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Applications of the Link Surgery Formula in Heegaard Floer Homology

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Abstract of the Dissertation

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Heegaard Floer homology is combinatorially computable, but a convenient computational scheme in general is still missing, especially for HF^- of hyperbolic manifolds. We aim to use the Manolescu-Ozsváth the link surgery formula for computing Heegaard Floer homology of surgeries on links and finding applications on L-space surgeries on links. The main difficulty is to reduce the complexity of the algorithms.

We give a polynomial time algorithm to compute the whole package of the completed Heegaard Floer homology $\mathbf{HF}^-$ of all surgeries on a two-bridge link of slope q/p, $L = b(p,q)$, by using nice diagrams and some algebraic rigidity results to simplify the link surgery formula.

We also initiate a general study of the definitions, properties, and examples of L-space links. In particular, we find many hyperbolic L-space links, including some chain links and two-bridge links; from them, we obtain many hyperbolic L-spaces by integral surgeries, including the Weeks manifold. We give bounds on the ranks of the link Floer homology of L-space links and on the coefficients in the multi-variable Alexander polynomials. We also describe the Floer homology of surgeries on any L-space link using the link surgery formula of Manolescu and Ozsváth.

As applications, we compute the graded Heegaard Floer homology of surgeries on 2-

component L-space links in terms of only the Alexander polynomial and the surgery framing. We also give a fast algorithm to classify L-space surgeries among them.

The dissertation of Yajing Liu is approved.

Robert Brown

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To my parents

Xining Liu and Yuming Zhang

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CHAPTER 1

Introduction

1.1 Background and Motivation.

The study of the topology of 4-manifolds was revolutionized in 1982 by the work of Donaldson, who has initiated the study of smooth structures on 4-manifolds using Yang-Mills theory. In order to distinguish different smooth structures on the same topological 4-manifold, the idea is to study the moduli space of solutions of certain PDE's on the smooth manifold, with some additional geometric structures prefixed such as a Riemannian metric. If one can derive some quantities from the moduli space of solutions which only depend on the smooth structure of the underlying 4-manifold, one call these quantities smooth 4-manifold invariants. However, it is usually hard to compute these invariants from the definitions and hard to see the internal relationship between these invariants and the smooth topology of the underlying manifold.

Instead of considering Yang-Mills equations, Seiberg and Witten wrote down a different set of equations, which are easier to work with. They defined their invariants and conjectured a relationship between their invariants and the Donaldson polynomial invariants. In order to better understand the Seiberg-Witten invariants, Ozsváth and Szabó defined Heegaard Floer homology for connected oriented closed 3-manifolds, which potentially are more computable.

Heegaard Floer homology is a package of invariants of 3-manifolds, and they are defined by using holomorphic disks and Heegaard splittings of the 3-manifold [39, 38]. Furthermore, it fits into a kind of 3+1 dimensional topological quantum field theory, which is important in the study of smooth structures on 4-manifolds. It also detects the Thurston norm and fiberedness of a 3-manifold [36, 8, 30]. Unlike other Floer homological invariants, Heegaard Floer homology is combinatorially computable, and there are several algorithms for computing various versions of it. Manolescu, Ozsváth and Sarkar described knot Floer homology combinatorially using grid diagrams in [48]. Sarkar and Wang in [48] found an algorithm for computing $\widehat{HF}(M^3)$ over $\mathbb{Z}/2\mathbb{Z}$ by using nice Heegaard diagrams. Lipshitz, Ozsváth and Thurston used bordered Floer homology to give another algorithm for computing $\widehat{HF}(M^3)$ in [19]. In [26], Manolescu, Ozsváth and Thurston showed that the plus and minus versions of Heegaard Floer homology (over $\mathbb{Z}/2\mathbb{Z}[[U]]$) can also be described combinatorially, by using link surgery and grid diagrams. (Admittedly, the MOT algorithm has a high time complexity.)

Despite these combinatorial algorithms, a convenient computational scheme in general is still missing, especially for HF^- of hyperbolic manifolds. Hence, improving these algorithms and developing new methods for computations are still important and interesting questions.

Throughout this paper, we use $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients and consider the completed version $\mathbf{HF}^-$.

1.2 Surgeries on Two-bridge Links.

1.2.1 The basic idea.

We are aiming to compute the full package of Heegaard Floer homologies of surgeries on two-bridge links using the link surgery formula due to Manolescu and Ozsváth. As a consequence, we provide new examples of hyperbolic 3-manifolds for which we can compute their Heegaard Floer homology.

When $\overrightarrow{L}$ is a two-bridge link $b(p, q)$, we find a fast algorithm for computing the Floer homology of surgeries on L, $\mathbf{HF}^-(S^3_\Lambda(\overrightarrow{L}))$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, where Λ is the framing matrix of a surgery. (Here, $\mathbf{HF}^-(S^3_\Lambda(\overrightarrow{L}))$ is the U-completion of HF^-. See [24] Section 2.) This algorithm uses genus-0 nice diagrams and algebraic arguments to simplify the Manolescu-

Ozsváth surgery formula. Its worst-case time complexity is a polynomial of p and $\det(\Lambda)$.

Let us mention some related work. In [45], Rasmussen studied Heegaard Floer homology of surgeries on two-bridge knots. In [43], Ozsváth and Szabó developed a formula for the Heegaard Floer homology of surgeries on knots. The paper [24] presents a generalization of this formula to the case of links. Two sets of data are needed in the surgery formula in [24]: the generalized Floer complexes $A_s^-(\overrightarrow{L})$'s and the maps in the surgery formula, namely the maps $\mathcal{I}_s^{\overrightarrow{L}'}, D_s^{\overrightarrow{L}'}$ connecting the complexes associated to oriented sublinks. In general, the Heegaard Floer homology of link surgeries is more difficult to compute, due to more involved algebraic structures. However, in some cases, computations using this surgery formula can be simplified.

The main complexity in the link surgery formula is the counting of the holomorphic domains on the Heegaard surface, which corresponds to holomorphic bigons and polygons in the symmetric product. For the special case of two-bridge links, we directly find a formula for the counts of holomorphic bigons. Furthermore, the general link surgery formula involves counting holomorphic polygons in the symmetric product for computing some cobordism maps, and this is of considerably high time complexity. Here we notice that, for two-bridge links, all these maps can be determined algebraically.

1.2.2 Main results.

Using the Schubert normal form of two-bridge links we get a class of nice Heegaard diagrams called *Schubert Heegaard diagrams*, in which every region is either a bigon or a square. We can explicitly describe all the composite bigons on a Schubert Heegaard diagram, and hence the Floer differentials. Further, we get a formula for the Alexander gradings of all intersection points, thus giving a formula for the multi-variable Alexander polynomial of a two-bridge link $b(p, q)$ in terms of p, q. See Theorem 3.1.22 and Proposition 3.1.17 below for the precise statements. This implies that $A_{\mathrm{s}}^-(\overrightarrow{L})$ can be directly computed from this diagram. For a two-bridge link $\overrightarrow{L} = b(p, q)$, we get an $O(p^2)$ time algorithm for computing

$A^-_{\mathrm{s}}(\overrightarrow{L})$. We also found different two-bridge links (modulo mirror and reorientation) sharing the same multi-variable Alexander polynomial, signature, and linking number.

Using algebraic arguments, we show some rigidity results of the destabilization maps $D^{\overrightarrow{M}}_{\mathrm{s}}$'s, $M \subset L$ up to chain homotopy, for two-bridge links. Further, if we perturb the destabilization maps $D^{\pm L_i}_{\mathrm{s}}$'s by chain homotopy, i.e. replace $D^{\pm L_i}_{\mathrm{s}}$ by $\tilde{D}^{\pm L_i}_{\mathrm{s}} \simeq D^{\pm L_i}_{\mathrm{s}}$, we can construct a new square of chain complexes called the *perturbed surgery complex.* Using the rigidity results, we show that the perturbed surgery complex is isomorphic to the original complex in the link surgery formula. Based on the perturbed surgery complex, we give the algorithm for computing $\mathbf{HF}^-(S^3_\Lambda(\overrightarrow{L}))$ mentioned before.

The main result we obtain is the following:

Theorem 1.2.1. *Suppose $\overrightarrow{L}$ is an oriented two-bridge link with framing Λ. Let $\mathcal{H}^L$ be a basic Heegaard diagram of $\overrightarrow{L}$ and let $\mathcal{H}$ be a primitive system induced by $\mathcal{H}^L$. After we determine the $\mathbb{F}[[U_1, U_2]]$-modules $A^-_{\mathrm{s}}(\overrightarrow{L})$'s sitting at the vertices of the square in the link surgery formula, any choices of*

- *$\mathbb{F}[[U_1]]$-linear chain homotopy equivalences $\tilde{D}^{-L_i}_{s_1,s_2}$ for the edge maps,*
- *$\mathbb{F}[[U_1]]$-linear chain homotopies for the diagonal maps*

yield a perturbed surgery complex $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ which is isomorphic to the original surgery complex in [24] as an $\mathbb{F}[[U_1]]$-module. By imposing the U_2-action to be the same as the U_1-action, the $\mathbb{F}[[U_1, U_2]]$-module $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}, \Lambda), \tilde{\mathcal{D}}^-)$ becomes isomorphic to the homology $\mathbf{HF}^-(S^3_\Lambda(\overrightarrow{L}))$. This isomorphism preserves the absolute grading.

See Theorem 3.2.13 below for a more precise statement and the proof.

Corollary 1.2.2. *For a two-bridge link $\overrightarrow{L}$, knowledge of the $A^-_{\mathrm{s}}(\overrightarrow{L})$ determines $\boldsymbol{HF}^-$ of all the surgeries on L.*

We also compute some examples explicitly: the surgeries on $b(4n, 2n + 1)$, $n \in \mathbb{N}$, which are two sequences of hyperbolic two-bridge links generalizing the Whitehead link and the torus link $T(2,4)$. See Proposition 3.3.9 and Theorem 3.3.10. Actually in the course of the computation, we also show that the Whitehead link is an L-space link, which means all of its large surgeries are L-spaces, i.e. $A^-_{s_1,s_2}(Wh)$'s all have homology $\mathbb{F}[[U]]$. This provides examples of hyperbolic L-spaces.

To compute these examples, we study the filtered homotopy type of $CFL^-(L)$. We prove that when $L = b(4n, 2n+1)$, the filtered chain homotopy type of $CFL^-(L)$ is determined by the filtered chain homotopy type of $\widehat{CFL}(L)$. See Proposition 3.3.5 and Proposition 3.3.6 for the precise statements. Basically, this is based on an observation that the Alexander polytope is simple and there are several symmetries on $CFL^-(L)$, which give constraints for the differentials in $CFL^-(L)$. From $CFL^-(L)$ we derive all the $A^-_{\mathrm{s}}(L)$'s and the inclusion maps. Finally, using the perturbed surgery complex, we compute the Floer homology of their surgeries and the associated d-invariants.

Since $CFL^-(L)$ is the same as $A^-_{+\infty,+\infty}(L)$ viewed as a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex (with the Alexander filtration), Corollary 1.2.2 means the filtered homotopy type of $CFL^-(L)$ contains all the information about the Floer homology of the surgeries on L, when L is a two-bridge link. In [42], it is shown that, for an alternating two-component link L, the filtered chain homotopy type of $\widehat{CFL}(L)$ is determined by the set of data:

- the multi-variable Alexander polynomial $\Delta_L(x,y)$,
- the signature $\sigma(L)$,
- the linking number $\mathrm{lk}(L)$,
- the filtered homotopy type of $\widehat{CFK}(L_i)$ of each component.

However, it is hard to determine the filtered homotopy type of $CFL^-(L)$ in general. For two-bridge links, the Schubert Heegaard diagrams show that the U_1, U_2-differentials in $CFL^-(L)$ are quite simple, since the bigons always contain exactly one basepoint. In addition, every component of a two-bridge link is the unknot. Thus, for two-bridge links, we raise the following question:

Question 1.2.3. *Given an oriented two-bridge link L, is the filtered homotopy type of $CFL^-(L)$ determined by the set of data $\{\Delta_L(x,y), \sigma(L), \mathrm{lk}(L)\}$?*

We notice that $\mathbf{HF}^-$ of surgeries on two-bridge links may also be computed by other methods. For example, as long as one of the framing coefficients is not 0, one can view one component as a knot in a lens space and compute using the grid diagram methods in [1]. Another method is to consider these surgeries as surgeries on (1,1)-knots in lens spaces and use the method of [9]. Nevertheless, the method in this paper is more conceptual. Some of the arguments here could be potentially used for other classes of links. In fact, Theorem 1.2.1 and Corollary 1.2.2 can be directly generalized to the two-component links with every component being an L-space knot.

1.3 L-space Links.

1.3.1 L-spaces and L-space knots.

Definition 1.3.1 ($\mathbb{Z}/2\mathbb{Z}$-L-space)**.** A 3-manifold M is called an L-space, if it is a rational homology sphere and $\dim_{\mathbb{F}}(\widehat{HF}(M)) = |H_1(M)|$.

Examples of L-spaces include all 3-manifolds with elliptic geometry and double branched covers over quasi-alternating links. L-spaces are of interests in 3-manifold topology. An L-space does not admit any co-oriented C^2 taut foliations; see Theorem 1.4 from [36]. While examples of closed hyperbolic manifolds admitting no taut foliations are very interesting and first found in [46] and [4] by considering their fundamental groups. In fact, any hyperbolic $\mathbb{Z}/2\mathbb{Z}$-L-space also provides an example of hyperbolic manifold admitting no co-oriented taut foliations. This is because in the proof of Theorem 1.4 of [36], it is pointed out that any $\mathbb{Z}/p\mathbb{Z}$-L-space does not admit a co-oriented taut foliation for all prime numbers p. There is also a conjecture of Boyer-Gordon-Watson from [2] relating L-spaces with left-orderability of the fundamental group.

In [40], L-space knots were introduced by Oszváth and Szabó, in order to study the

Berge conjecture on lens space surgeries on knots in S^3. For further results towards the Berge conjecture, see [12, 13].

Definition 1.3.2 (L-space knot)**.** A knot $K \subset S^3$ is called an L-space knot, if there is a positive integer n, such that the n-surgery on K is an L-space.

Since every 3-manifold is a surgery on a link in S^3, one can study L-spaces by surgeries on links. In this paper, we focus on a class of links called *L-space links*, whose large surgeries are all L-spaces. These links are natural generalizations of L-space knots. The terminology of L-space links was introduced by Gorsky and Némethi in [11] to study algebraic links. Actually, Ozsváth, Stipsicz and Szabó have shown that all plumbing trees are L-space links in [33]. The surgeries on algebraic links and plumbing trees are all graph manifolds. In this paper, we give many examples of hyperbolic L-space links, including some families of two-bridge links and chain links. In turn, these hyperbolic L-space links provide many examples of hyperbolic L-spaces, including the famous Weeks manifold; see Chapter 4. In fact, all the examples of hyperbolic L-spaces by considering large surgeries on L-space links are derived from elliptic L-spaces, by using the surgery exact triangle of Floer homology.

It turns out that L-space links are rich in geometry and simple in algebra. All the generalized Floer complexes are chain homotopy equivalent to $\mathbb{F}[[U]]$ and the link Floer homology are controlled by their Alexander polynomials; see Chapter 4 and 5. Moreover, there are L-space links of all kinds of geometry with arbitrarily many components, including non-prime links, torus links, satellite links, and hyperbolic links; see Example 1.3.13. There are also non-fibered prime L-space links, contrasting L-space knots.

Examples and properties of L-space knots have been extensively studied in the literature. We list some of them here.

Example 1.3.3. Examples of L-space knots include lens space knots such as Berge knots (up to mirror), algebraic knots (which are torus knots and their cables), and $(-2, 3, q)$ pretzel knots with $q > 1$ odd (which are hyperbolic). See [40, 14, 18, 16].

Fact 1.3.4 ([44]). *A positive rational L-space surgery implies a positive integer L-space surgery; a positive integer L-space surgery implies that all large surgeries are L-spaces.*

Fact 1.3.5 ([40]). *If K is an alternating L-space knot, then K is a $T(2, 2n+1)$ torus knot.*

Fact 1.3.6 ([30]). *An L-space knot is a fibered knot.*

Fact 1.3.7 ([40]). *Let K be an L-space knot. The knot Floer homology $\widehat{HFK}(K)$ is determined by the Alexander polynomial of K, and* $\mathrm{rank}(\widehat{HFK}(K,s)) \leq 1, \forall s \in \mathbb{Z}$.

These properties provide strong constraints on L-space knots. However, it turns out that none of the above properties extends to *L-space links* immediately.

1.3.2 Main results on L-space links.

In [11], Gorsky and Némethi define L-space links in terms of large surgeries.

Definition 1.3.8 (L-space link). An l-component link $L \subset S^3$ is called an *L-space link*, if all of its positive large surgeries are L-spaces, that is, there exist integers $p_1, ..., p_l$, such that $S^3_{n_1,...,n_l}(L)$ is an L-space for all $n_1, ..., n_l$ with $n_i > p_i, \forall 1 \leq i \leq l$. Note that whether L is an L-space link does not depend on the orientation of L. A link L is called a *non-L-space link*, if neither L nor its mirror is an L-space link.

The large surgeries on the link L are governed by the *generalized Floer complexes* $\mathfrak{A}^-_{\mathbf{s}}(L)$'s with $\mathbf{s} \in \mathbb{H}(L)$, which were introduced by Manolescu and Ozsváth in [24]. Here, $\mathbb{H}(L)$ is defined below. Also, see Definition 4.1.1 for the generalized Floer complexes.

Definition 1.3.9 ($\mathbb{H}(L)$). For an oriented link L with l components, we define $\mathbb{H}(L)$ to be the affine lattice over $\mathbb{Z}^l$,

$$\mathbb{H}(L) = \bigoplus_{i=1}^{l} \mathbb{H}(L)_i, \quad \mathbb{H}(L)_i = \mathbb{Z} + \frac{\mathrm{lk}(L_i, L - L_i)}{2}.$$

Based on the knowledge of $\mathfrak{A}^-_{\mathbf{s}}(L)$, we have the following necessary condition on L-space links.

Lemma 1.3.10. *If L is an L-space link, then all sublinks of L are L-space links.*

We also formulate L-space links in three other equivalent ways, which are easy to use. To this end, we study the relation between L-space surgeries and large surgeries on links. Using the *L-space surgery induction lemma* (Lemma 4.1.4) and the generalized Floer complexes, we give the following result.

Proposition 1.3.11. *The following conditions are equivalent:*

(i) *L is an L-space link;*

(ii) *there exists a surgery framing $\Lambda(p_1, ..., p_l)$, such that for all sublink $L' \subseteq L$, $\det(\Lambda(p_1, ..., p_l)|_{L'}) > 0$ and $S^3_{\Lambda|_{L'}}(L')$ is an L-space; (Notice that at this time Λ is positive definite.)*

(iii) $H_*(\mathfrak{A}^-_{\boldsymbol{s}}(L)) = \mathbb{F}[[U]], \forall \boldsymbol{s} \in \mathbb{H}(L)$;

(iv) $H_*(\hat{\mathfrak{A}}_{\boldsymbol{s}}(L)) = \mathbb{F}, \forall \boldsymbol{s} \in \mathbb{H}(L)$.

Using grid diagrams as in [25], one can compute $\mathfrak{A}^-_{\mathbf{s}}$ combinatorially and check condition (iii) or (iv). On the other hand, for special class of links, it is more convenient to use condition (ii). For instance, it follows immediately that an algebraically split link is an L-space link if and only if it admits a positive surgery Λ such that the surgeries restricted to sublinks are all L-spaces. Note that if we work with $\mathbb{Z}$ coefficients, conditions (i) and (ii) are also equivalent.

In contrast to Fact 1.3.4, a single L-space surgery (with positive surgery coefficients) on L fails to imply that all the large surgeries on L are L-spaces. See Example 4.1.3. It leads us to define *weak L-space links*.

Definition 1.3.12 (Weak L-space link)**.** A link L is called a *weak L-space link*, if there exists an L-space surgery on L.

There are generalizations of L-space links, called *generalized* $(\pm\cdots\pm)$*L-space links*, by considering the corresponding types of generalized large surgeries. There are also parallel theories of $\mathfrak{A}^-_{\mathbf{s}}$ for generalized large surgeries and the link surgery formula. See Chapter 4. An L-space link is literally a generalized $(+\cdots+)L$-space link. Note that there are generalized $(+-)L$-space links that are *non-L-space* links.

Example 1.3.13. We have the following examples of L-space links and generalized L-space links.

(A) Split disjoint unions of L-space knots are L-space links.

(B) Two-bridge links $b(rq-1,-q)$ with r,q being positive odd integers are all L-space links, which include $T(2,2n)$ torus links. See Theorem 4.2.8. Note that except for $T(2,2n)$, they are all hyperbolic links.

(C) A 2-component L-space link: $L7n1$ in the Thistlethwaite link table. See Example 4.2.17.

(D) Some 3-component L-space links: Borromean rings, $L6a5$, $L6n1$, $L7a7$ and a link in Example 4.2.3. See Example 4.2.17.

(E) A hyperbolic 4-chain L-space link: See Example 4.2.12.

(F) A hyperbolic 5-chain generalized $(++++-)L$-space link: See Example 4.2.13.

(G) Two families of hyperbolic L-space chain links: See Example 4.2.14 and Example 4.2.15.

(H) A sequence of plumbing graphs that are generalized L-space links: See Example 4.2.16.

(I) All plumbing trees of unknots are L-space links. This was proved by Ozsváth and Szabó in [35]. See Example 4.2.10.

(J) All algebraic links are L-space links. This was proved by Gorsky and Némethi in [11].

(K) See Table 4.2.2 for the list of which links with crossing number ≤ 7 are L-space links.

In contrast to Fact 1.3.5, there are alternating hyperbolic L-space links, for example, all two-bridge links $b(rq-1,-q)$ with $r,q>1$ being positive odd integers. Surgeries on these hyperbolic L-space links can give examples of hyperbolic L-spaces which are neither surgery nor double branched cover over any knot. See Example 4.2.1. In fact, surgeries on these L-space two-bridge links are always double branched covers over some links. It is not clear to us whether those links are quasi-alternating or not.

In relation to Example 1.3.13 (B), we make the following conjecture:

Conjecture 1.3.14. *The set of all L-space two-bridge links is*

$$\{b(rq-1,-q) : r, q \text{ are positive odd integers}\}.$$

Using the algorithm from [22] for computing $\hat{\mathfrak{A}}_{\mathbf{s}}(L)$ for two-bridge links, we verify that Conjecture 1.3.14 is true for all two-bridge links $b(p,q)$ with $p \leq 100$.

Compared with Fact 1.3.7, we study the Alexander polynomials of L-space links using $\mathfrak{A}^-_{\mathbf{s}}(L)$.

Theorem 1.3.15. *Suppose L is an l-component L-space link with $l \geq 2$, and has the multi-variable Alexander polynomial as follows*

$$\Delta_L(x_1, ..., x_l) = \sum_{i_1,...,i_l} a_{i_1,...,i_l} \cdot x_1^{i_1} \cdots x_l^{i_l}.$$

Then,

$$\operatorname{rank}_{\mathbb{F}}(HFL^-(L, \boldsymbol{s})) \leq 2^{l-1}, \forall \boldsymbol{s} \in \mathbb{H}(L), \tag{1.3.1}$$

$$-2^{l-2} \leq a_{i_1,...,i_l} \leq 2^{l-2}, \forall i_1, ..., i_l. \tag{1.3.2}$$

In particular, for a 2-component L-space link, the multi-variable Alexander polynomial has non-zero coefficients ± 1. Moreover, fixing i_1, the signs of non-zero $a_{i_1,*}$'s are alternating; similarly fixing i_2, the signs of non-zero a_{*,i_2}'s are alternating.

Remark 1.3.16. Inequality (1.3.1) is sharp for $l = 2$. For example, for the Whitehead link Wh, $HFL^-(Wh, 0, 1)$ equals to $\mathbb{F} \oplus \mathbb{F}$. Inequality (1.3.1) can also be deduced from a spectral sequence of Gorsky and Némethi from [11].

Inequality (1.3.2) is sharp for $l = 3$. The mirror of $L7a7$ is an L-space link with Alexander polynomial

$$\Delta_{L7a7}(u, v, w) = \frac{uvw - uv - uw + 2u - 2vw + v + w - 1}{\sqrt{uvw}}.$$

In contrast to knots, the Alexander polynomial condition does not give strong constraints for alternating links. In [40], it is shown that if K is an alternating knot with Alexander polynomial satisfying the condition in Fact 1.3.7, then K is a $T(2, 2n+1)$ torus knot; see Proposition 4.3.2 and Theorem 4.3.3. On the other hand, we find infinitely many hyperbolic alternating links with multi-variable Alexander polynomial satisfying Inequality (1.3.2), including L-space links and non-L-space links. See Chapter 4.

Theorem 1.3.15 also implies that a Floer homologically thin L-space 2-component link L has fibered link exterior.

In contrast to Fact 1.3.6, there are non-fibered L-space links. For example, the split disjoint union of two L-space knots is a non-fibered L-space link, since the complement is not irreducible any more. In fact, there are also many non-fibered L-space links among hyperbolic two-bridge links. See Example 4.2.9.

Actually, there are additional constraints on the Alexander polynomials of an L-space link; see Theorem 5.1.11 and Theorem 5.1.13 below for the precise statements. As a consequence, either of these theorems implies that $L7n2$ is not an L-space link, while Theorem 1.3.15 fails to do so.

1.4 Surgeries on L-space links.

Despite many algorithms on computing various versions of Heegaard Floer homology, explicit computations of plus/minus versions for 3-manifold invariants have only been done on a few cases, such as surgeries on knots and some mapping tori of surfaces, by exploiting surgery exact triangles. In [15], Hom pointed out that the result from [40] further implies that the whole package of Heegaard Floer homology of surgeries on an L-space knot K is determined by the Alexander polynomial of K and the surgery coefficients.

In this paper, we study the computation of Heegaard Floer invariants for integral surgeries on an L-space link L, including the completed Heegaard Floer homology $\mathbf{HF}^-$, absolute gradings, and the cobordism maps, using the link surgery formula of Manolescu-Ozsváth from [24]. The Manolescu-Ozsv'ath surgery complex is an object in the category of chain complexes of $\mathbb{F}[[U]]$-modules, while it can also be considered as an object in the homotopy category of chain complexes of $\mathbb{F}[[U]]$-modules. In [22], any representative in this chain homotopy equivalence class is called a *perturbed surgery complex*. Some algebraic rigidity results are established in [22], which imply that $\mathfrak{A}^-_{\mathbf{s}}(L)$ is chain homotopic to $\mathbb{F}[[U]]$ by a $\mathbb{F}[[U]]$-linear chain map preserving the $\mathbb{Z}$-grading.

Thus, for an L-space link L, the perturbed surgery complex turns out to be largely simplified. When L has 1 or 2 components, all the information needed in the perturbed surgery complex is completely determined by the Alexander polynomial and the surgery framing matrix.

Theorem 1.4.1. *For a 2-component L-space link $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$, all Heegaard Floer homology $\boldsymbol{HF}^-(S^3_\Lambda(L))$ together with the absolute gradings on them are determined by the following set of data:*

- *the multi-variable Alexander polynomial $\Delta_L(x, y)$,*
- *the Alexander polynomials $\Delta_{L_1}(t)$ and $\Delta_{L_2}(t)$,*
- *the framing matrix Λ.*

Remark 1.4.2. For L-space links with more components, besides the Alexander polynomials more information are needed to determine whether the higher diagonal maps vanish or not.

Furthermore, we explicitly describe $\widehat{HF}$ of surgeries on an L-space link $L = L_1 \cup L_2$ by a series of formulas in terms of the Alexander polynomials and the surgery framing matrix. These formulas give a fast algorithm computing $\widehat{HF}$ of these surgeries. We also give a fast algorithm for classifying L-space surgeries. As an application, we study the classification of

L-space surgeries on two-bridge links, and compute some examples explicitly: $(1, 1)$-surgery on a family of L-space links with linking number zero, $L_n = b(4n^2 + 4n, -2n - 1)$.

Instead of classifying L-space links with more than 2 components, we contend to show the prevalence of surgeries on L-space links among 3-manifolds:

Question 1.4.3. *Is every 3-manifold a surgery on a (generalized) L-space link?*

If Question 1.4.3 had a positive answer, one could hope to compute Heegaard Floer homology by L-space links. As a matter of fact, every 3-manifold M can be realized by a surgery on an algebraically split link after connect sum with several lens spaces; see Corollary 2.5 from [32]. It is also interesting to ask whether this algebraically split link can be chosen to be a generalized L-space link.

Regarding L-space surgeries, there is a more basic question:

Question 1.4.4. *Is every L-space a surgery on a (generalized) L-space link?*

CHAPTER 2

Review of the Manolescu-Ozsváth link surgery formula

2.1 Preliminaries

In this section, we give the precise definitions of what we need in the link surgery formula, including Heegaard diagrams, generalized Floer complexes, polygon maps, and nice diagrams. Here, the Heegaard diagram is adapted for a link inside a 3-manifold with multiple basepoints; the generalized Floer complexes of a link L are derived from the filtered complex $CFL^{-}(L)$, and they govern the large surgeries; the polygon maps are used in constructing cobordism maps and certain maps in the link surgery formula; knowledge of nice diagrams are also introduced to deal with two-bridge links.

2.1.1 Heegaard diagrams of links.

We give the most general definition of Heegaard diagrams for an oriented link $\overrightarrow{L}$ inside a 3-manifold M^3. When the link $\overrightarrow{L} = \emptyset$, the Heegaard diagram is simply for M^3.

Definition 2.1.1 (Heegaard diagram of links)**.** A *multi-pointed Heegaard diagram* for the oriented link $\overrightarrow{L}$ in M^3 is the data of the form $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, where:

- Σ is a closed, oriented surface of genus g;
- $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_{g+k-1}\}$ is a collection of disjoint, simple closed curves on Σ which span a g-dimensional lattice of $H_1(\Sigma; \mathbb{Z})$, hence specify a handlebody U_α; the same goes for $\boldsymbol{\beta} = \{\beta_1, ..., \beta_{g+k-1}\}$ specify a handlebody U_β.
- $\mathbf{w} = \{w_1, ..., w_k\}$ and $\mathbf{z} = \{z_1, ..., z_m\}$ (with $k \geq m$) are collections of points on Σ with

the following property. Let $\{A_i\}_{i=1}^k$ be the connected components of $\Sigma - \alpha_1 - \cdots - \alpha_{g+k-1}$ and $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \beta_1 - \cdots - \beta_{g+k-1}$. Then there is a permutation σ of $\{1, ..., m\}$ such that $w_i \in A_i \cap B_i$ for $i = 1, ..., k$, and $z_i \in A_i \cap B_{\sigma(i)}$ for $i = 1, ..., m$, such that connecting w_i to z_i inside A_i and connecting z_i to $w_{\sigma(i)}$ inside B_i will give rise to the link $\overrightarrow{L}$.

Definition 2.1.2 (Admissible diagrams)**.** A *periodic domain* is a two-chain ϕ on Σ which is a linear combination of components of $\Sigma - \boldsymbol{\alpha} \cup \boldsymbol{\beta}$ with integral coefficients, such that the local multiplicity of ϕ at every $w_i \in \mathbf{w}$ is 0 and the boundary of ϕ is a integral combination of α- and β-curves. A multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is called *admissible* if every non-trivial periodic domain has some positive local multiplicities and some negative local multiplicities.

Definition 2.1.3 (Basic diagrams of links)**.** An admissible Heegaard diagram of $\overrightarrow{L}$ is called *basic*, if $l = k = m$, meaning there are exactly two basepoints w_i, z_i for every component $\overrightarrow{L_i}$ and no free basepoints.

Remark 2.1.4. (1) The definitions of pointed Heegaard moves are systematically formulated in [24] section 4.

(2) In order to avoid the issue of naturality, we fix the Heegaard surface Σ as an embedded surface in the underlying 3-manifold M^3. Thus, a Heegaard diagram is equivalent to a self-indexed Morse function.

(3) In this paper we will only consider maximally colored diagrams in the sense of [24].

2.1.2 Generalized Floer complexes.

Here we define some chain complexes of a Heegaard diagram for an oriented link in S^3, which govern the large surgeries on this link. Suppose $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2} \cup \cdots \cup \overrightarrow{L_l}$, and $\overrightarrow{M}$ is an oriented sublink of $\overrightarrow{L}$, where $\overrightarrow{M}$ may not have the induced orientation from $\overrightarrow{L}$ on each component. By $\overrightarrow{L} - M$, we denote the oriented link obtained by deleting all the components of $\overrightarrow{M}$ from $\overrightarrow{L}$.

The identity $H_1(S^3 - \overrightarrow{L}) \cong \mathbb{Z}^l$ provides a way to record the Spin^c structures over S^3 relative to L as an affine lattice over $\mathbb{Z}^l$.

Definition 2.1.5 ($\mathbb{H}(L)$ and reduction maps)**.** Define the affine lattice $\mathbb{H}(\overrightarrow{L})$ over $H_1(S^3 - \overrightarrow{L})$ as follows:

$$\mathbb{H}(\overrightarrow{L})_i = \frac{\mathrm{lk}(\overrightarrow{L_i}, \overrightarrow{L} - \overrightarrow{L_i})}{2} + \mathbb{Z} \subset \mathbb{Q}, \mathbb{H}(\overrightarrow{L}) = \bigoplus_i^l \mathbb{H}(\overrightarrow{L})_i,$$

together with its completion

$$\overline{\mathbb{H}}(\overrightarrow{L})_i = \mathbb{H}(\overrightarrow{L})_i \cup \{-\infty, +\infty\}, \overline{\mathbb{H}}(\overrightarrow{L}) = \bigoplus_i^l \overline{\mathbb{H}}(\overrightarrow{L})_i.$$

The map $\psi^{\overrightarrow{M}} : \mathbb{H}(\overrightarrow{L}) \to \mathbb{H}(\overrightarrow{L} - M)$ is defined by $\psi^{\overrightarrow{M}}(\mathrm{s}) = \mathrm{s} - [\overrightarrow{M}]/2$. More precisely, let $M = L_{j_1} \cup ... \cup L_{j_m}$. Then for all i not in $\{j_1, ..., j_m\}$, let $L_i = (L - M)_{k_i}$, set

$$\psi_i^{\overrightarrow{M}} : \overline{\mathbb{H}}(\overrightarrow{L})_i \to \overline{\mathbb{H}}(\overrightarrow{L} - M)_{k_i}, s_i \to s_i - \frac{\mathrm{lk}(\overrightarrow{L_i}, \overrightarrow{M})}{2}. \tag{2.1.1}$$

The map $\psi^{\overrightarrow{M}}$ is defined to be the direct sum of the maps $\psi_i^{\overrightarrow{M}}$, for those i's with L_i not in M.

The reduction maps $\psi^{\overrightarrow{M}}$ are used in the definition of the destabilization maps in Section 2.2.

For convenience to define the generalized Floer complexes, here we focus on Heegaard diagrams with only one pair of basepoints w_i, z_i on each component and allow free basepoints. Given an admissible multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $\overrightarrow{L}$ with exactly two basepoints z_i and w_i for each link component L_i, we consider the Lagrangian pair $\mathbb{T}_\alpha, \mathbb{T}_\beta$ in $\mathrm{Sym}^{g+k-1}(\Sigma)$ and the Floer complex $CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. There is an Alexander multi-grading $A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{H}(L)$ characterized by the property

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = n_{z_i}(\phi) - n_{w_i}(\phi), \forall \phi \in \pi_2(\mathbf{x}, \mathbf{y})$$

and a normalization condition on the Alexander polynomial. The Alexander grading induces a filtration on $CF^-(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. Given a Spin^c structure on $S^3 - L$, i.e. an element $\mathrm{s} \in \mathbb{H}(L)$, we associate a chain complex $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ called the *generalized Heegaard Floer complex* using the Alexander filtration. We introduce variables U_i with $1 \le i \le l$ for each link component L_i, and U_i with $l+1 \le i \le k$ for each free basepoint w_i.

Definition 2.1.6 (Generalized Floer complex)**.** For $\mathrm{s} \in \mathbb{H}(L)$, the *generalized Floer complex* $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ is the free module over $\mathcal{R} = \mathbb{F}[[U_1, ..., U_l]]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \in \mathrm{Sym}^{g+k-1}(\Sigma)$, and equipped with the differential:

$$\partial^-_{\mathrm{s}} \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}(\alpha) \cap \mathbb{T}(\beta)} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{E^1_{s_1}(\phi)} \cdots U_l^{E^l_{s_l}(\phi)} \cdot U_{l+1}^{n_{w_{l+1}}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \cdot \mathbf{y}, \tag{2.1.2}$$

where $E^i_s(\phi)$ is defined by

$$E^i_s(\phi) = \max\{s - A_i(\mathbf{x}), 0\} - \max\{s - A_i(\mathbf{y}), 0\} + n_{z_i}(\phi) \tag{2.1.3}$$

$$= \max\{A_i(\mathbf{x}) - s, 0\} - \max\{A_i(\mathbf{y}) - s, 0\} + n_{w_i}(\phi). \tag{2.1.4}$$

For simplicity, we also write

$$\mathbf{U}^{E_{\mathrm{s}}(\phi)} = \prod_{i=1}^{l} U_i^{E^i_{s_i}(\phi)} \prod_{i=l+1}^{k} U_i^{n_{w_i}(\phi)}.$$

When the Heegaard diagram in the context is unique, we simply denote $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ by $A^-_{s_1,s_2}(L)$ or $A^-_{s_1,s_2}$, where $\mathrm{s} = (s_1, s_2)$. The direct product of all the generalized Floer complexes forms the first input of the surgery formula.

Remark 2.1.7. Let us explain the relation between $A^-_{\mathrm{s}}(L)$ and $CFL^-(L)$. First, the filtered chain complex $CFL^-(L)$ defined in [42] is the chain complex $CF^-(S^3)$ with the Alexander filtration. Second, the subcomplexes forming the Alexander filtration are isomorphic to the $A^-_{\mathrm{s}}(L)$'s. The Equation (4.1.4) is an explicit formulation of those differentials in $A^-_{\mathrm{s}}(L)$.

2.1.3 Polygon maps and homotopy equivalences between Floer complexes.

In the Fukaya category of a symplectic manifold (X, ω) (when it is well-defined), the product of morphisms

$$\mu^2 : \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_1)$$

is defined by counting holomorphic triangles. In general, higher products are defined by means of holomorphic polygons. In Heegaard Floer theory, the polygon maps are defined similarly. However, the technical issue is the compactness of moduli spaces of holomorphic polygons in the symmetric product of the Heegaard surface, i.e. whether the polygon counts are finite. This problem breaks down to periodic domains on the Heegaard surface. Admissibility of Heegaard multi-diagrams solves this problem. For more details, one can see Section 4.4 in [24].

Definition 2.1.8 (Strongly equivalent Heegaard diagrams)**.** (1) If two Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ have the same underlying Heegaard surface Σ, and their collections of curves β and β' are related by isotopies and handleslides only (supported away from the basepoints), we say that β and β' are *strongly equivalent*.

(2) Two multi-pointed Heegaard diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}, \tau), \mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{w}', \mathbf{z}', \tau')$ are called *strongly equivalent*, if $\Sigma = \Sigma', \mathbf{w} = \mathbf{w}', \mathbf{z} = \mathbf{z}', \tau = \tau'$, the curve collections $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are strongly equivalent, and $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are strongly equivalent as well.

(3) We say that two Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ differ by a *surface isotopy* if there is a self-diffeomorphism $\phi : \Sigma \to \Sigma$ isotopic to the identity and supported away from the link $\overrightarrow{L}$, such that $\Sigma = \Sigma'$ and ϕ takes all the attaching curves and basepoints on Σ to the corresponding one on Σ'. If $\mathcal{H}$ and $\mathcal{H}'$ are surface isotopic, we write $\mathcal{H} \cong \mathcal{H}'$.

Definition 2.1.9 (Triangle maps)**.** Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ be a generic, admissible Heegaard triple-diagram, where $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are strongly equivalent, such that $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}), \mathcal{H}'' = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ are both Heegaard diagrams of the link $\overrightarrow{L}$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is the

Heegaard diagram of the unlink in $\#^{g+k-1}(S^1 \times S^2)$. Then we can define the triangle map

$$f_{\alpha\beta\gamma} : \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathrm{s}) \otimes \mathfrak{A}^-(\mathbb{T}_\beta, \mathbb{T}_\gamma, \mathrm{s}') \to \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\gamma, \mathrm{s}{+}\mathrm{s}')$$

$$f_{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{z}) | \mu(\phi)=0\}} \#(M(\phi)) \cdot \mathbf{U}^{E_{\mathrm{s},\mathrm{s}'}(\phi)} \mathbf{z},$$

where

$$\begin{aligned}
\mathbf{U}^{E_{\mathrm{s},\mathrm{s}'}(\phi)} =& U_1^{E_{s_1,s_1'}(\phi)} \cdots U_l^{E_{s_l,s_l'}(\phi)} U_{l+1}^{n_{w_{l+1}}(\phi)} \cdots U_k^{n_{w_k}(\phi)}, \mathrm{s} = (s_1, \ldots, s_l), \mathrm{s}' = (s_1', \ldots, s_l'), \\
E^i_{s,s'}(\phi) =& \max\{A_i(\mathbf{x}) - s, 0\} + \max\{A_i(\mathbf{y}) - s', 0\} - \max\{A_i(\mathbf{z}) - s - s', 0\} + n_{w_i}(\phi).
\end{aligned}$$

Definition 2.1.10 (Quadrilateral maps)**.** Let $(\Sigma, \boldsymbol{\eta}^0, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\eta}^3, \mathbf{w}, \mathbf{z})$ be a generic, admissible multi-diagram, such that there are two equivalence classes of strongly equivalent attaching curves among $\{\boldsymbol{\eta}^i\}_i$, and $\boldsymbol{\eta}^0, \boldsymbol{\eta}^3$ are in different equivalent classes so that $(\Sigma, \boldsymbol{\eta}^0, \boldsymbol{\eta}^3, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram for the link $\overrightarrow{L}$. Now we can define the quadrilateral maps

$$\begin{aligned}
f_{\eta^0,\ldots,\eta^3} : \bigotimes_{i=1}^{3} \mathfrak{A}^-(\mathbb{T}_{\eta^{i-1}}, \mathbb{T}_{\eta^i}, \mathrm{s}_i) \to \mathfrak{A}^-(\mathbb{T}_{\eta^0}, \mathbb{T}_{\eta^3}, \mathrm{s}_1 + \mathrm{s}_2 + \mathrm{s}_3) \\
f_{\eta^0,\ldots,\eta^3}(\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3) = \sum_{\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^3}} \sum_{\{\phi \in \pi_2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{y}) | \mu(\phi)=-1\}} \#(M(\phi)) \cdot \mathbf{U}^{E_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3}(\phi)} \mathbf{y},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{U}^{E_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3}(\phi)} =& U_1^{E^1_{s_1^1,s_2^1,s_3^1}(\phi)} \cdots U_l^{E^l_{s_1^l,s_2^l,s_3^l}(\phi)} U_{l+1}^{n_{w_{l+1}}(\phi)} \cdots U_k^{n_{w_k}(\phi)}, \mathrm{s}_i = (s_i^1, ..., s_i^l), i = 1, 2, 3, \\
E^i_{s_1,s_2,s_3}(\phi) =& \max\{A_i(\mathbf{x}_1) - s_1, 0\} + \max\{A_i(\mathbf{x}_2) - s_2, 0\} \\
&+ \max\{A_i(\mathbf{x}_3) - s_3, 0\} - \max(A_i(\mathbf{y}) - s_1 - s_2 - s_3, 0) + n_{w_i}(\phi).
\end{aligned}$$

One can define higher polygon counts $f_{\eta^0 \ldots \eta^l}$ similarly, although the case $l > 3$ will not be needed in this paper. For simplicity, we ignore the subscripts of $f_{\eta^0 \ldots \eta^l}$. An important property of polygon maps is the so-called *quadratic A_∞-associativity equation*

$$\sum_{0\leq i<j\leq l} f(x_1,...x_i,f(x_{i+1},...,x_j),x_{j+1},...,x_l)=0. \tag{2.1.5}$$

2.2 Link surgery formula

In this section, we review the link surgery formula of Manolescu-Ozsváth for two-component links with basic diagrams. In Section 2.2.1, we review some algebra on *hyperboxes of chain complexes* and introduce *twisted gluing* of squares of chain complexes. In Section 2.2.2, we express the link surgery formula for a two-component link as a twisted gluing of certain squares of chain complexes derived from the link. These squares are elaborated in Section 2.2.3, by using *primitive systems of hyperboxes*. The primitive systems of hyperboxes are generalizations of the *basic systems of hyperboxes* used in [24]. One can consult [24] for the full generality of the link surgery formula with general Heegaard diagrams. We assume that the reader is familiar with Heegaard Floer homology [39, 38, 41, 42].

Throughout, $\overrightarrow{L}=\overrightarrow{L_1}\cup\overrightarrow{L_2}$ will be an oriented link in S^3, and $\overrightarrow{M}$ will denote an oriented sublink of $\overrightarrow{L}$ which may not have the induced orientation from $\overrightarrow{L}$ on each component.

2.2.1 Hyperboxes of chain complexes.

2.2.1.1 Hyperboxes of chain complexes.

Definition 2.2.1 (Hyperbox)**.** An n-dimensional *hyperbox* of size $\mathbf{d}=(d_1,...,d_n)\in\mathbb{Z}^n_{\geq 0}$ is the subset

$$\mathbb{E}(\mathbf{d})=\{(\varepsilon_1,...,\varepsilon_n)\in\mathbb{Z}^n_{\geq 0}:0\leq\varepsilon_i\leq d_i\}.$$

If $\mathbb{E}(\mathbf{d})=\{0,1\}^n$, then $\mathbb{E}(\mathbf{d})$ is called a *hypercube*, denoted by $\mathbb{E}_n$.

Definition 2.2.2 (Hyperbox of chain complexes)**.** Let R be an $\mathbb{F}$-algebra. An n-dimensional *hyperbox of chain complexes* of size $\mathbf{d} \in \mathbb{Z}^n_{\geq 0}$ is a collection of $\mathbb{Z}$-graded R-modules

$$(C^{\varepsilon})_{\varepsilon \in \mathbb{E}(\mathbf{d})}, C^{\varepsilon} = \bigoplus_{* \in \mathbb{Z}} C^{\varepsilon}_*,$$

together with a collection of R-linear maps

$$D^{\varepsilon}_{\varepsilon^0} : C^{\varepsilon^0}_* \to C^{\varepsilon^0+\varepsilon}_{*-1+||\varepsilon||},$$

one map for each $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$ and $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$. The maps are required to satisfy the relations

$$\sum_{\varepsilon' \leq \varepsilon} D^{\varepsilon - \varepsilon'}_{\varepsilon^0 + \varepsilon'} \circ D^{\varepsilon'}_{\varepsilon^0} = 0,$$

for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d}), \varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$.

By abuse of notation, we sometimes let D^{ε} stand for any of its map $D^{\varepsilon}_{\varepsilon^0}$. Note that a hypercube of chain complexes H gives rise to a total complex of the hypercube $\mathrm{Tot}(H)$.

Example 2.2.3 (1-dimensional hyperboxes)**.** A 1-dimensional hyperbox of chain complexes is a sequence of chain complexes C_n, together with a sequence of chain maps $f_n : C^{(n-1)} \to C^{(n)}$.

$$C^{(0)} \xrightarrow{f_1} C^{(1)} \xrightarrow{f_2} C^{(2)} \xrightarrow{f_3} \cdots \longrightarrow C^{(n-1)} \xrightarrow{f_n} C^{(n)}.$$

The total complex of a 1-dimensional hypercube of chain complexes can be regarded as a mapping cone. Therefore, we also call a 1-dimensional hyperbox of chain complexes a *sequence of chain complexes.*

Example 2.2.4 (2-dimensional hyperboxes)**.** A *square of chain complexes* is a 2-dimensional hypercube of chain complexes:

$$\begin{tikzcd} C^{(0,0)} \arrow[r, "D^{(1,0)}_{(0,0)}"] \arrow[d, "D^{(0,1)}_{(0.0)}"'] \arrow[dr, "D^{(1,1)}_{(0,0)}"] & C^{(1,0)} \arrow[d, "D^{(0,1)}_{(1,0)}"] \\ C^{(0,1)} \arrow[r, "D^{(1,0)}_{(0,1)}"'] & C^{(1,1)} \end{tikzcd} \quad = \quad \begin{tikzcd} C^{(0,0)} \arrow[r, "D^{(1,0)}"] \arrow[d, "D^{(0,1)}"'] \arrow[dr, "D^{(1,1)}"] & C^{(1,0)} \arrow[d, "D^{(0,1)}"] \\ C^{(0,1)} \arrow[r, "D^{(1,0)}"'] & C^{(1,1)} \end{tikzcd}.$$

Here $D^{(1,0)}, D^{(0,1)}$ are chain maps, and $D^{(1,1)}$ is a chain homotopy between $D^{(0,1)} \circ D^{(1,0)}$ and $D^{(1,0)} \circ D^{(0,1)}$. We can regard the total complex of this square as the mapping cone of

$$\operatorname{cone}(D^{(1,0)}_{(0,0)}) \xrightarrow{D^{(0,1)}_{(0,0)}+D^{(0,1)}_{(1,0)}+D^{(1,1)}_{(0,0)}} \operatorname{cone}(D^{(1,0)}_{(0,1)}).$$

A *rectangle of chain complexes* is a 2-dimensional hyperbox of chain complexes. It consists of squares of chain complexes. A rectangle of chain complexes of size $(m, 1)$ can also be regarded as a sequence of mapping cones, i.e. a size (m) 1-dimensional hyperbox of mapping cones.

Let $R = (C, D)$ be a hyperbox of chain complexes of size $(d_1, d_2, \ldots, d_n)$. Fixing $1 \leq i \leq n$, for any integer $0 \leq l \leq d_i$, we have a hyperbox $R^{\varepsilon_i = l} = (C^{\varepsilon_i = l}, D^{\varepsilon_i = l})$ of size $(d_1, ..., d_{i-1}, 0, d_{i+1}, ..., d_n)$, which consists of the chain complexes $C^{(\varepsilon_1, \ldots, \varepsilon_n)}$ with $\varepsilon_i = l$. The differentials $D^{\varepsilon_i = l}$ consist of all the differentials $D^{\varepsilon}_{\varepsilon^0}$ of (C, D) inside $R^{\varepsilon_i = l}$.

Remark 2.2.5. In general, a hyperbox of chain complexes is not a chain complex. But a hypercube is a chain complex considered as the total complex, and it can also be regarded as a mapping cone in many ways.

2.2.1.2 Compression.

From a hyperbox of chain complexes $H = ((C^{\varepsilon})_{\varepsilon \in \mathbb{E}(\mathrm{d})}, (D^{\varepsilon})_{\varepsilon \in \mathbb{E}_n})$, we can obtain a hypercube of chain complexes $\hat{H} = (\hat{C}^{\varepsilon}, \hat{D}^{\varepsilon})_{\varepsilon \in \mathbb{E}_n}$, thus generating a total complex $\operatorname{Tot}(\hat{H})$. The process of turning H into $\hat{H}$ is called *compression*.

Example 2.2.6 (Compression of 1-dimensional hyperboxes)**.** Let R be a hyperbox of dimension 1, see Example 2.2.3. The compression $\hat{R}$ is the mapping cone of the composition of the maps $f_1, \dots, f_n$

$$C^{(0)} \xrightarrow{f_n \circ \cdots \circ f_1} C^{(n)}.$$

Example 2.2.7 (Compression of 2-dimensional hyperboxes)**.** Consider a rectangle of chain complexes R of size $(n, 1)$:

$$\begin{tikzcd}
C^{(0,0)} \arrow[r, "f_1"] \arrow[d, "k_0"] \arrow[dr, "H_1"] & C^{(1,0)} \arrow[r, "f_2"] \arrow[d, "k_1"] \arrow[dr, "H_2"] & C^{(2,0)} \arrow[r, "f_3"] \arrow[d, "k_2"] \arrow[dr, "H_3"] & \cdots \arrow[r, "f_n"] \arrow[dr, "H_n"] & C^{(n,0)} \arrow[d, "k_n"] \\
C^{(0,1)} \arrow[r, "g_1"] & C^{(1,1)} \arrow[r, "g_2"] & C^{(2,1)} \arrow[r, "g_3"] & \cdots \arrow[r, "g_n"] & C^{(n,1)}.
\end{tikzcd}$$

As in 2.2.4, we can regard this rectangle as a 1-dimensional hyperbox of mapping cones $\mathrm{cone}(k_i), i = 0, 1, \dots, n$. The compression of 1-dimensional hyperboxes induces the compression of rectangles of chain complexes as follows

$$\begin{tikzcd}
C^{(0,0)} \arrow[r, "f_n \circ \cdots \circ f_1"] \arrow[d, "k_0"'] \arrow[dr, "\hat{H}"] & C^{(n,0)} \arrow[d, "k_n"] \\
C^{(0,1)} \arrow[r, "g_n \circ \cdots \circ g_1"'] & C^{(n,1)},
\end{tikzcd}$$

where

$$\hat{H} = \sum_{i=1}^{n} f_1 \circ \cdots \circ f_{i-1} \circ H_i \circ g_{i+1} \circ \cdots \circ g_n.$$

Similarly, we can compress a rectangle of chain complexes of size $(1, m)$. For a rectangle of chain complexes R of size (n, m), we can decompose this rectangle into a union of n vertical rectangles of size $(1, m)$. We first compress all of these n vertical rectangles, and thus get a rectangle R' of size $(n, 1)$. Then we keep compressing R' and get a square of chain complexes $\hat{R}$. Alternatively, we can also first compress every row, and then compress the column. So every ordering of the coordinate axes gives a different way to compress the rectangle.

For higher dimensional hyperbox, the compression is defined similarly by induction, once we fix an order of the coordinate axes. Let us describe this procedure using the language of composing *chain maps of hyperboxes*. One can check that it is the same as the compression by means of the *algebra of songs* introduced in [24].

Let ${}^0H = \Big(({}^0C^\varepsilon)_{\varepsilon\in\mathbb{E}(\mathbf{d})}, ({}^0D^\varepsilon)_{\varepsilon\in\mathbb{E}_n}\Big)$, ${}^1H = \Big(({}^1C^\varepsilon)_{\varepsilon\in\mathbb{E}(\mathbf{d})}, ({}^1D^\varepsilon)_{\varepsilon\in\mathbb{E}_n}\Big)$ be two hyperboxes of chain complexes, having the same size $\mathbf{d}\in\mathbb{Z}^n_{\geq 0}$. Let $(\mathbf{d},1)\in\mathbb{Z}^{n+1}_{\geq 0}$ be the sequence obtained from $\mathbf{d}$ by adding 1 at the end.

Definition 2.2.8 (Chain maps of hyperboxes)**.** A chain map $F : {}^0H \to {}^1H$ is a collection of linear maps

$$F^{\varepsilon}_{\varepsilon^0} : {}^0C^{\varepsilon^0}_{*} \to {}^1C^{\varepsilon^0+\varepsilon}_{*+\|\varepsilon\|},$$

satisfying

$$\sum_{\varepsilon'\leq\varepsilon}(D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'}\circ F^{\varepsilon'}_{\varepsilon^0} + F^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'}\circ D^{\varepsilon'}_{\varepsilon^0}) = 0,$$

for all $\varepsilon^0\in\mathbb{E}(\mathbf{d}), \varepsilon\in\mathbb{E}_n$ such that $\varepsilon^0+\varepsilon\in\mathbb{E}(\mathbf{d})$.

In other words, a chain map between the hyperboxes ${}^0H$ and ${}^1H$ is an $(n+1)$-dimensional hyperbox of chain complexes, of size $(\mathbf{d},1)$, such that the sub-hyperbox corresponding to $\varepsilon_{n+1}=0$ is ${}^0H$ and the one corresponding to $\varepsilon_{n+1}=1$ is ${}^1H$. The maps F are those maps D in the new hyperbox that increase ε_{n+1} by 1. Direct computations show the associativity $(F\circ G)\circ H = F\circ(G\circ H)$.

For a n-dimensional hyperbox H of size $\mathbf{d}=(d_1,...,d_n)$, we fix an order of the axes, say, the increasing order $1,2,...,n$. The hyperbox H can be decomposed into d_n pieces of hyperboxes of size $(d_1,...,d_{n-1},1)$, which is a chain map $F_i : H^{\varepsilon_{n+1}=i-1}\to H^{\varepsilon_{n+1}=i}$. Thus the composition $F_{d_n}\circ\cdots\circ F_1$ is a hyperbox of size $(d_1,...,d_{n-1},1)$, and we call it the compression along the n^{th}-axis $\mathrm{Comp}_n(H)$. If we keep doing compressions for the other axes, then we get the compression $\hat{H} = \mathrm{Comp}_1\circ\cdots\circ\mathrm{Comp}_n(H)$.

2.2.1.3 Gluing of squares.

In the link surgery formula, an algebraic operation occurs, which we could call a twisted gluing of hypercubes. It consists in repeatedly gluing mapping cones $A\xrightarrow{f}B, A\xrightarrow{g}B$ to get a new mapping cone $A \xrightarrow{f+g} B$. In this section we describe this operation in detail for the case of two-component links. We call it the *twisted gluing of framed product squares*.

Remark 2.2.9. A hypercube of chain complexes requires a $\mathbb{Z}$-grading on it. However, after gluing of hypercubes, it does not always admit a $\mathbb{Z}$-grading, but admits a $\mathbb{Z}/2\mathbb{Z}$-grading. Now we only require a $\mathbb{Z}/2\mathbb{Z}$-grading on each chain complex sitting at a vertex in the hypercube.

Definition 2.2.10 (Gluing of squares)**.** Suppose there are four squares of chain complexes $R_{i,j} = (C^{\varepsilon}_{i,j}, D^{\varepsilon}_{i,j}), i, j = 0, 1$ as listed below,

$$
R_{i,j}: \quad \begin{tikzcd}
C^{(0,0)}_{i,j} \arrow[r, "D^{(1,0)}_{i,j}"] \arrow[d, "D^{(0,1)}_{i,j}"'] \arrow[dr, "D^{(1,1)}_{i,j}"] & C^{(1,0)}_{i,j} \arrow[d, "D^{(0,1)}_{i,j}"] \\
C^{(0,1)}_{i,j} \arrow[r, "D^{(1,0)}_{i,j}"'] & C^{(1,1)}_{i,j}.
\end{tikzcd}
$$

The squares $\{R_{i,j}\}_{i,j}$ are called *gluable*, if $C^{\varepsilon}_{0,0} = C^{\varepsilon}_{0,1} = C^{\varepsilon}_{1,0} = C^{\varepsilon}_{1,1}$ for all $\varepsilon \in \mathbb{E}_2$ and $D^{(1,0)}_{i,0} = D^{(1,0)}_{i,1} = D^{(1,0)}_{i}, D^{(0,1)}_{0,j} = D^{(0,1)}_{1,j} = D^{(0,1)}_{j}$ for all $i, j = 0, 1$. Then we can define $R = (C^{\varepsilon}, D^{\varepsilon})$ to be the *gluing* of $R_{i,j}$'s as below, where we suppress the subscripts i, j of $C^{\varepsilon}_{i,j}$. One can check that R is a square of chain complexes.

$$
R := \quad \begin{tikzcd}
C^{(0,0)} \arrow[r, "D^{(1,0)}_0 + D^{(1,0)}_1"] \arrow[d, "D^{(0,1)}_0 + D^{(0,1)}_1"'] \arrow[dr, "\sum_{i,j} D^{(1,1)}_{i,j}"] & C^{(1,0)} \arrow[d, "D^{(0,1)}_0 + D^{(0,1)}_1"] \\
C^{(0,1)} \arrow[r, "D^{(1,0)}_0 + D^{(1,0)}_1"'] & C^{(1,1)}.
\end{tikzcd}
$$

Definition 2.2.11 (Framed product square)**.** A $\mathbb{Z}^2$*-product square of chain complexes* is a direct product of squares of chain complexes

$$R = \prod_{\mathrm{s} \in \mathbb{Z}^2} R_{\mathrm{s}},$$

where R_{s} is a square of chain complexes for all $\mathrm{s} \in \mathbb{Z}^2$. We call s the *coordinate* of any element $x \in R_{\mathrm{s}}$. The function

$$\mathcal{F} : R \to \mathbb{Z}^2, \quad \mathcal{F}(x) = \mathrm{s}, \forall \mathrm{s} \in R_{\mathrm{s}},$$

is called the *framing* of R. In order to denote the framing, we write a *framed product square* as a pair

$$(R, \mathcal{F}) = \prod_{\mathrm{s} \in \mathbb{Z}^2} (C^{\varepsilon}_{\mathrm{s}}, D^{\varepsilon}_{\mathrm{s}}).$$

We can shift the framing $\mathcal{F}$ by a set of vectors in $\mathbb{Z}^2$, $\mathbf{V} = \{v^{\varepsilon}\}_{\varepsilon \in \mathbb{E}_2}$, to get a new framing $\mathcal{F}^{\mathbf{V}}$, such that

$$\forall x \in C^{\varepsilon}_{\mathrm{s}}, \quad \mathcal{F}^{\mathbf{V}}(x) = \mathcal{F}(x) + v^{\varepsilon}.$$

We call the new framed product square $(R, \mathcal{F}^{\mathbf{V}})$ the *shifted square of R by* $\mathbf{V}$, and simply denote it by $R[\mathbf{V}]$. Thus, we can write $R[\mathbf{V}] = \prod_{\mathrm{s} \in \mathbb{Z}^2} (\tilde{C}^{\varepsilon}_{\mathrm{s}}, \tilde{C}^{\varepsilon}_{\mathrm{s}})$, where $\tilde{C}^{\varepsilon}_{\mathrm{s}+v^{\varepsilon}} = C^{\varepsilon}_{\mathrm{s}}, \forall \varepsilon \in \mathbb{E}_2, \forall \mathrm{s} \in \mathbb{Z}^2$.

Definition 2.2.12 (Framed gluable)**.** Let $(R_{i,j}, \mathcal{F}_{i,j})$ with $i, j = 0, 1$ be a set of framed product squares of chain complexes. The set of four squares $\{R_{i,j}\}_{i,j}$ is called *framed gluable*, if $\{R_{i,j}\}_{i,j}$ are gluable as squares of chain complexes and all the framings $\mathcal{F}_{i,j}, \forall i, j$ are the same. Then, the result is called the *framed gluing* of $(R_{i,j}, \mathcal{F}_{i,j})$'s.

Definition 2.2.13 (Twisted gluing)**.** Let $(R_{i,j}, \mathcal{F}_{i,j})$ with $i, j = 0, 1$ be a set of framed product squares of chain complexes. For any matrix $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^{2\times 2}$, let

$$\mathbf{V}_{i,j}(\Lambda) = \{v^{\varepsilon} = \Lambda \cdot (i\varepsilon_1, j\varepsilon_2)^t\}_{\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{E}_2}, \quad \forall i, j = 0, 1.$$

Then there are four shifted squares $R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]$, with $i, j = 0, 1$. As long as these four shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j=0,1}$ are framed gluable, we define the Λ-*twisted gluing of* $\{R_{i,j}\}_{i,j}$ to be the framed gluing of $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$, denoted by R^{Λ}. See Figure 2.2.1 for an example of twisted gluing.

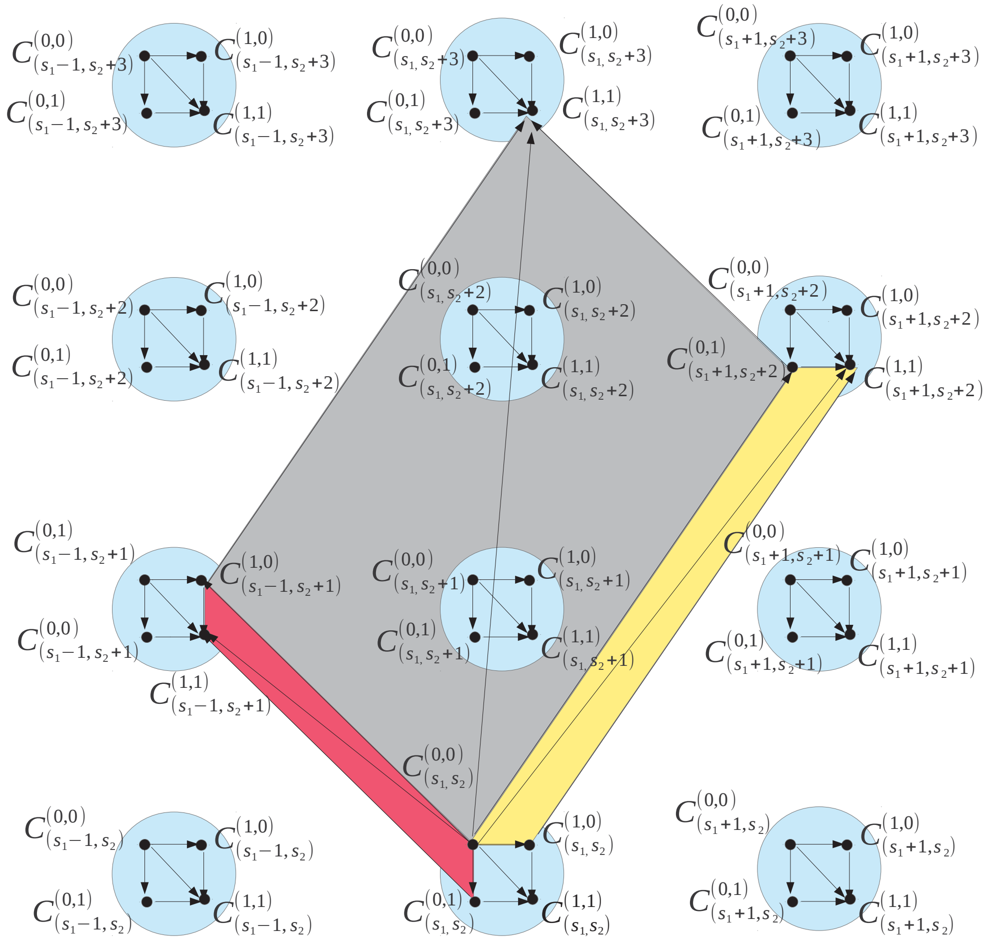


Figure 2.2.1: **An example of twisted gluing.** This is an example of $\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$-twisted gluing of four squares $\{R_{i,j} = \prod_{\mathrm{s}\in\mathbb{Z}^2} R_{\mathrm{s},i,j}\}_{i,j=0,1}$, where $R_{\mathrm{s},i,j} = (C^{\varepsilon}_{\mathrm{s},i,j}, D^{\varepsilon}_{\mathrm{s},i,j})$. Since $C^{\varepsilon}_{\mathrm{s},i,j}$ is identified with some $C^{\varepsilon}_{\mathrm{s}',0,0}$, we omit the subscripts i, j in the picture. Every shaded circle encloses a factor $R_{\mathrm{s},0,0}$ of the $\mathbb{Z}^2$-product square $R_{0,0}$ with some $\mathrm{s} \in \mathbb{Z}^2$. The yellow parallelogram indicates the D-maps of $R_{0,1}$, which is shifted by the vector Λ_2; whereas the red parallelogram indicates the D-maps of $R_{1,0}$, which is shifted by the vector Λ_1. The gray parallelogram indicates all the maps of the square $R_{1,1}$, which is shifted by using both Λ_1 and Λ_2.

Example 2.2.14 (Twisted gluing in the link surgery formula)**.** Suppose for any $(i,j) \in \mathbb{E}_2$,

$$R_{i,j} = \prod_{\mathrm{s}\in\mathbb{Z}^2} R_{\mathrm{s},i,j}$$

with $R_{\mathrm{s},i,j} = (C^{\varepsilon}_{\mathrm{s},i,j}, D^{\varepsilon}_{\mathrm{s},i,j})$ is a framed product square of chain complexes with the natural framing $\mathcal{F}_{i,j}(x) = \mathrm{s}, \forall x \in C_{\mathrm{s},i,j}$. Let $\overrightarrow{L}$ be a two-component link and lk be the linking number. Given any surgery framing matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \mathrm{lk} \\ \mathrm{lk} & \lambda_2 \end{pmatrix},$$

as long as the following identities hold for all $\mathrm{s} \in \mathbb{Z}^2$,

$$\begin{aligned}
&C^{(0,0)}_{\mathrm{s},0,0} = C^{(0,0)}_{\mathrm{s},0,1} = C^{(0,0)}_{\mathrm{s},1,0} = C^{(0,0)}_{\mathrm{s},1,1}, \\
&C^{(1,0)}_{\mathrm{s},0,0} = C^{(1,0)}_{\mathrm{s},0,1} = C^{(1,0)}_{\mathrm{s}-\Lambda_1,1,0} = C^{(1,0)}_{\mathrm{s}-\Lambda_1,1,1}, \\
&C^{(0,1)}_{\mathrm{s},0,0} = C^{(0,1)}_{\mathrm{s}-\Lambda_2,0,1} = C^{(0,1)}_{\mathrm{s},1,0} = C^{(0,1)}_{\mathrm{s}-\Lambda_2,1,1}, \\
&C^{(1,1)}_{\mathrm{s},0,0} = C^{(1,1)}_{\mathrm{s}-\Lambda_2,0,1} = C^{(1,1)}_{\mathrm{s}-\Lambda_1,1,0} = C^{(1,1)}_{\mathrm{s}-\Lambda_1-\Lambda_2,1,1},
\end{aligned} \tag{2.2.1}$$

the shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$ are framed gluable. Thus we define the twisted gluing of squares R^{Λ}.

Remark 2.2.15. The twisted glued square R^{Λ} no longer decomposes as a $\mathbb{Z}^2$ direct product. However, it decomposes as a direct sum $\bigoplus_{\mathfrak{u}\in\mathbb{Z}^2/\Lambda} R^{\Lambda}(\mathfrak{u})$, where Λ is viewed as a lattice spanned by Λ_1, Λ_2. The equivalence classes $\mathbb{Z}^2/\Lambda$ correspond to the Spin^c structures over the surgery manifold on a link in S^3.

2.2.2 Link surgery formula for a two-component link $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$.

In order to denote the orientations of the sublinks, we use $\pm$ signs to denote the positive and negative orientations, where the positive orientation is the induced orientation from $\overrightarrow{L}$ and the negative orientation is the opposite orientation. Let $L = +L_1 \cup +L_2$, that is, L has the same orientation as $\overrightarrow{L}$.

The the link surgery formula is the total complex of a square of chain complexes: the chain complexes at the vertices are the generalized Floer complexes described in Section 2.1.2, and the maps in the square are defined by means of *complete systems of hyperboxes* (see section 2.2.3 for the definition). From a complete system of hyperboxes of L, we get four sets of squares of chain complexes $R_{\mathrm{s},i,j}$, where $\mathrm{s} \in \mathbb{H}(L), i, j \in \{0,1\}$:

$$R_{\mathrm{s},0,0}: \quad \begin{tikzcd} \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \arrow[r, "\Phi_{\mathrm{s}}^{+L_1}"] \arrow[d, "\Phi_{\mathrm{s}}^{+L_2}"'] \arrow[dr, "\Phi_{\mathrm{s}}^{+L_1 \cup +L_2}"] & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{+L_1}(\mathrm{s})) \arrow[d, "\Phi^{+L_2}_{\psi^{+L_1}(\mathrm{s})}"] \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{+L_2}(\mathrm{s})) \arrow[r, "\Phi^{+L_1}_{\psi^{+L_2}(\mathrm{s})}"'] & \mathfrak{A}^-(\mathcal{H}^{\emptyset}, \psi^{+L_1 \cup +L_2}(\mathrm{s})); \end{tikzcd} \tag{2.2.2}$$

$$R_{\mathrm{s},1,0}: \quad \begin{tikzcd} \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \arrow[r, "\Phi_{\mathrm{s}}^{-L_1}"] \arrow[d, "\Phi_{\mathrm{s}}^{+L_2}"'] \arrow[dr, "\Phi_{\mathrm{s}}^{-L_1 \cup +L_2}"] & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{-L_1}(\mathrm{s})) \arrow[d, "\Phi^{+L_2}_{\psi^{-L_1}(\mathrm{s})}"] \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{+L_2}(\mathrm{s})) \arrow[r, "\Phi^{-L_1}_{\psi^{+L_2}(\mathrm{s})}"'] & \mathfrak{A}^-(\mathcal{H}^{\emptyset}, \psi^{-L_1 \cup +L_2}(\mathrm{s})); \end{tikzcd} \tag{2.2.3}$$

$$R_{\mathrm{s},0,1}: \quad \begin{tikzcd} \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \arrow[r, "\Phi_{\mathrm{s}}^{+L_1}"] \arrow[d, "\Phi_{\mathrm{s}}^{-L_2}"'] \arrow[dr, "\Phi_{\mathrm{s}}^{+L_1 \cup -L_2}"] & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{+L_1}(\mathrm{s})) \arrow[d, "\Phi^{-L_2}_{\psi^{+L_1}(\mathrm{s})}"] \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{-L_2}(\mathrm{s})) \arrow[r, "\Phi^{+L_1}_{\psi^{-L_2}(\mathrm{s})}"'] & \mathfrak{A}^-(\mathcal{H}^{\emptyset}, \psi^{+L_1 \cup -L_2}(\mathrm{s})); \end{tikzcd} \tag{2.2.4}$$

$$R_{\mathrm{s},1,1}: \quad \begin{tikzcd} \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \arrow[r, "\Phi_{\mathrm{s}}^{-L_1}"] \arrow[d, "\Phi_{\mathrm{s}}^{-L_2}"'] \arrow[dr, "\Phi_{\mathrm{s}}^{-L_1 \cup -L_2}"] & \mathfrak{A}^-(\mathcal{H}^{L_2}, \psi^{-L_1}(\mathrm{s})) \arrow[d, "\Phi^{-L_2}_{\psi^{-L_1}(\mathrm{s})}"] \\ \mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{-L_2}(\mathrm{s})) \arrow[r, "\Phi^{-L_1}_{\psi^{-L_2}(\mathrm{s})}"'] & \mathfrak{A}^-(\mathcal{H}^{\emptyset}, \psi^{-L_1 \cup -L_2}(\mathrm{s})), \end{tikzcd} \tag{2.2.5}$$

where $\psi^{\overrightarrow{M}}$ is defined in Equation (2.1.1). Thus, we have four framed product squares with the natural framings

$$R_{i,j} = \prod_{\mathrm{s} \in \mathbb{H}(L)} R_{\mathrm{s},i,j}.$$

Definition 2.2.16 (Link surgery formula)**.** For any surgery framing matrix Λ, the shifted squares $\{R_{i,j}[\mathbf{V}_{i,j}(\Lambda)]\}_{i,j}$ are framed gluable according to Equations (2.2.1). The link surgery formula for the framed link (L,Λ) is the total complex of the Λ-twisted gluing of $\{R_{i,j}\}_{i,j}$ as follows,

$$
(\mathcal{C}^-(\mathcal{H},\Lambda),\mathcal{D}^-) := \quad
\begin{array}{ccc}
\prod\limits_{\mathrm{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^L,\mathrm{s}) & \xrightarrow{\Phi^{+L_1}+\Phi^{-L_1}} & \prod\limits_{\mathrm{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_2},\psi^{+L_1}(\mathrm{s})) \\
\Big\downarrow {\scriptstyle \Phi^{+L_2}+\Phi^{-L_2}} & \searrow {\scriptstyle \Phi^{+L_1\cup -L_2}+\Phi^{-L_1\cup-L_2}+\Phi^{+L_1\cup+L_2}+\Phi^{-L_1\cup+L_2}} & \Big\downarrow {\scriptstyle \Phi^{+L_2}+\Phi^{-L_2}} \\
\prod\limits_{\mathrm{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_1},\psi^{+L_2}(\mathrm{s})) & \xrightarrow[\Phi^{+L_1}+\Phi^{-L_1}]{} & \prod\limits_{\mathrm{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{\emptyset},\psi^{+L_1\cup+L_2}(\mathrm{s})),
\end{array}
$$

where $\Phi^\circ = \prod_{\mathrm{s}\in\mathbb{H}(L)} \Phi^\circ_\mathrm{s}$ with $\circ = \pm L_1, \pm L_2, \pm L_1 \cup \pm L_2$.

The map

$$\Phi^{\vec{M}}_\mathrm{s} : \mathfrak{A}^-(\mathcal{H}^L,\mathrm{s}) \to \mathfrak{A}^-(\mathcal{H}^{L-M},\psi^{\vec{M}}(\mathrm{s}))$$

is defined by

$$\Phi^{\vec{M}}_\mathrm{s} = D^{\vec{M}}_{p^{\vec{M}}(\mathrm{s})} \circ \mathcal{I}^{\vec{M}}_\mathrm{s}. \tag{2.2.6}$$

We will spell out the constructions of $D^{\vec{M}}_{p^{\vec{M}}(\mathrm{s})}$ and $\mathcal{I}^{\vec{M}}_\mathrm{s}$ in the next sections by using *primitive systems of hyperboxes*.

For the Λ-twisted gluing of squares, there is a direct sum splitting of the complex

$$\mathcal{C}^-(\mathcal{H},\Lambda) = \bigoplus_{\mathfrak{u}\in\mathbb{H}(L)/H(L,\Lambda)} \mathcal{C}^-(\mathcal{H},\Lambda,\mathfrak{u}),$$

where we identify $\mathbb{H}(L)/H(L,\Lambda)$ with $\mathrm{Spin}^c(S^3_\Lambda(L))$.

Theorem 2.2.17 (Manolescu-Ozsváth Link Surgery Theorem, Theorem 7.7 of [24] for two–component links)**.** *Fix a primitive system of hyperboxes $\mathcal{H}$ for an oriented two-component link $\vec{L}$ in S^3, and fix a framing Λ of L. Then for any $\mathfrak{u} \in Spin^c(S^3_\Lambda(L)) \cong \mathbb{H}(L)/H(L,\Lambda)$,* *there is an isomorphism of* $\mathbb{F}[[U]]$*-modules*

$$H_*(\mathcal{C}^-(\mathcal{H}, \Lambda, \mathfrak{u}), \mathcal{D}^-) \cong \boldsymbol{HF}^-_*(S^3_\Lambda(L), \mathfrak{u}),$$

where $\mathbb{F}[[U]] = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$.

Here, we let $\mathbf{HF}^-$ denote the completion of HF^- with respect to the maximal ideal (U) in the ring $\mathbb{F}[U]$. Since completion is an exact functor, $\mathbf{HF}^-$ can be regarded as the homology of the complex $\mathbf{CF}^- = CF^- \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$, where $\mathbb{F}[[U]]$ is the completion of $\mathbb{F}[U]$. When s is a torsion Spin^c structure of a 3-manifold M, if

$$\mathbf{HF}^-(M, \mathrm{s}) = \mathbb{F}[[U]] \oplus T$$

with T a torsion $\mathbb{F}[[U]]$-module, then

$$HF^-(M, \mathrm{s}) = \mathbb{F}[U] \oplus T.$$

For more details, see Section 2 in [24].

Remark 2.2.18. The link surgery theorem states that all the U_i-actions are the same in the homology of the surgery complex.

Remark 2.2.19. Although all the squares in Equations (2.2.2) to (2.2.5) posses $\mathbb{Z}$-gradings, the surgery complex $\mathcal{C}^-(\mathcal{H}, \Lambda)$ does not always have a $\mathbb{Z}$-grading after the twisted gluing. In [24], an absolute grading was also given in Section 7.4, which is the same as the absolute grading on Heegaard Floer homology of the surgery manifold.

2.2.3 Inclusion maps and destabilization maps.

2.2.3.1 Inclusion maps.

In the link surgery formula, we need a set of chain maps $\mathcal{I}^{\overrightarrow{M}}_{\mathrm{s}}$ in (2.2.6) which are called *inclusion maps*. Here, we define the inclusion maps for all links with arbitrary number of components. In the knot case, the inclusion maps correspond to the maps h_s and v_s from [43].

Definition 2.2.20. Let $\overrightarrow{M}$ be an oriented sublink of $\overrightarrow{L}$. Define

$$I_+(\overrightarrow{L}, \overrightarrow{M}) = \{i : \overrightarrow{L} \text{ and } \overrightarrow{M} \text{ share the same orientation on } L_i\};$$
$$I_-(\overrightarrow{L}, \overrightarrow{M}) = \{i : \overrightarrow{L} \text{ and } \overrightarrow{M} \text{ have different orientations on } L_i\}.$$

A projection map $p^{\overrightarrow{M}} : \overline{\mathbb{H}}(L) \to \overline{\mathbb{H}}(L)$ is defined component-wisely as follows:

$$p_i^{\overrightarrow{M}}(s) = \begin{cases} +\infty & \text{if } i \in I_+(\overrightarrow{L}, \overrightarrow{M}), \\ -\infty & \text{if } i \in I_-(\overrightarrow{L}, \overrightarrow{M}), \\ s & \text{otherwise.} \end{cases}$$

Definition 2.2.21 (Inclusion maps)**.** Suppose $\overrightarrow{M} \subset \overrightarrow{L}$ is an oriented sublink, and $\mathrm{s} = (s_1, s_2) \in \mathbb{H}(L)$ satisfies $s_i \neq \mp\infty$ for those $i \in I_\pm(\overrightarrow{L}, \overrightarrow{M})$. Let $\mathcal{H}$ be a Heegaard diagram of L. The *inclusion map* $\mathcal{I}_{\mathrm{s}}^{\overrightarrow{M}} : \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \to \mathfrak{A}^-(\mathcal{H}^L, p^{\overrightarrow{M}}(\mathrm{s}))$ is defined by the formula

$$\mathcal{I}_{\mathrm{s}}^{\overrightarrow{M}}(x) = \prod_{i \in I_+(\overrightarrow{L}, \overrightarrow{M})} U_{\tau_i}^{\max(A_i(x) - s_i, 0)} \prod_{i \in I_-(\overrightarrow{L}, \overrightarrow{M})} U_{\tau_i}^{\max(s_i - A_i(x), 0)} x.$$

One can verify this is a chain map.

Example 2.2.22. Suppose $\overrightarrow{L}$ is a two-component link and $\mathcal{H}$ is a basic Heegaard diagram of L. We have the following inclusion maps

- $\mathcal{I}_{s_1,s_2}^{+L_1} : A_{s_1,s_2}^- \to A_{+\infty,s_2}^-$, $\quad \mathcal{I}_{s_1,s_2}^{-L_1} : A_{s_1,s_2}^- \to A_{-\infty,s_2}^-$.
- $\mathcal{I}_{s_1,s_2}^{+L_2} : A_{s_1,s_2}^- \to A_{s_1,+\infty}^-$, $\quad \mathcal{I}_{s_1,s_2}^{-L_2} : A_{s_1,s_2}^- \to A_{s_1,-\infty}^-$.
- $\mathcal{I}_{s_1,s_2}^{+L_1 \cup +L_2} : A_{s_1,s_2}^- \to A_{+\infty,+\infty}^-$, $\quad \mathcal{I}_{s_1,s_2}^{-L_1 \cup +L_2} : A_{s_1,s_2}^- \to A_{-\infty,+\infty}^-$,
  $\mathcal{I}_{s_1,s_2}^{+L_1 \cup -L_2} : A_{s_1,s_2}^- \to A_{+\infty,-\infty}^-$, $\quad \mathcal{I}_{s_1,s_2}^{-L_1 \cup -L_2} : A_{s_1,s_2}^- \to A_{-\infty,-\infty}^-$.

2.2.3.2 Destabilization maps.

Let $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$. We set

$$J(+L_i) = \{(s_1, s_2) \in \overline{\mathbb{H}}(L) | s_i = +\infty\}, \quad J(-L_i) = \{(s_1, s_2) \in \overline{\mathbb{H}}(L) | s_i = -\infty\}.$$

Now let $\overrightarrow{M} \subset L$ be an oriented sublink, and let $\{\overrightarrow{M_i}\}_i$ be all the oriented components of $\overrightarrow{M}$. Define

$$J(\overrightarrow{M}) = \bigcap_i J(\overrightarrow{M_i}).$$

For $s \in J(\overrightarrow{M})$, there is a *destabilization map*

$$D_{\mathrm{s}}^{\overrightarrow{M}} : \mathfrak{A}^-(\mathcal{H}^L, \mathrm{s}) \to \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^{\overrightarrow{M}}(\mathrm{s})),$$

which gives rise to the map $D^{\overrightarrow{M}}_{p^{\overrightarrow{M}}(\mathrm{s})}$ in (2.2.6). Note that $p^{\overrightarrow{M}}(s) \in J(\overrightarrow{M})$ for any $\mathrm{s} \in \mathbb{H}(L)$. In the knot case, the destabilization map corresponds to the map identifying $C\{i > 0\}$ and $C\{j > 0\}$. We will give the definition in the next section.

Example 2.2.23. Let $\mathrm{s} = (s_1, s_2) \in \mathbb{H}(L)$ and $\overrightarrow{M} = \pm L_1$, then $p^{\pm L_1}(\mathrm{s}) = (\pm\infty, s_2)$. The destabilization map

$$D^{\pm L_1}_{\pm\infty, s_2} : \mathfrak{A}^-(\mathcal{H}^L, \pm\infty, s_2) \to \mathfrak{A}^-(\mathcal{H}^{L_2}, s_2 - \frac{\mathrm{lk}(+L_1, \pm L_2)}{2})$$

is a chain homotopy equivalence.

If we consider sublinks $\overrightarrow{M} = \pm L_1 \cup \pm L_2$, then we will get destabilization maps from

$\mathfrak{A}^-(\mathcal{H}^L, \pm\infty, \pm\infty)$ to $\mathfrak{A}^-(\mathcal{H}^\emptyset, 0)$, namely,

$$\begin{aligned}
D^{+L_1\cup+L_2}_{+\infty,+\infty} &: \ \mathfrak{A}^-(\mathcal{H}^L, +\infty, +\infty) \to \mathfrak{A}^-(\mathcal{H}^\emptyset, 0),\\
D^{-L_1\cup+L_2}_{-\infty,+\infty} &: \ \mathfrak{A}^-(\mathcal{H}^L, -\infty, +\infty) \to \mathfrak{A}^-(\mathcal{H}^\emptyset, 0),\\
D^{+L_1\cup-L_2}_{+\infty,-\infty} &: \ \mathfrak{A}^-(\mathcal{H}^L, +\infty, -\infty) \to \mathfrak{A}^-(\mathcal{H}^\emptyset, 0),\\
D^{-L_1\cup-L_2}_{-\infty,-\infty} &: \ \mathfrak{A}^-(\mathcal{H}^L, -\infty, -\infty) \to \mathfrak{A}^-(\mathcal{H}^\emptyset, 0).
\end{aligned}$$

2.2.3.3 Primitive system of hyperboxes.

In [24], *complete system of hyperboxes* is defined in order to define the destabilization maps.

Definition 2.2.24 (Complete pre-system of hyperboxes)**.** A *complete pre-system of hyperboxes* $\mathcal{H}$ representing the link $\overrightarrow{L}$ consists of a collection of hyperboxes of Heegaard diagrams, subject to certain compatibility conditions as follows. For each pair of subsets $M \subseteq L' \subseteq L$, and each orientation $\overrightarrow{M}$ on M, the complete pre-system assigns a hyperbox $\mathcal{H}^{\overrightarrow{L'},\overrightarrow{M}}$ for the pair $(\overrightarrow{L'}, \overrightarrow{M})$, where $\overrightarrow{L'}$ has the induced orientation from $\overrightarrow{L}$. Moreover, the hyperbox $\mathcal{H}^{\overrightarrow{L'},\overrightarrow{M}}$ is required to be compatible with both $\mathcal{H}^{\overrightarrow{L'},\overrightarrow{M'}}$ and $\mathcal{H}^{\overrightarrow{L'}-\overrightarrow{M'},\overrightarrow{M}-\overrightarrow{M'}}$.

In the above definition, there is some compatibility condition we have not spelled out. A *complete system of hyperboxes* is a complete pre-system with some additional conditions regarding the surface isotopies connecting those hyperboxes. In a complete system of hyperboxes, every hyperbox of Heegaard diagrams induces a hyperbox of generalized Floer complexes. Instead of explaining these compatibility conditions, we give a special complete system of hyperboxes for two-component links satisfying these conditions, which illustrates the main idea. They are called *primitive system of hyperboxes*.

When the sublink $\overrightarrow{L'}$ has the induced orientation from $\overrightarrow{L}$, we simply denote it by L'. Thus, we use notation $\mathcal{H}^{L',\overrightarrow{M}} = \mathcal{H}^{\overrightarrow{L'},\overrightarrow{M}}$. In a complete pre-system $\mathcal{H}$, we have four zero dimensional hyperboxes of Heegaard diagrams, $\mathcal{H}^{L,\emptyset}, \mathcal{H}^{L_1,\emptyset}, \mathcal{H}^{L_2,\emptyset}, \mathcal{H}^{\emptyset,\emptyset}$, where $\mathcal{H}^{L,\emptyset}$ is a Heegaard diagram of L, $\mathcal{H}^{L_i,\emptyset}$ is a Heegaard diagram of L_i, and $\mathcal{H}^{\emptyset,\emptyset}$ is a Heegaard diagram of S^3. We denote the four Heegaard diagrams simply by $\mathcal{H}^L, \mathcal{H}^{L_1}, \mathcal{H}^{L_2}, \mathcal{H}^\emptyset$.

Given a basic Heegaard diagram $\mathcal{H}^L = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{w_1, w_2\}, \{z_1, z_2\})$ of L, from Equation (4.1.4), we see $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2))$ is counting $n_{w_1}(\phi)$ without using z_1, thus as the same as deleting z_1. Moreover $\mathcal{H}^{L_2} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{w_1, w_2\}, \{z_2\})$ is a Heegaard diagram of L_2 with one free basepoint w_1. We call this diagram the *reduction* of $\mathcal{H}^L$ at $+L_1$, denoted by $r_{+L_1}(\mathcal{H}^L)$; see [24] Definition 4.17. Hence, we have an identification between $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2))$ and $\mathfrak{A}^-(r_{+L_1}(\mathcal{H}^L), s_2 - \frac{\mathrm{lk}(+L_1,+L_2)}{2})$. Similarly, we define $r_{-L_1}(\mathcal{H}^L)$ to be the diagram obtained from $\mathcal{H}^L$ by deleting w_1 and relabeling z_1 as w_1. We have an identification between $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2))$ and $\mathfrak{A}^-(r_{-L_1}(\mathcal{H}^L), s_2 - \frac{\mathrm{lk}(-L_1,+L_2)}{2})$, since $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2))$ uses basepoints $\{z_1, w_2\} \subset \mathcal{H}^L$.

Moreover, the diagrams $r_{-L_1}(\mathcal{H}^L)$ and $r_{+L_1}(\mathcal{H}^L)$ are related by Heegaard moves, for they represent the same knot L_2. By definition, there is an arc c in $\Sigma - \boldsymbol{\alpha}$ connecting w_1 and z_1, so we can move z_1 along c to w_1, by a sequence of Heegaard moves. Moving a basepoint to cross some β-curve can be done by a sequence of handleslides and isotopies of β-curves, stabilizations, and destabilization followed by a surface isotopy. However, if we need stabilizations/destabilizations, we can modify the original Heegaard diagram $\mathcal{H}^L$ by these stabilizations in the beginning. Thus, we can always get a diagram $\tilde{\mathcal{H}}^L$, such that there is a sequence of Heegaard moves only of handleslides and isotopies of β-curves together with some surface isotopy from $r_{-L_1}(\tilde{\mathcal{H}}^L)$ to $r_{+L_1}(\tilde{\mathcal{H}}^L)$. In sum, there is an surface isotopy $h : \Sigma \to \Sigma$ supported in a small neighborhood of c (so h fixes other basepoints and all the α-curves), such that $h(w_1) = z_1$ and $h(r_{+L_1}(\tilde{\mathcal{H}}^L))$ is strongly equivalent to $r_{-L_1}(\tilde{\mathcal{H}}^L)$ via handleslides and isotopies of β-curves.

Definition 2.2.25 (Primitive Heegaard diagrams)**.** For any basic Heegaard diagram $\mathcal{H}$ of an oriented link $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$, there are surface isotopies $h_i^{\mathcal{H}} : \Sigma \to \Sigma$ supported in a small neighborhood of the arc c_i connecting w_i and z_i in $\Sigma - \boldsymbol{\alpha}$, such that $h_i^{\mathcal{H}}(w_i) = z_i$ and $h_i^{\mathcal{H}}$ preserves the α-curves and the other basepoints. They are unique up to isotopy. The basic Heegaard diagram $\mathcal{H}$ is called *primitive*, if it is admissible and $r_{-L_i}(\mathcal{H})$ is strongly equivalent to $h_i^{\mathcal{H}}(r_{+L_i}(\mathcal{H}))$ for both $i = 1, 2$.

From the above discussion, we can get the following lemma.

Lemma 2.2.26. *Let L be an oriented two-component link, and let $\mathcal{H}$ be a basic admissible Heegaard diagram of L. Then there is an index one/two stabilization turning $\mathcal{H}$ into a primitive Heegaard diagram $\tilde{\mathcal{H}}$.*

Fixing a primitive Heegaard diagram $\mathcal{H}^L$ for an oriented two-component link L, we can get two sequences of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L,-L_i}$:

$$\bar{\mathcal{H}}^{L,-L_1} : r_{-L_1}(\mathcal{H}^L) = \bar{\mathcal{H}}^{L,-L_1}(\emptyset) \to \bar{\mathcal{H}}^{L,-L_1}(L_1) = h_1^{\mathcal{H}}(r_{+L_1}(\mathcal{H}^L)), \tag{2.2.7}$$

$$\bar{\mathcal{H}}^{L,-L_2} : r_{-L_2}(\mathcal{H}^L) = \bar{\mathcal{H}}^{L,-L_2}(\emptyset) \to \bar{\mathcal{H}}^{L,-L_2}(L_2) = h_2^{\mathcal{H}}(r_{+L_2}(\mathcal{H}^L)). \tag{2.2.8}$$

These induce another two sequences of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L_i,-L_i}$:

$$\bar{\mathcal{H}}^{L_1,-L_1} : r_{-L_1}\Big(r_{+L_2}(\mathcal{H}^L)\Big) = \bar{\mathcal{H}}^{L_1,-L_1}(\emptyset) \to \bar{\mathcal{H}}^{L_1,-L_1}(L_1) = h_1^{\mathcal{H}}\Big(r_{+L_1}(r_{+L_2}(\mathcal{H}^L))\Big), \tag{2.2.9}$$

$$\bar{\mathcal{H}}^{L_2,-L_2} : r_{-L_2}\Big(r_{+L_1}(\mathcal{H}^L)\Big) = \bar{\mathcal{H}}^{L_2,-L_2}(\emptyset) \to \bar{\mathcal{H}}^{L_2,-L_2}(L_2) = h_2^{\mathcal{H}}(r_{+L_2}\Big(r_{+L_1}(\mathcal{H}^L))\Big), \tag{2.2.10}$$

together with a square of strongly equivalent Heegaard diagrams $\bar{\mathcal{H}}^{L,-L_1\cup -L_2}$:

$$\begin{tikzcd}
r_{-L_1\cup -L_2}(\mathcal{H}) = \bar{\mathcal{H}}^{L_1\cup L_2,-L_1\cup -L_2}(\emptyset) \arrow[r] \arrow[d] \arrow[dr] & \bar{\mathcal{H}}^{L_1\cup L_2,+L_1\cup -L_2}(L_1) = h_1^{\mathcal{H}}(r_{+L_1\cup -L_2}(\mathcal{H})) \arrow[d] \\
h_2^{\mathcal{H}}(r_{-L_1\cup +L_2}(\mathcal{H})) = \bar{\mathcal{H}}^{L_1\cup L_2,-L_1\cup -L_2}(L_2) \arrow[r] & \bar{\mathcal{H}}^{L_1\cup L_2,-L_1\cup -L_2}(L) = h_1^{\mathcal{H}} \circ h_2^{\mathcal{H}}(r_{+L_1\cup +L_2}(\mathcal{H})).
\end{tikzcd} \tag{2.2.11}$$

These almost produce a complete system of hyperbox $\bar{\mathcal{H}}$ except for the admissibility of $\bar{\mathcal{H}}$. We call this system a *primitive almost complete system of hyperbox*.

Definition 2.2.27 (Primitive almost complete system of hyperbox)**.** Given a primitive Heegaard diagram $\mathcal{H}^L = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{w_1, w_2\}, \{z_1, z_2\})$ of $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$, there exists a *primitive almost complete system of hyperbox* $\bar{\mathcal{H}}$ associated to $\mathcal{H}^L$ consisting of

- four 0-dimensional hyperboxes of Heegaard diagrams:

$$\bar{\mathcal{H}}^{L}=\mathcal{H},\ \bar{\mathcal{H}}^{L_1}=r_{+L_2}(\mathcal{H}),\ \bar{\mathcal{H}}^{L_2}=r_{+L_1}(\mathcal{H}),\ \bar{\mathcal{H}}^{\emptyset}=r_{+L_1\cup+L_2}(\mathcal{H});$$

- eight 1-dimensional hyperboxes of Heegaard diagrams:

$$\bar{\mathcal{H}}^{L,\pm L_i},\quad \bar{\mathcal{H}}^{L_i,\pm L_i},\forall i=1,2,$$

  where $\bar{\mathcal{H}}^{L,+L_i}$, $\bar{\mathcal{H}}^{L_i,+L_i}$ are trivial hyperboxes, i.e. just a Heegaard diagram, and $\bar{\mathcal{H}}^{L,-L_i}$,$\bar{\mathcal{H}}^{L_i,-L_i}$ are described in Equations from (2.2.7) to (2.2.10);

- four 2-dimensional hyperboxes of Heegaard diagrams: one trivial hyperbox $\bar{\mathcal{H}}^{L,+L_1\cup+L_2}$, two degenerate hyperboxes:

$$\bar{\mathcal{H}}^{L,+L_1\cup-L_2}=\bar{\mathcal{H}}^{L_2,-L_2},\ \ \bar{\mathcal{H}}^{L,-L_1\cup+L_2}=\bar{\mathcal{H}}^{L_1,-L_1},$$

  and a square of strongly equivalent Heegaard diagrams $\mathcal{H}^{L,-L_1\cup-L_2}$, which is described in Equation (2.2.11).

Definition 2.2.28 (Primitive system of hyperboxes)**.** Given a primitive diagram $\mathcal{H}^L$ and the induced primitive almost complete systems of hyperboxes $\bar{\mathcal{H}}$, if the admissibility of $\bar{\mathcal{H}}$ is not satisfied, we can enlarge the hyperbox in $\bar{\mathcal{H}}$ to achieve admissibility, thus getting a complete system of hyperboxes. We call the result a *primitive system of hyperboxes* $\mathcal{H}$.

Indeed, if $\bar{\mathcal{H}}^{L,-L_1}$ is not admissible, i.e. $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}',\mathbf{w},\mathbf{z})$ is not admissible, then we can insert an isotopy of $\boldsymbol{\beta}$, namely $\boldsymbol{\beta}''$, such that both $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'',\mathbf{w},\mathbf{z})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}'',\boldsymbol{\beta}',\mathbf{w},\mathbf{z})$ are admissible. Suppose $\{D_1,\dots,D_m\}$ is a basis of the $\mathbb{Q}$-vector space of the periodic domains in $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}',\mathbf{w},\mathbf{z})$ with only positive multiplicities . Let D_1^c be the union of all the regions which are not in D_1. Then $D_1^c\neq\emptyset$, since $n_{\mathbf{w}}D_1=0$. As $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}',\mathbf{w},\mathbf{z})$ are both admissible, the boundary of D_1 must contain a β-curve and a β'-curve. Thus there is a β-arc b and a β'-arc b' on $D_1\cap D_1^c$. So we can find a path γ in D_1 connecting b to b', and then do a finger move of the β-curve containing b along γ to get negative multiplicities for D_1 (see [48] for the definition of finger move). Similarly we deal with the other D_i's. Finally, the new $\boldsymbol{\beta}$ in the above process is chosen to be $\boldsymbol{\beta}''$. Similar arguments work for the case of the square $\bar{\mathcal{H}}^{L,-L_1\cup -L_2}$.

Therefore to achieve admissibility, we can enlarge the square of Heegaard diagram $\bar{\mathcal{H}}^{L,-L_1\cup -L_2}$:

$$\begin{tikzcd}
(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{11},\mathbf{w},\mathbf{z}) \arrow[r] \arrow[d] \arrow[dr] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{13},\mathbf{w},\mathbf{z}) \arrow[d] \\
(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{31},\mathbf{w},\mathbf{z}) \arrow[r] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{33},\mathbf{w},\mathbf{z})
\end{tikzcd}$$

into $\mathcal{H}^{L,-L_1\cup -L_2}$:

$$\begin{tikzcd}
(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{11},\mathbf{w},\mathbf{z}) \arrow[r] \arrow[d] \arrow[dr] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{12},\mathbf{w},\mathbf{z}) \arrow[r] \arrow[d] \arrow[dr] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{13},\mathbf{w},\mathbf{z}) \arrow[d] \\
(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{21},\mathbf{w},\mathbf{z}) \arrow[r] \arrow[d] \arrow[dr] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{22},\mathbf{w},\mathbf{z}) \arrow[r] \arrow[d] \arrow[dr] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{23},\mathbf{w},\mathbf{z}) \arrow[d] \\
(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{31},\mathbf{w},\mathbf{z}) \arrow[r] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{32},\mathbf{w},\mathbf{z}) \arrow[r] & (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{33},\mathbf{w},\mathbf{z}).
\end{tikzcd}$$

In order to send every hyperbox of Heegaard diagrams $\mathcal{H}^{\vec{L},\vec{M}}$ to a hyperbox of chain complexes $\mathfrak{A}^-(\mathcal{H}^{\vec{L},\vec{M}},\mathrm{s})$, we need a set of Θ chain elements. We call the choice of these Θ-elements a *filling* of the hyperboxes Heegaard diagrams. Let us explain Θ-elements case by case.

For $\mathcal{H}^{L,-L_1}$, we have a sequence of strongly equivalent Heegaard diagrams of $\vec{L}-L_1$,

$$\mathcal{H}^{L,-L_1}:(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1,\mathbf{w},\mathbf{z})\to(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_2,\mathbf{w},\mathbf{z})\to(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_3,\mathbf{w},\mathbf{z}).$$

We choose a cycle element Θ_{β_1,β_2} representing the maximal degree element in the homology of $\mathfrak{A}^-(\mathbb{T}_{\beta_1},\mathbb{T}_{\beta_2},0)$. Then we define a chain homotopy equivalence $D_{\beta_1,\beta_2}:\mathfrak{A}^-(\mathbb{T}_\alpha,\mathbb{T}_{\beta_1},s)\to\mathfrak{A}^-(\mathbb{T}_\alpha,\mathbb{T}_{\beta_2},s)$ by using triangle maps $D_{\beta_1,\beta_2}(\mathbf{x})=f_{\alpha\beta_1\beta_2}(\mathbf{x}\otimes\Theta_{\beta_1,\beta_2})$. Similarly, we get a chain homotopy equivalence $D_{\beta_2,\beta_3}:\mathfrak{A}^-(\mathbb{T}_\alpha,\mathbb{T}_{\beta_2},s)\to\mathfrak{A}^-(\mathbb{T}_\alpha,\mathbb{T}_{\beta_3},s)$ by choosing $\Theta_{\beta_2,\beta_3}\in\mathfrak{A}^-(\mathbb{T}_{\beta_2},\mathbb{T}_{\beta_3},0)$. Thus, $D^{-L_1}=D_{\beta_2\beta_3}\circ D_{\beta_1\beta_2}$ is also a chain homotopy equivalence. Let us put a subscript on D^{-L_1} for labeling the Spin^c structure. Since $\mathfrak{A}^-(\mathcal{H}^L, (+\infty, s_2)) = \mathfrak{A}(r_{+L_1}(\mathcal{H}^L), s_2 - \frac{\mathrm{lk}(+L_1,+L_2)}{2})$, $\mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2)) = \mathfrak{A}^-(r_{-L_1}(\mathcal{H}^L), s_2 - \frac{\mathrm{lk}(-L_1,+L_2)}{2})$, we write

$$D^{-L_1}_{-\infty,s_2} : \mathfrak{A}^-(\mathcal{H}^L, (-\infty, s_2)) \to \mathfrak{A}^-\bigg(\mathcal{H}^L, \Big(+ \infty, s_2 + \mathrm{lk}(+L_1, +L_2)\Big)\bigg),$$

or simply

$$D^{-L_1}_{-\infty,s_2} : A^-_{-\infty,s_2} \to A^-_{+\infty,s_2+\mathrm{lk}}.$$

Similarly we define $D^{-L_2}_{s_1,-\infty} : A^-_{s_1,-\infty} \to A^-_{s_1+\mathrm{lk},+\infty}$.

For the 2-dimensional hyperbox of Heegaard diagrams $\mathcal{H}^{L,-L_1\cup -L_2}$, we can get a square of chain complexes. Let us first look at the upper left quarter of $\mathcal{H}^{L,-L_1\cup -L_2}$:

$$\begin{tikzcd}
(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{11}, \mathbf{w}, \mathbf{z}) \arrow[r, "\Theta_{\beta_{11}\beta_{12}}"] \arrow[d, "\Theta_{\beta_{11}\beta_{21}}"'] \arrow[dr, "\Theta_{\beta_{11}\beta_{22}}"] & (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{12}, \mathbf{w}, \mathbf{z}) \arrow[d, "\Theta_{\beta_{12}\beta_{22}}"] \\
(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{21}, \mathbf{w}, \mathbf{z}) \arrow[r, "\Theta_{\beta_{21}\beta_{22}}"'] & (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{22}, \mathbf{w}, \mathbf{z}).
\end{tikzcd}$$

In the above the diagram, the Θ-elements on the edges are arbitrary cycles representing the maximal degree elements in their homology groups. Let $c = f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})$. The equation

$$\begin{gathered}
\partial(f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})) = \\
f_{\beta_{11}\beta_{12}\beta_{22}}((\partial\Theta_{\beta_{11}\beta_{12}}) \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes (\partial\Theta_{\beta_{21}\beta_{22}})) = 0
\end{gathered}$$

shows that c is a cycle in $\mathfrak{A}^-_\mu(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}}, 0)$, where μ equals to the maximal degree of the homology of $\mathfrak{A}^-(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}}, 0)$. Since the curves β_{**} are all strongly equivalent, up to chain homotopy equivalences, we can only consider the case when they are all small Hamiltonian isotopies of each other. By Lemma 9.7 in [39], we can see that $f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}})$, $f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}})$ both represent the maximal degree element in the homology of $CF(\mathbb{T}_{\beta_{11}}, \mathbb{T}_{\beta_{22}})$. Thus, c is 0 in the homology. So there is a $\Theta_{\beta_{11},\beta_{22}}$ such that

$\partial\Theta_{\beta_{11},\beta_{22}} = c$, where $\partial = f_{\beta_{11},\beta_{22}}$. In sum,

$$f_{\beta_{11}\beta_{12}\beta_{22}}(\Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\beta_{11}\beta_{21}\beta_{22}}(\Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}}) = f_{\beta_{11}\beta_{22}}(\Theta_{\beta_{11}\beta_{22}}).$$

From the quadratic A_∞ associativity Equation (2.1.5), we have a square of chain complexes

$$\begin{tikzcd}
\mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_{\beta_{11}}, \mathrm{s}) \arrow[r, "D_{\beta_{11}\beta_{12}}"] \arrow[d, "D_{\beta_{11}\beta_{21}}"'] \arrow[dr, "D_{\beta_{11}\beta_{22}}"] & \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_{\beta_{12}}, \mathrm{s}) \arrow[d, "D_{\beta_{12}\beta_{22}}"] \\
\mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_{\beta_{21}}, \mathrm{s}) \arrow[r, "D_{\beta_{21}\beta_{22}}"'] & \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_{\beta_{22}}, \mathrm{s}),
\end{tikzcd}$$

where $D_{\beta_{11}\beta_{22}}(x) = f_{\alpha\beta_{11}\beta_{12}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{12}} \otimes \Theta_{\beta_{12}\beta_{22}}) + f_{\alpha\beta_{11}\beta_{21}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{21}} \otimes \Theta_{\beta_{21}\beta_{22}}) + f_{\alpha\beta_{11}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{22}})$, and

$$D_{\beta_{11}\beta_{12}}(x) = f_{\alpha\beta_{11}\beta_{12}}(x \otimes \Theta_{\beta_{11}\beta_{12}}),\ \ D_{\beta_{12}\beta_{22}}(x) = f_{\alpha\beta_{12}\beta_{22}}(x \otimes \Theta_{\beta_{12}\beta_{22}}),$$
$$D_{\beta_{11}\beta_{21}}(x) = f_{\alpha\beta_{11}\beta_{21}}(x \otimes \Theta_{\beta_{11}\beta_{21}}),\ \ D_{\beta_{11}\beta_{22}}(x) = f_{\alpha\beta_{11}\beta_{22}}(x \otimes \Theta_{\beta_{11}\beta_{22}}).$$

Similarly, we can choose other Θ-elements on $\mathcal{H}^{L,-L_1\cup -L_2}$, and get a rectangle of chain complexes of size $(2,2)$. We denote it by $\mathfrak{A}^-(\mathcal{H}^{L,-L_1\cup -L_2}, \mathrm{s})$.

Definition 2.2.29. We define the destabilization map $D^{-L_1\cup -L_2}$ to be the diagonal map in the compression of $\mathfrak{A}^-(\mathcal{H}^{L,-L_1\cup -L_2}, \mathrm{s})$:

$$\begin{tikzcd}
\mathfrak{A}^-(r_{-L_1\cup -L_2}(\mathcal{H}^L), \mathrm{s}) \arrow[r] \arrow[d] \arrow[dr] & \mathfrak{A}^-(r_{-L_1\cup +L_2}(\mathcal{H}^L), \mathrm{s}) \arrow[d] \\
\mathfrak{A}^-(r_{+L_1\cup -L_2}(\mathcal{H}^L), \mathrm{s}) \arrow[r] & \mathfrak{A}^-(r_{+L_1\cup +L_2}(\mathcal{H}^L), \mathrm{s}).
\end{tikzcd}$$

Since $\mathfrak{A}^-(r_{-L_1\cup -L_2}(\mathcal{H}^L), \mathrm{s}) = \mathfrak{A}^-(\mathcal{H}^L, -\infty, -\infty)$, we denote it by $D^{-L_1\cup -L_2}_{-\infty,-\infty}$.

As all the other hyperboxes of Heegaard diagrams are trivial, the following identities hold

$$D^{+L_1}_{+\infty,s_2} = id,\ D^{+L_2}_{s_1,+\infty} = id,\ D^{-L_1\cup +L_2}_{-\infty,+\infty} = 0,\ D^{+L_1\cup -L_2}_{+\infty,-\infty} = 0,\ D^{+L_1\cup +L_2}_{+\infty,+\infty} = 0.$$

Now we can build all the rectangles of chain complexes as follows, where $\mathrm{lk} = \mathrm{lk}(\overrightarrow{L_1}, \overrightarrow{L_2})$.

$$\mathfrak{R}_{\mathrm{s},1,1} :=
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{-L_2}_{s_1,s_2}"] \arrow[d, "I^{-L_1}_{s_1,s_2}"'] \arrow[dr, "I^{-L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1,-\infty} \arrow[r, "D^{-L_2}_{s_1,-\infty}"] \arrow[d, "I^{-L_1}_{s_1,-\infty}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "I^{-L_1}_{s_1+\mathrm{lk},+\infty}"] \\
A^-_{-\infty,s_2} \arrow[r, "I^{-L_2}_{-\infty,s_2}"'] \arrow[d, "D^{-L_1}_{-\infty,s_2}"'] & A^-_{-\infty,-\infty} \arrow[r, "D^{-L_2}_{-\infty,-\infty}"] \arrow[d, "D^{-L_1}_{-\infty,-\infty}"'] \arrow[dr, "D^{-L_1\cup -L_2}_{-\infty,-\infty}"] & A^-_{-\infty,+\infty} \arrow[d, "D^{-L_1}_{-\infty,+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "I^{-L_2}_{+\infty,s_2+\mathrm{lk}}"'] & A^-_{+\infty,-\infty} \arrow[r, "D^{-L_2}_{+\infty,-\infty}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\qquad
\mathfrak{R}_{\mathrm{s},0,1} :=
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{-L_2}_{s_1,s_2}"] \arrow[d, "I^{+L_1}_{s_1,s_2}"'] \arrow[dr, "I^{+L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1,-\infty} \arrow[r, "D^{-L_2}_{s_1,-\infty}"] \arrow[d, "I^{+L_1\cup -L_2}_{s_1,-\infty}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "I^{+L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{-L_2}_{+\infty,s_2}"'] \arrow[d, "id"'] & A^-_{+\infty,-\infty} \arrow[r, "D^{-L_2}_{+\infty,-\infty}"'] \arrow[d, "id"'] & A^-_{+\infty,+\infty} \arrow[d, "id"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{-L_2}_{+\infty,s_2}"'] & A^-_{+\infty,-\infty} \arrow[r, "D^{-L_2}_{+\infty,-\infty}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\tag{2.2.12}$$

$$\mathfrak{R}_{\mathrm{s},1,0} :=
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{+L_2}_{s_1,s_2}"] \arrow[d, "I^{-L_1}_{s_1,s_2}"'] \arrow[dr, "I^{-L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[r, "id"] \arrow[d, "I^{-L_1}_{s_1,+\infty}"] & A^-_{s_1,+\infty} \arrow[d, "I^{-L_1}_{s_1,+\infty}"] \\
A^-_{-\infty,s_2} \arrow[r, "I^{+L_2}_{-\infty,s_2}"'] \arrow[d, "D^{-L_1}_{-\infty,s_2}"'] & A^-_{-\infty,+\infty} \arrow[r, "id"] \arrow[d, "D^{-L_1}_{-\infty,+\infty}"] & A^-_{-\infty,+\infty} \arrow[d, "D^{-L_1}_{-\infty,+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty} \arrow[r, "id"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\qquad
\mathfrak{R}_{\mathrm{s},0,0} :=
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{+L_2}_{s_1,s_2}"] \arrow[d, "I^{+L_1}_{s_1,s_2}"'] \arrow[dr, "I^{+L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[r, "id"] \arrow[d, "I^{+L_1}_{s_1,+\infty}"] & A^-_{s_1,+\infty} \arrow[d, "I^{+L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] \arrow[d, "id"'] & A^-_{+\infty,+\infty} \arrow[r, "id"] \arrow[d, "id"] & A^-_{+\infty,+\infty} \arrow[d, "id"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty} \arrow[r, "id"'] & A^-_{+\infty,+\infty}.
\end{tikzcd}
\tag{2.2.13}$$

The squares $R_{\mathrm{s},i,j}$'s used in Equations (2.2.2) to (2.2.5) are defined to be the compressions of $\mathfrak{R}_{\mathrm{s},i,j}$'s.

CHAPTER 3

Applications to surgeries on two-bridge links

3.1 Generalized Floer complexes of two-bridge links

In this section, we combinatorially compute the generalized Floer complexes $A^-_{s_1,s_2}(\overrightarrow{L})$ for all two-bridge links $\overrightarrow{L}$ by using nice diagrams.

3.1.1 Nice diagrams.

In [48], Sarkar and Wang use nice Heegaard diagrams to combinatorially compute the $\widehat{HF}(M)$. This algorithm is based on a fact: in a nice diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$, the index-1 pseudo-holomorphic disks in $\mathrm{Sym}^{g+k-1}(\Sigma)$ with $n_w = 0$ have simple domains on Σ and can be combinatorially counted.

Definition 3.1.1 (Nice diagrams)**.** A Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ is called *nice*, if any region (i.e. connect component of $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$) without any basepoint $w_i \in \mathbf{w}$ is either a bigon or a square. For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a domain $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is called an *empty embedded 2n-gon,* if it is an embedded disk with $2n$ vertices on its boundary, such that for each vertex v, $\mu_v(\phi) = \frac{1}{4}$, and it does not contain any x_i or y_i in its interior. An empty embedded 4-gon is also called empty embedded square.

Remark 3.1.2. The notation $\pi_2(\mathbf{x}, \mathbf{y})$ in [48] denotes the sets of domains, namely 2-chains ϕ on Σ such that $\partial(\partial\phi|_\alpha) = \mathbf{y} - \mathbf{x}$, whereas in this paper $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the homology classes of Whitney disks in $\mathrm{Sym}^{g+k-1}(\Sigma)$ from $\mathbf{x}$ to $\mathbf{y}$.

Theorem 3.1.3 ([48])**.** *Let $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ be a domain on a nice diagram such that $\mu(\phi) = 1$ and $n_{w_i}(\phi) = 0, \forall i$. If ϕ has a holomorphic representative, then ϕ is either an empty embedded* *bigon or an empty embedded square. Conversely, if* $\phi \in \pi_2(x, y)$ *with* $n_{w_i}(\phi) = 0, \forall i$ *is an empty embedded bigon or an empty embedded square, then the product complex structure on* $\Sigma \times D^2$ *achieves transversality for* ϕ *under a generic perturbation of the* α *and the* β *curves, and* $\mu(\phi) = 1$ *as well as* $\#M(\phi)/\mathbb{R} = 1 \pmod 2$.

The above theorem enables us to combinatorially count differentials in $\widehat{CF}$ on a nice diagram, by counting empty embedded bigons and squares. Following the same lines of the proof, we can obtain the following adaption.

Proposition 3.1.4. *Suppose* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ *is a Heegaard diagram such that any region of* Σ *is either a bigon or a square. Then, there is a 1-1 correspondence between the differentials in* $\mathfrak{A}^-(\mathcal{H}, \infty)$ *and the set of empty embedded bigons and empty embedded squares. Thus, the complex* $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ *can be described combinatorially.*

3.1.2 Schubert normal form.

A two-bridge link/knot can be obtained by closing a rational tangle. For the definition of rational tangles, one can see the reference [28] chapter 9 and [3] chapter 7E, 12D. Let us adopt the notations in [3]. By $b(p, q)$ where $\gcd(p, q) = 1$, we denote the two-bridge link/knot according to the rational tangle of slope $\frac{q}{p}$.

Definition 3.1.5 (Schubert normal form)**.** For a two-bridge link/knot $L = b(p, q)$, the *Schubert normal form* is a canonical projection of L with two over-bridges and two under-bridges, where we regard the projection plane as a sphere S in S^3. The two over-bridges O_1, O_2 are straight segments on the projection plane, and each component of the other two under-bridges U_1, U_2 crosses O_1, O_2 alternatively. Together with the lower half space (which is a ball in S^3), the under-bridges U_1, U_2 form a rational tangle of slope $\frac{q}{p}$. Moreover, if L has two components, we arrange the notation such that $L_i = O_i \cup U_i$. We denote the Schubert form by (S, O_1, O_2, U_1, U_2).

Concretely, the Schubert normal form can be obtained by gluing two disks D_1^α, D_2^α shown in Figure 3.1.1. The endpoint a_i is glued to a'_{q-i}, where all the subscripts are modulo $2p$.

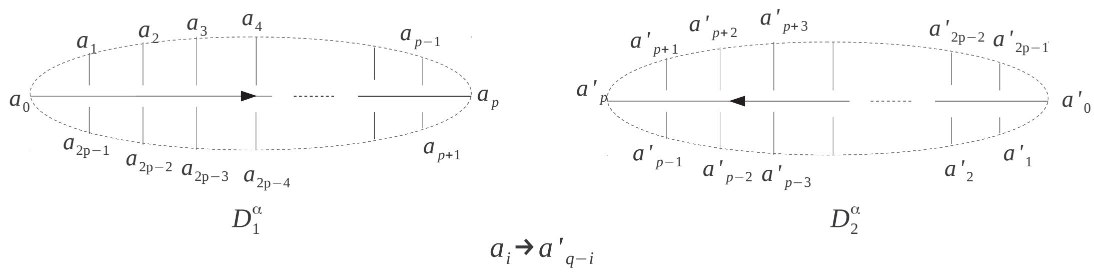


Figure 3.1.1: **The Schubert form: the neighborhoods of the two over-bridges.**

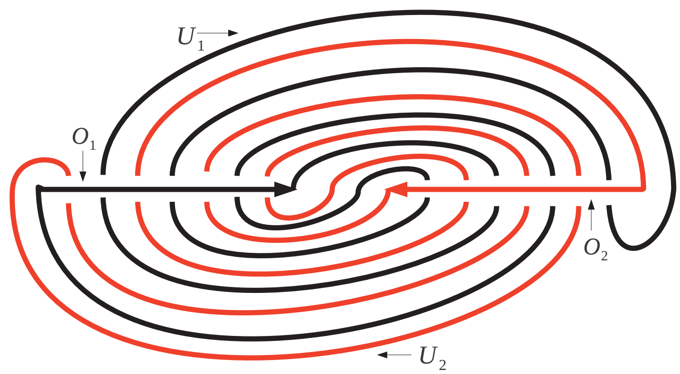


Figure 3.1.2: **The Schubert normal form of the Whitehead link.**

When L has two components, L can be endowed with a canonical orientation induced by the orientation of $\overrightarrow{O_1} = \overrightarrow{a_0 a_p}$, $\overrightarrow{O_2} = \overrightarrow{a'_0 a'_p}$, which is also shown in Figure 3.1.1.

Example 3.1.6. In Figure 3.1.2, we show the Schubert normal form of the Whitehead link $b(8, 3)$.

Remark 3.1.7. Here in the definition of Schubert normal form, we set the straight arcs to be over-bridges, while in other places the straight arcs are set to be under-bridges. However, given p, q, these two links are the same up to taking the mirror of each other.

Fact 3.1.8. *Let $b(p, q)$ denote the two-bridge link defined as above, where $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, $p > 0$. Then*

(1)(12.1. in [3]) *When p is odd, the link $b(p,q)$ is a knot; and when p is even, $b(p,q)$ has two components.*

(2)(Schubert, [3], Theorem 12.6.) *As an oriented link/knot, $b(p,q)$ is equivalent to $b(p',q')$ if and only if $p'=p, q'\equiv q^{\pm 1}(\text{mod } 2p)$; as an unoriented link/knot, $b(p,q)$ is equivalent to $b(p',q')$ if and only if $p'=p, q'\equiv q^{\pm 1}(\text{mod } p)$.*

(3)(12.8 in [3]) *When $b(p,q)$ has two components, the link $b(p,-q)$ is the mirror of $b(p,q)$, and the link $b(p,q+p)$ can be obtained by changing the orientation on one component of $b(p,q)$.*

(4)(Remark 12.7, [3]) *The linking number can be computed by the formula:*

$$\mathrm{lk}(b(p,q)) = -\sum_{i=1}^{\frac{p}{2}} (-1)^{\left\lfloor \frac{(2i-1)q}{p} \right\rfloor}.$$

(5)(Theorem 9.3.6, [28]) *The signature can be computed by the formula:*

$$\sigma(b(p,q)) = \sum_{i=1}^{p-1} (-1)^{\left\lfloor \frac{iq}{p} \right\rfloor}.$$

3.1.3 Heegaard diagrams of two-bridge links.

In this section, we construct nice Heegaard diagrams of two-bridge links by using their Schubert forms.

Definition 3.1.9. A *bridge presentation* of a link L is a topological pair (L,S) inside S^3, such that

- S is an embedded sphere transversely intersecting L,
- $S^3 - S \cong B_1 \cup B_2$, where B_1, B_2 are homeomorphic to the unit ball B^3,
- each pair $(L\cap B_i, B_i)$ with $i=1,2$ is homeomorphic to the pair $(\{P_j\}_{j=1}^k \times I, D^2\times I)$, where $\{P_j\}_{j=1}^k$ is a set of points in the interior of the unit disk D^2.

The minimum over all possible k is called the *bridge number* of the link, denoted by $\mathrm{br}(L)$.

Every bridge presentation of L gives rise to a genus-0 multi-pointed Heegaard diagram for L. Let the sphere S be the Heegaard surface. In each ball B_i, choose $k-1$ disjoint proper disks to divide B_i into k chambers, such that every component of the k bridges is in a distinct chamber. The boundaries of these disks are the alpha, beta curves. The basepoints w_i, z_i are the intersection points of L and S. In other words, by pushing all the bridges onto the sphere S, we obtain a projection of L consisting of $2k$ arcs $a_1, \cdots, a_k, b_1, \cdots, b_k$, such that the arcs $\{a_i\}$ are disjoint, the arcs $\{b_j\}$ are disjoint, and the arcs $\{a_i\}$ are always over the arcs $\{b_j\}$. Then the boundaries of tubular neighborhoods of the arcs $\{a_i\}_{i=1}^{k-1}, \{b_j\}_{j=1}^{k-1}$ are the alpha, beta curves.

Definition 3.1.10 (Schubert Heegaard diagrams)**.** Let $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$ be a two-bridge link, and let (S, O_1, O_2, U_1, U_2) be its Schubert form. The *Schubert Heegaard diagram* of L is the Heegaard diagram $\mathcal{H} = (S^2, \{\alpha\}, \{\beta\}, \{z_1, z_2\}, \{w_1, w_2\})$, where

- $\alpha = \partial N(O_1), \beta = \partial N(U_1)$ with $N(O_1), N(U_1)$ being disjoint tubular neighborhoods of O_1, U_1 on S,
- $\{z_1, w_1\} = \{L_1 \cap S\}$ and $\{z_2, w_2\} = \{L_2 \cap S\}$.

Concretely, regarding the Schubert form as the gluing of two disks D_1^α, D_2^α in Figure 3.1.1, we can take $\alpha = \partial D_1^\alpha$ and $\beta = \partial N(U_1)$. The basepoint z_1 can be any point in D_1^α near a_p, and the basepoint w_1 can be any point in D_1^α near a_0; whereas the basepoint z_2 can be any point in D_2^α near a'_p, and the basepoint w_2 can be any point in D_2^α near a'_0.

Example 3.1.11. The two-bridge link $b(8,3)$ is the Whitehead link *Wh* (or its mirror due to the convention). The Schubert Heegaard diagram of *Wh* is in Figure 3.1.3.

Notation 3.1.12. *Since we will repeatedly discuss the Schubert Heegaard diagram, it is convenient to make a notational convention for all the intersection points and regions as follows.*

- *The components of $S - \beta$ are both disks, denoted by D_1^β, D_2^β such that the disk D_i^β is a neighborhood of U_i.*

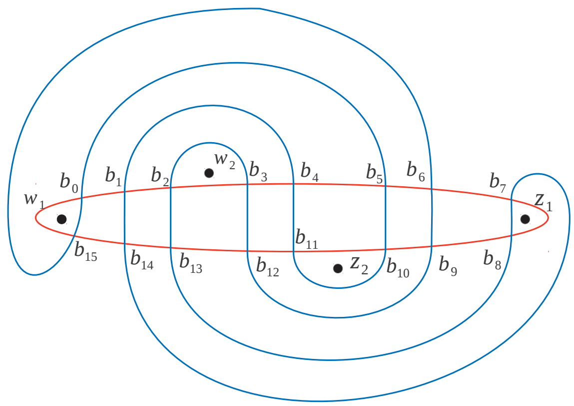


Figure 3.1.3: **The Schubert Heegaard diagram of the Whitehead link.** The red curve is α, and the blue curve is β.

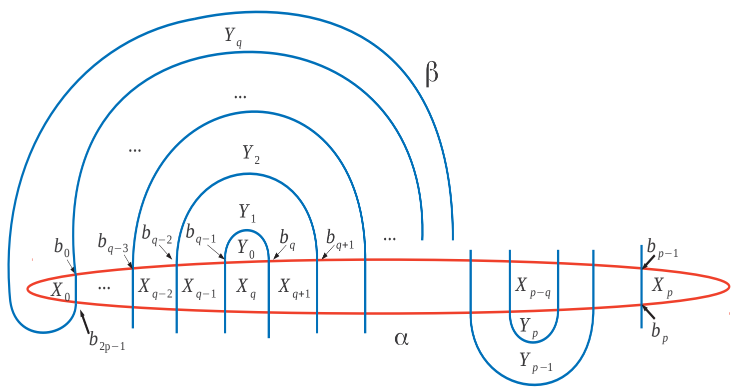


Figure 3.1.4: **The Schubert Heegaard diagram for the two-bridge link $b(p, q)$.**

- *There is a total of four bigons among the components of $S - (\alpha \cup \beta)$, and each of them contains a distinct basepoint in $\{z_1, z_2, w_1, w_2\}$. All the other components are squares.*

- *We label all the $p+1$ components of $D_1^\alpha - \beta$ by $X_0, X_1, ..., X_p$, and label all the $p+1$ components of $D_2^\alpha - \beta$ by $Y_0, Y_1, ..., Y_p$, such that X_0, X_p, Y_0, and Y_p are bigons and $w_1 \in X_0, w_2 \in Y_0, z_1 \in X_p, z_2 \in Y_p$.*

- *There is a total of $2p$ intersection points of α and β. We label them by $b_0, b_1, ..., b_{2p-1}$ clockwise, such that b_0, b_{2p-1} are vertices of X_0, and b_{p-1}, b_p are vertices of X_p. All the subscripts are modulo $2p$.*

The above properties and conventions are illustrated in Figure 3.1.4.

Lemma 3.1.13. *The Schubert Heegaard diagram of the two-bridge link $b(p,q)$ is a nice diagram. By Proposition 3.1.4, the generalized Floer complexes of the Schubert Heegaard diagram are combinatorial.*

From the property of Schubert normal form, it follows a direct description of the Schubert Heegaard diagram.

Lemma 3.1.14. *In the Schubert Heegaard diagram of $b(p,q)$,*

(1) in the disk D_1^α, the points b_i and b_{2p-1-i} are connected by a β-arc,

(2) in the disk D_2^α, the points b_i and b_j are connected by a β-arc if and only if $i+j \equiv 2q-1 \pmod{2p}$.

3.1.4 The multi-variable Alexander polynomial of two-bridge links.

With the help of link Floer homology, we can directly calculate the multi-variable Alexander polynomial of knots and links. In [42], there is a formula of the Euler characteristic of $\widehat{HFL}(L)$:

$$\sum_{h\in\mathbb{H}(L)} \chi(\widehat{HFL}_*(L,h)) \cdot e^h = \prod_{i=1}^{l}(T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}})\Delta_L. \tag{3.1.1}$$

Definition 3.1.15 (Thin complex and E_2-collapsed complex)**.** Suppose (C,∂) is a $\mathbb{Z}^2$-filtered chain complex of $\mathbb{F}$-vector spaces. Let (i,j) denote the filtration, and let g denote the internal grading. The complex (C,∂) is called *thin*, if $i+j-g$ is a constant for all elements in C. The chain complex C is called *E_2-collapsed*, if the differential can be decomposed as $\partial = \partial_1 + \partial_2$, such that $F(\partial_1(x)) = F(x) - (1,0)$ and $F(\partial_2(x)) = F(x) - (0,1)$, where $F(x)$ is the $\mathbb{Z}^2$-filtration of x.

Remark 3.1.16. A thin complex is E_2-collapsed. The classification of E_2-collapsed complexes of $\mathbb{F}$-vector spaces is shown in [42] Section 12.1.

Proposition 3.1.17. *Let $L = b(p,q)$ be a two-bridge link, where p is even and $-p<q<p$. Let $A(b_i)$ be the Alexander grading, and let q^{-1} be the number theoretical reciprocal of q*

modulo $2p$. Then

$$A(b_i) - A(b_{i-1}) = \begin{cases} ((-1)^{\lfloor q^{-1}\cdot i/p \rfloor}, 0), & i \textit{ is even,} \\ (0, (-1)^{\lfloor (q^{-1}\cdot i+1)/p \rfloor}), & i \textit{ is odd.} \end{cases}$$

Furthermore, we have $A_1(b_i) + A_2(b_i) - M(b_i)$ is a constant, i.e. not dependent on i, where $M(b_i)$ is the Maslov grading. In other words, the chain complex $A^-_{+\infty,+\infty}(L)$ is thin.

Proof. We use Notations 3.1.12. There is a set of bigons of Maslov index 1 connecting b_i and b_{i+1}, for $i = 0, 1, ..., 2p-2$. Each of these bigons is a part of one of the disks D^β_1 and D^β_2. In fact, these bigons can be obtained by chasing the under-bridges U_1 and U_2.

The under-bridge U_1 starts from $z_1 = a_p = a'_{q-p}$ and passes the disks D^α_2 and D^α_1 alternately. For i even, at the point a_i, if the under-bridge U_1 is pointing out of D^α_1, then $i \equiv p - 2kq \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\left\lfloor \frac{i\cdot q^{-1}}{p} \right\rfloor$ is even. In this case, there is a bigon ϕ from b_i to b_{i-1} with a single basepoint z_1 on it, and thereby

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor \frac{q^{-1}\cdot i}{p} \rfloor}, 0).$$

If the under-bridge U_1 is pointing into D^α_1, then $i \equiv 2kq - p \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\left\lfloor \frac{i\cdot q^{-1}}{p} \right\rfloor$ is odd. In this case, there is a bigon ϕ from b_{i-1} to b_i with a single basepoint z_1 on it, and still

$$A(b_i) - A(b_{i-1}) = ((-1)^{\lfloor \frac{q^{-1}\cdot i}{p} \rfloor}, 0).$$

Similarly, by keeping track of U_2, we can prove the other cases. For i odd, at the point a_i, if the under-bridge U_2 is pointing off D^α_1, then $i \equiv p - q - 2kq \pmod{2p}$ for some k with $0 < k < \frac{p}{2}$, which is equivalent to $\left\lfloor \frac{1+i\cdot q^{-1}}{p} \right\rfloor$ is even. In this case, there is a bigon ϕ of index 1 from b_i to b_{i-1} with a single basepoint z_2 on it. Thus, we have $A(b_i) - A(b_{i-1}) = (0, (-1)^{\lfloor \frac{q^{-1}\cdot i+1}{p} \rfloor})$ for i odd.

From Lipshitz's formula $\mu(\phi) = e(\phi) + \mu_{b_i}(\phi) + \mu_{b_{i-1}}(\phi)$, it follows $\mu(\phi) = 1$, and thereby for all i,

$$A_1(b_i) + A_2(b_i) - M(b_i) = A_1(b_{i-1}) + A_2(b_{i-1}) - M(b_{i-1}).$$

$\square$

Now we are able to compute the multi-variable Alexander polynomial of two-bridge links by Equation (3.1.1). Since the Floer chain complex for the Schubert Heegaard diagram is thin, there are no differentials in the associated graded complex of $\widehat{CFL}(L,h)$ for the Alexander filtration. That is, for this thin complex, $\widehat{HFL}(L,h) = \widehat{CFL}(L,h)$. Thus,

$$\prod_{i=1}^{l}(T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}}) \cdot \Delta_L(x,y) = \sum_{i=0}^{2p-1}(-1)^{A_1(b_i)+A_2(b_i)} \cdot x^{A_1(b_i)} \cdot y^{A_2(b_i)}.$$

By computer experiments, we have found two distinct two-bridge links $b(126,47)$ and $b(126,55)$ that share the same Alexander polynomial, signature, and linking number, but are not the same or mirror to each other.

$$\Delta_{b(126,47)}(x,y) = \Delta_{b(126,55)}(x,y) = -15 + \frac{8}{x} + 8x + \frac{y}{8} + 8y - \frac{4}{xy} - 4xy - \frac{4x}{y} - \frac{4y}{x},$$

$$\sigma(b(126,47)) = \sigma(b(126,55)) = 3,$$

$$\mathrm{lk}(126,47) = \mathrm{lk}(126,55) = 1.$$

3.1.5 The Floer complexes for two-bridge links.

Let $\mathcal{H} = (S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ be the Schubert Heegaard diagram of $b(p,q)$. By Lemma 3.9, the generalized Floer complex $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ is combinatorial. It consists of counting the empty embedded bigons of Maslov index 1 on S, since here $g + k - 1 = 1$.

The pattern of empty embedded bigons of Maslov index 1 is illustrated in Figure 3.1.5 as in [48]. In Schubert Heegaard diagram, these bigons are always in a similar form of Figure 3.1.5, where the function f_p is defined as follows.

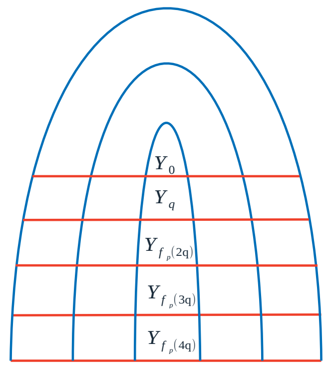


Figure 3.1.5: **A bigon in the Schubert Heegaard diagram of a two-bridge link.** The red lines are parts of α, and the blue curves are parts of β.

Definition 3.1.18. For all $n, m, k \in \mathbb{Z}$, let $\mathrm{Mod}(n, m, k)$ be the residue of n modulo m starting from k, that is,

$$\mathrm{Mod}(n, m, k) \equiv n \pmod{m} \quad \text{and} \quad k \leq \mathrm{Mod}(n, m, k) \leq k + m - 1.$$

Then f_p is defined by

$$f_p(n) = |\mathrm{Mod}(n, 2p, -p+1)| .$$

Lemma 3.1.19. *In the Schubert Heegaard diagram $(S, \{\alpha\}, \{\beta\}, \{w_1, w_2\}, \{z_1, z_2\})$ of the two-bridge link $\overrightarrow{L} = b(p, q)$, the regions in D_1^β are $X_0, Y_q, X_{f_p(2q)}, Y_{f_p(3q)}, \ldots, X_{f_p(pq)} = X_p$ consecutively, and the regions in D_2^β are $Y_0, X_q, Y_{f_p(2q)}, X_{f_p(3q)}, \ldots, Y_{f_p(pq)} = Y_p$ consecutively.*

Proof. Note that D_i^β is the regular neighborhood of the under-bridge U_i. The region X_i contains the arc $a_i a_{-i} \subset L$, and the region Y_j contains arc $a'_j a'_{-j} \subset L$. Conversely, the point a_i is contained in $X_{f_p(i)}$, and the point a'_j is contained in $Y_{f_p(j)}$. Thus since a_i, a'_{q-i} are glued together and a_{-i}, a'_{q+i} are glued together, $X_{f_p(i)}$ is adjacent to $Y_{f_p(q-i)}$ and $Y_{f_p(q+i)}$. Since Y_0 is in D_2^β and it is adjacent to X_q, the region X_q is adjacent to $Y_{f_p(2q)}$. Inductively, we can show in D_2^β, $Y_{f_p(kq)}$ is adjacent to $X_{f_p[(k-1)q]}$ and $X_{f_p[(k+1)q]}$. A similar argument applies to D_1^β. $\square$

Definition 3.1.20. In the bigon ϕ, denote the number of α arcs in ϕ by $n_\alpha(\phi)$, and denote the number β arcs in ϕ by $n_\beta(\phi)$.

Every bigon ϕ is uniquely determined by $n_\alpha(\phi), n_\beta(\phi)$ and the basepoint on it.

Lemma 3.1.21 (Patterns of bigons)**.** *In the Schubert Heegaard diagram of $b(p,q)$, suppose ϕ is an empty embedded bigon of index 1 in $\pi_2(b_i, b_j)$. Then*

$$
\begin{aligned}
(i,j) &= ((1-n_\alpha)q + n_\beta - 1, (1-n_\alpha)q - n_\beta), && \textit{if } w_1 \in \phi, n_\alpha \textit{ is odd,}\\
(i,j) &= (n_\alpha q - n_\beta, n_\alpha q + n_\beta - 1), && \textit{if } w_1 \in \phi, n_\alpha \textit{ is even,}\\
(i,j) &= (n_\alpha q - n_\beta, n_\alpha q + n_\beta - 1), && \textit{if } w_2 \in \phi, n_\alpha \textit{ is odd,}\\
(i,j) &= ((1-n_\alpha)q + n_\beta - 1, (1-n_\alpha)q - n_\beta), && \textit{if } w_2 \in \phi, n_\alpha \textit{ is even.}
\end{aligned}
$$

Furthermore, given $m, n \in \mathbb{Z}$ and a basepoint $pt \in \{w_1, w_2, z_1, z_2\}$, there exists at most one empty embedded bigon ϕ with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{pt}(\phi) = 1$.

There exists an empty embedded bigon ϕ of index 1 with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$, and $n_{w_2}(\phi) = 1$ if and only if the condition $P_1(m,n)$ holds.

The condition $P_1(m,n)$ is as follows:

1. *either $m = 1$, or if $m > 1$, then the set of intervals: $[0, n-1]$ and all intervals $[f_p(2iq) - n + 1, f_p(2iq) + n - 1]$ with $1 \leq 2i \leq m - 1$ are pairwise disjoint intervals in $[0, p]$;*

2. *either $m = 1$, or if $m > 1$, then the set of intervals: all intervals $[f_p((2i+1)q) - n + 1, f_p((2i+1)q) + n - 1]$ with $1 \leq 2i+1 \leq m-1$ are also pairwise disjoint intervals in $[1, p-1]$.*

Similarly, there exists an empty embedded bigon ϕ of index 1 with $n_\alpha(\phi) = m$, $n_\beta(\phi) = n$ and $n_{w_1}(\phi) = 1$ if and only if the condition $P_2(m,n)$ holds.

The condition $P_2(m,n)$ is as follows:

1. *either $n = 1$, or $n > 1$ and the set of intervals: $[0, m-1]$ and all intervals $[f_p(2iq) - m + 1, f_p(2iq) + m - 1]$ with $1 \leq 2i \leq n - 1$ are pairwise disjoint intervals in $[0, p]$;*

2. *either* $n = 1$, *or* $n > 1$ *and the set of intervals: all intervals* $[f_p((2i+1)q) - m + 1, f_p((2i+1)q) + m - 1]$ *with* $1 \leq 2i+1 \leq n-1$ *are also pairwise disjoint intervals in* $[1, p-1]$.

In addition, for $i = 1, 2$, *there is a one-to-one correspondence between the set of all the empty embedded bigons with* $n_{w_i} = 1$ *and the set of empty embedded bigons with* $n_{z_i} = 1$, *where the bigon* $\phi \in \pi_2(b_i, b_j)$ *with* $n_\alpha = m, n_\beta = n, n_{w_i} = 1$ *is sent to the bigon* $\phi' \in \pi_2(b_i+p, b_j+p)$ *with* $n_\alpha = m, n_\beta = n, n_{z_i} = 1$.

Proof. Suppose ϕ is a bigon of index 1 in $\pi_2(b_i, b_j)$ with $n_{w_2}(\phi) = 1$. Combining Lemma 3.1.14 and Lemma 3.1.19, we can get the formula of (i, j) out of Figure 3.1.5 by induction on n_α, n_β. The initial step is $(i, j) = (q-1, q)$, for $n_\alpha = n_\beta = 1$. Similarly, we can show the other case where $n_{w_1}(\phi) = 1$.

For the second part, the sufficient and necessary condition of when there exists an empty embedded bigon is that all the regions in the bigon are not overlapped. By Lemma 3.1.19, it is not hard to get the formulas by induction.

Finally, notice that there is a symmetry of the Heegaard Schubert diagram which sends b_k to b_{k+p} and exchanges w_i to z_i for all $1 \leq k \leq 2p, i = 1, 2$. This symmetry directly gives the one-to-one correspondence between the bigons with w_i and the ones with z_i. $\square$

Consequently, we get an algorithm for computing $A_s^-(b(p,q))$ as follows.

Theorem 3.1.22. *Let* $L = b(p,q)$ *be a two-bridge link and* $\mathcal{H}$ *be the Schubert Heegaard diagram. Define functions* $F_i : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}/2p\mathbb{Z}, i = 1, 2,$ *by*

$$F_1(m,n) = \begin{cases} ((1-m)q + n - 1, (1-m)q - n), & \text{if } m \text{ is odd,} \\ (mq - n, mq + n - 1), & \text{if } m \text{ is even.} \end{cases}$$

$$F_2(m,n) = \begin{cases} ((1-m)q + n - 1, (1-m)q - n), & \text{if } m \text{ is even,} \\ (mq - n, mq + n - 1), & \text{if } m \text{ is odd.} \end{cases}$$

The conditions $P_i(m,n), i=1,2$ are as in Lemma 3.1.21.

Then, the complex $\mathfrak{A}^-(\mathcal{H},+\infty,+\infty)$ is a free $\mathbb{F}[[U_1,U_2]]$-module generated by $g_0,...g_{2p-1}$ with differentials

$$\partial g_i = \sum_{j=0}^{2p-1}(\lambda_{i+p,j+p}+\mu_{i+p,j+p})g_j + \sum_{j=0}^{2p-1}\lambda_{i,j}U_1 g_j + \sum_{j=0}^{2p-1}\mu_{i,j}U_2 g_j,$$

where the coefficients $\lambda_{i,j},\mu_{i,j}\in\mathbb{Z}/2\mathbb{Z}$ are determined by the following equations

$$\lambda_{i,j} = \#\{(m,n)\in\mathbb{N}\times\mathbb{N} | 1\le m,n\le p, F_1(m,n)=(i,j), P_1(m,n) \textit{ is true}\} \pmod 2,$$
$$\mu_{i,j} = \#\{(m,n)\in\mathbb{N}\times\mathbb{N} | 1\le m,n\le p, F_2(m,n)=(i,j), P_2(m,n) \textit{ is true}\} \pmod 2.$$

Remark 3.1.23. One can get an algorithm of $O(p^2)$ time complexity for computing $A_{\mathrm{s}}^-(b(p,q))$. To compute $A_{\mathrm{s}}^-(b(p,q))$, we only need to know $A_{+\infty,+\infty}^-(b(p,q))$, which is determined by all the counting of bigons, i.e. those $\lambda_{i,j}$'s and $\mu_{i,j}$'s. Computing $\lambda_{i,j}$'s and computing $\mu_{i,j}$'s are similar. In order to get all the $\lambda_{i,j}$'s, one can nest two loops. The outer loop is indexed by $n\ge 1$, and the inner loop is indexed by $m\ge 1$ with a test condition $P_1(m,n)$. When $P_1(m,n)$ is true, we change the value of $\lambda_{F_1(m,n)}$ by 1 (mod 2) and keep running the inner loop; when $P_1(m,n)$ is false, we stop the inner loop and go back to the outer loop.

Let us estimate the time complexity. Switching the α and β roles converts the Heegaard diagram of $b(p,q)$ to its mirror $b(p,-q)$. Thus, we assume $0<q<p$. First, when $P_1(m,n)$ is true and $n>q$, m must be 1, as otherwise the second part of $P_1(m,n)$ would imply $f_p(q)-n+1=q-n+1\ge 1$. Thus we force the outer loop stop when $n=q+1$. Computing the other $\lambda_{F_1(m,n)}$'s with $(m,n)=(1,n), n>p$ can be done within $O(p)$ operations. Second, if $P_1(m,n)$ is true, then $m\le (p+1)/n$. This is because the first part of $P_1(m,n)$ implies that there are m pieces of open intervals $(f_p(2iq)-n+\frac{1}{2}, f_p(2iq)+n-\frac{1}{2})$ pairwise disjoint in $(-\frac{1}{2}, p+\frac{1}{2})$. Thus, at the n^{th} step of the outer loop, the inner loop stops within $\lfloor (p+1)/n\rfloor$ steps. Finally, testing $P_1(m,n)$ can be done within $2m$ steps. In fact, when m is even, we can check if the new interval $[f_p(mq)-n+1, f_p(mq)+n-1]$ is disjoint from the other $\frac{m}{2}-1$

intervals in the first part of $P_1(m,n)$ (which are already ordered in the previous step). If it is disjoint from the other intervals, we put it in the correct position in the order. This is done within $2m$ operations. It is similar when m is odd. Thus the time complexity is of the order

$$T(p,q) = (\sum_{n=1}^{q}\sum_{m=1}^{\lfloor (p+1)/n \rfloor} 2m) + O(p) \leq \left(\sum_{n=1}^{q}[(\frac{p+1}{n})^2 + \frac{p+1}{n}]\right) + O(p).$$

Since $n \leq q \leq p$, $(p+1)/n \geq 1$, thus

$$T(p,q) \leq 2[\sum_{n=1}^{q}\frac{(p+1)^2}{n^2}] + O(p) = O(p^2).$$

3.2 Applying the surgery formula to two-bridge links

In this section, we show some algebraic rigidity results for the chain maps between certain chain complexes up to chain homotopy. This provides a way to determine the destabilization maps in the surgery complex of two-bridge links up to chain homotopy. Using these maps to replace the original maps in the surgery complex, we construct a *perturbed surgery complex*. We further show that it has the same homology as the original one. Based on the perturbed surgery formula, we give an algorithm for computing the homology of surgeries on two-bridge links.

3.2.1 Algebraic rigidity results.

There is a short exact sequence in the Exercise 3.6.1 in [49] as follows. Suppose P_*, Q^* are (co)chain complexes of R-modules, and $P, d(P) = \mathrm{Im}(d)$ are both projective R-modules. Then there is an exact sequence

$$0 \to \prod_{p+q=n-1}\mathrm{Ext}^1_R(H_p(P_*), H^q(Q^*)) \to H_n(\mathrm{Hom}(P_*, Q^*)) \to \prod_{p+q=n}\mathrm{Hom}_R(H_p(P_*), H^q(Q^*)) \to 0.$$

For completeness, we give a proof here adapted to the setting of $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes.

Lemma 3.2.1. *Let P_*, Q_* be $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes of R-modules. If P and $d(P) = \mathrm{Im}(d)$ are projective modules, then there is a short exact sequence for any $n, p, q \in \mathbb{Z}/2\mathbb{Z}$,*

$$0 \to \bigoplus_{p+q=n+1} \mathrm{Ext}^1_R(H_p(P), H_q(Q)) \to H_n(\mathrm{Hom}(P,Q)) \to \bigoplus_{p+q=n} \mathrm{Hom}_R(H_p(P), H_q(Q)) \to 0. \tag{3.2.1}$$

Proof. First, all the indices n, p, q, i, j are in $\mathbb{Z}/2\mathbb{Z}$. Since $d(P)$ is projective, the short exact sequence $0 \to Z^P \to P \to d(P) \to 0$ splits, thus giving that $P = d(P) \oplus Z^P$. Thereby, Z^P is projective and thus $\mathrm{Ext}^1_R(Z^P, M) = 0$ for all R-module M. Also, by $\mathrm{Ext}^1_R(d(P), M) = 0, \forall M$ we get an exact sequence

$$0 \to \mathrm{Hom}_R(d(P_p), Q_q) \to \mathrm{Hom}_R(P_p, Q_q) \to \mathrm{Hom}_R(Z^P_p, Q_q) \to 0.$$

These assemble to a short exact sequence of chain complexes

$$0 \to \bigoplus_{p+q=n} \mathrm{Hom}_R(d(P_p), Q_q) \to (\mathrm{Hom}_R(P,Q))_n \to \bigoplus_{p+q=n} \mathrm{Hom}_R(Z^P_p, Q_q) \to 0. \tag{3.2.2}$$

Actually, it is not hard to check the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{p+q=n} \mathrm{Hom}_R(d(P_p), Q_q) \arrow[r] \arrow[d, "d"] & (\mathrm{Hom}_R(P,Q))_n \arrow[r] \arrow[d, "d"] & \bigoplus_{p+q=n} \mathrm{Hom}_R(Z^P_p, Q_q) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \bigoplus_{p+q=n+1} \mathrm{Hom}_R(d(P_p), Q_q) \arrow[r] & (\mathrm{Hom}_R(P,Q))_{n+1} \arrow[r] & \bigoplus_{p+q=n+1} \mathrm{Hom}_R(Z^P_p, Q_q) \arrow[r] & 0.
\end{tikzcd}$$

Since dP is projective, the short exact sequence $0 \to d(Q_{j-1}) \to Z^Q_j \to H_j(Q) \to 0$ gives a short exact sequence

$$0 \to \mathrm{Hom}_R(dP_i, d(Q_{j-1})) \to \mathrm{Hom}_R(dP_i, Z^Q_j) \to \mathrm{Hom}_R(dP_i, H_j(Q)) \to 0.$$

Furthermore, the differential in $\mathrm{Hom}_R(dP_i, Q_j)$ is d_Q, so from the above exact sequence it follows that $H_n(\mathrm{Hom}_R(d(P), Q)) = \bigoplus_{p+q=n} \mathrm{Hom}_R(d(P_p), H_q(Q)), p, q, n \in \mathbb{Z}/2\mathbb{Z}$. Since Z^P

is projective, similarly we have

$$H_n(\mathrm{Hom}_R(Z^P, Q)) = \bigoplus_{p+q=n} \mathrm{Hom}_R(Z^P_p, H_q(Q)).$$

Thus the long exact sequence of homology from Equation (3.2.2) is

$$\begin{aligned}\cdots \to \bigoplus_{p+q=n} \mathrm{Hom}_R(Z^P_{p+1}, H_q(Q)) &\xrightarrow{\partial_{n+1}} \bigoplus_{p+q=n} \mathrm{Hom}_R(d(P_p), H_q(Q)) \to H_n(\mathrm{Hom}_R(P,Q)) \\ \to \bigoplus_{p+q=n} \mathrm{Hom}_R(Z^P_p, H_q(Q)) &\xrightarrow{\partial_n} \bigoplus_{p+q=n} \mathrm{Hom}_R(d(P_{p+1}), H_q(Q)) \to \cdots . \end{aligned} \quad (3.2.3)$$

A diagram chasing shows that the connecting morphism $\partial_* : \mathrm{Hom}(Z^P_*, H_*(Q)) \to \mathrm{Hom}(d(P_*), H_*(Q))$ is the restriction.

Hence, the short exact sequence $0 \to dP_{i+1} \to Z^P_i \to H_i(P) \to 0$ can produce the exact sequence

$$\begin{aligned} 0 \;\to\; & \mathrm{Hom}_R(H_p(P), H_q(Q)) \to \mathrm{Hom}_R(Z^P_p, H_q(Q)) \xrightarrow{\partial_{p+q}} \mathrm{Hom}_R(dP_{p+1}, H_q(Q)) \\ \to\; & \mathrm{Ext}^1_R(H_p(P), H_q(Q)) \to \mathrm{Ext}^1_R(Z^P_p, H_q(Q)) = 0, \end{aligned}$$

thus $\mathrm{Ker}(\partial_{p+q}) \cong \mathrm{Hom}_R(H_p(P), H_q(Q))$ and $\mathrm{Coker}(\partial_{p+q}) \cong \mathrm{Ext}^1_R(H_p(P), H_q(Q))$. Finally, the exact sequence in Equation (3.2.1) comes from Equation (3.2.3). $\square$

Let (C_*, ∂_*) be a chain complex of $\mathbb{F}$-vector spaces, with U_1, U_2-actions which drop the $\mathbb{Z}$-grading by 2. Consider C as a $\mathbb{F}[[U_1, U_2]]$-module. Even though the U_1, U_2-actions do not preserve the $\mathbb{Z}$-grading, we will still call C a *complex of* $\mathbb{F}[[U_1, U_2]]$*-modules*.

Proposition 3.2.2. *Let A, B be complexes of $\mathbb{F}[[U_1, U_2]]$-modules with U_1, U_2-actions dropping grading by 2, and $A, d(A)$ are both free $\mathbb{F}[[U_1, U_2]]$-modules. Suppose $H_*(A) \cong H_*(B) \cong \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U_i \cdot H_{2k}(A) = H_{2k-2}(A), U_i \cdot H_{2k}(B) = H_{2k-2}(B)$ for both $i = 1, 2$. If $F, G : A \to B$ are both quasi-isomorphisms as $\mathbb{F}[[U_1, U_2]]$-modules, then F and G are homotopic as $\mathbb{F}[[U_1, U_2]]$-modules.*

Proof. First, the $\mathbb{Z}$-grading of A, B induces a $\mathbb{Z}/2\mathbb{Z}$-grading on both A and B, and U_1, U_2-action preserves the induced $\mathbb{Z}/2\mathbb{Z}$-grading, thus we regard A, B as $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes of $\mathbb{F}[[U_1, U_2]]$-modules. In order to distinguish these two gradings, we put brackets on the numbers to represent $\mathbb{Z}/2\mathbb{Z}$-gradings. Hence we have $H_{[0]}(A) = H_{[0]}(B) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2), H_{[1]}(A) = H_{[1]}(B) = 0$.

By Lemma 3.2.1, we have

$$
\begin{aligned}
0 \to \bigoplus_{[p]\in\mathbb{Z}/2\mathbb{Z}} & \mathrm{Ext}^1_{\mathbb{F}[[U_1,U_2]]}(H_{[p+1]}(A_*), H_{[p]}(B_*)) \to H_{[0]}(\mathrm{Hom}(A_*, B_*)) \\
\to \bigoplus_{[p]\in\mathbb{Z}/2\mathbb{Z}} & \mathrm{Hom}_{\mathbb{F}[[U_1,U_2]]}(H_{[p]}(A_*), H_{[p]}(B_*)) \to 0,
\end{aligned}
$$

thus

$$
\begin{aligned}
H_{[0]}(\mathrm{Hom}(A_*, B_*)) &= \mathrm{Hom}_{\mathbb{F}[[U_1,U_2]]}(\mathbb{F}[[U_1, U_2]]/(U_1 - U_2), \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)) \\
&= \mathbb{F}[[U_1, U_2]]/(U_1 - U_2).
\end{aligned}
$$

Since $H_{[0]}(\mathrm{Hom}(A_*, B_*))$ is the group of chain homotopy equivalence classes of chain maps from A to B, this means the chain maps from A to B are classified by their action on homology. Since F and G are both quasi-isomorphisms, they are homotopic as $\mathbb{F}[[U_1, U_2]]$-modules.

Let $H : A \to B$ be any homotopy such that $F - G = H\partial + \partial H, H \cdot U_i = U_i \cdot H$. Then H shifts the $\mathbb{Z}/2\mathbb{Z}$-grading by 1, thus shifting the $\mathbb{Z}$-grading by odd numbers. Thus, let $H = \sum_{i\in\mathbb{Z}} H_{2i+1}$, where $H_{2i+1} : A_* \to B_{*+2i+1}$. Since

$$
F - G = H\partial + \partial H = \sum_{i\in\mathbb{Z}}(H_{2i+1}\partial + \partial H_{2i+1})
$$

preserves the original $\mathbb{Z}$-grading, we have $H_{2i+1}\partial + \partial H_{2i+1} = 0, \forall i \neq 0$. So we can replace the homotopy H by $H_1 : A_* \to B_{*+1}$, thus being a chain homotopy of the original $\mathbb{Z}$-graded chain complexes. $\square$

Remark 3.2.3. The prototype of the complexes in the previous Proposition is the simplest Heegaard Floer chain complex of the unknot in S^3. Let C^{u} be the chain complex of $\mathbb{F}[[U_1, U_2]]$-modules generated by $\mathbf{x}, \mathbf{y}$ with differential $\partial \mathbf{x} = (U_1 - U_2)\mathbf{y}$, where $\mathbf{y}, \mathbf{x}$ are of gradings $0, -1$ respectively.

Corollary 3.2.4. *Let A, B be complexes of $\mathbb{F}[[U_1, U_2]]$-modules with U_1, U_2-actions dropping grading by 2. Suppose A is chain homotopy equivalent to the complex C^u as $\mathbb{F}[[U_1, U_2]]$-modules, and $H_*(B) \cong \mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$ as in Proposition 3.2.2. Then for any quasi-isomorphisms $F, G : A \to B$ of $\mathbb{F}[[U_1, U_2]]$-modules, F and G are chain homotopic as $\mathbb{F}[[U_1, U_2]]$-modules.*

Proof. Let $h_1 : A \to C^{\mathrm{u}}, h_2 : C^{\mathrm{u}} \to A$ be the chain homotopy equivalences, such that $h_1 \circ h_2 \simeq id_{C^{\mathrm{u}}}, h_2 \circ h_1 \simeq id_A$. Then by Proposition 3.2.2, $F \circ h_2$ is homotopic to $G \circ h_2$ as $\mathbb{F}[[U_1, U_2]]$-modules. Hence, $F \circ h_2 \circ h_1, G \circ h_2 \circ h_1$ are homotopic as $\mathbb{F}[[U_1, U_2]]$-modules, and thus so are F and G. □

Proposition 3.2.5. *Let A_*, B_* be complexes of $\mathbb{F}[[U]]$-modules with U-action dropping grading by 2, and A, B are both free $\mathbb{F}[[U]]$-modules. Suppose $H_*(A) = H_*(B) = \mathbb{F}[[U]]$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U \cdot H_{2k}(A) = H_{2k-2}(A), U \cdot H_{2k}(B) = H_{2k-2}(B)$. If $F, G : A \to B$ are both quasi-isomorphisms of $\mathbb{F}[[U]]$-modules, then F, G are chain homotopic as maps of $\mathbb{F}[[U]]$-modules.*

Moreover, if H, K are both chain homotopies as homomorphisms of $\mathbb{F}[[U]]$-modules between any two chain maps $f, g : A \to B$, i.e. $H\partial + \partial H = K\partial + \partial K = f - g$, then $H - K = \partial T + T\partial$, for some $\mathbb{F}[[U]]$-module homomorphism $T : A_* \to B_{*+2}$.

Proof. First, we regard A, B as $\mathbb{Z}/2\mathbb{Z}$-graded complexes of $\mathbb{F}[[U]]$-modules. Since a P.I.D. is hereditary, every submodule of a free module over a P.I.D. is a projective module. See [49] Definition 4.2.10 and Exercise 4.2.6. Thus, $d(A), d(B)$ are both projective $\mathbb{F}[[U]]$-modules.

Applying Lemma 3.2.1, we can compute

$$H_{[0]}(\mathrm{Hom}(A,B)) = \mathrm{Hom}(\mathbb{F}[[U]],\mathbb{F}[[U]]) = \mathbb{F}[[U]],$$
$$H_{[1]}(\mathrm{Hom}(A,B)) = \mathrm{Ext}^1_{\mathbb{F}[[U]]}(\mathbb{F}[[U]],\mathbb{F}[[U]]) = 0.$$

The first identity implies that the quasi-isomorphisms F, G are chain homotopic as $\mathbb{Z}/2\mathbb{Z}$-graded complexes via H. In order to get a homotopy between F and G preserving the $\mathbb{Z}$-grading, we decompose $H = \sum_{i\in\mathbb{Z}} H_{2i+1}$, where $H_{2i+1} : A_* \to B_{*+2i+1}$. Then similarly to Proposition 3.2.2, the map H_1 is also a chain homotopy between F, G.

Since $\partial(H-K)+(H-K)\partial = 0$, the second identity implies that $H-K \in Z_{[1]}(\mathrm{Hom}(A,B)) = B_{[1]}(\mathrm{Hom}(A,B))$. This means there is a homomorphism of $\mathbb{F}[[U]]$-modules $T : A \to B$ preserving the $\mathbb{Z}/2\mathbb{Z}$-grading, such that $H-K = \partial T + T\partial$. Thus, the map T can be decomposed as $T = \sum_{i\in\mathbb{Z}} T_{2i}$, where $T_{2i} : A_* \to B_{*+2i}$. From the fact that $H - K = \sum_{i\in\mathbb{Z}} \partial T_{2i} + T_{2i}\partial$ maps A_n into B_{n+1}, it follows that $\partial T_{2i} + T_{2i}\partial : A_* \to B_{*+2i-1}$ vanish for all $i \neq 1$. Thus $T = T_2 : A_* \to B_{*+2}$. □

Corollary 3.2.6. *Suppose the complexes A_* and B_* are as in Proposition 5.1.1. Then, A_* and B_* are chain homotopy equivalent as $\mathbb{F}[[U]]$-modules.*

Proof. From the proof of Proposition 5.1.1 we see $H_{[0]}(\mathrm{Hom}(A_*,B_*)) = \mathrm{Hom}(H_{[0]}(A), H_{[0]}(B)) = \mathbb{F}[[U]]$, which implies that there exists a quasi-isomorphism $h : A \to B$ as $\mathbb{Z}/2\mathbb{Z}$-graded chain complex of $\mathbb{F}[[U]]$-modules. Decompose h as $h = h_0 + h_1$ such that for all $a \in A_n$, $h_0(a) \in B_n, h_1(a) \in \bigoplus_{i\neq 0} B_{n+2i}$. Then $h\partial_A = \partial_B h$ implies that $h_0\partial_A = \partial_B h_0, h_1\partial_A = \partial_B h_1$, so h_0 is also a chain map preserving the $\mathbb{Z}$-grading. Since $Uh = hU$ and the U-action drops the $\mathbb{Z}$-grading by 2, we have $h_0U + Uh_0 = 0$. In addition, h_0 is also a quasi-isomorphism. Hence, on the homology level, $h_0(1) = 1 \in \mathbb{F}[[U]]$.

Similarly, we have another quasi-isomorphism $g_0 : B_* \to A_*$ preserving the $\mathbb{Z}$-grading, such that $g_0\partial_B = \partial_A g_0, g_0U = Ug_0$. Then, $g_0h_0 : A_* \to A_*$ is a quasi-isomorphism. From Proposition 5.1.1, it follows that $g_0h_0 - id_A = \partial H + H\partial$, where H is chain homotopy of $\mathbb{Z}$-

graded complexes commuting with the U-action. Similarly, we have that $h_0 g_0$ is homotopic to id_B. $\square$

3.2.2 Destabilization maps.

In the link surgery formula, the part of polygon counts for defining the destabilization maps is difficult to read off from the Heegaard diagram. However, in the case of two-bridge links we can use the algebraic rigidity result to avoid the difficulty.

In a primitive system of hyperboxes, all the destabilization maps we need are listed below:

$$\begin{aligned} D^{-L_1}_{-\infty,s_2} :&\mathfrak{A}^-(\mathcal{H}^L,-\infty,s_2)\to\mathfrak{A}^-(\mathcal{H}^L,+\infty,s_2+\mathrm{lk}(L_1,L_2)),\\ D^{-L_2}_{s_1,-\infty} :&\mathfrak{A}^-(\mathcal{H}^L,s_1,-\infty)\to\mathfrak{A}^-(\mathcal{H}^{L_2},s_1+\mathrm{lk}(L_1,L_2),+\infty),\\ D^{-L_1\cup -L_2}_{-\infty,-\infty} :&\mathfrak{A}^-(\mathcal{H}^L,-\infty,-\infty)\to\mathfrak{A}^-(\mathcal{H}^L,+\infty,+\infty). \end{aligned}$$

Second, notice that all the domains and targets of these maps have homology $\mathbb{F}[[U_1,U_2]]/U_1-U_2$, which is isomorphic to $\mathbb{F}[[U_1]]$ as an $\mathbb{F}[[U_1]]$-module. By Proposition 5.1.1, we can substitute $D^{-L_1}_{-\infty,s_2}, D^{-L_2}_{s_1,-\infty}$ by any $\mathbb{F}[[U_1]]$-linear homotopy equivalence, since they are all homotopic as homomorphisms of $\mathbb{F}[[U_1]]$-modules. We can also substitute the diagonal maps $D^{-L_1\cup -L_2}_{-\infty,-\infty}$ by any $\mathbb{F}[[U_1]]$-linear homotopy shifting grading by 1, since they are homotopic up to higher $\mathbb{F}[[U_1]]$-linear homotopy. We will show an invariance theorem of the surgery square under perturbations of the edge maps and the diagonal maps in the next sections.

3.2.3 Perturbed surgery complex for two-bridge links.

The rigidity results in Section 3.2.1 allow us to perturb the edge and diagonal maps up to homotopies, in the surgery square for two-bridge links. However, in order to obtain a square of chain complexes, we still need some more modifications of the square.

First, suppose we have a hypercube $(C^\varepsilon, D^\varepsilon)$. If we change $D^\varepsilon_{\varepsilon^0}$ to $D'^\varepsilon_{\varepsilon^0} = D^\varepsilon_{\varepsilon^0} + \Delta D^\varepsilon_{\varepsilon^0}$

for all ε with $\|\varepsilon\| > 0$, then in order to have a hypercube again, we need to have

$$\begin{aligned} \sum_{\varepsilon' \le \varepsilon} (D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'} + \Delta D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'}) \circ (D^{\varepsilon'}_{\varepsilon^0} + \Delta D^{\varepsilon}_{\varepsilon^0}) &= 0, \\ \sum_{\varepsilon' \le \varepsilon} \Delta D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'} \circ D^{\varepsilon'}_{\varepsilon^0} + D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'} \circ \Delta D^{\varepsilon}_{\varepsilon^0} + \Delta D^{\varepsilon-\varepsilon'}_{\varepsilon^0+\varepsilon'} \circ \Delta D^{\varepsilon}_{\varepsilon^0} &= 0. \end{aligned}$$

This formula provides a necessary condition to inductively perturb the maps from edges to the longest diagonal. Based on the above principles, we get the following procedures to construct the perturbed surgery square.

Suppose $\mathcal{H}$ be a primitive system of hyperboxes of a two-bridge link L and consider Equation (2.2.12). Now we choose an arbitrary $\mathbb{F}[[U_1]]$-linear quasi-isomorphism $\tilde{D}^{-L_i}_{s_1,s_2}$ for substituting $D^{-L_i}_{s_1,s_2}$. By Proposition 5.1.1, $\tilde{D}^{-L_i}_{s_1,s_2}$ and $D^{-L_i}_{s_1,s_2}$ are homotopic by a $\mathbb{F}[[U_1]]$-linear homotopy $H^{-L_i}_{s_1,s_2}$:

$$\tilde{D}^{-L_i}_{\mathrm{s}} = D^{-L_i}_{\mathrm{s}} + H^{-L_i}_{\mathrm{s}} \partial^-_{\mathrm{s}} + \partial^-_{p^{L_i}(\mathrm{s})} H^{-L_i}_{\mathrm{s}}.$$

Then, we choose any $\mathbb{F}[[U_1]]$-linear maps $\tilde{F}^{\pm L_1 \cup -L_2}_{s_1,-\infty}, \tilde{F}^{-L_1 \cup \pm L_2}_{-\infty,s_2}, \tilde{D}^{-L_1 \cup -L_2}_{-\infty,-\infty}$ which are homotopies in each square of Equation (2.2.12), such that the following rectangles are hyperboxes of chain complexes:

$$
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{-L_2}_{s_1,s_2}"] \arrow[d, "I^{-L_1}_{s_1,s_2}"'] \arrow[dr, "I^{-L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{s_1,-\infty}"] \arrow[d, "I^{-L_1}_{s_1,-\infty}"] \arrow[dr, "\tilde{F}^{-L_1\cup -L_2}_{s_1,-\infty}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "I^{-L_1}_{s_1+\mathrm{lk},+\infty}"] \\
A^-_{-\infty,s_2} \arrow[r, "I^{-L_2}_{-\infty,s_2}"'] \arrow[d, "\tilde{D}^{-L_1}_{-\infty,s_2}"'] \arrow[dr, "\tilde{F}^{-L_1\cup -L_2}_{-\infty,s_2}"] & A^-_{-\infty,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{-\infty,-\infty}"] \arrow[d, "\tilde{D}^{-L_1}_{-\infty,-\infty}"] \arrow[dr, "\tilde{D}^{-L_1\cup -L_2}_{-\infty,-\infty}"] & A^-_{-\infty,+\infty} \arrow[d, "\tilde{D}^{-L_1}_{-\infty,+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "I^{-L_2}_{+\infty,s_2+\mathrm{lk}}"'] & A^-_{+\infty,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{+\infty,-\infty}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\quad
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{-L_2}_{s_1,s_2}"] \arrow[d, "I^{+L_1}_{s_1,s_2}"'] \arrow[dr, "I^{+L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{s_1,-\infty}"] \arrow[d, "I^{+L_1}_{s_1,-\infty}"] \arrow[dr, "\tilde{F}^{+L_1\cup -L_2}_{s_1,-\infty}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "I^{+L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{-L_2}_{+\infty,s_2}"'] \arrow[d, "id"'] & A^-_{+\infty,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{+\infty,-\infty}"'] \arrow[d, "id"'] & A^-_{+\infty,+\infty} \arrow[d, "id"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{-L_2}_{+\infty,s_2}"'] & A^-_{+\infty,-\infty} \arrow[r, "\tilde{D}^{-L_2}_{+\infty,-\infty}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\tag{3.2.4}
$$

$$
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{+L_2}_{s_1,s_2}"] \arrow[d, "I^{-L_1}_{s_1,s_2}"'] \arrow[dr, "I^{-L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[r, "id"] \arrow[d, "I^{-L_1}_{s_1,+\infty}"] & A^-_{s_1,+\infty} \arrow[d, "I^{-L_1}_{s_1,+\infty}"] \\
A^-_{-\infty,s_2} \arrow[r, "I^{+L_2}_{-\infty,s_2}"'] \arrow[d, "\tilde{D}^{-L_1}_{-\infty,s_2}"'] \arrow[dr, "\tilde{F}^{-L_1\cup +L_2}_{-\infty,s_2}"] & A^-_{-\infty,+\infty} \arrow[r, "id"] \arrow[d, "\tilde{D}^{-L_1}_{-\infty,+\infty}"] & A^-_{-\infty,+\infty} \arrow[d, "\tilde{D}^{-L_1}_{-\infty,+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty} \arrow[r, "id"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\quad
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "I^{+L_2}_{s_1,s_2}"] \arrow[d, "I^{+L_1}_{s_1,s_2}"'] \arrow[dr, "I^{+L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[r, "id"] \arrow[d, "I^{+L_1}_{s_1,+\infty}"] & A^-_{s_1,+\infty} \arrow[d, "I^{+L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] \arrow[d, "id"'] & A^-_{+\infty,+\infty} \arrow[r, "id"] \arrow[d, "id"] & A^-_{+\infty,+\infty} \arrow[d, "id"] \\
A^-_{+\infty,s_2} \arrow[r, "I^{+L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty} \arrow[r, "id"'] & A^-_{+\infty,+\infty}.
\end{tikzcd}
$$

Definition 3.2.7 (Perturbed surgery square)**.** The above rectangles in Equation (3.2.4) are called *perturbed surgery rectangles* for the two-bridge link L. After compressing them, we get four sets of squares,

$$
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\tilde{\Phi}^{-L_2}_{s_1,s_2}"] \arrow[d, "\tilde{\Phi}^{-L_1}_{s_1,s_2}"'] \arrow[dr, "\tilde{\Phi}^{-L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "\tilde{\Phi}^{-L_1}_{s_1+\mathrm{lk},+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "\tilde{\Phi}^{-L_2}_{+\infty,s_2+\mathrm{lk}}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\quad
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\tilde{\Phi}^{+L_2}_{s_1,s_2}"] \arrow[d, "\tilde{\Phi}^{-L_1}_{s_1,s_2}"'] \arrow[dr, "\tilde{\Phi}^{-L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[d, "\tilde{\Phi}^{-L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2+\mathrm{lk}} \arrow[r, "\tilde{\Phi}^{+L_2}_{+\infty,s_2+\mathrm{lk}}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
$$

$$
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\tilde{\Phi}^{-L_2}_{s_1,s_2}"] \arrow[d, "\tilde{\Phi}^{+L_1}_{s_1,s_2}"'] \arrow[dr, "\tilde{\Phi}^{+L_1\cup -L_2}_{s_1,s_2}"] & A^-_{s_1+\mathrm{lk},+\infty} \arrow[d, "\tilde{\Phi}^{+L_1}_{s_1+\mathrm{lk},+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "\tilde{\Phi}^{-L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty};
\end{tikzcd}
\quad
\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\tilde{\Phi}^{+L_2}_{s_1,s_2}"] \arrow[d, "\tilde{\Phi}^{+L_1}_{s_1,s_2}"'] \arrow[dr, "\tilde{\Phi}^{+L_1\cup +L_2}_{s_1,s_2}"] & A^-_{s_1,+\infty} \arrow[d, "\tilde{\Phi}^{+L_1}_{s_1,+\infty}"] \\
A^-_{+\infty,s_2} \arrow[r, "\tilde{\Phi}^{+L_2}_{+\infty,s_2}"'] & A^-_{+\infty,+\infty}.
\end{tikzcd}
$$

After a Λ-twisted gluing of the above squares, we obtain *a perturbed surgery square* $(\tilde{\mathcal{C}}^-(\mathcal{H}^L,\Lambda),\tilde{\mathcal{D}}^-)$.

Remark 3.2.8. In the definition, a perturbed surgery square depends on the choices of the maps $\tilde{D}^{-L_i}_{s_1,s_2}, \tilde{F}^{\pm L_1\cup -L_2}_{s_1,-\infty}, \tilde{F}^{\pm L_1\cup -L_2}_{-\infty,s_2}, \tilde{D}^{-L_1\cup -L_2}_{-\infty,-\infty}$. However, we will show it is isomorphic to the original square as $\mathbb{F}[[U_1]]$-module.

3.2.4 Invariance of the perturbed surgery complex.

Now we establish the invariance of the perturbed surgery complex for two-bridge links under the change of edge maps and some diagonal maps up to chain homotopies.

Proposition 3.2.9. *Let R be a $\mathbb{F}$-algebra. Suppose $f,g: A\to B$ be two chain maps between two chain complexes of R-modules. If f,g are homotopic to each other by $f\xrightarrow{H} g$, then the mapping cones* $\mathrm{cone}(f),\mathrm{cone}(g)$ *are isomorphic.*

Proof. We directly construct the isomorphism between the mapping cones $\mathrm{cone}(f)$ and $\mathrm{cone}(g)$. Define $K_1:\mathrm{cone}(f)\to\mathrm{cone}(g), K_2:\mathrm{cone}(g)\to\mathrm{cone}(f)$ by

$$\begin{aligned} K_1|_A &= id_A\oplus H, K_1|_B = id_B,\\ K_2|_A &= id_A\oplus H, K_2|_B = id_B. \end{aligned}$$

In fact, K_1 is a chain map, since $\forall a\in A, b\in B$,

$$\begin{aligned} K_1\partial_f(a)+\partial_g K_1(a) &= K_1(\partial_A(a)+f(a))+\partial_g(a+H(a))\\ &= \partial(a)+H\partial_A(a)+f(a)+\partial_A(a)+g(a)+\partial_B H(a)=0,\\ K_1\partial_f(b)+\partial_g K_1(b) &= K_1\partial_B(b)+\partial_g(b)=\partial_B(b)+\partial_B(b)=0. \end{aligned}$$

Moreover, K_2K_1 is $id_{\mathrm{cone}(f)}$, since

$$\begin{aligned} K_2K_1(a) &= K_2(a+H(a))=a+H(a)+H(a)=a,\\ K_2K_1(b) &= K_2(b)=b, \forall a\in A, b\in B. \end{aligned}$$

$$
\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[d, "f"'] \arrow[dr, "H"] & A \arrow[r, "id_A"] \arrow[d, "g"'] \arrow[dr, "H"] & A \arrow[d, "f"'] \\
B \arrow[r, "id_B"'] & B \arrow[r, "id_B"'] & B
\end{tikzcd}
$$

$\square$

There is a hyperbox version of Proposition 3.2.9.

Definition 3.2.10. A hyperbox of chain complexes R is said to be *isomorphic* to another hyperbox R', if there are chain maps of hyperboxes $F : R \to R', G : R' \to R$, such that $F \circ G = id_{R'}, G \circ F = id_R$.

Proposition 3.2.11. *Let $R = ((C^{\varepsilon})_{\varepsilon \in \mathbb{E}((d,1))}, (D^{\varepsilon})_{\varepsilon \in \mathbb{E}(n+1)})$ be a hyperbox of chain complexes of size $(\boldsymbol{d}, 1) \in \mathbb{Z}^{n+1}_{\geq 0}$. If all the edge maps $D^{(\boldsymbol{0},1)} = id$, where $\boldsymbol{0} = (0, ..., 0) \in \mathbb{Z}^n$, then R induces an isomorphism from the subhyperbox $R^{\varepsilon_{n+1}=0}$ to the subhyperbox $R^{\varepsilon_{n+1}=1}$.*

Proof. We first show the case of hypercubes by induction, i.e., $\mathbf{d} = (1, ..., 1) \in \mathbb{Z}^n$.

When $n = 1$, this is exactly Proposition 5.8. When $n > 1$, let us make some notations at first. There is a $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 0$, denoted by R^{00}, and there is also a $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = 0, \varepsilon_{n+1} = 1$, denoted by R^{01}. Similarly, the subhypercube corresponding to $\varepsilon_n = 1, \varepsilon_{n+1} = 0$ is denoted by R^{10}, and the hypercube corresponding to $\varepsilon_n = \varepsilon_{n+1} = 1$ is denoted by R^{11}. Then we can view the hypercube R as the following square of hypercubes.

$$
\begin{tikzcd}
R^{00} \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, "H"] & R^{10} \arrow[d, "h_2"] \\
R^{01} \arrow[r, "f'"'] & R^{11}
\end{tikzcd}
$$

Notice that f, f', h_1, h_2 are chain maps of hypercubes, and H is a chain homotopy of hypercubes between the chain maps. In other words, we have

$$h_1 \circ D|_{R^{00}} = D|_{R^{10}} \circ h_1, h_2 \circ D|_{R^{01}} = D|_{R^{11}} \circ h_2,\ H \circ D|_{R^{00}} + D|_{R^{11}} \circ H = h_2 \circ f + f' \circ h_1.$$

By induction, the $(n-1)$-dimensional subhypercube corresponding to $\varepsilon_n = 0$ induces the isomorphism h_1. Thus, we have a chain map of hypercubes $h_1^{-1} : R^{01} \to R^{00}$, such that $h_1 h_1^{-1} = id_{R^{01}}$ and $h_1^{-1} h_1 = id_{R^{00}}$. Similarly, we have $h_2^{-1} : R^{11} \to R^{10}$ as the inverse of h_2. The hypercube R induces a chain map $h_1 + H + h_2$ from the subhypercube $R^{\varepsilon_{n+1}=0}$ to the subhypercube $R^{\varepsilon_{n+1}=1}$. We show that the chain map of hyperboxes $h_1 + H + h_2 : C^{\varepsilon_{n+1}=0} \to C^{\varepsilon_{n+1}=1}$ is an isomorphism, by directly constructing the inverse $h_1^{-1} + h_2^{-1} \circ H \circ h_1^{-1} + h_2^{-1} : R^{\varepsilon_{n+1}=1} \to R^{\varepsilon_{n+1}=0}$, which is induced by the following hypercube:

$$
K = \begin{tikzcd}
R^{01} \arrow[r, "f'"] \arrow[d, "h_1^{-1}"'] \arrow[dr, "h_2^{-1} \circ H \circ h_1^{-1}" description] & R^{11} \arrow[d, "h_2^{-1}"] \\
R^{00} \arrow[r, "f"'] & R^{10}.
\end{tikzcd}
$$

Here, the map $h_2^{-1} \circ H \circ h_1^{-1}$ is the composition of maps h_2^{-1}, H, h_1^{-1}.

The following two rectangles of hypercubes show that $h_1^{-1} + h_2^{-1} H h_1^{-1} + h_2^{-1}$ is the inverse of $h_1 + H + h_2$.

$$
\begin{tikzcd}
C^{00} \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, "H"] & C^{10} \arrow[d, "h_2"] \\
C^{01} \arrow[r, "f'"] \arrow[d, "h_1^{-1}"'] \arrow[dr, "h_2^{-1} H h_1^{-1}" description] & C^{11} \arrow[d, "h_2^{-1}"] \\
C^{00} \arrow[r, "f"'] & C^{10}
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{01} \arrow[r, "f'"] \arrow[d, "h_1^{-1}"'] \arrow[dr, "h_2^{-1} H h_1^{-1}" description] & C^{11} \arrow[d, "h_2^{-1}"] \\
C^{00} \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, "H"] & C^{10} \arrow[d, "h_2"] \\
C^{01} \arrow[r, "f'"] & C^{11}
\end{tikzcd}
$$

Next, to prove the general case for hyperboxes, we do induction on the size of H, while fixing n. We claim that for any k with $1 \leq k \leq n-1$, if the proposition is true for all hyperboxes R of size $(\mathbf{d}, 1)$ where $\mathbf{d} = (d_1, ..., d_k, 0, 0, ..., 0) \in \mathbb{Z}^n_{\geq 0}$, then the proposition is also true for all hyperboxes S of size $(\mathbf{d'}, 1)$ where $\mathbf{d'} = (d'_1, ..., d'_{k+1}, 0, ..., 0) \in \mathbb{Z}^n_{\geq 0}$.

Let $S^{i,j} = S^{\varepsilon_{k+1}=i, \varepsilon_{n+1}=j}$ with $i \in \{0, 1, ..., d'_{k+1}\}, j = \{0, 1\}$ be the subhyperbox corresponding to those complexes with $\varepsilon_{k+1} = i, \varepsilon_{n+1} = j$. Thereby, the subhyperbox $S^{i,j}$ is of size $\mathbf{d}'_k = (d'_1, ..., d'_k, 0, ..., 0) \in \mathbb{Z}^{n+1}_{\geq 0}$. So we can regard $S^{i,j}$ as a k-dimensional hyperbox of size $\bar{\mathbf{d}}'_k = (d'_1, ..., d'_k)$. For all $\varepsilon \in \mathbb{E}(\bar{\mathbf{d}}'_k)$, we denote the chain complex of S sitting at $(\varepsilon, i, 0, 0, ..., 0, j)$ by $(S^{i,j})^{\varepsilon}$.

We can decompose the hyperbox S as a rectangle of hyperboxes as the following diagram:

$$\begin{tikzcd}
S^{0,0} \arrow[r, "f_1"] \arrow[d, "h_0"] \arrow[dr, "H_1"] & S^{1,0} \arrow[r, "f_2"] \arrow[d, "h_1"] \arrow[dr, "H_2"] & S^{2,0} \arrow[r, "f_3"] \arrow[d, "h_2"] \arrow[dr, "H_3"] & \cdots \arrow[r, "f_{d'_{k+1}}"] \arrow[dr, "H_{d'_{k+1}}"] & S^{d'_{k+1},0} \arrow[d, "h_{d'_{k+1}}"] \\
S^{0,1} \arrow[r, "f'_1"'] & S^{1,1} \arrow[r, "f'_2"'] & S^{2,1} \arrow[r, "f'_3"'] & \cdots \arrow[r, "f'_{d'_{k+1}}"'] & S^{d'_{k+1},1},
\end{tikzcd}$$

where $f_1, ..., f_{d'_{k+1}}, f'_1, ..., f'_{d'_{k+1}}, h_0, ..., h_{d'_{k+1}}$ are chain maps of hyperboxes and $H_1, ..., H_{d'_{k+1}}$ are chain homotopies of hyperboxes.

By the induction hypothesis, the subhyperbox $S^{\varepsilon_{k+1}=j}, j \in \{0, 1, ..., d'_{k+1}\}$ is of size $(d'_1, ..., d'_k, 0, ..., 0, 1)$, and thereby induces the isomorphism of hyperboxes $h_j : S^{j,0} \to S^{j,1}$. Let the inverse of h_j be $h_j^{-1} : S^{j,1} \to S^{j,0}$. We define a set of homotopies of hyperboxes $h_j^{-1} \circ H_j \circ h_{j-1}^{-1} : S^{j-1,1} \to S^{j,0}$, for any $j \in \{1, 2, ..., d'_{k+1}\}$ by the following equations, for all $\varepsilon^0 \in \mathbb{E}(\bar{\mathbf{d}}'_k), \varepsilon \in \mathbb{E}(k)$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\bar{\mathbf{d}}'_k)$,

$$(h_j^{-1} \circ H_j \circ h_{j-1}^{-1})^{\varepsilon}_{\varepsilon^0} = \sum_{\{\varepsilon', \varepsilon'' \in \mathbb{E}(k) | \varepsilon' \leq \varepsilon'' \leq \varepsilon\}} (h_j^{-1})^{\varepsilon - \varepsilon''}_{\varepsilon^0 + \varepsilon''} \circ (H_j)^{\varepsilon'' - \varepsilon'}_{\varepsilon^0 + \varepsilon'} \circ (h_{j-1}^{-1})^{\varepsilon'}_{\varepsilon^0}.$$

We simply denote $h_j^{-1} \circ H_j \circ h_{j-1}^{-1}$ by $h_j^{-1} H_j h_{j-1}^{-1}$. From the definition of $h_j^{-1} H_j h_{j-1}^{-1}$, we can show the associativity of compositions of maps of hyperboxes. Thus, $H_j D|_{S^{j-1,0}} + D|_{S^{j,1}} H_j = h_j f_j + f'_j h_{j-1}$ and $h_j D|_{S^{j,0}} = D|_{S^{j,1}} h_j$ implies that

$$h_j^{-1} \circ H_j \circ h_{j-1}^{-1} \circ D|_{S^{j-1,1}} + D|_{S^{j,0}} \circ h_j^{-1} \circ H_j \circ h_{j-1}^{-1} = f_j \circ h_{j-1}^{-1} + h_j^{-1} \circ f'_j.$$

Therefore, we can construct the following the hypercube T

$$\begin{tikzcd}
S^{0,1} \arrow[r, "f'_1"] \arrow[d, "h_0^{-1}"'] \arrow[dr, "h_1^{-1}H_1h_0^{-1}"] & S^{1,1} \arrow[r, "f'_2"] \arrow[d, "h_1^{-1}"'] \arrow[dr, "h_2^{-1}H_2h_1^{-1}"] & S^{2,1} \arrow[r, "f'_3"] \arrow[d, "h_2^{-1}"'] \arrow[dr, "h_3^{-1}H_3h_2^{-1}"] & \cdots \arrow[r, "f'_{d'_{k+1}}"] \arrow[dr, "h^{-1}_{d'_{k+1}}H_{d'_{k+1}}h^{-1}_{d'_{k+1}-1}"] & S^{d'_{k+1},1} \arrow[d, "h^{-1}_{d'_{k+1}}"] \\
S^{0,0} \arrow[r, "f_1"'] & S^{1,0} \arrow[r, "f_2"'] & S^{2,0} \arrow[r, "f_3"'] & \cdots \arrow[r, "f_{d'_{k+1}}"'] & S^{d'_{k+1},0},
\end{tikzcd}$$

which induces a chain map from the subhyperbox $S^{\varepsilon_{n+1}=1}$ to the subhyperbox $S^{\varepsilon_{n+1}=0}$. Direct computations show that the chain maps induced by S and T are the inverses of each other.

□

Corollary 3.2.12. *On a rectangle of chain complexes $R = (C^\varepsilon, D^\varepsilon)$, if we change the diagonal maps $D_\varepsilon^{(1,1)}$ by a higher homotopy, i.e. $D'^{(1,1)}_\varepsilon = D_\varepsilon^{(1,1)} + H_\varepsilon^{(1,1)} D_\varepsilon^\varepsilon + D_{\varepsilon+(1,1)}^{\varepsilon+(1,1)} H_\varepsilon^{(1,1)}$ with $H_\varepsilon^{(1,1)} : C_*^\varepsilon \to C_{*+2}^{\varepsilon+(1,1)}$, then the new rectangle $R' = (C^\varepsilon, D'^\varepsilon)$ is isomorphic to R.*

Theorem 3.2.13. *Suppose L is an oriented two-bridge link with the framing Λ. For any $\mathbb{F}[[U_1]]$-linear quasi-isomorphisms $\tilde{D}^{-L_i}_{s_1,s_2}$ and $\mathbb{F}[[U_1]]$-linear homotopies $\tilde{F}^{-L_1\cup\pm L_2}_{s_1,s_2}, \tilde{F}^{\pm L_1\cup -L_2}_{s_1,s_2}$, $\tilde{D}^{-L_1\cup -L_2}_{-\infty,-\infty}$, the perturbed surgery complex $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ is isomorphic to the original surgery complex in [24] as $\mathbb{F}[[U_1]]$-module. By imposing the U_2-action to be the same as U_1-action, the $\mathbb{F}[[U_1, U_2]]$-module $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ is isomorphic to the homology $\boldsymbol{H}F^-(S^3_\Lambda(L))$. Furthermore, this isomorphism preserves the absolute grading.*

Proof. First, we restrict our scalars to $\mathbb{F}[[U_1]]$. By Proposition 3.2.11, the following cubes show that the top square in each cube is isomorphic to the bottom one.

$$\begin{tikzcd}
 & C \arrow[rr, "f_2"] \arrow[dd, "id"] & & D \arrow[dd, "id"] \\
A \arrow[ur, "f_1+K\partial_A+\partial_C K"] \arrow[rr, "f_3"] \arrow[urrr, "H+f_2K"] \arrow[dd, "id"] \arrow[dr, "K"] & & B \arrow[ur, "f_4"] \arrow[dd, "id"] & \\
 & C \arrow[rr, "f_2"] & & D \\
A \arrow[ur, "f_1"] \arrow[rr, "f_3"] \arrow[urrr, "H"] & & B \arrow[ur, "f_4"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \arrow[rr, "f_2"] \arrow[dd, "id"] & & D \arrow[dd, "id"] \\
A \arrow[ur, "f_1"] \arrow[rr, "f_3"] \arrow[urrr, "H+Kf_3"] \arrow[dd, "id"] & & B \arrow[ur, "f_4+K\partial_B+\partial_D K"] \arrow[dd, "id"] \arrow[dr, "K"] & \\
 & C \arrow[rr, "f_2"] & & D \\
A \arrow[ur, "f_1"] \arrow[rr, "f_3"] \arrow[urrr, "H"] & & B \arrow[ur, "f_4"] &
\end{tikzcd}$$

This means when an edge map is changed up to a $\mathbb{F}[[U_1]]$-linear chain homotopy in a square R, we are able to change the diagonal maps correspondingly to guarantee the new square is isomorphic to the original one as an $\mathbb{F}[[U_1]]$-module. By inductions on the edges of rectangles $\mathfrak{R}_{\mathrm{s},i,j}$ in Equation (2.2.12), we can show that after changing the edge maps $D^{-L_i}_{s_1,s_2}$ by $\tilde{D}^{-L_i}_{s_1,s_2}$ and changing some diagonal maps accordingly, the result rectangle $\mathfrak{R}'_{\mathrm{s},i,j}$ is isomorphic to $\mathfrak{R}_{\mathrm{s},i,j}$ as $\mathbb{F}[[U_1]]$-modules. In fact, we only have changed diagonal maps among the positions of $\tilde{F}^{\pm L_1 \cup \pm L_2}_{s_1,s_2}, \tilde{F}^{\pm L_1 \cup \pm L_2}_{s_1,s_2}, \tilde{D}^{\pm L_1 \cup \pm L_2}_{\pm\infty,\pm\infty}$ in (3.2.4), where we can keep applying the rigidity results in Proposition 5.1.1. Thereby, Corollary 3.2.12 implies the perturbed rectangles in Equation (3.2.4) are isomorphic to those rectangles $\mathfrak{R}'_{\mathrm{s},i,j}$'s, and thus isomorphic to the original $\mathfrak{R}_{\mathrm{s},i,j}$'s in Equation (2.2.12) as $\mathbb{F}[[U_1]]$-modules. After compressing these rectangles and gluing them together, the perturbed surgery complex is isomorphic to the original surgery complex as an $\mathbb{F}[[U_1]]$-module.

From Theorem 2.2.17, it follows that the U_1, U_2 actions in $H_*(\mathcal{C}^-(\mathcal{H}^L, \Lambda), \mathcal{D}^-)$ are the same. Thus by imposing the U_2-action as the same as the U_1-action on the $\mathbb{F}[[U_1]]$-module $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$, we get an isomorphism as $\mathbb{F}[[U_1, U_2]]$-module between $H_*(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ and $H_*(\mathcal{C}^-(\mathcal{H}^L, \Lambda), \mathcal{D}^-)$. As all the rigidity results respect the gradings, the above isomorphism also preserves the grading. $\square$

Remark 3.2.14. In the above theorem, the homology of the unknot is $\mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$

as an $\mathbb{F}[[U_1, U_2]]$-module. There is no analogue of the Proposition 5.1.1 for homotopies over the ring $\mathbb{F}[[U_1, U_2]]$. This is why we restrict our scalars to $\mathbb{F}[[U_1]]$. This idea is due to Ciprian Manolescu.

3.2.5 Algorithm for computing $\mathbf{HF}^-(S^3_\Lambda(L))$ for two-bridge links.

Let L be a two-bridge link.

First, we use the algorithm in Section 3.1.5 to compute all the $A^-_{\mathrm{s}}(L)$'s.

Second, by solving linear equations, we find $\mathbb{F}[[U_1, U_2]]$-linear quasi-homomorphisms

$$\tilde{D}^{-L_1}_{-\infty,s_2} : A^-_{-\infty,s_2} \to A^-_{-\infty,s_2+\mathrm{lk}}, \quad \tilde{D}^{-L_2}_{s_1,-\infty} : A^-_{s_1,-\infty} \to A^-_{s_1+\mathrm{lk},-\infty}.$$

Finding chain maps is a problem of solving linear equations modulo 2, which has fast algorithm in the case of sparse matrices. In order to find a quasi-isomorphism without computing the homology, we adopt an area filtration on the complexes as follows. From the Schubert diagram $\mathcal{H}^L$, one can see that the diagram $r_{-L_1}(\mathcal{H}^L)$ is isotopic to the standard genus-0 diagram of the unknot with one free basepoint and two intersection points $\mathbf{x}, \mathbf{y}$ of attaching curves. Let C^{u} be the chain complex of $\mathbb{F}[[U_1, U_2]]$-modules generated by $\mathbf{x}, \mathbf{y}$, with differential $\partial \mathbf{x} = (U_1 - U_2)\mathbf{y}$. Thus there is a chain homotopy equivalence by counting holomorphic triangles from $A^-_{\pm\infty,s_2}$ to C^{u}, denoted by $F : A^-_{\pm\infty,s_2} \to C^{\mathrm{u}}$.

Consider the Heegaard diagram by removing z_1, then an area filtration argument shows this chain homotopy equivalence $F : A^-_{+\infty,s_2} \to C^{\mathrm{u}}$ is in the form of

$$F(b_0) = \mathbf{x} + \text{lower terms},$$
$$F(b_{-1}) = \mathbf{y} + \text{lower terms},$$

where the lower terms are referred to the area filtration. In fact, as long as a chain map

$G : C^{\mathrm{u}} \to A^-_{+\infty,s_2}$ is in the form of

$$G(\mathbf{x}) = b_0 + \text{lower terms},$$
$$G(\mathbf{y}) = b_{-1} + \text{lower terms},$$

then it is a quasi-isomorphism. This is because $F \circ G : C^{\mathrm{u}} \to C^{\mathrm{u}}$ is in the form of

$$F \circ G(\mathbf{x}) = \mathbf{x} + \text{lower terms},$$
$$F \circ G(\mathbf{y}) = \mathbf{y} + \text{lower terms},$$

which is an isomorphism of groups by Lemma 9.10 in [39]. In order to find an area filtration, we can set every bigon and square to be of the same area 1 on the Schubert Heegaard diagram so that every periodic domain has area 0. We can also similarly determine F.

Third, we plug in all the maps $I^{\overrightarrow{M}}_{\mathrm{s}}$ and $\tilde{D}^{-L_i}$ to (3.2.4). By Corollary 3.2.4, these $\tilde{D}^{-L_i}$'s are chain homotopic to D^{-L_i}'s as maps of $\mathbb{F}[[U_1, U_2]]$-modules. Thus, following the same line in the proof of Theorem 3.2.13, we can find $\mathbb{F}[[U_1, U_2]]$-linear diagonal maps $\tilde{F}$'s and $\tilde{D}^{-L_1 \cup -L_2}_{-\infty,-\infty}$ to make those rectangles to be hyperboxes of chain complexes. Finding such maps is also a problem of solving linear equations.

Finally, after compressing all the rectangles and doing Λ-twisted gluing of these squares, we obtain the perturbed surgery complex $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$. Then, we compute the homology over the polynomial ring $\mathbb{F}[[U_1, U_2]]$ and there are several algorithms of polynomial time. This $\mathbb{F}[[U_1, U_2]]$-module might not be isomorphic to $\mathbf{HF}^-(S^3_\Lambda(L))$. However, by Theorem 3.2.13, as an $\mathbb{F}[[U_1]]$-module, it is isomorphic to $\mathbf{HF}^-(S^3_\Lambda(L))$. So we impose the U_2-action on the homology of $(\tilde{\mathcal{C}}^-(\mathcal{H}^L, \Lambda), \tilde{\mathcal{D}}^-)$ to be the same as the U_1-action. By Theorem 3.2.13, now it is isomorphic to $\mathbf{HF}^-(S^3_\Lambda(L))$ as an $\mathbb{F}[[U_1, U_2]]$-module.

We note that the surgery complex is infinitely generated over $\mathbb{F}[[U_1, U_2]]$. Hence, before finding the perturbed surgery complex, we need to do truncations for a fixed framing matrix Λ, as described in [24] Section 8.3. The time complexity of doing truncations is a polynomial of $\det(\Lambda)$.

Remark 3.2.15. Indeed, in the second and third steps, we only need $\mathbb{F}[[U_1]]$-linear quasi-isomorphisms $\tilde{D}^{-L_i}$'s and $\mathbb{F}[[U_1]]$-linear diagonal maps $\tilde{F}$'s and $\tilde{D}^{-L_1\cup -L_2}$ to replace those $\mathbb{F}[[U_1,U_2]]$-linear maps. The reason we use $\mathbb{F}[[U_1,U_2]]$-linear maps is that over $\mathbb{F}[[U_1,U_2]]$ the module A^-_{s} is finitely generated and thus easier to use in computer programs.

Example 3.2.16. Consider $(0,0)$ surgery on the unlink $L = L_1 \cup L_2$ and look at the $(0,0)$ Spin^c structure s_0. The general Floer complexes of the unlink $A^-_{\mathrm{s}}(L)$'s are all C^{u}, where C^{u} is defined in Remark 3.2.3. Since the Alexander gradings $A(\mathbf{x}) = A(\mathbf{y}) = (0,0)$, the inclusion maps $I^{\pm L_i}_{\mathrm{s}_0}$ are all the identities for $i = 1,2$. It follows that $\Phi^{+L_i}_{\mathrm{s}_0}$ and $\Phi^{-L_i}_{\mathrm{s}_0}$ are chain homotopic by Proposition 3.2.2. Hence, we can get the perturbed surgery complex for the Spin^c structure s_0 as follows

$$\begin{tikzcd} C^{\mathrm{u}} \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, "F"] & C^{\mathrm{u}} \arrow[d, "0"] \\ C^{\mathrm{u}} \arrow[r, "0"'] & C^{\mathrm{u}}, \end{tikzcd}$$

where F is an $\mathbb{F}[[U_1,U_2]]$-linear map shifting the gradings by 1 satisfying $\partial F = F\partial$. Thus, F can either be 0 or the following map $f : C^{\mathrm{u}} \to C^{\mathrm{u}}$, where $f(\mathbf{x}) = \mathbf{y}, f(\mathbf{y}) = 0$.

Case I For $F = 0$, the homology of the perturbed complex is $\Big(\mathbb{F}[[U_1,U_2]]/(U_1 - U_2)\Big)^{\oplus 4}$.

Case II For $F = f$, the homology is $\Big(\mathbb{F}[[U_1,U_2]]/(U_1 - U_2)\Big)^{\oplus 2} \oplus \Big(\mathbb{F}[[U_1,U_2]]/(U_1 - U_2)^2\Big)$.

However, as $\mathbb{F}[[U_1]]$-modules, both homology groups are isomorphic to $\mathbb{F}[[U_1]]^{\oplus 4}$. Thus, by imposing the U_2-action to be the same as U_1, we obtain the correct homology is $\Big(\mathbb{F}[[U_1,U_2]]/(U_1 - U_2)\Big)^{\oplus 4}$.

3.2.6 Further discussions of the perturbed surgery complex.

Besides replacing the maps in a hypercube of chain complexes up to homotopy, we can also replace the chain complexes sitting at the vertices in the hypercube up to chain homotopy equivalences. Sometimes this procedure allows us to simplify the computations of the homology of a hypercube.

Lemma 3.2.17. *Let $A, \tilde{A}, B, \tilde{B}$ be chain complexes. Suppose $h_A : A \to \tilde{A}, h_{\tilde{A}} : \tilde{A} \to A$ are the chain homotopy equivalences, with $h_A h_{\tilde{A}} \overset{K_{\tilde{A}}}{\simeq} id_{\tilde{A}}, h_{\tilde{A}} h_A \overset{K_A}{\simeq} id_A$. Similarly, we have the chain homotopy equivalences $h_B, h_{\tilde{B}}$ and the chain homotopies $K_B, K_{\tilde{B}}$ with $h_B h_{\tilde{B}} \overset{K_{\tilde{B}}}{\simeq} id_{\tilde{B}}, h_{\tilde{B}} h_B \overset{K_B}{\simeq} id_B$. Let $f : A \to B$ be any chain map. Then, the mapping cones* $\text{cone}(f)$ *and* $\text{cone}(h_B f h_{\tilde{A}})$ *are chain homotopy equivalent via the chain maps $H_1 : \text{cone}(f) \to \text{cone}(h_B f h_{\tilde{A}}), H_2 : \text{cone}(h_B f h_{\tilde{A}}) \to \text{cone}(f)$ induced by the following squares of chain complexes*

$$H_1 := \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_A"'] \arrow[dr, "h_B f K_A"] & B \arrow[d, "h_B"] \\ \tilde{A} \arrow[r, "h_B f h_{\tilde{A}}"'] & \tilde{B}; \end{tikzcd} \quad H_2 := \begin{tikzcd} \tilde{A} \arrow[r, "h_B f h_{\tilde{A}}"] \arrow[d, "h_{\tilde{A}}"'] \arrow[dr, "K_B f h_{\tilde{A}}"] & \tilde{B} \arrow[d, "h_{\tilde{B}}"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Proof. We directly compute the compositions $H_1 \circ H_2$ and $H_2 \circ H_1$ to check that they are both chain homotopic to the identities. The composition $H_2 \circ H_1$ is the compression of the juxtaposition of the two squares, which is the square

$$H_2 \circ H_1 = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_{\tilde{A}} h_A"'] \arrow[dr, "\tilde{F}"] & B \arrow[d, "h_{\tilde{B}} h_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $\tilde{F} = K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A$. The following cube shows that $H_2 \circ H_1$ is homotopic to $id_{\text{cone}(f)}$

$$\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dl, "h_{\tilde{A}} h_A"'] \arrow[dr, "\tilde{F}"] \arrow[dd, "id_A"] \arrow[dddl, "K_A"] \arrow[dddr, "K_B f K_A"] & & B \arrow[dl, "h_{\tilde{B}} h_B"'] \arrow[dd, "id_B"] \arrow[dddl, "K_B"] \\
A \arrow[rr, "f"] \arrow[dd, "id_A"'] & & B \arrow[dd, "id_B"] & \\
& A \arrow[rr, "f"] \arrow[dl, "id_A"'] & & B \arrow[dl, "id_B"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

where $\tilde{F} = K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A$. Direct computation verifies that the above cube is a hypercube chain complex. We only check the longest diagonal here:

$$\begin{aligned} K_B f + f K_A + K_B f h_{\tilde{A}} h_A + h_{\tilde{B}} h_B f K_A &= K_B f(\partial_A K_A + K_A \partial_A) + (\partial_B K_B + K_B \partial_B) f K_A) \\ &= K_B f K_A \partial_A + \partial_B K_B f K_A. \end{aligned}$$

Similarly, $H_1 \circ H_2$ is chain homotopic to $id_{\mathrm{cone}(h_B f h_{\tilde{A}})}$. □

Now we generalize this lemma to a hypercube version. The proof is similar, so we omit it.

Proposition 3.2.18. *Suppose A and $\tilde{A}$ are two chain homotopy equivalent n-dimensional hypercubes, and so do B and $\tilde{B}$. Then the $(n+1)$-dimensional hypercube* $\mathrm{cone}(A \xrightarrow{f} B)$ *is chain homotopy equivalent to* $\mathrm{cone}(\tilde{A} \xrightarrow{h_B f h_{\tilde{A}}} \tilde{B})$, *where $h_B : B \to \tilde{B}, h_{\tilde{A}} : \tilde{A} \to A$ are chain homotopy equivalences.*

Iterating these conjugation constructions, we have the following proposition.

Proposition 3.2.19. *Let $H = (C^\varepsilon, D^\varepsilon)$ be a n-dimensional hypercube of chain complexes. Suppose we have that $\tilde{C}^\varepsilon$ is chain homotopy equivalent to C^ε for all $\varepsilon \in \mathbb{E}_n$. Then there exists a hypercube $\tilde{H} = (\tilde{C}^\varepsilon, \tilde{D}^\varepsilon)$ which is chain homotopy equivalent to H.*

For the purpose of this paper, we give an example of the 2-dimensional case.

Example 3.2.20. Let H be the following square of chain complexes

$$H = \begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, "F"] & C_2 \arrow[d, "f_2"] \\ C_3 \arrow[r, "g_2"'] & C_4. \end{tikzcd}$$

Then suppose we have a set of chain homotopy equivalences $h_i : C_i \to \tilde{C}_i, \tilde{h}_i : \tilde{C}_i \to C_i$, where $h_i \circ \tilde{h}_i$ is homotopic to $id_{\tilde{C}_i}$ via $\tilde{K}_i$ and $\tilde{h}_i \circ h_i$ is homotopic to id_{C_i} via K_i. Then compressing the following rectangle, we obtain the desired square $\tilde{H}$.

$$\begin{tikzcd}
\tilde{C}_1 \arrow[r, "\tilde{h}_1"] \arrow[d, "h_3 g_1 \tilde{h}_1"'] \arrow[dr, "K_3 g_1 \tilde{h}_1"] & C_1 \arrow[r, "f_1"] \arrow[d, "g_1"] \arrow[dr, "F"] & C_2 \arrow[r, "h_2"] \arrow[d, "f_2"'] \arrow[dr, "h_4 f_2 K_2"] & \tilde{C}_2 \arrow[d, "h_4 f_2 \tilde{h}_2"] \\
\tilde{C}_3 \arrow[r, "\tilde{h}_3"'] & C_3 \arrow[r, "g_2"'] & C_4 \arrow[r, "h_4"'] & \tilde{C}_4.
\end{tikzcd}$$

Thus,

$$\tilde{H} = \quad \begin{tikzcd}
\tilde{C}_1 \arrow[r, "h_2 f_1 \tilde{h}_1"] \arrow[d, "h_3 g_1 \tilde{h}_1"'] \arrow[dr, "\tilde{F}"] & \tilde{C}_2 \arrow[d, "h_4 f_2 \tilde{h}_2"] \\
\tilde{C}_3 \arrow[r, "h_4 g_2 \tilde{h}_3"'] & \tilde{C}_4.
\end{tikzcd}$$

where $\tilde{F} = h_4 g_2 K_3 g_1 \tilde{h}_1 + h_4 F \tilde{h}_1 + h_4 f_2 K_2 f_1 \tilde{h}_1$.

If we know the chain homotopy types of all the A^-_{s}'s, we can also replace the chain complexes in the perturbed surgery complex by the conjugation construction. We will still call it the perturbed surgery complex. This is used in simplifying the computations in the next section.

3.3 Examples

3.3.1 The complexes $\widehat{CFL}(L)$ for two-bridge links L.

We recall from [42] that for a link L, the filtered chain complex $\widehat{CFL}(L)$ is a chain complex of S^3 with a filtration induced from L. More precisely, fixing a Heegaard diagram $\mathcal{H}^L$ of $L \subset S^3$, we obtain a chain complex of $\mathbb{F}$-modules $\widehat{CF}(\mathcal{H}^L)$, generated by the intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in the symmetric product. There is an Alexander filtration on $\widehat{CF}(\mathcal{H}^L)$. It is shown that given different Heegaard diagrams of L, $\mathcal{H}^L_1$ and $\mathcal{H}^L_2$, there is a chain homotopy equivalence from $\widehat{CF}(\mathcal{H}^L_1)$ to $\widehat{CF}(\mathcal{H}^L_2)$, which preserves the Alexander filtration. Thus, the filtered chain homotopy equivalence class of these chain complexes is called the *filtered chain homotopy type* of $\widehat{CFL}(L)$. By abuse of notation, we also let $\widehat{CFL}(L)$ be some filtered complex in this equivalence class. Similarly, we define the filtered chain homotopy type of $CFL^-(L)$, by looking at the Alexander filtered chain complex $CF^-(\mathcal{H}^L)$.

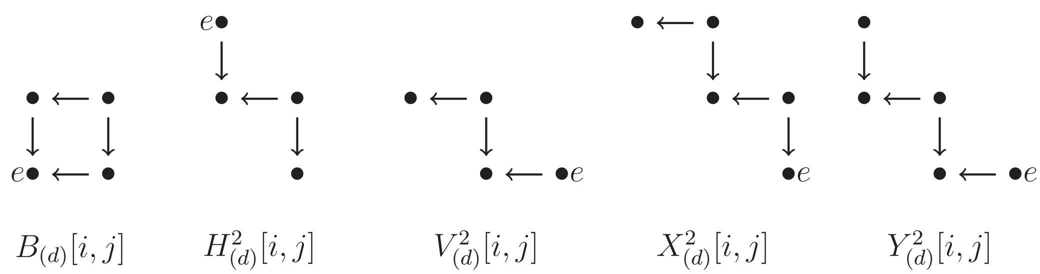


Figure 3.3.1: **Examples of the $\mathbb{Z}^2$-filtered chain complexes** B, H, V, X, Y**.** The labelled dots e in $B_{(d)}[i,j], H^l_{(d)}[i,j], V^l_{(d)}[i,j], X^l_{(d)}[i,j], Y^l_{(d)}[i,j]$ are of grading d and with the filtrations $(i,j), (i,j), (i,j), (i+l,j), (i+l,j)$ respectively.

We represent $\mathbb{Z}^2$-filtered complexes graphically by dots and arrows on the x-y coordinate plane, with the dots representing generators, the arrows representing differentials, and the coordinates representing filtrations.

Theorem 3.3.1 (Theorem 12.1 in [42])**.** *Suppose $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$ is an oriented two-component alternating link. Then the filtered chain homotopy type of $\widehat{CFL}(L)$ is determined by the following data:*

(1)the multi-variable Alexander polynomial of L, Δ_L;

(2)the signature of L, $\sigma(L)$, and the linking number of L, $\mathrm{lk}(L)$;

(3)the filtered chain homotopy type of $\widehat{CFK}(L_1)$ and $\widehat{CFK}(L_2)$.

In fact, for alternating two-component links, $\widehat{CFL}(L)$ is filtered chain homotopy equivalent to a simplified filtered chain complex $\widehat{CFL}_{\mathrm{OS}}(L)$. The simplified complex is a direct sum of five different types of $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complexes $B_{(d)}[i,j]$,$H^l_{(d)}[i,j]$,$V^l_{(d)}[i,j]$,$X^l_{(d)}[i,j]$ and $Y^l_{(d)}[i,j]$. These basic filtered complexes are described in Section 12.1 of [42]; the filtered complex $B_{(d)}[i,j]$ looks like a box and the others look like zigzags. See Figure 3.3.1.

Corollary 3.3.2. *If L is an oriented two-bridge link, then the filtered homotopy type of $\widehat{CFL}(L)$ is determined by $\sigma(L)$, $\mathrm{lk}(L)$ and the multi-variable Alexander polynomial $\Delta_L(x,y)$. More concretely, let*

$$l = \mathrm{lk}(\overrightarrow{L}) + \frac{\sigma(\overrightarrow{L}) - 1}{2},$$

(1) if $l \geq 0$, let $a = \frac{1-\sigma-\mathrm{lk}}{2}$, $b = \frac{-1-\sigma-\mathrm{lk}}{2}$, then we have that $\widehat{CFL}(L)$ is filtered chain homotopic to

$$Y^{l}_{(0)}[a,a] \oplus Y^{l+1}_{(-1)}[b,b] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k's are determined by the Alexander polynomial Δ_L;

(2) if $l < 0$, then we have that the $\widehat{CFL}(L)$ is filtered chain homotopic to

$$X^{|l|}_{(0)}[\frac{\mathrm{lk}}{2}, \frac{\mathrm{lk}}{2}] \oplus X^{|l|-1}_{(-1)}[\frac{\mathrm{lk}}{2}, \frac{\mathrm{lk}}{2}] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k's are determined by the Alexander polynomial Δ_L.

Example 3.3.3. Let Wh denote the Whitehead link. Since $\mathrm{lk}(Wh) = 0$, $\sigma(Wh) = -1$, we get $l = -1$. Notice that $\mathrm{lk} = 0$ implies that the signature doesn't depend on the orientations of the link. Thus the filtered chain homotopy type of $\widehat{CFL}(Wh)$ is

$$X^{1}_{(0)}[0,0] \oplus X^{0}_{(-1)}[0,0] \oplus \bigoplus_k B_{(d_k)}[i_k, j_k],$$

where those d_k, i_k, j_k are determined by the Alexander polynomial. If we consider the mirror of Wh, we have $\sigma(\overline{Wh}) = 1$. Similarly, the filtered chain homotopy type of $\widehat{CFL}(\overline{Wh})$ is

$$Y^{0}_{(0)}[0,0] \oplus Y^{1}_{(-1)}[-1,-1] \oplus \bigoplus_k B_{(d'_k)}[i'_k, j'_k].$$

In the following diagram, $\widehat{CFL}(Wh)$ and $\widehat{CFL}(\overline{Wh})$ are illustrated, where each dot represents a generator and each arrow represents a differential.

$$\sigma(\overline{Wh}) = 1 : \quad \begin{tikzcd}[row sep=small, column sep=small]
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[d] & \bullet & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet & \bullet \arrow[l]
\end{tikzcd}, \qquad \sigma(Wh) = -1 : \quad \begin{tikzcd}[row sep=small, column sep=small]
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet & \bullet \\
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet & \bullet \arrow[l]
\end{tikzcd}. \tag{3.3.1}$$

We find that it is easier to work with Wh with $\sigma = -1$, since all the $A^{-}_{\mathrm{s}}(Wh)$ have homology

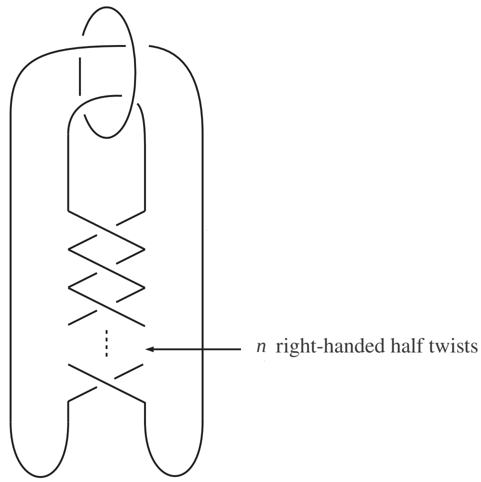


Figure 3.3.2: **The two-bridge links $b(4n+4, 2n+3)$.** If $n = 2k-1$, then the linking number is 0; if $n = 2k$, then the linking number is 2.

$\mathbb{F}[[U_1, U_2]]/(U_1 - U_2)$, so that we can apply the rigidity results. (One can compare this to the case of the right-handed trefoil knot versus the left-handed trefoil knot.)

Remark 3.3.4. The way of decomposing the complex $\widehat{CFL}(\overrightarrow{L})$ into direct sums of B, H, V, X, Y is not canonical. We can do some base-changes to change the above direct sum decomposition of $\widehat{CFL}$ such that the patterns of the arrows don't change.

3.3.2 The filtered homotopy type of $CFL^-(L)$ for some two-bridge links.

Given a two-bridge link L, we can use the Schubert Heegaard diagram to combinatorially find the filtered complex $CFL^-(L)$. However, this description is too cumbersome. Instead, here we use algebraic arguments to determine the filtered homotopy type of $CFL^-(L)$ in some special examples.

Consider the Schubert Heegaard diagram $\mathcal{H}$ for L and the $\mathbb{Z}^2$-filtered chain complex $\widehat{CF}(\mathcal{H})$. Then there is a filtered chain homotopy equivalence $F : \widehat{CF}(\mathcal{H}) \to \widehat{CFL}_{\mathrm{OS}}(L)$. Thus, F induces an isomorphism on the homology of their associated graded, i.e. the link Floer homology. In fact, the homology of their associated graded are just the chain groups themselves, because $\widehat{CF}(\mathcal{H})$ and $\widehat{CFL}_{\mathrm{OS}}(L)$ are both thin (with no differentials in their associated graded). So F is an isomorphism. In other words, we can change the basis of

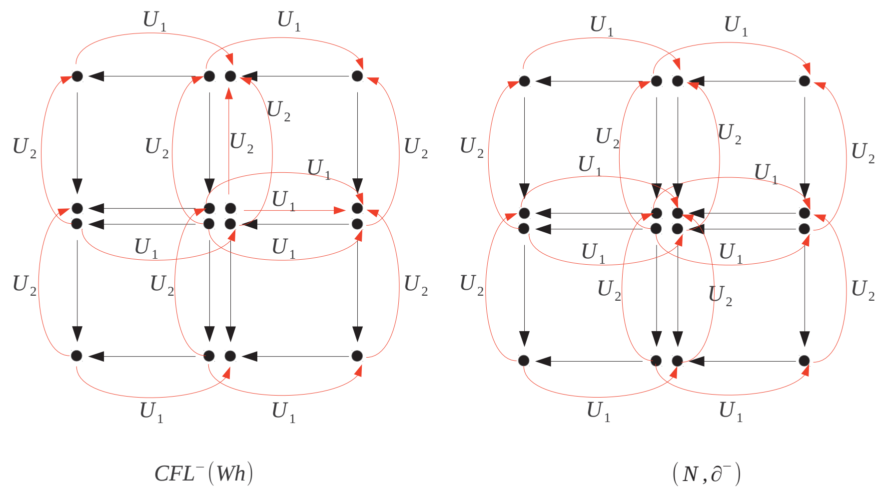


Figure 3.3.3: **The filtered complex $CFL^-(Wh)$ and (N, ∂^-).** The horizontal red arrows and vertical red arrows have U_1 and U_2 coefficients respectively.

$\widehat{CF}(\mathcal{H})$ preserving the filtration, such that the arrows are pruned as in the Ozsváth-Szabó simplified pattern. We use this new basis to consider $CF^-(\mathcal{H})$. Since every bigon in $\mathcal{H}$ contains a basepoint, those ∂_{U_1,U_2} arrows in $CF^-(\mathcal{H})$ are either upward or rightward of length 1. This property is repeatedly used later.

In this section, we show that for the two-bridge links $b(4n, 2n+1)$, the filtered homotopy type of $CFL^-(L)$ is determined by $\widehat{CFL}(L)$. Since $\widehat{CFL}(L)$ can be decomposed as direct sums of B, X, Y's, our goal is to show that $CFL^-(L)$ can be viewed as a square of chain complexes of these B, X, Y's.

Using continuous fractions, we can get the 4-plat presentations of $b(4n, 2n+1)$, thus providing the diagram in Figure 3.3.2. In addition, there is a convention issue of signs of the signature. We adopt the convention compatible with Corollary 3.3.2, so that $\sigma\Big(b(8k, 4k + 1)\Big) = -1$.

Proposition 3.3.5. *For the two-bridge link $L = b(8k, 4k+1)$, the filtered homotopy type of $CFL^-(L)$ is determined by the Alexander polynomial, signature and linking number of L, or equivalently by $\widehat{CFL}(L)$. Precisely, we have $CFL^-(L) = CFL^-(Wh) \oplus \bigoplus_{i=1}^{k-1}(N, \partial^-)$, where $CFL^-(Wh)$ and (N, ∂^-) are described in Figure 3.3.3.*

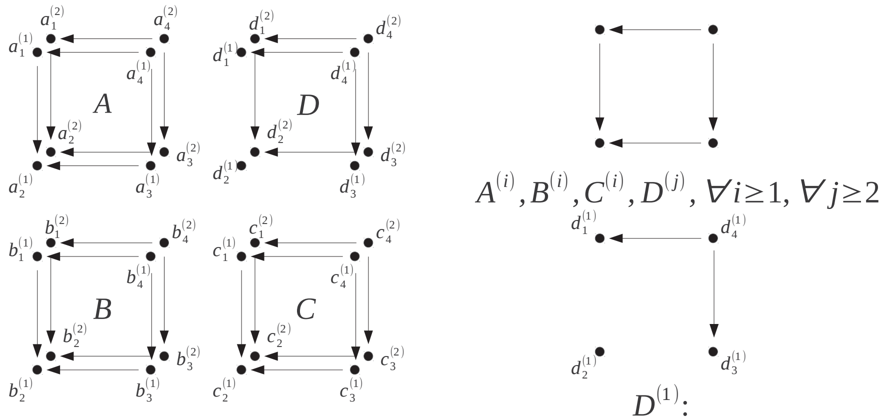


Figure 3.3.4: $\widehat{CFL}_{\mathrm{OS}}(b(8k, 4k \pm 1))$. On the left side, the figure illustrates the Alexander grading of A, B, C, D summands, where $k = 2$. On the right side, it indicates the filtered homotopy types of $(A^{(i)}, \hat{\partial}), (B^{(i)}, \hat{\partial}), (C^{(i)}, \hat{\partial}), (D^{(i)}, \hat{\partial})$, which all have the filtered homotopy types as boxes, except for $(D^{(1)}, \hat{\partial})$.

Proof. By Theorem 3.3.1, the Ozsváth-Szabó simplified complex $\widehat{CFL}_{\mathrm{OS}}(L)$ can be computed in terms of $\Delta_L(x, y) = k\Delta_{Wh}(x, y) = k\frac{(x-1)(y-1)}{\sqrt{xy}}$, lk $= 0$, and $\sigma(L) = -1$. Then, we compute that

$$\widehat{CFL}_{\mathrm{OS}}(L) = A \oplus B \oplus C \oplus D = \bigoplus_{i=1}^{k} A^{(i)} \oplus \bigoplus_{i=1}^{k} B^{(i)} \oplus \bigoplus_{i=1}^{k} C^{(i)} \oplus \bigoplus_{i=1}^{k} D^{(i)}.$$

See Figure 3.3.4 for the filtered homotopy type of $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$, where we denote the generators in $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$ by $a_j^{(i)}, b_j^{(i)}, c_j^{(i)}, d_j^{(i)}$, $j = 1, 2, 3, 4$, respectively.

Given $\widehat{CFL}_{\mathrm{OS}}(L)$, let us investigate the possibilities for $CFL^-(L)$. The differential ∂^- in $CFL^-(L)$ decomposes into

$$\partial^- = \hat{\partial} + \partial_{U_1,U_2} = \partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2},$$

where $\partial_{U_1,U_2}(x) = \partial_{U_1}(x) + \partial_{U_2}(x)$ consists of the components in $\partial^-(x)$ with coefficients of U_1, U_2 powers, and $\hat{\partial}(x) = \partial_{A_1}(x) + \partial_{A_2}(x)$ is decomposed by the Alexander filtration. As stated before, here ∂_{U_i} has the form of $\partial_{U_i}(x) = U_i y$ for $i = 1, 2$, i.e. the ∂_{U_i}-arrows are all of length 1. A close examination of U_1, U_2 powers and the Alexander filtrations in the coefficients of the following identity provides that

$$0 = (\partial^-)^2 = (\hat{\partial} + \partial_{U_1,U_2})^2 = (\partial_{A_1} + \partial_{A_2} + \partial_{U_1} + \partial_{U_2})^2 \implies [\hat{\partial}, \partial_{U_1,U_2}] = 0,\ \partial^2_{U_1,U_2} = \hat{\partial}^2 = 0 \implies$$
$$[\partial_{A_1}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_1}] = [\partial_{A_2}, \partial_{U_2}] = [\partial_{U_1}, \partial_{U_2}] = 0,\ \ \partial^2_{A_1} = \partial^2_{U_1} = \partial^2_{A_2} = \partial^2_{U_2} = 0.$$

where $[f, g] = fg + gf$.

At this point, we first consider the Whitehead link. The $\widehat{CFL}(Wh)$ is shown by the right term in Equation (3.3.1), and the bullets are labeled as in Figure 3.3.4. By looking at the vertical arrows only, the equations $\partial^2_{U_2} = \partial^2_{A_2} = [\partial_{A_2}, \partial_{U_2}] = 0$ give rise to the two possibilities of the rightmost column as follows, according to whether $\partial_{U_2}(c_4^{(1)})$ is 0 or not.

$$\begin{tikzcd} d_4^{(1)} \arrow[d] \\ d_3^{(1)} \\ c_4^{(1)} \arrow[d] \arrow[uu, bend left, "U_2"] \\ c_3^{(1)} \arrow[uu, bend left, "U_2"] \end{tikzcd} \qquad\qquad\qquad \begin{tikzcd} d_4^{(1)} \arrow[d] \\ d_3^{(1)} \\ c_4^{(1)} \arrow[d] \\ c_3^{(1)} \end{tikzcd}$$

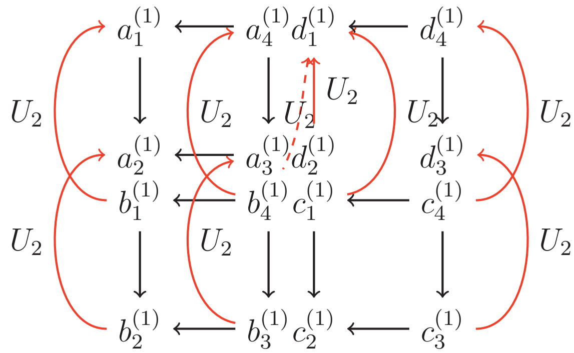
Consider the Heegaard diagram of L_1 obtained from $\mathcal{H}$ by deleting w_2, i.e. the reduction $r_{-L_2}(\mathcal{H})$. The differentials in $\widehat{CF}(r_{-L_2}(\mathcal{H}))$ count the bigons without basepoints w_1, z_1, z_2 on $\mathcal{H}$, which are the same as bigons with basepoint w_2. Thus, the complex $\widehat{CF}(r_{-L_2}(\mathcal{H}))$ can be obtained by ignoring the arrows ∂_{A_2} and setting $U_2 = 1$. So the vertical homology of $CFL^-(L)$ using only the ∂_{U_2} arrows is the knot Floer homology of the unknot L_1, $\mathbb{F} \oplus \mathbb{F}$, supported in the filtration $A_1 = \tau(L_1) + \frac{\mathrm{lk}(L)}{2} = 0$. Thus, the right-hand side in the above diagram is ruled out. A similar argument applies to the leftmost column. In sum,

$$\partial_{U_2}(b_1^{(1)}) = U_2 a_1^{(1)},\ \partial_{U_2}(b_2^{(1)}) = U_2 a_2^{(1)},\ \partial_{U_2}(c_4^{(1)}) = U_2 d_4^{(1)},\ \partial_{U_2}(c_3^{(1)}) = U_2 d_3^{(1)}.$$

Together with $\partial_{A_1}\partial_{U_2} = \partial_{U_2}\partial_{A_1}$, we get

$$\partial_{A_1}\partial_{U_2}(b_4^{(1)}) = \partial_{U_2}\partial_{A_1}(b_4^{(1)}) = \partial_{U_2} b_1^{(1)} = U_2 a_1^{(1)} \implies \partial_{U_2}(b_4^{(1)}) = U_2 a_4^{(1)} \text{ or } U_2(a_4^{(1)} + d_1^{(1)}).$$

Thus, $\partial_{A_2}\partial_{U_2} = \partial_{U_2}\partial_{A_2}$ implies that $\partial_{U_2}(b_3^{(1)}) = a_3^{(1)}$ and $\partial_{U_2}(c_1^{(1)}) = d_1^{(1)}$, $\partial_{U_2}(c_2^{(1)}) = 0$.



Next, $\partial_{U_2}\partial_{A_2} = \partial_{A_2}\partial_{U_2}$ implies that $\partial_{U_2}(d_2^{(1)}) \in U_2 \cdot D$. To determine $\partial_{U_2}(d_2^{(1)})$, we consider the complex $CFL^-(L) \otimes_{\mathbb{F}[[U_1,U_2]]} (\mathbb{F}[[U_1,U_2]]/U_1) = CFL^-(L)/(U_1 \cdot CFL^-(L))$, i.e. setting $U_1 = 0$. The homology of this complex can be computed from the long exact sequence of homologies, and it is $\mathbb{F}[[U_2]]/U_2$ as an $\mathbb{F}[[U_2]]$-module. Meanwhile, to compute this homology we can also use the A_1-filtration and kill acyclic subcomplexes and acyclic quotient complexes. Taking the vertical homology of this complex with respect to ∂_{A_2} leaves only $d_2^{(1)}$ and $d_1^{(1)}$. Thus, from the homology constraint we have computed, it follows that $\partial_{U_2}(d_2^{(1)}) = U_2 d_1^{(1)}$. Thus, we recover all the U_2-arrows. See the above figure, where the dashed arrow is undetermined. Similarly, we can get all the U_1-arrows. By changing basis, $\tilde{a}_4^{(1)} = a_4^{(1)} + d_1^{(1)}$ and $\tilde{c}_4^{(1)} = c_4^{(1)} + d_3^{(1)}$, we can get rid of the dashed arrows, which gives the picture of $CFL^-(Wh)$ in Figure 3.3.3.

When $k > 1$, we follow the same line of argument, together with doing more changes of basis to prune the arrows. First, consider the rightmost column, i.e. $R = \mathrm{Span}_{\mathbb{F}[[U_2]]}\{d_3^{(i)}, d_4^{(i)}, c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ with the differentials $\partial_{A_2} + \partial_{U_2}$. Assume $\partial_{U_2}(c_4^{(i)}) = U_2 \cdot \sum_{m=1}^k \lambda_{i,m} d_4^{(m)}$. Then $\partial_{U_2}\partial_{A_2} = \partial_{A_2}\partial_{U_2}$ implies that $\partial_{U_2}(c_3^{(i)}) = U_2 \cdot \sum_{m=1}^k \lambda_{i,m} d_3^{(m)}$. So the matrix $D = (\lambda_{i,m})$ represents the differential ∂_{U_2} in the upside down vertical complex $\big(R \otimes_{\mathbb{F}[[U_2]]} \mathbb{F}[[U_2]]/(U_2 - 1), \partial_{U_2}\big)$. Since its homology is 0, the matrix D is invertible. In other words, the ∂_{U_2}-arrows form an isomorphism from $\mathrm{Span}_{\mathbb{F}}\{c_3^{(i)}, c_4^{(i)}\}_{i=1}^k$ to $\mathrm{Span}_{\mathbb{F}}\{d_3^{(i)}, d_4^{(i)}\}_{i=1}^k$. Thus, we can find a new basis of C, namely $\{\tilde{c}_1^{(i)}, \tilde{c}_2^{(i)}, \tilde{c}_3^{(i)}, \tilde{c}_4^{(i)}\}_{i=1}^k$, such that

$$\partial_{U_2}(\tilde{c}_3^{(i)}) = U_2 \cdot d_3^{(i)}, \quad \partial_{U_2}(\tilde{c}_4^{(i)}) = U_2 \cdot d_4^{(i)}, \forall 1 \leq i \leq k,$$

while the pattern of the $\hat{\partial}$ is preserved. In addition, $[\partial_{U_2}, \partial_{A_1}] = 0$ implies that

$$\partial_{U_2}(\tilde{c}_1^{(i)}) = U_2 \cdot d_1^{(i)}, \quad \partial_{U_2}(\tilde{c}_2^{(i)}) = U_2 \cdot d_2^{(i)}, \forall 2 \leq i \leq k,$$
$$\partial_{U_2}(\tilde{c}_1^{(1)}) = U_2 \cdot d_1^{(1)}, \quad \partial_{U_2}(\tilde{c}_2^{(1)}) = 0.$$

From the fact that the vertical homology of $CFL^-(L)$ with respect to the differential $\partial_{A_2}+\partial_{U_2}$ is $\mathbb{F}[[U_2]]/U_2$, it follows that $\partial_{U_2}(d_2) = U_2 \cdot d_1$.

We may as well keep using the notations $c_j^{(i)}$ for the new basis. Applying similar arguments for the leftmost column with respect to vertical arrows, we can change the basis of A without changing the pattern of $\hat{\partial}$, such that

$$\partial_{U_2}(b_j^{(i)}) = U_2 a_j^{(i)}, \forall j = 1, 2, \forall i = 1, ..., k.$$

Then $\partial_{A_1}\partial_{U_2} = \partial_{U_2}\partial_{A_1}$ implies

$$\partial_{U_2}(b_4^{(i)}) = U_2 a_4^{(i)} + \sum_{m=1}^{k} \varepsilon_{i,m} U_2 d_1^{(m)}, \forall i = 1, ..., k, \varepsilon_{i,m} \in \mathbb{F}.$$

Thus, $\partial_{U_2}(b_3^{(i)}) = U_2 a_3^{(i)} + \sum_{m=2}^{k} \varepsilon_{i,m} U_2 d_2^{(m)}, \forall i = 1, ..., k$. Do base-changes:

$$\tilde{a}_4^{(i)} = a_4^{(i)} + \sum_{m=1}^{k} \varepsilon_{i,m} d_1^{(m)}, \quad \tilde{a}_3^{(i)} = a_3^{(i)} + \sum_{m=2}^{k} \varepsilon_{i,m} d_2^{(m)}.$$

We can preserve the pattern of $\hat{\partial}$, such that under the new basis (where we keep using the notations $a_j^{(i)}$) all the vertical arrows are pruned as $\partial_{U_2}(b_j^{(i)}) = U_2 a_j^{(i)}, \forall j = 1, 2, 3, 4, \forall i = 1, ..., k$. Similarly, by changing the bases of A and B simultaneously, we can prune the horizontal arrows in the top row, while preserving the pattern of $\hat{\partial}, \partial_{U_2}$ on A and B, such that $\partial_{U_1}(b_j^{(i)}) =$

$U_1c_j^{(i)}, \forall j=1,4, \forall i=1,...,k$. Then $\partial_{A_2}\partial_{U_1}=\partial_{U_1}\partial_{A_2}$ implies that $\partial_{U_1}(b_2^{(i)}), \partial_{U_1}(b_3^{(i)})$ are determined.

Similarly, all the horizontal arrows from B can be pruned by changing the basis of C. Suppose

$$\partial_{U_1}(b_4^{(i)})=U_1(\sum_{m=1}^{k}\lambda_{i,m}c_4^{(m)}+\sum_{m=1}^{k}\mu_{i,m}d_3^{(m)}).$$

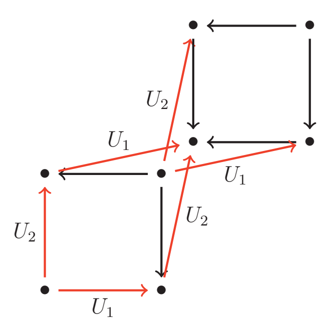
Then $\partial_{U_1}\partial_{U_2}(b_4^{(i)})=\partial_{U_2}\partial_{U_1}(b_4^{(i)})$ implies that $U_1U_2(\sum_{m=1}^{k}\lambda_{i,m}d_4^{(m)})=U_1U_2d_4^{(i)}$. Thus, $\lambda_{i,i}=1, \lambda_{i,m}=0, \forall i=1,...,k, \forall m\neq i$. Thus, $\partial_{U_1}(b_4^{(i)})=U_1(c_4^{(i)}+\sum_{m=1}^{k}\mu_{i,m}d_3^{(m)})$. Do base-changes

$$\tilde{c}_4^{(i)}=c_4^{(i)}+\sum_{m=1}^{k}\mu_{i,m}d_3^{(m)},\ \ \tilde{c}_1^{(i)}=c_1^{(i)}+\sum_{m=2}^{k}\mu_{i,m}d_2^{(m)}, \forall i=1,...,k.$$

Under the new basis (where we keep using the notations $c_j^{(i)}$), the patterns of all the $\hat{\partial}$ and ∂_{U_2} arrows are preserved, while $\partial_{U_1}(b_4^{(i)})=c_4^{(i)}$. Moreover, $[\partial_{A_1},\partial_{U_1}]=[\partial_{A_2},\partial_{U_1}]=0$ implies that $\partial_{U_1}b_j^{(i)}=U_1c_j^{(i)}, \forall j=1,2,3,4, \forall i=1,...,k$. Finally, all the arrows are as in Figure 3.3.3. Thus, $CFL^{-}(L)$ can be viewed as a square of chain complexes of A,B,C,D. $\square$

Similar arguments apply to the case of $L=b(8k+4,4k+3)$.

Proposition 3.3.6. *For the two-bridge link $L=b(8k+4,4k+3)$, the filtered homotopy type of $CFL^{-}(L)$ is determined by the filtered homotopy type of $\widehat{CFL}(L)$ (and hence by the Alexander polynomial, signature and linking number). Furthermore, $CFL^{-}(L)=CFL^{-}(T(2,4))\oplus\bigoplus_{i=1}^{k-1}(N,\partial^{-})$, where (N,∂^{-}) is as in Figure 3.3.3 and $CFL^{-}(T(2,4))$ is as follows.*



3.3.3 Computations of surgeries on $b(8k, 4k+1)$.

In this section, we compute the homology of surgeries on the two-bridge link $b(8k, 4k+1)$ and their d-invariants explicitly. Here, we make a convention of the d-invariants of $\mathbf{HF}^-$ different from [41]. We require that $d(\mathbf{HF}^-(S^3)) = 0$. Thus, the d-invariants computed here are the same as the d-invariants for HF^+.

We will first compute for the Whitehead link, following three steps: computations of $A^-_{\mathrm{s}}(Wh)$, computations of the inclusion maps $I^{\vec{M}}_{\mathrm{s}}$, and the computations of the homology of the surgeries on Wh.

Lemma 3.3.7. *$H_*(A^-_{\mathrm{s}}(Wh)) = \mathbb{F}[[U_1, U_2]]/(U_1 - U_2) = \mathbb{F}[[U]]$ for all $s \in \mathbb{H}(Wh) = \mathbb{Z}^2$.*

Proof. By Proposition 3.3.5, we can decompose $A^-_{+\infty,+\infty}(Wh)$ into a square of chain complexes, i.e.

$$\begin{array}{ccc} A & \rightarrow & D \\ \uparrow & & \uparrow \\ B & \rightarrow & C, \end{array}$$

where the summands A, B, C, D are described in Proposition 3.3.5. Since $A^-_{+\infty,+\infty}$ can be viewed as a mapping cone from $A \oplus B \oplus C$ to D, we get a short exact sequence of chain complexes

$$0 \to D \xrightarrow{i} A^-_{+\infty,+\infty} \to A \oplus B \oplus C \to 0.$$

From the fact that $H_*(A \oplus B \oplus C) = 0$, it follows that i is a quasi-isomorphism. Hence $H_*(A^-_{+\infty,+\infty}) = \mathbb{F}[[U]]$ and $[d_1] = [d_3] = 1 \in H_*(A^-_{+\infty,+\infty})$.

All the other complexes $A^-_{s_1,s_2}(Wh)$ can be actually obtained by taking various reflections on $A^-_{+\infty,+\infty}(Wh)$. Note that Equation (4.1.4) implies that the differentials in A^-_{s} are only changed by U_1, U_2 powers from $A^-_{+\infty,+\infty}$. In order to read off the correct powers of U_1, U_2-coefficients, we can change the $\mathbb{Z}^2$-filtration of A^-_{s}, such that the upward and rightward arrows in A^-_{s} are with U_1 and U_2 coefficients respectively. For instance, when $s_1 > 0, s_2 = 0$, we can flip the summands A and D about the A_1-axis to obtain the complex $A^-_{s_1,0}$. For convenience, we denote the vertical reflections of A, B, C, D by $\overline{A}, \overline{B}, \overline{C}, \overline{D}$. Thus, the complex $A^-_{s_1,0}$ is still a square of chain complexes as follows:

$$\begin{array}{ccc} \overline{A} & \rightarrow & \overline{D} \\ \uparrow & & \uparrow \\ B & \rightarrow & C. \end{array}$$

Thus, the fact that $\overline{A}, B, C$ are acyclic implies that $H_*(A^-_{s_1,0}) = H_*(\overline{D}) = \mathbb{F}[[U]]$. Similarly, we denote the horizontal reflections of A, B, C, D by $|A, |B, |C, |D$ respectively. Thus, the complex A^-_{0,s_1} with $s_1 > 0$ is the following square of chain complexes

$$\begin{array}{ccc} A & \rightarrow & |D \\ \uparrow & & \uparrow \\ B & \rightarrow & |C. \end{array}$$

Following the same line, we list all the filtered homotopy types of A^-_{s}'s together with some generators of their homologies in Table 3.3.1. Since $\max A_1 = \max A_2 = 1$, $\min A_1 = \min A_2 = -1$, the notation $+\infty$ means a positive integer s, while $-\infty$ means a negative integer s.

$A^-_{-\infty,+\infty} = \begin{array}{ccc} \lvert A & \rightarrow & \lvert D \\ \uparrow & & \uparrow \\ \lvert B & \rightarrow & \lvert C, \end{array}$ $[d_1] = 1 \in H_*(A^-_{-\infty,+\infty})$;	$A^-_{0,+\infty} = \begin{array}{ccc} A & \rightarrow & \lvert D \\ \uparrow & & \uparrow \\ B & \rightarrow & \lvert C, \end{array}$ $[d_1] = 1 \in H_*(A^-_{0,+\infty})$;	$A^-_{+\infty,+\infty} = \begin{array}{ccc} A & \rightarrow & D \\ \uparrow & & \uparrow \\ B & \rightarrow & C, \end{array}$ $[d_1] = [d_3] = 1 \in H_*(A^-_{+\infty,+\infty})$;
$A^-_{-\infty,0} = \begin{array}{ccc} \lvert\overline{A} & \rightarrow & \lvert\overline{D} \\ \uparrow & & \uparrow \\ \lvert B & \rightarrow & \lvert C, \end{array}$ $[a_2] = 1 \in H_*(A^-_{-\infty,0})$;	$A^-_{0,0} = \begin{array}{ccc} \overline{A} & \rightarrow & \lvert\overline{D} \\ \uparrow & & \uparrow \\ B & \rightarrow & \lvert C, \end{array}$ $[d_1] = [d_3] = 1 \in H_*(A^-_{0,0})$;	$A^-_{+\infty,0} = \begin{array}{ccc} \overline{A} & \rightarrow & \overline{D} \\ \uparrow & & \uparrow \\ B & \rightarrow & C, \end{array}$ $[d_3] = 1 \in H_*(A^-_{+\infty,0})$;
$A^-_{-\infty,-\infty} = \begin{array}{ccc} \lvert\overline{A} & \rightarrow & \lvert\overline{D} \\ \uparrow & & \uparrow \\ \lvert\overline{B} & \rightarrow & \lvert\overline{C}, \end{array}$ $[a_2] = [c_2] = 1 \in H_*(A^-_{-\infty,-\infty})$;	$A^-_{0,-\infty} = \begin{array}{ccc} \overline{A} & \rightarrow & \lvert\overline{D} \\ \uparrow & & \uparrow \\ \overline{B} & \rightarrow & \lvert\overline{C}, \end{array}$ $[c_2] = 1 \in H_*(A^-_{0,-\infty})$;	$A^-_{+\infty,-\infty} = \begin{array}{ccc} \overline{A} & \rightarrow & \overline{D} \\ \uparrow & & \uparrow \\ \overline{B} & \rightarrow & \overline{C}, \end{array}$ $[d_3] = [c_2] = 1 \in H_*(A^-_{+\infty,-\infty})$.

Table 3.3.1: $A^-_{\mathrm{s}}(Wh)$ **and generators of their homology.**

Note that $A, \overline{A}, |A, B, \overline{B}, |B, C, \overline{C}, |C$ are all acyclic, and $D, \overline{D}, |D, |\overline{D}$ all have the same homology $\mathbb{F}[[U]]$. We can use the same argument for $A^-_{+\infty,+\infty}$ to show that $A^-_{-\infty,+\infty}$, $A^-_{0,+\infty}$, $A^-_{+\infty,+\infty}$, $A^-_{0,0}$, $A^-_{+\infty,0}$, and $A^-_{+\infty,-\infty}$ all have the same homology $\mathbb{F}[[U]]$. For those other A^-_{s}, we can use the conjugation symmetry, that is, $H_*(A^-_{\mathrm{s}}(L)) = H_*(A^-_{-\mathrm{s}}(L)), \forall \mathrm{s} \in \mathbb{H}(L), \forall L$. This is because A^-_{s}'s are quasi-isomorphic to the Floer complexes of large surgeries on L.

Now we explain the generators of their homologies in Table 3.3.1. The chain complex $A^-_{0,-\infty}$ can be viewed as a mapping cone of a chain map from $\mathrm{cone}(\overline{B} \to \overline{A})$ to $\mathrm{cone}(|\overline{C} \to |\overline{D})$. Because $\mathrm{cone}(\overline{B} \to \overline{A})$ is acyclic, the generator of $H_*\Big(\mathrm{cone}(|\overline{C} \to |\overline{D})\Big)$ is also a generator of $H_*(A^-_{0,-\infty})$. Since $H_*(|\overline{C}) = \mathbb{F}[[U]]/U, H_*(|\overline{D}) = \mathbb{F}[[U]]$, we derive a short exact sequence

$$0 \to \mathbb{F}[[U]] \to \mathbb{F}[[U]] \to \mathbb{F}[[U]]/U \to 0$$

from the long exact sequence of the homologies $\cdots \to H_*(|\overline{D}) \to H_*\Big(\mathrm{cone}(|\overline{C} \to |\overline{D})\Big) \to H_*(|\overline{C}) \to \cdots$. Because $[c_2] = 1 \in H_*(|\overline{C}) = \mathbb{F}[[U]]/U$ and $[c_2] \in \mathrm{cone}(|\overline{C} \to |\overline{D})$ is mapped to $[c_2] \in H_*(|\overline{C})$, the above short exact sequence implies that $[c_2] = 1 \in H_*\Big(\mathrm{cone}(|\overline{C} \to |\overline{D})\Big)$, and thus $[c_2] = 1 \in H_*(A^-_{0,-\infty})$.

Similar arguments show that $[a_2] = [c_2] = 1 \in H_*(A^-_{-\infty,-\infty})$ and $[d_3] = 1 \in H_*(A^-_{+\infty,-\infty})$. Moreover, in the complex $A^-_{+\infty,-\infty}$, the equations $\partial^- c_1 = U_2 c_2 + d_1, \partial^- d_2 = d_1 + U_1 d_3$ imply that $[c_2] = [d_3] = 1 \in H_*(A^-_{+\infty,-\infty})$. $\square$

Taking the grading into account, we adopt the formula of the $\mathbb{Z}/\mathfrak{d}(\mathfrak{u})\mathbb{Z}$-grading defined on the surgery complex for a Spin^c structures $\mathfrak{u} \in \mathbb{H}(L)/H(L,\Lambda)$ in [24] Section 7.4,

$$\mu(\mathrm{s}, \mathbf{x}) = \mu^M_{\mathrm{s}}(\mathbf{x}) + \nu(\mathrm{s}) - \|M\|, \mathbf{x} \in \mathfrak{A}^-(\mathcal{H}^{L-\overrightarrow{M}}, \psi^{\overrightarrow{M}}(\mathrm{s})), \tag{3.3.2}$$

where $\mathrm{s} \in \mathfrak{u}$ and $\mu^M_{\mathrm{s}} = \mu_{\psi^{\overrightarrow{M}}(s)}$ is a natural $\mathbb{Z}$-grading defined on $\mathfrak{A}^-\Big(L - \overrightarrow{M}, \psi^{\overrightarrow{M}}(s)\Big)$. In the torsion case, the quadratic function ν can be chosen as 0. The natural $\mathbb{Z}$-grading $\mu^{\emptyset}_{\mathrm{s}} = \mu_{s_1,s_2}$

on each $A^-_{s_1,s_2}$ is given by

$$\mu_{s_1,s_2}(\mathbf{x}) = M(\mathbf{x}) - 2\sum_{i=1}^{2} \max\{A_i(\mathbf{x}) - s_i, 0\},$$

where $M(\mathbf{x})$ is the Maslov grading. When we use the Schubert Heegaard diagram, $A_1(\mathbf{x}) + A_2(\mathbf{x}) - M(\mathbf{x})$ is constant. Thus, up to a shift of a constant number, we can take $M(\mathbf{x}) = A_1(\mathbf{x}) + A_2(\mathbf{x})$ for $\forall \mathbf{x} \in A^-_{s_1,s_2}$. In the primitive system we identify $\mathfrak{A}^-\left(L - \overrightarrow{M}, \psi^{\overrightarrow{M}}(\mathrm{s})\right)$ with some $A^-_{s_1',s_2'}$ (where s_1', s_2' can evaluate $+\infty$), so the grading μ^M_{s} is actually $\mu_{\mathrm{s}'}$. We define some rules of ∞ as follows:

$$0 \cdot (+\infty) = +\infty;\ s + (+\infty) = +\infty, \forall s \in \mathbb{R};\ s + (-\infty) = -\infty, \forall s \in \mathbb{R};\ (\pm 1) \cdot (+\infty) = \pm\infty.$$

Recall the notations in Example 2.2.14. The complexes $C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)} = A^-_{s_1+\varepsilon_1\cdot\infty, s_2+\varepsilon_2\cdot\infty}, \varepsilon_i \in \{0,1\}$ are setting at the position $(\varepsilon_1, \varepsilon_2)$ in the square and with the index (s_1, s_2) in the product complex $C^{(\varepsilon_1,\varepsilon_2)} = \prod_{s_1,s_2} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}$.

We define the grading $\mu^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$ on the complex $C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}$ by the formula:

$$\mu^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}(\mathbf{x}) = M(\mathbf{x}) - 2\sum_{i=1}^{2} \max\{A_i(\mathbf{x}) - s_i - \varepsilon_i(+\infty), 0\} - \varepsilon_1 - \varepsilon_2.$$

Here $\mu^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$ plays the role as μ in Equation (3.3.2).

Let W be the four-manifold cobordism corresponding to the surgery from S^3 to $S^3_\Lambda(L)$. In [24], it is shown that the cobordism map $F^-_{W,\mathrm{s}}$ corresponds to the inclusion $\iota : \mathfrak{A}^-(\mathcal{H}^\emptyset) \to \mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{\overrightarrow{L}}(\mathrm{s})) \subset \mathcal{C}^-(\mathcal{H}, \Lambda)$, $\overrightarrow{L} = +L_1 \cup +L_2$. So we need to shift the grading such that ι is of the degree $\deg(F^-_{W,\mathrm{s}}) = \frac{c_1(\mathrm{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$. In our case, the complex $\mathfrak{A}^-(\mathcal{H}^\emptyset, \psi^{\overrightarrow{L}}(\mathrm{s})) = C^{(1,1)}_{(s_1,s_2)} = A^-_{+\infty,+\infty}(Wh)$ has a generator $[d_1] = 1 \in H_*(A^-_{+\infty,+\infty})$ of Alexander grading $A(d_1) = (0,1)$. Finally, the grading formula turns out to be

$$\mu^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}(\mathbf{x}) = A_1(\mathbf{x}) + A_2(\mathbf{x}) - 2\sum_{i=1}^{2} \max\{A_i(\mathbf{x}) - s_i - \varepsilon_i(+\infty), 0\} - \varepsilon_1 - \varepsilon_2 + \frac{c_1(\mathrm{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + 1, \tag{3.3.3}$$

where $\mathrm{c}_1(\mathrm{s}) = [2\mathrm{s}] - \Lambda_1 - \Lambda_2 = (2s_1 - p_1, 2s_2 - p_2) \in \mathbb{Z}^2/\Lambda$. In the perturbed surgery complex, since all the perturbed maps have the same degrees as the original, we can compute the gradings still using Equation (3.3.3).

Now we restrict our scalars to $\mathbb{F}[[U_1]]$. By Proposition 5.1.1 and Lemma 3.3.7, up to $\mathbb{F}[[U_1]]$-linear chain homotopy, all the edge maps $\Phi_{\mathrm{s}}^{\pm L_i}$ are classified by their actions on the homologies. The actions of $\Phi_{\mathrm{s}}^{\pm L_i}$ on homologies are determined by the corresponding inclusion maps $I_{\mathrm{s}}^{\pm L_i}$. We denote the induced maps on homologies by $(I_{\mathrm{s}}^{\pm L_i})_* : \mathbb{F}[[U]] \to \mathbb{F}[[U]]$.

Lemma 3.3.8. *Regarding the inclusion maps, we have the following results for $I_s^{\pm L_1}$, where $s = (s_1, s_2)$.*

- *If $s_1 > 0$, then $(I_s^{+L_1})_* = id$.*
- *If $s_1 = 0, s_2 \neq 0$, then $(I_s^{+L_1})_* = id$.*
- *If $s_1 = s_2 = 0$, then $(I_s^{+L_1})_* = U \cdot id$.*
- *If $s_1 < 0$, then $(I_s^{+L_1})_* = U^{-s_1} \cdot id$.*
- *If $s_1 > 0$, then $(I_s^{-L_1})_* = U^{s_1} \cdot id$.*
- *If $s_1 = 0, s_2 \neq 0$, then $(I_s^{-L_1})_* = id$.*
- *If $s_1 = s_2 = 0$, then $(I_s^{-L_1})_* = U \cdot id$.*
- *If $s_1 < 0$, then $(I_s^{-L_1})_* = id$.*

Proof. In fact, when $s_1 > 0$, by definition $I_{s_1,s_2}^{+L_1} = id$. When $s_1 = 0, s_2 > 0$, we have $I_{0,s_2}^{+L_1}(d_1) = d_1$. Therefore by Table 3.3.1, the inclusion map $I_{0,s_2}^{+L_1}$ acts on the homology as $id : \mathbb{F}[[U]] \to \mathbb{F}[[U]]$. When $s_1 = 0, s_2 < 0$, we have $I_{0,s_2}^{+L_1}(c_2) = c_2$. Therefore by Table 3.3.1, the inclusion map $I_{0,s_2}^{+L_1}$ acts on homology as the identity. When $s_1 = s_2 = 0$, we have $I_{0,0}^{+L_1}(d_3) = U_1 \cdot d_3$. Therefore by Table 3.3.1, $I_{0,0}^{+L_1}$ acts on homology as $U \cdot id$. When $s_1 < 0, s_2 > 0$, we have $I_{s_1,s_2}^{+L_1}(d_1) = U_1^{-s_1} \cdot d_1$. Thus, $(I_{\mathrm{s}}^{+L_1})_* = U^{-s_1} \cdot id$. When $s_1 < 0, s_2 \leq 0$, we have $I^{+L_1}_{s_1,s_2}(a_2) = U_1^{-1-s_1} \cdot a_2$. In the complex $A^-_{+\infty,s_2}$, $s_2 \leq 0$, the equation $\partial^- a_3 = a_2 + U_1 d_3$ implies that $[a_2] = U_1[d_3] \in H_*(A^-_{+\infty,s_2})$. Thus, $[a_2] = U \in \mathbb{F}[[U]] = H_*(A^-_{+\infty,s_2})$. Therefore, it follows that $(I^{+L_1}_{s_1,s_2})_* = U^{-s_1} \cdot id$, when $s_1 < 0, s_2 \leq 0$.

In the same way, we get the following results for $I^{-L_1}_{\mathrm{s}}$, where $\mathrm{s} = (s_1, s_2)$. $\square$

Now we can compute the homology of surgeries on S. In each case, we write down the d-invariants, which are the gradings of the top element in each $\mathbb{F}[[U]]$ summand.

Proposition 3.3.9. *Let Wh be the Whitehead link, $\Lambda = diag(p_1, p_2)$ and Y be the surgery manifold $S^3_\Lambda(Wh)$. Then* $\mathrm{Spin}^c(Y)$ *can be identified with $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$, so we use $(t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ to denote the* Spin^c *structures over Y. Then, the Floer homology of Y is as follows.*

- *If $p_1 = p_2 = 0$, then* $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 4}, & (t_1, t_2) = (0, 0); \\ 0, & \textit{otherwise}, \end{cases}$ *with $d = -1, -1, 0, 0$.*

- *If $p_1 > 0, p_2 = 0$, then* $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2}, & (t_1, t_2) = (t_1, 0); \\ 0, & \textit{otherwise}. \end{cases}$ *Their d-invariants are $d(Y, (0, 0)) = \frac{p_1}{4} - \frac{7}{4}, \frac{p_1}{4} - \frac{3}{4}$, and $d(Y, (t_1, 0)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{1}{4}, \frac{(2s_1+p_1)^2}{4p_1} - \frac{3}{4}$, when $t_1 \neq 0$, where s_1 is an integer in the class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $-p_1 < s_1 \leq 0$.*

- *If $p_1 < 0, p_2 = 0$, then* $\mathbf{HF}^-(Y, (t_1, t_2)) = \begin{cases} \mathbb{F}[[U]]^{\oplus 2} \oplus (\mathbb{F}[[U]]/U), & (t_1, t_2) = (0, 0); \\ \mathbb{F}[[U]]^{\oplus 2}, & t_1 \neq 0, t_2 = 0; \\ 0, & \textit{otherwise}. \end{cases}$ *Their d-invariants are $d(Y, (t_1, 0)) = \frac{(2s_1-p_1)^2}{4p_1} + \frac{3}{4}, \frac{(2s_1-p_1)^2}{4p_1} - \frac{1}{4}$, where s_1 is an integer in the class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $p_1 < s_1 \leq 0$.*

- *If $p_1 > 0, p_2 > 0$, then* $\mathbf{HF}^-(Y, (t_1, t_2)) = \mathbb{F}[[U]]$, $\forall (t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$. *Their d-invariants are $d(Y, (0, 0)) = \frac{p_1+p_2-10}{4}$, and $d(Y, (t_1, t_2)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{(2s_2+p_2)^2}{4p_2} - \frac{1}{2}$, when $(t_1, t_2) \neq (0, 0)$, where s_i is an integer in the class $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $-p_i < s_i \leq 0$.*

- *If* $p_1 > 0, p_2 < 0$, *then* $\mathbf{HF}^-(Y,(t_1,t_2)) = \begin{cases} \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U), & (t_1,t_2) = (0,0); \\ \mathbb{F}[[U]], & \textit{otherwise}. \end{cases}$
  Their d-invariants are $d(Y,(t_1,t_2)) = \frac{(2s_1+p_1)^2}{4p_1} + \frac{(2s_2-p_2)^2}{4p_2}$, *where* s_i *is an integer in the class* $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ *such that* $-|p_i| < s_i \leq 0$.
- *If* $p_1, p_2 < 0$, *then* $\mathbf{HF}^-(Y,(t_1,t_2)) = \begin{cases} \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U), & (t_1,t_2) = (0,0); \\ \mathbb{F}[[U]], & \textit{otherwise}. \end{cases}$ *Their d-invariants are* $d(Y,(t_1,t_2)) = \frac{(2s_1-p_1)^2}{4p_1} + \frac{(2s_2-p_2)^2}{4p_2} + \frac{1}{2}$, *where* s_i *is an integer in the class* $t_i \in \mathbb{Z}/p_i\mathbb{Z}$ *such that* $p_i < s_i \leq 0$.

Proof. First, let's look at the $(0,0)$-surgery on the Whitehead link. The surgery complex splits into a direct product of squares of chain complexes according to Spin^c structures. See Figure 3.3.5. In the (s_1,s_2) Spin^c structure, the factor of the direct product is the following square of chain complexes:

$$\begin{tikzcd} A^-_{s_1,s_2} \arrow[r, "\Phi^{+L_1}_{s_1,s_2}+\Phi^{-L_1}_{s_1,s_2}"] \arrow[d, "\Phi^{+L_2}_{s_1,s_2}+\Phi^{-L_2}_{s_1,s_2}"'] \arrow[dr, "\sum \Phi^{\pm L_1 \cup \pm L_2}_{s_1,s_2}"] & A^-_{+\infty,s_2} \arrow[d, "\Phi^{+L_2}_{+\infty,s_2}+\Phi^{-L_2}_{+\infty,s_2}"] \\ A^-_{s_1,+\infty} \arrow[r, "\Phi^{+L_1}_{s_1,+\infty}+\Phi^{-L_1}_{s_1,+\infty}"'] & A^-_{+\infty,+\infty}. \end{tikzcd}$$

For the torsion Spin^c structure $(0,0) \in \mathbb{Z}^2$, since $\Phi^{+L_1}_{0,0} \simeq \Phi^{-L_1}_{0,0}$, $\Phi^{+L_2}_{0,0} \simeq \Phi^{-L_2}_{0,0}$, $\Phi^{+L_1}_{0,+\infty} \simeq \Phi^{-L_1}_{0,\infty}$, $\Phi^{+L_2}_{+\infty,0} \simeq \Phi^{-L_2}_{+\infty,0}$, the perturbed surgery complex is as follows:

$$\begin{tikzcd} A^-_{0,0} \arrow[r, "0"] \arrow[d, "0"'] & A^-_{+\infty,0} \arrow[d, "0"] \\ A^-_{0,+\infty} \arrow[r, "0"] & A^-_{+\infty,+\infty}. \end{tikzcd}$$

Therefore, the homology is $\mathbb{F}[[U]]^{\oplus 4}$ generated by $d_1 \in C^{(0,0)}_{(0,0)}, d_3 \in C^{(1,0)}_{(0,0)}, d_1 \in C^{(0,1)}_{(0,0)}, d_1 \in C^{(1,1)}_{(0,0)}$. Since $\mathrm{c}_1\big((0,0)\big) = (0,0)$, $\chi(W) = 2$, $\sigma(W) = 0$, from Equation (3.3.3), we also get their absolute gradings $\mu^{0,0}_{0,0}([d_1]) = -1, \mu^{1,0}_{0,0}([d_3]) = 0, \mu^{0,1}_{0,0}([d_1]) = 0, \mu^{1,1}_{0,0}([d_3]) = -1$.

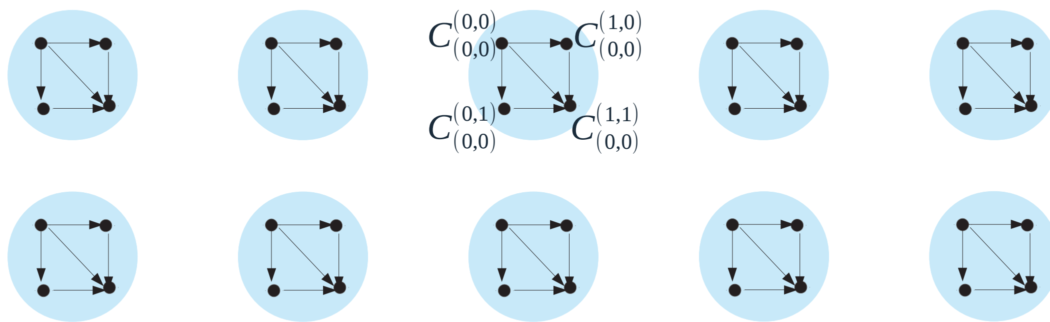


Figure 3.3.5: **The surgery complex for $\Lambda = (0,0)$.** Every dot represents a complex $C^{\varepsilon}_{\mathrm{s}}$ which is a certain generalized Floer complex $A^{-}_{\mathrm{s}'}(Wh)$, and every arrow represents a Φ-map according to the endpoints of the arrow. We only label the four complexes $C^{\varepsilon}_{\mathrm{s}}$ for the Spin^c structure $\mathrm{s} = (0,0)$, and the others are similar.

For the non-torsion Spin^c structure $(s_1, s_2) \in \mathbb{Z}^2, s_1 > 0$, since $\Phi^{+L_1}_{s_1,s_2} + \Phi^{-L_1}_{s_1,s_2}$ acts on homology as $id + U^{s_1} \cdot id = (1 + U^{s_1}) \cdot id$, which is a quasi-isomorphism, it follows that the homology of this Spin^c structure is 0. Indeed, one can consider the horizontal filtration for this square, whose associated graded is the direct sum of the two acyclic horizontal rows. A similar argument applies to all the other non-torsion Spin^c structures.

Second, let's look at the $(p_1, 0)$-surgery with $p_1 \neq 0$, which gives rise to a manifold with $b_1 = 1$. Suppose $p_1 > 0$. In order to compute the homology, we need some filtrations to kill acyclic subcomplexes and quotient complexes. Let $\mathcal{F}_1(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = -s_1, \mathcal{F}_2(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1$. Without loss of generality, see Figure 3.3.6 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1,0)$.

For any $(t_1, t_2) \in \mathrm{Spin}^c(Y) = \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}$ with $t_2 \neq 0$, the Floer homology is 0. Indeed, we can consider the union of all these Spin^c structures, which corresponds to the subcomplex

$$\mathcal{R}_1 = \prod_{s_2 \neq 0} (C^{(0,0)}_{(s_1,s_2)} \oplus C^{(1,0)}_{(s_1,s_2)} \oplus C^{(0,1)}_{(s_1,s_2)} \oplus C^{(1,1)}_{(s_1,s_2)}).$$

Since $\Phi^{+L_2}_{s_1,s_2} + \Phi^{-L_2}_{s_1,s_2}, s_2 \neq 0$ acts on homology as $id + U^{|s_2|} \cdot id = (1 + U^{|s_2|}) \cdot id$, which is a quasi-isomorphism, the following square is acyclic:

$$\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\Phi^{+L_1}_{s_1,s_2}"] \arrow[d, "\Phi^{+L_2}_{s_1,s_2}+\Phi^{-L_2}_{s_1,s_2}"'] \arrow[dr, "\Phi^{+L_1\cup+L_2}_{s_1,s_2}+\Phi^{+L_1\cup-L_2}_{s_1,s_2}" description] & A^-_{+\infty,s_2} \arrow[d, "\Phi^{+L_2}_{+\infty,s_2}+\Phi^{-L_2}_{+\infty,s_2}"] \\
A^-_{s_1,+\infty} \arrow[r, "\Phi^{+L_1}_{s_1,+\infty}"'] & A^-_{+\infty,+\infty}.
\end{tikzcd}$$

The associated graded complex of $\mathcal{F}_1$ splits as a direct product of the above squares, so $\mathcal{R}_1$ is acyclic.

For the Spin^c structure $(t_1,0)$, we first kill the acyclic subcomplex

$$\mathcal{R}_2=\prod_{s_1>0}C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,0)}.$$

Since the inclusion map $I^{+L_1}_{s_1,0}$ is id for all $s_1>0$, the associated graded complex of the filtration $\mathcal{F}_1$ splits as a direct product of acyclic complexes in the form of

$$\begin{tikzcd}
C^{(0,0)}_{(s_1,0)} \arrow[r, "\Phi^{+L_1}_{s_1,0}"] \arrow[d, "\Phi^{+L_2}_{s_1,0}+\Phi^{-L_2}_{s_1,0}"'] \arrow[dr, "\Phi^{+L_1\cup+L_2}_{s_1,s_2}+\Phi^{+L_1\cup-L_2}_{s_1,s_2}" description] & C^{(1,0)}_{(s_1,0)} \arrow[d, "\Phi^{+L_2}_{+\infty,0}+\Phi^{-L_2}_{+\infty,0}"] \\
C^{(0,1)}_{(s_1,0)} \arrow[r, "\Phi^{+L_1}_{s_1,+\infty}"'] & C^{(1,1)}_{(s_1,0)}.
\end{tikzcd}$$

Thus $\mathcal{R}_2$ is acyclic.

On the other hand, we have another acyclic subcomplex

$$\mathcal{R}_3=\prod_{\mathcal{F}_2\leq 0}C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,0)}.$$

In fact, since the inclusion maps $I^{-L_1}_{s_1,0}$ and $I^{-L_1}_{s_1,+\infty}$ are both id when $s_1<0$, the associated graded complex of the filtration $\mathcal{F}_2$ splits as a direct product of acyclic complexes in the form of

$$\begin{tikzcd}
C^{(0,0)}_{(s_1,0)} \arrow[r, "\Phi^{-L_1}_{s_1,0}"] \arrow[d, "\Phi^{+L_2}_{s_1,0}+\Phi^{-L_2}_{s_1,0}"'] \arrow[dr, "\Phi^{-L_1\cup+L_2}_{s_1,s_2}+\Phi^{-L_1\cup-L_2}_{s_1,s_2}"] & C^{(1,0)}_{(s_1+p_1,0)} \arrow[d, "\Phi^{+L_2}_{+\infty,0}+\Phi^{-L_2}_{+\infty,0}"] \\
C^{(0,1)}_{(s_1,0)} \arrow[r, "\Phi^{-L_1}_{s_1,+\infty}"'] & C^{(1,1)}_{(s_1+p_1,0)}.
\end{tikzcd}$$

Thus $\mathcal{R}_3$ is acyclic. So the quotient complex $\mathcal{Q} = \mathcal{C}^-/\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ is a direct product of

$$C^{(0,0)}_{(s_1,0)} \xrightarrow{\Phi^{+L_2}_{s_1,0}+\Phi^{-L_2}_{s_1,0}} C^{(0,1)}_{(s_1,0)},$$

where $-p_1 + 1 \leq s_1 \leq 0$. From the computations of the inclusion maps, we know that $\Phi^{+L_2}_{s_1,0} \simeq \Phi^{-L_2}_{s_1,0}$. Thus the homology of each Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}$ is $\mathbb{F}[[U]]^{\oplus 2}$. Note that $\chi(W) = 2, \sigma(W) = 1$. When $-p_1 + 1 \leq s_1 < 0$, the complex $C^{(0,0)}_{(s_1,0)} = A^-_{s_1,0}$ has a_2 as a generator of its homology of grading $\mu^{0,0}_{s_1,0}(a_2) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1}{4} + \frac{1}{4}$, and the complex $C^{(0,1)}_{(s_1,0)} = A^-_{s_1,+\infty}$ has d_1 as a generator of its homology of grading $\mu^{0,1}_{s_1,0}(d_1) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1}{4} - \frac{3}{4}$. While for the $(0,0)$ Spin^c structure, $C^{(0,0)}_{(0,0)} = A^-_{0,0}$ has d_1 as a generator of its homology with grading $\mu^{0,0}_{0,0}(d_1) = \frac{p_1}{4} - \frac{7}{4}$, and $C^{(0,1)}_{(0,0)} = A^-_{0,+\infty}$ has d_1 as a generator of its homology with grading $\mu^{0,1}_{0,0}(d_1) = \frac{p_1}{4} - \frac{3}{4}$.

The case of $p_1 < 0$ is similar. We first kill the acyclic subcomplex

$$\mathcal{R}_1 = \prod_{s_2 \neq 0} (C^{(0,0)}_{(s_1,s_2)} \oplus C^{(1,0)}_{(s_1,s_2)} \oplus C^{(0,1)}_{(s_1,s_2)} \oplus C^{(1,1)}_{(s_1,s_2)}).$$

Thus, the homology for the Spin^c structure (t_1, t_2) with $t_2 \neq 0$ is 0. Next, we kill the acyclic quotient complexes

$$\mathcal{R}_2 = \prod_{s_1 > 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,0)}, \quad \mathcal{R}_3 = \prod_{s_1 - \varepsilon_1 p_1 < 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,0)}.$$

In the $(0,0)$ Spin^c structure, the remaining complexes are as follows

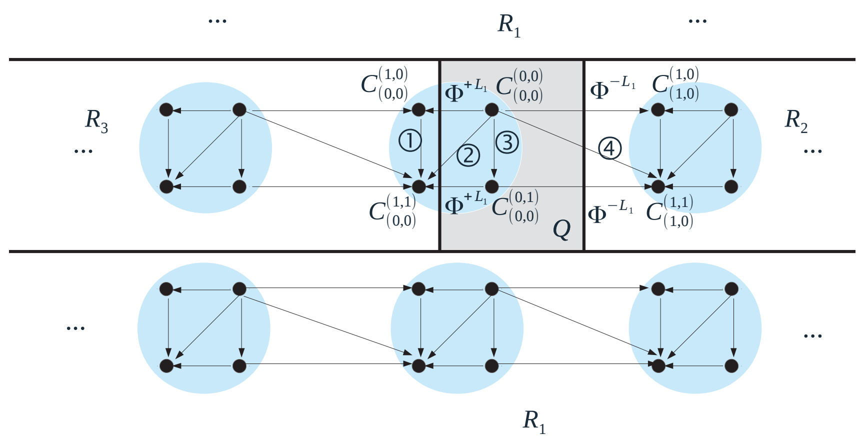


Figure 3.3.6: **The surgery complex for $\Lambda = (1, 0)$.** Every dot represents a complex $C^{\varepsilon}_{\mathrm{s}}$ which is a certain generalized Floer complex $A^{-}_{\mathrm{s}'}(Wh)$, and in every shaded circle the complexes $C^{\varepsilon}_{\mathrm{s}}$'s have the same subscript s. Every arrow represents a Φ-map according to the endpoints of the arrow, where we omit the subscripts. All the parallel arrows share the same type of $\Phi^{\overrightarrow{M}}$, i.e. having the same superscript $\overrightarrow{M}$. The arrows with circled numbers $1, 2, 3, 4$ are $\Phi^{+L_2}_{+\infty,0} + \Phi^{-L_2}_{+\infty,0}$, $\Phi^{+L_2\cup+L_1}_{0,0} + \Phi^{-L_2\cup+L_1}_{0,0}$, $\Phi^{+L_2}_{0,0} + \Phi^{-L_2}_{0,0}$, and $\Phi^{+L_2\cup-L_1}_{0,0} + \Phi^{-L_2\cup-L_1}_{0,0}$ respectively. The regions R_1, R_2, R_3 divided by the (thicker) lines are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. The shaded region Q corresponds to the truncated complex $\mathcal{Q}$.

$$\begin{tikzcd}
C^{(1,0)}_{(p_1,0)} \arrow[d, "\Phi^{+L_2}_{+\infty,0}+\Phi^{-L_2}_{+\infty,0}"'] & C^{(0,0)}_{(0,0)} \arrow[l, "\Phi^{-L_1}_{0,0}"'] \arrow[r, "\Phi^{+L_1}_{0,0}"] \arrow[dl, "\Phi^{-L_1\cup+L_2}_{0,0}+\Phi^{-L_1\cup-L_2}_{0,0}"'] \arrow[dr, "\Phi^{+L_1\cup+L_2}_{0,0}+\Phi^{+L_1\cup-L_2}_{0,0}"] \arrow[d, dashed, "\Phi^{+L_2}_{0,0}+\Phi^{-L_2}_{0,0}"] & C^{(1,0)}_{(0,0)} \arrow[d, "\Phi^{+L_2}_{+\infty,0}+\Phi^{-L_2}_{+\infty,0}"] \\
C^{(1,1)}_{(p_1,0)} & C^{(0,1)}_{(0,0)} \arrow[l, "\Phi^{-L_1}_{0,+\infty}"] \arrow[r, "\Phi^{+L_1}_{0,+\infty}"'] & C^{(1,1)}_{(0,0)}.
\end{tikzcd}$$

Since $\Phi^{+L_2}_{+\infty,0}, \Phi^{-L_2}_{+\infty,0}$ are chain homotopic, we can replace $\Phi^{+L_2}_{+\infty,0} + \Phi^{-L_2}_{+\infty,0}$ by 0 in the perturbed surgery complex. Therefore, we can also replace the diagonal maps by 0. Thus, we have two split complexes,

$$C^{(1,0)}_{(p_1,0)} \xleftarrow{\Phi^{-L_1}_{0,0}} C^{(0,0)}_{(0,0)} \xrightarrow{\Phi^{+L_1}_{0,0}} C^{(1,0)}_{(0,0)},$$

$$C^{(1,1)}_{(p_1,0)} \xleftarrow[\Phi^{-L_1}_{0,+\infty}]{} C^{(0,1)}_{(0,0)} \xrightarrow[\Phi^{+L_1}_{0,+\infty}]{} C^{(1,1)}_{(0,0)}.$$

Since $C^{(1,0)}_{(p_1,0)} = C^{(1,0)}_{(0,0)}$ and $\Phi^{+L_1}_{0,0} \simeq \Phi^{-L_1}_{0,0}$, we can replace $\Phi^{+L_1}_{0,0}$ by $\Phi^{-L_1}_{0,0}$ in the perturbed complex. By changing basis, we can split the first row as $\text{cone}(\Phi^{+L_1}_{0,0}) \oplus C^{(1,0)}_{(p_1,0)}$ with homology $\mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U)$. Similarly, from that $\Phi^{\pm L_1}_{0,+\infty}$ are quasi-isomorphisms, it follows that the second row is quasi-isomorphic to $C^{(1,1)}_{(0,0)}$ by changing basis. Thus in the $(0,0)$ Spin^c structure, the homology is $\mathbb{F}[[U]]^{\oplus 2} \oplus (\mathbb{F}[[U]]/U)$.

For the other Spin^c structures $(t_1, 0), t_1 \neq 0$, the remaining complexes are as follows

$$C^{(1,0)}_{(s_1,0)} \xrightarrow{\Phi^{+L_2}_{+\infty,0}+\Phi^{-L_2}_{+\infty,0}} C^{(1,1)}_{(s_1,0)},$$

where the integer s_1 is in the residue class $t_1 \in \mathbb{Z}/p_1\mathbb{Z}$ such that $p_1 < s_1 < 0$. Similarly, we replace $\Phi^{+L_2}_{+\infty,0} + \Phi^{-L_2}_{+\infty,0}$ by 0, and get that the homology is $\mathbb{F}[[U]]^{\oplus 2}$. The correction terms can be computed similarly with $\chi(W) = 2, \sigma(W) = -1$.

Finally, let's look at the (p_1, p_2)-surgery, where $p_1 p_2 \neq 0$. This breaks down to three cases: $p_1, p_2 > 0$, $p_1, p_2 < 0$, and $p_1 p_2 < 0$. We apply the truncation tricks shown in [24] Section 8.3.

(1) When $p_1, p_2 > 0$, the (p_1, p_2)-surgery is actually a large surgery, so its homology can be derived from A^-_{s} directly. However, we still compute them by elementary methods. We construct two filtrations,

$$\mathcal{F}_{00}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = -s_1 - s_2, \quad \mathcal{F}_{11}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1 + s_2 - (\varepsilon_2 - 1)p_2.$$

Without loss of generality, see Figure 3.3.7 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1, 1)$.

We first consider an acyclic subcomplex

$$\mathcal{R}_1 = \prod_{\max\{s_1,s_2\}>0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

In fact, since the inclusion maps $I^{+L_1}_{s_1,s_2}, s_1 > 0$ and $I^{+L_2}_{s_1,s_2}, s_2 > 0$ are both id's, the associated graded complex of the filtration $\mathcal{F}_{00}$ splits as a direct product of acyclic squares $R_{\mathrm{s},0,0}$ in Equation 2.2.2:

$$\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\Phi^{+L_1}_{s_1,s_2}"] \arrow[d, "\Phi^{+L_2}_{s_1,s_2}"'] \arrow[dr, "\Phi^{+L_1\cup+L_2}_{s_1,s_2}"] & A^-_{+\infty,s_2} \arrow[d, "\Phi^{+L_2}_{+\infty,s_2}"] \\
A^-_{s_1,+\infty} \arrow[r, "\Phi^{+L_1}_{s_1,+\infty}"'] & A^-_{+\infty,+\infty}.
\end{tikzcd}$$

Thus $\mathcal{R}_1$ is acyclic.

There is another acyclic subcomplex

$$\mathcal{R}_2 = \prod_{\max\{s_1-(\varepsilon_1-1)p_1, s_2-(\varepsilon_2-1)p_2\}\leq 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

One can directly check $\mathcal{R}_2$ is a subcomplex by computation. Because the inclusion maps $I^{-L_1}_{s_1,s_2}, s_1 < 0$ and $I^{-L_2}_{s_1,s_2}, s_2 < 0$ are both id's, the associated graded complex of $\mathcal{F}_{11}$ splits as a product of acyclic squares $R_{\mathrm{s},1,1}$:

$$\begin{tikzcd}
A^-_{s_1,s_2} \arrow[r, "\Phi^{-L_1}_{s_1,s_2}"] \arrow[d, "\Phi^{-L_2}_{s_1,s_2}"'] \arrow[dr, "\Phi^{-L_1\cup -L_2}_{s_1,s_2}"] & A^-_{+\infty,s_2} \arrow[d, "\Phi^{-L_2}_{+\infty,s_2}"] \\
A^-_{s_1,+\infty} \arrow[r, "\Phi^{-L_1}_{s_1,+\infty}"'] & A^-_{+\infty,+\infty},
\end{tikzcd}$$

where $s_1+p_1 \leq 0, s_2+p_2 \leq 0$. Thus $\mathcal{R}_2$ is acyclic.

Let $\mathcal{C}_1 = \mathcal{C}/(\mathcal{R}_1+\mathcal{R}_2)$. Inside $\mathcal{C}_1$, there are two acyclic subcomplexes

$$\mathcal{R}_3 = \{ \prod_{s_1-(\varepsilon_1-1)p_1\leq 0, -p_2+1\leq s_2\leq 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)} \} \cap \mathcal{C}_1 = \prod_{s_1+p_1\leq 0, 0\geq s_2 > -p_2} \Big(C^{(0,0)}_{(s_1,s_2)} \oplus C^{(1,0)}_{(s_1+p_1,s_2)} \Big),$$

$$\mathcal{R}_4 = \{ \prod_{s_2-(\varepsilon_2-1)p_2\leq 0, -p_1+1\leq s_1\leq 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)} \} \cap \mathcal{C}_1 = \prod_{s_2+p_2\leq 0, 0\geq s_1 > -p_1} \Big(C^{(0,0)}_{(s_1,s_2)} \oplus C^{(1,1)}_{(s_1,s_2+p_2)} \Big).$$

In fact, the associated graded complex of $\mathcal{F}_{11}$ on $\mathcal{R}_3$ splits as a direct product of acyclic complexes $C^{(0,0)}_{(s_1,s_2)} \xrightarrow{\Phi^{-L_1}_{s_1,s_2}} C^{(1,0)}_{(s_1+p_1,s_2)}$, because the inclusion map $I^{-L_1}_{s_1,s_2}, s_1<0$ is id. Thus $\mathcal{R}_3$ is acyclic. Similar argument applies to $\mathcal{R}_4$.

At last, we look at the quotient complex

$$\mathcal{Q} = \mathcal{C}_1/(\mathcal{R}_3+\mathcal{R}_4) = \bigoplus_{-p_1<s_1\leq 0, -p_2<s_2\leq 0} C^{(0,0)}_{(s_1,s_2)},$$

where $C^{(0,0)}_{(s_1,s_2)} = A^-_{s_1,s_2}$. There is only one A^-_{s} left in each Spin^c structure Y with homology $\mathbb{F}[[U]]$. For $(s_1,s_2)=(0,0)$, the complex $C^{(0,0)}_{(0,0)} = A^-_{0,0}$ has d_1 as a generator of its homology with grading $\mu^{0,0}_{0,0}(d_1) = \frac{p_1+p_2-10}{4}$. For $-p_1<s_1<0$, the complex $C^{(0,0)}_{(s_1,s_2)} = A^-_{s_1,s_2}$ has a_2 as a generator of its homology with grading $\mu^{0,0}_{s_1,s_2}(a_2) = \frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 + s_2 + \frac{p_1+p_2-2}{4}$. Similarly, we have the same formula for $-p_2<s_2<0, -p_1<s_1\leq 0$.

(2) When $p_1p_2<0$, we might as well suppose $p_1>0, p_2<0$ due to the symmetry of the

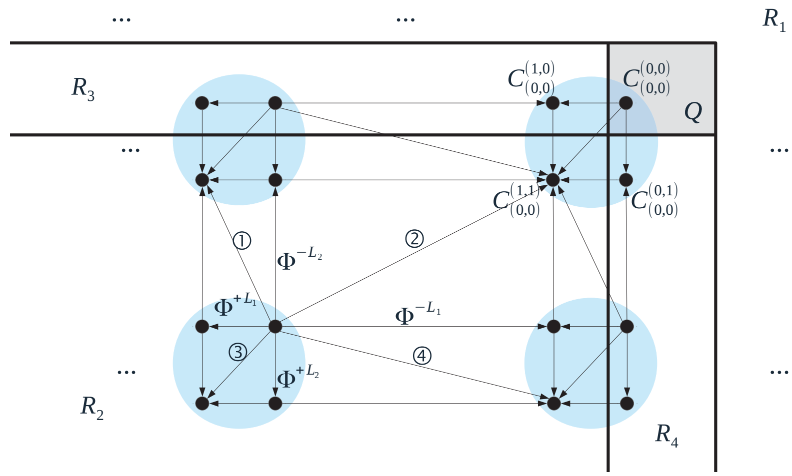


Figure 3.3.7: **The surgery complex for** $\Lambda = (1,1)$**.** The arrows with circled numbers $1,2,3,4$ are $\Phi^{+L_1\cup+L_2}$, $\Phi^{+L_1\cup-L_2}$,$\Phi^{-L_1\cup-L_2}$, and $\Phi^{-L_1\cup+L_2}$ respectively. The regions R_1,R_2,R_3,R_4 divided by the (thicker) lines are corresponded to the acyclic subcomplexes $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4$. The shaded region Q corresponds to the truncated complex $\mathcal{Q}$.

two components. We construct four filtrations

$$\mathcal{F}_{00}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = -s_1 + s_2,\;\; \mathcal{F}_{01}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = -s_1 - s_2 + (\varepsilon_2 - 1)p_2,$$
$$\mathcal{F}_{10}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1 + s_2,\;\; \mathcal{F}_{11}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = s_1 - (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2.$$

Without loss of generality, see Figure 3.3.8 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (1,-1)$. We first kill an acyclic subcomplex $\mathcal{R}_1$ composed of $C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}$ with $s_1 > 0$. Indeed, the associated graded complex of $\mathcal{F}_{00}$ on $\mathcal{R}_1$ splits as a direct product of acyclic squares, since the inclusion map $I^{+L_1}_{s_1,s_2}, s_1 > 0$ is id.

We have another acyclic subcomplex

$$\mathcal{R}_2 = \prod_{s_1-(\varepsilon_1-1)p_1\leq 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

In fact, since $\Phi^{-L_1}_{s_1,s_2}$ are quasi-isomorphisms when $s_1 < 0$, the associated graded of the filtration $\mathcal{F}_{10}$ for $\mathcal{R}_2$ splits as a direct product of acyclic squares $R_{s,1,0}$. Thus $\mathcal{R}_2$ is acyclic.

Thus, $\mathcal{C}$ is quasi-isomorphic to the quotient complex

$$\mathcal{C}_1 = \mathcal{C}/(\mathcal{R}_1 + \mathcal{R}_2) = \prod_{-p_1 < s_1 \leq 0} \Big(C^{(0,0)}_{(s_1,s_2)} \oplus C^{(0,1)}_{(s_1,s_2)} \Big).$$

We have an acyclic quotient complex $\mathcal{R}_3$ of $\mathcal{C}_1$

$$\mathcal{R}_3 = \prod_{-p_1 < s_1 \leq 0, s_2 > 0} \Big(C^{(0,0)}_{(s_1,s_2)} \oplus C^{(0,1)}_{(s_1,s_2)} \Big),$$

since the inclusion maps $I^{+L_2}_{s_1,s_2}, s_2 > 0$ are all the identities. Furthermore, we have another acyclic quotient complex $\mathcal{R}_4$ of $\mathcal{C}_1$

$$\mathcal{R}_4 = \prod_{-p_1 < s_1 \leq 0, s_2 < 0} \Big(C^{(0,0)}_{(s_1,s_2)} \oplus C^{(0,1)}_{(s_1,s_2-p_2)} \Big).$$

Thus $\mathcal{C}$ is quasi-isomorphic to

$$\mathcal{Q} = \mathcal{C}_1 \backslash (\mathcal{R}_3 \cup \mathcal{R}_4) = \{ \bigoplus_{-p_1 < s_1 \leq 0} \Big(C^{(0,0)}_{(s_1,0)} \oplus C^{(0,1)}_{(s_1,0)} \oplus C^{(0,1)}_{(s_1,p_2)} \Big) \} \oplus \{ \bigoplus_{-p_1 < s_1 \leq 0, p_2 < s_2 < 0} C^{(0,1)}_{(s_1,s_2)} \}.$$

In the Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$, we have the complex as follows,

$$\begin{tikzcd}
C^{(0,0)}_{(s_1,0)} = A^-_{s_1,0} \arrow[r, "\Phi^{+L_2}_{s_1,0}"] \arrow[dr, "\Phi^{-L_2}_{s_1,0}"'] & A^-_{s_1,+\infty} = C^{(0,1)}_{(s_1,0)} \\
 & A^-_{s_1,+\infty} = C^{(0,1)}_{(s_1,p_2)},
\end{tikzcd}$$

where s_1 is an integer such that $-p_1 < s_1 \leq 0$ and $s_1 \equiv t_1 (\text{mod } p_1)$. Since the inclusion maps $I^{\pm L_2}_{0,0}$ induce the same action on homology, $\Phi^{\pm L_2}_{0,0}$ are chain homotopic to each other. By Corollary 3.2.6, we can replace $A^-_{0,0}, A^-_{0,+\infty}$ by the complex $\mathbb{F}[[U_1, U_2]] \xrightarrow{U_1 - U_2} \mathbb{F}[[U_1, U_2]]$, where the generators are g_1, g_2. Then, we can replace the chain maps $I^{\pm L_2}_{0,0}$ by the same chain map $\tilde{I}$, where $\tilde{I}(g_i) = U_1 g_i$. Thus, the homology of the $(0,0)$ Spin^c structure can be computed by this perturbed complex, which is $\mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U$. From above computation, the generator corresponding to $\mathbb{F}[[U]]$ is actually the generator of $H_*(C^{(0,1)}_{(0,0)})$, which is $d_1 \in$

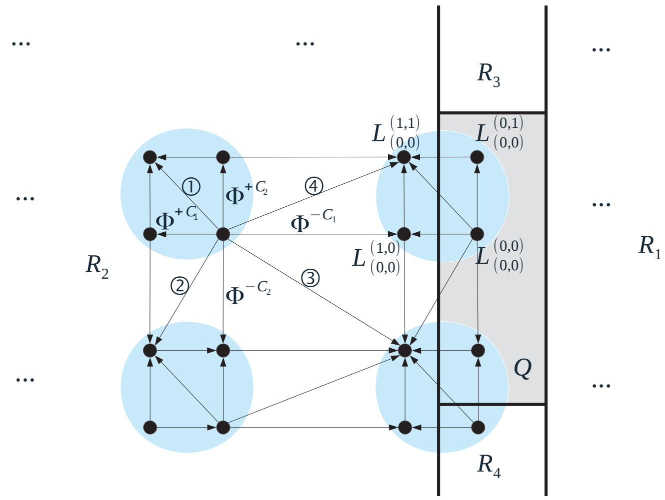


Figure 3.3.8: **The surgery complex for** $\Lambda = (1, -1)$**.** The arrows with circled numbers $1, 2, 3, 4$ are $\Phi^{+L_1 \cup +L_2}$, $\Phi^{+L_1 \cup -L_2}$,$\Phi^{-L_1 \cup -L_2}$, and $\Phi^{-L_1 \cup +L_2}$ respectively. The regions R_1, R_2, R_3, R_4 divided by the (thicker) lines are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$. The shaded region Q corresponds to the truncated complex $\mathcal{Q}$.

$A^-_{s_1,+\infty}$ with grading $\mu^{0,1}_{0,0}(d_1) = \frac{p_1+p_2}{4}$ by Equation (3.3.3).

On the other hand, since the inclusion maps $I^{-L_2}_{s_1,0}, s_1 < 0$ are all quasi-isomorphisms, we can kill the acyclic quotient complex $A^-_{s_1,0} \xrightarrow{\Phi^{-L_2}} A^-_{s_1,+\infty}$. Thus, the homology for the Spin^c structure $(t_1, 0) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ with $t_1 \neq 0$ is $\mathbb{F}[[U]]$ generated by $d_1 \in A^-_{s_1,+\infty}$ of grading $\mu^{0,1}_{s_1,0}(d_1) = \frac{s_1^2}{p_1} + s_1 + \frac{p_1+p_2}{4}$, where s_1 is an integer with $-p_1 < s_1 < 0$ in the class t_1 modulo p_1.

In every other Spin^c structure in the complex $\mathcal{Q}$, there is only one complex $C^{(0,1)}_{(s_1,s_2)} = A^-_{s_1,+\infty}, -p_1 < s_1 \leq 0$ of homology $\mathbb{F}[[U]]$ of grading $\frac{s_1^2}{p_1} + \frac{s_2^2}{p_2} + s_1 - s_2 + \frac{p_1+p_2}{4}$.

(3) The last case is when p_1, p_2 are both negative integers. We use two filtrations

$$\mathcal{F}_{00}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = s_1 + s_2, \quad \mathcal{F}_{11}(C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}) = -s_1 + (\varepsilon_1 - 1)p_1 - s_2 + (\varepsilon_2 - 1)p_2.$$

Without loss of generality, see Figure 3.3.9 for the illustration of the surgery complex and the truncation in the case of $\Lambda = (-1, -1)$.

We first kill an acyclic quotient complex

$$\mathcal{R}_1 = \prod_{\max\{s_1,s_2\}>0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

By considering the filtration $\mathcal{F}_{00}$, we can see that $\mathcal{R}_1$ is acyclic. We also have another acyclic quotient complex

$$\mathcal{R}_2 = \prod_{\min\{s_1-\varepsilon_1 p_1, s_2-\varepsilon_2 p_2\}<0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

In fact, the inclusion maps $I^{-L_i}_{s_1,s_2}, s_i < 0$ are quasi-isomorphisms. Thus the associated graded complex of the filtration $\mathcal{F}_{11}$ splits as a direct product of acyclic complexes

$$R_{\mathrm{s},1,1} \cap (\mathcal{C} \backslash \mathcal{R}_1)$$

where $\min\{s_1, s_2\} < 0$ and $R_{\mathrm{s},1,1}$ is in Equation (2.2.5). Therefore $\mathcal{R}_2$ is acyclic.

Hence, the subcomplex $\mathcal{Q} = \mathcal{C}\backslash(\mathcal{R}_1 \cup \mathcal{R}_2)$ is quasi-isomorphic to $\mathcal{C}$, where

$$\mathcal{Q} = \bigoplus_{\max\{s_1,s_2\}\le 0, \min\{s_1-\varepsilon_1 p_1, s_2-\varepsilon_2 p_2\}\ge 0} C^{(\varepsilon_1,\varepsilon_2)}_{(s_1,s_2)}.$$

In the Spin^c structure (t_1, t_2), $t_1 \neq 0, t_2 \neq 0$, there is only one complex $C^{(1,1)}_{(s_1,s_2)}$ in $\mathcal{Q}$, thus having the homology $\mathbb{F}[[U]]$ with grading $\frac{(2s_1-p_1)^2}{4p_1} + \frac{(2s_2-p_2)^2}{4p_2} + \frac{1}{2}$, where s_1, s_2 are negative integers in the residue classes t_1, t_2 such that $s_i \ge p_i + 1, i = 1, 2$. In the Spin^c structure $(0, t_2), t_2 \neq 0$, there are three complexes $C^{(0,1)}_{(0,s_2)}, C^{(1,1)}_{(0,s_2)}, C^{(1,1)}_{(p_1,s_2)}$ in $\mathcal{Q}$, where s_2 is an integer in the residue class t_2 such that $p_2 < s_2 < 0$. Since the inclusion map $I^{\pm L_1}_{0,s_2}, s_2 \neq 0$ are quasi-isomorphisms, we can replace $\Phi^{+L_1}_{0,s_2}$ by $\Phi^{-L_1}_{0,s_2}$ in the perturbed complex and thus split it as a direct sum, $\mathrm{cone}(\Phi^{+L_1}_{0,s_2}) \oplus C^{(1,1)}_{(0,s_2)}$, by changing the basis. Thus the homology is the same as the homology of $C^{(1,1)}_{(0,s_2)} = A^{-}_{+\infty,+\infty}$, which is $\mathbb{F}[[U]]$ generated by $[d_1]$ with grading $\frac{p_1}{4} + \frac{(2s_2-p_2)^2}{4p_2} + \frac{1}{2}$. It is similar for $(t_1, 0) \in \mathrm{Spin}^c(Y), t_1 \neq 0$.

The most interesting Spin^c structure is $(0, 0)$. It consists of nine complexes, which are also illustrated in Figure 3.3.9. By Corollary 3.2.6 and the discussion in Section 3.2.6, in

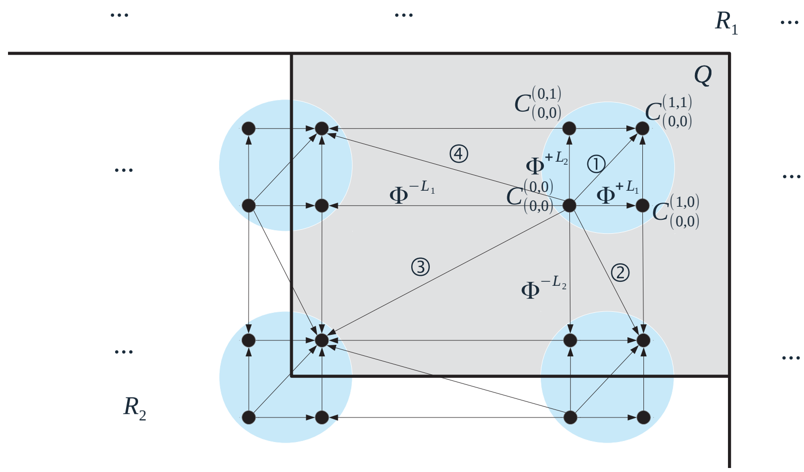


Figure 3.3.9: **The surgery complex for** $\Lambda = (-1, -1)$**.** The arrows with circled numbers $1, 2, 3, 4$ are $\Phi^{+L_1\cup+L_2}$, $\Phi^{+L_1\cup-L_2}$,$\Phi^{-L_1\cup-L_2}$, and $\Phi^{-L_1\cup+L_2}$ respectively. The regions R_1, R_2 divided by the (thicker) lines are corresponded to the acyclic subcomplexes $\mathcal{R}_1, \mathcal{R}_2$. The shaded region Q corresponds to the truncated complex $\mathcal{Q}$.

the perturbed complex we can replace all the A^-_s by the complex $\mathbb{F}[[U]]$ with 0 differential and replace the edge maps by the corresponding maps on homology. Finally, the perturbed complex is the following chain complex

$$\begin{tikzcd}
\mathbb{F}[[U]] & \mathbb{F}[[U]] \arrow[l, "1"'] \arrow[r, "1"] & \mathbb{F}[[U]] \\
\mathbb{F}[[U]] \arrow[u, "1"'] \arrow[d, "1"] & \mathbb{F}[[U]] \arrow[l, "U"'] \arrow[r, "U"] \arrow[u, "U"] \arrow[d, "U"'] & \mathbb{F}[[U]] \arrow[u, "1"] \arrow[d, "1"'] \\
\mathbb{F}[[U]] & \mathbb{F}[[U]] \arrow[l, "1"'] \arrow[r, "1"] & \mathbb{F}[[U]].
\end{tikzcd}$$

Direct computation shows that

$$\mathbf{HF}^-(S^3_\Lambda(Wh), (0,0)) = \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U),$$

when $\Lambda = \mathrm{diag}(p_1, p_2)$ with $p_1, p_2 < 0$. Thereby, $[d_1] = 1 \in H_*(C^{(1,1)}_{(p_1,p_2)})$ is a generator of the $\mathbb{F}[[U]]$ summand with the absolute grading $\frac{p_1+p_2+2}{4}$. $\square$

Theorem 3.3.10. *Let $\overrightarrow{L}$ be the two-bridge link $b(8k, 4k+1), k \in \mathbb{N}$ and $\Lambda = diag(p_1, p_2), p_1, p_2 \in \mathbb{Z}$ be the framing matrix of an integer surgery on $\overrightarrow{L}$. As in Proposition 3.3.9, we use*

$(t_1, t_2) \in \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}$ *to denote the* Spin^c *structures over* $S^3_\Lambda(\overrightarrow{L})$*. Then, we have the Floer homology*

$$\boldsymbol{HF}^-(S^3_\Lambda(\overrightarrow{L}), (t_1, t_2)) = \begin{cases} \boldsymbol{HF}^-(S^3_\Lambda(Wh), (0,0)) \oplus \mathbb{F}^{\oplus(k-1)}, & (t_1, t_2) = (0,0), \\ \boldsymbol{HF}^-(S^3_\Lambda(Wh), (t_1, t_2)), & \textit{otherwise.} \end{cases} \tag{3.3.4}$$

The correction terms of the elements in the $\boldsymbol{HF}^-(S^3_\Lambda(Wh))$*-summand are the same as in* $\boldsymbol{HF}^-(S^3_\Lambda(Wh))$.

Proof. By Proposition 3.3.5, $CFL^-(\overrightarrow{L}) = CFL^-(Wh) \oplus \bigoplus_{i=1}^{k-1}(N, \partial^-)$. Let $\mathscr{N} = \bigoplus_{i=1}^{k-1}(N, \partial^-)$. We define $\mathscr{N}_{\mathrm{s}}$ similarly as A^-_{s} in (4.1.4). Concretely, suppose G be a set of homogeneous generators of $\mathscr{N}$ as a $\mathbb{F}[[U_1, U_2]]$-module, and for $x \in G$,

$$\partial x = \sum_{y \in G} k_{xy} y,$$

where $k_{xy} \in \mathbb{F}[[U_1, U_2]]$. Let $A(x) = (A_1(x), A_2(x))$ denote the Alexander filtration of $x \in G$. Define $\mathscr{N}_{\mathrm{s}}$ by

$$\partial x = \sum_{y \in G} k_{xy} \cdot U_1^{\max\{A_1(x)-s_1,0\}-\max\{A_1(y)-s_1,0\}} U_2^{\max\{A_2(x)-s_2,0\}-\max\{A_2(y)-s_2,0\}} \cdot y.$$

Thus $A^-_{\mathrm{s}}(\overrightarrow{L}) = A^-_{\mathrm{s}}(Wh) \oplus \mathscr{N}_{\mathrm{s}}$. Thus all the inclusion maps $I^{\pm L_i}, i = 1, 2$ preserve this direct sum decomposition. Since the complexes $\mathscr{N}_{s_1, \pm\infty}$ are acyclic complexes, we can choose $\tilde{D}^{\pm L_2}_{s_1, -\infty} : A^-_{s_1, \pm\infty}(\overrightarrow{L}) \to A^-_{s_1, +\infty}(\overrightarrow{L})$ to be

$$\tilde{D}^{\pm L_2}_{s_1, -\infty} = D(Wh)^{\pm L_2}_{s_1, -\infty} \oplus 0,$$

where $D(Wh)^{\pm L_2}_{s_1, -\infty}$ is the destabilization map for Wh. Therefore $\tilde{\Phi}^{\pm L_i}_{\mathrm{s}} = \Phi(Wh)^{\pm L_i}_{\mathrm{s}} \oplus \Phi^{\pm L_i}_{\mathscr{N}, \mathrm{s}}$, where $\Phi^{\pm L_i}_{\mathscr{N}, \mathrm{s}} = 0 : \mathscr{N}_{\mathrm{s}} \to \mathscr{N}_{\psi^{\pm L_i}(\mathrm{s})}$.

Thus the perturbed surgery complex $(\tilde{\mathcal{C}}^-(\overrightarrow{L}, \Lambda), \tilde{\mathcal{D}}^-)$ is a direct sum of two twisted gluing of squares

$$(\tilde{\mathcal{C}}^-(\overrightarrow{L},\Lambda),\tilde{\mathcal{D}}^-) = (\mathcal{C}^-(Wh,\Lambda),\mathcal{D}^-) \oplus \prod_{\mathrm{s}=(s_1,s_2)\in\mathbb{Z}^2} (\mathscr{N}_{\mathrm{s}} \oplus \mathscr{N}_{s_1,+\infty} \oplus \mathscr{N}_{+\infty,s_2} \oplus \mathscr{N}_{+\infty,+\infty}, \tilde{\mathcal{D}}^-).$$

From the fact that any $\mathscr{N}_{\mathrm{s}}$ with $\mathrm{s} \neq (0,0)$ is acyclic, it follows that $H_*(\tilde{\mathcal{C}}^-(\overrightarrow{L}),\tilde{\mathcal{D}}^-) = H_*(\mathcal{C}^-(Wh),\mathcal{D}^-)\oplus H_*(\mathscr{N}_{0,0})$. For that $\mathscr{N}_{0,0}$ belongs to the $(0,0)$ Spin^c structure and $H_*(\mathscr{N}_{0,0}) = \mathbb{F}[[U]]/U$, we have the equations (3.3.4). The absolute gradings are inherited from $H_*(\mathcal{C}^-(Wh),\mathcal{D}^-)$.

$\square$

CHAPTER 4

L-space links

4.1 L-space links

In this section, we study the L-space surgeries on a link L and the large surgeries. Then, we introduce various notions of L-space links.

4.1.1 L-space links.

Let us recall the definition of generalized Floer complexes of a link L in S^3 in [24] Section 4, which govern the large surgeries on L. For simplicity, we only consider generic admissible multi-pointed Heegaard diagrams with each component L_i having only two basepoints w_i, z_i. Here, we allow free basepoints.

Definition 4.1.1 (Generalized Floer complexes)**.** Let L be a link in S^3 and choose a Heegaard diagram $\mathcal{H}$. For $\mathrm{s} \in \mathbb{H}(L)$, the *generalized Floer complex* $\mathfrak{A}^-(\mathcal{H}, \mathrm{s})$ is the free module over $\mathcal{R} = \mathbb{F}[[U_1, ..., U_l]]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \in \mathrm{Sym}^{g+k-1}(\Sigma)$, and equipped with the differential:

$$\partial^-_{\mathbf{s}} \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}(\alpha) \cap \mathbb{T}(\beta)} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{E^1_{s_1}(\phi)} \cdots U_l^{E^l_{s_l}(\phi)} \cdot U_{l+1}^{n_{w_{l+1}}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \cdot \mathbf{y}, \tag{4.1.1}$$

where $E_s^i(\phi)$ is defined by

$$E_s^i(\phi) = \max\{s - A_i(\mathbf{x}), 0\} - \max\{s - A_i(\mathbf{y}), 0\} + n_{z_i}(\phi) \tag{4.1.2}$$

$$= \max\{A_i(\mathbf{x}) - s, 0\} - \max\{A_i(\mathbf{y}) - s, 0\} + n_{w_i}(\phi). \tag{4.1.3}$$

Here, $\mathcal{M}(\phi)$ denotes the moduli space of the pseudo-holomorphic disk ϕ, and $A_i(\mathbf{x})$ denotes the ith Alexander grading of the intersection point $\mathbf{x}$. The stable quasi-isomorphism type of $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ is an invariant of L. For simplicity, we also write $\mathfrak{A}^-(L, \mathbf{s})$, $\mathfrak{A}^-_{\mathbf{s}}(L)$, or $\mathfrak{A}^-_{\mathbf{s}}$, when the context is clear.

Notation 4.1.2. *Let L be an l-component link in S^3. In order to simplify the notation, we denote the $(p_1, ..., p_l)$-surgery on L by $S^3_{p_1,...,p_l}(L)$ and the surgery framing matrix by $\Lambda(p_1, ..., p_l)$, where $p_1, ..., p_l$ are surgery coefficients on the link; i.e. $\Lambda(p_1, ..., p_l)$ is the matrix with $p_1, ..., p_l$ on the diagonal and linking numbers off the diagonal.*

Proof of Lemma 1.3.10. First, let us recall Theorem 10.1 in [24].

Theorem. *For $\tilde{\Lambda}$ sufficiently large, there exist quasi-isomorphisms of relative $\mathbb{Z}$-graded complexes*

$$\boldsymbol{CF}^-(S^3_{\Lambda}(L), \boldsymbol{s}) \to \mathfrak{A}^-_{\boldsymbol{s}}(L)$$

for all $\boldsymbol{s}$.

Thus, L is an L-space link if and only if $\mathfrak{A}^-_{\mathbf{s}}(L)$ has the homology $\mathbb{F}[[U]]$ for all $\mathbf{s} \in \mathbb{H}(L)$. When the ith component of $\mathbf{s}$, say s_i, equals to $+\infty$, there is a destabilization map between $\mathfrak{A}^-(L, \mathbf{s})$ and $\mathfrak{A}^-(L - L_i, \psi^{+L_i}(\mathbf{s}))$, which is a quasi-isomorphism. See Example 7.2 in [24]. Roughly, this is because the generalized Floer complexes of $L - L_i$ can be computed from the Heegaard diagram of L by deleting the basepoint z_i, which is the same as putting $s_i = +\infty$ in $\mathfrak{A}^-(L, \mathbf{s})$. Thus, $\mathfrak{A}^-(L - L_i, \mathbf{s}')$ has homology $\mathbb{F}[[U]]$ for all $\mathbf{s}' \in \mathbb{H}(L - L_i)$. So $L - L_i$ is an L-space link for $L_i \subset L$. An induction will show that all sublinks are L-space links. $\square$

In contrast to knots, a weak L-space link L might be a non-L-space link.

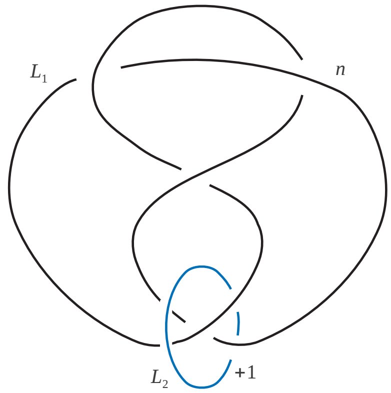


Figure 4.1.1: **An example of weak L-space link.**

Example 4.1.3. Let $L = L_1 \cup L_2$ be the link consisting of a Figure-8 knot L_1 and an unknot L_2 as in Figure 4.1.1. Then by blowing down the unknot, the Figure-8 knot is then unknotted, and thus the surgery $S^3_{n,1}(L)$ is the lens space $L(n-4,1)$, when $n \neq 4$. However, the Figure-8 knot is not an L-space knot. Thus, by Lemma 1.3.10, L is a weak L-space link but not an L-space link. Similarly, the mirror of L is not a L-space link neither.

4.1.2 L-space induction and generalized large surgeries.

In this subsection, we study how to characterize L-space links, by exploiting induction in light of surgery exact triangles.

Lemma 4.1.4 (L-space surgery induction)**.** *Let $L = L_1 \cup \ldots \cup L_n$ be a link with n components, and $L' = L - L_1$. Let Λ be the framing matrix of L for the surgery $S^3_{p_1,\ldots,p_n}(L)$, and denote by Λ' the restriction of Λ on L'. Suppose $S^3_{p_1,\ldots,p_n}(L)$ and $S^3_{p_2,\ldots,p_n}(L')$ are both L-spaces. Then,*

Case I *if $\det(\Lambda) \cdot \det(\Lambda') > 0$, then for all $k > 0$, $S^3_{p_1+k,p_2,\ldots,p_n}(L)$ is an L-space;*

Case II *if $\det(\Lambda) \cdot \det(\Lambda') < 0$, then for all $k > 0$, $S^3_{p_1-k,p_2,\ldots,p_n}(L)$ is an L-space.*

Proof. Let Λ_k be the framing matrix of the surgery $S^3_{p_1+k,p_2,\ldots,p_n}(L)$. Notice that $\det(\Lambda_k) = \det(\Lambda) + k\det(\Lambda')$.

For the case $\det(\Lambda) \cdot \det(\Lambda') > 0$, consider the following exact triangle of surgeries:

$$\begin{tikzcd}
\widehat{HF}(S^3_{p_1,p_2,...,p_n}(L)) \arrow[rr] & & \widehat{HF}(S^3_{p_1+1,p_2,...,p_n}(L)) \arrow[dl] \\
& \widehat{HF}(S^3_{p_2,...,p_n}(L')). \arrow[ul] &
\end{tikzcd}$$

Thus, from that $\det(\Lambda_1) = \det(\Lambda) + \det(\Lambda')$, it follows that $S^3_{p_1+1,p_2,...,p_n}(L)$ is also an L-space. Iterating this argument for all $k > 0$, we can obtain that $S^3_{p_1+k,p_2,...,p_n}(L)$ is an L-space for all $k \geq 0$. The case where $\det(\Lambda) \cdot \det(\Lambda') < 0$ is similar. $\square$

Lemma 4.1.5 (Positive L-space surgery criterion). *An l-component link L is an L-space link if and only if there exists a surgery framing $\Lambda(p_1, ..., p_l)$, such that for all sublink $L' \subseteq L$, $\det(\Lambda(p_1, ..., p_l)|_{L'}) > 0$ and $S^3_{\Lambda|_{L'}}(L')$ is an L-space.*

In particular, if the surgery framing $\Lambda(p_1, ..., p_l)$ satisfies the above condition, then for any surgery framing $\Lambda' = \Lambda(n_1, ..., n_l)$ with $n_i \geq p_i$ for all i, the surgery $S^3_{\Lambda'}(L)$ is an L-space.

Proof. If L is an L-space link, then every sublink L' is an L-space link, by Lemma 1.3.10. Thus, there is a large $(p_1, ..., p_l)$-surgery on L such that for all $L' \subseteq L$, $\det(\Lambda(p_1, ..., p_l)) > 0$ and $S^3_{\Lambda|_{L'}}(L')$ is an L-space.

Conversely, let $\Lambda(p_1, ..., p_l)$ be the surgery framing satisfying the condition in the proposition. Let $\Lambda' = \Lambda(p_1, ..., p_i + 1, ..., p_l)$. By the L-space surgery induction lemma, we have that for all $L' \subseteq L$, $S^3_{\Lambda'|_{L'}}(L')$ is an L-space and $\det(\Lambda'|_{L'}) = \det(\Lambda|_{L'}) + \varepsilon \det(\Lambda|_{L'-L_i})$, where $\varepsilon = 1$ if $L_i \subset L'$ and $\varepsilon = 0$ if $L_i \not\subseteq L'$. Thus, by induction, we can show that for any surgery framing $\Lambda'' = \Lambda(n_1, ..., n_l)$ with $n_i \geq p_i$, the surgery $S^3_{\Lambda''|_{L'}}(L')$ is an L-space for all sublinks $L' \subset L$. In particular, $S^3_{\Lambda''}(L)$ is an L-space, and this finishes the proof. $\square$

Definition 4.1.6. A link is called *algebraically split*, if all the pairwise linking numbers are 0.

Corollary 4.1.7. *Let $L = L_1 \cup ... \cup L_l$ be an algebraically split link. Then L is an L-space link if and only if $\exists\ p_i > 0, i = 1, ..., l$, such that $S^3_{\Lambda|_{L'}}(L')$ is an L-space for all $L' \subseteq L$, where $\Lambda = \Lambda(p_1, ..., p_l)$.*

Now we can prove Proposition 1.3.11.

Proof of Proposition 1.3.11. Lemma 4.1.5 implies that condition (i) and (ii) are equivalent. By Theorem 10.1 from [24], it follows L is an L-space link if and only if $\mathfrak{A}^-_{\mathbf{s}}(L)$ has homology $\mathbb{F}[[U]]$ for all $\mathbf{s} \in \mathbb{F}[[U]]$. Thus, condition (i) is equivalent to (iii) as well as (iv). $\square$

4.1.3 Generalized L-space links.

We can enlarge our scope to generalized large surgeries on a link L. Let us use $\pm$ signs to denote the type of the generalized large surgeries.

Definition 4.1.8 (Generalized L-space links)**.** A 2-component link $L = L_1 \cup L_2$ is called a *generalized* $(\pm\pm)$*L-space link*, if there exist integers p_1, p_2, such that for all positive integers $k_1, k_2 > 0$, $S^3_{p_1 \pm k_1, p_2 \pm k_2}(L)$ is an L-space. Similarly, we define an l-component *generalized* $(\pm \cdots \pm)$*L-space link.*

Example 4.1.9. The split disjoint union of the left-handed trefoil and the right-handed trefoil is a generalized $(+-)L$-space link. However, it is not an L-space link, and neither is its mirror.

Let us look at some examples of 2-component generalized L-space links.

Proposition 4.1.10. *Suppose L is a 2-component link $L = L_1 \cup L_2$ with L_1, L_2 both being the unknots, and $S^3_{p_1,p_2}(L)$ is an L-space. Then,*

1. *if $p_1 p_2 > \mathrm{lk}^2, p_1 > 0, p_2 > 0$, then $S^3_{p_1+k_1,p_2+k_2}(L)$ are L-spaces for all $k_1, k_2 \in \mathbb{N}$;*
2. *if $p_1 p_2 > \mathrm{lk}^2, p_1 < 0, p_2 < 0$, then $S^3_{p_1-k_1,p_2-k_2}(L)$ are L-spaces for all $k_1, k_2 \in \mathbb{N}$;*
3. *if $p_1 > 0, p_2 < 0$, then $S^3_{p_1+k_1,p_2-k_2}(L)$ are L-spaces for all $k_1, k_2 \in \mathbb{N}$;*
4. *if $p_1 p_2 < \mathrm{lk}^2, p_1 > 0, p_2 > 0$, then the surgeries $S^3_{p_1+k_1,-1-k_2}(L), S^3_{-1-k_1,p_2+k_2}(L)$ with $k_1 \geq 0, k_2 \geq 0$ and $S^3_{p'_1,p'_2}(L)$ with $0 < p'_1 \leq p_1, 0 < p'_2 \leq p_2$ are all L-spaces;*

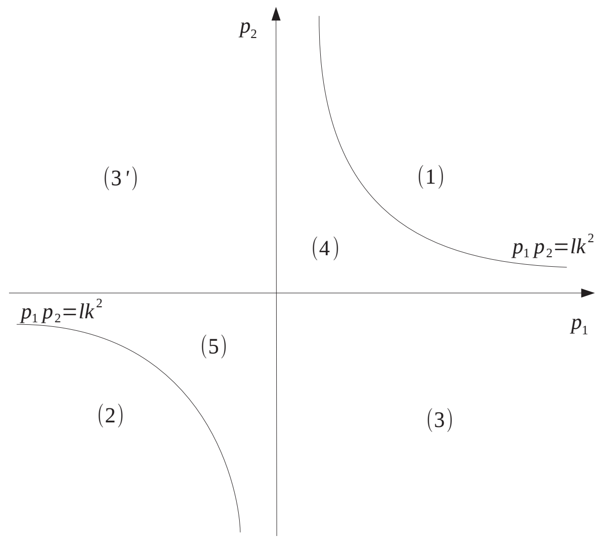


Figure 4.1.2: We illustrate the cases of the (p_1, p_2)-surgeries in Proposition 4.1.10 on the (p_1, p_2) plane, where the case $(3')$ is similar to case (3).

5. *if* $p_1p_2 < \mathrm{lk}^2, p_1 < 0, p_2 < 0$, *then the surgeries* $S^3_{p_1-k_1,k_2}(L), S^3_{k_1,p_2-k_2}(L)$ *with* $k_1 > 0, k_2 > 0$ *and* $S^3_{p_1',p_2'}(L)$ *with* $0 > p_1' \geq p_1, 0 > p_2' \geq p_2$ *are all L-spaces.*

The above cases are shown in Figure 4.1.2.

Proof. The cases (1), (2), and (3) are proved by induction using the long exact sequences for the surgery triple $\left(S^3_{p,q}(L), S^3_{p+1,q}(L), S^3_q(L_2)\right)$.

For the case (4), first by Lemma 4.1.4, we have that $S^3_{p_1,-1}(L), S^3_{-1,p_2}(L)$ are both L-space spaces. From (3), it follows that $S^3_{p_1+k_1,-1-k_2}(L), S^3_{-1-k_1,p_2+k_2}(L)$ are all L-spaces for all non-negative integers k_1, k_2. Second, we can do induction to prove that $S^3_{p_1',p_2'}(L)$ with $0 < p_1' \leq p_1, 0 < p_2' \leq p_2$ are all L-spaces. The case (5) is similar to the case (4). $\square$

Proposition 4.1.10 says that if L is a 2-component link with unknotted components, then L is a weak L-space link if and only if L is a generalized L-space link. The following proposition gives another example of generalized L-space links.

Proposition 4.1.11. *Let L be an algebraically split link. If there exists a surgery framing $\Lambda(p_1, ..., p_l)$ on L, such that for all sublink $L' \subseteq L$, $S^3_{\Lambda|_{L'}}(L')$ is an L-space, then L is a generalized L-space link of "$\epsilon_1 \cdots \epsilon_l$"-type, where ϵ_i is the sign of p_i.*

4.1.4 Subcomplexes of $CFL^{\infty}(L)$ governing generalized large surgeries.

In this section, we demonstrate that parallel theory of $\mathfrak{A}^-_{\mathbf{s}}$ can be done by considering generalized large surgeries of different types. Here, we illustrate the idea by using 2-component links.

The generalized Floer complexes $\mathfrak{A}^-_{\mathbf{s}}(L)$ governs the positive large surgeries on L. In fact, there are also subcomplexes of $CFL^{\infty}(L)$ governing the other types of large surgeries on L respectively.

For any basic Heegaard diagram of an l-component link L, there is an Alexander grading on the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{H}(L),$$

which is characterized by

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = n_{z_i}(\phi) - n_{w_i}(\phi), \quad \forall \phi \in \pi_2(\mathbf{x}, \mathbf{y})$$

and a normalization condition of the Alexander polynomial.

Definition 4.1.12 (CFL^{∞})**.** Let L be an l-component link and $\mathcal{H}$ be a basic Heegaard diagram of L. Then, $CFL^{\infty}(\mathcal{H})$ is a chain complex of $\mathbb{F}[[U_1, ..., U_l, U_1^{-1}, ..., U_l^{-1}]]$-modules freely generated by $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) | \mu(\phi) = 1\}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{n_{w_1}(\phi)} \cdots U_l^{n_{w_l}(\phi)} \cdot \mathbf{y},$$

where the U_i's lower the $\mathbb{Z}$-grading by 2. There is an Alexander filtration on $CFL^{\infty}(\mathcal{H})$ extended from $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ by $A_i(U_j^k \cdot \mathbf{x}) = A_i(\mathbf{x}) - \delta_{ij} k$. The filtered chain homotopy type of the filtered complex $(CFL^{\infty}(\mathcal{H}), A)$ is an invariant of L, where A denotes the Alexander filtration. We denote this filtered homotopy type by $CFL^{\infty}(L)$. By abuse of notation, we also use $CFL^{\infty}(L)$ to denote a chain complex in this class.

To obtain the subcomplexes of $CFL^{\infty}(L)$ governing the large surgeries, let us first recall some facts of knots. Let K be a knot in S^3. Following [37], we can also regard $CFL^{\infty}(K)$ as a chain complex C of $\mathbb{F}$ vector spaces generated by triples

$$[\mathbf{x}, i, j], \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i, j \in \mathbb{Z}, \text{ with } A(x) = j - i.$$

The triple $[\mathbf{x}, i, j]$ is corresponding to $U^{-i}\mathbf{x}$. Then, Heegaard Floer homology of the positive large surgeries $HF^-(S^3_p(K))$ with $p \gg 0$ can be computed from the subcomplexes $C\{\max(i, j-m) \leq s\}$'s, whereas, Heegaard Floer homology of the negative large surgeries $S^3_p(K)$ with $p \ll 0$ can be computed from the subcomplexes $C\{\min(i, j-m) \leq s\}$'s. See Theorem 4.1 and Theorem 4.4 in [37] or Section 2.2 in [34]. Thus, $C\{\min(i, j-s) \leq 0\}$ is corresponded to the subcomplex of $CFL^{\infty}(K)$:

$$^{-}\mathfrak{A}^-_s(K) = \{U^k\mathbf{x} | \min(-k, A(\mathbf{x}) - k - s) \leq 0\}.$$

One can also formulate this complex more explicitly by using a similar approach in [24]. For simplicity, we consider basic Heegaard diagram without free basepoints, i.e. a doubly-pointed Heegaard diagram of a knot K.

Definition 4.1.13 ($^{-}\mathfrak{A}^-_s(K)$)**.** Let $\mathcal{H}$ be a doubly-pointed Heegaard diagram of K. For any $s \in \mathbb{Z}$, the complex $^{-}\mathfrak{A}^-(\mathcal{H}, s)$ is the free module over $\mathcal{R} = \mathbb{F}[[U]]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and equipped with the differential:

$$\partial^-_{\mathbf{s}}\mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}(\alpha) \cap \mathbb{T}(\beta)} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{\bar{E}_s(\phi)} \cdot \mathbf{y}, \tag{4.1.4}$$

where $\bar{E}_s(\phi)$ is defined by

$$\bar{E}_s(\phi) = \max(0, s - A(\mathbf{y})) - \max(0, s - A(\mathbf{x})) + n_w(\phi), \quad \forall \phi \in \pi_2(\mathbf{x}, \mathbf{y}).$$

The chain homotopy type of ${}^{-}\mathfrak{A}^{-}(\mathcal{H}, s)$ is an invariant of K and s, and we denote it by ${}^{-}\mathfrak{A}^{-}_{s}(K)$.

Now we can pass from knots to links.

Definition 4.1.14 (${}^{\pm\pm}\mathfrak{A}^{-}_{s_1,s_2}$)**.** Let L be a 2-component link, and $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a basic Heegaard diagram of L. Then we define the following subcomplexes of the Alexander filtered complex $CFL^{\infty}(L)$

- ${}^{++}\mathfrak{A}^{-}_{s_1,s_2} := \{U_1^{k_1}U_2^{k_2}\mathbf{x} \in CFL^{\infty}(L) : \mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}, \max(-k_i, A_i(\mathbf{x}) - k_i - s_i) \leq 0, \forall i = 1, 2\}$;
- ${}^{+-}\mathfrak{A}^{-}_{s_1,s_2} := \{U_1^{k_1}U_2^{k_2}\mathbf{x} \in CFL^{\infty}(L) : \mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}, \max(-k_1, A_1(\mathbf{x}) - k_1 - s_1) \leq 0, \min(-k_2,$
  $A_2(\mathbf{x}) - k_2 - s_2) \leq 0\}$;
- ${}^{-+}\mathfrak{A}^{-}_{s_1,s_2} := \{U_1^{k_1}U_2^{k_2}\mathbf{x} \in CFL^{\infty}(L) : \mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}, \min(-k_1, A_1(\mathbf{x}) - k_1 - s_1) \leq 0, \max(-k_2,$
  $A_2(\mathbf{x}) - k_2 - s_2) \leq 0\}$;
- ${}^{--}\mathfrak{A}^{-}_{s_1,s_2} := \{U_1^{k_1}U_2^{k_2}\mathbf{x} \in CFL^{\infty}(L) : \mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}, \min(-k_i, A_i(\mathbf{x}) - k_i - s_i) \leq 0, \forall i = 1, 2\}$.

Thus, ${}^{++}\mathfrak{A}^{-}_{\mathbf{s}}(L)$ is isomorphic to $\mathfrak{A}^{-}_{\mathbf{s}}(L)$ defined in [24]. One can also formulate these complexes by using $E^i_s(\phi)$'s and $\bar{E}^i_s(\phi)$'s. In fact, these four sets of complexes are equivalent data of L. The advantage of considering $\mathfrak{A}^{-}_{\mathbf{s}}(L)$'s is that they can be identified with the subcomplexes of $CFL^{-}(L)$ which form the Alexander filtration of $CFL^{-}(L)$.

Remark 4.1.15. The original $CFK^{\infty}(K)$ defined in [37] is slightly different from the formulation of $CFL^{\infty}(K)$ here. We adopt the formulation of $CFK^{\infty}(K)$ in [23], and it is the same as $CFL^{\infty}(K)$ here.

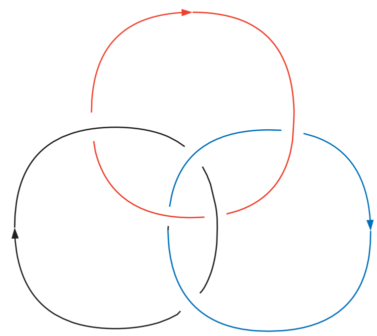


Figure 4.2.1: **The Borromean ring.** The $(1,1,1)$-surgery on the Borromean link is the Poincaré sphere.

4.2 Examples of L-space links and generalized L-space links

In this section, we use the lemmas and propositions in Section 4.1 to show some examples of L-space links and generalized L-space links.

Example 4.2.1 (Two hyperbolic links: the Whitehead link and the Borromean rings)**.** The Whitehead link and the Borromean rings are two well-known hyperbolic links. In fact, they are both L-space links.

The $(1,1)$-surgery on the Whitehead link is the Poincaré sphere. See Example 8 on Page 263 in [47]. The $(1,1,1)$-surgery on the Borromean rings is also the Poincaré sphere. See Exercise 4 on Page 269 in [47]. By Corollary 4.1.7, they are both L-space links.

Remark 4.2.2. There are no alternating hyperbolic L-space knots. See Theorem 4.3.3 below cited from [40]. However, Example 4.2.1 shows that there are L-space alternating hyperbolic links. In fact, there are many, see Theorem 4.2.8.

Moreover, these hyperbolic links provide many examples of hyperbolic L-spaces which are neither surgery over knots nor double branched cover over knots. For example, surgeries on the Whitehead link $S^3_{n,2n}(Wh)$ with $n>0$ are all L-spaces but not surgeries nor double branched cover on a knot. The reason is that the first homology of these surgeries is neither cyclic nor of odd order.

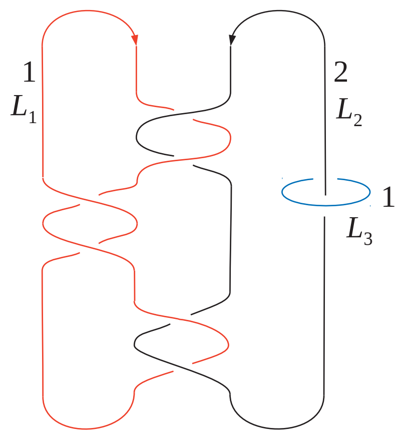


Figure 4.2.2: **An L-space link giving the Weeks manifold.**

Example 4.2.3 (An L-space link providing the Weeks manifold)**.** Consider the link $L = L_1 \cup L_2 \cup L_3$ in Figure 4.2.2, where $L_1 \cup L_2$ is the Whitehead link (using the convention in [47]) and L_3 is the meridian of L_2. The $(1, 2, 1)$-surgery is the Poincaré sphere, and it satisfies the positive L-space surgery criterion. Thus, it is an L-space link.

By Lemma 4.1.5, we have that for any $n_1 \geq 1, n_2 \geq 2, n_3 \geq 1$, the (n_1, n_2, n_3)-surgery on L is an L-space. Thus, the $(5, 3, 2)$-surgery is an L-space, which is the $(5, 5/2)$-surgery on the Whitehead link. This surgery is the Weeks manifold; see [4]. The Weeks manifold has the smallest hyperbolic volume among closed hyperbolic 3-manifolds; see [7]. Thus, we confirm that the Weeks manifold does not admit a taut foliation.

The fact that the Weeks manifold is an L-space was already known by experts such as [17] and [5].

Example 4.2.4 ($T(2, 2n)$ torus links)**.** The oriented torus links $T(2, 2n)$ are L-space links as Corollary 4.2.6 below shows. We need to distinguish them from their mirrors, so see Figure 4.2.3 for the precise definitions of $T(2, 2n)$.

Lemma 4.2.5. *For the torus links $T(2, 2n)$, we have the following identifications of surgeries*

$$S^3_{n+1,n-1}(T(2, 2n)) = S^3, \; S^3_{n+1,n+1}(T(2, 2n)) = L(2n + 1, 2), \; S^3_{n,n+1}(T(2, 2n)) = L(n, 1).$$

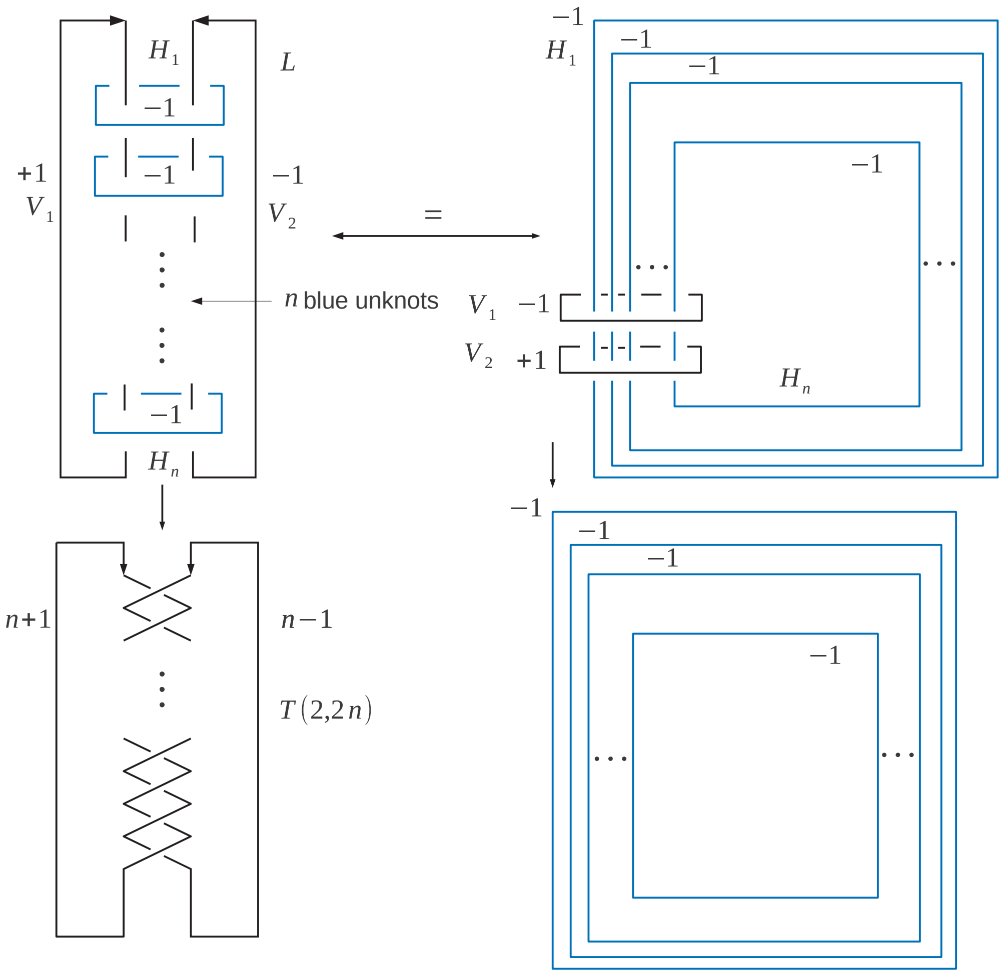


Figure 4.2.3: **The** $(n+1, n-1)$**-surgery on** $T(2, 2n)$**.** Consider the surgery on the upper-left link L, which is a plumbing of unknots. By blowing down the horizontal unknots H_i's, we get the surgery on the lower-left link $T(2, 2n)$. While blowing down the black unknots V_j's, we can get the surgery on the lower-right link, which is S^3.

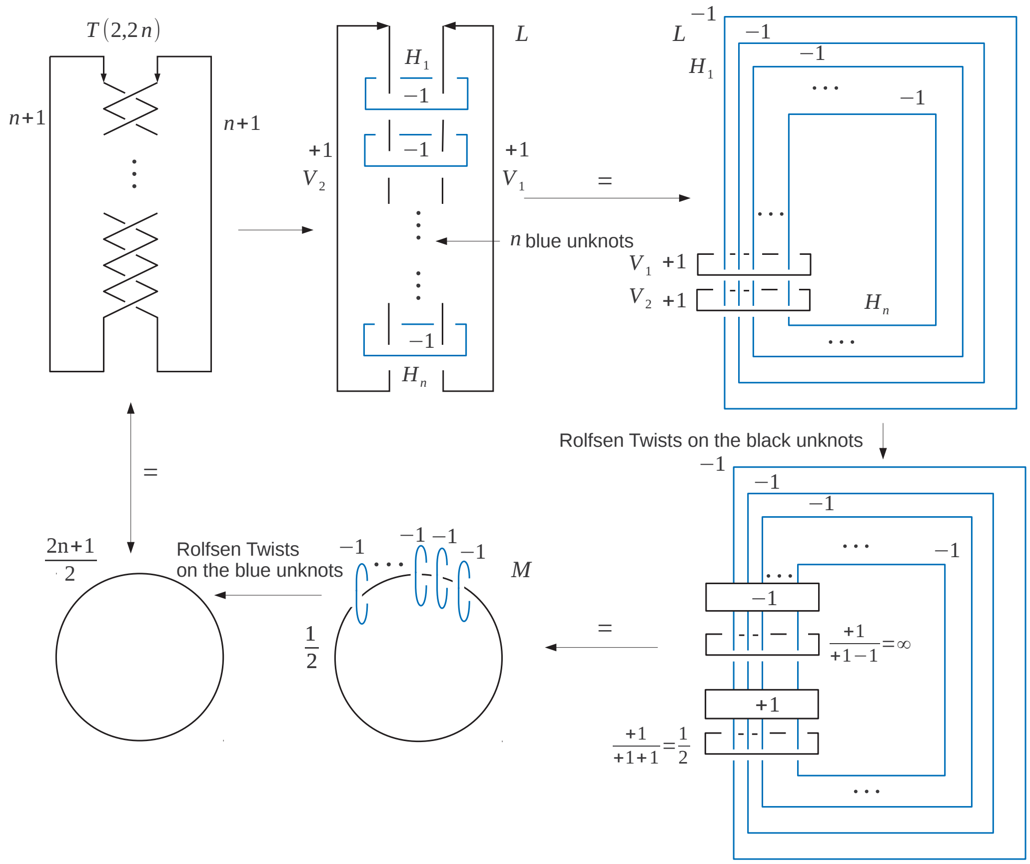


Figure 4.2.4: **The** $(n+1, n+1)$**-surgery on the** $T(2, 2n)$ **torus link.** Consider the surgery on upper-middle link L, which is a plumbing of unknots. After blowing down the horizontal (blue) unknots H_i's, we get the $(n+1, n+1)$-surgery on the upper-left link $T(2, 2n)$. While after doing Rolfsen twists on the black unknots V_j's, we can get a rational surgery on the lower-middle link M, which is a lens space by blowing-down the blue unknots using Rolfsen twists again.

Proof. First, for the $(n+1, n-1)$-surgery on $T(2, 2n)$, we consider a surgery on the upper-left link L in Figure 4.2.4, where L is a plumbing of unknots. After two different blowing-down procedures, we get the identification of $S^3_{n+1,n-1}(T(2, 2n))$ with S^3.

Second, for the $(n+1, n+1)$-surgery on $T(2, 2n)$, we similarly consider a different surgery on L, which is drawn in Figure 4.2.4. After two different processes of doing Rolfsen twists, we can obtain the identification of $S^3_{n+1,n+1}(T(2, 2n))$ with $L(2n+1, 2)$. See Figure 4.2.4. As is similar to the $(n + 1, n + 1)$-surgery, the $(n, n + 1)$-surgery is $L(n, 1)$. $\square$

Corollary 4.2.6. *The following surgeries on the torus link $T(2, 2n)$ are all L-spaces:*

- $S^3_{n+1+k_1,n+1+k_2}(T(2, 2n))$, $\forall k_1 \geq 0, \forall k_2 \geq 0$,
- $S^3_{n+1-k_1,n-1}(T(2, 2n))$, $\forall k_1 \geq 0$,
- $S^3_{-1-k_1,n-1+k_2}(T(2, 2n))$, $\forall k_1 \geq 0, \forall k_2 \geq 0$,
- $S^3_{n,q}(T(2, 2n))$ *with* $q \neq n$.

Proof. We combine Proposition 4.1.10 and Lemma 4.2.5.

From $S^3_{n+1,n+1}(T(2, 2n)) = L(2n + 1, 2)$, it follows that $S^3_{n+1+k_1,n+1+k_2}(T(2, 2n))$ are all L-spaces for $k_1, k_2 \geq 0$.

From $S^3_{n+1,n-1}(T(2, 2n)) = S^3$, it follows that $S^3_{n+1-k_1,n-1}(T(2, 2n))$ are all L-spaces by Lemma 4.1.4. Thus, $(-1, n-1)$-surgery is an L-space, and so is any $S^3_{-1-k_1,n-1+k_2}(T(2, 2n))$ with $k_1, k_2 \geq 0$.

From $S^3_{n+1,n-1}(T(2, 2n)) = S^3$, it follows that $(n, n - 1)$-surgery is an L-space and thus all (n, q)-surgeries with $q \leq n - 1$ are L-spaces.

From $S^3_{n,n+1}(T(2, 2n)) = L(n, 1)$, it follows that all (n, q)-surgery with $q \geq n + 1$ are L-spaces. $\square$

Example 4.2.7 (Algebraic links)**.** Gorsky and Némethi showed in [11] that every algebraic link is an L-space link. An algebraic knot can be obtained by iterated cabling of the unknot.

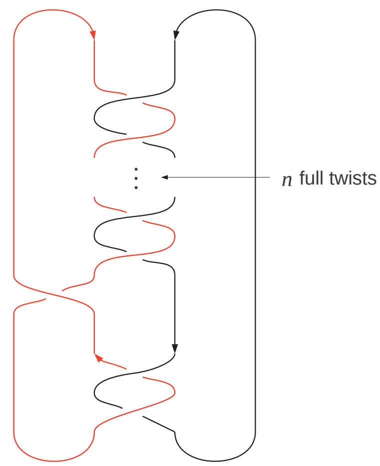


Figure 4.2.5: **The two-bridge link** $b(6n+2,-3)$**.**

In [14], Hedden proved that algebraic knots are L-space knots. Note that algebraic knots include all torus knots.

For hyperbolic L-space links, we have the following theorem by considering the two-bridge links.

Theorem 4.2.8. *For all positive odd integers r, q, the two-bridge link $b(rq-1,-q)$ is an L-space link.*

Before proving this theorem, let us clarify some conventions for two-bridge links. First, the notation $b(p,q)$ denotes an oriented two-bridge link of slope $\frac{q}{p}$. For any continued fraction of $\frac{q}{p}$:

$$\frac{q}{p} = [a_1, a_2, ..., a_m] = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}},$$

a 4-plat projection of $b(p,q)$ can be obtained in the following ways:

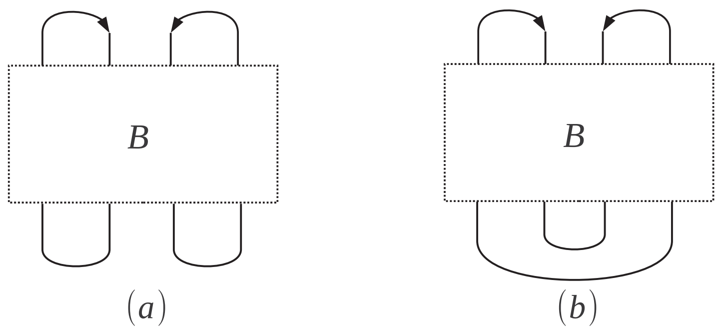


Figure 4.2.6: **The 4-plat presentations of two-bridge links.** For any continued fraction $[a_1, ..., a_m] = q/p$, there is a 4-plat projection of the two-bridge link $b(p, q)$. When m is odd, we use (a) to close the 4-braid B in the box; when m is even, we use (b) to close the 4-braid B.

Case I If m is odd, then the 4-plat is obtained by closing the 4-braid

$$B = \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_2^{a_m}$$

in the way shown in Figure 4.2.6(a).

Case II If m is even, then the 4-plat is obtained by closing the 4-braid

$$B = \sigma_2^{a_1}\sigma_1^{-a_2}\cdots\sigma_1^{-a_m}$$

in the way shown in Figure 4.2.6(b).

Here, we follow [3] Chapter 12B. We prescribe an orientation on $b(p, q)$ shown in Figure 4.2.6. Note that this orientation convention is different from [3].

Proof. Let $r = 2n + 1$ and $q = 2k + 1$. Let us do induction on k.

First, for $k = 1$, we need to show the family of two-bridge links $b(6n + 2, -3)$ drawn in Figure 4.2.5 are all L-space links. We claim that for any integer $n \geq 1$, the $(n + 2, n + 2)$-surgery on the two-bridge link $b(6n + 2, -3)$ is an L-space. Consider the 3-component link $L = L_1 \cup L_2 \cup L_3$ drawn in Figure 4.2.7. We see that $L_1 \cup L_2$ is the $T(2, 2n)$ torus link with the linking number n. Now consider the $(n+1, n+1, 1)$-surgery on L, $S^3_{n+1,n+1,1}(L)$. By blowing

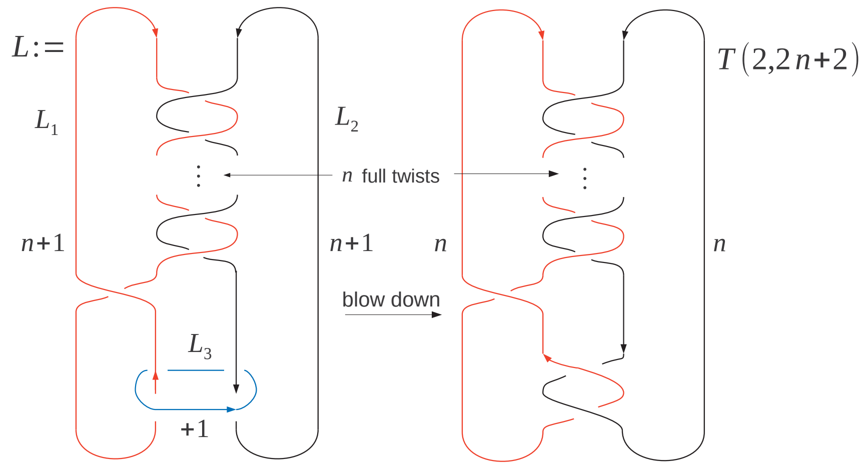


Figure 4.2.7: **A 3-component link used to study the surgeries on** $b(6n+2,-3)$**.** The left link L is used to study the surgeries on $b(6n+2,-3)$. After blowing down the (-1)-framed L_3, we can get the two-bridge link $b(6n+2,-3)$. While if we consider the $(n+1,n+1,1)$-surgery on L, after blowing down the $(+1)$-framed component L_3, we get the (n,n)-surgery on $T(2,2n+2)$, which is an L-space.

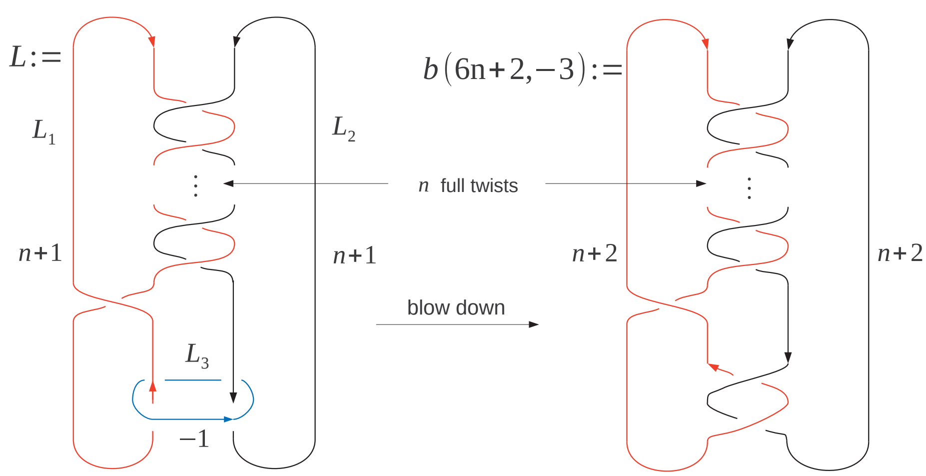


Figure 4.2.8: **The** $(n+2,n+2)$**-surgery on the two-bridge link** $b(6n+2,-3)$**.** Consider the $(n+1,n+1,-1)$-surgery on the left 3-component link L. After blowing down the (-1)-framed component L_3, we get the $(n+2,n+2)$-surgery on the two-bridge link $b(6n+2,-3)$.

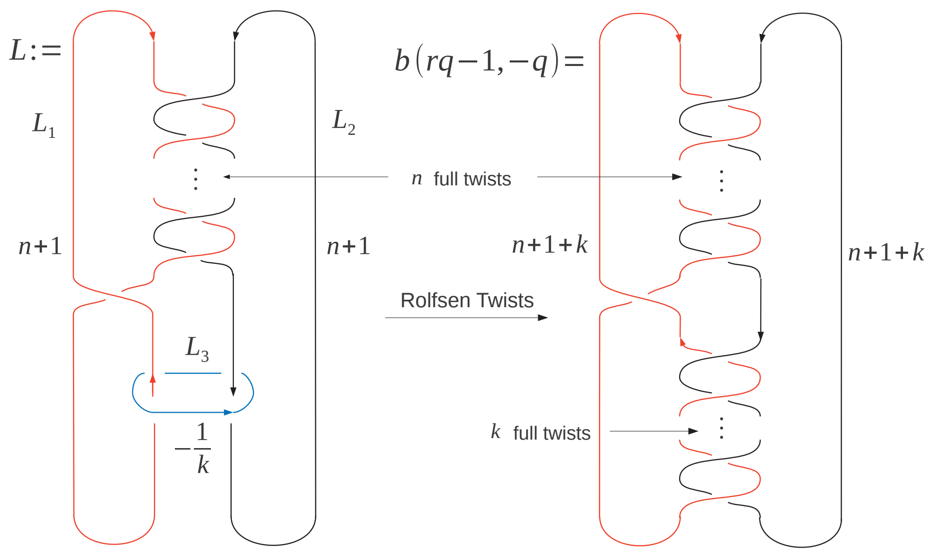


Figure 4.2.9: **The $(n+1+k, n+1+k)$-surgery on the two-bridge link $b(rq-1,-q)$ with $r = 2n+1, q = 2k+1$.** Consider the $(n+1, n+1, -\frac{1}{k})$-surgery on the left 3-component link L. After doing the Rolfsen twists on the (-1)-framed component L_3, we get the $(n+1+k, n+1+k)$-surgery on the two-bridge link $b(rq-1,-q)$.

down the $(+1)$-framed component L_3, we get the (n,n)-surgery on the $T(2,2n+2)$ torus link, $S^3_{n,n}(T(2,2n+2))$, which is an L-space by Corollary 4.2.6. While the $(n+1,n+1)$-surgery on the $T(2,2n)$ torus link $L_1 \cup L_2$, $S^3_{n+1,n+1}(L_1 \cup L_2)$, is also an L-space. In addition, since

$$\det\begin{pmatrix} n+1 & n & 1 \\ n & n+1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = -1-2n,$$

$$\det\begin{pmatrix} n+1 & n \\ n & n+1 \end{pmatrix} = 2n+1,$$

from Lemma 4.1.4 it follows that the surgeries $S^3_{n+1,n+1,0}(L), S^3_{n+1,n+1,-1}(L)$ are both L-spaces. By blowing down the (-1)-framed component L_3 on the $(n+1,n+1,-1)$-surgery on L, we get the $(n+2,n+2)$-surgery on the two-bridge link $b(6n+2,-3)$. See Figure 4.2.8.

Since $\det\begin{pmatrix} n+2 & n-1 \\ n-1 & n+2 \end{pmatrix} > 0$ and $n+2 > 0$, it follows from Lemma 4.1.4 that the two-bridge links $b(6n+2,-3)$ are all L-space links.

Fixing n, for $k > 1$, consider rational surgeries on the 3-component link L in Figure 4.2.7, with a rational coefficient on L_3. Then we have an exact triangle for the triple $S^3_{n+1,n+1,0}(L)$, $S^3_{n+1,n+1,-1/k}(L)$, and $S^3_{n+1,n+1,-1/(k+1)}$,

$$\begin{tikzcd}
\widehat{HF}\left(S^3_{n+1,n+1,0}(L)\right) \arrow[rr] & & \widehat{HF}\left(S^3_{n+1,n+1,-\frac{1}{k}}(L)\right) \arrow[dl] \\
& \widehat{HF}\left(S^3_{n+1,n+1,-\frac{1}{k+1}}(L)\right) . \arrow[ul] &
\end{tikzcd}$$

We claim that $S^3_{n+1,n+1,-\frac{1}{k+1}}(L)$ is an L-space for all positive integers k. We have shown that $S^3_{n+1,n+1,0}(L)$ is an L-space in the first step, and by the induction hypothesis, we can assume $S^3_{n+1,n+1,-\frac{1}{k}}(L)$ is an L-space. Moreover, we have

$$\left|H_1(S^3_{n+1,n+1,-\frac{1}{k+1}}(L))\right| = \left|H_1(S^3_{n+1,n+1,-\frac{1}{k}}(L))\right| + \left|H_1(S^3_{n+1,n+1,0}(L))\right|,$$

since

$$\left|H_1(S^3_{n+1,n+1,-\frac{1}{k}}(L))\right| = \det\begin{pmatrix} n+1 & n & 1 \\ n & n+1 & -1 \\ k & -k & -1 \end{pmatrix} = -1 - 2n - 2k - 4kn.$$

Hence, from the above exact triangle it follows that $S^3_{n+1,n+1,-\frac{1}{k+1}}(L)$ is an L-space.

Now by doing Rolfsen twists on L_3, we get a $(n+1+k, n+1+k)$-surgery on the two-bridge link $b(pq-1, -q) = b(4kn+2k+2n, -2k-1)$. See Figure 4.2.9. Since the linking number of $b(4kn+2k+2n, -2k-1)$ is $\pm(n-k)$, the determinant $\det\begin{pmatrix} n+1+k & \pm(n-k) \\ \pm(n-k) & n+1+k \end{pmatrix}$ is positive. Thus, by Lemma 4.1.5, we get $b(rq-1, -q)$ is an L-space link for all positive odd integers r, q. $\square$

Example 4.2.9 (Non-fibered hyperbolic L-space links)**.** The two-bridge links $b(10n+4, -5)$ with $n \in \mathbb{N}$ are L-space links, by Theorem 4.2.8. At least for $2 \leq n \leq 6$, they are not fibered links, i.e., there does not exist any Seifert surface F such that the link complement fibers

$\overrightarrow{L} = L_1 \cup L_2$	$\Delta_{-L_1 \cup L_2}(t) =$	$\Delta_{L_1 \cup L_2}(t) =$
$b(24,-5)$	$2t^2 - 3t + 2 - \frac{3}{t} + \frac{2}{t^2}$	$\frac{1}{t^3}(2t^6 - 3t^5 + 2t^4 - 3t^3 + 2t^2 - 3t + 2)$
$b(34,-5)$	$3t^2 - 4t + 3 - \frac{4}{t} + \frac{3}{t^2}$	$\frac{-1}{t^4}(2t^8 - 3t^7 + 2t^6 - 3t^5 + 2t^4 - 3t^3 + 2t^2 - 3t + 2)$
$b(44,-5)$	$4t^2 - 5t + 4 - \frac{5}{t} + \frac{4}{t^2}$	$\frac{1}{t^5}(2t^{10} - 3t^9 + 2t^8 - 3t^7 + 2t^6 - 3t^5 + 2t^4 - 3t^3 +$ $2t^2 - 3t + 2)$
$b(54,-5)$	$5t^2 - 6t + 5 - \frac{6}{t} + \frac{5}{t^2}$	$\frac{-1}{t^6}(2t^{12} - 3t^{11} + 2t^{10} - 3t^9 + 2t^8 - 3t^7 + 2t^6 - 3t^5 +$ $2t^4 - 3t^3 + 2t^2 - 3t + 2)$
$b(64,-5)$	$6t^2 - 7t + 6 - \frac{7}{t} + \frac{6}{t^2}$	$\frac{1}{t^7}(2t^{14} - 3t^{13} + 2t^{12} - 3t^{11} + 2t^{10} - 3t^9 + 2t^8 - 3t^7 +$ $2t^6 - 3t^5 + 2t^4 - 3t^3 + 2t^2 - 3t + 2)$

Table 4.2.1: **Alexander polynomials of non-fibered hyperbolic L-space links.** Here, we consider the single variable Alexander polynomials for the two different orientations on the above L-space links. None of them has leading coefficient 1, although the multi-variable Alexander polynomials do have coefficients ± 1. Thus, they are not fibered with any orientation.

over circle with fiber F. The fiberedness of links is detected by the knot Floer homology. See Corollary 1.2 in [30]: An oriented link $\overrightarrow{L}$ in S^3 is fibered if and only if the knot Floer homology $\widehat{HFK}(\overrightarrow{L})$ has a single copy of $\mathbb{Z}$ at the top Alexander grading. Thus, for a homologically thin link L, the link L is fibered if and only if its single-variable Alexander polynomial has leading coefficient ± 1. Note that two-bridge links are alternating and thus homologically thin; see Theorem 1.3 in [42]. We compute the multi-variable polynomials $\Delta_L(x, y)$ using the algorithm in [22] , and plug t or t^{-1} for x, y so as to get the single-variable Alexander polynomials. It turns out that $b(10n+4, -5)$ is not fibered with any orientation, when $2 \leq n \leq 6$. See Table 4.2.1. In fact, the fibered two-bridge knots and links are also classified by using continued fractions due to Gabai. See [6]. One should be able to generalize this to all $n \geq 2$ using number theoretic arguments.

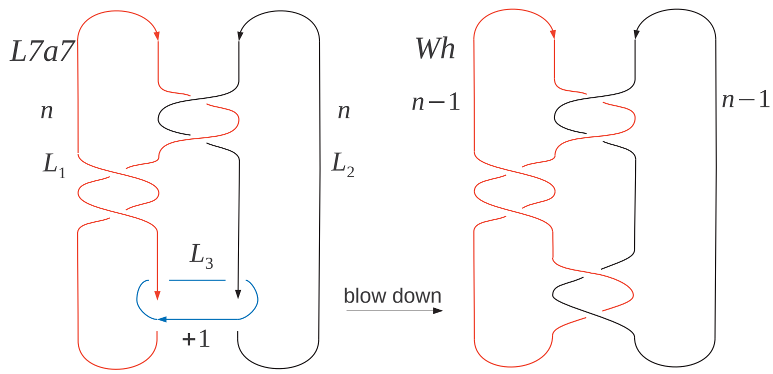


Figure 4.2.10: **The 3-component link $L7a7$.** The 3-component link L drawn above on the left is the mirror of $L7a7$ drawn in the Thistlethwaite Link Table on Knot Atlas. Consider the $(n, n, 1)$-surgery on L. After blowing down the 1-framed component L_3, we get the $(n-1, n-1)$-surgery on the Whitehead link Wh.

Example 4.2.10 (Plumbing trees)**.** Any plumbing tree L of unknots is an L-space link. In fact, any sufficiently negative surgery on L is a negative definite graph without bad vertices, and thus is an L-space, by [35] Lemma 2.6. Since the plumbing tree is amphichiral, the sufficiently positive surgeries are also L-spaces. Note that if M is an L-space, then so is $-M$.

The surgeries on a plumbing tree are generally Seifert manifolds. Actually, we also have examples of plumbing graph of unknots to be generalized L-space links, which give rise to hyperbolic manifolds. But one should be very careful about the types of generalized L-space links. Also note that for the same graph there are many different plumbings.

Example 4.2.11 ($L7a7$ in the Thistlethwaite Link Table)**.** The link L drawn in Figure 4.2.10 is an L-space link. It is actually the mirror of $L7a7$ drawn in the Thistlethwaite Link Table. Consider the $(n, n, 1)$-surgery on L. It is an L-space when $n \gg 0$. This is because after blowing down the $(+1)$-framed knot L_3, we get the L-space Whitehead link $b(8, -3)$. Then, it follows from Lemma 4.1.5 that L is an L-space link.

Example 4.2.12. The plumbing of unknots L in Figure 4.2.11 is a hyperbolic L-space link. In fact, consider the $(3, 1, 3, 1)$-surgery on L, which is S^3. By Lemma 4.1.5, L is an L-space

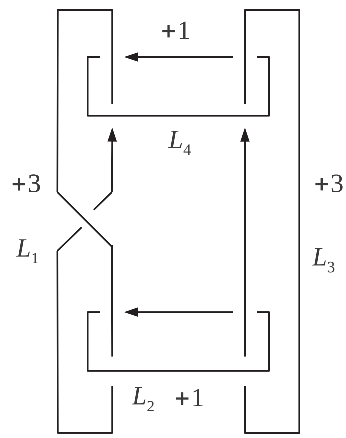


Figure 4.2.11: **A plumbing graph L-space link.** Consider the link $L = L_1 \cup ... \cup L_4$ in the figure which is a plumbing of unknots. By blowing down L_2, L_4, we see that the surgery shown is S^3.

link. In fact, this link is derived by resolving the Whitehead link. Thus, all the surgeries on the Whitehead link are surgeries on this link.

Example 4.2.13. The plumbing shown in Figure 4.2.12 is a generalized $(++++-)L$-space link. The $(1,1,1,1,1)$-surgery is the Poincaré sphere. See [47] page 309. In fact, every proper sublink is an L-space link, since the surgeries on them are lens spaces. Thus, by Lemma 4.1.4, the $(p_1,1,1,1,1)$-surgery is an L-space for all $p_1 \geq 1$, since $\det(\Lambda(1,1,1,1,1)) = -1$ and $\det(\Lambda(1,1,1,1,1)|_{L-L_1}) = -1$. Next, from that $S^3_{p_1,1,1,1}(L - L_2) = L(p_1, 1)$ and $\det(\Lambda(p_1,1,1,1,1)) = \det(\Lambda(p_1,1,1,1,1)|_{L-L_2}) = -p_1$, it follows that $(p_1,p_2,1,1,1)$-surgery on L is an L-space for all $p_1 \geq 1, p_2 \geq 1$. Similarly, we can get the $(p_1,p_2,p_3,1,1)$-surgery is an L-space for all $p_1 \geq 1, p_2 \geq 1, p_3 \geq 1$. This is because $S^3_{p_1,p_2,1,1}(L - L_3) = L(p_1, 1)$, and $\det(\Lambda(p_1,p_2,1,1,1)) = -p_1p_2$, $\det(\Lambda(p_1,p_2,1,1,1)|_{L-L_3}) = -p_1$. Now, we can get that the $(p_1,p_2,p_3,p_4,1)$-surgery on L is an L-space, for all $p_1 \geq 3, p_2 \geq 3, p_3 \geq 2, p_4 \leq 1$, since $\det(\Lambda(p_1,p_2,p_3,1,1)) = p_2 - p_1p_2 - p_2p_3 < 0$, $\det(\Lambda(p_1,p_2,p_3,1,1)|_{L-L_4}) = 1 - p_1 - p_3 - p_1p_2 + p_1p_2p_3 > 0$. Finally, we can obtain that the (p_1,p_2,p_3,p_4,p_5)-surgery on L is an L-space for

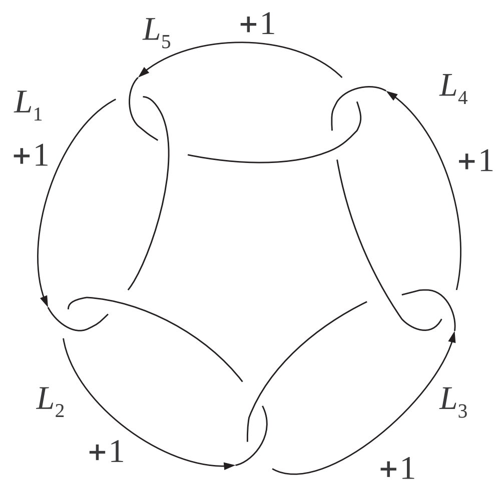


Figure 4.2.12: **A generalized** $(++++-)$$L$**-space link.** Consider the link $L = L_1 \cup ... \cup L_5$ in the figure which is a plumbing of unknots. The surgery shown is the Poincaré sphere.

all $p_1 \gg 0, p_2 \gg 0, p_3 \gg 0, p_4 \ll 0, p_5 \geq 1$, due to

$$\det(\Lambda(p_1, p_2, p_3, p_4, 1)) = p_1 p_2 p_3 p_4 + \text{ lower terms} < 0,$$

$$\det(\Lambda(p_1, p_2, p_3, p_4, 1)|_{L-L_5}) = p_1 p_2 p_3 p_4 + \text{ lower terms} < 0.$$

Example 4.2.14 (A family of L-space chain links)**.** A n-chain link consists of n unknotted circles, linked together in a closed chain. Hyperbolic structures on n-chain link complements have been studied, for example, by Neumann and Reid [29]. They show that when $l \geq 5$ they are hyperbolic links.

The family of l-component chain links in Figure 4.2.14 are all L-space links. In fact, the $(1, 2, ..., 2, l-2)$-surgery satisfies the positive L-space surgery criterion. First, if we blow down $L_1, L_2, ..., L_{l-2}$ successively, then we get the $(1, 1)$-surgery on the Whitehead link, the Poincaré manifold. Moreover, every proper sublink is a union of linear plumbings of unknots, and their surgeries are all connected sum of lens spaces. Thus, we only need to check the positive determinant condition.

Since a handle slide does not change the determinants of the surgery framing matrices, blowing down a $+1$ framed unknot does not change the determinants of the surgery framing matrices. Thus, after successively blowing down $L_1, ..., L_{l-2}$ from L, we see that $\det(\Lambda(1, 2, ..., 2, l-1)) = 1$. For the proper sublinks, we only need to consider a linear plumbing $L' \subset L$. Since the determinant of the surgery framing matrix does not depend on the orientations, we can always orient L' such that all the linking numbers of adjacent components are -1. Let $M(k, n)$ denote the following $k \times k$ matrix

$$M(k,n) = \begin{pmatrix} n & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & \cdots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix},$$

which is the surgery framing matrix of the linear plumbing in Figure 4.2.13. There are four cases for computing $\det(\Lambda|_{L'})$:

- if $L_1 \notin L', L_l \notin L'$, then $\Lambda|_{L'} = M(k, 2)$ with k being the number of components in L';
- if $L_1 \notin L', L_l \in L'$, then $\Lambda|_{L'} = M(k, l-1)$;
- if $L_1 \in L', L_l \notin L'$, then after successively blowing down $L_1, L_2, ...,$ we can see $\det(\Lambda|_{L'}) = 1$;
- if $L_1 \in L', L_l \in L'$, then after successively blowing down $L_1, L_2, ...$ inside L', we can see $\det(\Lambda|_{L'})$ equals to $\det(M(k,n))$ with $k \leq l-2, n \geq 1$.

It is not hard to see $\det(M(k, 2)) = k+1$ by induction, and thus $\det(M(k, n)) = nk - k + 1$. Therefore, all determinants are positive.

Example 4.2.15 (Another sequence of L-space chain links)**.** Similarly, the family of l-component chain links in Figure 4.2.16 are also all L-space links for $l \geq 3$. In fact, when n_1, n_2 are large enough, the $(1, 2, ..., 2, n_2, n_1)$-surgery satisfies the positive L-space surgery

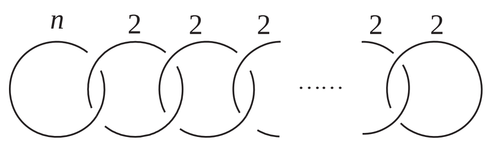


Figure 4.2.13: **A linear plubming.**

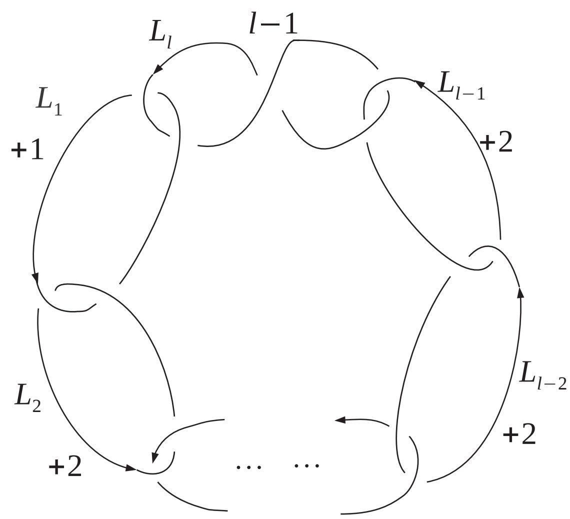


Figure 4.2.14: **A family of hyperbolic L-space chain links.** The surgery labelled above satisfies the positive L-space surgery criterion.

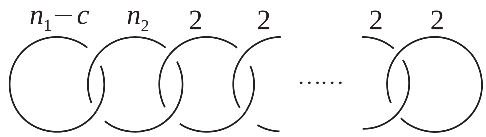


Figure 4.2.15: **A linear plubming.**

criterion. This is because after blowing down $L_1, ..., L_{l-2}$, we have an $(n_1 - l + 2, n_2 - 1)$ framed $T(2,4)$ torus link. Thus, when n_1, n_2 are both large, this surgery is an L-space, since $T(2,4)$ is an L-space link. As is similar in Example 4.2.14, we only need to show when n_1, n_2 are large enough, $\det(\Lambda(1, 2, ..., 2, n_2, n_1)|_{L'})$ is positive for any sublink L'. For any sublink L', we can blow down the circles on the side of L_1, and then obtain a linear plumbing as in Figure 4.2.15. The surgery matrix is a $k \times k$ matrix in the form of

$$\begin{pmatrix} n_1 - c & -1 & & & & \\ -1 & n_2 & -1 & & & \\ & -1 & 2 & & & \\ & & & \cdots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix},$$

where c is the number of times for blowing down $+1$-framed unknots. The determinant of the above matrix is a polynomial of n_1, n_2, and the leading term is $\det(M(k-2,2))n_1 n_2 = (k-1)n_1 n_2$. Thus, for n_1, n_2 large enough, all the determinants are positive.

Note that the link in Example 4.2.13 is the same as the link here for $l = 5$.

Example 4.2.16. The link $L^{(n)} = V_1 \cup V_2 \cup H_1 \cup ... \cup H_n$ shown in Figure 4.2.17 is a generalized L-space link of "$++-\cdots-$" type, for any $n \geq 1$. One can do similar induction as in Example 4.2.13 to show the following claim.

Claim: For any $0 \leq k \leq n$ and all integers $p_1 \gg 0, p_2 \gg 0, q_1 \ll 0, \cdots, q_k \ll 0$, the $(p_1, p_2, q_1, \ldots, q_k, -1, \ldots, -1)$-surgery on $L^{(n)}$ is an L-space. Notice that the determinant of

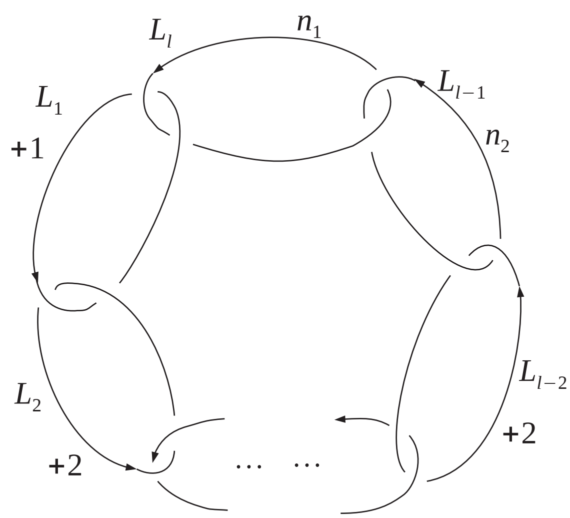


Figure 4.2.16: **Another family of hyperbolic L-space chain links.** The surgery labelled above satisfies the positive L-space surgery criterion, when n_1, n_2 are large enough.

framing matrix

$$\det(\Lambda((p_1, p_2, q_1, \ldots, q_k, -1, \ldots, -1))) = (-1)^{n-k} p_1 p_2 q_1 \cdots q_k + \text{ lower terms.}$$

The claim will follow from two induction on n and on k.

Notice that surgeries on $L^{(n)}$ are mostly graph manifolds.

Example 4.2.17 (Thistlethwaite Link Table with crossing number ≤ 7)**.** We examine the links in the Thistlethwaite Link Table with crossing number ≤ 7 and list the results in Table 4.2.2.

Using the conditions of Alexander polynomials in Theorem 1.3.15, we conclude that $L6a1$, $L7a1$, $L7a2$, $L7a4$, and $L7a5$ are all non-L-space links.

The link $L6a2$ is the two-bridge link $b(10, 7)$. Conjecture 1.3.14 has been verified for all two-bridge links $b(p, q)$ with $p \leq 100$ using the algorithm from [22]. So $L6a2$ is a non-L-space link.

The link $L6a5$ is the mirror of the left link in Figure 4.2.7 with $n = 1$, on which the

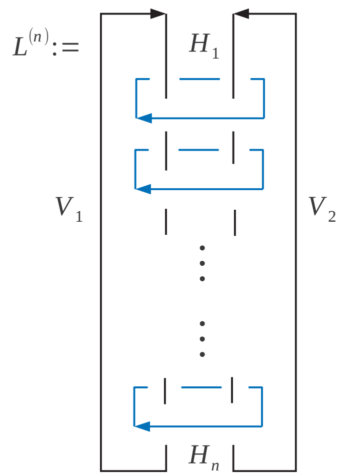


Figure 4.2.17: **Another sequence of generalized L-space link.** Consider the link $L^{(n)}$ used in the proof of Lemma 4.2.5. It is in fact a generalized L-space link.

$(2,2,1)$-surgery is an L-space. Then, it quickly follows from the positive L-space surgery criterion that the mirror of $L6a5$ is an L-space link.

For the link $L6n1$, after blowing down a $+1$-framed component from it (all three components are symmetric), we get the unlink. So the $(10,10,1)$-surgery on $L6n1$ satisfies the positive surgery criterion, and thus showing that $L6n1$ is an L-space link.

The mirror of $L7a3$ consists of two components L_1 and L_2, where L_1 is the right-handed trefoil and L_2 is the unknot. Consider the $(n,1)$-surgery on the mirror of $L7a3$ with n large. After blowing down the unknot, we have a large surgery on the right-handed torus knot $T(2,5)$. This is an L-space, since the right-handed torus knot $T(2,5)$ is an L-space knot. Then it follows from the positive surgery criterion that the mirror of $L7a3$ is an L-space link.

The link $L7a6$ is the two-bridge link $b(14,-9)$, and it is not L-space link by direct computation.

The link $L7n1$ has two components L_1 and L_2, where L_1 is the right-handed trefoil knot and L_2 is the unknot. Consider the $(10,1)$-surgery on this link. After blowing down the unknot, the trefoil is unknotted and we obtain a lens space surgery. Since the right-handed trefoil is an L-space knot, from the positive surgery criterion it follows that $L7n1$ is an L-space link.

Links	L-space link	Alexander polynomial	Comments
$L2a1$	Yes	Yes	The Hopf link
$L4a1$	Yes	Yes	The $T(2,4)$ torus link
$L5a1$	Yes	Yes	Mirror of the L-space Whitehead link
$L6a1$	No	No	
$L6a2$	No	Yes	
$L6a3$	Yes	Yes	The $T(2,6)$ torus link
$L6a4$	Yes	Yes	The Borromean link
$L6a5$	Yes	Yes	The mirror is an L-space link
$L6n1$	Yes	Yes	
$L7a1$	No	No	
$L7a2$	No	No	
$L7a3$	Yes	Yes	The mirror is an L-space link
$L7a4$	No	No	
$L7a5$	No	No	
$L7a6$	No	Yes	The two-bridge link $b(14,-9)$
$L7a7$	Yes	Yes	The mirror is an L-space link
$L7n1$	Yes	Yes	
$L7n2$	No	Yes	*Generalized* $(+-)$*L-space link*

Table 4.2.2: **Thistlethwaite Link Table with crossing number** ≤ 7. Here, by "Yes" in the column "L-space link", it means either the link or its mirror is an L-space link; by "Yes" in the column "Alexander polynomial", it means the conditions on the multi-variable Alexander polynomial in Theorem 1.3.15 are satisfied.

The link $L7n2$ is not an L-space link; see Proposition 5.1.14 for the proof. Its mirror is not an L-space link neither, since the left-handed trefoil is not an L-space knot. However, $L7n2$ is a generalized $(+-)L$-space link. The link $L7n2$ consists of two components L_1 and L_2, with L_1 being the right-handed trefoil and L_2 being the unknot. Consider the $(n,-1)$-surgery on $L7n2$ with n large. After blowing down the unknot, we get the unknot, thus getting a lens space surgery. Then, since the right-handed trefoil is an L-space knot, $(n,-k)$-surgery is an L-space for all $k>0$ and large n by Lemma 4.1.4.

4.3 Floer homology and Alexander polynomials of L-space links

In this section, we study the link Floer homology and the multi-variable Alexander polynomials of L-space links with $l \geq 2$ components. The Alexander polynomial of L is determined by the Euler characteristics of the link Floer homology $HFL^{-}(L, \mathbf{s})$, due to Equation (2) in [42]

$$\Delta_L(x_1, ..., x_l) \doteq \sum_{(s_1,...,s_l)\in \mathbb{H}(L)} \chi(HFL^{-}(L, s_1, ..., s_l)) \cdot x_1^{s_1} \cdots x_l^{s_l}, \tag{4.3.1}$$

where $f \doteq g$ denotes that f and g differ by multiplication by units. Here, we use $CFL^{-}(L)$ rather than $\widehat{CFL}(L)$ as in [40]. Note that $CFL^{-}(L, s_1, s_2)$ is a finite dimensional $\mathbb{F}$-vector space, and thus $\chi(CFL^{-}(L, s_1, s_2)) = \chi(HFL^{-}(L, s_1, s_2))$.

Now we are ready to prove Theorem 1.3.15 from the introduction.

4.3.1 Proof of Theorem 1.3.15.

Proof. Fixing $\{s_i\}_i$, we denote the following successive quotient complexes by

$$\begin{aligned} C_k^{(1)} =& \{x \in \mathbf{CF}^{-}(S^3) | A_i(x) = s_i, 1 \leq i \leq k, A_j(x) \leq s_j, k+1 \leq j \leq l\}, \\ C_k^{(2)} =& \{x \in \mathbf{CF}^{-}(S^3) | A_i(x) = s_i, 1 \leq i \leq k, A_{k+1}(x) \leq s_{k+1} - 1, A_j(x) \leq s_j, k+2 \leq j \leq l\}. \end{aligned}$$

Then, $C_0^{(1)} = \mathfrak{A}^{-}_{\mathbf{s}}$, $C_0^{(2)} = \mathfrak{A}^{-}_{s_1-1, s_2, ..., s_l}$, $C_l^{(1)} = CFL^{-}(L, \mathbf{s})$, and $C_{k+1}^{(1)} = C_k^{(1)}/C_k^{(2)}$.

Consider the short exact sequence of chain complexes

$$0 \to C_0^{(2)} \xrightarrow{\iota} C_1^{(1)} \to C_1^{(1)} \to 0,$$

where the map ι is the inclusion map. It induces another short exact sequence of homologies

$$0 \to \operatorname{coker}(\iota_*) \to H_*(C_1^{(1)}) \to \ker(\iota_*) \to 0.$$

The map $\iota_* : \mathbb{F}[[U]] \to \mathbb{F}[[U]]$ is either 0 or a multiplication of U^k for some integer $k \geq 0$.

Since U_1 acts on the chain complex $C_1^{(1)}$ as 0, it also acts as 0 on homology. Thus, $H_*(C_1^{(1)})$ is either 0 or $\mathbb{F}[[U]]/U$. Note here $\mathbb{F}[[U]]$ denotes $\mathbb{F}[[U_1, U_2, ..., U_l]]/(U_1 - U_2, ..., U_1 - U_l)$ as an $\mathbb{F}[[U_1, U_2, ..., U_l]]$-module and the U-action denotes any action of U_i. Furthermore, $\chi(H_*(C_1^{(1)}))$ is either 0 or 1. In fact, if $H_*(C_1^{(1)}) = 0$, then the grading of $1 \in \mathbb{F}[[U]] = H_*(C_0^{(1)})$ equals to the grading of $1 \in \mathbb{F}[[U]] = H_*(C_0^{(2)})$; while if $H_*(C_1^{(1)}) = \mathbb{F}[[U]]/U$, then the grading of $1 \in H_*(C_0^{(1)})$ equals to the grading of $1 \in H_*(C_0^{(2)})$ plus 2, and the grading of $1 \in \mathbb{F}[[U]]/U = H_*(C_1^{(1)})$ equals to the grading of $1 \in H_*(C_1^{(1)})$. Moreover, the complex $\mathfrak{A}^-_{+\infty,...,+\infty}$ is just $\mathbf{CF}^-(S^3)$ and the absolute gradings of elements in $H_*(\mathfrak{A}^-_{+\infty,...,+\infty})$ are all even integers. An induction will show that all the absolute gradings of elements in the homologies of $\mathfrak{A}^-_{s_1,s_2,...,s_l}$ and any successive quotients of them are all even integers.

Thus, we have

$$\chi(H_*(C_1^{(1)})) = 0 \text{ or } 1.$$

Notice that $C_k^{(1)}$ and $C_k^{(2)}$ are defined similarly, just with different $\mathbf{s}$ values. Thus, we can similarly show that

$$\chi(H_*(C_1^{(2)})) = 0 \text{ or } 1.$$

Since $\chi(H_*(C_{k+1}^{(1)})) = \chi(H_*(C_k^{(1)})) - \chi(H_*(C_k^{(2)}))$, we can inductively prove that

$$\left|\chi(H_*(C_k^{(1)}))\right| \leq 2^{k-2}, \forall k = 2, ..., l,$$

$$\left|\chi(H_*(C_k^{(2)}))\right| \leq 2^{k-2}, \forall k = 2, ..., l.$$

Hence, we prove Inequality (1.3.2) by letting $k = l$.

Since $C_{k+1}^{(1)} = C_k^{(1)}/C_k^{(2)}$, we have

$$\mathrm{rank}_{\mathbb{F}}(C_{k+1}^{(1)}) \leq \mathrm{rank}_{\mathbb{F}}(C_k^{(1)}) + \mathrm{rank}_{\mathbb{F}}(C_k^{(2)}).$$

From $\mathrm{rank}_{\mathbb{F}} H_*(C_1^{(1)}) \leq 1$, it follows that $\mathrm{rank}_{\mathbb{F}} HFL^-(L, \mathbf{s}) \leq 2^{l-1}$. Thus, Inequality (1.3.1) holds.

Let us look at the signs of the multi-variable Alexander polynomial when $l = 2$. Suppose $\chi(CFL^-(L, s_1, s_2))$ and $\chi(CFL^-(L, s_1 + k, s_2)$ are the consecutive non-zero Euler characteristics among the horizontal Alexander gradings, that is,

- $|\chi(CFL^-(L, s_1, s_2))| = 1$,
- $|\chi(CFL^-(L, s_1 + k, s_2))| = 1$,
- $\chi(CFL^-(L, s_1 + i, s_2)) = 0, \forall i = 1, 2, ..., k-1$.

Then, we have

$$\begin{aligned} &\chi(H_*(\mathfrak{A}^-_{s_1+k,s_2}/\mathfrak{A}^-_{s_1+k,s_2-1})) - \chi(H_*(\mathfrak{A}^-_{s_1-1,s_2}/\mathfrak{A}^-_{s_1-1,s_2-1})) \qquad (*) \\ =&\sum_{i=0}^{k} \chi(CFL^-(L, s_1 + i, s_2)) \\ =&\chi(CFL^-(L, s_1, s_2)) + \chi(CFL^-(L, s_1 + k, s_2)). \end{aligned}$$

Since $\chi(H_*(\mathfrak{A}^-_{s_1,s_2}/\mathfrak{A}^-_{s_1,s_2-1})) = 0$ or 1, for all $(s_1, s_2) \in \mathbb{H}(L)$, the top row of Equation $(*)$ is 0 or ± 1. Whereas by the assumption, the bottom row of Equation $(*)$ is 0 or ± 2. Thus, we have

$$\chi(CFL^-(L, s_1, s_2)) + \chi(CFL^-(L, s_1 - k, s_2)) = 0.$$

□

Corollary 4.3.1. *A homological thin L-space 2-component prime link $L = L_1 \cup L_2$ has fibered link exterior.*

Proof. The homological thin condition means that the homology $\widehat{HFL}(L, \mathbf{s})$ is supported in a single Maslov grading, and thus is determined by its Euler characteristic. Thus, the link Floer homology is determined by the multi-variable Alexander polynomial. However, here we need to consider the hat version link Floer homology for the discussions of fiberedness.

Let the symmetrized Alexander polynomial be

$$\Delta_L(x,y) = \sum_{i,j} a_{i,j} \cdot x^i \cdot y^j.$$

We choose

$$x_0 = \max\{i | a_{i,j} \neq 0\}, \quad y_0 = \max\{j | a_{x_0,j} \neq 0\}.$$

Since

$$\sum_{(s_1,s_2)\in\mathbb{H}(L)} \chi(\widehat{HFL}(L,s_1,s_2)) \cdot x^{s_1} \cdot y^{s_2} = \pm\frac{(x-1)(y-1)}{\sqrt{xy}}\Delta_L(x,y),$$

we have that $(x_0+\frac{1}{2}, y_0+\frac{1}{2})$ is an extreme point of the polytope for $\widehat{HFL}(L)$, and $\chi(\widehat{HFL}(L, x_0+\frac{1}{2}, y_0+\frac{1}{2})) = \pm 1$. Furthermore, since L is homological thin, we have that $\mathrm{rank}\widehat{HFL}(L, x_0 + \frac{1}{2}, y_0 + \frac{1}{2}) = 1$, and thereby the link exterior of L is fibered. □

4.3.2 Examples.

Let us use Theorem 1.3.15 to filter L-space links among two-bridge links. Notice that in the knot case the Alexander polynomial gives a strong obstruction for an alternating knot to be an L-space knot. In [40], it is shown that alternating L-space knots are only $(2, 2n+1)$ torus knots.

Proposition 4.3.2 (Ozsváth-Szabó, [40], Proposition 4.1)**.** *If K is an alternating knot with the property that all the coefficients a_i of its Alexander polynomial Δ_K have $|a_i| \leq 1$, then K is the $(2, 2n+1)$ torus knot.*

Theorem 4.3.3 (Ozsváth-Szabó, [40], Theorem 1.5)**.** *If $K \subset S^3$ is an alternating knot with the property that there is some integral surgery along K is an L-space, then K is a $(2, 2n+1)$ torus knot for some integer n.*

In contrast to the knot case, by computer experiments, we find many hyperbolic two-bridge non-L-space links whose Alexander polynomials satisfy the constraints in Theorem 1.3.15. We list some interesting phenomena in the two-bridge links $b(p,q)$ below, where

$0 < p \leq 100$. Note that all these phenomena should presumably be true for all positive even integers p.

Links	Alexander polynomial condition:	Hyperbolic link:	L-space link:
$b(2n, -1)$	Yes	Torus link $T(2, 2n)$	Yes
$b(6n+2, -3)$	Yes	Hyperbolic link	Yes
$b(6n+4, -3)$	Yes	Hyperbolic link	No, when $6n+2 \leq 100$.
$b(10n \pm 2, 5)$	No	Hyperbolic link	No

4.3.3 HFL^- of L-space links.

Let L be an L-space link. In general, $CFL^-(L, \mathbf{s})$ is an iterated quotient complex of $\mathfrak{A}^-_{\mathbf{s}}$.

For every subcomplex $C_1 \subset C$, the quotient complex C/C_1 is quasi-isomorphic to the mapping cone of the inclusion map $i : C_1 \to C$. Thus, it leads to an iterated mapping cone construction of $CFL^-(L, \mathbf{s})$ by using $\mathfrak{A}^-_{\mathbf{s}}$. This provides a spectral sequence converging to $HFL^-(L, \mathbf{s})$ considered as $\mathbb{F}$-vector spaces, which is stated in [10]. This spectral sequence also implies Inequality (1.3.1).

CHAPTER 5

Applications to surgeries L-space links

5.1 Surgeries on L-space links

Using the knot surgery formula from [43], the graded Heegaard Floer homology of surgeries on L-space knots are determined by the Alexander polynomial and the surgery coefficient. Using the Manolescu-Ozsváth link surgery formula from [24] and algebraic rigidity results from [22], we prove Theorem 1.4.1 and give some explicit formulas in this section.

The generalized Floer complexes $\mathfrak{A}^-_{\mathbf{s}}$'s are $\mathbb{F}[[U_1, ..., U_l]]$-models, and all the U_i multiplications are homotopic to the U_1 multiplication. In fact, when L is an L-space link, $\mathfrak{A}^-_{\mathbf{s}}(L)$ is chain homotopic to $\mathbb{F}[[U_1]]$ preserving the $\mathbb{Z}$-grading. This is done by restricting our scalars to $\mathbb{F}[[U_1]]$ and applying the algebraic rigidity results Proposition 5.5 and Corollary 5.6 in [22]. There is an absolute $\mathbb{Z}$-grading on $\mathfrak{A}^-_{\mathbf{s}}$. However, the U_1 action lows down it by 2, and thus it is not a chain complex of $\mathbb{F}[[U_1]]$-modules. So, the complexes here are considered as $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes of $\mathbb{F}[[U_1]]$-modules together with a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading, where U_1 lowers the $\mathbb{Z}$-grading by 2.

Proposition 5.1.1 (Proposition 5.5, [22])**.** *Let A_*, B_* be $\mathbb{Z}$-graded complexes of $\mathbb{F}$-modules with U-action dropping grading by 2 and commuting with the differential. Suppose A, B are both free $\mathbb{F}[[U]]$-modules, and $H_*(A) = H_*(B) = \mathbb{F}[[U]]$, precisely, $H_{2k}(A) \cong H_{2k}(B) \cong \mathbb{F}$ for all $k \leq 0$ and $H_i(A) = H_i(B) = 0$ otherwise, where $U \cdot H_{2k}(A) = H_{2k-2}(A), U \cdot H_{2k}(B) = H_{2k-2}(B)$.*

Then, if $F, G : A \to B$ are both quasi-isomorphisms of $\mathbb{F}[[U]]$-modules, then F, G are chain homotopic as maps of $\mathbb{F}[[U]]$-modules. Moreover, if H, K are both chain homotopies *as homomorphisms of* $\mathbb{F}[[U]]$*-modules between any two chain maps* $f, g : A \to B$*, i.e.* $H\partial + \partial H = K\partial + \partial K = f - g$*, then* $H - K = \partial T + T\partial$*, for some* $\mathbb{F}[[U]]$*-module homomorphism* $T : A_* \to B_{*+2}$*.*

Using these chain homotopy equivalences, we replace $\mathfrak{A}^-_{\mathbf{s}}(L)$ by $\mathbb{F}[[U_1]]$ in the Manolescu-Ozsváth link surgery complex and replace the maps up to homotopies. In [22], we call this new complex the *perturbed surgery formula.* Thus, we only need to determine the map $\Phi^{\overrightarrow{M}}_{\mathbf{s}}$ in the perturbed surgery formula, where are either 0 or multiplications of U^k. For the definition of those Φ maps, one can see [24] Section 7 or [22] Section 4.

Combining this with conjugation symmetry, we determine the maps $\Phi^{\pm L_i}_{\mathbf{s}}$ by the coefficients in the multi-variable Alexander polynomials of the sublinks in L and the linking numbers. We also show that in the perturbed surgery complex, $\Phi^{\pm L_1 \cup \pm L_2}_{\mathbf{s}} = 0$ for all $\mathbf{s} \in \mathbb{H}(L)$. For higher diagonal maps, more information is needed. For 2-component case, we write down explicit formulas.

5.1.1 Conjugation symmetry of inclusion maps.

Definition 5.1.2 ($p^{\overrightarrow{M}}(\mathbf{s})$)**.** For $\mathbf{s} \in \bar{\mathbb{H}}(L)$ and $\overrightarrow{M} \subset L$, we define $p^{\overrightarrow{M}}(\mathbf{s}) = (p^{\overrightarrow{M}}_1(s_1), ..., p^{\overrightarrow{M}}_l(s_l))$ by the following formulas

$$p^{\overrightarrow{M}}_i(s) = \begin{cases} +\infty & \text{if } L_i \subset M \text{ has the induced orientation from } L; \\ -\infty & \text{if } L_i \subset M \text{ has the opposite orientation from } L; \\ s & \text{if } L_i \not\subset M. \end{cases}$$

Definition 5.1.3 ($n^{\overrightarrow{M}}_{\mathbf{s}}(L)$)**.** Suppose $\overrightarrow{L}$ is an oriented l-component L-space link and $\overrightarrow{M} \subset \overrightarrow{L}$ is a sublink which might not have the induced orientation. Choose a Heegaard diagram $\mathcal{H}$ of L. The inclusion map $I^{\overrightarrow{M}}_{\mathbf{s}} : \mathfrak{A}^-(\mathcal{H}, \mathbf{s}) \to \mathfrak{A}^-(\mathcal{H}, p^{\overrightarrow{M}}(\mathbf{s}))$ is a chain map shifting the $\mathbb{Z}$-grading by a definite amount, which is explicitly expressed in Equation (57) in [24]. Thus, the map induced on homologies $(I^{\overrightarrow{M}}_{\mathbf{s}})_* : H_*(\mathfrak{A}^-(\mathcal{H}, \mathbf{s})) \to H_*\left(\mathfrak{A}^-(\mathcal{H}, p^{\overrightarrow{M}}(\mathbf{s}))\right)$ is a multiplication of a monomial $U^k : \mathbb{F}[[U]] \to \mathbb{F}[[U]]$ or 0 rather than a multiplication of a polynomial. In fact, this map is not 0. Consider the short exact sequence

$$0 \to \mathfrak{A}^-_{p^{\vec{M}}(\mathbf{s})} \to \mathfrak{A}^-_{\mathbf{s}} \to \mathfrak{A}^-_{\mathbf{s}}/\mathfrak{A}^-_{p^{\vec{M}}(\mathbf{s})} \to 0$$

and the induced exact triangle on homology. The homology $\mathfrak{A}^-_{\mathbf{s}}/\mathfrak{A}^-_{p^{\vec{M}}(\mathbf{s})}$ is a torsion U_1 module, which is argued similarly as in the proof of Theorem 1.3.15.

The integer k does not depend on the choice of $\mathcal{H}$, and thus we define it to be $n^{\vec{M}}_{\mathbf{s}}(L)$. When the context is clear, we simply denote it by $n^{\vec{M}}_{\mathbf{s}}$.

Remark 5.1.4. When L is a L-space knot K, these $n^{\pm K}_s(K)$'s are just the same as V_s's and H_s's defined for knots in [31].

Lemma 5.1.5 (Conjugation symmetry of $n^{\vec{M}}_{\mathbf{s}}(L)$)**.** *Suppose L is an oriented n-component L-space link. Then*

$$n^{\vec{M}}_{\boldsymbol{s}} = n^{-\vec{M}}_{-\boldsymbol{s}}, \quad \forall \boldsymbol{s} \in \mathbb{H}(L), \forall \vec{M} \subset L.$$

Proof. Choose an admissible basic Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}^H, \mathbf{z}^H)$ for $\vec{L}$. In order to distinguish the basepoints in different Heegaard diagrams, we put a superscript H on w and z. Then, $\mathcal{H}' = (-\Sigma, \beta, \alpha, \mathbf{w}^{H'}, \mathbf{z}^{H'})$ is also a Heegaard diagram for $\vec{L}$, where $\mathbf{w}^H = \mathbf{z}^{H'}, \mathbf{z}^H = \mathbf{w}^{H'}$.

There is an $\mathbb{F}[[U_1, \dots, U_n]]$-linear isomorphism of chain complexes

$$\begin{aligned} h_{\mathbf{s}} : \ \mathfrak{A}^-(\mathcal{H}, \mathbf{s}) &\longrightarrow \mathfrak{A}^-(\mathcal{H}', -\mathbf{s}), \\ \mathbf{x} &\longmapsto \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta. \end{aligned}$$

Actually, for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and a class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, the moduli space of holomorphic disks $M(\phi, \mathcal{H})$ is identical to $M(\phi, \mathcal{H}')$. See Theorem 2.4 in [38]. Moreover, it is not hard to see that the Alexander gradings are of opposite signs

$$A(\mathbf{x}, \mathcal{H}) = -A(\mathbf{x}, \mathcal{H}').$$

Thus, we just need to show $h_{\mathbf{s}}$ is a chain map, i.e.

$$\partial^{\mathcal{H}'}_{-\mathbf{s}}(h_{\mathbf{s}}(\mathbf{x})) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\phi\in\pi_2(x,y),\mu(\phi)=1} \#(M(\phi)/\mathbb{R}) \cdot U_1^{E^{\mathcal{H}'}_{-s_1}(\phi)} \cdots U_n^{E^{\mathcal{H}'}_{-s_n}(\phi)} \cdot \mathbf{y}$$

$$= h_{\mathbf{s}}(\partial^{\mathcal{H}}_{\mathbf{s}}(\mathbf{x})) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\phi\in\pi_2(x,y),\mu(\phi)=1} \#(M(\phi)/\mathbb{R}) \cdot U_1^{E^{\mathcal{H}}_{s_1}(\phi)} \cdots U_n^{E^{\mathcal{H}}_{s_n}(\phi)} \cdot \mathbf{y}.$$

In fact, by Equation (4.1.3), $\forall \phi \in \pi_2(\mathbf{x}, \mathbf{y}),\ \forall 1 \le i \le n,$

$$\begin{aligned} E^{\mathcal{H}'}_{-s_i}(\phi) &= \max(-s_i - A_i^{\mathcal{H}'}(\mathbf{x}), 0) - \max(-s_i - A_i^{\mathcal{H}'}(\mathbf{y}), 0) + n_{z_i^{\mathcal{H}'}}(\phi) \\ &= \max(-s_i + A_i^{\mathcal{H}}(\mathbf{x}), 0) - \max(-s_i + A_i^{\mathcal{H}}(\mathbf{y}), 0) + n_{w_i^{\mathcal{H}}}(\phi) \\ &= E^{\mathcal{H}}_{s_i}(\phi). \end{aligned}$$

Moreover, by direct computation, we have the following commuting diagram

$$\begin{tikzcd} \mathfrak{A}^-(\mathcal{H}, \mathbf{s}) \arrow[r, "h_{\mathbf{s}}"] \arrow[d, "I^{\vec{M}}_{\mathbf{s}}(\mathcal{H})"'] & \mathfrak{A}^-(\mathcal{H}', -\mathbf{s}) \arrow[d, "I^{-\vec{M}}_{-\mathbf{s}}(\mathcal{H}')"] \\ \mathfrak{A}^-(\mathcal{H}, \mathrm{p}^{\vec{M}}(\mathbf{s})) \arrow[r, "h_{p^{\vec{M}}(\mathbf{s})}"'] & \mathfrak{A}^-(\mathcal{H}', -\mathrm{p}^{\vec{M}}(\mathbf{s})). \end{tikzcd}$$

Thus, it follows that

$$n^{\vec{M}}_{\mathbf{s}} = n^{-\vec{M}}_{-\mathbf{s}}, \quad \forall \mathbf{s} \in \mathbb{H}(L),\ \forall \vec{M} \subset L.$$

□

5.1.2 Perturbed the link surgery formula for 2-component L-space links.

We review the link surgery formula of Manolescu-Ozsváth for a 2-component link L. See [24] and Section 4 in [22]. We need some notations. Denote the set of orientations on a link N by $\Omega(N)$. We define some projection maps by $p^{\pm L_1}(s_1, s_2) = (\pm\infty, s_2)$, $p^{\pm L_2}(s_1, s_2) = (s_1, \pm\infty)$, and $p^{\pm L_1 \cup \pm L_2}(s_1, s_2) = (\pm\infty, \pm\infty)$.

Choose an admissible basic Heegaard diagram $\mathcal{H}$ and denote $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ by $\mathfrak{A}^-_{\mathbf{s}}$. Then, the

Manolescu-Ozsváth surgery complex $(C^-(\mathcal{H},\Lambda), D^-(\Lambda))$ is as follows:

$$(C^-(\mathcal{H},\Lambda), D^-(\Lambda)) := \begin{tikzcd} \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \mathfrak{A}^-_{s_1,s_2} \arrow[r, "D^{10}_{00}(\Lambda)"] \arrow[d, "D^{01}_{00}(\Lambda)"'] \arrow[dr, "D^{11}_{00}(\Lambda)"] & \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \mathfrak{A}^-_{+\infty,s_2} \arrow[d, "D^{01}_{10}(\Lambda)"] \\ \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \mathfrak{A}^-_{s_1,+\infty} \arrow[r, "D^{10}_{01}(\Lambda)"'] & \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \mathfrak{A}^-_{+\infty,+\infty}, \end{tikzcd} \tag{5.1.1}$$

where $\forall \delta_1,\delta_2,\varepsilon_1,\varepsilon_2 \in \{0,1\}$,

$$D^{\delta_1\delta_2}_{\varepsilon_1\varepsilon_2}(\Lambda) = \prod_{(s_1,s_2)\in\mathbb{H}(L)} \left(\sum_{\overrightarrow{M}\in\Omega(\delta_1 L_1\cup\delta_2 L_2)} \Phi^{\overrightarrow{M}}_{p^{+\varepsilon_1 L_1\cup+\varepsilon_2 L_2}(s_1,s_2)} \right). \tag{5.1.2}$$

The Manolescu-Ozsváth surgery complex is in the category of complexes of $\mathbb{F}[[U_1]]$-modules, **Ch**. Inspired by the idea of homotopy category **K** of $\mathbb{F}[[U_1]]$-modules, we can replace the complexes on the vertices of the hypercube by its chain homotopy type and replace the maps on the edges by its homotopy type. Then, the Manolescu-Ozsváth surgery complex becomes a *perturbed surgery formula.*

Lemma 5.1.6. *Let $\overrightarrow{L} = \overrightarrow{L_1} \cup \overrightarrow{L_2}$ be an L-space link. Then the Heegaard Floer homologies on all the surgeries $\boldsymbol{HF}^-(S^3_\Lambda(L))$ and their absolute gradings are determined by $\{n^{+L_1}_{\boldsymbol{s}}(L)\}_{\boldsymbol{s}\in\mathbb{H}(L)}$ and $\{n^{+L_2}_{\boldsymbol{s}}(L)\}_{\boldsymbol{s}\in\mathbb{H}(L)}$.*

Proof. We restrict our scalars to $\mathbb{F}[[U_1]]$ from now on. Consider the chain complex $\mathbb{F}[[U_1]]$, which is freely generated by a single element over $\mathbb{F}[[U_1]]$ with 0 differential. Since L is an L-space link, i.e. $H_*(\mathfrak{A}^-_{\mathbf{s}}(L)) = \mathbb{F}[[U]], \forall \mathbf{s} \in \mathbb{H}(L)$, $\mathfrak{A}^-_{\mathbf{s}}(L)$ is in fact chain homotopic to $\mathbb{F}[[U_1]]$ by Corollary 5.6 in [22] as a $\mathbb{Z}$-graded $\mathbb{F}[[U_1]]$-module with U_1 lowering grading by 2.

Thus, we can replace every $\mathfrak{A}^-_{\mathbf{s}}$ by $\tilde{\mathfrak{A}}^-_{\mathbf{s}}$ which is isomorphic to C^{u} and replace the maps correspondingly so as to get a new complex $(\tilde{C}^-(\mathcal{H},\Lambda), \tilde{D}^-(\Lambda))$. We call it the *perturbed surgery complex*, and it is chain homotopic to the original one.

More concretely, we first replace the edge maps in the squares in Equation (5.1.1) $\Phi^{\pm L_i}_{\mathbf{s}}$

by

$$\tilde{\Phi}^{\pm L_i}_{\mathbf{s}} = U_1^{n^{\pm L_i}_{\mathbf{s}}} : \mathbb{F}[[U_1]] \to \mathbb{F}[[U_1]].$$

Next, we replace the diagonal maps $\Phi^{\pm L_1 \cup \pm L_2}_{\mathbf{s}}$ by

$$\tilde{\Phi}^{\pm L_1 \cup \pm L_2}_{\mathbf{s}} = 0.$$

The reason we replace the diagonal maps by 0 is that, in the link surgery complex, the $\mathbb{F}[[U_1]]$-linear diagonal maps always shift the $\mathbb{Z}$-gradings by an odd number.

Finally, we get the new perturbed surgery complex $\tilde{C}(\Lambda)$ as follows:

$$(\tilde{C}^-(\mathcal{H},\Lambda), \tilde{D}^-(\Lambda)) := \begin{tikzcd} \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \tilde{\mathfrak{A}}^-_{s_1,s_2} \arrow[r, "\tilde{D}^{10}_{00}(\Lambda)"] \arrow[d, "\tilde{D}^{01}_{00}(\Lambda)"'] \arrow[dr, "\tilde{D}^{11}_{00}(\Lambda)"] & \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \tilde{\mathfrak{A}}^-_{+\infty,s_2} \arrow[d, "\tilde{D}^{01}_{10}(\Lambda)"] \\ \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \tilde{\mathfrak{A}}^-_{s_1,+\infty} \arrow[r, "\tilde{D}^{10}_{01}(\Lambda)"'] & \prod\limits_{(s_1,s_2)\in\mathbb{H}(L)} \tilde{\mathfrak{A}}^-_{+\infty,+\infty}, \end{tikzcd} \tag{5.1.3}$$

where

$$\tilde{D}^{\delta_1\delta_2}_{\varepsilon_1\varepsilon_2}(\Lambda) = \prod_{(s_1,s_2)\in\mathbb{H}(L)} \left(\sum_{\overrightarrow{M}\in\Omega(\delta_1 L_1 \cup \delta_2 L_2)} \tilde{\Phi}^{\overrightarrow{M}}_{p^{+\varepsilon_1 L_1 \cup +\varepsilon_2 L_2}(s_1,s_2)} \right), \quad \delta_1,\delta_2,\varepsilon_1,\varepsilon_2 \in \{0,1\}. \tag{5.1.4}$$

The perturbed complex $\tilde{C}(\Lambda)$ is chain homotopy equivalent to the original surgery complex as $\mathbb{F}[[U_1]]$-modules. Moreover, this chain homotopy equivalence is preserving the $\mathbb{Z}$-grading on it. For more details, see Section 5.6 in [22].

Hence, we have $H_*(\tilde{C}(\Lambda)) \cong \mathbf{HF}^-(S^3_\Lambda(L))$ as an $\mathbb{F}[[U_1]]$-module. By Link Surgery Theorem in [24], we have U_i actions on the homology of $\mathbf{HF}^-(S^3_\Lambda(L))$ are all the same, i.e.

$$\mathbf{HF}^-\left(S^3_\Lambda(L)\right) = H_*(\tilde{C}(\Lambda)) \otimes_{\mathbb{F}[[U_1]]} \mathbb{F}[[U_1,U_2]]/(U_1 - U_2).$$

All the inputs of $\tilde{C}(\Lambda)$ are $\{n^{\pm L_1}_{\mathbf{s}}(L)\}_{s\in\mathbb{H}(L)}$ and $\{n^{\pm L_2}_{\mathbf{s}}(L)\}_{s\in\mathbb{H}(L)}$. Thus, the proof is done by Lemma 5.1.5. To compute the absolute grading for $\mathbf{HF}^-$, we only need to shift the absolute $\mathbb{Z}$-grading by $\frac{c_1{}^2(\mathbf{s})-2\chi-3\sigma}{4}$ which can be computed from Λ. $\square$

5.1.3 Redefining knot Floer homology.

We redefine the knot Floer homology by using slightly generalized Heegaard diagrams with extra basepoints. The reason we consider these diagrams is that they are used in the proof of Theorem 1.4.1. In [23], there are many generalized versions of knot Floer complex and homology discussed. Since the version in this subsection is not presented in [23], we define it here.

1. Heegaard diagram: We choose a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, ..., w_k\}, \{z_1\})$.

2. Alexander grading: For any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$A(U_1^{n_1} \cdots U_k^{n_k}\mathbf{x}) = A(\mathbf{x}) - n_1.$$

3. Alexander filtration: The complex $\mathbf{CF}^-(S^3)$ is freely generated by $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ over $\mathbb{F}[[U_1, U_2, ..., U_k]]$ and the differentials are counting holomorphic disks. For $\forall \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have $A(\partial\mathbf{x}) \leq A(\mathbf{x})$. This is because for a pseudo-holomorphic disk in $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, $n_{z_1}(\phi) \geq 0$ and

$$A(\mathbf{x}) = A(\mathbf{y}) + n_{z_1}(\phi) - n_{w_1}(\phi) = A(U_1^{n_{w_1}(\phi)} \cdot \ldots U_k^{n_{w_k}(\phi)} \cdot \mathbf{y}) + n_{z_1}(\phi).$$

   Thus, the Alexander grading induces a filtration on $\mathbf{CF}^-(S^3)$. We define the subcomplex

$$\mathfrak{A}_s^-(K) := \{x \in \mathbf{CF}^-(S^3) | A(x) \leq s\}.$$

4. The *filtered minus knot Floer homology*: We define the chain complex $CFK^-(K, s) = \mathfrak{A}_s^-/\mathfrak{A}_{s-1}^-$ and $HFK^-(K, s) = H_*(CFK^-(K, s))$.

5. The *total minus knot Floer homology*: We define the chain complex $gCFK^-(K)$ to be freely generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $\forall \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1, n_{z_1}(\phi)=0}} \#(M(\phi)/\mathbb{R}) \cdot U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \cdot \mathbf{y}.$$

The homology $HFK^-(K)$ is defined to be the homology of $gCFK^-(K)$.

Remark 5.1.7. Considered only as $\mathbb{F}$-vector spaces, $HFK^-(K) = \bigoplus_{s \in \mathbb{Z}} HFK^-(K, s)$. However, considered as $\mathbb{F}[[U_1, ..., U_k]]$-modules, $HFK^-(K, s)$ is the associated graded of a filtration on $HFK^-(K)$. Note that $HFK^-(K, s)$'s are always torsion modules.

Proposition 5.1.8. *Suppose $K \subset S^3$ is a knot. For a multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, ..., w_k\}, \{z_1\})$ for K, we have the following:*

1. *The knot Floer homology $HFK^-(K, s)$ is an $\mathbb{F}[[U]] := \mathbb{F}[[U_1, ..., U_k]]/(U_2, ..., U_k)$-module, and does not depend on $\mathcal{H}$ considered as an $\mathbb{F}[[U]]$-module.*

2. *We have the following identity*

$$\sum_{s \in \mathbb{Z}} \chi\left(HFK^-(K, s)\right) \cdot t^s \doteq \frac{1}{t-1} \Delta_K(t). \tag{5.1.5}$$

Proof. This is actually a direct corollary of Theorem 4.10 in [24]. There are six types of Heegaard moves according to [24],

(i) 3-manifold isotopy;

(ii) α-curve isotopy and β-curve isotopy;

(iii) α-handleslide and β-handleslide;

(iv) index one/two stabilizations;

(v) free index zero/three stabilizations;

(vi) free index zero/three link stabilizations.

By Proposition 4.13 in [24], we only need to check how the knot Floer homology changes under these Heegaard moves and their inverses.

The Heegaard moves of types (i) to (iv) are chain homotopy equivalences preserving the Alexander filtration, and thus do not change the knot Floer homology.

A Heegaard move of type (v) changes the chain complex $\mathbf{CF}^-(\mathcal{H})$ into $\mathbf{CF}^-(\mathcal{H}')$, which is the mapping cone $\mathbf{CF}^-(\mathcal{H})[[U_{k+1}]] \xrightarrow{U_{k+1}-U_{i_0}} \mathbf{CF}^-(\mathcal{H})[[U_{k+1}]]$. Notice that U_{k+1} does not change the Alexander grading. Thus, if $i_0 \neq 1$, then $CFK^-(\mathcal{H}', s)$ is the mapping cone

$$CFK^-(\mathcal{H}, s)[[U_{k+1}]] \xrightarrow{U_{k+1}-U_{i_0}} CFK^-(\mathcal{H}, s)[[U_{k+1}]].$$

If $i_0 = 1$, then $CFK^-(\mathcal{H}', s)$ is the mapping cone

$$CFK^-(\mathcal{H}, s)[[U_{k+1}]] \xrightarrow{U_{k+1}} CFK^-(\mathcal{H}, s)[[U_{k+1}]].$$

In both cases, we have that the homology of the mapping cone is

$$HFK^-(\mathcal{H}, s) \otimes_{\mathcal{R}} \mathbb{F}[[U_1, ..., U_{k+1}]]/(U_2, ..., U_{k+1}),$$

where $\mathcal{R} = \mathbb{F}[[U_1, ..., U_k]]$.

The Heegaard move of type (vi) changes the complex $CFK^-(\mathcal{H}, s)$ by

$$CFK^-(\mathcal{H}, s) \otimes H_*(S^1) \cong CFK^-(\mathcal{H}, s) \oplus CFK^-(\mathcal{H}, s).$$

However, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent Heegaard diagrams both with a single pair of basepoints on K, then total number of copies of $HFK^-(\mathcal{H}_1, s)$'s in $HFK^-(\mathcal{H}_2, s)$ is one. $\square$

5.1.4 Reduction of Heegaard diagrams.

Let $\mathcal{H}$ be Heegaard diagram for a link L. Then there are several Heegaard diagrams $r_{\overrightarrow{M}}(\mathcal{H})$ of the sublinks of L reduced from $\mathcal{H}$. See Definition 4.17 in [24].

Lemma 5.1.9. *Let $\overrightarrow{L} = \overrightarrow{L}_1 \cup \overrightarrow{L}_2$ be a link and $\mathcal{H} = (\Sigma, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ be a Heegaard diagram for $\overrightarrow{L}$. Denote $\mathfrak{A}^-(\mathcal{H}, (s_1, s_2))$ by $\mathfrak{A}^-_{s_1,s_2}$, for all $(s_1, s_2) \in \mathbb{H}(L)$. Then*

$$H_*(\mathfrak{A}^-_{+\infty,s_2}/\mathfrak{A}^-_{+\infty,s_2-1}) = HFK^-(L_2, s_2 - \frac{\mathrm{lk}}{2}).$$

In particular, $\chi\left(H_*(\mathfrak{A}^-_{+\infty,s_2}/\mathfrak{A}^-_{+\infty,s_2-1})\right)$ is determined by the Alexander polynomial $\Delta_{L_2}(t)$.

Proof. By Proposition 5.1.8, we can use $\mathfrak{A}^-_{+\infty,s_2}/\mathfrak{A}^-_{+\infty,s_2-1}$ to compute the knot Floer homology of L_2. The only issue is on the Alexander grading. From Equation (36) in [24], there is an identification

$$\mathfrak{A}^-(\mathcal{H}, p^{\overrightarrow{M}}(\mathbf{s})) \xrightarrow{\cong} \mathfrak{A}^-(r_{\overrightarrow{M}}(\mathcal{H}), \psi^{\overrightarrow{M}}(\mathbf{s})).$$

Note that the definition of $\psi^{\overrightarrow{M}}(\mathbf{s})$ involves the linking numbers. Thus, we have the following commuting diagram

$$\begin{tikzcd}
\mathfrak{A}^-_{+\infty,s_2-1}(L) \arrow[r, "\cong"] \arrow[d, "\iota^{+L_2}_{+\infty,s_2-1}"'] & \mathfrak{A}^-_{s_2-1-\frac{\mathrm{lk}}{2}}(L_2) \arrow[d, "\iota^{+L_2}_{s_2-1-\frac{\mathrm{lk}}{2}}"] \\
\mathfrak{A}^-_{+\infty,s_2}(L) \arrow[r, "\cong"] & \mathfrak{A}^-_{s_2-\frac{\mathrm{lk}}{2}}(L_2),
\end{tikzcd}$$

where $\iota^{+L_2}_{+\infty,s_2-1}$ and $\iota^{+L_2}_{s_2-1-\frac{\mathrm{lk}}{2}}$ are both the inclusions of subcomplex. Thus, we have

$$\frac{\mathfrak{A}^-_{+\infty,s_2}(L)}{\mathfrak{A}^-_{+\infty,s_2-1}(L)} \cong \frac{\mathfrak{A}^-_{s_2-\frac{\mathrm{lk}}{2}}(L_2)}{\mathfrak{A}^-_{s_2-1-\frac{\mathrm{lk}}{2}}(L_2)} = CFK^-(L_2, s_2 - \frac{\mathrm{lk}}{2}).$$

Thus, the lemma follows. $\square$

5.1.5 Proof of Theorem 1.4.1.

Proof. Consider the following factorization of inclusion maps of subcomplexes

$$I^{+L_2}_{s_1,s_2} : \mathfrak{A}^-_{s_1,s_2} \xrightarrow{\iota^{+L_2}_{s_1,s_2}} \mathfrak{A}^-_{s_1,s_2+1} \xrightarrow{I^{+L_2}_{s_1,s_2+1}} \mathfrak{A}^-_{s_1,+\infty}.$$

It induces a factorization of the maps on homology $(I^{+L_2}_{s_1,s_2})_* = (I^{+L_2}_{s_1,s_2+1})_* \circ (\iota^{+L_2}_{s_1,s_2})_*$. As is discussed in the proof of Theorem 1.3.15, we see $(\iota^{+L_2}_{s_1,s_2})_*$ is a multiplication of $U^{k^{+L_2}_{s_1,s_2}}$, where

$$k^{+L_2}_{s_1,s_2} = n^{+L_2}_{s_1,s_2} - n^{+L_2}_{s_1,s_2+1}.$$

Moreover, $k = 0$ if and only if $H_*(\mathfrak{A}^-_{s_1,s_2+1}/\mathfrak{A}^-_{s_1,s_2}) = 0$, and $k = 1$ if and only if $H_*(\mathfrak{A}^-_{s_1,s_2+1}/\mathfrak{A}^-_{s_1,s_2}) = \mathbb{F}$ with an even grading. Then, we have

$$\chi\left(H_*(\mathfrak{A}^-_{s_1,s_2+1}/\mathfrak{A}^-_{s_1,s_2})\right) = n^{+L_2}_{s_1,s_2} - n^{+L_2}_{s_1,s_2+1}.$$

Whereas,

$$\begin{aligned}
&\chi\left(H_*(\mathfrak{A}^-_{s_1+k,s_2+1}/\mathfrak{A}^-_{s_1+k,s_2})\right)\\
=&\chi\left(H_*(\mathfrak{A}^-_{s_1,s_2+1}/\mathfrak{A}^-_{s_1,s_2})\right) + \sum_{i=1}^{k}\chi\left(HFL^-(L, s_1+i, s_2+1)\right), \forall k > 0.
\end{aligned}$$

Let $k \to \infty$, and then we have $\chi\left(H_*(\mathfrak{A}^-_{k,s_2}/\mathfrak{A}^-_{k,s_2-1})\right) = \chi\left(H_*(\mathfrak{A}^-_{+\infty,s_2}/\mathfrak{A}^-_{+\infty,s_2-1})\right)$ determined by $\Delta_{L_2}(t)$, by Lemma 5.1.9. Thus, all the $n^{+L_2}_{s_1,s_2}$ are determined by the Alexander polynomials. Similar results hold for L_1. The theorem follows from Lemma 5.1.6 and Theorem 5.1.10. □

In fact, when the linking number is not 0, the Alexander polynomials of L_1 and L_2 are determined by the Alexander polynomial of $L = L_1 \cup L_2$ and the linking number:

Theorem 5.1.10 (Murasugi, Proposition 4.1 in [27]). *Let $\Delta_L(x, y)$ and $\Delta_{L_1}(t)$ be the Alexan-*

der polynomial of a link $L = L_1 \cup L_2$ and L_1 respectively in S^3. Then

$$\Delta_L(t,1) \doteq \frac{1-t^{\mathrm{lk}}}{1-t}\Delta_{L_1}(t),$$

where lk *is the linking number of L.*

5.1.6 Formulas for $n_{\mathbf{s}}^{\pm L_i}(L)$'s.

Using the Alexander polynomials of L, L_1, L_2, we can get formulas for $n_{\mathbf{s}}^{\pm L_i}(L)$'s.

First of all, we fix the overall signs of these Alexander polynomials to get normalization of Equation (5.1.5) and Equation (4.3.1):

$$\textstyle\sum_{s\in\mathbb{Z}} \chi\left(HFK^-(K,s)\right)\cdot t^s = \frac{t}{t-1}\Delta_K(t), \tag{5.1.6}$$

$$\textstyle\sum_{(s_1,s_2)\in\mathbb{H}(L)} \chi(HFL^-(L,s_1,s_2))\cdot x_1^{s_1}\cdot x_2^{s_2} = x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}\Delta_L(x_1,x_2). \tag{5.1.7}$$

For an L-space knot K, to get Equation (5.1.6), we require that $\frac{t}{t-1}\Delta_K(t)$ has finitely many non-zero positive powers and all the non-zero coefficients of $\frac{t}{t-1}\Delta_K(t)$ are 1, which is equivalent to $\Delta_K(1) = 1$.

Theorem 5.1.11. *Suppose $L = L_1 \cup L_2$ is an L-space link. Let $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$, and $\Delta_L(x_1,x_2)$ be the symmetrized Alexander polynomials, such that $\Delta_{L_1}(1) = \Delta_{L_2}(1) = 1$. Let*

$$\frac{t}{t-1}\Delta_{L_1}(t) = \sum_{k\in\mathbb{Z}} a_k^{L_1}\cdot t^k,$$
$$\frac{t}{t-1}\Delta_{L_2}(t) = \sum_{k\in\mathbb{Z}} a_k^{L_2}\cdot t^k,$$
$$\Delta_L(x_1,x_2) = \sum_{i,j} a_{i,j}^L\cdot x_1^i\cdot x_2^j.$$

Suppose (i_0,j_0) satisfies that $a_{i_0,j_0}^L \neq 0$, $a_{i,j_0}^L = 0$ for all $i > i_0$, and $a_{i_0,j}^L = 0$ for all $j > j_0$. Then,

- $\chi\left(HFL^{-}(L, i_0+\frac{1}{2}, j_0+\frac{1}{2})\right) = 1$ *if and only if* $a^{L_1}_{i_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = a^{L_2}_{j_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = 1$;
- $\chi\left(HFL^{-}(L, i_0+\frac{1}{2}, j_0+\frac{1}{2})\right) = -1$ *if and only if* $a^{L_1}_{i_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = a^{L_2}_{j_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = 0$.

Proof. Notice that $\chi\left(\mathfrak{A}^{-}_{s_1,s_2}/\mathfrak{A}^{-}_{s_1,s_2-1}\right)$ can only be 0 or 1 for all $(s_1, s_2) \in \mathbb{H}(L)$. By Equation (**??**), we have two possible cases:

(a) $\chi\left(HFL^{-}(L, s_1, s_2)\right) = 1$ if and only if $\chi\left(\mathfrak{A}^{-}_{s_1,s_2}/\mathfrak{A}^{-}_{s_1,s_2-1}\right) = 1$ and $\chi\left(\mathfrak{A}^{-}_{s_1-1,s_2}/\mathfrak{A}^{-}_{s_1-1,s_2-1}\right) = 0$;

(b) $\chi\left(HFL^{-}(L, s_1, s_2)\right) = -1$ if and only if $\chi\left(\mathfrak{A}^{-}_{s_1,s_2}/\mathfrak{A}^{-}_{s_1,s_2-1}\right) = 0$ and $\chi\left(\mathfrak{A}^{-}_{s_1-1,s_2}/\mathfrak{A}^{-}_{s_1-1,s_2-1}\right) = 1$.

In addition, we have

$$\chi\left(\mathfrak{A}^{-}_{i_0+\frac{1}{2},j_0+\frac{1}{2}}/\mathfrak{A}^{-}_{i_0+\frac{1}{2},j_0-\frac{1}{2}}\right) = \chi\left(\mathfrak{A}^{-}_{+\infty,j_0+\frac{1}{2}}/\mathfrak{A}^{-}_{+\infty,j_0-\frac{1}{2}}\right) = \chi\left(HFK^{-}(L_2, j_0+\frac{1}{2}-\frac{\mathrm{lk}}{2})\right).$$

So $\chi\left(HFL^{-}(L, s_1, s_2)\right) = 1$ if and only if $a^{L_2}_{j_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = 1$. Symmetrically, we have $\chi\left(HFL^{-}(L, s_1, s_2)\right) = 1$ if and only if $a^{L_1}_{i_0+\frac{1}{2}-\frac{\mathrm{lk}}{2}} = 1$. Similar argument applies to the case (b). □

Definition 5.1.12 (Normalization of Alexander polynomials for L-space links)**.** Suppose $L = L_1 \cup L_2$ is an L-space link. Let the symmetrized Alexander polynomial of L be

$$\Delta_L(x_1, x_2) = \sum_{i,j} a^L_{i,j} \cdot x_1^i \cdot x_2^j,$$

where x_i corresponds to the component L_i for $i = 1, 2$. Let the symmetrized Alexander polynomials of L_1, L_2 be $\Delta_{L_1}(t), \Delta_{L_2}(t)$ in the forms of

$$\frac{t}{t-1}\Delta_{L_1}(t) = \sum_{k\in\mathbb{Z}} a^{L_1}_k \cdot t^k, \quad \frac{t}{t-1}\Delta_{L_2}(t) = \sum_{k\in\mathbb{Z}} a^{L_2}_k \cdot t^k.$$

Let (i_0, j_0) be such that

$$j_0 = \max\{j \in \mathbb{Z} + \frac{\mathrm{lk}-1}{2} \mid a^L_{i,j} \neq 0\} \text{ and } i_0 = \max\{i \in \mathbb{Z} + \frac{\mathrm{lk}-1}{2} \mid a^L_{i,j_0} \neq 0\}.$$

Then, these Alexander polynomials are called **normalized**, if

1. the leading coefficient of $\Delta_{L_i}(t)$ is 1 for both $i = 1, 2$, which is equivalent to $\Delta_{L_i}(1) = 1$;
2. if $a^{L_2}_{j_0-\frac{\mathrm{lk}}{2}+\frac{1}{2}} = 1$, then $a^{L}_{i_0,j_0} = 1$; while if $a^{L_2}_{j_0-\frac{\mathrm{lk}}{2}+\frac{1}{2}} = 0$, then $a^{L}_{i_0,j_0} = -1$.

After normalization, we have $\chi(HFL^-(L, s_1, s_2)) = a^L_{s_1-\frac{1}{2},s_2-\frac{1}{2}}$ and $\chi(HFK^-(L_i, s)) = a^{L_i}_s$ for $i = 1, 2$. Therefore,

$$\chi(H_*(\mathfrak{A}^-_{s_1,s_2}/\mathfrak{A}^-_{s_1,s_2-1})) = a^{L_2}_{s_2-\frac{\mathrm{lk}}{2}} - \sum_{i=1}^{\infty} a^L_{s_1-\frac{1}{2}+i,s_2-\frac{1}{2}} = 0 \text{ or } 1.$$

Hence, we have

$$n^{+L_2}_{s_1,s_2} = \sum_{j=1}^{\infty}\left(a^{L_2}_{s_2+j-\frac{\mathrm{lk}}{2}} - \sum_{i=1}^{\infty} a^L_{s_1+i-\frac{1}{2},s_2+j-\frac{1}{2}}\right). \tag{5.1.8}$$

Similarly, we have

$$n^{+L_1}_{s_1,s_2} = \sum_{i=1}^{\infty}\left(a^{L_1}_{s_1+i-\frac{\mathrm{lk}}{2}} - \sum_{j=1}^{\infty} a^L_{s_1+i-\frac{1}{2},s_2+j-\frac{1}{2}}\right). \tag{5.1.9}$$

Theorem 5.1.13. *Suppose $L = L_1 \cup L_2$ is an L-space link. Under the normalization in Definition 5.1.12, we have that the formulas in Equation* (5.1.8) *and Equation* (5.1.9) *are non-negative for all* $(s_1, s_2) \in \mathbb{H}(L)$.

In fact, both of Theorem 5.1.11 and Theorem 5.1.13 give additional constraints for the Alexander polynomials of an L-space 2-component link.

Proposition 5.1.14. *The link $L7n2$ is not an L-space link.*

Proof. We give two proofs based on Theorem 5.1.11 and Theorem 5.1.13 respectively. Suppose $L = L7n2$ is an L-space link with components L_1 and L_2, where L_1 is the unknot and L_2 is the right-handed trefoil. Then, we get the normalized Alexander polynomials of L_1

and L_2:

$$\frac{t}{t-1}\Delta_{L_1}(t) = 1 + t^{-1} + t^{-2} + \cdots,$$
$$\frac{t}{t-1}\Delta_{L_2}(t) = t + t^{-1} + t^{-2} + \cdots.$$

Since $\Delta_L(x, y) = \frac{(x-1)(y-1)}{\sqrt{xy}}$ and lk $= 0$, by Theorem 5.1.11, we have $a_1^{L_1} = a_1^{L_2}$. This is a contradiction to $a_1^{L_1} = 0$ and $a_1^{L_2} = 1$.

Another proof is as follows. If we used the normalization in Definition 5.1.12 for $L7n2$, then we get $n_{0,0}^{+L_1} = -1$ by Equation (5.1.9). This is a contradiction to Theorem 5.1.13. □

5.2 Applications: classification of L-space surgeries.

Classifying L-space surgeries on an L-space link L is usually challenging. One difficulty is the lack of criterion for hyperbolic L-spaces. In [11], Gorsky and Némethi studied L-space surgeries on the torus links $T(pr, qr)$ with $p, q > 1$ and $r \geq 1$ using Lisca-Stipsicz characterization of Seifert L-spaces. Let us look at the case where $p = 1$ and $r = 2$, i.e. the torus links $L = T(2, 2n)$. We assume $n \geq 2$, since the $T(2, 2)$ torus link is the Hopf link and its surgeries are lens spaces.

When both of p and q are not equal to n, the (p, q)-surgery on $T(2, 2n)$ is a Seifert manifold with three singular fibers over the base S^2. Using the notational convention in [21], we can write $S^3_{p,q}(T(2, 2n)) = -M(0; \frac{1}{n}, \frac{1}{p-n}, \frac{1}{q-n})$. In [21], Lisca and Stipsicz give a characterization of L-space Seifert manifolds.

Theorem 5.2.1 (Theorem 1.1, [21])**.** *Suppose M is an oriented rational homology sphere which is Seifert fibered over S^2. Then, M is an L-space if and only if either M or $-M$ carries no positive, transverse contact structures.*

Theorem 5.2.2 ([20])**.** *An oriented Seifert rational homology sphere $M = M(e_0; r_1, ..., r_k)$ with $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ admits no positive, transverse contact structure if and only if*

- $e_0(M) \geq 0$, *or*

- $e_0(M) = -1$ *and there are no relatively prime integers* $m > a$ *such that*

$$mr_1 < a < m(1 - r_2), \text{ and } mr_i < 1, \ i = 3, ..., k.$$

While the (n, q)-surgery on $T(2, 2n)$ is usually a graph manifold. The (n, q)-surgeries are discussed in Corollary 4.2.6. Direct computation gives the following result.

Proposition 5.2.3 (Classification of L-space surgeries on $T(2, 2n)$ with $n \geq 2$)**.** *For all* $q \neq n$, *the* (n, q)*-surgery on* $T(2, 2n)$ *is an* L*-space.*

When $p \neq n, q \neq n$ *and* $p \geq q$, $S^3_{p,q}(T(2, 2n))$ *is an* L*-space with if and only if one of the following conditions holds:*

1. $n + 2 \leq p, n + 1 \leq q$;

2. $2n \leq p, n - 2 \geq q$, *and there are no relatively prime integers* $m > a > 0$ *such that*

$$m\frac{n - q - 1}{n - q} < a < m(1 - \frac{1}{n}) \text{ and } \frac{m}{p - n} < 1;$$

3. $n + 2 \leq p \leq 2n, q \leq n - 2$, *and there are no relatively prime integers* $m > a > 0$ *such that*

$$m\frac{n - q - 1}{n - q} < a < m(1 - \frac{1}{p - n}) \text{ and } \frac{m}{n} < 1;$$

4. $p = n + 1, q \leq n + 1$, *and* $q \neq n$;

5. $p = n - 1, q \leq n - 1$;

6. $p \leq n - 2, q \leq p$, *and there are no relatively prime integers* $m > a > 0$ *such that*

$$m(1 - \frac{1}{n}) < a < m\frac{1}{n - p} \text{ and } \frac{m}{n - q} < 1.$$

See Figure 5.2.1 for the example of $T(2, 20)$. Compare this result with Theorem 7 in [11].

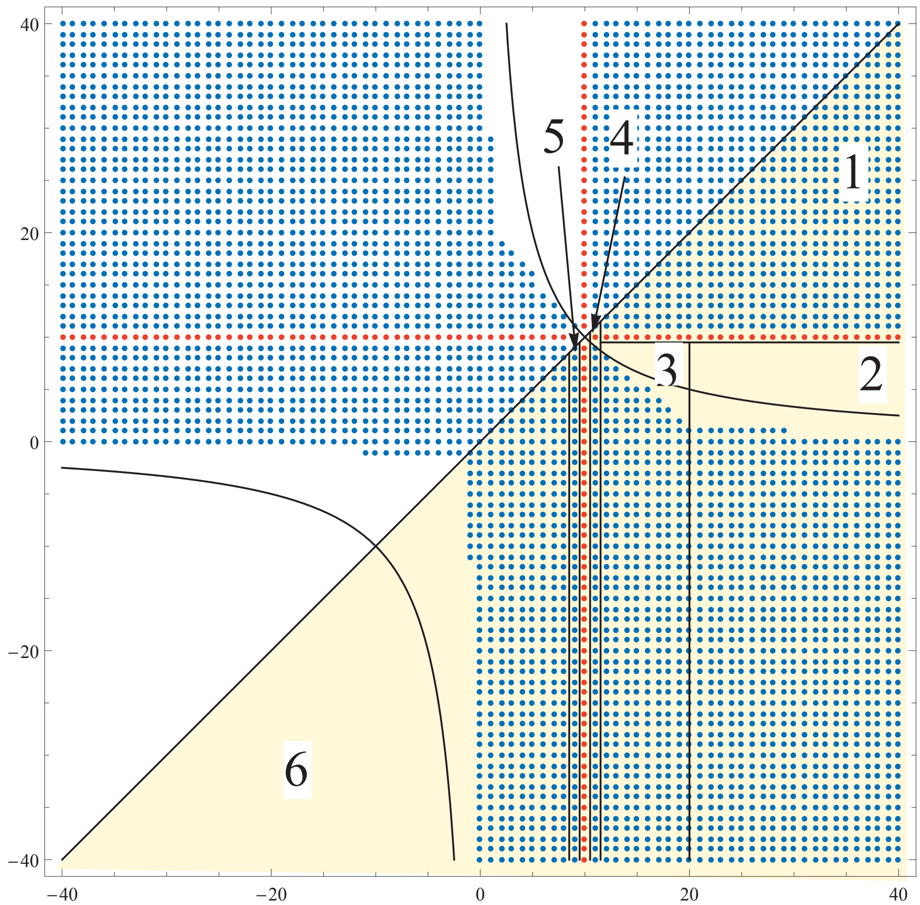


Figure 5.2.1: **The L-space surgeries on $T(2,20)$.** We draw the L-space surgeries of $T(2,20)$ on the x-y plane within the range $[-40,40]\times[-40,40]$. Every dot (p,q) represents an L-space surgery (p,q). The blue points are Seifert L-space surgeries determined by the characterization of Lisca-Stipsicz, while the red points are determined by induction. The six labelled regions correspond to the six conditions (1) to (6) in Proposition 5.2.3. The drawn hyperbola indicates the positions of the surgeries with $b_1 = 1$.

Nevertheless, the links $T(2,2n)$ are the simplest two-bridge links. In order to generally study L-space surgeries on L, we give an algorithm computing $\widehat{HF}(S^3_\Lambda(L))$ using the Alexander polynomials.

Another example is the Whitehead link. By the results in Section 6 in [22] or the method introduced in this section, we can obtain the following proposition. In order to distinguish it with its mirror, we call it the *L-space Whitehead link*.

Proposition 5.2.4. *The (p_1,p_2)-surgery on the L-space Whitehead link is an L-space if and only if $p_1>0, p_2>0$.*

5.2.1 Truncated perturbed surgery complex.

The the link surgery formula is an infinitely generated $\mathbb{F}[[U_1,U_2]]$-module. A truncation procedure is introduced in Section 8.3 in [24] to reduce it to finitely generated $\mathbb{F}[[U_1,U_2]]$-module. It is called *horizontal truncation* in [24], and we just call it *truncation* here. A truncation for the Λ-surgery on a 2-component link L is described by four finite subsets of $\mathbb{H}(L)$,

$$S^{00}(\Lambda), S^{01}(\Lambda), S^{10}(\Lambda), S^{11}(\Lambda).$$

The way of doing truncation is not unique. Later, we will describe an explicit way which depends on L and Λ.

Define

$$\bar{C}^{\delta_1\delta_2}(\Lambda) = \bigoplus_{\mathbf{s}\in S^{\delta_1\delta_2}} \mathfrak{A}^-_{p^{+\delta_1 L_1\cup+\delta_2 L_2}(\mathbf{s})}, \quad \delta_1,\delta_2\in\{0,1\}.$$

Then, the truncated perturbed complex $\bar{C}(\Lambda)$ for an L-space link is defined as follows:

$$(\bar{C}^-(\mathcal{H},\Lambda), \bar{D}^-(\Lambda)) := \begin{tikzcd} \bar{C}^{00}(\Lambda) \arrow[r, "\bar{D}^{10}_{00}(\Lambda)"] \arrow[d, "\bar{D}^{01}_{00}(\Lambda)"'] & \bar{C}^{10}(\Lambda) \arrow[d, "\bar{D}^{01}_{10}(\Lambda)"] \\ \bar{C}^{01}(\Lambda) \arrow[r, "\bar{D}^{10}_{01}(\Lambda)"'] & \bar{C}^{11}(\Lambda), \end{tikzcd} \tag{5.2.1}$$

where $\bar{D}^{\delta_1\delta_2}_{\varepsilon_1\varepsilon_2}(\Lambda)$ are the restrictions of $\tilde{D}^{\delta_1\delta_2}_{\varepsilon_1\varepsilon_2}(\Lambda)$ on the truncated complexes. See Equation (5.1.2) and (5.1.4) for the definitions of $\tilde{D}^{\delta_1\delta_2}_{\varepsilon_1\varepsilon_2}(\Lambda)$. They are determined by the set of integers $n^{\pm L_i}_{\mathbf{s}}$.

The surgery complex naturally splits as a direct sum corresponding to Spin^c structures. For the Λ-surgery on L, there is an identification $\mathrm{Spin}^c(S^3_\Lambda(L)) = \mathbb{H}(L)/H(L,\Lambda)$, where $H(L,\Lambda)$ is the lattice spanned by Λ. For $\mathfrak{u} \in \mathbb{H}(L)/H(L,\Lambda)$, choose $\mathbf{s} = (s_1, s_2) \in \mathfrak{u}$. Denote

$$\bar{C}^{\delta_1\delta_2}(\Lambda,\mathfrak{u}) = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{\substack{j \in \mathbb{Z} \\ s+i\Lambda_1+j\Lambda_2 \in S^{\delta_1\delta_2}}} \tilde{\mathfrak{A}}^-_{s+i\Lambda_1+j\Lambda_2}.$$

Then, the summand $\bar{C}(\Lambda,\mathfrak{u})$ is as follows:

$$(\bar{C}^-(\mathcal{H},\Lambda,\mathfrak{u}), \bar{D}^-(\Lambda,\mathfrak{u})) := \begin{tikzcd} \bar{C}^{00}(\Lambda,\mathfrak{u}) \arrow[r, "\bar{D}^{10}_{00}(\Lambda,\mathfrak{u})"] \arrow[d, "\bar{D}^{01}_{00}(\Lambda,\mathfrak{u})"'] & \bar{C}^{10}(\Lambda,\mathfrak{u}) \arrow[d, "\bar{D}^{01}_{10}(\Lambda,\mathfrak{u})"] \\ \bar{C}^{01}(\Lambda,\mathfrak{u}) \arrow[r, "\bar{D}^{10}_{01}(\Lambda,\mathfrak{u})"'] & \bar{C}^{11}(\Lambda,\mathfrak{u}). \end{tikzcd} \tag{5.2.2}$$

By putting $U_1 = 0$, we can get the chain complex of $\mathbb{F}$-vector spaces $\bar{C}^\wedge(\Lambda,\mathfrak{u})$, whose homology is isomorphic to $\widehat{HF}(S^3_\Lambda(L),\mathfrak{u})$.

Lemma 5.2.5. *Suppose A, B, C, D are finite dimensional $\mathbb{F}$-vector spaces and the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "v_1"'] & B \arrow[d, "v_2"] \\ C \arrow[r, "h_2"'] & D. \end{tikzcd}$$

We form a chain complex (R_*, d_*) supported on degrees $0,1,2$, $R : A \xrightarrow{d_2} B \oplus C \xrightarrow{d_1} D$ with $d_2 = h_1 + v_1$ and $d_1 = h_2 \oplus v_2$. Then, we have the following conclusions

(1) $\dim H_*(R) = 2\dim(\mathrm{Ker} h_1 \cap \mathrm{Ker} v_1) - 2\dim(\mathrm{Im}\ v_2 + \mathrm{Im}\ h_2) - \dim A + \dim B + \dim C + \dim D$;

(2) $\dim H_*(R) = 1$ *iff one of the following is true*

a. $\chi = \dim A - \dim B - \dim C + \dim D = 1$, *and* $\dim(\mathrm{Ker}(h_1) \cap \mathrm{Ker}(v_1)) + \dim \mathrm{Coker}(v_2 + h_2) = 1$.

b. $\chi = \dim A - \dim B - \dim C + \dim D = -1$, *and* $\dim(\mathrm{Ker}(h_1) \cap \mathrm{Ker}(v_1)) + \dim \mathrm{Coker}(v_2 + h_2) = 0$.

Proof. Part (1) is a straightforward computation. Notice that $H_0 = \mathrm{Coker}(h_2 \oplus v_2)$, $H_2 = \mathrm{Ker}(h_1 + v_1)$.

For Part (2), there are only three cases when $H_*(R) = \mathbb{F}$ happens,

1. $H_0(R) = \mathbb{F}, H_1(R) = H_2(R) = 0$;

2. $H_1(R) = \mathbb{F}, H_0(R) = H_2(R) = 0$;

3. $H_2(R) = \mathbb{F}, H_0(R) = H_1(R) = 0$.

In cases (1) and (3), we have that $\chi = 1$ and $\dim H_0 + \dim H_2 = 1$; in case (2), we have that $\chi = -1$ and $\dim H_0 + \dim H_1 = 0$. It is not hard to check the converse. $\square$

If $\mathrm{Ker}(v_1), \mathrm{Ker}(h_1)$ are both known, then computing $\dim(\mathrm{Ker}(v_1) \cap \mathrm{Ker}(h_1))$ is equivalent to computing $\dim(\mathrm{Ker}(v_1) + \mathrm{Ker}(h_1))$, which can be done by Gauss Elimination.

While computing $\mathrm{Coker}(v_2 + h_2)$ is the dual question for computing $\mathrm{Ker}(v_2^*) \cap \mathrm{Ker}(h_2^*)$. While the dual maps v_2^* and h_2^* can be obtained by reversing the arrows, since we are working over $\mathbb{F}$.

We can directly apply the above lemma for each truncated perturbed complex $\bar{C}^{\wedge}(\Lambda, \mathfrak{u})$ for each Spin^c structure. Thus, we only need to describe the truncated regions $S^{00}(\Lambda)$, $S^{01}(\Lambda)$, $S^{10}(\Lambda)$,$S^{11}(\Lambda)$ and the kernels of the maps $\bar{D}^{\wedge **}_{**}(\Lambda, \mathfrak{u})$ and their dual.

Proposition 5.2.6. *Suppose L is an L-space link. Fix a surgery framing Λ and a Spin^c structure $\mathfrak{u}$. Then, $\widehat{HF}(\Lambda, \mathfrak{u}) = \mathbb{F}$ iff in the truncated complex $\bar{C}^{\wedge}(\Lambda, \mathfrak{u})$, one of the following is true,* *(A)* $\#S^{00}(\Lambda,\mathfrak{u}) - \#S^{01}(\Lambda,\mathfrak{u}) - \#S^{10}(\Lambda,\mathfrak{u}) + \#S^{11}(\Lambda,\mathfrak{u}) = 1$, *and* $\dim(\mathrm{Ker}(\hat{D}^{01}_{00}) \cap \mathrm{Ker}(\hat{D}^{10}_{00})) + \dim \mathrm{Coker}(\hat{D}^{10}_{01} + \hat{D}^{01}_{10}) = 1$.

(B) $\#S^{00}(\Lambda,\mathfrak{u}) - \#S^{01}(\Lambda,\mathfrak{u}) - \#S^{10}(\Lambda,\mathfrak{u}) + \#S^{11}(\Lambda,\mathfrak{u}) = -1$, *and* $\dim(\mathrm{Ker}(\hat{D}^{01}_{00}) \cap \mathrm{Ker}(\hat{D}^{10}_{00})) + \dim \mathrm{Coker}(\hat{D}^{10}_{01} + \hat{D}^{01}_{10}) = 0$.

5.2.2 Truncations.

We explicitly describe the truncated regions $S^{00}(\Lambda), S^{01}(\Lambda), S^{10}(\Lambda), S^{11}(\Lambda)$ here. Let us briefly recall the procedure to form these truncated regions for a general two-component link L in Section 8.3 [24].

1. Choose a number $b \in \mathbb{N}$, such that the inclusion maps $I^{\pm L_i}_{s_1,s_2}$'s are quasi-isomorphisms whenever $\pm s_i \geq b$.

2. Determine a parallelogram Q in the plane, with vertices P_1, P_2, P_3, P_4 counterclockwise labelled, satisfying the following condition: The point P_i has the coordinate (x_i, y_i) such that
$$\begin{cases} x_1 > b \\ y_1 > b, \end{cases} \quad \begin{cases} x_2 < -b \\ y_2 > b, \end{cases} \quad \begin{cases} x_3 < -b \\ y_3 < -b, \end{cases} \quad \begin{cases} x_4 > b \\ y_4 < -b. \end{cases} \tag{5.2.3}$$
We also require that every edge is either parallel to the vector Λ_1 with length greater than $\|\Lambda_1\|$ or parallel to Λ_2 with length greater than $\|\Lambda_2\|$.

3. Decide which is the case among the six cases of the surgeries described in Figure 22 in [24]. Then, we can decide the corresponding truncated regions according to Section 8.3 in [24].

The way of doing truncation is not unique. One explicit way to choose the parallelogram Q to be centered at the origin as follows. See Figure 5.2.2.

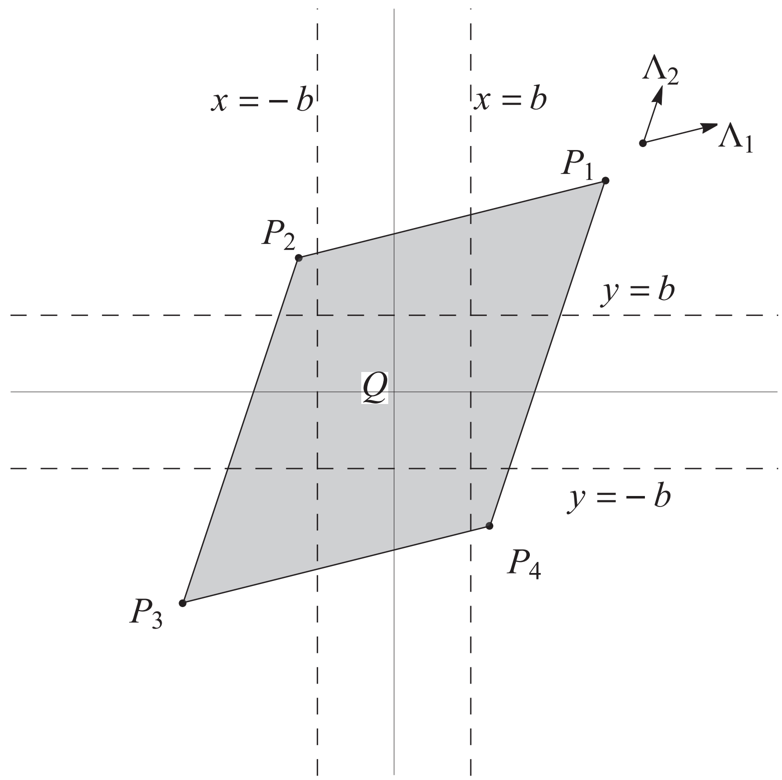


Figure 5.2.2: **The truncation.** The vectors Λ_1 and Λ_2 are determined by the surgery framing matrix. The edges of the parallelogram Q are parallel to Λ_1 and Λ_2, and they indicate the border lines of various acyclic subcomplexes or quotient complexes. Thus, the parallelogram Q roughly indicates the support of the truncated complex.

Let

$$\{P_1, P_2, P_3, P_4\} = \left\{ \frac{i_0\Lambda_1 + j_0\Lambda_2}{2}, \frac{-i_0\Lambda_1 + j_0\Lambda_2}{2}, \frac{i_0\Lambda_1 - j_0\Lambda_2}{2}, \frac{-i_0\Lambda_1 - j_0\Lambda_2}{2} \right\},$$

with i_0, j_0 being positive integers, such that Equations (5.2.3) hold.

Fix Λ and $\mathfrak{u} \in \mathbb{H}(L)/H(L,\Lambda)$. Suppose

$$\mathbf{s} = \theta_1\Lambda_1 + \theta_2\Lambda_2 \in \mathfrak{u}, \quad P_1 = a_1\Lambda_1 + a_2\Lambda_2.$$

We denote

$$\begin{aligned} A_1 &= \lceil -\theta_1 - |a_1| \rceil, & A_2 &= \lfloor -\theta_1 + |a_1| \rfloor, \\ B_1 &= \lceil -\theta_2 - |a_2| \rceil, & B_2 &= \lfloor -\theta_2 + |a_2| \rfloor. \end{aligned}$$

Then, the truncated regions in the six cases are as follows.

Case I

$$\begin{aligned} S^{00}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q, \\ S^{10}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q \cap (Q + \Lambda_1), \\ S^{01}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q \cap (Q + \Lambda_2), \\ S^{11}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q \cap (Q + \Lambda_1 + \Lambda_2). \end{aligned}$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{ s + i\Lambda_1 + j\Lambda_2 | A_1 + \delta_1 \le i \le A_2, B_1 + \delta_2 \le j \le B_2. \}.$$

Case II

$$S^{00}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap Q,$$
$$S^{10}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cup (Q + \Lambda_1)\},$$
$$S^{01}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cup (Q + \Lambda_2)\},$$
$$S^{11}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cup (Q + \Lambda_1) \cup (Q + \Lambda_2) \cup (Q + \Lambda_1 + \Lambda_2)\}.$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{s + i\Lambda_1 + j\Lambda_2 | A_1 - \delta_1 \leq i \leq A_2, B_1 - \delta_2 \leq j \leq B_2.\}.$$

Case III

$$S^{00}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap Q,$$
$$S^{10}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cap (Q + \Lambda_1)\},$$
$$S^{01}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cup (Q + \Lambda_2)\},$$
$$S^{11}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{[Q \cup (Q + \Lambda_2)] \cap ([Q \cup (Q + \Lambda_2)] + \Lambda_1)\}.$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{s + i\Lambda_1 + j\Lambda_2 | A_1 + \delta_1 \leq i \leq A_2, B_1 \leq j \leq B_2 + \delta_2.\}.$$

Case IV

$$S^{00}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap Q,$$
$$S^{10}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cup (Q + \Lambda_1)\},$$
$$S^{01}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{Q \cap (Q + \Lambda_2)\},$$
$$S^{11}(\Lambda, \mathfrak{u}) = \mathfrak{u} \cap \{[Q \cap (Q + \Lambda_2)] \cup ([Q \cap (Q + \Lambda_2)] + \Lambda_1)\}.$$

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{s + i\Lambda_1 + j\Lambda_2 | A_1 \le i \le A_2 + \delta_1, B_1 + \delta_2 \le j \le B_2.\}.$$

Case V This case is similar to **Case I**, but the regions $S^{10}(\Lambda, \mathfrak{u}), S^{01}(\Lambda, \mathfrak{u})$ have two more points at the corners.

$$\begin{aligned}
S^{00}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q, \\
S^{10}(\Lambda, \mathfrak{u}) &= (\mathfrak{u} \cap Q \cap (Q + \Lambda_1)) \cup T^{10}, \\
S^{01}(\Lambda, \mathfrak{u}) &= (\mathfrak{u} \cap Q \cap (Q + \Lambda_2)) \cup T^{01}, \\
S^{11}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q \cap (Q + \Lambda_1 + \Lambda_2),
\end{aligned}$$

where $T^{10} = \{\mathbf{s} + A_2\Lambda_1 + B_1\Lambda_2\}, T^{10} = \{\mathbf{s} + A_1\Lambda_1 + B_2\Lambda_2\}$.

In other words, for $\delta_1, \delta_2 \in \{0, 1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{s + i\Lambda_1 + j\Lambda_2 | A_1 + \delta_1 \le i \le A_2, B_1 + \delta_2 \le j \le B_2.\} \cup T^{\delta_1\delta_2},$$

where $T^{00} = T^{11} = \emptyset$.

Case VI This is similar to **Case V**.

$$\begin{aligned}
S^{00}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q \cap (Q - \Lambda_1 - \Lambda_2), \\
S^{10}(\Lambda, \mathfrak{u}) &= (\mathfrak{u} \cap Q \cap (Q - \Lambda_1)) \cup T^{10}, \\
S^{01}(\Lambda, \mathfrak{u}) &= (\mathfrak{u} \cap Q \cap (Q - \Lambda_2)) \cup T^{01}, \\
S^{11}(\Lambda, \mathfrak{u}) &= \mathfrak{u} \cap Q,
\end{aligned}$$

where $T^{10} = \mathbf{s} + A_1\Lambda_1 + B_2\Lambda_2, T^{01} = \mathbf{s} + B_1\Lambda_2 + A_1\Lambda_1$.

In other words, for $\delta_1, \delta_2 \in \{0,1\}$,

$$S^{\delta_1\delta_2}(\Lambda, \mathfrak{u}) = \{s + i\Lambda_1 + j\Lambda_2 | A_1 + 1 - \delta_1 \leq i \leq A_2, B_1 + 1 - \delta_2 \leq j \leq B_2.\} \cup T^{\delta_1\delta_2},$$

where $T^{00} = T^{11} = \emptyset$.

Remark 5.2.7. In all of the above cases, $\#S^{00}(\Lambda,\mathfrak{u}) - \#S^{01}(\Lambda,\mathfrak{u}) - \#S^{10}(\Lambda,\mathfrak{u}) + \#S^{11}(\Lambda,\mathfrak{u}) = \pm 1$.

5.2.3 Kernel of $\bar{D}^{\wedge **}_{**}(\Lambda, \mathfrak{u})$

In fact, all the mapping cones of $\bar{D}^{\wedge **}_{**}(\Lambda, \mathfrak{u})$ split as a direct sum of mapping cones in a common form. They look like the mapping cones in computing +1-surgery on knots. Since this type of mapping cones looks like zigzags, we just call them "zigzags". We denote the set of integers in $[a,b]$ by $[a;b]$, where we allow $a = b$.

Definition 5.2.8 (Zigzags)**.** A *zigzag mapping cone* C is a mapping cone of $\mathbb{F}$-vector spaces:

$$\bigoplus_{a_1 \leq s \leq a_2} A_s \xrightarrow{f+g} \bigoplus_{b_1 \leq t \leq b_2} B_t,$$

where

$$\begin{aligned}
&A_s = \mathbb{F}, \ \forall a_1 \leq s \leq a_2,\\
&B_t = \mathbb{F}, \ \forall b_1 \leq t \leq b_2,\\
&f = \bigoplus f_s, \quad f_s : A_s \to B_s,\\
&g = \bigoplus g_s, \quad g_s : A_s \to B_{s+1}.
\end{aligned}$$

The *code* of the zigzag C is a set of data $\{[a_1;a_2], [b_1;b_2], S_1, S_2\}$, where

$$S_1 = \{s \in \mathbb{Z} | f_s \neq 0\}, \tag{5.2.4}$$

$$S_2 = \{s \in \mathbb{Z} | g_s \neq 0\}. \tag{5.2.5}$$

We define $\mathrm{Ker}(C)$ (resp. $\mathrm{Coker}(C)$) to be $\mathrm{Ker}(f+g)$ (resp. $\mathrm{Coker}(f+g)$).

Definition 5.2.9. For any element x in $\bigoplus_{a_1 \le s \le a_2} \mathbb{F} \cdot e_s$, we can represent it uniquely by $x = \sum_{s \in \Gamma} e_s$. We call Γ the *support* of x, and denote it by $\mathrm{Supp}(x)$. Similarly, for $X = \{x_1, ..., x_n\}$, we denote $\{\mathrm{Supp}(x_1), ..., \mathrm{Supp}(x_n)\}$ by $\mathrm{Supp}(X)$.

Proposition 5.2.10. *For a zigzag C with the code $\{[a_1, a_2], [b_1, b_2], S_1, S_2\}$, we represent $S_1 \cap S_2$ by a minimal disjoint unions*

$$S_1 \cap S_2 = \coprod_{i \in [1;K]} [\alpha_i; \beta_i],$$

with $\beta_i \le \alpha_{i+1} + 2, \forall i$. Then, $\mathrm{Ker}(C)$ *has a basis with the following support*

$$\Big\{\{s\}\Big| s \in [a_1, a_2] \backslash (S_1 \cup S_2)\Big\} \cup \Big\{[\alpha_j - 1, \beta_j + 1]\Big| \alpha_j - 1 \in S_2, \beta_j + 1 \in S_1\Big\}.$$

Proof. Straightforward. □

Definition 5.2.11. Let $L = L_1 \cup L_2$ be an L-space link. For all $s_1 \in \mathbb{H}_1(L), s_2 \in \mathbb{H}_2(L)$, we define

$$\nu_{s_1}^{+L_2}(L) = \min\{s_2 \in \mathbb{H}_2(L) \Big| n_{s_1,s_2}^{+L_2} \neq 0\}, \tag{5.2.6}$$

$$\nu_{s_2}^{+L_1}(L) = \min\{s_1 \in \mathbb{H}_1(L) \Big| n_{s_1,s_2}^{+L_1} \neq 0\}. \tag{5.2.7}$$

It is easy to see that in Section 5.1.3 we can let $b = \max\{\max\{\nu_{s_1}^{+L_2}(L)\}_{s_1}, \max\{\nu_{s_2}^{+L_1}(L)\}_{s_2}\}$. Moreover, the truncated perturbed complex $\bar{C}^{\wedge}(\Lambda)$ is determined by these $\nu_{s_1}^{+L_2}(L)$'s and $\nu_{s_2}^{+L_1}(L)$'s, and thereby so are the zigzag mapping cones corresponding to $\bar{D}^{\wedge **}_{**}(\Lambda, \mathfrak{u})$'s.

For example, suppose

$$\Lambda = \begin{pmatrix} p_1 & \mathrm{lk} \\ \mathrm{lk} & p_2 \end{pmatrix}$$

and choose $\mathbf{s} = (s_1, s_2) \in \mathfrak{u}$. Before truncation, we have

$$\mathrm{cone}(\hat{D}^{01}_{00}(\Lambda, \mathfrak{u})) = \prod_{i \in \mathbb{Z}} \mathrm{cone}(\prod_{j \in \mathbb{Z}} (\hat{\Phi}^{+L_2}_{\mathbf{s}+i\Lambda_1+j\Lambda_2} + \hat{\Phi}^{-L_2}_{\mathbf{s}+i\Lambda_1+j\Lambda_2})).$$

After truncation, $\mathrm{cone}(\bar{D}\hat{}^{01}_{00}(\Lambda, \mathfrak{u}))$ splits into direct sums of zigzags in form of

$$\mathrm{cone}(\prod_{j \in \mathbb{Z}} (\hat{\Phi}^{+L_2}_{\mathbf{s}+i\Lambda_1+j\Lambda_2} + \hat{\Phi}^{-L_2}_{\mathbf{s}+i\Lambda_1+j\Lambda_2})) \cap \bar{C}\hat{}(\Lambda).$$

Let us figure out the codes of these zigzags. Suppose the code of the above zigzag is

$$\{[a_1; a_2], [b_1; b_2], S_1, S_2\}.$$

Then, it is not hard to get the following formulas for the code,

$$\begin{cases} [a_1; a_2] = \{j \in \mathbb{Z} | \mathbf{s} + i\Lambda_1 + j\Lambda_2 \in S^{00}(\Lambda, \mathfrak{u})\}, \\ [b_1; b_2] = \{j \in \mathbb{Z} | \mathbf{s} + i\Lambda_1 + j\Lambda_2 \in S^{01}(\Lambda, \mathfrak{u})\}, \\ S_1 = \{j \in \mathbb{Z} | s_2 + i \cdot \mathrm{lk} + j \cdot p_2 \geq \nu^{+L_2}_{s_1+i\cdot p_1+j\cdot \mathrm{lk}}(L)\}, \\ S_2 = \{j \in \mathbb{Z} | s_2 + i \cdot \mathrm{lk} + j \cdot p_2 \leq -\nu^{+L_2}_{-s_1-i\cdot p_1-j\cdot \mathrm{lk}}(L)\}. \end{cases}$$

5.2.4 Examples: L-space surgeries on two-bridge links

From Proposition 4.1.10, we see that if a two-bridge link has an L-space surgery, then it is a generalized L-space link. By taking mirrors, we can reduce these links to two types: L-space links and generalized $(+-)L$-space links. We have discussed two-bridge L-space links in Chapter 4. By the method in this section, it is convenient to make computer programs for computing $\widehat{HF}$ of their surgeries and give characterizations of L-space surgeries. For example, regarding the surgeries on the Whitehead link, we can do truncations as in Proposition 6.9 in [22] and then use the method of zig-zags in Section 5.2.3 to recover the results in Proposition 6.9 [22] for the hat version. Thus, we can obtain Proposition 5.2.4.

However, to find a general formula of $\widehat{HF}$ is not easy.

In fact, finding L-space homology spheres is more interesting. Let us try some examples here, by looking at the $(1,1)$-surgeries on a sequence of two-bridge links $L_n = b(4n^2 + 4n, -2n-1)$ for all positive integers n. This sequence of L-space links have linking numbers 0. Note that L_1 is the Whitehead link.

Proposition 5.2.12. *For all $n \geq 2$, the $(1,1)$-surgery on $b(4n^2+4n, -2n-1)$ is not an L-space.*

With the help of a computer program, we get the Alexander polynomials of L_n:

$$\Delta_{L_n}(x,y) = \sum_{j=-n}^{n-1} \sum_{i=-n-\frac{1}{2}+|j+\frac{1}{2}|}^{n-\frac{1}{2}-|j+\frac{1}{2}|} (-1)^{i+j} x^{i+\frac{1}{2}} y^{j+\frac{1}{2}}.$$

After normalizing $\Delta_{L_n}(x,y)$ by Definition 5.1.12, we can get formulas for $n^{+L_1}_{s_1,s_2}$ by Equation (5.1.9), (5.1.8). We list the numbers $\{n^{+L_2}_{s_1,s_2}(L_n)\}_{-4\leq s_1\leq 4, -4\leq s_2\leq 4}$ for $n = 1,2,3,4$ as follows:

$$\{n^{+L_2}_{s_1,s_2}(L_1)\} : \left\{\begin{matrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&2&2\\ 3&3&3&3&3&3&3&3&3\\ 4&4&4&4&4&4&4&4&4 \end{matrix}\right\} ; \{n^{+L_2}_{s_1,s_2}(L_2)\} : \left\{\begin{matrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 1&1&1&1&2&1&1&1&1\\ 2&2&2&2&2&2&2&2&2\\ 3&3&3&3&3&3&3&3&3\\ 4&4&4&4&4&4&4&4&4 \end{matrix}\right\}.$$

$$\{n^{+L_2}_{s_1,s_2}(L_3)\}: \left\{\begin{matrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&1&1&2&1&1&0&0\\ 1&1&1&2&2&2&1&1&1\\ 2&2&2&2&3&2&2&2&2\\ 3&3&3&3&3&3&3&3&3\\ 4&4&4&4&4&4&4&4&4 \end{matrix}\right\}; \{n^{+L_2}_{s_1,s_2}(L_4)\}: \left\{\begin{matrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&1&1&2&1&1&0&0\\ 0&1&1&2&2&2&1&1&0\\ 1&1&2&2&3&2&2&1&1\\ 2&2&2&3&3&3&2&2&2\\ 3&3&3&3&4&3&3&3&3\\ 4&4&4&4&4&4&4&4&4 \end{matrix}\right\}.$$

In particular, we get the following formulas for all $s_1 \in \mathbb{Z}$,

$$\nu^{+L_2}_{s_1}(L_n) = \begin{cases} n - |s_1|, & |s_1| \leq n, \\ 0, & |s_1| \geq n. \end{cases}$$

Since L_n is a two-bridge link, we have the symmetry $\nu^{+L_1}_{s_2} = \nu^{+L_2}_{s_1}$, when $s_1 = s_2$.

Thus, we can let $b(L_n) = n$. Then, as described in Section 5.2.3, the truncation regions are determined by the parallelogram Q, with vertices $P_1 = (n, n), P_2 = (-n, n), P_3 = (-n, -n), P_4 = (n, -n)$. The surgery framing is in Case I, so we have the truncated regions

$$S^{\delta_1\delta_2} = \left\{(i, j) \in \mathbb{Z}^2 \middle| -n + \delta_1 \leq i \leq n, -n + \delta_2 \leq j \leq n\right\}.$$

Now we can see

$$\hat{\Phi}^{\pm L_i}_{s_1,s_2} = 0, \forall -n+1 \leq s_1 \leq n-1, -n+1 \leq s_2 \leq n-1, i = 1, 2.$$

So $\hat{\mathfrak{A}}_{s_1,s_2} \in \bar{C}^{00}$ with $-n < s_1 < n, -n < s_2 < n$ are all in the kernel of $\hat{D}^{10}_{00}$ and $\hat{D}^{01}_{00}$. So when $n \geq 2$, we have that $\mathrm{Ker}(\hat{D}^{01}_{00}) \cap \mathrm{Ker}(\hat{D}^{10}_{00})$ has rank at least $n^2 + (n-1)^2 > 1$. Thus, by Proposition 5.2.6, the $(1, 1)$-surgeries on L_n with $n \geq 2$ are never L-spaces. Similar arguments apply to $(\pm1, \pm1)$-surgeries on these links.

Proposition 5.2.13. *On the two-bridge L-space links $L_n = b(4n^2+4n, -2n-1)$ with $n \geq 2$, there are no L-space homology sphere surgeries.*

In fact, direct computations using the zigzags give that $\widehat{HF}(S^3_{1,1}(L_n))$ has dimension $(2n-1)^2$.

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