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Spacetimes with Torsion

DISSERTATION

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DOCTOR OF PHILOSOPHY

in Philosophy

by

Helen Meskhidze

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DEDICATION

To my parents, for enabling and encouraging me to pursue my dreams
and to Ashton, for being there every step of the way.

TABLE OF CONTENTS

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Chapter 1 and a portion of Chapter 3 are collaborative work with Jim Weatherall.

VITA

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ABSTRACT OF THE DISSERTATION

Spacetimes with Torsion

By

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Doctor of Philosophy in Philosophy

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How should we understand gravitational influence? In traditional formulations of Newtonian Gravity, gravitational influence is understood through forces; massive bodies attract one another through gravitational force. Our current best theory of gravity, General Relativity (GR), presents a different understanding of gravitational influence. GR is thought to have taught us that gravitational influence should be properly understood as a manifestation of spacetime curvature. This lesson, however, is complicated by the existence of a gravitational theory that is empirically equivalent to General Relativity and represents gravitational influence again through forces: Teleparallel Gravity (TPG). In contrast to Newtonian Gravity, the forces of TPG feature torsion (or, twisting). TPG raises both questions regarding underdetermination and more fundamental conceptual questions: Which theory, General Relativity or Teleparallel Gravity, describes our world? And how should we understand the torsional forces posited by Teleparallel Gravity?

The first question mentioned above has been the subject of philosophical study (see, e.g., Knox 2011). Addressing the second question mentioned above will be the main goal of this dissertation. Since we are familiar with how forces operate in the non-relativistic context, Chapter 1 begins by formulating a novel non-relativistic theory of gravity that features torsional forces. To build this theory, we discuss how to incorporate torsion in the non-

relativistic context and what we would expect of such a theory. We state and prove a theorem that establishes the relation between models of Newton-Cartan theory and torsional models.

With a non-relativistic, torsional theory in hand, in Chapter 2, I turn to consider the non-relativistic limit of Teleparallel Gravity. I show how to take the classical limit using the tetrad formalism of Teleparallel gravity. I prove that Teleparallel gravity reduces not to the previously outlined non-relativistic, torsional theory but, rather, to standard Newtonian Gravity.

In Chapter 3, I discuss and contextualize these results. I first present the similarities between my results and those derived in the torsion-free context. Malament (1986b) shows that taking the classical limit of General Relativity results in Newton-Cartan theory, a theory that is spatially flat. In other words, taking the classical limit "squeezes out" the spatial curvature of General Relativity. I discuss how my results similarly show that taking the classical limit of TPG "squeezes out" the torsion.

Next, I consider recent efforts by physicists to develop classical, torsional theories of gravity. It is commonly claimed in this literature that one cannot have both non-vanishing torsion and a closed temporal metric. Having formulated a classical theory with both non-vanishing torsion and a closed temporal metric, I reflect on why this claim is made and where the argument went awry.

Finally, I discuss projects that use other methods of relating relativistic theories to classical, torsional ones. Some argue that torsional gravity is the correct framework to describe the non-relativistic limit of General Relativity while others claim it is the non-relativistic limit of Teleparallel Gravity. I show how we might understand these projects so that their claims are not inconsistent with one another or with the results presented here.

Chapter 1

Torsion in the Classical Spacetime Context

1.1 Introduction

How should we understand gravitational influence? In traditional formulations of Newtonian Gravity (NG), gravitational influence is understood as a force. Gravitational force is mediated by a gravitational potential, which is itself related to the distribution of matter. This means that, in Newtonian Gravity, massive bodies exert attractive gravitational forces on one another. Our current best theory of gravity, General Relativity (GR), presents a different understanding of gravitational influence. GR is thought to have taught us that gravitational influence should be properly understood as a manifestation of spacetime curvature. In particular, massive bodies curve spacetime, and gravitational influence is a manifestation of this curvature. This means that, unlike the flat spacetime background of traditional Newtonian Gravity, GR posits a curved spacetime that depends dynamically on the distribution of matter.

Interestingly, the lessons of GR can be applied in the non-relativistic context. It is possible to formulate a non-relativistic (i.e., classical) theory of gravity with curvature. This theory goes by the name of Newton-Cartan theory (NCT; sometimes referred to as "geometrized Newtonian gravity") and, while space is flat in this theory, space*time* is curved. The curvature in NCT is dynamically determined by the matter distribution, and gravitational influences are a manifestation of the curvature of spacetime. Models of NCT are systematically related to models of GR as well as Newtonian Gravity.

The above picture is complicated by the existence of a gravitational theory that is empirically equivalent to General Relativity, but represents gravitational influence as a force and is set on a flat spacetime background: Teleparallel Gravity (TPG). In contrast to Newtonian Gravity, the forces of TPG feature torsion (or, spacetime twisting). TPG raises both questions regarding underdetermination and more fundamental conceptual questions: Which theory, General Relativity or Teleparallel Gravity, describes our world?[1] How should we understand the torsional forces posited by Teleparallel Gravity? And what is the relation between TPG and the other gravitational theories mentioned above?

Addressing these questions will be the goal here. We suggest that to better understand the forces of TPG, a natural place to begin is another gravitational theory that employs forces, namely, Newtonian Gravity. Newtonian Gravity, however, is a non-relativistic theory and gravitational force does not involve torsion. A classical theory of gravity with torsional forces will prove to be a more informative comparison. Developing such a theory to address the above questions will be the goal of this paper.

Beyond the motivations already outlined, formulating a classical spacetime with torsion will also have implications for various projects in the physics literature. There have been proposals for torsional classical theories to describe the fractional quantum Hall effect (Geracie, Son,

[1]This question has also been addressed by Eleanor Knox (2011). Our discussion here is influenced by her insights about TPG.

Wu, and Wu 2015) and to serve as the lower-dimensional reductions to 5D quantum gravity (Christensen, Hartong, Obers, and Rollier 2014b; Bergshoeff, Hartong, and Rosseel 2014; Hartong and Obers 2015; Afshar, Bergshoeff, Mehra, Parekh, and Rollier 2016; Figueroa-O'Farrill 2020). Though we will not discuss these projects in detail, we will discuss the classical spacetimes with torsion that they develop. In so doing, we will trouble one assumption they share and propose an alternative treatment of time. In particular, it is widely claimed that a classical spacetime with torsion cannot have a temporal metric that is closed, in the sense of differential forms. As we discuss, this is not true—or rather, it holds only in the presence of a further condition that is motivated only by specific applications of classical spacetimes with torsion.

To build a non-relativistic theory with torsion, we begin with some background on the theories mentioned above. We then discuss how to incorporate torsion in the non-relativistic context and what we would expect of such a theory in terms of how it represents space and time, as well as how it treats sources and forces. We next consider the relation of the proposed theory to other classical theories with torsion in the literature. Then, we finally state and prove a theorem, analogous to Trautman's (1965) degeometrization theorem, that establishes that, associated with every model of Newton-Cartan theory, there exists a (non-unique) model of Newtonian gravitation with torsion with the same mass density and particle trajectories.

1.1.1 Background

Let us begin by making the claims of the introduction more precise. As mentioned, our current best theory of gravity, General Relativity, represents gravitational influence through the curvature of spacetime. We take a model of that theory to be a pair, (M, g_{ab}) where M is a smooth, connected, four-dimensional, paracompact, Hausdorff manifold, and g_{ab} is a smooth, Lorentz-signature metric on M. As a relativistic theory, GR places an upper bound on the speed of light. This feature of GR is formalized by the metric, g_{ab}, which determines a light cone structure. Light-like particles (i.e., photons) follow trajectories along the cone while massive particles follow trajectories that lie inside the light cones.

In contrast, Teleparallel Gravity is set on a flat spacetime background and represents gravitational influence through forces using torsion. Like GR, TPG is a relativistic theory (it posits a Lorentz-signature metric) and, as mentioned, TPG is empirically equivalent to GR, at least locally. We present the formal apparatus for understanding torsion below, but let us first develop an intuition for it. The torsion tensor characterizes the twisting of the tangent space as it is parallel transported along a curve. One can imagine parallel transporting two end-to-end vectors along one other. When the torsion is vanishing, this procedure yields a parallelogram. However, in spaces with torsion, the parallelograms break because the vectors do not end up tip-to-tip. As Cai, Capozziello, De Laurentis, and Saridakis put it:

> ...while in curved spaces considering two bits of geodesics and displacing one along the other will form an infinitesimal parallelogram, in twisted spaces the above procedure of displacing one geodesic bit along the other leads to a gap between the extremities, i.e. the infinitesimal parallelogram breaks. This implies that in performing the parallel transportation of a vector field in a space with torsion, an intrinsic length—related to torsion—appears. (2016, 5)

A non-relativistic spacetime (i.e., that of Newton-Cartan theory and Newtonian gravity) can, in general, be expressed as $(M, t_a, h^{ab}, \tilde{\nabla})$. M is, as before, a smooth, connected four-dimensional, Hausdorff, paracompact manifold. The metric of GR, however, is replaced by two degenerate metrics: the temporal metric, t_a, and the spatial metric, h^{ab}. These metrics are orthogonal to one another (i.e., $h^{ab}t_b = 0$). If the temporal length of a vector is non-vanishing, we characterize it as "timelike" (else, "spacelike"). Finally, we require the derivative operator to be metric compatible ($\tilde{\nabla}_a t_b = \mathbf{0}$ and $\tilde{\nabla}_a h^{bc} = 0$). These conditions ensure that the metric and affine structures agree.[2]

1.2 A classical gravitational theory with torsion

We now describe the features of a gravitational theory set in (flat) spacetime with classical metrics and torsion.

1.2.1 Torsion

We begin with the most pressing issue: how to represent torsion in a classical spacetime. Since the displacement of vectors along one another is given by the derivative operator, to derive a theory of gravity with possibly non-vanishing torsion in the classical context, we must adjust the conditions on our derivative operator.

We define a generic (i.e., not specific to the classical context) derivative operator following Malament's (2012) conventions but drop the last requirement ($DO6$)—that the action of two derivative operators on a scalar field commute—to allow torsion. This leaves:

> **Definition 1.** ∇ is a (covariant) derivative operator on M if it satisfies the following conditions:

(D01) ∇ commutes with addition on tensor fields.

(D02) ∇ satisfies the Leibniz rule with respect to tensor multiplication.

(D03) ∇ commutes with index substitution.

[2]In GR, the metric uniquely picks out a derivative operator and so it need not be explicitly specified. Since this is not the case in classical spacetimes, one needs to specify the derivative operator explicitly.

(D04) ∇ commutes with contraction.

(D05) For all smooth scalar fields α and all smooth vector fields, ξ^n:

$$\xi^n \nabla_n \alpha = \xi(a).$$

Instead of requiring that ∇ commute in its action on scalar fields ($D06$), we set this to be the torsion tensor.

Definition 2. Let ∇ be a covariant derivative operator on the manifold M. Then, there exists a smooth tensor field $T^a{}_{bc}$, the torsion tensor, which is defined by:

$$2\nabla_{[a}\nabla_{b]}\alpha = \nabla_a \nabla_b \alpha - \nabla_b \nabla_a \alpha = T^c{}_{ab} \nabla_c \alpha$$

for all smooth scalar fields α.

Note, from the definition above, that $T^a{}_{bc}$ is anti-symmetric in its lowered indices, b and c:

$$T^a{}_{bc} = -T^a{}_{cb}$$

As in the torsion-free case, the action of any two (possibly-torsional) derivative operators, ∇ and $\tilde{\nabla}$, can be related by a smooth tensor field $C^a{}_{bc}$, with the property that for any smooth vector field ξ^a, $\nabla_a \xi^b = \tilde{\nabla}_a \xi^b - C^b{}_{an} \xi^n$ (and likewise for other tensor fields). In this case we write $\nabla = (\tilde{\nabla}, C^a{}_{bc})$ Unlike in the torsion-free case, however, this field $C^a{}_{bc}$ need not be symmetric in its lower indices. Instead we have

$$2C^a{}_{[bc]} = T^a{}_{bc} - \tilde{T}^a{}_{bc}.$$

1.2.2 Time and space

We now consider how to represent time and space in a classical spacetime theory with torsion. As in standard Newtonian gravitation, we will assume that spacetime has a temporal metric t_a and spatial metric h^{ab}, both of which will be compatible with the possibly-torsional derivative operator ∇. In standard models of Newtonian gravitation, without torsion, it follows from the compatibility of the temporal metric with the (torsion-free) derivative operator that t_a is closed, i.e., $d_a t_b = 0$, where d is the exterior derivative. This implies that t_a is locally exact, i.e., $t_a = \nabla_a t$ for some smooth time function, t. Physically, the availability of a time function means that we can have a well-defined notion of the temporal distance between points.

Indeed, if M is simply connected, then we will have a global time function, $t : M \to \mathbb{R}$. This means our spacetime consists of global simultaneity slices stacked through time and any two global time functions will differ only in their assignment of the zero-point for the time scale. Compatibility with a torsional derivative operator no longer implies that t_a is closed in general. However, we will assume t_a is closed and we will only consider derivative operators compatible with closed temporal metrics in what follows.

The curvature of a spacetime is formalized by the Riemann curvature tensor. Intuitively, the Riemann tensor measures the degree to which a vector fails to return to its original value when parallel transported around a closed loop. More formally, it measures the degree to which the second covariant derivatives fail to commute. For a torsion-free spacetime, it is defined as

$$R^a{}_{bcd}\xi^b = -2\tilde{\nabla}_{[c}\tilde{\nabla}_{d]}\xi^a.$$

In spaces with torsion, we adjust the definition of the Riemann tensor to include the contribution from the torsion tensor. This yields

$$R^a{}_{bcd}\xi^b = -2\nabla_{[c}\nabla_{d]}\xi^a + T^n{}_{cd}\nabla_n\xi^a,$$

(see Appendix §A.1 for a derivation). There is a valuable formula relating the curvatures of two derivative operators with torsion. If $\nabla = (\tilde{\nabla}, C^a{}_{bc})$, then

$$R^a{}_{bcd} = \tilde{R}^a{}_{bcd} + 2\tilde{\nabla}_{[c}C^a{}_{d]b} + 2C^p{}_{[c|b|}C^a{}_{d]p} - \tilde{T}^m{}_{cd}C^a{}_{mb}$$

(see Appendix §A.3 for a derivation). Note only the torsion of $\tilde{\nabla}$ appears in this equation.

Let us compare the spatiotemporal geometry of NG and NCT. NG posits that space and time are both flat (i.e., "spacetime is flat"), implying that the Riemann tensor, $R^a{}_{bcd}$, vanishes entirely. NCT, by contrast, only requires spatial flatness (i.e., "space is flat"). We formalize this condition as $R^{abcd} = \mathbf{0}$, where indices are raised using h^{ab}, and interpret it as saying that the parallel transport of spacelike vectors in spacelike directions is, at least locally, path independent.[3]

To develop a theory most like Teleparallel Gravity in the classical context, we will require the curvature of our spacetime to vanish, $R^a{}_{bcd}$.[4] This is because TPG is set on a flat spacetime background, and we are seeking a classical theory analogous to it.

We have not, thus far, placed any constraints on the torsion tensor. Recall that in NCT, the spatial curvature vanishes. Analogously, we propose that the spatial torsion of our spacetime vanish (i.e., $T^{abc} = 0$). The vanishing of the spatial torsion will yield a theory like NCT but with torsion, not curvature, encoding gravitational influence.

[3]Often, a stronger condition is adopted in NCT, that $R^{ab}{}_{cd} = \mathbf{0}$. $R^{ab}{}_{cd} = \mathbf{0}$ is equivalent to $R^{abcd} = 0$ if and only if $R^{abcd} = \mathbf{0}$ *and* there exists a local, unit timelike vector field ξ^a that is rigid and twist-free (Malament 2012, Proposition 4.3.1). Note that $R^{ab}{}_{cd} = \mathbf{0}$ implies that $R^{abcd} = \mathbf{0}$ as we can simply raise the indices: $\mathbf{0} = R^{ab}{}_{pq}h^{pc}h^{qd} = R^{abcd}$).

[4]Allowing both curvature and torsion would allow one to consider the classical analog of a Poincaré Gauge Theory, a worthwhile project in its own right but beyond our scope.

1.2.3 Sources and forces

Finally, let us consider how we expect sources to exert (torsional) force in our theory. It will be instructive to consider the treatment of sources and forces in the non-torsional, classical context first. In Newtonian Gravity, bodies are subject to gravitational forces and force is mediated by a gravitational potential (ϕ). The four-velocity, ξ^a, of a particle satisfies

$$-\nabla_a \phi = \xi^n \nabla_n \xi_a, \tag{1.1}$$

where ϕ is a smooth, scalar field and ∇ denotes the flat, torsion-free derivative operator of standard NG. The right-hand side of the equation describes the acceleration that the test point particle undergoes in the presence of the gravitational potential, ϕ. The gravitational potential further satisfies Poisson's equation, relating it to the distribution of matter

$$\nabla_a \nabla^a \phi = 4\pi\rho, \tag{1.2}$$

where ρ is the Newtonian mass density function.

In Newton-Cartan theory, like in GR, the curvature of spacetime means that inertial motion is governed by the geodesic principle: in the absence of external (non-gravitational) forces, bodies move along the geodesics of (curved) spacetime. The equation of motion is given as

$$\xi^n \tilde{\nabla}_n \xi^a = \mathbf{0}, \tag{1.3}$$

where $\tilde{\nabla}$ is the curved derivative operator of NCT.

To account for spatiotemporal curvature, NCT adopts a geometrized form of Poisson's equation, relating the distribution of matter to the curvature of spacetime,

$$R_{ab} = 4\pi\rho t_a t_b. \tag{1.4}$$

As it turns out, models of NG and NCT are systematically related. The Trautman geometrization lemma and degeometrization theorem describe these relations. Let us consider the recovery of models of NG from NCT. This is the direction in which force terms arise and so it will be instructive in formulating torsional forces. To build up to the degeometrization theorem, we will first consider the derivative operators of each theory. One can show that in the non-torsional context, one has the following result.

> **Proposition 1.** (Malament 2012, Proposition 4.1.3) Let $(M, t_a, h^{ab}, \tilde{\nabla})$ be a classical spacetime. Let $\nabla = (\tilde{\nabla}, C^a{}_{bc})$ be a second derivative operator on M. Then, ∇ is compatible with t_a and h^{ab} if and only if $C^a{}_{bc}$ is of the form:
>
> $$C^a{}_{bc} = 2h^{an} t_{(b}\kappa_{c)n}$$
>
> where κ_{ab} is a smooth anti-symmetric field on M and the parentheses denote symmetrization.

If we permit derivative operators with torsion, a broader class of derivative operators are compatible with the classical metrics. We now have the following generalization of the preceding proposition (see Appendix B for a proof).

Proposition 2. Let $(M, t_a, h^{ab}, \tilde{\nabla})$ be a classical spacetime with (possibly) non-vanishing torsion. Let $\nabla = (\tilde{\nabla}, C^a{}_{bc})$ be another derivative operator on M also with (possibly) non-vanishing torsion (i.e., $2C^a{}_{[bc]} = T^a{}_{bc} - \tilde{T}^a{}_{bc}$). Then ∇ is compatible with t_a and h^{ab} if and only if $C^a{}_{bc}$ is of the form:

$$C^a{}_{bc} = 2h^{ar}\kappa_{[r|b|c]}.$$

If, in addition, we require $\tilde{T}^{abc} = T^{abc} = \mathbf{0}$, then

$$C^a{}_{bc} = 2h^{ar}(x_{rb}t_c + y_{rc}t_b).$$

where x_{ab} is an arbitrary smooth tensor field and y_{rc} is any antisymmetric field.

As we can see, in the presence of torsion, there is considerable freedom to define metric-compatible derivative operators. Below, we will limit attention to flat, metric-compatible derivative operators whose torsion has the form $T^a{}_{bc} = 2F^a{}_{[b}t_{c]}$ where $F^a{}_b$ is a smooth rank (1,1) tensor field, spacelike in the a index. This is tantamount to stipulating that y_{ab} in the Prop. 2 vanishes. This restriction clearly satisfies the above-outlined general form and ensures that the spatial torsion vanishes. Furthermore, as will be seen in the below theorem, we can recover the standard, torsion-free connecting field assumed for the degeometrization theorem as a special case of the above.

In describing the difference between the derivative operators, the connecting field is closely related to the force field that arises in the degeometrization of a model of Newton-Cartan theory. In the torsion-free context, one typically assumes a connecting field of the form

$C^a{}_{bc} = t_b t_c \tilde{\nabla}^a \phi$.[5] The force term is then just the contracted connecting field: $C^a{}_{rn}\xi^r\xi^n = \tilde{\nabla}^a\phi$.

To adapt this to the torsional context, we want to consider the connecting field relating a non-torsional, flat derivative operator to a torsional one. We will capture the impact of the torsion on the trajectories of test bodies using the above-mentioned tensor field, $F^a{}_b$. In other words, we want $F^a{}_b$ to play the role of a torsional force term. Given a timelike geodesic of $\tilde{\nabla}$ with unit tangent field ξ^a, the force equation we expect to be satisfied is

$$\xi^n\nabla_n\xi^a = \xi^n\tilde{\nabla}_n\xi^a - C^a{}_{rn}\xi^r\xi^n = -F^a{}_n t_c \xi^n \xi^c = -F^a{}_n \xi^n, \tag{1.5}$$

where ∇ is the derivative operator of our torsional spacetime.

Finally, we want to relate the torsional force term to gravitational sources. In other words, we want to formulate a field equation that is the torsional analog to Poisson's Equation. It will turn out to be

$$\delta_a{}^n \nabla_{[n} F^a{}_{b]} = 2\pi\rho t_b. \tag{1.6}$$

Again, Poisson's equation will be recovered as a special case of Eq. (1.6), but Eq. (1.6) more generally establishes a relation between the first-derivative of the force term and the mass distribution along the temporal direction.

[5] A connecting field of this form satisfies the more general constraint for the connecting field between the derivative operators of any two classical spacetime models if we take κ_{cn} from above to be $t_{[c}\tilde{\nabla}_{n]}\phi$. That some ϕ exists with the necessary properties to make this derivative operator flat depends on several background assumptions that we suppress for reasons of space.

1.3 Degeometrization with Torsion

We now state and prove a theorem analogous to the Trautman degeometrization theorem. This result establishes that for every model of Newton-Cartan theory, there is a corresponding model of the classical analog to teleparallel gravity described above.

Theorem 1. Let $(M, t_a, h^{ab}, \tilde{\nabla})$ be a classical spacetime (without torsion) satisfying:

$$\tilde{R}_{ab} = 4\pi\rho t_b t_c \tag{1.7}$$

$$\tilde{R}^{ab}{}_{cd} = \mathbf{0} \tag{1.8}$$

for some smooth scalar field ρ. Then given any point p in M, there is an open set O containing p and a pair $(\nabla, F^a{}_b)$ on O, where ∇ is a derivative operator and $F^a{}_b$ is a smooth rank (1,1) tensor field, which together satisfy the following conditions:

1. ∇ is compatible with t_a and h^{ab};
2. ∇ is flat;
3. ∇ has torsion $T^a{}_{bc} = 2F^a{}_{[b}t_{c]}$;
4. For all timelike curves with unit tangent field ξ^a, $\xi^n\tilde{\nabla}_n\xi^a = \mathbf{0}$ if and only if $\xi^n\nabla_n\xi^a = -F^a{}_n\xi^n$; and
5. $(\nabla_a, F^a{}_b)$ together satisfy the field equations $\delta^n{}_a\nabla_{[n}F^a{}_{b]} = 2\pi\rho t_b$.

The pair $(\nabla, F^a{}_b)$ is not unique. Moreover, there exist pairs $(\nabla, F^a{}_b)$, satisfying the conditions above, for which the torsion is non-vanishing.

Proof. Existence follows from the Trautman degeometrization theorem (Malament 2012, Proposition 4.2.5). Fix any classical spacetime $(M, t_a, h^{ab}, \tilde{\nabla})$ satsifying $\tilde{R}^{ab}{}_{cd} = \mathbf{0}$ and $\tilde{R}_{ab} = 4\pi\rho t_a t_b$ for some smooth scalar field ρ. Choose a point p and a rigid and twist-free field η^a defined on some neighborhood of p, and let $\varphi^a = \eta^n \tilde{\nabla}_n \eta^a$ be the acceleration field associated with η^a. Then the pair $(\nabla, F^a{}_b)$, where $F^a{}_b = \varphi^a t_b$ and $\nabla = (\tilde{\nabla}, F^a{}_b t_c)$, satisfies conditions 1-5, with torsion $T^a{}_{bc} = 2\varphi^a t_{[b} t_{c]} = \mathbf{0}$, by arguments given in Malament's proof. Indeed, in this case, the field equation $\delta^n{}_a \nabla_{[n} F^a{}_{b]} = 2\pi t_b$ reduces to

$$2\pi\rho t_b = \frac{1}{2}\delta^n{}_a \left(\nabla_n \varphi^a t_b - \nabla_b \varphi^a t_n\right) = \frac{1}{2} t_b \nabla_a \varphi^a$$

and the resulting structure is a model of ordinary Newtonian gravitation with gravitational field φ^a. (If one assumed further that $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$, one could conclude that $\varphi^a = \nabla^a \varphi$ for some smooth scalar field φ, possibly on a subneighborhood of O.)

Non-uniqueness also follows from the Trautman degeometrization theorem. We wish to show, however, that there exist pairs $(\nabla, F^a{}_b)$ satisfying conditions 1-5 with non-vanishing torsion. We do so by direction construction. Let $\nabla = (\tilde{\nabla}, \varphi^a t_b t_c)$ be the flat derivative operator (without torsion) considered above. Choose any spacelike vector x^a at p, and extend it to a neighborhood of p by parallel transport via ∇. Finally, let ψ be any smooth scalar field defined near p whose gradient is non-vanishing, spacelike, and normal to x^a. Now define $\hat{F}^a{}_b = x^a \nabla_b \psi$ and $\check{F}^a{}_b = \varphi^a t_b + \hat{F}^a{}_b$. Then the pair $(\check{\nabla}, \check{F}^a{}_b)$, where $\check{\nabla} = (\nabla, \hat{F}^a{}_b t_c)$, satisfies conditions 1-5 with torsion $T^a{}_{bc} = 2x^a t_{[c} \nabla_{b]} \psi \neq 0$.

To see that 1 is satisfied, observe that $\check{\nabla}_a t_b = t_n \check{F}^n{}_a t_b = 0$; and $\check{\nabla}_a h^{bc} = F^b{}_a t_n h^{nc} + F^c{}_a t_n h^{bn}$. For 2, note that since ∇ is flat and torsion-free, and x^a is constant with respect to ∇, we have

$$\check{R}^a{}_{bcd} = 2x^a t_b \nabla_{[c} \nabla_{d]} \psi + 2x^p t_p x^a t_b \nabla_{[c} \psi \nabla_{d]} \psi = \mathbf{0}$$

where the first term vanishes because ∇ is torsion-free and the second because x^a is spacelike. 3 follows from the definition of $\check{\nabla}$ and the fact that ∇ is torsion-free. 4 follows because for all unit timelike vector fields ξ^a,

$$\xi^n \tilde{\nabla}_n \xi^a = \mathbf{0} \iff \quad \xi^n \nabla_n \xi^a = -F^a{}_n \xi^n \iff \quad \xi^n \check{\nabla}_n \xi^a = -\varphi^a - \hat{F}^a{}_n \xi^n = -\check{F}^a{}_n \xi^n.$$

Finally, 5 is satisfied because

$$\begin{aligned}
\delta^n{}_a \check{\nabla}_{[n} F^a{}_{b]} &= \delta^n{}_a \check{\nabla}_{[n} (\varphi^a t_{b]} + x^a \nabla_{b]} \psi) \\
&= 2\pi \rho t_b + \delta^n{}_a \check{\nabla}_{[n} x^a \check{\nabla}_{b]} \psi) \\
&= 2\pi \rho t_b + x^n \check{\nabla}_{[n} \check{\nabla}_{b]} \psi \\
&= 2\pi \rho t_b + \frac{1}{2} x^n T^a{}_{nb} \check{\nabla}_a \psi \\
&= 2\pi \rho t_b + x^n x^a t_{[b} \nabla_{n]} \psi \check{\nabla}_a \psi \\
&= 2\pi \rho t_b
\end{aligned}$$

where in the first equality we use the facts that $0 = \nabla_a x^b = \check{\nabla}_a x^b - \hat{F}^b{}_a t_b x^b = \check{\nabla}_a x^b$ and that ∇ and $\check{\nabla}$ agree on scalar fields (because all derivative operators do); while in the final equality we use the fact that $\check{\nabla}_a \psi$ is normal to x^a. $\square$

1.4 Discussion

The general proof strategy is to leverage the original Trautman degeometrization theorem results. We show that NG can be recovered as a special (torsion-free) case of the theorem presented above. By broadening the class of allowed derivative operators, the non-uniqueness results establish the possibility of a classical spacetime with non-vanishing torsion.

There are some important differences between our result and the Trautman theorem. We do not give necessary and sufficient conditions to construct new pairs $(\nabla, F^a{}_b)$ from old ones satisfying 1-5. This is because, unlike the situation with vanishing torsion, the derivative operators associated with models of the torsional theory (for some model of NCT) do not appear to form an affine space. Nonetheless, we are able to establish the non-uniqueness of torsional models, and we give a general strategy for constructing alternative models with torsion associated with a given model of NCT. It would be interesting to provide a complete description of this space. To construct a new model, we can add the field $\hat{F}^a{}_b$ to a flat (degeometrized) derivative operator where $\hat{F}^a{}_b = x^a \nabla_b \psi$. Here, x^a is a spacelike vector at p; ∇ is a flat, torsion-free derivative operator corresponding to a model of NG; and ψ is a smooth scalar field near p whose gradient is spacelike and normal to x^a. Thus, we can derive new torsional solutions by adding these spacelike fields to existing solutions of NG.

We also do not require our model of Newton-Cartan theory to satisfy $R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b$, as the Trautman theorem does This is because the role of that condition is to ensure that a certain field φ^a is closed, and therefore locally exact. We do not invoke that field in the result, and so we drop the condition. In that sense, we generalize the Trautman theorem. Finally, we note that more general versions of the theory (and theorem) discussed here are almost certainly possible. For instance, one might consider force fields for which y_{ab} from Prop. 2 is non-vanishing, among other variations (Malament 2012, c.f. §4.5).

Chapter 2

The Classical Limit of Teleparallel Gravity

2.1 Introduction and Motivation

In Chapter 1, we developed a classical theory of gravity with torsion. The goal now is to understand the classical limit of Teleparallel Gravity. As discussed, Teleparallel Gravity is a relativistic theory of gravity with torsion.[1] Considering the classical theory of gravity with torsion developed in the previous chapter and TPG together raises a natural question: what

[1]The first formulation of teleparallel gravity is often attributed to Einstein. In a paper published in June 1928 ("Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus" or "Riemannian Geometry with Maintaining the Notion of Distant Parallelism," often called "Fernparallelismus" for short since "Fernparallelismus" translates as either "distant parallelism" or "absolute parallelism."), he begins developing a gravitational theory with torsion using the tetrad approach. The motivation for the project was to unify gravity and electromagnetism, the idea being that the (six) extra degrees of freedom afforded by torsion could be used to represent the electromagnetic field. Just one week later, he publishes "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität" ("New Possibility for a Unified Field Theory of Gravitation and Electricity") which presents the field equations of the new theory. After corresponding with Weitzenböck and Cartan, Einstein abandoned the project, finding himself unable to attribute physical meaning to the structures posited by the theory (see, especially, his 1932 letter to Cartan, reprinted in Debever 2015, 209-10). The theory of absolute parallelism remained abandoned until it was taken back up nearly 30 years later by Møller in 1961, and, completely independently, by Hayashi and Nakano in 1967. After some further work in this area, Hayashi and Shirafuji bring together these distinct projects in their paper "New general relativity" published in 1979. The contemporary formulation TPG began to emerge in

is the relation between these theories? In particular, is there a way to recover the latter from the former in the non-relativistic limit?

Examining this relation will be instructive for a few reasons. First, such projects of intertheoretic reduction have, historically, been instructive and explanatory. Understanding, for instance, the non-relativistic limit of General Relativity helped us gain insight into the similarities in structure between Newtonian Gravity and GR. Relatedly, such projects can help us appreciate why the former theory was (empirically) successful. One way of doing this is by showing under what circumstances—under what limiting conditions or in what domain of application—one can recover the empirically successful but false theory from the current one (see Fletcher (2019) for similar points and Weatherall (2011) for a project along these lines).

In this context, understanding the limiting relation will be especially instructive as the two theories are standardly written in different formalisms. TPG is formulated using the tetrad formalism while the torsional classical theory was formulated using a differential geometry approach. Thus, the conceptual similarities/differences between these theories are not immediately obvious. Studying their relation formally will help clarify their features.

Finally, such a project will help us situate the theories, not only in relation to one another but also in relation to other nearby gravitational theories. We know some aspects of the conceptual landscape. We know, for instance, how to construct models of TPG from models of GR. We also know that NCT is the classical limit of GR. We have shown how to construct a classical torsional theory from NCT. By studying the classical limit of TPG, we will address whether the theories form a commuting diagram. Specifically, we will address whether, if one starts with GR, taking the classical limit and then allowing torsion commutes with first allowing torsion and then taking the classical limit.

the 1990s with work by de Andrade, Pereira, Obukhov, and Aldrovandi (see especially (de Andrade and Pereira 1997)).

To address these issues, I discuss the limiting behavior we are interested in by sketching the limiting proof for GR. I then turn to the torsional context, presenting some preliminaries about the tetrad formulation of TPG. I conclude with a proof of the limiting proposition for the torsional context.

2.2 What does it mean to take the classical limit?

To relate a relativistic theory to a classical one, we must first formulate a limiting procedure. Given that one of the differences between the relativistic and classical contexts is their respective lightcone structure, it seems natural to develop a limiting procedure featuring this lightcone structure. In particular, relativistic theories place a limit on the speed of light, represented by the edge of the light cones. Classical theories place no such limit. Thus, it seems natural to develop a limiting procedure that allows the speed of light to become unbounded. We can visualize this as "opening up" the lightcones of a relativistic theory and thus allowing the speed of light to become unbounded.

Allowing the lightcones to "open up" is the method adopted in Malament's proof showing the relation between GR and NCT (1986b). There, he writes:

> ...the work under discussion provides the means with which to make clear geometric sense of the standard claim that Newtonian gravitational theory is the "classical limit" of general relativity. One considers an appropriate one-parameter family of relativistic models $(M, g_{ab}(\lambda), T_{ab}(\lambda))$ satisfying Einstein's equation, defined for $\lambda > 0$, and the proves that in the limit as $\lambda \to 0$ a classical model $(M, t_a, h^{ab}, \nabla_a, \rho)$ satisfying (the recast version of) Poisson's equation is defined.[2]

[2]The recast version of Poisson's equation referred to here is geometrized Poisson's equation (Eq. 1.4) presented above.

> Intuitively, as $\lambda \to 0$, the null cones of the $g_{ab}(\lambda)$ "flatten" until they become degenerate. (1986b, 182)

Some authors critique the procedure of "opening up the lightcones" on the grounds that it is difficult to interpret physically (see (Fletcher 2019, §1) for discussion of various positions in this debate). They question what it means to allow a constant of nature to vary and to study the relation of counterfactual models. Especially if the limit is intended to do explanatory work (i.e., explain how the classical theory was empirically successful), they question this geometric approach.

Aside from the geometric approach, there are two other limiting procedures that have been applied to this context: post-Newtonian ($1/c$ or $1/c^2$) expansions and dimensional reduction. I will consider these limits, and the results obtained with them, in more detail in Chapter 3 but briefly outline them now. For the case of $1/c$ or $1/c^2$ expansions, the physical interpretation is thought to be clearer and it does not require thinking of counterfactuals. One typically expands the metric in powers of $1/c$ or $1/c^2$ and rewrites the field equations with these expansions. The result corresponds to the slow-velocity, weak-field limit. In the case of dimensional reduction, one can formulate a $(D + 1)$-dimensional gravitational wave solution of GR and show that it reduces to a D-dimensional solution of NCT. This is the methodology adopted by Read and Teh (2018).

Overall, each approach seems to offer a different perspective on the relations amongst theories. The post-Newtonian expansion of a relativistic theory is the most common method encountered in the physics literature and does seem to provide the most explanatory upshot regarding the empirical success of the non-relativistic theory. Nonetheless, I would argue that the geometric approach taken here seems most suitable to consider the conceptual relations amongst *theories*. Instead of considering the weak-field, slow-velocity expansion of a particular model of the relativistic theory, we consider how a collection of models of a theory, parameterized by the speed of light, relate to one another and recover a model of the non-relativistic theory in the limit. In so doing, we are considering the relations amongst the structures of each theory and how those structures relate. Though, as its critics claim, it may be less physically explanatory than the post-Newtonian expansion regarding the empirical successes of the theory we reduce to, it still offers a valuable perspective about the structures of that theory. For instance, as Malament (1986b) argues, we learn why Newtonian Gravity was successful in positing that space is flat as we consider the classical limit of the theory. Finally, dimensional reduction has been very useful for studying the symmetries of various models of the theories it is applied to. However, it is unclear what the physical interpretation ought to be and what the corresponding explanatory upshot could be. The motivation for considering a gravitational wave solution and dimensional reduction in general seems to be lacking. Put differently, whatever the difficulties associated with explanations on the geometric approach taken here, dimensional reduction seems to confront many more.

2.2.1 The classical limit of GR

Let us begin by considering what the limiting process looks like with GR (for the complete proofs, see Malament 1986b; the propositions are reproduced in §3.1). Here, we proceed in two steps. The first step involves specifying the process of opening up the light cones and showing that, in the limit, the metric and derivative operator of GR converge to those of NCT (i.e., the GR metric converges to the temporal and spatial metrics of NCT and the derivative operator convergences to the classical derivative operator with vanishing spatial curvature). More specifically, we allow $g_{ab}(\lambda)$ to be a one-parameter family of non-degenerate Lorentz metrics where λ ranges over some interval $(0, k)$. Given that a classical spacetime has a temporal and spatial metric, we will require the metric of GR to reduce to these in the limit. Thus, we place two conditions on the limit:

1. $g_{ab}(\lambda) \to t_a t_b$ as $\lambda \to 0$ for some closed field t_a;

2. $\lambda g^{ab}(\lambda) \to -h^{ab}$ as $\lambda \to 0$ for some field h^{ab} of signature (0,1,1,1).

where λ is $\frac{1}{c^2}$. To understand the scaling in the second condition, consider that as the light cones open up, spacelike vectors will begin to diverge. Thus, we must "rescale" the spatial metric with $\frac{1}{c^2}$ to ensure convergence.

To show that the derivative operator converges to one of NCT, we use the connecting fields. Consider the derivative operators in the limit, $\overset{\lambda}{\nabla}_a$. We want to show that these derivative operators converge to ∇_a, the spatially flat derivative operator of NCT. To do so, we will consider an arbitrary, intermediary derivative operator, $\tilde{\nabla}_a$, that relates any two derivative operators "along the way," i.e., as the light cones are opening up. Specifically, we take $\overset{\lambda}{\nabla}_a = (\tilde{\nabla}_a, \overset{\lambda}{C}{}^a{}_{bc})$ and $\nabla_a = (\tilde{\nabla}_a, C^a{}_{bc})$. It suffices to show that $\overset{\lambda}{C}{}^a{}_{bc} \to C^a{}_{bc}$ in order to demonstrate the convergence of the derivative operators. Finally, we demonstrate that the recovered spacetime model is a model of NCT by showing it satisfies the curvature conditions required (i.e., conditions placed on the Riemann tensor).

The second step in the process involves considering how the matter content behaves in the limit. Here, we consider the limit of Einstein's equation and want to show that Einstein's equation will reduce to the geometrized Poisson's equation in the limit. Placing conditions on the behavior of Einstein's equation for each step of the limit will ensure this:

3. $\overset{\lambda}{R}_{ab} = 8\pi(T_{ab}(\lambda) - \frac{1}{2}g_{ab}(\lambda)\overset{\lambda}{T})$ holds for all λ;

4. $T^{ab}(\lambda) \to T^{ab}$ as $\lambda \to 0$ for some field T^{ab}.

Here, the second condition arises from the requirement that the limiting process assigns limiting values to various components of the energy-momentum tensor (i.e., the mass-energy density, three-momentum density, and three-dimensional stress tensor).

2.2.2 Teleparallel Gravity

Before considering the classical limit of Teleparallel Gravity, let me first sketch the formalism. The tetrad approach used for TPG generalizes the coordinate basis approach: instead of requiring (holonomic) coordinate bases as the bases for the tangent bundle, one only requires a locally defined set of linearly independent vector fields as the basis for the tangent bundle. When the vector bundle has four-dimensional fibers, it is referred to as a "tetrad" or a "vierbein" (where "veir" means "four"). Though in general, there may not be such bases (also called "frames" or "frame fields") across all the manifold, they do always exist locally.

Formally, let us fix an arbitrary manifold, M, and let $E \xrightarrow{\pi} M$ be a vector bundle over the manifold with $n-$dimensional fibers. We denote elements of the bundle with capital Latin indices. We can define a frame field (and coframe field) as follows:

> **Definition 3.** A frame field for E on a neighborhood of the manifold ($O \subseteq M$) is a collection of n vector fields $\{(e_i)^A\}$, $(i = 1, \ldots, n)$, that form a basis for the fiber at E at each point $p \in O$. A *coframe field* on O is a collection of n covector fields $\{(e^i)_A\}$ forming a basis for the dual bundle at each point $p \in O$.

Note that the lowercase indices are counting, not abstract, indices. Taking i to range from 1 to $n = 4$ yields the tetrads.

From the above definition, we have the proposition:

> **Proposition 3.** Given any frame field $\{(e_i)^A\}$ for E on $O \subseteq M$, there exists a unique coframe field $\{(e^i)_A\}$ such that $(e^i)_A(e_j)^A$ equals 1 if $i = j$ and $\mathbf{0}$ otherwise.

In TPG, the metric is standardly expressed in terms of the tetrads as

$$g_{ab} = \sum_{i=1}^{4}\sum_{j=1}^{4} \eta_{ij}(e^i)_a(e^j)_b = \sum_{i=1}^{4} \eta_{ii}(e^i)_a(e^i)_b, \tag{2.1}$$

where η_{ij} are the Minkowski metric components and the second equality follows because $\eta_{ij} = 0$ when $i \neq j$ (c.f., Aldrovandi and Pereira 2013, Eq. 1.27). For the metric with raised indices, we have

$$g^{ab} = \sum_{i=1}^{4} \sum_{j=1}^{4} \eta^{ij} (e_i)^a (e_j)^b = \sum_{i=1}^{4} \eta^{ii} (e_i)^a (e_i)^b. \tag{2.2}$$

Finally, let us also consider the torsional derivative operator. Assume that ∂ is a coordinate derivative operator relative to some local coordinate system on M; we know ∂ is both flat and torsion-free (symmetric). Next, consider any frame field, $\{(e_i)^a\}$, as defined above. It turns out that there always exists a *unique* derivative operator D relative to which all n frame elements are constant: $D_a(e^i)^b = \mathbf{0}$. This derivative operator can be defined relative to ∂ by taking $D = (\partial, C^a{}_{bc})$, where for each $i = 1, \ldots, n$:

$$C^a{}_{bc}(e_i)^c = \partial_b(e_i)^a,$$

or, equivalently,

$$C^a{}_{bc} = \sum_{i=1}^{n} (e^i)_c \partial_b (e_i)^a.$$

The torsion of this derivative operator is given in terms of the tetrads as

$$T^a{}_{bc} = \sum_{i=1}^{n} (e^i)_{[c} \partial_{b]} (e_i)^a. \tag{2.3}$$

2.2.3 Tetrads in the Limit

Let us now consider the tetrad expressions of the metrics of a classical spacetime. The goal is to mimic the decomposition of the standard metric into the temporal and spatial metrics of a classical spacetime with tetrads. On the tetrad formalism, we have, at each point, a collection of four orthonormal vector fields. Suppose that we fix the first "leg" of the cotetrad to correspond to the temporal metric of a classical spacetime. Then, we require it to be closed. The remaining cotetrad elements will vanish for a classical spacetime while the other tetrad elements (besides the first) will compose the spatial metric. Let us call such a tetrad a "classical tetrad."

Definition 4. Call a tetrad $\{(f_i)^a\}$ a "classical tetrad" if and only if an element of its cotetrad is closed.[3] If this is the case, then classical metrics t_a and h^{ab} can be defined such that the cotetrad element that is closed corresponds to t_a and the following conditions are satisfied:

1. $(f_1)^a t_a = 1$,
2. $(f_i)^a t_a = 0$ for $i = 2, 3, 4$,
3. $(f^1)_a (f^1)_b h^{ab} = \mathbf{0}$, and
4. $(f^i)_a (f^j)_b h^{ab} = 1$ for $i = j = 2, 3, 4$ and $\mathbf{0}$ otherwise.

With this understanding of a classical tetrad in hand, we now we turn to the desired limiting behavior. We want to consider the limit of a family of tetrads, $\{(e_i)^a(\lambda)\}$, on a manifold, M, as the speed of light, c, is allowed to become unbounded (i.e., $c \to \infty$). Allow λ to range over some interval $0 < \lambda < k$. To recover a "classical tetrad," we will consider the convergence of the tetrad components. We need the first "leg" of the cotetrad to yield the temporal metric in the limit, i.e. $(e^1)_a \to t_a$. We will need to appropriately rescale the tetrad components yielding the spatial metric to ensure that they converge in the limit. We rescale these components with $\frac{1}{c} = \sqrt{\lambda}$ so the desired limiting behavior is $\sqrt{\lambda}(e_i)^a \to (f_i)^a$ for $i = 2, 3, 4$. We formalize these considerations with the following conditions.

[3]From the fact that a cotetrad element is closed, we know that it is locally exact. So taking the first cotetrad element to be $\nabla_{[a}(f^1)_{b]} = 0$ implies $(f^1)_b = \nabla_b t$ for some smooth function t. Contracting this with the spatial metric would yield $\mathbf{0}$ since $(\nabla_a \nabla_b t) h^{ab} = \mathbf{0}$.

C1 $\sum_{i=1}^{4}(\overset{\lambda}{e}{}^{i})_a \to \sum_{i=1}^{4}(f^i)_a = (f^1)_a = t_a$ as $\lambda \to 0$ for some closed field t_a.

C2 $\sqrt{\lambda}\sum_{i=1}^{4}(\overset{\lambda}{e}_i)^a \to \sum_{i=2}^{4}(f_i)^a$ as $\lambda \to 0$.

For C1, the equality follows because only the first "leg" of the cotetrad (i.e., $i = 1$) is non-vanishing since $\{(f^i)_a\}$ is a classical tetrad. For C2, we will take $\sum_{i=1}^{4}\eta^{ii}(f_i)^a(f_i)^b = -h^{ab}$ for some field h^{ab} of signature $(0, 1, 1, 1)$. This condition ensures that we can recover the spatial metric in the limit. These two conditions recover Malament's (1986b) limiting conditions as follows.

C1, Malament $g_{ab}(\lambda) = \sum_{i=1}^{4}\eta_{ii}(\overset{\lambda}{e}{}^{i})_a(\overset{\lambda}{e}{}^{i})_b \to (f^1)_a(f^1)_b = t_a t_b$ as $\lambda \to 0$.

C2, Malament $\lambda g^{ab}(\lambda) = \lambda\sum_{i=1}^{4}\eta^{ii}(\overset{\lambda}{e}_i)^a(\overset{\lambda}{e}_i)^b \to \sum_{i=2}^{4}\eta^{ii}(f_i)^a(f_i)^b = -h^{ab}$ as $\lambda \to 0$.

2.3 The Classical Limit of TPG

We are now in a position to prove the classical limit of TPG.

> **Theorem 2.** Suppose that $\{(e_i)^a(\lambda)\}$ is a one-parameter family of tetrads on a manifold, M. Suppose $\{(f_i)^a\}$ satisfies conditions C1 and C2 from above and that there is a derivative operator ∇_a on M satisfying $\overset{\lambda}{\nabla}_a \to \nabla_a$ as $\lambda \to 0$. Then
>
> 1. ∇_a is such that $(M, \{(f_i)^a\}, \nabla_a)$ is a classical spacetime model where $\{(f_i)^a\}$ is a classical tetrad and ∇_a is flat.
> 2. For any derivative operator, ∇_a, satisfying the above, the torsion vanishes.

Proof. Suppose, for the sake of contradiction, that the torsion does not vanish in the limit (i.e., $\overset{\lambda=0}{T}{}^a{}_{bc} \neq 0$). We know that $\{(\overset{\lambda}{e}{}^{i})_a\}) \to (f^1)_a$ smoothly. Like Malament, let us require that the limit be twice differentiable. This means that there must exist smooth fields m_a, $n_a(\lambda)$, and n_a satisfying

$$(\overset{\lambda}{e}{}^1)_a \to (f^1)_a - \lambda m_a + \lambda^2 n_a(\lambda), \tag{2.4}$$

$$n_a(\lambda) \to n_a \text{ as } \lambda \to 0.$$

Similarly, since $\sqrt{\lambda}\{(\overset{\lambda}{e}_i)^a\} \to \{(f_i)^a\}$ for $i = 2, 3, 4$, there must exist some smooth fields r^a, $s^a(\lambda)$, and s^a satisfying

$$\sum_{i=1}^{4} \sqrt{\lambda}(\overset{\lambda}{e}_i)^a \to \Big(\sum_{i=2}^{4}(f_i)^a\Big) - \lambda r^a + \lambda^2 s^a(\lambda), \tag{2.5}$$

$$s^a(\lambda) \to s^a \text{ as } \lambda \to 0.$$

We begin by fixing some flat, torsion-free ∂ on M. (The existence of such a ∂ is always guaranteed locally; for more, see the discussion above.) Now suppose

$$\overset{\lambda}{\nabla} = (\partial, C^a{}_{bc}(\lambda))$$

for all λ. The expression for the torsion in terms of the tetrads and this derivative operator is given as

$$\overset{\lambda}{T}{}^a{}_{bc} = \sum_{i=1}^{n} (\overset{\lambda}{e}{}^i)_{[c}\partial_{b]}(\overset{\lambda}{e}_i)^a$$

for all λ. Since we are providing a proof by contradiction, we assume that the torsion is non-vanishing for all λ.

Relative to ∂ and for each λ, the C-fields defining the tetrad derivative operators can be expressed as[4]

$$C^a{}_{bc}(\lambda) = \frac{1}{2} g^{ad}(\lambda) \Big[\partial_d g_{bc}(\lambda) - \partial_b g_{dc}(\lambda) - \partial_c g_{db}(\lambda) \\ - T_{dcb}(\lambda) + T_{bdc}(\lambda) + T_{cdb}(\lambda) \Big].$$

Rewriting the last three terms using $T_{abc}(\lambda) = g_{am}(\lambda) T^m{}_{bc}(\lambda)$ yields the expression

$$C^a{}_{bc}(\lambda) = \frac{1}{2} g^{ad}(\lambda) \Big[\partial_d g_{bc}(\lambda) - \partial_b g_{dc}(\lambda) - \partial_c g_{db}(\lambda) \\ - g_{dm}(\lambda) T^m{}_{cb}(\lambda) + g_{bm}(\lambda) T^m{}_{dc}(\lambda) + g_{cm}(\lambda) T^m{}_{db}(\lambda) \Big].$$

We now consider each (set of) terms and their tetrad formulations. Using C2, we rewrite $g^{ad}(\lambda)$ in terms of the tetrads as

$$g^{ad}(\lambda) = \sum_{i=1}^{4} n^{ii} (\overset{\lambda}{e}_i)^a (\overset{\lambda}{e}_i)^d.$$

Then, in the limit, we have

$$\begin{aligned} \sum_{i=1}^{4} n^{ii} (\overset{\lambda}{e}_i)^a (\overset{\lambda}{e}_i)^d \rightarrow & \frac{1}{\lambda} \Big[\Big(\sum_{i=2}^{4} (f_i)^a - \lambda r^a + \lambda^2 s^a(\lambda) \Big) \Big(\sum_{i=2}^{4} (f_i)^d - \lambda r^d + \lambda^2 s^d(\lambda) \Big) \Big] \\ &= \Big[\frac{1}{\lambda} \Big(\sum_{i=2}^{4} (f_i)^a (f_i)^d \Big) - \Big(\sum_{i=2}^{4} (f_i)^a r^d + \sum_{i=2}^{4} (f_i)^d r^a \Big) \\ &+ \lambda \Big(r^a r^d + \sum_{i=2}^{4} (f_i)^a s^d(\lambda) + \sum_{i=2}^{4} (f_i)^d s^a(\lambda) \Big) \\ &- \lambda^2 \Big(r^d s^a(\lambda) + r^a s^d(\lambda) \Big) + \lambda^3 \Big(s^a(\lambda) s^d(\lambda) \Big) \Big]. \end{aligned}$$

[4]C.f. Jensen 2005, Eq. 3.1.27.

Next, let us consider the first three terms inside the square brackets. In the limit, we have

$$\begin{aligned}\partial_d g_{bc}(\lambda) - &\partial_b g_{dc}(\lambda) - \partial_c g_{db}(\lambda) \\ &= \partial_d \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_b (\overset{\lambda}{e}{}^i)_c - \partial_b \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_c - \partial_c \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_b \\ &\to \partial_d \Big((f^1)_b - \lambda m_b + \lambda^2 n_b(\lambda) \Big) \Big((f^1)_c - \lambda m_c + \lambda^2 n_c(\lambda) \Big) \\ &- \partial_b \Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_c - \lambda m_c + \lambda^2 n_c(\lambda) \Big) \\ &- \partial_c \Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_b - \lambda m_b + \lambda^2 n_b(\lambda) \Big).\end{aligned}$$

We now consider the terms arising from the torsion. Let us just consider the first of these terms. The expansion of the first torsion term yields

$$g_{dm}(\lambda) T^m{}_{cb} = \sum_{i=1}^{4} \eta_{ii} (\overset{\lambda}{e}{}^i)_d (\overset{\lambda}{e}{}^i)_m \sum_{j=1}^{4} (\overset{\lambda}{e}{}^j)_{[b} \partial_{c]} (\overset{\lambda}{e}_j)^m.$$

In the limit, we would have

$$\begin{aligned}g_{dm}(\lambda) T^m{}_{cb} \to &\Big((f^1)_d - \lambda m_d + \lambda^2 n_d(\lambda) \Big) \Big((f^1)_m - \lambda m_m + \lambda^2 n_m(\lambda) \Big) \\ &\Big((f^1)_{[b} - \lambda m_{[b} + \lambda^2 n_{[b}(\lambda) \Big) \partial_{c]} \frac{1}{\sqrt{\lambda}} \Big(\sum_{i=2}^{4} (f_i)^m - \lambda r^m + \lambda^2 s^m(\lambda) \Big).\end{aligned}$$

2.3.1 Simplifications

With the expression for the connecting fields expanded, we now consider the behavior of all the relevant terms in the limit. Since $\lambda \to 0$ in the limit, any terms that are multiplied by $\sqrt{\lambda}$, λ, or higher orders of λ will vanish. Any terms divided by $\sqrt{\lambda}$, λ, or higher orders of λ will become unbounded. Let us work from left to right of our expression. The first term yielded an expression with a $\frac{1}{\lambda}$ term, a term with no λ dependence, and terms with λ and higher-orders of λ dependence. When these terms are multipled through and the limit is considered, the $\frac{1}{\lambda}$ term and the term with no λ dependence will remain. The remainder will vanish since the only term in the rest of the expression with an inverse λ dependence is just $\frac{1}{\sqrt{\lambda}}$ which will not be enough to keep the other terms from vanishing. Thus, all that we will need to consider from the first term in the limit is $\sum_{i=1}^{4}\left[\frac{1}{\lambda}\Big((f_i)^a(f_i)^d\Big)-\Big((f_i)^a r^d+(f_i)^d r^a\Big)\right]$.

Consider the limiting behavior of the next three terms. There, we have terms that have no λ dependence, some with λ dependence, and some with higher-orders of λ dependence. Only terms with no λ dependence or just λ dependence will remain (the latter because one of the terms from above has a λ^{-1} dependence). Removing the λ^2 terms yields

$$\begin{aligned}&\partial_d\Big((f^1)_b-\lambda m_b\Big)\Big((f^1)_c-\lambda m_c\Big)-\partial_b\Big((f^1)_d-\lambda m_d\Big)\Big((f^1)_c-\lambda m_c\Big)\\&-\partial_c\Big((f^1)_d-\lambda m_d\Big)\Big((f^1)_b-\lambda m_b\Big).\end{aligned}$$

We multiply through, again dropping the λ^2 terms

$$\begin{aligned}=&\partial_d(f^1)_b(f^1)_c-\partial_b(f^1)_d(f^1)_c-\partial_c(f^1)_d(f^1)_b\\&-2\lambda\partial_d(f^1)_{(b}m_{c)}+2\lambda\partial_b(f^1)_{(d}m_{c)}+2\lambda\partial_c(f^1)_{(d}m_{b)}.\end{aligned}$$

Consider just the first three terms in each expansion above (i.e., all the terms without λ in front). Let us use the Leibniz rule to expand these terms. Then, we can use the fact that, in the limit, the temporal metric will be closed (i.e., $\partial_{[a}(f^1)_{b]}=0$) to simplify this expression.

$$\begin{aligned}&\partial_d(f^1)_b(f^1)_c-\partial_b(f^1)_d(f^1)_c-\partial_c(f^1)_d(f^1)_b\\&\quad=(f^1)_b\partial_d(f^1)_c+(f^1)_c\partial_d(f^1)_b\\&\qquad-(f^1)_d\partial_b(f^1)_c-(f^1)_c\partial_b(f^1)_d-(f^1)_d\partial_c(f^1)_b-(f^1)_b\partial_c(f^1)_d\\&\quad=-2(f^1)_d\partial_{(b}(f^1)_{c)}=-2(f^1)_d\partial_b(f^1)_c.\end{aligned}$$

To move from line two to three, notice that the first and last terms cancel and the second and fourth terms cancel. This leaves only the third and fifth terms. Then, since $\partial_{[a}(f^1)_{b]} = 0$, we can drop the symmetrization parentheses. (One can see that this is the corresponding tetrad expression to Malament's $-2t_d\tilde{\nabla}_b t_c$).

The three terms we have been considering simplify in the limit to

$$-2(f^1)_d\partial_b(f^1)_c - 2\lambda\Big(\partial_d(f^1)_{(b}m_{c)} - \partial_b(f^1)_{(d}m_{c)} - \partial_c(f^1)_{(d}m_{b)}\Big).$$

Finally, we consider the limiting behavior of the terms corresponding to the torsion. We expanded only one of these terms and the expansion yielded some terms will have a $\frac{1}{\sqrt{\lambda}}$ dependence, some just $\sqrt{\lambda}$, some $\lambda^{\frac{3}{2}}$, and others with higher orders of λ dependence. All these terms are then multiplied by the expression out front which has a λ^{-1} and a term with no λ dependence (again, all terms of higher-order in λ are disregarded in the limit). Thus, when multiplied through, we have terms with $\frac{1}{\lambda^{3/2}}$, $\frac{1}{\sqrt{\lambda}}$ dependence, $\sqrt{\lambda}$ dependence, and higher-orders of λ dependence. To show that the torsion must vanish in the limit, it suffice to consider the $\frac{1}{\lambda^{3/2}}$ terms. These are

$$\sum_{i=2}^{4}\frac{1}{\lambda}(f_i)^a(f_i)^d\Big(- (f^1)_d(f^1)_m(f^1)_{[b}\partial_{c]}\frac{1}{\sqrt{\lambda}}\sum_{j=2}^{4}(f_j)^m$$
$$+ (f^1)_b(f^1)_m(f^1)_{[c}\partial_{d]}\frac{1}{\sqrt{\lambda}}\sum_{i=2}^{4}(f_i)^m + (f^1)_c(f^1)_m(f^1)_{[b}\partial_{d]}\frac{1}{\sqrt{\lambda}}\sum_{i=2}^{4}(f_i)^m\Big).$$

The first term in the parentheses above will vanish as $\sum_{i=2}^{4}(f_i)^d(f^1)_d$ yields 0. Similarly, when the anti-symmetrization yields $(f^1)_d$, it will mean that the whole term vanishes. However, we have no way of constraining the remainder: the $\frac{1}{\lambda^{3/2}}$ dependence means that they will become unbounded in the limit. Specifically, let us define

$$\overset{\lambda}{Z}{}^{a}{}_{bc} := \frac{1}{\lambda^{3/2}} \sum_{i=2}^{4} (f_i)^a (f_i)^d \Big(\frac{1}{2} (f^1)_b (f^1)_m (f^1)_c \partial_d \sum_{i=2}^{4} (f_i)^m \\ + \frac{1}{2} (f^1)_c (f^1)_m (f^1)_b \partial_d \sum_{i=2}^{4} (f_i)^m \Big).$$

Or,

$$\overset{\lambda}{Z}{}^{a}{}_{bc} = \frac{1}{\lambda^{3/2}} \sum_{i=2}^{4} (f_i)^a (f_i)^d (f^1)_b (f^1)_m (f^1)_c \partial_d \sum_{i=2}^{4} (f_i)^m .$$

Recall that we require that the derivative operators converge in the limit. Since the C-fields relate the derivative operators, for them to converge in the limit, we require that $\overset{\lambda}{Z}{}^{a}{}_{bc}$ be bounded. The only way for it to be bounded is if it vanishes. If $\overset{\lambda}{Z}{}^{a}{}_{bc}$, we find that $C^a{}_{bc}(\lambda) \to C^a{}_{bc}$ as $\lambda \to 0$ where

$$C^a{}_{bc} = \sum_{i=1}^{4} \Big[\frac{1}{\lambda} \Big((f_i)^a (f_i)^d \Big) - \Big((f_i)^a r^d + (f_i)^d r^a \Big) \Big] \\ \Big[-2 (f^1)_d \partial_b (f^1)_c - 2\lambda \Big(\partial_d (f^1)_{(b} m_{c)} - \partial_b (f^1)_{(d} m_{c)} - \partial_c (f^1)_{(d} m_{b)} \Big) \Big].$$

If $\overset{\lambda}{Z}{}^{a}{}_{bc}$ vanishes, then the torsion will also vanish since it is the only remaining contribution to the torsion. Thus, we find that *torsion must vanish in the limit.* □

2.4 Discussion

The proof strategy here was to consider the behavior of the tetrads in $c \to \infty$ limit. It turned out that this behavior, together with the required convergence of the derivative operators, was sufficient to demonstrate the vanishing of torsion entirely in the limit. In other words, any derivative operator that the limiting process convergences to has vanishing torsion. The proof does not demonstrate uniqueness of the limiting model; rather, it shows that for any derivative operator that the one-parameter family of tetrads convergences to, the associated torsion must vanish. Unlike the case of curvature, we cannot maintain *any* torsion in the limit.

One important and striking aspect of the reduced theory is regarding the behavior of test bodies. In TPG, the trajectories of massive test particles are influenced by the presence of mass. Their acceleration is given as

$$\xi^n \nabla_n \xi^a = K^a{}_{bc} \xi^b \xi^c, \tag{2.6}$$

where ξ^a is tangent to the particle's trajectory, $K^a{}_{cd}$ is the contorsion tensor which relates any metric-compatible connection to the unique Levi-Civita connection,[5] and ∇ is the torsional derivative operator. (This expression is sometimes given in terms of the torsion tensor instead of the contorsion tensor as in (Knox 2011, Eq. 28); see (Aldrovandi and Pereira 2013, Comment 6.4) for further discussion.)

Recall that in Newtonian gravity, the trajectories of massive test particles are also influenced by the presence of matter. The acceleration is governed by

$$\xi^n \nabla_n \xi^a = -\nabla^a \phi, \tag{2.7}$$

where ξ^a is again tangent to the particle's trajectory, ϕ is the gravitational potential, and ∇ is flat.

The torsion vanishing in the classical theory means that the contorsion, too, will vanish. This means that Eq. 2.6 will reduce simply to $\xi^n \nabla_n \xi^a = 0$. One can think of this reduced theory as a special case of Newtonian gravity where the scalar field ϕ representing the gravitational

[5](On the conventions adopted throughout, $K^a{}_{bc} = \frac{1}{2}(T^a{}_{bc} + T_{cb}{}^a - T_{bc}{}^a)$)

potential vanishes. This means that in the reduced theory, particles are not accelerated in the presence of matter. Put simply, in this reduced theory, *there are no gravitational effects.*

Chapter 3

Alternative Formulations and Approaches to the Limit

The literature discussing torsional theories of gravity contains many claims that may strike one as surprising given the results of Chapters 1 and 2. Some claim that a classical theory with both torsion and a closed temporal metric cannot be constructed (Christensen, Hartong, Obers, and Rollier 2014a). Others argue that a particular form of Torsional Newton-Cartan geometry is "the correct framework to describe General Relativity (GR) in the non-relativistic limit" (Hansen, Hartong, and Obers 2020, 1). And, finally, some claim to show that a teleparallel formulation of Newton-Cartan theory is the large-speed-of-light limit of the TPG (Schwartz 2023). These claims are surprising because in Chapter 1, we have constructed a classical theory with torsion and a closed temporal metric and, in Chapter 2, I have shown the vanishing of torsion in the non-relativistic limit. So how are the results here consistent with the claims made in the literature?

In this chapter, I discuss and compare the results presented in the previous two chapters to the aforementioned projects. I begin by comparing the results of the first two chapters to

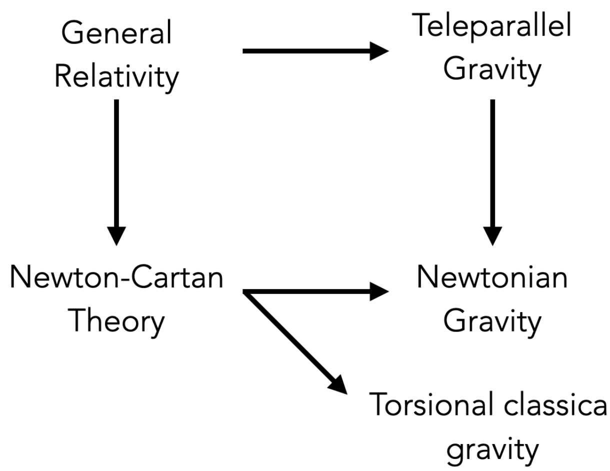

Figure 3.1: The relationships amongst the theories discussed above. The vertical arrows depict the procedure of taking the classical limit. The top-most horizontal arrow shows the process of "teleparallelization." The middle horizontal arrow shows the result of the Trautman degeometrization theorem while the bottom-most horizontal arrow shows the result of the more general degeometrization procedure outlined in Chapter 1. As depicted, the classical limit of GR is NCT. Introducing torsion allows one to formulate TPG from GR and a torsional classical theory from NCT. However, NG is the classical limit of TPG, not this torsional classical theory.

standard results in the torsion-free context. I then turn to various proposals in the literature, some that offer alternative classical theories of gravity and others that use alternative methods for taking the classical limit. In each instance, I show how to reconcile their claims with the results shown here.

3.1 Comparison to the torsion-free context

In Chapter 1, we constructed a classical theory of gravity with a closed temporal metric and non-vanishing torsion. In Chapter 2, I showed that torsion vanishes in the classical limit of Teleparallel Gravity and one recovers standard Newtonian Gravity. In other words, the theory from Chapter 1 is not the classical limit of TPG: teleparallelization does not commute with taking the classical limit (see Fig. 3.1).

The inability to preserve torsion in the classical limit seems quite surprising, especially considering that we have shown it is possible to construct a classical theory of gravity with torsion. One might have expected the theory developed in Chapter 1 to arise as the classical limit of TPG; after all, if it is possible to construct a torsional classical theory, why would such a theory not arise in the classical limit of TPG? In light of the results presented in Chapter 2, let us reconsider this expectation. Indeed, I hope to show how this expectation was unfounded. First, let us consider the classical limit of GR.

What happens when one takes the classical limit of GR by allowing the speed of light to become unbounded? As is well-known, the limiting procedure forces the curvature to vanish and one recovers flat space. As Malament puts it, "...the limiting process which effects the transition from general relativity to Newtonian gravitational theory 'squeezes out' all spatial curvature" (Malament 1986a, 406). The spacetime recovered in the limit is Euclidean, i.e., spatially flat.

Two propositions are required to formalize the above. First, Malament proves a proposition about the existence of a classical spacetime model that satisfies the requirements of the limiting procedure as well as the requirements required of a model of NCT (1986b, 194). Formally, the proposition is as follows.

Proposition 4. (Malament 1986b, Proposition on Limits (1)) Suppose $g_{ab}(\lambda)$ is a one-parameter family of Lorentz metrics on a manifold, M. Suppose also that t_a and h^{ab} satisfy

C1 $g_{ab}(\lambda) \rightarrow t_a t_b$ as $\lambda \rightarrow 0$ for some closed field t_a;

C2 $\lambda g^{ab}(\lambda) \rightarrow -h^{ab}$ as $\lambda \rightarrow 0$ for some closed field h^{ab} of signature $(0, 1, 1, 1)$.

Then

1. There is a derivative operator ∇_a on M satisfying $\overset{\lambda}{\nabla} \rightarrow \nabla_a$ as $\lambda \rightarrow 0$.

2. $(M, t_a, h^{ab}, \nabla_a)$ is a classical spacetime model satisfying $R^{[a}{}_{(b}{}^{c]}{}_{d)} = 0$.

As is evident in the above proposition, all that is known about the curvature at this point is that it satisfies $R^{[a}{}_{(b}{}^{c]}{}_{d)} = 0$. In this context, conditions (C1) and (C2) are not sufficient to show that the resultant spacetime is spatially flat. Spatial flatness is only shown with the second proposition; this second proposition considers the behavior of the material content of the spacetime in the limit.

Proposition 5. (Malament 1986b, Proposition on Limits (2)) Suppose $g_{ab}(\lambda)$ is a one-parameter family of Lorentz metrics on a manifold, M which, together with the symmetric family $T_{ab}(\lambda)$, satisfies conditions (C1) and (C2) as well as (C3) and (C4) given below.

C3 $\overset{\lambda}{R}_{ab} = 8\pi(T_{ab}(\lambda) - \frac{1}{2}g_{ab}(\lambda)\overset{\lambda}{T})$ holds for all λ

C4 $T^{ab}(\lambda) \to T^{ab}$ as $\lambda \to 0$ for some field T^{ab}

Further suppose $(M, t_a, h^{ab}, \nabla_a)$ is the classical spacetime model described in the previous proposition. Then there is a function ρ n M satisfying

1. $T_{ab}(\lambda) \to \rho t_a t_b$ as $\lambda \to 0$.

2. $R_{ab} = 4\pi\rho t_a t_b$.

As it turns out, Einstein's equation reduces to the geometrized formulation of Poisson's equation in the limit. Poisson's equation, in turn, is what ensures that space is flat (see (Malament 2012, Proposition 4.1.5) for detailed discussion on this point). In other words, the behavior of the matter fields is what entails that space is flat in the classical limit of GR. Malament writes:

> If at every intermediate stage of the collapse process [i.e., the opening up of the lightcones] spacetime structure is in conformity with the dynamic constraints of general relativity (as embodied in Einstein's field equation), then the resulting induced hyperspaces are necessarily flat, i.e., have vanishing Riemann curvature. (1986a, 406).

From the perspective of the reduction of GR to NCT, one can argue that the result derived in Chapter 2 is unsurprising. Since the classical limit squeezes out curvature, it is unsurprising that it would squeeze out the torsion as well. Put differently, the two results seem consistent: the limiting procedure applied to General Relativity returns a spatially flat theory. When applied to Teleparallel gravity, it returns a torsion-free theory. Notably, in the context of the reduction of GR to NCT, the space*time* is not flat; while the spatial curvature is squeezed out, one does still find temporal curvature. In the classical limit of TPG, we find that the torsion vanishes entirely.

One might find it more surprising that, in this context, the failure to derive a classical torsional spacetime was not a result of the behavior of matter fields, but arose from the requirement that the derivative operators converge. Furthermore, this requirement had to be made independently. These features seems to set the two classical limits apart: in Malament's proof, the recovery of flat space came from considering the behavior of Einstein's equation in the limit and the convergence of the metrics entailed the convergence of their associated derivative operators. Here, we independently require the derivative operators to converge and that constrains the allowed tetrads in the limit.[1]

[1]Another way of understanding the disanalogy between the two situations is in terms of the strength of the requirement that the C-fields be symmetric. The symmetric components of the C-fields have a well-behaved $\lambda \to 0$ limit while the anti-symmetric components turn out not to. Therefore, taking the limit requires that those anti-symmetric components—here, the torsion—vanish entirely.

3.2 Classical torsional theories of gravity

How do the results of Chapters 1 and 2 compare to other projects in the literature that focus on torsional theories? There are two main avenues of comparison here: projects aimed at developing classical torsional theories and projects studying the limit(s) of relativistic theories. Let us begin by discussing approaches to constructing classical torsional theories.

A small literature has recently emerged in physics surrounding torsional classical spacetime theories (generally referred to as "torsional Newton-Cartan" theories). Many in this literature are interested in incorporating torsion in the classical context to address the different notions of time proposed by GR and Quantum Gravity (QG): though GR does not admit a global notion of simultaneity, in some formulations, QG does. To resolve this difference, some authors have proposed taking the notion of time presented in QG as fundamental and allowing relativistic time to emerge at large distances. Then, motivated by the holographic principle (i.e., that a volume of space can be thought of as encoded in the lower dimensional boundary of that volume),[2] this literature considers 5D QG and its 4D reduction. The holography considered is between Hořava–Lifshitz gravity and a specific type of torsional Newton-Cartan theory: twistless torsional Newton-Cartan (TTNC) theory.[3]

Typically in the classical spacetime context, we take t_a to be closed (i.e., $\partial_a t_b = 0$), which means it is locally exact and determines a local time function. We adopted this assumption in the torsional gravitational theory developed in Chapter 1. The torsional Newton-Cartan theory formalism, and TTNC in particular, starts with NC theory but claims that *taking* $\partial_\mu t_\nu = 0$, where ∂ is a (torsion-free) coordinate derivative operator *will always result in a*

[2]The projects in this literature are also sometimes motivated as attempts to find further holographic correspondences beyond the AdS/CFT correspondence (see (Christensen, Hartong, Obers, and Rollier 2014b, 1)).

[3]The first paper developing this theory was (Christensen, Hartong, Obers, and Rollier 2014a). A slew of others followed including (Christensen, Hartong, Obers, and Rollier 2014b; Bergshoeff, Hartong, and Rosseel 2014; Hartong and Obers 2015; Afshar, Bergshoeff, Mehra, Parekh, and Rollier 2016; Figueroa-O'Farrill 2020).

torsion-free spacetime. Those working in this literature do not require temporal metrics to be compatible with any torsion-free derivative operator; more generally, they do not require that t_a is closed. Consider the following characteristic passage:

> The absence of torsion implies that the temporal vielbein[4] τ_μ corresponds to a closed one-form and that it can be used to define an absolute time in the space–time...TTNC geometry is characterized by the fact that the temporal vielbein is hypersurface orthogonal but not necessarily closed. (Bergshoeff, Hartong, and Rosseel 2014, 3)

In order to derive a hypersurface orthogonal temporal metric, such authors appeal to Frobenius' theorem. This allows them to argue that a spacetime admits a foliation with a time flow orthogonal to the Riemannian spacelike slices if and only if it satisfies the hypersurface orthogonality condition (i.e., $t_{[a}\partial_b t_{c]} = 0$). Notably, the "hypersurface orthogonality condition" is a weaker condition than the condition that the temporal metric be closed.

A series of questions emerge from the above discussion: Why does the TTNC literature claim that closed temporal metrics, and metric compatibility more generally, are in tension with torsion? And how does the theory described above incorporate torsion and a closed temporal metric, and thus a notion of absolute time? The answers lie in the form of the connection assumed by the TTNC literature.

Geracie and collaborators define a spacetime derivative operator $\nabla = (\partial, \Gamma^a{}_{bc})$, where they require the form of the connecting field $\Gamma^a{}_{bc}$ to be

$$\Gamma^a{}_{bc} = v^a \partial_b t_c + \frac{1}{2} h^{an} (\partial_b \hat{h}_{cn} + \partial_c \hat{h}_{bn} - \partial_n \hat{h}_{bc}),$$

[4] It is common to see formulations of classical gravity with torsion presented with the vielbein formalism typical of presentations of TPG. One can simply think of the temporal vielbein here as the temporal metric.

where v^a is a unit timelike field, and $\hat{h}_{ab}$ is a spatial projection field determined by v^a, such that $h^{an}\hat{h}_{nb} = \delta^a{}_b - v^a t_b$ (2015, Eq. 77).[5] This definition is motivated by the standard definition of a Levi-Civita derivative operator, and the terms in the parentheses are always symmetric in b, c. It follows that the torsion is given by $T^a{}_{bc} = 2\Gamma^a{}_{[bc]} = 2v^a \partial_{[b} t_{c]}$ (see, e.g., Geracie, Son, Wu, and Wu 2015, Eq. 79). (Indeed, the name "twistless torsional NCT," then, comes from the fact that torsion vanishes on spacelike hypersurfaces but not in general.)

And so it is true that if t_a is closed, the torsion of this derivative operator would vanish. However, *this is only because they have adopted such a strict definition for their connection.* Put differently, their connection ensures that the only way to allow torsion is to sacrifice having a closed temporal metric. But there are many other torsional derivative operators that are compatible with a closed temporal metric. Once we allow for a broader class of connections, as is done in the present paper, we recover metric compatibility and a notion of absolute time. In particular, Chapter 1 shows how to construct a torsional classical theory of gravity with the assumption that the temporal metric is closed. This claim, that a closed temporal metric is incompatible with torsion, is also found in discussions regarding the classical limit of relativistic (torsional) theories which I discuss next.

3.3 Alternative limits

Finally, we turn to projects considering the limit(s) of relativistic theories. There are a couple of approaches to consider here, as mentioned in §2.2: post-Newtonian expansions—$1/c^2$ as well as $1/c$—and dimensional reductions. Let us now turn to investigating each limiting approach and reconcile some of the seemingly surprising claims in this literature with the results presented in Chapters 1 and 2.[6]

[5]Their formalism has been adapted here to match the notation used throughout this dissertation.

[6]This section has benefited immensely from discussions with James Read. Needless to say, any errors in the below presentation are my own.

3.3.1 $1/c^2$ expansion

Let us begin by considering the claim that torsional Newton-Cartan theory is "the correct framework to describe General Relativity in the non-relativistic limit" (Hansen, Hartong, and Obers 2020, 1). This claim is made by a series of papers (see, e.g., Van den Bleeken 2017; Hansen, Hartong, and Obers 2020; Hartong, Obers, and Oling 2023), all of which perform a $1/c^2$ expansion to derive a non-relativistic theory. Considering the results from Malament discussed above—that standard NCT is the non-relativistic limit of GR—, how can such a claim be substantiated?

These results are based on Van den Bleeken's (2017) paper. There, he considers the non-relativistic limit of GR and, as claimed above, argues that one recovers a torsional theory in the limit. His project differs from ours in methodology. He is considering the $\frac{1}{c^2}$ expansion of GR. On this approach, one typically allows the metric to diverge in the large c limit but the associated Levi-Cevita connection remains finite. However, Van den Bleeken takes this approach further, allowing the connection to diverge as well. As he puts it

> [In previous work,] it is assumed that the relativistic metric is such that the associated Levi–Cevita connection remains finite in the large c limit. Although this might appear a natural assumption at first, one should keep in mind that the metric is allowed to diverge as $c \to \infty$. So why not the connection one could ask. In this work we relax this assumption and find that it leads to rather interesting observations, both mathematically and physically. (2017, 2)

Allowing the connection to diverge, he argues that the standard connection of NCT is not "the most natural connection for the expanded theory, as it is not compatible with the structure provided by the [spatial and temporal metrics]" (2017, 6). The connection he advocates ends up being that of twistless-torsional Newton-Cartan theory, discussed above. Thus, the failure of the temporal metric to be closed is what allows for torsion in the recovered spacetime. As Van den Bleeken puts it, the torsion corresponds to "a non-trivial timelike warpfactor already at the leading c^2 order" (2017, 2).

There are two things of note here. First, the different methodology used allows Van den Bleeken to derive torsion in the limit. On the methodology used by Malament, one considers a sequence of models, parameterized by λ, but all of one theory. This means that if one begins with GR, the limiting spacetime cannot have torsion as, at each step along the limit, one still has a model of GR. Put differently, insofar as the sequence of models we are considered are models of GR, their connections—and, correspondingly, the connection of the recovered spacetime—are required to be symmetric. Thus, Malament's methodology cannot yield a torsional spacetime model from one without torsion. On the other hand, Van den Bleenken's style of $1/c^2$ expansion seemingly does allow one to recover a spacetime with torsion from one without. This is because one does not require a sequence of models of a theory at all. Van den Bleenken does not impose the constraints one does when considering the limit of one theory and whether its models converge to a model of another theory in the limit. Instead, one simply expands the relevant equations or quantities of interest in powers of c.

Second, it is unclear how to understand Van den Bleeken's methodology in light of Malament's results. Malament's (Malament 1986b) proofs demonstrate that if one has a sequence of metrics of GR parameterized by λ and these converge, their derivative operators converge as well. In the above quotation, Van den Bleenken suggests that one can consider allowing the connections to diverge. However, given Malament's results, if the metrics converge, then their connections must as well. Thus, it seems that Van den Bleenken is entertaining a contradictory methodology.

3.3.2 Null Reduction

Another limiting procedure, one that has received philosophical treatment, is null reduction. In their (2018) paper, James Read and Nicholas Teh develop a method for "teleparallelizing" in the classical context and show the relation between teleparallel gravity and their classical, teleparallelized theory using null reduction. Their argument draws on a notion of "extended torsion" which I describe next.

The notion of "extended torsion" was introduced by Geracie, Prahbu, and Roberts (2015) but the general approach dates back to the 1980s (see, e.g., Duval and Künzle 1984). One standardly takes the symmetries of Newton-Cartan theory to be those described by the Galilean group. Recent projects have argued that, properly considered, the symmetry group of NCT is not the Galilean group, but rather the Bargmann group. The (inhomogenous) Galilei group (**IGal**) includes space and time translations and rotations as well as Galilei boosts. The Bargmann group is the one-dimensional central extension of inhomogenous Galilei group

$$\mathbf{Barg} = \mathbf{Gal} \ltimes (\mathbb{R}^4 \times \mathbf{U}(1)).^{7}$$

Using the Bargmann group as the symmetry group of NCT yields an "extended vielbein": ${e_\mu}^I = (\tau_\mu, {e_\mu}^a, m_\mu)$ where τ_μ is dubbed the clock torsion, ${e_\mu}^a$ is the spatial torsion,[8] and m_μ is the mass torsion (see, e.g., Geracie, Prabhu, and Roberts 2015, Eq. 2.14). As explained by Geracie, Prabhu, and Roberts, the mass torsion introduces "an additional gauge-field which couples to the mass of matter fields" (2015, 4). Importantly, the mass torsion cannot be converted into spacetime torsion, as noted by Read and Teh (2018, 2).

[7] The unitary extension, $\mathbf{U}(1)$, corresponds to "translations along a 'mass dimension"' (Read and Teh 2018, 2).

[8] Note that though the terminology is shared, this notion of "spatial torsion" is not the same as that used in Chapter 2.

With this broadened notion of torsion, Read and Teh construct the teleparallel equivalent of Newton-Cartan theory. They then study the relation between this theory and TPG using null reduction. Null reduction is a limiting procedure outlined in Duval (1985) wherein one considers the reduction of a $(D + 1)$-dimensional gravitational wave solution of a relativistic theory. Read and Teh show that the null reduction of TPG is their teleparallelized Newtonian Gravity.

These results offer another way of investigating the relationships amongst these theories. By broadening the notion of torsion that is at play to include "mass torsion," Read and Teh show that a commuting diagram can be constructed. Namely, either one can start with GR, teleparallelize to get TPG, and null reduce to get NG; or one can start with GR, null reduce to get NCT, and teleparallelize to get NG.

The methods adopted in this dissertation are more continuous with those typically used in the torsion-free context (e.g., those of Wald 2010; Malament 2012). With such methods, we have shown that a commuting diagram cannot be constructed; in other words, if one only allows spacetime torsion and considers the classical limit as the speed of light becomes unbounded, the addition of torsion in the classical context does not commute with the classical limit of TPG. Taking the classical limit of GR yields NCT and allowing torsion yields the classical theory with torsion outlined in Chapter 1. However, starting with GR, allowing torsion to get TPG, and then considering the classical limit yields standard Newtonian Gravity. Each proposal and set of methods is, of course, interesting in its own right.

3.3.3 $1/c$ expansion

The final comparison is to a recent paper by Philip Schwartz (2023). Like the present dissertation, Schwartz is interested in investigating the limit of Teleparallel Gravity but his project combines the two methods discussed above: he constructs the classical limit by performing a $1/c$ expansion while also considering the 'gauge-theoretic' description of Newton-Cartan gravity in terms of the Bargmann group.[9] In the end, Schwartz claims that teleparallel Newton-Cartan gravity is the large-speed-of-light limit of TPG. Let us next turn to his approach.

The classical, torsional theory developed by Schwartz closely resembles that developed by Read and Teh but is intended to be more general. As Schwartz puts it

> [Read and Teh's theory] is constructed only in a restricted 'gauge-fixed' situation; in the present paper, we develop instead a completely general teleparallel description of Newton–Cartan gravity, without introducing arbitrary assumptions on the connection or the frame. (2023, 2)

The gauge-fixing indicated by Schwartz is regarding the so-called spatial torsion of the extended vielbein (i.e., $e_\mu{}^a$). Whereas Read and Teh assume that this quantity vanishes for the torsional Newton-Cartan gravity they construct, Schwartz does not.[10]

Notably, in contrast to the typical approach taken by those performing a $1/c$ or $1/c^2$ expansion, Schwartz adopts a notion of absolute time in the classical context. The form for the connection that he adopts is more general and so he can assume both torsion and a closed temporal metric (see his discussion in §2, especially his expression for the connection in Eq. 2.13).

[9] The salient difference between this methodology and the $1/c^2$ expansion presented above is not in what powers of c are used to expand the quantities of interest. Rather, I hope to present the salient differences in the discussion below.

[10] Though he does not assume it at the outset, Schwartz does ultimately take the spatial torsion to vanish. As he puts it

> Let us stress here again that this 'gauge-fixing' assumption of vanishing purely spatial torsion is, differently to the situation considered in [Read and Teh], not part of the formulation of the theory, but only added afterwards for the recovery of standard Newtonian gravity. (2023, 20)

Given that the spatial torsion ultimately does not seem to play any meaningful role in the theory, it is not clear what the significance of this generalization to the theory is supposed to be.

With this torsional equivalent to Newton-Cartan theory in hand, Schwartz turns to the $1/c$ expansion of TPG. The tetrads are expanded as follows (see 2023, Equations 3.2a and 3.2b)

$$E^0_\mu = c\tau_\mu + c^{-1}a_\mu + O(c^{-3}), \qquad E^a_\mu = e^a_\mu + O(c^{-2}), \tag{3.1}$$

and

$$E^\mu_0 = c^{-1}v^\mu + O(c^{-3}), \qquad E^\mu_a = e^\mu_a + O(c^{-2}). \tag{3.2}$$

Here, a_μ is eventually related to the mass torsion, τ_μ corresponds to the temporal metric, and the spatial metric is defined as $h := \delta^{ab} e_a \otimes e_b$.

With these expansions in hand, Schwartz claims to recover a torsional classical theory from TPG. As he writes

> This means that *as the formal $c \to \infty$ limit of the Lorentzian manifold we started with, we obtain a Galilei manifold with a Bargmann structure.* We stress again that the only assumption that is needed for this result is an expansion of the Lorentzian tetrad and dual tetrad as in [(3.1-3.2)], with a nowhere vanishing τ. (2023, 15)

Using this limiting procedure, Schwartz claims to recover teleparallel Newton-Cartan theory from TPG and, eventually, shows how one might derive the field and force equations of standard Newtonian gravity in the recovered theory.

One can understand his project as another way of capturing the results of Read and Teh. On the one hand, Read and Teh use null reduction to show how to recover a torsional classical theory from TPG. On the other hand, Schwartz shows how this theory arises as the large-speed-of-light limit of TPG. In a sense, then, the two methods converge on similar results. Importantly, however, both proposals require generalizing the notion of torsion with the extended vielbein formalism. Indeed, the results in Chapter 2 indicate that one *must* generalize the notion of torsion to recover a classical spacetime with torsion from TPG. Without this more general notion, the torsion is proven to vanish in the limit. That said, once you do generalize the torsion, the limiting methods ($1/c$ expansion and null reduction) seem to agree. This helps make sense of the seemingly surprising claim that torsional Newton-Cartan theory is the large-speed-of-light limit of the TPG. Only once allows this extended notion of torsion is such a claim plausible.

3.4 Conclusion

In this chapter, I have demonstrated how to reconcile the claims made in the literature with the results presented here. First, I outlined some key similarities and differences between the limiting results with and without torsion. I showed that those claiming the incompatibility of torsion and a closed temporal metric do so on the basis of their chosen connection. Once one allows for a more general form for the connection, this purported incompatibility disappears.

Next, I discussed various other means for taking the classical limit of a relativistic spacetime. This discussion was fruitful for understanding how the various methods complement one another. The discussion highlighted that without extending one's notion of torsion beyond standard spacetime torsion, it is impossible to maintain torsion in the classical limit. Other projects performing a $1/c$ expansion or a null reduction have all adopted a notion of "extended torsion," which is what has allowed them to maintain torsion in the limit.

Naturally, some open and interesting questions still remain. One might wonder: Is there a relativistic theory for which the classical spacetime developed in Chapter 1 would be the classical limit? Is there a way of performing the geometric limiting process with the extended notion of torsion proposed in the literature and would the results of this limiting procedure then agree with the others? And, finally, were one to begin with a relativistic theory featuring both curvature and torsion, is there a way to maintain spacetime torsion in the limit?

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Appendix A

Curvature with Torsion

A.1 Generalizing the Riemann tensor definition

The Riemann curvature measures the failure of successive differentiation operations to commute. When there is torsion present, the torsion will contribute to this failure.[1] In the absence of torsion, the action of the Riemann tensor on any smooth field ξ^b is given by: $R^a{}_{bcd}\xi^b = -2\nabla_{[c}\nabla_{d]}\xi^a$. In the absence of torsion, this quantity vanishes when ξ^a vanishes. To derive the expression for the Riemann tensor in the presence of non-vanishing torsion, we follow the same methodology as Wald (see Eq. 3.2.1). Let α and ξ^a be smooth and consider

$$-\nabla_{[c}\nabla_{d]}(\alpha\xi^a) = -\alpha\nabla_{[c}\nabla_{d]}\xi^a - \xi^a\nabla_{[c}\nabla_{d]}\alpha - \nabla_{[c}\xi^a\nabla_{d]}\alpha - \nabla_{[c}\alpha\nabla_{d]}\xi^a.$$

[1] I begin by following Jensen's derivation of the Riemann curvature tensor in the presence of torsion but use Malament's conventions.

The last two terms cancel and the second term on the right-hand side is just the expression for torsion (see Defn. 2). This leaves

$$-\nabla_{[c}\nabla_{d]}(\alpha\xi^a) = -\,\alpha\nabla_{[c}\nabla_{d]}\xi^a - \frac{1}{2}(T^n{}_{cd}\nabla_n\alpha)\xi^a.$$

Using the Leibniz rule, we know $T^n{}_{cd}\nabla_n(\alpha\xi^a) = (T^n{}_{cd}\nabla_n\alpha)\xi^a + \alpha T^n{}_{cd}\nabla_n\xi^a$. Substituting this into the above, we have

$$(-2\nabla_{[c}\nabla_{d]} + T^n{}_{cd}\nabla_n)(\alpha\xi^a) = \alpha(-2\nabla_{[c}\nabla_{d]}\xi^a + T^n{}_{cd}\nabla_n\xi^a).$$

By the same reasoning as in the torsion-free case, the expression in the parentheses on the right-hand side defines a tensor. And since α was arbitrary, we have:

$$R^a{}_{bcd}\xi^b = -2\nabla_{[c}\nabla_{d]}\xi^a + T^n{}_{cd}\nabla_n\xi^a. \tag{A.1}$$

For a covector field α_a, we have

$$R^a{}_{bcd}\alpha_a = 2\nabla_{[c}\nabla_{d]}\alpha_b - T^n{}_{cd}\nabla_n\alpha_b. \tag{A.2}$$

A.2 Symmetries of the Riemann Tensor

We next consider the symmetries of the Riemann tensor in the presence of torsion. Some of the symmetry properties are retained while others need to be adjusted.[2]

Skew Symmetry To begin, note that it is still the case that $R^a{}_{bcd} = -R^a{}_{bdc}$ as the torsion tensor is also anti-symmetric in c, d. In GR, $R_{abcd} = -R_{bacd}$ even when the torsion is non-

[2] I again loosely follow Jensen's approach in this section. Note, however, that Jensen is working in the context of GR with non-vanishing torsion and thus has a metric to raise and lower indices. This will not hold in the context of classical spacetimes, so I adjust some of the derivations accordingly.

vanishing (Jensen 2005, 7). In the classical spacetime context, we cannot lower indices with the temporal metric but we can raise them with the spatial metric. Thus, instead of R_{abcd} we will be considering $R^{ab}{}_{cd}$ and whether it is equal to $-R^{ba}{}_{cd}$ (i.e., $R^{(ab)}{}_{cd} = \mathbf{0}$). Here I follow the derivation given in Malament (Eq. 4.1.26) adding the contribution from the torsion:

$$
\begin{aligned}
R^{(ab)}{}_{cd} =& \frac{1}{2}(R^{ab}{}_{cd} + R^{ba}{}_{cd}) = \frac{1}{2}(R^{a}{}_{mcd}h^{mb} + R^{b}{}_{mcd}h^{ma}) \\
&= -\nabla_{[c}\nabla_{d]}h^{ab} + \frac{1}{2}T^{m}{}_{cd}\nabla_m h^{ab} - \nabla_{[c}\nabla_{d]}h^{ba} + \frac{1}{2}T^{m}{}_{cd}\nabla_m h^{ba} \\
&= -2\nabla_{[c}\nabla_{d]}h^{ab} + T^{m}{}_{cd}\nabla_m h^{ab} \\
&= \mathbf{0}.
\end{aligned}
$$

where the final equality holds (i.e., both terms drop out) due to the compatibility condition: $\nabla_a h^{bc} = 0$. Thus, we have confirmed that even with non-vanishing torsion, $R^{(ab)}{}_{cd} = \mathbf{0}$.

Bianchi symmetry Though the skew symmetries still hold, $R^{a}{}_{[bcd]} \neq \mathbf{0}$. To derive $R^{a}{}_{[bcd]}$ in the case of non-vanishing torsion, we begin by noting that any covariant vector can be realized in the form $\nabla_a \alpha$ and so we replace α_a in Eq. A.2 with $\nabla_a \alpha$:[3]

$$R^{a}{}_{bcd}\nabla_a \alpha = 2\nabla_{[c}\nabla_{d]}\nabla_b \alpha - T^{n}{}_{cd}\nabla_n \nabla_b \alpha$$

Anti-symmetrizing on b, c, d yields

$$R^{a}{}_{[bcd]}\nabla_a \alpha = 2\nabla_{[[c}\nabla_{d]}\nabla_{b]}\alpha - T^{n}{}_{[cd}\nabla_{|n|}\nabla_{b]}\alpha.$$

[3]See (Malament 2012, Proposition 1.8.2(3)) for a derivation of the below in the context of vanishing torsion.

Since $2\nabla_{[[c}\nabla_{d]}\nabla_{b]}\alpha = 2\nabla_{[c}\nabla_{d}\nabla_{b]}\alpha = 2\nabla_{[c}\nabla_{[d}\nabla_{b]]}\alpha = \nabla_{[c}T^{n}{}_{db]}\nabla_{n}\alpha$, we have

$$= \nabla_{[c}T^{n}{}_{db]}\nabla_{n}\alpha - T^{n}{}_{[cd}\nabla_{|n|}\nabla_{b]}\alpha.$$

We expand the first term using the Leibniz rule

$$= (\nabla_{[c}T^{n}{}_{db]})\nabla_{n}\alpha + T^{n}{}_{[db}\nabla_{c]}\nabla_{n}\alpha - T^{n}{}_{[cd}\nabla_{|n|}\nabla_{b]}\alpha \tag{A.3}$$

To simplify, consider the following expression: $T^{n}{}_{cd}T^{e}{}_{bn}\nabla_{e}\alpha = T^{n}{}_{cd}(\nabla_{b}\nabla_{n} - \nabla_{n}\nabla_{b})\alpha = T^{n}{}_{cd}\nabla_{b}\nabla_{n}\alpha - T^{n}{}_{cd}\nabla_{n}\nabla_{b}\alpha$. Note that the last term in this expression is the same as the last term in the previous equation. Adding in the appropriate anti-symmetrization brackets, we replace the term in Eq. A.3

$$= (\nabla_{[c}T^{n}{}_{db]})\nabla_{n}\alpha + T^{n}{}_{[db}\nabla_{c]}\nabla_{n}\alpha + T^{n}{}_{[cd}T^{e}{}_{b]n}\nabla_{e}\alpha - T^{n}{}_{[cd}\nabla_{b]}\nabla_{n}\alpha.$$

The second and last terms cancel one another leaving

$$R^{a}{}_{[bcd]}\nabla_{a}\alpha = (\nabla_{[c}T^{n}{}_{db]})\nabla_{n}\alpha + T^{n}{}_{[cd}T^{e}{}_{b]n}\nabla_{e}\alpha.$$

Finally, we have[4]

$$R^{a}{}_{[bcd]} = \nabla_{[c}T^{a}{}_{db]} + T^{n}{}_{[cd}T^{a}{}_{b]n}$$

In sum, the symmetry properties of the Riemann tensor in the presence of torsion are

$$R^{a}{}_{bcd} = -R^{a}{}_{bdc}, \tag{A.4}$$

[4]C.f. 4.2.39 in Penrose and Ridler or 3.2.14 of Jensen, though both have a different sign convention.

$$R^{ab}{}_{cd} = -R^{ba}{}_{cd}, \tag{A.5}$$

and

$$R^{a}{}_{[bcd]} = \nabla_{[c}T^{a}{}_{db]} + T^{n}{}_{[cd}T^{a}{}_{b]n}\,. \tag{A.6}$$

A.3 Relating Riemann tensors

The relation between two derivative operators with possibly non-vanishing torsion can still be expressed with a smooth tensor field $C^{a}{}_{bc}$ but this field will no longer be symmetric in b, c as it is in the torsion-free context. Instead, given two derivative operators ∇ and ∇' with associated torsion tensors $T'^{a}{}_{bc}$ and $T^{a}{}_{bc}$, we have that

$$C^{a}{}_{[bc]} = T'^{a}{}_{bc} - T^{a}{}_{bc}. \tag{A.7}$$

Proposition 6. Let ∇ and ∇' be derivative operators on a manifold M with $\nabla' = (\nabla, C^{a}{}_{bc})$. Suppose that ∇ and ∇' both have possibly non-vanishing torsion. Suppose also that they have curvatures $R^{a}{}_{bcd}$ and $R'^{a}{}_{bcd}$ respectively. The relation between $R^{a}{}_{bcd}$ and $R'^{a}{}_{bcd}$ is given as

$$R'^{a}{}_{bcd}\alpha_a = R^{a}{}_{bcd}\alpha_a + 2\alpha_a\nabla_{[c}C^{a}{}_{d]b} + 2\alpha_a C^{a}{}_{[d|n|}C^{n}{}_{c]b} - C^{a}{}_{nb}T^{n}{}_{cd}\alpha_a. \tag{A.8}$$

Proof: To derive the relation between the Riemann tensors of each derivative operator, we will need to derive the expression for $\nabla'_c\nabla'_d\alpha_b$. To do so, substitute β_{db} in for the right-hand side of $\nabla'_d\alpha_b = \nabla_d\alpha_b + \alpha_a C^{a}{}_{db}$ (i.e., suppose $\beta_{db} = \nabla_d\alpha_b + \alpha_a C^{a}{}_{db}$). Then, using Proposition

1.7.3 (Malament 2012),

$$\nabla'_c \nabla'_d \alpha_b = \nabla'_c(\beta_{db}) = \nabla_c \beta_{db} + \beta_{nb} C^n{}_{cd} + \beta_{dn} C^n{}_{cb}.$$

Substituting the expression for β_{db} back in:

$$\begin{aligned}
\nabla'_c \nabla'_d \alpha_b &= \nabla_c(\nabla_d \alpha_b + \alpha_a C^a{}_{db})(\nabla_n \alpha_b + \alpha_a C^a{}_{nb}) C^n{}_{cd} \\
&\quad + (\nabla_d \alpha_n + \alpha_a C^a{}_{dn}) C^n{}_{cb} \\
&= \nabla_c \nabla_d \alpha_b + C^a{}_{db} \nabla_c \alpha_a + \alpha_a \nabla_c C^a{}_{db} + C^n{}_{cd} \nabla_n \alpha_b \\
&\quad + \alpha_a C^a{}_{nb} C^n{}_{cd} + C^n{}_{cb} \nabla_d \alpha_n + \alpha_a C^a{}_{dn} C^n{}_{cb}.
\end{aligned}$$

Anti-symmetrizing this expression in c and d yields

$$\begin{aligned}
\nabla'_{[c} \nabla'_{d]} \alpha_b =& \nabla_{[c} \nabla_{d]} \alpha_b + C^a{}_{[d|b|} \nabla_{c]} \alpha_a + \alpha_a \nabla_{[c} C^a{}_{d]b} + C^n{}_{[cd]} \nabla_n \alpha_b \\
&+ \alpha_a C^a{}_{nb} C^n{}_{[cd]} + C^n{}_{[c|b} \nabla_{d]} \alpha_n + \alpha_a C^a{}_{[d|n|} C^n{}_{c]b},
\end{aligned}$$

where the second and sixth terms on the right-hand side cancel each other since their order of antisymmetrized indices is reversed. We know $\nabla_{[c} \nabla_{d]} \alpha_b = \frac{1}{2} R^a{}_{bcd} \alpha_a + \frac{1}{2} T^n{}_{cd} \nabla_n \alpha_a$ and $C^n{}_{[cd]} = \frac{1}{2} T'^n{}_{cd} - \frac{1}{2} T^n{}_{cd}$. We thus have

$$\begin{aligned}
\nabla'_{[c} \nabla'_{d]} \alpha_b = \frac{1}{2} R^a{}_{bcd} \alpha_a + \frac{1}{2} T^n{}_{cd} \nabla_n \alpha_a + \alpha_a \nabla_{[c} C^a{}_{d]b} \\
+ \frac{1}{2} T'^n{}_{cd} \nabla_n \alpha_b - \frac{1}{2} T^n{}_{cd} \nabla_n \alpha_b + \frac{1}{2} \alpha_a C^a{}_{nb} (T'^n{}_{cd} - T^n{}_{cd}) + \alpha_a C^a{}_{[d|n|} C^n{}_{c]b}.
\end{aligned} \tag{A.9}$$

Using the expression for the Riemann tensor with torsion (Eq. A.2), we have

$$\begin{aligned}
R'^a{}_{bcd}\ \alpha_a =& 2\nabla'_{[c} \nabla'_{d]} \alpha_b - T'^n{}_{cd} \nabla'_n \alpha_b \\
&= 2\nabla'_{[c} \nabla'_{d]} \alpha_b - T'^n{}_{cd} (\nabla_n \alpha_n + C^p{}_{nb} \alpha_p) \\
&= 2\nabla'_{[c} \nabla'_{d]} \alpha_b - T'^n{}_{cd} \nabla_n \alpha_n - T'^n{}_{cd} C^p{}_{nb} \alpha_p.
\end{aligned}$$

Substituting the above in for the left-hand side of Eq. A.9, we have

$$\begin{aligned}
R'^a{}_{bcd}\alpha_a &= \frac{1}{2}R^a{}_{bcd}\alpha_a + \frac{1}{2}T^n{}_{cd}\nabla_n\alpha_a + \alpha_a\nabla_{[c}C^a{}_{d]b} + \frac{1}{2}T'^n{}_{cd}\nabla_n\alpha_b - \frac{1}{2}T^n{}_{cd}\nabla_n\alpha_b \\
&+ \frac{1}{2}\alpha_a C^a{}_{nb}(T'^n{}_{cd} - T^n{}_{cd}) + \alpha_a C^a{}_{[d|n|}C^n{}_{c]b} - \frac{1}{2}T'^n{}_{cd}\nabla_n\alpha_n - \frac{1}{2}T'^n{}_{cd}C^p{}_{nb}\alpha_p.
\end{aligned}$$

Canceling terms and simplifying the above leaves

$$R'^a{}_{bcd}\alpha_a = R^a{}_{bcd}\alpha_a + 2\alpha_a\nabla_{[c}C^a{}_{d]b} + 2\alpha_a C^a{}_{[d|n|}C^n{}_{c]b} - C^a{}_{nb}T^n{}_{cd}\alpha_a. \tag{A.10}$$

Appendix B

Connecting Fields

Let us begin by noting a few conditions on the connecting fields (fields that relate derivative operators) in the presence of torsion. Suppose that we have a classical spacetime (M, t_a, h^{ab}, ∇) with (possibly) non-vanishing torsion and let $\nabla' = (\nabla, C^a{}_{bc})$ be another derivative operator on M also with (possibly) non-vanishing torsion. We know $2C^a{}_{[bc]} = T'^a{}_{bc} + T^a{}_{bc}$, where $T^a{}_{bc}$ and $T'^a{}_{bc}$ are the torsion tensors associated with ∇ and ∇' respectively. We require ∇' to satisfy the metric-compatibility conditions and thus for $C^a{}_{bc}$ to be compatibility preserving. This means

$$0 = \nabla'_a t_b = \nabla_a t_b + C^r{}_{ab} t_r = C^r{}_{ab} t_r \tag{B.1}$$

where the final step follows from the fact that since (M, t_a, h^{ab}, ∇) is a classical spacetime, we know ∇ is compatible with t_a. The above indicates that the first index of the torsion tensor must be spacelike since

$$T^a{}_{bc} t_a = 0. \tag{B.2}$$

We also require compatibility with h^{bc}

$$0 = \nabla'_a h^{bc} = \nabla_a h^{bc} - C^b{}_{ar} h^{rc} - C^c{}_{ar} h^{br} = -C^b{}_{ar} h^{rc} - C^c{}_{ar} h^{br}, \tag{B.3}$$

where we have again used the known compatibility of ∇ with h^{bc}. This indicates

$$C^b{}_{ar} h^{rc} = -C^c{}_{ar} h^{br}. \tag{B.4}$$

We are now in a position to prove a proposition about the general form of the connecting fields.

> **Proposition 7.** Let (M, t_a, h^{ab}, ∇) be a classical spacetime with (possibly) non-vanishing torsion. Let $\nabla' = (\nabla, C^a{}_{bc})$ be another derivative operator on M also with (possibly) non-vanishing torsion (i.e., $2C^a{}_{[bc]} = T'^a{}_{bc} + T^a{}_{bc}$, where $T^a{}_{bc}$ and $T'^a{}_{bc}$ are the torsion tensors associated with ∇ and ∇' respectively). Then ∇' is compatible with t_a and h^{ab} if and only if $C^a{}_{bc}$ is of the form:
>
> $$C^a{}_{bc} = 2h^{ar} \kappa_{[r|b|c]} \tag{B.5}$$

Proof: Since (M, t_a, h^{ab}, ∇) is a classical spacetime, ∇ is compatible with t_a and h_{ab}, i.e., $\nabla_a t_b = 0$ and $\nabla_a h^{bc} = 0$. We also know that the relations expressed by Eqs. B.1 and B.4 must hold.

Consider the raised index tensor field $C^{abc} = C^a{}_{mn} h^{mb} h^{nc}$. We no longer have the symmetries expressed by Malament's equations (4.1.18) since the C-field is not symmetric in its last two indices. Instead, we have

$$C^{abc} = C^{acb} + (T'^{abc} - T^{abc}). \tag{B.6}$$

However, Malament's equation (4.1.19) stays intact. Based on Eq. B.4

$$C^{bac} = -C^{cab}. \tag{B.7}$$

We thus require that our C^{abc} be orthogonal to t_a (as described by Eq. B.1) and that it satisfies the symmetry property expressed by Eq. B.6. By repeated application of these two relations, we have

$$C^{abc} = C^{acb} + (T'^{abc} - T^{abc}) = -C^{bca} + (T'^{abc} - T^{abc})$$

$$= -C^{bac} + (T'^{abc} - T^{abc}) - (T'^{bca} - T^{bca}) = C^{cab} + (T'^{abc} - T^{abc}) - (T'^{bca} - T^{bca})$$

$$= C^{cba} + (T'^{abc} - T^{abc}) - (T'^{bca} - T^{bca}) + (T'^{cab} - T^{cab})$$

$$= -C^{abc} + (T'^{abc} - T^{abc}) - (T'^{bca} - T^{bca}) + (T'^{cab} - T^{cab}).$$

We know that the torsion tensor is anti-symmetric in its last two indices. Let us define another tensor, $K^a{}_{bc}$ as

$$K^a{}_{bc} = \frac{1}{2}(T^a{}_{bc} + T_{cb}{}^a - T_{bc}{}^a).$$

Or,

$$K^{bac} = \frac{1}{2}(T^{bac} + T^{cab} - T^{acb}).$$

This tensor is anti-symmetric in its first and last indices ($K^{abc} = -K^{cba}$). We will use this tensor to simplify the above expression. First, we note

$$C^{abc} = -C^{abc} + (-T'^{acb} + T'^{bac} + T'^{cab}) - (-T^{acb} + T^{bac} + T^{cab}).$$

Using K^{abc}, we have

$$C^{abc} = -C^{abc} + 2(K'^{bac} - K^{bac}). \tag{B.8}$$

The field does not vanish in the presence of torsion. Rather, $C^{abc} = (K'^{bac} - K^{bac})$. Since K^{bac} is anti-symmetric in its first and last index, we have shown that C^{abc} must be anti-symmetric in a and c. This concludes the "only if" direction of the proof.

For the "if" direction, suppose that we have a tensor of the form indicated above—namely, $C^a{}_{bc} = 2h^{ar}\kappa_{[r|b|c]}$. We want to show that it is sufficient that $C^a{}_{bc}$ take this general form for it be to compatible with t_a and h^{bc}. We know that $t_a C^a{}_{bc} = 0$ from the orthogonality condition.

From equation B.4, we know

$$\nabla'_a h^{bc} = -C^b{}_{ar} h^{rc} - C^c{}_{ar} h^{br} \tag{B.9}$$

Hence

$$\nabla'_a h^{bc} = -2h^{bn}\kappa_{nar} h^{rc} - 2h^{cn}\kappa_{nar} h^{br},$$

and

$$\nabla'_a h^{bc} = -2(\kappa^b{}_a{}^c + \kappa^c{}_a{}^b) = 0.$$

Where the last expression follows because κ is anti-symmetric in its first and last index. Thus, we have shown that $C^a{}_{bc} = 2h^{ar}\kappa_{[r|b|c]}$ is sufficient for ∇' to be compatible with t_a and h^{bc}, concluding the proof.