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**Maximal Functions, Incidence Theorems,
and Efficient Partitions of Euclidean Space**

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Joshua Norbert Zahl

2013

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ABSTRACT OF THE DISSERTATION

**Maximal Functions, Incidence Theorems,
and Efficient Partitions of Euclidean Space**

by

Joshua Norbert Zahl

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor Terence Tao, Chair

We establish several new results in incidence geometry and harmonic analysis. Each of the problems we consider is about objects in Euclidean space, and we make essential use of partitioning theorems that efficiently cut \mathbb{R}^d into pieces that have desirable combinatorial properties.

The dissertation of Joshua Norbert Zahl is approved.

John Garnett

Rowan Killip

Amit Sihal

Terence Tao, Committee Chair

University of California, Los Angeles

2013

To my parents, for all their love and support

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PUBLICATIONS

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On universal cycles for multisets. (with G. Hurlbert and T. Johnson). *Discrete Math.*, 309(17):5321–5327, 2009.

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CHAPTER 1

Introduction

In this thesis we will discuss several results in incidence geometry and Kakeya-type maximal functions. The latter type of problem can also be phrased as an incidence geometry-type problem. A common theme for all of these problems is that they consider objects (points, lines, surfaces) in \mathbb{R}^d , and the properties of Euclidean space will play an important role in the theorems that follow. One key property of \mathbb{R} is that it is ordered—if we select a point $x \in \mathbb{R}$, then $\mathbb{R} \setminus \{x\}$ consists of two distinct connected sets. This property does not hold for either \mathbb{C} or finite fields \mathbb{F}_p . This property of \mathbb{R} is an important part of the *discrete polynomial partitioning theorem*, which will be discussed further in Chapter 2, and which will play an important role in each of the results below.

While the problems we consider originated in combinatorics and harmonic analysis, the tools used to solve them rely heavily on real algebraic geometry, and these tools are discussed in Chapter 2. In Chapter 3, we will discuss a bound on the number of incidences between points and surfaces in \mathbb{R}^3 . This is based on the author's work in [Zah13a]. In chapter 4, we will discuss an analogue of the Szemerédi-Trotter theorem for 2-flats in \mathbb{R}^4 . This is based on the author's work in [Zah12b]. Finally, in Chapter 5, we will discuss a L^3 bound on a variable-coefficient analogue of the Wolff circular maximal function. This is based on the author's work in [Zah12a] and [Zah13b].

CHAPTER 2

Preliminaries

2.1 Notation

Throughout this thesis, c and C will denote sufficiently small and large constants, respectively, which are allowed to vary from line to line. We will write $A \lesssim B$ to mean $A < CB$, and we say that a quantity is $O(A)$ if it is $\lesssim A$.

2.2 Graph theory

Throughout our arguments, we will frequently make use of the following Turan-type bound:

Theorem 1 (Kővari, Sós, Turan [KST54]). *Let s, t be fixed, and let $G = G_1 \sqcup G_2$ be a bipartite graph with $|G_1| = m$, $|G_2| = n$ that contains no copy of $K_{s,t}$. Then G has at most $O(nm^{1-1/s} + m)$ edges. Symmetrically, G has at most $O(mn^{1-1/t} + n)$ edges. All implicit constants depend only on s and t .*

In Chapter 4 we will require the crossing lemma. This is described below. Let G be a graph, and let H be a drawing of G , i.e. a collection of points and curves in \mathbb{R}^2 such that every vertex of G corresponds to a distinct point of H , and every edge of G corresponds to a curve in H such that every two curves intersect in a discrete set, and no points (i.e. vertices) are contained in the relative interior of any curve (i.e. edge).

Definition 2. We define $\mathcal{V}(G)$ to be the number of vertices of G and $\mathcal{E}(G)$ to be the number of edges, and similarly for H . We define $\mathcal{C}(H)$ to be the *crossing number* of H , i.e. the

number of times two curves cross each other. Since the intersection of any two curves is a discrete set, $\mathcal{C}(H)$ is finite.

Theorem 3 (Ajtai, Chvatal, Newborn, Szemerédi [ACN82]; Leighton [Lei83]; Székely [Sz97]).

Let H be a drawing of a graph. Then either $\mathcal{E}(H) < 5\mathcal{V}(H)$, or

$$\mathcal{C}(G) \geq \frac{\mathcal{E}(H)^3}{100\mathcal{V}(H)^2}. \quad (2.1)$$

2.3 Real algebraic geometry

2.3.1 Semi-algebraic sets

See e.g. [BPR06, BCR98] for additional information.

Definition 4. A set $S \subset \mathbb{R}^n$ is *semi-algebraic* if it can be expressed in the form

$$S = \bigcup_{i=1}^n \{x: f_{i,1}(x) = 0, \dots, f_{i,\ell_i}(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,m_i}(x) > 0\} \quad (2.2)$$

for $\{f_{i,j}\}$ and $\{g_{i,j}\}$ polynomials.

Definition 5. For S a semi-algebraic set, the *complexity* of S is

$$\inf \left(\sum \deg f_{i,j} + \sum \deg g_{i,j} \right), \quad (2.3)$$

where the infimum is taken over all representations of S of the form (2.2).

Definition 6. For S a semi-algebraic set, we define the *boundary* $\partial(S) = \overline{S} \setminus S$, where \overline{S} is the closure of S in the Euclidean topology.

Proposition 7. $\partial(S)$ is semi-algebraic, $\dim(\partial(S)) \leq \dim(S) - 1$, and the complexity of $\partial(S)$ is controlled by a polynomial function of the complexity of S .

Definition 8. For S a semi-algebraic set, we define its *Zariski closure* $\text{Zar}(S)$ to be the closure of S in the (real) Zariski topology

Proposition 9.

- (i) $\text{Zar}(S)$ is an algebraic set.
- (ii) $\dim(\text{Zar}(S)) = \dim(S)$.
- (iii) $\deg(\text{Zar}(S))$ is bounded by a polynomial function of the complexity of S .

Proof. Statements (i) and (ii) are standard. Statement (iii) follows from the standard properties of the cylindrical algebraic decomposition (see e.g. [BPR06, BCR98]). \square

Proposition 10 (Effective Tarski-Seidenberg Theorem [Col75]). Let $S \subset \mathbb{R}^d$ be a semi-algebraic set of complexity k and let $\pi: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ be the projection onto the first $d-1$ coordinates. Then $\pi(S)$ is a semi-algebraic set of complexity at most k^C for some constant C that depends only on d .

2.3.2 Sign conditions

Definition 11. Let $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_d]$ be a collection of non-zero real polynomials. A *strict sign condition* on \mathcal{Q} is a map $\sigma: \mathcal{Q} \rightarrow \{\pm 1\}$. If $Q \in \mathcal{Q}$, we will denote the evaluation of σ at Q either by σ_Q or $\sigma(Q)$, depending on context. If σ is a strict sign condition on \mathcal{Q} we define its *realization* by

$$\text{Reali}(\sigma, \mathcal{Q}) = \{x \in \mathbb{R}^d: Q(x)\sigma_Q > 0 \text{ for all } Q \in \mathcal{Q}\}. \quad (2.4)$$

If $\text{Reali}(\sigma, \mathcal{Q}) \neq \emptyset$ then we say that σ is *realizable*. We define

$$\Sigma_{\mathcal{Q}} = \{\sigma: \text{Reali}(\sigma, \mathcal{Q}) \neq \emptyset\}, \quad (2.5)$$

and

$$\text{Reali}(\mathcal{Q}) = \{\text{Reali}(\sigma, \mathcal{Q}): \sigma \in \Sigma_{\mathcal{Q}}\}. \quad (2.6)$$

We call $\text{Reali}(\mathcal{Q})$ the collection of “realizations of realizable strict sign conditions of \mathcal{Q} .”

If $Z \subset \mathbb{R}^d$ is a variety, and σ is a strict sign condition on \mathcal{Q} , then we can define the *realization of σ on Z* by

$$\text{Reali}(\sigma, \mathcal{Q}, Z) = \{x \in Z: Q(x)\sigma_Q > 0 \text{ for all } Q \in \mathcal{Q}\}, \quad (2.7)$$

and we can define analogous sets

$$\Sigma_{\mathcal{Q}, Z} = \{\sigma : \text{Reali}(\sigma, \mathcal{Q}, Z) \neq \emptyset\}, \quad (2.8)$$

and

$$\text{Reali}(\mathcal{Q}, Z) = \{\text{Reali}(\sigma, \mathcal{Q}, Z) : \sigma \in \Sigma_{\mathcal{Q}, Z}\}. \quad (2.9)$$

We call $\text{Reali}(\mathcal{Q}, Z)$ the collection of “realizations of realizable strict sign conditions of \mathcal{Q} on Z .” Note that if some $Q \in \mathcal{Q}$ vanishes identically on Z then $\Sigma_{\mathcal{Q}, Z} = \emptyset$ and thus $\text{Reali}(\mathcal{Q}, Z) = \emptyset$.

2.3.3 Algebra

All ideals and varieties will be assumed to be affine. Unless otherwise specified, all ideals are subsets of $\mathbb{R}[x_1, \dots, x_d]$, and all varieties are defined over \mathbb{R} and thus are subsets of \mathbb{R}^d , though sometimes we will specialize to the case $d = 3$. If P is a polynomial, $(P) \subset \mathbb{R}[x_1, \dots, x_d]$ is the ideal generated by P .

Special emphasis will be placed on “real ideals.” These are described in Definition 12 below, and they should not be confused with ideals that are merely subsets of $\mathbb{R}[x_1, \dots, x_d]$. On the other hand, a “real variety” is merely a variety defined over \mathbb{R} (as opposed to \mathbb{C}).

If I is an ideal, we use

$$\mathbf{Z}(I) = \{x \in \mathbb{R}^d : P(x) = 0 \text{ for all } P \in I\}$$

to denote the zero set of I . If P is a polynomial we shall abuse notation and use $\mathbf{Z}(P)$ to denote $\mathbf{Z}((P)) = \{x \in \mathbb{R}^d : P(x) = 0\}$. If $Z \subset \mathbb{R}^d$ is a real variety, then we define

$$\mathbf{I}(Z) = \{P \in \mathbb{R}[x_1, \dots, x_d] : P(x) = 0 \text{ for all } x \in Z\}$$

to be the ideal of polynomials that vanish on Z .

Definition 12. An ideal $I \subset \mathbb{R}[x_1, \dots, x_d]$ is *real* if for every sequence $a_1, \dots, a_\ell \in \mathbb{R}[x_1, \dots, x_d]$, $a_1^2 + \dots + a_\ell^2 \in I$ implies $a_j \in I$ for each $j = 1, \dots, \ell$.

The following proposition shows that real principal prime ideals and their corresponding real varieties have some of the “nice” properties of ideals and varieties defined over \mathbb{C} .

Proposition 13 (see [BCR98, §4.5]). Let $P \in \mathbb{R}[x_1, \dots, x_d]$ be irreducible. Then the following are equivalent:

1. (P) is real.
2. $(P) = \mathbf{I}(\mathbf{Z}(P))$.
3. $\dim(\mathbf{Z}(P)) = d - 1$.
4. ∇P does not vanish identically on $\mathbf{Z}(P)$.
5. The sign of P changes somewhere on \mathbb{R}^d (i.e. from strictly positive to strictly negative).

While not every polynomial $P \in \mathbb{R}[x_1, \dots, x_d]$ is a product of real ideals, the following lemma shows that for our applications, we can always modify our polynomials to ensure that this is the case.

Lemma 14. *Let $P \in \mathbb{R}[x_1, \dots, x_d]$ be a real polynomial. Then there exists a real polynomial \tilde{P} such that $\deg \tilde{P} \leq \deg P$, $\mathbf{Z}(P) \subset \mathbf{Z}(\tilde{P})$, and the irreducible components of \tilde{P} generate real ideals.*

Proof. Let $\mathcal{Q} = \emptyset$. Write $P = P_1 \dots P_a$ as a product of irreducible factors. Place each irreducible factor that generates a real ideal in \mathcal{Q} . If P_j is a factor that does not generate a real ideal then consider $\nabla_v P_j$ for v a generic vector. We have $\deg \nabla_v P_j < \deg P$, and $\mathbf{Z}(P_j) \subset \mathbf{Z}(\nabla_v P_j)$. Apply the above procedure to $\nabla_v P_j$. This process will terminate after finitely many iterations. Let $\tilde{P} = \prod_{Q \in \mathcal{Q}} Q$. □

We will make essential use of the real Bézout’s theorem and a version of Harnack’s theorem for space curves.

Proposition 15 (Real Bézout). Let $P_1, \dots, P_d \in \mathbb{R}[x_1, \dots, x_d]$ be real polynomials of degrees D_1, \dots, D_d . Then the number of nonsingular intersection points of $\mathbf{Z}(P_1) \cap \dots \cap \mathbf{Z}(P_d)$ is at most $D_1 \dots D_d$.

For a proof of this proposition, see e.g. [BPR06, §4.7].

We will also require a variant of Harnack's theorem for space curves \mathbb{R}^4 .

Proposition 16 (Harnack's theorem for space curves). Let $P_1, \dots, P_{d-1} \in \mathbb{R}[x_1, \dots, x_d]$, and suppose that $\deg P_j = O(1)$ for each $1 \leq j \leq d-2$, and that for each $1 \leq j \leq d-1$, $\text{codim}(\mathbf{Z}(P_1) \cap \dots \cap \mathbf{Z}(P_j)) = j$. Then the number of connected components of $\mathbf{Z}(P_1) \cap \dots \cap \mathbf{Z}(P_{d-1})$ is $O((\deg P_{d-1})^2)$.

This proposition is Theorem 11 of [BB13].

2.3.4 Real and complex varieties

If $Z \subset \mathbb{R}^d$ is a real variety, then $Z^* \subset \mathbb{C}^d$ denotes the smallest complex variety containing Z . Conversely, if $Z \subset \mathbb{C}^d$ is a complex variety, then $\Re(Z) \subset \mathbb{R}^d$ is its set of real points.

As noted in Section 3.1.3, the number of intersection points of a collection of real polynomials may exceed the product of their degrees, even if those polynomials intersect completely. Over \mathbb{C} things are much better behaved, so there will be times when we will wish to embed everything into \mathbb{C} . The following proposition relates the properties of a variety defined over \mathbb{R} and the corresponding variety defined over \mathbb{C} :

Proposition 17 (see [Whi57, §10]). Let $Z \subset \mathbb{R}^d$ be a real variety and let $(Z^*)_1, \dots, (Z^*)_l$ be the irreducible components of Z^* . Then $\Re((Z^*)_1), \dots, \Re((Z^*)_l)$ are the irreducible components of Z . Furthermore, for each $j = 1, \dots, l$, $\Re((Z^*)_j)^* = (Z^*)_j$, so in particular $\Re((Z^*)_j)$ is non-empty.

Corollary 18. If $P, Q \in \mathbb{R}[x_1, \dots, x_d]$ are irreducible and $(P), (Q)$ are real ideals such that $\dim(\mathbf{Z}(P) \cap \mathbf{Z}(Q)) = d-2$, then $\mathbf{Z}(P)^* \cap \mathbf{Z}(Q)^*$ is a complete intersection.

2.4 Discrete polynomial partitioning theorems

Recall the discrete polynomial partitioning theorem from [GK11, Theorem 4.1]:

Theorem 19. *Let \mathcal{P} be a collection of points in \mathbb{R}^d , and let $D > 0$. Then there exists a*

non-zero polynomial P of degree at most D such that each connected component of $\mathbb{R}^d \setminus \mathbf{Z}(P)$ contains $O(|\mathcal{P}|/D^d)$ points of \mathcal{P} .

Remark 20. Without loss of generality, we can assume that P is square-free. Indeed if P is not square-free then we can replace P by its square-free part, and the new polynomial still has all of the desired properties.

After applying Lemma 14, we can ensure that the irreducible components of P generate real ideals:

Corollary 21. Let \mathcal{P} be a collection of points in \mathbb{R}^4 , and let $D > 0$. Then there exists a non-zero polynomial P of degree at most D such that each connected component of $\mathbb{R}^4 \setminus \mathbf{Z}(P)$ contains at most $O(|\mathcal{P}|/D^4)$ points of \mathcal{P} , and each irreducible component of P generates a real ideal.

Example 22. Consider the set of 24 points

$$\mathcal{P}_1 = \{(0, \pm 1, \pm 1), (0, \pm 2, \pm 2), (\pm 1, \pm 1, \pm 1), (\pm 2, \pm 2, \pm 2)\} \subset \mathbb{R}^3,$$

and let $D = 3$. Then the polynomial $P_1(x_1, x_2, x_3) = x_1 x_2 x_3$ partitions \mathbb{R}^3 into 8 octants, each of which contains 2 points from \mathcal{P}_1 .

Remark 23. Note as well that in the above example, the 8 points $\{0, \pm 1, \pm 1\}$, $\{0, \pm 2, \pm 2\}$ lie on the set $\mathbf{Z}(P_1)$ and thus they do not lie inside any of the open components of $\mathbb{R}^3 \setminus \mathbf{Z}(P_1)$. This is not merely a consequence of us choosing P_1 poorly; it is an unavoidable phenomena that occurs when performing the discrete polynomial partitioning decomposition. In order to control the number of incidences between points lying on $\mathbf{Z}(P_1)$ and surfaces in \mathcal{S} , we shall have to perform a second polynomial partitioning decomposition “on” the surface $\mathbf{Z}(P_1)$. For technical reasons, we cannot simply consider the complement of the zero set of our second partitioning polynomial as a union of relatively open subsets of $\mathbf{Z}(P_1)$. Instead, we need to perform a somewhat more detailed decomposition that partitions $\mathbf{Z}(P_1)$ into sets that are realizations of realizable strict sign conditions of a certain family of polynomials. This is made precise in the theorem below.

Theorem 24 (Discrete polynomial partitioning on a hypersurface). *Let \mathcal{P} be a collection of points in \mathbb{R}^d lying on the set $Z = \mathbf{Z}(P)$ for P an irreducible polynomial of degree D such that P generates a real ideal. Let $\rho > 0$ be a small constant, and let $E \geq \rho D$. Then there exists a collection of polynomials $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_d]$ with the following properties:*

1. $|\mathcal{Q}| \leq \log_2(DE^{d-1}) + O(1)$.
2. $\sum_{Q \in \mathcal{Q}} \deg Q \lesssim E$.
3. None of the polynomials in \mathcal{Q} vanish identically on Z .
4. The realization of each of the $O(DE^{d-1})$ strict sign conditions of \mathcal{Q} on Z contains $O(\frac{|\mathcal{P}|}{DE^{d-1}})$ points of \mathcal{P} .
5. Each irreducible component of each polynomial $Q \in \mathcal{Q}$ generates a real ideal.

All implicit constants depend only on ρ and the dimension d .

In our applications, we will always have $d = 3$ or $d = 4$.

Example 25. Let us continue Example 22. The polynomial P_1 from Example 22 was not irreducible, but we can factor it into the three irreducible factors x_1, x_2, x_3 . All of the points lying on $\mathbf{Z}(P_1)$ actually lie on the irreducible component $\mathbf{Z}(x_1)$, so we let $P_2(x_1, x_2, x_3) = x_1$. Note that $(P_2) = (x_1)$ is a real ideal, and $D = \deg(P_2) = 1$. Select $E = 2$ (which is larger than D). Then the collection of polynomials $\mathcal{Q} = \{x_2, x_3\}$ satisfies the requirements of Theorem 24. The realizations of realizable strict sign conditions of \mathcal{Q} on Z are the 4 sets of the form

$$\{(x_1, x_2, x_3) : x_1 = 0, \pm x_2 > 0, \pm x_3 > 0\}. \quad (2.10)$$

Note that each of these sets contains 2 points of $\mathcal{P}_1 \cap \mathbf{Z}(P_2)$. Two coincidences occur in this example that are not present in general. First, in this example the realizations of the four strict sign conditions of \mathcal{Q} on Z correspond to the four connected components of $Z \setminus \bigcup_{Q \in \mathcal{Q}} \mathbf{Z}(Q)$. In general, each realization of a strict sign condition may be a union of multiple connected

components of $Z \setminus \bigcup_{\mathcal{Q}} \mathbf{Z}(Q)$. Second, each of the polynomials in \mathcal{Q} were irreducible factors of P_1 . In general this does not occur.

The proof of Theorem 24 will be similar to the original proof of the discrete polynomial ham sandwich theorem in [Gut10, §4], which can be stated as follows:

Proposition 26 (Discrete polynomial ham sandwich theorem). Let $V \subset \mathbb{R}[x_1, \dots, x_d]$ be a vector space of dimension ℓ , and let $F_1, \dots, F_\ell \subset \mathbb{R}^d$ be finite families of points. Then there exists a polynomial $P \in V$ such that

$$\begin{aligned} |F_j \cap \{x \in \mathbb{R}^d : P(x) > 0\}| &\leq |F_j|/2, \text{ and} \\ |F_j \cap \{x \in \mathbb{R}^d : P(x) < 0\}| &\leq |F_j|/2, \quad j = 1, \dots, \ell. \end{aligned}$$

Proposition 26 is proved in [Gut10] only in the special case where V is the vector space of all polynomials of degree at most e (where e is chosen large enough to ensure that V has the required dimension). However, the proof carries over verbatim to the general case where V is arbitrary. To prove Theorem 24, we will iterate the following lemma:

Lemma 27. *Let $Z = \mathbf{Z}(P) \subset \mathbb{R}^d$ for P an irreducible polynomial of degree D such that (P) is a real ideal. Let $E > 0$, and let F_1, \dots, F_ℓ , $\ell = c \min(E^d, DE^{d-1})$ be finite families of points in \mathbb{R}^d , with $F_j \subset Z$ for each j . Then provided c is sufficiently small (depending only on d), there exists a polynomial Q of degree at most E that does not vanish identically on $\mathbf{Z}(P)$ such that*

$$\begin{aligned} |F_j \cap \{x \in \mathbb{R}^d : Q(x) > 0\}| &\leq |F_j|/2, \text{ and} \\ |F_j \cap \{x \in \mathbb{R}^d : Q(x) < 0\}| &\leq |F_j|/2, \quad j = 1, \dots, \ell. \end{aligned} \tag{2.11}$$

Proof. Let $\mathbb{R}[x_1, \dots, x_d]_{\leq E}$ be the vector space of all polynomials in d variables of degree at most E , and let $(P)_{\leq E}$ be the vector space of all polynomials in the ideal (P) that have degree at most E (of course, if $E < \deg P$ then $(P)_{\leq E} = 0$). We have

$$\dim(\mathbb{R}[x_1, \dots, x_d]_{\leq E}) - \dim((P)_{\leq E}) > c \min(E^d, DE^{d-1})$$

for some (explicit) constant c depending only on the dimension d . Thus, we can find a vector space $V \subset \mathbb{R}[x_1, \dots, x_d]_{\leq E}$ with $\dim(V) > c \min(E^d, DE^{d-1})$ such that $V \cap (P)_{\leq E} = 0$. By

Proposition 26, we can find a polynomial $Q \in V$ satisfying (2.11). Since $Q \in \mathbb{R}[x_1, \dots, x_d]_{\leq E}$ but $Q \notin (P)_{\leq E}$, we have $Q \notin (P)$. Since P is irreducible and generates a real ideal, by Item 2 of Proposition 13, Q does not vanish identically on $\mathbf{Z}(P)$. \square

Proof of Theorem 24. Let $\mathcal{Q}_0 = \{1\}$. For each $i = 1, \dots, t$, with

$$t = \lceil \log_2(DE^{d-1}) \rceil, \quad (2.12)$$

use Lemma 27 to find a polynomial Q_i with

$$\deg(Q_i) \lesssim \max\left((2^i/D)^{1/(d-1)}, 2^{i/d}\right)$$

(the implicit constant depends only on d) such that for each $\sigma \in \Sigma_{\mathcal{Q}_{i-1}}$ we have

$$\begin{aligned} \left| \{x \in \mathbb{R}^d : Q_i(x) > 0\} \cap (\mathcal{P} \cap \text{Reali}(\sigma, \mathcal{Q}_{i-1})) \right| &\leq \frac{1}{2} |\mathcal{P} \cap \text{Reali}(\sigma, \mathcal{Q}_{i-1})|, \\ \left| \{x \in \mathbb{R}^d : Q_i(x) < 0\} \cap (\mathcal{P} \cap \text{Reali}(\sigma, \mathcal{Q}_{i-1})) \right| &\leq \frac{1}{2} |\mathcal{P} \cap \text{Reali}(\sigma, \mathcal{Q}_{i-1})|. \end{aligned} \quad (2.13)$$

Some of the above sets may be empty, but this does not pose a problem. Let $\mathcal{Q}_i = \mathcal{Q}_{i-1} \cup \{Q_i\}$.

None of the polynomials in $\mathcal{Q} = \mathcal{Q}_t$ vanish on P , so Item 3 of the theorem is satisfied.

Since $E \geq D$ we have

$$\begin{aligned} \sum_{\mathcal{Q}} \deg Q &\lesssim \sum_{i=1}^t (2^i/D)^{1/(d-1)} + \sum_{i=1}^t 2^{i/d} \\ &\lesssim (DE^{d-1}/D)^{1/(d-1)} + (DE^{d-1})^{1/d} \\ &\lesssim E, \end{aligned}$$

which satisfies Item 2. By (2.13), for each $\sigma \in \Sigma_{\mathcal{Q}}$,

$$\begin{aligned} |\mathcal{P} \cap \text{Reali}(\sigma, \mathcal{Q})| &\lesssim 2^{-t} |\mathcal{P}| \\ &\lesssim \frac{|\mathcal{P}|}{DE^{d-1}}, \end{aligned} \quad (2.14)$$

which satisfies Item 4. Finally, Item 1 follows from (2.12). \square

2.5 Intersection theory

Definition 28. Let $\gamma \in \mathbb{R}^d$ be an algebraic curve, and let $x \in \mathbb{R}^d$. Then we define the *geometric multiplicity* of the curve γ at the point x , $\text{mult}_g(\gamma, x)$, as follows. Let γ' be a generic projection of γ onto \mathbb{R}^2 , let x' be the image of x under the same projection, and let g be the square-free polynomial with $\mathbf{Z}(g) = \gamma'$. Then $\text{mult}(\gamma, x)$ is the order of vanishing of g at x' . In particular, if $x \notin \gamma$ then $\text{mult}(\gamma, x) = 0$, while if x is a smooth point of γ , then $\text{mult}(\gamma, x) = 1$.

Definition 29. An ideal $I \subset \mathbb{C}[x_1, \dots, x_d]$ is said to be *radical* if $I = \mathbf{I}(\mathbf{Z}(I))$.

Proposition 30. if $I = (f_1, \dots, f_\ell) \subset \mathbb{C}[x_1, \dots, x_d]$ is a complete intersection, then I is radical

if and only if $\begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_\ell \end{pmatrix}$ has full rank at every smooth point of Z .

Proposition 31. If $f_1, \dots, f_\ell \in \mathbb{C}[x_1, \dots, x_d]$ with $\ell \leq d$, then for any $\epsilon > 0$, we can find a set of numbers $0 < \epsilon_j < \epsilon$, $j = 1, \dots, \ell$ so that $(f_1 + \epsilon_1, \dots, f_\ell + \epsilon_\ell)$ is a radical ideal.

We will need several elementary results from intersection theory. Further details can be found in standard textbooks such as [Ful98, Har83].

We shall frequently make use of the embedding $\mathbb{C}^d \rightarrow \mathbb{C}\mathbf{P}^d$, $(x_1, \dots, x_d) \rightarrow [x_1 : \dots : x_d : 1]$. This embedding allows us to identify points in \mathbb{C}^d with those in $\mathbb{C}\mathbf{P}^d$, and to identify (complex, affine) polynomials with complex homogeneous polynomials.

Definition 32. If $f \in \mathbb{C}[x_1, \dots, x_d]$, we will let I_f denote the projective ideal generated by f (here as elsewhere, f is identified with its corresponding homogeneous polynomial). If $f \in \mathbb{C}[x_1, \dots, x_d]$, let $\mathbf{Z}^*(f)$ be the zero set of f (either in \mathbb{C}^d or in $\mathbb{C}\mathbf{P}^d$, depending on context). If $x \in \mathbb{C}^d$, then $\mathcal{O}_{\mathbb{C}\mathbf{P}^d, x}$ is the local ring obtained by localizing $\mathbb{C}\mathbf{P}^d$ at the point x (again, we have identified x with its image in $\mathbb{C}\mathbf{P}^d$). If f_1, f_2, \dots, f_k are polynomials, we say that f_1, \dots, f_k intersect properly if $\text{codim}(\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)) = k$.

If f_1, \dots, f_k intersect properly, and V is an irreducible variety contained in $\mathbf{Z}^*(f_1) \cap \dots \cap$

$\mathbf{Z}^*(f_k)$, then we define

$$\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V) = \dim \mathcal{O}_{\mathbb{C}\mathbf{P}^d, x} / (I_{f_1} + \dots + I_{f_j}),$$

where x is a generic point of V .

This definition of multiplicity has several useful properties.

1. If V_1, V_2 are irreducible varieties, V_2 is a component of $\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)$, $V_1 \subset V_2$, and $V_1 \not\subset (V_2)_{\text{sing}}$, then

$$\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V_2) = \text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V_1)$$

2. Let V be a codimension- k irreducible variety contained in $\mathbf{Z}^*(f_1) \cap \dots \cap \mathbf{Z}^*(f_k)$. Then if x is a generic point of V and H is a generic hyperplane of codimension k such that $x \in H$. Then we can find a small ball $B \subset \mathbb{C}^d$ containing x such that x is the only point in $B \cap H \cap V$. Now, let $f'_1 = f_1 + \epsilon_1, \dots, f'_k = f_k + \epsilon_k$, where $\epsilon_1, \dots, \epsilon_k$ are generic real numbers with sufficiently small magnitude. Then $B \cap H \cap \mathbf{Z}^*(f'_1) \cap \dots \cap \mathbf{Z}^*(f'_k)$ is a union of $\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V)$ points. Informally, if B is a small ball centered around a generic point of V , then if we perturb f_1, \dots, f_k , the intersection “splits” into $\text{mult}(\mathbf{Z}^*(f_1), \dots, \mathbf{Z}^*(f_k); V)$ sheets.

2.6 Differential geometry

Definition 33. Let $\text{Gr}(d, k; \mathbb{C})$ be the Grassmanian of (complex) k -dimensional subspaces of \mathbb{C}^d . Note that this is a smooth manifold.

Definition 34. Let $M \subset \mathbb{C}^d$ be a smooth manifold of dimension k . Then we define the k -dimensional Grassmannian bundle over M as

$$G(M, k) = \{(z, T_z(M)) : z \in M\}. \quad (2.15)$$

Then $G(M, k)$ is a smooth k -dimensional sub-manifold of $\mathbb{C}^d \times \text{Gr}(d, k; \mathbb{C})$ (here a k -dimensional subspace $\Pi \in \text{Gr}(d, k; \mathbb{C})$ is considered a 0-dimensional manifold, not a k -dimensional manifold, since it is a “point” in $\Pi \in \text{Gr}(d, k; \mathbb{C})$.)

Definition 35. If M_1, M_2 are smooth sub-manifolds of a smooth manifold M , and $\dim(M_1) + \dim(M_2) = \dim M$, we say that M_1, M_2 intersect transversely at $x \in M_1 \cap M_2$ if $T_x(M_1) + T_x(M_2) = T_x(M)$. We say that M_1, M_2 intersect transversely if they intersect transversely at every point $x \in M_1 \cap M_2$.

Definition 36. Let $M \subset \mathbb{R}^d$ be a smooth manifold that is contained in a compact set. We say that M' is an ϵ -perturbation of M if there exists a diffeomorphism $\psi: \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\psi(M) = M'$, and $\|\psi - I\|_{C^1} < \epsilon$. If $M \subset \mathbb{C}^d$, we define an ϵ -perturbation similarly.

Proposition 37. Let M_1, M_2 be smooth sub-manifolds of a smooth manifold $M \subset \mathbb{R}^d$ or $M \subset \mathbb{C}^d$. Suppose that $\dim M_1 + \dim M_2 = \dim M$, and that M is contained in a compact set. Then if M_1, M_2 intersect transversely, there exists an $\epsilon_0 > 0$ such for all smooth sub-manifolds $M'_1, M'_2 \subset M$, if M'_1 is an ϵ -perturbation of M_1 and M'_2 is an ϵ -perturbation of M_2 , then $|M_1 \cap M_2| = |M'_1 \cap M'_2|$. Note that since M is contained in a compact set and M_1, M_2 are transverse and have complimentary dimension (in M), the intersections have finite cardinality.

Proposition 38. Let $(f_1, f_2) \subset \mathbb{C}[x_1, x_2, x_3, x_4]$ be a (scheme-theoretic) complete intersection, and let $B \subset \mathbb{C}^4$ be a ball. Suppose that $Z(f_1) \cap Z(f_2) \cap B$ is a smooth 2 (complex) dimensional manifold. Then for every $\epsilon > 0$, there exists a $\delta > 0$ so that if $0 < \delta_1, \delta_2 < \delta$, then $Z(f_1 + \delta_1) \cap Z(f_2 + \delta_2) \cap B$ is an ϵ -perturbation of $Z(f_1) \cap Z(f_2) \cap B$.

CHAPTER 3

An improved bound on the number of point-surface incidences in three dimensions

3.1 Introduction

In [CEG90], Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl obtained the following bound on the number of incidences between points and spheres in \mathbb{R}^3 :

Theorem 39 (Clarkson et al.). *The number of incidences between m points and n spheres in \mathbb{R}^3 with no three spheres meeting at a common circle is*

$$O((mn)^{3/4}\beta(m,n) + m + n), \quad (3.1)$$

where $\beta(m,n)$ is a very slowly growing function of m and n . In particular, $\beta(m,n) \leq 2^{C\alpha(m^3/n)^2}$, where $\alpha(s)$ is the inverse Ackerman function and C is a large constant.

We obtain the following slight sharpening:

Theorem 40. *Let $k \geq 3$, and let $\mathcal{P} \subset \mathbb{R}^3$ be a collection of m points and \mathcal{S} a collection of n smooth algebraic surfaces of bounded degree (the degree is allowed to depend on k) such that for some constant C we have $|S \cap S' \cap S''| \leq C$ for all $S, S', S'' \in \mathcal{S}$, and for any collection of k points in \mathbb{R}^3 , there are at most C surfaces that contain all k points. Then the number of incidences between points in \mathcal{P} and surfaces in \mathcal{S} is*

$$O(m^{\frac{2k}{3k-1}} n^{\frac{3k-3}{3k-1}} + m + n), \quad (3.2)$$

where the implicit constant depends only on k , C , and the degree of the algebraic surfaces.

In particular, the number of incidences between m points and n spheres in \mathbb{R}^3 with no three spheres meeting at a common circle is

$$O((mn)^{3/4} + m + n). \quad (3.3)$$

Remark 41. The requirement that every three surfaces meet in a complete intersection, or some variant thereof, is necessary to prevent the situation in which all of the surfaces meet in a common curve and all of the points lie on that curve, yielding mn incidences (i.e. if we don't place any restrictions on how the surfaces can intersect, then the trivial bound of mn incidences is sharp).

Remark 42. When $k = 2$ and $n = m$, the following example shows that Theorem 40 is sharp. Let \mathcal{P} be the set $[-2k, 2k]^2 \times [0, 2k^2]$, and let

$$\mathcal{S} = \{z = (x - x_0)^2 + (y - y_0)^2 + z_0 : x_0, y_0 = -k, \dots, k; z_0 = 0, \dots, k^2\}.$$

Then $|\mathcal{P}| = 32k^4$, $|\mathcal{S}| = k^4$, and we can verify that for every triple S, S', S'' of surfaces in \mathcal{S} , we have $|S \cap S' \cap S''| \leq 8$, and for every three points of \mathcal{P} , there are at most four surfaces from \mathcal{S} that contain all three. Since each $S \in \mathcal{S}$ hits $\geq k^2$ points from \mathcal{P} . Thus there are $\geq k^6$ incidences total.

Remark 43. The requirements that every three surfaces meet in C points and that every k points have at most C surfaces passing through them are analogous to the definition of “curves with k degrees of freedom” from [PS98], though in [PS98] the curves do not need to be algebraic.

Remark 44. Theorem 39 can be extended to the more general case of bounded degree algebraic surfaces using the decomposition techniques described in [SA96, §8.3] to obtain an analogue of (3.2). Doing so yields a bound of

$$O(m^{\frac{2k}{3k-1}} n^{\frac{3k-3}{3k-1}} \beta(m, n) + m + n),$$

where β a slowly growing function.

3.1.1 Previous results

Concurrently with (and independently of) this work, Kaplan, Matoušek, Safernová, and Sharir in [KMS12a] obtained results similar to the bound (3.3) using similar methods. Kaplan et. al. are able to avoid some of the technical difficulties present in this paper by using an explicit parameterization of the sphere by rational functions.

Similar results to Theorem 3.1 and 3.2 have been obtained by Łaba and Solymosi in [IS07] and by Iosevich, Jorati, and Łaba in [IJL09]. In [IS07] and [IJL09], however, the authors consider a more general class of surfaces (they need not be algebraic), but they require that the point set be “homogeneous” in a suitable sense.

Our techniques do not work well when $k = 2$, i.e. for obtaining bounds on point-hyperplane incidences, but this case has been studied by other authors (see e.g. [ET05], where the authors obtain sharp bounds on point-hyperplane incidences under a slightly different set of non-degeneracy conditions).

3.1.2 Proof sketch

Clarkson et al. obtain Theorem 39 through their “Canham threshold plus divide and conquer” technique: the arrangement of spheres in \mathbb{R}^3 is subdivided into smaller collections through a careful partitioning of \mathbb{R}^3 , and the number of incidences between these smaller collections of spheres and points is controlled by a Turan-type bound on the number of edges in a bipartite graph with certain forbidden subgraphs.

In this paper, we employ similar ideas, except instead of dividing the problem into smaller subproblems by partitioning \mathbb{R}^3 into cells using a decomposition adapted to the collection of spheres (or more general nonsingular algebraic surfaces), we employ a partition adapted to the collection of points. This partition is obtained from the discrete polynomial ham sandwich theorem recently used to great effect by Guth and Katz in [Gut10] and more recently by Solymosi and Tao in [ST12] and by Kaplan, Matoušek, and Sharir in [KMS12b]. Specifically, we find a polynomial P such that the complement of the zero set of P consists

of open “cells,” none of which contain too many points. We can then apply a Turan-type bound to the points and surfaces inside each cell. However, some points may lie on the zero set of P , and thus do not lie in any of the cells. To deal with these points, we perform a second polynomial ham sandwich decomposition to find a polynomial Q whose zero set partitions the zero set of P into cell-like objects, and we apply the Turan-type bound to each of these “cells.” While it is possible that a point could lie in the zero set of both P and Q , we can use Bézout-type theorems to control how often this can occur.

3.1.3 Some difficulties with real algebraic sets

There are several technical difficulties that have to be dealt with while executing the above strategy. In contrast to the situation over \mathbb{C} , there exist polynomials $P_1, \dots, P_d \in \mathbb{R}[x_1, \dots, x_d]$ of degrees D_1, \dots, D_d such that $\{P_1 = 0\} \cap \dots \cap \{P_d = 0\}$ contains more than $D_1 \dots D_d$ isolated points, i.e. the naïve analogue of Bézout’s theorem fails over \mathbb{R} . To deal with this problem, we will sometimes be forced to embed our varieties into \mathbb{C} and use the (usual) Bézout’s theorem (though we have to be careful that the intersection of the embedded varieties does not contain new, unexpected components of positive dimension).

A second difficulty concerns the failure of the Nullstellensatz for varieties defined over \mathbb{R} . In contrast to the complex case, if (P) is a principal prime ideal and Q is a real polynomial, it need not be the case that if Q vanishes identically on $\{x \in \mathbb{R}^d : P = 0\}$ then $Q \in (P)$. Luckily, there is a special type of ideal known as a “real ideal” for which an analogue of the Nullstellensatz does hold. Frequently we will be required to replace our polynomials with new polynomials that generate real ideals.

Finally, if $P \in \mathbb{R}[x_1, \dots, x_d]$ then the dimension of $\{x \in \mathbb{R}^d : P = 0\}$ may be less than $d - 1$, and even if P is square-free, ∇P may vanish on $\{P = 0\}$. Again, we can remedy this problem by working with (irreducible) polynomials that generate real ideals.

3.2 Main Result

We are now ready to prove Theorem 40. For the reader's convenience, we will restate the theorem below.

Theorem 40. Let $k \geq 3$, and let $\mathcal{P} \subset \mathbb{R}^3$ be a collection of m points and \mathcal{S} a collection of n smooth algebraic surfaces of bounded degree (the degree is allowed to depend on k) such that for some constant C we have $|S \cap S' \cap S''| \leq C$ for all $S, S', S'' \in \mathcal{S}$, and for any collection of k points in \mathbb{R}^3 , there are at most C surfaces that contain all k points. Then the number of incidences between points in \mathcal{P} and surfaces in \mathcal{S} is

$$O(m^{\frac{2k}{3k-1}} n^{\frac{3k-3}{3k-1}} + m + n), \quad (3.2)$$

where the implicit constant depends only on k , C , and the degree of the algebraic surfaces.

Proof. We will begin with a few definitions that will be useful throughout the proof.

Definition 45. If $\tilde{\mathcal{S}}$ is a collection of smooth (real) surfaces and $\tilde{\mathcal{P}}$ a collection of points, let $\mathcal{I}(\tilde{\mathcal{P}}, \tilde{\mathcal{S}})$ be the number of incidences between the surfaces in $\tilde{\mathcal{S}}$ and the points in $\tilde{\mathcal{P}}$. If $S \in \tilde{\mathcal{S}}$ is a surface, then f_S is the polynomial whose zero set is S .

In our case, we have that $|S \cap S' \cap S''| \leq C$ for every three surfaces S, S', S'' , and any k points have at most C surfaces passing through all of them. Thus we have the bounds

$$\mathcal{I}(\mathcal{P}, \mathcal{S}) \lesssim |\mathcal{P}| |\mathcal{S}|^{1-1/k} + |\mathcal{S}|, \quad (3.4)$$

$$\mathcal{I}(\mathcal{P}, \mathcal{S}) \lesssim |\mathcal{P}|^{2/3} |\mathcal{S}| + |\mathcal{P}|. \quad (3.5)$$

From (3.4) and (3.5), we have that if $n > cm^k$ or $m > cn^3$ for some fixed small constant $c > 0$ to be specified later, then Theorem 40 immediately holds. Thus we may assume

$$\begin{aligned} n &< cm^k, \\ m &< cn^3. \end{aligned} \quad (3.6)$$

We may also assume that the surfaces in \mathcal{S} are irreducible varieties. Indeed, if this was not the case then we could let \mathcal{S}' be the set of all irreducible components of surfaces in \mathcal{S} . We have $|\mathcal{S}'| \lesssim |\mathcal{S}|$, and the surfaces in \mathcal{S}' satisfy the same bounds as the surfaces in \mathcal{S} . We could then run our arguments below with \mathcal{S}' in place of \mathcal{S} .

Let P be a square-free polynomial of degree at most D (D will be determined later, but the impatient reader can jump to (3.25)) that cuts \mathbb{R}^3 into $O(D^3)$ cells with $O(m/D^3)$ points in each cell, and let $Z = \mathbf{Z}(P)$. Let m_i be the number of points lying in the i -th cell of the above decomposition, and let n_i be the number of surfaces that meet the interior of the i -th cell.

Lemma 46.

$$\sum n_i \lesssim D^2 n. \quad (3.7)$$

Proof. Let S be a surface that is not contained in Z . Since there are finitely many cells, we can select a large closed ball $B \subset \mathbb{R}^3$ so that the number of cells that meet S is equal to the number of cells that meet $S \cap B$. We can apply a small generic translation to S , and doing so can only increase the number of cells that meet $S \cap B$ (and thus can only increase the number of cells that meet S). Select a generic vector $v \in \mathbb{R}^3$ and let $T(x) = v \wedge \nabla f_S(x) \wedge \nabla P(x)$, so if $x \in S \cap Z$ and $\nabla f_S(x)$ and $\nabla P(x)$ are non-zero and non-collinear, then $T(x) = 0$ if the curve $S \cap Z$ is tangent at x to a plane with normal vector v .

For every cell Ω that meets S , either Ω contains an entire connected component of S (since S has bounded degree, at most $O(1)$ cells can contain an entire connected component of S), or there is a point $x \in \partial\Omega \cap S$ satisfying the following properties.

1. x is a smooth point of the space curve $Z \cap S$.
2. x is a non-singular intersection point of $\mathbf{Z}(T) \cap Z \cap S$.
3. x is a smooth point of $\partial\Omega$.

These three properties follow from the fact that v is generic and we picked a generic translation of S . From Item 3, each point x satisfying the above properties can be associated to at

most two distinct cells. By Item 2 and the real Bézout inequality (see e.g. [BPR06, §4.7]), there can be at most $\deg(P) \deg(T) \deg(f_s) = O(D^2)$ such points, and thus S can enter at most $O(D^2)$ such cells. Since there are n surfaces in \mathcal{S} , the result follows. \square

Using Lemma 46 and the bound from (3.4) we can control the number of incidences between points not lying in Z and surfaces in \mathcal{S} :

$$\begin{aligned}
\mathcal{I}(\mathcal{P} \setminus Z, \mathcal{S}) &= \sum_i \mathcal{I}(\mathcal{P} \cap \Omega_i, \mathcal{S}) \\
&\lesssim \sum_i m_i n_i^{1-1/k} + n_i \\
&\lesssim \left(\sum_i m_i^k \right)^{1/k} \left(\sum_i n_i \right)^{1-1/k} + D^2 n \\
&\lesssim \left(D^3 \frac{m^k}{D^{3k}} \right)^{1/k} (D^2 n)^{1-1/k} + D^2 n \\
&\lesssim \frac{mn^{1-1/k}}{D^{1-1/k}} + D^2 n.
\end{aligned} \tag{3.8}$$

We must now control $\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S})$. We have

$$\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}) = \mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_1) + \mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_2), \tag{3.9}$$

where \mathcal{S}_1 is the set of surfaces contained in Z , and \mathcal{S}_2 are the remaining surfaces. Since Z has degree D , Z can contain at most D surfaces from \mathcal{S} , i.e. $|\mathcal{S}_1| \leq D$. By (3.5),

$$\begin{aligned}
\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_1) &\lesssim |\mathcal{S}_1| |\mathcal{P}|^{2/3} + |\mathcal{P}| \\
&\lesssim Dm^{2/3} + m.
\end{aligned} \tag{3.10}$$

Thus it remains to control $\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_2)$. Write $P = P_1 \dots P_\ell$, where each P_j is irreducible of degree D_j , and let $Z_j = \mathbf{Z}(P_j)$. Thus we have $D_1 + \dots + D_\ell \leq D$, and $Z = \bigcup Z_j$. We would like to use Lemma 24 to perform a second discrete polynomial ham sandwich decomposition on each variety Z_j , but if (P_j) is not a real ideal then we cannot apply the lemma. Luckily, the following lemma lets us remedy this situation.

Lemma 47. *Let $\mathcal{A} \subset \mathbb{R}[x_1, \dots, x_d]$ be a collection of irreducible polynomials. Then we can find a new collection \mathcal{A}' of irreducible polynomials such that:*

1. $\bigcup_{P \in \mathcal{A}} \mathbf{Z}(P) \subset \bigcup_{P \in \mathcal{A}'} \mathbf{Z}(P)$.
2. $\sum_{P \in \mathcal{A}} \deg P \leq \sum_{P \in \mathcal{A}'} \deg P$.
3. (P) is a real ideal for each $P \in \mathcal{A}'$.

Proof. We shall proceed by induction on $\sum_{P \in \mathcal{A}} \deg P$. If the sum is 1 then the result is trivial since in that case \mathcal{A} consists of a single linear polynomial, so we can let $\mathcal{A}' = \mathcal{A}$. Suppose the lemma has been established for all families $\tilde{\mathcal{A}}$ with $\sum_{P \in \tilde{\mathcal{A}}} \deg P < w$, and let $\sum_{P \in \mathcal{A}} \deg P = w$. If (P) is a real ideal for every $P \in \mathcal{A}$ then the result is immediate. If not, select $P_0 \in \mathcal{A}$ such that (P_0) is not a real ideal. By Proposition 13 in Chapter 2, ∇P_0 vanishes on $\mathbf{Z}(P_0)$. Let $v \in \mathbb{R}^d$ be a generic unit vector. Then $\mathbf{Z}(P_0) \subset \mathbf{Z}(\nabla_v P_0)$ and $\deg(\nabla_v P_0) < \deg P_0$. Write $\nabla_v P_0 = Q_1 \dots Q_a$ as a product of irreducible components, and let $\tilde{\mathcal{A}} = \mathcal{A} \setminus \{P_0\} \cup \{Q_1, \dots, Q_a\}$. We have $\sum_{P \in \tilde{\mathcal{A}}} \deg P < \sum_{P \in \mathcal{A}} \deg P = w$, and $\bigcup_{P \in \mathcal{A}} \mathbf{Z}(P) \subset \bigcup_{P \in \tilde{\mathcal{A}}} \mathbf{Z}(P)$. Apply the induction hypothesis to $\tilde{\mathcal{A}}$ to obtain a family $\tilde{\mathcal{A}}'$ satisfying Properties 1–3 with $\tilde{\mathcal{A}}$ in place of \mathcal{A} . We can verify that $\tilde{\mathcal{A}}'$ has the desired properties. \square

After applying Lemma 47, we can assume that each irreducible polynomial P_j in the factorization of P generates a real ideal. Write $\mathcal{P} \cap Z = \bigsqcup \mathcal{P}_j$, where \mathcal{P}_j consists of those points lying in Z_j . If a point lies on two or more such varieties, place it into only one of the sets. We need to distinguish between several cases. Let

$$\begin{aligned}
\mathcal{A}_1 &= \{j : |\mathcal{P}_j|^k < D_j^k n\}, \\
\mathcal{A}_2 &= \{j : D_j^k n \leq |\mathcal{P}_j|^k < D_j^{3k-1} n\}, \\
\mathcal{A}_3 &= \{j : |\mathcal{P}_j|^k \geq D_j^{3k-1} n\}.
\end{aligned} \tag{3.11}$$

For each $j \in \mathcal{A}_1$ we have

$$\begin{aligned}
\mathcal{I}(\mathcal{P}_j, \mathcal{S}_2) &\lesssim |\mathcal{P}_j| n^{1-1/k} + n \\
&\lesssim D_j n,
\end{aligned} \tag{3.12}$$

where the second inequality uses the assumption $|\mathcal{P}_j| < D_j n^{1/k}$. Summing (3.12) over all $j \in \mathcal{A}_1$, we obtain

$$\begin{aligned} \mathcal{I}\left(\bigcup_{j \in \mathcal{A}_1} \mathcal{P}_j, \mathcal{S}_2\right) &\lesssim \sum_{\mathcal{A}_1} D_j n \\ &\leq Dn. \end{aligned} \tag{3.13}$$

Now we must control the incidences between surfaces and points lying on varieties Z_j , $j \in \mathcal{A}_2$ or \mathcal{A}_3 . If $j \in \mathcal{A}_2$, use Theorem 19 to select a square-free polynomial Q_j of degree at most E_j ,

$$E_j = \left(\frac{|\mathcal{P}_j|^k}{nD_j^k}\right)^{1/(2k-1)}, \tag{3.14}$$

that cuts \mathbb{R}^3 into $O(E_j^3)$ cells, each of which contains $O(|\mathcal{P}_j|/E_j^3)$ points of \mathcal{P}_j . Recall that P_j is irreducible, (P_j) is real, and $j \in \mathcal{A}_2$ implies $\deg(Q_j) \leq E_j < \deg(P_j)$. Thus Q_j does not vanish identically on Z_j . Let $\mathcal{Q}_j = \{Q_j\}$ and let $W_j = \mathbf{Z}(Q_j)$.

If $j \in \mathcal{A}_3$, let E_j be as in (3.14) and use Theorem 24 (with $E = E_j$) to find a family \mathcal{Q}_j of polynomials satisfying properties 1–4 of the theorem. In particular, the realizations of the realizable strict sign conditions of \mathcal{Q}_j on Z_j partition Z_j into $O(D_j E_j^2)$ (not necessarily connected) sets, each of which contains $O(|\mathcal{P}_j|/D_j E_j^2)$ points, plus the “boundary” $Z_j \cap \bigcup_{\mathcal{Q}_j} \mathbf{Z}(Q)$. Define $W_j = \bigcup_{\mathcal{Q}_j} \mathbf{Z}(Q)$ (thus the definition of W_j depends on whether $j \in \mathcal{A}_2$ or $j \in \mathcal{A}_3$).

Regardless of whether $j \in \mathcal{A}_2$ or \mathcal{A}_3 , have

$$\mathcal{I}(\mathcal{P}_j, \mathcal{S}_2) = \mathcal{I}(\mathcal{P}_j \setminus W_j, \mathcal{S}_2) + \mathcal{I}(\mathcal{P}_j \cap W_j, \mathcal{S}_2). \tag{3.15}$$

We shall begin by bounding the first term of (3.15). If $j \in \mathcal{A}_2$, then through the same computation performed in (3.8) we have

$$\begin{aligned} \mathcal{I}(\mathcal{P}_j \setminus W_j, \mathcal{S}_2) &\lesssim \frac{|\mathcal{P}_j| n^{1-1/k}}{E_j^{1-1/k}} + nE_j^2 \\ &\leq \frac{|\mathcal{P}_j| n^{1-1/k}}{E_j^{1-1/k}} + nD_j E_j. \end{aligned} \tag{3.16}$$

If $j \in \mathcal{A}_3$, then let Ω_{ij} be the realization of the i -th realizable strict sign condition of \mathcal{Q}_j on Z_j . Recall that there are $O(D_j E_j^2)$ such realizable strict sign conditions. Let $m_{ij} = |\mathcal{P}_j \cap \Omega_{ij}|$, and let n_{ij} be the number of surfaces in \mathcal{S}_2 that intersect Ω_{ij} .

Lemma 48.

$$\sum_i n_{ij} \lesssim n D_j E_j. \quad (3.17)$$

Proof. If a surface $S \in \mathcal{S}_2$ lies in W_j then it does not contribute to the above sum, so we need only consider those surfaces that do not lie in Z_j or W_j . First, we can replace each $Q \in \mathcal{Q}$ by the polynomial $Q + \epsilon$ for $\epsilon > 0$ a sufficiently small constant. If $S \cap \{x \in \mathbb{R}^3 : Q(x) > 0\} \cap Z_j \neq \emptyset$, then there must be a point on $S \cap Z_j$ where Q is positive, so $S \cap \{x \in \mathbb{R}^3 : Q(x) + \epsilon > 0\} \cap Z_j \neq \emptyset$ for ϵ sufficiently small, and similarly for $S \cap \{x \in \mathbb{R}^3 : Q(x) < 0\} \cap Z_j$. Thus replacing each $Q \in \mathcal{Q}$ by $Q + \epsilon$ does not increase the number of realizations of realizable strict sign conditions that meet S . We shall select a small generic (with respect to S and Z_j) choice of ϵ .

By Corollary 56 in Appendix A below, we can assume that each irreducible component of each polynomial in \mathcal{Q}_j generates a real ideal.

Instead of counting $\sum_i n_{ij}$ directly, we shall bound the number of times a surface S enters a connected component of $Z_j \setminus W_j$, as this quantity controls $\sum_i n_{ij}$ (i.e. if the same surface enters multiple connected components of the same realization of a realizable strict sign condition then we will over-count, but this is acceptable). The proof is essentially topological.

Let $S \in \mathcal{S}_2$ with S not contained in W_j . As in Lemma 46, we can select a large closed ball B so that the number of connected components of $Z_j \setminus W_j$ that S enters is equal to the number of connected components that $S \cap B$ enters. Now, replace S by $S' = \mathbf{Z}((f_S + \epsilon)(f_S - \epsilon))$ for $\epsilon > 0$ a sufficiently small generic number. Provided ϵ is sufficiently small, if S meets a connected component Δ of $Z \setminus W_j$ then S' also meets Δ , since f_S is a continuous function on the (relatively) open set Δ , so f_S vanishes somewhere on Δ but does not vanish identically on Δ . Thus it suffices to count the number of times S' meets a connected component of

$Z_j \setminus W_j$. After replacing S by S' (and recalling that we applied a small generic perturbation to each $Q \in \mathcal{Q}$), every point in $Z_j \cap W_j \cap S'$ is a point of non-singular intersection.

Now, if S meets a connected component Δ of $Z_j \setminus W_j$, then one of the following two things must occur:

1. Δ contains (all of) a connected component of $S' \cap Z_j$.
2. $S' \cap \Delta$ contains a (topological) curve that meets the boundary of Δ at a point $x \in S' \cap Z_j \cap W_j$. Furthermore, there is at most one other connected component Δ' for which Item 2 holds for the same point x .

We will first bound the number of times Item 1 can occur by showing that $S' \cap Z_j$ contains $O(D_j^2) = O(D_j E_j)$ connected components. Apply a generic rotation to the coordinate axes, and consider the plane curve $\gamma = \mathbf{Z}(\text{res}_{x_3}(f_{S'}, P_j))$, where res_{x_3} is the resultant of $f_{S'}$ and P_j in the x_3 variable. Since neither of the (two) irreducible components of S' are contained in Z_j , γ is indeed a plane curve, and γ contains the image of the projection of $S' \cap Z_j$ in the x_3 direction. Thus, the number of connected components of $S' \cap Z_j$ is bounded by the number of connected components of γ plus the number of singular points of γ . Since γ has degree $O(D_j)$, both these quantities are $O(D_j^2)$ (this follows from Bézout's theorem in the plane and the Harnack curve theorem).

We will now bound the number of times Item 2 can occur. By the real Bézout's inequality, $S' \cap Z_j \cap W_j$ contains $O(D_j E_j)$ points of non-singular intersection, and thus Item 2 can occur at most $O(D_j E_j)$ times.

Thus S' can enter at most $O(D_j E_j)$ connected components of $Z_j \setminus W_j$. Since there are at most n surfaces, the result follows. □

Remark 49. A similar result to Lemma 48 can be obtained from the recent work of Barone and Basu in [BB12] and Solymosi and Tao in [ST12].

Using Lemma 48, we have

$$\begin{aligned}
\mathcal{I}(\mathcal{P}_j \setminus W_j, \mathcal{S}_2) &= \sum_i \mathcal{I}(\mathcal{P}_j \cap \Omega_{ij}, \mathcal{S}_2) \\
&\lesssim \sum_i m_{ij} n_{ij}^{1-1/k} + n_{ij} \\
&\leq \left(\sum_i m_{ij}^k \right)^{1/k} \left(\sum_i n_{ij} \right)^{1-1/k} + n_{ij} \\
&\lesssim \left(D_j E_j^2 \frac{|\mathcal{P}_j|^k}{(D_j E_j^2)^k} \right)^{1/k} (n D_j E_j)^{1-1/k} + n D_j E_j \\
&= \frac{|\mathcal{P}_j| n^{1-1/k}}{E_j^{1-1/k}} + n D_j E_j.
\end{aligned} \tag{3.18}$$

Our analysis of the second term of (3.15) will be the same regardless of whether $j \in \mathcal{A}_2$ or \mathcal{A}_3 . We shall express this bound as a lemma.

Lemma 50. *For $j \in \mathcal{A}_2 \cup \mathcal{A}_3$, let Z_j , W_j , \mathcal{P}_j , and \mathcal{S}_2 be as above. Then*

$$\mathcal{I}(\mathcal{P}_j \cap W_j, \mathcal{S}_2) \lesssim n D_j E_j + |\mathcal{P}_j|. \tag{3.19}$$

Proof. We shall write

$$\mathcal{I}(\mathcal{P}_j \cap W_j, \mathcal{S}_2) = \mathcal{I}_1(\mathcal{P}_j \cap W_j, \mathcal{S}_2) + \mathcal{I}_2(\mathcal{P}_j \cap W_j, \mathcal{S}_2), \tag{3.20}$$

where \mathcal{I}_1 counts those incidences between points $p \in \mathcal{P}_j \cap W_j$ and surfaces $S \in \mathcal{S}_2$ such that p^* lies on a 1 (complex) dimensional component of $S^* \cap Z_j^* \cap W_j^*$, and \mathcal{I}_2 counts the remaining incidences. To control \mathcal{I}_2 , note that by Bézout's inequality (over \mathbb{C}), for each $S \in \mathcal{S}_2$, $S^* \cap Z_j^* \cap W_j^*$ contains $O(D_j E_j)$ isolated points. Since $|\mathcal{S}_2| \leq n$ we obtain

$$\mathcal{I}_2(\mathcal{P}_j \cap W_j, \mathcal{S}_2) \lesssim n D_j E_j. \tag{3.21}$$

Thus it remains to control \mathcal{I}_1 . First, we shall replace \mathcal{Q}_j with a new family of polynomials $\tilde{\mathcal{Q}}_j$ with the following properties:

1. $Z_j \cap W_j \subset Z_j \cap \bigcup_{Q \in \tilde{\mathcal{Q}}_j} \mathbf{Z}(Q)$.
2. $\sum_{Q \in \tilde{\mathcal{Q}}_j} \deg Q \leq E_j$.

3. Each $Q \in \tilde{\mathcal{Q}}_j$ is irreducible.
4. For each $Q \in \tilde{\mathcal{Q}}_j$, every irreducible component of $Z_j^* \cap \mathbf{Z}(Q)^*$ that contains a real point has (complex) dimension 1.

The procedure will be similar to that in the proof of Lemma 47: For each $Q \in \mathcal{Q}_j$, write $Q = Q_1 \cdots Q_a$ as a product of irreducible factors. Discard those factors Q_b with $\mathbf{Z}(Q_b) \cap Z_j = \emptyset$. Of the remaining factors, place each irreducible factor that generates a real ideal in $\tilde{\mathcal{Q}}_j$. If Q_b is a factor that does not generate a real ideal then consider $\nabla_v Q_b$ for v a generic vector. By assumption, Q_b does not vanish identically on Z_j , but it does vanish on at least one point on Z_j . Thus Q_b is not constant on Z_j , so ∇Q_b does not vanish identically on Z_j and hence if v is a generic vector then $\nabla_v Q_b$ does not vanish identically on Z_j . Thus we can repeat the above procedure with $\nabla_v Q_b$ in place of Q_b . This process will eventually terminate, and the resulting collection of polynomials $\tilde{\mathcal{Q}}_j$ has the desired properties; Properties 1–3 are immediate. To obtain Property 4, suppose that for some $Q \in \tilde{\mathcal{Q}}_j$, $Z_j^* \cap \mathbf{Z}(Q)^*$ fails to be a complete intersection. Then there exists some variety Y that is an irreducible component of both Z_j^* and $\mathbf{Z}(Q)^*$. By Proposition 17 in Chapter 2, $\mathcal{R}(Y)$ is an irreducible component of Z_j and $\mathbf{Z}(Q)$, and thus either $\mathcal{R}(Y) = \emptyset$ or $\mathcal{R}(Y) = Z_j = \mathbf{Z}(Q)$. The latter is impossible since Z_j and $\mathbf{Z}(Q)$ have dimension 2, while $Z_j \cap \mathbf{Z}(Q)$ has dimension at most 1.

Let

$$\tilde{W}_j = \bigcup_{Q \in \tilde{\mathcal{Q}}_j} \mathbf{Z}(Q).$$

We can write

$$Z_j^* \cap \tilde{W}_j^* = \bigcup Y_j \tag{3.22}$$

as a union of irreducible (complex) varieties. By Property 4 above, we need only consider those components with (complex) dimension 1. We shall discard all components that have dimension 2. Let

$$\begin{aligned} \tilde{\mathcal{P}}_j = \{p \in \mathcal{P}_j : \text{there exists a (Euclidean) neighborhood } U \subset \mathbb{C}^3 \text{ of } p^* \text{ such} \\ \text{that } Z_j^* \cap \tilde{W}_j^* \cap U \text{ is a (topological) 1-complex-dimensional curve}\}. \end{aligned}$$

We shall establish several claims.

1. $Z_j^* \cap \tilde{W}_j^*$ is a union of $O(D_j E_j)$ irreducible varieties.
2. If $p \in \tilde{\mathcal{P}}_j$ then p^* lies on at most one of the irreducible components from (3.22).
3. Let Y be a variety from the above decomposition. If there exist three surfaces S_1, S_2, S_3 in \mathcal{S}_2 such that $Y \subset S_i^*$, $i = 1, 2, 3$, then $|\mathcal{P}_j \cap \mathfrak{R}(Y)| \leq C$.
4. If $S \in \mathcal{S}_2$, then there are $O(D_j E_j)$ points $p \notin \tilde{\mathcal{P}}_j$ such that p^* is contained in a 1-dimensional component of $S^* \cap Z_j^* \cap W_j^*$.

For Item 1, see e.g. [Ful98]. Item 2 follows from the assumption that every variety in the decomposition (3.22) has dimension 1. Item 3 follows from the requirement that any three surfaces intersect in at most C points. To obtain Item 4, suppose that $D_j \leq E_j$ (if not, we can interchange the roles of Z_j and W_j). Note that if p satisfies the requirements of Item 4, then $S^* \cap Z_j^* \cap W_j^*$ fails to be a complex (C^0) curve in a small neighborhood of p^* (i.e. in a small neighborhood of p^* , $S^* \cap Z_j^* \cap W_j^*$ is a union of several complex curves all passing through p^*), and thus $S^* \cap Z_j^*$ fails to be a complex (C^0) curve in a small neighborhood of p^* . Thus after a generic rotation of the coordinate axis, the image of p^* under the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ is a singular point of the (complex) plane curve $\mathbf{Z}(\text{res}_{x_3}(f_S, P_j))^*$, where res_{x_3} is the bivariate polynomial obtained by taking the resultant of f_S and P_j in the x_3 variable. This curve has degree $O(D_j)$ and thus has $O(D_j^2) = O(D_j E_j)$ singular points.

Now, for each $S \in \mathcal{S}_2$, at most $O(D_j E_j)$ points $p \in \mathcal{P}_j \setminus \tilde{\mathcal{P}}_j$ can contribute to $\mathcal{I}_1(\mathcal{P}_j, \{S\})$, so the total contribution from all surfaces in \mathcal{S}_2 is $O(nD_j E_j)$. To control the remaining incidences, use Item 3 to write $\{Y_j\} = \{Y_j'\} \sqcup \{Y_j''\}$, where the first set consists of varieties that are contained in at most 2 surfaces $S \in \mathcal{S}_2$, and the second consists of varieties that contain at most C points. Each point $p \in \tilde{\mathcal{P}}_j$ with $p^* \in \bigcup Y_j'$ can be incident to at most two surfaces, so the total contribution from such points is $O(|\mathcal{P}_j|)$. On the other hand, by Item 1 at most $O(D_j E_j)$ points can be contained in $\mathfrak{R}(\bigcup Y_j'')$, so these points can contribute at most $O(nD_j E_j)$ incidences. \square

Combining (3.16), (3.18), and (3.19) and optimizing in E_j , we see that our choice of E_j from (3.14) yields the bound

$$\mathcal{I}(\mathcal{P}_j, \mathcal{S}_2) \lesssim |\mathcal{P}_j|^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D_j^{\frac{k-1}{2k-1}} + m_j. \quad (3.23)$$

Summing (3.23) over all $j \in \mathcal{A}_2 \cup \mathcal{A}_3$ and noting that $\frac{2k-1}{k}$ and $\frac{2k-1}{k-1}$ are conjugate exponents, we obtain

$$\begin{aligned} \mathcal{I}\left(\bigcup_{j \in \mathcal{A}_2 \cup \mathcal{A}_3} \mathcal{P}_j, \mathcal{S}_2\right) &\lesssim \sum_{\mathcal{A}_2 \cup \mathcal{A}_3} |\mathcal{P}_j|^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D_j^{\frac{k-1}{2k-1}} + |\mathcal{P}_j| \\ &\lesssim n^{\frac{2k-2}{2k-1}} \left(\sum_j |\mathcal{P}_j|\right)^{\frac{k}{2k-1}} \left(\sum_j D_j\right)^{\frac{k-1}{2k-1}} + m \\ &\lesssim m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D^{\frac{k-1}{2k-1}} + m. \end{aligned} \quad (3.24)$$

Finally, selecting

$$D = m^{\frac{k}{3k-1}} n^{\frac{-1}{3k-1}}, \quad (3.25)$$

which by (3.6) satisfies $D > C$, and combining (3.6), (3.8), (3.10), (3.13), and (3.24), we obtain

$$\begin{aligned} \mathcal{I}(\mathcal{P}, \mathcal{S}) &\lesssim D^2 n + m + \frac{mn^{1-1/k}}{D^{1-1/k}} + Dm^{2/3} \\ &\quad + nD + m + m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D^{\frac{k-1}{2k-1}} \\ &\lesssim m^{\frac{2k}{3k-1}} n^{\frac{3k-3}{3k-1}} + m^{\frac{2}{3} + \frac{k}{3k-1}} n^{\frac{-1}{3k-1}} + m + n \\ &\lesssim m^{\frac{2k}{3k-1}} n^{\frac{3k-3}{3k-1}} + m + n. \end{aligned} \quad (3.26)$$

□

3.3 Applications

In [Erd46, Erd60], Erdős asked how many unit distances there could be amongst m points in the plane or in \mathbb{R}^3 . Theorem 40 yields new bounds for the \mathbb{R}^3 version of this question. Let \mathcal{P} be a collection of m points in \mathbb{R}^3 , and let \mathcal{S} be a collection of unit spheres centered

about the points in \mathcal{P} . We can immediately verify that any three spheres have at most eight points in common, so Theorem 40 tells us that there are $O(m^{3/2})$ point-sphere incidences, i.e.

Theorem 51. *The maximum number of unit-distance pairs in a set of m points in \mathbb{R}^3 is $O(m^{3/2})$.*

This is a slight improvement over the previous bound of $O(m^{3/2}\beta(m))$ from [CEG90], where β is a very slowly growing function.

As observed in [CEG90], Theorem 40, combined with the method outlined in [Chu88] can be used to establish bounds on the number of incidences between points and spheres in \mathbb{R}^d . Specifically, we have the following theorem:

Theorem 52. *The maximum number of incidences between m points and n spheres in \mathbb{R}^d is*

$$O(m^{d/(d+1)}n^{d/(d+1)} + m + n), \tag{3.27}$$

provided no d of the spheres intersect in a common circle.

Again, this is a slight improvement (by a $\beta(m, n)$ factor) from the analogous bounds established in [CEG90]. See [CEG90, §6.5] for additional applications of Theorem 40. In each case, we are able to slightly sharpen the bound from [CEG90] by removing the $\beta(m)$ factor.

3.4 Generalizations to higher dimension

It is reasonable to ask whether Theorem 40 can be generalized to incidences between points and hypersurfaces in higher dimensions. This task appears to be quite involved, as the necessary algebraic geometry becomes more difficult. In particular, it appears that in order to generalize the proof of Theorem 40 to (say) spheres in \mathbb{R}^d , we need to perform $d - 1$ polynomial ham sandwich decompositions, with each successive decomposition performed

on the variety defined by the previous decompositions. As d increases, the number of cases to be considered increases dramatically, and certain difficulties such as the failure of the connected components of a complete intersection to themselves be a complete intersection, the failure of an arbitrary complete intersection to be a nonsingular complete intersection, etc. become increasingly problematic.

One could also consider two dimensional surfaces in \mathbb{R}^d , $d > 3$, and this appears to be more promising. However, the analogues of (3.10) and Lemma 50 become more difficult: an algebraic variety of dimension $d - 1$ can contain many 2-dimensional surfaces without obvious constraints being imposed on its structure, and in higher dimensions there are more (and more complicated) ways in which varieties can fail to intersect completely. Nevertheless, this is certainly a promising area for future work.

Appendix A: Removing non-real components from a polynomial

Definition 53. If $P \in \mathbb{R}[x_1, \dots, x_d]$ is a polynomial and $P = P_1 \dots P_\ell$ is its factorization, we define \hat{P} to be the polynomial obtained by removing those irreducible components that generate ideals that aren't real. If every irreducible component of P generates an ideal that is not real, then we define $\hat{P} = 1$.

Example 54. Let $P = (x_1^2 + x_2^2 + x_3^2 - 1)(x_1^2 + x_2^2)$. Then $\hat{P} = x_1^2 + x_2^2 + x_3^2 - 1$. Geometrically, if $\hat{P} \neq 1$, then $\mathbf{Z}(P)$ is a $(d-1)$ -dimensional (real) variety, but some of the components of $\mathbf{Z}(P)$ may have dimension less than $d - 1$. \hat{P} keeps only those factors that generate components that have dimension $d - 1$, and discards the rest. Note that $\mathbf{Z}(\hat{P})$ may still contain points whose local dimension is less than $d - 1$.

The existence of polynomials that do not generate real ideals complicates our analysis, but since the zero sets of such polynomials have codimension at least 2, we can ignore them when we are computing the number of times a surface meets the realization of a realizable strict sign condition of a family of polynomials. The following theorem helps make this statement precise.

Theorem 55. Let $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_d]$, $d \geq 3$ be a collection of real polynomials and let $\hat{\mathcal{Q}} = \{\hat{Q} : Q \in \mathcal{Q}\}$. Then there exists a bijection

$$\tau: \text{Reali}(\mathcal{Q}) \rightarrow \text{Reali}(\hat{\mathcal{Q}})$$

such that

$$X \subset \tau(X) \text{ for every } X \in \text{Reali}(\mathcal{Q}). \quad (3.28)$$

Similarly, if $Z = \mathbf{Z}(P)$ where $P \in \mathbb{R}[x_1, \dots, x_d]$ generates a real ideal and no polynomial $Q \in \mathcal{Q}$ vanishes identically on Z , then there exists a bijection

$$\tau: \text{Reali}(\mathcal{Q}, Z) \rightarrow \text{Reali}(\hat{\mathcal{Q}}, Z)$$

such that

$$X \subset \tau(X) \text{ for every } X \in \text{Reali}(\mathcal{Q}, Z). \quad (3.29)$$

Proof. First, by Item 5 of Proposition 13, for each $Q \in \mathcal{Q}$ we have that $Q/\hat{Q} \geq 0$ or $Q/\hat{Q} \leq 0$ on all of \mathbb{R}^d . Choose $\varepsilon_Q \in \{\pm 1\}$ so that $\varepsilon_Q Q/\hat{Q} \geq 0$. Now, note that if there exist $Q_1, Q_2 \in \mathcal{Q}$ with $\hat{Q}_1 = \hat{Q}_2$ and if σ is a strict sign condition on \mathcal{Q} , then either $\varepsilon_{Q_1}\sigma(Q_1) = \varepsilon_{Q_2}\sigma(Q_2)$ or $\text{Reali}(\sigma, \mathcal{Q}) = \emptyset$. Thus if σ is a realizable strict sign condition on \mathcal{Q} , then we can define $\hat{\sigma}: \hat{\mathcal{Q}} \rightarrow \{\pm 1\}$ by $\hat{\sigma}(T) = \varepsilon_Q\sigma(Q)$, where $Q \in \mathcal{Q}$ satisfies $T = \hat{Q}$, and $\hat{\sigma}$ is well-defined.

We shall show that the map $\Sigma_{\mathcal{Q}} \rightarrow \Sigma_{\hat{\mathcal{Q}}}$, $\sigma \mapsto \hat{\sigma}$ is a bijection. To prove injectivity, note that if distinct σ_1, σ_2 both map to the same element $\hat{\sigma}$, then $\varepsilon_Q\sigma_1(Q) = \varepsilon_Q\sigma_2(Q)$ for all $Q \in \mathcal{Q}$, so clearly $\sigma_1 = \sigma_2$. To establish surjectivity, note that each $\sigma_1 \in \Sigma_{\hat{\mathcal{Q}}}$ has a pre-image under the map $\sigma \mapsto \hat{\sigma}$. Thus every element of $\Sigma_{\hat{\mathcal{Q}}}$ may be written as $\hat{\sigma}$ for some strict sign condition σ on \mathcal{Q} . All that we must establish is that σ is realizable. For each $Q \in \mathcal{Q}$, we have

$$\dim(\{\hat{Q} > 0\} \setminus \{\varepsilon_Q Q > 0\}) \leq d - 2, \quad (3.30)$$

(see i.e. [BCR98] for the dimension of a semi-algebraic set). On the other hand, the realization of each realizable strict sign condition of $\hat{\mathcal{Q}}$ has dimension d . Thus if $\text{Reali}(\hat{\sigma}, \hat{\mathcal{Q}}) \neq \emptyset$ then $\text{Reali}(\sigma, \mathcal{Q})$ can be written as a (non-empty) dimension d semi-algebraic set minus a dimension $d - 2$ semi-algebraic set, and in particular, $\text{Reali}(\sigma, \mathcal{Q}) \neq \emptyset$.

Thus the map $\text{Reali}(\mathcal{Q}) \rightarrow \text{Reali}(\hat{\mathcal{Q}})$ given by $\text{Reali}(\sigma, \mathcal{Q}) \mapsto \text{Reali}(\hat{\sigma}, \hat{\mathcal{Q}})$ is well-defined and is a bijection. Now, note that by Items 3 and 5 of Proposition 13, $\{\varepsilon_{\mathcal{Q}}Q > 0\} \subset \{\hat{\mathcal{Q}} > 0\}$, and similarly with “>” replaced by “<”. Thus

$$\text{Reali}(\sigma, \mathcal{Q}) \subset \text{Reali}(\hat{\sigma}, \hat{\mathcal{Q}}),$$

so (3.28) holds.

The same arguments establish the second part of the theorem. The only new thing that must be verified is that the map $\Sigma_{\mathcal{Q}, Z} \rightarrow \Sigma_{\hat{\mathcal{Q}}, Z}$, $\sigma \mapsto \hat{\sigma}$ is onto. However, this is established by (3.30) plus the fact that the realization of each realizable strict sign condition of \mathcal{Q} on Z has dimension $d - 1$. \square

Corollary 56. Let $S \subset \mathbb{R}^3$ be a smooth surface, let \mathcal{Q} be a collection of polynomials, and let $\hat{\mathcal{Q}}$ be as in Theorem 55. Then

$$|\{X \in \text{Reali}(\mathcal{Q}): X \cap S \neq \emptyset\}| \leq |\{X \in \text{Reali}(\hat{\mathcal{Q}}): X \cap S \neq \emptyset\}|. \quad (3.31)$$

Similarly, let $S \subset \mathbb{R}^3$ be a smooth surface, let $Z = \mathbf{Z}(P)$ where $P \in \mathbb{R}[x_1, x_2, x_3]$ generates a real ideal, let \mathcal{Q} be a collection of polynomials, none of which vanish identically on Z , and let $\hat{\mathcal{Q}}$ be as in Theorem 55. Then

$$|\{X \in \text{Reali}(\mathcal{Q}, Z): X \cap S \neq \emptyset\}| \leq |\{X \in \text{Reali}(\hat{\mathcal{Q}}, Z): X \cap S \neq \emptyset\}|. \quad (3.32)$$

CHAPTER 4

A Szemerédi-Trotter type theorem in \mathbb{R}^4

4.1 Introduction

In [ST83], Szemerédi and Trotter proved the following theorem:

Theorem 57 (Szemerédi-Trotter). *The number of incidences between m points and n lines in \mathbb{R}^2 is $O(m^{2/3}n^{2/3} + m + n)$.*

Theorem 57 has seen a number of generalizations. In [T03], Tóth generalized Theorem 57 to complex points and lines in \mathbb{C}^2 . However, as of this writing (2013), Tóth's paper is still in the midst of a lengthy review process while awaiting publication. Solymosi and Tardos [ST07] gave a simpler proof of the same result in the special case where the point set is a Cartesian product of the form $A \times B \subset \mathbb{C}^2$. Edelsbrunner and Sharir [ES91] obtained incidence results for certain configurations of points and codimension-1 hyperplanes in \mathbb{R}^4 , and Laba and Solymosi [LS07] obtained incidence bounds for points and a general class of 2-dimensional surfaces in \mathbb{R}^3 , provided the points satisfied a certain homogeneity condition. Elekes and Tóth [ET05] and later Solymosi and Tóth [Sol05] obtained incidence results between points and hyperplanes in \mathbb{R}^d , again provided the points satisfied various non-degeneracy and homogeneity conditions.

In [ST12], Solymosi and Tao used the discrete polynomial ham sandwich theorem to obtain bounds for the number of incidences between points and flats. Aside from an ϵ loss in the exponent, Solymosi and Tao's result resolved a conjecture of Tóth on the number of incidences between points and d -flats in \mathbb{R}^{2d} . The discrete polynomial ham sandwich theorem was also used by the author in [Zah13a] to obtain incidence results between points

and 2–dimensional surfaces in \mathbb{R}^3 (with no homogeneity condition), and by Kaplan et al. in [KMS12a] to obtain similar bounds on the number of incidences between points and spheres in \mathbb{R}^3 .

Here, we combine the crossing lemma and the discrete polynomial ham sandwich theorem to obtain a new result which can be seen either as a sharpening of the \mathbb{R}^4 version of the Solymosi-Tao result from [ST12] or a generalization of Tóth’s result from [T03]. To the best of the author’s knowledge, this is the first time these two techniques have been used together.

4.1.1 New Results

Definition 58. Let \mathcal{P} be a collection of points in \mathbb{R}^4 and \mathcal{S} a collection of 2–flats. We define the set of incidences of \mathcal{P} and \mathcal{S} to be

$$\mathcal{I}(\mathcal{P}, \mathcal{S}) = |\{(p, S) \in \mathcal{P} \times \mathcal{S} : p \in S\}|.$$

In [Zah12b], the author proved the following theorem.

Theorem 59. *Let $\mathcal{P} \subset \mathbb{R}^4$ be a collection of points, with $|\mathcal{P}| = m$. Let \mathcal{S} be a collection of 2–flats, with the property that no two 2–flats meet in a common line. Suppose that $m \leq n$. Then*

$$\mathcal{I}(\mathcal{P}, \mathcal{S}) \lesssim m^{2/3}n^{2/3} + m + n. \tag{4.1}$$

In this section, we will give an abbreviated proof of the theorem (the original theorem in [Zah12b] was more general).

4.1.2 Corollaries and applications of Theorem 59

We can use Theorem 59 to recover the Szemerédi-Trotter theorem for complex lines in \mathbb{C}^2 , which was originally proved by Tóth in [T03]. Note that by point-line duality in \mathbb{C}^2 , we can always assume that the number of lines is at least as great as the number of points (i.e. we can always assume $\rho \leq 1$). Thus we have:

Corollary 60 (Complex Szemerédi-Trotter). Let \mathcal{P} be a collection of m points and let \mathcal{S} be a collection of n (complex) lines in \mathbb{C}^2 . Then the number of incidences between points in \mathcal{P} and complex lines in \mathcal{S} is $O(m^{2/3}n^{2/3} + m + n)$.

4.1.3 Proof sketch

The basic idea is as follows. Since each pair of points has at most two 2-planes passing through both of them, we can use the Cauchy-Schwarz inequality to obtain a rudimentary bound on the cardinality of any collection of point-flat incidences. We will call this the Cauchy-Schwarz bound. Using the discrete polynomial ham sandwich theorem, we find a polynomial P of controlled degree whose zero set cuts \mathbb{R}^4 into open “cells,” such that each cell contains roughly the same number of points from \mathcal{P} , and each 2-flat from \mathcal{S} does not enter too many cells. We can then apply the Cauchy-Schwarz bound within each cell. This allows us to count the incidences occurring between flats and points in $\mathcal{P} \setminus Z$. In order to count the remaining incidences, we perform a “second level” polynomial ham sandwich decomposition on the variety Z . This gives us a polynomial Q which cuts Z into a collection of 3-dimensional cells, which are open in the relative (Euclidean) topology of Z . We then apply the Cauchy-Schwarz bound to each of these 3-dimensional cells. The only incidences left to count are those between flats in \mathcal{S} and points in $\mathcal{P} \cap Z \cap \{Q = 0\}$. Let $Y = Z \cap \{Q = 0\}$.

We can choose P and Q in such a way that Y is a 2-dimensional variety in \mathbb{R}^4 . Let S be a 2-flat in \mathcal{S} . Then S will intersect Y in a union of isolated points (proper intersections) and 1-dimensional curves (non-proper intersections) (the case where S meets Y in a 2-dimensional variety can be dealt with easily). The number of isolated points in the intersection can be controlled by the degrees of the polynomials P and Q (we are working over \mathbb{R} , where Bézout’s theorem need not hold, so we need to be a bit careful), and thus the number of incidences between points $p \in \mathcal{P} \cap Y$ and flats $S \in \mathcal{S}$ such that p is an isolated point of $S \cap Y$ can be controlled.

The only remaining task is to control the number of incidences between points of $\mathcal{P} \cap Y$

and 1–dimensional curves arising from the intersection of Y and flats $S \in \mathcal{S}$. To simplify the exposition, pretend Y is a disjoint union of N 2–planes, i.e. $Y = \Pi_1 \sqcup \dots \sqcup \Pi_N$ (of course, we will not make this assumption when we prove the actual result). Then for each plane Π_i , $\Pi_i \cap S = L_{S,i}$ is a line on Π_i . It remains to count the number of incidences between $\mathcal{P} \cap \Pi_i$ and $\{L_{S,i}\}_{S \in \mathcal{S}}$. The Szemerédi-Trotter theorem for lines in \mathbb{R}^2 would give us the bound

$$I(\mathcal{P} \cap \Pi_i, \{L_{S,i}\}_{S \in \mathcal{S}}) \leq C|\mathcal{P} \cap \Pi_i|^{2/3}|\mathcal{S}|^{2/3} + |\mathcal{P} \cap \Pi_i| + |\mathcal{S}|, \quad (4.2)$$

but if we sum (4.2) over the N values of i , we have only bounded the number of incidences by $N^{1/3}|\mathcal{P}|^{2/3}|\mathcal{S}|^{2/3} + |\mathcal{P}| + |\mathcal{S}|$. Since N can be quite large, this is not sufficient. Instead, recall Székely’s proof in [Sz97] of the Szemerédi-Trotter theorem, which uses the crossing lemma (the crossing lemma and all other graph-related quantities are defined in Section 2.2). Loosely speaking, we consider the graph drawing G_i on Π_i whose vertices are the points of $\mathcal{P} \cap \Pi_i$, and two vertices are connected by an edge if there is a line segment from $\{L_{i,S}\}_{S \in \mathcal{S}}$ passing through the two points. Then the number of edges of the graph is comparable to the number of incidences between points and lines, and this is controlled by $\mathcal{C}(G_i)^{1/3}\mathcal{V}(G_i)^{2/3}$, where $\mathcal{C}(G_i)$ is the number of edge crossings and $\mathcal{V}(G_i)$ is the number of vertices of G_i . Thus in place of (4.2), we have

$$I(\mathcal{P} \cap \Pi_i, \{L_{S,i}\}_{S \in \mathcal{S}}) \leq C|\mathcal{P} \cap \Pi_i|^{2/3}|\mathcal{C}(G_i)|^{1/3} + |\mathcal{P} \cap \Pi_i| + |\mathcal{S}|. \quad (4.3)$$

The key insight is that

$$\sum_i |\mathcal{C}(G_i)| \leq |\mathcal{S}|^2. \quad (4.4)$$

Indeed, every pair of 2–planes $S, S' \in \mathcal{S}$ can intersect in at most one point, and since we assumed the planes $\{\Pi_i\}$ composing Y were disjoint, the intersection point of $S \cap S'$ can occur on Π_i for at most one index i . Summing (4.3) over all choices of i , applying Hölder’s inequality, and inserting (4.4), we obtain the correct bound on the number of incidences between 2–flats in \mathcal{S} and points lying on Y .

Unfortunately, the assumption that Y is a disjoint union of 2–planes need not be true. Thus we must cut Y up into pieces, each of which behaves like a 2–plane, and we need to

prove a more general form of the Szemerédi-Trotter theorem for families of curves and points that lie on suitable domains. This is a purely topological argument, and it is done using the crossing lemma. The decomposition of Y into suitable domains relies on results from real algebraic geometry, and it works well provided Y is not of too high degree. The degree of Y depends on the value of $(\log |\mathcal{P}|)/(\log |\mathcal{S}|)$. If $|\mathcal{P}|$ is too large compared to $|\mathcal{S}|$, the degree of Y is too big, and we cannot cut Y into suitable pieces without introducing error terms that are larger than the bounds we are trying to prove. This is why we impose the requirement $m \leq n$. In [Zah12b], we use more sophisticated techniques to get around this problem,

4.2 Proof of Theorem 59 step 1: cell partitionings

4.2.1 Initial reductions

We shall prove the statement by induction on m . Thus, we may assume that (4.1) holds for all collections $|\mathcal{P}'|$, $|\mathcal{S}'|$ with $|\mathcal{P}'| \leq m$. Let C_0 be the implicit constant in (4.1).

From Lemma 1, we have

$$|\mathcal{I}| \lesssim mn^{1/2} + n, \tag{4.5}$$

$$|\mathcal{I}| \lesssim m^{1/2}n + m \tag{4.6}$$

Thus we may assume

$$\begin{aligned} n &< cm^2, \\ m &< cn^2, \end{aligned} \tag{4.7}$$

where c is a small constant (we may make c as small as we like by making the implicit constant in (4.1) larger). Thus we may assume

$$m + n < c_0 m^{2/3} n^{2/3}, \tag{4.8}$$

where we may make c_0 arbitrarily small at the cost of increasing the implicit constant in (4.1).

4.2.2 First polynomial partition

Let

$$D = m^{1/3}n^{-1/6}, \quad (4.9)$$

which by (4.7) satisfies

$$C < D < n^{1/2}. \quad (4.10)$$

Let P be a square-free polynomial of degree at most D such that $Z = \mathbf{Z}(P)$ cuts \mathbb{R}^4 into $O(D^4)$ cells $\{\Omega_i\}$, such that $|\mathcal{P} \cap \Omega_i| \lesssim m/D^4$. Let m_i be the number of points in the i -th cell, and let n_i be the number of 2-flats in \mathcal{S} that meet the i -th cell. We have $m_i \lesssim m/D^4$.

Lemma 61.

$$\sum n_i \lesssim D^2 n \quad (4.11)$$

Proof. The proof is similar to the proof of Lemma 46 in Chapter 3, so we will only give a brief sketch here. Let $S \in \mathcal{S}$. Select a large number R so that every cell that meets S does so within the ball centered at the origin of radius R . Let $\tilde{P} = P \cdot f_B$, where $f_B(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - R^2$. Thus the number of cells of $\mathbf{Z}(P)$ that S meets is at most the number of bounded cells of $\mathbf{Z}(\tilde{P})$ that S meets. Since the property of S meeting a cell is open, we can apply a small generic translation to S and a small generic perturbation to \tilde{P} , and doing so can only increase the number of bounded cells that S meets. Now, we can find f_1, f_2 such that $S \subset \mathbf{Z}(f_1) \cap \mathbf{Z}(f_2)$, (f_1, f_2) is a reduced ideal, and f_1 and f_2 are of bounded degree. Let v be a generic vector in \mathbb{R}^4 , and let $T(x) = v \wedge \nabla f_1 \wedge \nabla f_2 \wedge \nabla \tilde{P}$. Then $\deg T(x) \lesssim D$. Now, the number of cells of $\mathbf{Z}(\tilde{P})$ that S enters is controlled by the number of (necessarily non-singular) intersection points of S , $\mathbf{Z}(\tilde{P})$, and $\mathbf{Z}(T)$ (again, see Lemma 46 in Chapter 3 for details), and this is $O(D^2)$. \square

We have

$$\begin{aligned}
\sum_i \mathcal{I}(\mathcal{P} \cap \Omega_i, \mathcal{S}) &\lesssim \sum_i (|\mathcal{P} \cap \Omega_i| n_i^{1/2} + n_i) \\
&\leq \left(\sum_i \left(\frac{m}{D^4}\right)^2 \right)^{1/2} \left(\sum_i n_i \right)^{1/2} + \sum_i n_i \\
&\lesssim \frac{mn^{1/2}}{D} + nD^2.
\end{aligned} \tag{4.12}$$

Selecting D as in (4.9), we have

$$\sum_i \mathcal{I}(\mathcal{P} \cap \Omega_i, \mathcal{S}) \lesssim m^{2/3} n^{2/3}. \tag{4.13}$$

4.2.3 Boundary incidences of the first partition

Let

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \tag{4.14}$$

where \mathcal{S}_1 (resp. \mathcal{S}_2) consists of those 2-flats that are contained (resp. not contained) in Z .

Lemma 62.

$$|\mathcal{I} \cap I(\mathcal{P} \cap Z_{\text{smooth}}, \mathcal{S}_1)| \lesssim m. \tag{4.15}$$

Proof. Let $p \in \mathcal{P}$, and let $H = T_p(Z)$. Suppose there existed flats $S_1, S_2 \in \mathcal{S}_1$ with $p \in S, p \in S'$. Since $S \subset Z$, we have $S \subset T_p(Z) = H$. Similarly, $S' \subset H$. Recall that $T_p(S) \cap T_p(S') = p$. Thus we have two affine 2-planes, S and S' which meet only at the point p , but both are contained in the affine 3-plane H . This cannot occur. Thus for each point $p \in \mathcal{P}$, there can exist at most one flat $S \in \mathcal{S}_1$ with $p \in S$. \square

Thus it suffices to consider incidences between 2-flats and points lying on Z_{sing} . Let R be the square-free part of $|\nabla P|^2$, i.e. (R) is radical and $\mathbf{Z}(R) = \mathbf{Z}(|\nabla P|^2)$. Since P was square-free, $Z \cap \mathbf{Z}(R)$ is a complete intersection. If we let $\mathcal{S}'_1 \subset \mathcal{S}_1$ be those 2-flats contained in $Z \cap \mathbf{Z}(R)$, then we must have $|\mathcal{S}'_1| \lesssim D^2$, and thus applying Lemma 1, we have

$$\mathcal{I}(\mathcal{P} \cap Z_{\text{sing}}, \mathcal{S}'_1) \lesssim D^2 m^{1/2} + m. \tag{4.16}$$

Let $\mathcal{S}'_2 \subset \mathcal{S}_1$ be those 2-flats (contained in Z) that are not contained in $\mathbf{Z}(R)$. We must now control $\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_2)$ and $\mathcal{I}(\mathcal{P} \cap \mathbf{Z}(R), \mathcal{S}'_2)$. But note that Z and $\mathbf{Z}(R)$ are both the zero-set of polynomials of degree $O(D)$, and thus the two collections of incidences can be dealt with in the same fashion.

4.2.4 Second ham sandwich decomposition

We shall now control $|\mathcal{I} \cap I(\mathcal{P} \cap Z, \mathcal{S}_2)|$. Factor $P = P_1 \dots, P_\ell$, with each P_j irreducible of degree D_j , and let $Z_j = \{P_j\}$. Let $\mathcal{P}_j \subset Z_j \cap \mathcal{P}$ so that $\bigsqcup \mathcal{P}_j = \mathcal{P}$; if the same point p lies on several Z_j , place p into just one of the sets \mathcal{P}_j (the choice of set does not matter).

Let

$$\begin{aligned} \mathcal{A}_0 &= \{j: |\mathcal{P}_j|^2 \leq c_1 n D_j^6\}, \\ \mathcal{A}_1 &= \{j: |\mathcal{P}_j|^2 \geq c_1 n D_j^6\}. \end{aligned}$$

4.2.4.1 Incidences on varieties in \mathcal{A}_0

We have

$$\begin{aligned} \left| \bigcup_{j \in \mathcal{A}_0} \mathcal{P}_j \right| &\leq c_1 \sum_{j \in \mathcal{A}_0} n^{1/2} D_j^3 \\ &\leq c_1 n^{1/2} D^3 \\ &\leq c_1 m. \end{aligned} \tag{4.17}$$

Thus by our induction hypothesis, we have

$$\begin{aligned} \mathcal{I}\left(\bigcup_{j \in \mathcal{A}_0} \mathcal{P}_j, \mathcal{S}\right) &\leq C_0 c_1^{2/3} m^{2/3} n^{2/3} + m + n \\ &\leq C_0 (c_0 + c_1) m^{2/3} n^{2/3}, \end{aligned} \tag{4.18}$$

where c_1 is the constant from (4.8). If we choose c_0 and c_1 small enough, then the contribution from (4.18) is acceptable.

4.2.4.2 Incidences on varieties in \mathcal{A}_1

For each $j \in \mathcal{A}_1$, define

$$E_j = |\mathcal{P}_j|^{1/2} n^{-1/4} D_j^{-1/2}. \quad (4.19)$$

Use Theorem 24 to find a collection of polynomials $\mathcal{Q} \subset \mathbb{R}[x_1, \dots, x_4]$ whose strict sign conditions partition Z_j into $O(D_j E_j^3)$ “cells” (note that a cell need not be connected), such that each cell contains $O(|\mathcal{P}_j|/D_j E_j^3)$ points from \mathcal{P} . Let m_{ij} be the number of points from \mathcal{P}_j in the i -th strict sign condition, and let n_{ij} be the number of flats from \mathcal{S}_2 that meet the i -th strict sign condition.

Lemma 63.

$$\sum_i n_{ij} \lesssim n D_j E_j. \quad (4.20)$$

Proof. This proof is similar to the proof of Lemma 15 in [Zah13a], so we will only provide a brief sketch here. Let $S \in \mathcal{S}$. Write $S \cap Z$ as a union of irreducible curves, and denote this collection of irreducible curves by Γ . By Harnack’s theorem (Proposition 16), $\bigcup_{\gamma \in \Gamma} \gamma$ can have at most $O((\deg P)^2)$ components. Now, for each irreducible curve $\gamma \in \Gamma$ and each $Q \in \mathcal{Q}$, either $\gamma^* \subset \mathbf{Z}^*(Q)$ or $|\gamma^* \cap \mathbf{Z}^*(Q)| \lesssim \deg \gamma \deg Q$, and thus $|\gamma \cap \mathbf{Z}(Q)| \lesssim \deg \gamma \deg Q$. We will call intersections of this type “important” intersections between S , Z and $\mathbf{Z}(Q)$. If $\gamma^* \subset \mathbf{Z}^*(Q)$, then since $\gamma^* \subset \mathbf{Z}^*(Q)$, $\gamma \subset \mathbf{Z}(Q)$, and thus γ does not enter *any* realizations of realizable strict sign conditions of \mathcal{Q} on Z . Now, if Ω is a realization of a realizable sign condition of \mathcal{Q} on Z , and $S \cap \Omega \neq \emptyset$, then we can associate to the pair (S, Ω) an important intersection of S , Z , and $\mathbf{Z}(Q)$ for some $Q \in \mathcal{Q}$ in such a way that every important intersection is assigned to at most 2 pairs (S, Ω) . Thus the number of realizations of realizable strict sign conditions of \mathcal{Q} on Z is at most $O(\deg P \sum_{Q \in \mathcal{Q}} \deg Q) = O(DE)$. \square

Remark 64. A similar result to Lemma 63 can be obtained from the recent work of Barone and Basu in [BB12].

For each index j , we now have

$$\begin{aligned}
\sum_i \mathcal{I}(\mathcal{P}_j \cap \Omega_{ij}, \mathcal{S}_2) &\lesssim \sum_i m_{ij} n_{ij}^{1/2} + \sum_i n_{ij} \\
&\lesssim \left(\sum_i \left(\frac{|\mathcal{P}_j|}{D_j E_j^3} \right)^2 \right)^{1/2} (D_j E_j n)^{1/2} + D_j E_j n \\
&\lesssim \frac{|\mathcal{P}_j| n^{1/2}}{E_j} + D_j E_j n.
\end{aligned} \tag{4.21}$$

Thus, if $W_j = \bigcup_{Q \in \mathcal{Q}_j} \mathbf{Z}(Q)$, then

$$\begin{aligned}
\sum_j \mathcal{I}(\mathcal{P}_j \setminus W_j, \mathcal{S}_2) &\lesssim \sum_j \left(\frac{|\mathcal{P}_j| n^{1/2}}{E_j} + D_j E_j n \right) \\
&\lesssim n^{3/4} \sum_j |\mathcal{P}_j|^{1/2} D_j^{1/2} \\
&\lesssim n^{3/4} m^{1/2} D^{1/2} \\
&\lesssim m^{2/3} n^{2/3}.
\end{aligned} \tag{4.22}$$

It remains to control the incidences that occur between flats in \mathcal{S}_2 and points that lie on $\bigcup_{j \in \mathcal{A}_1} \mathcal{P}_j \cap W_j$. We will do this in the next section.

4.3 Proof of Theorem 59 step 2: incidences on a surface in \mathbb{R}^4

Let

$$Y = \bigcup_{j \in \mathcal{A}_1 \cup \mathcal{A}_2} Z_j \cap W_j. \tag{4.23}$$

Recall that

$$\deg Z_j = D_j, \quad \deg W_j \leq E_j, \tag{4.24}$$

where $\sum D_j \leq D$ (D is specified in (4.9), and E_j is specified in (4.19)). Our task is now to establish the bound

$$\mathcal{I}(\mathcal{P} \cap Y, \mathcal{S}_2) \lesssim m^{2/3} n^{2/3} + m + n. \tag{4.25}$$

Once this has been done, Theorem 59 will be complete.

Remark 65. Before we begin, let us first recall Székely's proof in [Sz97] of the Szemerédi-Trotter theorem, which uses the crossing lemma. Let $\tilde{\mathcal{P}}$ be a collection of points and \mathcal{L} a collection of lines in \mathbb{R}^2 . Let $I(\tilde{\mathcal{P}}, \mathcal{L})$ be the number of incidences between points in $\tilde{\mathcal{P}}$ and lines in \mathcal{L} . Suppose that all of the points (and thus all of the incidences) are contained in some large disk $U \subset \mathbb{R}^2$. Consider the following graph drawing H : the vertices of H are the points of $\tilde{\mathcal{P}}$ and the points where a line from \mathcal{L} meets ∂U . The edges of H are the line segments connecting two vertices that arise from lines in \mathcal{L} . To each incidence $(p, L) \in I$, we can associate an edge of H in such a way that the same edge is assigned to at most two incidences. Thus $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim \mathcal{E}(H)$. Now, delete all of the edges involving a vertex on ∂U , and delete the vertices on ∂U . We have deleted at most $2|\mathcal{L}|$ edges. Let H' be the resulting graph drawing. Then $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim \mathcal{E}(H') + 2|\mathcal{L}|$. By the crossing lemma,

$$\begin{aligned} \mathcal{E}(H') &\lesssim \mathcal{V}(H') + \mathcal{C}(H')^{1/3} \mathcal{V}(H')^{2/3} \\ &\lesssim |\tilde{\mathcal{P}}| + |\mathcal{L}|^{2/3} |\tilde{\mathcal{P}}|^{2/3}. \end{aligned}$$

Thus $I(\tilde{\mathcal{P}}, \mathcal{L}) \lesssim |\tilde{\mathcal{P}}|^{2/3} |\mathcal{L}|^{2/3} + |\tilde{\mathcal{P}}| + |\mathcal{L}|$.

We wish to do the same thing on a 2-dimensional real algebraic variety in \mathbb{R}^4 . In this section, we will develop the tools to do this.

Definition 66. Recall that $Y_j = Z_j \cap W_j$. We define

$$\begin{aligned} \mathcal{I}_0 = \{(p, S) \in \mathcal{P} \times \mathcal{S}_2 : p \in S, \text{ there exists an index } j \text{ such that } p \\ \text{is an isolated point of } S \cap Y_j\}, \end{aligned} \tag{4.26}$$

$$\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0. \tag{4.27}$$

Note that since $S \not\subset Y_j$ for any $S \in \mathcal{S}$, \mathcal{I}_1 consists of those pairs (p, S) such that for each index j , either $p \notin S \cap Y_j$ or p lies on a 1-dimensional component of $S \cap Y_j$.

We shall first deal with \mathcal{I}_0 . Fix a choice of S and j . Since $S \not\subset Z_j$, $S \cap Z_j$ consists of 0 and 1-dimensional irreducible components. By Harnack's theorem (Proposition 16), $S \cap Z_j$ can contain at most D_j^2 0-dimensional components. Now, consider the collection Υ

of 1-dimensional irreducible components of $S \cap Z_j$. We have $\sum_{\gamma \in \Upsilon} \deg \gamma \lesssim D_j$. For each curve $\gamma \in \Upsilon$, either $\gamma^* \subset W_j^*$, or $\gamma^* \cap W_j^*$ is a discrete set. Let Υ_1 and Υ_2 denote those sets where the former (resp. latter) occurs. If $\gamma \in \Upsilon_1$, then we must have $\gamma \subset Y_j$, and thus for any point $p \in \mathcal{P}$ that lies on γ , we must have $(p, S) \in \mathcal{I}_1$ (we will deal with these incidences later). If $\gamma \in \Upsilon_2$, then by Bézout's theorem (over \mathbb{C}), $|\gamma^* \cap W_j^*| \leq \deg \gamma \cdot E_j$, and thus there are at most $O(\deg \gamma \cdot E_j)$ points $p \in \mathcal{P}$ for which $p \in \gamma$ and $(p, S) \in \mathcal{I}_0$. Summing over all curves in Υ_2 , we conclude

$$\sum_{\gamma \in \Upsilon_2} \{(p, S) \in \mathcal{I}_0 : p \in \gamma\} \lesssim D_j E_j. \quad (4.28)$$

Summing (4.28) over all indices j and all choices of $S \in \mathcal{S}$ and adding back the incidences that lie on 0-dimensional components of $S \cap Z_j$ for all choices of $S \in \mathcal{S}$, we obtain

$$\begin{aligned} |\mathcal{I}_0| &\lesssim n \sum_j (D_j^2 + D_j E_j) \\ &\lesssim m^{2/3} n^{2/3}. \end{aligned} \quad (4.29)$$

It remains to control $|\mathcal{I}_1|$. We will do this in the next section.

4.3.1 Building an incidence model: $\mathbb{R}\text{-Alg} \rightarrow C^0$

Lemma 67. *Let Y , $\{Y_j\}$, \mathcal{P} , \mathcal{S}_2 , and \mathcal{I}_1 be as above.*

Then there exists an “error set” $\mathcal{I}'_1 \subset \mathcal{I}_1$ and an “incidence model”

$$\mathcal{M} = \{(U_i, A_i, B_i, \iota_i)\}_{i=1}^M,$$

where for each i ,

- $U_i \subset \mathbb{R}^4$ is homeomorphic to an open subset of \mathbb{R}^2 .
- A_i is a collection of open curves (homeomorphic to $(0, 1)$) contained in U_i .
- $B_i \subset U_i$ is a collection of points.

- $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ is an embedding.

\mathcal{M} and \mathcal{I} have the following properties:

(i) \mathcal{M} does not increase crossing number:

$$\sum_i \sum_{\alpha, \alpha' \in A_i} |\alpha \cap \alpha'| \leq \sum_{S, S' \in \mathcal{S}} |S \cap S'|. \quad (4.30)$$

(ii) \mathcal{M} counts incidences: If $(p, S) \in \mathcal{I}_1 \setminus \mathcal{I}'_1$, then there exists some index i , some $\tilde{p} \in B_i$, and some $\alpha \in A_i$ such that $\iota_i(\tilde{p}) = p$, $p \in \iota_i(\alpha)$, $\alpha = \iota_i^{-1}(S)$, and $\tilde{p} \in \bar{\alpha}$.

(iii) The curves in A_i are quasi-lines: For each index i , and for pair of points $p_1, p_2 \in B_i$, there is at most one curve in A_i passing through both p_1 and p_2 .

(iv) \mathcal{M} does not contain too many curves:

$$\sum_i |A_i| \lesssim |\mathcal{S}| \sum_j D_j E_j. \quad (4.31)$$

(v) \mathcal{I}' is not too big:

$$|\mathcal{I}'| \lesssim m^{2/3} n^{2/3}. \quad (4.32)$$

Proof. We shall begin by constructing the “error set” \mathcal{I}'_1 :

$$\mathcal{I}'_1 = \{(p, S) \in \mathcal{I} : p \in Y_{\text{sing}}, p \text{ lies on a 1-dimensional component of } S \cap Y_{\text{sing}}\}. \quad (4.33)$$

First, note that Y_{sing} is an algebraic curve, and

$$\deg(Y_{\text{sing}}) \leq \left(\sum_j D_j E_j \right)^2.$$

Now, if $p \in (Y_{\text{sing}})_{\text{smooth}}$ and there exist two distinct flats $S, S' \in \mathcal{S}$ such that both (p, S) and (p, S') are in \mathcal{I}'_1 , then $S \cap S'$ must contain a 1-dimensional component of Y_{sing} , so in particular $S \cap S'$ is not a discrete set, which contradicts the requirement that every two flats

meet at a single point. Thus for each $p \in (Y_{\text{sing}})_{\text{smooth}}$, there exists at most one $S \in \mathcal{S}$ for which $(p, S) \in \mathcal{I}'_1$. Thus it suffices to control

$$\sum_{p \in (Y_{\text{sing}})_{\text{sing}}} \# \text{ of curves of } Y_{\text{sing}} \text{ passing through } p. \quad (4.34)$$

However, (4.34) is bounded by the degree of $(Y_{\text{sing}})_{\text{sing}}$, which is

$$O\left(\left(\sum_j D_j E_j\right)^4\right).$$

Since $m \leq n$, this establishes (4.32).

Now for each $S \in \mathcal{S}$, we shall partition the curves of $S \cap Y$ that do not lie in Y_{sing} into classes $\Gamma_{S,j}$. Write $S \cap Y$ as a union of irreducible real curves and isolated points. For each irreducible real curve γ , either $\gamma \subset Y_{\text{sing}}$, or there is a unique index j for which every 1-dimensional connected component of γ lies in Y_j (note that γ may have 0-dimensional components, which need not lie in Y_j , but since we are only counting incidences in \mathcal{I}_1 , we do not care about incidences involving 0-dimensional components). If $\gamma \not\subset Y_{\text{sing}}$, place γ in $\Gamma_{S,j}$ for this choice of j .

Similarly, partition \mathcal{P} into collections $\{\mathcal{P}_j\}$, where $p \in \mathcal{P}_j$ if j is the minimal index for which $p \in Y_j$. Then if $p \in \mathcal{P}$, $S \in \mathcal{S}$, $(p, S) \in \mathcal{I}_1 \setminus \mathcal{I}'_1$, then there is a unique index j such that $p \in \mathcal{P}_j$ and there exists a curve $\gamma \in \Gamma_{S,j}$ with $p \in \gamma$. Indeed, the only way this can fail to be the case is if $p \in \mathcal{P}_j$ and every irreducible curve $\gamma \subset S \cap Y$ containing p lies in $\Gamma_{S,j'}$ for some index $j' \neq j$. But this would imply that p is an isolated point of $S \cap Y_j$, and by assumption \mathcal{I}_1 does not contain any pairs (S, p) for which this can occur.

Thus

$$\mathcal{I}_1 \setminus \mathcal{I}'_1 \subset \bigcup_j \mathcal{I}_1 \cap \{(p, S) : p \in \gamma \text{ for some } \gamma \in \Gamma_{S,j}\}. \quad (4.35)$$

We wish to consider each variety Y_j and the associated points and curves on that variety separately. However, before we do this we need to ensure that the same intersection $S \cap S'$ does not get counted as two distinct ‘‘crossings’’ on two different varieties (say Y_j and $Y_{j'}$).

To prevent this from happening, we will keep track of the points where such an accident might occur. To this end, for each $S \in \mathcal{S}$ and each index j let

$$\Xi_{S,j} = \bigcup_{\gamma \in \Gamma_{S,j}} \{x \in \gamma : \text{there exists an index } j' \neq j \text{ such that } x \text{ is a discrete point of } \gamma \cap Y_j\}. \quad (4.36)$$

Recall the definition of the geometric multiplicity of a curve at a point (Definition 28) By the same arguments used above to obtain (4.29),

$$\sum_{x \in \Xi_{S,j}} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}_g(\gamma, x) \leq E_j \sum_{\gamma \in \Gamma_{S,j}} \text{deg } \gamma. \quad (4.37)$$

Once we have removed the points $\Xi_{S,j}$, we can control the number of times two curves intersect:

$$\sum_j \sum_{S, S'} \sum_{\substack{\gamma \in \Gamma_{S,j} \\ \gamma' \in \Gamma_{S',j}}} |(\gamma \setminus \Xi_{S,j}) \cap (\gamma' \setminus \Xi_{S',j})| \leq \sum_{S, S'} |S \cap S'|. \quad (4.38)$$

Indeed, the only way (4.38) could fail is if the following occurs—there exist:

- Two flats S and S' ,
- A point $x \in S \cap S'$,
- Two indices j and \tilde{j} ,
- Four curves $\gamma \in \Gamma_{S,j}$, $\tilde{\gamma} \in \Gamma_{S,\tilde{j}}$, and $\gamma' \in \Gamma_{S',j}$, $\tilde{\gamma}' \in \Gamma_{S',\tilde{j}}$

such that

$$x \in \gamma \cap \gamma', \quad x \in \tilde{\gamma} \cap \tilde{\gamma}', \quad (4.39)$$

and

$$x \notin \Xi_{S,j} \cup \Xi_{S,\tilde{j}} \cup \Xi_{S',j} \cup \Xi_{S',\tilde{j}}. \quad (4.40)$$

But if (4.39) holds then x is an isolated point of $\gamma \cap Y_{\tilde{j}}$, so $x \in \Xi_{S,j}$ and thus (4.40) fails. This establishes (4.38) .

Thus, we can consider each variety Y_j and its associated point and curve sets $\mathcal{P}_j, \{\Gamma_{S,j}\}_{S \in \mathcal{S}}$ individually. We are reduced to proving the following lemma.

Lemma 68. *Suppose we are given the following data:*

- A 2-dimensional real variety of the form $Y_j = Z_j \cap W_j$ with $Z_j = \mathbf{Z}(P_j)$, $W_j = \mathbf{Z}(Q_j)$, $\deg(P_j) = D_j$, $\deg(Q_j) = E_j$.
- A collection of points \mathcal{P} , a collection of 2-flats \mathcal{S} , and for each $S \in \mathcal{S}$, a collection $\Gamma_{S,j}$ of irreducible curves in $S \cap Y_j$ such that for each $\gamma \in \Gamma_{S,j}$, $\gamma \not\subset (Y_j)_{\text{sing}}$.
- A collection of incidences $\mathcal{I} \subset \{(p, S) : p \in S\}$ such that for each $(p, S) \in \mathcal{I}$, there exists $\gamma \in \Gamma_{S,j}$ with $p \in \gamma$.
- For each $S \in \mathcal{S}$, a finite collection of “bad” points Ξ_S .

Then there exists an incidence model $\mathcal{M}_j = \{(U_i, A_i, B_i, \iota_i)\}_{i=1}^M$ satisfying:

(i) For each i , A_i is a collection of simple open curves, B_i is a collection of points, U_i is homeomorphic to a open subset of \mathbb{R}^2 , and $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ is an embedding.

(ii)

$$\begin{aligned} & \sum_i \sum_{\alpha, \alpha' \in A_i} |\alpha \cap \alpha'| \\ & \leq \sum_{S, S' \in \mathcal{S}} \left| \left(\bigcup_{\gamma \in \Gamma_{S,j}} \gamma \setminus \Xi_S \right) \cap \left(\bigcup_{\gamma' \in \Gamma_{S',j}} \gamma' \setminus \Xi_{S'} \right) \right|, \end{aligned} \quad (4.41)$$

i.e. the number of crossings between pairs of curves in the incidence model is controlled by the number of crossings of the curves from $\Gamma_{S,j}$ and $\Gamma_{S',j}$ that do not occur on the “bad” sets Ξ_S or $\Xi_{S'}$, as S and S' range over all pairs of flats.

(iii) If $p \in \mathcal{P}$, $S \in \mathcal{S}$, and $\gamma \in \Gamma_{S,j}$, then there exists some index i , some $\tilde{p} \in B_i$, and some $\alpha \in A_i$ such that $\iota_i(\tilde{p}) = p$, $\alpha = \iota_i^{-1}(S)$, and $\tilde{p} \in \bar{\alpha}$, *i.e. the incidence model counts curve-point incidences.*

(iv) For each i , the curves in A_i are quasi-lines (in the sense of Item (iii) from Lemma 67).

(v)

$$\sum_i |A_i| \lesssim (D_j^2 + D_j E_j) n + \sum_{S \in \mathcal{S}} \sum_{x \in X_{i_s}} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}_g(\gamma, x), \quad (4.42)$$

where $\text{mult}_g(\gamma, x)$ is as defined in Definition 28.

Once we have Lemma 68, we can prove Lemma 67 as follows. For each index j , apply Lemma 68 to each collection $(Y_j, \mathcal{S}, \{\Gamma_{S,j}\}_{S \in \mathcal{S}}, \mathcal{I} \cap I(\mathcal{P}_j, \mathcal{S}), \{\Xi_{S,j}\})$, and denote the resulting incidence model $\mathcal{M}_j = \{(U_i^{(j)}, A_i^{(j)}, B_i^{(j)}, \iota_i^{(j)})\}$. From (4.38) and (4.41), we have

$$\sum_j \sum_i \sum_{\alpha, \alpha' \in A_i^{(j)}} |\alpha \cap \alpha'| \lesssim |\mathcal{S}|^2. \quad (4.43)$$

If $(p, S) \in \mathcal{I}$, then as noted above there exists a unique index j and a curve $\gamma \in \Gamma_{S,j}$ with $p \in \mathcal{P}_j$ and $p \in \gamma$. But then by Property (iii), there exists some index i , some $\tilde{p} \in B_i^{(j)}$, some $\alpha \in A_i^{(j)}$ such that $p \in \iota_i^{(j)}(\tilde{p})$.

Finally,

$$\begin{aligned} \sum_j \sum_i |A_i^{(j)}| &\lesssim n \sum_j (D_j^2 + D_j E_j) + \sum_{S \in \mathcal{S}} |\Xi_S^{(j)}| \\ &\lesssim n \sum_j D_j E_j, \end{aligned} \quad (4.44)$$

where on the final line we used the fact that $D_j \lesssim E_j$.

Thus the incidence model $\mathcal{M} = \bigcup_j \mathcal{M}_j$ verifies the requirements of Lemma 67. This concludes the proof of Lemma 67, modulo the proof of Lemma 68. \square

Proof of Lemma 68. We shall select a very large ball B_0 containing all of the points from \mathcal{P} , and we shall decompose $B_0 \cap (Y_j)_{\text{smooth}}$ into a union of 2-dimensional C^∞ manifolds. The manifolds will have the property that for a suitably chosen 2-plane

$$\Pi_0 = \langle e_1, e_2 \rangle, \quad (4.45)$$

any affine translate of Π_0 will intersect a given manifold at most once (such manifolds are called “monotone” in the computational geometry literature).

Indeed, let

$$X_j = \{z \in (Y_j)_{\text{smooth}} : \dim(\Pi_0 \cap T_z(Y_j)) \geq 1\}. \quad (4.46)$$

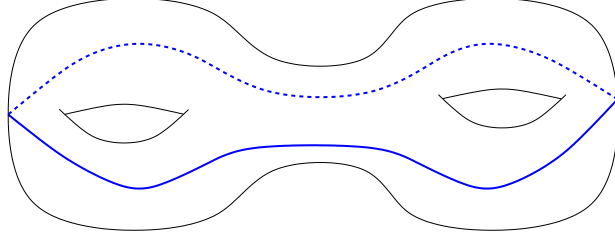


Figure 4.1: Here, Y_j is a 2-torus in \mathbb{R}^4 . In the figure, we have projected Y into \mathbb{R}^3 with a projection π chosen so that $\pi(\Pi_0)$ is a vertical line passing through the origin. The set X_j is denoted by the blue line.

If our choices for e_1 and e_2 in (4.45) are generic, then X_j may be empty or it may be a union of isolated points and 1-dimensional curves. We can see that each (necessarily smooth) connected component of $(Y_j)_{\text{smooth}} \setminus X_j$ is monotone: First, let $\pi: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a generic projection in the direction of some vector $v \in \Pi_0$, so $\pi(\Pi_0) \subset \mathbb{R}^3$ is a line passing through the origin (see Figure 4.1). Let e be a vector so that $\langle e \rangle = \pi(\Pi)$. Then if U is a (necessarily bounded) connected component of $B_0 \cap (Y_j)_{\text{smooth}} \setminus X_j$ then $\pi(U)$ is also bounded and connected. It suffices to show that for any $z \in \pi(U)$, z is a smooth point of $\pi(U)$, and the line $z + \langle e \rangle$ meets $\pi(U)$ solely at the point z . First, suppose $\pi(U)$ is not smooth, and let z be a singular point. Then since U is smooth and π is a local diffeomorphism in a neighborhood of each pre-image point of $\pi^{-1}(z)$, in a small neighborhood of z , $\pi(U)$ is a union of distinct, smooth 2-manifolds, each of which contains z . Thus, we can find a nearby point z' for which $z' + \langle e \rangle$ meets $\pi(U)$ in at least two distinct points, call them z'_1 and z'_2 . Now, let β be a smooth path in U connecting a pre-image of z'_1 to a pre-image of z'_2 . Then $\pi(\beta)$ is a smooth curve in $\pi(U)$, and $d(\pi(\beta))$ always lies in $TC_g(\pi(U))$, the geometric tangent cone of $\pi(U)$. But $TC_g(\pi(U))$ never contains $\langle e \rangle$, which is a contradiction. Thus each compact connected component of $(Y_j)_{\text{smooth}} \setminus X_j$ is monotone.

We must now count how frequently a curve $\gamma \in \Gamma_{S,j}$ intersects $(Y_j)_{\text{sing}} \cup X_j$. This will be done in the next two lemmas.

Lemma 69. *Let $\mathcal{P}, \mathcal{S}, Y_j, Z_j, W_j, P_j, Q_j$ and $\{\Gamma_{S,j}\}_{S \in \mathcal{S}}$ be as in the statement of Lemma 68.*

Select $\gamma \in \Gamma_{S,j}$ and let X_j be as in (4.46). Then

$$|\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \lesssim \deg \gamma \cdot (D_j + E_j). \quad (4.47)$$

Proof. First, note that if our choices of e_1 and e_2 in (4.45) are generic, then for each curve $\gamma \in \bigcup_S \Gamma_{S,j}$, we have that $X_j \cap \gamma$ will be a discrete set of points. Thus, we can guarantee the following things:

1. Every point of $X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$ is a smooth point of γ and a smooth point of X_j . In particular, no point of $X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$ is an isolated point of X_j .
2. At every point $z \in X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$, we have the following property: let $B \subset \mathbb{C}^4$ be a sufficiently small ball centered at z^* , then

$$G(B^* \cap Z_j^* \cap W_j^*, 2) = \{(z, T_z(Y_j)) : z \in B \cap (Y_j)_{\text{smooth}}\}$$

is a smooth 2-(complex)-dimensional sub-manifold of $\mathbb{C}^4 \times \text{Gr}(4, 2; \mathbb{C})$. Let

$$J = \mathbb{C}^4 \times \{\Pi \in \text{Gr}(4, 2; \mathbb{C}) : \dim(\Pi \cap \Pi_0) \geq 1\}. \quad (4.48)$$

Then if $z \in X_j \cap (Y_j)_{\text{smooth}} \cap \gamma$, we have that $T(B \cap Z_j^* \cap W_j^*)$ and J intersect transversely at the point $(z, T_z(Y_j)) \in \mathbb{C}^4 \times \text{Gr}(4, 2; \mathbb{C})$. Thus (since e_1 and e_2 were chosen generically), $T(B \cap Z_j^* \cap W_j^*) \cap J$ is a smooth curve in $\mathbb{C}^4 \times \text{Gr}(4, 2; \mathbb{C})$. Furthermore, if we apply a small C^1 perturbation to the surface $B \cap Z_j^* \cap W_j^*$, then the image of the perturbed surface in $\mathbb{C}^4 \times \text{Gr}(4, 2; \mathbb{C})$ will still intersect J transversely.

First, observe: If (P_j, Q_j) is a radical ideal (which by Proposition 30 is equivalent to $\dim \left(\frac{\nabla P_j}{\nabla Q_j} \right) = 2$ on $(Y_j)_{\text{smooth}}$), then

$$(Y_j)_{\text{sing}} \cup X_j = Y_j \cap \mathbf{Z}(\Psi(P_j, Q_j; \cdot)), \quad (4.49)$$

where

$$\Psi(P_j, Q_j; z) = \det \begin{bmatrix} e_1 \\ e_2 \\ \nabla P_j \\ \nabla Q_j \end{bmatrix} (z). \quad (4.50)$$

Then, to compute $|\gamma \cap (X_j \cup (Y_j)_{\text{sing}})|$, it suffices to count the number of intersection points in $\gamma \cap \mathbf{Z}(\Psi(P_j, Q_j; \cdot))$, and this is a (set-theoretic) complete intersection. We are working over \mathbb{R} , so we cannot appeal directly to Bézout’s theorem, but we can use arguments similar to those used to obtain (4.29).

In order to make this argument work, we will need to perturb P_j and Q_j to make (P_j, Q_j) a radical ideal. Doing so will cause $\mathbf{Z}(P_j)^* \cap \mathbf{Z}(Q_j)^*$ to “split” (in a small neighborhood of a smooth point) into several sheets, and (locally) there will be one or more copy or copies of γ on each sheet. Through careful counting, we can recover the above result.

The first difficulty is that $S \cap Y_j$ is not a proper intersection; S and Y_j each have codimension 2, so we would expect $S \cap Y_j$ to have dimension 0, but instead it has dimension 1. This will complicate our attempts to use tools from intersection theory. To deal with this, we will replace S with a larger variety that does intersect Y_j properly. We will then recover the intersection properties of S from those of the larger variety.

Let S be the 2–flat and let $\gamma \in \Gamma_{S,j}$ be the curve in the statement of Lemma 69. We define

$$S^\dagger = \bigcup_{x \in S} (x + \langle e \rangle), \tag{4.51}$$

where e is a generic vector in \mathbb{R}^4 . We can verify that S^\dagger is an irreducible 3–dimensional variety of bounded degree: in short, select a rotation of \mathbb{R}^4 so that e is the x_1 –direction. By the Tarski-Seidenberg theorem (see e.g. [BCR98]), the projection $\pi_{x_1}(S)$ is a bounded degree algebraic variety, with ideal $I \subset \mathbb{R}[x_2, x_3, x_4]$. Let $S^\dagger = \mathbf{Z}(I^\dagger)$, where $I^\dagger \in \mathbb{R}[x_1, \dots, x_4]$ is the canonical embedding of I into $\mathbb{R}[x_1, \dots, x_4]$. S^\dagger has codimension 1, so we can write $S^\dagger = \mathbf{Z}(f_S)$ for some polynomial f_S that generates a real ideal.

Now, S^\dagger has codimension 1, Y_j has codimension 2, and $S^\dagger \cap Y_j$ is a 1–dimensional curve. Thus $S^\dagger \cap Y_j$ is a proper intersection (this is the entire point of introducing S^\dagger). We also have $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ is a 1–dimensional (complex) curve and so $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ is a proper intersection. Since $\gamma \subset S \cap Y_j$ we also have $\gamma \subset S^\dagger \cap Y_j$, so there exists an irreducible component $\gamma^\dagger \subset S^\dagger \cap Y_j$ such that (as sets) $\gamma^\dagger = \gamma$. We define $(\gamma^\dagger)^* \subset (S^\dagger)^* \cap Z_j^* \cap W_j^*$

similarly. We define

$$\text{mult}(Z_j^*, W_j^*; (\gamma^\dagger)^*) = \dim \mathcal{O}_{\mathbb{P}^4, x} / (I_{P_j} + I_{Q_j}), \quad (4.52)$$

where x is a generic point of $(\gamma^\dagger)^*$. By Proposition 17, $(\gamma^\dagger)^*$ is irreducible, and thus $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ is well-defined.

Since $(\gamma^\dagger)^*$ lies generically in $(Z_j^* \cap W_j^*)_{\text{smooth}}$, there exists a unique irreducible component V of $(Z_j^* \cap W_j^*)_{\text{smooth}}$ that contains $(\gamma^\dagger)^*$, and

$$\text{mult}(Z_j^*, W_j^*; \gamma) = \text{mult}(Z_j^*, W_j^*; V),$$

where the latter multiplicity is given by Definition 32.

Let $P'_j = P_j - \epsilon_1$, $Q'_j = Q_j - \epsilon_2$, where ϵ_1, ϵ_2 are chosen generically from the interval $(0, \epsilon)$; ϵ will be chosen later. Then by Proposition 31, (P'_j, Q'_j) is a radical ideal.

We claim: Suppose z and B satisfy the following:

- z is a smooth point of γ (and thus z^* is a smooth point of $(\gamma^\dagger)^*$),
- z is a smooth point of Y_j ,
- z is a smooth point of $S^\dagger \cap Y_j$,
- $B \subset \mathbb{C}^4$ is a sufficiently small ball centered at z^*

then:

- $B \cap (Z'_j)^* \cap (W'_j)^*$ is a union of $\text{mult}(Z_j^*, W_j^*; (\gamma^\dagger)^*)$ smooth disjoint 2-manifolds, and each of these 2-manifolds is a $O(\epsilon)$ -perturbation of $B \cap Z_j^* \cap W_j^*$.
- $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ is a union of $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ smooth curves, and each of these curves is a $O(\epsilon)$ -perturbation of $B \cap (\gamma^\dagger)^*$. These curves lie on the various connected components of $B \cap (Z'_j)^* \cap (W'_j)^*$.

The key observation is the following.

Lemma 70. *Select $z \in \gamma \cap X_j \cap (Y_j)_{\text{smooth}}$ and let $\rho > 0$. Let B be the ball centered at z of radius ρ . Then provided ϵ is sufficiently small (depending on ρ), we have*

$$\begin{aligned} & |B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))| \\ & \geq \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*). \end{aligned} \tag{4.53}$$

Proof. The idea is to show that each of the 1–dimensional curves in $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ (there are $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ such curves) intersects some 1–dimensional curve from $B \cap (Z'_j)^* \cap (W'_j)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$. We expect this to happen because the (unperturbed) curves intersect transversely, and the perturbation is very small.

Indeed, let $\zeta \subset B$ be a simple curve from $B \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. Then since $\gamma \cap B$ is smooth, ζ is a $O(\epsilon)$ –perturbation of $\gamma \cap B$, in the sense of Definition 36. Let $U \subset B \cap (Z'_j)^* \cap (W'_j)^*$ be the smooth 2–manifold containing ζ . Then since $B \cap X_j$ is smooth, $U \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$ is a $O(\epsilon)$ perturbation of $B \cap X_j^*$. Now, z is one of finitely many intersection points of γ and X_j , and each of these intersections are transverse. Thus if we select ϵ sufficiently small (depending on both the transversality of the intersection and ρ), then ζ and $U \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$ must intersect. Thus in particular, ζ intersects $B \cap (Z'_j)^* \cap (W'_j)^* \cap \mathbf{Z}^*(\Psi(P'_j, Q'_j; \cdot))$. But there are $\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)$ such curves ζ . This establishes Lemma 70. \square

We claim: if ϵ is sufficiently small, then there exists a curve $\zeta_\gamma \subset (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ which is the “image” of $(\gamma^\dagger)^*$ under the above perturbation. Indeed, we can select a small constant c so that if $G \subset (\gamma^\dagger)^*$ is the set of points that are distance at least $2c$ from any point of $Z_j^* \cap W_j^* \cap (S^\dagger)^*$ that does not lie in $(\gamma^\dagger)^*$, then G contains an open interval. Let $z \in G \cap (\gamma^\dagger)_{\text{smooth}}^* \cap (Z_j^* \cap W_j^*)_{\text{smooth}}$ be a point contained in the relative interior of this open interval, and let B_1 be a small ball centered at z . Then if we select ϵ sufficiently small, then $B_1 \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$ is a union of curves, each of which is a c –perturbation of $B_1 \cap (\gamma^\dagger)^*$. Let ζ_γ be the smallest algebraic set containing $B_1 \cap (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. We have $\zeta_\gamma \subset (Z'_j)^* \cap (W'_j)^* \cap (S^\dagger)^*$. ζ_γ corresponds to the intuitive notation of the “image” of $(\gamma^\dagger)^*$ under the perturbation.

We can now bound the degree of ζ_γ . Let $H \subset \mathbb{C}^4$ be a generic 3-plane. Then $|H \cap \gamma| = \deg \gamma$. But by the definition of multiplicity above, to each point $x \in H \cap \gamma$, we can find a small ball B centered at x such that

$$B \cap \zeta_\gamma \cap H = \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*).$$

Thus

$$\deg \zeta_\gamma = \deg \gamma \cdot \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*). \quad (4.54)$$

Lemma 70 can be rephrased as the statement

$$\begin{aligned} & |\zeta_\gamma \cap \mathbf{Z}^*(\Phi(P'_j, Q'_j; \cdot))| \\ & \geq |\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \cdot \text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*), \end{aligned} \quad (4.55)$$

i.e.

$$|\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| \leq \frac{|\zeta_\gamma \cap \mathbf{Z}^*(\Phi(P'_j, Q'_j; \cdot))|}{\text{mult}(Z_j^*, W_j^*, (S^\dagger)^*; (\gamma^\dagger)^*)}. \quad (4.56)$$

But this and (4.54) imply that

$$\begin{aligned} |\gamma \cap X_j \cap (Y_j)_{\text{smooth}}| & \leq \deg \gamma \cdot \deg \Phi(P'_j, Q'_j; \cdot) \\ & \lesssim \deg \gamma \cdot (D_j + E_j). \end{aligned} \quad (4.57)$$

This concludes the proof of Lemma 69. □

We can now complete the proof of Lemma 68. Fix a choice of $S \in \mathcal{S}$. For each $\gamma \in \Gamma_{S,j}$, Delete the following points:

- The points that lie in Ξ_S ,
- The points of $\gamma \cap (Y_j)_{\text{sing}}$,
- The points of $\gamma \cap X_j$,
- The points that lie on some $\gamma' \neq \gamma$.

We can verify that after these points have been removed, the remaining set is a disjoint union of connected 1–dimensional manifolds. Furthermore, the number of 1–manifolds is

$$\begin{aligned}
& O\left((D_j + E_j) \sum_{\gamma \in \Gamma_{S,j}} \deg \gamma + \sum_{\gamma \in \Gamma_{S,j}} \sum_{x \in \gamma \cap (Y_j)_{\text{sing}}} \text{mult}(\gamma, x)\right) \\
& + \left(\sum_{\gamma \in \Gamma_{S,j}} \deg \gamma \right)^2 + \sum_{x \in \Xi_S} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x)
\end{aligned} \tag{4.58}$$

Indeed, every time a point x is removed, the number of manifolds can increase by at most $\sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x)$. If $x \in \gamma \cap (Y_j)_{\text{smooth}} \cap X_j$, then $\text{mult}(\gamma, x) = 1$, so the number of curves added by removing points of this type is at most $O\left((D_j + E_j) \sum_{\gamma \in \Gamma_{S,j}} \deg \gamma\right) = O(D_j^2 + D_j E_j)$.

The only term remaining to bound is the following

Lemma 71. *Let $\Gamma_{S,j}$ and Y_j be as above. Then*

$$\sum_{\gamma \in \Gamma_{S,j}} \sum_{x \in \gamma \cap (Y_j)_{\text{sing}}} \text{mult}(\gamma, x) \lesssim D_j E_j + D_j^2. \tag{4.59}$$

Proof. Let S^\dagger be as defined in (4.51). For each $\gamma \in \Gamma_{S,j}$, let γ^\dagger and $(\gamma^\dagger)^*$ be defined as above. Then if $x \in \gamma \cap (Y_j)_{\text{sing}}$, there must be smooth points of Y_j in every (Euclidean) neighborhood of x ($\gamma \in \Gamma_{S,j}$ implies γ lies generically on $(Y_j)_{\text{smooth}}$). Thus x^* is a singular point of $Z_j^* \cap W_j^*$, so

$$\dim \mathcal{O}_{\mathbb{C}\mathbb{P}^4, x} / (I_{f_S} + I_{P_j} + I_{Q_j}) > \text{mult}((S^\dagger)^*, Z_j^*, W_j^*; (\gamma^\dagger)^*). \tag{4.60}$$

From this we can conclude that x^* is a singular point of the (not necessarily irreducible) curve $(S^\dagger)^* \cap Z_j^* \cap W_j^*$. Furthermore, the number of intersection points of $(\gamma^\dagger)^*$ with $(Z_j^* \cap W_j^*)_{\text{sing}}$ (counting multiplicity) as γ ranges over all curves in $\Gamma_{S,j}$ is controlled by the number of singular points of $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that occur on S^* (again, counting multiplicity). Let $\zeta \subset (S^\dagger)^* \cap Z_j^* \cap W_j^*$ be the union of all irreducible curves in $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that are not contained in S^* . Then the number of singular points of $(S^\dagger)^* \cap Z_j^* \cap W_j^*$ that occur on S^* is at most the number of singular points of $S^* \cap Z_j^* \cap W_j^*$ plus the number of intersection points of ζ with S^* (again, counting multiplicity). The former quantity is at most D_j^2 , and the latter is at most $D_j E_j$. \square

After the points described above have been deleted from γ , we are left with a collection of 1-manifolds. Each of these manifolds is homeomorphic to either the interval $(0, 1)$ or the circle S^1 . For those manifolds that are homeomorphic to circles, remove two points at random to obtain two curves homeomorphic to $(0, 1)$. Our new collection of simple curves has the same cardinality (up to a factor of two), and each curve lies entirely within a single component of $(Y_j)_{\text{smooth}} \setminus X_j$. Let $\{U_i\}$ be the connected components of $(Y_j)_{\text{smooth}} \setminus X_j$, and let A_i consist of those simple curves lying in U_i , as γ ranges over $\bigcup_{S \in \mathcal{S}} \Gamma_{S,j}$. Let $B_i = U_i \cap \mathcal{P}_j$ and let $\iota_i: U_i \hookrightarrow \mathbb{R}^4$ be the canonical embedding of U_i into \mathbb{R}^4 .

From (4.58) we have

$$\sum_i |A_i| \lesssim D_j^2 + D_j E_j + \sum_{S \in \mathcal{S}} \sum_{x \in \Xi_S} \sum_{\gamma \in \Gamma_{S,j}} \text{mult}(\gamma, x), \quad (4.61)$$

so Requirement (v) is satisfied. In order to verify the remaining requirements we shall introduce some notation.

Definition 72. Let $\alpha \in A_i$ for some index i . Then there exists a unique 2-flat $S_0 \in \mathcal{S}$ and a unique curve $\gamma_0 \in \Gamma_{S_0,j}$ such that $\alpha \subset \iota_i^{-1}(\gamma_0)$. We will define $S(\alpha)$ to be S_0 and we will define $\gamma(\alpha)$ to be γ_0 .

We can verify that $\mathcal{P}_j = \bigcup_i B_i$, i.e. that $X_j \cap \mathcal{P}_i = \emptyset$, since the vectors e_1 and e_2 from (4.45) were chosen generically. Recall as well that by assumption, $\mathcal{P}_j \cap (Y_j)_{\text{sing}} = \emptyset$. If $\gamma \in \Gamma_{S,j}$, $p \in \mathcal{P}$, $p \in \gamma$, then we can immediately verify that either there exists an index i and a curve $\alpha \in A_i$ with $\gamma(\alpha) = \gamma$ and $p \in \partial(\iota_i(\alpha))$, or there exists an index i , a curve $\alpha \in A_i$, and a point \tilde{p} in B_i with $\iota_i(\tilde{p}) = p$, $\gamma(\alpha) = \gamma$, and $\tilde{p} \in \alpha$. Thus Requirement (iii) is satisfied.

To verify Requirement (ii), note that if $\alpha, \alpha' \in A_i$ and $x \in \alpha \cap \alpha'$, Then $\iota_i(x) \in \iota_i(\alpha) \cap \iota_i(\alpha')$, and $x \notin \Xi_{S(\alpha)} \cup \Xi_{S(\alpha')}$. Furthermore, (α, α') are the unique pair of curves in $\{A_i\}$ corresponding to the triple $(\iota_i(x), \gamma(\alpha), \gamma(\alpha'))$.

To obtain Requirement (iv), we shall apply a slight perturbation to the curves of $\{A_i\}$ as follows: for each curve $\alpha \in A_i$ and point $p \in B_i$, if $p \in \alpha$ but $(S(\alpha), \iota(p)) \notin \mathcal{I}$, modify α in a small neighborhood of p so that $p \notin \alpha$. We can always do this in such a way that the number

of crossings between α and the other curves is not affected, and α remains unchanged in a small neighborhood of every point $p' \in B_i$ distinct from p . After this perturbation has been performed for every curve, then the collection $\{(A_i, B_i)\}$ satisfies Requirement (iv). Indeed, since \mathcal{I} is k -admissible, if there exists an index i , a collection of k points $p_1, \dots, p_k \in B_i$, and a collection of $C_0 + 1$ curves in A_i such that each curve is incident to each of p_1, \dots, p_k , then there must exist two curves α, α' from this collection of curves such that $S(\alpha) = S(\alpha')$. However, this implies that $\iota_i(\alpha)$ either contains a singular point of $\gamma(\alpha)$, or it contains a point of $\gamma(\alpha')$ (if $\gamma(\alpha') \neq \gamma(\alpha)$). But by our construction of A_i , no such points may lie on any curve in A_i . Thus Requirement (iv) is satisfied.

Finally, recall that we already established Requirement (i) above (in the discussion preceding Lemma 69). □

4.3.2 Szemerédi-Trotter on a domain

Use Lemma 67 on the data $(Y, \mathcal{P} \cap Y, \mathcal{S}, \mathcal{I}_1)$. Denote the resulting incidence model \mathcal{M} and the resulting error set \mathcal{I}' . We have

$$\sum_i |I(A_i, B_i)| \geq |\mathcal{I}_1|, \tag{4.62}$$

$$\sum_i |B_i| \leq \sum_j |\mathcal{P}_j|, \tag{4.63}$$

$$\sum_i |A_i| \leq |\mathcal{S}| \sum_j D_j E_j, \tag{4.64}$$

$$\sum_i \mathcal{C}(U_i) \leq \sum_{S, S'} |S \cap S'|. \tag{4.65}$$

(4.62), (4.63), and (4.65) follow from the definition of \mathcal{M}_j and the observation that if $\alpha \in A_i$, $|\partial\alpha| = 2$, i.e. an open curve only has two ends.

We will now control the number of incidences that occur on the surfaces $\{U_i\}$.

Definition 73. Let A be a collection of simple curves (i.e. homeomorphic images of $(0, 1)$) and B a collection of points on a planar domain $U \subset \mathbb{R}^2$. Then we define the number of

incidences between curves in A and points in B to be

$$I(A, B) = \{(\alpha, p) \in A \times B : p \in \bar{\alpha}\},$$

and we define the number of crossings of curves in B to be

$$\mathcal{C}(U) = |\{(\alpha, \alpha', p) \in A^2 \times U : p \in \alpha \cap \alpha'\}|.$$

Lemma 74. *Let:*

- U be a smooth 2-dimensional manifold that is homeomorphic to an open subset of \mathbb{R}^2 .
- A be a collection of 1-dimensional open curves lying on U .
- B be a collection of points on U .
- Let C_0 be a constant so that for any two points $p_1, p_2 \in B$, there are at most C_0 curves from A that contain p_1 and p_2

Then

$$|I(A, B)| \lesssim |B|^{2/3} \mathcal{C}(U)^{1/3} + |B| + |A|. \quad (4.66)$$

where the implicit constant depends only on C_0 .

Remark 75. Note that in Definition 73, a point is incident to an (open) curve if it lies on the closure of that curve. On the other hand, a crossing of two curves is a point common to the relative interior of both curves. Similarly, in Lemma 74 we require that for any two points, there are at most $O(1)$ curves which contain those points in their relative interiors—there may be arbitrarily many curves whose closures contain the two points.

Proof. Since all of the quantities we wish to consider are invariant under homeomorphism, without loss of generality we can assume U is an open subset of \mathbb{R}^2 . Now, replace each curve $\gamma \in A$ with a slightly “shrunk” curve γ' , so that $\partial(\gamma')$ does not meet any point from B nor any curve from A . If A' denotes the set of shrunk curves, then $|I(A', B)| \geq |I(A, B)| - 2|A|$.

Delete from A' those curves γ' that are incident to fewer than 2 points from B , and denote the resulting set of curves A'' . Then $|I(A'', B)| \geq |I(A, B)| - 4|A|$.

Consider the drawing H of the graph whose vertices are the points of B and where two vertices are connected by an edge if the two corresponding points are joined by a curve from A'' . Then H is an admissible graph drawing. Note that $\mathcal{E}(H) \geq \frac{1}{2}|I(A'', B)|$, so by Theorem 3,

$$|I(A, B)| \lesssim |B|^{2/3} \mathcal{C}(U)^{1/3} + |B| + 4|A|. \quad \square$$

By Lemma 67, Item *ii*, we have

$$\mathcal{I}_1 \leq m^{2/3} n^{2/3} + \sum_i I(A_i, B_i), \quad (4.67)$$

where the indices i run over the elements of the incidence model \mathcal{M} . Applying Lemma 74, we thus have

$$\begin{aligned} \mathcal{I}_1 &\leq m^{2/3} n^{2/3} + \sum_i |B_i|^{2/3} \mathcal{C}(U_i)^{1/3} + \sum_i |B_i| + \sum_i |A_i| \\ &\lesssim m^{2/3} n^{2/3} + \left(\sum_i |B_i| \right)^{2/3} \left(\sum_i \mathcal{C}(U_i) \right)^{1/3} + m^{2/3} n^{2/3} \\ &\lesssim m^{2/3} n^{2/3}. \end{aligned} \quad (4.68)$$

This establishes the bound (4.25), and thus concludes the proof of Theorem 59.

CHAPTER 5

A variable coefficient Wolff circular maximal function

5.1 Introduction

In [Wol97a], Wolff considered the following maximal function:

$$M^\delta f(r) = \sup_{x \in \mathbb{R}^2} \frac{1}{|C^\delta(x, r)|} \int_{C^\delta(x, r)} |f(y)| dy, \quad (5.1)$$

where $C^\delta(x, r)$ is the δ -neighborhood of a circle centered at x of radius r . This maximal function has the same relationship to Besicovich-Rado-Kinney (BRK) sets (compact subsets of the plane containing a circle of every radius $1/2 \leq r \leq 1$) as the Kakeya maximal function has to Kakeya sets. In particular, a bound of the form

$$\|M^\delta f\|_{L^p([1/2, 1])} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^2)} \quad (5.2)$$

for some value of p and all $\epsilon > 0$ would imply that every BRK set has Hausdorff dimension 2. See [Wol99] for further details. By considering the examples where f is the characteristic function of a ball of radius δ and a rectangle of dimensions $\delta \times \sqrt{\delta}$, we can see that $p = 3$ is the smallest value of p for which (5.2) can hold. In [Wol97a], Wolff proved (5.2) for $p = 3$.

In a similar vein, Wolff and Kolasa considered the more general class of maximal functions

$$M_\Phi^\delta f(r) = \sup_{x \in U_1} \frac{1}{|\Gamma^\delta(x, r)|} \int_{\Gamma^\delta(x, r)} |f(y)| dy. \quad (5.3)$$

Here, U_1 is a sufficiently small neighborhood of a point $a \in \mathbb{R}^2$, and $\Gamma^\delta(x, r)$ is the δ -neighborhood of the curve

$$\Gamma(x, r) = \{y \in U_2 : \Phi(x, y) = r\}, \quad (5.4)$$

where U_2 is a sufficiently small neighborhood of a point $b \in \mathbb{R}^2$ and

$$\Phi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

is a smooth function satisfying Sogge's *cinematic curvature* conditions at the point (a, b) :

- $$\nabla_y \Phi(a, b) \neq 0. \tag{5.5}$$

- $$\det \left(\nabla_x \begin{bmatrix} e \cdot \nabla_y \Phi(x, y) \\ e \cdot \nabla_y \left(\frac{e \cdot \nabla_y \Phi(x, y)}{|\nabla_y \Phi(x, y)|} \right) \end{bmatrix} \Big|_{(x, y) = (a, b)} \right) \neq 0, \tag{5.6}$$

where e is a unit vector orthogonal to $\nabla_y \Phi(a, b)$.

See [Sog91] for further discussion of cinematic curvature and its properties. Cinematic curvature was first introduced when studying the Bourgain circular maximal function (see e.g. [Bou86]), and it appears that replacing the circles $C(x, r)$ in (5.1) by families of curves satisfying the cinematic curvature condition is the most natural variable-coefficient generalization of the Wolff circular maximal function. In particular, geodesic circles for a Riemannian metric satisfy the cinematic curvature condition provided that the injectivity radius is larger than the diameter of the circles.

In [KW99], Wolff and Kolasa established the bound

$$\|M_{\Phi}^{\delta} f\|_{L^q([1/2, 1])} \leq C_{p, q} \delta^{-\frac{1}{2}(\frac{3}{p}-1)} \|f\|_{L^p(\mathbb{R}^2)}, \quad p < \frac{8}{3}, \quad q \leq 2p'. \tag{5.7}$$

In particular, (5.7) implies that any compact set containing a curve of the form $\{y: \Phi(x, y) = r\}$ for each $0 < r < 1$ must have Hausdorff dimension at least $11/6$. We shall call such sets *Cinematic BRK sets*.

5.1.1 New results

In [Zah13b], the author proved the following theorem:

Theorem 76. *Let Φ satisfy the cinematic curvature conditions (5.5) and (5.6). Then for all $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\|M_\Phi^\delta f\|_{L^3([1/2,1])} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^3(\mathbb{R}^2)}. \quad (5.8)$$

In particular, every cinematic BRK set must have Hausdorff dimension 2.

Corollary 77. Equation (5.2) holds with $p = 3$.

Theorem 76 improves upon a previous result of the author in [Zah12a] in which a similar statement is proved under the additional restriction that the function Φ be algebraic. We follow a similar proof strategy in this proof as in [Zah12a], but at a key step we use the discrete polynomial ham sandwich theorem rather than the vertical algebraic decomposition.

5.1.2 Proof sketch

Through standard techniques, it suffices to obtain certain weak-type bounds on $\sum \chi_{F^\delta}$ for a collection of curves $\{F\}$ with δ -separated “radii.” The main difficulty arises when many pairs of curves are almost tangent, and indeed a result due to Schlag in [Sch03] shows that we can obtain the desired bounds on M_Φ if we can control the number of such almost-tangencies. More specifically, if \mathcal{W} and \mathcal{B} are collections of curves such that all curves in \mathcal{W} (resp. \mathcal{B}) are close to each other in a suitable parameter space, and all curves in \mathcal{W} are far from curves in \mathcal{B} (again in a suitable parameter space), then we need to control the number of near-tangencies between curves in \mathcal{W} and curves in \mathcal{B} . We shall do this with an induction argument.

First, we shall use Jackson’s theorem to replace the curves $\{F\}$ by algebraic curves that closely approximate them. The degree of the algebraic curves will depend on δ , but the dependence is mild enough to be controllable. We will then identify the curves in \mathcal{W} with points in \mathbb{R}^3 (if the curves were actually circles, we could use the center and radius of the circle to perform this identification). We then use the discrete polynomial ham sandwich theorem to find a low degree trivariate polynomial P whose zero set partitions \mathbb{R}^3 into open “cells,” such that the points are evenly split up amongst the cells. To each curve $F \in \mathcal{B}$ we

associate a semi-algebraic set $Q(F) \subset \mathbb{R}^3$ (of controlled degree), such that if $F \in \mathcal{B}$ is almost tangent to $\tilde{F} \in \mathcal{W}$, then $Q(F)$ must intersect the cell containing (the point associated with) \tilde{F} . The bounds on the degree of P and $Q(F)$ yield bounds on the number of cells that $Q(F)$ can intersect. We then apply the induction hypothesis within each cell. Summing over all cells, we obtain the desired bound on the total number of almost-tangencies between curves in \mathcal{W} and \mathcal{B} .

The key innovation is the use of the discrete polynomial ham sandwich theorem. While the partition of \mathbb{R}^3 described above could be done with the vertical algebraic decomposition instead of the polynomial ham sandwich theorem, the resulting control on the number of cells that $Q(F)$ can intersect is so poor that we cannot run the induction argument except in the special case where the defining function Φ is algebraic (and thus the algebraic curves F have degree that does not depend on δ).

5.2 Proof of Theorem 76

5.2.1 Initial reductions

The first step will be to replace the defining function Φ by an algebraic approximation. This idea was suggested to the author by Larry Guth, and it appears in a similar form in [BG11]. Throughout the proof, we shall assume that Φ satisfies the cinematic curvature conditions at the point $(a, b) = (0, 0)$ and that U_1, U_2 are small balls centered at 0. By Jackson's theorem (see e.g. [BBL02]), for each $K > 0$, $A > 0$, and $\delta > 0$, we can find a polynomial $\Psi(x, y): \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\deg \Psi \leq C_K \delta^{-1/K}, \tag{5.9}$$

$$\|\Psi - \Phi\|_{C^2(B(0,100))} < \delta/A, \tag{5.10}$$

where C_K depends on K , A , and Φ . If A is chosen sufficiently large depending on the infimum of the quantities in (5.5) and (5.6), then Ψ satisfies (5.5) and (5.6).

Since $\|\nabla_y \Phi\|$ and $\|\nabla_y \Psi\|$ are bounded from below for $y \in U_1$ (after possibly shrinking U_1),

we have that if A is chosen sufficiently large in (5.10) then for each $x_0 \in U_2$ and $1/2 < r_0 < 1$ we have that $\{y \in U_1: \Phi(x_0, y) = r_0\}$ and $\{y \in U_1: \Psi(x_0, y) = r_0\}$ are contained in $\delta/100$ neighborhoods of each other. Thus if f is supported in $B(0, 1)$ then $M_\Phi^\delta f \sim M_\Psi^\delta f$, so it suffices to obtain bounds on $M_\Psi^\delta f$.

Remark 78. If the reader is only interested in the original Wolff circular maximal function, then this step can be omitted, and every instance of Ψ can be replaced by $\Phi(x, y) = \|x - y\|$. In this case, Ψ -circles are arcs of genuine circles. Throughout the proof, we shall refer to this situation as the “circles” case.

Fix $\alpha > 0$ sufficiently small depending on the quantities appearing in (5.5) and (5.6) and on $\|\Phi\|_{C^3(B(0,100))}$. For $x \in B(0, \alpha)$ and $r \in [1/2, 1]$, we define

$$\Gamma(x_0, r_0) = \{y \in B(0, \alpha): \Psi(x_0, y) = r_0\}. \quad (5.11)$$

We shall call these sets Ψ -circles, and if Γ is a Ψ -circle then Γ^δ will denote its δ -neighborhood. If $\Gamma, \tilde{\Gamma}$, etc. are Ψ -circles, then unless otherwise noted, x_0, r_0 and \tilde{x}_0, \tilde{r}_0 will refer to their respective centers and radii. The Ψ -circles defined here are strict subsets of the analogous sets F defined in the introduction. However, if the function f is supported on a sufficiently small neighborhood of the origin then we can define a maximal function analogous to (5.3) with Γ in place of F , and the two maximal functions will agree. Thus we shall henceforth work with curves Γ defined by (5.11).

We shall restrict our attention to those Ψ -circles Γ with $x_0 \in B(0, \alpha_1)$, $\alpha_1 = C_0^{-1}\alpha$, and $r_0 \in (1 - \tau, 1)$ where C_0 and τ are sufficiently small constants that depends only on the quantities appearing in (5.5) and (5.6) and on $\|\Phi\|_{C^2(B(0,100))}$. By standard compactness arguments, we can recover $L^p([1/2, 1])$ bounds on M_Ψ from those on the “restricted” version of M_Ψ by considering the supremum over a finite number of scaled versions of the function.

Using standard reductions (see e.g. [Sch03], §4), in order to prove Theorem 76 it suffices to prove the following lemma:

Lemma 79. *For $\eta > 0$ and δ sufficiently small depending on η , let \mathcal{A} be a collection of Ψ -circles with δ -separated radii, with each radius lying in $(1 - \tau, 1)$. Then there exists $\tilde{\mathcal{A}} \subset \mathcal{A}$*

with $|\tilde{\mathcal{A}}| \geq \frac{1}{C}|\mathcal{A}|$ such that for all $\Gamma \in \tilde{\mathcal{A}}$ and $\delta < \lambda < 1$,

$$\left| B(0, \alpha_1) \cap \{y \in \Gamma^\delta : \sum_{\tilde{\Gamma} \in \mathcal{A}} \chi_{\tilde{\Gamma}^\delta}(y) > \delta^{-n} \lambda^{-2}\} \right| \leq \lambda |\Gamma^\delta|. \quad (5.12)$$

5.2.2 Schlag's reduction

We shall recall a result due to Schlag that shows that Lemma 79 is implied by a combinatorial lemma controlling the number of almost-incidences between Ψ -circles. In order to state Schlag's result, we will first need several definitions.

Definition 80. Let Γ and $\tilde{\Gamma}$ be two Ψ circles. We define

$$\Delta(\Gamma, \tilde{\Gamma}) = \inf_{\substack{y \in B(0, \alpha_1) : \Psi(x_0, y) = r_0 \\ \tilde{y} \in B(0, \alpha_1) : \Psi(\tilde{x}_0, \tilde{y}) = \tilde{r}_0}} |y - \tilde{y}| + \left| \frac{\nabla_y \Psi(x_0, y)}{\|\nabla_y \Psi(x_0, y)\|} - \frac{\nabla_y \Psi(\tilde{x}_0, \tilde{y})}{\|\nabla_y \Psi(\tilde{x}_0, \tilde{y})\|} \right|. \quad (5.13)$$

Informally, if $\Delta(\Gamma, \tilde{\Gamma})$ is small then there is a point $y \in B(0, \alpha_1)$ where Γ and $\tilde{\Gamma}$ pass close to each other and are nearly parallel (i.e. they are nearly tangent).

Let

$$d(\Gamma, \tilde{\Gamma}) = |x_0 - \tilde{x}_0| + |r_0 - \tilde{r}_0|. \quad (5.14)$$

$d(\cdot, \cdot)$ is a metric on the space of curves.

Definition 81. Let \mathcal{W} and \mathcal{B} be collections of Ψ -circles. We say that $(\mathcal{W}, \mathcal{B})$ is a (δ, t) -bipartite pair if

$$|r_0 - \tilde{r}_0| \geq \delta \text{ for all } \Gamma, \tilde{\Gamma} \in \mathcal{W} \cup \mathcal{B}, \quad (5.15)$$

$$d(\Gamma, \tilde{\Gamma}) \in (t, 2t) \text{ if } \Gamma \in \mathcal{W}, \tilde{\Gamma} \in \mathcal{B}, \quad (5.16)$$

$$d(\Gamma, \tilde{\Gamma}) \in (0, t) \text{ if } \Gamma, \tilde{\Gamma} \in \mathcal{W} \text{ or } \Gamma, \tilde{\Gamma} \in \mathcal{B}. \quad (5.17)$$

Definition 82. A (δ, t) -rectangle R is the δ -neighborhood of an arc of length $\sqrt{\delta/t}$ of a Ψ -circle Γ . We say that a Ψ -circle Γ is *incident* to R if R is contained in the $C_1\delta$ neighborhood of Γ . We say that R is of type $(\gtrsim \mu, \gtrsim \nu)$ relative to a (δ, t) -bipartite pair $(\mathcal{W}, \mathcal{B})$ if R is incident to at least μ curves in \mathcal{W} and at least ν curves in \mathcal{B} . We say that R is of type

$(\sim \mu, \sim \nu)$ if it is of type $(\gtrsim \mu, \gtrsim \nu)$, but is neither of type $(\gtrsim C\mu, \gtrsim \nu)$ nor $(\gtrsim \mu, \gtrsim C\nu)$ for some absolute constant C which shall be determined later. We say that two (δ, t) -rectangles R_1, R_2 are *comparable* if R_1 is contained in a $A_0\delta$ -neighborhood of R_2 and vice versa, where A_0 is an absolute constant. Otherwise, we say R_1 and R_2 are *incomparable*.

We are now able to state Schlag's result.

Proposition 83 (Schlag). Let \mathcal{A} be a family of Ψ -circles with δ -separated radii that satisfy the following requirements:

$$(i) \quad |\Gamma^\delta \cap \tilde{\Gamma}^\delta \cap B(0, \alpha)| \lesssim \frac{\delta^2}{(d(\Gamma, \tilde{\Gamma}) + \delta)^{1/2}(\Delta(\Gamma, \tilde{\Gamma}) + \delta)^{1/2}}. \quad (5.18)$$

(ii) Fix $\epsilon > 0$. Then there exists a constant C_ϵ so that for any (δ, t) -bipartite pair $(\mathcal{W}, \mathcal{B})$, with $t > C\delta$ for an appropriate choice of C ; $\mathcal{W}, \mathcal{B} \subset \mathcal{A}$; $|\mathcal{W}| = m$; and $|\mathcal{B}| = n$, the maximum number of pairwise incomparable (δ, t) -rectangles of type $(\gtrsim \mu, \gtrsim \nu)$ relative to $(\mathcal{W}, \mathcal{B})$ is at most

$$C_\epsilon \delta^{-\epsilon} \left(\left(\frac{mn}{\mu\nu} \right)^{3/4} + \frac{m}{\mu} + \frac{n}{\nu} \right). \quad (5.19)$$

Then Lemma 79 holds for the collection \mathcal{A} .

Remark 84. Schlag uses the stronger bound

$$C_\epsilon (mn)^\epsilon \left(\left(\frac{mn}{\mu\nu} \right)^{3/4} + \frac{m}{\mu} + \frac{n}{\nu} \right) \quad (5.20)$$

in place of (5.19). However, an examination of the proof in [Sch03] reveals that the bound (5.19) suffices. If we restrict our attention to the original Wolff circular maximal function (i.e. if we are only concerned with the circles case), then we obtain the bound (5.20), so Schlag's result can be used as a black box.

Property (i) follows from [KW99, Lemma 3.1(i)], but if the reader is only interested in the original Wolff circular maximal function, a shorter proof can be found in [Wol99, §3]. Property (ii) follows from the following lemma, which is an analogue of Lemma 1.4 in [Wol97b]:

Lemma 85. *Let $\Psi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a (multivariate) polynomial of degree k satisfying the cinematic curvature requirements. Then for every $\epsilon > 0$ there exists a constant C_ϵ such that if $(\mathcal{W}, \mathcal{B})$ is a (δ, t) -bipartite pair of Ψ -circles with $|\mathcal{W}| = m$, $|\mathcal{B}| = n$, and if \mathcal{R} is a collection of pairwise incomparable (δ, t) -rectangles of type $(\gtrsim \mu, \gtrsim \nu)$ relative to $(\mathcal{W}, \mathcal{B})$, then*

$$|\mathcal{R}| \leq C_\epsilon k^{C_\epsilon} (mn)^\epsilon \left(\left(\frac{mn}{\mu\nu} \right)^{3/4} + \frac{m}{\mu} + \frac{n}{\nu} \right). \quad (5.21)$$

To obtain Property (ii) from Lemma 85, select $K > C_\epsilon/\epsilon$ in (5.9) and note that $(mn)^\epsilon \leq \delta^{2\epsilon}$. In the case of circles we have $k = O(1)$, and (5.21) becomes (5.20).

Thus all that remains is to prove Lemma 85. First, we shall recall several properties of curves satisfying the cinematic curvature condition.

5.3 Cinematic Curvature and its Implications

Many of Wolff's arguments from [Wol97a] rely on the local differential properties of families of circles. The relevant properties are captured by the notion of cinematic curvature defined in the introduction. In [KW99], Kolasa and Wolff establish several key properties of families of curves with cinematic curvature which we shall recall below.

Property 86 (Straightening out). Let $x_0 \in U_1$. Then we can find a diffeomorphism $\psi_{x_0}: U'_2 \rightarrow U_2$ and a choice of $r_0 = r_0(x_0)$ such that

$$\Psi(x_0, \psi_{x_0}(y)) - r_0 = y^{(2)}$$

where U'_2 is an appropriately chosen domain (which may no longer be a disk). Furthermore for fixed y_0 ,

$$\psi_{x_0}(y_0) \text{ and } r_0(x_0) \text{ are continuous functions of } x_0. \quad (5.22)$$

This is discussed in [KW99, p 136]. To simplify notation, we shall say that Ψ has been *straightened out* around x_0 if we (temporarily) replace the function $\Psi(x_0, \cdot)$ with $\Psi(x_0, \phi_{x_0}(\cdot)) - r_0(x_0)$, i.e. in "straightened out" coordinates, $\Psi(x_0, y) = y^{(2)}$. Note that if we straighten out around x_0 then in this new coordinate system Ψ might no longer be algebraic.

This will not pose any problems to our analysis below; we shall only be straightening out to simplify the proofs of certain diffeomorphism-invariant statements, and the statement can then be “pulled back” to the original (algebraic) Ψ . This process may change some of the constants involved in the relevant statements. However (5.22) will guarantee that the constants are worsened by at most a bounded amount so we can safely ignore this problem.

Property 87 (Derivative bounds). If we straighten out Ψ at x_0 then for $y \in B(0, \alpha)$,

$$|\partial_{y^{(1)}}\Psi(x, y)| + |\partial_{y^{(1)}}^2\Psi(x, y)| \sim |x - x_0|, \quad (5.23)$$

$$|\partial_{y^{(2)}}\Psi(x, \psi_{x_0, r_0}(y))| \sim 1, \quad (5.24)$$

where $\partial_{y^{(1)}}$ denotes the partial derivative in the $y^{(1)}$ -direction, etc. The constants in the quasi-equalities above are uniform in all variables. Indeed, since the cinematic curvature condition is diffeomorphism invariant, (5.23) and (5.24) are equivalent to the cinematic curvature condition. This is addressed in Equation (21) of [KW99] and the surrounding discussion.

Property 88 (Unique point of parallel normals). Let $\Gamma, \tilde{\Gamma}$ be Ψ -circles with

$$\Delta(\Gamma, \tilde{\Gamma}) \leq C'^{-1}|x_0 - \tilde{x}_0|$$

for a sufficiently large constant C' . Then there is a unique point

$$\xi = \xi(x_0, r_0, \tilde{x}_0) \in \Gamma \cap B(0, \alpha)$$

such that

$$\nabla_y \Phi(x_0, \xi) \wedge \nabla_y \Phi(\tilde{x}_0, \xi) = 0. \quad (5.25)$$

Furthermore,

$$|\Phi(\tilde{x}_0, \xi) - \tilde{r}_0| \lesssim \Delta(\Gamma, \tilde{\Gamma}), \quad (5.26)$$

and

$$\Gamma \cap \tilde{\Gamma} \cap B(0, \alpha_1) \subset B\left(\xi, C\left(\frac{\Delta(\Gamma, \tilde{\Gamma})}{|x_0 - \tilde{x}_0|}\right)^{1/2}\right). \quad (5.27)$$

Equations (5.26) and (5.27) are Equations (26) and (27) in [KW99].

Property 89 (Appolonius-type bounds). Let $t > C\delta$. Fix three Ψ -circles $\Gamma_1, \Gamma_2, \Gamma_3$, let $B_0 = B(b, \alpha_1)$, and let

$$\begin{aligned}
Y = \left\{ \Gamma: \Delta(\Gamma, \Gamma_i) < C_1\delta, \ i = 1, 2, 3; \right. \\
d(\Gamma \cap B_0, \Gamma_i \cap B_0) > t, \ i = 1, 2, 3; \\
\Gamma^\delta \cap \Gamma_i^\delta \cap B_0 \neq \emptyset, \ i = 1, 2, 3; \\
\left. \text{dist}(\Gamma^{C_1\delta} \cap \Gamma_i^{C_1\delta} B_0, \Gamma^\delta \cap \Gamma_j^\delta \cap B_0) > C_3\sqrt{\delta/t}, \ i \neq j \right\}.
\end{aligned} \tag{5.28}$$

Informally, Y is the collection of curves that are almost tangent to each of the curves $\Gamma_1, \Gamma_2, \Gamma_3$, with the additional requirement that the three regions of almost-tangency not be too close to each other. If we identify Ψ -circles Γ with points $(x_0, r_0) \in \mathbb{R}^3$ then

$$Y \text{ is the union of two sets, each of diameter } \lesssim t. \tag{5.29}$$

This is is Lemma 3.1(ii) in [KW99].

Property 90. For three fixed curves $\Gamma_1, \Gamma_2, \Gamma_3$, and a given curve $\Gamma = \Gamma(x_0, r_0)$, we say that Ψ is Γ -*adapted* if there exists points a_1, a_2, a_3 , with $a_j \in \Gamma_j$ such that

$$|a_j - \xi_j(x_0)| \leq C^{-1}\sqrt{\delta/t},$$

and

$$\begin{aligned}
\Phi(x, a_1) &= 0, \\
\nabla_x \Phi(x, a_2) &= (e \cdot (a_2 - a_1))\beta
\end{aligned}$$

for all x , where e is a unit tangent vector to Γ_1 at a_1 , β is a vector independent of y with $|\beta| \sim 1$, and

$$\xi_i(x_0) = \xi(x_i, r_i, x_0).$$

Remark 91. Informally, the notion of a Γ -adapted defining function is a way of getting around the problem that we are forced to work with a defining function Ψ , but we are actually interested in its level sets $\{\Psi(x, \cdot) = r\}$. Thus we are free (within certain constraints to be dealt with below) to modify Ψ provided that our new defining function has the same

level sets as the old one. Choosing a Γ -adapted defining function (provided a suitable one exists) simplifies many of the technicalities in our estimates.

Lemma 3.6 in [KW99] tells us that if $\Gamma \in Y$ then by pre-composing Ψ with suitable diffeomorphisms, a Γ -adapted defining function Ψ exists which satisfies uniform derivative bounds, and this function Ψ has the same level sets as our original Ψ (i.e. it gives rise to the same Ψ -circles), so the corresponding maximal functions are identical (the adapted defining function may not be algebraic, but this will not affect our analysis).

Now, if Ψ is Γ -adapted, define

$$T(x) = \begin{pmatrix} \nabla_x \Psi(x, \xi_1(x)) & -1 \\ \nabla_x \Psi(x, \xi_2(x)) & -1 \\ \nabla_x \Psi(x, \xi_3(x)) & -1 \end{pmatrix}. \quad (5.30)$$

Informally, if we fix a choice of Γ and select a defining function adapted to Γ , then for x in a neighborhood of x_0 , $T(x)$ describes how changing x affects how close $\Gamma(x, r_0)$ is to being tangent with each of $\Gamma_1, \Gamma_2, \Gamma_3$.

Lemma 3.8 in [KW99] tells us that when restricted to each connected component of Y (individually), T is boundedly conjugate to its linear part, i.e. if Γ , and $\tilde{\Gamma}$ lie in the same connected component of Y , then

$$T(x_0)T(\tilde{x}_0)^{-1} = I + E(\tilde{x}_0), \quad (5.31)$$

where (say) $\|E(\tilde{x}_0)\| < 1/100$. Furthermore, for the same choice of $\Gamma, \tilde{\Gamma}$,

$$|\xi_1(\tilde{x}_0) - \xi_1(x_0)| \lesssim \sqrt{\delta/t}. \quad (5.32)$$

Equation (5.32) is a consequence of Equation (45) in [KW99] once we note that if $\tilde{\Gamma} \in Y$ is in the same connected component as $\Gamma \in Y$, then since T is boundedly conjugate to its linear part, $|T(x_0)(\tilde{x}_0 - x_0, \tilde{r}_0 - r_0)| < C\delta$.

Property 92 (Bounds on intersection area). Let $\Gamma, \tilde{\Gamma}$ be Φ circles. Then

$$|\Gamma^\delta \cap \tilde{\Gamma}^\delta \cap B(b, C^{-2}\alpha)| \lesssim \frac{\delta^2}{(d(\Gamma, \tilde{\Gamma}) + \delta)^{1/2} (\Delta_{B(b, C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) + \delta)^{1/2}}, \quad (5.33)$$

This is Lemma 3.1(i) in [KW99].

5.4 Some elementary incidence bounds on bipartite pairs of curve families

Recall the definition of a t -bipartite pair $(\mathcal{W}, \mathcal{B})$, a (δ, t) -rectangle, and a rectangle of type $(\gtrsim \mu, \gtrsim \nu)$ relative to $(\mathcal{W}, \mathcal{B})$ (Definitions 81 and 82)

Definition 93. If $(\mathcal{W}, \mathcal{B})$ is a (δ, t) -bipartite pair, then we define $\mathbf{R}_{\mu, \nu}(\mathcal{W}, \mathcal{B})$ to be the maximum cardinality of a collection of pairwise incomparable rectangles of type $(\gtrsim \mu, \gtrsim \nu)$ relative to $(\mathcal{W}, \mathcal{B})$. Define $\mathbf{R}(\mathcal{W}, \mathcal{B})$ to be $\mathbf{R}_{1,1}(\mathcal{W}, \mathcal{B})$.

Definition 94. If $(\mathcal{W}, \mathcal{B})$ is a (δ, t) -bipartite pair, then we define

$$\mathcal{I}(\mathcal{W}, \mathcal{B}) = |\{(R, \Gamma, \tilde{\Gamma}) : \Gamma \in \mathcal{W}, \tilde{\Gamma} \in \mathcal{B}, R \text{ is incident to } \Gamma \text{ and } \tilde{\Gamma}\}|.$$

We shall state and prove a series of lemmas that are analogous to Lemmas 1.5–1.16 in [Wol97b]. If the proof of a lemma is the same as that of the corresponding lemma in [Wol97b] we shall omit it. Throughout the discussion below, $(\mathcal{W}, \mathcal{B})$ is a t -bipartite pair with $|\mathcal{W}| = m$, $|\mathcal{B}| = n$.

Lemma 95.

- (i) *If $\Delta(\Gamma, \tilde{\Gamma}) < \delta$, then there exists a (δ, t) -rectangle $R \subset B(b, \alpha)$ such that Γ and $\tilde{\Gamma}$ are tangent to any (δ, t) -rectangle in the 2-fold dilate of R .*
- (ii) *Conversely, if $\Gamma, \tilde{\Gamma}$ are tangent to a common (δ, t) -rectangle $R \in B(b, \alpha)$, then $\Delta(\Gamma, \tilde{\Gamma}) \leq C\delta$, and if $\Gamma, \tilde{\Gamma}$ are tangent to comparable (δ, t) -rectangles $R, R' \in B(b, \alpha)$ then $\Delta(\Gamma, \tilde{\Gamma}) \lesssim \delta$.*

Lemma 96. *Let $\Gamma \in \mathcal{W}, \tilde{\Gamma} \in \mathcal{B}$. Then there are at most $O(1)$ incomparable (δ, t) -rectangles $R \subset B(0, \alpha)$ tangent to both Γ and $\tilde{\Gamma}$.*

Proof. Since $d(\Gamma, \tilde{\Gamma}) \sim t$, (5.33) gives us the bound

$$|B(b', \alpha_1) \cap \Gamma \cap \tilde{\Gamma}| \lesssim \delta^{3/2} t^{-1/2} \tag{5.34}$$

Each (δ, t) -rectangle has area $\sim \delta^{3/2}t^{-1/2}$ and incomparable (δ, t) -rectangles are pairwise disjoint. \square

Lemma 97. *There exists a collection \mathcal{R} of pairwise incomparable (δ, t) -rectangles $R \in B(b, \alpha)$ such that*

$$\mathcal{I}(\mathcal{W}, \mathcal{B}) \lesssim |\{(R, \Gamma, \tilde{\Gamma}) \in \mathcal{R} \times \mathcal{B} \times \mathcal{W} : \Gamma \text{ and } \tilde{\Gamma} \text{ are tangent to } R\}|.$$

Proof. This can be proved in the same way as Lemma 1.7 in [Wol97b] with (5.25) and (5.26) used in place of the analogous equations in [Wol97b]. \square

Lemma 98. *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three Ψ -circles. Let \mathcal{R} be a collection of pairwise incomparable rectangles $R \in B(b, \alpha)$ with the property that for each $R \in \mathcal{R}$ there is a Φ -circle Γ such that:*

- $d(\Gamma, \Gamma_i) \geq t, i = 1, 2, 3.$
- Γ, Γ_1 are tangent to $R.$
- *There exist two (δ, t) -rectangles $R_2, R_3 \in B(b, \alpha)$ such that Γ and Γ_i are tangent to $R_i, i = 2, 3$ and such that R_1, R_2, R_3 are pairwise incomparable.*

Then $|\mathcal{R}| \lesssim 1.$

Proof. We shall establish the proof with the additional restriction that R must lie in $B(b, C^{-2}\alpha)$ for b in a sufficiently small neighborhood of 0. Once this has been established, we can recover the full result by selecting $O(1)$ choices of b' such that $B(0, \alpha) \subset \bigcup_b B(b, C^{-2}\alpha).$

Let $R \in \mathcal{R}$ and let Γ be a Ψ -circle satisfying the above conditions. Then we must have $\Gamma \in Y$, where Y is as defined in (5.28); indeed the above requirements on Γ are precisely those needed to ensure that $\Gamma \in Y$. By (5.33),

$$\Gamma \cap \Gamma_1 \cap B(b, C^{-2}\alpha) \subset B(\xi(x_0, r_0, x_1), C\delta^{1/2}t^{-1/2}). \quad (5.35)$$

Now, let $\Gamma_0 \in Y$ and let $\tilde{\Psi}$ be a Γ_0 -adapted defining function with the same level sets as Ψ . Since $\tilde{\Psi}$ has the same level sets as Ψ and the gradient of $\tilde{\Psi}$ is comparable to that of Ψ ,

it suffices to prove the lemma for $\tilde{\Psi}$. However, by (5.32) we have that if Γ is in the same connected component of Y as Γ_0 then

$$|\xi(x_1, r_1, x_0) - \xi(x_1, r_1, x)| \lesssim \sqrt{\delta/t}. \quad (5.36)$$

Since Y contains only two connected components, (5.35) and (5.36) imply that

$$\begin{aligned} \bigcup_{(x_0, r_0) \in Y} \Gamma(x_0, r_0) \cap \Gamma_1 \cap B(b, C^{-2}\alpha) \\ \subset \left(B(z_0, C\delta^{1/2}t^{-1/2}) \cap \Gamma_1 \right) \cup \left(B(z_1, C\delta^{1/2}t^{-1/2}) \cap \Gamma_1 \right), \end{aligned} \quad (5.37)$$

where z_0, z_1 are points in the two connected components of Y respectively. In particular, the set on the right hand side of (5.37) has measure $\lesssim \delta^{3/2}t^{-1/2}$. Since every $R \in \mathcal{R}$ must lie in this set, and pairwise incomparable rectangles must be disjoint, we obtain $|\mathcal{R}| \lesssim 1$. \square

Lemma 99. *Let $\Gamma, \tilde{\Gamma}$ be Ψ -circles with $d(\Gamma, \tilde{\Gamma}) = t > C\delta$ and $r_0 \geq \tilde{r}_0$. Let $R, \tilde{R} \in B(0, \alpha_1)$ be comparable (δ, t) -rectangles with $\Gamma, \tilde{\Gamma}$ tangent to R, \tilde{R} respectively. Then*

(i) $\tilde{\Gamma} \cap B(0, \alpha_1)$ is contained in the $C\delta$ -neighborhood of

$$\{y \in B(0, \alpha) : \Phi(x_0, y) \leq r_0\}.$$

(ii) For any constant A there is a constant $C(A)$ such that the cardinality of any set of pairwise incomparable (δ, t) -rectangles $R \in B(0, \alpha_1)$ each of which is tangent to Γ and intersects the $A\delta$ -neighborhood of

$$\{y \in B(0, \alpha) : \Phi(\tilde{x}_0, y) \leq r_0\}$$

does not exceed $C(A)$.

Proof. Straighten Φ around x_0 . By Lemma 95.(ii), with α replaced by $C_0^{-1}\alpha$, we have $\Delta(\Gamma, \tilde{\Gamma}) \leq CC_0^{-1}\delta$. Thus if we choose the value of $C(A)$ in the statement of the Lemma 99.(ii) to be sufficiently large (depending on C'), then $|x_0 - \tilde{x}_0| > C''\Delta(\Gamma, \tilde{\Gamma})$, so by Property 88 of cinematic curvature, there exists a unique point $\xi(\tilde{x}_0, \tilde{r}_0, x_0) \in \tilde{\Gamma}$ satisfying (5.25), i.e.

$$\nabla_y \Psi(\tilde{x}_0, \xi) = (0, \pm 1),$$

so $\xi^{(1)}$ is the point where the function $y^{(1)} \mapsto \Psi(\tilde{x}_0, (y^{(1)}, y^{(2)}))$ achieves its maximum in the domain $(y^{(1)}, y^{(2)}) \in B(0, \alpha)$, where $y^{(2)} = y^{(2)}(y^{(1)})$ is implicitly defined by $(y^{(1)}, y^{(2)}(y^{(1)})) \in \tilde{\Gamma}$ (we can verify without difficulty that this is well-defined). By (5.26) (noting that in the straightened out coordinate system, $\Gamma = \{y^{(2)} = 0\} \cap U'_2$),

$$\begin{aligned} \Psi(\tilde{x}_0, \xi) &\lesssim \Delta(\Gamma, \tilde{\Gamma}) \\ &\lesssim \delta, \end{aligned}$$

and thus for an appropriate choice of C ,

$$\tilde{\Gamma} \cap U'_2 \subset \{y^{(2)} < C\delta\}.$$

Returning to our original coordinate system, this is Statement (i) of the lemma.

To obtain the second statement, note that by the same reasoning as above,

$$\begin{aligned} \Gamma^{C\delta} \cap \left(\{y \in B(b, \alpha) : \Phi(\tilde{x}_0, y) \leq \tilde{r}_0\} + B(0, A\delta) \right) \\ \subset \Gamma^{C(A)\delta} \cap \tilde{\Gamma}^{C(A)\delta} \cap B(b, \alpha) \end{aligned} \tag{5.38}$$

for a suitable constant $C(A)$, where the $+$ in the above equation denotes the Minkowski sum. The result then follows from (5.33) and the fact that incomparable rectangles are disjoint. \square

Lemma 100.

(i) *The cardinality of any set of $(\sim \mu, \sim \nu)$ rectangles is*

$$O\left(\frac{mn^{2/3}}{\mu\nu^{2/3}}\right) \tag{5.39}$$

(ii) *The cardinality of any set of $(\gtrsim \mu, \gtrsim \nu)$ rectangles is*

$$O\left(\frac{mn^{2/3}}{\mu\nu^{2/3}} + \frac{n}{\nu} \log \frac{m}{\mu}\right) \tag{5.40}$$

Remark 101. Recall that a rectangle of type $(\gtrsim \mu, \gtrsim \nu)$ is a rectangle that is incident to at least $C\mu$ curves in \mathcal{W} and at least $C\nu$ curves in \mathcal{B} for some absolute constant C (a rectangle of type $(\sim \mu, \sim \nu)$ is defined similarly), so the statement of the lemma is well defined.

Proof. Combined with the previous lemmas, Statement (i) is just the graph-theoretic statement, due to Kóvari, Sós, and Turan in [KST54], that a $m \times n$ matrix with entries 0 and 1 which has a forbidden 2×3 sub-matrix of 1s has $\lesssim mn^{2/3}$ 1s in total. Statement (ii) is obtained from Statement (i) by dyadic summation. \square

If every Ψ -circle from \mathcal{W} and \mathcal{B} are incident to some common rectangle R then $\mathcal{I}(\mathcal{W}, \mathcal{B}) = |\mathcal{W}||\mathcal{B}|$. However, if neither \mathcal{W} nor \mathcal{B} contain large clusters then this cannot occur. The following lemma (which is an analogue of Lemma 1.11 in [Wol97b]) is a quantitative version of this statement.

Lemma 102. *Let $(\mathcal{W}, \mathcal{B})$ be a t -bipartite pair that has no $(\gtrsim 1, \gtrsim \nu_0)$ or $(\gtrsim \mu_0, \gtrsim 1)$ rectangles $R \in B(b, \alpha)$. Then*

$$\mathcal{I}(\mathcal{W}, \mathcal{B}) \lesssim \mu_0^{1/3} nm^{2/3} \log \nu_0 + \nu_0 m \log \mu_0. \quad (5.41)$$

Definition 103. We define a *cluster* of Ψ -circles analogously to Wolff's definition in [Wol97b]: A cluster is a subset $\mathcal{C} \subset \mathcal{W}$ (or \mathcal{B}) with the property that there exists a (δ, t) -rectangle R such that every $\Gamma \in \mathcal{C}$ is tangent to a (δ, t) -rectangle comparable to R .

Lemma 104. *Let $\mathcal{C} \subset \mathcal{W}$ be a cluster and let $\Gamma \in \mathcal{B}$. Then then any set of pairwise incomparable (δ, t) -rectangles each of which is tangent to some circle in \mathcal{C} and to Γ has cardinality $O(1)$.*

Remark 105. Lemma 99 is used to prove this lemma. See Lemma 1.14 of [Wol97b] for details.

Lemma 106. *Given a value of μ_0 , we can write*

$$\mathcal{W} = \mathcal{W}_g \sqcup \mathcal{W}_b, \quad (5.42)$$

where

(i) \mathcal{W}_g and \mathcal{B} have no (δ, t) -rectangles of type $(\gtrsim \mu_0, \gtrsim 1)$.

(ii) \mathcal{W}_b is the union of $\lesssim \frac{|\mathcal{W}|}{\mu_0} (\log m)(\log n)$ clusters.

5.4.1 Algebraic considerations

We shall identify the Ψ -circle Γ with the point $(x_0, r_0) \in \mathbb{R}^3$ (actually in $B(0, \alpha) \times (1 - \tau, 1) \subset \mathbb{R}^3$). Thus if \mathcal{W} is a collection of Ψ -circles, we shall abuse notation and simultaneously consider \mathcal{W} as a subset of \mathbb{R}^3 .

Lemma 107. *Let $\Psi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a (multivariate) polynomial of degree k that satisfies the cinematic curvature conditions. For each Ψ -circle Γ , there exists a set $Q(\Gamma) \subset \mathbb{R}^3$ with the following properties:*

- (i) $\partial Q(\Gamma)$ is contained in an algebraic set S_Γ of dimension 2 and complexity $O(k^C)$ (see Appendix 2.3 for relevant definitions).
- (ii) Let $\tilde{\Gamma}$ be a Ψ -circle with $d(\Gamma, \tilde{\Gamma}) > A\delta$ for A a sufficiently large constant. If $\tilde{\Gamma} \in Q(\Gamma)$ then $\Delta(\Gamma, \tilde{\Gamma}) \leq 100\delta$. Conversely, if $\Delta(\Gamma, \tilde{\Gamma}) < \delta$ then $\tilde{\Gamma} \in Q(\Gamma)$.

Remark 108. Informally, $Q(\Gamma)$ can be understood as follows. If $\gamma_1 = C(x_1, r_1)$, $\gamma_2 = C(x_2, r_2)$ are two circles, then γ_1 and γ_2 are tangent if and only if (x_2, r_2) lies on the right-angled light-cone $Z_{\gamma_1} = \{(y, t): |r - t| = \|x - y\|\}$, and γ_1 and γ_2 are almost tangent if (x_2, r_2) lies in the δ -neighborhood of Z_{γ_1} . $Q(\Gamma)$ is the analogue of the δ -neighborhood of the light cone Z_{γ_1} for general curves Γ .

Proof. Define

$$V_\Gamma = V_{1,\Gamma} \cap V_{2,\Gamma} \cap V_{3,\Gamma} \cap V_{4,\Gamma}, \quad (5.43)$$

where

$$\begin{aligned} V_{1,\Gamma} &= \{(\tilde{x}_0, \tilde{r}_0, y, \tilde{y}): \|\tilde{x}_0\|^2 < \alpha^2, 0 < 1 - \tilde{r}_0 < \tau, \|y\|^2 < \alpha^2, \|\tilde{y}\|^2 < \alpha^2\}, \\ V_{2,\Gamma} &= \{(\tilde{x}_0, \tilde{r}_0, y, \tilde{y}): \Psi(x_0, y) = r_0, \Psi(\tilde{x}_0, \tilde{y}) = \tilde{r}_0\}, \\ V_{3,\Gamma} &= \{(\tilde{x}_0, \tilde{r}_0, y, \tilde{y}): \|y - \tilde{y}\|^2 < \delta^2\}, \\ V_{4,\Gamma} &= \{(\tilde{x}_0, \tilde{r}_0, y, \tilde{y}): \|\nabla_y \Psi(x_0, y) \wedge \nabla_y \Psi(\tilde{x}_0, \tilde{y})\|^2 \\ &\quad < 4\delta^2 \|\nabla_y \Psi(x_0, y)\|^2 \|\nabla_y \Psi(\tilde{x}_0, \tilde{y})\|^2\}. \end{aligned}$$

Each $V_{j,\Gamma}$, $j = 1, 2, 3, 4$ is a semi-algebraic set of complexity $O(k^C)$ (see Appendix 2.3 for relevant definitions), and thus so is V_Γ . Let

$$Q(\Gamma) = (\pi_{(\tilde{x}_0, \tilde{r}_0)} V_\Gamma) \cap \{\tilde{x}_0: \|x_0 - \tilde{x}_0\|^2 > A^2 \delta^2\}, \quad (5.44)$$

where $\pi_{(\tilde{x}_0, \tilde{r}_0)}: (\tilde{x}_0, \tilde{r}_0, y, \tilde{y}) \mapsto (\tilde{x}_0, \tilde{r}_0)$ is the projection map.

An examination of the definition of $\Delta(\Gamma, \tilde{\Gamma})$ verifies that $Q(\Gamma)$ satisfies Property (ii), so all that remains is to verify Property (i). Since V_Γ is a semi-algebraic set of complexity $O(k^C)$, by the Tarski-Seidenberg theorem (see Proposition 10 in Appendix 2.3), so is $Q(\Gamma)$. Thus by Proposition 7 in Appendix 2.3, either $Q(\Gamma)$ is empty or $\partial(Q(\Gamma))$ has dimension at most 2 and complexity $O(k^C)$, so by Proposition 9, its Zariski closure, $Z_\Gamma = \text{Zar}(\partial(Q(\Gamma)))$, is an algebraic set of dimension at most 2 and degree $O(k^C)$. If $\dim(Z_\Gamma) = 2$ then let $S_\Gamma = Z_\Gamma$. If not, we can find an algebraic set of dimension 2 containing Z_Γ whose degree is controlled by a polynomial function of the degree of Z_Γ and we shall let this set be S_Γ . \square

Definition 109. Let $\mathcal{W} \subset \mathbb{R}^N$ be a finite collection of points. We say that \mathcal{W} is *hypersurface generic* if for every polynomial $P \in \mathbb{R}[x_1, \dots, x_N]$ of degree D we have

$$|\{P = 0\} \cap \mathcal{W}| \leq \binom{D}{N} - 1.$$

Lemma 110. *Let $\mathcal{W} \subset \mathbb{R}^3$ be finite. Then after an infinitesimal perturbation, \mathcal{W} is hypersurface generic*

Proof. Identify the space of all sets $H \subset \mathbb{R}^3$ of cardinality ℓ with $(\mathbb{R}^N)^\ell$. Let $|\mathcal{W}| = m$. Then the subset of $(\mathbb{R}^N)^m$ corresponding to sets of cardinality m that are not hypersurface generic is Zariski closed—it is a finite union of determinantal varieties. \square

5.4.2 Proof of Lemma 85

In order to prove Lemma 85, it suffices to consider the case where $\mu = \nu = 1$ and establish the following bound:

Lemma 111. *Let $(\mathcal{W}, \mathcal{B})$ be as in Lemma 85. Then for all $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\mathbf{R}(\mathcal{W}, \mathcal{B}) \leq C_\epsilon k^{C_\epsilon} (mn)^\epsilon ((mn)^{3/4} + m + n). \quad (5.45)$$

To obtain (5.21) from (5.45) we apply a random sampling argument. The details can be found in [Wol97b, p1253], so we shall not reproduce them here.

Proof of Lemma 111. We shall proceed by induction on the quantity $|\mathcal{W}||\mathcal{B}|$. To handle the base case, we may assume

$$mn > C_\epsilon k^{C_\epsilon}, \quad (5.46)$$

since otherwise we can use the trivial bound $\mathbf{R}(\mathcal{W}, \mathcal{B}) \lesssim mn$. Now suppose Lemma 111 has been established for all (δ, t) -bipartite pairs $(\mathcal{W}', \mathcal{B}')$ with $|\mathcal{W}'||\mathcal{B}'| < mn$.

We may assume

$$Am^{1/3+\epsilon} < n < m, \quad (5.47)$$

for a large constant A (depending on ϵ) to be determined later, since if the first inequality fails then the result follows from (5.40) (and after selecting a sufficiently large value of C_ϵ , depending on A), while if the second inequality fails we can reverse the roles of \mathcal{W} and \mathcal{B} .

Let $\mu_0 = (mn)^{1/4}$, and use Lemma 106 to write $\mathcal{W} = \mathcal{W}_g \sqcup \mathcal{W}_b$ and similarly $\mathcal{B} = \mathcal{B}_g \sqcup \mathcal{B}_b$. Using Lemma 104, we have

$$\mathbf{R}(\mathcal{W}_g, \mathcal{B}) \leq \frac{1}{100} (mn)^{3/4+\epsilon}, \quad (5.48)$$

$$\mathbf{R}(\mathcal{W}, \mathcal{B}_g) \leq \frac{1}{100} (mn)^{3/4+\epsilon}. \quad (5.49)$$

See [Wol97b, p1251-2] for details. Thus in order to prove Lemma 111, it suffices to establish the following bound:

$$\mathbf{R}(\mathcal{W}_g, \mathcal{B}_g) < \frac{1}{2} C_\epsilon k^{C_\epsilon} (mn)^{3/4+\epsilon} + C_\epsilon k^{C_\epsilon} (mn)^\epsilon (m + n). \quad (5.50)$$

Use Proposition 19 to select a polynomial $P \in \mathbb{R}[x_1, x_2, r]$ of degree at most D (D shall be chosen later, but it should be thought of as $\delta^{-\epsilon}$) so that the set $\mathbb{R}^3 \setminus \{P = 0\}$ is a union of $\lesssim D^3$ cells, each of which contains $\lesssim |\mathcal{W}_g|/D^3$ Ψ -circles $\Gamma \in \mathcal{W}_g$.

Lemma 112. *Let Ω be a cell from the above decomposition. If $\Gamma \in \mathcal{B}_g$, $\tilde{\Gamma} \in \Omega$, and $\Delta(\Gamma, \tilde{\Gamma}) \leq \delta$, then at least one of the following must hold.*

(i) $\partial Q(\Gamma) \cap \Omega \neq \emptyset$.

(ii) $\Omega \subset Q(\Gamma)$.

Indeed, since $\Delta(\Gamma, \tilde{\Gamma}) \leq \delta$, by Property (ii) of $Q(\Gamma)$ from Lemma 107, $\tilde{\Gamma} \in Q(\Gamma)$ and thus $\Omega \cap Q(\Gamma) \neq \emptyset$. Since Ω is an open connected set, it must either be contained in $Q(\Gamma)$ or it must meet the (topological) boundary of $Q(\Gamma)$.

Now, for each cell Ω , let

$$\mathcal{B}_g = \mathcal{B}_1^\Omega \sqcup \mathcal{B}_2^\Omega \sqcup \mathcal{B}_3^\Omega,$$

where \mathcal{B}_1^Ω (resp. \mathcal{B}_2^Ω) contains those $\Gamma \in \mathcal{B}_g$ for which Item (i) (resp. Item (ii)) occurs, and $\Gamma \in \mathcal{B}_3^\Omega$ if $\Delta(\Gamma, \tilde{\Gamma}) > \delta$ for all $\tilde{\Gamma} \in \Omega$.

We shall first consider incidences involving \mathcal{B}_2^Ω .

Lemma 113. *Suppose D satisfies*

$$D < n^{\epsilon/6}. \tag{5.51}$$

Then if m and n are sufficiently large, at least one of the following must hold:

$$\left| \bigcup_{\Omega} \mathcal{B}_2^\Omega \right| < n/1000, \tag{5.52}$$

$$|\mathcal{W}_g| < m/1000. \tag{5.53}$$

Proof. Suppose (5.53) fails. By (5.47), (5.51), and the fact that \mathcal{W} is hyperplane generic, we have that for each cell Ω ,

$$\begin{aligned} |\mathcal{W}_g \cap \Omega| &\gtrsim |\mathcal{W}_g| D^{-3} \\ &\gtrsim m D^{-3}. \end{aligned}$$

Thus each $\Gamma \in \bigcup_{\Omega} \mathcal{B}_2^\Omega$ is incident to $\gtrsim m D^{-3}$ Ψ -circles from \mathcal{W}_g , so

$$\mathcal{I}(\mathcal{W}_g, \mathcal{B}_g) \gtrsim m D^{-3} \left| \bigcup_{\Omega} \mathcal{B}_2^\Omega \right|. \tag{5.54}$$

On the other hand, by Lemma 102 (with $\mu_0 = \nu_0 = (mn)^{1/4}$),

$$\mathcal{I}(\mathcal{W}_g, \mathcal{B}_g) \lesssim m^{5/4} n^{1/4} \log m + m^{3/4} n^{13/12} \log n. \quad (5.55)$$

Combining (5.54), (5.55), and (5.47), we obtain

$$\left| \bigcup_{\Omega} \mathcal{B}_2^{\Omega} \right| \lesssim D^3 n^{1-\epsilon} \log n. \quad (5.56)$$

This and (5.46), (5.51) gives us (5.52). \square

If either (5.52) or (5.53) holds, then we can apply the induction hypothesis to the pair $(\mathcal{W}_g, \bigcup_{\Omega} \mathcal{B}_2^{\Omega})$ and conclude that

$$\begin{aligned} \mathbf{R}(\mathcal{W}_g, \bigcup_{\Omega} \mathcal{B}_2^{\Omega}) &\leq \frac{1}{100} C_{\epsilon} k^{C_{\epsilon}} (mn)^{\epsilon} ((mn)^{3/4} + m + n) \\ &\leq \frac{1}{10} C_{\epsilon} k^{C_{\epsilon}} (mn)^{3/4+\epsilon}, \end{aligned} \quad (5.57)$$

where on the second line we used (5.47).

Remark 114. Lemma 113 is an analogue of Equation (5.23) from [Zah12a]. In essence, both state that if $|\mathcal{W}_g|$ were too big then that would force an illegally large number of incidences to occur. However, the current formulation is much simpler. In [Zah12a], the analogue of $Q(\Gamma)$ was defined differently and thus we needed statements of the form “if two curves Γ_1, Γ_2 are almost tangent then after a slight perturbation they are exactly tangent.” Making statements such as this rigorous introduced many technical difficulties that have been avoided in the present proof.

We shall now control incidences involving \mathcal{B}_2^{Ω} . Let

$$n_{\Omega} = |\{\Gamma \in \mathcal{B}_g : \partial(Q(\Gamma)) \cap \Omega \neq \emptyset\}|.$$

Since $\partial(Q(\Gamma)) \subset S_{\Gamma}$, we have

$$n_{\Omega} \leq |\{\Gamma \in \mathcal{B}_g : S_{\Gamma} \cap \Omega \neq \emptyset\}|.$$

By a Thom-Milnor type theorem (see e.g. [BB12, Theorem 1.1]), we have that for each $\Gamma \in \mathcal{B}_g$, $S_{\Gamma} \setminus \{P = 0\}$ contains $O(k^C D^2)$ connected components. Since the number of cells

that intersect $\partial(Q(\Gamma))$ is bounded by the number of connected components of $S_\Gamma \setminus \{P = 0\}$, we have

$$\sum_{\Omega} n_{\Omega} \leq C_1 D^2 k^C n. \quad (5.58)$$

Let $m_{\Omega} = |\mathcal{W}_g \cap \Omega|$. Applying the induction hypothesis,

$$\begin{aligned} & \sum_{\Omega} \mathbf{R}(\mathcal{W}_g \cap \Omega, \mathcal{B}_1^{\Omega}) \\ & \leq k^{C_{\epsilon}} C_{\epsilon} \left[\sum m_{\Omega}^{3/4+\epsilon} n_{\Omega}^{3/4+\epsilon} + (mn)^{\epsilon} \sum m_{\Omega} + (mn)^{\epsilon} \sum n_{\Omega} \right] \\ & \leq C_{\epsilon} k^{C_{\epsilon}} \left[\left(\sum m_{\Omega}^{\frac{4}{1-4\epsilon}(3/4+\epsilon)} \right)^{\frac{1-4\epsilon}{4}} \left(\sum n_{\Omega} \right)^{3/4+\epsilon} + (mn)^{\epsilon} \sum m_{\Omega} + \sum (mn)^{\epsilon} n_{\Omega} \right] \quad (5.59) \\ & \leq C_{\epsilon} k^{C_{\epsilon}} \left[\left(D^3 m^{\frac{3+4\epsilon}{1-4\epsilon}} D^{-\frac{9+12\epsilon}{1-4\epsilon}} \right)^{\frac{1-4\epsilon}{4}} (C_1 D^2 k^C n)^{3/4+\epsilon} + (mn)^{\epsilon} m + (mn)^{\epsilon} C_1 D^2 k^C n \right] \\ & = C_{\epsilon} k^{C_{\epsilon}} (mn)^{\epsilon} \left[\frac{C_1 (mn)^{3/4} k^C}{D^{2\epsilon}} + m + C_1 D^2 k^C n \right]. \end{aligned}$$

Finally, since the points of \mathcal{W} are hypersurface generic, we have that

$$|\mathcal{W}_g \cap \{P = 0\}| \lesssim D^3,$$

and thus

$$\mathbf{R}(\mathcal{W}_g \cap \{P = 0\}, \mathcal{B}_g) \leq C_2 D^3 n. \quad (5.60)$$

We have

$$\mathbf{R}(\mathcal{W}_g, \mathcal{B}_g) = \sum_{\Omega} \mathbf{R}(\mathcal{W}_g \cap \Omega, \mathcal{B}_1^{\Omega}) + \sum_{\Omega} \mathbf{R}(\mathcal{W}_g \cap \Omega, \mathcal{B}_2^{\Omega}) + \mathbf{R}(\mathcal{W}_g \cap \{P = 0\}, \mathcal{B}_g). \quad (5.61)$$

Combining (5.57), (5.59), and (5.60), we conclude that there there exists an absolute constant C_0 such that

$$\mathbf{R}(\mathcal{W}_g, \mathcal{B}_g) \leq C_{\epsilon} k^{C_{\epsilon}} (mn)^{\epsilon} \left(\frac{C_1 (mn)^{3/4} k^{C_0}}{D^{2\epsilon}} + C_2 D^3 k^{C_0} n + m \right). \quad (5.62)$$

Now, select $D > 1$ satisfying (5.51) and also

$$\frac{C_1 k^{C_0}}{D^{2\epsilon}} < \frac{1}{100}, \quad (5.63)$$

$$C_2 D^3 k^{C_0} n < \frac{(mn)^{3/4}}{100}. \quad (5.64)$$

The existence of such a D is guaranteed by (5.46) and (5.47) provided we select the constants C_ϵ (from (5.46)) and A (from (5.47)) to be sufficiently large (depending on the constant C_0 from (5.62) and the ϵ that appears in the statement of Lemma 111). With such a choice of D , (5.50) is satisfied. This completes the proof of Lemma 111 and hence also Theorem 76. □

Remark 115. The use of a “low degree” partitioning polynomial to prove incidence theorems was first introduced by Solymosi and Tao in [ST12]. What we do here is very similar, except instead of using a bounded degree variety and the general heuristic that operations such as projection, etc. send bounded degree varieties to bounded degree varieties, we use a variety of “sub-polynomial” degree, and we rely on the heuristic that projections, etc. send sub-polynomial degree varieties to sub-polynomial degree varieties.

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