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Some Results on Fillings in Contact Geometry

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Some Results on Fillings in Contact Geometry

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by

Michael Menke

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ABSTRACT OF THE DISSERTATION

Some Results on Fillings
in Contact Geometry

by

Michael Menke
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2018
Professor Ko Honda, Chair

In this thesis we prove some classification results for symplectic and exact Lagrangian fillings in contact geometry. First we prove a classification result for symplectic fillings of certain contact manifolds. Let (M, ξ) be a contact 3-manifold and $T^2 \subset (M, \xi)$ a mixed torus. We prove a JSJ-type decomposition theorem for strong and exact symplectic fillings of (M, ξ) when (M, ξ) is cut along T^2. As an application we prove the uniqueness of exact fillings when (M, ξ) is obtained by Legendrian surgery on a knot in (S^3, ξ_{std}) which is stabilized both positively and negatively. Second we show a classification result for Lagrangian fillings of Legendrian representatives of positive braid closures in S^3. This second result follows from an injectivity result for augmentation categories of positive braids.

The dissertation of Michael Menke is approved.

Zvi Bern

Ciprian Manolescu

Ko Honda, Committee Chair

University of California, Los Angeles

2018

To Loki and Miso,

for making work bearable.

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CHAPTER 1

A JSJ Decomposition Theorem for Symplectic Fillings

1.1 Introduction

A fundamental question in contact geometry is to determine the symplectic fillings of a given contact manifold, i.e. to what extent does the boundary determine its interior? The goal of this paper is to explain how to decompose the symplectic filling (W, ω) of a contact manifold (M, ξ) when we decompose $M = \partial W$ along a convex torus of a special type which we call a *mixed torus*, and to use this decomposition to show the uniqueness of some fillings of contact manifolds obtained as Legendrian surgeries.

Recall that a *strong symplectic filling* of a contact manifold (M, ξ) is a symplectic manifold (W, ω) such that $\partial W = M$, $\omega = d\alpha$ near M, and α is a positive contact form for (M, ξ). An *exact symplectic filling* of (M, ξ) is a strong symplectic filling (W, ω) such that $\omega = d\alpha$ on all of W.

Let us start with a partial list of known results classifying the number of exact symplectic fillings of a given contact manifold.

- (Eliashberg [El]) (S^3, ξ_{std}) has a unique exact filling up to symplectomorphism.
- (Wendl [We]) (T^3, ξ_1), where ξ_1 is canonical contact structure on the unit cotangent bundle of T^2, has a unique exact filling up to symplectomorphism (Stipsicz [St] had previously shown that, up to homeomorphism, there is a unique exact filling on $\Sigma(2, 3, 5)$ and (T^3, ξ_1)).
- (McDuff [MD]) The standard tight contact structure on $L(p, 1)$ has a unique exact filling up to diffeomorphism for $p \neq 4$ and for $p = 4$ there are two.

- (Lisca [Li]) Lisca classified the fillings for $L(p, q)$ with the canonical contact structure.

- (Plamenevskaya and Van Horn-Morris [PV], Kaloti [Ka]) There is a unique filling for lens spaces of the form $L(p(m+1)+1, m+1)$ with virtually overtwisted contact structures. The case $L(p, 1)$ is shown in [PV] and the general case in [Ka].

- (Sivek and Van Horn-Morris [SV]) Fillings for the unit cotangent bundle of an orientable surface are unique up to s-cobordism, and similar results for non-orientable surfaces were proven by Li and Ozbagci [LO].

- (Akhmedov, Etnyre, Mark, Smith [AEMS]) It is not always the case that there is a unique exact filling, or even finitely many.

Our main theorem is the following (see Section 1.2.3 for the definition of a *mixed torus*):

Theorem 1.1.1. *Let (M, ξ) be a closed, cooriented 3-dimensional contact manifold and let (W, ω) be a strong (resp. exact) symplectic filling of (M, ξ). Let $T^2 \subset (M, \xi)$ be a mixed torus. Then there exists a (possibly disconnected) symplectic manifold (W', ω') such that:*

- *$\exists$ Legendrian knots $L_1, L_2 \subset \partial W'$ with standard neighborhoods $N(L_1), N(L_2)$ and $T_i = \partial N(L_i)$.*

- *$M = \partial W' - int(N(L_1)) - int(N(L_2))/(T_1 \simeq T_2)$ where T_i are glued such that the dividing sets are identified and the meridian of $N(L_1)$ is mapped to the meridian of $N(L_2)$.*

- *(W', ω') is a strong (resp. exact) filling of its boundary (M', ξ').*

- *W can be recovered from W' by attaching a symplectic handle in the sense of Avdek [A].*

Remark 1.1.2. The condition that T^2 be a mixed torus is essential; the analogous proof fails if one assumes that T^2 is just a convex torus with two homotopically essential dividing curves.

We can use Theorem 1.1.1 to prove:

Theorem 1.1.3. *Let L be an oriented Legendrian knot in a closed cooriented 3-manifold (M, ξ). Let (M', ξ') be the manifold obtained from (M, ξ) by Legendrian surgery on $S_+S_-(L)$, where S_+ and S_- are positive and negative stabilizations, respectively. Then every exact filling of (M', ξ') is obtained from a filling of (M, ξ) by attaching a Lagrangian 2-disk along $S_+S_-(L)$.*

In particular the following corollary holds when $(M, \xi) = (S^3, \xi_{std})$, since (S^3, ξ_{std}) has a unique exact filling.

Corollary 1.1.4. *If (M', ξ') is obtained from (S^3, ξ_{std}) by Legendrian surgery on $S_+S_-(L)$, then (M', ξ') has a unique exact filling up to symplectomorphism.*

Kaloti and Li [KL] had previously shown the uniqueness up to symplectomorphism of exact fillings on manifolds obtained from Legendrian surgery along certain 2-bridge and twist knots and their stabilizations.

Related results were shown by Lazarev for higher dimensions in [La]. While not stated in quite the same manner, the main result of Lazarev involves surgery on loose Legendrians. We observe that in dimensions ≥ 5 all stabilized Legendrians are loose and that their analog in dimension 3 is a Legendrian which has been stabilized both positively and negatively.

1.2 Background

1.2.1 Contact geometry preliminaries

A knot in $L \subset (M, \xi)$ is called *Legendrian* if it is everywhere tangent to the contact structure ξ. The front projection of a Legendrian knot in $\subset (\mathbb{R}^3, ker(dz - ydx))$ is its projection to the xz-plane. The stabilization of $L \subset (M, \xi)$ is obtained by locally adding a zigzag in the front projection, there are two possibilities S_+ and S_- as given in Figure 1.1.

An oriented properly embedded surface Σ in (M, ξ) is called *convex* if there is a vector field v transverse to Σ whose flow preserves ξ.

A convex surface Σ which is closed or compact with Legendrian boundary has a *dividing*

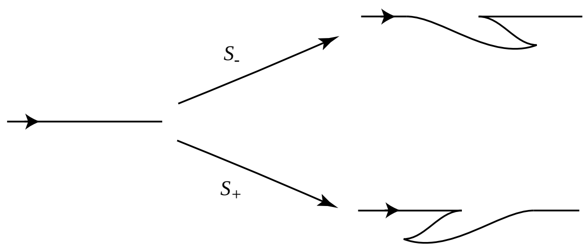

Figure 1.1: Stabilizations.

set Γ_Σ: The *dividing set* $\Gamma_\Sigma(v)$ of Σ with respect to v is the set of points $x \in \Sigma$ where $v(x) \in \xi(x)$. $\Gamma_\Sigma(v)$ is a disjoint union of properly embedded smooth curves and arcs which are transverse to the *characteristic foliation* $\xi|_\Sigma$. If Σ is closed, there will only be closed curves $\gamma \subset \Gamma_\Sigma(v)$. The isotopy type of $\Gamma_\Sigma(v)$ is independent of the choice of v — hence we will slightly abuse notation and call it *the dividing set of* Σ and denote it Γ_Σ. We will write Γ for Γ_Σ when there is no ambiguity in Σ. Denote the number of connected components of Γ_Σ by $\#\Gamma_\Sigma$. $\Sigma\backslash\Gamma_\Sigma = R_+ - R_-$, where R_+ is the subsurface where the orientations of v (coming from the normal orientation of Σ) and the normal orientation of ξ coincide, and R_- is the subsurface where they are opposite.

A convex surface has a standard neighborhood $\Sigma \times [-\epsilon, \epsilon] \subset (M, \xi)$ such that $\Sigma = \Sigma \times \{0\}$ and on this neighborhood α can be written as $\alpha = gdt + \beta$, where $g : \Sigma \to \mathbb{R}$ is a smooth function, β is a 1-form on Σ, and $\Gamma = \{g = 0\}$

The *standard neighborhood* $N(L)$ of a Legendrian knot L is a sufficiently small tubular neighborhood of L whose torus boundary is convex and whose dividing set we may take to have 2 components. If $S_\pm(L)$ is the stabilization of L, then $N(S_\pm(L))$ can be viewed as a subset of $N(L)$. Fix an oriented identification $\partial N(L) \simeq \mathbb{R}^2/\mathbb{Z}^2$ such that $\text{slope}(\Gamma_{\partial N(L)}) = \infty$ and $\text{slope}(\textit{meridian}) = 0$. Then $\text{slope}(\Gamma_{\partial N(S_\pm(L))}) = -1$.

Let $S_\pm(L)$ be a stabilization of L. Then $S_\pm(L)$ and L cobound a disk D, called the *stabilizing disk* of L, such that

1. $S_\pm(L) - \partial D = L - \partial D$.

2. $L \cap D$ contains 3 singularities of the same sign, two elliptic with one hyperbolic between

them.

3. $S_{\pm}(L) \cap D$ contains the same 2 elliptic singularities and has an elliptic singularity between them of the opposite sign.

4. The stabilization is positive (resp. negative) if the elliptic singularity from (3) is positive (resp. negative).

1.2.2 Bypasses

A *bypass disk* D for a Legendrian knot L is a convex disk whose boundary is the union of two Legendrian arcs a and b such that

- $a = L \cap D \subset L$.
- Along a there are three elliptic singularities, two at the endpoints of a with the same sign, and one in the middle with the opposite sign.
- Along b there are at least 3 singularities all of the same sign.
- There are no other singularities in D.

Remark 1.2.1. A bypass disk for L is a stabilizing disk for $L' = (L - a) \cup b$.

The following theorem due to Honda [H] shows how a bypass changes the dividing set of a surface:

Theorem 1.2.2 ([H, Lemma 3.12]). *Let Σ be a convex surface, D a bypass disk along $a \subset \Sigma$. Inside any open neighborhood of $\Sigma \cup D$ there is a one-sided neighborhood $\Sigma \times [0, 1]$ such that $\Sigma = \Sigma \times \{0\}$ and Γ_{Σ} is related to $\Gamma_{\Sigma \times \{1\}}$ by Figure 1.2.*

We say $\Sigma \times \{1\}$ is obtained from Σ by a bypass attachment. If the endpoints of the Legendrian arc a lie on the dividing set Γ of Σ then we say the bypass is attached along Γ.

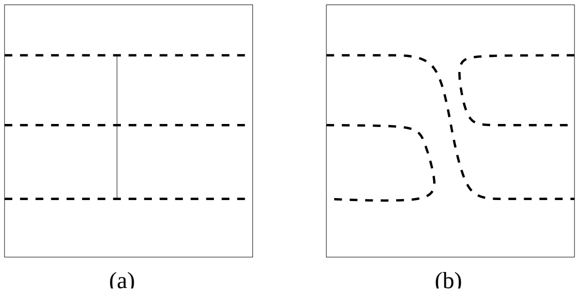

Figure 1.2: On the left is the dividing set of Σ with solid attaching arc a. On the right is the result of bypass attachment.

1.2.3 Basic slices

Identify $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. Consider a tight $(T^2 \times I, \xi)$, where $I = [0,1]$, with convex boundary where both boundary components have two homotopically non-trivial dividing curves. Let s_0 and s_1 be the slopes of the dividing curves on $T^2 \times \{0\}$ and $T^2 \times \{1\}$ respectively. If the slopes of the dividing curves are connected by a single edge on the Farey tessellation and the slopes of all dividing curves on convex tori parallel to $T^2 \times \{0\}$ and $T^2 \times \{1\}$ have slopes on $[s_1, s_0]$ if $s_1 < s_0$ and on $[s_1, \infty] \cup [-\infty, s_0)$ if $s_0 < s_1$ then $(T^2 \times I, \xi)$ is called a *basic slice*. It was shown by Honda [H] that there are exactly two tight contact structures on a given basic slice. They are distinguished by their relative Euler class.

We would like to know when $T^2 \times [0,2]$ is universally tight given that $T^2 \times [0,1]$ and $T^2 \times [1,2]$ are basic slices. Let s_0, s_1, s_2, the slopes of the dividing sets on $T^2 \times \{0,1,2\}$, be $-2, -1, 0$ respectively. Then $T^2 \times [0,2]$ is universally tight if the relative Euler class $e(\xi, s)$ satisfies $PD(e(\xi, s)) = \pm(0,2)$.

Definition 1.2.3. A convex torus $T^2 \times \{1\} = T^2 \subset (M, \xi)$ is a *mixed torus* if there exist basic slices $T^2 \times [0,1]$ and $T^2 \times [1,2]$ such that $T^2 \times [0,2]$ is not universally tight.

If T^2 is a mixed torus then the basic slices $T^2 \times [0,1]$ and $T^2 \times [1,2]$ can be obtained from bypasses of opposite sign.

1.2.4 Contact handles

Let D be a bypass disk. Then attaching D to a convex surface can be represented by attaching a pair of index 1 and 2 contact handles which cancel topologically. Full details can be found in [O].

Let ξ be the contact structure in $\mathbb{R}^3$ defined by the contact form $\alpha_1 = dz + ydx + 2xdy$ and $Z_1 = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. A model for a contact 1-handle consists of the following data:

Let $H_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + z^2 \leq \epsilon, y^2 \leq 1\}$. A model for a contact 1-handle is (H_1, ξ). Then ∂H_1 is convex with dividing set $\partial H_1 \cap \{z = 0\}$. The attaching disks of the handle are $\partial H_1 \cap \{y = \pm 1\}$. The handle is attached using Z_1.

A contact 2-handle is (H_2, ξ) where $H_2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + z^2 \leq 1, y^2 \leq \epsilon\}$ ∂H_2 is convex with dividing set $H_2 \cap \{z = 0\}$ and the attaching disk is $H_2 \cap \{x^2 + z^2 = 1\}$. The contact 2-handle is attached using $-Z_1$.

1.2.5 Legendrian surgery

Let L be a Legendrian knot in (M, ξ) with standard neighborhood $N(L)$. On the 3-manifold level, Legendrian surgery is a $tb(L) - 1$ Dehn surgery on L, where we have taken care that the contact structures agree on the boundary.

More precisely, pick an oriented identification of $\partial N(L)$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\pm(1, 0)^T$ is the meridian and $\pm(0, 1)^T$ corresponds to slope of $\Gamma_{N(L)}$. Identifying $\partial M \setminus N(L)$ with $-\partial N(L)$ we can define maps

$$\phi_\pm : \partial(D^2 \times S^1) \to \partial(M \setminus N(L))$$

on the topological level by

$$\phi(x, y) = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $M_\pm(L)$ be the manifold obtained by gluing $D^2 \times S^1$ to $M \setminus N(L)$ using this map. The contact structure ξ restricts to a contact structure $\xi|_{M \setminus N(L)}$ on $M \setminus N(L)$ and the two dividing curves on $\partial(M \setminus N(L))$, as seen on $\partial(D^2 \times S^1)$, represent $(\mp 1, 1)$ curves. Thus, according to [H], there is a unique tight contact structure on $D^2 \times S^1$ having convex boundary with these dividing curves. Hence we may extend $\xi|_{M\setminus N(L)}$ to a contact structure $\xi_\pm$ on $M_\pm$. The contact manifold $(M_\pm, \xi_\pm)$ is said to be obtained from (M, ξ) by ± 1-*contact surgery* on L. The term *Legendrian surgery* refers to -1-contact surgery.

1.2.6 Symplectization

Let (M, ξ) be a 3-dimensional contact manifold with contact form α. The symplectization of (M, ξ) is the symplectic manifold $(\mathbb{R} \times M, d(e^s\alpha))$, where s is the $\mathbb{R}$ coordinate. Given a strong symplectic filling (W, ω) of (M, ξ) we can form the completion $(\hat{W}, \hat{\omega})$ of W by attaching $([0, \infty) \times M, d(e^s\alpha))$ to $M = \partial W$, where $\omega = d\alpha$ on $M \times \{0\}$. We will refer to $([0, \infty) \times M, d(e^s\alpha))$ as the *symplectization part* and (W, ω) as the *cobordism part* of the completion.

1.2.7 Liouville hypersurfaces and convex gluing

Theorem 1.1.1 relies on a result of Avdek [A]. This section reviews the necessary background for a 3-dimensional contact manifold (M, ξ).

A *Liouville domain* is a pair (Σ, β) where

1. Σ is a smooth, compact manifold with boundary,
2. $\beta \in \Omega^1(\Sigma)$ is such that $d\beta$ is a symplectic form on Σ, and
3. the unique vector field Z_β satisfying $d\beta(Z_\beta, *) = \beta$ points out of $\partial\Sigma$ transversely.

The vector field Z_β on Σ described above is called the *Liouville vector field* for (Σ, β).

Let (M, ξ) be a 3-dimensional contact manifold and let (Σ, β) be a 2-dimensional Liouville domain. A *Liouville embedding* $i : (\Sigma, \beta) \to (M, \xi)$ is an embedding $i : \Sigma \to M$ such that there exists a contact form α for (M, ξ) for which $i^*\alpha = \beta$. The image of a Liouville embedding will be called a *Liouville submanifold* and will be denoted by $(\Sigma, \beta) \subset (M, \xi)$. We say that $(\Sigma, \beta) \subset (M, \xi)$ is a *Liouville hypersurface* in (M, ξ).

One example of a Liouville hypersurface is the positive region of a convex surface.

Every Liouville hypersurface $(\Sigma, \beta) \subset (M, \xi)$ admits a neighborhood of the form

$$N(\Sigma) = \Sigma \times [-\epsilon, \epsilon] \quad \text{on which} \quad \alpha = dt + \beta$$

where t is a coordinate on $[-\epsilon, \epsilon]$. After rounding the edges $\partial\Sigma \times (\partial[-\epsilon, \epsilon])$ of $\Sigma \times [-\epsilon, \epsilon]$, we obtain a neighborhood $\mathcal{N}(\Sigma)$ of Σ for which $\partial\mathcal{N}(\Sigma)$ is a smooth convex surface in (M, ξ) with contact vector field $t\partial_t + Z_\beta$ and dividing set $\{0\} \times \partial\Sigma$.

Fix a 2-dimensional Liouville domain (Σ, β) and a (possibly disconnected) 3-dimensional contact manifold (M, ξ). Let i_1 and i_2 be Liouville embeddings of (Σ, β) into (M, ξ) whose images, which we will denote by Σ_1 and Σ_2, are disjoint. Let α be a contact form for (M, ξ) satisfying $\alpha|_{T\Sigma_1} = \alpha|_{T\Sigma_2} = \beta$.

Consider neighborhoods $\mathcal{N}(\Sigma_1), \mathcal{N}(\Sigma_2) \subset M$ as described above. Taking coordinates (z, x) on the boundary of each such neighborhood, where $x \in \Sigma$ we may consider the mapping

$$\Upsilon : \partial\mathcal{N}(\Sigma_1) \to \partial\mathcal{N}(\Sigma_2), \quad \Upsilon(z, x) = (-z, x).$$

The map Υ sends

1. the positive region of $\partial\mathcal{N}(\Sigma_2)$ to the negative region of $\partial\mathcal{N}(\Sigma_1)$,
2. the negative region of $\partial\mathcal{N}(\Sigma_1)$ to the positive region of $\partial\mathcal{N}(\Sigma_2)$, and
3. the dividing set of $\partial\mathcal{N}(\Sigma_1)$ to the dividing set of $\partial\mathcal{N}(\Sigma_2)$

in such a way that we may perform a *convex gluing*. In other words, the map Υ naturally determines a contact structure $\#_{((\Sigma,\beta),(i_1,i_2))}\xi$ on the manifold

$$\#_{(\Sigma,(i_1,i_2))}M := \Big(M \setminus \big(N(\Sigma_1) \cup N(\Sigma_2)\big)\Big)/\sim$$

where $p \sim \Upsilon(p)$ for $p \in N(\Sigma_1)$.

1.3 Proof of Theorem 1.1.1

Let (M, ξ) be a contact manifold with a strong (resp. exact) symplectic filling (W, ω) and mixed torus $T^2 \subset M$. Let $(\hat{W}, \hat{\omega})$ be the completion of (W, ω) and J an adapted almost complex structure on $\hat{W}$ (i.e. on $(R \times M, d(e^s\alpha))$, J is s-invariant, takes ∂_s to R_α, and $\xi = \ker \alpha$ to itself and on W is ω-positive). During the proof we will impose additional conditions on J but the regularity will still be ensured by the automatic transversality results of Wendl [We3]. The proof of Theorem 1.1.1 proceeds as follows. First we will construct a 1-parameter family $\mathcal{S} = \{u_t : (\mathbb{R} \times S^1, j) \to (\hat{W}, J) | du_t \circ j = J \circ du_t, t \in \mathbb{R}\}$ of finite energy embedded holomorphic cylinders in $(\hat{W}, \hat{\omega})$ such that

(C1) When $t \gg 0$ the images Σ_t and Σ_{-t} of the curves u_t and u_{-t} are in the symplectization $[0, \infty) \times M$.

(C2) When $t \gg 0$ their projections under the map $\pi : [0, \infty) \times M \to M$ are $R_+(T^2)$ and $R_-(T^2)$ respectively.

(C3) $\mathrm{Im}(u_t) \cap \mathrm{Im}(u_{t'}) = \emptyset$ if $t \neq t'$.

We then show that $S = \cup_{t \in \mathbb{R}} \Sigma_t$ sweeps out a properly embedded solid torus in $(\hat{W}, \hat{\omega})$. We finally cut W along $S' = W \cap S$ and modify the result to obtain a strong (resp. exact) filling.

Our first step is to standardize the contact form and almost complex structure on a neighborhood of T^2. We will essentially follow the holomorphic curve construction coming from open book decompositions of Wendl [We2]. We also note that this is essentially the same as the construction in [V, Section 4] except that Vaugon uses a sutured boundary condition instead of a convex boundary condition.

Lemma 1.3.1 ([We2])**.** *There is a choice of contact form α defined on a neighborhood of T^2 such that the components of Γ_{T^2} are non-degenerate elliptic Reeb orbits of Conley-Zehnder index 1 with respect to the framing coming from T^2.*

Proof. By the flexibility theorem, modulo a perturbation of the convex surface T^2, it suffices to construct an explicit model subject to the condition that Γ_{T^2} consists of two parallel curves of slope ∞.

Let $N(\Gamma_{T^2})$ be a small neighborhood of Γ_{T^2} and let $S^1 \times D^2_{\rho_0}$ (here $D^2_{\rho_0} = \{(\rho,\phi) | \rho \leq \rho_0\}$ with $\rho_0 > 0$ small) be a component of $N(\Gamma_{T^2})$. On $S^1 \times D^2_{\rho_0}$, let $\alpha = f(\rho)d\theta + g(\rho)d\phi$ such that the following conditions hold:

- The path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ is a straight line segment in first quadrant with $(f(0), g(0)) = (c, 0)$ for some $c > 0$.
- $0 < -f'(\rho) \ll g'(\rho)$
- The maps $D^2_{\rho_0} \to \mathbb{R}$ defined by $(\rho,\phi) \mapsto f(\rho)$ and $(\rho,\phi) \mapsto g(\rho)/\rho^2$ are smooth at the origin.

Then the Reeb vector field is $R_\alpha = \frac{g'}{D}\partial_\theta - \frac{f'}{D}\partial_\phi$ where $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$. At $\rho = 0$ the Reeb field is ∂_θ. Under these conditions $\rho = 0$ is a nondegenerate Reeb orbit of Conley-Zehnder index 1 with respect to the framing coming from T^2 and all other orbits in $S^1 \times D^2_{\rho_0}$ have much larger action.

On $N' := (T^2 \times [-\epsilon, \epsilon]) - N(\Gamma_{T^2})$ let $\alpha = dt + \beta$ such that $t \in [-\epsilon, \epsilon]$ and $\ker(\beta)$ directs the characteristic foliation on T^2. We can choose coordinates (x, y) on T^2 such that $R'_+ := R_+ - N(\Gamma_{T^2}) \simeq [-1, 1] \times S^1$ and $\beta = -y dx$. In order to match the contact forms on the overlaps of N' and $N(\Gamma(T^2))$ we may need to take a diffeomorphism of N' which restricts to the identity on R'_+. □

Let e_1 and e_2 be the elliptic Reeb orbits constructed in Lemma 1.3.1. We now show how to extend α to the 1-sided neighborhood $N(T^2 \cup D)$ where D is a bypass.

Lemma 1.3.2. *Let $T^2 \subset (M, \xi)$ be a mixed torus with dividing set Γ. There exists a decomposition $N(T^2 \cup D) = N_1 \cup_\Sigma N_2 \simeq T^2 \times [0, 1]$ and an extension of α to $N(T^2 \cup D)$ such that:*

1. *N_i corresponds to the i-handle;*

2. $T^2 = T^2 \times \{0\}$ *is convex with dividing set* $e_1 \cup e_2$*;*

3. $T^2 \times \{1\}$ *is convex with dividing set* $e_4 \cup e_5$ *which are elliptic orbits of Conley-Zehnder index* 1 *with respect to* T^2*;*

4. Σ *is a genus 2 convex surface which separates* N_1 *and* N_2*, intersects* $T^2 \times \{0\}$ *along* e_1 *and* $T^2 \times \{1\}$ *along* e_4*, has corners along* e_1 *and* e_4*, and contains one other orbit, an elliptic orbit* e_3 *of Conley-Zehnder index 1 with respect to* Σ*;*

5. the Reeb vector field R_α *is positively transverse to* R_+ *and negatively transverse to* R_- *for each of* $T^2 \times \{0\}$*,* $T^2 \times \{1\}$*, and* Σ*;*

6. $\mathcal{A}_\alpha(e_3), \mathcal{A}_\alpha(e_4) > \mathcal{A}_\alpha(e_1), \mathcal{A}_\alpha(e_2), \mathcal{A}_\alpha(h_2), \mathcal{A}_\alpha(h_5)$*;*

7. there exist hyperbolic orbits h_2 *and* h_5 *in* N_1 *and* N_2*, respectively; they have Conley-Zehnder index* 0 *with respect to* T^2*;*

8. all other orbits contained in N_1 *or* N_2 *have arbitrarily large action.*

A schematic picture of the Reeb orbits in $N(T^2 \cup D)$ is given in Figure 1.3.

Remark 1.3.3. There may be other Reeb orbits which intersect $N(T^2 \cup D)$, but they will have action larger than e_1 and e_2 as was shown in [V, Theorem 2.1].

Proof. By Section 1.2.4 we know that a bypass neighborhood can be viewed as a canceling pair of contact 1- and 2-handles. We will show how to extend the α (and R_α) after the attachment of each handle.

We start by attaching the contact 1-handle along an arc of attachment whose endpoints lie on e_2. In order to attach the handle we first apply a convex-to-sutured boundary modification to $T^2 \times [-\epsilon, \epsilon]$ as in [CGHH, Section 4]. This is done by introducing a canceling hyperbolic orbit h_2 for e_2 as in Figure 1.5. After attaching the 1-handle we apply the sutured-to-convex boundary modification to obtain e_3 and e_4 as in Figure 1.4. It is easy to take $\mathcal{A}_\alpha(e_3)$ and $\mathcal{A}_\alpha(e_4)$ to be much larger than $\mathcal{A}_\alpha(e_1), \mathcal{A}_\alpha(e_2), and \mathcal{A}_\alpha(h_2)$.

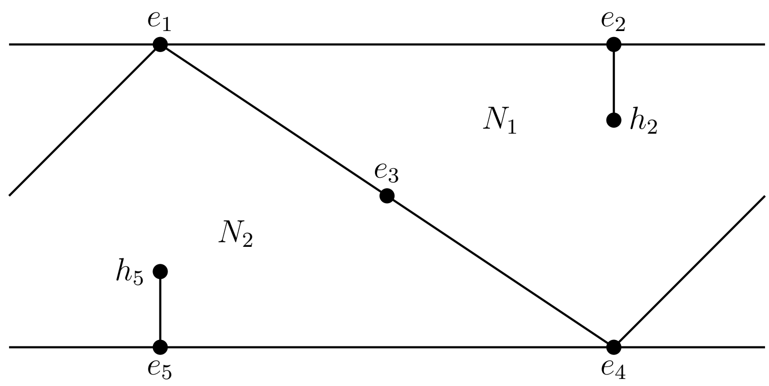

Figure 1.3: Sufficiently short Reeb orbits in $N(T^2 \cup D)$ which are strictly contained in N_1 and N_2. The e_i are elliptic orbits and the h_i are canceling hyperbolic orbits. We label the closed region corresponding to the 1-handle N_1 and the region corresponding to the 2-handle N_2.

The orbits e_1, e_3 and e_4 lie in the middle line in Figure 1.3 which represents Σ. Attaching the contact 2-handle can be viewed as attaching a contact 1-handle from the bottom layer which, from the above, gives the middle to the bottom portion of Figure 1.3. □

Lemma 1.3.4. *Let $(B, \beta = -df \circ j)$ be a 2-dimensional Weinstein domain, where $f : W \to \mathbb{R}$ is a Morse function such that ∂B is a level set of f, and let $\alpha = dt + \beta$ be a contact form on $[-\epsilon, \epsilon] \times B$, where t is the ϵ coordinate. Then there is an adapted almost complex structure on $\mathbb{R} \times [-\epsilon, \epsilon] \times B$ such that we can lift B to a holomorphic curve by the map $u(\mathbf{x}) = (f(\mathbf{x}), 0, \mathbf{x})$.*

Proof. The Liouville vector field X for β directs the characteristic foliation on $B = \{0\} \times B$ and satisfies $d\beta(X, \cdot) = \beta$ and $\beta(X) = 0$. The Reeb vector field on $[-\epsilon, \epsilon] \times B$ is ∂_t. The contact structure $\ker(\alpha)$ is spanned by X and $jX + g\partial_t$ for some function $g : B \to \mathbb{R}$. Since $0 = \alpha(jX + g\partial_t) = g + \beta(jX) = g + df(X)$ we have that $g = -df(X)$.

We want the almost complex structure J to lift j so we specify

$$J(X) = X - df(X)\partial_t \qquad J(\partial_s) = \partial_t.$$

In order to verify that $u(\mathbf{x}) = (f(\mathbf{x}), 0, \mathbf{x})$ is J-holomorphic we verify

$$J(df(X), 0, X) = (df(jX), 0, jX).$$

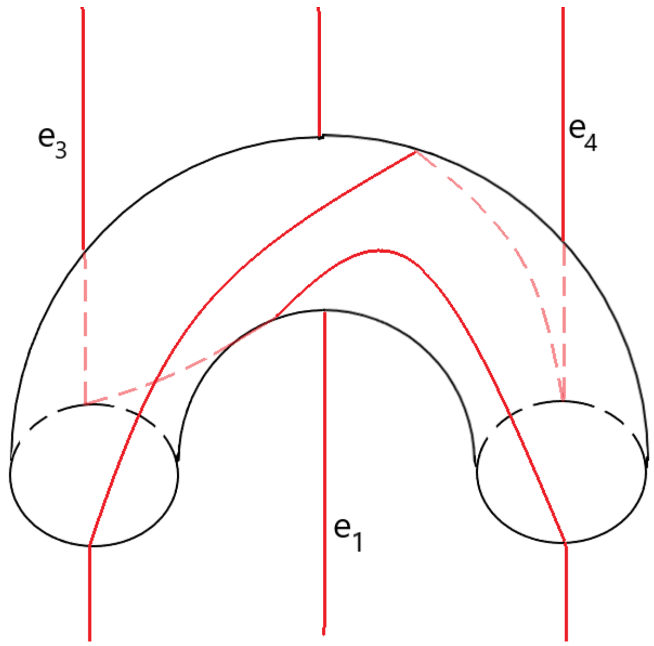

Figure 1.4: Attaching the contact 1-handle.

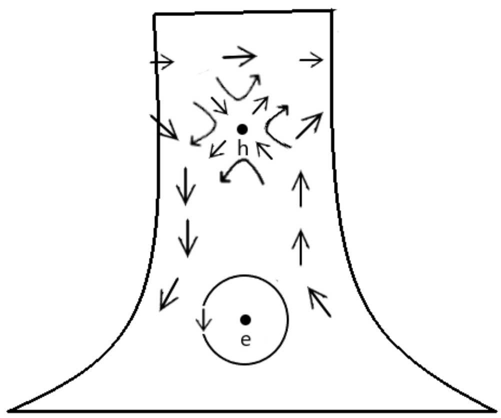

Figure 1.5: Canceling pair of hyperbolic and elliptic orbits in the convex-to-sutured boundary modification.

Indeed,

$$(df(jX), 0, jX) = (-\beta(X), 0, jX) = (0, 0, jX)$$

and

$$J(df(X), 0, X) = (0, df(X), 0) + (0, -df(X), jX) = (0, 0, jX).$$

This shows that u is J-holomorphic. □

We can lift the components R_+ and R_- of T^2 to Fredholm index 2 holomorphic curves in the symplectization $\mathbb{R} \times M$ with positive ends at e_1 and e_2.

Lemma 1.3.5 ([We2, Section 3]). *There are embedded holomorphic curves $u_\pm : \mathbb{R} \times S^1 \to [0, \infty) \times M$ such that:*

- *$u_\pm$ are Fredholm regular and index 2.*
- *$u_\pm$ are positively asymptotic to e_1 and e_2.*
- *The image of $u_\pm$ under the projection $\pi : [0, \infty) \times M \to M$ is $R_\pm(T^2)$.*

Proof. Consider the standard tight neighborhood $[-\epsilon, \epsilon] \times T^2$ of T^2. Let $R'_\pm$ be $R_\pm$ minus small collar neighborhoods. Then $\{0\} \times R'_+$ and $\{0\} \times R'_-$ are Weinstein domains. By Lemma 1.3.4 they lift to holomorphic curves in the symplectization which have constant s coordinate at the boundary.

We will construct holomorphic half cylinders in the standard neighborhood of Lemma 1.3.1 which are asymptotic to e_1 and e_2 which will glue to these lifts.

The vectors $v_1 = \partial_\rho$ and $v_2 = -g(\rho)\partial_\theta + f(\rho)\partial_\phi$ span the contact structure on $S^1 \times D^2$. Pick a smooth function $\beta(\rho) > 0$ and define J by the condition $Jv_1 = \beta(\rho)v_2$. We will assume that $\beta(\rho) = 1$ outside a neighborhood of $\rho = 0$.

In conformal coordinates (s, t), a map

$$u(s, t) = (a(s, t), \theta(s, t), \rho(s, t), \phi(s, t))$$

is J-holomorphic if

$$a_s = f\theta_t + g\phi_t \qquad \rho_s = \frac{1}{\beta D}(f'\theta_t + g'\phi_t)$$
$$a_t = -f\theta_s - g\phi_s \qquad \rho_t = -\frac{1}{\beta D}(f'\theta_s + g'\phi_s)$$

where f, g, D and β are all functions of $\rho(s,t)$. At the boundary the two equations on the right become

$$\rho_s = -\theta_t, \qquad \rho_t = \theta_s.$$

There are then solutions of the form

$$u_{\phi_0} : [0,\infty) \times S^1 \to \mathbb{R} \times (S^1 \times \mathbb{D}) : (s,t) \mapsto (a(s), t, \rho(s), \phi_0)$$

for any choice of ϕ_0, where $a(s)$ and $\rho(s)$ solve the ordinary differential equations

$$\frac{da}{ds} = f(\rho), \qquad \frac{d\rho}{ds} = \begin{cases} -1 & \text{if } \rho > \rho_0 \\ \frac{f'(\rho)}{\beta(\rho)D(\rho)} & \text{otherwise} \end{cases} \tag{1.3.1}$$

Therefore there are holomorphic half cylinders u_{ϕ_0} for any choice of ϕ_0. The conditions imposed on $f(\rho)$ and $g(\rho)$ imply that the curve u_{ϕ_0} with $\rho(0) = 1$ yields a holomorphic half-cylinder which is positively asymptotic to e_1 or e_2 as $s \to \infty$ and which has $a(s,t)$ and $\phi(s,t)$ constant near the boundary.

We want to glue these half cylinders to the lifts of R'_+ and R'_- to create the curves in the lemma. Consider $([-\epsilon,\epsilon]\times T^2) - N(\Gamma)$ where $N(\Gamma)$ is the union of the standard neighborhood from Lemma 1.3.1. There is a diffeomorphism from $[-\epsilon,\epsilon] \times T^2 \to ([-\epsilon,\epsilon] \times T^2) - N(\Gamma)$ such that near the boundary $t \to \phi$. Using this diffeomorphism we can then glue $N(\Gamma)$ to $[-\epsilon,\epsilon] \times T^2$ such that the contact structures and Reeb orbits match at the boundary of each.

Let ϕ_0 correspond to $t = 0$ under this diffeomorphism. Then we can glue the half cylinders asymptotic to e_1 and e_2 to the lifts of R'_+ and R'_- by specifying that $a(1) = f_\pm(\partial R'_\pm)$, where $f_\pm$ is a Morse function on $R'_\pm$. These curves are Fredholm regular by automatic transversality cf. [We2, Proposition 7]. □

Since T^2 is mixed there is another bypass layer $T^2 \times [-1,0]$ stacked "on top" with $T^2 \times [0,1]$ as the "bottom layer", see Figure 1.6. The orientation of the top layer is reversed because the bypass has opposite sign. Let P be a thrice-punctured sphere. We will construct holomorphic curves which represent the solid lines in Figure 1.6.

Lemma 1.3.6. *There are embedded holomorphic curves*

$$u^{\pm}_{i,j,k} : P \to [0,\infty) \times T^2 \times [-1,1]$$

and

$$u^{\pm}_{i,j} : \mathbb{R} \times S^1 \to [0,\infty) \times T^2 \times [-1,1]$$

for admissible $\{i,j,k\}$ *and* $\{i,j\}$ *such that:*

- $u^{\pm}_{i,j,k}$ *and* $u^{\pm}_{i,j}$ *are Fredholm regular and have index 2 and*
- $u^{\pm}_{i,j,k}$ *are positively asymptotic to* e_i, e_j*, and* e_k *and* $u^{\pm}_{i,j}$ *are positively asymptotic to* e_i *and* e_j*.*

The admissible $\{i,j,k\}$ *and* $\{i,j\}$ *are* $\{1,7,6\}, \{1,3,4\}, \{1,6\}, \{1,4\}, \{4,5\}, \{6,8\}$ *and the* u^+ *and* u^- *are distinguished by whether the orientations of their projections to* M *agree with* R_+ *or* R_- *with respect to the orientation coming from* T^2*.*

These curves represent solid lines in Figure 1.6.

Proof. Recall that in a neighborhood of an elliptic orbit e_i there are holomorphic half cylinders of the form

$$u_{\phi_i} : [0,\infty) \times S^1 \to \mathbb{R} \times (S^1 \times \mathbb{D})$$

$$(s,t) \mapsto (a(s), t, \rho(s), \phi_i).$$

Choose $\phi_i \neq \phi_0$ and let A_{ϕ_i} be the image of ϕ_i. If P is a thrice-punctured sphere we can repeat the procedure of Lemma 1.3.5 to lift P minus the three ends to a holomorphic curve and glue the boundary to A_{ϕ_i}. These curves have ind $= 2$ by a straightforward index calculation and are Fredholm regular by [We2, Prop. 7]. □

Let $\mathcal{M}(e_1, e_2)$ denote the moduli space of ind $= 2$ curves $u : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ which are positively asymptotic to e_1 and e_2 and represent the same homology class as u_+ or u_- and let

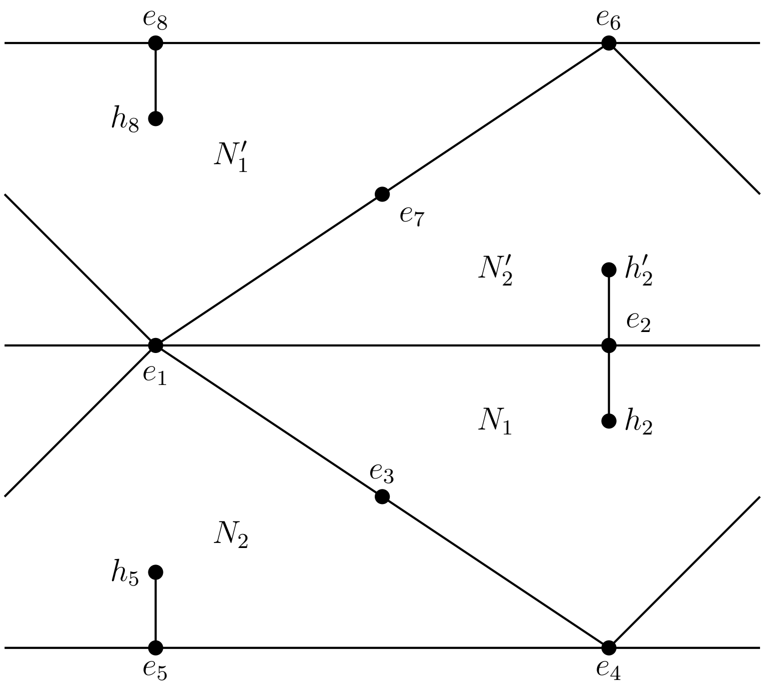

Figure 1.6: Orbits in a neighborhood of a mixed torus. The solid lines represent holomorphic curves. The regions N_i and N'_i, $i = 1, 2$ correspond to the i-handle attachments as in Figure 1.3. For each solid line there are two holomorphic curves, one whose orientation agrees with R_+ and one whose orientation agrees with R_-.

$\mathcal{M}(e_1, e_2)/\mathbb{R}$ be the quotient by the $\mathbb{R}$-translation. We can now describe the compactification of this moduli space.

Lemma 1.3.7. *The compactification $\overline{\mathcal{M}(e_1, e_2)/\mathbb{R}}$ is the disjoint union of two components $\mathcal{N}_\pm$ containing the equivalence classes of $u_\pm$ up to $\mathbb{R}$ translation. The boundary $\partial\mathcal{N}_\pm$ consists of*

- *a two-level building $v_{1,\pm} \cup v_{0,\pm}$, where $v_{1,\pm}$ is the top level consisting of a cylinder positively asymptotic to e_2 and negatively asymptotic to h_2 and $v_{0,\pm}$ is the bottom level consisting of a cylinder positively asymptotic to e_1 and h_2 and*
- *another two-level building $v'_{1,\pm} \cup v'_{0,\pm}$ with h_2 replaced by h'_2.*

Let $\mathcal{A}_\alpha$ denote the α-action of a Reeb orbit.

Proof. We may assume $\mathcal{A}_\alpha(e_1) = \mathcal{A}_\alpha(e_2)$. By [V] the only Reeb orbits that may have smaller action than $\mathcal{A}_\alpha(e_i)$, $i = 1, 2$, are those in Figure 1.6. We see that $\partial\mathcal{N}_\pm$ can contain a cylinder positively asymptotic to e_2 and negatively asymptotic to h_2 followed by a cylinder positively asymptotic to e_1 and h_2. The same is true for h_2 replaced by h'_2.

The images of the curves $u^\pm_{i,j,k}$ for admissible $\{i, j, k\}$ are embedded and do not intersect $u_\pm$. Their projections to M are embedded and disjoint from the projections to M of any curve in $\mathcal{N}_\pm$. From [We, Appendix A] we see that the images in the symplectization of $u^\pm_{i,j,k}$ are disjoint from any curve in $\mathcal{N}_\pm$. These curves act as walls so that curves in $\mathcal{N}_\pm$ cannot break into curves asymptotic to orbits outside of the regions labeled N'_2 and N_1.

Finally we claim that there are no other curves in $\overline{\mathcal{M}(e_1, e_2)/\mathbb{R}}$ contained in the regions N'_2 and N_1. We note that the orbit e_2 is contained in the interior of the projections of all curves in $\mathcal{N}_+ \cup \mathcal{N}_-$. Any other holomorphic buildings in $\overline{\mathcal{M}(e_1, e_2)/\mathbb{R}}$ would need to have at least one level with a curve asymptotic to e_2 for at least one end, but we have already enumerated the possibilities above. □

In order to cut along T^2 we need to push this index 2 family of curves into the filling (W, ω).

Lemma 1.3.8. *There is a regular 1-parameter family*

$$\mathcal{S} = \{u_t : \mathbb{R} \times S^1 \to (\hat{W}, J) | du_t \circ j = J \circ du_t\}$$

of embedded holomorphic cylinders in $(\hat{W}, \hat{\omega})$ parametrized by $t \in \mathbb{R}$ satisfying Conditions (C1)—(C3).

Proof. Consider the ind $= 1$ family $\mathcal{M}_{\hat{W}}(e_1, h_2)$ consisting of holomorphic cylinders in $\hat{W}$ that limit to e_1 and h_2 at the positive ends and represent the same homology class as $v_{0,+}$ or $v'_{0,+}$ from Lemma 1.3.7.

We first claim that $\partial\mathcal{M}_{\hat{W}}(e_1, h_2)$ can only consist of curves $v_{0,+}$ and $v'_{0,+}$; this implies that there is one noncompact component of $\mathcal{M}_{\hat{W}}(e_1, h_2)$, which we take to be $\mathcal{S}$. Bubbling is a codimension 2 phenomenon and can be safely ignored since we are only considering an ind $= 1$ family. Let w be the topmost level of an element of $\partial\mathcal{M}_{\hat{W}}(e_1, h_2)$; it has image in $\mathbb{R} \times M$. By the positivity of intersections and the existence of "walls" $u_\pm, u^\pm_{1,7,6}, u^\pm_{1,3,4}, u^\pm_{4,5}, u^\pm_{6,8}$ (and their $\mathbb{R}$-translations) which are disjoint from elements of $\mathcal{M}_{\hat{W}}(e_1, h_2)$, it follows that $\pi \circ w$ must be contained in N_1, N'_2, N_2, or N'_1. By the description of the Reeb orbits from Lemma 1.3.2, the only possibilities are $w = v_{0,+}$ and $v'_{0,+}$: Assume without loss of generality that the slopes of $\Gamma_{T^2\times\{0\}}$ and $\Gamma_{T^2\times\{1\}}$ are 0 and 1, respectively. Under the identification $H_1(T^2 \times [-1, 1]) \simeq H_1(T^2) \simeq \mathbb{Z}^2$, we can take $[e_1] = (0, -1)$ and $[e_2] = (0, 1)$. Then $[h'_2] = [h_2] = (0, 1), [e_3] = (-1, 0)$, and $[e_4] = [e_5] = [h_5] = (1, 1)$. If $\mathrm{Im}(\pi \circ w) \subset N_2$, then w must have e_1 at the positive end; however, no nonnegative linear combination of $[e_3], [e_4], [h_5], [e_5]$ is homologous to $[e_1]$. If $\mathrm{Im}(\pi \circ w) \subset N_1$, then either

1. e_1 is at the positive end
2. h_2 is at the positive end, or
3. both e_1 and h_2 are at the positive end.

The only possibility is $[h_2] = [e_3] + [e_4]$, but we are taking $\mathcal{A}_\alpha(h_2) < \mathcal{A}_\alpha(e_3) + \mathcal{A}_\alpha(e_4)$ which is a contradiction. This implies the claim.

For $t \gg 0$, take u_t (resp. u_{-t}) to be a translation of $v_{0,+}$ (resp. $v'_{0,+}$) by some $t+c$, where c is a constant, viewed inside the symplectization part $[0,\infty) \times M$. This implies (C1). (C2) is not met precisely on the nose, but we may isotop T^2 so that $R_+(T^2) = \mathrm{Im}(\pi \circ u_t)$ and $R_-(T^2) = \mathrm{Im}(\pi \circ u_{-t})$ for $t \gg 0$.

We now prove (C3). For large $t \neq t'$ the images of $u(t)$ and $u(t')$ are disjoint so their intersection number $i(u_+(t); u_+(t')) = 0$. The intersection number is a relative homology invariant, so we need to show that no new intersections occur near the ends as we push into W. If any intersections did occur they would be negative which contradicts the positivity of intersections, hence the intersection number continues to be 0 cf. [We, Lemma A.3] □

Lemma 1.3.9. *$S = \sqcup_{t\in\mathbb{R}} \Sigma_t$ sweeps out a properly embedded solid torus in $\hat{W}$.*

Proof. The curve u_t is an embedding for every $t \in \mathbb{R}$, hence all nearby curves can be described as sections of the normal bundle N_{u_t}. The first Chern class of the normal bundle has the following form, cf. [We3, Section 1]:

$$2c_1(N_{u_t}) = \mathrm{ind}(u_t) - \chi(\dot{\Sigma}) + \#\Gamma_0,$$

where $\dot{\Sigma}$ is the domain of u_t and $\#\Gamma_0$ is the number of punctures asymptotic to orbits with even Conley-Zehnder index.

Since S consists of an ind $= 1$ family we have $\mathrm{ind}(u_t) = 1$, $\chi(\dot{\Sigma})$, and $\#\Gamma_0 = 1$ hence $c_1(N_{u_t}) = 0$ and so sections must be zero-free and the total family S is also an embedding. □

We want to remove $S \cap W$ from W. In order to do this we first modify W slightly. Consider $W_R = W \cup ([0,R] \times M)$, where R is large so that there exist u_T and u_{-T} whose images are in $[0,\infty) \times M$ and whose π-projections after to restricting to $[0,R] \times M$ are R'_+ and R'_- which are $R_\pm$ minus small collar neighborhoods. Then form $W'_R = W_R - \widetilde{N}(\Gamma_{T^2})$, where $\widetilde{N}(\Gamma_{T^2})$ is a small (half-)tubular neighborhood of $\{R\} \times \Gamma_{T^2}$ in W_R. Note that W'_R has corners, and $\partial_h W'_R = S^1 \times D^2 = \partial W'_R - \partial W_R$ is analogous to the horizontal boundary of a Lefschetz fibration for a Weinstein domain, and $\partial_v W'_R = \partial W'_R - \partial_h W'_R$ is analogous to the vertical boundary. We assume that $\{R\} \times R'_\pm = \{R\} \times R_\pm - \widetilde{N}(\Gamma_{T^2})$.

Lemma 1.3.10. *There exists an embedding $\Sigma \times [-T-1, T+1] \subset W'_R$ such that:*

1. *Σ is an annulus and is a symplectic submanifold of W'_R;*
2. *$\Sigma \times \{\pm(T+1)\} = \{R\} \times R'_\pm$;*
3. *for $t \in [-T-1, T+1], \partial\Sigma \times \{t\} = S^1 \times \gamma(t) \subset \partial_h W'_R$, where $\gamma(t)$ is a straight arc from $(-1, 0)$ to $(1, 0)$ in D^2.*

Proof. First note that the family $\Sigma_t, t \in [-T, T]$, restricted to W'_R, gives rise to an embedding $\Sigma \times [-T, T] \subset W'_R$ that satisfies the conditions of the lemma except for $\Sigma \times \{\pm T\} = \{R\} \times R'_\pm$. For $t \gg 0$ the curves $u_{\pm t}$ have the form $u_{\pm t}(\mathbf{x}) = (f(\mathbf{x}), 0, \mathbf{x})$ in $\mathbb{R} \times \mathbb{R} \times R'_\pm$ by Lemma 1.3.4. We can interpolate symplectically from $\Sigma_{\pm T} = \mathrm{Im}(u_{\pm T})$ to $\Sigma_{\pm(T+1)} = R''_\pm$ through symplectic subsurfaces of the form $(cf(\mathbf{x}), 0, \mathbf{x})$ for $c \in [0, 1]$. A slight modification of $\Sigma \times [-T-1, T+1]$ near $\partial\Sigma \times [-T-1, T+1]$ yields the lemma. □

Let $S' = \Sigma \times [-T-1, T+1]$ with coordinates (x, t).

Lemma 1.3.11. *After a slight adjustments of S' and W'_R, there exists a neighborhood $N(S') = S' \times [-\epsilon, \epsilon] \subset W'_R$ and a 1-form $\lambda = \lambda_B + \lambda_\Sigma$ (here $B = [-T-1, T+1] \times [-\epsilon, \epsilon]$ has coordinates (t, w)) on $N(S')$ such that:*

1. *$\Sigma \times \{-T-1, T+1\} \times [-\epsilon, \epsilon] \subset \partial_v W'_R$ and $(\partial\Sigma) \times [-T-1, T+1] \times [-\epsilon, \epsilon] \subset \partial_h W'_R$;*
2. *λ_Σ is the Liouville form for R'_+ and, after adjusting $\partial_v W'_R$, also agrees with the Liouville form for R'_-;*
3. *$\lambda_B = t dw$;*
4. *$d\lambda$ agrees with the symplectic form on W'_R;*
5. *λ agrees with the Liouville form on W'_R near $\partial W'_R$.*

Proof. Let λ_Σ be the Liouville form for $\{R\} \times R'_+ = \Sigma \times \{T+1\}$. After a slight adjustment of $\partial_h W'_R$ we may assume that the restriction of the Liouville form β on W'_R to each $\partial\Sigma \times \{t\}$ is the same ($= \lambda_\Sigma$ on $\partial\Sigma \times \{T+1\}$).

Using a relative version of the Moser technique, we normalize $d\beta$ on S' so that each $\Sigma\times\{t\}$ has symplectic form $d\lambda_\Sigma$. Viewing S' as a symplectic fibration with base $[-T-1, T+1]$, there is a symplectic connection Ω; by applying fiberwise diffeomorphisms (with fixed boundary), we can "straighten out" the connection so that Ω is given by ∂_t and we use λ_Σ on each $\Sigma \times \{t\}$. We need to apply the Giroux flexibility theorem to R'_- so that the Liouville form on R'_- agrees with the Liouville form λ_Σ after flowing along the connection.

Finally, using the Moser-Weinstein neighborhood theorem, we can normalize $d\beta$ so it equals $d\lambda = d(tdw + \lambda_\Sigma)$ on $N(S')$. Also, λ agrees with β near $\partial W'_R$. □

By the following lemma, we can cut along S' to obtain a strong filling of a contact manifold.

Lemma 1.3.12. *There exists a modification*

$$\lambda' = \lambda + d(tw) = 2tdw + wdt + \lambda_\Sigma,$$

whose Liouville vector field $Z' = 2t\partial_t - w\partial_w + X_\Sigma$ *(here* X_Σ *is the Liouville vector field for* λ_Σ*) points into* $N(S')$ *along* $w = \pm\epsilon$*. Hence* $W' := W'_R - N(S')$ *is a strong filling of its boundary.*

If the original filling is exact then we need to construct a global Liouville form on $W' = W'_R - N(S')$.

Lemma 1.3.13. *If* (W, β) *is an exact filling, then there exists a 1-parameter family of Liouville forms* $\beta_\tau, \tau \in [0, 1]$*, on* W'_R *such that* $\beta_0 = \beta$ *and* $\beta_1 = \lambda'$ *on* $N(S') \cap \{-\epsilon/2 \leq w \leq \epsilon/2\}$*.*

Proof. Since $d\beta$ and $d\lambda'$ agree on $N(S')$, there exists a function f on $N(S')$ such that $\lambda' - \beta = df$. We can choose f such that $f = 0$ on $\partial W'_R$. Next modify f to g on $N(S')$ such that $g = f$ for $w \in [-\epsilon/2, \epsilon/2]$ and $g = 0$ for $w = \pm\epsilon$; then extend g by 0 to all of W'_R. Now consider the 1-parameter family of Liouville forms $\beta_\tau = \beta + \tau dg$. Clearly $\beta_0 = \beta$ and $\beta_1 = \lambda'$ on $N(S') \cap \{-\epsilon/2 \leq w \leq \epsilon/2\}$. □

Finally we explain how to obtain W from W'. For this construction we will use the following result from [A]:

Theorem 1.3.14 ([A, Theorem 1.9]). *Let (M, ξ) be a closed, possibly disconnected, $(2n+1)$-dimensional contact manifold. Suppose that there are two Liouville embeddings $i_1, i_2 : (\Sigma, \beta) \to (M, \xi)$ with disjoint images. Then there is an exact symplectic cobordism (W, ω) whose negative boundary is (M, ξ) and whose positive boundary is $\#_{(\Sigma,\beta)}\ (M, \xi)$. Moreover, if (Σ, β) admits a Stein structure, then so does the cobordism.*

The manifold $\#_{(\Sigma,\beta)}\ (M, \xi)$ is obtained by the convex gluing operation defined in Section 1.2.7.

After cutting, we can find two disjoint copies of Σ inside W'. By construction Σ is a Liouville domain. The proof of Theorem 1.3.14 involves attaching a symplectic handle to a collar neighborhood of (M, ξ) in (W', ω). After attaching this handle we obtain (W, ω) with convex boundary $\#_{(\Sigma,\beta)}\ (M, \xi)$ as desired.

1.4 Proof of Theorem 1.1.3

We will now prove Theorem 1.1.3 using Theorem 1.1.1. Let (M', ξ') be the contact manifold obtained from (M, ξ) by Legendrian surgery on $S_+S_-(L)$.

Let (W, ω) be an exact filling of (M', ξ'). Consider the standard neighborhood $N(S_-(L)) \subset M$ of $S_-(L)$. Let V_1 be the solid torus obtained from $N(S_-(L))$ by Legendrian surgery along $S_+S_-(L)$. Let $V_2 = M - N(S_-(L))$. Then $M' = V_1 \cup V_2$.

The torus $T = \partial N(S_-(L))$ is a mixed torus because stabilizing twice with opposite signs is equivalent to performing two bypasses with opposite signs. Theorem 1.1.1 then guarantees that we can decompose W into a disconnected manifold W' such that $\partial W' = M_1 \cup M_2$, where $M_1 = V_1 \cup_{\partial S'} S'$ and $M_2 = V_2 \cup_{\partial S'} S'$. The contact structures on M_1 and M_2 are obtained by using the canonical tight contact structure on the solid torus S'.

The choice of S' is not unique and we want to enumerate the possibilities for S'. Take an oriented identification of $\partial N(S_-(L))$ with $\mathbb{R}^2/\mathbb{Z}^2$ such that the meridian of $N(S_-(L))$ has slope 0 and $\Gamma_{\partial N(S_-(L))}$ has slope ∞. With respect to this identification, $\Gamma_{\partial N(L)}$ has slope 1 and $\Gamma_{\partial N(S_+S_-(L))}$ has slope -1. The meridian μ_{V_1} of V_1 has slope $-1/2$. The boundary of the solid torus S' has the same dividing set as V_1, but the meridian $\mu(S')$ is undetermined. Since the shortest integer vector representing the meridian must form an integer basis with the shortest integer vector representing the dividing set, the possible choices for $\mu(S')$ are of the form $(1, m)$ for $m \in \mathbb{Z}$.

Observe that since M_i is fillable it must be tight. We want to compute which choices of $\mu(S')$ yield tight contact structures on M_1 and M_2 using the classification of tight contact structures from [H]. The choices for $\mu(S')$ are compiled in Table 1.1. First consider M_1.

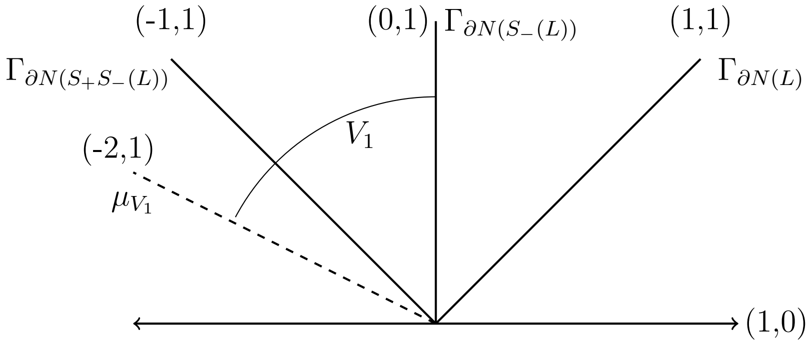

Figure 1.7: Dividing set and meridian for V_1.

On the S' part the contact planes rotate from the meridian of S' to the dividing set Γ in a counterclockwise manner viewed using the identification with $\mathbb{R}^2/\mathbb{Z}^2$ as in Figure 1.7, and on the V_1 part they rotate from Γ to the meridian. Rotation by more than π results in an overtwisted contact structure which contradicts the fillability of M_1. From Figure 1.7 we see that this eliminates the possibility $m \leq -1$.

On M_2 we see that if $m > 1$ then the slopes of the dividing curves rotate more than π. If $m = 1$ then we can find a solid torus with convex boundary and boundary slope 0 by taking the union of $N(L) - N(S_-(L))$ with S', which is then overtwisted by Giroux's flexibility theorem. This leaves $\mu(S') = (1, 0)$ as the only option. With this choice, $M_1 \simeq (S^1 \times S^2, \xi_{std})$ and $M_2 \simeq (M, \xi)$ and M_1 has a unique exact filling.

From Theorem 1.1.1 we know there is a cobordism from (M', ξ') to $(S^1 \times S^2, \xi_{std}) \sqcup (M, \xi)$.

Therefore any exact filling of (W, ω) of (M', ξ') is obtained from an exact filling of (M, ξ) by attaching $S^1 \times D^3$. This proves Theorem 1.1.3. □

	M_1	M_2
⋮		X
(1,2)		X
(1,1)		X
(1,0)	$(S^1 \times S^2, \xi_{std})$	(M, ξ)
(1,-1)	X	
(1,-2)	X	
⋮	X	

Table 1.1: Choices of meridian for $\partial S'$ using the identification $N(S_-(L)) \simeq \mathbb{R}^2/\mathbb{Z}^2$. X's correspond to overtwisted contact structures.

CHAPTER 2

Exact Lagrangian Fillings of Positive Braid Closures

2.1 Introduction

The A_∞ augmentation category $Aug_-(\Lambda)$ of a Legendrian link $\Lambda \subset \mathbb{R}^3_{xyz}$ was first introduced in [BC]. The objects of this category are augmentations of the link and the morphisms are chain complexes whose cohomology is known as bilinearized Legendrian contact cohomology. Bilinearized Legendrian contact cohomology can be used to distinguish exact Lagrangian fillings of a Legendrian knot up to Hamiltonian isomorphism and it provides an obstruction to Lagrangian concordances [C]. In [NRSSZ] a unital A_∞ augmentation category $Aug_+(\Lambda)$ was constructed and used to show that augmentations of a Legendrian knot correspond to constructible sheaves on $\mathbb{R}^2_{xz}$ whose singular support at infinity lies on Λ.

In this chapter we will focus on the class of knots which are the closures of positive braids. Positive braid closures are a particular well-behaved class of knots whose Legendrian contact homology has been studied extensively in [K]. Our goal is to study the effects of 0-resolution on the augmentation categories of positive braid closures. We will show that Lagrangian cobordisms corresponding to 0-resolution give rise to cohomologically faithful functors on augmentation categories.

We will use $\mathbb{Z}/2$ coefficients throughout. Our main result is the following:

Theorem 2.1.1. *Let $\Lambda_+ \subset \mathbb{R}^3$ be the Legendrian closure of a positive braid and let Λ_- be the link obtained after resolving a crossing of Λ_+. Then there are cohomologically faithful A_∞ functors $F^+ : Aug_+(\Lambda_-) \to Aug_+(\Lambda_+)$ and $F^- : Aug_-(\Lambda_-) \to Aug_-(\Lambda_+)$ which are induced by 0-resolution.*

Using the Aug_- version of Theorem 2.1.1 we obtain the following result for the bilinearized Legendrian contact cohomology of positive braids:

Corollary 2.1.2. *Let F^- be the functor from Theorem 2.1.1. Let ϵ_1 and ϵ_2 be $\mathbb{Z}/2$ graded augmentations of Λ_-. Then*

$$LCH^*_{F^-(\epsilon_1),F^-(\epsilon_2)}(\Lambda_+) \simeq LCH^*_{\epsilon_1,\epsilon_2}(\Lambda_-) \oplus \mathbb{Z}/2[0]$$

where $\mathbb{Z}/2[0]$ denotes a copy of $\mathbb{Z}/2$ in degree 0.

2.2 Preliminaries

2.2.1 Positive Braids and Legendrian Contact Homology

A positive braid β is a braid whose braid word consists entirely of right-handed twists. An example of a positive braid is the (p, q) torus braid which can be represented by a diagram with p strands and whose braid word is $(\sigma_1\sigma_2...\sigma_{p-1})^q$. The closure of a positive braid is obtained by attaching arcs which connect the beginning of strand i to the right end of the strand which leaves the braid at position i. The closure of any positive braid has a Legendrian representative whose Lagrangian projection is shown in Figure 2.2. We will refer to this representative as the Legendrian representative.

We label the crossings in the braid portion of a torus link as in Figure 2.1. The crossings are $b_{i,j}$ where $1 \leq i \leq q$ and $1 \leq j \leq p - 1$. Since every positive braid can be obtained as a sequence of 0-resolutions from a torus braid, every positive braid will inherit a labeling on its crossings from the corresponding torus knot.

To a Lagrangian diagram Λ of a Legendrian knot we can associate a differential graded algebra $\mathcal{A}(\Lambda)$ with $\mathbb{Z}/2$ coefficients known as the Chekanov-Eliashberg DGA [Ch]. An alternative introduction can be found in [Et]. Let $\Lambda \subset \mathbb{R}^3$ where $\mathbb{R}^3$ has coordinates x, y, z. The projection of Λ to the xy-plane is known as the Lagrangian projection. The generators of this DGA are the crossings of Λ when viewed in the Lagrangian projection. These crossings correspond to Reeb chords of the knot. The differential of a crossing a counts immersed

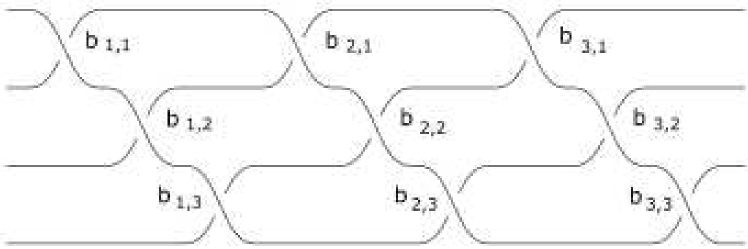

Figure 2.1: The (4,3) Torus Braid

disks in the xy-plane with punctures $a_1, b_1, \ldots, b_n$ oriented counter-clockwise such that the a_1 puncture is at a positive convex corner of the crossing a, and the punctures $b_1, \ldots, b_n$ are at negative convex corners of crossings in the diagram. The corners of a crossing are labeled as in Figure 2.2. The homology of this DGA is known as Legendrian contact homology. In the Chekanov-Eliashberg DGA of a positive braid the crossings $c_{i,j}, b_{i,j}$ have degree 0 and the crossings $s_{i,j}$ have degree 1. More details on positive braid closures and their Legendrian contact homology can be found in [K].

A $\mathbb{Z}/2$-graded augmentation of a DGA (A, ∂) is a map $\epsilon : A \to \mathbb{Z}/2$ such that $\epsilon \circ \partial = 0$ and $\epsilon(a) = 0$ unless $deg(a) = 0$. Throughout this paper by augmentation we will always mean a $\mathbb{Z}/2$-graded augmentation.

2.2.2 0-resolution

In [EHK, Prop 6.17] it is shown that the Lagrangian cobordism corresponding to 0-resolution of a crossing a of a link Λ_+ gives rise to a DGA map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$, where $\Lambda_+ = \Lambda_- \cup \{a\}$. This map is obtained by counting holomorphic disks with boundary on Λ_+ and two positive punctures, one of which is at the crossing a.

Let $d \neq a$ and denote by $M(d, a^k; \mathbf{b})$ the moduli space of holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ and boundary on $\mathbb{R} \times \Lambda_+$ modulo biholomorpism, with one positive puncture at d, k positive punctures at a, and negative punctures at $\mathbf{b}$. The index of a curve $u \in M(d, a^k; \mathbf{b})$ is

$$ind(u) = |d| + k|a| - |\mathbf{b}| + k.$$

A contractible crossing a is simple if $ind(u) \geq k$ for $u \in M(d, a^k; \mathbf{b})$ and $k > 1$. The crossings $b_{i,j}$ of a positive braid are all simple crossings.

In [EHK] it is shown for a simple crossing that the DGA map $\Psi : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-)$ is given by $\Psi(d) = \psi_0(d) + \psi_1(d)$ where:

- $\psi_0(a) = 1$,
- $\psi_0(d) = d$ for all $d \in \mathcal{C}(\Lambda_-)$,
- $\psi_1(a) = 0$,
- $\psi_1(d) = \sum_{dim(M(d,a;\mathbf{b}))=1} |M(d, a; \mathbf{b})/\mathbb{R}| \cdot \psi_0(\mathbf{b})$ for all $d \in \mathcal{C}(\Lambda_-)$.

Where $C(\Lambda)$ denotes the set of crossings of Λ. We note that Ψ is a surjective map.

As an example consider the $(4, 3)$ torus braid from Figure 2.1 whose Legendrian closure is shown in Figure 2.2. If we resolve at $b_{1,1}$ then we see that the following crossings have non-zero ψ_1:

- $\psi_1(b_{2,2}) = b_{1,2}$
- $\psi_1(b_{2,3}) = 1$
- $\psi_1(c_{4,1}) = c_{4,2}$
- $\psi_1(c_{3,1}) = c_{3,2}$
- $\psi_1(c_{2,1}) = 1$

2.2.3 Augmentation Categories

We now give a basic overview of $Aug_+(\Lambda)$. We note that everything in Sections 2.2.3 and 2.2.4 also holds for $Aug_-(\Lambda)$ with small modifications. For full details the reader should consult [NRSSZ] and [BC]. Let Λ be a the Legendrian closure of a positive braid and let (A, ∂) be the Chekanov-Eliashberg DGA associated to Λ.

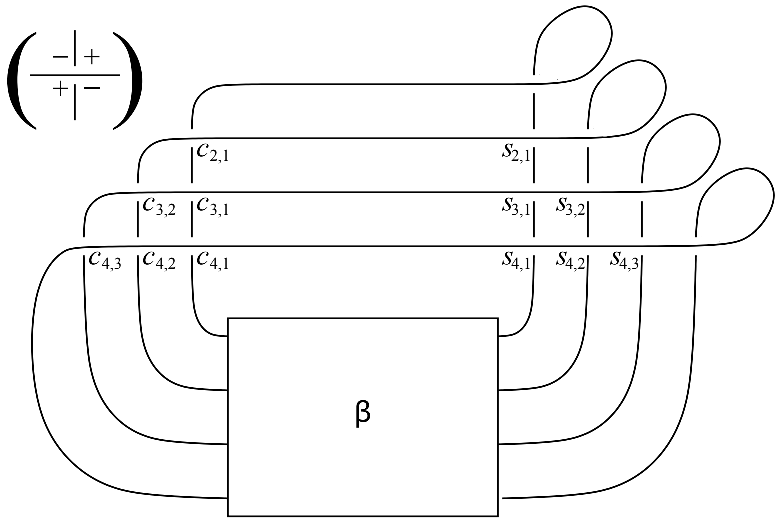

Figure 2.2: Lagrangian diagram of the Legendrian closure of the positive braid. The positive and negative corners of a crossing are labeled at the top left.

Our goal will be to construct an A_∞ category for (A, ∂). To do this we will need the m-copy Λ^m of Λ along with a Morse perturbation f of Λ. The m-copy is obtained by $m-1$ small distinct pushoffs of Λ in the Reeb direction of $\mathbb{R}^3_{xyz}$. The Morse perturbation insures that the Reeb chords of Λ^m are non-degenerate. We will require that f has k maxima and k minima, where k is the number of strands of Λ. Let Λ^m_f denote the m-copy together with the Morse perturbation f. Label the copies $\Lambda^1, \ldots, \Lambda^m$ from top to bottom. Then there is a DGA (A^m, ∂^m) which is generated by the following:

- Crossings a^{ij} where $1 \leq i, j \leq m$ and a is a crossing of Λ.
- Crossings x^{ij}_l where $1 \leq i < j \leq m$ and $1 \leq l \leq k$. These are the crossings associated to the maxima of f.
- Crossings y^{ij}_l where $1 \leq i < j \leq m$ and $1 \leq l \leq k$. These are the crossings associated to the minima of f.

Throughout the rest of the paper, upper indices will denote components of the m-copy and lower indices will be used to distinguish crossings of the knot.

A diagonal augmentation $\boldsymbol{\epsilon}$ of A^m is an m-tuple $(\epsilon_1, ..., \epsilon_m)$ of augmentations of Λ such that $\epsilon_i(b^{jk}) = \delta_{i,j}\delta_{i,k}\epsilon_i(b)$. In this paper all augmentations of A^m will be diagonal.

Inside A^m there are special elements known as composable words. A word $b_{i_1 j_1} ... b_{i_n j_n}$ is called **composable** if $j_k = i_{k+1}$ for all i, j. We will not describe the differential here; we only need to know that $\partial(b_{ij})$ is a sum of composable words from i to j.

We are interested in a linearized version of the above chain complexes. Let $\phi_\epsilon : A^m_\epsilon \to A^m_\epsilon$ be the map defined by $\phi_\epsilon(a) = a + \epsilon(a)$. We define a new differential on A^m by $\partial^m_\epsilon = \phi_\epsilon^{-1} \circ \partial^m \circ \phi_\epsilon$. We note that ∂^m_ϵ has no constant terms.

Let C^{ij} denote the $\mathbb{Z}/2$ submodule of (A^m, ∂^m) generated by the crossings a^{ij}, x^{ij}, y^{ij}. Then since ∂ is made up of composable words we have that $\partial^m_\epsilon|_{C^{ij}}$ breaks into a direct sum of maps of the form

$$\partial^m_\epsilon : C^{ij} \to C^{ij_1} \otimes ... \otimes C^{i_m j}$$

In particular we are interested in $\partial^m_\epsilon|_{C^{1m}}$. The dual of this map will be denoted μ^m_ϵ. The multiplication maps $\mu^2_{\epsilon_1,\epsilon_2}$ satisfy $(\mu^2_{\epsilon_1,\epsilon_2})^2 = 0$, hence $(C^{*12}, \mu^2_{\epsilon_1,\epsilon_2})$ is a cochain complex where C^{*12} denotes the dual complex.

We can now define the A_∞ augmentation category $Aug_+(\Lambda)$ as follows:

- *Objects:* The objects are graded augmentations $\epsilon : \Lambda \to \mathbb{Z}_2$.
- *Morphisms:* The morphisms between two augmentations ϵ_1 and ϵ_2 are the cochain complexes $(C^{*12}, \mu^2_{\epsilon_1,\epsilon_2})$.
- *Multiplication Maps:* The multiplication maps are given by the μ^m_ϵ.

As the morphisms of $Aug_+(\Lambda)$ are cochain complexes we can form the cohomology category $HAug_+(\Lambda)$ as follows:

- *Objects:* The objects are graded augmentations $\epsilon : \Lambda \to \mathbb{Z}_2$
- *Morphisms:* The morphisms between ϵ_1 and ϵ_2 are the cohomology groups of the chain complex $(C^{*12}, \mu^2_{\epsilon_1,\epsilon_2})$.

The cohomology groups for $Aug_-(\Lambda)$ will be denoted $LCH_{\epsilon_1,\epsilon_2}(\Lambda)$

2.2.4 Functors

We now show how to construct a functor between two augmentation categories. This brings us to the notion of a consistent sequence. A sequence of DGA maps $\Psi^m : (A^m, \partial^m) \to (B^m, \partial^m)$ is called **consistent** if $\Psi^m(a^{ij})$ is a sum of composable words from i to j in B^m. We note that in general, stronger conditions are required for a sequence to be consistent. The reader should consult [NRSSZ] for details.

Given a consistent sequence as above, we construct an A_∞ functor

$$F : Aug_+(B) \to Aug_+(A)$$

as follows:

On objects we have $F(\epsilon) = \epsilon \circ \Psi$. We denote the submodules of (A^m, ∂^m) and (B^m, ∂^m) as C_A^{ij} and C_B^{ij} respectively.

For each k we need to define maps

$$F_k : C_B^{*k,k+1} \otimes ... \otimes C_B^{*12} \to C_A^{*1,k+1}.$$

Consider the diagonal augmentation $\epsilon = (\epsilon_1, ..., \epsilon_{k+1})$ of B^{k+1} and let

$$\Psi_\epsilon^{k+1} = \Phi_\epsilon \circ \Psi^{k+1} \circ \Phi^{-1}_{\Psi^*(\epsilon)}.$$

Note that this map contains no constant term. We then define F_k by dualizing the component of Ψ_ϵ^{k+1} that maps

$$C_A^{1,k+1} \to C_B^{12} \otimes ... \otimes C_B^{k,k+1}.$$

The A_∞ functor F is defined as the collection of maps $\{F_k\}$.

We note that F_1 descends to a well-defined map on cohomology, hence a consistent sequence also induces a functor on cohomology categories.

2.3 Results

Recall that 0-resolution induces a DGA map $\Psi : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-)$. The following lemma gives the structure Ψ for Legendrian closures of positive braids when a crossing in the braid portion is resolved.

Lemma 2.3.1. *Let $b_{m,n}$ be the crossing to be resolved. Then we claim the following:*

1. *Let $(i,j) < (m,n)$ in the lexicographic ordering. Then $\psi_1(b_{i,j})$ is a polynomial, possibly with constant term 1, whose monomials are products of elements of the form $b_{l,k}$ for $(i,j) < (l,k) < (m,n)$.*

2. *Let $(i,j) > (m,n)$ in the lexicographic ordering. Then $\psi_1(b_{i,j})$ is a polynomial, possibly with constant term 1, whose monomials are products of elements of the form $b_{l,k}$ for $(i,j) > (l,k) > (m,n)$.*

3. *$\psi_1(c_{i,j})$ is a polynomial, possibly with constant term 1, whose monomials are products of elements of the form $b_{l,k}$ for $(l,k) < (m,n)$ and $c_{i,w}$ for $w > j$.*

4. *$\psi_1(s_{i,j})$ is a polynomial, possibly with constant term 1, whose monomials are products of elements of the form $b_{l,k}$ for $(l,k) > (m,n)$ and $a_{i,w}$ for $w < j$.*

The essential content of the lemma is that the disks which contribute to $\psi_1(d)$ for a crossing d of the link only have negative corners at crossings which are between d and $b_{m,n}$ in the ordering given in the proof of Theorem 2.3.1.

Proof: We prove Case 3, which is slightly more difficult than Cases 1 and 2 and analogous to Case 4. Let u be a disk which contributes to $\psi(c_{i,j})$. We first make the observation that the positive puncture of u at $c_{i,j}$ lies in the bottom left quadrant of $c_{i,j}$.

Starting at $c_{i,j}$ the top strand of u tavels left, either turning down at some c crossing or none and entering the braid. The bottom strand travels down until it enters the braid. From here both strands continue to the right, possible turning at negative punctures but always continuing to the right. We want to show that the strands cannot extend farther right than $b_{m,n}$. Assume they extend farther to the right. If the strands meet somewhere in the positive braid past $b_{m,n}$ then this is a third positive puncture, which contradicts u contributing to $\psi_1(c_{i,j})$. If they do not meet then they must exit the positive braid on the right side. We then claim that u has a third positive corner at some $s_{i,j}$. Let the bottom strand exit the braid at the i^{th} level. This strand must turn right on or before $s_{i+1,i}$. Then we see that u has a positive corner at $s_{j,j}$ for some $j > i$. □

We are now able to prove our main theorem:

Theorem 2.3.1. *Let Λ_+ be the Legendrian closure of a positive braid and let Λ_- be the link obtained after resolving a crossing of Λ_+. Then there are cohomologically faithful A_∞ functors $F^+ : Aug_+(\Lambda_-) \to Aug_+(\Lambda_+)$ and $F^- : Aug_-(\Lambda_-) \to Aug_-(\Lambda_+)$ which are induced by 0-resolution.*

Proof: We prove Theorem 2.3.1 for Aug_+. The proof for Aug_- is similar.

Let $\Psi : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-)$ be the map induced by 0-resolution. We will construct a consistent sequence of DGA maps $\Psi^m : A^m(\Lambda_+) \to A^m(\Lambda_-)$. Let $a \in A(\Lambda_+)$. Let I_m^{ij} be the set of all composable sequences from i to j whose superscripts are at most m. For a word $b_1 b_2 ... b_l$ in $\Psi(a)$ and $J \in I_m^{ij}$ we denote by $b_1 b_2 ... b_l^J$ the element of $A^m(\Lambda_-)$ corresponding to $b_1 b_2 ... b_l$ indexed by the composable sequence J. Then we set

$$\Psi^m(a^{ij}) = \sum_{J \in I_m^{ij}} \Psi(a)^J.$$

In other words, $\Psi^m(a^{ij})$ is the sum of all possible composable sequences from i to j of $\Psi(a)$. We also set Ψ^m to be the identity on the mixed chords x^{ij}, y^{ij}.

This consistent sequence gives rise to an A_∞ functor F^+ as in Section 2.2.4.

To show that F^+ is cohomologically faithful, we will show that $F_1^+ : C_B^{*12} \to C_A^{*12}$ is injective. We will abuse notation and use a^{ij} to denote both the element of C^{ij} as well as its dual. It is clear the F_1^+ is the identity on x^{12}, y^{12}. We will order the non-Morse elements of C^{*12} in the following way:

1. $c_{i,j} < c_{m,n}$ if $n < j$, or if $n = j$, $m < i$.

2. $b_{i,j} < b_{m,n}$ if $(i,j) < (m,n)$ in the lexicographic ordering.

3. $s_{i,j} < s_{m,n}$ if $n < j$ or if $n = j$, $i < m$.

4. $c_{i,j} < b_{i,j} < s_{i,j}$ for all i, j.

Then by Lemma 3.1, the linear part of Ψ_ϵ^2 on the non-Morse chords of C_A^{12} is a block triangular matrix of the form:

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & * & * & 1 \end{pmatrix}$$

The switch from upper triangular to lower triangular occurs at $b_{m,n}$, the resolved crossing. Then F_1 is obtained by taking the transpose of this matrix and removing the row and column associated to $b_{m,n}$. This matrix is block triangular and hence is an invertible matrix. Therefore F_1 descends to an injective map on cohomology. The proof of Corollary 3.3 will show that this implies F is a cohomologically faithful functor. □

Using Theorem 2.3.1 we can prove:

Corollary 2.3.2. *Let F^- be the functor from Theorem 2.3.1. Let ϵ_1 and ϵ_2 be augmentations of Λ_-. Then*

$$LCH^*_{F^-(\epsilon_1),F^-(\epsilon_2)}(\Lambda_+) \simeq LCH^*_{\epsilon_1,\epsilon_2}(\Lambda_-) \oplus \mathbb{Z}/2[0].$$

Proof: Let C_- denote the chain complex associated to Λ_- and C_+ the chain complex associated to Λ_+. From the proof of Theorem 2.3.1 we know that there is an injective map F_1^- from C_-^* to C_+^*. This gives rise to a short exact sequence of chain complexes:

$$0 \to C_-^* \xrightarrow{F_1^-} C_+^* \to C_+^*/C_-^* \to 0.$$

The complex C_+^*/C_-^* consists of one element in degree 0, the crossing $b_{m,n}$ which was resolved. The differential of this complex is identically 0. Therefore we have that its cohomology is isomorphic to $\mathbb{Z}/2$ in degree 0 and is 0 elsewhere. This short exact sequence gives rise to the following long exact sequence in cohomology:

$$\begin{aligned} 0 &\to H_0(C_+^*/C_-^*) \to LCH_0^{F(\epsilon_1),F(\epsilon_2)}(C_+^*) \to LCH_0^{\epsilon_1,\epsilon_2}(C_-^*) \\ &\to H_1(C_+^*/C_-^*) \to LCH_1^{F(\epsilon_1),F(\epsilon_2)}(C_+^*) \to LCH_1^{\epsilon_1,\epsilon_2}(C_-^*) \to 0. \end{aligned}$$

It is then immediate from the above sequence that

$$LCH_0^{F(\epsilon_1),F(\epsilon_2)}(C_+^*) \simeq LCH_0^{\epsilon_1,\epsilon_2}(C_-^*) \oplus \mathbb{Z}/2.$$

and

$$LCH_1^{F(\epsilon_1),F(\epsilon_2)}(C_+^*) \simeq LCH_1^{\epsilon_1,\epsilon_2}(C_-^*).$$

□

2.3.1 Example

In this section we show an small example to illustrate the maps for Aug_-. Let Λ_+ be the positive braid with 3 strands whose crossings are labeled $b_{2,1}, b_{1,2}, b_{2,2}, b_{3,2}$. We will resolve $b_{2,1}$ which will result in the $(3,2)$ torus knot union a copy of the unknot. The Legendrian closure of Λ_+ also has crossings $c_{2,1}, c_{3,1}, c_{3,2}$ and $s_{2,1}, s_{3,1}, s_{3,2}, s_{1,1}, s_{2,2}, s_{3,3}$.

ψ_0 is the identity on all chords except for $b_{2,1}$ for which $\psi_0(b_{2,1}) = 1$. The map ψ_1 is given as follows:

- $\psi_1(b_{2,1}) = 0$
- $\psi_1(b_{1,2}) = 0$
- $\psi_1(b_{2,2}) = 0$
- $\psi_1(b_{3,1}) = 0$
- $\psi_1(c_{2,1}) = b_{1,2}$
- $\psi_1(c_{3,1}) = 1 + c_{3,2} \cdot b_{1,2}$
- $\psi_1(c_{3,2}) = 0$

- $\psi_1(s_{3,1}) = 0$
- $\psi_1(s_{3,2}) = s_{3,1} + s_{3,1} \cdot b_{2,2} \cdot b_{3,1}$
- $\psi_1(s_{3,3}) = s_{3,1} \cdot b_{2,2}$
- $\psi_1(s_{2,1}) = 0$
- $\psi_1(s_{2,2}) = s_{2,1} + s_{2,1} \cdot b_{2,2} \cdot b_{3,1}$
- $\psi_1(s_{1,1}) = 0$

The map $\Psi : \Lambda_+ \to \Lambda_-$ is the sum of ψ_0 and ψ_1.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 are augmentations of Λ_-. In order to compute F_1^- we will need to know Ψ_ϵ^2. $(\psi_0)_\epsilon^2$ is the identity on every crossing except $b_{2,1}$ for which $(\psi_1)_\epsilon^2(b_{2,1}) = 0$.

The non-trivial parts of $(\psi_1)_\epsilon^2$ are as follows:

- $c_{2,1} \mapsto b_{1,2}$
- $c_{3,1} \mapsto \epsilon_1(c_{3,2}) \cdot b_{1,2} + c_{3,2} \cdot \epsilon_2(b_{1,2})$
- $s_{3,2} \mapsto s_{3,1} + \epsilon_1(s_{3,1}) \cdot \epsilon_1(b_{2,2}) \cdot b_{3,1} + \epsilon_1(s_{3,1}) \cdot b_{2,2} \cdot \epsilon_2(b_{3,1}) + s_{3,1} \cdot \epsilon_2(b_{2,2}) \cdot \epsilon_2(b_{3,1})$
- $s_{3,3} \mapsto \epsilon_1(s_{3,1}) \cdot b_{2,2} + s_{3,1} \cdot \epsilon_2(b_{2,2})$
- $s_{2,2} \mapsto \epsilon_1(s_{2,1}) \cdot \epsilon_1(b_{2,2}) \cdot b_{3,1} + \epsilon_1(s_{2,1}) \cdot b_{2,2} \cdot \epsilon_2(b_{3,1}) + s_{2,1} \cdot \epsilon_2(b_{2,2}) \cdot \epsilon_2(b_{3,1})$

Using the ordering given in Theorem 2.3.1, this is a matrix of the form given in the proof of Theorem 2.3.1. Hence removing the row and column corresponding to $b_{2,1}$ and taking the transpose gives F_1^-. □

The bilinearized Legendrian contact cohomology of the $(3, 2)$ torus knot is calculated in [BC]. The extra copy of the unknot contributes a $\mathbb{Z}/2$ factor in degree 1. Using Corollary 2.3.2 we see that if ϵ_1 and ϵ_2 are distinct then:

$$LCH^*_{F^-(\epsilon_1),F^-(\epsilon_2)}(\Lambda_+) \simeq (\mathbb{Z}/2)^2[0] \oplus (\mathbb{Z}/2)^2[1].$$

□

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