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Finitely-Generated Projective Modules over θ -deformed Spheres

by

Mira Alexander Peterka

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

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of the

University of California, Berkeley

Committee in charge:

Professor Marc A. Rieffel, Chair
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Mira Alexander Peterka

Abstract

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Abstract: We investigate the “ θ -deformed spheres” $C(S_\theta^3)$ and $C(S_\theta^4)$ for the case θ an irrational number. We show that all finitely-generated projective modules over $C(S_\theta^3)$ are free, and that $C(S_\theta^4)$ has the cancellation property. We classify and construct all finitely-generated projective modules over $C(S_\theta^4)$ up to isomorphism. An interesting feature is that there are nontrivial “rank-1” modules over $C(S_\theta^4)$. Every finitely-generated projective module over $C(S_\theta^4)$ is a sum of rank-1 modules. This is because the group of path-components of the invertible elements of $C(S_\theta^3)$ is \mathbb{Z} and maps isomorphically onto $K_1(C(S_\theta^3))$ under the natural map.

To Yancy, Fortinbra, and Lothar.

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Introduction

Abstract: We investigate the “ θ -deformed spheres” $C(S_\theta^3)$ and $C(S_\theta^4)$ for the case θ an irrational number. We show that all finitely-generated projective modules over $C(S_\theta^3)$ are free, and that $C(S_\theta^4)$ has the cancellation property. We construct and classify all finitely-generated projective modules over $C(S_\theta^4)$ up to isomorphism. An interesting feature is that there are nontrivial “rank-1” modules over $C(S_\theta^4)$. Every finitely-generated projective module over $C(S_\theta^4)$ is a sum of rank-1 modules. This is because the group of path-components of the invertible elements of $C(S_\theta^3)$ is \mathbb{Z} and maps isomorphically onto $K_1(C(S_\theta^3))$ under the natural map.

The noncommutative geometry of Connes [7] generalizes topological and geometric structures to situations in which either a classical topological or geometric space fails to exist, or in which a classical space does exist but certain geometric aspects of the situation are undetected by purely classical techniques (e.g. in transverse geometry). The noncommutative n -tori $C(T_\theta^n)$ are the most studied examples of “noncommutative differential manifolds” (i.e. spectral triples satisfying the axioms given in [8]), and have been found to arise naturally in a number of contexts in mathematics and string theory (see [7], [9], [15]). Particularly interesting connections exist between string theory and gauge theory on noncommutative spaces (see [28]). In turn, there is a transformation between gauge theory on noncommutative spaces and that of their classical limiting cases (see [41]). Classical gauge theory was first formulated on spheres, and numerous quantum analogs of spheres have been discovered (e.g. [50], [30], [47], [32], [5], [14]). Of these, the so-called θ -deformed spheres $C(S_\theta^n)$ of Connes and Landi [12] most closely resemble the noncommutative tori. There has been interest in calculating a Yang-Mills theory for $C(S_\theta^4)$, and much has already been done in this regard [6]. However, a complete classification and construction of the “vector bundles” over $C(S_\theta^4)$ has not yet appeared. Such a classification and construction is the main result of the present work.

Given a space X with homotopy type that of a CW-complex, there are various approaches (classifying spaces, clutching constructions, etc.) one can use to determine the vector bundles over X . These methods are only superficially different; In any case, the problem is purely homotopy-theoretic in nature, and reduces to calculating the homotopy classes of maps from one space to another. These calculations involve analyzing the cell-structures of both source

and target spaces and using the techniques of obstruction theory as one adds cells to the skeleta of a given level. This typically is very difficult, even using all the tools of homotopy theory at one's disposal (e.g. spectral sequences), though at least one knows how to proceed in principle.

By Swan's theorem, the problem of classifying the vector bundles over a space generalizes to that of classifying the finitely-generated projective modules over a unital ring. In the context of C^* -algebras, this has been done for relatively few examples (see [42], [49]), presumably in part due to a lack of general techniques. Rieffel [37] classified the modules over $C(T_\theta^n)$, for all n and all non-rational choices of θ . Because the noncommutative tori are simple for generic choices of θ , the noncommutative tori cannot be thought of as having any sort of "cell-structure" and thus the local homotopy-theoretic methods of vector bundle theory are of little use. Rieffel resorted to entirely different techniques, many of which rely on very specific aspects of the structure of the $C(T_\theta^n)$.

For irrational θ , we are able to classify and construct the finitely-generated projective modules over $C(S_\theta^4)$ by combining essentially classical techniques with data concerning the path-classes of the group of invertible elements in the noncommutative tori. Specifically, we proceed as follows: First we show that all finitely-generated projective modules over $C(S_\theta^3)$ are free by using a clutching construction to equate the isomorphism classes of the modules with " $GL_n(C(T_\theta^2))$ -cocycle classes". Next we investigate the algebra $TC(S_\theta^3)$ of continuous functions from the circle into the algebra $C(S_\theta^3)$. We calculate the isomorphism classes of finitely-generated projective modules over $TC(S_\theta^3)$ in terms of " $GL_n(TC(T_\theta^2))$ -cocycles". This allows us to completely determine the path-classes of elements of $GL_n(C(S_\theta^3))$. As $C(S_\theta^4)$ is isomorphic to a pullback of two copies of the θ -deformed 4-ball $C(D_\theta^4)$ over $C(S_\theta^3)$, this information suffices to classify and construct the finitely-generated projective modules over $C(S_\theta^4)$ using a clutching construction.

We find that for irrational θ , the semigroup $V(C(S_\theta^4))$ of isomorphism classes of finitely-generated projective-modules over $C(S_\theta^4)$ is $\{0\} \cup (\mathbb{N} \times \mathbb{Z})$. For each $(n, s) \in \mathbb{N} \times \mathbb{Z}$, we construct a $C(S_\theta^4)$ -module $N(n, s)$. The two parameters can be thought of as "rank" and "index". Finitely-generated free $C(S_\theta^4)$ -modules have the form $N(n, 0)$. From the isomorphism $N(n, s) \cong N(1, s) \oplus N(n-1, 0)$, we see that every finitely-generated projective $C(S_\theta^4)$ -module splits as a direct sum of a rank-1 module and a free module. In particular, the noncommutative "instanton bundle of charge -1" e of Connes and Landi [12] must be isomorphic to $N(2, 1)$, and so splits as $N(1, 1) \oplus C(S_\theta^4)$. However, the K_0 -class of Connes and Landi's e , together with the class of the rank-1 free $C(S_\theta^4)$ -module, does generate $K_0(C(S_\theta^4))$. The module $p_{(n)}$ of Landi and Van Suijlekom [20] is isomorphic to $N(n+1, (1/6)n(n+1)(n+2))$ for $n \geq 1$. This contrasts somewhat with the situation for the commutative algebra $C(S^4)$. There are no nontrivial complex line bundles over S^4 , since $\pi_3(U(1)) \cong 0$. However, for each $n \geq 2$, there are \mathbb{Z} -many complex rank- n vector bundles over S^4 , up to isomorphism. These isomorphism classes of bundles are classified by the integral of their second Chern-classes ("charge") or, equivalently, by the index of any associated clutching function for the bundle. The K -group $K^0(S^4)$ is generated by the classes of the trivial line bundle and the rank-2

“instanton bundle” of charge -1.

The results we obtain regarding the finitely-generated projective modules over $C(S_\theta^4)$ reflect the fact that, if θ is irrational, then the group of path-components $\pi_0(GL_n(C(S_\theta^3)))$ of $GL_n(C(S_\theta^3))$ turns out to be isomorphic to \mathbb{Z} , for all $n \geq 1$. The group $\pi_0(GL_n(C(S_\theta^3)))$ is isomorphic to $K_1(C(S_\theta^3)) \cong \mathbb{Z}$ under the natural map. We show that, if $|\theta| < 1$, then the generator of $\pi_0(GL_n(C(S_\theta^3)))$ is $X := \exp(2\pi it)p+1-p$, where p is a Reiffel projection of trace $|\theta|$. This contrasts with the classical situation. Indeed, the group $\pi_0(GL_1(C(S^3))) \cong \pi_3(S^1)$ is trivial. The natural map $\pi_0(GL_n(C(S^3))) \rightarrow K_1(C(S^3)) \cong \mathbb{Z}$ is an isomorphism, however, for $n \geq 2$. The group $K_1(C(S^3))$ is generated by the matrix $\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$. One can show that $K_1(C(S_\theta^3))$ is generated by the matrix $\begin{pmatrix} z_1 & z_2 \\ -\lambda z_2^* & z_1^* \end{pmatrix}$. From this it follows that $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$ must be homotopic through a path in $GL_2(C(S_\theta^3))$ to $\begin{pmatrix} z_1 & z_2 \\ -\lambda z_2^* & z_1^* \end{pmatrix}$.

Chapter 1

Preliminary Material

Before continuing, we remark that since we will mostly be working with noncommutative algebras, we must make a distinction between left and right modules. We will always use right modules in this work, and will generally drop the term “right”. Our doing so is not intended to imply that a given right module has an (A, A) -bimodule structure. Also, if A is a normed algebra, and Y is a compact space, we will use the notation YA to denote the algebra of continuous functions from Y into A with the supremum norm. Thus TA denotes the continuous maps from the circle T into A , the notation T^2A denotes the continuous maps from the 2-torus T^2 into A etc.

1.1 The θ -deformed Spheres $C(S_\theta^3)$ and $C(S_\theta^4)$

In this section we describe the θ -deformed spheres $C(S_\theta^3)$ and $C(S_\theta^4)$.

The θ -deformed 3-sphere $C(S_\theta^3)$ can be obtained in several equivalent ways:

- 1) As a noncommutative analog of the genus-1 Heegaard splitting of S^3 into two solid tori.
- 2) As the universal C^* -algebra generated by a particular solution of certain homological equations studied by Connes and Landi.
- 3) As the result of the “ θ -deformation” procedure of Connes and Landi on S^3 .
- 4) As an example of Rieffel’s general deformation quantization by actions of \mathbb{R}^n .
- 5) As a certain fixed-point subalgebra.

The θ -deformed 4-sphere $C(S_\theta^4)$ also has descriptions in terms of 2)-5) above.

We will only use descriptions 1) and 5) in obtaining the results in this work, but we will now provide the reader with at least a very brief account of these different descriptions in order to motivate the interest in the example. We also use this section to fix some of the terminology and notation that we will use.

1) Matsumoto [24] introduced $C(S_\theta^3)$ as a deformed version of the well-known genus-1 Heegaard splitting of S^3 into two solid tori [40]. We briefly review this splitting: The 3-sphere embeds into \mathbb{C}^2 as the subspace

$$S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}.$$

For each $t \in [0, 1]$ consider the subspace

$$T_t := \{(z_1, z_2) \in S^3 : |z_1|^2 = t\}.$$

The space T_t is a 2-torus if $t \in (0, 1)$, and is a circle if $t \in \{0, 1\}$. The spaces T_t are mutually disjoint and their union is all of S^3 . The 3-sphere S^3 thus splits into two solid tori $\cup_{t \leq 1/2} T_t$ and $\cup_{t \geq 1/2} T_t$ with common boundary $T_{1/2}$.

Recall that a base-pointed loop α on the boundary of a solid torus is called a *meridian* if it is contractible in the solid torus. A base-pointed loop β on the boundary of a solid torus is called a *longitude* if it generates the fundamental group of the solid torus. For the genus-1 Heegaard splitting of S^3 , choosing the base-point to be $(1/\sqrt{2}, 1/\sqrt{2})$, say, the loop $\alpha = \{(1/\sqrt{2}, z_2) \in T_{1/2}\}$ is a meridian for the solid torus $\cup_{t \leq 1/2} T_t$, but a longitude for $\cup_{t \geq 1/2} T_t$. The loop $\beta = \{(z_1, 1/\sqrt{2}) \in T_{1/2}\}$ is a longitude for the solid torus $\cup_{t \leq 1/2} T_t$, but a meridian for the solid torus $\cup_{t \geq 1/2} T_t$.

We would like to express the Heegaard splitting more abstractly: Take a solid torus $F_v^2 = D^2 \times S^1$ in which some loop α is a meridian and some loop β is a longitude. Take a second solid torus $F_h^2 = S^1 \times D^2$ in which some loop β' is a meridian and some loop α' is a longitude. The 3-sphere S^3 is then homeomorphic to the splitting $F_v^2 \cup_f F_h^2$, where $f : \partial F_v^2 \rightarrow \partial F_h^2$ is any homeomorphism that identifies the meridian α with the longitude α' , and identifies the longitude β with the meridian β' . We note that the concrete Heegaard splitting given above makes the assignments:

$$F_v^2 = \cup_{t \leq 1/2} T_t, \quad F_h^2 = \cup_{t \geq 1/2} T_t$$

$$\alpha = \alpha' = \{(1/\sqrt{2}, z_2) \in T_{1/2}\}, \quad \beta = \beta' = \{(z_1, 1/\sqrt{2}) \in T_{1/2}\}, \quad f = id.$$

Matsumoto noted that by using the duality between compact Hausdorff spaces and the algebras of continuous functions on them, an abstract description of the genus-1 Heegaard splitting can be expressed at the level of C^* -algebras as the isomorphism

$$C(S^3) \cong \{(a, b) \in C(F_v^2) \oplus C(F_h^2) : \pi_v(a) = f^*(\pi_h(b))\},$$

where $(\pi_v(a), \pi_h(b)) = (a \upharpoonright (S^1 \times S^1), b \upharpoonright (S^1 \times S^1))$. Matsumoto deformed this splitting by deforming the commutative function algebras $C(F_v^2)$ and $C(F_h^2)$ into (noncommutative) crossed products.

Let θ be an arbitrary real number. Let α_θ be the automorphism of $C(D^2)$ induced by rotating points of the 2-disk D^2 by the angle $2\pi\theta$.

Definition 1.1.1. The noncommutative solid torus $C(F_\theta^2)$ (called D_θ by Matsumoto) is the crossed product $C(D^2) \times_\theta Z$ of $C(D^2)$ by the action of Z that comes from α_θ .

Since the dual group of Z is S^1 , we recover that $C(F_\theta^2) \cong C(D^2 \times S^1)$ by the Fourier transform.

Matsumoto observed that $C(F_\theta^2)$ is isomorphic to the universal C^* -algebra generated by y and v , satisfying the relations

$$yy^* = y^*y, \|y\| = 1, \quad vv^* = v^*v = 1, vy = \lambda yv,$$

where $\lambda = \exp(2\pi i\theta)$. Note $C^*(y, 1) \cong C(D^2)$ and $C^*(v) \cong C(S^1)$.

Definition 1.1.2. The noncommutative 2-torus $C(T_\theta^2)$ is the universal C^* -algebra generated by U, V satisfying the relations

$$UU^* = U^*U = 1, \quad VV^* = V^*V = 1, \quad VU = \lambda UV,$$

where $\lambda = \exp(2\pi i\theta)$.

There is an evident surjection $\pi_\theta : C(F_\theta^2) \rightarrow C(T_\theta^2)$, with $\pi_\theta(v) = V$ and $\pi_\theta(y) = U$. Now take $C(F_{-\theta}^2)$ generated by x, u such that $xx^* = x^*x$, $uu^* = u^*u = 1$, $\|x\| = 1$, with commutation relation $ux = \bar{\lambda}xu$. There is a corresponding surjection $\pi_{-\theta} : C(F_{-\theta}^2) \rightarrow C(T_{-\theta}^2)$, with $\pi_{-\theta}(x) = U'$ and $\pi_{-\theta}(u) = V'$, where $V'U' = \bar{\lambda}U'V'$, and U', V' are unitary generators of $C(T_{-\theta}^2)$. Now define the isomorphism $f_\theta^* : C(T_{-\theta}^2) \rightarrow C(T_\theta^2)$ by the assignments

$$f_\theta^*(U') = V, \quad f_\theta^*(V') = U.$$

Definition 1.1.3. The noncommutative 3-sphere $C(S_\theta^3)$ is the pullback

$$\{(a, b) \in C(F_\theta^2) \oplus C(F_{-\theta}^2) : \pi_\theta(a) = f_\theta^*(\pi_{-\theta}(b))\}.$$

In this definition, the algebra $C(F_\theta^2)$ corresponds to the solid torus F_v^2 . The algebra $C(F_{-\theta}^2)$ corresponds to F_h^2 . The unitary U corresponds to the meridian α and the unitary V corresponds to the longitude β . The unitary U' corresponds to the meridian β' and the unitary V' corresponds to the longitude α' . The proof of Proposition 2.1.3 will give these last statements further sense.

Matsumoto and Tomiyama proved the following theorem:

Theorem 1.1.4. (Theorem 7.6 of [26]). *The following C^* -algebras are isomorphic:*

- (1) *Matsumoto's algebra $C(S_\theta^3)$.*
- (2) *The universal C^* -algebra generated by the relations*

$$SS^* = S^*S, \quad TT^* = T^*T, \quad TS = \lambda ST, \quad (1 - T^*T)(1 - S^*S) = 0, \quad \|S\| = \|T\| = 1.$$

- (3) *The universal C^* -algebra generated by the relations*

$$z_1 z_1^* = z_1^* z_1, \quad z_2 z_2^* = z_2^* z_2, \quad z_2 z_1 = \lambda z_1 z_2, \quad z_1^* z_1 + z_2^* z_2 = 1.$$

If $\theta = 0$, the equations in (3) give exactly the relationships holding between the coordinate functions z_1 and z_2 for S^3 , and so $C(S_\theta^3) = C(S^3)$. This fact and the equivalence of (1) and (3) supports thinking of $C(S_\theta^3)$ as truly being a sort of deformation of S^3 that preserves the Heegaard splitting. The subsequent realization of $C(S_\theta^3)$ as an example of a θ -deformation in Connes and Landi's sense [12] (and in turn as an example of Rieffel's deformation quantization by actions of \mathbb{R}^n [38]) gives this statement a precise meaning.

Matsumoto also showed that $C(S_\theta^3)$ can be viewed as a continuous field of C^* -algebras over the unit interval $[0, 1]$. This mirrors the identification of S^3 with a fibered space over $[0, 1]$. Specifically, there is an obvious action of the 2-torus T^2 on S^3 given by

$$(\exp(2\pi i\phi_1), \exp(2\pi i\phi_2)) \cdot (z_1, z_2) = (\exp(2\pi i\phi_1)z_1, \exp(2\pi i\phi_2)z_2).$$

The orbit space of S^3 for this action is the unit interval $[0, 1]$. The fiber over $t := |z_1|^2 \in [0, 1]$ is simply the space T_t from the Heegaard splitting, i.e. a torus with generating circles of radii \sqrt{t} and $\sqrt{1-t}$, respectively. Thus, there are degenerate circles over the endpoints of the interval, and S^3 is not locally-trivial as a fibered space over the interval. Since the fiber over t is the space T_t from the Heegaard splitting, the restrictions of the fibered space to the intervals $[0, 1/2]$ and $[1/2, 1]$ are homeomorphic to the solid tori $\cup_{t \leq 1/2} T_t$ and $\cup_{t \geq 1/2} T_t$.

At the level of function algebras, if V is the coordinate function of the degenerate circle over the point 0 and U is the coordinate function of the degenerate circle over the point 1, we may identify the generating circle of radius \sqrt{t} over t with the circle $\sqrt{t}U$, and we may identify the generating circle of radius $\sqrt{1-t}$ with $\sqrt{1-t}V$. Viewing z_1 and z_2 as coordinate functions in $C(S^3)$, we see that z_1 corresponds to the map $t \mapsto \sqrt{t}U$, and z_2 corresponds to the map $t \mapsto \sqrt{1-t}V$. So $C(S^3)$ is a continuous field of C^* -algebras over $[0, 1]$. The following theorem generalizes this to the case where θ is any real number.

Theorem 1.1.5. *(Proposition 2 of [25]). The noncommutative solid torus $C(F_\theta^2)$ is a continuous field of C^* -algebras over $[0, 1/2]$. The fibers over each point in $(0, 1/2]$ are isomorphic to $C(T_\theta^2)$. The fiber over 0 is isomorphic to $C^*(V)$. The field is locally constant except over 0. The noncommutative 3-sphere $C(S_\theta^3)$ is a continuous field of C^* -algebras over $[0, 1]$. Each fiber is isomorphic to $C(T_\theta^2)$, except over the endpoints $\{0, 1\}$, over which the fibers are respectively $C^*(V), C^*(U)$. The field is locally constant except over the end points. The generator z_1 corresponds to the map $t \mapsto \sqrt{t}U$ under this isomorphism, and the generator z_2 corresponds to the map $t \mapsto \sqrt{1-t}V$.*

Matsumoto also calculated the K -theory of $C(S_\theta^3)$.

Theorem 1.1.6. *(Proposition 5.2 of [24]). $K_0(C(S_\theta^3)) \cong \mathbb{Z}$, $K_1(C(S_\theta^3)) \cong \mathbb{Z}$.*

This was proved by using the Mayer-Vietoris sequence for K -theory. We remark that the result also follows from [39] by noting that $C(S_\theta^3)$ is a deformation of $C(S^3)$ by an action of \mathbb{R}^2 . Matsumoto did not, however, explicitly describe the generator of $K_1(C(S_\theta^3))$. We explicitly give the generator of the K_1 -group in the case that $|\theta| < 1$ is irrational in our

Theorem 2.4.4.

We now introduce the θ -deformed 4-sphere $C(S_\theta^4)$.

2) Looking for examples of “noncommutative differentiable manifolds” in the sense of [8], Connes and Landi [12] investigated the solutions of certain homological equations. More precisely, for each n (thought of as dimension), Connes and Landi associated a set of equations. These were chosen so that any solution of the equations for dimension n would have homology closely resembling that of the n -sphere S^n . Specifically, let e be any matrix projection with entries from any $*$ -algebra. The matrix e satisfies Connes and Landi’s equations for the case $n = 2k$ if the components $ch_j(e)$ of it’s Chern character in (algebraic) cyclic homology [22] vanish (as chains in the complex for cyclic homology) for $j < k$, and if also $ch_k(e)$ is nontrivial (as a cyclic cycle). These conditions entail that $ch_k(e)$ is in fact a nontrivial Hochschild cycle. The possible sizes of the matrix projections that satisfy the equations turns out to be controlled by n . Satisfying the equations forces relations between the entries of the matrix projection e . One may consider the polynomial $*$ -algebra A generated by the entries of e and subject to the relations imposed by e being a projection and a solution to Connes and Landi’s equations for $n = 2k$. The algebra A generates a C*-algebra in the usual way. For the case $n = 2$, the function algebra $C(S^2)$ is the only solution. For $n = 4$, Connes and Landi considered projections of the form

$$e := \frac{1}{2} \begin{pmatrix} 1+x & 0 & z_1 & z_2 \\ 0 & 1+x & -\lambda z_2^* & z_1^* \\ z_1^* & -\bar{\lambda} z_2 & 1-x & 0 \\ z_2^* & z_1 & 0 & 1-x \end{pmatrix},$$

where $\lambda = \exp(2\pi i\theta)$. The matrix e is a projection and satisfies Connes and Landi’s equations if the relations imposed on the entries of e are

$$z_i z_i^* = z_i^* z_i, \quad x = x^*, \quad x z_1 = z_1 x, \quad x z_2 = z_2 x, \quad z_2 z_1 = \lambda, \quad z_1 z_2, z_1 z_1^* + z_2 z_2^* + x^2 = 1.$$

Definition 1.1.7. The polynomial $*$ -algebra generated by z_1, z_2, x subject to the relations

$$z_i z_i^* = z_i^* z_i, \quad x = x^*, \quad x z_1 = z_1 x, \quad x z_2 = z_2 x, \quad z_2 z_1 = \lambda, \quad z_1 z_2, z_1 z_1^* + z_2 z_2^* + x^2 = 1,$$

is denoted $C_{alg}(S_\theta^4)$. The universal C*-algebra generated by $C_{alg}(S_\theta^4)$ is denoted $C(S_\theta^4)$.

If $\theta = 0$, so $\lambda = 1$, the module $e(C(S^4)^4)$ is the space of sections of a rank-2 complex vector bundle over S^4 with “topological charge” (integral over S^4 of it’s second Chern class in DeRham cohomology), or “index”, equal to -1. The vector bundle supports an anti-self-dual connection, or “instanton”, satisfying the $SU(2)$ -Yang-Mills equations [3]. Connes and Landi calculated that for arbitrary θ , the index pairing between the Chern character of e in cyclic homology with the Chern character of an appropriate Fredholm module in cyclic

cohomology is 1. The pairing is a noncommutative generalization of Atiyah's index pairing [2] between the K -theory and K -homology of spaces. Connes and Landi also show that the associated Levi-Civita connection ede is anti-self-dual for an appropriate " θ -deformed" algebra of differential forms and Hodge $*$ -operation. Furthermore, the partial trace of e is 2 (By definition, the partial trace of e is the sum in $C(S_\theta^4)$ of the diagonal entries of e). So if tr is a normalized trace for $C(S_\theta^4)$, then $tr(e)=2$. As discussed further in the section "Trace and Dimension", this means that e can be thought of as having "rank" equal to 2. Putting together these considerations, it is appropriate to think of e as a noncommutative rank-2 "instanton bundle" over $C(S_\theta^4)$ of charge -1.

Subsequent work by Landi and Van Suijlekom [20] exhibits $C(S_\theta^4)$ -modules of rank $n+1 \geq 2$ and index $(1/6)n(n+1)(n+2)$ with corresponding anti-self-dual connections. Brain and Landi [6] have obtained a substantial amount of information concerning the moduli spaces of some solutions of a Yang-Mills theory for $C(S_\theta^4)$, though this is still very much a theory in development.

We remark that one might suspect that e will be a basic module for $C(S_\theta^4)$, as in the case $\theta = 0$, in that e won't split as a direct sum of nonzero $C(S_\theta^4)$ -modules. However, the example of the noncommutative tori shows that the semigroup $V(A)$ for a noncommutative space A need not resemble that of a corresponding classical case very closely at all, as any nonzero finitely-generated projective $C(T_\theta^2)$ -module in $V(C(T_\theta^2))$ can be split into a nontrivial direct sum (see [37]). For θ irrational, we will see that e is special in that it splits into a direct sum of two modules that cannot be further split and together generate $K_0(C(S_\theta^4))$.

For the case $n = 2k + 1$, Connes and Landi's equations instead take a unitary matrix U as argument, and the requirement is that the components $ch_{\frac{j}{2}}(U)$ vanish (again at the chain level) for $j < k$, while $ch_{\frac{k}{2}}(U)$ is required to be nontrivial. For the case $n = 3$, Connes and Landi considered matrices of the form $U := \begin{pmatrix} z_1 & z_2 \\ -\lambda z_2^* & z_1^* \end{pmatrix}$. The relations imposed on the entries of U by the condition that U be unitary and satisfy Connes and Landi's equations are

$$z_1 z_1^* = z_1^* z_1, \quad z_2 z_2^* = z_2^* z_2, \quad z_2 z_1 = \lambda z_1 z_2, \quad z_1^* z_1 + z_2^* z_2 = 1.$$

Thus the universal C^* -algebra generated by the entries of U subject to these relations is isomorphic to Matsumoto's algebra $C(S_\theta^3)$. We thus denote the polynomial $*$ -algebra generated by the entries of U subject to the above relations $C_{alg}(S_\theta^3)$.

If $\theta = 0$, so $\lambda = 1$, viewing z_1 and z_2 as coordinates, the corresponding unitary matrix U is simply the homeomorphism from S^3 onto $SU(2)$, and therefore is also the generator of $K^1(S^3) \cong \pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$. The complex vector bundle over S^4 obtained from the clutching construction by using U to glue together rank-2 trivial bundles over the northern and southern hemisphere is the instanton bundle of charge -1.

Later Connes and Dubois-Violette found all solutions of Connes and Landi's equations for $n = 3$ as a three parameter family of algebras. Connes and Dubois-Violette describe the moduli space of solutions in [11]. For certain values in the moduli space of solutions,

one obtains algebras that are isomorphic to the $C(S_\theta^3)$. In the generic case, one recovers the algebras of Sklyanin [45]. This is very interesting, as it is a link between operator-algebraic noncommutative geometry and algebraic geometry.

3) In showing that the algebras $C(S_\theta^4)$ satisfy the axioms of a “noncommutative differentiable manifold” [8], Connes and Landi showed that they can be constructed through a type of deformation procedure they call θ -deformation. These deformations were introduced in the setting in which a spin- \mathbb{C} manifold M (equipped with its usual charge-conjugation and Dirac operator) has an abelian isometry group of rank $n \geq 2$. In that case, the corresponding spectral triple can be deformed by an action of the 2-torus without deforming the Dirac operator (“isospectral deformation”). The resulting spectral triple satisfies the axioms of a noncommutative differentiable manifold. As the charge-conjugation and Dirac operator for M play no role in the deformation of the algebra of smooth functions $C^\infty(M)$ itself, we will ignore their role in deforming the spectral triple, and instead refer the reader to [12].

Let M be a compact Riemannian manifold with an isometric action of the 2-torus T^2 . Then any element of $C^\infty(M)$ can be written as a series whose terms are indexed by the spectral subspaces for the induced action of T^2 on $C^\infty(M)$. The series converges rapidly in the usual Fréchet topology on $C^\infty(M)$. The spectral subspaces will be indexed by the space of characters of T^2 i.e. \mathbb{Z}^2 .

Definition 1.1.8. Let θ be any real number. Define a product on the vector space $C^\infty(M)$ by $f_r \times_\theta g_s := \exp(-2\pi i \theta r_1 s_2) f_r g_s$, where f_r and g_s are in the r th and s th spectral subspaces respectively. The θ -deformed algebra $C^\infty(M_\theta)$ is defined to be the $C^\infty(M)$ together with the product \times_θ .

For the case where m is the 4-sphere

$$S^4 = \{(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} : z_1 z_1^* + z_2 z_2^* + x^2 = 1\},$$

we use the action of T^2 given by

$$(\exp(2\pi i \phi_1), \exp(2\pi i \phi_2)) \cdot (z_1, z_2, x) = (\exp(2\pi i \phi_1) z_1, \exp(2\pi i \phi_2) z_2, x).$$

Notice that the original involution of complex conjugation of functions in $C^\infty(M)$ is not an involution for the product \times_θ . However, as was quickly pointed out independently by Várilly [48] and Sitarz [44], one can define a product \times_J on the vector space of functions $C^\infty(M)$ so as to obtain an algebra that is isomorphic to $C^\infty(M_\theta)$, but for which the complex conjugation of functions is still an involution. First, one observes that if the rank of the isometry group of M is $n \geq 2$, one can generalize the θ -deformation to a deformation by the action of the n -torus. In that case, the spectral subspaces for the action will be indexed by \mathbb{Z}^n , and if one replaces the real number θ with an $n \times n$ matrix $(\theta)_{ij}$, the formula $f_r \times_\theta g_s := \exp(-2\pi i \sum_{i \leq j} r_i \theta_{ij} s_j) f_r g_s$ will generalize the definition given for the product for

the case of an action of the 2-torus. But the 2-cocycle $\rho(r, s) := \exp(-2\pi i \sum_{i \leq j} \theta_{ij} r_i s_j)$ is cohomologous to the skew-symmetrized 2-cocycle $\sigma(r, s) := \exp(-\pi i \sum_{i=j} r_i \theta_{ij} s_j)$, and thus, as explained in [37], replacing ρ with σ in the formula for the product results in an isomorphic algebra on the underlying set $C^\infty(M)$. If the matrix $(\theta)_{ij}$ is skew-symmetric, then complex conjugation of functions will be an involution for this new product. One completes to a C*-algebra $C(M_\theta)$ by taking the largest possible C*-norm on the vector space $C^\infty(M)$ endowed with the product from σ and with complex conjugation as the involution.

4) The above observation is the main insight in Várilly and Sitarz's recognition that θ -deformations can be seen as special instances of Rieffel's deformation quantization in which the action of the vector group $V = \mathbb{R}^n$ on $C^\infty(M)$ factors through the action of the compact abelian group T^n .

The analysis involved in Rieffel's very general procedure for deforming a C*-algebra A by an action of the group $V = \mathbb{R}^n$ is quite technical. The reader is referred to Rieffel's monograph [38] (which we follow closely) for more detail.

Let α be an action of V on A , and let J be any skew-symmetric matrix in $M_n(\mathbb{R})$. Let A^∞ be the space of vectors in A that are smooth for the action α (i.e. the set of $a \in A$ such that the map $v \mapsto \alpha_v(a)$ is C^∞ for the norm on A). The vector space A^∞ is in fact necessarily a dense *-subalgebra of A . Introduce a new product on A^∞ by the twisted Fourier transform

$$a \times_J b := \iint_{V \times V} \alpha_{J \cdot u}(a) \alpha_v(b) \exp(2\pi i u \cdot v) du dv.$$

Here the integral must be taken as an oscillatory integral in order to make sense. Because J is skew-symmetric, the original involution on A is still an involution for the product \times_J . A C*-norm for the product \times_J can be placed on A^∞ , by first introducing \mathcal{S}^A , the right A -module of A -valued Schwartz functions on V . The A -valued inner product

$$\langle f, g \rangle_A = \int f(x)^* g(x) dx$$

is placed on \mathcal{S}^A . For $a \in A^\infty$, define an operator L_a on \mathcal{S}^A by the formula

$$(L_a f)(x) = \iint_{V \times V} \alpha_{x+J \cdot u}(a) f(x+v) \exp(2\pi i u \cdot v) du dv.$$

One checks that L is a *-homomorphism from A^∞ with the product \times_J into the algebra of operators on \mathcal{S}^A . Rieffel showed that the operators L_a are in fact bounded for the A -valued inner product on \mathcal{S}^A . The C*-norm $\|a\| := \|L_a\|_{\mathcal{B}(\mathcal{S}^A)}$ is defined on A^∞ . The algebra A^∞ is completed in this norm to a C*-algebra A_J .

The construction is functorial in that if A and B are C*-algebras equipped with actions of V , and $\phi : A \rightarrow B$ is a homomorphism that intertwines the actions of V , then the restriction $\phi : A^\infty \rightarrow B^\infty$ extends to a homomorphism $\phi_J : A_J \rightarrow B_J$. A fundamental result of Rieffel's that we will need is:

Theorem 1.1.9. *Given an α -invariant ideal I of A and an equivariant short exact sequence*

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0,$$

the induced sequence

$$0 \rightarrow I_J \rightarrow A_J \rightarrow B_J \rightarrow 0$$

is also exact.

If the V -action α on A factors through an action of a compact abelian group G , then for each character s of G , there is a spectral subspace

$$A_r = \{a \in A : \alpha_v(a) = \exp(2\pi i r \cdot v)a \text{ for all } a \in V\}.$$

The direct sum of these spectral subspaces is dense in A . Rieffel calculates in proposition 2.22 of [38] that in this case $a \times_J b = \exp(-2\pi i r \cdot Js)ab$ for $a \in A_r, b \in A_s$. Thus, if G is the n -torus, and A is $C(M)$, taking J to be the skew-symmetric matrix $(\theta)_{ij}/2$, we recover the product formula for the θ -deformation. One can show that the norm $\|\cdot\|_J$ is the largest C^* -norm on $C^\infty(M_\theta)$.

5) Connes and Dubois-Violette [10] give an equivalent description of the algebras $C^\infty(M_\theta)$ as a fixed-point subalgebra. The fixed-point description is in terms of the smooth manifold M and the noncommutative torus. Specifically, following [10], suppose that σ is a smooth action of the torus T^n on M . We use σ to also denote the induced action of T^n on the Fréchet algebra $C^\infty(M)$. Let τ be the natural action of T^n on $C(T_\theta^n)$, and let $C^\infty(T_\theta^n)$ be the set of smooth vectors for the action τ . Define a Fréchet topology on $C^\infty(T_\theta^n)$ as the locally-convex topology generated by the seminorms $|a|_r = \sup_{r_1+\dots+r_n \leq r} \|X_1^{r_1} \dots X_n^{r_n}(a)\|$, where the X_k are the infinitesimal generators of the action of T^n on $C(T_\theta^n)$, and $\|\cdot\|$ is the norm on $C(T_\theta^n)$. Form the projective tensor product of $C^\infty(M)$ and $C^\infty(T_\theta^n)$, and take its completion $C^\infty(M) \hat{\otimes} C^\infty(T_\theta^n)$. Note that $C^\infty(T_\theta^n)$ is nuclear in the sense of Grothendieck [16]. Since the actions σ and τ are continuous for the respective Fréchet topologies on $C^\infty(M)$ and $C^\infty(T_\theta^n)$, the fixed-point subalgebra $(C^\infty(M) \hat{\otimes} C^\infty(T_\theta^n))^{\sigma \times \tau^{-1}}$ is a Fréchet space. The Fréchet algebra $C^\infty(M_\theta)$ is then isomorphic to $(C^\infty(M) \hat{\otimes} C^\infty(T_\theta^n))^{\sigma \times \tau^{-1}}$. The universal C^* -algebra generated by $(C^\infty(M) \hat{\otimes} C^\infty(T_\theta^n))^{\sigma \times \tau^{-1}}$ is the C^* -completion of $(C(M) \otimes_{alg} C(T_\theta^n))^{\sigma \times \tau^{-1}}$.

By using similar constructions in terms of fixed-point algebras, Connes and Dubois-Violette [10] provide deformed versions of differential forms and various matrix groups. It should be noted that $C(S_\theta^3)$ cannot be viewed as a deformation of the group structure of $SU(2)$. Várilly [48] showed that θ -deformed spheres can, however, be viewed as “quantum homogeneous spaces” for actions of θ -deformed special orthogonal groups. Using the fixed-point description of θ -deformed spaces, Connes and Dubois-Violette show that they preserve *Hochschild dimension*, in that the highest nonvanishing Hochschild homology group occurs at the same level for all θ -deformations of an algebra. Indeed the impetus for θ -deformations was to find noncommutative manifolds whose cohomology agrees with that of their classical

versions as closely as possible. The homology of algebras resulting from other types of deformation procedures may not resemble that of the deformed classical space. For example, Masuda, Nakagami, and Watanabe's [23] calculation of the cyclic homology of the coordinate ring $A(SL_q(2))$ shows that the Hochschild dimension of $A(SL_q(2))$ is one. In fact, for a general deformation of a manifold by \mathbb{R}^2 , it can happen that all Hochschild homology groups above level zero collapse [10].

Hanfeng Li [21] showed that θ -deformed compact metric spaces are compact quantum metric spaces in Rieffel's sense.

1.2 Non-stable K-theory

Definition 1.2.1. A commutative semigroup S is said to be *cancellative* if for all $x, y, z \in S$, if $x + z = y + z$, then $x = y$.

A commutative semigroup S generates a Grothendieck or universal abelian enveloping group $G = K(S)$. Perhaps the most straightforward way to construct G is to consider the quotient of the semigroup $S \times S$ by the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if and only if there is a $z \in S$ so that $x_1 + y_2 + z = x_2 + y_1 + z$. It is easy to check that the addition in $S \times S$ descends to a well-defined associative and commutative addition on G . For any x , the equivalence class $[(x, x)]$ of (x, x) is the identity element of G , and the inverse of $[(x, y)]$ is $[(y, x)]$. Intuitively, one may think of $[(x, y)] \in G$ as $x - y$ in a formal group of differences of S . Indeed, applying the enveloping construction to the semigroup of natural numbers results in the group of integers.

There is a canonical semigroup homomorphism from S into G given by $x \mapsto [(x + x, x)]$. It is very easy to verify that this homomorphism is injective if and only if S is cancellative. The image of S in G under this mapping is a cancellative semigroup and is often referred to as the *positive cone* of G . Identifying $(x + x, x)$ with the difference $x - 0$ we regard the image of x in the positive cone as "x up to stabilization".

The construction is universal in that any semigroup homomorphism from S into any abelian group factors through G , and is unique in the appropriate sense. The process of taking Grothendieck groups is a covariant functor from the category of commutative semigroups to abelian groups, and in many cases of interest, forms a (extraordinary) homology theory. As a consequence of this, the group $K(S)$ can often be computed without detailed information about S , or without information about how the positive cone of S is embedded in $K(S)$. However, if S is not cancellative, it is possible for $K(S)$ to provide little, or even no information about S , as we will see below. The investigation of S rather than $K(S)$ is called *non-stable K-theory*. See [36] for a list of interesting questions of non-stable K-theory appropriate for C^* -algebras.

Example 1.2.2. For $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , the isomorphism classes $V(X)$ of \mathbb{F} -vector bundles over a compact space X form a commutative semigroup (with identity element the class of the zero bundle over X) under the direct (Whitney) sum operation. In this setting, the group $K(V)$ is denoted $K^0(X)$. Given a map $X \rightarrow Y$ between compact spaces X and Y , and a vector bundle E over Y , the pullback $f^*(E)$ is a vector bundle over X . Since homotopic maps yield isomorphic pullbacks, taking K^0 gives a contravariant functor from the category of compact spaces to the category of abelian groups. This is the topological K -theory of Atiyah [1], [19].

If R is a (not necessarily commutative) unital ring, then the set $V(R)$ of isomorphism classes of right finitely-generated projective R -modules forms a commutative semigroup under direct sum. The Grothendieck group of $V(R)$ is denoted $K_0(R)$ and is called the (algebraic) K_0 -group of R . By Swan's theorem, in the case of the algebra $C(X)$ of continuous complex-valued functions on a compact space X , each finitely-generated projective right $C(X)$ -module is the space of continuous sections of some complex vector bundle over X , and the space of continuous sections of any complex vector bundle over X is a right finitely-generated projective $C(X)$ -module. The correspondence is in fact an equivalence of categories, and an analogous statement also holds between the smooth subcategories. This justifies using the notation $V(R)$ to denote the semigroup of isomorphism classes of right finitely-generated projective R -modules. Thus $V(C(X))$ identifies with $V(X)$, and $K_0(C(X))$ identifies with $K^0(X)$.

Definition 1.2.3. Let R be a unital ring. If the semigroup $V(R)$ is cancellative, we say that R is K -cancellative.

Thus R is K -cancellative if and only if $V(R)$ injects onto its positive cone in $K_0(R)$.

Example 1.2.4. The tangent bundle $T_*(S^n)$ consists of all tangent vectors to the n -sphere S^n . The tangent vectors to a point $x \in S^n$ are the vectors in \mathbb{R}^{n+1} based at x that are orthogonal to S^n at x . However, $T_*(S^2)$ is not isomorphic to the real rank-2 trivial bundle $S^2 \times \mathbb{R}^2$ because if it were, then any constant nonzero section of the trivial bundle could be pulled back to the tangent bundle to give a nonzero vector field on S^2 , which famously cannot exist by a classical theorem of algebraic topology. In fact, $T_*(S^n)$ is trivial if and only if $n = 1, 3, 7$, by a theorem of Adams.

The normal bundle to S^n in \mathbb{R}^{n+1} consists of vectors based on the n -sphere that are perpendicular to the tangent bundle, so the normal bundle consists of elements of the form (x, tx) , $t \in \mathbb{R}$. The normal bundle of S^n is thus a trivial line bundle over S^n via the isomorphism $(x, tx) \mapsto (x, t)$. The sum $T_*(S^n) \oplus N(S^n)$ is isomorphic to $X \times \mathbb{R}^{n+1}$ via the map $(x, v) + (x, tx) \mapsto (x, v + tx)$. Thus $T_*(S^n)$ is stably trivial, and the semigroup of isomorphism classes of real vector bundles over S^n is not cancellative in the case $n \neq 1, 3, 7$, and so the real K^0 -group of S^n does not detect the tangent bundle over S^n if $n \neq 1, 3, 7$ (or rather, we might say it believes that the tangent bundle is trivial).

The 5-sphere S^5 is an example of a compact space for which its semigroup of complex vector bundles fails cancellation. By the clutching construction, there is a bijection between the set of isomorphism classes of complex rank- k vector bundles over S^n and the homotopy group $\pi_{n-1}(U(k))$. As $\pi_4(U(k)) \cong 0$ for $k \neq 2$, the 5-sphere has no nontrivial complex vector bundles of rank $k \neq 2$. But since $\pi_4(U(2)) \cong \pi_4(SU(2)) \cong \pi_4(S^3) \cong \mathbb{Z}_2$, there is exactly one nontrivial complex vector bundle over S^5 , up to isomorphism. Its direct sum with a trivial line bundle must be trivial, and so S^5 is not K -cancellative. Indeed, by Bott periodicity, one has that $K^0(S^5) \cong \mathbb{Z}$.

In a positive direction, it can be shown using only very elementary results from homotopy theory that if X is an n -dimensional CW-complex, and if $E \oplus F \cong E' \oplus F$ where E is a complex bundle over X of rank $k \geq n/2$, and E', F are arbitrary complex bundles over X , then $E \cong E'$ (see [18]). Thus cancellation automatically holds for vector bundles of high enough rank compared to the dimension of the base space. One can also show that any vector bundle of sufficiently high rank can be decomposed into the sum of a bundle of lower rank and a trivial bundle. Using these results along with a by-hand examination of the direct sums of lower rank bundles, it may be possible to fully understand the structure of the semigroup of vector bundles over a CW-complex of sufficiently low dimension.

Rieffel [34] introduced the notion of topological stable rank (tsr) for Banach algebras as a generalization of the complex dimension of a topological space. Rieffel also introduced the related notions of connected stable rank (csr) and general stable rank (gsr). These bear upon the nonstable K -theory of a Banach algebra in various ways (see [36]). We mention that statements can be made about cancelling modules of high enough rank in comparison to the topological stable rank of the endomorphism ring of the module (e.g. theorem 3 of [36]). The topological stable rank of a C^* -algebra was shown to agree with its algebraic Bass stable rank as a ring by Herman and Vaserstein [17]. If the topological stable rank of a unital C^* -algebra A is 1, then A is K -cancellative. Putnam [31] showed that the topological stable rank of any simple noncommutative 2-torus is equal to 1. The topological stable rank of any non-simple noncommutative torus is 2, for θ any skew-symmetric matrix with at least one irrational entry. However, these results were only obtained after Rieffel first fully described the nonstable K -theory of the noncommutative tori (he did, however, use upper bounds on topological stable ranks to obtain some of his results). In general it is not very easy to calculate the topological stable rank of a C^* -algebra, and some questions of nonstable K -theory require more information than the tsr , gsr , and csr of relevant algebras for their solution. We will be able to give easy proofs that $tsr(C(S_\theta^3)) \leq 2$ (Sudo [46] has shown that $tsr(C(S_\theta^3)) = 2$) and $csr(C(S_\theta^3)) = 2$.

1.3 Clutching Construction

Throughout this section we work in the category of unital rings.

Our classification of the right finitely-generated projective $C(S_\theta^3)$ and $C(S_\theta^4)$ -modules relies on a generalization of the familiar clutching construction (theorem 3.1 of [19]) of vector bundle theory to the setting of unital rings. We follow Milnor [27]:

Given M a right R -module, a unital ring homomorphism $f : R \rightarrow S$ induces a right S -module $M \otimes_R S$ denoted $f_\# M$. There is a canonical R -linear map $f_* : M \rightarrow f_\# M$ defined by $f_*(m) := m \otimes_R 1$.

Consider the pullback diagram

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ i_2 \downarrow & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & S. \end{array}$$

Now suppose P_1 is a projective right R_1 -module, P_2 is a projective right R_2 -module, and that $h : j_{1\#} P_1 \rightarrow j_{2\#} P_2$ is an isomorphism.

Definition 1.3.1. The right R -module $M(P_1, P_2, h)$ is defined to be the additive group

$$\{(p_1, p_2) \in P_1 \times P_2 : h j_{1*}(p_1) = j_{2*}(p_2)\}$$

together with an R -module structure given by $(p_1, p_2) \cdot r = (p_1 \cdot i_1(r), p_2 \cdot i_2(r))$.

Assume also that at least one of the j_k is surjective. The following three theorems are found in Milnor [27]:

Theorem 1.3.2. (Theorem 2.1 of [27]). *The module $M(P_1, P_2, h)$ is projective over R . If P_1 and P_2 are finitely-generated over R_1 and R_2 respectively, then $M(P_1, P_2, h)$ is finitely-generated over R .*

Theorem 1.3.3. (Theorem 2.2 of [27]). *Any projective R -module M is isomorphic to $M(P_1, P_2, h)$ for some P_1, P_2 and h .*

Theorem 1.3.4. (Theorem 2.3 of [27]). *The modules P_1 and P_2 are canonically isomorphic to $i_{1\#} M$ and $i_{2\#} M$ respectively.*

We note that an immediate consequence of Theorem 1.3.2, Theorem 1.3.3, and Theorem 1.3.4 is that any projective R -module M is isomorphic to $M(i_{1\#} M, i_{2\#} M, f)$, where f is the canonical isomorphism

$$f(e \otimes_R 1 \otimes_{R_1} 1) = e \otimes_R 1 \otimes_{R_2} 1, \quad e \in M.$$

This is the algebraic analog of the fact that the restrictions of any vector bundle over a space X to two intersecting subspaces of X can be viewed as glued together over the intersection by the identity map.

We also need a slight extension of the well-known result (theorem 3.4 of [19]) that the rank- n complex vector bundles over a compact space X are in bijective correspondence with the set of so-called “ GL_n -cocycle classes of X ”. Of course that result itself is just another version of the clutching construction.

Specifically, if both maps j_k are surjective, and if $i_{1\#}M$ and $i_{2\#}M$ are free modules over R_1 and R_2 , respectively, such that $i_{1\#}M \cong R_1^n$ and $i_{2\#}M \cong R_2^n$, we will then associate to $M \cong M(i_{1\#}M, i_{2\#}M, f)$ an element $g \in GL_n(S)$ so that, viewing g as a map, we have that $M \cong M(R_1^n, R_2^n, g)$. The matrix g will be unique up to the following notion of equivalence:

Definition 1.3.5. We write $g \sim g'$ if there exist $g_1 \in GL_n(R_1)$ and $g_2 \in GL_n(R_2)$ such that $g' = j_{2*}(g_2) \cdot g \cdot j_{1*}(g_1^{-1})$. We then say that g and g' are in the same $GL_n(S)$ -cocycle class \bar{g} .

It is trivial to verify that \sim is an equivalence relation. We now describe how to construct g : Let $\psi_k : R_k^n \xrightarrow{\cong} i_{k\#}M$ be trivializations for $i_{k\#}M$, $k = 1, 2$. Let $e_{R_k, i}$ denote the i -th element $(0, \dots, 1_{R_k}, \dots, 0)$ of the standard ordered basis for R_k^n . Define maps $j_{k*}(\psi_k) : S^n \rightarrow j_{k\#}i_{k\#}M$ by the formula

$$j_{k*}(\psi_k)(e_{S, i}) = j_{k*}(\psi_k)(e_{R_k, i}).$$

We have the following diagram:

$$\begin{array}{ccc}
 i_{1\#}M & \xleftarrow[\cong]{\psi_1} & R_1^n \\
 \downarrow & & \downarrow \\
 j_{1\#}i_{1\#}M & \xleftarrow[\cong]{j_{1*}(\psi_1)} & S^n \\
 \downarrow f & & \downarrow g \\
 j_{2\#}i_{2\#}M & \xleftarrow[\cong]{j_{2*}(\psi_2)} & S^n \\
 \uparrow & & \uparrow \\
 i_{2\#}M & \xleftarrow[\cong]{\psi_2} & R_2^n
 \end{array}$$

where the map $R_k^n \rightarrow S^n$ takes $e_{R_k, i}$ to $(e_{S, i})$, and the map f is the canonical isomorphism defined by $f(m \otimes_R 1 \otimes_{R_1} 1) = m \otimes_R 1 \otimes_{R_2} 1$. Requiring that the diagram commutes forces the definition

$$g := j_{2*}(\psi_2^{-1}) \circ f \circ j_{1*}(\psi_1),$$

where $j_{2*}(\psi_2^{-1})$ is defined to be $(j_{2*}(\psi_2))^{-1}$. Given our choice of the standard ordered basis for S^n , the map g identifies uniquely with an element of $GL_n(S)$, which by abuse of notation we also refer to as g .

Proposition 1.3.6. *The cocycle class of $g \in GL_n(S)$ obtained from the above construction is independent of the choice of trivializations $\psi_k : R_k^n \xrightarrow{\cong} i_{k\#}M$.*

Proof. Suppose $\phi_k : R_k^n \rightarrow \cong i_{k\#}M$ are trivializations for which the above construction yields the map g' . Define g_k as the isomorphism $\phi_k^{-1} \circ \psi_k$. Then

$$\begin{aligned} j_{2*}(g_2) \circ g \circ j_{1*}(g_1^{-1}) &= j_{2*}(\phi_2^{-1} \circ \psi_2) \circ j_{2*}(\psi_2^{-1}) \circ f \circ j_{1*}(\psi_1) \circ j_{1*}(\psi_1^{-1} \circ \phi_1) \\ &= j_{2*}(\phi_2^{-1}) \circ f \circ j_{1*}(\phi_1) = g', \end{aligned}$$

by commutativity of the diagram. Expressing this calculation in terms of the matrices that represent the maps $j_{k*}(g_k)$, g , and g' in the basis $\{e_{S,i}\}_{i \leq n}$ gives the result. \square

Proposition 1.3.7. *If $g \sim g'$, then the R -modules $M(R_1^n, R_2^n, g)$ and $M(R_1^n, R_2^n, g')$ are isomorphic.*

Proof. If $g' = j_{2*}(g_2) \circ g \circ j_{1*}(g_1^{-1})$, then viewing g and g' as maps, we see that $M(R_1^n, R_2^n, g) \cong M(R_1^n, R_2^n, g')$ via the map $(p_1, p_2) \mapsto (g_1 p_1, g_2 p_2)$, where $(p_1, p_2) \in M(R_1^n, R_2^n, g)$. \square

Thus we have shown how to obtain a $GL_n(S)$ -cocycle class from a R -module N for which $i_{1\#}N \cong R_1^n$ and $i_{2\#}N \cong R_2^n$. Conversely, we can construct an R -module M by using the clutching map corresponding to any representative of a $GL_n(S)$ -cocycle class. We need to verify that these are inverse processes up to cocycle equivalence on the one hand, and isomorphism of R -modules on the other.

Proposition 1.3.8. *If g is the result of applying the construction immediately preceding Proposition 1.3.6 to an R -module N for which $i_{1\#}N \cong R_1^n$ and $i_{2\#}N \cong R_2^n$, then $N \cong M(R_1^n, R_2^n, g)$.*

Proof. By the construction, the following diagram commutes:

$$\begin{array}{ccc} i_{1\#}N & \xleftarrow[\cong]{\psi_1} & R_1^n \\ \downarrow & & \downarrow \\ j_{1\#}i_{1\#}N & \xleftarrow[\cong]{j_{1*}(\psi_1)} & S^n \\ \downarrow f & & \downarrow g \\ j_{2\#}i_{2\#}N & \xleftarrow[\cong]{j_{2*}(\psi_2)} & S^n \\ \uparrow & & \uparrow \\ i_{2\#}N & \xleftarrow[\cong]{\psi_2} & R_2^n. \end{array}$$

Hence if $(i_{1*}e, i_{2*}e) \in N$, then $(\psi_1^{-1}(i_{1*}e), \psi_2^{-1}(i_{2*}e)) \in M(R_1^n, R_2^n, g)$. This correspondence is an isomorphism. \square

Conversely, we have:

Proposition 1.3.9. *Applying the construction immediately preceding Proposition 1.3.6 to the R -module $M := M(R_1^n, R_2^n, g)$ yields a matrix g' in the same cocycle class as the matrix g .*

Proof. By Theorem 1.3.4, we have the commutative diagram

$$\begin{array}{ccc}
 i_{1\#}M & \xleftarrow{\cong} & R_1^n \\
 \downarrow & & \downarrow \\
 j_{1\#}i_{1\#}M & \xleftarrow{\cong} & S^n \\
 f \downarrow & & \downarrow g \\
 j_{2\#}i_{2\#}M & \xleftarrow{\cong} & S^n \\
 \uparrow & & \uparrow \\
 i_{2\#}M & \xleftarrow{\cong} & R_2^n.
 \end{array}$$

Now, in constructing g' , if we choose the canonical maps from Theorem 1.3.4 as our trivializations of the modules $i_{k\#}(M)$, we then obtain exactly the same commutative diagram, only with g' replacing g . So the constructed g' is precisely g in this case. So by Proposition 1.3.6, any other choices of trivializations would result in different elements of $GL_n(S)$ all in the same cocycle class. \square

Proposition 1.3.10. *There is a bijective correspondence between the set of those isomorphism classes of finitely-generated projective R -modules that contain a representative M for which $i_{1\#}M \cong R_1^n$ and $i_{2\#}M \cong R_2^n$, and the set of equivalence classes of $GL_n(S)$ -cocycles.*

Proof. Immediate from Propositions 1.3.8 and 1.3.9. \square

Definition 1.3.11. A unital ring R has the *invariance of dimension* property if for each free R -module F , every basis for F has the same cardinality.

Invariance of dimension is equivalent to the condition that for any n and m , the R -modules R^n and R^m are isomorphic if and only if $n = m$. It is easy to see that if S has the invariance of dimension property, and $j : R \rightarrow S$ is an epimorphism, then R has the invariance of dimension property. Any unital commutative ring has the invariance of dimension property.

Theorem 1.3.12. *Consider the pullback of unital rings*

$$\begin{array}{ccc}
 R & \xrightarrow{i_1} & R_1 \\
 i_2 \downarrow & & \downarrow j_1 \\
 R_2 & \xrightarrow{j_2} & S,
 \end{array}$$

where the maps j_1, j_2 are surjective. Suppose R_1, R_2 and S have the invariance of dimension property. Suppose in addition that whenever P_1 and P_2 are finitely-generated projective R_1 and R_2 -modules respectively, such that the induced S -modules $j_{1\#}P_1$ and $j_{2\#}P_2$ are isomorphic, then P_1 and P_2 must be free modules over R_1 and R_2 respectively. Then, there is a bijective correspondence between the isomorphism classes of finitely-generated projective R -modules, and the set of all cocycles-classes $\{\bar{g} \mid g \in GL_n(S) \text{ some } n\}$.

Proof. This follows from Proposition 1.3.10, Theorems 1.3.3 and 1.3.4, and the fact that if M is a finitely-generated projective R -module, then, by our hypotheses, $i_{1\#}M \cong R_1^n$ for a unique n and also $i_{2\#}M \cong R_2^n$ for only this same n . \square

We remark that the results from this section can be generalized to some extent in certain directions in a manner resembling sheaf cohomology, but doing so is unnecessary for our present purposes.

1.4 Trace and Dimension

In this section, we review the well-known generalization of the concept of the dimension of the fibers of a vector bundle to finitely-generated projective modules over unital Banach algebras that carry a trace. We do this so as to later give a clear meaning to the statement that for θ any irrational number, every finitely-generated projective $C(S_\theta^4)$ -module not only has a well-defined integral “rank”, but moreover, splits (up to isomorphism) into a direct sum of a “rank-1” $C(S_\theta^4)$ -module and a free $C(S_\theta^4)$ -module.

Definition 1.4.1. A *trace* on an algebra A is a positive \mathbb{C} -valued linear functional on A .

Thus for us a trace on A necessarily takes finite values everywhere on A by definition. This differs from uses of the term that do not require a trace to take on finite values on all of A .

Let R be a unital ring. Let P be any idempotent in $M_n(R)$ for any n . If one identifies P with its image in each matrix ring $M_k(R)$, for all $k \geq n$, then one obtains the well-known semigroup isomorphism between the isomorphism classes of right finitely-generated projective R -modules, and similarity classes of the idempotent elements of the direct limit of all the $M_n(R)$ (See section 1.7.1 of [4]). Now if tr is a trace on an algebra A , then it extends to a trace on each $M_n(A)$, and so also to the direct limit of all the $M_n(A)$. In particular, if M is a right finitely-generated projective module over an algebra A , and if tr is a trace on A , then the induced trace on the direct limit of all the $M_n(A)$ will be equal for all choices of P that represent M . Thus tr gives rise to a (usually incomplete) isomorphism invariant tr_* for the right finitely-generated projective modules over A .

If A is the Banach algebra $C(X)$ of continuous complex valued-functions on a compact space X , then any finitely-generated projective $C(X)$ -module M is the space of continuous

sections of some complex vector bundle E over X , by Swan's theorem. View elements of $M \cong PC(X)^n$ as elements of $C(X, M_n(\mathbb{C}))$. If we then evaluate at any point x , we obtain a vector space $P_x\mathbb{C}^n$ that is isomorphic to the fiber E_x . The usual matrix trace of $P_x \in M_n(\mathbb{C})$ is the dimension of the fiber E_x . Thus, if X is connected, evaluation at any point x gives a trace on $C(X)$ whose induced trace on any right finitely-generated projective $C(X)$ -module gives the rank of the corresponding complex vector bundle.

Definition 1.4.2. Let A be a unital Banach algebra with a trace tr . Let M be a right finitely-generated projective A -module. The value $tr_*(M)$ will be called the *tr-rank* of M .

If a unital algebra A supports a trace that is nonzero on the identity element of A , then A has the invariance of dimension property. The tr -rank of any free module M over A agrees with the usual notion (cardinality of any basis for M) of the rank of M if and only if tr is a normalized trace.

The noncommutative n -torus $C(T_\theta^n)$ has a canonical faithful normalized trace Tr if θ is irrational. It is the unique trace on $C(T_\theta^n)$ that is invariant under the action of T^n on $C(T_\theta^n)$. The trace takes any homogeneous polynomial in the generators of $C(T_\theta^n)$ that is not a scalar multiple of the identity to zero. Interestingly, if $|\theta| < 1$ is irrational, there are projections of trace $|\theta|$ in the algebra $C(T_\theta^2)$ itself (see [33]). Any finitely-generated projective $C(T_\theta^2)$ -module can be split as a nontrivial direct sum of $C(T_\theta^2)$ -modules. Thus, the notion of rank need not be rational.

Suppose A is any unital C^* -algebra that surjects onto a unital C^* -algebra B via a unital map j . Suppose also that tr is a normalized trace on B . We can then define a normalized trace $tr_{B,j}^A$ on A by $tr \circ j$. We note that if j has a nontrivial kernel, the trace $tr_{B,j}^A$ will not be faithful. The definition of $tr_{B,j}^A$ is rigged so that $(tr_j^A)_*(M) = tr_*(j_\#(M))$ for any finitely-generated projective A -module M . Moreover, given a trace tr on an algebra C and a surjection from A to C that factors through a surjection from B to C , the construction gives compatible traces in the obvious sense.

Let θ be irrational and let A be any of the θ -deformed algebras

$$C(S_\theta^3), \quad C(F_\theta^2), \quad TC(S_\theta^3), \quad TC(F_\theta^2), \quad TC(T_\theta^2), \quad C(S_\theta^4), \quad C((D^4)_1)_\theta$$

defined in this work, with their corresponding surjections onto $B = C(T_\theta^2)$. We can then define a normalized trace $Tr_{B,j}^A$ on A by $Tr \circ j$, where Tr is the unique normalized trace on $C(T_\theta^2)$. In particular, we have:

Proposition 1.4.3. *The C^* -algebras $C(T_\theta^2)$, $C(F_\theta^2)$, $C(S_\theta^3)$ have the invariance of dimension property.*

Proof. If θ is irrational, then the noncommutative torus $C(T_\theta^2)$ has a unique normalized trace Tr . Composing Tr with the surjections from $C(F_\theta^2)$ and $C(S_\theta^3)$ show that $C(F_\theta^2)$ and $C(S_\theta^3)$ also have normalized traces. \square

In this work, we will construct modules $N(n, s)$ over $A = C(S_\theta^4)$, for θ irrational, by clutching together rank- n free modules over two copies of $C(D_\theta^4)$. In this very intuitive sense, the “rank” of such a $N(n, s)$ should be n . The $Tr_{B,j}^A$ -rank of $N(n, s)$ is trivially n for the map j from $C(S_\theta^4)$ to $B = C(T_\theta^2)$ that factors through $C(S_\theta^3)$. We will thus define the “rank” of an $C(S_\theta^4)$ -module to be its $Tr_{B,j}^A$ -rank. We will show that up to isomorphism, every finitely-generated projective $C(S_\theta^4)$ module is isomorphic to one of the $N(n, s)$. But also $N(n, s)$ will be isomorphic to $N(1, s) \oplus C(S_\theta^4)^{n-1}$. Thus, for θ irrational, every finitely-generated projective $C(S_\theta^4)$ -module splits as a direct sum of a “rank-1” module and a free module. Interestingly, the “instanton projection” e of Connes and Landi trivially has $Tr_{B,j}^A$ -rank equal to 2. Thus it automatically splits as $N(1, s) \oplus C(S_\theta^4)$ for some s (in fact, for $s = 1$).

Chapter 2

Finitely-Generated Projective Modules over the θ -deformed Spheres

2.1 Finitely-Generated Projective Modules over $C(S_\theta^3)$

We assume that θ is irrational throughout this section. We prove that for θ irrational, all right finitely-generated projective $C(S_\theta^3)$ -modules are free, and $V(C(S_\theta^3)) \cong \mathbb{N}$. Our strategy is to use our clutching Theorem 1.3.12. Towards this, we first need some information concerning the group $\pi_0(GL_n(C(T_\theta^2)))$ of path-classes of $n \times n$ invertible $C(T_\theta^2)$ -valued matrices.

Theorem 2.1.1. *(Theorem 3.3 of [35]).*

$$\pi_k(GL_n(C(T_\theta^m))) \cong \begin{cases} K_1(C(T_\theta^m)) & \text{for } k \text{ even} \\ K_0(C(T_\theta^m)) & \text{for } k \text{ odd} \end{cases} \cong \mathbb{Z}^{2^{m-1}}$$

for all $k \geq 0$, $n \geq 1$, for $C(T_\theta^m)$ a noncommutative m -torus with not all entries of θ rational.

We note that the isomorphism is given by composing the natural map

$$\pi_k(GL_n(C(T_\theta^m))) \rightarrow \pi_k(GL_\infty(C(T_\theta^m)))$$

induced by the usual embedding of $GL_n(C(T_\theta^m))$ into $GL_\infty(C(T_\theta^m))$ with Bott periodicity.

Corollary 2.1.2. *Let θ be irrational. Then the group $\pi_0(GL_n(C(T_\theta^2)))$ of path-components of $GL_n(C(T_\theta^2))$ is generated by the path-classes of the images of U and V in $GL_n(C(T_\theta^2))$, where U and V are the generators of $C(T_\theta^2)$ given in Definition 1.1.2.*

Proof. Pimsner and Voiculescu show in corollary 2.5 of [29] that $K_1(C(T_\theta^2))$ is generated by the K_1 -classes of U, V . Combining this with Theorem 2.1.1 shows that $\pi_0(GL_n(C(T_\theta^2)))$ is generated by the path-classes of the images of U, V in $GL_n(C(T_\theta^2))$. \square

Notice that Theorem 2.1.1 is false in the commutative case $\theta = 0$ for $m \geq 3$, since $\pi_0(U_1(A_0)) \cong [T^m, U(1)] \cong H^1(T^m; \mathbb{Z}) \cong \mathbb{Z}^m$. In another direction, one can show using Postnikov approximation that $\pi_0(GL_2(T^2C(T_0^2))) \cong [T^4, U(2)] \cong \mathbb{Z}^8 \oplus \mathbb{Z}_2$.

We still need two small propositions before we can apply Theorem 1.3.12 to see that, if θ is irrational, then all finitely-generated projective $C(S_\theta^3)$ -modules are free.

Proposition 2.1.3. *For all real θ , all right finitely-generated projective $C(F_\theta^2)$ -modules are free.*

Proof. Recall that homomorphisms of Banach algebras $\phi, \psi : A \rightarrow B$ are said to be *homotopic* (denoted $\phi \sim_h \psi$) if there is a homomorphism $\gamma : A \rightarrow C([0, 1], B)$ such that $\varepsilon_0 \circ \gamma = \phi$ and $\varepsilon_1 \circ \gamma = \psi$, where $\varepsilon_t : C([0, 1], B) \rightarrow B$ is evaluation at $t \in [0, 1]$. A homomorphism $\phi : A \rightarrow B$ is a *homotopy equivalence* if there is a morphism $\psi : B \rightarrow A$ such that $\psi \circ \phi \sim_h id_A$ and $\phi \circ \psi \sim_h id_B$. If additionally, $\phi \circ \psi = id_B$, then ϕ is said to be a *deformation retraction* of A onto B . These are the dual notions of the familiar definitions for continuous maps between topological spaces. We observe that $C(F_\theta^2)$ deformation retracts onto the commutative C^* -subalgebra $C^*(v) \cong C(S^1)$: First define the epimorphism $j : C(F_\theta^2) \rightarrow C(S^1)$ by the assignments $j(v) = v$ and $j(y) = 0$. Define the homomorphism $\ell : C(S^1) \rightarrow C(F_\theta^2)$ by $\ell(v) = v$. Since $j\ell = id_{C(S^1)}$, we need only see that $\ell j \sim_h id_{C(F_\theta^2)}$. To this end, define $\gamma : C(F_\theta^2) \rightarrow C([0, 1], C(F_\theta^2))$ as

$$\begin{aligned} (\gamma(v))(t) &= \gamma_t(v) = v \\ (\gamma(y))(t) &= \gamma_t(y) = ty \end{aligned}$$

for $t \in [0, 1]$.

As $\gamma_0 = \ell j$ and $\gamma_1 = id_{C(F_\theta^2)}$, the map j is a deformation retraction of $C(F_\theta^2)$ onto $C(S^1)$. But as $j : C(F_\theta^2) \rightarrow C(S^1)$ is a homotopy equivalence of Banach algebras, the induced map $j_* : V(C(F_\theta^2)) \rightarrow V(C(S^1))$ is a semigroup isomorphism, by the homotopy invariance of the functor V . Thus, since all finitely-generated projective modules over $C(S^1)$ are free, the same is true for all finitely generated projective modules over $C(F_\theta^2)$. \square

Proposition 2.1.4. *Suppose*

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ i_2 \downarrow & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & S \end{array}$$

is a pullback in the category of unital Banach algebras, and that g_0 and g_1 are path-connected in $GL_n(S)$. Then $M(R_1^n, R_2^n, g_0) \cong M(R_1^n, R_2^n, g_1)$.

Proof. Suppose $\{g_t\}$ is a path from g_0 to g_1 . Then $\{g_t\}$ may be regarded as an element of

$GL_n(C(I, S))$. Form the pullback diagram

$$\begin{array}{ccc} C(I, R) & \xrightarrow{i_1} & C(I, R_1) \\ i_2 \downarrow & & \downarrow j_1 \\ C(I, R_2) & \xrightarrow{j_2} & C(I, S) \end{array}$$

and consider the $C(I, R)$ -module $E = M(C(I, R_1)^n, C(I, R_2)^n, \{g_t\})$. By homotopy invariance of the functor V , we have $\varepsilon_{0*} = \varepsilon_{1*} : V(C(I, R)) \rightarrow V(R)$, where $\varepsilon_t : C(I, R) \rightarrow R$ is evaluation at t . So $\varepsilon_{0*}(E) \cong \varepsilon_{1*}(E)$. But simply by inspecting their definitions, we see that $\varepsilon_{0*}(E) \cong M(R_1^n, R_2^n, g_0)$ and $\varepsilon_{1*}(E) \cong M(R_1^n, R_2^n, g_1)$. \square

Theorem 2.1.5. *If θ is irrational, then all right finitely-generated projective $C(S_\theta^3)$ -modules are free. The free $C(S_\theta^3)$ -modules $C(S_\theta^3)^n$ are mutually non-isomorphic.*

Proof. As given in Definition 1.1.3, the C^* -algebra $C(S_\theta^3)$ is the pullback

$$\begin{array}{ccc} C(S_\theta^3) & \xrightarrow{i_1} & C(F_\theta^2) \\ i_2 \downarrow & & \downarrow j_1 \\ C(F_{-\theta}^2) & \xrightarrow{j_2} & C(T_\theta^2), \end{array}$$

where $i_1(a_1, a_2) = a_1$, $i_2(a_1, a_2) = a_2$, $j_1 = \pi_\theta$, $j_2 = f_\theta^* \circ \pi_{-\theta}$.

All finitely-generated projective modules over $C(F_\theta^2)$ or $C(F_{-\theta}^2)$ are free by Proposition 2.1.3. Thus, by Proposition 1.4.3 and Theorem 1.3.12, each finitely-generated projective module over $C(S_\theta^3)$ is of the form $M(C(F_\theta^2)^n, C(F_{-\theta}^2)^n, g)$ where $g \in GL_n(C(T_\theta^2))$. But, by Corollary 2.1.2, if we identify $U^k V^l$ with its image in $GL_n(C(T_\theta^2))$, it must be that g is path-connected in $GL_n(C(T_\theta^2))$ to some $U^k V^l$. But also $U^k V^l = j_{2*}(u^k) \cdot 1_n \cdot j_{1*}(((v^{-1})^l)^{-1})$, so $U^k V^l$ and 1_n are in the same cocycle class in $GL_n(C(T_\theta^2))$. Therefore, by Proposition 2.1.4,

$$\begin{aligned} M(C(F_\theta^2)^n, C(F_{-\theta}^2)^n, g) &\cong M(C(F_\theta^2)^n, C(F_{-\theta}^2)^n, U^k V^l) \\ &\cong M(C(F_\theta^2)^n, C(F_{-\theta}^2)^n, 1_n) \cong C(S_\theta^3)^n. \end{aligned}$$

By Proposition 1.4.3, the free $C(S_\theta^3)$ -modules $C(S_\theta^3)^n$ are mutually non-isomorphic. \square

We note that the above proof also is valid in the case that $\theta = 0$, since $\pi_0(C(T^2)) \cong K^1(T^2)$ is generated by U and V , and we have only used the condition that θ is irrational in this section for Theorem 2.1.1 and Corollary 2.1.2. We thus obtain a proof of the classical result that all complex vector bundles over S^3 are trivial.

2.2 Finitely-Generated Projective Modules over $TC(S_\theta^3)$

We assume that $|\theta| < 1$ is irrational in this section. We do this because we will crucially use the fact that under this assumption there is a projection in $C(T_\theta^2)$ of trace $Tr = |\theta|$ (see [33]) in order to explicitly find a formula for the generator X given in Proposition 2.2.6. However, none of the results of this section depend upon actually knowing a formula for X , so the condition that $|\theta| < 1$ could be dropped (we do use that θ is irrational). In any event, we lose little by imposing this condition, as the commutation relations given in Definition 1.1.2 show that $C(T_\theta^2)$ depends only on the value $\theta \bmod 1$.

In this section we classify the finitely-generated projective modules over $TC(S_\theta^3)$ where $|\theta| < 1$ is irrational. We do not do this for its own sake, but because the classification will allow us to conclude that the natural map $\pi_0(GL_1(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3)) \cong \mathbb{Z}$ is an isomorphism. That fact will be essential to classifying the finitely-generated $C(S_\theta^4)$ -modules.

We will view the C^* -algebra $TC(S_\theta^3)$ as the pullback

$$\begin{array}{ccc} TC(S_\theta^3) & \xrightarrow{i_1} & TC(F_\theta^2) \\ i_2 \downarrow & & \downarrow j_1 \\ TC(F_{-\theta}^2) & \xrightarrow{j_2} & TC(T_\theta^2), \end{array}$$

and form the finitely-generated projective $TC(S_\theta^3)$ -modules by gluing together $TC(F_\theta^2)$ and $TC(F_{-\theta}^2)$ -modules over $TC(T_\theta^2)$. Not all finitely-generated projective modules over these later algebras are free, but we will see that only free modules over them can be glued together over $TC(T_\theta^2)$ to obtain finitely-generated projective $TC(S_\theta^3)$ -modules. This greatly simplifies the analysis.

Theorem 2.2.1. *If P_1 and P_2 are respectively finitely-generated projective $TC(F_\theta^2)$ and $TC(F_{-\theta}^2)$ -modules such that $j_{1\#}(P_1) \cong j_{2\#}(P_2)$, then both P_1 and P_2 are free modules.*

Proof. We will need a few lemmas and independent propositions to prove Theorem 2.2.1. First, given a unital C^* -algebra A , we would like to characterize the finitely-generated TA -modules in terms of the finitely-generated A -modules. We use a construction which is itself a form of clutching.

Definition 2.2.2. Let A be a unital C^* -algebra. Let M be an A -module, and let h be an automorphism of M . We let $X(h)$ denote the TA -module

$$X(h) := \{f : I \rightarrow M : f(1) = hf(0), \quad h \in \text{Aut}_A(M), \quad f \text{ continuous}\},$$

Lemma 2.2.3. *(Theorem 8.4 of [37]) Suppose A is a unital C^* -algebra. Then, any finitely-generated projective TA -module is isomorphic to one of the form $X(h)$ defined in Definition*

2.2.2, where M is a finitely-generated projective A -module. Lemma 8.10 of [37] shows that if $X(h_1) \cong X(h_2)$, where h_1 and $h_2 \in \text{Aut}_A(M)$, then there is a $g \in \text{Aut}_A(M)$ with h_2 and gh_1g^{-1} path-connected in $\text{Aut}_A(M)$. Also, if h_1 and h_2 are path-connected in $\text{Aut}_A(M)$, then $X(h_1) \cong X(h_2)$.

By noticing that $TC(F_\theta^2)$ retracts to $TC^*(v) \cong C(T^2)$, and likewise $TC(F_{-\theta}^2)$ retracts to $TC^*(u) \cong C(T^2)$, for the purposes of classification we may regard the projective modules over each as projective modules over $C(T^2)$. The consequence is a lemma that we state after a preliminary definition.

Definition 2.2.4. We denote by $X(v^k, n)$ the $TC(F_\theta^2)$ -module

$$X(v^k, n) := \{f : I \rightarrow C(F_\theta^2)^n : f(1) = v^k f(0), \quad f \text{ continuous}\},$$

where v^k is the image in $GL_n(C(F_\theta^2))$ of the k -th power of the generator $v \in C(F_\theta^2)$.

Lemma 2.2.5. *Every finitely-generated projective $TC(F_\theta^2)$ -module is isomorphic to a module of the form $X(v^k, n)$ given in Definition 2.2.4.*

Proof. Recall from the proof of Proposition 2.1.3 that $C(F_\theta^2)$ deformation retracts onto $C^*(v) \cong C(S^1)$, which has only free finitely-generated projective modules over it. Now $\pi_0(GL_1(C^*(v))) \cong \pi_1(GL_1(\mathbb{C})) \cong \pi_1(S^1) \cong \mathbb{Z}$ is generated by v . Furthermore, the natural map $\pi_0(GL_1(C^*(v))) \rightarrow K_1(C^*(v))$ is an isomorphism. So, by Lemma 2.2.3, taking the image of v^k in $GL_n(C(F_\theta^2))$, each finitely-generated projective module over $TC(F_\theta^2)$ is isomorphic to a unique module $X(v^k, n) := X(v^k)$. The module $X(v^k, n)$ corresponds to the well-known complex rank- n vector bundle over T^2 of “twist” (integral of the first Chern class), $-k$. Since $\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$ is homotopic through invertibles to $\begin{pmatrix} zw & 0 \\ 0 & 1 \end{pmatrix}$ if z and w are invertible (proposition 3.4.1 of [4]), we see that the bijection $[X(v^k, n)] \mapsto (k, n)$ gives a semigroup isomorphism from $V(TC(F_\theta^2))$ onto $\{0\} \cup (\mathbb{Z} \times \mathbb{N})$.

Of course similar remarks apply for $TC(F_{-\theta}^2)$, so there is a bijection between the isomorphism classes of finitely-generated projective modules over $TC(F_{-\theta}^2)$ and finitely-generated projective modules of the form $X(u^l, m)$, with the definition of the $X(u^l, m)$ paralleling that of the $X(v^k, n)$, but now employing the generator u and a rank- m free module over $C(F_{-\theta}^2)$. \square

For the next lemma, we will need to know the generators of $\pi_0(GL_n(TC(T_\theta^2)))$.

Proposition 2.2.6. *Let U and V be as given in Definition 1.1.2. Let $W := \exp(2\pi it) \cdot 1$ and $X := \exp(2\pi it) \cdot p + 1 - p$, where $p \in C(T_\theta^2)$ is a projection of trace $|\theta| < 1$ irrational. Then, for any $n \geq 1$, the group $\pi_0(GL_n(TC(T_\theta^2))) \cong \mathbb{Z}^4$ is generated by the path-classes of the images of U, V, W, X in $GL_n(TC(T_\theta^2))$. So the image of each monomial $U^l V^k W^r X^s$ in $GL_n(TC(T_\theta^2))$ is in a distinct path-class.*

Proof. From the split-exact sequence

$$0 \rightarrow SC(T_\theta^2) \rightarrow TC(T_\theta^2) \rightarrow C(T_\theta^2) \rightarrow 0,$$

we observe that $K_1(TC(T_\theta^2)) \cong K_1(C(T_\theta^2)) \oplus K_1(SC(T_\theta^2))$. The invertibles U and V generate $K_1(C(T_\theta^2)) \cong \mathbb{Z}^2$, by Corollary 2.5 of [29]. Recall that for A any Banach algebra, the assignment $e \mapsto \exp(2\pi it) \cdot e + 1 - e$, where e is an idempotent in any matrix algebra $M_n(A)$, induces the Bott map $K_0(A) \cong K_1(SA)$ (see section 9.1 of [4]). Now for $A = C(T_\theta^2)$, this assignment takes 1 to W and p to X , by definition. But the K_σ -classes of 1 and p generate $K_0(C(T_\theta^2))$ (see page 116 of [29]). Thus U, V, W , and X generate $K_1(TC(T_\theta^2))$. So the proposition follows by Theorem 2.1.1. \square

We remark that Proposition 2.2.6 can also be proved by using the Mayer-Vietoris sequence of K -theory, but the proof is more complicated.

We still need a small proposition which is a generalization of the fact that vector bundles over connected spaces have isomorphic fibers.

Proposition 2.2.7. *Let Y be any path-connected space, and let A be any unital Banach algebra. Let YA denote the Banach algebra of continuous functions from Y into A . Let M and N be isomorphic finitely-generated projective YA -module. Then each fiber of M is isomorphic to each fiber of N as A -modules.*

Proof. Since the fiber of M over y is $(\varepsilon_y)_\#(M)$, and the fiber of N over y is $(\varepsilon_y)_\#(N)$, the fiber of M over y and the fiber of N over y must be isomorphic by functoriality of the induced module construction. So we only need to show that for any points y_1 and y_2 in Y , the fiber of M over y_1 is isomorphic to the fiber over y_2 . So suppose y_1 and y_2 are connected by a path γ_t . The evaluation map $\varepsilon_y : YA \rightarrow A$ induces maps $(\varepsilon_{y_1})_*, (\varepsilon_{y_2})_* : V(YA) \rightarrow V(A)$. But $\varepsilon_{y_1} \sim_h \varepsilon_{y_2}$ via ε_{γ_t} . So, by homotopy invariance of the functor V , the maps $(\varepsilon_{y_1})_*$ and $(\varepsilon_{y_2})_*$ are identical. \square

Lemma 2.2.8. *The $TC(T_\theta^2)$ -modules $j_{1\#}X(v^k, n)$ and $j_{2\#}X(u^l, m)$ are isomorphic if and only if $m = n$ and $k = l = 0$.*

Proof. Since $j_1 : TC(F_\theta^2) \rightarrow TC(T_\theta^2)$ is surjective, any element of

$$j_{1\#}X(v^k, n) := X(v^k, n) \otimes_{TC(F_\theta^2)} TC(T_\theta^2)$$

can be written as an elementary tensor $g \otimes 1$. Viewing g as in $C([0, 1], C(F_\theta^2)^n)$ and j_1 as mapping $C([0, 1], C(F_\theta^2)^n)$ onto $C([0, 1], C(T_\theta^2)^n)$, it is easily verified that the map $g \otimes 1 \mapsto j_1(g)$ restricts to a well-defined isomorphism from $j_{1\#}X(v^k, n)$ onto $X(V^k, n)$. Repeating this argument for $TC(F_{-\theta}^2)$ gives $j_{2\#}X(u^l, m) \cong X(U^l, m)$.

Now suppose that $X(V^k, n) \cong X(U^l, m)$. Then, firstly, by Proposition 2.2.7, any fiber of $X(V^k, n)$ over any point of T must be isomorphic to any fiber of $X(U^l, m)$ over any point of T . But since these fibers are $C(T_\theta^2)^n$ and $C(T_\theta^2)^m$ respectively, and since $C(T_\theta^2)$ has the

invariance of dimension property, this is possible only if $n = m$. Now, by Lemma 2.2.3, if $X(V^k, n) \cong X(U^l, n)$, then $\begin{pmatrix} V^k & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ must be path-connected in $GL_n(TC(T_\theta^2))$ to $g \begin{pmatrix} U^l & 0 \\ 0 & 1_{n-1} \end{pmatrix} g^{-1}$ for some $g \in GL_n(TC(T_\theta^2))$. But since $\pi_0(GL_n(TC(T_\theta^2))) \cong \mathbb{Z}^4$ is abelian, this is equivalent to $\begin{pmatrix} V^k & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ being path-connected to $\begin{pmatrix} U^l & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ in $GL_n(TC(T_\theta^2))$. But that is the case only if $k = l = 0$, by Proposition 2.2.6. \square

Theorem 2.2.1 follows immediately by Lemmas 2.2.5 and 2.2.8. \square

Corollary 2.2.9. *Every finitely-generated projective $TC(S_\theta^3)$ -module is (up to isomorphism) of the form $M((TC(F_\theta^2))^n, (TC(F_{-\theta}^2))^n, U^l V^k W^r X^s)$, for the pullback diagram*

$$\begin{array}{ccc} TC(S_\theta^3) & \xrightarrow{i_1} & TC(F_\theta^2) \\ i_2 \downarrow & & \downarrow j_1 \\ TC(F_{-\theta}^2) & \xrightarrow{j_2} & TC(T_\theta^2), \end{array}$$

where U, V, W, X are the images in $GL_n(TC(T_\theta^2))$ of the generators of $\pi_0(GL_n(TC(T_\theta^2))) \cong K_1(TC(T_\theta^2))$ defined in Proposition 2.2.6.

Proof. Immediate from Theorem 2.2.1, Proposition 2.2.6, and Theorem 1.3.3. \square

We are now in position to prove the following theorem, after introducing some notation.

Definition 2.2.10. We denote the $TC(S_\theta^3)$ -module $M((TC(F_\theta^2))^n, (TC(F_{-\theta}^2))^n, X^s)$ by the notation $M(n, s)$.

Theorem 2.2.11. *Every finitely-generated projective $TC(S_\theta^3)$ -module is (up to isomorphism) of the form $M(n, s)$ for some (n, s) . Every choice of pair (n, s) results in a distinct isomorphism class of $TC(S_\theta^3)$ -modules.*

Proof. We need two lemmas.

Lemma 2.2.12. *Fix n and s . Then for all l, k , and r*

$$M((TC(F_\theta^2))^n, (TC(F_{-\theta}^2))^n, U^l V^k W^r X^s) \cong M(n, s)$$

as $TC(S_\theta^3)$ -modules.

Proof. By Theorem 1.3.12, it suffices to see that $U^l V^k W^r X^s$ and X^s are in the same cocycle class of $GL_n(TC(T_\theta^2))$. Clearly, $U^l V^k W^r X^s$ and $U^l X^s V^k W^r$ are homotopic (they are equal up to multiplication by a phase factor in S^1), and hence in the same cocycle class. But of course $U^l X^s V^k W^r = j_{2*}(u^l) X^s j_{1*}(((v^k w^r)^{-1})^{-1})$, so $U^l X^s V^k W^r \sim X^s$. \square

Lemma 2.2.13. *The $TC(S_\theta^3)$ -modules $M((TC(F_\theta^2))^n, (TC(F_{-\theta}^2))^n, X^s)$ are mutually non-isomorphic for different choices of s .*

Proof. The statement follows from Theorem 1.3.12, after observing that X^{s_1}, X^{s_2} are in different cocycle classes if $s_1 \neq s_2$. To see this, suppose that $X^{s_1} \sim X^{s_2}$. As $\pi_0(GL_n(TC(T_\theta^2))) \cong \mathbb{Z}^4$ is abelian, this is equivalent to the formula $X^{s_1} = j_{2*}(g_2)j_{1*}(g_1^{-1})X^{s_2}$ for some $g_1 \in GL_n(TC(F_\theta^2))$ and $g_2 \in GL_n(TC(F_{-\theta}^2))$. Hence $X^s = j_{2*}(g_2)j_{1*}(g_1^{-1})$ for $s = s_1 - s_2$. But $\pi_0(GL_n(TC(F_\theta^2))) \cong \mathbb{Z}^2$ is generated by w and v and $\pi_0(GL_n(TC(F_{-\theta}^2))) \cong \mathbb{Z}^2$ is generated by z and u . So $j_{2*}(g_2)j_{1*}(g_1^{-1})$ must be homotopic in $GL_n(TC(T_\theta^2))$ to an invertible of the form $W^r U^k V^l$, for some r, k, l , since $j_{1*}(w) = W, j_{1*}(v) = V$ and $j_{2*}(z) = W, j_{2*}(u) = U$. But by Proposition 2.2.6, the invertible X^s cannot be homotopic to such an element, unless $r = k = l = s = 0$. \square

Theorem 2.2.11 is now immediate from Corollary 2.2.9, and Lemmas 2.2.12 and 2.2.13. \square

Corollary 2.2.14. *$V(TC(S_\theta^3)) \cong \{0\} \cup (\mathbb{Z} \times \mathbb{N})$ is a cancellative semigroup. It is generated by the isomorphism classes of the modules $M(1, s)$ and the zero module $\{0\}$.*

Proof. Since $\begin{pmatrix} X^s & 0 \\ 0 & X^t \end{pmatrix}$ is homotopic through invertibles to $\begin{pmatrix} X^{s+t} & 0 \\ 0 & 1 \end{pmatrix}$, we have an isomorphism $M(m+n, s+t) \cong M(m, s) \oplus M(n, t)$.

Suppose now that $M_1 \oplus N \cong M_2 \oplus N$, where $M_1 \cong M(k, s_1)$ and $M_2 \cong M(l, s_2)$. Since N is projective, there is an N' so that $N \oplus N' \cong (TC(S^3))^n \cong M(n, 0)$ for some n . So we must merely see that when $M(k, s_1) \oplus M(n, 0) \cong M(k+n, s_1+0)$ is isomorphic to $M(l, s_2) \oplus M(n, 0) \cong M(l+n, s_2+0)$, then $M(k, s_1)$ must be isomorphic to $M(k, s_2)$. But $M(k+n, s_1)$ can be isomorphic to $M(l+n, s_2)$ only if $k+n = l+n$ and $s_1 = s_2$, by Theorem 2.2.11. So $M_1 \cong M_2$, and $V(TC(S_\theta^3))$ is cancellative. \square

Let us pause to note that the classical case is a bit different. Since $\pi_0(GL_1(C(S^3))) \cong \pi_3(GL_1(\mathbb{C})) \cong \pi_3(S^1) \cong 0$, and the only complex line bundle over S^3 is the trivial one, the only complex line bundle over $T \times S^3$ is the trivial one. On the other hand, $\pi_0(GL_n(C(S^3))) \cong \pi_3(GL_n(\mathbb{C})) \cong \pi_3(SU_2(\mathbb{C})) \cong \pi_3(S^3) \cong \mathbb{Z}$ for $n \geq 2$. This, together with the fact that all complex vector bundles over $C(S^3)$ are trivial, means that there are exactly \mathbb{Z} -many complex vector bundles of rank- n over $T \times S^3$ for $n \geq 2$. These bundles are indexed by their ranks and second Chern classes, and one can easily calculate that the complex rank-2 bundle with charge $c_2 = -1$ is precisely the bundle $X(y)$, where $y = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ is the generator of $\pi_3(SU_2(\mathbb{C}))$. Every bundle over $T \times S^3$ other than the trivial line bundle must be of the form $X\left(\begin{pmatrix} y^k & 0 \\ 0 & 1_{n-2} \end{pmatrix}, n\right)$ for some $n \geq 2$, and some k . By way of contrast, if θ is irrational, we have just shown that there are \mathbb{Z} -many ‘‘line bundles’’ $M(1, s) = M(TC(F_\theta^2), TC(F_{-\theta}^2), X^s)$ over the (nonexistent) noncommutative space $T \times S_\theta^3$. Moreover, our work shows that all

“vector bundles” over this noncommutative space are obtained by adding trivial bundles to one of these noncommutative line bundles.

2.3 The Group $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$

In this section we show that if θ is irrational, then $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$, and the natural map

$$\pi_0(GL_1(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3))$$

is an isomorphism. We will assume also that $|\theta| < 1$, since we will use the description of the generator X from Lemma 2.2.6. But as we will note later, we could obtain the results of this section without assuming $|\theta| < 1$.

By Lemma 2.2.3, the $TC(S_\theta^3)$ -module $M(1, 1) := M(TC(F_\theta^2), TC(F_{-\theta}^2), X^s)$ is isomorphic to a module of the form

$$X(u, V) := \{f : I \rightarrow V : f(1) = uf(0), \quad u \in \text{Aut}_A(V), \quad f \text{ continuous}\},$$

where V is some $C(S_\theta^3)$ -module, and u is an automorphism of V . We claim that $V \cong C(S_\theta^3)$ and that u can then be represented as an element of $GL_1(C(S_\theta^3))$. Of course the invertible u will not be path-connected to the identity through $GL_1(C(S_\theta^3))$, since $M(1, 1)$ is not a free $TC(S_\theta^3)$ -module. In fact, we will show in Theorem 2.3.3 that $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$.

Theorem 2.3.1. *The projective $TC(S_\theta^3)$ -module $M(1, 1) := M(TC(F_\theta^2), TC(F_{-\theta}^2), X^s)$ is isomorphic to $X(u, C(S_\theta^3))$ for some invertible $u \in C(S_\theta^3)$.*

Proof. The proof will use theorem 3.2 of [13] as a technical lemma:

Lemma 2.3.2. *(Theorem 3.2 of [13]). Suppose R is a pullback*

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ i_2 \downarrow & & \downarrow j_1 \\ R_2 & \xrightarrow{j_2} & S \end{array}$$

of unital rings with j_1 surjective, and suppose that E_1 and E_2 are free rank- n modules over respectively R_1 and R_2 . Then $M(E_1, E_2, g) \cong P(R)^{2n}$, where the idempotent $P \in M_{2n}(R)$ is given by

$$\begin{pmatrix} (1, c(2 - dc)d) & (0, c(2 - dc)(1 - dc)) \\ (0, (1 - dc)d) & (0, (1 - dc)^2) \end{pmatrix}$$

for c a lift of g to $M_n(R_1)$, and d a lift of g^{-1} to $M_n(R_1)$.

Proof. The formula for P is not at all obvious, but the derivation in [13] consists of simple calculations based on the proof of Theorem 1.3.3 given in [27]). \square

Returning to the proof of Theorem 2.3.1, we first apply Lemma 2.3.2 to find an idempotent 2×2 matrix $P \in M_2(TC(S_\theta^3))$ with $M(1, 1) := M(TC(F_\theta^2), TC(F_{-\theta}^2), X) \cong P(TC(S_\theta^3))^2$. To lift $X = \exp(2\pi it) \cdot p + 1 - p$ to $c \in TC(F_\theta^2)$, first lift the projection $p \in C(T_\theta^2)$ to a self-adjoint (but not idempotent) element $q \in C(F_\theta^2)$. Take $c = \exp(2\pi it) \cdot q + 1 - q$. Since $X^{-1} = \exp(-2\pi it) \cdot p + 1 - p$, we can take d to be $\exp(-2\pi it) \cdot q + 1 - q$. Of course then $d = c^*$, and c is normal (though not invertible). Regarding P, c, d as functions on the interval $[0, 1]$, we have $c(0) = c(1) = d(0) = d(1) = 1 \in C(F_\theta^2)$, and $P(0) = P(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(C(S_\theta^3))$.

Let us review the argument that every projective module M over TA for A a C^* -algebra is isomorphic to one of the form $X(u, V)$, where V is some projective A -module and u is some automorphism of V . We follow lemma 8.11 of [37]: Since M is finitely-generated and projective, there is an idempotent P in some $M_n(TA)$, so that $M \cong P(TA)^n$. But P can be considered a path of idempotents in $M_n(A)$ with $P(0) = P(1)$. Take $V = P(0)A^n$. One can construct a path $U := \{U(t)\}$ through $GL_n(A)$ with $U(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $P(t) = U(t)^{-1}P(0)U(t)$ (see, for instance, proposition 4.3.3 of [4]). Now $P(1) = P(0)$, so $P(0)$ commutes with $U(1)$, and $u := P(0)U(1)$ is an automorphism of V . The assignment $\phi(t) = U(t)f(t)$ supplies an isomorphism $P(TA)^n \cong X(u, V)$, here regarding $f \in P(TA)^n$ as a function from T into A^n .

For $M(1, 1) \cong P(TC(S_\theta^3))^2$, we have that $V = P(0)(C(S_\theta^3))^2 \cong C(S_\theta^3)$ as a $C(S_\theta^3)$ -module, via the identification $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow x$. Thus we may identify u with some element of $GL_1(C(S_\theta^3))$. Indeed, using that $P(0) = P(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P(1) = U(1)^{-1}P(0)U(1)$, we notice that $U(1)$ must be a diagonal matrix $\begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix}$ with u_{11} and u_{22} necessarily invertible in $C(S_\theta^3)$. It is u_{11} that we identify with u . \square

Although we won't need this fact, we note that u_{22} is path-connected to u_{11}^{-1} in $GL_1(C(S_\theta^3))$. This is because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix} \sim_h \begin{pmatrix} u_{11}u_{22} & 0 \\ 0 & 1 \end{pmatrix}$, and the natural map

$$\pi_0(GL_1(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3)) \cong \mathbb{Z}$$

is injective (moreover, an isomorphism, see discussion below).

We remark that although a concrete procedure for constructing the path $U(t)$ is well-known (see the proof of proposition 4.3.3 of [4]), doing so involves estimating norms in $M_n(TA)$ and then splitting the interval into an appropriate number of subintervals to insure an inequality. One then multiplies together a number of matrix elements for each of these subintervals to produce $U(t)$. We could attempt to do this in our case to obtain an explicit formula for u_1 . However, we will instead later guess a generator for $\pi_0(GL_1(C(S_\theta^3)))$. This

generator must be homotopic through invertibles in $GL_1(C(S_\theta^3))$ to either u or u^{-1} . However, even without an explicit formula for u we may at this point already easily prove the following theorem:

Theorem 2.3.3. *The natural map $\pi_0(GL_1(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3)) \cong \mathbb{Z}$ is an isomorphism.*

Proof. Theorem 2.2.11 shows that each $TC(S_\theta^3)$ -module is isomorphic to exactly one of the $M(n, s) = M((TC(F_\theta^2))^n, (TC(F_{-\theta}^2))^n, X^s)$. Using Lemma 2.3.2, we obtain an idempotent $P \in M_{2n}(TC(S_\theta^3))$ so that $M(n, s) \cong P(TC(S_\theta^3))^{2n}$. By Lemma 2.2.3, we write $M(n, s) \cong X(v, V)$, where $V = P(0)(C(S_\theta^3))^{2n}$. Since $P(0) = P(1) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}$, we have that $V \cong C(S_\theta^3)^n$, and so v can be identified with some element of $GL_n(C(S_\theta^3))$. Since $C(S_\theta^3)$ and $TC(S_\theta^3)$ are both cancellative, the natural map

$$\pi_0(GL_k(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3))$$

is injective, for all $k \geq 0$, by theorem 8.4 of [37]. Theorem 2.3.1 shows that $M(1, 1) \cong X(u, C(S_\theta^3))$, for some $u \in GL_1(C(S_\theta^3))$. But $u \sim_h 1$ in $GL_1(C(S_\theta^3))$, since $M(1, 1)$ is not a free $TC(S_\theta^3)$ -module, so $\pi_0(GL_1(C(S_\theta^3)))$ is not the trivial group. Thus $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$, since the map $\pi_0(GL_1(C(S_\theta^3))) \rightarrow K_1(C(S_\theta^3)) \cong \mathbb{Z}$ is injective. We must prove that the map is onto: Let v_1 be an arbitrary member of $GL_n(C(S_\theta^3))$ for $n \geq 2$. Let $V = C(S_\theta^3)^n$ and form the $TC(S_\theta^3)$ -module $X(v_1, V)$. As a $TC(S_\theta^3)$ -module, $X(v_1, V)$ is isomorphic to some $M(k, s)$ for a unique pair (k, s) . But applying Lemma 2.3.2 to find an idempotent P so that $M(k, s) \cong P(TC(S_\theta^3))^{2k} \cong X(v_2, P(0)C(S_\theta^3)^{2k}) \cong X(v_2, C(S_\theta^3)^k)$, for some $v_2 \in Aut(P(0)C(S_\theta^3)^{2k})$, we see that k must be n , because isomorphic $TC(S_\theta^3)$ -modules must have isomorphic fibers over each point of T , by Proposition 2.2.7. (Although we don't need this fact, we remark that we can conclude that $v_1 \sim_h v_2$ in $GL_n(C(S_\theta^3))$). Also, $M(n, s) \cong M(1, s) \oplus (TC(S_\theta^3))^{n-1}$, by Corollary 2.2.14. Now $M(1, s)$ is itself isomorphic to a module $X(v', C(S_\theta^3))$ for some $v' \in GL_1(C(S_\theta^3))$, and $(TC(S_\theta^3))^{n-1}$ is isomorphic to $X(1_{n-1}, C(S_\theta^3)^{n-1})$ as a $TC(S_\theta^3)$ -module. Thus $X(v_1, C(S_\theta^3)^n) \cong X\left(\begin{pmatrix} v' & 0 \\ 0 & 1_{n-1} \end{pmatrix}, C(S_\theta^3)^n\right)$, and v_1 and $\begin{pmatrix} v' & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ are path-connected in $GL_n(C(S_\theta^3))$, by Lemma 2.2.3. Since $\pi_0(GL_n(C(S_\theta^3))) \cong \mathbb{Z}$ is abelian, we conclude that $v_1 \sim_h \begin{pmatrix} v' & 0 \\ 0 & 1_{n-1} \end{pmatrix}$ in $GL_n(C(S_\theta^3))$, so the map $\pi_0(GL_1(C(S_\theta^3))) \rightarrow \pi_0(GL_n(C(S_\theta^3)))$ is onto for all $n \geq 2$. \square

We remark that a similar proof of Theorem 2.3.3 can be given by considerations of compatibility of Tr -ranks, without invoking Lemma 2.3.2. Such an approach is simpler and also does not require an explicit formula for the generator X . Thus such an approach does not require the restriction that $|\theta| < 1$. The proof we give, however, has the advantage of being somewhat more constructive.

We pause to remark that by Proposition 1.4.3 and Theorem 2.1.5, it must be that the general stable rank (or gsr) of $C(S_\theta^3)$ is equal to 1. Therefore, for irrational θ , we conclude

that the connected stable rank (or *csr*) of $C(S_\theta^3)$ equals 2, by proposition 2.6 of [35], since $GL_1(C(S_\theta^3))$ is not connected. Moreover, by proposition 8.2 of [37] and proposition 2.6 of [35], we conclude that $csr(TC(S_\theta^3)) = 2$. However, I suspect that $\pi_0(GL_1(TC(S_\theta^3))) \cong \mathbb{Z}^4$, while we have just shown that $\pi_0(GL_n(TC(S_\theta^3))) \cong K_1(TC(S_\theta^3)) \cong \mathbb{Z}^2$, for all $n \geq 2$, so $T^2(C(S_\theta^3))$ would fail cancellation. I can show that this would imply that $C(S_\theta^5)$ also fails cancellation, but in contrast to the commutative case, $C(S_\theta^5)$ would then have nontrivial “noncommutative line bundles”, which when added together in any combination always result in free modules. Furthermore, the algebra $C(S_\theta^5)$ would not support any nontrivial finitely-generated projective modules of “rank” 2 or higher.

Proposition 2.3.4. *Let θ be irrational. Then $tsr(C(S_\theta^3)) \leq 2$.*

Proof. Consider the short exact sequence

$$0 \rightarrow Cone(C(T_\theta^2)) \rightarrow C([0, 1], C(T_\theta^2)) \rightarrow C(T_\theta^2) \rightarrow 0.$$

Putnam [31] proved that $tsr(C(T_\theta^2)) = 1$. So $tsr(C([0, 1], C(T_\theta^2))) \leq 2$, by corollary 7.2 of [34]. But then $tsr(Cone(C(T_\theta^2))) \leq 2$, by theorem 4.4 of [34].

We deduce the short exact sequence

$$0 \rightarrow Cone(C(T_\theta^2)) \rightarrow C(D^2) \times_\theta Z \rightarrow C(S^1) \rightarrow 0$$

from Theorem 1.1.5. As $tsr(C(S^1)) = 1$, we conclude that $tsr(C(D^2) \times_\theta Z) \leq 2$, by corollary 4.12 of [34]. Finally, since $C(S_\theta^3)$ is a pullback of the C*-algebras $C(F_\theta^2) = C(D^2) \times_\theta Z$ and $C(F_{-\theta}^2) = C(D^2) \times_{-\theta} Z$, it must be that $tsr(C(S_\theta^3)) \leq 2$, by corollary 3.16 of [43]. \square

I have discovered that Sudo [46] has already given a proof that $tsr(C(S_\theta^3)) = 2$, but the proof is less elementary.

2.4 The Generator of $\pi_0(GL_1(C(S_\theta^3)))$

In this section, we explicitly describe the generator of $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$ for the case $|\theta| < 1$ is irrational.

Let $|\theta| < 1$ be irrational. Recall from the proof of Proposition 2.2.6 that $W := \exp(2\pi it) \cdot 1$ and $X := \exp(2\pi it)p + 1 - p$ generate $K_1(SC(T_\theta^2)) \cong \mathbb{Z}^2$, where $p \in C(T_\theta^2)$ is a Rieffel projection of trace θ . We will observe that the unitisation of $SC(T_\theta^2)$ is contained in $C(S_\theta^3)$. (In fact $SC(T^2)$ is an ideal of $C(S_\theta^3)$). This suggests that X might generate $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$. Indeed, this is the content of Theorem 2.4.4, towards which we first obtain some lemmas.

Lemma 2.4.1. *The suspension $SC(T_\theta^2)$ is an ideal of $C(S_\theta^3)$.*

Proof. The C*-algebra $C(S_\theta^3)$ is isomorphic to the algebra

$$\{f \in C([0, 1], C(T_\theta^2)) : f(0) \in C^*(V), f(1) \in C^*(U)\}$$

by Theorem 1.1.5. The suspension $SC(T_\theta^2)$ is obviously an ideal of the later algebra. \square

Lemma 2.4.2. *The induced map $i_* : K_1(SC(T_\theta^2)) \rightarrow K_1(C(S_\theta^3))$ is surjective.*

Proof. That we have the short exact sequence

$$0 \rightarrow SC(T_\theta^2) \xrightarrow{i} C(S_\theta^3) \xrightarrow{q} C^*(V) \oplus C^*(U) \rightarrow 0$$

follows from Lemma 2.4.1. The standard six-term exact sequence of Banach algebra K-theory (theorem 9.3.1 of [4]) becomes:

$$\begin{array}{ccccc} K_0(SC(T_\theta^2)) & \xrightarrow{i_*} & K_0(C(S_\theta^3)) & \xrightarrow{q_*} & K_0(C^*(V)) \oplus K_0(C^*(U)) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(C^*(V)) \oplus K_1(C^*(U)) & \xleftarrow{q_*} & K_1(C(S_\theta^3)) & \xleftarrow{i_*} & K_1(SC(T_\theta^2)) \end{array}$$

or,

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z} & \xrightarrow{q_*} & \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow \partial & & & & \downarrow \partial \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{q_*} & \mathbb{Z} & \xleftarrow{i_*} & \mathbb{Z} \oplus \mathbb{Z}. \end{array}$$

The above values of the K -groups, the condition of exactness, and the fact that the map $q_* : K_0(C(S_\theta^3)) \rightarrow K_0(C^*(V)) \oplus K_0(C^*(U))$ is injective (established below), jointly suffice to completely determine the images and kernels (at least up to isomorphism as abstract discrete groups) of all of the other maps in the six-term sequence.

Indeed, from the Grothendieck construction, the map

$$q_* : K_0(C(S_\theta^3)) \rightarrow K_0(C^*(V)) \oplus K_0(C^*(U))$$

is induced by the semigroup homomorphism $q_* : V(C(S_\theta^3)) \rightarrow V(C^*(V)) \oplus V(C^*(U))$. The algebras $C(S_\theta^3)$, $C^*(V)$, and $C^*(U)$ are all K -cancellative, and every finitely-generated projective module over one of these algebras is free. Thus, identifying the rank of a free module over one of these algebras with the isomorphism class of that free module, it is clear both that the map $q_* : V(C(S_\theta^3)) \rightarrow V(C^*(V)) \oplus V(C^*(U))$ is simply $[k] \mapsto ([k], [k])$, for k any non-negative integer, and, moreover, it's extension $q_* : K_0(C(S_\theta^3)) \rightarrow K_0(C^*(V)) \oplus K_0(C^*(U))$ is $[n] \mapsto ([n], [n])$, for n any integer. Therefore, by exactness, the map $i_* : K_0(SC(T_\theta^2)) \rightarrow K_0(C(S_\theta^3))$ is the zero map, and the index map $\partial : K_1(C^*(V)) \oplus K_1(C^*(U)) \rightarrow K_0(SC(T_\theta^2))$ is surjective. But given exactness and the values (as abstract discrete groups) of the K -groups in the six-term sequence, the index map cannot be surjective unless the map $q_* : K_1(C(S_\theta^3)) \rightarrow K_1(C^*(V)) \oplus K_1(C^*(U))$ is the zero map. So $i_* : K_1(SC(T_\theta^2)) \rightarrow K_1(C(S_\theta^3))$ is surjective. \square

The inclusion map $i : SC(T_\theta^2) \rightarrow C(S_\theta^3)$ from Lemma 2.4.1 extends to an inclusion of the unitisation of $SC(T_\theta^2)$ into $C(S_\theta^3)$. We will abuse notation and refer to this map as i as well.

Lemma 2.4.3. *The image $i(W)$ is path-connected through $GL_1(C(S_\theta^3))$ to the identity element.*

Proof. Viewing $C(S_\theta^3)$ as the continuous field of C^* -algebras given in Theorem 1.1.5, it is evident that $i(W) = \exp(2\pi\sqrt{-1}z_1z_1^*)$. \square

Theorem 2.4.4. *The image $i(X)$ generates $\pi_0(GL_1(C(S_\theta^3))) \cong \mathbb{Z}$.*

Proof. Immediate by Theorem 2.3.3, Lemma 2.4.2, and Lemma 2.4.3. \square

2.5 Finitely-Generated Projective Modules over $C(S_\theta^4)$

In this section, we classify and construct all finitely-generated projective modules over $C(S_\theta^4)$, where θ is irrational. In order to obtain an explicit formula for the generator X of $\pi_0(GL_1(C(S_\theta^3)))$ we need to impose the condition that $|\theta| < 1$.

The 4-sphere embeds into $\mathbb{C}^2 \times \mathbb{R}$ as the subspace

$$S^4 = \{(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} : |z_1|^2 + |z_2|^2 + x^2 = 1\}.$$

We define the northern hemisphere of S^4 to be the subspace

$$D_1^4 = \{(z_1, z_2, x) \in S^4 : x \geq 0\}$$

and the southern hemisphere to be the subspace

$$D_2^4 = \{(z_1, z_2, x) \in S^4 : x \leq 0\}.$$

We recall that there is an action of T^2 on S^4 that trivially extends the action of T^2 on S^3 . Explicitly it is given by

$$(\exp(2\pi i\phi_1), \exp(2\pi i\phi_2)) \cdot (z_1, z_2, x) = (\exp(2\pi i\phi_1)z_1, \exp(2\pi i\phi_2)z_2, x).$$

The action of T^2 restricts to actions on D_1^4 and D_2^4 and further restricts to the subspace $\{(z_1, z_2, 0) \in S^4\} \approx S^3$. In other words, we have a pullback diagram

$$\begin{array}{ccc} C(S^4) & \xrightarrow{i_1} & C(D_1^4) \\ i_2 \downarrow & & \downarrow j_1 \\ C(D_2^4) & \xrightarrow{j_2} & C(S^3), \end{array}$$

where the maps i_k, j_k are equivariant for the T^2 -actions.

We claim that the θ -deformation process respects the pullback structure. This is an easy consequence of the fact that the θ -deformed spaces are special instances of Rieffel's deformation quantization by actions of $V = \mathbb{R}^n$.

Proposition 2.5.1. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ i_2 \downarrow & & \downarrow j_1 \\ A_2 & \xrightarrow{j_2} & B \end{array}$$

is a pullback diagram of C^* -algebras carrying V -actions for which the $*$ -homomorphisms $i_k, j_k, k = 1, 2$, are equivariant. Let J be any skew-symmetric matrix. There is then an induced pullback diagram

$$\begin{array}{ccc} A_J & \xrightarrow{(i_1)_J} & (A_1)_J \\ (i_2)_J \downarrow & & \downarrow (j_1)_J \\ (A_2)_J & \xrightarrow{(j_2)_J} & B_J. \end{array}$$

If the maps j_k are surjective, then so are the maps $(j_k)_J$.

Proof. Suppose $a \in A^\infty$. Then $i_k(a) \in A_k^\infty$, so $(i_k)_J(a) = i_k(a)$, and

$$(j_1)_J((i_1)_J(a)) = j_1(i_1(a)) = j_2(i_2(a)) = (j_2)_J((i_2)_J(a)).$$

So the homomorphisms $(j_1)_J \circ (i_1)_J : A_J \rightarrow B_J$ and $(j_2)_J \circ (i_2)_J : A_J \rightarrow B_J$ agree on A^∞ , and hence on all of A_J , since A^∞ is dense in A_J . \square

We also note that equivariant homotopies between C^* -algebras equipped with V -actions induce homotopies on their deformations:

Proposition 2.5.2. *Suppose A and B are C^* -algebras carrying V -actions α_A and α_B , respectively, and that $\phi, \psi : A \rightarrow B$ are $*$ -homomorphisms. Suppose that $\phi \sim_h \psi$ via a map $\gamma : A \rightarrow C([0, 1], B)$. Let α be the V -action on $C([0, 1], B)$ defined by $(\alpha \cdot f)(t) = \alpha_B \cdot f(t)$. Suppose additionally that $\gamma(\alpha_A \cdot a) = \alpha \cdot (\gamma(a))$, and that J is any skew-symmetric matrix. Then $\phi_J \sim_h \psi_J$ via a map $A_J \rightarrow C([0, 1], B_J)$ that identifies with γ_J under the isomorphism $C([0, 1], B)_J \cong C([0, 1], B_J)$.*

Proof. Let $I = [0, 1]$. The evaluation map $\varepsilon : C(I, B) \rightarrow B$ is equivariant for the actions α, α_B . So ϕ and ψ are equivariant (of course it would have been natural and harmless to have assumed this as hypotheses anyway) since γ is. Thus, the maps ϕ_J, ψ_J, γ_J and ε_J all exist. By hypothesis, $\varepsilon_0 \cdot \gamma = \psi$ on A and $\varepsilon_1 \cdot \gamma = \phi$ on A , and so in particular on the dense subalgebra A^∞ . Since each of the homomorphisms involved in these equations map smooth elements to smooth elements, and since A^∞ is dense in A_J , we obtain $(\varepsilon_0)_J \cdot (\gamma)_J = (\psi)_J$ and $(\varepsilon_1)_J \cdot (\gamma)_J = (\phi)_J$ on A_J . We now need only observe that $C(I, B)_J \cong C(I, B_J)$, and that $(\varepsilon_t)_J$ corresponds to the evaluation map $C(I, B_J) \rightarrow B_J$ under this isomorphism.

Under the isomorphism $C(I, B) \cong C(I) \otimes B$, the action α is trivial on the subalgebra $C(I) \otimes \mathbb{C} \cong C(I)$. The action α also restricts to the action $1 \otimes \alpha_B$ on $\mathbb{C} \otimes B \cong B$. Thus

$(C(I) \otimes B)_J$ contains $C(I)$ and B_J as subalgebras. Moreover, the copies of $C(I)$ and B_J generate $(C(I) \otimes B)_J$, since the later algebra is generated by $C(I) \otimes_{alg} B^\infty$. The subalgebras $C(I) \otimes \mathbb{C}$ and $\mathbb{C} \otimes B_J$ meet only on $\mathbb{C} \otimes \mathbb{C}$. Also, since all elements of $C(I) \otimes \mathbb{C}$ are fixed points for the action α , the product in $(C(I) \otimes B)_J$ of any element of $C(I) \otimes \mathbb{C}$ with any element of $\mathbb{C} \otimes B^\infty$ is just their original undeformed product in $C(I) \otimes_{alg} B$. So the copies of $C(I)$ and B_J in $(C(I) \otimes B)_J$ commute with each other. So $(C(I) \otimes B)_J \cong C(I) \otimes B_J$. That $(\varepsilon_t)_J$ corresponds to the evaluation map $C(I, B_J) \rightarrow B_J$ under this isomorphism is clear, since $(\varepsilon_t)_J$ restricted to the dense subalgebra $C(I) \otimes B^\infty$ of $(C(I) \otimes B)_J$ is evaluation. \square

Since $id_J : A_J \rightarrow A_J$ is the identity map from A_J to itself, and is trivially equivariant for any action on A , an equivariant homotopy that is a homotopy equivalence or deformation retraction between A and B will be a homotopy equivalence or deformation retraction between A_J and B_J .

The dual statement to the fact that D_1^4 is contractible, is the fact that the function algebra $C(D_1^4)$ deformation retracts onto its subalgebra of scalar multiples of the identity element. Explicitly, a retraction is given by the map

$$\begin{aligned} F_t(z_1) &= (1-t)z_1 \\ F_t(z_2) &= (1-t)z_2 \\ F_t(x) &= \sqrt{1 - (1-t)^2(z_1z_1^* + z_2z_2^*)} \end{aligned}$$

where z_1, z_2 , and x are the coordinate functions for $C(D_1^4)$. This retraction is clearly equivariant for the action of T^2 on $C(D_1^4)$ that by the restriction of the action of T^2 on S^4 to D_1^4 .

Proposition 2.5.3. *All finitely-generated projective $C((D_1^4)_\theta)$ -modules are free.*

Proof. The algebra $C((D_1^4)_\theta)$ deformation retracts onto $\mathbb{C} \cdot 1$ by Proposition 2.5.2. \square

Of course the same is true for $C((D_2^4)_\theta)$, since it is obviously isomorphic to $C((D_1^4)_\theta)$.

Definition 2.5.4. We define $N(n, s)$ to be the finitely-generated projective $C(S_\theta^4)$ -module $M(C((D_1^4)_\theta)^n, C((D_2^4)_\theta)^n, X^s)$, where X^s is the image of the s th-power of $X = \exp(2\pi it)p + 1 - p$ in $GL_n(C(S_\theta^3))$.

Theorem 2.5.5. *There is a semigroup isomorphism*

$$V(C(S_\theta^4)) \cong \{0\} \cup (\mathbb{N} \times K_1(C(S_\theta^3))) \cong \{0\} \cup (\mathbb{N} \times \mathbb{Z}).$$

The module $N(n, s)$ is a representative for the element (n, s) of $V(C(S_\theta^4))$.

Proof. We have just observed that all finitely-generated projective $C((D_k^4)_\theta)$ -modules are free. So by Theorem 1.3.3, every finitely-generated projective $C(S_\theta^4)$ -module is isomorphic to one of the form $M(P_1, P_2, h)$, where P_1 and P_2 are free $C((D_1^4)_\theta)$ and $C((D_2^4)_\theta)$ -modules

respectively. But since the free $C(S_\theta^3)$ -modules $j_{1\#}P_1$ and $j_{2\#}P_2$ are assumed isomorphic, and since $C(S_\theta^3)$ has the invariance of dimension property, the free modules P_1 and P_2 must have the same rank, say n , and we may identify h with an element of $GL_n(C(S_\theta^3))$. But, by Theorems 2.3.3 and 2.4.4, each h is path-connected in $GL_n(C(S_\theta^3))$ to $\begin{pmatrix} X^s & 0 \\ 0 & 1_{n-1} \end{pmatrix}$, for some unique integer s . Thus each projective $C(S_\theta^4)$ -module is isomorphic to one of the modules $N(n, s)$ by Proposition 2.1.4. Suppose that $N(n_1, s_1)$ is isomorphic to $N(n_2, s_2)$. Then, by Theorem 1.3.4, it must be that

$$C((D_1^4)_\theta)^{n_1} \cong i_{1\#}N(n_1, s_1) \cong i_{1\#}N(n_2, s_2) \cong C((D_1^4)_\theta)^{n_2}$$

as free $C((D_1^4)_\theta)$ -modules. But $C((D_1^4)_\theta)$ has the invariance of dimension property, since it has a normalized trace. Thus $n_1 = n_2$. Suppose $N(n, s_1)$ is isomorphic to $N(n, s_2)$. Then, by Theorem 2.5.5, the matrix

$$\begin{pmatrix} X^{s_1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} = j_{2*}(g_2) \begin{pmatrix} X^{s_2} & 0 \\ 0 & 1_{n-1} \end{pmatrix} j_{1*}(g_1^{-1}),$$

for some $g_1, g_2 \in GL_n(C(D_\theta^4))$. But $GL_n(C(D_\theta^4))$ is path-connected, so $j_{2*}(g_2), j_{1*}(g_1^{-1})$ are both path-connected in $GL_n(C(S_\theta^3))$ to the identity matrix. But this means that

$$\begin{pmatrix} X^{s_1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} = j_{2*}(g_2) \begin{pmatrix} X^{s_2} & 0 \\ 0 & 1_{n-1} \end{pmatrix} j_{1*}(g_1^{-1}) \sim_h \begin{pmatrix} X^{s_2} & 0 \\ 0 & 1_{n-1} \end{pmatrix},$$

which is possible only if $s_1 = s_2$, by Theorems 2.4.4 and 2.3.3. Finally, since $\begin{pmatrix} X^{s_1} & 0 \\ 0 & X^{s_2} \end{pmatrix}$ is homotopic through invertibles to $\begin{pmatrix} X^{s_1} X^{s_2} & 0 \\ 0 & 1 \end{pmatrix}$, we see that:

$$N(n_1, s_1) \oplus N(n_2, s_2) \cong N(n_1 + n_2, s_1 + s_2) \cong N(1, s_1 + s_2) \oplus (C(S_\theta^4))^{n_1+n_2-1}.$$

Thus we have completely characterized the set of isomorphism classes of finitely-generated projective $C(S_\theta^4)$ -modules as a semigroup. \square

We immediately obtain:

Corollary 2.5.6. *The algebra $C(S_\theta^4)$ is K -cancellative.*

We note that the noncommutative instanton bundle e of Connes and Landi trivially has $Tr_{C(S_\theta^3)}^{C(S_\theta^4)}$ -rank equal to 2, and thus is isomorphic to some $N(2, s)$. One can show that the matrix $U := \begin{pmatrix} z_1 & z_2 \\ -\lambda z_2^* & z_1^* \end{pmatrix}$ generates $K_1(C(S_\theta^3))$, and that correspondingly the index s must be 1 or -1 . The modules $P_{(n)}$ constructed by Landi and Van Suijlekom [20] must then be of

the form $N(n+1, (1/6)n(n+1)(n+2))$, where $n \geq 1$. Thus e and the rank-1 free module generate $K_0(C(S_\theta^4))$. Yet e is not really the most basic nontrivial $C(S_\theta^4)$ -module, since e is isomorphic to $N(1, 1) \cong C(S_\theta^4)$.

We end this work by observing that the finitely-generated projective modules over $C(S_\theta^4)$ are in bijective correspondence with their restrictions as $C(S^2)$ -modules, for a certain subalgebra of $C(S_\theta^4)$ that is isomorphic to $C(S^2)$. We first need the following proposition:

Proposition 2.5.7. $C((D_1^4)_\theta)$ is isomorphic to the unitisation of $\text{Cone}(C(S_\theta^3))$.

Proof. We work in the context of Theorem 2.5.2. Let $A = C(S^3)$, and write A_J for $C(S_\theta^3)$, with similarly write $C(D_1^4)_J$ for $C((D_1^4)_\theta)$. Recall that $\text{Cone}(A)$ is defined as $C_0((0, 1], A)$. Let IA denote $C([0, 1], A)$. We have the short exact sequence

$$0 \rightarrow \text{Cone}(A_J) \rightarrow I(A_J) \rightarrow A_J \rightarrow 0.$$

Since $\text{Cone}(A)$ is an equivariant ideal of IA for the action of T^2 on A , we have the short exact sequence

$$0 \rightarrow \text{Cone}(A)_J \rightarrow (IA)_J \rightarrow A_J \rightarrow 0$$

by Theorem 1.1.9. But $I(A_J) \cong (IA)_J$, by the proof of Proposition 2.5.2. Combining these exact sequences we obtain the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Cone}(A_J) & \longrightarrow & I(A_J) & \longrightarrow & A_J \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Cone}(A)_J & \longrightarrow & (IA)_J & \longrightarrow & A_J \longrightarrow 0, \end{array}$$

where the map $\text{Cone}(A_J) \rightarrow \text{Cone}(A)_J$ is given by the restriction of the map $I(A_J) \rightarrow (IA)_J$ to the ideal $\text{Cone}(A_J)$. Thus, the map $\text{Cone}(A_J) \rightarrow \text{Cone}(A)_J$ is an isomorphism, by the 5-lemma of algebra. Consider the north-pole $N := (0, 0, 1) \in D_1^4$. We have an isomorphism $C(D_1^4) \cong C(A) \oplus C(N) \cong C(A) \oplus \mathbb{C}$. The action of T^2 on D_1^4 acts trivially on the point N . Thus $C(D_1^4)_J \cong (\text{Cone}(A) \oplus \mathbb{C})_J \cong \text{Cone}(A)_J \oplus \mathbb{C} \cong \text{Cone}(A_J) \oplus \mathbb{C}$. \square

For each $N(1, s)$, we can use Lemma 2.3.2 to find an idempotent matrix $P \in M_2(C(S_\theta^4))$ such that $N(1, s) \cong PC(S_\theta^4)^2$. Consider the element $X = \exp(2\pi it)p + 1 - p$ in $C(S_\theta^3)$. Lift p to a self-adjoint element q in $C((D_1^4)_\theta)$ (for example $q = t \otimes p \in C_0((0, 1]) \otimes C(S_\theta^3) \subset C((D_1^4)_\theta)$, this containment by Proposition 2.5.7). Let $c = \exp(2\pi it)q + 1 - q$, which is then a lift of X to $C((D_1^4)_\theta)$. Then $d = c^*$ will be a lift of X^{-1} and c^s will be a lift of X^s . Also, the element c is normal. Now $N(1, s) \cong P_s C(S_\theta^4)^2$, where the idempotent $P_s \in M_2(C(S_\theta^4))$ is given by

$$\begin{pmatrix} (1, c^s(2 - d^s c^s)d^s) & (0, c^s(2 - d^s c^s)(1 - d^s c^s)) \\ (0, (1 - d^s c^s)d^s) & (0, (1 - d^s c^s)^2) \end{pmatrix}.$$

The module $N(n, s)$ is then isomorphic to $PC(S^4)^{2n}$, where $P = \begin{pmatrix} P_s & 0 \\ 0 & 1_{2n-2} \end{pmatrix}$.

Now, since p is a projection, the spectrum $\sigma(X)$ is the full circle S^1 . If we take $q = t \otimes p$ as suggested, then $\sigma(q) = [0, 1]$. For each fixed s , the spectrum of $\exp(2\pi is)p + 1 - p$ will be the chord in the unit disk in the complex plane that connects the point 1 to the point $\exp(2\pi is)$, so $\sigma(c) = D^2$. Similarly, we lift p to a self-adjoint element q' in $C((D^4)_2)_\theta$ such that $\sigma(q') = [0, 1]$, and define $c' = \exp(2\pi it)q' + 1 - q'$, which also has spectrum D^2 . Now consider the pullback \mathcal{P} of the algebras $C^*(c, 1)$ and $C^*(c', 1)$ over $C^*(X, 1)$. By the above remarks, the pullback \mathcal{P} is clearly isomorphic to $C(S^2)$ and is a C^* -subalgebra of $C(S^4_\theta)$. Now each entry of $P_s \in M_2(C(S^4_\theta))$ is in \mathcal{P} , so we can consider the \mathcal{P} -module $\begin{pmatrix} P_s & 0 \\ 0 & 1_{2n-2} \end{pmatrix} \mathcal{P}^{2n} \cong M(C^*(c, 1)^n, C^*(c', 1)^n, X^s)$, where here X^s is viewed as in $GL_n(C^*(X, 1))$. But under the isomorphism $C^*(X, 1) \cong C(S^1)$, multiplication by X^s corresponds to multiplication of a function in $C(S^1)$ by the function z^s given by $z = \exp(2\pi it) \mapsto z^s = \exp(2\pi ist)$. So under the isomorphism $\mathcal{P} \cong C(S^2)$, the module $M(C^*(c, 1)^n, C^*(c', 1)^n, X^s)$ is $M(C(D^2)^n, C(D^2)^n, z^s)$ (here z^s is viewed as in $GL_n(C(S^1))$). But the later module is the space of continuous sections of the complex rank- n vector bundle over S^2 of index or charge (integral of the first Chern class) equal to $-s$, since that bundle is formed by clutching rank- n trivial bundles over the two hemispheres of S^2 via the map z^s . Thus, since each module over $C(S^4_\theta)$ is of the form $N(n, s)$, the inclusion $i : C(S^2) \cong \mathcal{P} \hookrightarrow C(S^4)$ induces an isomorphism $i_* : V(C(S^2)) \cong V(C(S^4_\theta))$ given by $M(C(D^2)^n, C(D^2)^n, z^s) \mapsto N(n, s)$. Speaking heuristically, every complex vector bundle over the noncommutative space S^4_θ is the pullback of a complex vector bundle over S^2 .

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