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Weak Solutions to the Muskat Problem with Surface Tension via Optimal Transport

A dissertation submitted in partial satisfaction of the requirements for the degree

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by

Ryan Carlton Wallace

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ABSTRACT OF THE DISSERTATION

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Professor Christina Kim, Chair

The Muskat problem, which models the flow of immiscible viscous fluids in a porous medium, has been studied extensively in recent years from a number of perspectives. While most studies have focused on harmonic analysis techniques in the graphical setting, the formation of topological singularities makes it desirable to have a notion of weak solution that exists globally in time. In the case of the Muskat problem, this is possible due to the *gradient flow* structure of the problem relative to the quadratic Wasserstein distance from optimal transport theory, which makes available the JKO or Minimizing Movements scheme [JKO98; AGS05]. This scheme constructs discrete-in-time solutions for a given timestep $h > 0$, then finds a continuous-time solution by taking the limit as $h \to 0$. In this dissertation, we study the corresponding weak solutions in two settings.

In the first chapter, we study the *discrete* (in time) JKO solutions to the Muskat problem in a two-dimensional square domain. We show that if the energy of the initial configuration is sufficiently small, then asymptotically any discrete solution is close in C^1 to the global equilibrium consisting of a flat interface separating the heavier fluid on

bottom from the lighter on top, modulo possible “drops”. Further, if the surface tension is sufficiently strong, and the discrete solution converges to the global equilibrium, then this convergence occurs exponentially fast.

The second chapter uses a modified JKO scheme to model a situation in which a source and sink are present. Using estimates similar to those from the standard scheme, we show convergence of the discrete solutions to a continuous solution, though with some caveats.

The results of the first chapter of this thesis will be published in a mathematical journal.

The dissertation of Ryan Carlton Wallace is approved.

Andrea Bertozzi

Wilfrid Dossou Gangbo

Georg Menz

Christina Kim, Committee Chair

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For my family

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CHAPTER 1

Long-term Behavior of Discrete Solutions in Two-Dimensional Square Container

1.1 Introduction

1.1.1 Background

The Muskat problem, first introduced in [Mus34], models the flow of two incompressible, immiscible fluids moving in a porous medium. Because of its importance to problems in applied mathematics such as oil flows in reservoirs and groundwater flows in aquifers, it has been studied extensively from a variety of perspectives since its introduction. In this chapter we study the long-time behavior of discrete-in-time approximations to the flow in a particular two-dimensional setting.

The mathematical setup is as follows: the porous medium will be represented by the open unit square $\Omega := (0,1)^2 \subseteq \mathbf{R}^2$, a generic point of which we denote by $x = (x_1, x_2)$. We will think of the x_2 direction as vertical. We assume, crucially for our purposes, that there is surface tension between the two fluids, with coefficient σ. The gravitational potential corresponding to fluid i is $g_i x_2$, where $g_1 > g_2$ (i.e. the first fluid is heavier). We denote by E_i the set occupied by fluid i; since the fluids are immiscible and fill Ω, we have $E_1 \cap E_2 = \emptyset, E_1 \cup E_2 = \Omega$ up to measure zero. The velocity of fluid i is denoted by

v_i and is given by Darcy's law

$$v_i = -\frac{k}{\mu_i}(\nabla p_i + g_i \vec{e_2}), \tag{1.1.1}$$

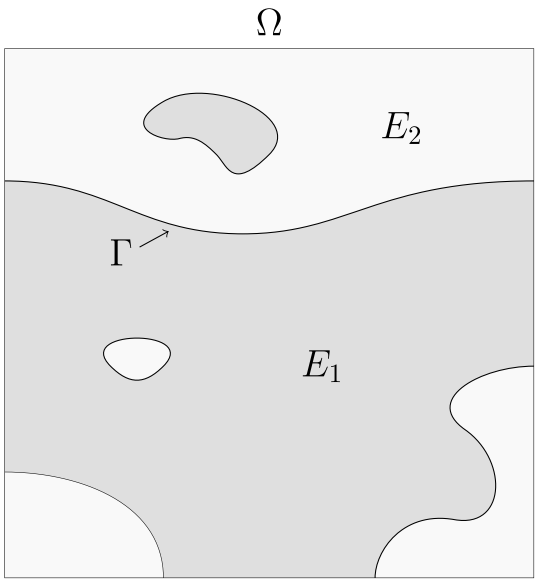
where p_i is the pressure acting on phase i, $k > 0$ is the permeability of the medium, μ_i is the viscosity of phase i, and $\vec{e_2} = (0,1)$ is the vertical unit vector. We call $b_i := k/\mu_i$ the mobility of phase i. Finally, the pressures solve the following free boundary problem:

$$\begin{cases} -\Delta p_i = \nabla \cdot (g_i \vec{e_2}) = 0 & \text{in } E_i \\ \frac{\partial}{\partial \nu}(p_i + g_i y) = 0 & \text{on } \partial\Omega \\ V := b_1 \frac{\partial}{\partial \nu}(p_1 + g_1 x_2) = b_2 \frac{\partial}{\partial \nu}(p_2 + g_2 x_2) & \text{on } \Gamma \\ [\![p]\!] := p_1 - p_2 = \sigma H & \text{on } \Gamma \\ \nu = \tilde{\nu} & \text{on } \partial\Gamma \cap \partial\Omega. \end{cases} \tag{1.1.2}$$

Here, Γ denotes the interface between the fluids, ν is the outer unit normal on Γ (relative to E_1) and $\partial\Omega$ (relative to Ω), V is the normal velocity of the interface in the ν direction, $\tilde{\nu}$ is the co-normal vector tangent to Γ and pointing outside of Ω, and H is the mean curvature of Γ, positive if E_1 is convex. The first condition encodes incompressibility, the second says the fluids cannot enter or exit Ω, and the third says that the fluid interface is advected by the fluids (i.e., there is no exchange of mass as in phase transitions). The fourth condition is known in the static case as the Young-Laplace equation, and is a balance-of-forces equation on the interface, the forces being pressure and surface tension. The last condition, known in the static case as the Young equation, says that the interface must intersect the container walls at right angles (at non-corner points). Mathematically, the problem is equivalent to a two-fluid vertical Hele-Shaw cell. A typical example is shown in Figure 1.

The Muskat problem, including variants with only one fluid, has been studied intensively by a number of authors. Most results have focused on graphical solutions with

Figure 1.1: Example configuration of fluids; the heavier fluid is dark gray, the lighter fluid is light gray.



sufficient regularity, which allows the use of harmonic analysis methods. For a (very nonexhaustive) sample of such studies, including without surface tension, see [Amb04; Con+12; SCH04; Con+16; Con+17; CCG11]. Somewhat different approaches can be found in [PS16b; PS16a; EM11]. These approaches generally do not allow for topological changes in the phases E_1 and E_2, such as pinching, coalescing, etc. However, in general topological singularities are unavoidable [EM11; Cas+13], so it is desirable to have a notion of weak solution which can accommodate such changes. Such an approach has been studied by various authors, especially Otto [Ott98; GO01], and is based crucially on the observation that the Muskat problem turns out to be the *gradient flow* of a free energy functional with respect to the (weighted product) Wasserstein metric. This allows the use of the Euler implicit scheme, also known as de Giorgi's minimizing movements scheme [AGS05] or JKO scheme in the Wasserstein case [JKO98], to construct discrete-in-time approximate solutions for a chosen timestep $h > 0$. By letting $h \to 0$ and taking the limit, one obtains a candidate for a weak solution to the Muskat problem starting from general initial data. To show that this is a genuine weak solution, it must be assumed that there is no loss of perimeter when taking the limit; i.e., the time-integrated perimeters of the approximate solutions converge to that of the limiting solution [Ott98].

Solutions obtained in this way are often called *flat flows* in other settings, such as the mean curvature flows and Mullins-Sekerka flows (see below).

This scheme was recently modified in [JKM21], in which the authors replace the energy functional (1.1.3) with an approximation inspired by the Merriman-Bence-Osher (MBO) scheme for mean curvature flow [MBO92] in order to make the scheme amenable to numerical simulations. See also [CL21], which gives a complete proof of global existence for the original scheme of Otto [Ott98], with a slightly generalized notion of solution involving varifolds that allows the perimeter convergence assumption to be dropped.

So far, the JKO scheme has only been applied to show existence of global-in-time weak solutions to the Muskat/Hele-Shaw problem for general intitial conditions, usually under conditional perimeter convergence. However, to the knowledge of the author, no in-depth studies of particular settings have been published. In this chapter, we focus on a particular simple 2D geometry, and study the *discrete* solutions coming from the JKO scheme for a fixed timestep $h > 0$. In particular, we show that if the initial configuration has energy sufficiently close to the global energy minimizer (heavier fluid on bottom, lighter on top, flat interface between), then eventually the discrete solution is also close to the global minimizer in a C^1 sense, modulo the possible presence of a small amount of displaced fluid at the top and bottom of the container. Additionally, if surface tension is sufficiently strong, and if there is no displaced fluid present after some point, then the solution converges exponentially fast to the global minimizer, both in terms of energy and in Hausdorff distance.

The methods and results of this chapter are inspired by similar ones for the volume-preserving mean curvature flow (MCF) in $\mathbf{R}^d$, another important geometric motion that has been studied from the perspective of minimizing movements [LS95; MSS16; ATW93]. In this motion we have a family of moving sets E_t whose outer normal velocity $V(x,t)$ at $x \in \partial E_t$ is given by the negative of the mean curvature $H_{\partial E_t}(x)$ of ∂E_t at x, adjusted by a constant to preserve volume: $V(x,t) = \hat{H}(t) - H_{\partial E_t}(x)$, where $\hat{H}(t)$ is the average of $H_{\partial E_t}$ over ∂E_t. In particular, our results are inspired by [MPS22], in which the authors prove that any discrete solution must converge in C^k, $k \geq 1$, exponentially fast to a disjoint union of balls of equal volume. The statement of our result is clearly similar, although in general we cannot say whether the full discrete solution converges to anything. Instead, we examine subsequential limits of the discrete solution, and show that these must be smooth critical points of the energy functional (under locally volume preserving perturbations). By energy dissipation, they must have energy lower than the initial data. Elementary geometric arguments show that smooth critical points of the energy functional with sufficiently low energy are constrained to have a particular form, which is essentially the same as the global energy minimizer, modulo possible displaced fluid on the top and bottom of the container. Finally, by proving a uniform minimality result for the discrete sets E_1^n, we upgrade convergence in L^1 to convergence in C^1, as in [MPS22]. In both cases, the key for the exponential convergence is to prove an "energy-energy dissipation" inequality, bounding the energy dissipated at each step by the difference in energy between the current step and the global minimum. This essentially boils down to certain PDE inequalities, which in [MPS22] takes the form of a "Quantitative Alexandrov Theorem".

Regarding the structure of smooth critical points of the energy functional, we mention the paper [DM19], in which the authors show that critical points of the perimeter functional *among sets of finite perimeter*, i.e. with no regularity assumptions, are disjoint unions of balls. Since these more general sets appear as the limits of solutions in the continuous-time case, analyzing their structure is important for the asymptotic study of weak solutions to the MCF, and likewise for other gradient flows. It would be interesting to see if similar results can be proved in the more complicated situation of added potential energy due to gravity, as in our setting.

Another related geometric motion that has been studied via minimizing movements is the Mullins-Sekerka equation [LS95; Rög05; Jul+21], which models phase transitions in materials. Here, the moving interface Γ is the boundary separating the two phases, and its normal velocity V is given by the jump in the normal derivative of the temperature function u across the interface, where u is harmonic away from the interface and satisfies $u = H_\Gamma$ on Γ. Recently, the authors in [Jul+21] were able to analyze the long-term behavior of the limiting time-continuous flow of both the volume-preserving mean curvature flow and the Mullins-Sekerka flow in $\mathbf{R}^2$, improving earlier work for MCF in [JN20; FJM22]. As in the discrete MCF case [MPS22], they prove exponential convergence to a disjoint union of discs. The key in both cases is to use the natural estimates coming from the minimizing movements scheme to deduce geometric information about the moving interface. In our case this is more difficult, in part due to the higly nonlocal nature of the flow (and the appearance of the difficult Wasserstein distance), but is an interesting possibility for further research. (See remark 1.2.13 below.)

Partly because the implicit scheme for mean curvature flow is in some respects more tractable than in the Muskat case, some more detailed results have been proved, including analyzing special situations such as convex or graphical sets [DL18; CC06; Log16]. We hope that the present work will be a first step towards extending these efforts to the Muskat setting. In particular, we mention the geometric Theorem 1.2.7, which is used to prove that the discrete sets $\{E_1^n\}_{n=1}^\infty$ satisfy a uniform (in n) almost-minimality condition involving perimeter, which is a key step for upgrading L^1 convergence to C^1 convergence. In the MCF case this is much easier, because the metric term appearing in the minimization step is linear in the test set E and is well-defined for sets E of any volume, not just sets of the same size as the previous set E^{n-1}, making it easy to compare E^n directly to any other set E. (In the Mullins-Sekerka case the metric term is not quite linear, but is still given by an explicit formula that is amenable to manipulation.) By contrast, the Wasserstein term in (1.1.4) below is highly nonlinear and only makes sense for sets E with the same measure as E_1^{n-1}. Thus, to study the perimeter-minimizing properties of E_1^n, we must first modify a competing set E to have the same measure as E_1^n, while keeping good control on the change in perimeter induced by this modification. In Theorem 1.2.7, we do this by a simple, constructive method in which the perimeter change in E is controlled by the diameter of the largest component of E_1^n. By the perimeter bounds coming from the minimizing movements scheme, this diameter must be at least some constant l independent of n. Hence, the uniform minimality of the sets E_1^n is established.

The outline of the chapter is as follows: in the first section, we introduce the discrete scheme and establish its regularity properties, and then prove the result about the structure of low-energy critical points alluded to above, establishing subsequential convergence in L^1 to critical points similar to the global minimizer. Then we prove the geometric Theorem 1.2.7, which establishes uniform almost-minimality of the discrete sets and consequently (by the classical regularity theory) convergence in C^1. In the second section, we upgrade this to exponential convergence by proving an energy dissipation inequality as in [MPS22], assuming convergence to the global minimizer.

1.1.2 Statement of Results

Again, our setting is the open unit square $\Omega = (0,1)^2$. In general, (E_1, E_2) denotes a partition of Ω: $|E_1 \cap E_2| = 0$, $|\Omega \backslash (E_1 \cup E_2)| = 0$, and E_i represents fluid i. We will restrict to the special case where both fluids occupy equal volumes: $|E_1| = |E_2| = 1/2$. Since surface tension is proportional to area, we will restrict to sets of finite perimeter. Recall that a set E is said to have finite perimeter in the open set A if the *perimeter of E in A*

$$P(E; A) := \sup\{\int_E \nabla \cdot \xi \,\mathrm{d}x : \xi \in C_c^1(A; \mathbf{R}^2)\} < \infty.$$

We will abbreviate $P(E; \Omega)$ as $P(E)$, and will always assume we have chosen the Lebesgue representative of E. (See [Mag12] for more on finite perimeter sets.) As mentioned in the introduction, the Muskat flow is the gradient flow of a particular energy functional, denoted by $\mathcal{E}$:

$$\mathcal{E}(E_1) = \mathbf{\Phi}(E_1) + \sigma P(E_1), \tag{1.1.3}$$

where

$$\mathbf{\Phi}(E_1) := \int_{E_1} g_1 x_2 \,\mathrm{d}x + \int_{E_2} g_2 x_2 \,\mathrm{d}x$$

is the total *potential energy* and $P(E)$ is as above. Thus $\sigma P(E)$ is the total *interfacial energy.*

The metric associated with this gradient flow structure is the Wasserstein metric. Recall that the quadratic Wasserstein distance $W_2(\mu, \nu)$ between two positive measures

μ, ν of the same mass on Ω is defined by

$$W_2^2(\mu,\nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\Omega\times\Omega} |x-y|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y),$$

where $\Pi(\mu,\nu)$ is the set of all couplings between μ and ν, i.e. measures γ on the product space $\Omega\times\Omega$ with marginals μ and ν: $\pi_{1\#}\gamma = \mu$ and $\pi_{2\#}\gamma = \nu$, where $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ are the coordinate projections and $\#$ denotes the pushforward operation. If μ is absolutely continuous with respect to Lebesgue measure, then Brenier's theorem says there is a unique minimizer $\gamma \in \Pi(\mu,\nu)$ which has the form of a transport map: $\gamma = (Id, T)_\#$, where T pushes μ onto ν: $T_\#\mu = \nu$. Moreover, $T(x) = x - \nabla\varphi(x)$, where φ is a Lipschitz function (a "Kantorovich potential") determined μ-a.e. up to an additive constant. See [Vil03] for more on the Wasserstein distance.

In our case, the measures involved are simply uniform measures on the sets E_i, rescaled to have total mass $|E_i| = 1/2$. Thus, the measure associated with the set E_i is $\mathbf{1}_{E_i}(x)\,\mathrm{d}x$. Since there are two phases, each with different mobility, the total squared distance between two fluid configurations (E_1,E_2) and (F_1,F_2) is

$$\frac{1}{b_1}W_2^2(\mathbf{1}_{E_1}\mathrm{d}x, \mathbf{1}_{F_1}\mathrm{d}x) + \frac{1}{b_2}W_2^2(\mathbf{1}_{E_2}\mathrm{d}x, \mathbf{1}_{F_2}\mathrm{d}x).$$

We will abbreviate $W_2(\mathbf{1}_E\,\mathrm{d}x, \mathbf{1}_F\,\mathrm{d}x)$ by $W_2(E,F)$ throughout the chapter.

Let (E_1^0, E_2^0) be the initial configuration of fluids, and let $h > 0$ be a chosen time-step. Given (E_1^{n-1}, E_2^{n-1}), we recursively define

$$E_1^n \in \operatorname*{arg\,min}_{|E_1| = |E_1^{n-1}|} \mathcal{F}(E_1; E_1^{n-1}), \tag{1.1.4}$$

where

$$\mathcal{F}(E_1; E_1^{n-1}) = \frac{1}{2hb_1}W_2^2(E_1, E_1^{n-1}) + \frac{1}{2hb_2}W_2^2(E_2, E_2^{n-1}) + \mathcal{E}(E_1);$$

and we let E_2^n be the complementary set to E_1^n, i.e. $E_2^n = (E_1^n)^c$. Such a sequence $\{(E_1^n, E_2^n)\}_{n=1}^\infty$ will be called a *discrete flow* or *discrete solution* to the Muskat problem with initial data (E_1^0, E_2^0).

Let $E_\infty = (0,1)\times(0,\frac{1}{2})$ be the global minimizer of $\mathcal{E}$ among sets of volume $1/2$. Define $g = g_1 - g_2 > 0$. Then our main result is as follows:

Theorem 1.1.1 (Main Theorem)**.** *(1) For any $\epsilon > 0$, there is an energy threshold $L = L(\sigma, g, \epsilon)$ such that if $\mathcal{E}(E_1^0) < L$, then for any $h > 0$ and any discrete solution $\{(E_1^n, E_2^n)\}_{n=1}^\infty$, there exists N such that for $n \geq N$, the configuration (E_1^n, E_2^n) has the following form: there is a C^2 function $f_n : [0,1] \to (0,1)$, with $||f_n - 1/2||_{L^\infty} < \epsilon/2$ and $f_n'(0) = f_n'(1) = 0$, such that*

$$\{(x_1, x_2) : \epsilon < x_2 < f_n(x_1)\} \subseteq E_1^n$$

and

$$\{(x_1, x_2) : f_n(x_1) < x_2 < 1 - \epsilon\} \subseteq E_2^n.$$

(See figure 1.2.) Moreover, $||f_n'||_{L^\infty} \to 0$ as $n \to \infty$. In particular, any subsequential limit point (E_1, E_2) of $\{(E_1^n, E_2^n)\}_{n=1}^\infty$ has the same form, with f_n replaced with a constant $f(x) = c$, for some c with $|c - 1/2| < \epsilon$.

(2) For any $h > 0$ and discrete solution $\{(E_1^n, E_2^n)\}_{n=1}^\infty$, if

$$\sigma > 2g/\pi^4, \tag{1.1.5}$$

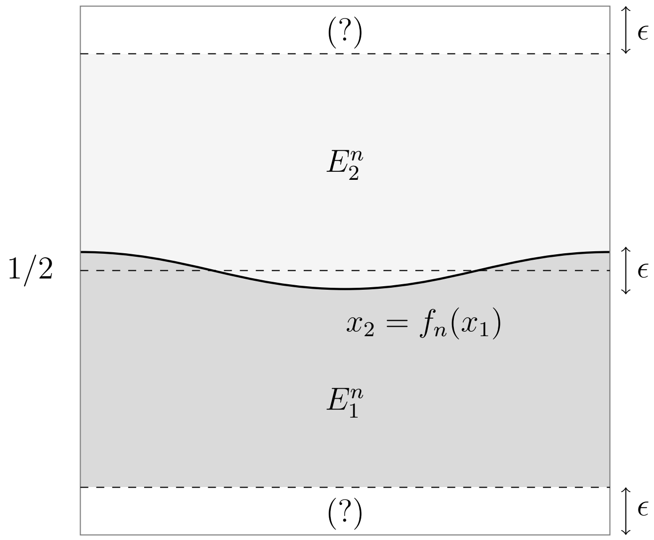
and if $E_1^n \to E_\infty$ as $n \to \infty$, then E_1^n converges exponentially fast (relative to $\mathcal{E}$ and in Hausdorff distance) to E_∞. Also, $E_1^n \to E_\infty$ in C^2 (though not necessarily exponentially fast), meaning that for large n, E_1^n is the subgraph of a C^2 function $f_n : [0,1] \to (0,1)$,

$$E_1^n = \{(x_1, x_2) : 0 < x_2 < f_n(x_1)\},$$

and we have $f_n \to 1/2$ in $C^2([0,1])$.

Remark 1.1.2. It remains an interesting open problem to prove quantitative convergence rates which are independent of the timestep h, which would allow to prove convergence

Figure 1.2: Structure of configuration for large n.



for the continuous-time solution obtained by taking the limit as $h \to 0$. This is done for the volume-preserving mean curvature flow and the Mullins-Sekerka flow in $\mathbf{R}^2$ in [Jul+21] (as noted above). See also remark 1.2.13 below.

Remark 1.1.3. The specific choice of the square domain Ω is mostly for simplicity and convenience; the proofs would apply equally well to many other domains, the main requirement being the existence of a structural theorem for low-energy critical points as in Proposition 1.2.5 below. In fact, the proofs should extend to the three-dimensional case, on domains such as the cylinder $\Omega = B(0,1) \times (0,2)$; however, this would require some boundary regularity theory for almost-minimal surfaces.

Remark 1.1.4. This theorem can be interpreted as a sort of well-posedness or stability result for discrete solutions near the global minimizer E_∞: any initial configuration sufficiently close to E_∞ (as measured by energy) must eventually be close to E_∞ in C^1, modulo possible displaced fluid on top and bottom. This accords with the fact that E_∞ is a stable configuration in the classical (smooth) case, since the heavier fluid is beneath the lighter one.

Notation 1.1.5. The following notation will be used throughout chapters one and two:

- ∂E = the topological boundary of E as a subset of $\mathbf{R}^2$;
- $\overline{E}$ = the closure of E in $\mathbf{R}^2$;

- $\mathcal{H}^1$ = one-dimensional Hausdorff measure on $\mathbf{R}^2$;
- $A \lesssim B$ means $A \leq C \cdot B$ for some constant $C > 0$ independent of A, B (where A and B are variable quantities).

1.2 Proofs

1.2.1 Convergence in C^1

We will collect here some properties of the discrete flow $\{(E_1^n, E_2^n)\}_{n=1}^{\infty}$. First, note that the existence of minimizers for (1.1.4) is immediate by the direct method of the calculus of variations. Next, we have the usual estimates coming from the minimizing movements scheme [Vil03]: using E_1^{n-1} as a competitor to E_1^n in (1.1.4), we get

$$\mathcal{E}(E_1^n) + \frac{1}{2hb_1} W_2^2(E_1^{n-1}, E_1^n) + \frac{1}{2hb_2} W_2^2(E_2^{n-1}, E_2^n) \leq \mathcal{E}(E_1^{n-1}), \tag{1.2.1}$$

and iterating this inequality and summing yields

$$\begin{aligned} \frac{1}{2h} \sum_{n=1}^{N} \left\{ \frac{1}{b_1} W_2^2(E_1^{n-1}, E_1^n) + \frac{1}{b_2} W_2^2(E_2^{n-1}, E_2^n) \right\} &\leq \mathcal{E}(E_1^0) - \mathcal{E}(E_1^N) \\ &\leq \mathcal{E}(E_1^0) - \mathcal{E}(E_\infty). \end{aligned}$$

In particular,

$$\mathcal{E}(E_1^n) \leq \mathcal{E}(E_1^0) \tag{1.2.2}$$

for all $n \geq 1$ and

$$\frac{1}{2h} \sum_{n=1}^{\infty} \left\{ \frac{1}{b_1} W_2^2(E_1^{n-1}, E_1^n) + \frac{1}{b_2} W_2^2(E_2^{n-1}, E_2^n) \right\} < \infty. \tag{1.2.3}$$

Moreover, from optimal transport theory we know [Vil03]

$$W_2^2(E_i^{n-1}, E_i^n) = \int_{E_i^n} |\nabla \varphi_i^n|^2 \, \mathrm{d}x, \tag{1.2.4}$$

where $T(x) = x - \nabla\varphi_i^n(x)$ is the optimal transport map from E_i^n to E_i^{n-1}; hence (1.2.3) becomes

$$\frac{1}{2h}\sum_{n=1}^{\infty}\left\{\frac{1}{b_1}\int_{E_1^n}|\nabla\varphi_1^n|^2\,\mathrm{d}x + \frac{1}{b_2}\int_{E_2^n}|\nabla\varphi_2^n|^2\,\mathrm{d}x\right\} < \infty. \tag{1.2.5}$$

Next, we examine some regularity properties of discrete solutions:

Proposition 1.2.1 (Regularity of Discrete Solutions)**.** Let $\{(E_1^n, E_2^n)\}_{n=1}^{\infty}$ be a discrete flow defined by the minimizing movements scheme (1.1.4).

1. There exist constants $\Lambda, r_0 > 0$, *a priori* depending on n*, such that

$$P(E_1^n; B(x,r)) \leq P(E; B(x,r)) + \Lambda|E_1^n \triangle E| \tag{1.2.6}$$

   whenever $\overline{E_1^n \triangle E} \subseteq B(x,r) \cap \Omega$ and $r < r_0$. In the language of [Mag12], E_1^n is a (Λ, r_0)**-perimeter minimizer in** Ω.

2. $\partial E_1^n \cap \Omega$ is in fact a locally $C^{2,\alpha}$ curve for every $0 < \alpha < 1/2$, and the mean curvature H therefore is well-defined in the classical sense on $\partial E_1^n \cap \Omega$, and for each connected component Γ_c of $\partial E_1^n \cap \Omega$ there is a constant λ such that†

$$\sigma H + gx_2 + \frac{\varphi_1^n}{hb_1} - \frac{\varphi_2^n}{hb_2} = \lambda \tag{1.2.7}$$

   on Γ_c, where φ_i^n is the Kantorovich potential for the optimal transport of E_i^n onto E_i^{n-1} (see (1.2.4)). Consequently, H extends continuously to the closure $\overline{\Gamma}$.

3. $\partial E_1^n \cap \Omega$ has bounded curvature for each $n \geq 1$ and therefore has a well-defined tangent space at each point of $\partial(\partial E_1^n \cap \Omega)$. If ∂E_1^n intersects $\partial\Omega$ at a non-corner point, then the two curves intersect orthogonally.‡

Proof. (1) Arguing exactly as in [Mag12, pp. 279-80], we can find $C, r_0 > 0$, depending on E_1^n, Ω, such that if $\overline{E_1^n \triangle F} \subseteq B_r \cap \Omega$ with $r < r_0$, then there exists $G \subseteq \Omega$ with

*In 1.2.7 below, we will se that the dependence on n can be omitted.
†This is the discrete analogue of the Young-Laplace equation in (1.1.2).
‡This is the discrete analogue of the Young equation in (1.1.2).

$|G| = |E_1^n|$ and $\overline{E_1^n \triangle G} \subseteq \Omega$ such that $|E_1^n \triangle G| = 2|E_1^n \triangle F|$ and

$$P(G) \leq P(F) + C\big||E_1^n| - |F|\big|. \tag{1.2.8}$$

Using G as a competitor to E_1^n in the minimality of $\mathcal{F}(\cdot, E_1^{n-1})$, we get

$$\mathcal{F}(E_1^n, E_1^{n-1}) \leq \mathcal{F}(G, E_1^{n-1}),$$

or

$$\begin{aligned}
P(E) \leq P(G) &+ \frac{1}{2\sigma h b_1}(W_2^2(G, E_1^{n-1}) - W_2^2(E_1^n, E_1^{n-1})) \\
&+ \frac{1}{2\sigma h b_2}(W_2^2(G^c, E_2^{n-1}) - W_2^2(E_2^n, E_2^{n-1})) \\
&+ \frac{1}{\sigma}(\int_{G \setminus E_1^n} g_1 x_2 \, \mathrm{d}x - \int_{E_1^n \setminus G} g_1 x_2 \, \mathrm{d}x) \\
&+ \frac{1}{\sigma}(\int_{E_1^n \setminus G} g_2 x_2 \, \mathrm{d}x - \int_{G \setminus E_1^n} g_2 x_2 \, \mathrm{d}x).
\end{aligned}$$

From the definition of the W_2 metric, we have

$$|W_2^2(E_1^n, A) - W_2^2(E_1^n, B)| \leq 2|A \triangle B|$$

(since the diameter of Ω is $\sqrt{2}$). Hence,

$$\begin{aligned}
P(E_1^n) &\leq P(G) + \frac{1}{\sigma h b_1}|E_1^n \triangle G| + \frac{1}{\sigma h b_2}|E_2^n \triangle G^c| + \frac{g_1}{\sigma}|E_1^n \triangle G| + \frac{g_2}{\sigma}|E_1^n \triangle G| \\
&= P(G) + C'|E_1^n \triangle G|
\end{aligned}$$

where

$$C' = \frac{1}{\sigma}(g_1 + g_2 + \frac{1}{hb_1} + \frac{1}{hb_2}).$$

Therefore, using $|E_1^n \triangle G| = 2|E_1^n \triangle F|$ and (1.2.8) (and the trivial inequality $||E_1^n| - |F|| \leq |E_1^n \triangle F|$), we get

$$P(E_1^n) \leq P(F) + \Lambda|E_1^n \triangle F|,$$

where $\Lambda = C + 2C'$.

(2) By the classical regularity theory for almost-minimal surfaces [Mag12], it follows from part (1) that $\Gamma := \partial E_1^n \cap \Omega$ is locally $C^{1,\alpha}$ for every $0 < \alpha < 1/2$. To upgrade this to $C^{2,\alpha}$ and prove (2), we have to look at the optimality condition (1.1.4) and derive the associated Euler-Lagrange equation. Let $p \in \Gamma$. By the local $C^{1,\alpha}$ regularity, there is a direction $\nu \in S^1$ and radius $\delta > 0$ such that Γ is the graph of a $C^{1,\alpha}((-\delta, \delta))$ function within the cylinder $\mathbf{C} \subset \Omega$ defined by $\mathbf{C} := \{x : |(x - p) \cdot \nu| < \delta\}$. By rotating we can assume $\nu = (0, 1)^t$, so that

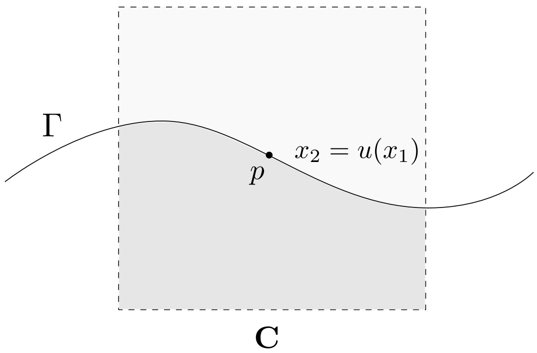
$$E_1^n \cap \mathbf{C} = p + S(u),$$

where $S(u)$ is the subgraph of a $C^{1,\alpha}((-\delta, \delta))$ function u:

$$S(u) := \{(x_1, x_2) : -\delta < x_2 < u(x_1), -\delta < x_1 < \delta\},$$

$u(0) = 0$. (See figure 1.2.1.)

Figure 1.3: Local representation of Γ as a graph.



We now want to modify E_1^n by perturbing its boundary within $\mathbf{C}$. Let $\eta \in C_c^\infty((-\delta, \delta))$ satisfy $\int_{-\delta}^{\delta} \eta \, \mathrm{d}x_1 = 0$, and given $\epsilon > 0$, replace E_1^n by the set E_ϵ defined by replacing E_1^n inside $\mathbf{C}$ by

$$p + S(u + \epsilon\eta).$$

(Of course, ϵ must be small enough so that this new set is contained in $\mathbf{C}$.) We have

$|E_\epsilon| = 1/2$, hence

$$\mathcal{F}(E_1^n; E_1^{n-1}) \le \mathcal{F}(E_\epsilon; E_1^{n-1})$$

for small ϵ. In particular,

$$\frac{d}{d\epsilon}\mathcal{F}(E_\epsilon; E_1^{n-1})\Big|_{\epsilon=0} = 0, \tag{1.2.9}$$

provided the derivative exists. By standard calculations

$$\frac{d}{d\epsilon}\mathcal{E}(E_\epsilon)\Big|_{\epsilon=0} = \int_{-\delta}^{\delta} \sigma \frac{u'\eta'}{\sqrt{1+(u')^2}} + gu\eta \,\mathrm{d}x_1,$$

since u is $C^{1,\alpha}$. To differentiate the W_2 terms, it is more natural to flow E_1^n along a divergence-free field. We define such a field v as follows:

$$v = \begin{cases} 0 & \Omega\backslash\mathbf{C} \\ \nabla p & E_1^n \cap \mathbf{C} \\ \nabla q & E_2^n \cap \mathbf{C}, \end{cases}$$

where p, q solve the boundary value problems

$$\begin{cases} \Delta p = 0 & \text{on } E_1^n \cap \mathbf{C} \\ \frac{\partial p}{\partial \nu} = \frac{\eta}{\sqrt{1+(u')^2}} & \text{on } \Gamma \\ \frac{\partial p}{\partial \nu} = 0 & \text{on } \partial(\overline{E_1^n \cap \mathbf{C}})\backslash\Gamma, \end{cases}$$

$$\begin{cases} \Delta q = 0 & \text{on } E_2^n \cap \mathbf{C} \\ \frac{\partial q}{\partial \nu} = -\frac{\eta}{\sqrt{1+(u')^2}} & \text{on } \Gamma \\ \frac{\partial q}{\partial \nu} = 0 & \text{on } \partial(\overline{E_2^n \cap \mathbf{C}})\backslash\Gamma, \end{cases}$$

ν being the unit outer normal on the domain in question (either $E_1^n \cap \mathbf{C}$ or $E_2^n \cap \mathbf{C}$). Let E'_ϵ be the image of E_1^n under the flow of v for time $\epsilon > 0$[§]. Since the normal component

[§]Note that although v is in general discontinuous across Γ, the flow is still well-defined and volume-preserving because the component of v normal to Γ is continuous across Γ, and v is divergence-free on of v on Γ coincides with that of $\eta\vec{e_2}$, we have $|E_\epsilon \triangle E'_\epsilon| = o(\epsilon)$ and hence

$$\frac{d}{d\epsilon} W_2^2(E_1^{n-1}, E_\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon} W_2^2(E_1^{n-1}, E'_\epsilon)\Big|_{\epsilon=0},$$

again provided one (and hence both) derivative exists. But the derivative on the right can be evaluated, and equals¶

$$\int_{E_1^n \cap \mathbf{C}} \nabla \varphi_1^n \cdot v \, \mathrm{d}x,$$

$\nabla \varphi_1^n$ being the optimal displacement from E_1^n to E_1^{n-1}. As φ_1^n is Lipschitz, we can use the Gauss-Green formula for vector fields of bounded variation, to get

$$\int_{E_1^n \cap \mathbf{C}} \nabla \varphi_1^n \cdot v \, \mathrm{d}x = \int_{-\delta}^{\delta} \varphi_1^n(x_1, u(x_1)) \eta(x_1) \, \mathrm{d}x_1.$$

Performing the same calculation but for the flow of E_2^n, we finally see that (1.2.9) is equivalent to

$$\int_{-\delta}^{\delta} \sigma \frac{u'\eta'}{\sqrt{1+(u')^2}} + gu\eta + \frac{b_1^{-1}\varphi_1^n(x_1, u(x_1)) - b_2^{-1}\varphi_2^n(x_1, u(x_1))}{h} \eta(x_1) \, \mathrm{d}x_1 = 0.$$

This is an elliptic equation in divergence form, with Lipschitz coefficients. From here Schauder theory easily gives that u' is $C^{1,\alpha}$, i.e. u is $C^{2,\alpha}$, which means $\partial E_1^n \cap \Omega$ is locally $C^{2,\alpha}$. Hence, we may convert the equation to differential (pointwise) form, to get (1.2.7). (See [Mag12, §27] for analogous calculations.) Since φ_i is Lipschitz continuous on E_i^n, it extends to a continuous Lipschitz function on the closure $\overline{E_i^n}$. Therefore, from the equation (1.2.7) we deduce that H is in fact continuous on $\overline{\Gamma}$, i.e. it is continuous up to the boundary $\partial\Omega$.

(3) See below, 1.2.1. □

Remark 1.2.2. The (Λ, r_0)-minimality and consequent regularity of sets appearing in

E_1^n and E_2^n separately.

¶The proof given in [Vil03] assumes that the vector field in question is C^1; however, since our v is bounded, one can easily verify that the same proof applies *mutatis mutandis* to our situation.

minimization problems involving the perimeter and the Wasserstein distance has been studied before, in [Mil06; Xia05]. However, these studies do not examine the higher C^2-regularity as we did above, which is crucially needed for our purposes.

We now investigate the long-term behavior of the sets E_1^n, and prove the first part of Theorem 1.1.1. To do so, we will first establish the last statement: that any subsequential W_2-limit point (E_1, E_2) of $\{(E_1^n, E_2^n)\}_{n=1}^\infty$ satisfies $(0,1)\times(\epsilon, c) \subseteq E_1$ and $(0,1)\times(c, 1-\epsilon) \subseteq E_2$, where c is a constant with $|c - 1/2| < \epsilon$. See figure 1.2.1 for an example. In other words, E_1 is essentially the same as the global equilibrium E_∞, except for the fact that some mass may be displaced to the top and bottom of Ω. Then, we will show that the sets (E_1^n, E_2^n) satisfy a stronger almost-minimality property that is uniform in n, and therefore, by the classical regularity theory for almost minimal surfaces, any subsequence $\{(E_1^{n_k}, E_2^{n_k})\}_{k=1}^\infty$ with $(E_1^{n_k}, E_2^{n_k}) \to (E_1, E_2)$ in W_2 actually converges in C^1, and hence must have the form described in Theorem 1.1.1(1). Since any subsequence has a further subsequence which is convergent, this establishes the first part of the theorem.

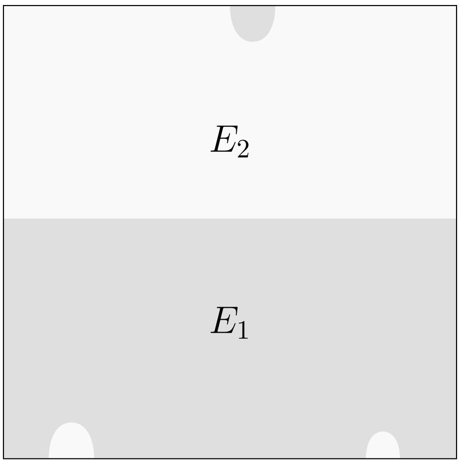


Figure 1.4: An example critical point. Note that some fluid is displaced to the top and bottom.

To prove the first statement above, that any limit point of the discrete solution has the form described, we argue as follows: first, any limit point E_1 of the discrete solution $\{E_1^n\}_{n=1}^\infty$ must be a C^2 critical point for the energy functional $\mathcal{E}$ under locally volume-preserving variations, and hence must satisfy the associated Euler-Lagrange equations. On the other hand, by energy dissipation, E_1 must have lower energy than E_1^0. But, by elementary geometric considerations, critical points of $\mathcal{E}$ with C^2 boundary and sufficiently low energy are very constricted, and in fact must have the form described in the theorem.

Remark 1.2.3. The study of the precise shapes of the critical points of energy functionals involving perimeter and gravity ("equilibrium capillary surfaces") is a classical subject, see [Fin12]. Since we do not need any detailed information about these shapes beyond what is contained in Proposition 1.2.5, we will not go into more detail.

Lemma 1.2.4 (Limit points are critical points)**.** Let $\{E_1^{n_k}\}_{k=1}^\infty$ be a subsequence of $\{E_1^n\}_{n=1}^\infty$ such that $E^{n_k} \to E_1$ in W_2. Then E_1 is a C^2 critical point of $\mathcal{E}$, meaning the following: $|E_1| = 1/2$, $\partial E_1 \cap \Omega$ has finitely many components and is locally the graph of a C^1 function, and for each component Γ_c of ∂E_1 and all continuous functions $s : \Gamma_c \to \mathbf{R}$ with $\int_{\Gamma_c} s \, \mathrm{d}\mathcal{H}^1 = 0$, we have

$$\int_{\Gamma_c} (\sigma H + g x_2) s \, \mathrm{d}\mathcal{H}^1 = 0,$$

where H is the mean curvature of Γ_c, with positive sign where E_1 is convex. In particular, there is a constant λ such that $\sigma H + g x_2 = \lambda$ on Γ_c.

Proof. We first claim that E_1 minimizes the functional

$$F_1 \mapsto \frac{1}{2hb_1} W_2^2(E_1, F_1) + \frac{1}{2hb_2} W_2^2(E_2, F_2) + \mathcal{E}(F_1). \tag{1.2.10}$$

For suppose not, i.e. suppose there were some $F_1 \neq E_1$ with

$$\frac{1}{2hb_1} W_2^2(E_1, F_1) + \frac{1}{2hb_2} W_2^2(E_2, F_2) + \mathcal{E}(F_1) < \mathcal{E}(E_1).$$

Since $W_2(E_1^{n_k}, E_1) \to 0$, for large enough k we have

$$\frac{1}{2hb_1} W_2^2(E_1^{n_k}, F_1) + \frac{1}{2hb_2} W_2^2(E_2^{n_k}, F_2) + \mathcal{E}(F_1) < \mathcal{E}(E_1).$$

Adding

$$\frac{1}{2hb_1}W_2^2(E_1^{n_k}, E_1^{n_k+1}) + \frac{1}{2hb_2}W_2^2(E_2^{n_k}, E_2^{n_k+1})$$

to the right side, and using $\mathcal{E}(E_1) \le \mathcal{E}(E_1^{n_k+1})$, we deduce

$$\begin{aligned}\frac{1}{2hb_1}W_2^2(E_1^{n_k}, F_1) + \frac{1}{2hb_2}W_2^2(E_2^{n_k}, F_2) + \mathcal{E}(F_1)& \\ < \frac{1}{2hb_1}W_2^2(E_1^{n_k}, E_1^{n_k+1}) + \frac{1}{2hb_2}W_2^2(E_2^{n_k}, E_2^{n_k+1}) + \mathcal{E}(E_1^{n_k+1}),&\end{aligned}$$

contradicting the definition of $E_1^{n_k+1}$.

Thus, E_1 minimizes the functional (1.2.10), and so, using the same reasoning as in Proposition 1.2.1, we deduce $\partial E_1 \cap \Omega$ is locally $C^{2,\alpha}$ and that for each component Γ_c of $\partial E_1 \cap \Omega$ and all continuous functions $s : \Gamma_c \to \mathbf{R}$ with $\int_{\Gamma_c} s \, \mathrm{d}\mathcal{H}^1 = 0$, we have

$$\int_{\Gamma_c} (\sigma H + g x_2) s \, \mathrm{d}\mathcal{H}^1 = 0,$$

as desired. □

Now, we show that critical points with sufficiently low energy must have the form described in Theorem 1.1.1(1).

Proposition 1.2.5 (Structure of critical points)**.** Let $\epsilon > 0$. Then there is an energy threshold $L = L(\sigma, g, \epsilon)$ such that, if E_1 is a C^2 critical point of $\mathcal{E}$ in the sense above, with complementary set E_2, then $\mathcal{E}(E_1) < L$ implies

$$(0,1) \times (\epsilon, c) \subseteq E_1$$

and

$$(0,1) \times (c, 1-\epsilon) \subseteq E_2,$$

where c is a constant with $|c - 1/2| < \epsilon$.

Proof. We first determine what kinds of curves the connected components of $\Gamma := \partial E_1 \cap \Omega$ can be. Let Γ_c be such a component. If Γ_c is either a closed curve or a curve with both endpoints on the left or right wall of Ω, then by shifting Γ_c vertically up or down we see that E_1 cannot be a critical point of $\mathcal{E}$. Thus, Γ_c must be one of three types: both endpoints on the top wall or bottom wall of Ω, endpoints on adjacent walls (spanning a "corner" of Ω), or endpoints on opposite walls (left and right or top and bottom).

Since E_∞ minimizes both the potential energy $\Phi(E)$ and the perimeter $P(E)$ among $E \subseteq \Omega$ with $|E| = 1/2$, in order for $\mathcal{E}(E_1)$ to be close to $\mathcal{E}(E_\infty)$, $\Phi(E_1)$ must be close to $\Phi(E_\infty)$ and $P(E_1)$ must be close to $P(E_\infty) = 1$. $\Phi(E_1)$ being close to $\Phi(E_\infty)$ forces $|E_1 \triangle E_\infty|$ to be small. In order to achieve this using the allowable curves above, while still keeping $P(E_1)$ close to 1, there must be exactly one curve Γ_c with endpoints on $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$, with E_1 immediately below it. (Γ may of course have other curves, but it must have precisely one of this form, connecting the left and right walls.)

We claim that Γ_c must be a straight horizontal line. Suppose not. Then we can find two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ on Γ_c, with $p_2 > q_2$, such that $H(p) \geq 0$ and $H(q) \leq 0$. Indeed, we can simply choose p to be the highest point on Γ_c, and q the lowest; then since E_1 is below Γ_c, we have $H(p) \geq 0 \geq H(q)$. But this contradicts the equation $\sigma H + g x_2 = \lambda$ which must hold on Γ_c by Lemma 1.2.4, which implies that H strictly decreases with height. (See figure 1.5.) Thus, Γ_c is a straight line.

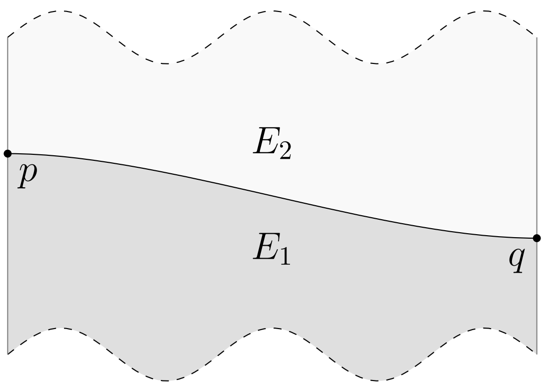


Figure 1.5: Forbidden configuration: $H(p) \geq 0 \geq H(q)$ but $p_2 > q_2$.

The only other curves left to account for are those that have both endpoints on the top or bottom walls of Ω, or that connect two adjacent walls at the corners. Since we already have a curve of length 1, each of the remaining curves can have diameter at most $\delta := P(E_1) - 1 = P(E_1) - P(E_\infty)$, and since they touch either the bottom or top wall of Ω, they must be contained within the strips $(0,1)\times(0,\delta)$ and $(0,1)\times(1-\delta,1)$. Hence, we simply choose L small enough so that $P(E_1)-P(E_\infty) < \epsilon$. This completes the proof. $\square$

Remark 1.2.6. By examining the above proof more closely, it is easy to see how L could be calculated approximately; but since we do not need this information, we will not investigate it further.

All of the above proofs dealt with convergence in W_2, and hence convergence in L^1 (since Ω is bounded). To upgrade to C^1 convergence, we will exploit the fact that the sets E_i^n actually satisfy a stronger *uniform* minimality condition, in contrast to the statement of Proposition 1.2.1(1), which is *a priori* dependent on n. As explained in the introduction, this involves a geometric construction for changing the volume of a set while controlling the associated change in perimeter.

Theorem 1.2.7 (Uniform Almost-Minimality). *Let $\mathcal{E}(E_1^0) \leq M$. There are constants $\Lambda, r_0 > 0$, depending only on M, g_1, g_2, σ, h (but not on n), such that E_1^n is a (Λ, r_0)-perimeter minimizer in Ω for all $n \geq 1$.*

Proof. From above, $\partial E_1^n \cap \Omega$ is locally $C^{2,\alpha}$. Thus $\partial E_1^n \cap \Omega$ consists of a finite[‖] number of locally $C^{2,\alpha}$ curves Γ_i (its connected components). Since $|E_1^n| = 1/2$ and $P(E_1^n) \leq M/\sigma$, there is a constant $l > 0$, depending only on M/σ, such that at least one Γ_i, say Γ_c, has diameter at least l. Define $r_0 = l/2\sqrt{2}$, and suppose $F \subseteq \Omega$ satisfies $\overline{F \triangle E_1^n} \subseteq B(x,r) \cap \Omega$ for some $x \in \Omega, r < r_0$. To apply the minimality condition (1.1.4) satisfied by E_1^n, we have to compare E_1^n to a set of the same size 1/2. If $|F| = 1/2$ already, then there is nothing to do. In general, however, $|F|$ will not equal 1/2, and hence we must construct another set F' such that $|F'| = 1/2$ and such that we have good control on the excess $(P(F')-P(F))_+$ in terms of the difference $|F \triangle F'|$. The construction used in Proposition

[‖]Finiteness follows easily from the perimeter and curvature bounds: infinitely many components with diameter at least some positive constant would yield infinite perimeter, while on the other hand allowing arbitrarily small diameter components would lead to arbitrarily large curvature.

1.2.1 above is taken from [Mag12] and depends on the set E_1^n; since we want the result to be independent of n, this will not work. Similarly, the studies mentioned above [Mil06; Xia05] use approaches that either are dependent on the set in question or else do not seem to apply to our situation. Hence, we must use another procedure.

Without loss of generality we may assume $|F| < 1/2$, since the two cases are completely symmetric (just apply the same reasoning to E_2^n instead of E_1^n). We will construct a set F' with the following properties:

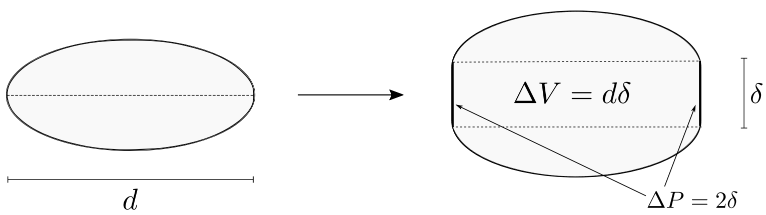
1. $|F'| = 1/2$;

2. $F \subseteq F'$;

3. $P(F') \leq P(F) + \frac{8\sqrt{2}}{l}|F' \triangle F|$.

The basic idea is very simple: if we have a nice connected set (say convex) with diameter d, then we can increase its volume by "thickening" it in the direction perpendicular to its diameter by some distance δ (see figure 1.6). The volume added by this change is $d\delta$, whereas the perimeter has increased by 2δ. Thus, the ratio of perimeter increase ΔP to volume increase ΔV is $\Delta P/\Delta V = 2/d$. Hence, if we have a lower bound l on the diameter d, then we can bound $\Delta P/\Delta V \leq 2/l$, exactly the kind of bound we are looking for.

Figure 1.6: Enlarging a set by "thickening".



We want to apply this strategy to F, or rather to one of its connected components. However, F may not have a nice, convex component like the set in figure 1.6 above. Hence, we will first have to apply some modifications, which amount to "filling in" holes in F and various concave portions of ∂F. Moreover, we have to take into consideration the presence of other components of F, as well as the container wall $\partial\Omega$, both of which may obstruct the extension we are trying to make. Most of the rest of the proof is devoted to these technicalities; however, they should not overshadow the basically simple geometric idea behind them.

Throughout the following, we will be continually modifying the set F', which initially is F. Hence, F' will simply stand for the current modification of F, at the given stage of the construction.

Since $r_0 = l/4$, we can find a rectangle $\mathcal{R}$, of the form $(a,b) \times (0,1)$ or $(0,1) \times (a,b)$ for some $0 < a < b < 1$, such that

1. $\mathcal{R} \cap B(x,r) = \emptyset$;

2. the width $b - a$ is at least $l/4\sqrt{2}$;

3. $\overline{\mathcal{R}} \cap \Gamma_c$ contains at least one curve, call it $\mathcal{B}$, with endpoints on opposite sides of $\mathcal{R}$.

Indeed, since Γ_c has diameter at least l, its projection onto one of the coordinate axes must have diameter at least $l/\sqrt{2}$; since we want to avoid the ball $B(x,r)$, we subtract $r_0 = l/2\sqrt{2}$. The remaining projection then has at least one interval of length $\geq l/4\sqrt{2}$, which we use as (a,b). Since Γ_c is connected, the preimage of (a,b) has at least one connected curve joining opposite sides of $\mathcal{R}$.

For concreteness, assume w.l.o.g. that $\mathcal{R}$ is vertical, $(a,b) \times (0,1)$. See figure 1.7. Note that since $\mathcal{R}$ does not intersect $B(x,r)$, $E_1^n \cap \mathcal{R} = F \cap \mathcal{R}$; in particular, ∂F is smooth within $\mathcal{R}$.

Our strategy is to enlarge F within $\mathcal{R}$, using the method outlined above, by "thickening" the portion of F within $\mathcal{R}$ in the vertical direction. Let E be the connected component of $E_1^n \cap \mathcal{R} = F \cap \mathcal{R}$ touching $\mathcal{B}$. We will first fill in any holes in E. Let $\mathcal{C}$ be a closed curve in ∂E not containing $\mathcal{B}$, and call the region it encloses $R_{\mathcal{C}} \subseteq \mathcal{R}$. (Note: it may be that F has other components within $R_{\mathcal{C}}$; this is fine.) If $|R_{\mathcal{C}} \backslash F| \leq |E_1^n| - |F|$, then add $R_{\mathcal{C}}$ to F to get $F' = F \cup R_{\mathcal{C}}$; this clearly decreases $P(F)$. Otherwise, we partially "fill in" $R_{\mathcal{C}}$ from the left, i.e. we add to F the set $A_t := ((0,t) \times (0,1)) \cap R_{\mathcal{C}}$, where t is chosen so that $|A_t \backslash F| = |E_1^n| - |F|$, i.e. so that the new set $|F'| = 1/2$. See figure 1.8.

Figure 1.7: An example configuration, with rectangle $\mathcal{R}$.

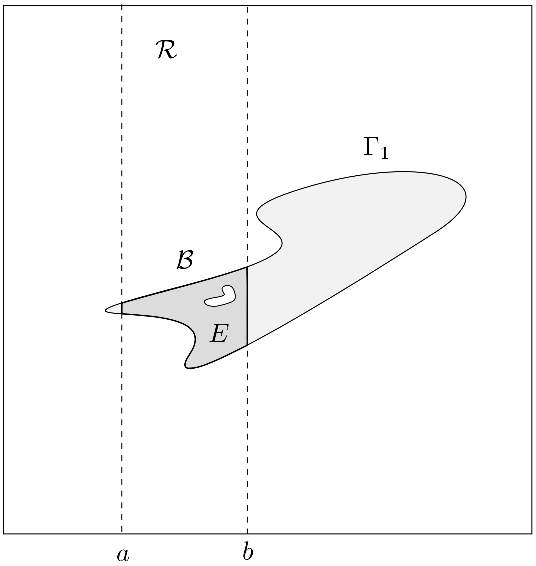
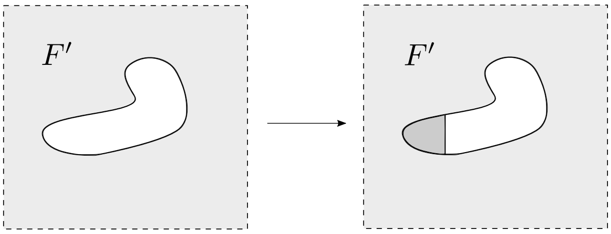


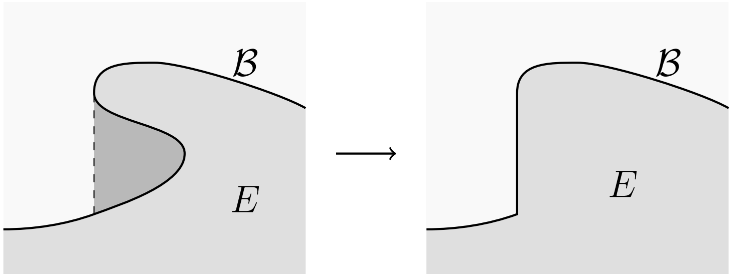
Figure 1.8: Partially filling in a hole.



Clearly, the new F' has strictly smaller perimeter than the original F. If we added all of $R_{\mathcal{C}}$ to F and still have $|F'| < 1/2$, then move on to the next closed curve in ∂E disjoint from $\mathcal{B}$, and repeat the procedure. If, after adding all such "holes" $R_{\mathcal{C}}$ to F, we still have $|F'| < 1/2$, then we do the same thing for any "side holes", i.e. components of $E_2^n \cap \mathcal{R}$ bounded by a curve $C \subseteq \partial E$ with both endpoints on $\{a\} \times (0,1)$ or both on $\{b\} \times (0,1)$.

Suppose after adding all such "holes" we still have $|F'| < 1/2$. We now want to apply to E the "thickening" strategy in the vertical direction. Recall that $\mathcal{B} \subseteq \partial E$ connects opposite sides of $\mathcal{R}$; suppose without loss of generality that it is an upper boundary cuve of E (i.e. E lies immediately below it). In order to have $F \subseteq F'$, $\mathcal{B}$ must not "fold over itself", i.e., its outer normal vector $\vec{n}$ must satisfy $\vec{n} \cdot \vec{e_2} \geq 0$ everywhere. If this is not the case, we simply "fill in" the offending areas from the left or right, as with the holes above, until either we have overcome the volume deficit $1/2 - |F'|$ or $\mathcal{B}$ no longer folds over itself, as in figure 1.9.

Figure 1.9: Filling in the "fold" on $\mathcal{B}$ (dark gray region).



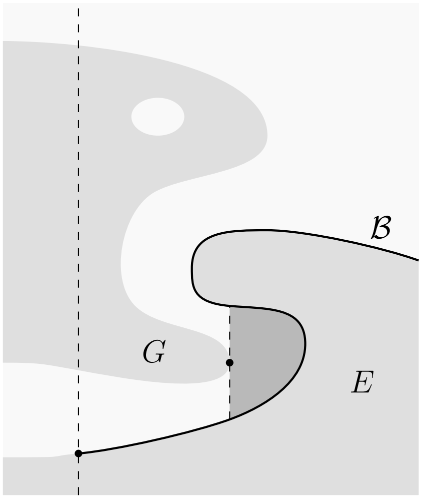
We continue to use 'E' to denote this enlarged version of the original E. As before, this only decreases the perimeter of F'. It may happen that as we are filling in portions of $\mathcal{B}$, we intersect the boundary of another component G of $F \cap \mathcal{R}$. If this occurs, we simply add G to E, and repeat all of the above procedure, beginning with filling in holes, to the enlarged E. This is best illustrated visually, in figure 1.10. Again, all of this only decreases $P(F')$.

Suppose after all the above that we still have $|F'| < 1/2$. We now have a (possibly new) upper boundary curve $\mathcal{B}$, connecting opposite sides of $\mathcal{R}$, which does not fold over itself. After these preparations, we can finally apply to E the extension procedure. We enlarge E by translating $\mathcal{B}$ upwards by a distance δ. Thus, we replace the curve $\mathcal{B}$ with the translated curve $\mathcal{B}' := \mathcal{B} + \delta\vec{e_2}$, for some $\delta > 0$, along with the vertical segments between p and $p + \delta\vec{e_2}$ and between q and $q + \delta\vec{e_2}$, where p, q are the endpoints of $\mathcal{B}$ (see figure 1.11).

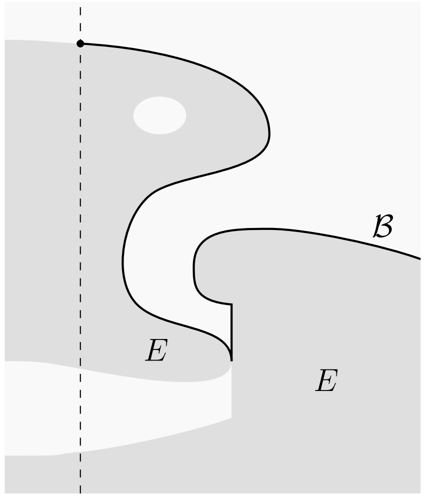
Note that since $\mathcal{B}'$ doesn't fold over itself, the new F' strictly contains the old F'. If the extended region does not intersect any other components of $F \cap \mathcal{R}$ or the container wall $\partial\Omega$, then this increases the volume of F' by $\Delta V = (b - a) \cdot \delta$ and the perimeter by $\Delta P = 2\delta$. Thus, $\Delta P/\Delta V = 2/(b - a)$. If, on the other hand, we run into the boundary of another component G of $F \cap \mathcal{R}$, and we still have a volume deficit, then we do the same thing we did above: we add G to E, then fill in any holes, then fill in the upper boundary curve until it doesn't fold over itself, to get a new upper boundary curve, which we then extend again if necessary. Similarly if we run into $\partial\Omega$, as illustrated in figure 1.12.

If a deficit remains after all this, we do the same procedure with the lower boundary

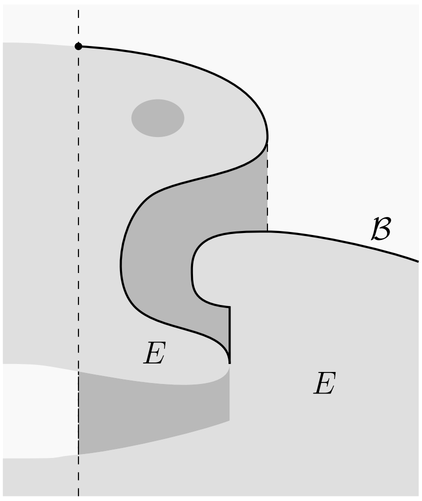
Figure 1.10: Hitting another component.



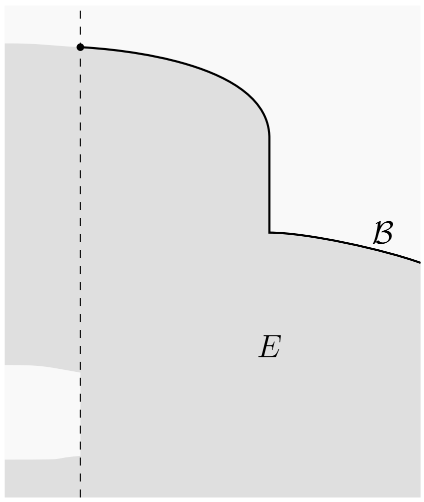
(a) While trying to fill in the fold on $\mathcal{B}$, we hit another component G.



(b) We add G to the expanded E, to get a larger connected set E, with new upper boundary $\mathcal{B}$.



(c) We apply the procedures described previously to the new E: filling in holes and side holes, and the fold on the new $\mathcal{B}$ (middle dark gray region).



(d) The final result, with new upper boundary $\mathcal{B}$.

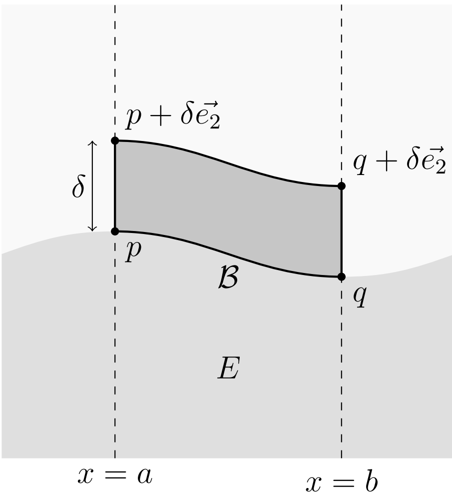


Figure 1.11: Extending $\mathcal{B}$ upward by distance δ.

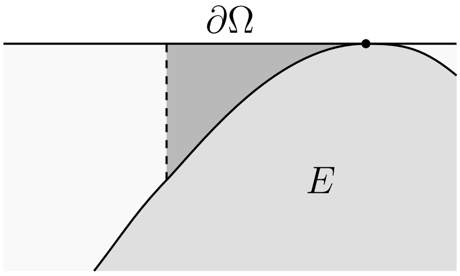


Figure 1.12: If the extended boundary hits $\partial\Omega$, we simply fill in on the left and/or right (dark gray region), until either the volume deficit is overcome or the entire region between $\partial\Omega$ and the translated $\mathcal{B}$ is added to E.

curve (if any) of E, so that, if necessary, we end up adding the entire rectangle $\mathcal{R} = (a, b) \times (0, 1)$ to F'.

Finally, if a deficit still remains when $F' = \mathcal{R}$, then we let

$$F' = [(a - \delta, b + \delta) \times (0, 1)] \cap \Omega,$$

where δ is chosen so that $|F'| = 1/2$. In other words, we simply expand the sides of $\mathcal{R}$ until the deficit is finally eliminated. Again, this clearly decreases $P(F')$.

All of the above procedures either add volume while decreasing perimeter, or else (during an extension) sastify $\Delta P/\Delta V \leq 2/(b - a)$. Thus, since $F \subseteq F'$, we have

$$P(F') \leq P(F) + \frac{2}{b - a}|F' \triangle F| \leq P(F) + \frac{8\sqrt{2}}{l}|F' \triangle F|, \tag{1.2.11}$$

where we used the fact that $b - a \leq l/4\sqrt{2}$.

Thus, we eventually arrive at a modified set F', satisfying $|F'| = 1/2$ and

$$P(F') \leq P(F) + \frac{8\sqrt{2}}{l}|F' \triangle F|. \tag{1.2.12}$$

Also, $|F' \triangle E_1^n| \leq 2|F \triangle E_1^n|$, since $F \subseteq F'$ and $|F'| - |F| = |E_1^n| - |F|$. Now, just as in the proof of Proposition 1.2.1, since $|F'| = 1/2$, we have

$$\mathcal{F}(E_1^n, E_1^{n-1}) \leq \mathcal{F}(F', E_1^{n-1}),$$

which after rearranging terms implies

$$P(E_1^n) \leq P(F') + C|E_1^n \triangle F'|,$$

with $C = \sigma^{-1}(g_1 + g_2 + h^{-1}b_1^{-1} + h^{-1}b_2^{-1})$. Combining this with 1.2.12 and $|F' \triangle E_1^n| \leq 2|F \triangle E_1^n|$, we get

$$P(E_1^n) \leq P(F) + \Lambda|E_1^n \triangle F|, \tag{1.2.13}$$

where

$$\Lambda = \frac{16C\sqrt{2}}{l}. \tag{1.2.14}$$

Thus, E_1^n is a (Λ, r_0)-perimeter minimizer in Ω, where Λ and r_0 only depend on l, which in turn only depends on $P(E_1^n) \leq M/\sigma$ and not on n. Hence, the sets $\{E_1^n\}_{n=1}^\infty$ are uniformly (Λ, r_0)-perimeter minimizers. $\square$

Remark 1.2.8.

With this stronger regularity at hand, we can now complete the proof of Theorem 1.1.1(1). First, however, we finish the proof of Proposition 1.2.1:

Proof of Proposition 1.2.1(3). Note that the proof of uniform minimality given above did not depend on the perturbation $F \triangle E_1^n$ being compactly contained in $B(x, r) \cap \Omega$, but only on the size of $r < r_0$. Thus, the inequality (1.2.13) applies whenever $F \triangle E_1^n \subseteq$

$B(x,r) \cap \Omega$ with $r < r_0$, including when F is a perturbation of E_1^n at the boundary. Now suppose p is a non-corner intersection point of Γ and $\partial\Omega$. Since Γ has bounded curvature, Γ and $\partial\Omega$ intersect at a well-defined angle at p. If this is not orthogonal, then by modifying E_1^n in a small enough neighborhood of p, we can produce a set F such that $(P(E_1^n) - P(F))/|F \triangle E_1^n| > 0$ is as large as desired, in particular larger than Λ from 1.2.14, a contradiction. Likewise, since the limit critical points appearing in Lemma 1.2.4 minimize the functional (1.2.10), the same proof and conclusion applies to them as well, to deduce orthogonal intersection. □

Remark 1.2.9. The orthogonal intersection can also be proved in a similar manner to the proof of Proposition 1.2.1(2), using divergence-free vector fields; we chose to prove it this way in order to illustrate another use of the geometric method introduced in Theorem 1.2.7. It is also interesting to note that the same reasoning used in the previous proof shows that corner intersections are in fact not possible, both for the discrete sets and their limits; but we do not need this fact.

Proof of Theorem 1.1.1(1). We claim that the threshold of Proposition 1.2.5 works. Suppose not; that is, suppose that for some E_1^0 with $\mathcal{E}(E_1^0) < L = L(\sigma, g, \epsilon)$, there is a subsequence $\{(E_1^{n_k}, E_2^{n_k})\}_{k=1}^{\infty}$ such that no $(E_1^{n_k}, E_2^{n_k})$ has the form described in 1.1.1(1). By compactness, we can find a subsequence, not relabeled, so that $(E_1^{n_k}, E_2^{n_k}) \to (E_1, E_2)$ for some sets $E_1, E_2 \subseteq \Omega$. By Lemma 1.2.4, E_1 is a C^2 critical point of $\mathcal{E}$, and since $\mathcal{E}(E_1) < L$ by energy dissipation, Proposition 1.2.5 implies that

$$(0,1) \times (\epsilon, c) \subseteq E_1$$

and

$$(0,1) \times (c, 1-\epsilon) \subseteq E_2$$

for some constant c, $|c - 1/2| < \epsilon/2$. By Theorem 1.2.7 and the regularity theory for almost-minimal sets [Mag12], we have $E_1^{n_k} \to E_1$ in C^1. In particular, for large k, there is a C^2 function $f_{n_k} : [0,1] \to (0,1)$ such that $f_{n_k} \to c$ in $C^1((0,1))$ and

$$\{(x_1, x_2) : \epsilon < x_2 < f_n(x_1)\} \subseteq E_1^n$$

and

$$\{(x_1, x_2) : f_n(x_1) < x_2 < 1 - \epsilon\} \subseteq E_2^n.$$

Also, by Lemma 1.2.1, $f'_{n_k}(0) = f'_{n_k}(1) = 0$. Thus, for such k, $(E_1^{n_k}, E_2^{n_k})$ has the structure described in Theorem 1.1.1(1), contrary to assumption. The statement on limit sets follows immediately from the first part. $\square$

1.2.2 Exponential Convergence

We now assume that E_1^n converges to the global energy minimizer E_∞, and want to show that if in addition (1.1.5) holds, then the convergence is exponentially fast, both relative to energy and in Hausdorff distance, and that $E_1^n \to E_\infty$ in C^2. We do this by first estimating how quickly the energy is dissipated at each step, $\mathcal{E}(E_1^{n-1}) - \mathcal{E}(E_1^n)$, in terms of the (weighted) W_2-gradient of $\mathcal{E}$ at E_1^n, which comes naturally from the minimizing movement scheme. Then, we show that this energy dissipation is bounded by a constant times the energy difference $\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty)$, which immediately yields that $\mathcal{E}(E_1^n) \to \mathcal{E}(E_\infty)$ exponentially fast. This second step is the main goal of this section, and boils down to certain basic PDE inequalities. Finally, the Hausdorff convergence follows from the energy convergence by examining the convergence of the perimeter.

Since $E_1^n \to E_\infty$ in C^1, for large n we know that E_1^n is the subgraph of a C^2 function $f_n : [0,1] \to (0,1)$:

$$E_1^n = \{(x_1, x_2) : 0 < x_2 < f_n(x_1)\}. \tag{1.2.15}$$

Since $\Gamma := \partial E_1^n \cap \Omega$ intersects $\partial\Omega$ orthogonally, we have $f'_n(0) = f'_n(1) = 0$. Furthermore

$$\int_0^1 f_n(x_1)\, \mathrm{d}x_1 = 1/2,$$

and

$$||f_n'||_{L^\infty([0,1])} \to 0$$

as $n \to \infty$. Moreover, by direction computation we have the following expression for the energy difference $\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty)$:

$$\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty) = \frac{g}{2}\int_0^1 |f_n - \frac{1}{2}|^2 \,\mathrm{d}x_1 + \sigma \int_0^1 \sqrt{1 + (f_n')^2} - 1 \,\mathrm{d}x_1.$$

(Note also that $\mathcal{E}(E_1^n) \to \mathcal{E}(E_\infty)$ by the C^1 convergence $f_n \to 1/2$.)

Throughout this section, we use the notation $\Gamma := \partial E_1^n \cap \Omega$.

Our first goal is to prove the energy dissipation inequality (1.2.16), which is another one of the basic estimates coming from the minimizing movements scheme [Vil03, §8]. Given $n > 0$, we proved in Proposition 1.2.1 that Γ is locally $C^{2,\alpha}$ and intersects $\partial\Omega$ orthogonally at non-corner points. Consequently, it is possible to solve the boundary value problem (1.1.2) with $E_i := E_i^n$ for the pressures p_1, p_2. Let $v_i = -b_i \nabla(p_i + g_i x_2)$ be the corresponding velocity field on E_i^n, and let E_i^ϵ be the image of E_i^n under the flow of $v := v_1 \chi_{E_1^n} + v_2 \chi_{E_2^n}$ for time ϵ. (This vector field v is the (weighted) W_2-*gradient* of $\mathcal{E}$ at (E_1^n, E_2^n).) Then by Gauss's theorem

$$\begin{aligned}
\frac{1}{b_1}\int_{E_1^n} |v_1|^2 \,\mathrm{d}x + \frac{1}{b_2}\int_{E_2^n} |v_2|^2 \,\mathrm{d}x &= \int_{\partial E_1^n} (p_1 - p_2 + (g_1 - g_2)x_2) v_1 \cdot \nu \,\mathrm{d}\mathcal{H}^1 \\
&= \int_{\partial E_1^n} (\sigma H + g x_2) v_1 \cdot \nu \,\mathrm{d}\mathcal{H}^1 \\
&= \left.\frac{d\mathcal{E}(E_1^\epsilon)}{d\epsilon}\right|_{\epsilon=0}.
\end{aligned}$$

With this fact in hand, we can prove:

Proposition 1.2.10 (One-step energy dissipation)**.** For $n \geq 1$, let v be the W_2-gradient of $\mathcal{E}$ at (E_1^n, E_2^n) defined above. Then

$$\frac{1}{b_1}\int_{E_1^n} |v|^2 \,\mathrm{d}x + \frac{1}{b_2}\int_{E_2^n} |v|^2 \,\mathrm{d}x \leq \frac{\mathcal{E}(E_1^{n-1}) - \mathcal{E}(E_1^n)}{h}. \tag{1.2.16}$$

Proof. The proof in the general case is given in [Vil03, §8.4.1]; for the convenience of the reader, we reproduce it here in our specific case. Given two sets $E, F \subseteq \Omega$ with $|E| = |F| = 1/2$, we abbreviate

$$\operatorname{dist}(E, F) := \sqrt{\frac{1}{b_1} W_2^2(E, F) + \frac{1}{b_2} W_2^2(E^c, F^c)}.$$

We also use $||v||_{W_2}$ to denote the weighted W_2-norm of v:

$$||v||_{W_2} := \sqrt{\frac{1}{b_1} \int_{E_1^n} |v_1|^2 \, \mathrm{d}x + \frac{1}{b_2} \int_{E_2^n} |v_2|^2 \, \mathrm{d}x}.$$

Let E_i^ϵ be the image of E_i^n under the flow of v for time ϵ. By minimality of E_1^n in (1.1.4),

$$\mathcal{F}(E_1^n; E_1^{n-1}) \leq \mathcal{F}(E_1^\epsilon; E_1^{n-1}),$$

or

$$\mathcal{E}(E_1^n) + \operatorname{dist}(E_1^n, E_1^{n-1}) \leq \mathcal{E}(E_1^\epsilon) + \operatorname{dist}(E_1^\epsilon, E_1^{n-1}). \tag{1.2.17}$$

From the discussion above, we have

$$\mathcal{E}(E_1^\epsilon) = \mathcal{E}(E_1^n) + \epsilon ||v||_{W_2}^2 + o(\epsilon). \tag{1.2.18}$$

Next, elementary manipulations yield

$$\begin{aligned}
&\operatorname{dist}^2(E_1^{n-1}, E_1^\epsilon) - \operatorname{dist}^2(E_1^{n-1}, E_1^n) \\
&\quad = [\operatorname{dist}(E_1^{n-1}, E_1^\epsilon) + \operatorname{dist}(E_1^{n-1}, E_1^n)][\operatorname{dist}(E_1^{n-1}, E_1^\epsilon) - \operatorname{dist}(E_1^{n-1}, E_1^n)] \\
&\quad \leq [\operatorname{dist}(E_1^{n-1}, E_1^\epsilon) + \operatorname{dist}(E_1^{n-1}, E_1^n)] \operatorname{dist}(E_1^n, E_1^\epsilon) \\
&\quad \leq [2 \operatorname{dist}(E_1^{n-1}, E_1^n) + \operatorname{dist}(E_1^n, E_1^\epsilon)] \operatorname{dist}(E_1^n, E_1^\epsilon).
\end{aligned} \tag{1.2.19}$$

Since

$$\operatorname{dist}(E_1^n, E_1^\epsilon) = \epsilon ||v||_{W_2} + o(\epsilon),$$

we get

$$\frac{\operatorname{dist}(E_1^{n-1}, E_1^{\epsilon})^2}{2h} \le \frac{\operatorname{dist}(E_1^{n-1}, E_1^{n})^2}{2h} + \epsilon \operatorname{dist}(E_1^{n-1}, E_1^{n}) \frac{||v||_{W_2}}{h} + o(\epsilon). \tag{1.2.20}$$

Combining the inequalities (1.2.17), (1.2.18), (1.2.20) and taking $\epsilon \to 0$, we arrive at the desired inequality. □

The following proposition is the main result of this section:

Proposition 1.2.11 (Energy-energy dissipation inequality)**.** There is a constant $C > 0$ depending only on Ω such that for sufficiently large n, and for the vector field v of Proposition (1.2.10), we have

$$\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty) \le C\left(\frac{1}{b_1}\int_{E_1^n} |v|^2 \,\mathrm{d}x + \frac{1}{b_2}\int_{E_2^n} |v|^2 \,\mathrm{d}x\right). \tag{1.2.21}$$

For this we will need a lemma:

Lemma 1.2.12 (Vanishing curvature)**.** We have

$$||H||_{L^\infty(\Gamma)} \to 0$$

as $n \to \infty$.

Proof. Suppose n is large enough that E_1^n is of the form (1.2.15). In equation (1.2.7), we may assume $\int \varphi_i^n = 0$ for $i = 1, 2$, by adjusting φ_i^n by a constant if necessary. Using test functions then yields that $\lambda = g/2$, so that (1.2.7) takes the form

$$\sigma H + g x_2 + \frac{\varphi_1^n}{h b_1} - \frac{\varphi_2^n}{h b_2} = g/2.$$

We know that $||g x_2 - g/2||_{L^\infty(\Gamma)} \to 0$ as $n \to \infty$; therefore, if we can show that $||\varphi_i^n||_{L^\infty(\Gamma)} \to 0$ for $i = 1, 2$, the lemma will follow. Since $E_1^n \to E_\infty$ in C^1, for large n there is a single constant C, independent of n, so that for all $w \in H^1(E_i^n)$, $i = 1, 2$, we have the Poincare inequality

$$||w - (w)_{E_i^n}||_{L^2(E_i^n)} \leq C||\nabla w||_{L^2(E_i^n)},$$

where $(w)_{E_i^n}$ denotes the average of w over E_i^n: $(w)_{E_i^n} := 2\int_{E_i^n} w$. In particular,

$$||\varphi_i^n||_{L^2(E_i^n)} \leq C||\nabla \varphi_i^n||_{L^2(E_i^n)}.$$

By (1.2.5), this implies $||\varphi_i^n||_{L^2(E_i^n)} \to 0$ as $n \to \infty$. Hence, given any subsequence of $\{(E_1^n, E_2^n)\}_{n=1}^{\infty}$, we can find a further subsequence along which $\varphi_i^n \to 0$ almost everywhere. On the other hand, we know that the family $\{\varphi_i^n\}_{n,i}$ is equicontinuous, since $|\nabla \varphi_i^n| \leq \sqrt{2}$ (as $\nabla \varphi_i^n$ is a displacement vector within Ω). Consequently, we must have $\varphi_i^n \to 0$ uniformly along this subsequence. As the original subsequence was aribtrary, we conclude that $||\varphi_i^n||_{L^\infty} \to 0$. $\square$

Proof of Proposition 1.2.11. The proof is divided into two steps. First, we want to show that there is a constant C_1 such that for large n we get

$$\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty) \leq C_1 \int_\Gamma |\sigma H + g x_2 - (\sigma H + g x_2)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1, \tag{1.2.22}$$

where $(f)_\Gamma$ denotes the average of f over Γ: $(f)_\Gamma := (1/|\Gamma|)\int_\Gamma f \, \mathrm{d}\mathcal{H}^1$. Second, we will show there is a constant C_2 such that for large enough n,

$$\int_\Gamma |\sigma H + g x_2 - (\sigma H + g x_2)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1 \leq C_2 \int_\Omega \frac{|v_1|^2}{b_1} + \frac{|v_2|^2}{b_2} \, \mathrm{d}x. \tag{1.2.23}$$

We assume n is large enough so that the E_1^n's are of the form (1.2.15).

To prove (1.2.22), we split it into two parts, corresponding to the two types of energy:

$$P(E_1^n) - P(E_\infty) \lesssim \int_\Gamma |H - (H)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1, \tag{1.2.24}$$

$$\Phi(E_1^n) - \Phi(E_\infty) \lesssim \int_\Gamma |x_2 - (x_2)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1, \tag{1.2.25}$$

again for large enough n. To prove (1.2.24), note first that $(H)_\Gamma = 0$, since $f_n'(0) = f_n'(1) = 0$ and H is the rate of change of the angle of inclination of the tangent line with respect to arclength. Using coordinates, the inequality therefore takes the form

$$\int_0^1 \sqrt{1+(f_n')^2} - 1 \, \mathrm{d}x_1 \leq \int_0^1 \frac{(f_n''(x_1))^2}{(1+(f_n'(x_1))^2)^{5/2}} \, \mathrm{d}x_1.$$

Using $\sqrt{1+t} \leq 1 + t/2$, and the Sobolev inequality for $H_0^1([0,1])$ with optimal constant $1/\pi^2$, we get

$$\int_0^1 \sqrt{1+(f_n')^2} - 1 \, \mathrm{d}x_1 \leq \frac{1}{2\pi^2} \int_0^1 (f_n''(x_1))^2 \, \mathrm{d}x_1.$$

Since $||f_n'||_{L^\infty([0,1])} \to 0$, we have (e.g.)

$$\int_0^1 (f_n''(x_1))^2 \, \mathrm{d}x_1 \leq 2 \int_0^1 \frac{(f_n''(x_1))^2}{(1+(f_n'(x_1))^2)^{5/2}} \, \mathrm{d}x_1$$

for large enough n, which proves (1.2.24).**

To prove (1.2.25), note that

$$\int_0^1 |f_n(x_1) - \frac{1}{2}|^2 \, \mathrm{d}x_1 \leq \int_0^1 |f_n(x_1) - (y)_\Gamma|^2 \, \mathrm{d}x_1,$$

since $\int_0^1 f_n = 1/2$. Combined with the trivial inequality

$$\int_0^1 |f_n(x_1) - (x_2)_\Gamma|^2 \, \mathrm{d}x_1 \leq \int_0^1 |f_n(x_1) - (x_2)_\Gamma|^2 \sqrt{1+(f_n'(x_1))^2} \, \mathrm{d}x_1,$$

we get (1.2.25) with implicit constant $= 1$.

To combined these to get (1.2.22), we expand:

$$\int_\Gamma |\sigma H + g \cdot [x_2 - (x_2)_\Gamma]|^2 \, \mathrm{d}\mathcal{H}^1 = \sigma^2 \int_\Gamma H^2 \, \mathrm{d}\mathcal{H}^1 + g^2 \int_\Gamma |x_2 - (x_2)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1 + 2\sigma g \int_\Gamma H \cdot [x_2 - (x_2)_\Gamma] \, \mathrm{d}\mathcal{H}^1 \tag{1.2.26}$$

**This is an easier version of [MPS22, Theorem 1.3], our setting being a one-dimensional interval rather than a sphere of arbitrary dimension.

$$
\begin{aligned}
&= \sigma^2 \int_\Gamma H^2 \,\mathrm{d}\mathcal{H}^1 + g^2 \int_\Gamma |x_2 - (x_2)_\Gamma|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\quad + 2\sigma g \int_\Gamma H x_2 \,\mathrm{d}\mathcal{H}^1.
\end{aligned}
$$

The last term equals (using integration by parts)

$$
\int_\Gamma H x_2 \,\mathrm{d}\mathcal{H}^1 = \int_0^1 \frac{f_n f_n''}{1+(f_n')^2} \,\mathrm{d}x_1 = -\int_0^1 (f_n')^2 \frac{1+(f_n')^2 - 2 f_n f_n''}{(1+(f_n')^2)^2} \,\mathrm{d}x_1.
$$

By Lemma 1.2.12, we know $||H||_{L^\infty} \to 0$; and since $||f_n'||_{L^\infty} \to 0$, it follows that $||f_n''||_{L^\infty} \to 0$ as well. Hence, for any $\epsilon > 0$, we have

$$
\Big| \int_\Gamma H x_2 \,\mathrm{d}\mathcal{H}^1 \Big| \leq (1+\epsilon) \int_0^1 (f_n')^2 \,\mathrm{d}x_1
$$

for large enough n. Combined with the Sobolev Inequality, this gives

$$
\Big| \int_\Gamma H x_2 \,\mathrm{d}\mathcal{H}^1 \Big| \leq \frac{1+\epsilon}{\pi^2} \int_0^1 (f_n'')^2 \,\mathrm{d}x_1
$$

for large n. Converting f_n'' back to H, we conclude that for large n,

$$
\Big| \int_\Gamma H x_2 \,\mathrm{d}\mathcal{H}^1 \Big| \leq \frac{1+\epsilon}{\pi^2} \int_\Gamma H^2 \,\mathrm{d}\mathcal{H}^1.
$$

In particular, since $\sigma > 2g/\pi^4$, we can choose ϵ so that

$$
\delta := \sigma^2 - (1+\epsilon)\frac{2\sigma g}{\pi^2} > 0.
$$

Consequently, from (1.2.2), we get

$$
\begin{aligned}
\int_\Gamma |\sigma H + g(x_2 - (x_2)_\Gamma)|^2 \,\mathrm{d}\mathcal{H}^1 &\geq \delta \int_\Gamma H^2 \,\mathrm{d}\mathcal{H}^1 + g^2 \int_\Gamma |x_2 - (x_2)_\Gamma|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\geq \Delta \left\{ \sigma \int_\Gamma H^2 \,\mathrm{d}\mathcal{H}^1 + g \int_\Gamma |x_2 - (x_2)_\Gamma|^2 \,\mathrm{d}\mathcal{H}^1 \right\}
\end{aligned}
$$

for $\Delta = \min\{\delta/\sigma, g\}$. Combined with (1.2.24) and (1.2.25), this immediately yields

(1.2.22).

To prove (1.2.23), we note (as in the proof of Lemma 1.2.12) that, since $E_1^n \to E_\infty$ in C^1, for large n there is a single constant C, independent of n, so that for all $w \in H^1(E_i^n)$, $i = 1, 2$, we have the trace and Poincare inequalities

$$||Tw||_{L^2(\partial E_i^n)} \le C||w||_{H^1(E_i^n)},$$

$$||w - (w)_{E_i^n}||_{L^2(E_i^n)} \le C||\nabla w||_{L^2(E_i^n)},$$

where $T : H^1(E_i^n) \to L^2(\partial E_i^n)$ denotes the trace operator (we use the same symbol for the trace operator on the separate domains E_1^n, E_2^n). In particular, we can combine the two inequalities to get

$$||T(w - (w)_{E_i^n})||_{L^2(\partial E_i^n)} \le C||\nabla w||_{L^2(E_i^n)}$$

for $i = 1, 2$, where we are using the same letter to denote a different constant C. We let $w_i = p_i + g_i x_2$ for $i = 1, 2$, where p_i is the pressure in the definition of the W_2-gradient of $\mathcal{E}$ above. Then $Tw_1 - Tw_2 = \sigma H + gx_2$, and thus

$$\begin{aligned}
\int_\Gamma |\sigma H + g(x_2 - (x_2)_\Gamma)|^2 \, \mathrm{d}\mathcal{H}^1 &= \int_\Gamma |Tw_1 - Tw_2 - g(x_2)_\Gamma|^2 \, \mathrm{d}\mathcal{H}^1 \\
&\le \int_\Gamma |Tw_1 - Tw_2 - ((w_1)_{E_1^n} - (w_2)_{E_2^n})|^2 \, \mathrm{d}\mathcal{H}^1 \\
&= \int_\Gamma |T(w_1 - (w_1)_{E_1^n}) - T(w_2 - (w_2)_{E_2^n})|^2 \, \mathrm{d}\mathcal{H}^1 \\
&\le 2\int_\Gamma |T(w_1 - (w_1)_{E_1^n})|^2 \, \mathrm{d}\mathcal{H}^1 \\
&\qquad + 2\int_\Gamma |T(w_2 - (w_2)_{E_2^n})|^2 \, \mathrm{d}\mathcal{H}^1 \\
&\le 2C^2||\nabla w_1||^2_{L^2(E_1^n)} + 2C^2||\nabla w_2||^2_{L^2(E_2^n)}.
\end{aligned}$$

Since $v_i = b_i \nabla w_i$, this immediately gives (1.2.23) with

$$C_2 = 2C^2 \cdot \min\{b_1^{-1}, b_2^{-1}\}.$$

This completes the proof of Proposition 1.2.11. □

Remark 1.2.13. The idea used in this proof of deducing an inequality on ∂E_1^n involving mean curvature from Sobolev-type bounds on a harmonic function on E_1^n is used also in [Che93] for a similar problem in the plane $\mathbf{R}^2$, in the continuous time setting. By contrast, in the case of mean curvature flow, the natural estimate from the gradient flow structure is already in the form of an L^2 inequality involving mean curvature on the boundary of the evolving set, which carries over (with some technical modifications) to the discrete minimizing movements scheme; see [JN20, Proposition 4.1.(ii)] and [Jul+21]. This allows the authors of [Jul+21] to prove uniform estimates that are independent of h and thus deduce the convergence of the limiting solution. Similarly, in the case of the minimizing movements scheme for Mullins-Sekerka, the natural estimates come in the form of uniform $\dot{H}^1$ bounds on the entire domain Ω (rather than the evolving set), which can then be transferred to the evolving interface using a theorem of Schatzle [Sch01] to again deduce convergence of the limiting flow [Jul+21]. Since our estimates, by contrast, come from using the trace and Poincare inequalities on the evolving domain, they involve a distortion factor coming from the shape of the domain, which is why we only apply them after the domain is C^1-close to E_∞. It remains an interesting open problem to deduce similar uniform estimates for the Muskat flow, which would allow to prove the convergence of the limiting flow.

Proof of Theorem 1.1.1(2). We have already proved that $E_1^n \to E_\infty$ in C^1; by Lemma 1.2.12, we also have convergence in C^2. If (1.1.5) holds, then combining Propositions 1.2.10 and 1.2.11 yields the inequality

$$\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty) \leq \frac{C}{h}(\mathcal{E}(E_1^{n-1}) - \mathcal{E}(E_1^n)),$$

for sufficiently large n. This immediately implies that $\mathcal{E}(E_1^n) \to \mathcal{E}(E_\infty)$ exponentially fast.

To show that $E_1^n \to E_\infty$ exponentially fast in Hausdorff distance, we need to show that

$||f_n - 1/2||_{L^\infty} \to 0$ exponentially fast. Since the total energy difference $\mathcal{E}(E_1^n) - \mathcal{E}(E_\infty)$ decreases exponentially, so does the perimeter difference $P(E_1^n) - 1$. Since $\sqrt{1+t} - 1 \geq t/2 - t^2/4$, we have

$$P(E_1^n) - 1 \geq \int_0^1 (f_n')^2/2 - (f_n')^4/4 \, \mathrm{d}x_1,$$

and since $f_n' \to 0$ uniformly, for large n this gives

$$P(E_1^n) - 1 \geq \int_0^1 (f_n')^2/4 \, \mathrm{d}x_1$$

(say). Also, we know that $||f_n''||_{L^\infty}$ is bounded (e.g. from 1.2.12), say by 1 (for large enough n). Hence, if x is a point at which $|f_n'|$ achieves its maximum, then there is an interval around x of length at least $||f_n'||_{L^\infty}/2$ on which $|f_n'| \geq ||f_n'||_{L^\infty}/2$. Therefore, for large n,

$$\int_0^1 (f_n')^2/4 \, \mathrm{d}x_1 \geq (||f_n'||_{L^\infty}/2) \cdot (||f_n'||_{L^\infty}^2/16) = ||f_n'||_{L^\infty}^3/32.$$

Finally, since each f_n equals $1/2$ at some x_1 in $[0, 1]$, we have $||f_n||_{L^\infty} \leq ||f_n'||_{L^\infty}$. Putting it all together, for sufficiently large n

$$||f_n||_{L^\infty} \leq ||f_n'||_{L^\infty} \leq 32^{\frac{1}{3}} \left(P(E_1^n) - 1\right)^{\frac{1}{3}}.$$

Since $P(E_1^n) \to 1$ exponentially fast, this finishes the proof. $\square$

CHAPTER 2

Muskat Problem with Source

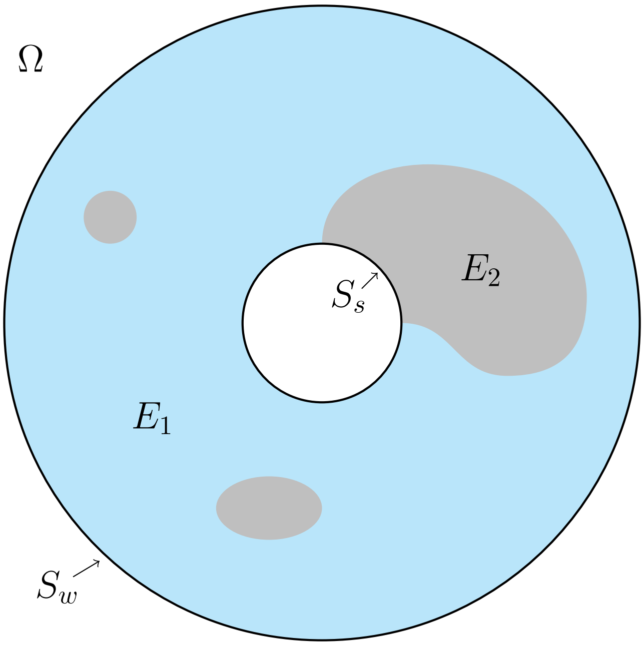
2.1 Introduction

2.1.1 Background

In this chapter we adapt the minimizing movement scheme for the Muskat problem to the case when a source and sink are involved. Specifically, we model the extraction of oil from an oil sand, in which water is pumped into the sand to force oil out. In recent years several studies have appeared which adapt the JKO scheme to similar situations where mass is not conserved. For example, in [JKT21], the authors modify the JKO scheme to model Darcy's law for tumor growth with a pressure-dependent source term. Most relevant to our situation is [MRS10], in which the authors study the motion of a crowd exiting a domain subject to a density constraint. The exit is represented by a subset Γ of the domain boundary, and a measure on Γ represents the portion of the crowd that has exited. In our model, we will have two boundary curves, one where water enters the domain, the other where oil and water exit.

The oil sand is represented by the domain $\Omega \subseteq \mathbf{R}^2$, the open region between two smooth simple closed curves S_w and S_s, where S_s is enclosed by S_w (see figure 2.1 for an example). For example, S_w and S_s could be concentric spheres with radii $R_1 > R_2$ respectively. Water is pumped into Ω through S_w at a constant rate of one unit volume per unit time, uniformly distributed across S_w, causing oil and water to flow out of Ω through

Figure 2.1: Example domain with fluids. The blue region is water, the gray region oil.



S_s. The boundary S_s is assumed to be uniformly permeable, so that fluid flows just as easily through any portion of it. We assume that water and oil completely fill Ω at all times and that they are incompressible and immiscible. We will describe the configuration of fluids at any given time both by their domains of saturation and their density functions, which by incompressibility are simply the indicator functions of the saturation domains. We will denote by E_1 the saturation domain of water and by E_2 the saturation domain of oil. Similarly the density functions of water and oil will be denoted by ρ_1 and ρ_2 respectively; by the above remarks, $\rho_i = \mathbf{1}_{E_i}$. Since the fluids are immiscible and fill Ω, we have $\rho_1 + \rho_2 = 1$ and $\rho_1\rho_2 = 0$ a.e., or equivalently $|E_1 \cap E_2| = |\Omega \backslash (E_1 \cup E_2)| = 0$.

As in chaper 1, we assume that surface tension is present between the fluids, and that each fluid is subject to a conservative force. The total interfacial energy due to surface tension is $\sigma P(E_1)$, where σ is the surface tension coefficient; and the total potential energy of fluid i is $\int_{E_i} \Phi_i \,\mathrm{d}x$, where Φ_i is Lipschitz continuous on $\bar{\Omega}$ and $-\nabla\Phi_i$ is the force on fluid i. Thus the total energy of the configuration is

$$\mathcal{E}(E_1) := \mathcal{E}(E_1, E_2) := \sigma P(E_1) + \int_{E_1} \Phi_1 \,\mathrm{d}x + \int_{E_2} \Phi_2 \,\mathrm{d}x. \tag{2.1.1}$$

Finally, b_i again denotes the mobility of fluid i, as in chapter 1.

2.1.2 Construction of Scheme and Statement of Results

We first describe the scheme intuitively, then define it more precisely. Fix a timestep $h > 0$. At each step in the discrete scheme, we put a uniform measure of total mass h on the outer curve S_w, representing the water that flows into Ω in the next h units of time. We then push this measure into Ω, and require a total volume of h of water and oil to exit from Ω onto the inner curve S_s. The water and oil pushed out of Ω are each represented by a measure supported on S_s. As in chapter 1, we measure the cost of this procedure in terms of optimal transport, while simultaneously trying to minimize the total energy of the new configuration.

Let E_1^0, E_2^0 be the initial configuration of oil and water in Ω. For $n > 0$, we define

$$(E_1^n, E_2^n, \nu_1^n, \nu_2^n) \in \arg\min \left\{ \frac{1}{2hb_1} \Big(W_2^2(h\mu + \mathbf{1}_{E_1^{n-1}} \mathrm{d}x, \mathbf{1}_{E_1} \mathrm{d}x + \nu_1) \right.$$
$$\left. + \frac{1}{2hb_2} W_2^2(\mathbf{1}_{E_2^{n-1}} \mathrm{d}x, \mathbf{1}_{E_2} \mathrm{d}x + \nu_2) + \mathcal{E}(E_1) \right\}, \quad (2.1.2)$$

where E_1, E_2, ν_1, ν_2 are subject to the constraints

$$\begin{cases} |\Omega \backslash (E_1 \cup E_2)| = 0, \\ |E_1 \cap E_2| = 0, \\ \mathcal{N}(S_w, h^2) \subseteq E_1, \\ \nu_1, \nu_2 \geq 0, ||\nu_1|| + ||\nu_2|| = h, \\ h + |E_1^{n-1}| = |E_1| + ||\nu_1||, \\ |E_2^{n-1}| = |E_2| + ||\nu_2||. \end{cases} \quad (2.1.3)$$

Here $\mathcal{N}(S_w, \delta) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \delta\}$ is the (closed) δ-neighborhood of S_w in Ω; ν_1, ν_2 are Radon measures on S_w; and μ is uniform measure on S_w, rescaled to have mass h. Thus, ν_1 represents the water that accumulates on S_s after exiting Ω during the timestep, and similarly ν_2 represents the corresponding measure for oil. The reason for requiring $\mathcal{N}(S_w, h^2) \subseteq E_1$ is because, if water is actually being pumped in continuously, then it should not be able to jump instantaneously away from its source S_w; moreover, the total distance covered by the water should be $\sim h$, so to give enough room we just require the water to occupy at least a distance of h^2 adjacent to S_w.

From the discrete scheme we create piecewise-constant interpolations by setting

$$\rho_i^h(t) := \rho_i^{\lfloor t/h \rfloor},$$

$t \geq 0$, $i = 1, 2$. We then take $h \to 0$ and hope to extract a subsequence converging to some form of solution to the Muskat problem. The main result states that this is the case, under conditional perimeter convergence. To state it, we first recall that the space $BV(\Omega)$ of functions of bounded variation on Ω is defined as all $u \in L^1(\Omega)$ such that there exists a finite Radon measure Du on Ω with

$$\int_\Omega \xi \cdot Du \,\mathrm{d}x = -\int_\Omega u \nabla \cdot \xi \,\mathrm{d}x$$

for all $\xi \in C_c^\infty(\Omega; \mathbf{R}^2)$. Given $u \in BV(\Omega)$, its BV-norm is

$$||u||_{BV(\Omega)} := ||u||_{L^1(\Omega)} + |Du|(\Omega),$$

where $|Du|$ is the absolute variation of Du. Thus, a set $E \subseteq \Omega$ is of finite perimeter if and only if $\mathbf{1}_E \in BV(\Omega)$.

Theorem 2.1.1. *Let $\rho_1^0, \rho_2^0 \in BV(\Omega; \{0, 1\})$ be initial data for the Muskat problem. For any time $T > 0$, there exist*

$$\rho_1, \rho_2 \in L^\infty([0, T]; BV(\Omega; \{0, 1\})) \cap C^{0, \frac{1}{4}}([0, T]; L^1(\Omega)), \tag{2.1.4}$$

$$v_1, v_2 \in L^2([0, T] \times \Omega; \mathbf{R}^2), \tag{2.1.5}$$

such that for every t, we have $\rho_1(t) + \rho_2(t) = 1$ and $\rho_1(t)\rho_2(t) = 0$ a.e.; $\rho_1(0) = \rho_1^0$, $\rho_2(0) = \rho_2^0$; and

$$\partial_t \rho_i + \nabla \cdot (\rho_i v_i) = 0, \tag{2.1.6}$$

$i = 1, 2$, in the distributional sense on Ω: i.e., for all $\psi \in C_c^\infty([0,T] \times \Omega)$

$$\left[\int_\Omega \psi(t,x)\rho_i(t,x)\,\mathrm{d}x\right]_{t=0}^{t=T} - \int_0^T \int_\Omega (\partial_t \psi + v_i \cdot \nabla\psi)\rho_i \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.1.7}$$

($i = 1, 2$). Moreover, if the perimeter convergence assumption (2.2.15) *holds, then ρ_i, v_i solve the Muskat problem in weak formulation: for all $\xi \in C_c^\infty([0,T] \times \Omega; \mathbf{R}^2)$ with $\nabla \cdot \xi = 0$, we have*

$$\begin{aligned} -\int_0^T \int_\Omega \sum_{i=1}^2 b_i^{-1} v_i \cdot \xi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \sum_{i=1}^2 \Phi_i \xi \cdot D\rho_i \,\mathrm{d}t \\ + \sigma \int_0^T \int_\Omega (\nabla \cdot \xi - \mathbf{n} \cdot \nabla \xi \mathbf{n}) |D\rho_1| \,\mathrm{d}t, \end{aligned} \tag{2.1.8}$$

where

$$\mathbf{n} = \frac{D\rho_1}{|D\rho_1|}$$

is the Radon-Nikodym derivative of $D\rho_1$ with respect to its absolute variation $|D\rho_1|$.

Remark 2.1.2. Note that the continuity equation (2.1.7) only gives information about what happens within Ω, and not about the creation and loss of mass on $\partial\Omega$. See below, section 2.3 for discussion.

2.2 Proofs

First, we prove the existence of minimizers to (2.1.2), using the direct method of the calculus of variations. Specifically, let $\{(E_1^m, E_2^m, \nu_1^m, \nu_2^m)\}_{m=1}^\infty$ be a minimizing sequence. By the perimeter bounds, we can find finite-perimeter sets E_1, E_2 with $E_1^m \to E_1$ and $E_2^m \to E_2$ in $L^1(\Omega)$ after taking subsequences. Clearly $\mathcal{N}(S_w, h^2) \subseteq E_1$, since $\mathcal{N}(S_w, h^2) \subseteq E_1^m$ for each m. By the Banach-Aloagula theorem, after taking a further subsequence we can also assume $\nu_1^m \to \nu_1$ and $\nu_2^m \to \nu_2$ weakly-star, for nonnegative Radon measures ν_1, ν_2 on S_o. Since S_s is compact, we have $||\nu_i^m|| \to ||\nu_i||$, and hence $||\nu_1|| + ||\nu_2|| = h$ and (E_1, E_2, ν_1, ν_2) is admissible. Since all terms in (2.1.2) are lower semicontinuous, we conclude that (E_1, E_2, ν_1, ν_2) is a minimizer.

Let M_0 be the minimum appearing in (2.1.2). Since we cannot simply use the current configuration as a competitor (as in the usual minimizing movements situation), we instead estimate M directly by constructing a feasible candidate (E_1, E_2, ν_1, ν_2). To do this, let $\xi = \nabla q$, where q solves

$$\begin{cases} \Delta q = 0 \text{ in } \Omega \\ \frac{\partial q}{\partial n} = -(\mathcal{H}^1(S_w))^{-1} \text{ on } S_w \\ \frac{\partial q}{\partial n} = -(\mathcal{H}^1(S_s))^{-1} \text{ on } S_s. \end{cases} \tag{2.2.1}$$

Then ξ is bounded and C^1 on $\bar{\Omega}$. We extend ξ to $\mathbf{R}^2$ by setting $\xi = 0$ outside Ω. Let the measures θ_1, θ_2 be the images of the measures $\rho_1^{n-1}\mathrm{d}x, \rho_2^{n-1}\mathrm{d}x$ under the flow of ξ for time h, respectively. Then we define E_2, ν_2 by the equation $\mathbf{1}_{E_2}\mathrm{d}x + \nu_2 = \theta_2$, and we then define $E_1 := \Omega \backslash E_2$ and ν_1 as the singular part of θ_1. It is clear that (E_1, E_2, ν_1, ν_2) is admissible for (2.1.2). Since ξ is bounded, the W_2^2 distances appearing in (2.1.2) are $O(h^2)$, with an implicit constant independent of (E_2^{n-1}, E_2^{n-1}). Likewise, because ξ is bounded and C^1, and because $\mathcal{N}(S_w, h^2) \subseteq E_1^{n-1}$ for $n \geq 2$, it follows that for $n \geq 2$ the total energy increase $[\mathcal{E}(E_1^n) - \mathcal{E}(E_1^{n-1})]_+$ is $O(h)$, again with constant independent of n. (Note: the reason we need $n \geq 2$ is that if oil is initially adjacent to S_w [i.e., if $\partial E_2^0 \cap S_w$ has positive Hausdorff measure], then when water is pushed in during the next timestep, a new interface is created, causing the perimeter component of $\mathcal{E}$ to increase suddenly. However, after the first timestep, no oil is adjacent to S_w, so this situation no longer occurs.) Similarly, one can verify that $[\mathcal{E}(E_1^1) - \mathcal{E}(E_1^0)]_+ = O(1)$. Therefore,

$$\sum_{n=1}^{N} \left\{ \frac{1}{2hb_1} W_2^2(h\mu + \rho_1^{n-1}\mathrm{d}x, \rho_1^n\mathrm{d}x + \nu_1^n) + \frac{1}{2hb_2} W_2^2(\rho_2^{n-1}\mathrm{d}x, \rho_2^n\mathrm{d}x + \nu_2^n) \right\} \leq C_1 T \tag{2.2.2}$$

and

$$\mathcal{E}(E_1^n) \leq C_2(1+T) \tag{2.2.3}$$

for some constants $C_1, C_2 > 0$ independent of h.

We now define the discrete velocities v_i^n. Let $\tilde{E}_i^{n-1}$ be the subset of E_i^{n-1} that is pushed onto E_i^n in (2.1.2) (as opposed to being pushed onto the boundary S_s). Since $\rho_i^n \, \mathrm{d}x$ is absolutely continuous with respect to Lebesgue measure, Brenier's theorem gives a transport map $T_i^n : E_i^n \to \tilde{E}_i^{n-1}$ such that $(T_i^n)_\#(\rho_i^n \, \mathrm{d}x) = \mathbf{1}_{\tilde{E}_i^{n-1}} \, \mathrm{d}x$ and

$$\int_\Omega |T_i^n(x) - x|^2 \rho_i^n(x) \mathrm{d}x = W_2^2(\mathbf{1}_{\tilde{E}_i^{n-1}} \, \mathrm{d}x, \rho_i^n \, \mathrm{d}x).$$

We define v_i^n by

$$v_i^n(x) := \frac{T_i^n(x) - x}{h},$$

$i = 1, 2$, and the corresponding interpolations

$$v_i^h(t) := v_i^{\lfloor t/h \rfloor}.$$

From (2.2.2) we conclude

$$||v_i^h||^2_{L^2([0,T]\times\Omega)} = \int_0^T \int_\Omega |v_i^h(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C_2 T. \tag{2.2.4}$$

Now we take $h \to 0$ and attempt to extract a convergent subsequence. For this, we need to show pre-compactness of $\{\rho_i^h\}_{h>0}$ in $L^1([0,T] \times \Omega)$ for $i = 1, 2$. For any fixed t, we have by the uniform perimeter bound (2.2.3)

$$||\rho_i^h(t,\cdot) - \rho_i^h(t, \cdot - y)||_{L^1(\Omega)} \leq |y| \cdot |D\rho_i^h(t,\cdot)|(\Omega) \lesssim |y|, \tag{2.2.5}$$

for a constant independent of h.

To get equicontinuity in time, we note that since $\rho_1^h(t,\cdot) + \rho_2^h(t,\cdot) = 1$ for all t, h, it is enough to show equicontinuity for ρ_2^h. To compare ρ_2^m and ρ_2^n for $m < n$, we split E_2^m into the part $C^{m,n}$ which goes into E_2^n, and the remainder $D^{m,n}$ which goes into S_s. Specifically, $C^{m,n} := T^{-1}(E_2^n)$, where

$$T := T_2^n \circ T_2^{n-1} \circ \cdots \circ T_2^{m+1},$$

and $D^{m,n} := E_2^m \backslash C^{m,n}$. By construction $|C^{m,n}| = |E_2^n|$. Also, $|D^{m,n}| \leq (n-m)\cdot h$, since $||\nu_2^n|| \leq h$ for all n. We next define the measure $\theta^{m,n}$ as

$$\theta^{m,n} := \rho_2^n \,\mathrm{d}x + \sum_{k=m+1}^{n} \nu_2^k,$$

and the associated interpolation $\theta^h(s,t) := \theta^{\lfloor s/h \rfloor, \lfloor t/h \rfloor}$. In other words, $\theta^{m,n}$ represents the current oil in Ω at timestep n plus the oil that has accumulated on S_s since timestep m. In particular, $|E_2^m| = ||\theta^{m,n}||$ for all $n > m$, and $|D^{m,n}| = ||\theta^{m,n} - \rho_2^n \,\mathrm{d}x||$. Moreover

$$W_2(\theta^{m,n}, \theta^{m,n+1}) \leq W_2^2(\rho_2^n \,\mathrm{d}x, \rho_2^{n+1} \,\mathrm{d}x + \nu_2^{n+1}).$$

Therefore, if $0 \leq s < t \leq T$, then setting $n_1 = \lfloor s/h \rfloor, n_2 = \lfloor t/h \rfloor$ we get

$$\begin{aligned}
W_2(\rho_2^h(s,\cdot)\,\mathrm{d}x, \theta^{s,t}) &= W_2(\rho_2^{n_1}\,\mathrm{d}x, \theta^{n_1,n_2}) \\
&\leq W_2(\rho_2^{n_1}\,\mathrm{d}x, \theta^{n_1,n_1+1}) + \sum_{k=n_1+2}^{n_2} W_2(\theta^{n_1,k}, \theta^{n_1,k-1}) \\
&\leq \sum_{k=n_1+1}^{n_2} W_2(\rho_2^{k-1}\,\mathrm{d}x, \rho_2^k\,\mathrm{d}x + \nu_2^k) \\
&\leq \left(\sum_{k=n_1+1}^{n_2} W_2^2(\rho_2^{k-1}\,\mathrm{d}x, \rho_2^k\,\mathrm{d}x + \nu_2^k) \right)^{1/2} \cdot (n_2 - n_1)^{1/2} \\
&\lesssim (hT)^{1/2} \cdot \Big(\frac{t-s+h}{h} \Big)^{1/2} \\
&= T^{1/2}(t-s+h)^{1/2}.
\end{aligned}$$

Since $W_2(\mathbf{1}_{C^{m,n}}\,\mathrm{d}x, \rho_2^n\,\mathrm{d}x) \leq W_2(\rho_2^m\,\mathrm{d}x, \theta^{m,n})$, we get

$$W_2(\mathbf{1}_{C^{m,n}}\,\mathrm{d}x, \rho_2^n\,\mathrm{d}x) \lesssim T^{1/2}(t-s+h)^{1/2}. \tag{2.2.6}$$

We now quote a result from [CL21]:

Lemma 2.2.1. *Let A, B have finite perimeter in Ω. Then*

$$|A \triangle B| \lesssim \sqrt{P(A)+P(B)}\sqrt{W_1(\chi_A, \chi_B)},$$

where the implicit constant depends only on Ω.

Since $W_1 \leq W_2$ by Cauchy-Schwarz, Lemma 2.1 combined with (2.2.6) gives

$$\begin{aligned} ||\rho_2^h(s,\cdot) - \rho_2^h(t,\cdot)||_{L^1(\Omega)} &= ||\rho_2^{n_1} - \rho_2^{n_2}||_{L^1(\Omega)} \\ &\leq |D^{n_1,n_2}| + |C^{n_1,n_2} \triangle E^{n_2}| \\ &\lesssim (n_2 - n_1)\cdot h + T^{1/2}\cdot \left\{T^{1/2}(t-s+h)^{1/2}\right\}^{1/2} \\ &\leq \left(\frac{t-s}{h}+1\right)\cdot h + T^{1/2}\cdot \left\{T^{1/2}(t-s+h)^{1/2}\right\}^{1/2} \\ &= (t-s+h) + T^{3/4}(t-s+h)^{1/4}. \end{aligned} \tag{2.2.7}$$

Thus $\{\rho_2^h\}_h$, and therefore also $\{\rho_1^h\}_h$, is equicontinuous in time. Combining this with (2.2.5), we get

$$\begin{aligned} ||\rho_i^h(s,\cdot) - \rho_i^h(t,\cdot-y)||_{L^1(\Omega)} &\leq ||\rho_i^h(s,\cdot) - \rho_i^h(s,\cdot-y)||_{L^1(\Omega)} \\ &\quad + ||\rho_i^h(s,\cdot-y) - \rho_i^h(t,\cdot-y)||_{L^1(\Omega)} \\ &\lesssim |y| + (t-s+h) + T^{3/4}(t-s+h)^{1/4}. \end{aligned}$$

Thus, $\{\rho_i^h\}_{h>0}$ is equicontinuous in $L^1([0,T]\times\Omega)$ for $i=1,2$, and so by the Riesz-Frechet-Kolmogorov theorem is precompact. In particular, after taking a subsequence, we have $\rho_i^h \to \rho_i$ in $L^1([0,T]\times\Omega)$ for some ρ_i. This implies $\rho_i^h(t,\cdot) \to \rho_i(t,\cdot)$ in $L^1(\Omega)$ for almost every t. This $\rho_i(t,\cdot)$ is necessarily a characteristic function, and lower semicontinuity of the perimeter/total variation implies $\rho_i(t,\cdot) \in BV(\Omega;\{0,1\})$, with $\rho_i(t,\cdot)$ bounded in BV norm uniformly in t. Thus

$$\rho_i \in L^\infty([0,T]; BV(\Omega;\{0,1\})).$$

Moreover, from (2.2.7) we see that

$$||\rho_i(s,\cdot) - \rho_i(t,\cdot)||_{L^1(\Omega)} \lesssim (t-s) + T^{3/4}(t-s)^{1/4} \lesssim (t-s)^{1/4}, \tag{2.2.8}$$

i.e.

$$\rho_i \in C^{0,\frac{1}{4}}([0,T]; L^1(\Omega)).$$

(Thus we actually have $\rho_i^h(t,\cdot) \to \rho_i(t,\cdot)$ in $L^1(\Omega)$ for *every* t.) It is clear that $\rho_i(0,\cdot) = \rho_i^0$ for each i, since $\rho_i^h(0,\cdot) = \rho_i^0$ for each h; and similarly $\rho_1(t,x) + \rho_2(t,x) = 1$ a.e.

Next, we derive the continuity equation (2.1.7). For simplicity we assume T/h is an integer; the general case is similar. Likewise, we assume that ψ is actually defined on $[-\epsilon, T+\epsilon] \times \bar{\Omega}$ for some small $\epsilon > 0$; this can easily be removed, and is simply to make the proof easier to follow. First, by the uniform L^2 bounds (2.2.4), and the weak compactness of $L^2([0,T]\times\Omega)$, we deduce that there exist $v_i \in L^2([0,T]\times\Omega)$ with $v_i^h \rightharpoonup v_i$ as $h \to 0$ (along a further subsequence of the subsequence above, again not relabeled). Let $\psi \in C^\infty([0,T]\times\Omega)$. Since $-\nabla\phi_i^n = -hv_i^h$ pushes $\rho_i^n \,\mathrm{d}x$ onto $\rho_i^{n-1}\,\mathrm{d}x$,

$$\begin{aligned}
\int_\Omega \psi(\rho_i^n - \rho_i^{n-1})\,\mathrm{d}x &= \int_\Omega \rho_i^n[\psi(t,x) - \psi(t, x - hv_i^n(x))]\,\mathrm{d}x \\
&= \int_\Omega \rho_i^n \nabla\psi(t,x)\cdot(hv_i^n)\,\mathrm{d}x
\end{aligned}$$

up to $O(||\nabla^2\psi||_{L^\infty} h^2 ||v_i^n||_{L^2})$ error. Therefore

$$\sum_{n=1}^{T/h} h \int_\Omega \rho_i^n v_i^n \cdot \nabla\psi(nh,x)\,\mathrm{d}x = \sum_{n=1}^{T/h} \int_\Omega \psi(nh,x)(\rho_i^n - \rho_i^{n-1})\,\mathrm{d}x + O(h).$$

(The $O(h)$ comes from the fact that we summed $\sim h^{-1}$-many terms.) On the other hand

$$h\int_\Omega \rho_i^n \partial_t\psi(t,x)\,\mathrm{d}x = \int_\Omega \rho_i^n[\psi(t+h,x)-\psi(t,x)]\,\mathrm{d}x + O(h^2||\partial_{tt}\psi||_{L^\infty});$$

hence

$$\sum_{n=1}^{T/h} h\int_\Omega \rho_i^n \partial_t\psi(nh,x)\,\mathrm{d}x = \sum_{n=1}^{T/h}\int_\Omega \rho_i^n[\psi((n+1)h,x)-\psi(nh,x)]\,\mathrm{d}x + O(h).$$

Adding, we get

$$\sum_{n=1}^{T/h} h\int_\Omega \rho_i^n[\partial_t\psi(nh,x)+v_i^n\cdot\nabla\psi(nh,x)]\,\mathrm{d}x = \sum_{n=1}^{T/h}\int_\Omega \psi(nh,x)(\rho_i^n-\rho_i^{n-1})$$
$$+\rho_i^n[\psi((n+1)h,x)-\psi(nh,x)]\,\mathrm{d}x + O(h).$$

But the sum on the right simplifies to

$$\sum_{n=1}^{T/h}\int_\Omega [\rho_i^n\psi((n+1)h,x)-\rho_i^{n-1}\psi(nh,x)]\,\mathrm{d}x =$$
$$\int_\Omega \rho_i^{T/h+1}\psi(T,x)\,\mathrm{d}x - \int_\Omega \rho_i^1\psi(0,x)\,\mathrm{d}x.$$

Thus,

$$\int_0^T\int_\Omega \rho_i^h(\partial_t\psi + v_i^h\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_i^{T/h+1}\psi(T,x)\,\mathrm{d}x - \int_\Omega \rho_i^1\psi(0,x)\,\mathrm{d}x + O(h).$$

Since $\rho_i^h v_i^h = v_i^h$ (a.e.) and $v_i^h \rightharpoonup v_i$, and since $\rho_i^{T/h+1}\to\rho_i(T,\cdot)$ and $\rho_i^1\to\rho_i^0$ in $L^1(\Omega)$, we get

$$\int_0^T\int_\Omega \rho_i(\partial_t\psi + v_i\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_i(T,x)\psi(T,x)\,\mathrm{d}x - \int_\Omega \rho_i^0(x)\psi(0,x)\,\mathrm{d}x. \qquad (2.2.9)$$

Finally, we derive the curvature equation (2.1.8). Let $\xi \in C_c^\infty([0,T] \times \Omega; \mathbf{R}^2)$ be divergence free: $\nabla \cdot \xi = 0$. For each $\tau \in \mathbf{R}$, let E_i^τ be the image of E_i^h under the flow of ξ for time τ, and define

$$\mathcal{F}(\tau) = \frac{1}{2hb_1} W_2^2(h\mu + \rho_1^{n-1}\mathrm{d}x, \mathbf{1}_{E_1^\tau}\mathrm{d}x + \nu_1^n) \\ + \frac{1}{2hb_2} W_2^2(\rho_2^{n-1}\mathrm{d}x, \mathbf{1}_{E_2^\tau}\mathrm{d}x + \nu_2^n) + \mathcal{E}(E_1^\tau). \quad (2.2.10)$$

Then, since $\mathcal{F}(\tau)$ achieves its minimum at $\tau = 0$, we have $\mathcal{F}'(0) = 0$, or

$$\int_\Omega (b_1^{-1} v_1^h + b_2^{-1} v_2^h) \cdot \xi \,\mathrm{d}x = \int_\Omega \Phi_1 \xi \cdot D\rho_1^h + \Phi_2 \xi \cdot D\rho_2^h \\ + \sigma \int_\Omega (\nabla \cdot \xi - \mathbf{n}^h \cdot \nabla \xi \mathbf{n}^h) |D\rho_1^h| \quad (2.2.11)$$

(see [Mag12, §17.3] for the formula for first variation of perimeter). Here $\mathbf{n}^h$ is the Radon-Nikodym derivative of $D\rho_1^h$ with respect to its absolute variation $|D\rho_1^h|$. Integrating both sides over $[0,T]$ with respect to t yields

$$\int_0^T \int_\Omega (b_1^{-1} v_1^h + b_2^{-1} v_2^h) \cdot \xi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \left[\int_\Omega \Phi_1 \xi \cdot D\rho_1^h + \Phi_2 \xi \cdot D\rho_2^h \right] \mathrm{d}t \\ + \sigma \int_0^T \int_\Omega (\nabla \cdot \xi - \mathbf{n}^h \cdot \nabla \xi \mathbf{n}^h) |D\rho_1^h| \,\mathrm{d}t. \quad (2.2.12)$$

We now want to take $h \to 0$ for each of the three terms in this equation, to recover (2.1.8). Since $v_i^h \rightharpoonup v_i$ in $L^2([0,T] \times \Omega)$, we immediately get

$$\int_0^T \int_\Omega (b_1^{-1} v_1^h + b_2^{-1} v_2^h) \cdot \xi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega (b_1^{-1} v_1 + b_2^{-1} v_2) \cdot \xi \,\mathrm{d}x\,\mathrm{d}t. \quad (2.2.13)$$

For the second term, we note that since $\rho_i^h(t,\cdot) \to \rho_i(t,\cdot)$ in $L^1(\Omega)$ for each t, we have

$$\int_\Omega \Phi_i \xi \cdot D\rho_i^h \to \int_\Omega \Phi_i \xi \cdot D\rho_i$$

for each t, by definition of $D\rho_i^h, D\rho_i$. Since the integrals $\int_\Omega \Phi_1 \xi \cdot D\rho_1^h$ are uniformly bounded by virtue of the perimeter bounds, dominated convergence yields

$$\int_0^T \left[\int_\Omega \Phi_1\xi \cdot D\rho_1^h + \Phi_2\xi \cdot D\rho_2^h\right] \mathrm{d}t \to \int_0^T \left[\int_\Omega \Phi_1\xi \cdot D\rho_1 + \Phi_2\xi \cdot D\rho_2\right] \mathrm{d}t. \tag{2.2.14}$$

Finally, for the curvature term we will need the following *perimeter convergence assumption*:

$$\int_0^T P(E_1^h(t))\,\mathrm{d}t \to \int_0^T P(E_1(t))\,\mathrm{d}t. \tag{2.2.15}$$

Note that since the perimeter is automatically lower-semicontinuous, perimeter convergence simply means that no perimeter is lost in the limit, for example through disappearing interfaces (see for example [LO16, Fig. 1] and discussion, as well as [Rög05; LS95]). Since $P(E_1(t)) \leq P(E_1^h(t))$ for every h, t by lower-semicontinuity, (2.2.15) implies that $P(E_1(t)) \to P(E_1^h(t))$ for almost every t. Therefore, by a theorem of Reshetnyak [Mag12, §20.12],

$$\int_\Omega (\nabla\cdot\xi - \mathbf{n}^h\cdot\nabla\xi\mathbf{n}^h)|D\rho_1^h| \to \int_\Omega (\nabla\cdot\xi - \mathbf{n}\cdot\nabla\xi\mathbf{n})|D\rho_1|.$$

Again, the perimeter bounds imply that the integrals on the left are uniformly bounded, and so dominated convergence gives

$$\int_0^T \int_\Omega (\nabla\cdot\xi - \mathbf{n}^h\cdot\nabla\xi\mathbf{n}^h)|D\rho_1^h|\,\mathrm{d}t \to \int_0^T \int_\Omega (\nabla\cdot\xi - \mathbf{n}\cdot\nabla\xi\mathbf{n})|D\rho_1|\,\mathrm{d}t. \tag{2.2.16}$$

Combining (2.2.13), (2.2.14), and (2.2.16), we get (2.1.8), which concludes the proof of Theorem 2.1.1. □

2.3 Remarks and Open Questions

Many questions remain about the more detailed behavior of the weak solution in Theorem 2.1.1. In particular, the continuity equation (2.1.7) only gives information about what happens inside Ω, since it only involves test functions vanishing near $\partial\Omega$; it does not capture the source and sink behavior on $\partial\Omega$. One expects that there exist measures

$\nu_1(t), \nu_2(t) \geq 0$ on S_s such that

$$\partial_t \rho_1 + \nabla \cdot (\rho_1 v_1) = \mu - \nu_1$$

and

$$\partial_t \rho_2 + \nabla \cdot (\rho_2 v_2) = -\nu_2$$

distributionally on $\bar{\Omega}$. In other words, ν_i represents the instantaneous flux of fluid i across S_s in the direction out of Ω. The distributional form of the above equations is

$$\begin{aligned}\left[\int_\Omega \psi(t,x)\rho_1(t,x)\,\mathrm{d}x\right]_{t=0}^{t=T} + \int_0^T \int_{\partial\Omega} \psi(\rho_1 v_1)\cdot \mathbf{n}_\Omega \,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ - \int_0^T \int_\Omega (\partial_t \psi + v_1 \cdot \nabla \psi)\rho_1 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\partial\Omega} \psi\, d(\mu - \nu_1)\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\left[\int_\Omega \psi(t,x)\rho_2(t,x)\,\mathrm{d}x\right]_{t=0}^{t=T} + \int_0^T \int_{\partial\Omega} \psi(\rho_2 v_2)\cdot \mathbf{n}_\Omega \,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t \\ - \int_0^T \int_\Omega (\partial_t \psi + v_2 \cdot \nabla \psi)\rho_2 \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\partial\Omega} \psi\, d\nu_2\,\mathrm{d}t,\end{aligned}$$

where $\mathbf{n}_\Omega$ is the outer unit normal to $\partial\Omega$. However, the integrals over $\partial\Omega$ involving $\rho_i v_i$ require us to know the boundary values of $\rho_i v_i = v_i$, which for general $v_i \in L^2(\Omega; \mathbf{R}^2)$ does not make sense without further regularity assumptions, e.g. $v_i \in W^{1,p}(\Omega)$ for some $p \geq 1$. Since the natural estimates only give $v_i \in L^2(\Omega; \mathbf{R}^2)$, it is not clear whether this extra regularity holds. Moreover, it is not very clear what the correct boundary condition on S_s is to capture the uniform permeability of S_s; heuristic calculations suggest that the velocities v_i should be normal to S_s, but again this requires evaluating them on the boundary.

Beyond these technical issues, many natural questions arise concerning the qualitative behavior of solutions. For example, does all oil eventually exit Ω? If so, how long does it take? Does this depend on the initial data? Also, how does this change if we alter the rate at which water is pumped in? Is there an optimal rate that extracts oil fastest?

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