Kubuntu, a recognized flavor, have been forked as well. A select few of these forks are given by name, date, and popularity in Table 1.2. A complete

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[22]In practice, there are instances where the original project dies out post-fork, though in these cases, the fork is usually a response to an underlying failure that leads to the original project's death.

[23]Slackware itself was forked from SLS, which was the first comprehensive Linux distribution.

[24]Rankings are determined by an algorithm based on page traffic.

[25]http://www.ubuntu.com/about/about-ubuntu, accessed August 26, 2015.

[26]https://wiki.ubuntu.com/UbuntuFlavors, accessed August 26, 2015.
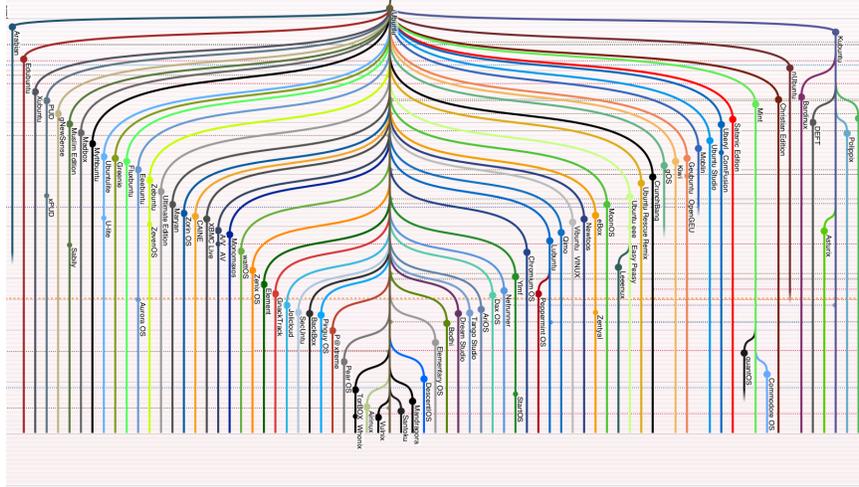
Figure 1.12: Ubuntu forks: origins to 2012.

Name of fork*	Date	Name of fork	Date
Kubuntu (34)	2005	Easy Peasy (NR)	2008
Edibuntu (164)	2005	Zorin OS (15)	2008
Mint (1)	2006	Lubuntu (17)	2009
Xubuntu (27)	2006	Peppermint OS (38)	2010
Ultimate Edition (48)	2008	Bio-Linux (170)	2012

* Number in parentheses indicates ranking according to distrowatch.com. NR corresponds to not ranked.

Table 1.2: Summary of select Ubuntu forks.

history from Ubuntu's origin through 2012 is given in Figure 1.12.[27] It is clear by looking at the names of these forks that they occurred due to heterogeneity among the contributors. I illustrate this point with a brief description of a subset of the forks reported in table 1.2. Kubuntu and Edibuntu are two of the recognized flavors. Kubuntu took Ubuntu, removed the desktop environment GNOME (distance-based depreciation) and replaced it with the KDE desktop environment. Designed for use in classrooms, Edibuntu adds an array of eduction-oriented software and is an example where there is minimal distance-based depreciation. Mint Linux removed Ubuntu's default desktop environment GNOME and replaced it with a fork known as Cinnamon which has been customized for Linux Mint. It also includes some proprietary software such as Adobe Flash. Zorin OS takes the Ubuntu distribution,

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[27]Image licensed under the creative commons license, https://commons.wikimedia.org/wiki/File:Linux_Distribution_Timeline.svg, accessed September 1, 2015.

removes the imbedded userland software (distance-based depreciation) and replaces it with a Windows clone, designed for newcomers to Linux to make the transition easier. The same line of reasoning applies for all of the forks listed in Table 1.2.

Incubating and forking software for horizontal reasons is not unique to Ubuntu or Linux distributions in general. The BSD family of distributions have also experienced their fair share of forks. The BSD distributions are perceived as direct descendants of AT&T's Unix operating system.[28] Unlike Linux, no distinction is made between the kernel and the distribution. That is, the BSD kernel and associated software are developed and released together, so each BSD distribution has its own unique kernel. There are many BSD distributions. Most modern BSD distributions are derivatives of the 386BSD distribution. Two well known distributions, FreeBSD and NetBSD, emerged in the early 1990s from 386BSD. FreeBSD represents the continuation of 386BSD under a new name while NetBSD is a distinct fork. The mission (as defined in section 1.2) of FreeBSD development is features, speed, and stability. The mission of NetBSD development is portability. Portability refers to the ability of software to run many different hardware platforms. Other well known forks of BSD systems include Darwin OS, which was forked from FreeBSD. Darwin OS is developed by Apple and is an input of OS X.[29]

OpenBSD is a distribution forked by Theo de Raadt, a long time developer of NetBSD. OpenBSD was formed with portability in mind, but also emphasizes standardization, correctness, proactive security, and integrated cryptography. It was on these points that de Raadt disagreed with fellow contributors, eventually leading to the fork.[30]

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[28]BSD was released as open source only after a series of lawsuits between developers and AT&T were settled.

[29]Darwin was developed as a combination of forks from FreeBSD and BSD 4.4, as well as NextStep 3.3, which was developed by Steve Jobs and his team at NeXT.

[30]`https://www.freebsd.org/doc/en/articles/explaining-bsd/comparing-bsd-and-linux.html`, accessed August 26, 2015.

1.5.2 Proprietary Software Development: Internal Incubation

Beyond OSS, similar development patterns can be found within individual firms developing PS. It is not uncommon to find proprietary software vendors developing multiple, related products, referred to as product lines. Microsoft's Office Suite consists of a word processor, a spreadsheet program, a database manager, and an email manager. Apple develops video and audio editing software such as Final Cut Pro and GarageBand. Autodesk develops several products useful in the rendering of three dimensional CGI, including Civil 3D, 3DS Max, and Maya. The important feature of product lines, such as the ones described above, is that the component products consist of shared features (modules). When developing product lines, the vendor must decide how to internally structure development of the software.

In general, there are three internal development structures that can be used to develop software lines, each with its own benefits and costs. The vendor can utilize multiple development teams, where each team specializes in developing a single program. Each development team codes both the common features and the features of their program independently. Alternatively, the vendor could have a single development team manufacture the programs sequentially (Taylor et al., 2010).[31] The first approach resembles the underlying strategy in the cohesive equilibrium and the second resembles the underlying strategy in the splitting equilibrium. The third option is a form of strategic membership, only the split is internal. The developers code modules of each program together as a cohesive unit, then split up to finish each program independently, while still remaining under the umbrella of the vendor.

First suppose that, as in the splitting equilibrium, there are M development teams, each responsible for independently developing one of M programs. Under this structure, development of the software proceeds rapidly since the programs are developed simultaneously. The rapidity comes at a cost, namely through a duplication of effort with the possibility of

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[31]A single team developing the various programs simultaneously is isomorphic to separate teams developing each.

(a) Simultaneous & Sequential.
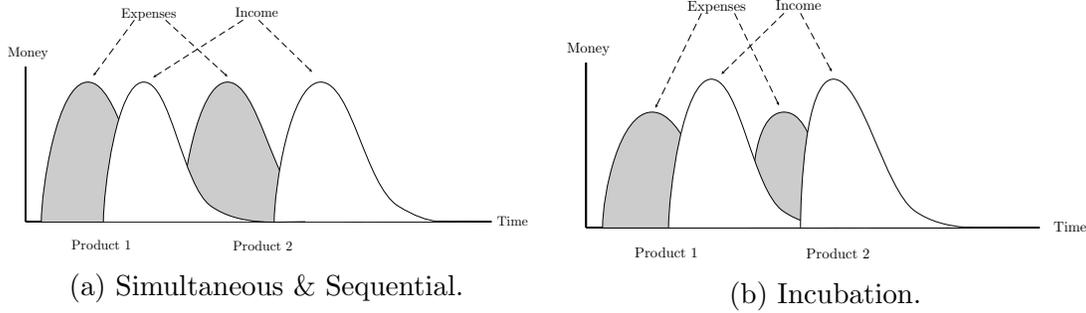
(b) Incubation.

Figure 1.13: Profitability of the three development models with $M = 2$, adapted from Taylor et al. (2010).
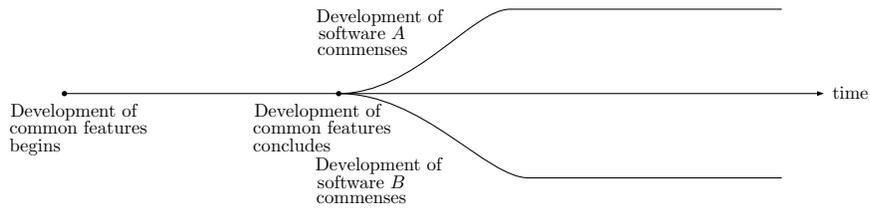
Figure 1.14: Development timeline under incubation with $M = 2$.

incompatibilities. Since each product is developed independently, the common features are also coded independently, M times.

A similar relationship exists when there is a single team developing the software, as in the cohesive equilibrium, only development proceeds at a slower pace. Each program is developed separately and sequentially, delaying the income from sales, while still suffering from the inefficiencies of duplication. Figure 1.13 illustrates how these two structures relate to the vendor's income and expenditure. During development, expenditure initially exceeds income. Once the software reaches market, income increases and eventually the software becomes profitable. The process is repeated for each of the M programs.

Incubation provides a profitable alternative for the vendor. At the outset, the entire development team works together to code the common features. Once the common features are coded, the team divides up and codes the independent modules for each program. A sample timeline is given in Figure 1.14. When utilizing the incubation strategy, the vendor eliminates duplicative waste. Not only is the cost of developing each program decreased,

but the total cost of development is decreased as well since the development cost is spread among all of the projects. This relationship is illustrated for $M = 2$ in Figure 1.13b.

In summary, incubation can benefit software vendors in the software industry developing product lines. To end with a concrete example, suppose a software vendor has two programs it wishes to develop: a voice-to-text/text-to-voice program (V2T), and visual voicemal (VV). Each product has its own independent features. The V2T program takes voice and turns it into a text output. The VV program collects voicemails and stores them on the user's mobile phone, making them available for immediate playback while also creating text versions of each voicemail. However, these two programs also share common features, namely converting voice to text. The vendor must choose between developing these two products separately or incubating, where the development team first works together to develop the text-to-voice libraries, and then splits up. One group then finishes the V2T software while the other finishes the VV software. If the vendor were to develop the projects separately, then there would likely be duplicative waste, as both projects require the development of text-to-voice libraries. By jointly developing these libraries, the vendor eliminates the duplicative waste and is able to save costly resources.

1.6   Discussion and Concluding Remarks

To summarize, this paper develops a dynamic club goods framework and explores three related equilibria. In the cohesive equilibrium, a single club persists in the long run. In the splitting equilibrium, multiple clubs exist at both the outset and the long run. The third strategy, strategic membership, and its associated equilibrium, the incubation equilibrium, is somewhat of a hybrid between the two. There is a single cohesive club at the outset, but at some point in time, the club splits and individuals partition themselves according to their mission. In the long run, splitting is always preferred to incubation, but due to the

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impatience of individuals, a tradeoff exists where it is profitable to exchange future utility for greater utility in the present via incubation.

The parent club can alter the relative payoffs of the various equilibria through the use of various instruments. The two instruments discussed in the paper are taxation/subsidization and intellectual property (exclusionary) restrictions. Through taxation and subsidization, the parent club can incentivize those practicing strategic membership to adjust the length of the incubation period to match the preferred length of the parent club, so long as the impact of the change in the incubation period is greater for the parent club. If the members of the parent club are sufficiently patient, then the parent club can also capture all rents gained by the child club through incubation. Intellectual property restrictions can be used to adjust the excludability of the club, making incubation less desirable as the proportion of installed base retained by the child club can be taken to zero by the use of restrictions including copyrights, patents, and non-compete clauses. All that is necessary for these avenues to be effective is an external enforcement mechanism, such as the court system.

Many of the features and results discussed in this paper are not unique to the software industry. Strategic membership is a strategy prevalent in many environments in which collaborative production occurs and installed base is able to accumulate over time. In some environments, such as R&D, it would be difficult to sustain strategic membership in research joint ventures between firms due to the stochastic nature of payoffs from such ventures. However, when it comes to within-firm R&D, adopting the strategy identified in the proprietary software application in section 1.5.2 could be worthwhile to a firm. When the R&D program is in its early stages and less focused, the firm concentrates its efforts to a single project, where its researchers work as a single, cohesive team. As the program grows, multiple directions may present themselves, at which point the firm splits the team to pursue each avenue. Alternatively, it could be that the firm splits the team simply to maximize the likelihood of success of the venture. If the probability of success exhibits diminishing marginal returns to

group size, two groups of 10 researchers have a better chance of success than one group of 20. As long as the cost of maintaining two groups over one does not exceed the increased expected payoff, incubation can improve the expected outcome in R&D development.

Moving outside of the IT industry and R&D, the model and ideas developed within can also be useful in understanding problems in political economy. One avenue worth exploring is the use of incubation to develop political capital in understanding the splitting of political parties in multiparty systems. Furthermore, this paper can provide an explanation for the emergence of extremist groups from mainline groups over time, e.g., the emergence of the Islamic State from Al Qaeda and large scale religious schisms, where religious capital must be built up prior to forming a successful religious denomination.

From a theoretical point of view, there are several avenues for future research. This paper relies on heterogeneity and capital accumulation to generate the underlying incentives. However, there are often multiple groups that appear to be homogeneous. The only way to generate such a situation in equilibrium is if there is congestion. Congestion is often modeled on the demand (consumption) side, but there is an under-explored alternative. The clubs could exhibit supply-side congestion, which opens up the possibility for strategic membership and the splitting of clubs, even with homogeneous individuals.

Chapter 2

Public and Club Goods When Private Costs Have Negative Externalities

2.1 Introduction

In many economic contexts, public and club goods exhibit non-monotonicities that are not accounted for in existing theoretical work. I show that these non-monotonicities significantly alter the equilibrium structure, comparative statics, and welfare implications. For example, the free-rider problem can become less severe and even vanish entirely when the number of investors increases. As the number of investors continues to increase, over-provision becomes a concern. The phenomenon of over-provision is particularly noticeable in innovation economics, for instance in the development of open source software. This paper examines the equilibrium and welfare properties of voluntarily provided public and club goods exhibiting non-monotonicities and illustrates how the typical policy prescriptions can lead to both unanticipated and undesirable outcomes. I then suggest how these policies may be altered to better suit public and club goods exhibiting non-monotonicities.

The consequences of non-monotonicity vary greatly depending on the nature of provision. When investing in the provision of the good is solely a burden (the utility from provision is strictly decreasing in investment while the consumption benefits are non-monotonic), the free-rider problem persists as expected and there is under-provision. However, in contrast to previous work such as Cornes and Sandler (1996), Scotchmer (2002), and the references therein, the severity of the free-rider problem is decreasing in the number of investors. Moreover, as the number of investors increases, the total equilibrium investment converges to the welfare-maximizing investment, which converges to the consumption-value-maximizing investment. The decrease in each individual's investment is outweighed by the increase in total investment due to the increase in the number of contributors. In other words, provision is socially optimal in the limit. The result is in stark contrast to existing models, including the well studied congestion phenomenon, which often exhibits the non-monotonicity in consumption discussed above.[1]

Open source software is often modeled as a public (Johnson, 2002; Polanski, 2007; Athey and Ellison, 2014) or club (Sacks, 2016) good. The prior literature has primarily treated the value of both open source (and proprietary) software as monotonic: increased investments imply increased quality, which imply increased utility from consumption.[2] Learning costs imply that advances in technology need not increase the utility from consumption, or alternatively, high-tech software could require costly upgrades to the underlying hardware negating any benefits of added quality. Linus Torvalds, the founding developer and namesake of Linux, once said, "given enough eyeballs, all bugs are shallow (Raymond, 1999, p. 29)" to illustrate the benefits of having a large developer base in open source software, echoing the above consequence of non-monotonicity in consumption. However, the statement neglects the fact that across many public and club goods, including open source software, developers privately

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[1]For examples, see Olson (1965), Brown, Jr. (1974), Arnott and Small (1994), and Cornes and Sandler (1996, ch. 8), among others.

[2]Exceptions include Sen (2007) and Sacks (2015), where non-monotonicities in consumption enter via learning costs.

benefit from the provision.[3] The above quote, dubbed by Raymond as "Linus's Law," need not be true when accounting for these private benefits. In fact, over-provision may occur with added investments having deleterious effects on quality.

Non-monotonicity in the utility from providing public and club goods, when coupled with non-monotonicity in the utility from consuming these goods, can lead to over-provision. If the private benefits to provision are small, then the free-rider problem persists only when there are few contributors. If the private benefits to provision are large, then instead of free-riding, there is over-provision of the good: the equilibrium investment is greater than the welfare-maximizing investment, which is greater than consumption-value-maximizing level of investment. As the number of contributors increases, the over-provision result persists, even when the private benefits to provision are small. That is to say, there is always over-provision in the limit. The incentives for provision induce a negative externality on other contributors, which is increasing in the number of contributors to the point that it dominates the positive externality of free-riding. Contributors do not internalize the impact of their own investments on the consumption value of other contributors. While in contrast to the typical congestion problem, this phenomenon can be interpreted as a supply-side corollary to the congestion problem, and can aptly be thought of as *supply-side congestion*. Supply-side congestion occurs due to the strategic tension between individual and aggregate incentives. Each participant places value not only the total quality of the good, but on her own participation in its provision.

Stamelos et al. (2002) find such a relationship in open source software after analyzing quality characteristics of one hundred open source projects written for Linux. The authors show that the relationship between the size of a component of an open source application (aggre-

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[3]These benefits can range from altruism (Andreoni, 1990), to signaling (Glazer and Konrad, 1996), to other common economic incentives (Lerner and Tirole, 2002). Lerner and Tirole (2002), Feller et al. (2005), Bitzer et al. (2007), Myatt and Wallace (2008), Fang and Neufeld (2009), and Athey and Ellison (2014), among others, have argued that the primary reasons individuals contribute to the development of open source software are not for consumption benefits, but for provision benefits, such as altruism, signaling skill, and the joy of coding.

gate investment) is negatively related to user satisfaction for the application (utility from consumption). In cases where consumers reported only superficial errors at worst, the mean length of the program is strictly lower than in cases where (*a*) consumers reported either all major program functions are working but at least one minor function is disabled or incorrect, (*b*) at least one major function is disabled or incorrect, and (*c*) the program is inoperable. Their result is robust to other metrics, including the number of statements per module. The model developed here offers a mechanism explaining the results of Stamelos et al. (2002).

A policymaker seeking to solve supply-side congestion will find the task to be quite difficult. In the public good case, a policymaker able to adjust the outcome through changing the number of investors is not likely to maximize the quality of the good, even though maximizing the quality involves maximizing both the public benefits for everyone and the private benefits for investors. Instead, the policymaker opts to increase the number of investors, decreasing both the public and private benefits each individual receives while increasing the number of individuals receiving private benefits. In the club good case, the same policymaker will maximize quality only if non-investors cannot purchase the club good without investing. If the good is priced such that non-investors can pay for the benefits, then the policymaker minimizes the number of investors, which serves to maximize quality. Thus from a quality-maximization standpoint, non-monotonic club goods can be made to be more efficient than non-monotonic public goods.

When a policymaker is able to adjust the outcome through manipulating the value of the private benefits, the results are counterintuitive and in fact make the externality worse. Welfare is maximized by increasing the private benefits without bound, sacrificing quality for the benefit of investors. All surpluses are transferred to the investors, leading to the worst possible outcome from the viewpoint of the non-investing population. This outcome can be avoided only if the policymaker can commit to mandating a minimal quality. The private benefits channel has a very interesting and important consequence in the context of

open source software: the poison pill.

It is well known that proprietary software developers support the development of open source software (Mustonen, 2005; Economides and Katsamakas, 2006b; Kumar et al., 2011; Llanes and de Elejalde, 2013). These works make arguments revolving around network effects and complementarities through spillover effects. I propose a more insidious alternative. A proprietary software developer can inject capital into an open source project thereby increasing the private benefits to provision, which can lead to decreases in the quality of its open source competition. The decrease in open source software quality increases the amount consumers are willing to pay for the proprietary software (by increasing the net quality of the proprietary software relative to the open source software). Thus the supporters are implementing a "poison pill" in the open source project. Moreover it is individually rational for the developers to accept and take the pill.

This paper contributes to several strands of literature, including the theory of public and club goods, public economics, industrial organization, and the economics of innovation. Adam Smith not withstanding, the modern literature on public and club goods can be traced back to Samuelson (1954), Olson (1965), and Buchanan (1965), with the literature on the private provision of public goods beginning with Bernheim (1986) and Bergstrom et al. (1986). Summaries of the core theories are given by Cornes and Sandler (1996), Sandler and Tschirhart (1997), and Scotchmer (2002). While many papers have been written expanding these benchmark models, the current paper is the first to analyze the effects of structural non-monotonicities in both the consumption and provision benefits, though some work has been conducted in the context of regulatory complexity.

Quandt (1983) and Kearl (1983) show that there exists conditions such that there is an over-investment in regulation (a public good). This paper generalizes these results and broadens their applicability. In a similar vein, Nitzan et al. (2013) argue that the increasing volume and velocity of complexity in regulation over time leads to inefficiencies induced by implementing

barriers to entry that allow only the largest established firms to survive, which leads to monopolistic behaviors and outcomes. The authors reference accounting regulations, corporate governance and securities regulations, and the market for corporate charters. Combining the results of Nitzan et al. (2013), Quandt (1983), Kearl (1983), and the corresponding citations therein with those of the current paper lead to a more complete understanding of the non-monotonic effect of complexity in regulation.[4]. Galasso and Schankerman (2014) find that whether or not patents hinder subsequent innovation depends on the complexity of the industry and that negative effects are concentrated in more complex industries. If one accepts the claim that it is relatively more costly to implement IP in more complex industries, then Galasso and Schankerman (2014) can be viewed as evidence in support of the non-monotonicity argument presented above.

Within the industrial organization and economics of innovation literature, this paper is particularly useful in understanding the structure and outcomes of industries (and more generally, economic actors) whose interactions are characterized by collaborative production, such as the software industry. Sacks (2015) argues that open source software developers will target the upper-end of the technology spectrum due to the nature of the user-developer model of open source software. I show more generically that collaborative groups will likely over-invest in technological investment and can also shed light on the nature of research joint ventures, building upon works such as Kamien et al. (1992) and those outlined in Caloghirou et al. (2003). Along a similar line, the model developed in the following sections provides a new alternative mechanism for understanding over-investment in patent races (Gilbert and Newbery, 1982; Fudenberg et al., 1983; Shapiro, 1985; Harris and Vickers, 1987; Aoki, 1991; Baye and Hoppe, 2003).

The remainder of the paper is structured as follows. Section 2.2 provides a simple analytical

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[4]For examples of regulatory complexity in finance, see Berglund (2014). For examples in intellectual property, see Allison and Lemley (2002). Evidence of a negative impact of the regulations is given in Murray and Stern (2007) and Williams (2013)

example outlining the results. Section 2.3 formally outlines the model. Section 2.4 presents the main results. Section 2.5 illustrates the implications of the previous section and presents a normative analysis of how policy should be conducted. Section 2.6 relates the findings to open source software with a particular focus on the poison pill scenario. Section 2.7 concludes.

2.2   Analytical Example

Before introducing the formal model and results, I use a simple quadratic public good framework to preview the underlying mechanisms. Suppose there are N individuals indexed by i, with N large. Each individual receives utility

$$u(x_i, x_{-i}; \theta, \sigma) = \left(\sum_{j=1}^{N} x_j \right) - \frac{\theta}{2} \left(\sum_{j=1}^{N} x_j \right)^2 + \sigma x_i - \frac{1}{2} x_i^2.$$

I refer to $X = \sum_{j=1}^{n} x_j$ as the aggregate investment. The typical public good model can be obtained by setting $\theta = \sigma = 0$. Setting $\theta > 0$ yields non-monotonicity in consumption and setting $\sigma > 0$ yields non-monotonicity in provision. It follows that there exists a Nash equilibrium with $M \leq N$ contributors, each investing

$$x_i^*(M, \theta, \sigma) = \frac{1 + \sigma}{1 + \theta M},$$

where $M < N$ requires that if i does not invest, then $\sum_{j \neq i}^{N} x_j^* > \frac{1}{\theta}$. The individual welfare-maximizing investment is given by

$$\tilde{x}_i(M, N, \theta, \sigma) = \frac{N + \sigma}{1 + \theta M N}.$$

The counter-veiling effects of the two forms of non-monotonicity can be seen by the fact that x_i^* and $\tilde{x}_i$ are increasing in σ but decreasing in θ. However, it is the relative size and direction of these impacts that influences the comparative statics, not the absolute size or direction.

Firstly consider the monotonic model with $\theta = \sigma = 0$. It follows that $x_i^* = 1$ and $\tilde{x}_i = N$ with $\tilde{X} - X^* = M(N-1)$. Thus as either the population size or the number of investors increases, the free-rider problem becomes more severe. Now suppose that non-monotonicity enters only in consumption, so $\theta > \sigma = 0$. The difference between the aggregate welfare-maximizing investment and the aggregate equilibrium investment is given by

$$\tilde{X} - X^* = \frac{M(N-1)}{(1+\theta M)(1+\theta M N)}.$$

The above value is strictly positive; however, note that as M increases, the above converges to zero. Thus the free-rider problem becomes less severe.

Next suppose that both forms of non-monotonicity enter, so $\theta, \sigma > 0$. The difference between the aggregate welfare-maximizing investment and the aggregate equilibrium investment is given by

$$\tilde{X} - X^* = -\frac{M(1+\sigma)}{1+\theta M} + \frac{M(N+\sigma)}{1+\theta M N} = \frac{M(N-1)(1-\sigma\theta M)}{(1+\theta M)(1+\theta M N)}.$$

Note that as the private benefits increase (σ increases), the above value decreases and eventually becomes negative. Furthermore as M (and by extension N) tends to infinity, $\tilde{X} - X^* \to -\frac{\sigma}{\theta}$, which is negative for all $\sigma > 0$.

From the above three scenarios, two conclusions can be drawn. Firstly the non-monotonicity in consumption benefits diminish the size of the free-rider externality. Secondly the non-monotonicity in consumption, coupled with the non-monotonicity in provision can reduce

and even eliminate the free-rider problem in its entirety; however, the result is not efficient. Rather, the opposite type of inefficiency, over-provision, emerges as a result. To illustrate that non-monotonicity in provision alone is not enough to induce over-provision, suppose that $\sigma > \theta = 0$. The difference between the aggregate welfare-maximizing investment and the aggregate equilibrium investment is given by

$$\tilde{X} - X^* = M(N-1).$$

Thus the free-rider problem is not negated at all by the non-monotonicity in provision, but rather by the interaction of the non-monotonicity in consumption with the non-monotonicity in provision. In what follows, I show that these results are not sensitive to the functional forms selected in the current section and provide further details on the trade-offs and underlying mechanisms.

2.3   Formal Model

The economy consists of N individuals, a single inside good (henceforth good), and an outside option. The good is either a public good or a club good.[5] Each individual i chooses her investment level $x_i \geq 0$. If the good is a public good, then the individual has the option to either free-ride or utilize the outside option and if the good is a club good, then the individual either chooses a contribution $x_i > 0$ and benefits from the good, or chooses $x_i = 0$ and consumes the outside option. Denote by $\mathbf{x} = \langle x_1, \ldots, x_N \rangle$ the vector of investments in the good. The investments are aggregated according to the production function $X = g(\mathbf{x})$.

Assumption 2.1. *The production function is continuous everywhere, unboundedly increasing, weakly concave, and symmetric. That is:*

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[5]Both cases are analyzed simultaneously.

(i) $\frac{\partial g(\mathbf{x})}{\partial x_i} > 0$ *for all* i *with* $g(\mathbf{x}) \to \infty$ *as* $x_j \to \infty$ *for any* j;

(ii) $\frac{\partial^2 g(\mathbf{x})}{\partial (x_i)^2} \leq 0$ *for all* i;

(iii) $\frac{\partial g(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x})}{\partial x_j}$ *for all* i, j.

Assumption 2.1(iii) imposes anonymity among the contributors (abstracting away from horizontal and vertical heterogeneity).[6]

Each individual's preferences are represented by a utility function consisting of two additively-separable components. The first component is a public consumption benefit that depends only on the public quality (henceforth quality) of the good as a function of aggregate investment and is common for all i not choosing the outside option. Denote the quality of the good by $Q(g(\mathbf{x}))$.

Assumption 2.2. $Q(\cdot)$ *is* C^2 *and strictly concave with* $Q(0) = 0$. *Furthermore there exists a value* $\hat{X} = g(\hat{\mathbf{x}}) > 0$ *such that* $\frac{\partial Q(X)}{\partial X} > 0$ *for all* $X < \hat{X}$ *and* $\frac{\partial Q(X)}{\partial X} < 0$ *for all* $X > \hat{X}$.

The aggregate investment level $\hat{X}$, which is independent of the number of affiliates, is interpreted as the quality-maximizing investment level. Non-monotonicity in the utility from consumption can be interpreted in multiple ways, such as costly implementation or technological limitations. Initial investments are valuable to the good; however, there comes a point where providing too much can have deleterious effects. For instance, if the good is an innovation and implementing the innovation is costly, then too large an investment in R&D can lead to a less valuable innovation, as it becomes increasingly costly to implement.

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[6]Many production functions are permitted under Assumption 2.1, such as the CES production function. In circumstances where $x_j = 0 \Rightarrow g(\mathbf{x}) = 0$ for some j, $\mathbf{x}$ can be augmented to $\langle x_j > 0 \rangle$ (only nonzero investments are counted), in which case every result still holds. All of the major results also hold under the Leontief (perfect complements) production function as well, though the proof strategies and assumptions must be altered due to the associated discontinuities. However, bounded functions such as $g(\mathbf{x}) = A - (1 + \iota \cdot \mathbf{x})^{-1}$, where $A > 0$, ι is the $1 \times N$ unit vector, and $\iota \cdot \mathbf{x}$ is the inner product are not permitted, as they violate Assumption 2.1(i).

A similar story can be told using technological limitations, e.g. in open source software development, where an over-investment in features induces lower quality software through less usability.

The second component of the utility function is a private provision (production) benefit, which is unique to each individual investment x_i. The private benefit depends on two elements: the individual's investment x_i and a benefit parameter $\sigma \geq 0$. If $\sigma = 0$, then investing is viewed as a burden and any benefits from investment arise solely from the quality of the good provided, so the private provision benefit reduces to a standard cost function; however, when $\sigma > 0$, individuals benefit not only from the quality of the good being provided, but also from investing in its provision. For example, when developing open source software, contributors benefit not only from the quality of the software being developed, but also from contributing for reasons such as signaling ability or the joy of coding.[7] Thus σ determines the relative strength of the private benefits. Regardless of the relative strength of the private benefits a large enough investment will lead to the cost of investing dominating any private benefits Denote the private benefit function by $v(x_i; \sigma)$.

Assumption 2.3. *$v(x_i; \sigma)$ is C^2 and strictly concave in x_i. Furthermore:*

(i) If $\sigma > 0$, then there exists a value $\overline{x}(\sigma) > 0$ such that $\frac{\partial v(x_i; \sigma)}{\partial x_i} > 0$ for all $x_i < \overline{x}(\sigma)$ and $\frac{\partial v(x_i; \sigma)}{\partial x_i} < 0$ for all $x_i > \overline{x}(\sigma)$;

(ii) $\frac{d\overline{x}(\sigma)}{d\sigma} > 0$ with $\overline{x}(0) = 0$;

(iii) $\frac{\partial v(x_i; \sigma)}{\partial \sigma} > 0$ and $\frac{\partial^2 v(x_i; \sigma)}{\partial x_i \partial \sigma} > 0$.

Panels (a) and (b) of Figure 2.1 diagrammatically illustrates Assumptions 2.2 and 2.3, respectively. Individual indices have been omitted from panel (b) for expositional purposes.

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[7]See Lerner and Tirole (2002) or Sacks (2016).

(a) Assumption 2.2. (b) Assumption 2.3.
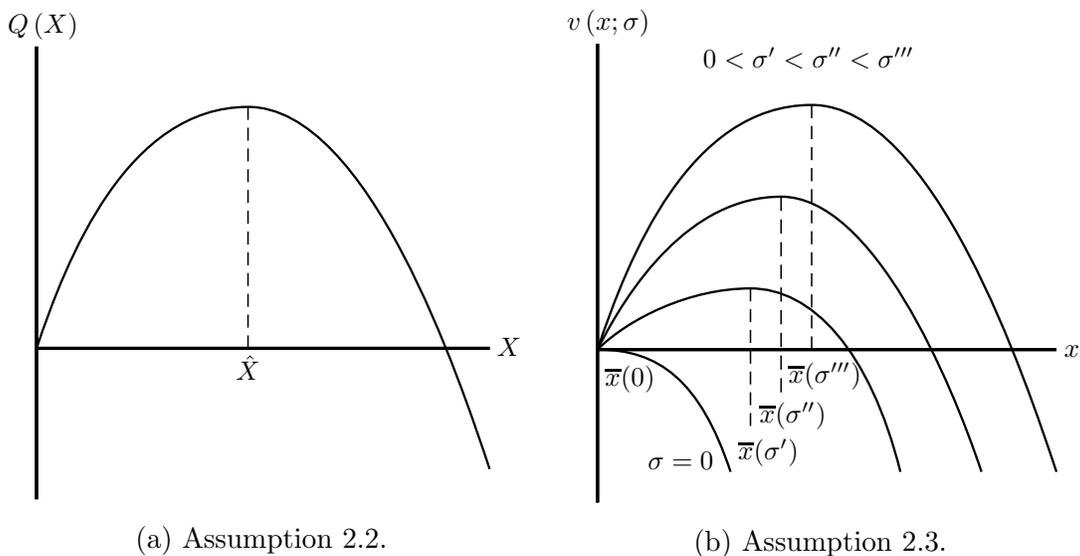
Figure 2.1: Diagrammatic representation of Assumptions 2.2 and 2.3.

Given Assumptions 2.1-2.3, and the additive-separability, individual i's utility is given by

$$u_i \equiv u\left(X, x_i; \sigma\right) = \begin{cases} Q\left(g\left(\mathbf{x}\right)\right) + v\left(x_i; \sigma\right) & \text{if public good} \\ \mathbf{1}\left\{x_i > 0\right\} Q\left(g\left(\mathbf{x}\right)\right) + v\left(x_i; \sigma\right) & \text{if club good.} \end{cases} \tag{2.1}$$

where $\mathbf{1}\left\{\cdot\right\}$ is an indicator function equaling one when the argument is true and zero otherwise. Therefore when the good is a public good, utility is given by the unconditional sum of the quality benefits and the provision benefits and when the good is a club good, the utility is given by the sum only if the individual is an investor and zero otherwise. Under Assumptions 2.1-2.3, utility is C^2 and strictly concave in x_i.[8] The utility from consuming the outside option is normalized to zero. The game is structured in a static fashion as follows. Each individual simultaneously and independently makes an investment decision.

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[8]It is worth noting that additive separability, continuous differentiability, and strict concavity are not necessary for the main results. A generic utility function $u(X, x_i; \sigma)$ that satisfies upper semi-continuity and weaker versions of Assumptions 2.1-2.3 is sufficient.

2.4 Equilibrium Analysis

As this game is static with complete information, the relevant solution concept is the Nash equilibrium. Furthermore individuals are anonymous (Assumption 2.1), so the equilibria exhibit symmetry and all individuals who invest do so identically. Accordingly I drop individual-indexing subscripts. Denote by x^* the individual equilibrium investment in the good and by $X^* = g(\mathbf{x}^*)$ the aggregate equilibrium investment. Throughout the paper, comparisons will be made between three values:

- the aggregate equilibrium investment level X^*;

- the quality-maximizing investment $\hat{X}$;

- the welfare-maximizing investment level $\tilde{X}$.

Define a team player as an individual who maximizes the objective

$$\sum_{j=1}^{n} u(X, x_j; \sigma),$$

but is still endowed with utility u_i and the associated individual rationality constraints.[9] The welfare-maximizing investment represents the investments made by team players. Denote by $\tilde{x}$ the individual welfare-maximizing investment. Unlike the quality-maximizing investment, the welfare-maximizing investment varies according to the number of investors M and the benefit parameter σ. Without loss of generality, I order the individuals by investors followed by non-investors, so if $M < N$ individuals invest, individuals $1, \ldots, M$ are the investors and individuals $M+1, \ldots, N$ are the non-investors.

The equilibrium investments are defined by the following system of first-order necessary

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[9]See (2.6) and (2.7).

conditions:

$$\frac{\partial Q\left(g\left(\mathbf{x}^{*}\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}^{*}\right)}{\partial x}+\frac{\partial v\left(x^{*};\sigma\right)}{\partial x}=0,\quad \forall i=1,\ldots,M$$
$$\frac{\partial Q\left(g\left(\mathbf{x}^{*}\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}^{*}\right)}{\partial x}+\frac{\partial v\left(0;\sigma\right)}{\partial x}<0,\quad \forall i=M+1,\ldots,N. \tag{2.2}$$

By the implicit function theorem, $x^{*}=x(M,\sigma)$, $\mathbf{x}^{*}=\mathbf{x}(M,\sigma)$, and $X^{*}=X(M,\sigma)$ and by the theorem of the maximum, $\frac{\partial x(M,\sigma)}{\partial\sigma}$ exists and is well defined.[10] There are two candidate unilateral deviations to consider. Denote by $\mathbf{x}'$ the vector of investments in which $M-1$ individuals each invest $x(M,\sigma)$ and the remaining $N-M+1$ investors each invest zero. In words, $\mathbf{x}'$ denotes the vector of investments given the deviation in which an individual i who is originally investing $x(M,\sigma)$ switches to $x_i'=0$ while all other strategies remain fixed. Denote by $\mathbf{x}''$ the vector of investments in which M individuals each invest $x(M,\sigma)$, $N-M-1$ individuals each invest zero, and one individual invests a value $x''>0$ satisfying

$$\frac{\partial Q\left(g\left(\mathbf{x}''\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}''\right)}{\partial x}+\frac{\partial v\left(x'';\sigma\right)}{\partial x}=0.$$

In words, $\mathbf{x}''$ denotes the vector of investments given the deviation in which an individual originally investing $x=0$ switches to x''. For the public good case, there exists a Nash equilibrium in which the first M individuals invest and the remaining individuals do not if

$$Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right)+v\left(x(M,\sigma);\sigma\right)\geq\max\left\{Q\left(g\left(\mathbf{x}'\right)\right),0\right\} \tag{2.3}$$

$$\max\left\{Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right),0\right\}\geq Q\left(g\left(\mathbf{x}''\right)\right)+v\left(x'';\sigma\right). \tag{2.4}$$

Replacing the right-hand side of (2.3) and the left-hand side of (2.4) with zeros yields the club good case:

$$Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right)+v\left(x(M,\sigma);\sigma\right)\geq 0\geq Q\left(g\left(\mathbf{x}''\right)\right)+v\left(x'';\sigma\right).$$

---

[10]For $\sigma=0$, define the derivative as $\lim_{\Delta\to 0^{+}}\frac{x(M,\Delta)-x(M,0)}{\Delta}$.

Inequality (2.3) states that there is no profitable unilateral deviation in which an investor deviates by not investing and (2.4) states that there is no profitable unilateral deviation in which a non-investor deviates by investing. Inequality (2.4) need only hold when $M < N$; otherwise, such a deviation does not exist. Denote by $M^*(N)$ the values of M such that $\mathbf{x}(M^*(N), \sigma)$ constitutes a Nash equilibrium.[11] The concavity of the utility function guarantees that every other potential unilateral deviation yields a payoff no greater than the two mentioned above.

The welfare-maximizing investments conditional on M investors are defined by the system

$$
\begin{aligned}
\frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X}\frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x} + \frac{1}{K}\frac{\partial v\left(\tilde{x};\sigma\right)}{\partial x} &= 0, \quad \forall\, i = 1, \ldots, M \\
\frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X}\frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x} + \frac{1}{K}\frac{\partial v\left(0;\sigma\right)}{\partial x} &< 0, \quad \forall\, i = M+1, \ldots, N,
\end{aligned}
\tag{2.5}
$$

where

$$
K = \begin{cases} N & \text{if public good} \\ M & \text{if club good.} \end{cases}
$$

The second-order sufficient conditions for both the equilibria and welfare maximizers are globally satisfied by the strict concavity of u_i. By the implicit function theorem and the theorem of the maximum, $\tilde{x}$, $\tilde{\mathbf{x}}$, and $\tilde{X}$ can all be represented by functions of M, N, and σ. Inequalities isomorphic to (2.3) and (2.4) can be derived for the welfare-maximizing case by augmenting the maximal deviation payoffs appropriately and replacing $x(M, \sigma)$ and $\mathbf{x}(M, \sigma)$ with $\tilde{x}(M, N, \sigma)$ and $\tilde{\mathbf{x}}(M, N, \sigma)$, respectively. Denote by $\tilde{\mathbf{x}}'$, $\tilde{\mathbf{x}}''$, $\tilde{x}''$ the respective deviations, where $\tilde{x}''$ satisfies

$$
\frac{\partial Q\left(g\left(\tilde{\mathbf{x}}''\right)\right)}{\partial X}\frac{\partial g\left(\tilde{\mathbf{x}}''\right)}{\partial x} + \frac{1}{K}\frac{\partial v\left(\tilde{x}'';\sigma\right)}{\partial x} = 0.
$$

---

[11]Details on the existence of such a value will be provided shortly.

The augmented versions of (2.3) and (2.4) are given by

$$Q\left(g\left(\tilde{\mathbf{x}}(M, N, \sigma)\right)\right) + v\left(\tilde{x}(M, N, \sigma); \sigma\right) \geq \max\left\{Q\left(g\left(\tilde{\mathbf{x}}'\right)\right), 0\right\} \tag{2.6}$$

$$\max\left\{Q\left(g\left(\tilde{\mathbf{x}}(M, N, \sigma)\right)\right), 0\right\} \geq Q\left(g\left(\tilde{\mathbf{x}}''\right)\right) + v\left(\tilde{x}''; \sigma\right)., \tag{2.7}$$

with the club good case following accordingly. As with (2.4), (2.7) need only hold for $M < N$. Denote by $\tilde{M}(N)$ the values of M such that $\tilde{\mathbf{x}}\left(\tilde{M}(N), N, \sigma\right)$ satisfies conditions (2.6) and (2.7). In what follows, $M^*(N)$ and $\tilde{M}(N)$ are used to denote the number of equilibrium and team-player investors, while M will be used to represent an arbitrary number of investors.

To isolate the effect of the non-monotonicity in quality, suppose for the moment that $\sigma = 0$, so conditional on the quality (public benefit), investing is solely a burden. The following proposition characterizes the relationship between the number of investors and the welfare-maximizing investment.

Proposition 2.1. *Suppose that investing is solely a burden* $(\sigma = 0)$ *and* $M \leq N$ *individuals invest. It follows that for any finite* M *and* N, $\tilde{X}(M, N, \sigma) < \hat{X}$, *and*

(i) *if the good is a public good, then as* $N \to \infty$, $\tilde{X}(M, N, \sigma) \to \hat{X}$.

(ii) *If the good is a club good, then as* $M \to \infty$, $\tilde{X}(M, N, \sigma) \to \hat{X}$.

When maximizing welfare, each individual places weight on both quality and private benefits. Because every investor receives an identical quality benefit, as do the free-riders in the public good case, the quality benefit is allocated K times the weight of the private benefit. Maximizing quality is akin to each individual placing no weight on the private benefits (or infinite weight on the quality benefit). Thus for any finite K, less weight is being placed on quality relative to the quality-maximizing case, which implies that the welfare-maximizing investment is strictly below the quality-maximizing investment. However, as

K increases (the population size in the public good case or both the population size and the number of investors in the club good case), the welfare-maximizing investment tends towards the quality-maximizing investment. When M tends to infinity, the club good may or may not resemble a public good, as the result is valid when M and N converge to infinity at different rates, with $M \leq N$. While the above result is not overly surprising given the single-peakedness of the quality benefits coupled with the monotonicity of the private benefits, it serves as a benchmark to understand the effect of non-monotonicity in provision and the relationship between the quality-maximizing investment, welfare-maximizing investment, and the equilibrium investments. The next step is to uncover how the welfare-maximizing case described in Proposition 2.1 compares to the equilibrium case.

Proposition 2.2. *If investing is solely a burden ($\sigma = 0$), then $M^*(N) = N$ individuals investing with $\mathbf{x}^* = \mathbf{x}(N, 0)$ constitutes the unique Nash equilibrium. Furthermore $X(N, \sigma) \leq \tilde{X}(N, N, \sigma)$ for all finite $N \geq 1$, with equality iff $N = 1$. Lastly as $N \to \infty$, $x(N, \sigma) \to 0$ and $X(N, \sigma) \to \hat{X}$.*

When investing is solely a burden, Proposition 2.2 has two implications. Firstly and consistent with previous work, for small N the free-rider problem persists. As is the case with any public good with voluntary contributions, investors do not internalize the impact of their own contributions on the quality benefits of other investors. Secondly and in contrast to previous work, as N increases the free-rider problem becomes less severe and the Nash equilibrium converges to the welfare-maximizing investment, which converges to the quality-maximizing investment. As more individuals invest, the positive externality induced by the quality benefits leads to free-riding and each investor contributes less. The single-peakedness of the quality benefits diminishes the effects of free riding, allowing the Nash investors to "catch up" with the team players. By Proposition 2.1, the aggregate investments of the team players (the welfare-maximizing investment) is already tending towards the quality-maximizing

69
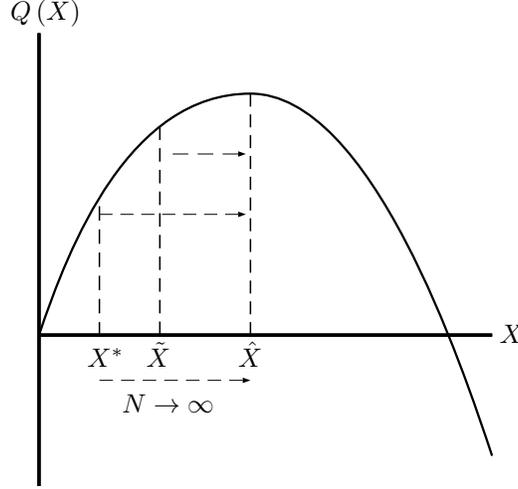
Figure 2.2: Relationship between the symmetric Nash equilibrium, the welfare-maximizing investment, and the quality-maximizing investment for $N > 1$ and $\sigma = 0$.

investment. Proposition 2.2 is diagrammatically depicted in Figure 2.2. According Proposition 2.2, simply increasing the number of investors is welfare-improving. It turns out that this relationship is false in general, but for different reasons than the typical model of the voluntary provision of a public good.

Now suppose that there may be private benefits to investing, so $\sigma \geq 0$. In any equilibrium with $M^*(N)$ investors, it must be that (2.2) and (2.3) hold for each of the $M^*(N)$ investors and (2.4) holds for the $N - M^*(N)$ non-investors (free-riders in the public good case). However, the condition $\frac{\partial v(x(M^*(N),\sigma);\sigma)}{\partial x} < 0$ need not hold as it did in Propositions 2.1 and 2.2. Rather, one of three mutually exclusive and exhaustive equilibrium conditions must hold.

Condition 2.1. (C1) $\frac{\partial Q(g(\mathbf{x}(M;\sigma)))}{\partial X} > 0$ and $\frac{\partial v(x(M,\sigma);\sigma)}{\partial x} < 0.$

Condition 2.2. (C2) $\frac{\partial Q(g(\mathbf{x}(M;\sigma)))}{\partial X} < 0$ and $\frac{\partial v(x(M,\sigma);\sigma)}{\partial x} > 0.$

Condition 2.3. (C3) $\frac{\partial Q(g(\mathbf{x}(M;\sigma)))}{\partial X} = \frac{\partial v(x(M,\sigma);\sigma)}{\partial x} = 0.$

70

It is worth noting that $C2.3$ occurs only under very restrictive conditions, so attention is focused on $C2.1$ and $C2.2$ for the remainder of the analysis, with $C2.3$ discussed only when relevant.

Lemma 2.1. *There exists a decreasing function $\overline{\sigma}(M) : \mathbb{N} \setminus \{0\} \to \mathbb{R}_+ \cup \{0\}$ such that if $\sigma < \overline{\sigma}(M)$, then $C2.1$ holds and if $\sigma > \overline{\sigma}(M)$, then $C2.2$ holds. As $M \to \infty$, $\overline{\sigma}(M) \to 0$.*

Lemma 2.1 follows from the fact that the equilibrium and welfare-maximizing individual investments are both decreasing in the number of investors, while the aggregate equilibrium investment and welfare-maximizing investment are both increasing in M.[12] That is,

$$x(M+1, \sigma) < x(M, \sigma) \qquad \tilde{x}(M+1, \sigma) < \tilde{x}(M, \sigma)$$

$$X(M+1, \sigma) > X(M, \sigma) \qquad \tilde{X}(M+1, \sigma) > \tilde{X}(M, \sigma).$$

The result of Lemma 2.1 is important, as it shows that both conditions $C2.1$ and $C2.2$ are possible in equilibrium. By continuity, $C2.3$ holds only at $\sigma = \overline{\sigma}(M)$. Moreover, over-investment relative to the quality-maximizing investment will occur if the private benefit from investing is sufficiently large ($\sigma > \overline{\sigma}(M)$). A statement isomorphic to Lemma 2.1 can be derived for the welfare maximizing case. When σ is small, the typical under-provision result stemming from the free-rider problem persists. Thus private incentives in collaborative production impose a negative externality on the contributors, which can be severe enough to induce over-investment. An important consequence of over-provision is that in equilibrium, some individuals may choose to free-ride or consume the outside option. To illustrate, consider the following simple club good example.

Example 2.1. *Suppose that $g(\mathbf{x}) = \sum_{j=1}^{M} x_j$, $Q(g(\mathbf{x})) = g(\mathbf{x}) - \frac{\theta}{2} g(\mathbf{x})^2$ for some small positive θ, and $v(x_j; \sigma) = \sigma x_j - \frac{1}{2} x_j^2$. It follows that with M affiliates, each affiliate invests*

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[12]See the Proof of Lemma 2.1.

(a) Utility.

(b) Region for $u_i \geq 0$.
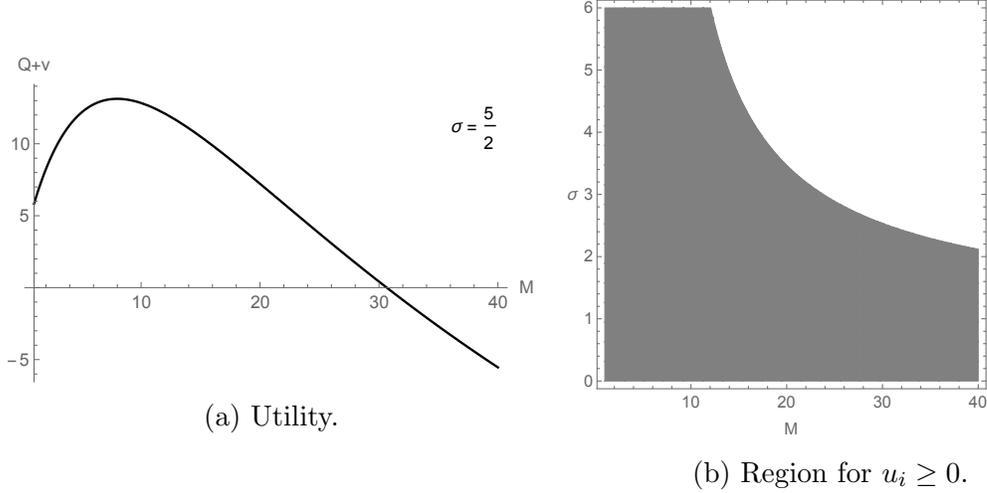
Figure 2.3: Maximal club size under Example 2.1.

$x(M, \sigma) = \frac{1+\sigma}{1+\theta M}$. *Substituting this value into the quality function yields*

$$Q(g(\mathbf{x}(M, \sigma))) = \frac{[2 - \theta M(\sigma - 1)](1 + \sigma)M}{2(1 + \theta M)^2}.$$

Note that $\overline{\sigma}(M) = \frac{1}{\theta M}$, which for fixed θ tends to zero as M increases, and $Q(\cdot)$ is strictly decreasing in σ for all $\sigma > \overline{\sigma}(M)$. Furthermore $Q(g(\mathbf{x}(M, \sigma)))$ is decreasing in M when $\sigma > \overline{\sigma}(M)$, while $v(x(M, \sigma); \sigma)$ is bounded from above. Thus for $\sigma > 0$, the equilibrium club size is finite. Figure 2.3 diagrammatically presents the above argument.

Denote by $M^U \in \mathbb{R}_+$ the maximal club size, which corresponds to the M such that (2.3) holds with equality. As individuals are atomistic, the feasible maximum number of investors is given by $\lfloor M^U \rfloor$. Under Example 2.1, if $\sigma = \frac{5}{2}$, then $\lfloor M^U \rfloor = 30$. The shaded region of Figure 2.3(B) corresponds to all pairs (M, σ) such that $u_i \geq 0$ given $x = x(M, \sigma)$ and $\mathbf{x} = \mathbf{x}(M, \sigma)$.

A similar upper bound can be derived in the case of a public good. If there are M investors and $\sigma < \overline{\sigma}(M)$, then (2.4) cannot be satisfied as both $Q(\cdot)$ and $v(\cdot; \sigma)$ are increasing for small x_i. Thus the public good must satisfy either $C2.2$ or $C2.3$. It follows that if $\sigma \leq \overline{\sigma}(N)$,

then all N individuals invest. If the public good is developed under $C2.2$ or $C2.3$, then the value of M that makes (2.3) hold with equality (M^U) indicates the upper bound on the number of investors in equilibrium. To see this relationship, note that under $C2.2$ and $C2.3$, $v(x(M, \sigma); \sigma)$ is decreasing in M while $Q(g(\mathbf{x}')) - Q(g(\mathbf{x}(M, \sigma)))$ is increasing in M, so (2.3) is violated for all $M > M^U$. The value of M, denoted by M^L, that makes (2.4) hold with equality indicates the lower bound on the number of investors in equilibrium under both the public good case and the club good case. It follows that (2.4) is violated for all $M < M^L$. Isomorphic relationships can be derived for constraints (2.6) and (2.7). Denote by $\tilde{M}^U$ and $\tilde{M}^L$ these values, respectively. This result is stated formally below.

Lemma 2.2. *The following statements are true.*

(i) *There is at most one value such that (2.3) holds with equality.*

(ii) *There is at most one value such that (2.4) holds with equality.*

(iii) *If (2.3) holds with equality, then (2.4) is strictly satisfied.*

(iv) *If (2.4) holds with equality, then (2.3) is strictly satisfied.*

(v) $M^U - M^L \geq 1$.

The implications of Lemma 2.2 are threefold. Firstly by statement (iii), M^U does indeed define an upper bound and by statement (iv), M^L is indeed a lower bound. Secondly statements (i) and (ii) guarantee that there can be no Nash equilibrium with $M \notin [M^L, M^U]$. Thirdly statement (v) states that the range must contain an integer value, guaranteeing the existence of a Nash equilibrium.

Consequently in equilibrium, the maximal club size is determined by $\min \{\lfloor M^U \rfloor, N\} \in \mathbb{N} \setminus \{0\}$. If the club consists of $M > \lfloor M^U \rfloor$ individuals, then (a) $\frac{\partial Q(g(\mathbf{x}(M;\sigma)))}{\partial X} < 0$ and

(b) $\lim_{\varepsilon \to 0} u\left(g\left(\mathbf{x}'\right), \varepsilon; \sigma\right) < 0$, where each element j in $\mathbf{x}'$ equals each element j in $\mathbf{x}(M, \sigma)$ for all but the ith element, which is replaced by ε. Statement (a) asserts that there is an over-investment in the good with respect to quality. Statement (b) expands upon statement (a), illustrating that the over-investment is so severe that if an individual had to choose between being a part of the club by contributing approximately zero or choosing the outside option, then the outside option is strictly preferred. Naturally if $N > M^U$, then $N - \lfloor M^U \rfloor$ individuals remain unaffiliated in equilibrium. The maximal number of investors when the good is public is given by $\min\left\{\lfloor M^U \rfloor, N\right\}$, while the minimal number of investors in both cases is also given by $\lceil M^L \rceil$. The correspondence

$$M^*(N) : N \to \left\{Z \in \mathbb{N} \,|\, M^L \leq Z \leq \min\left\{M^U, N\right\}\right\}$$

defines the equilibrium number of investors as a function of the population size. A similar statement for $\tilde{M}(N)$ can be derived corresponding to the welfare-maximizing investment.

Proposition 2.3. *For every $M^*(N)$, there exists a Nash equilibrium with $M^*(N)$ investors, each investing $x^* = x(M^*(N), \sigma)$. Furthermore the good is produced under C2.1 only if $M^*(N) = N$.*

While multiple equilibria may persist, all equilibria share the same characteristics so the analysis can be conducted as if $M^*(N)$ were unique. Unlike the case with $\sigma = 0$, the equilibrium aggregate investment $X\left(M^*(N), \sigma\right)$ need not be lower than the welfare-maximizing investment, nor need it converge to the quality-maximizing investment. The relationships between the equilibrium contributions, the welfare-maximizing contributions conditional on the club size equaling the equilibrium club size, and the quality-maximizing contributions for any $\sigma \geq 0$ are outlined in the following theorem.

Theorem 2.1. *If $\sigma < \bar{\sigma}(N)$, then $X\left(M^*(N), \sigma\right) \leq \tilde{X}\left(\tilde{M}(N), N, \sigma\right) < \hat{X}$ and if $\sigma > \bar{\sigma}(N)$, then $\hat{X} < \tilde{X}\left(\tilde{M}(N), N, \sigma\right) \leq X\left(M^*(N), \sigma\right)$, where in either case, $X\left(M^*(N), \sigma\right) = \tilde{X}\left(\tilde{M}(N), N, \sigma\right)$ only if $K = 1$. That is, when the private benefits are small, there is under-provision and the free-rider problem persists and when the private benefits are large, supply-side congestion occurs. Furthermore as $K \to \infty$, $\tilde{X}\left(\tilde{M}(N), N, \sigma\right) \to \hat{X}$ for any σ, $X\left(M^*(N), \sigma\right) \to \hat{X}$ for $\sigma = 0$, and for all $\sigma > 0$, $X\left(M^*(N), \sigma\right) > \hat{X}$. That is, supply side congestion always occurs with a large enough population.*

If $\sigma = \bar{\sigma}(M)$, then it immediately follows from $C2.3$ that the equilibrium investment equals the welfare-maximizing investment, which equals the quality-maximizing investment. The following corollary emerges as a direct consequence of Theorem 2.1.

Corollary 2.1. *If $\sigma' > \sigma''$, then*

$$g\left(\mathbf{x}\left(M^*(N), \sigma'\right)\right) > g\left(\mathbf{x}\left(M^*(N), \sigma''\right)\right)$$

$$g\left(\mathbf{x}\left(\tilde{M}(N), \sigma'\right)\right) > g\left(\mathbf{x}\left(\tilde{M}(N), \sigma''\right)\right).$$

In words, the aggregate equilibrium investment is increasing in σ, which implies that as M becomes increasingly large, the distance between the equilibrium investment and the quality-maximizing investment is increasing in the relative strength of the private benefit.

A diagrammatic representation of Theorem 2.1 is provided in Figure 2.4, where X^∞ denotes the limiting value of $X(M^*(N), \sigma)$ for $\sigma > 0$. Similar to Proposition 2.2, the welfare-maximizing investment always converges to the quality-maximizing investment; however, in contrast to Proposition 2.2, when $\sigma > 0$ the aggregate equilibrium investment does not not converge to the quality-maximizing investment, instead always converging to a level strictly greater than the quality-maximizing investment (X^∞ in the example presented in the figure).
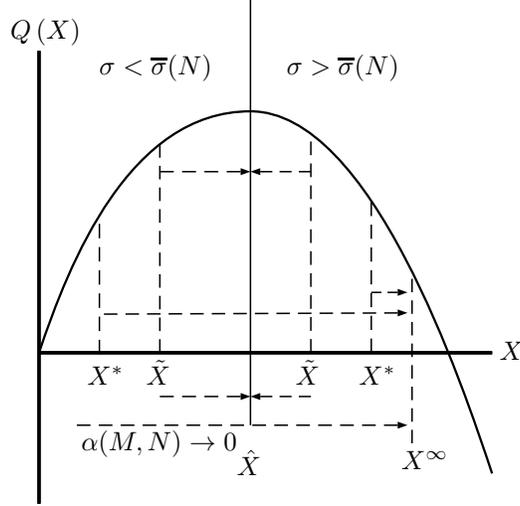
Figure 2.4: Relationship of the aggregate contributions between the Nash equilibrium and welfare-maximizing solution.

This result follows from Lemma 2.1 as $\overline{\sigma}(M)$ tends to zero as M becomes large, which implies that the equilibrium always falls under $C2.2$. Thus there is always over-provision in the limit whenever $\sigma > 0$.

Example 2.2. *Using the simple example quadratic example presented in Example 2.1, recall that*

$$Q\left(g\left(\mathbf{x}\left(M,\sigma\right)\right)\right) = \frac{[2 - \theta M(\sigma - 1)](1+\sigma)M}{2(1+\theta M)^2}.$$

Differentiating the above with respect to M (supposing for the moment that M is continuous) yields

$$\frac{\partial Q\left(g\left(\mathbf{x}\left(M,\sigma\right)\right)\right)}{\partial M} = \frac{(1+\sigma)(1-\sigma\theta M)}{(1+\theta M)^3} \lessgtr 0 \Leftrightarrow \sigma \gtrless \frac{1}{\theta M}.$$

Therefore over-investment relative to the quality-maximizing investment $\left(\hat{X} = \frac{1}{2\theta}\right)$ occurs whenever $\sigma > \overline{\sigma}(M)$ and $\lim_{M\to\infty} Q(g(\mathbf{x}(M,\sigma))) = \frac{1-\sigma^2}{2\theta}$. The equilibrium investment equals the quality-maximizing investment only at $\sigma = 0$.

The lack of convergence in $X(M^*(N), \sigma)$ follows from the negative externality imposed by $\sigma > 0$ dominating the positive externality (free-riding). In other words, quality is monotonically increasing in the number of investors whenever $\sigma = 0$. If $\sigma > 0$, then the relationship between quality and the number of investors is non-monotonic.

2.5   Policy Perspectives

Depending on the context, there may be many options available to policymakers with respect to public and club goods. For both tractability and concreteness, I assume that policymakers cannot directly affect investments, but can indirectly affect them through two channels: the number of investors M and the private benefit parameter σ. Call the first channel the investment channel and the second the benefits channel. Policies that manipulate the number of investors induce changes on the intensive margin via changes on the extensive margin. Policies that manipulate the private benefit parameter induce changes on the extensive margin via altering decisions on the intensive margin. In either case, I allow for both individually rational investors and team players. This section is divided into two subsections. The first outlines the feasible welfare-maximizing policies through the investment channel. The second outlines the welfare-maximizing policies through the benefits channel.

2.5.1   Investment Channel

Suppose that the policymaker has no influence over the individuals' benefits σ or decisions on the intensive margin, but is able to selectively restrict or expand investment on the extensive margin, thereby rendering (2.3), (2.4), (2.6), and (2.7) obsolete. It turns out that even though maximizing the quality of the good will maximize the utility of both any individual investor and any individual non-investor, a policymaker interested in maximizing total welfare rarely

has the incentive to maximize quality. Instead of maximizing quality, a policymaker can shift the number of investors and non-investors and transfer non-investor surpluses from quality to investors in the form of increasing private benefits. The remainder of the section details the incentives behind the above result, first for the public good case and followed by the club good case.

2.5.1.1 Public Good

The objective of the planner is given by

$$\max_M \left\{ (N - M) \max \{ Q \left(g \left(\mathbf{x}(M, \sigma) \right) \right), 0 \} + \sum_{j=1}^{M} \left[Q \left(g \left(\mathbf{x}(M, \sigma) \right) \right) + v \left(x_j(M, \sigma); \sigma \right) \right] \right\}, \quad (2.8)$$

subject to

$$x_j(M, \sigma) = \arg\max \{ Q \left(g \left(\mathbf{x} \right) \right) + \beta(M, N) v \left(x_j; \sigma \right) \} \quad \forall j \le M \quad (2.9)$$

with $\beta(M, N) \in \{\alpha(M, N), 1\}$ and $M \in \{1, \ldots, N\}$. The function $\beta(M, N)$ allows for comparisons with both the Nash and welfare-maximizing investment of Section 2.4. Denote by $\hat{M}$ the solution to (2.8).

Firstly note that there can be no interior solution $\hat{M} < \overline{\sigma}^{-1}(\sigma)$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\hat{M} + \left(\hat{M} + 1 \right)}{2};$$

otherwise, adding one more investor increases the quality for the free-riders and both the quality and private value for the original $\hat{M}$ investors while also increasing the utility of the new investor by the marginal change in the quality plus the entirety of the private value, leading to a strict welfare gain.[13] Counter-intuitively, this logic cannot be extended to the

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[13]If $\sigma = 0$, then the private value is a net loss on the new contributor, but by (2.4), the marginal impact

other side of $\overline{\sigma}^{-1}(\sigma)$. Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists a value $\hat{M} - 1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\left(\hat{M} - 1\right) + \hat{M}}{2}.$$

By decreasing $\hat{M}$ to $\hat{M} - 1$, the quality increases, as does the value for each of the $\hat{M}$ investors. Yet, this change may actually lead to a decrease in total welfare if the change in quality, weighted by the number of consumers plus the change in value, weighted by the number of investors, is less than the loss in private value by the $\hat{M}^{th}$ investor:

$$
\begin{aligned}
(N - M) &\left[\max \left\{ Q\left(g\left(\mathbf{x}\left(\hat{M} - 1, \sigma \right) \right) \right), 0 \right\} - \max \left\{ Q\left(g\left(\mathbf{x}\left(\hat{M}, \sigma \right) \right) \right), 0 \right\} \right] \\
+ M &\left[Q\left(g\left(\mathbf{x}\left(\hat{M} - 1, \sigma \right) \right) \right) - Q\left(g\left(\mathbf{x}\left(\hat{M}, \sigma \right) \right) \right) \right] \\
+ \left(\hat{M} - 1 \right) &\left[v\left(x\left(\hat{M} - 1, \sigma \right) ; \sigma \right) - v\left(x\left(\hat{M}, \sigma \right) ; \sigma \right) \right] \\
&\qquad\qquad\qquad\qquad < v\left(x\left(\hat{M}, \sigma \right) ; \sigma \right). \quad (2.10)
\end{aligned}
$$

That is to say, increasing the quality of the good by decreasing the number of investors increases the utility for each investor and each free-rider, but may decrease welfare due to the decrease in the utility of the investor turned non-investor. Thus depending on how welfare is defined, there are different policy protocols.

Theorem 2.2. *If (2.10) is satisfied, then the optimal strategy for a policymaker interested in maximizing total welfare need not match the optimal strategy for a policymaker interested in maximizing quality.*

Theorem 2.2 has bite due to the fact that maximizing quality is akin to currently maximizing both the public quality and private value for any given investor. A tension remains on the extensive margin, whereby adding an additional investor may decrease both the quality for

---

of the loss is less than the marginal increase in quality leading to a positive change in utility.

the entire economy and the private value for the original investors, but increase the new investors payoff enough that aggregate welfare of the economy is increased. Even if there is an integer M such that $M = \overline{\sigma}^{-1}(\sigma)$, it may not be the welfare maximizer. In other words, supply-side congestion exists on both the intensive and extensive margins. I illustrate the results of this section with the following example.

Example 2.3. *Consider the quadratic framework from Examples 2.1 and 2.2 and note that for a public good,*

$$x(M, \sigma) = \frac{1 + \sigma}{1 + \theta M}$$
$$\tilde{x}(M\sigma) = \frac{N + \sigma}{1 + \theta MN}.$$

It follows from Example 2.2 that the M that maximizes quality is an element of the set $\{\lfloor \frac{1}{\theta\sigma} \rfloor, \lceil \frac{1}{\theta\sigma} \rceil\}$ or $\sigma > 0$ and N for $\sigma = 0$. The M that maximizes total welfare when individuals are individually rational is given by the solution to

$$\max_{M} \frac{N(1 + \sigma)[\sigma + 2M(1 + \theta\sigma) - 1 - \theta M^2(\sigma - 1)]}{2(1 + \theta M)^2},$$

which is given by

$$\hat{M} \in \begin{cases} \left\{ \left\lfloor \frac{2N + \sigma - 1}{\theta(2N\sigma - 3\sigma - 1)} \right\rfloor, \left\lceil \frac{2N + \sigma - 1}{\theta(2N\sigma - 3\sigma - 1)} \right\rceil \right\} & \text{if } \sigma > \overline{\sigma}\left(\hat{M}\right) \\ \{N\} & \text{if } \sigma < \overline{\sigma}(N). \end{cases}$$

Note that the above only equals the quality-maximizing level for $\sigma = 0$ or as $N \to \infty$.

The M that maximizes total welfare when individuals are team players is given by the solution to

$$\max_{M} \frac{M(N + \sigma)^2}{2(1 + \theta MN)},$$

80

which is given by $\hat{M} = N$.

Example 2.3 shows that even when individuals are team players, a policymaker will choose a welfare-maximizing number of investors that also maximizes quality if $\sigma < \overline{\sigma}(N)$, or equivalently, N is sufficiently large. Regardless of the type of investor, the above shows that a policymaker is willing to sacrifice the utility of a consumer in order to increase the utility of a producer in order to maximize total welfare.

Corollary 2.2. *If the set of potential investors is a strict subset of the population and σ is small, then, for a finite population, a policymaker seeking to maximize total welfare will never choose the number of investors that maximizes quality.*

2.5.1.2 Club Good

Now suppose that the good is a club good. Many of the results from the prior section remain with only minor changes. The ultimate question is how to treat those who do not affiliate and invest. There are two options. Firstly individuals who do not invest in its development can purchase membership. That is, for some price $p > 0$, individuals receive the quality benefits without contributing. If $p = 0$, then the good is a public good; however, for all $p > 0$, the relevant trade-offs are all but identical. Individuals would be willing to pay any price $p \le Q(g(\mathbf{x}))$. If the club is given the choice, the affiliates will select $p = Q(g(\mathbf{x}))$. While seemingly innocuous, such a pricing scheme can actually be welfare enhancing. Under this scheme, each non-investor receives utility

$$
\begin{cases}
Q(g(\mathbf{x})) - p & \text{if purchase} \\
0 & \text{if no purchase}
\end{cases}
$$

Each investor receives utility

$$Q(g(\mathbf{x})) + v(x_i; \sigma) + \text{share of sales to non-investors.}$$

In equilibrium, $p = Q(g(\mathbf{x}))$ if $Q(g(\mathbf{x})) \geq 0$. If $Q(g(\mathbf{x})) < 0$, then p is free to take on any value, as no non-investor will purchase, instead opting for the outside option. Denote $s_i(N - M) : \mathbb{N} \to [0, 1]$ by the share of profits from the $N - M$ non-investors allocated to each investor i, where $\sum_{j=1}^{M} s_j(N - M) = N - M$. In equilibrium, each investor receives

$$[1 + s_i(N - M)]Q(\mathbf{x}(M, \sigma))) + v(x(M, \sigma); \sigma).$$

Compared to Section 2.4, more weight is placed on the quality relative to the private value. Without loss of generality, suppose that revenues are split evenly, so that $s_i(N - M) = \frac{N-M}{M}$.

Proposition 2.4. *For any M, the pricing scheme described above is both quality-improving and welfare-improving relative to the public good case.*

While not incentive compatible, if a policymaker has the ability to choose a $p \in (0, Q(g(\mathbf{x})))$, then the quality of the club good will be strictly greater than that of Section 2.4 and the outcome is welfare-improving.

If such a pricing scheme cannot be initiated, then a policymaker with the ability to influence the number of affiliates should set the number of investors such that production occurs as close to $C2.3$ as feasibly possible. Unlike the public good case, there are no externalities associated with free-riders, so the trade-off described in the previous section between decreasing quality and increasing the private value does not apply. Therefore the optimal value $\hat{M}$ must

be such that

$$\frac{\hat{M} + \left(\hat{M} + 1\right)}{2} < \bar{\sigma}^{-1}(\sigma) < \frac{\left(\hat{M} - 1\right) + \hat{M}}{2}.$$

In words, there can be no integer closer to $\bar{\sigma}^{-1}(\sigma)$ (the quality-maximizing number of investors) than $\hat{M}$.

Theorem 2.3. *When the pricing scheme described above can be implemented, the club good case is identical to Theorem 2.2. When the pricing scheme cannot be implemented, a policymaker always maximizes quality.*

The immediate implication of Theorem 2.3 is that the outcome that can be achieved by a policymaker in the public good case can be no better than the outcome in the club case. However, the gains in total welfare in the club good case come at a cost to the non-investors. In the public good case, total welfare may be lower, but every individual receives a positive payoff. While total welfare is greater in the club good case, non-investors end up with zero utility, so all of the excess surplus is captured by the investors.

2.5.2 Benefits Channel

Now suppose that the policymaker has no influence on the extensive margin, so (2.3), (2.4), (2.6), and (2.7) all hold, but can now influence the individuals' benefit parameter σ. It turns out that a policymaker interested in maximizing total welfare by manipulating the benefits parameter does so by rather counterintuitive means. Instead of adjusting the incentives such that the good is produced under $C2.3$, or at least as close to $C2.3$ as is feasibly possible due to the integer nature of investing on the extensive margin, the policymaker can unboundedly increase welfare by increasing the value of the private benefits at the expense of decreasing

the quality, thereby transferring all surpluses to the investors and leading all non-investors to consume the outside option. The remainder of the section is devoted to detailing the incentives leading to this result and providing an alternative welfare maximizing program.

For both the public and club good cases, the policymaker's objective function is given by

$$
\max_{\sigma \geq 0} \left\{ \mu \mathbf{1} \{\text{public}\} (N - M) \max \{Q (g (\mathbf{x}(M, \sigma))), 0\} \right.
$$
$$
\left. + (1 - \mu) \sum_{j=1}^{M} [Q (g (\mathbf{x}(M, \sigma))) + v (x_j(M, \sigma); \sigma)] \right\}, \quad (2.11)
$$

subject to

$$
x_j(M, \sigma) = \arg \max \{Q (g (\mathbf{x})) + \beta(M, N) v (x_j; \sigma)\} \quad \forall j \leq M
$$
$$
M \in \left\{ M^*(N), \tilde{M}(N) \right\},
$$

where $\mu \in [0, 1]$ is the weight placed on the non-investors and investors, respectively. Although σ is continuous and $v(\cdot, \sigma)$ is continuously differentiable with respect to σ, (2.11) is not continuously differentiable in σ. The objective function has discontinuous jumps in σ because $M^*(N)$ and $\tilde{M}(N)$ are discontinuous functions of σ (as they must be integers). It is worth noting that the objective function can be altered to include the club good pricing strategy described in the previous section without qualitatively altering the results.

By inspection of (2.3)-(2.7), $\lceil M^L \rceil$, $\lceil \tilde{M}^L \rceil$, $\lfloor M^U \rfloor$, and $\lfloor \tilde{M}^U \rfloor$ are all decreasing step functions of σ. It follows that as σ increases, each investor responds by investing a greater amount until σ hits a threshold, at which point an investor drops out, leading to a positive jump in each individual's investment. The change in total investment, however, can be non-monotonic due to diminishing, and eventual decreasing, marginal returns of investments.

Recall from (2.4) and (2.7) that there always exists the incentive for at least one investor.

(a) Full figure. (b) Zoomed region.
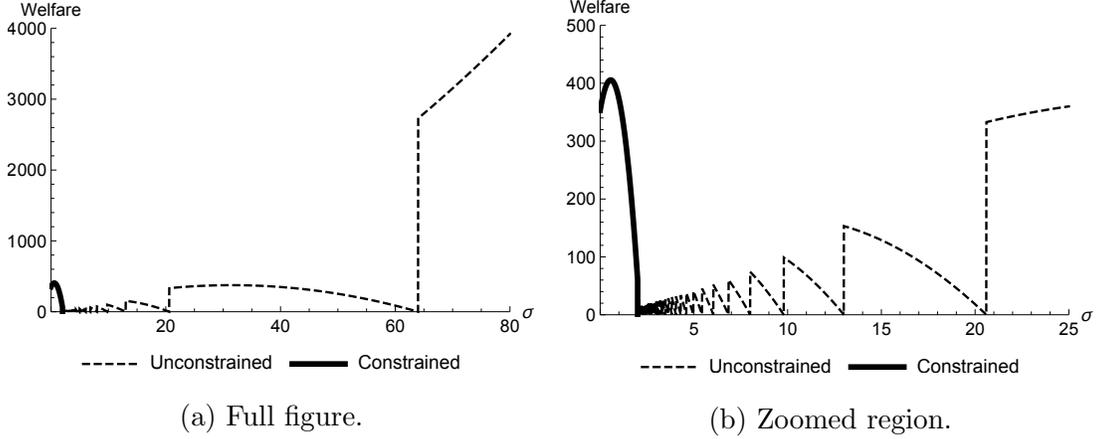
Figure 2.5: Welfare maximization, Example 2.4.

Therefore for at least one investor,

$$\frac{\partial Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(M,\sigma)\right)}{\partial x} + \alpha(M,N)\frac{\partial v\left(x(M,\sigma);\sigma\right)}{\partial x} = 0$$

with $x(M,\sigma) > 0$ and $M \in \left\{M^*(N), \tilde{M}(N)\right\}$. As u_i is strictly concave in x and $v(x;\sigma)$ is strictly increasing in x, indirect utility is also increasing in σ for σ large. Because the non-investors' utilities are bounded below by zero, welfare can be made arbitrarily large by increasing σ.

There are two options available to the policymaker that lead to a solution that does not transfer all surpluses to the investors: ban the outside option or directly consider quality in the optimization program. The second solution is both more efficient and practical than the first, so I focus attention on this option. A natural consideration is to ensure that the quality is no worse than the outside option, which is accomplished by adding the condition that $Q\left(g\left(\mathbf{x}\left(M,\sigma\right)\right)\right) \geq \underline{Q}$ for some $\underline{Q} \in \mathbb{R} \cup \{-\infty\}$ to (2.11). The most natural value is $\underline{Q} = 0$, though the optimization can be conducted with any feasible $\underline{Q}$. I conclude the section with the formal statement of the result and an illustrative example.

Theorem 2.4. *For every $\mu \in [0,1)$ and $\underline{Q} = -\infty$, a policymaker can make total welfare arbitrarily large by increasing σ. For $\mu = 1$ and $\underline{Q} = -\infty$, the optimal policy is arbitrary*

85

in the club good case and is given by $\hat{\sigma} = \overline{\sigma}\left(M^*(N)\right)$ if investors are individually rational and $\hat{\sigma} = \overline{\sigma}\left(\tilde{M}(N)\right)$ if investors are team players in the public good case. For every finite $\underline{Q} \leq Q\left(\hat{X}\right)$ and $\mu \in [0,1]$, there exists a finite $\hat{\sigma} \geq 0$ that maximizes total welfare.

There are two effects at work: a direct effect and an indirect effect. Increasing σ increases the benefits of all investors. However, by Corollary 2.1, increasing the benefits decreases quality through increasing the aggregate investments. Thus non-investors are impacted negatively by such a policy. Because non-investors and free-riders will eventually choose the outside option, a large enough increase in the private benefits will allow the net effect to grow arbitrarily large and positive. The following example illustrates this result.

Example 2.4. *Consider the quadratic setup from the previous examples and suppose that the good is a public good. Panel (A) of Figure 2.5 plots the welfare objective function for $\mu = \frac{1}{2}$ and panel (B) zooms in towards the origin to provide a more detailed view. The constrained line corresponds to $\underline{Q} = 0$ and the unconstrained line corresponds to $\underline{Q} = -\infty$. Each jump corresponds to an investor switching to not-invest. The final jump, which occurs at $\sigma \approx 64$, corresponds to the point at which the number of investors settles. In the quadratic case, there is always a minimum of five investors.*

2.6   Open Source Software and the Poison Pill

The benefits channel described in the previous section has interesting and important implications with respect to competition between open source and proprietary software. In particular, a proprietary software vendor may support open source software developers not because of network effects or knowledge spillovers, but rather to deteriorate the quality of open source software. As open source software is available at no cost (with respect to money), proprietary software vendors must compete on other dimensions such as quality.

If an open source option has a lower quality than its proprietary counterpart, then assuming a homogeneous set of consumers with unity demand, the proprietary software vendor can charge a price up to the difference in qualities and sell to those individuals who do not invest in the development of open source software. Alternatively, if the open source software is of equal or higher quality than the proprietary software, then the proprietary software vendor must do something in order to charge a positive price and face positive demand. The vendor has two options: improve the quality of its own product or find a way to reduce the quality of the open source software. Section 2.5.2 provides such a mechanism for the second option. It may be more cost effective for the firm to decrease the quality of open source software by inducing supply-side congestion as opposed to investing in the quality of its own product. Diminishing marginal returns to quality near the peak (Figure 2.1(A)) imply that investments in quality will have smaller effects while inducing the open source software developers to over-invest will have increasingly large effects. Thus by injecting capital into the open source software and decreasing the quality of the public good, the proprietary software vendors can benefit.

The effects are three-fold. The open source software developers are better off, as increasing σ increases their utility from investing. The consumers can be no better off. At best the consumers (free-riders) are indifferent, which occurs only if they received zero utility pre and post investment. Otherwise, the consumers' utility strictly decreases through the deterioration of the open source software quality or the transition from consuming the open source software with a positive surplus to consuming the proprietary software with zero economic surplus. Naturally, the proprietary software developer is better off, but the economy is left with the average quality of software decreasing.

2.7 Discussion and Concluding Remarks

This paper has outlined several novel and empirically relevant results. When extending the standard theories of public and club goods to non-monotonic public and club goods, the standard equilibrium and comparative static results may no longer apply. While inefficiencies due to free-riding may still persist, over-provision due to supply-side congestion also presents a concern, but is not dealt with in the same manner as free-riding. Over-provision is made more severe by large populations of individually rational individuals and made less severe by larger populations of individuals that are team players. Moreover, policy prescriptions typically used to internalize the externalities driving these inefficiencies no longer have the same desirable properties. Without further consideration, maximizing welfare by controlling the number of investors can lead to a slight deterioration of quality to benefit the investors, while maximizing welfare by controlling the private benefits leads to a severe deterioration in the quality to benefit the investors unless the policymaker also implements a minimum-quality requirement.

While a few assumptions were included in the model for tractability, it is worth noting that heterogeneity can be introduced into the model without qualitatively changing results. If Assumption 2.1(iii) were to be augmented such that the partials need not be symmetric, then all of the results carry over with the exception that the equilibrium is no longer symmetric. For example, suppose that heterogeneity is introduced such that the ratio $\frac{\partial g(\mathbf{x}^k)}{\partial x_i^k}\Big/\frac{\partial g(\mathbf{x}^k)}{\partial x_j^k} = \frac{\eta_i}{\eta_j}$ for some positive constants η_i and η_j. It is straightforward to see that the only change is in the first-order necessary conditions (for both the equilibrium and welfare-maximizing program), where the marginal effect of a change in contributions on the quality function is scaled. The resulting values are scaled accordingly, where individuals with a greater η invest more, but their relationships remain qualitatively unchanged (i.e., the welfare-maximizing investment is closer to the quality-maximizing investment than the equilibrium investment

for each i). Notation would be more cumbersome but no new insights would be gained.

An interesting extension to consider is the dynamic aspects of public and club good provision. Regulations, statutes, and other legal guidelines such as tax codes tend to be developed and evolve over time with investors (regulators) who have their own private and public incentives. The quality-enhancing properties of early investments can induce a feedback loop that increases the investments made in these regulations over time. The early investments may lead to improvements in the quality of these regulations, but over time, these improvements diminish and eventually have deleterious effects, decaying the quality of these regulations. Such a dynamic analysis focusing on the efficiency of provision over time presents an interesting and important avenue for future research.

Chapter 3

Competition Between Open Source and Proprietary Software: Strategies for Survival

3.1 Introduction

The software industry is typically characterized by two distinct development methodologies: proprietary software (PS) and open source software (OSS).[1] By its nature, OSS is "free" in that the source code (human readable code) is made publicly available at no cost, so any individual can conceivably download the source code, compile it into binary code (machine readable code), and run it on her computer. To the contrary, the only component made available by the PS developer is the binary code. In this paper, I address a fundamental question concerning competition: which development methodology will survive in the long run?

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[1]More recently, a third methodology - mixed source - has been identified as a hybrid between open source and proprietary (closed source). See Casadesus-Masanell and Llanes (2011) for details.

Popular media suggests that open source software is the future (Pirillo, 2008; Noyes, 2013). I argue that it is irrelevant which development methodology represents the future. The developers of PS and the developers of OSS need not compete over the same market. There are, of course, overlaps between the target markets; however, the underlying nature of the OSS development process leads to self-selection, where the PS developer differentiates its product to focus on the market not targeted by the OSS. Therefore, one should expect both development methodologies to survive in the long run, as long as the underlying markets that the two methodologies target continue to exist.

Consider the market for scientific typesetting software, specifically two products: the proprietary Scientific Word (SW) and the open source GNU Emacs (Emacs). SW is a "what you see is what you get (WYSIWYG)" editor, akin to Microsoft Word, whereas Emacs is a "what you see is what you mean (WYSIWYM)" editor, which more closely resembles computer programming to document typesetting. SW is designed for those who routinely typeset documents heavy in mathematical notation and collaborate with others who do the same, such as LaTeX users. Emacs is fully capable of replicating every feature of SW, but can also be used in other applications, including computer programming and editing system files. Furthermore, Emacs is known for its immense customizability and steep learning curve.

Thus two more appropriate questions to ask are: (1) who does each type of developer target, and (2) are there externalities to competition between the two processes? In this paper, I address the first question. There is a large literature involving network effects and positive externalities in open research and development (R&D) (R&D without, or with minimal intellectual property restrictions) addressing the second question, including work directly on software competition, e.g., Mustonen (2005), Economides and Katsamakas (2006a,b), Sen (2007), Kumar et al. (2011), Cheng et al. (2011), Llanes and de Elejalde (2013), and Athey and Ellison (2014). For a review of the theory of network industries, see Shy (2001) and Shy (2011). A review of the empirical literature is provided by Birke (2009).

Before researchers began analyzing OSS, many of the features of software competition were captured by the standard models of quality competition and durable goods markets (Spence, 1975; Bulow, 1982, 1986; Waldman, 1996). Once OSS came to fruition, researchers began developing new models of competition. Two primary classes of models emerged from this research program. The first class analyzes externalities and spillover effects and the second class determines long-run shares and survival. The second class has received considerably less attention than the first. See von Krogh and von Hippel (2006) for a review of these avenues of research in OSS. A brief survey of the economics of OSS is provided by Fershtman and Gandal (2011).

Mitchell and Skrzypacz (2006) and Chen et al. (2009) develop general results related to long-run market shares in industries with network effects and show that there exists conditions under which multiple firms survive. Several analyses addressed the long-run market shares in the software industry when there is competition between PS and OSS (Bitzer, 2004; Casadesus-Masanell and Ghemawat, 2006; Raju, 2007; Sen, 2007; Jaisingh et al., 2008-9; Cheng et al., 2011; Christiaans, 2013; Llanes and de Elejalde, 2013). Each paper arrives at a similar conclusion: it depends on either initial conditions or exogenous parameters. That is, there are scenarios in which OSS overtakes the market, those in which PS is the lone survivor, and those where both coexist.

Sen (2007) models software usability when there is competition between PS, OSS, and a commercial OSS developer, where consumers vary according to their preference for usability, but have an identical inherent value of the software, regardless of the developer. The PS (OSS) is assumed to be the most (least) usable software. Only the commercial OSS developer is strategic in its usability decision, and must locate somewhere in between the PS and the OSS. The author finds that, if network effects are weak, then the commercial OSS developer has the incentive to make its software less usable than the PS. If network effects are strong, then the commercial OSS developer develops software as usable as the PS, which drives the

PS developer out of the market. However, this equilibrium emerges due to two factors: the consumers value all software identically, and the usability of the PS is exogenously set to the minimal value. If these assumptions are dropped, and both the PS and commercial OSS producers endogenously select their usability, then multiple equilibria emerge and the conclusions no longer apply.

Jaisingh et al. (2008-9) provide an elegant analysis of competition between OSS and PS, focusing on the vertical quality and pricing decisions of the PS developer. The authors find that quality is inversely related to competition from OSS; that is, the presence of OSS decreases the quality of PS, regardless of the initial quality of the OSS. The effect is intensified as the quality of the OSS is increased. On the other hand, when facing competition from other PS developers, quality can go in any direction depending on how closely substitutable the programs are. Consumers are assumed similar in their valuations of quality. That is, an increase in quality necessarily leads to an increase in demand. It is reasonable to assume that quality has a horizontal component, where different consumers have different needs to be satisfied, and quality is determined by how well those needs are met. Under a horizontal valuation such as the one described above, the same comparative statics can break down. For instance, an increase in the quality of the OSS can in fact lead to an increase in the quality of the PS, depending on the distribution of consumers.

In this paper, I treat the degree of product differentiation as a strategic choice and analyze consumer choice through this endogenous process, developing a model in which there is a single firm developing PS and a potential consumer base. The firm sequentially chooses how sophisticated the software should be via an investment in technology, and then sets the price in order to maximize profits. Competition arises through the development of OSS, which is developed by a subset of the potential consumer base. The community producing the OSS chooses the level of sophistication of the OSS through a similar investment in technology at the same time as the firm. Rather than maximizing profits, the community

seeks to cooperatively find an efficient level of sophistication of the OSS.[2] Each individual has a preferred level of technological sophistication. Individuals then choose which option to purchase and face two costs: the price of the software (which is zero for the OSS) and a learning cost, which is determined by the consumers' preferred level of technology and the actual employed level of technology.

Competition is modeled as a mixed duopoly, though this model differs from the standard mixed duopoly model.[3]. Typically, mixed duopolies are modeled with a private, profit-maximizing firm competing with a government-owned entity, concerned with maximizing social welfare. More recently, researchers have developed models of partial-privatization, where a fraction of the firm is owned by individual shareholders concerned with maximizing profits while the remaining shares are held by a welfare-maximizing government (Anderson et al., 1997; Matsumura, 1998). In this paper, the OSS developers are interesting in maximizing their own welfare. It is similar to a model of privatization except that there are distinct asymmetries between the objective function of the OSS developers and the objective function of the firm.

I find that, when there is competition between a firm developing PS and a community developing OSS, the firm will cater the software towards individuals with lower technological ability, a phenomenon I call *catering to ignorance.* Furthermore, this unique equilibrium emerges endogenously. It is worth noting that this notion is not new. Lerner and Tirole (2002) reference a quote from an open source developer, who states:

> [I]n every release cycle Microsoft always listens to its *most ignorant customers.* This is the key to dumbing down each release cycle of software for further assaulting the non personal-computing population. Linux and OS/2 developers,

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[2]Equivalently, a sequential non-cooperative bargaining process in which individuals seek to maximize their own utility generates an equivalent outcome if the individuals are sufficiently patient.

[3]See De Fraja and Delbono (1990) and citations therein for a survey of the early models of mixed oligopoly.

on the other hand, tend to listen to their *smartest* customers...The good that Microsoft does in bringing computers to non-users is outdone by the curse that they bring on experienced users (Nadeau, 1999).

Lerner and Tirole (2002) illustrate the importance of the argument made in the above quote to license selection. To my knowledge, however, the present paper is the first to provide an analytical microeconomic foundation leading to this endogenous differentiation under consumer heterogeneity. Johnson (2002) alludes to this idea in a brief extension by considering an environment with modular (piecewise) development, in which as the number of modules grows, the probability of OSS developing all modules converges to zero while the probability converges to a positive value for the PS.

The OSS applies downward pressure, that is, the firm's choice of software sophistication decreases relative to a monopolist. In reference to the above quote, it can be said that Microsoft listens to its least technological savvy consumer base because Linux developers listen to the most technologically savvy individuals, namely themselves.

The endogenous equilibrium emerges from either one of two possible mechanisms: heterogeneous costs of technology or the intrinsic motivations of OSS contributors. Much of the work on software competition has ignored the fact that technology is not easy to use, or equivalently, it is costly (an exception being Dey et al. (2013)). Therefore, the real price of using software consists of two components, the purchase price and a learning cost. The learning cost is not the same for all individuals. Some individuals are more technologically savvy and are able to learn to use technology at a lower cost, in terms of time and effort. Thus any two individuals purchasing the same software likely face different prices (in real terms), which even compels a monopolist PS developer to choose a degree of technology below the median ability of the consumer population.

Even without heterogeneous learning costs, this same pattern can emerge. The intrinsic

motivations of contributors to OSS tends to bias the chosen technology level of the software above the median, which limits the market available to the firm, since it needs to compete with a zero-price competitor to capture high-skill individuals. In response, the firm best responds by targeting the group most ignored by the OSS, the technologically ignorant individuals.

3.2   The Open Source Software Paradigm

3.2.1   Open Source Software: Definition and Background

The Open Source Initiative (OSI), a non-profit 501(c)3, was founded in 1998 to formalize the open source definition (OSD). OSI is the community-recognized body for reviewing and approving licenses as conforming to OSD.[4] Those licenses furthest from the standard copyright licenses are typically referred to as copyleft. Among the most copyleft licenses is the GNU General Purpose License (GPL) and among the most copyright, but still considered conforming to the OSD is the Berkeley Software Distribution 2-Clause (BSD) (Open Source Initiative, n.d.b,n).

The implication is simple: free software is not equivalent to OSS. For a consumer, there is still a cost of consuming OSS. In its entirety, the source code principle in the OSD states:

> The program must include source code, and must allow distribution in source code as well as compiled form. Where some form of a product is not distributed with source code, there must be a well-publicized means of obtaining the source code for no more than a reasonable reproduction cost preferably, downloading via the Internet without charge. The source code must be the preferred form in

---

[4]http://opensource.org/about, accessed January 27, 2015.

which a programmer would modify the program. Deliberately obfuscated source code is not allowed. Intermediate forms such as the output of a preprocessor or translator are not allowed (Open Source Initiative, n.d.*c*).

At a minimum, the source code must be made available; however, computers cannot directly read source code. It must be compiled into binary, which computers are then able to interpret. This implies that even in instances where the source code is made freely available, the software is not free in the economic sense. It is costly in terms of time and knowledge. The same holds true in PS, where users must invest time and energy in learning to use the software. To capture this feature, I treat all software as costly, where the cost is characterized by the sophistication of the software's technology. The cost of learning to use the software depends on how technologically skilled the user is and how technologically sophisticated the software is. For the purposes of this paper, I assume that consumers are able to freely obtain pre-compiled binaries of OSS. Thus individuals face both a price and learning cost when purchasing the PS and only a learning cost when using the OSS. The environment can easily be extended to one in which consumers must expend extra effort to compile the source code; however, this would only make the product differentiation more extreme, as the PS becomes more valuable to the marginal consumer, relative to the OSS.

3.2.2 Open Source Software: Development

This section's focus is on the primary characteristics relevant to this paper, namely who contributes, how development decisions are made, and why individuals contribute (Ajila and Wu, 2007; Bitzer et al., 2007; Aksulu and Wade, 2010).

In general, there are three kinds of contributors: institutional contributors, individuals representing institutions, and hobbyists. Institutional contributors can further be subdivided into two types: professional open source entities and commercial supporters. Note that these

97

types need not be mutually exclusive. For example, Red Hat Inc. provides OSS and, for a fee, related services to enterprises. In this respect, it is acting as a professional OSS entity. However, Red Hat also sponsors the Fedora Project, where it is acting as a commercial supporter.[5] While Fedora is primarily a community-driven endeavor, Red Hat has veto power over all Fedora development decisions.

Institutional contributors typically support large-scale OSS projects. A notable example is Mozilla's OSS (such as Thunderbird and Firefox). The Mozilla Suite is made up of all types of contributors. There are three subsidiaries to consider: the Mozilla community, the Mozilla Corporation, and the Mozilla Foundation. The Mozilla community consists of over 10,000 contributors.[6] The Mozilla Corporation is a taxable entity that serves its non-profit, public parent (the Mozilla Foundation). Its leadership, known as the "Steering Committee" is responsible for guiding the direction of the projects. In short, the Mozilla Corporation works to fund the Mozilla Foundation, which guides the direction of the projects developed by the Mozilla community.

The preceding two examples illustrate the underlying nature of how development decisions are made. Typically collaboration occurs online through email lists and message boards through hosting servers, such as Github or Sourceforge. The project's leadership facilitates communication back and forth between various contributors and a establishes a plan of action according to the input of contributors. Rather than explicitly modeling this bargaining process, I employ the results of the median voter theorem, as preferences are assumed to be single peaked.

Lerner and Tirole (2001) identify several theoretical reasons why individuals contribute to OSS projects without remuneration. The reasons can be classified into two categories: immediate rewards and delayed benefits. Immediate rewards include productivity gains, enjoyment

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[5]The Fedora Project coordinates the development of Fedora, a Linux-based operating system.
[6]https://www.mozilla.org/en-US/contribute/, accessed January 28, 2015.

from working on the project, and peer recognition, while delayed rewards include various signaling effects and future employment opportunities. These effects are expounded upon in various works, such as Lerner and Tirole (2002), Lerner et al. (2006), Maurer and Scotchmer (2006), Bitzer et al. (2007), Fershtman and Gandal (2007), Saurer (2007), and Fang and Neufeld (2009). Based on a survey conducted by Hars and Ou (2001), some programmers claimed that signaling was a motivating factor in their decision to contribute. According to Holmström (1999), individuals expend more effort signaling when performance is more visible to the firms, performance has a greater impact, and performance is highly informative about skill. Myatt and Wallace (2008) develop an evolutionary model explaining voluntary contributions to OSS. To capture these features, I model an increase in the technological sophistication of both the programmer and the software as the benefit of contributing to OSS.

3.3   The Model

There is a single firm developing PS utilizing technology level $t \in [0, 1]$ and charging price $p \geq 0$. The firm is not alone in the market. It faces competition from OSS, which uses technology level $\tau \in [0, 1]$ and is sold at a price of zero. I refer to an arbitrary technological investment by ζ. The firm does not face a development cost; however, the developers of OSS do face one, to be specified shortly. This assumption on the cost function of the firm is fairly innocuous, and I will show that the results are qualitatively unchanged for both a cost function increasing in t and a U-shaped cost function.

There is a continuum of consumers normalized to unit mass. Each consumer is identified by her technological prowess $x \in [0, 1]$. Technological prowess is assumed to be distributed amongst the consumers according to the distribution $G(\cdot)$ with associated log-concave pdf $g(\cdot)$. Individuals with $x \leq G^{-1}\left(\frac{1}{2}\right)$ are said to be *technologically ignorant*, while individuals

with values $x > G^{-1}\left(\frac{1}{2}\right)$ are considered to be *technologically elite*. Every consumer faces unit demand for software with single-peaked valuation $V = v(x, \zeta)$, and chooses either the PS, the OSS, or an outside option normalized to zero value. V is twice differentiable in each argument, symmetric in ζ and x, strictly concave in x and ζ, and obtains a maximum at $v(x, x) = \overline{V}$ at any x. Each individual has an ideal technology level suited for her own technological prowess, and obtains equal loss when the software's location varies from this value, regardless of the side of x that ζ falls on.[7] The symmetry assumption does not imply that utility itself is symmetric. In fact, utility is asymmetric due to the increasing learning cost, to be described.

Consumers also face a cost of learning to use the software, which depends on their own technological prowess x, and the level of technology employed by their chosen software ζ (as illustrated in Dey et al. (2013)). Define this cost as $\ell(x, \zeta)$, which is assumed twice differentiable in each argument, decreasing and weakly convex in x and both increasing and strictly convex in ζ. At $x = 1$, $\ell(1, \zeta) = 0$ for all ζ and at $\zeta = 0$, $\ell(x, 0) = 0$ for all x.

That is, the most technologically elite individual can learn even the most sophisticated software at no cost, while any individual can learn to use the least sophisticated software at no cost. Figure 3.1 graphically illustrates this relationship: panel (a) with respect to the technological investment and panel (b) with respect to individual prowess. Any individual can use any level of software at a cost. For any given level ζ, the cost of learning to use the software is decreasing in x. Thus a technologically elite individual can learn to use a technologically advanced software at a lower cost than a technologically ignorant individual, and can also learn to use a low technology software at a lower cost than a technologically ignorant individual.

OSS is developed by a subset of the individuals in the economy. Contributors receive two

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[7]Symmetry allows a wide variety of functions, such as the power loss function $V(x, \zeta) = \overline{V} - \theta(x - \zeta)^\alpha$, $0 < \theta \leq \overline{V}$, $\alpha = 2k, \forall k \in \mathbb{N}$.
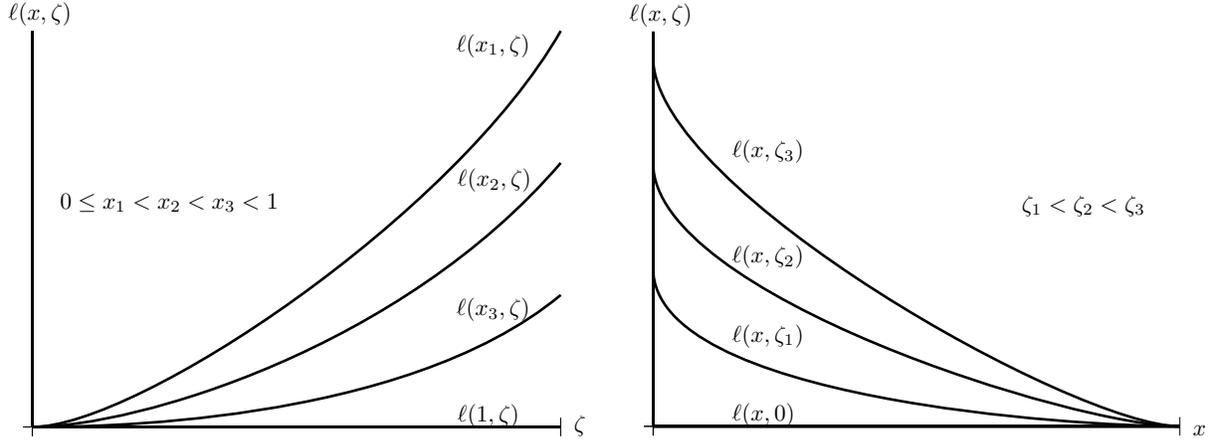
(a) Learning cost at $\{x_1, x_2, x_3, 1\}$, $\forall \zeta \in [0, 1]$.   (b) Learning cost at $\{0, \zeta_1, \zeta_2, \zeta_3\}$, $\forall x \in [0, 1]$.

Figure 3.1: Learning cost.

types of benefits: a benefit from using the software $V = v(x, \tau)$, and a benefit from contribut-ing. Define the contribution payoff as $b(x, \tau)$, which is assumed to be twice differentiable in each argument, nondecreasing and weakly concave in x, and increasing and weakly concave in τ. The cost of contributing is defined by the function $K(x, \tau)$, where $\frac{\partial}{\partial x} K(x, \tau) < 0$, $\frac{\partial^2}{\partial x^2} K(x, \tau) > 0$, and $\frac{\partial}{\partial \tau} K(x, \tau) > 0$. Whereas variables such as time/effort do not enter the cost/benefit function, under the rational choice framework employed in this paper, any individual with ability x contributing to software τ would choose an identical effort level because the values that equate marginal benefit and marginal cost would be identical. Thus the model can be seen as isomorphic to one in which time/effort is explicitly modeled, with the output/notation suppressed. Furthermore, I assume that

$$b(1, \tau) > K(1, \tau), \quad \forall \tau. \tag{3.1}$$

This condition guarantees that a nonzero measure of individuals are willing to contribute to developing the OSS.

Utility is assumed quasilinear, which implies that the utility function of an individual x can

be written as

$$u(x,t,\tau,p) = \begin{cases} v(x,t) - \ell(x,t) - p & \text{if consuming PS} \\ v(x,t) + b(x,\tau) - \ell(x,t) - K(x,\tau) & \text{if consuming PS \& producing OSS} \\ v(x,\tau) + b(x,\tau) - \ell(x,\tau) - K(x,\tau) & \text{if consuming \& producing OSS} \\ v(x,\tau) - \ell(x,\tau) & \text{if only consuming OSS} \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

It also follows that $u(x,t,\tau,p)$ is strictly concave and single-peaked in x.

3.3.1 Outline of the Game

Interactions occur over the course of three stages. In the first stage, the firm and OSS contributors simultaneously select t and τ, respectively. The firm's objective is to maximize profits and the OSS developers' objective is to maximize the total utility of contributors. In the second stage, the firm chooses price $p(t)$ given the technology investment and the OSS price is fixed at zero. In the third stage, individuals make their purchase/usage decision. Since information is complete, I will be looking for a subgame-perfect Nash equilibrium.

3.4 Equilibrium Analysis

The equilibrium analysis consists of two parts. I first analyze a monopolist firm developing PS, choosing both location and price, to establish a benchmark to characterize the preferred firm location. I then add an OSS and compare the firm's decision under each scenario. Throughout the analysis, I will be using three terms: the lower neighborhood about x, defined as $N_\eta^-(x) \equiv \{y : 0 \le x - y < \eta\}$, the upper neighborhood about x, defined as $N_\eta^+(x) \equiv \{y : 0 \le y - x < \eta\}$, and the neighborhood about x, defined as $N_\eta(x) \equiv N_\eta^-(x) \cup N_\eta^+(x)$.

3.4.1 Monopoly Benchmark with only Proprietary Software

Consider the third stage of the game. For a given equilibrium price and location $(p(t^m), t^m)$, consumers will purchase the PS if

$$v(x, t^m) - p(t^m) - \ell(x, t^m) \geq 0. \tag{3.3}$$

Taking (3.3) into account, the firm chooses the price p. The following lemma characterizes the demand function faced by the firm.

Lemma 3.1. *For any given technology investment t^m, the support of the demand function of the firm is a non-degenerate convex subset of $[0, 1]$.*

All proofs can be found in Appendix C. Lemma 3.1 implies that there are at most two critical individuals.[8] Order these individuals by their technological prowess x and define the lower-critical individual by $\underline{x}(p, t^m)$ and the upper-critical individual by $\overline{x}(p, t^m)$. The demand function takes the form

$$D(p, t^m) = \int_{\underline{x}(p, t^m)}^{\overline{x}(p, t^m)} g(x) dx, \tag{3.4}$$

where $0 \leq \underline{x}(p, t^m) \leq \overline{x}(p, t^m) \leq 1$. The firm can either target the entire market, an interior subset $[\underline{x}(p, t^m), \overline{x}(p, t^m)]$, or a subset including the boundary; either $[0, \underline{x}(p, t^m)]$ or $[\overline{x}(p, t^m), 1]$, with $\underline{x}(p, t^m) > 0$ and $\overline{x}(p, t^m) < 1$.

The next step in describing the demand function is to define properties of $\underline{x}(p, t^m)$ and $\overline{x}(p, t^m)$. Lemma 3.2 does this and provides a relevant implication for the demand function.

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[8]When interior, these individuals are indifferent between purchasing the proprietary software and choosing the outside option.

Lemma 3.2. $\frac{\partial \underline{x}(p,t^m)}{\partial p} \geq 0$ and $\frac{\overline{x}(p,t^m)}{\partial p} \leq 0$. If $0 < D(p,t^m) < 1$, then at least one of the inequalities must hold strictly, which implies that $\frac{\partial D(p,t^m)}{\partial p} \leq 0$.

Together, Lemmas 3.1 and 3.2 show that demand is downward sloping. Furthermore, there is a region in which the demand is inelastic and a region in which demand is elastic. In the second stage, the firm's objective is to choose the price that maximizes profits for every possible technology investment t.

Proposition 3.1. In equilibrium, the pair $(p(t^m), t^m)$ must satisfy $\varepsilon(p(t^m)) = -1$, where $\varepsilon(p(t^m))$ is the demand elasticity at price $p(t^m)$.

This is the standard profit maximizing result. A firm with zero marginal cost of production sets the price such that demand elasticity equals unity.[9]

Peeling the game tree back to the first stage, the firm optimally selects $t = t^m$, understanding that it will then choose price $p(t^m)$. That is, price is treated as a function of location. This is in contrast to Sen (2007), where the location t^m is assumed to equal zero nonstrategically.

Proposition 3.2. The equilibrium technological investment t^m is interior and satisfies $\mu(t^m) = 0$, where $\mu(t^m)$ is the elasticity of technology, defined by $t^m \frac{\frac{\partial D(p(t^m),t^m)}{\partial t}}{D(p(t^m),t^m)}$.

The choice of t^m affects demand through two channels: a demand (direct) effect through t^m and a strategic (indirect) effect through $p(t^m)$. Given the sequential nature of the problem, the firm need only consider the demand effect since the price will self adjust in the second

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[9]If there was a positive marginal cost of production, then Proposition 3.1 would require $\varepsilon(p(t^m)) < -1$, as is standard.

stage. Thus the objective of the firm collapses to choosing the t^m which, conditional on $p(t^m)$, maximizes the firm's market share.

The intuition that t^m is interior is straightforward. Suppose that the firm chooses location $t^m = 0$ and price $p(0)$. At this location and price, the individual with the highest valuation (and thus willingness to pay) is the one located at $x = 0$. Now consider the upper neighborhood about $x = 0$. These individuals must also have a strictly positive valuation. It follows that the monopolist could increase the technological level to $t^m = 0 + \xi$ for some small, positive ξ and increase the size of its market while still charging a price of $p(0)$, thereby increasing profits since $\underline{x}(p(0), 0) = \underline{x}(p(0), \xi)$ and $\overline{x}(p(0), 0) < \overline{x}(p(0), \xi)$ for small enough ξ.[10] A similar argument holds at $t^m = 1$ and $p(1)$.

It immediately follows that the individuals located at $x = 0$ and $x = 1$ should never obtain positive surplus. If the individual at $x = 1$ has a positive surplus at technology level t, then it must be that $x = 1$ has positive surplus at $t - \xi$ for small enough ξ. Thus, not only are $t = 0$ and $t = 1$ ruled out, but also all values of t arbitrarily close to zero and one. Furthermore, since $g(\cdot)$ is log-concave, there is higher mass placed at values of x near the median than the extremes.[11] Thus $G^{-1}\left(\frac{1}{2}\right) \in [\underline{x}(p(t^m), t^m), \overline{x}(p(t^m), t^m)]$ must be true. By continuing the analysis, the following result emerges.

Proposition 3.3. *In equilibrium,* $t^m < G^{-1}\left(\frac{1}{2}\right)$.

Compared to the typical Hotelling model, a bias is introduced through the learning cost, which pushes t^m to the left of the median, towards technological ignorance. Therefore the open source developer quoted by Lerner and Tirole (2002) was correct in saying that the proprietary firm "targets" the technologically ignorant consumers.

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[10]This line of reasoning assumes that the market share is currently a strict subset of $[0, 1]$. A similar argument can be made when the firm's market share covers the entire consumer base.

[11]The exception is the uniform distribution, which satisfies log-concavity and places equal mass across all x in the support.
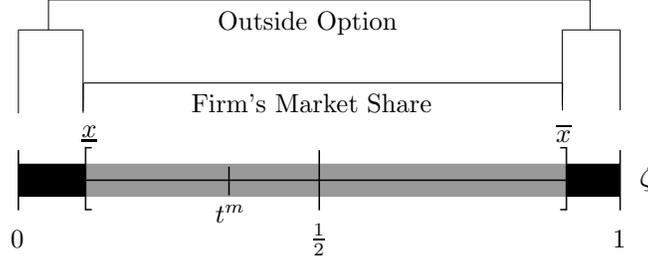
Figure 3.2: Summary of Propositions 3.1-3.3.

Thus the subgame-perfect equilibrium outcome can be characterized as follows. The firm chooses location $t^m < G^{-1}\left(\frac{1}{2}\right)$ and price $p(t^m)$ such that $p(t^m)$ and t^m satisfy the conditions in Propositions 3.1 and 3.2. All consumers $x \in [\underline{x}(p(t^m), t^m), \overline{x}(p(t^m), t^m)]$ purchase the PS. The remaining individuals choose the outside option. Figure 3.2 provides a unidimensional illustration of the equilibrium.

3.4.2 Mixed Duopoly

The analysis again proceeds through a standard application of backward induction. Taking equilibrium locations t^d, τ^d and price $p\left(t^d, \tau^d\right)$ as given, there are six possible combinations: an individual can choose to purchase the PS, with or without contributing to the OSS, use the OSS without contributing, both contribute to and use the OSS, only contribute to the OSS and use the outside option, or just choose the outside option.

In equilibrium, an individual with technological prowess x prefers to purchase the proprietary software if

$$v\left(x, t^d\right) - p\left(t^d, \tau^d\right) - \ell\left(x, t^d\right) \geq \max\left\{v\left(x, \tau^d\right) - \ell\left(x, \tau^d\right), 0\right\}, \tag{3.5}$$

which, as in the monopolist case, is linear in price. Note that $b\left(x, \tau^d\right)$ and $K\left(x, \tau^d\right)$ are not included. This omission is because the contribution decision is made in a previous stage, and therefore both the benefit and cost are sunk. Consider the critical individuals, characterized

106

by technological prowesses $\underline{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right) \geq 0$ and $\overline{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right) \leq 1$. Lemmas 3.1 and 3.2 apply in this case as well, so $\underline{x}\left(p, t, \tau\right)$ and $\overline{x}\left(p, t, \tau\right)$ have the same properties as in the monopoly case.

Given these critical consumers, the demand function for the firm, given equilibrium technology levels t^d and τ^d, is:

$$D\left(p, t^d, \tau^d\right) = \int_{\underline{x}\left(p, t^d, \tau^d\right)}^{\overline{x}\left(p, t^d, \tau^d\right)} g(x) dx. \tag{3.6}$$

The associated profit function is $\pi\left(p, t^d, \tau^d\right) = pD\left(p, t^d, \tau^d\right)$. Profits are zero when either price is zero or demand is zero, so in order for the firm to obtain positive profits, it must be that $p\left(t^d, \tau^d\right) > 0$ and $D\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right) > 0$. That is, the proprietary firm is profitable only when it charges a positive price and it faces positive demand. The following proposition begins to characterize the conditions under which this statement is true.

Proposition 3.4. *In any equilibrium where $p\left(t^d, \tau^d\right) > 0$, the firm faces positive demand only if $t^d \neq \tau^d$.*

In other words, product differentiation endogenously arises through competition. Suppose a rational consumer is given a choice between two identical options, where one is offered at a positive price, while the other is offered at zero price. The consumer always chooses the zero price option under this circumstance. The firm could choose to set $t^d = \tau^d$ and $p = 0$ in an effort to split the market with the OSS developers, but it always has a more profitable option via product differentiation.

Proposition 3.4 shows that all consumers who do not choose the outside option choose the OSS when there is no product differentiation, so the firm has the incentive to distinguish its PS from the OSS. The next logical question is when there is product differentiation, how is

the market segmented? I proceed by jointly analyzing the two classes of subgames induced by the technology choices of the firm and the OSS developers, $t^d < \tau^d$ and $t^d > \tau^d$.

First suppose that $t^d < \tau^d$. It is obvious that $\bar{x}\left(p, t^d, \tau^d\right) < \tau^d$ since any individual in the neighborhood of $x = \tau^d$ consumes the OSS. Now suppose that $t^d > \tau^d$. In this case, $\underline{x}\left(p, t^d, \tau^d\right) > \tau^d$ since the individuals located in the neighborhood of $x = \tau^d$ consume the OSS. The following proposition characterizes the price under either of the two cases.

Proposition 3.5. *The equilibrium price $p\left(t^d, \tau^d\right)$ must satisfy $\varepsilon\left(p\left(t^d, \tau^d\right)\right) = -1$ for all feasible t^d, τ^d.*

Proposition 3.5 is analogous to Proposition 3.1.

The final two stages of the game can be summarized as follows:

(i) In stage 3, all consumers on the interval $\left[\underline{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right), \bar{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right)\right]$ purchase the PS. If $t^d < \tau^d$, then every individual on the interval $\left[0, \underline{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right)\right)$ chooses the outside option and every individual on the interval $\left(\bar{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right), 1\right]$ chooses the OSS. If $t^d > \tau^d$, then every individual on the interval $\left[0, \underline{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right)\right)$ chooses the OSS and every individual on the interval $\left(\bar{x}\left(p\left(t^d, \tau^d\right), t^d, \tau^d\right), 1\right]$ chooses the outside option.[12]

(ii) In stage 2, the firm sets the price $p\left(t^d, \tau^d\right)$ according to Proposition 3.5.

Focusing on the first stage, the firm and the OSS developers simultaneously choose t^d, τ^d to maximize their respective objective functions. At the extensive margin, a consumer x will contribute to the OSS if:

$$b\left(x, \tau^d\right) \geq K\left(x, \tau^d\right). \tag{3.7}$$

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[12]Define $[0,0) = (1,1] = \emptyset$, i.e., no individual consumes the associated software.

Let $\hat{x}\left(\tau^d\right)$ represent the value of x such that (3.7) holds with equality. Every individual $x \geq \hat{x}\left(\tau^d\right)$ will contribute to the OSS. The remaining individuals do not.

The development of OSS is a cooperative social process, so the collaboration that leads to the choice of location for the OSS can thus be described by a social choice function. Define $X(t) = \{x | x \geq \hat{x}(t)\}$ as the set of all individuals who contribute to the OSS. The social choice function $F : X(t) \to [0, 1]$ prescribes a location τ^d based on the individuals $x \in X(t)$. A reasonable strategy-proof social choice function is the smallest median peak.

While the imposition of this social choice function may seem *ad-hoc*, Cho and Duggan (2009) show that in a noncooperative bargaining problem, the set of proposals that can be passed in any pure strategy subgame-perfect equilibrium collapses to the ideal point of the median individual, conditional on a sufficiently low discount rate. The social choice function can be interpreted as either the level of sophistication which maximizes total contributor welfare, or equivalently as the outcome of an unmodeled, game theoretic bargaining process in which each individual seeks to maximize her own utility. In either case, the OSS developers' choice τ^d satisfies the following two equations:

$$G\left(\tau^d\right) - \frac{1}{2}G\left(\hat{x}\left(\tau^d\right)\right) = \frac{1}{2} \tag{3.8}$$

$$b\left(\hat{x}\left(\tau^d\right), \tau^d\right) = K\left(\hat{x}\left(\tau^d\right), \tau^d\right). \tag{3.9}$$

Equation (3.8) pins down the conditional median, conditional on the indifferent individual $\hat{x}\left(\tau^d\right)$ and equation (3.9) pins down the indifferent individual. It remains to be shown that such a value $\hat{x}\left(\tau^d\right)$ exists.

Lemma 3.3. *For every t, there exists a value $\hat{x}(t)$ such that $b\left(\hat{x}(t), t\right) = K\left(\hat{x}(t), t\right)$.*

The benefit of contributing is increasing in x while the cost of contributing is decreasing in x. Figure 3.3 illustrates the indifference point.
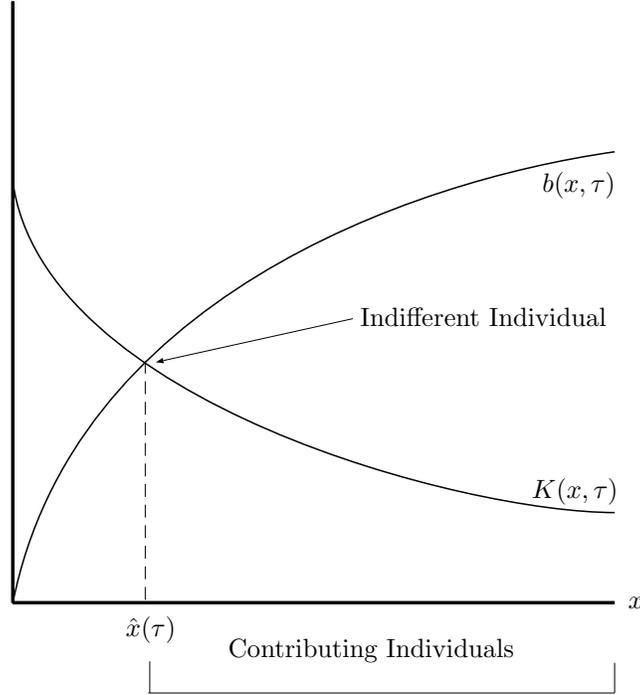
Figure 3.3: The individual indifferent between contributing and not.

Note that $\tau^d \geq G^{-1}\left(\frac{1}{2}\right)$ since the conditional median cannot be lower than the unconditional median given the cost of contributing $K(x,t)$. Choosing $\tau^d \geq G^{-1}\left(\frac{1}{2}\right)$ is what I call *catering to elites*, where the OSS is designed specifically with the technologically elite individuals in mind. Also the OSS developers' optimal choice is independent of the firm's choice t^d. That is, no consideration is given to any individual who is not a participant in the development process, implying that the firm's problem can be reduced to maximizing profits, conditional on τ^d, which simplifies the analysis since expectations do not need to be taken over the distribution of the conditional median. As in the monopolist case, the firm's equilibrium location choice must satisfy

$$\frac{\partial D\left(p\left(t^d, \tau^d\right), t^d\right)}{\partial t} = 0,$$

if interior.

The important question is where does the firm locate, relative to the OSS? It turns out that

it is always more profitable for the firm to locate to the left of the OSS. That is, $t^d < \tau^d$. I call this relationship *catering to ignorance*, where the firm targets the technologically ignorant individuals while the OSS is designed for the technologically elite.

Proposition 3.6. *In equilibrium, the firm locates to the left of the OSS developers:* $t^d < G^{-1}\left(\frac{1}{2}\right) < \tau^d$.

Since $\tau^d \geq G^{-1}\left(\frac{1}{2}\right)$, the market share of the firm at any given price is at most $\frac{1}{2}$ if the firm locates at a point such that $t^d \geq \tau^d$. By moving just slightly to the left of τ^d, the firm weakly increases its market share without needing to change its price. In summary, given that the OSS caters to the elite, the firm is strictly better off catering to ignorance.

The next step is to examine how t^d and t^m relate.

Proposition 3.7. *The equilibrium technological sophistication of the PS in the mixed duopoly is weakly lower than the location of the PS in the monopoly.*

Demand for the proprietary software is constrained from the right under the presence of OSS. Therefore, either one of two events must be true if the PS in the duopoly is located to the right of the PS under the monopoly. The monopolist firm would always have the incentive to replicate the duopoly outcome if the duopoly firm is doing better, or vice versa. The catering to ignorance result is thus twofold. First, the PS is developed with the technologically ignorant individuals in mind while the OSS is developed with the technologically elite individuals in mind. Second, the presence of OSS actually pushes the PS further toward the most technologically ignorant individuals. Figure 3.4 provides a unidimensional representation of the subgame-perfect equilibrium.

The easiest market to notice this location choice is in the market for computer operating systems. Consider Mac OS X. At its current stage, OS X is relatively easy to learn, use, and
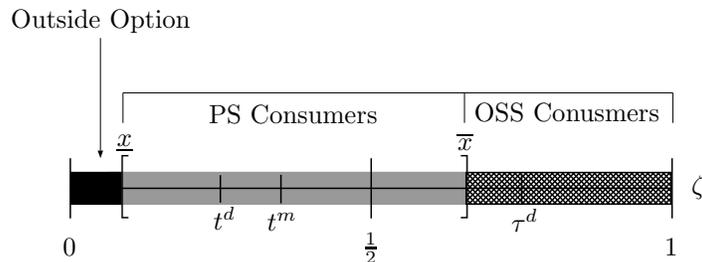
Figure 3.4: Summary of Propositions 3.4-3.7.

maintain. Compare this (proprietary) option to another Unix-based open source operating system, such as Arch Linux. Installing OS X is as simple as pushing an install button. To install Arch Linux, first the user must download and verify the install medium, then change the boot order so that the install medium (typically a USB drive or CD) is read before the HDD (hard disk drive) or SSD (solid state drive). Once the medium is opened, there is no graphical user interface (GUI); all commands are entered in a Unix-like environment. Even when the install is finished, the user must manually install a GUI. The difficulty of use for a technologically ignorant user does not end with the installation. System maintenance and upgrades must also be conducted manually in a Unix-like text environment. If the user is not careful, upgrading can damage the system when certain incompatible packages are combined and dependencies are broken.

Lastly, I analyze how changes in $\hat{x}\left(\tau^d\right)$ impact the firm's choice t^d.

Proposition 3.8. t^d *is weakly increasing in* τ^d. *The inequality is strict if* $t^d < t^m$.

Proposition 3.8 is an implication of the firm's desire to profit as if it were a monopolist. If the OSS locates itself toward more sophisticated individuals, then those individuals to the left of the lower neighborhood about $x = \tau^d$ are more willing to purchase the PS, allowing the duopolist firm to capture a market that more closely resembles its monopolist counterpart. This result is in contrast to Jaisingh et al. (2008-9), who find an inverse relationship between the quality (location) of PS and OSS, as opposed to the positive relationship this paper

finds. The difference can be attributed to the way heterogeneity is modeled. Jaisingh et al. (2008-9) assume purely vertical differentiation, while I allow for both horizontal and vertical differentiation through v and ℓ.

Consider the evolution of Microsoft Word over time, namely with typesetting mathematical equations. Prior to Word 2013, the Equation Editor was required to add mathematical equations to documents. As open source LaTeX options have become easier to use, Microsoft has responded by including a built-in equation menu, where equations can be added with the push of a button.

3.4.3 Analytical Example

To better illustrate the results, consider the following example. Suppose that

$$G \sim U(0,1)$$
$$v(x, \zeta) = 1 - (x - \zeta)^2$$
$$\ell(x, \zeta) = (1 - x)\zeta^2.$$

For expositional convenience, I will suppress the monopoly and duopoly equilibrium superscripts m and d when there is no confusion.

If the firm holds a monopoly, then in the third stage, consumer x will purchase the PS if

$$1 - (x - t)^2 - p - (1 - x)t^2 \geq 0.$$

Since G is uniform, $D(p, t) = \overline{x}(p, t) - \underline{x}(p, t)$. Solving the above inequality for x yields the associated demand function

$$D(p,t) = \min\left\{1, \frac{1}{2}\left(2t + t^2 + \sqrt{4 - 4p - 4t^2 + 4t^3 + t^4}\right)\right\}$$
$$- \max\left\{0, \frac{1}{2}\left(2t + t^2 - \sqrt{4 - 4p - 4t^2 + 4t^3 + t^4}\right)\right\},$$

which is clearly weakly decreasing in price as required. The first line represents $\bar{x}(p,t)$ and the second line represents $\underline{x}(p,t)$.

Fixing $t = t^m$ and using standard optimization techniques, it is straightforward to show that the equilibrium strategy is to choose p such that

$$p(t^m) = 1 - 2\left(t^m\right)^2,$$

which implies that

$$t^m = \frac{1}{1 + \sqrt{2}} < \frac{1}{2}.$$

The median of $G(x)$ is $\frac{1}{2}$, so t^m is strictly below the median as expected. The reason is the learning cost $\ell(x,t)$. Suppose that $\ell(x,t) = 0$, for all x and t. Then a consumer x will purchase if $1 - (x - t)^2 \geq 0$. Now suppose that the monopolist targets the entire consumer base. This implies that

$$1 - (1 - t)^2 - p = 0$$
$$1 - t^2 - p = 0.$$

Therefore $t = \frac{1}{2}$ if $\ell(x,t) = 0$.[13]

Now suppose that the firm faces competition from OSS. Each individual x chooses the PS

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[13]Verifying the other boundary conditions shows that profits are indeed maximized by targeting the entire market with $t = \frac{1}{2}$.

if:

$$1 - (x - t)^2 - p - (1 - x)t^2 \geq \max \left\{ 1 - (x - \tau)^2 - (1 - x)\tau^2, 0 \right\}.$$

For the lower-critical individual, attention can be restricted to $1 - (x-t)^2 - p - (1-x)t^2 \geq 0$, while attention can be restricted to $1 - (x - t)^2 - p - (1 - x)t^2 \geq 1 - (x - \tau)^2 - (1 - x)\tau^2$ for the upper-critical individual. Therefore the firm faces demand

$$D(p, t, \tau) = \begin{cases} \frac{2(\tau - t)(t + \tau) - p}{(\tau - t)(2 + t + \tau)} - \max\left\{ 0, \frac{1}{2} \left(2t + t^2 - \sqrt{4 - 4p - 4t^2 + 4t^3 + t^4} \right) \right\} & \text{if } t < \tau \\ 1 - \frac{p + 2(t - \tau)(t + \tau)}{(t - \tau)(2 + t + \tau)} & \text{if } t > \tau. \end{cases}$$

Fix $t = t^d$ and $\tau = \tau^d$. It follows that a profit maximizing firm sets

$$p(t^d, \tau^d) = \begin{cases} \left(\tau^d \right)^2 - \left(t^d \right)^2 & \text{if } t^d < \tau^d \\ \frac{1}{2}(t^d - \tau^d)(2 - t^d - \tau^d) & \text{if } t^d > \tau^d. \end{cases}$$

Peeling the game tree back to the first stage,

$$t^d(\tau) = \begin{cases} \frac{1}{2} \left(\sqrt{9 + 10\tau + \tau^2} - 3 - \tau \right) & \text{if } t^d < \tau \\ 1 & \text{if } t^d > \tau. \end{cases}$$

To verify that the firm chooses $t^d < \tau^d$, consider the most strict case: $\tau^d = \frac{1}{2}$. Since this is the smallest possible value τ^d takes, if the firm chooses to position itself to the left of the open source software, it will for all other values of $\tau^d > \frac{1}{2}$.

When the firm locates to the left of the OSS, it receives profits of approximately 0.056. When the firm locates to the right of the OSS, it receives profits of approximately 0.0089.

Thus the PS is developed with technology:

$$t^d\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{\sqrt{57}}{2} - \frac{7}{2}\right) < \frac{1}{1+\sqrt{2}}.$$

Thus far, the analytical example has illustrated the results from Propositions 3.1-3.7. To illustrate Proposition 3.8, note that when $t^d < \tau^d$,

$$\frac{dt^d\left(\tau\right)}{d\tau} = \frac{10 + 2\tau}{4\sqrt{9 + 10\tau + \tau^2}} - \frac{1}{2} > 0, \quad \forall \tau \in \left[\frac{1}{2}, 1\right].$$

As the OSS increases its location, the PS follows suit, as stated in Proposition 3.8.

3.5   Extensions

3.5.1   Cost Structures

The learning cost is not essential for the main results. To see this, suppose that $\ell(x, \zeta) = 0$ for all x, ζ, so technology becomes purely horizontal. Participating in a project that attracts other high-quality programmers is one of the benefits of contributing to OSS. This fact has a very important implication: the location of the OSS will still be above the median due to the fact that contributing to the development of OSS is costly. Therefore, the firm still has the incentive to choose a technology level no larger than the median.

The analysis was conducted under assumptions that are least favorable to the results. For example, the firm could choose any location at zero cost, which means it can develop the most technologically advanced software at the same cost as the simplest software. Including this cost would only serve to strengthen the above conclusions. Suppose that the firm faces the strictly increasing and convex cost function $c(t)$, where $c(0) = 0$. It is obvious that

nothing on the consumer side changes, nor does the firm's pricing decision. The condition outlined in Proposition 3.2 and its associated result becomes

$$\frac{\partial D\left(p\left(t^d, \tau^d\right), t^d\right)}{\partial t} = c'\left(t^d\right).$$

Proposition 3.6 follows immediately, and becomes stronger since moving slightly to the left of τ^d not only weakly increases the firm's market share, but strictly decreases the firm's costs, leading to a strict increase in profits. Now consider a U-shaped cost function $c(t)$, where $G^{-1}\left(\frac{1}{2}\right) \equiv \arg\min c(t)$. Since $\tau^d \geq G^{-1}\left(\frac{1}{2}\right)$, any value $t^d = \tau^d - \kappa$ is strictly preferred to $t^d = \tau^d + \kappa$ for small, positive κ. The argument presented above still holds, as does Proposition 3.6. By invoking the same line of reasoning, the result holds for any U-shaped cost function where $G^{-1}\left(\frac{1}{2}\right) \geq \arg\min c(t)$. Things are more complicated when $G^{-1}\left(\frac{1}{2}\right) < \arg\min c(t)$. In this case, the result holds so long as $|c'(t)|$ is sufficiently small at all t. Otherwise, the cost of developing easy to use (low technology) software is excessively costly to the point that it cannot be profitable to do so, regardless of the demand.

Now consider the valuation $v(x, \zeta)$. It may be the case that the sophistication of the software has no impact on the valuation of the software, that is, a program with technology level ζ gives each individual the same value V, for all x, or $v(x, \zeta) = v(x, x) = \overline{V}$ for all x, ζ (V takes on the maximal value for all individuals), as in Sen (2007). In this case, the results become more extreme. The interpretation of $b(x, \zeta)$ changes slightly, as now there is no direct benefit from use. That is, the contributors have no "need" to be satisfied that is not already satisfied by the PS. Thus the benefits of contributing consist solely of the joy of programming and the delayed benefits (such as signaling).

The first implication of this change is in the pricing and location decision of the firm. Since every individual has the same value $\overline{V}$, heterogeneity arises solely in the learning cost. An

individual x will purchase the PS over the OSS if:

$$\overline{V} - \ell(x,t) - p \geq \max\left\{\overline{V} - \ell(x,\tau), 0\right\},$$

which implies that the firm has a dominant strategy in choosing $t = 0$. Thus the assumption that $t = 0$ from Sen (2007) is actually an endogenous result of a special case of this model. Since $t = 0$, the learning cost for all individuals is zero. The firm then chooses price to maximize profits, given the OSS location choice τ. Recall that the equilibrium choice of τ is independent of t. Therefore the optimal choice τ is unchanged. The firm chooses a price $p\left(0, \tau^d\right) > p\left(t^d, \tau^d\right)$ and its profits are strictly greater and the catering to ignorance result is more severe, with the firm producing to maximize the value of the most technologically ignorant individual. As a result, the market is fully covered. The location of the OSS remains unchanged while the firm locates closer to the most technologically ignorant individual.

Recall that the OSS developers need not make pre-compiled binaries available. Suppose that there are no binaries available, but the source code is available. It is then up to the individual to download, the source code, then compile and install it. This procedure is both time- and knowledge-intensive. Those individuals who are technologically savvy can easily accomplish this task, but those who are not face a higher barrier. Formally, denote $I(x)$ as the compilation/installation cost. It is assumed unchanging in ζ, but strictly decreasing in x, with $I(1) = 0$. An individual x will then choose the proprietary software over the open source software if

$$v(x,t) - p - \ell(x,t) \geq \max\{v(x,\tau) - \ell(x,\tau) - I(x), 0\}. \tag{3.10}$$

Comparing condition (3.10) to condition (3.5), it is obvious that for any given price p, (3.10) is easier to satisfy. Without changing its price, the firm captures every individual it had when $I(x) = 0$, plus the marginal individuals who, when $I(x) = 0$, consumed the OSS but

now purchase the PS. Thus the firm is better off, or in other words, the firm strictly prefers a more costly OSS. This is not surprising, as a higher cost diminishes the degree of competition the firm faces from the OSS developers by shifting the critical consumer, who is indifferent between choosing PS and OSS, rightward.

3.5.2 Network Effects and R&D Spillovers

It is well-known that in industries with network effects, under certain conditions, multiple firms persist in the long run, especially when compatibility is involved (Mitchell and Skrzypacz, 2006). Chen et al. (2009) show that when coupled with strategic pricing, compatibility can prevent one product from exhibiting market dominance. It is fairly common in the software world to see compatibility; for example, there is compatibility between Microsoft Office, Google Docs, Apache OpenOffice, and LibreOffice. Similarly, documents written in one of the many LaTeX typesetters are compatible with proprietary editors, such as Scientific Workplace. Thus including Network effects is unlikely to affect the main results. In Casadesus-Masanell and Ghemawat (2006), the demand side learning is related to network externalities, and the authors show that these externalities can lead to only having a single survivor in the long run; however, their model does not include heterogeneous consumers. The remaining papers mentioned in the introduction that discuss network effects do so under the context of proprietary firms supporting OSS initiatives. Cheng et al. (2011) show that the impact of network externalities can lead either the firm or OSS developers to support network effects, but quality (product differentiation) is modeled as in Sen (2007), where the location is treated as exogenous. Thus, Cheng et al. (2011) are unable to illustrate the impact of network externalities on location choice. However, they do show that in the presence of network externalities, both PS and OSS coexist.

R&D spillovers are very important in the context of competition between OSS and PS. These

spillovers are often one-sided, moving from the OSS to the PS, and typically depend on the license choice of the OSS. Llanes and de Elejalde (2013) show that when there are large spillovers, both PS and OSS coexist. Counterintuitively, large R&D spillovers occur when the open source license is less open. The GNU GPL license is the most open, and requires that any additions or changes to the source code must also carry the GNU GPL license. This implies that a firm, if using OSS code that is protected by the GNU GPL license, would be required to release its software under the same license, making the software open source. On the other hand, if the OSS is developed under the BSD 2-clause license, then any changes need not be branded under the same license, so the firm can include open source code and the resulting software can still be licensed as PS.

The effect of introducing R&D spillovers to this model is straightforward. Suppose that the OSS is developed under the BSD 2-clause license, so the firm can appropriate any spillovers. Furthermore, assume that the closer the firm locates to the OSS, the more valuable the spillovers are. The firm would then have the incentive to move its software closer to the OSS, but this incentive is tempered by the demand effect. Moving too close would lead the consumers of the PS to jump ship and choose either the outside option or the OSS. In particular, those far to the left of the PS would choose the outside option while those to the right would choose the OSS.

3.5.3 Open Source Monopoly and Oligopoly

Very little changes when OSS is the only option. The location decision of the OSS is determined by the system

$$G\left(\tau^d\right) - \frac{1}{2}G\left(\hat{x}\left(\tau^d\right)\right) = \frac{1}{2}$$
$$b\left(\hat{x}\left(\tau^d\right), \tau^d\right) = K\left(\hat{x}\left(\tau^d\right), \tau^d\right),$$

which is independent of t. Therefore the OSS's location is independent of competition from PS. What changes is the resulting market share. Instead of the values derived under the mixed duopoly, the critical consumers are defined by the solutions to the equation

$$v(x, \tau) - \ell(x, \tau) = 0$$

when interior, or the associated boundary conditions when either $v(1, \tau) - \ell(1, \tau) > 0$ or $v(0, \tau) - \ell(0, \tau) > 0$. Contributors' motivations include solving one's own needs, intellectual curiosity, and the joy of programming. These intrinsic motivations support the notion that an individual's decision on whether or not to contribute, and where the individual's preferred location is, is independent of the PS's location. Maximizing market share/profits is a rarely cited reason and is typically reserved for commercial developers of OSS.

Independence, as defined above, need not always be the case. In many instances, contributions to and the direction of OSS are motivated (often financially) by for profit entities. Take the Linux-based Fedora operating system. Red Hat Inc. has veto power over all of Fedora's development decisions. While clearly not a monopoly, this example illustrates the important points which still hold under a monopoly. This idea can be included directly into the associated framework by modeling Red Hat as an atomic agent in the bargaining problem defined in analysis. While each individual developer has zero bargaining power, Red Hat has enough bargaining power to influence the outcome. Since releases of Red Hat Enterprise Linux are branches of the Fedora operating system, Red Hat has a strong incentive to exert this influence. However, in OSS guided solely by community interests, the development decisions remain independent of competition from PS.

On the other hand, introducing competition from other OSS will change the locations. The community is divided and each project locates at the median point of its developers' interests. This closely resembles the development of LaTeX typesetters. There are many open source

options, such as TexMaker, TexStudio, Emacs, each varying in its capabilities and ease of use.

3.6 Discussion

Product differentiation arises endogenously through competitive forces, where the developers of PS cater to the technologically ignorant consumers while the developers of OSS cater to the technologically elite consumers. Catering to ignorance should not be seen as a surprise. It may be less expensive to develop simple software as opposed to more complex software with many technical features (though it need not be). Furthermore, more individuals are technologically ignorant than technologically elite. In the modeling setup, I assume an equal measure of technologically ignorant individuals and technologically elite individuals. This assumption represents the strictest setting; if the firm has the incentive to target the less sophisticated individuals, then that incentive only increases as the measure of less sophisticated individuals increases.

The mere presence of OSS pushes the PS further away from the technologically elite consumers, toward the technologically ignorant consumers. These results have significant competitive and policy implications. For instance, if entry by an OSS competitor cannot be blocked, a firm developing PS may have the incentive to "push" the OSS toward the most elite individuals, for example, through a steering committee as in the Mozilla example. Doing so would allow the market to closely resemble the monopoly market.

Raju (2007) shows that governments around the world have been implementing policies in support of OSS, but claims that it is unclear whether this is efficiency improving. Efficiency should be improved through the entry of OSS; however, it is still unclear whether or not costly government investment would be beneficial. Government investment could either help solve

122

the potential free-rider problem that exists when there is community-based development, or it could crowd out investments made by firms to OSS entities, such as those discussed in Mustonen (2005) and Riehle (2009). Engelhardt and Freytag (2013) provide evidence indicating that the supply side of OSS benefits from the enforcement of intellectual property rights, while some regulation has led to an increase on the number of OSS developers.

Some OSS projects are developed because those technologically elite individuals felt that the PS was designed not for them, but for the technologically ignorant consumers. In response to the firm's behavior, OSS is developed for the elite. However, this development further pushes the PS towards the technologically ignorant consumers. This process becomes a self-propelling mechanism, where the firm's location decision induces competition from OSS, which in turn induces the firm to reevaluate its location, which induces more OSS competition, *ad infinitum.* The previous illustration is an extrapolation of the static model as a dynamic process, where entry occurs over time and what follows is a firm located near $t = 0$, with OSS located at various intervals across the spectrum of technological savviness. The degree to which the firm targets the most ignorant consumer is increasing in how costly the OSS is, relative to the PS. If the consumer is required to compile the software herself, then the cost of using the OSS increases. It is, as if, the OSS moves further to the right. The firm is thus able to capture the critical individuals who would have used the OSS had they not faced this added cost.

3.7   Conclusion

There is still much work to be done in studying competition in the software industry. OSS is not the same as free software. Many companies offer free OSS and sell complementary services, such as customer and user guides. In some cases, such as the Red Hat Enterprise Linux distribution, firms sell the actual software at a price. While any individual can tech-

nically obtain any OSS for free (monetarily), there may be a significant opportunity cost in doing so, typically in terms of time and effort. Thus one could model OSS by considering a firm practicing second-degree price discrimination, where consumers with a low opportunity cost choose to download the source code and expend costly effort to convert it into usable software, while those with a higher opportunity cost of time pay a premium to receive the same OSS in a more readily usable condition. This price discrimination leaves the OSS developer with an interesting strategic decision. Is there an optimal "difficulty" the developer should set that maximizes the developers' profits, and how does this compare to the level which maximizes total welfare?

Bibliography

Abreu, Dilip and David Pearce (2015), 'A dynamic reinterpretation of nash bargaining with endogenous threats', *Econometrica* **83**(4), 1641–1655.

Ahn, T. K., R. Mark Isaac and Timothy C. Salmon (2008), 'Endogenous group formation', *Journal of Public Economic Theory* **10**(2), 171–194.

Aimone, Jason A., Laurence R. Iannaccone, Michael D. Makowsky and Jared Rubin (2013), 'Endogenous group formation via unproductive costs', *Review of Economic Studies* **80**(4), 1215–1236.

Ajila, Samuel A. and Di Wu (2007), 'Empirical study of the effects of open source adoption on software development economics', *Journal of Systems and Software* **80**(9), 1517–1529.

Aksulu, Altay and Michael Wade (2010), 'A comprehensive review and synthesis of open source research', *Journal of the Association for Information Systems* **11**(11/12), 576–656.

Alesina, Alberto and Eleina La Ferrara (2000), 'Participation in heterogeneous communities', *The Quarterly Journal of Economics* **115**(3), 847–904.

Alesina, Alberto and Enrico Spolaore (1997), 'On the number and size of nations', *The Quarterly Journal of Economics* **112**(4), 1027–1056.

Alesina, Alberto, Reza Baqir and William Easterly (1999), 'Public goods and ethnic divisions', *The Quarterly Journal of Economics* **114**(4), 1243–1284.

Allison, John R. and Mark A. Lemley (2002), 'The growing complexity of the united states patent system', *Boston University Law Review* **82**(1), 77–144.

Anderson, Simon, André de Palma and Jacques-François Thisse (1997), 'Privatization and efficiency in a differentiated industry', *European Economic Review* **41**(9), 1635–1654.

Andreoni, James (1990), 'Impure altruism and donations to public goods: A theory of warm-glow giving', *The Economic Journal* **100**(401), 464–477.

Aoki, Reiko (1991), 'R&d competition for product innovation: An endless race', *American Economic Review* **81**(2), 252–256.

Arnold, Tone and Myrna Wooders (2005), Dynamic Club Formation with Coordination, Vanderbilt University Department of Economics Working Papers 0522, Vanderbilt University Department of Economics.
URL: *http://ideas.repec.org/p/van/wpaper/0522.html*

Arnold, Tone and Myrna Wooders (2009), Dynamic club formation with coalitional deviations.
URL: *http://wrap.warwick.ac.uk/46082/*

Arnott, Richard and Kenneth Small (1994), 'The economics of traffic congestion', *American Scientist* **82**(5), 446–455.

Athey, Susan and Glenn Ellison (2014), 'Dynamics of open source movements', *Journal of Economics and Management Strategy* **23**(2), 294–316.

Austin, D. Andrew (1995), 'Coordinated action in local public goods models: The case of secession without exclusion', *Journal of Public Economics* **58**(2), 235–256.

Baye, Michael and Heidrun Hoppe (2003), 'The strategic equivalence of rent-seeking, innovation, and patent-race games', *Games and Economic Behavior* **44**(2), 217–226.

Berglund, Tom (2014), 'Incentives for complexity in financial regulation', *Journal of Risk Finance* **15**(2), 102–109.

Bergstrom, Theodore, Lawrence Blume and Hal Varian (1986), 'On the private provision of public goods', *Journal of Public Economics* **29**(1), 25 – 49.

Bernheim, B. Douglas (1986), 'On the voluntary and involuntary provision of public goods', *The American Economic Review* **76**(4), pp. 789–793.

Birke, Daniel (2009), 'The economics of networks: A survey of the empirical literature', *Journal of Economic Surveys* **23**(4), 762–793.

Bitzer, Jürgen (2004), 'Commercial versus open source software: the role of product heterogeneity in competition', *Economic Systems* **28**(4), 369–381.

Bitzer, Jürgen, Wolfram Schrettl and Phillip J.H. Schröder (2007), 'Intrinsic motivation in open source software development', *Journal of Comparative Economics* **35**(1), 160–169.

Brown, Jr., Gardner (1974), 'An optimal program for managing common property resources with congestion externalities', *Journal of Political Economy* **82**(1), 163–173.

Buchanan, James M. (1965), 'An economic theory of clubs', *Economica* **32**(125), 1–14.

Bulow, Jeremy (1982), 'Durable-goods monopolists', *Journal of Political Economy* **90**(2), 314–332.

Bulow, Jeremy (1986), 'An economic theory of planned obsolescence', *Quarterly Journal of Economics* **101**(4), 729–750.

Caloghirou, Yannis, Stavros Ionnides and Nicholas S. Vonortas (2003), 'Research joint ventures', *Journal of Economic Surveys* **17**(4), 541–570.

Carvalho, Jean-Paul and Mark Koyama (2015), 'Jewish emancipation and schism: Economic development and religious change', *Working Paper* .

Casadesus-Masanell, Ramon and Gastón Llanes (2011), 'Mixed source', *Management Science* **57**(7), 1212–1230.

Casadesus-Masanell, Ramon and Pankaj Ghemawat (2006), 'Dynamic mixed duopoly: A model motivated by linux vs. windows', *Management Science* **52**(7), 1072–1084.

Casella, Alessandra (2001), 'The role of market size in the formation of jurisdictions', *Review of Economic Studies* **68**(1), 83–108.

Chen, Jiawei, Ulrich Doraszelski and Joseph E. Harrington Jr. (2009), 'Avoiding market dominance: Product compatibility in markets with network effects', *RAND Journal of Economics* **40**(3), 455–485.

Cheng, Hsing, Yipeng Liu and Qian Tang (2011), 'The impact of network externalities on the competition between open source and proprietary software', *Journal of Management Information Systems* **27**(4), 201–230.

Cho, Seok-ju and John Duggan (2009), 'Bargaining foundations of the median voter theorem', *Journal of Economic Theory* **144**(2), 851–868.

Christiaans, Thomas (2013), 'On the dynamics of competition between commercial and free software', *Studies in Microeconomics* **1**(1), 37–58.

Cornes, Richard and Todd Sandler (1996), *The Theory of Externalities, Public Goods, and Club Goods*, 2 edn, Cambridge University Press.

De Fraja, Giovanni and Flavio Delbono (1990), 'Game theoretic models of mixed oligopoly', *Journal of Economic Surveys* **4**(1), 1–17.

Debreu, Gerard (1952), 'A social equilibrium existence theorem', *Proceedings of the National Academy of Sciences* **38**(10), 886–893.

Dey, Debabrata, Atanu Lahiri and Dengpan Liu (2013), 'Consumer learning and time-locked trials of software products', *Journal of Management Information Systems* **30**(2), 239–268.

Economides, Nicholas and Evangelos Katsamakas (2006*a*), *The Economics of Open Source Software Development*, Elsevier B.V., chapter 10 - Linux vs. Windows: A Comparison of Application and Platform Innovation Incentives for Open Source and Proprietary Software Platforms, pp. 207–218.

Economides, Nicholas and Evangelos Katsamakas (2006*b*), 'Two-sided competition of proprietary vs. open source technology platforms and the implications for the software industry', *Management Science* **52**(7), 1057–1071.

Engelhardt, Sebastian and Andreas Freytag (2013), 'Institutions, culture, and open source', *Journal of Economic Behavior and Organization* **95**, 90–110.

Fan, Ky (1952), 'Fixed-point and minimax theorems in locally convex topological linear spaces', *Proceedings of the National Academy of Sciences* **38**(2), 121–126.

Fang, Yulin and Derrick Neufeld (2009), 'Understanding sustained participation in open source software projects', *Journal of Management Information Systems* **25**(4), 9–50.

Feller, Joseph, Brian Fitzgerald and Karim Lakhani, eds (2005), *Perspectives on Open Source Software*, MIT Press, London, England.

Fershtman, Chaim and Neil Gandal (2007), 'Open source software: Motivation and restrictive licensing', *Journal of International Economics and Economic Policy* **4**(2), 209–225.

Fershtman, Chaim and Neil Gandal (2011), 'A brief survey of the economics of open source software', *CEPR Discussion Paper No. DP8434* .

Fershtman, Chaim and Shmuel Nitzan (1991), 'Dynamic voluntary provision of public goods', *European Economic Review* **35**(5), 1057 – 1067.

Fudenberg, Drew, Richard Gilbert, Joseph Stiglitz and Jean Tirole (1983), 'Preemption, leapfrogging and competition in patent races', *European Economic Review* **22**(1), 3–31.

Fujiwara, Kenji and Norimichi Matsueda (2009), 'Dynamic voluntary provision of public goods: A generalization', *Journal of Public Economic Theory* **11**(1), 27–36.

Galasso, Alberto and Mark Schankerman (2014), 'Patents and cumulative innovation: Causal evidence from the courts', *NBER Working Paper* **20269**, 1–77.

Georgiadis, George (2015), 'Projects and team dynamics', *Review of Economic Studies* **82**(1), 187–218.

Gilbert, Richard and David Newbery (1982), 'Preemptive patenting and the persistence of monopoly', *American Economic Review* **72**(3), 514–526.

Glazer, Amihai, Esko Niskanen and Suzanne Scotchmer (1997), 'On the uses of club theory: Preface to the club theory symposium', *Journal of Public Economics* **65**(1), 3–7.

Glazer, Amihai and Kai Konrad (1996), 'A signaling explanation for private charity', *American Economic Review* **86**(4), 1019–1028.

Glicksberg, Irving (1952), 'A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points', *Proceedings of the American Mathematical Society* **3**(1), 170–174.

Glomm, Gerhard and Roger Lagunoff (1999), 'A dynamic tiebout theory of voluntary vs. involuntary provision of public goods', *Review of Economic Studies* **66**(3), 659–677.

Haimanko, Ori, Michel Le Breton and Shlomo Weber (2004), 'Voluntary formation of communities for the provision of public projects', *Journal of Economic Theory* **115**(1), 1 – 34.

Harris, Christopher and John Vickers (1987), 'Racing with uncertainty', *Review of Economic Studies* **54**(1), 1–21.

Hars, Alexander and Shaosong Ou (2001), Working for free? - motivations of participating in open source projects, Proceedings of the 34th Hawaii International Conference on System Sciences.

Holmström, Bengt (1999), 'Managerial incentive problems: A dynamic perspective', *Review of Economic Studies* **66**(1), 169–182.

Iannaccone, Laurence R. (1992), 'Sacrifice and stigma: Reducing free-riding in cults, communities, and other collectives', *Journal of Political Economy* **100**(2), 271–291.

Iannaccone, Laurence R. (1994), 'Why strict churches are strong', *American Journal of Sociology* **99**(5), 1180–1211.

Jaisingh, Jeevan, Eric W.K. See-To and Kar Yan Tam (2008-9), 'The impact of open source software on the strategic choices of firms developing proprietary software', *Journal of Management Information Systems* **25**(3), 241–275.

Jaramillo, Fernando, Hubert Kempf and Fabien Moizeau (2003), 'Inequality and club formation', *Journal of Public Economics* **87**(5-6), 931 – 955.

Johnson, Justin P. (2002), 'Open source software: Private provision of a public good', *Journal of Economics and Management Strategy* **11**(4), 637–662.

Kalai, Ehud and Meir Smorodinsky (1975), 'Other solutions to nash's bargaining problem', *Econometrica* **43**(3), 513–518.

Kamien, Morton I., Eitan Muller and Israel Zang (1992), 'Research joint ventures and r&d cartels', *American Economic Review* **82**(5), 1293–1306.

Kearl, J.R. (1983), 'Rules, rule intermediaries and the complexity and stability of regulation', *Journal of Public Economics* **22**(2), 215–226.

Kumar, Vineet, Brett R. Gordon and Kannan Srinivasan (2011), 'Competitive strategy for open source software', *Marketing Science* **30**(6), 1066–1078.

Kuran, Timur (1987), 'Preference falsification, policy continuity and collective conservatism', *The Economic Journal* **97**(387), 642–665.

Kuran, Timur (1995), *Private Truths, Public Lies: The Social Consequences of Preference Falsification*, Harvard University Press.

Lancaster, Kelvin (1966), 'A new approach to consumer theory', *Journal of Political Economy* **74**(2), 132–157.

Lerner, Josh and Jean Tirole (2001), 'The open source movement: Key research questions', *European Economic Review* **45**(4-6), 819–826.

Lerner, Josh and Jean Tirole (2002), 'Some simple economics of open source', *Journal of Industrial Economics* **L**(2), 197–234.

Lerner, Josh, Parag A. Pathak and Jean Tirole (2006), 'The dynamics of open-source contributors', *American Economic Review* **96**(2), 114–118.

Llanes, Gastón and Ramiro de Elejalde (2013), 'Industry equilibrium with open-source and proprietary firms', *International Journal of Industrial Organization* **31**(1), 36–49.

Marx, Leslie M. and Steven A. Matthews (2000), 'Dynamic voluntary contribution to a public project', *Review of Economic Studies* **67**(2), 327–358.

Matsumura, Toshihiro (1998), 'Partial privatization in mixed duopoly', *Journal of Public Economics* **70**(3), 473–483.

Maurer, Stephen M. and Suzanne Scotchmer (2006), 'Open source software: The new intellectual proprety paradigm', *NBER Working Paper Series* .

McBride, Michael (2007*a*), 'Club mormon: Free-riders, monitoring, and exclusion in the lds church', *Rationality and Society* **19**(4), 395–424.

McBride, Michael (2007*b*), 'Why churches need free-riders: Religious capital formation and religious group survival', *University of California, Irvine Working Paper* pp. 1–34.

McBride, Michael (2008), 'Religious pluralism and religious participation: A game theoretic analysis', *American Journal of Sociology* **114**(1), 77–108.

Mitchell, Matthew F. and Andrzej Skrzypacz (2006), 'Network externalities and long-run market shares', *Journal of Economic Theory* **29**(3), 621–648.

Murray, Fiona and Scott Stern (2007), 'Do formal intellectual property rights hinder the free flow of scientific knowledge?: An empirical test of the anti-commons hypothesis', *Journal of Economic Behavior & Organization* **63**(4), 648–687.

Mustonen, Mikko (2005), 'When does a firm support substitute open source programming?', *Journal of Economics and Management Strategy* **14**(1), 121–139.

Myatt, David P. and Chris Wallace (2008), 'An evolutionary analysis of the volunteer's dilemma', *Games and Economic Behavior* **62**(1), 67–76.

Nadeau, Tom (1999), 'Learning from linux'.
  URL: *http://www.os2hq.com/archives/linmemo1.htm*

Nash, Jr., John F. (1950), 'The bargaining problem', *Econometrica* **18**(2), 155–162.

Nitzan, Shmuel, Uriel Procaccia and Joseph Tzur (2013), 'On the political economy of complexity', *CESifo Working Paper Series No. 4547* .

Noyes, Katherine (2013), 'Open source is taking over the software world, survey says'.
URL: *http://www.pcworld.com/article/2035651/open-source-is-taking-over-the-software-world-survey-says.html*

Olson, Mancur (1965), *The Logic of Collective Action: Public Goods and the Theory of Groups*, Oxford University Press.

Open Source Initiative (2015), 'Apache license 2.0'.
URL: *http://opensource.org/licenses/Apache-2.0*

Open Source Initiative (n.d.*a*), 'The bsd 2-clause license'.
URL: *http://opensource.org/licenses/BSD-2-Clause*

Open Source Initiative (n.d.*b*), 'Gnu general purpose license v3'.
URL: *http://opensource.org/licenses/GPL-3.0*

Open Source Initiative (n.d.*c*), 'The open source definition'.
URL: *http://opensource.org/osd*

Pirillo, Chris (2008), 'The future of software is open source'.
URL: *http://chris.pirillo.com/the-future-of-software-is-open-source/*

Polanski, Arnold (2007), 'Is the general public license a rational choice', *Journal of Industrial Economics* **55**(4), 691–714.

Polborn, Mattias K. (2008), 'Competing for recognition through public good provision', *The B.E. Journal of Theoretical Economics* **8**(1), Article 22.

Quandt, Richard (1983), 'Complexity in regulation', *Journal of Public Economics* **22**(2), 199–214.

Raju, K.D. (2007), 'Is the future of software development in open source software? proprietary vs. open source software: A cross country analysis', *Journal of Intellectual Property Rights* **12**(2), 1–20.

Raymond, Eric (1999), 'The cathedral and the bazaar', *Knowledge, Technology & Policy* **12**(3), 23–49.

Riehle, Dirk (2009), 'The commercial open source business model', *Lecture Notes in Business Information Processing* **36**, 18–30.

Rubinstein, Ariel (1982), 'Perfect equilibrium in a bargaining model', *Econometrica* **51**(1), 97–109.

Sacks, Michael (2015), 'Competition between open source and proprietary software: Strategies for survival', *Journal of Management Information Systems* **32**(3), 268–295.

Sacks, Michael (2016), 'In the club: strategic admission, membership, and endogenous splits', *Working Paper* pp. 1–73.

Samuelson, Paul (1954), 'The pure theory of public expenditure', *Review of Economics and Statistics* **36**(4), 387–389.

Sandler, Todd and John Tschirhart (1997), 'Club theory: Thirty years later', *Public Choice* **93**(3/4), 335–355.

Saurer, Robert M. (2007), 'Why develop open-source software? the role of non-pecuniary benefits, monetary rewards and open-source license type', *Oxford Review of Economic Policy* **23**(4), 605–619.

Schelling, Thomas (1978), *Micromotives and Macrobehavior*, W.W. Norton and Company.

Scotchmer, Suzanne (2002), 'Chapter 29 local public goods and clubs', *Hanbook of Public Economics* **4**, 1997–2042.

Sen, Ravi (2007), 'A strategic analysis of competition between open source and proprietary software', *Journal of Management Information Systems* **24**(1), 233–257.

Shapiro, Carl (1985), 'Patent licensing and r&d rivalry', *American Economic Review* **75**(2), 25–30.

Shy, Oz (2001), *The Economics of Network Industries*, Cambridge: Cambridge University Press.

Shy, Oz (2011), 'A short survey of network economics', *Review of Industrial Organization* **38**(2), 119–149.

Spence, Michael (1975), 'Monopoly, quality, and regulation', *Bell Journal of Economics* **6**(2), 417–429.

Stamelos, Ioannis, Lefteris Angelis, Apostolos Oikonomou and Georgios L. Bleris (2002), 'Code quality analysis in open source software development', *Information Systems Journal* **12**(1), 43–60.

Stewart, Katherine and Sanjay Gosain (2006), 'The impact of ideology on efectiveness in open source software development teams', *MIS Quarterly* **30**(2), 291–314.

Taylor, Richard, Nenad Medvidović and Eric Dashofy (2010), *Software Architectrure: Foundations, Theory, and Practice*, John Wiley & Sons, Hoboken, NJ.

von Krogh, Georg and Eric von Hippel (2006), 'The promise of research on open source software', *Management Science* **52**(7), 975–983.

Waldman, Michael (1996), 'Durable goods pricing when quality matters', *Journal of Business* **69**(4), 489–510.

Williams, Heidi (2013), 'Intellectual property rights and innovation: Evidence from the human genome', *Journal of Political Economy* **121**(1), 1–27.

Wirl, Franz (1996), 'Dynamic voluntary provision of public goods: Extension to nonlinear strategies', *European Journal of Political Economy* **12**(3), 555 – 560.

Appendix A

Appendix to Chapter 1

A.1   Dynamic Analysis with a Discrete Strategy Space

In this extension, I alter the model by restricting $x_i^k(t) \in \{0, 1\}$, so that contributing is a binary decision. For instance, when developing OSS, a contributor cannot simply submit a single character, word, or line. Rather, she must contribute a complete working module. I then impose the following four assumptions related to discounting and the cost of contributing. First $\delta \to \infty$, so individuals are myopic and only interested in maximizing instantaneous utility. I restrict ψ by assuming that

$$\psi > 2(1 + \sigma), \tag{A.1}$$

$$\psi < d(A(\mu), A_1)(N_1 + \sigma), \tag{A.2}$$

$$\psi < 2d(A(\mu), A_2)\left[\frac{1}{\gamma}N_1 + 1 + \sigma\right]. \tag{A.3}$$

Inequality (A.1) ensures that individuals receive negative utility when acting alone at the outset, so the club is not a singleton, (A.2) ensures that majority individuals are willing to

contribute to a cohesive community at the outset, and (A.3) ensures that minority individuals find it profitable to contribute to a cohesive community at some time T, and all subsequent $t > T$.

Cohesive communities come in two forms when contributions are discrete. Every individual can contribute to a single club good, or alternatively, a subset of individuals can contribute to a single club good while the remaining individuals choose nonmembership. If an individual i contributes to a cohesive club, she receives utility

$$d(A(\mu), A_i) \left[(1 - \gamma)C(t) + \sum_{j=1}^{N} x_j(t) + \sigma \right] - \frac{\psi}{2},$$

$\sum_j x_j(t) \leq N$, as not all individuals may be willing to contribute. At the outset, any individual who contributes receives utility

$$d(A(\mu), A_i) \left[\sum_{j=1}^{N} x_j(0) + \sigma \right] - \frac{\psi}{2},$$

which requires one of the following two statements to be true:

(i) Only those with the highest match value (majority individuals) contribute at the outset.

(ii) Every individual contributes at the outset.

If (i) is satisfied, then installed base is going to accumulate over time according to

$$\dot{C}(t) = N_1 - \gamma C(t), \quad t < T.$$

By (A.3), there exists a finite time T at which minority individuals will begin contributing as well, altering the dynamic to

$$\dot{C}(t) = N_1 + N_2 - \gamma C(t), \quad t \geq T.$$
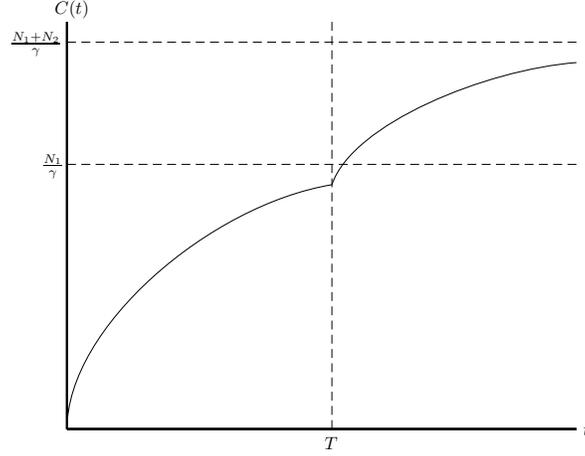
Figure A.1: Cohesive dynamic path

Therefore, the complete evolution of installed base is characterized by

$$
C(t) = \begin{cases} \frac{N_1}{\gamma} \left(1 - e^{-\gamma t}\right) & \text{if } t < T \\ \frac{N_1 + N_2}{\gamma} - \left(\frac{N_2}{\gamma} e^{-\gamma(t-T)} + \frac{N_1}{\gamma} e^{-\gamma t}\right) & \text{if } t \geq T. \end{cases}
$$

Figure A.1 diagrammatically illustrates the dynamic path. This pattern can be generalized to any number of types of individuals. At the outset, only the individuals with the highest match values contribute. As the installed base grows, the next best-matched individuals join, and so on. I treat the scenarios in which either (i) all individuals contribute to a single club at the outset or (ii) all individuals contribute to a single club over time as cohesive equilibria.

The splitting equilibrium can be characterized by the relationship between the number of minority individuals, the cost of contributing, and the relative marginal benefit of contributing. This is in contrast to both the static and dynamic case, where the existence of a splitting equilibrium was independent of ψ. Since contributions are binary, they cannot be adjusted along the intensive margin to changes in ψ, driving this difference.

Proposition A.1. *Under the discrete dynamic structure, there exists a splitting equilibrium*

if and only if $\psi \leq 2(N_2 + \sigma)$.

If $\psi > 2(N_2 + \sigma)$, then it is not profitable for the minority individuals to form a club at the outset, and there are three possibilities. The minority individuals can choose to not contribute at the outset, as described in the second cohesive equilibria defined above, or they can practice strategic membership. Strategic membership is practiced in one of two ways. The minority individuals can either contribute to the parent club for a finite period of time and then orchestrate a split, or never contribute, but still orchestrate a split. The latter has been known to occur in the development of OSS. The first form of strategic membership is the familiar form of active strategic membership. I refer to the second new form as *passive strategic membership*.

Without loss of generality, suppose that $r = 0$ and $\rho(0) = 1$. Therefore, if a split occurs, the installed base of the parent club is unaffected. Let T^* represent the time at which a split occurs. At time T^* the child club begins with installed base

$$C^c\left(T^*\right) = (1 - \gamma)\rho(1 - d(A(\mu), A^c))C\left(T^*\right),$$

where

$$C(T^*) = \begin{cases} \frac{N_1}{\gamma}\left(1 - e^{-\gamma T^*}\right) & \text{if } T^* < T \\ \frac{N_1 + N_2}{\gamma} - \left(\frac{N_2}{\gamma}e^{-\gamma(T^* - T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right) & \text{if } T^* \geq T. \end{cases}$$

First consider passive strategic membership. This is equivalent to T being sufficiently large, such that no minority individual contributes to the parent club.

Proposition A.2. (Passive Strategic Membership) *Suppose that T is sufficiently large.*

(i) An incubation equilibrium exists if and only if

$$\sigma \in \left(\frac{\psi}{2d(A^c, A_2)} - N_2 - \frac{1-\gamma}{\gamma}\rho(1 - d(A(\mu), A^c))N_1, \frac{\psi}{2d(A^c, A_2)} - N_2 \right].$$

(ii) The equilibrium incubation period is $[0, T^*]$, where $0 \leq T^* < T$ and T^* is given by

$$T^* \geq -\frac{1}{\gamma} \ln \left(1 + \frac{\gamma \left[2d(A^c, A_2)(N_2 + \sigma) - \psi \right]}{2d(A^c, A_2)(1 - \gamma)\rho(1 - d(A(\mu), A^c))N_1} \right).$$

If σ is at or above the upper bound given in Proposition A.2, then the time at which a split occurs is zero, and the equilibrium is not an incubation equilibrium, but a splitting equilibrium. If σ is too small, then the effects of economies of scale dominate the contribution effect, as described in the contribution-scale tension, so individuals would rather remain in a cohesive equilibrium. A unique T^* cannot be identified due to the myopic behavior of individuals. Since individuals are not internalizing the continuation payoff from any strategy, any time in which there is no incentive to deviate can be sustained in equilibrium. Therefore only a lower bound of T^* can be characterized. The time T^* can be uniquely pinned down only if individuals are forward looking. Passive strategic membership is summarized as follows:

1. At the outset, majority individuals contribute to the development of the club good, while minority individuals sit on the sidelines and wait.

2. Over time, the installed base is growing solely due to majority contributions.

3. At time T^*, the installed base grows large enough that the minority leader has the incentive to enter the group and immediately leave with a portion of the installed base under the expectation that all other minority individuals will contribute to the development of the new club good.

4. From time T^* onward, all majority individuals continue to contribute to the parent while all minority type individuals contribute to the new child club.

When practicing active strategic membership, minority individuals actively contribute to the parent club during the incubation period (pre-split). Then at time T^*, the split occurs. As in proposition A.2, σ must be bounded in order for incubation to occur.

Proposition A.3. (Active Strategic Membership) *Suppose that T is sufficiently small and that $\rho(1 - d(A(\mu), A^c)) \neq \frac{d(A(\mu), A^c)}{d(A^c, A_2)}$.*

(i) *An incubation equilibrium exists if and only if $\sigma \in (\underset{\sim}{\sigma}, \tilde{\sigma}]$.[1]*

(ii) *The optimal incubation period is $[0, T^*]$, where $0 \leq T \leq T^*$ and T^* is given by*

$$T^* \geq -\frac{1}{\gamma} \ln \left(\frac{(1 - \gamma)(N_1 + N_2 + \gamma\sigma)[d(A^c, A_2) - d(A(\mu), A_2)] - \gamma d(A(\mu), A_2)N_1}{(1 - \gamma)(N_1 + N_2 e^{\gamma T})[d(A^c, A_2)\rho(1 - d(A(\mu), A^c)) - d(A(\mu), A_2)]} \right).$$

The condition on $\rho(\cdot)$ is used to ensure that T^* is interior. If $\rho(1 - d(A(\mu), A^c)) = \frac{d(A(\mu), A^c)}{d(A^c, A_2)}$, then it follows that $T^* = 0$ or $T^* \to \infty$, which is to say that the incubation equilibrium reduces to either the splitting equilibrium or the cohesive equilibrium.

It is straightforward to see how the mission-scope tension influences incubation. In both passive and active strategic membership, as the distance between the parent and child club increases, the minimal equilibrium incubation time increases. That is to say, as the degree of distance-based depreciation increases, the necessary incubation period increases to compensate. There is one notable difference between Proposition A.2 and Proposition A.3: ψ does not enter into the equilibrium timing decision when active strategic membership is practiced, but does so when passive strategic membership is practiced. This difference is attributed to the fact that active strategic membership and passive strategic membership

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[1] The values of $\tilde{\sigma}$ and $\underset{\sim}{\sigma}$ can be found in the proof.

operate on different margins. Under passive strategic membership, minority individuals only contribute post-split. Therefore, when timing a split, the cost of contributing must be taken into account. Under active strategic membership, contributions by minority individuals are independent of a split occurring. Therefore, when timing a split, the cost of contributing is irrelevant. This distinction is due to of the discreteness of contributions.

A.2 Proofs of Propositions

A.2.1 Proof of Proposition 1.1

The indirect utility under the cohesive equilibrium is given by

$$
u(x_i^*, x_{-i}^*) = \left(\frac{1+\sigma}{2\psi}\right)\left[2\hat{N} - (1-\sigma)d(A(\mu), A_i)\right]d(A(\mu), A_i).
$$

If an individual were to unilaterally deviate, she does so by forming her own club with $A_k = A_i$. It follows that the indirect utility from such an endeavor is derived by taking the above utility and setting $\hat{N} = 1$ and the match value to 1:

$$
\frac{(1+\sigma)^2}{2\psi}.
$$

Since $d(A(\mu), A_1) > d(A(\mu), A_2)$, if an individual has the incentive to deviate, she will be a type 2 individual. Thus there exists a unilateral deviation if and only if

$$
1 + \sigma > \left[2\hat{N} - (1-\sigma)d(A(\mu), A_2)\right]d(A(\mu), A_2).
$$

The above is a quadratic function of $d(A(\mu), A_2)$. When rearranged, the condition can be written as

$$[2N_2 - (1 - \sigma)]d(A(\mu), A_2)^2 + 2\hat{N}_1 d(A(\mu), A_2) - (1 + \sigma) < 0.$$

Note that the discriminant is positive, so there exists two real roots. Since $2N_2 - (1-\sigma)] > 0$, the above is a convex quadratic function, which is only less then zero if $d(A(\mu), A_i)$ lies between the lower and upper root. Solving for the roots yields

$$d = \frac{-\hat{N}_1 \pm \sqrt{\left(\hat{N}_1\right)^2 + 2N_2(1 + \sigma) - (1 - \sigma^2)}}{2N_2 - (1 - \sigma)}.$$

The lower root is strictly negative. Thus setting $\overline{d}_2$ equal to the upper root completes (i). For (ii), reconsider the condition for the existence of a unilateral deviation.

$$1 + \sigma > \left[2\hat{N} - (1 - \sigma)d(A(\mu), A_2)\right] d(A(\mu), A_2)$$

$$\sigma \left(1 - d(A(\mu), A_2)^2\right) > 2\hat{N}d(A(\mu), A_2) - 1 - d(A(\mu), A_2)^2$$

$$\sigma \left(1 - d(A(\mu), A_2)^2\right) > N_1 d(A(\mu), A_1)d(A(\mu), A_2) + (N_2 - 1) d(A(\mu), A_2)^2 - 1$$

$$\sigma > \frac{N_1 d(A(\mu), A_1)d(A(\mu), A_2) + (N_2 - 1) d(A(\mu), A_2)^2 - 1}{1 - d(A(\mu), A_2)^2}.$$

Setting

$$\overline{\sigma}_2 = \frac{N_1 d(A(\mu), A_1)d(A(\mu), A_2) + (N_2 - 1) d(A(\mu), A_2)^2 - 1}{1 - d(A(\mu), A_2)^2}$$

completes statement (ii) and the proof. ∎

A.2.2 Proof of Proposition 1.2

The indirect utility under the splitting equilibrium is given by

$$u\left(x_i^*, x_{-i}^*\right) = \left(\frac{1+\sigma}{2\psi}\right)[2N^k - 1 + \sigma].$$

If an individual were to unilaterally deviate, she does so by joining a different club, namely the larger one due to economies of scale. Thus attention can be restricted to a type 2 individual deviating and joining the type 1 club. If she were to deviate, she receives utility of

$$\left(\frac{1+\sigma}{2\psi}\right)d(A_1, A_2)[2N_1 + (1-\sigma)d(A_1, A_2)].$$

The deviation payoff is greater than the equilibrium payoff if

$$(1+\sigma)d(A_1, A_2)^2 + 2N_1 d(A_1, A_2) - [2N_2 - (1-\sigma)] > 0.$$

For (i), note that the above is a convex quadratic function with a positive discriminant, so both roots are real. Thus the function is positive only if $d(A_1, A_2)$ is greater than the upper root or smaller than the lower root, where the roots are given by

$$d = \frac{-N_1 \pm \sqrt{N_1^2 + (1+\sigma)[2N_2 - (1-\sigma)]}}{1+\sigma}.$$

Since the lower root is negative, setting $\hat{d}_2$ equal to the upper root completes (i). For (ii), solving

$$(1 + \hat{\sigma}(2,1))\, d(A_1, A_2)^2 + 2N_1 d(A_1, A_2) - [2N_2 - (1 - \hat{\sigma}(2,1))] = 0$$

141

yields the cutoff value

$$\hat{\sigma}_2 = \frac{(2N_1 + d(A_1, A_2)d(A_1, A_2) - 2N_2 + 1}{1 - d(A_1, A_2)^2}.$$

Thus there is no incentive to deviate if $\sigma \geq \hat{\sigma}_2$.   ∎

A.2.3   Proof of Proposition 1.3

For a minority individual, the utility from playing the splitting equilibrium is greater than the utility from playing the cohesive equilibrium if

$$2\hat{N}d(A(\mu), A_2) - (1 - \sigma)d(A(\mu), A_2)^2 < 2N_2 - (1 - \sigma)$$

$$2\left[N_2 d(A(\mu), A_2)^2 + \hat{N}_1 d(A(\mu), A_2)\right] - (1 - \sigma)d(A(\mu), A_2)^2 < 2N_2 - (1 - \sigma).$$

At $d(A(\mu), A_2) = 0$,

$$0 < 2N_i - (1 - \sigma),$$

which is true. At $d(A(\mu), A_2) = 1$,

$$2N_2 + \hat{N}_1 - (1 - \sigma) < 2N_2 - (1 - \sigma)$$

$$\hat{N}_1 < 0,$$

which is false. Furthermore, note that the derivative of the left-hand side with respect to $d(A(\mu), A_2)$ is

$$2(2N_2 d(A(\mu), A_2) + \hat{N}_1) - 2(1 - \sigma)d(A(\mu), A_2) \gtrless 0$$

$$2(N_2 - 1)d(A(\mu), A_2) + \hat{N}_1 > -\sigma d(A(\mu), A_2).$$

The left-hand side is monotonically increasing in $d(A(\mu), A_2)$ while the the right-hand side is unchanging in $d(A(\mu), A_2)$. Thus there is unique threshold which determines the equilibrium that yields the highest utility. Finding this cutoff point amounts finding the roots of

$$2 \left[N_2 d(A(\mu), A_2)^2 + \hat{N}_1 d(A(\mu), A_2) \right] - (1 - \sigma) d(A(\mu), A_2)^2 = 2N_2 - (1 - \sigma).$$

It follows that

$$d = \frac{-\hat{N}_2 \pm \sqrt{\hat{N}_1^2 + [2N_2 - (1 - \sigma)]^2}}{2N_2 - (1 - \sigma)}.$$

Since the lower root is negative, setting $\tilde{d}_2(\mu)$ equal to the upper root completes the proof. An identical argument holds for a majority individual. ∎

A.2.4 Proof of Proposition 1.4

As $\delta \to 0$, an individual's indirect utility function can be divided into two components: a finite-horizon component

$$\int_0^T e^{-\delta t} u_i dt,$$

and an infinite-horizon component

$$+ \int_T^\infty e^{-\delta t} u_i dt,$$

where u_i is the instantaneous utility for individual i. For δ near zero, the value of the infinite component dominates the finite component. Thus for δ sufficiently small, attention can be restricted to the steady-state. I claim that the steady-state utility from the splitting equilibrium is greater than the steady-state utility from the incubation equilibrium. The

difference is given by

$$\frac{1+\sigma(\delta+\gamma)}{\psi(\delta+\gamma)}\left[\frac{N_i}{\gamma}+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}\right] > \frac{1+\sigma(\delta+\gamma)}{\psi(\delta+\gamma)}\left[\frac{d(A^k,A_i)^2 N_i}{\gamma}+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}d(A^k,A_i)^2\right],$$

which simplifies to

$$\frac{N_i}{\gamma}(1-d(A^k,A_i)^2)+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}(1-d(A^k,A_i)^2) > 0$$

$$\frac{N_i}{\gamma}+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)} > 0$$

$$\frac{2\delta N_i}{\gamma}+2N_i+\sigma(\delta+\gamma) > 1.$$

Since $N_i \geq 1$ for all i, the inequality holds, which implies that the steady state payoff is greater in the splitting equilibrium than in the incubation equilibrium, which implies that for δ small enough, total discounted utility in the splitting equilibrium exceeds the total discounted utility in the incubation equilibrium.

The next step is to do a similar comparison of the steady-state splitting payoff and the steady-state cohesive payoff. The difference is given by

$$\frac{1+\sigma(\delta+\gamma)}{\psi(\delta+\gamma)}\left[\frac{N_i}{\gamma}+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}\right] > \frac{1+\sigma(\delta+\gamma)}{\psi(\delta+\gamma)}\left[\frac{d(A(\mu),A_i)\hat{N}}{\gamma}+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}d(A(\mu),A_i)^2\right],$$

which simplifies to

$$\frac{N_i}{\gamma}\left(1-d(A(\mu),A_i)^2)\right)-\frac{N_{-i}}{\gamma}d(A(\mu),A_{-i})d(A(\mu),A_i)+\frac{\sigma(\delta+\gamma)-1}{2(\delta+\gamma)}\left(1-d(A(\mu),A_i)^2\right) > 0$$

$$\left(\frac{2N_i(\delta+\gamma)}{\gamma}+\sigma(\delta+\gamma)-1\right)\left(1-d(A(\mu),A_i)^2\right)-\frac{2N_{-i}(\delta+\gamma)}{\gamma}d(A(\mu),A_i)d(A(\mu),A_{-i}) > 0.$$

Since $\frac{2N_i(\delta+\gamma)}{\gamma}+\sigma(\delta+\gamma)-1 > 0$, the above is monotonically decreasing in $d(A(\mu),A_i)$. At $d(A(\mu),A_i) = 0$, the above inequality holds. At $d(A(\mu),A_i) = 1$, the above is strictly negative. Thus there is a unique cutoff point such that the splitting equilibrium payoff dominates the cohesive equilibrium (and *vice versa*) for player i.

The last step is to compare the incubation payoff to the cohesive payoff. The difference is given by

$$\frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} \left[\frac{d(A^k, A_i)^2 N_i}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{2(\delta + \gamma)} d(A^k, A_i)^2 \right] >$$
$$\frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} \left[\frac{d(A(\mu), A_i)\hat{N}}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{2(\delta + \gamma)} d(A(\mu), A_i)^2 \right],$$

which simplifies to

$$\left(\frac{N_i}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{2(\delta + \gamma)} \right) \left(d(A^k, A_i)^2 - d(A(\mu), A_i)^2 \right) > \frac{N_{-i}}{\gamma} d(A(\mu), A_i) d(A(\mu), A_{-i}).$$

Note that the left-hand side is strictly decreasing in $d(A(\mu), A_i)$ while the right-hand side is strictly increasing in $d(A(\mu), A_i)$. At $d(A(\mu), A_i) = 0$, the inequality is necessarily satisfied since $\frac{N_i}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{2(\delta + \gamma)} > 0$. At $d(A(\mu), A_i) = 1$,

$$\left(\frac{N_i}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{2(\delta + \gamma)} \right) \left(d(A^k, A_i)^2 - 1 \right) > \frac{N_{-i}}{\gamma} d(A(\mu), A_{-i}),$$

which is false since the right-hand side is positive and the left-hand side is negative. Thus by the intermediate value theorem, there exists conditions under which the incubation equilibrium payoff dominates the cohesive equilibrium for player i. $\blacksquare$

A.2.5   Proof of Proposition 1.5

As in proposition 1.4, lifetime utility can be decomposed into two components, a finite-horizon component and an infinite-horizon component. Thus for δ sufficiently small, attention can be restricted to the steady-state. Take $\delta \to 0$. When practicing active strategic membership, a minority individual receives steady-state utility

$$\frac{1 + \sigma(\delta + \gamma)}{2\psi(\delta + \gamma)} \left[\frac{2N_2 d(A^c, A_2)^2}{\gamma} + \left(\frac{\sigma(\delta + \gamma) - 1}{(\delta + \gamma)} \right) d(A^c, A_2)^2 \right].$$

If a minority individual were to deviate, she would do so in one of two ways. She could either leave to join the parent club, or leave and form a new club. The majority individual would only deviate by forming her own club with a perfect match value, as joining the child club gives her a worse match value and fewer fellow contributors, thus decreasing her utility. If a minority individual were to form a new club, then she receives utility

$$\frac{1 + \sigma(\delta + \gamma)}{2\psi(\delta + \gamma)} \left[\frac{2}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{\delta + \gamma} \right].$$

There is no incentive to unilaterally deviate if

$$\frac{2N_2 d(A^c, A_2)^2}{\gamma} + \left(\frac{\sigma(\delta + \gamma) - 1}{(\delta + \gamma)} \right) d(A_c, A_2)^2 \geq \frac{2}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{\delta + \gamma}$$

$$\frac{2(N_2 d(A^c, A_2)^2 - 1)}{\gamma}(\delta + \gamma) \geq [\sigma(\delta + \gamma) - 1] \left(1 - d(A^c, A_2)^2 \right).$$

As $d(A^c, A_2) \to 1$, the right-hand side is strictly positive since $N_2 > 1$ and the left-hand side is zero. Thus for $d(A^c, A_2)$ sufficiently large there is no incentive for a minority individual to unilaterally deviate. By the same argument, no majority individual has the incentive to unilaterally deviate if $d(A(\mu), A_1)$ is sufficiently large.

Now consider the deviation in which the minority individual joins the parent club. The deviation payoff is given by

$$\frac{1 + \sigma(\delta + \gamma)}{2(\delta + \gamma)} \left[\frac{2(d(A(\mu), A_2) + N_1 d(A(\mu), A_1) d(A(\mu), A_2))}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{\delta + \gamma} d(A(\mu), A_2)^2 \right].$$

There is no incentive to unilaterally deviate if

$$\frac{2N_2 d(A^c, A_2)^2}{\gamma} + \left(\frac{\sigma(\delta + \gamma) - 1}{(\delta + \gamma)} \right) d(A^c, A_2)^2$$

$$\geq \frac{2(d(A(\mu), A_2) + N_1 d(A(\mu), A_1) d(A(\mu), A_2))}{\gamma} + \frac{\sigma(\delta + \gamma) - 1}{\delta + \gamma} d(A(\mu), A_2)^2,$$

which when rearranged yields

$$\frac{2}{\gamma}(N_2 d(A^c, A_2)^2 - N_1 d(A(\mu), A_1) d(A(\mu), A_2))$$

$$+ \frac{\sigma(\delta + \gamma) - 1}{\delta + \gamma}(d(A^c, A_2) - d(A(\mu), A_2))(d(A^c, A_2) + d(A(\mu), A_2))$$

$$\geq \frac{2}{\gamma}d(A(\mu), A_2)^2.$$

As $d(A(\mu), A_2) \to 0$, the left-hand side is strictly positive while the left-hand side tends to zero. Thus for $d(A(\mu), A_2)$ small, there is no incentive to deviate. Combining the three conditions completes the proof. $\blacksquare$

A.2.6   Proof of Result (Strategic Admission)

Define $V = \sum_{j(T') \in p}(V_{j(T')T'} - V_{j(T')T''})$ and $W = \sum_{j(T') \in c}(W_{j(T')T'} - W_{j(T')T''})$. Suppose that $|V| \geq |W|$ and $\text{sign}\{V\} \neq \text{sign}\{W\}$. This implies that the interval $[-W, V]$ exists and is not empty. For (i), suppose that $\text{sign}\{V\} = +$. $V > 0$ implies that those who do not split prefer the splitters to incubate. $W < 0$ implies that the splitters prefer to leave immediately rather than incubate. Given that $[W, V]$ exists and is nonempty, there exists a value $\overline{\phi} \in [W, V]$ such that $V - \overline{\phi} = 0$ and $W + \overline{\phi} > 0$. This represents the largest ϕ that the non-splitters are wiling to offer the splitters to incubate for a time of $T' - T''$. The smallest value of ϕ is defined by the conditions $V - \underline{\phi} > 0$ and $W + \underline{\phi} = 0$. This represents the point at which the splitters are indifferent between incubating and not. Thus a transfer $\phi \in [-W, V]$ is sufficient to delay splitting by a time of $T' - T''$. The proof is identical for (ii). $\blacksquare$

A.2.7 Proof of Proposition A.1

A splitting equilibrium exists only if the utility of contributing to a club with $d(A^k, A_i) = 1$ exceeds zero at time $t = 0$. At the outset, the utility from contributing is

$$(N_2 + \sigma) - \frac{\psi}{2}.$$

This value is nonnegative if

$$\psi \leq 2(N_2 - \sigma).$$

∎

A.2.8 Proof of Proposition A.2

Suppose that the minority individuals orchestrate a split, choosing mission A^c. If they orchestrate the split and all minority individuals join, then each minority individual receives a benefit of

$$d(A^c, A_2) \left[(1 - \gamma)\rho(1 - d(A(\mu), A^c))\frac{N_1}{\gamma} \left(1 - e^{-\gamma T^*}\right) + N_2 + \sigma \right] - \frac{\psi}{2}.$$

Since $T^* < T$, the status quo yields zero utility. Thus there is no incentive to deviate if

$$d(A^c, A_2) \left[(1 - \gamma)\rho(1 - d(A(\mu), A^c))\frac{N_1}{\gamma} \left(1 - e^{-\gamma T^*}\right) + N_2 + \sigma \right] \geq \frac{\psi}{2}.$$

Any time T^* such that the above holds constitutes an incubation equilibrium, conditional on such a value existing.

Assigning equality and rearranging the above equation gives

$$d(A^c, A_2)\frac{(1-\gamma)}{\gamma}\rho(1 - d(A(\mu), A^c))N_1 e^{-\gamma T^*}$$

$$= d(A^c, A_2)\left[\frac{(1-\gamma)}{\gamma}\rho(1 - d(A(\mu), A^c))N_1 + N_2 + \sigma\right] - \frac{\psi}{2},$$

which simplifies to

$$e^{-\gamma T^*} = 1 + \frac{\gamma[2d(A^c, A_2)(N_2 + \sigma) - \psi]}{2d(A^c, A_2)(1-\gamma)\rho(1 - d(A(\mu), A^c))N_1}.$$

The left-hand side is monotonically decreasing in T^*, where at $T^* = 0$, the left-hand side equals 1 and as $T^* \to \infty$, the left-hand side tends to zero. Since the right-hand side is independent of T^*, if the right-hand side is bounded between 0 and 1, then there exists a finite T^* at which the split occurs. The right-hand side is weakly less than one if

$$2d(A^c, A_2)(N_2 + \sigma) - \psi \le 0,$$

which simplifies to

$$\sigma \le \frac{\psi}{2d(A^c, A_2)} - N_2.$$

The right-hand side is strictly greater than zero if

$$\frac{\gamma[2d(A^c, A_2)(N_2 + \sigma) - \psi]}{2d(A^c, A_2)(1-\gamma)\rho(1 - d(A(\mu), A^c))N_1} > -1,$$

which simplifies to

$$\sigma > \frac{\psi}{2d(A^c, A_2)} - N_2 - \frac{1-\gamma}{\gamma}\rho(1 - d(A(\mu), A^c))N_1.$$

149

If σ is in the range defined above, then T^* can be found by solving

$$e^{-\gamma T^*} = 1 + \frac{\gamma\left[2d(A^c, A_2)(N_2 + \sigma) - \psi\right]}{2d(A^c, A_2)(1 - \gamma)\rho(1 - d(A(\mu), A^c))N_1}.$$

Thus

$$T^* \geq -\frac{1}{\gamma}\ln\left(1 + \frac{\gamma\left[2d(A^c, A_2)(N_2 + \sigma) - \psi\right]}{2d(A^c, A_2)(1 - \gamma)\rho(1 - d(A(\mu), A^c))N_1}\right).$$

∎

A.2.9 Proof of Proposition A.3

Suppose that the minority individuals orchestrate a split, choosing mission A^c. If they orchestrate the split and all minority individuals join, then each minority individual receives a benefit of

$$d(A^c, A_2)\left\{(1 - \gamma)\rho(1 - d(A(\mu), A^c))\left[\frac{N_1 + N_2}{\gamma}\right.\right.$$
$$\left.\left. - \left(\frac{N_2}{\gamma}e^{-\gamma(T^* - T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_2 + \sigma\right\} - \frac{\psi}{2}.$$

If an individual were to unilaterally deviate, she receives a payoff of

$$d(A(\mu), A_2)\left\{(1 - \gamma)\left[\frac{N_1 + N_2}{\gamma}\right.\right.$$
$$\left.\left. - \left(\frac{N_2}{\gamma}e^{-\gamma(T^* - T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_1 + 1 + \sigma\right\} - \frac{\psi}{2}.$$

Furthermore, the optimal time to split is given by the solution to

$$d(A^c, A_2)\left\{(1 - \gamma)\rho(1 - d(A(\mu), A^c))\left[\frac{N_1 + N_2}{\gamma}\right.\right.$$

$$
\begin{aligned}
&\left. -\left(\frac{N_2}{\gamma}e^{-\gamma(T^*-T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_2 + \sigma\Bigg\} - \frac{\psi}{2} \\
&\geq d(A(\mu), A_2)\Bigg\{(1-\gamma)\left[\frac{N_1+N_2}{\gamma}\right. \\
&\left.\left. -\left(\frac{N_2}{\gamma}e^{-\gamma(T^*-T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_1 + N_2 + \sigma\Bigg\} - \frac{\psi}{2}.
\end{aligned}
$$

Since the right-hand side of the above is strictly greater than the deviation payoff, analysis of this condition is sufficient to pinpoint both existence of an incubation equilibrium and the optimal time to split:

$$
\begin{aligned}
&d(A^c, A_2)\Bigg\{(1-\gamma)\rho(1 - d(A(\mu), A^c))\left[\frac{N_1+N_2}{\gamma}\right. \\
&\left.\left. -\left(\frac{N_2}{\gamma}e^{-\gamma(T^*-T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_2 + \sigma\Bigg\} - \frac{\psi}{2} \\
&\geq d(A(\mu), A_2)\Bigg\{(1-\gamma)\left[\frac{N_1+N_2}{\gamma}\right. \\
&\left.\left. -\left(\frac{N_2}{\gamma}e^{-\gamma(T^*-T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_1 + N_2 + \sigma\Bigg\} - \frac{\psi}{2} \\
&> d(A(\mu), A_2)\Bigg\{(1-\gamma)\left[\frac{N_1+N_2}{\gamma}\right. \\
&\left.\left. -\left(\frac{N_2}{\gamma}e^{-\gamma(T^*-T)} + \frac{N_1}{\gamma}e^{-\gamma T^*}\right)\right] + N_1 + 1 + \sigma\Bigg\} - \frac{\psi}{2}.
\end{aligned}
$$

Rearranging the optimality condition and assigning equality yields

$$
e^{-\gamma T^*} = \frac{(1-\gamma)(N_1 + N_2 + \gamma\sigma)(d(A^c, A_2) - d(A(\mu), A_2)) - \gamma d(A(\mu), A_2)N_1}{(1-\gamma)\left(N_1 + N_2 e^{\gamma T}\right)\left(d(A^c, A_2)\rho(1 - d(A(\mu), A^c)) - d(A(\mu), A_2)\right)}.
$$

Since $\rho(1 - d(A(\mu), A^c)) \neq \frac{d(A(\mu), A^c)}{d(A^c, A_2)}$, there is no dividing by zero issue. The left-hand side is bounded between $(0, 1]$. Thus for a finite T^* to exist, the right-hand side must fall in the same interval. Let $\tilde{\sigma}$ satisfy

$$
\frac{(1-\gamma)N_1 + N_2 + \gamma\tilde{\sigma})(d(A^c, A_2) - d(A(\mu), A_2)) - \gamma d(A(\mu), A_2)N_1}{(1-\gamma)\left(N_1 + N_2 e^{\gamma T}\right)\left(d(A^c, A_2)\rho(1 - d(A(\mu), A^c)) - d(A(\mu), A_2)\right)} = 1
$$

and $\underline{\sigma}$ satisfy

$$\frac{(1-\gamma)N_1 + N_2 + \gamma\underline{\sigma})(d(A^c, A_2) - d(A(\mu), A_2)) - \gamma d(A(\mu), A_2)N_1}{(1-\gamma)(N_1 + N_2 e^{\gamma T})(d(A^c, A_2)\rho(1 - d(A(\mu), A^c)) - d(A(\mu), A_2))} = 0.$$

If $\sigma \in (\underline{\sigma}, \tilde{\sigma}]$, then by the intermediate value theorem, a lower bound for T^* exists and is given by the value presented in the proposition. ∎

A.3   Derivations of Dynamic Contribution Functions

I derive the individual contribution schedules using Pontryagin's maximum principle (PMP). I then show, for the contribution schedule under the cohesive equilibrium, that the solution found via Pontryagin's maximum principle is identical to a solution found via solving the system of Hamilton-Jacobi-Bellman (HJB) equations. In particular, the solution found via PMP is identical to a solution to the HJB system in linear strategies $(x(t) = a + bC(t)$ for some $a, b)$. Since each equilibrium problem varies only in its boundary conditions, match values, and club size I omit the Hamilton-Jacobi-Bellman derivation, as they follow from a procedure identical to the one illustrated under the cohesive equilibrium.

A.3.1   Cohesive Contribution Schedule

Each individual i faces the optimization problem

$$\max_{x_i} \left\{ d(A(\mu), A_i)[C + \sigma x_i] - \frac{\psi}{2}x_i^2 + \lambda_i \left(\sum_{j=1}^{N} x_j - \gamma C \right) \right\},$$

where λ_i is the shadow price of a marginal increase in installed base for individual i. The first-order necessary condition (which is also sufficient due to the concavity of the instantaneous

utility function) and adjoint equation are given by

$$\lambda_i = \psi x_i^* - \sigma d(A(\mu), A_i)$$

$$(\delta + \gamma)\lambda_i - \dot{\lambda}_i = d(A(\mu), A_i)$$

Substituting the first-order condition into the adjoint equation yields

$$\psi(\delta + \gamma)x_i^* - \psi\dot{x}_i^* = (1 + \sigma(\delta + \gamma))d(A(\mu), A_i).$$

The particular solution to the above first-order differential equation is

$$\bar{x}_i = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A(\mu), A_i).$$

The homogeneous solution is given by

$$x_i(t) = \Omega e^{(\delta + \gamma)t},$$

where Ω is a constant to be determined. Thus the general solution is

$$x_i(t) = \bar{x}_i + \Omega e^{(\delta + \gamma)t}.$$

Since the time horizon is infinite, Pontryagin's maximum principle requires that the transversality condition

$$\lim_{t \to \infty} e^{-\delta t} \lambda(t) = 0$$

is satisfied. Making the appropriate substitution implies that

$$\lim_{t \to \infty} \left\{ \frac{1 + \sigma(\delta + \gamma)}{\delta + \gamma} e^{-\delta t} + \psi\Omega e^{\gamma t} - \psi\sigma d(A(\mu), A_i)e^{-\delta t} \right\} = 0$$

153

$$\lim_{t \to \infty} \psi \Omega e^{\gamma t} = 0.$$

Since $\gamma > 0$, it must be that $\Omega = 0$. Therefore, as stated in the text $x_i(t) = \overline{x}_i$.

Now consider the Hamilton-Jacobi-Bellman formulation. Since the objective function is autonomous, each individual i faces the condition

$$\delta V^i = \max_{x_i} \left\{ d(A(\mu), A_i)[C + \sigma x_i] - \frac{\psi}{2} x_i^2 + V_C^i \left(\sum_{j=1}^{N} x_j - \gamma C \right) \right\},$$

Where V^i is individual i's continuation value function and $V_C^i = \frac{dV^i}{dC}$. Evaluating the right-hand side implies that

$$x_i^* = \psi^{-1}(\sigma d(A(\mu), A_i) + V_C^i).$$

Since the instantaneous utility function is linear in C, I propose a solution that is linear in C. Let

$$V^i = \Omega_1^i + \psi \Omega_2^i C,$$

$$V_C^i = \psi \Omega_2^i,$$

where Ω_1^i and Ω_2^i are individual-specific constants to be determined. Substituting the proposed value function and its derivative into the Hamilton-Jacobi-Bellman yields

$$\delta \Omega_1^i + \delta \psi \Omega_2^i C = d(A(\mu), A_i)[C + \psi^{-1} \sigma^2 d(A(\mu), A_i) + \sigma \Omega_2^i]$$
$$- \frac{\psi}{2} (\psi^{-1} \sigma d(A(\mu), A_i) + \Omega_2^i)^2 + \psi \Omega_2^i \left(\psi^{-1} \sigma \hat{N} + \sum_{j=1}^{N} \Omega_2^j - \gamma C \right)$$

The above must hold for all C. Isolating terms with a "C" on the right-hand side yields

$$[d(A(\mu), A_i) - \psi\gamma\Omega_2^i]C$$

Therefore it follows that

$$\delta\psi\Omega_2^i = d(A(\mu), A_i) - \psi\gamma\Omega_2^i$$
$$\Omega_2^i = \frac{d(A(\mu), A_i)}{\psi(\delta + \gamma)}.$$

Substituting Ω_i^2 into the expression for x_i^* yields

$$x_i^* = \left(\frac{\sigma}{\psi} + \frac{1}{\psi(\delta + \gamma)}\right) d(A(\mu), A_i)$$
$$x_i^* = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A(\mu), A_i),$$

which corresponds to the exact value found using Pontryagin's method. Now to find the installed base dynamic, recall that

$$\dot{C}(t) = \sum_{j=1}^{N} x_j(t) - \gamma C(t).$$

Substituting values found using either method yields

$$\dot{C}(t) + \gamma C(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} \hat{N}.$$

The particular solution to this first-order differential equation is given by

$$\overline{C} = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)} \hat{N}.$$

155

The homogeneous solution is

$$C(t) = \Omega e^{-\gamma t},$$

where Ω is a constant to be determined. The general solution is given by the sum of the particular and homogeneous solutions:

$$C(t) = \overline{C} + \Omega e^{-\gamma t}.$$

Since $C(0) = 0$, it follows that $\Omega = -\overline{C}$. Thus $C(t)$ is given by the function in the text. The indirect utility function follows immediately from integrating the objective function, evaluated at these equilibrium values.

A.3.2 Splitting Contribution Schedule

This process closely resembles the cohesive structure, only with $d(A^k, A_i) = 1$ for all $i \in k$, $k = p, c$. Each individual i faces the problem

$$\max_{x_i^k} \left\{ C^k + \sigma x_i^k - \frac{\psi}{2} \left(x_i^k \right)^2 + \lambda_i \left(\sum_{j \in k} x_j^k - \gamma C^k \right) \right\},$$

which has the first-order necessary and adjoint conditions

$$\sigma - \psi x_i^{k*} + \lambda_i = 0$$
$$(\delta + \gamma)\lambda_i - \dot{\lambda}_i = 1.$$

Substituting the first-order necessary condition and its time derivative into the adjoint condition yields the first-order differential equation

$$\psi(\delta + \gamma)x_i^{k*} - \psi\dot{x}_i^{k*} = 1 + \sigma(\delta + \gamma).$$

Akin to the cohesive case, the general solution to this differential equation is of the form

$$x_i^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} + \Omega e^{(\delta+\gamma)t},$$

where Ω is a constant to be determined. The transversality condition $\lim_{t\to\infty} e^{-\delta t}\lambda(T) = 0$ implies that $\Omega = 0$, so

$$x_i^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)}$$

Substituting this value into the dynamic yields the first-order differential equation

$$\dot{C}^k + \gamma C^k = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)}N_k$$

The general solution to this differential equation is

$$C^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}N_k + \Omega e^{-\gamma t},$$

where Ω is pinned down by the initial condition $C^k(0) = 0$. Thus $\Omega = -\frac{1+\sigma(\delta+\gamma)}{\gamma\psi(\delta+\gamma)}N_k$.

A.3.3 Post-Split Incubation Contributions

Suppose that a split has occurred at some time $T^* > 0$. Each individual i's optimization problem is represented by the current value Hamiltonian

$$d(A^k, A_i)\left[C + \sigma x_i^k\right] - \frac{\psi}{2}\left(x_i^k\right)^2 + \lambda_i\left(\sum_{j \in k} x_j^k - \gamma C\right).$$

Following a procedure identical to the previous derivations,

$$x_i^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} d(A^k, A_i).$$

Thus the installed base dynamic is governed by

$$\dot{C}^k(t) + \gamma C^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)}\hat{N}^k.$$

The general solution to this first-order differential equation is given by

$$C^k(t) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}\hat{N}^k + \Omega e^{-\gamma t},$$

where Ω is a constant to be determined. The initial condition is given by $C^k(T^*) = (1 - \gamma)\alpha_k C(T^*)$, where $C(T^*)$ is the accumulated pre-split installed base. Substituting the initial condition yields

$$C^k(T^*) = (1 - \gamma)\alpha_k C(T^*) = \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}\hat{N}^k + \Omega e^{-\gamma T^*}$$

$$\Omega = \left[(1 - \gamma)\alpha_k C(T^*) - \frac{1 + \sigma(\delta + \gamma)}{\gamma\psi(\delta + \gamma)}\hat{N}^k\right]e^{\gamma T^*}.$$

Making this substitution yields the result in the text.

A.3.4 Pre-Split Incubation Contributions

Each individual i's optimization problem is represented by the current value Hamiltonian

$$d(A(\mu), A_i) [C + \sigma x_i] - \frac{\psi}{2} x_i^2 + \lambda_i \left(\sum_{j=1}^{N} x_j - \gamma C \right)$$

The first-order necessary and adjoint conditions are given by

$$d(A(\mu), A_i)\sigma - \psi x_i + \lambda_i = 0$$

$$(\delta + \gamma)\lambda_i - \dot{\lambda}_i = d(A(\mu), A_i).$$

Since nether $C(t)$ nor $C(0)$ enter into either condition, the Markov property is satisfied, as is subgame perfection. Substitution of the first-order necessary condition into the adjoint equation yields the first-order differential equation

$$\psi(\delta + \gamma)x_i - \psi\dot{x}_i = [1 + \sigma(\delta + \gamma)]d(A(\mu), A_i).$$

The general solution to this differential equation is given by

$$\underbrace{\frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)}d(A(\mu), A_i)}_{\equiv \bar{x}_i} + \Omega_i e^{(\delta + \gamma)t},$$

where Ω_i is a constant to be determined. Since this is a finite horizon problem, the transversality condition requires that $\lambda(T^*) = \frac{dB_i(C(T^*))}{dC(T^*)}$, or equivalently

$$\psi x_i(T^*) - \sigma d(A(\mu), A_i) = \frac{\alpha^k(1 - \gamma)d(A^k, A_i)}{\delta + \gamma},$$

where k is the club that i belongs to post-split. By substitution, it follows that

$$\Omega_i = \left[\frac{(1 - \gamma)\alpha^k d(A^k, A_i) - d(A(\mu), A_i)}{\psi(\delta + \gamma)} \right] e^{-(\delta+\gamma)(T^*-t)}.$$

Note that

$$\sum_{j=1}^{N} x_j(t) = \sum_{j=1}^{N} \overline{x}_j + \left[\frac{(1 - \gamma) \sum_k \alpha^k \hat{N}^k - \hat{N}}{\psi(\delta + \gamma)} \right] e^{-(\delta+\gamma)(T^*-t)}.$$

Substituting this value into the dynamic yields

$$\dot{C} = \frac{1 + \sigma(\delta + \gamma)}{\psi(\delta + \gamma)} \hat{N} + \left[\frac{(1 - \gamma) \sum_k \alpha^k \hat{N}^k - \hat{N}}{\psi(\delta + \gamma)} \right] e^{-(\delta+\gamma)(T^*-t)} - \gamma C.$$

The homogeneous solution is given by $\Omega e^{-\gamma t}$ and the particular solution is of the form $C(t) = a + b e^{-(\delta+\gamma)(T^*-t)}$, for unknown constants Ω, a, and b. a and b are found by using the fact that

$$\dot{C} + \gamma C = \sum_{j=1}^{N} x_j$$

$$(\delta + 2\gamma)b e^{-(\delta+\gamma)(T^*-t)} + \gamma a = \sum_{j=1}^{N} \overline{x}_j + \left[\frac{(1 - \gamma) \sum_k \alpha^k \hat{N}^k - \hat{N}}{\psi(\delta + \gamma)} \right] e^{-(\delta+\gamma)(T^*-t)}.$$

Thus

$$a = \frac{1}{\gamma} \sum_{j=1}^{N} \overline{x}_j$$

$$b = \frac{1}{(\delta + 2\gamma)} \left[\frac{(1 - \gamma) \sum_k \alpha_k \hat{N}_k - \hat{N}}{\psi(\delta + \gamma)} \right].$$

The general solution is thus $C(t) = \Omega e^{-\gamma t} + a + b e^{-(\delta+\gamma)(T^*-t)}$. Using the condition $C(0) = 0$ implies that

$$0 = \Omega + a + b e^{-(\delta+\gamma)T^*}$$

$$\Omega = -a - b e^{-(\delta+\gamma)T^*},$$

so

$$C(t \leq T^*) = a \left(1 - e^{-\gamma t}\right) + b e^{-(\delta+\gamma)T^*} \left[e^{(\delta+\gamma)t} - e^{-\gamma t}\right].$$

A.4 Numerical Analysis of Active Strategic Membership

A.4.1 Procedure

The numerical algorithm proceeds as follows:

1. Create a grid/grids for the parameter/s to be varied

2. At a given value of the parameter/s on the grid/s, evaluate the pre-split and post-split indirect utilities as functions of T^* using the values given in table A.1.

3. Evaluate the indirect utilities under the cohesive equilibrium and the splitting equilibrium using the values given in table A.1.

4. Find the equilibrium T^* by minimizing the negative post-split indirect utility.

5. Evaluate and the pre- and post-split indirect utilities at the equilibrium T^*, then sum the two values to find the total indirect utility.

6. Evaluate the deviation payoff at the given values and the given T^*.

7. If the equilibrium payoff is greater than the deviation payoff, record the value of T^* as an equilibrium.

8. If the given T^* is an equilibrium, compare the equilibrium payoffs and record payoff dominance as the difference between the indirect utility under the incubation equilibrium and the maximum of the indirect utility under the splitting and cohesive equilibria.

9. Move to the next parameter combination on the grid and repeat until all parameter values on the grid have been considered.

A.4.2 Parameter Values

Parameter	Value when fixed	Grid
γ	0.025	$\{0.025, 0.05, 0.075, \ldots, 1\}$
σ	1	$\{0, 30, 60, \ldots, 3000\}$
α^p	1	$\{0, 0.05, 0.1, \ldots, 1\}$
α^c	0.8	$\{0, 0.05, 0.1, \ldots, 1\}$
$d(A(\mu), A_1)$	0.9	N/A
$d(A(\mu), A_2)$	0.3	$\{0.01, 0.02, \ldots, d(A(\mu), A_1)\}$
$d(A^c, A_2)$	0.8	$\{d(A(\mu), A_2), d(A(\mu), A_2) + 0.01, \ldots, 1\}$
N_1	50	N/A
N_2	20	N/A
ψ	10	N/A

Table A.1: Summary of parameter values

Appendix B

Appendix to Chapter 2

B.1 Proof of Proposition 2.1

The welfare-maximizing problem is given by

$$
\max_{x_j} \sum_{k=1}^{N} u\left(X, x_k; 0\right)
$$

for each j. For each contributing individual j, the first-order necessary condition for this problem is given by

$$
\frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X} \frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_j} = -\frac{1}{K} \frac{\partial v\left(\tilde{x}; \sigma\right)}{\partial x_j} \tag{B.1}
$$

for each j, where

$$
K = \begin{cases} N & \text{if public good} \\ M & \text{if club good.} \end{cases}
$$

Case 1: *public good.* Taking the limit of both sides as N tends to infinity requires that

$$\lim_{N\to\infty} \frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X} \frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_j} = -\lim_{N\to\infty} \frac{1}{N} \frac{\partial v\left(\tilde{x};\sigma\right)}{\partial x_j}.$$

By Assumption 2.3, $\frac{\partial v(x;\sigma)}{\partial x_j} \leq 0$ for all x_j when $\sigma = 0$. Thus the above holds if and only if $\lim_{N\to\infty} \frac{\partial Q(g(\tilde{\mathbf{x}}))}{\partial X} \frac{\partial g(\tilde{\mathbf{x}})}{\partial x_j} = 0$, which by Assumption 2.2, occurs only if $\lim_{N\to\infty} g\left(\tilde{\mathbf{x}}\right) = \hat{X}$.

Case 2: *club good.* Taking the limit of both sides as $M \to \infty$ requires that

$$\lim_{M\to\infty} \frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X} \frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_j} = -\lim_{M\to\infty} \frac{1}{M} \frac{\partial v\left(\tilde{x};\sigma\right)}{\partial x_j}.$$

The result follows from an identical argument to that presented in case 1.

The second statement follows from the fact that $\frac{\partial v(x;\sigma)}{\partial x} < 0$, which implies that the right-hand side of (B.1) is positive. Thus the left-hand side must also be positive, which holds only for $\tilde{X} < \hat{X}$. ∎

B.2 Proof of Proposition 2.2

Set $\sigma = 0$ and suppose that $M < N$ individuals affiliate. Assumption 2.3 requires that $\frac{\partial v(x^*;0)}{\partial x} < 0$ for $x^* > 0$. Thus for an interior solution, it must be that

$$\frac{\partial Q\left(g\left(\mathbf{x}^*\right)\right)}{\partial X} \frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;0\right)}{\partial x_i} = 0 \tag{B.2}$$

for $i = 1, \ldots, M$ with $\frac{\partial Q(g(\mathbf{x}^*))}{\partial X} \frac{\partial g(\mathbf{x}^*)}{\partial x_i} > 0$. For there to exist a Nash equilibrium with $M < N$, (2.3) and (2.4) require that for each i,

$$Q\left(g\left(\mathbf{x}(M,0)\right)\right) + v\left(x(M,0);0\right) \geq \max\left\{\mathbf{1}\left\{\text{public good}\right\} Q\left(g\left(\mathbf{x}'\right)\right), 0\right\}$$

$$\max \left\{ \mathbf{1} \left\{ \text{public good} \right\} Q\left(g\left(\mathbf{x}(M,0) \right) \right), 0 \right\} \geq Q\left(g\left(\mathbf{x}'' \right) \right) + v\left(x''; 0 \right).$$

Without loss of generality, consider a deviation by individual $i = M + 1$. Inequality (2.4) can be rewritten as

$$\max \left\{ \mathbf{1} \left\{ \text{public good} \right\} Q\left(g\left(\mathbf{x} \right) \right), 0 \right\} + v(0; 0) \geq \max_{x_{M+1}} Q\left(g\left(\mathbf{x} \right) \right) + v\left(x_{M+1}; 0 \right),$$

where $\mathbf{x}$ is an $N \times 1$ vector given by $\langle x(M,0), \ldots, x(M,0), x_{M+1}, 0, \ldots, 0 \rangle$ and $\mathbf{x}(M,0)$ is equal to $\mathbf{x}$ with $x = 0$. It follows that (2.4) can only be satisfied with equality, which means x'' must equal zero. Differentiating the right-hand side and evaluating at $x = 0$ yields

$$\frac{\partial Q\left(g\left(\mathbf{x}^* \right) \right)}{\partial X} \frac{\partial g\left(\mathbf{x}^* \right)}{\partial x_{M+1}} > 0,$$

a contradiction. Thus there can be no equilibrium with $M < N$ as at least one non-investing individual has a profitable unilateral deviation. As (2.4) is not defined at $M = N$ (no such deviation exists), it remains to be shown that at $M = N$, (2.3) is satisfied. Given $M = N$, (2.3) can be rewritten as

$$\max_x \left\{ Q\left(g\left(\mathbf{x} \right) \right) + v\left(x; 0 \right) \right\} \geq \max \left\{ \mathbf{1} \left\{ \text{public good} \right\} Q\left(g\left(\mathbf{x}' \right) \right), 0 \right\} + v(0; 0),$$

where $\mathbf{x}$ is an $N \times 1$ vector with $N - 1$ elements equaling $x(N, 0)$ and the Nth element equaling x, chosen by agent $i = N$ (without loss of generality). $\mathbf{x}'$ is equal to $\mathbf{x}$ with $x = 0$. As $\frac{\partial Q(g(\mathbf{x}'))}{\partial X} \frac{\partial g(\mathbf{x}')}{\partial x_N} > 0$, the inequality is necessarily satisfied so no investing individual has a unilateral incentive to deviate. Thus there exists a Nash equilibrium with N investors. The relevant first-order necessary condition is now

$$\frac{\partial Q\left(g\left(\mathbf{x}(N,0) \right) \right)}{\partial X} \frac{\partial g\left(\mathbf{x}(N,0) \right)}{\partial x_i} + \frac{\partial v\left(x(N,0); 0 \right)}{\partial x_i} = 0, \quad \forall\, i. \tag{B.3}$$

To prove the last statement, suppose to the contrary that $x^* \to z > 0$ as $N \to \infty$. This positive convergence implies that $X^* \to \infty$. It follows that $\frac{\partial Q(\infty)}{\partial X}\frac{\partial g(\mathbf{x}^*)}{\partial x} < 0$ by Assumption 2.2 and $\frac{\partial v(z;0)}{\partial x} < 0$, contradicting (B.3). Thus $x^* \to 0$ as $N \to \infty$. As x^* approaches zero and $\lim_{N \to \infty} \frac{\partial v(x^*;0)}{\partial x} = 0$, (B.3) holds only if $\lim_{N \to \infty} \frac{\partial Q(g(\mathbf{x}^*))}{\partial X}\frac{\partial g(\mathbf{x}^*)}{\partial x} = 0$, which by Assumption 2.2 is true when $\lim_{N \to \infty} X^* = \hat{X}$.

Lastly to prove that $X^* < \tilde{X}$ for all finite $N > 1$, note that by Assumption 2.3, $\frac{\partial v(x;0)}{\partial x} < 0$ for all $x > 0$. Evaluating the LHS of (B.3) at $x = x(N,0)$ yields

$$\frac{\partial Q\left(g\left(\mathbf{x}(N,0)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}.$$

Since $\frac{\partial v(x(N,0);0)}{\partial x} < 0$,

$$\frac{\partial v\left(x(N,0);0\right)}{\partial x} < \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}.$$

From (B.3), it follows that

$$\frac{\partial Q\left(g\left(\mathbf{x}(N,0)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{1}{N}\frac{\partial v\left(x(N,0);0\right)}{\partial x}$$
$$> \frac{\partial Q\left(g\left(\mathbf{x}(N,0)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(N,0)\right)}{\partial x} + \frac{\partial v\left(x(N,0);0\right)}{\partial x} = 0,$$

so $\tilde{x} > x(N,0)$ and $\tilde{X} > X^*$ for all finite $N > 1$. If $N = 1$, then $x(1,0) = \tilde{x}$, so $X^* = \tilde{X}$ by default. $\blacksquare$

B.3   Proof of Lemma 2.1

The first step is to show that $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$. Fix M and recall that, for each investor $i = 1,\ldots,M$, it must be that

$$\frac{\partial Q\left(g\left(\mathbf{x}^*\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} = 0 \tag{B.4}$$

holds in equilibrium. Differentiating (B.4) with respect to σ yields

$$M\left[\underbrace{\frac{\partial^2 Q(g(\mathbf{x}(M,\sigma)))}{\partial X^2}\left(\frac{\partial g(\mathbf{x}(M,\sigma))}{\partial x_i}\right)^2}_{A} + \underbrace{\frac{\partial Q(g(\mathbf{x}(M,\sigma)))}{\partial X}\frac{\partial^2 g(\mathbf{x}(M,\sigma))}{\partial x_i{}^2}}_{B}\right]\frac{\partial x(M,\sigma)}{\partial \sigma}$$

$$+ \underbrace{\frac{\partial^2 v(x(M,\sigma);\sigma)}{\partial x_i{}^2}}_{C}\frac{\partial x(M,\sigma)}{\partial \sigma} = -\frac{\partial^2 v(x(M,\sigma);\sigma)}{\partial x_i\partial\sigma} \tag{B.5}$$

The RHS of (B.5) is strictly negative by Assumption 2.3, so $\frac{\partial x(M,\sigma)}{\partial \sigma} \neq 0$ for any $\sigma \geq 0$. Therefore $x(M,\sigma)$ is monotonic in σ. Suppose that $\frac{\partial x(M,\sigma)}{\partial \sigma} < 0$. A and C are negative by Assumptions 2.2 and 2.3, respectively. Thus it must be that $A + B > 0$. The second term in B is non-positive by Assumption 2.1, so the first term must be strictly negative for all σ including $\sigma \to 0$, which is false by Proposition 2.2. Thus $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$.

Since $\frac{\partial x(M,\sigma)}{\partial \sigma} > 0$, $g(\mathbf{x}(M,\sigma))$ is also increasing in σ by Assumption 2.1. At $\sigma = 0$, $\frac{\partial Q(g(\mathbf{x}(M,0)))}{\partial X} > 0$ by Assumption 2.2, so there must exist a value $\bar{\sigma}(M) > 0$ for every $M \geq 1$ such that for all $\sigma < \bar{\sigma}(M)$, condition $C2.1$ holds and for all $\sigma > \bar{\sigma}(M)$, condition $C2.2$ holds.

Proving that $\bar{\sigma}(M)$ is decreasing in M first requires showing that X^* is increasing in M. Fix σ and suppose, to the contrary, that X^* is decreasing in M on any subset of $\mathbb{N}\setminus\{0\}$. By (B.4),

$$\frac{\partial Q\left(g\left(\mathbf{x}(M,\sigma)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(M,\sigma)\right)}{\partial x} - \frac{\partial Q\left(g\left(\mathbf{x}(M+1,\sigma)\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}(M+1,\sigma)\right)}{\partial x}$$
$$= \frac{\partial v\left(x(M+1,\sigma);\sigma\right)}{\partial x} - \frac{\partial v\left(x(M,\sigma);\sigma\right)}{\partial x}. \quad \text{(B.6)}$$

As X^* is decreasing in M, x^* must also be decreasing in M, so the RHS of (B.6) is strictly positive. Thus the LHS must also be positive. However, note that by concavity, $\frac{\partial Q(g(\mathbf{x}(M,\sigma)))}{\partial X} < \frac{\partial Q(g(\mathbf{x}(M+1,\sigma)))}{\partial X}$ and by weak concavity, $0 < \frac{\partial g(\mathbf{x}(M,\sigma))}{\partial x} \leq \frac{\partial g(\mathbf{x}(M+1,\sigma))}{\partial x}$, which implies the LHS is negative, a contradiction. Thus X^* is increasing in M.

Now to show that $\overline{\sigma}(M)$ is decreasing in M, suppose that $g(\mathbf{x}(M,\sigma))$ is such that $\frac{\partial Q(g(\mathbf{x}(M,\sigma)))}{\partial X}$ is nonnegative and $\frac{\partial Q(g(\mathbf{x}(M+1,\sigma)))}{\partial X}$ is strictly negative. It follows that $\sigma \leq \overline{\sigma}(M)$ but $\sigma > \overline{\sigma}(M+1)$, which implies that $\overline{\sigma}(M)$ is decreasing in M. To show that as $M \to \infty$, $\overline{\sigma}(M) \to 0$ recall that by Proposition 2.2, $x^* \to 0$ as $M \to \infty$, while $\lim_{M\to\infty} \frac{\partial v(x(M,\sigma))}{\partial x} > 0$ for all $\sigma > 0$. The result follows from $C2.2$. ∎

B.4   Proof of Lemma 2.2

Lemma 2.2 follows immediately from Kakutani's fixed point theorem and the extensions in Debreu (1952), Glicksberg (1952), and Fan (1952). ∎

B.5   Proof of Proposition 2.3

The result follows from the argument in the text. ∎

B.6 Proof of Theorem 2.1

The proof proceeds in two cases. Firstly suppose that $\sigma < \overline{\sigma}(N)$. By Lemma 2.1, $C2.1$ applies, and the result follows immediately from an identical argument to that in Proposition 2.2.

Next suppose that $\sigma > \overline{\sigma}(N)$, so by Lemma 2.1, $C2.2$ applies for $M > \overline{\sigma}^{-1}(\sigma)$. The first-order necessary conditions for a maximum are given by

$$\frac{\partial Q\left(g\left(\mathbf{x}^*\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} = 0, \quad \forall\, i = 1,\ldots,M \tag{B.7}$$

while the welfare maximizing conditions with M investors is given by

$$\frac{\partial Q\left(g\left(\tilde{\mathbf{x}}\right)\right)}{\partial X}\frac{\partial g\left(\tilde{\mathbf{x}}\right)}{\partial x_i} + \frac{1}{K}\frac{\partial v\left(\tilde{x};\sigma\right)}{\partial x_i} = 0, \quad \forall\, i \leq M. \tag{B.8}$$

Given (B.7),

$$\frac{\partial Q\left(g\left(\mathbf{x}^*\right)\right)}{\partial X}\frac{\partial g\left(\mathbf{x}^*\right)}{\partial x_i} + \frac{1}{K}\frac{\partial v\left(x^*;\sigma\right)}{\partial x_i} < 0$$

by Lemma 2.1. Thus $x^* > \tilde{x}$ and by extension, $X^* > \tilde{X}$. Both being greater than $\hat{X}$ follows immediately. $\tilde{X} \to \hat{X}$ as $K \to \infty$ follows from taking the limit of (B.8) as $K \to \infty$. By Proposition 2.2, $x^* \to 0$ as $K \to \infty$. Since $\lim_{x\to 0^+} \frac{\partial v(x;\sigma)}{\partial x} > 0$ for $\sigma > 0$, $\frac{\partial Q(g(\mathbf{x}(\infty,\sigma)))}{\partial x} < 0$, which implies that $X^* > \hat{X}$ for all $\sigma > 0$. ∎

B.7 Proof of Corollary 2.1

The result follows immediately from Assumption 2.3 and Theorem 2.1. ∎

B.8 Proof of Theorem 2.2

Suppose that $\hat{M} > \overline{\sigma}^{-1}(\sigma)$ and there exists an integer $\hat{M} - 1$ such that

$$\overline{\sigma}^{-1}(\sigma) < \frac{\left(\hat{M} - 1\right) + \hat{M}}{2},$$

so producing the good with $\hat{M} - 1$ investors leads to a good with a higher quality than having $\hat{M}$ investors. It immediately follows that if (2.10) holds, then $\hat{M}$ leads to greater aggregate welfare than $\hat{M} - 1$. ∎

B.9 Proof of Corollary 2.2

The result follows from Lemma 2.1 and Theorem 2.2. ∎

B.10 Proof of Proposition 2.4

In the both the public and club good case, free-riders/non-investors receive zero utility. However, each investor's first-order necessary condition is now given by

$$\frac{\partial Q\left(g\left(\mathbf{x}^{**}\right)\right)}{\partial X} \frac{\partial g\left(\mathbf{x}^{**}\right)}{\partial x} + \frac{M}{N} \frac{\partial v\left(x_i^{**};\sigma\right)}{\partial x} = 0.$$

As $\frac{M}{N} < 1$, it immediately follows from Lemma 2.2 that $x(M,\sigma) > x^{**}$ so $X(M,\sigma) > X^{**}$, which implies that $Q(g(\mathbf{x}(M,\sigma))) < Q\left(g\left(\mathbf{x}^{**}\right)\right)$. Because x^{**} requires that $u(X^{**},x^{**};\sigma) < u(X(M,\sigma),x(M,\sigma);\sigma)$ (otherwise a deviation mimicking the outcome of the strategy $x(M,\sigma)$ exists), each investors utility is strictly greater. Therefore both quality and welfare improve. ∎

B.11 Proof of Theorem 2.3

Suppose the pricing scheme described in the text can be implemented. The objective function becomes identical to the public good case, so Theorem 2.2 applies. When the pricing scheme cannot be implemented, the objective function is given by

$$\max_{M} \; M \left[Q \left(g \left(\mathbf{x}(M, \sigma) \right) \right) + v \left(x(M, \sigma); \sigma \right) \right].$$

This function is maximized when $Q \left(g \left(\mathbf{x}(M, \sigma) \right) \right) + v \left(x(M, \sigma); \sigma \right)$ is maximized, which occurs under $C2.3$. The result follows. ■

B.12 Proof of Theorem 2.4

Suppose that $\mu \in [0, 1)$ and $\underline{Q} = -\infty$. As u_i is strictly concave and by Assumption 2.3, strictly increasing in σ, the indirect utility of each investor is strictly increasing in σ. The utility of each free-rider is non-monotonic in σ. As σ grows large, $Q \left(g \left(\mathbf{x}(M, \sigma) \right) \right)$ experiences non-monotonic spikes, where Q will decrease. By (2.3), there exists a finite set of cutoff values $\sigma_1, \ldots, \sigma_Z$, where at each σ_z, an individual stops contributing. Between each cutoff value, Q is decreasing with a sharp jump at each cutoff point until there is a fixed number of investors, which by By (2.3) and By (2.4) is at least one. As Q decreases, free-riders minimum payoff is zero given the availability of the outside option. Thus aggregate payoffs are bounded below by zero and can be made arbitrarily large by increasing σ and benefiting the investors, while all non-investors receive zero utility.

Now suppose that $\mu = 1$ and $\underline{Q} = -\infty$. If the good is a club good, then the choice of σ is arbitrary because the policy function equals zero for all σ. If the good is a public good, then $\mu = 1$ implies quality maximization, which occurs under $C2.3$, which implies $\hat{\sigma} = \overline{\sigma}(M^*(N))$.

The last statement follows immediately from the argument above coupled with the added constraint on $\underline{Q}$. Because Q is strictly decreasing for σ large, imposing a minimum quality constraint requires that the set of feasible optimal parameters $\hat{\sigma}$ is compact. ∎

Appendix C

Appendix to Chapter 3

C.1   Proof of Lemma 3.1

To the contrary, suppose not. This implies that there exists three individuals x_1, x_2 and x_3, with $0 \leq x_1 < x_2 < x_3 \leq 1$ such that

$$v(x_1, t^m) - p - \ell(x_1, t^m) > 0$$

$$v(x_2, t^m) - p - \ell(x_2, t^m) < 0$$

$$v(x_3, t^m) - p - \ell(x_3, t^m) > 0.$$

By the intermediate value theorem, there must exist some $w \in (x_1, x_2)$ such that

$$v(w, t^m) - p - \ell(w, t^m) = 0.$$

Similarly, there must be some $z \in (x_2, x_3)$ such that

$$v(z, t^m) - p - \ell(z, t^m) = 0.$$

This implies that there exists some point $y \in (w, z)$ such that

$$v_x(y, t^m) - \ell_x(y, t^m) = 0$$

and

$$v_{xx}(y, t^m) - \ell_{xx}(y, t^m) > 0,$$

a contradiction since $v(x, t^m) - \ell(x, t^m)$ is strictly concave in x. Thus the support of the demand is a non-degenerate convex subset of $[0, 1]$. ∎

C.2 Proof of Lemma 3.2

Recall that $v(x, t) - \ell(x, t) - p$ is strictly concave and single peaked. Set $t = t^m$. For prices p and p', $p \neq p'$, the difference between $v(x, t^m) - \ell(x, t^m) - p$ and $v(x, t^m) - \ell(x, t^m) - p'$ is constant and equal to $p' - p$. At $p = 0$, $v(x, t^m) - \ell(x, t^m) > 0$ for a subset of individuals. Thus for a fixed t, it must be that the indifferent individual on each side of t must (weakly) move closer to t as p increases. The following diagram graphically illustrates the proof.

∎
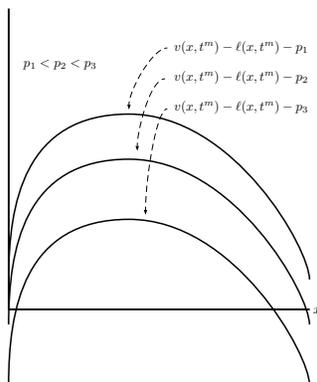
C.3   Proof of Proposition 3.1

The objective function can be written as

$$\max_{p \in \mathbb{R}_+} p \int_{\underline{x}(p,t)}^{\overline{x}(p,t)} g(x) dx.$$

The first-order necessary condition is

$$\int_{\underline{x}(p^*,t)}^{\overline{x}(p^*,t)} g(x) dx + p^* \left[g(\overline{x}(p^*,t)) \frac{\partial \overline{x}(p^*,t)}{\partial p} - g(\underline{x}(p^*,t)) \frac{\partial \underline{x}(p^*,t)}{\partial p} \right] = 0.$$

The second-order sufficient condition is satisfied by the log-concavity assumption. By the implicit function theorem, $p^* = p(t)$. The first-order necessary condition can thus be rewritten as

$$\frac{1}{D(p(t),t)} \left(1 + \underbrace{p(t) \frac{\frac{\partial D(p(t),t)}{\partial p}}{D(p(t),t)}}_{\equiv \varepsilon(p(t))} \right) = 0.$$

Note that $D(p(t),t)$ must be positive in equilibrium for any chosen t. The result follows directly. ■

C.4   Proof of Proposition 3.2

The objective function can be written as

$$\max_{t \in [0,1]} p(t) D(p(t),t).$$

Note that by the theorem of the maximum and concavity of the utility function, $p(t)$ is continuously differentiable about $t = t^m$ if $t^m \in (0, 1)$. If $t^m \in \{0, 1\}$, then I define

$$
p'(t) \equiv
\begin{cases}
\lim_{\Delta \to +0} \frac{p(\Delta) - p(0)}{\Delta} & \text{if } t = 0 \\[2mm]
\lim_{\Delta \to -1} \frac{p(1) - p(1-\Delta)}{1-\Delta} & \text{if } t = 1,
\end{cases}
$$

where $\lim_{\Delta \to +0}$ is the limit from the right and $\lim_{\Delta \to -1}$ is the limit from the left. The first-order necessary condition for an interior solution can be written as

$$
p'(t^m) D(p(t^m), t^m) + p(t^m) \left(\frac{\partial D(p(t^m), t^m)}{\partial p} p'(t^m) + \frac{\partial D(p(t^m), t^m)}{\partial t} \right) = 0.
$$

The second-order sufficient condition holds by assumption. With a bit of rearranging and using the fact that $\varepsilon(p(t^m)) = -1$ from Proposition 3.1, the first-order necessary condition can be rewritten as

$$
\underbrace{t^m \frac{\frac{\partial D(p(t^m), t^m)}{\partial t}}{D(p(t^m), t^m)}}_{\equiv \mu(t^m)} = 0.
$$

To rule out $t^m = 0$, it must be that

$$
\frac{\partial D(p(0), 0)}{\partial t} \le 0,
$$

for $t^m = 0$, which cannot be true. To see this, note that at $t^m = 0$, $\underline{x}(p(0), 0) = 0$ since this product is most valuable to the consumer located at $x = 0$. For a small enough $\varepsilon > 0$, $\underline{x}(p(0), \varepsilon) = 0$. However, any individual to the right of $t^m = \varepsilon$ is strictly better off than when $t^m = 0$. Thus $\bar{x}(p(0), \varepsilon) > \bar{x}(p(0), 0)$, which means $\frac{\partial D(p(0), 0)}{\partial t} \equiv \lim_{\Delta \to +0} \frac{D(p(0), \Delta) - D(P(0), 0)}{\Delta} > 0$. An identical argument holds for $t^m = 1$. $\blacksquare$

C.5 Proof of Proposition 3.3

Lemma C.1. *Suppose $\ell(x, \zeta) = 0$ for all x, ζ. Then $t^m = G^{-1}\left(\frac{1}{2}\right)$.*

Proof. From Proposition 3.2, t^m exists and is both unique and interior. Suppose $t^m > G^{-1}\left(\frac{1}{2}\right)$. Then, by log concavity of $G(\cdot)$, there exists an $\epsilon > 0$ such that

$$\int_{\underline{x}(p(t^m), t^m - \epsilon)}^{\overline{x}(p(t^m), t^m - \epsilon)} g(y) dy \geq \int_{\underline{x}(p(t^m), t^m)}^{\overline{x}(p(t^m), t^m)} g(y) dy.$$

The inequality is strict for all non-uniform log-concave distributions. Thus $t^m > G^{-1}\left(\frac{1}{2}\right)$ cannot possibly be an equilibrium. A symmetric argument holds for $t^m < G^{-1}\left(\frac{1}{2}\right)$ by replacing each $-\epsilon$ with $+\epsilon$. □

To see this result, recall that $\ell(x, t)$ is strictly increasing and convex in t. Thus making an arbitrarily small increase in t leads to an increase in the learning cost and this increase is itself increasing in t. Therefore the reverse movement (a slight decrease in t) leads to a decrease in costs, with that decrease decreasing in magnitude the larger the change. However, the effect on value is symmetric. Take two individuals x_1 and x_2 and the median technology level t, where $x_1 < t < x_2$ that satisfies $v(x_1, t) = v(x_2, t)$. It follows for a small, positive value ν that

$$v(x_2, t + \nu) - v(x_2, t) = v(x_1, t - \nu) - v(x_1, t).$$

Thus a decrease in t has a stronger demand effect than an increase in t by the same amount, *ceteris paribus*. This implies the existence of a value $t^* = t - \nu$ such that, for the price p that satisfies $\varepsilon(p(t)) = -1$, $p(t)D(p(t), t) < p(t)D(p(t), t^*)$. Therefore the equilibrium must be less than the median. ∎

C.6   Proof of Proposition 3.4

Recall that firm demand is positive only if

$$v\left(x, t^d\right) - p\left(t^d, \tau^d\right) - \ell\left(x, t^d\right) \geq v\left(x, \tau^d\right) - \ell\left(x, \tau^d\right)$$

for a positive mass of individuals. Suppose, to the contrary, that $t^d = \tau^d = t^d$. This implies that

$$-\ell\left(x, t^d\right) = -\ell\left(x, \tau^d\right)$$

Adding $v(x, t^d)$ to both sides has no effect on the inequality. Thus

$$v\left(x, t^d\right) - \ell\left(x, t^d\right) < v\left(x, \tau^d\right) - \ell\left(x, \tau^d\right)$$

This inequality holds for all x, which in turn implies that

$$v\left(x, t^d\right) - p\left(t^d\right) - \ell\left(x, t^d\right) < v\left(x, \tau^d\right) - \ell\left(x, \tau^d\right)$$

for all x. Therefore, there is no demand for the proprietary good when $t^d = \tau^d$.   ∎

C.7   Proof of Proposition 3.5

The proof is identical to that of Proposition 3.1, making the appropriate substitution for demand.   ∎

C.8   Proof of Lemma 3.3

By assumption, since $b(1, \tau) > K(1, \tau)$, the most technologically sophisticated individual is willing to contribute. Since $b(x, \tau)$ is weakly concave in x and $K(x, \tau)$ is strictly convex in x, there must be a positive mass of individuals with $b(x, \tau) \geq K(x, \tau)$. Thus there exists a value $\hat{x}(\tau)$ such that $b\left(\hat{x}(\tau), \tau\right) = K\left(\hat{x}(\tau), \tau\right)$.   ∎

C.9   Proof of Proposition 3.6

Consider the most extreme case $\tau^d = G^{-1}\left(\frac{1}{2}\right)$. If the firm locates at a point $\hat{t} > G^{-1}\left(\frac{1}{2}\right)$, then the firm's quantity demanded at $p = 0$ is strictly less than $\frac{1}{2}$, while the firm's quantity demanded at at $t^d < G^{-1}\left(\frac{1}{2}\right)$ and $p = 0$ is strictly greater than the quantity demanded at $\hat{t}$. Also note that utility is both continuous and linear in price. Thus for any price p, there exists a $t^d < G^{-1}\left(\frac{1}{2}\right)$ such that for any $t > G^{-1}\left(\frac{1}{2}\right)$, $pG\left(\underline{x}\left(p, t^d, \tau^d\right)\right) \geq p\left[1 - G\left(\overline{x}\left(p, t, \tau^d\right)\right)\right]$. Therefore, the firm prefers to locate to the left of the open source software.   ∎

C.10   Proof of Proposition 3.7

Denote π^m as the firm's profits under the monopoly and π^d as the firm's profits under the mixed duopoly. To the contrary, suppose that $t^m < t^d < \tau^d$. Under the mixed duopoly, the firm obtains zero demand from consumers located in the neighborhood about $x = \tau^d$ and zero demand from all values of $x \geq \tau^d$. If $\pi^m > \pi^d$, then the firm in the mixed duopoly can replicate the monopolist and see a strict increase in profits. Thus $t^d > t^m$ is not an equilibrium when $\pi^m > \pi^d$. Similarly, if $\pi^d > \pi^m$, then the monopolist can replicate the duopolist and thus $t^m < t^d$ was not an equilibrium, furthermore, it cannot be that $t^m < t^d$

and $\pi^m = \pi^d$ since the monopolist could set $t^m = t^d$, and see a strict increase in profits since it faces no competition from an open source option. ■

C.11 Proof of Proposition 3.8

The result immediately follows from Proposition 3.7. ■