Title
Low Regularity Solutions of Korteweg-de Vries and Chern-Simons-Schr\"{o}dinger Equations

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Low Regularity Solutions of Korteweg-de Vries and Chern-Simons-Schrödinger Equations

by

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University of California, Berkeley

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Abstract

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The aim of this thesis is to understand the locall wellposedness theory for some nonlinear dispersive equations at low regularity.

The Korteweg-de Vries equation has sharp wellposedness at $H^{-\frac{3}{4}}$ if we are concerned about the Lipschitz dependence of solutions on the initial data. For lower regularity, one might still have a weaker form of wellposedness only with continuous dependence on data. Here we prove that the smooth solutions satisfy a-priori local in time H^s bound in terms of the H^s size of the initial data for $s \geq -\frac{4}{5}$. Together with the bounds we obtained on the nonlinearity, the result here ensures that the equation is satisfied in the sense of distributions even for weak limits.

The Chern-Simons-Schrödinger equation is a planar gauged Schrödinger equation which has some similarity to the derivative formulation of the Schrödinger map problem. We work on to prove local wellposedness in the full subcritical range $H^s(\mathbb{R}^2), s > 0$.

One important idea in working on these problems is to find a suitable space to characterize the solution. We use $X^{s,b}$ spaces introduced by Bourgain, and U^2, V^2 spaces introduced by Koch and Tataru. For the Chern-Simons-Schrödinger equation, we also need to fix a suitable gauge to make the problem well-posed. The heat gauge is a variation of Coulomb gauge, and it serves as a good candidate for this problem.

I would like to dedicate this thesis to my family.

Contents

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Chapter 1

Background and Preliminary Tools

In this chapter, we introduce some basic tools that are commonly used for dispersive equations. Precisely, we will introduce Littlewood-Paley decomposition which serves as the main technique to handle problems in this thesis. We also define $X^{s,b}$ and U^p, V^p spaces that are well suited for each equation.

1.1 Littlewood-Paley decomposition

We define the Fourier Transform $\mathcal{F}$ on Schwartz space $S_x(\mathbb{R}^n)$

$$\mathcal{F}f = \hat{f}(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,$$

The definition of Fourier Transform varies slightly in different settings in the literature: the constant in front of the integral might be taken differently and $e^{-2\pi ix\cdot\xi}$ may be used instead of $e^{-ix\cdot\xi}$ here.

The Fourier transform gives an automorphism on the Schwartz space, moreover, it can be extended to wider function spaces, such as tempered distributions [30].

Let $\psi : \mathbb{R} \to [0,1]$ be a smooth even function compactly supported in $[-2,2]$ and equal to 1 on $[-1,1]$. For dyadic integers $N = 2^k, k \in \mathbb{Z}+$, set

$$\psi_N(\xi) = \psi\Big(\frac{|\xi|}{N}\Big) - \psi\Big(\frac{2|\xi|}{N}\Big), \quad \text{for } N \geq 2 \quad \text{and} \quad \psi_1(\xi) = \psi(|\xi|). \tag{1.1}$$

So ψ_N is a smooth function supported in region $\{\xi \in \mathbb{R}^n, \frac{N}{2} \leq |\xi| \leq 2N\}$ when $N > 1$.

For each such $N \geq 1$ dyadic, define the Littlewood-Paley projection operator P_N as the Fourier multiplier with symbol ψ_N.

$$\widehat{P_N f}(\xi) := \psi_N(\xi)\hat{f}(\xi). \tag{1.2}$$

Moreover, let

$$P_{\leq N} := \sum_{1\leq M\leq N} P_M, \quad P_{\geq N} := \sum_{M\geq N} P_M$$

$$P_{M_1 < \cdot \leq M_2} = P_{\leq M_2} - P_{\leq M_1}.$$

All summations are taken over dyadic numbers. We set $u_N = P_N u$ for short, similar for other notations $u_{\leq N}, u_{\geq N}$.

The Littlewood-Paley decomposition helps us to decompose a function, on the frequency side, to pieces that have almost disjoint frequency supports. Now low frequency components are slowly varying and have higher regularity while the high frequency parts oscillate rapidly and have low regularity. Thus when we are proving estimates, especially when we are handling bilinear and multilinear interactions of functions, we can identify the worst case scenario.

Now we recall the following Bernstein inequalities and Littlewood-Paley inequality, which helps to turn our heuristics above into rigorous analysis.

Lemma 1.1.1. (Berstein's inequality)[56] *For $s > 0$ and $1 \leq p \leq q \leq \infty$, the following inequalities hold true.*

$$\begin{aligned}
\|f_N\|_{L^q_x(\mathbb{R}^n)} &\lesssim_{p,q,n} N^{\frac{n}{p}-\frac{n}{q}} \|f_N\|_{L^p_x(\mathbb{R}^n)} \\
\||\nabla|^{\pm s} f_N\|_{L^p_x(\mathbb{R}^n)} &\sim_{p,s,n} N^{\pm s} \|f_N\|_{L^p_x(\mathbb{R}^n)} \\
\|f_{\geq N}\|_{L^p_x(\mathbb{R}^n)} &\lesssim_{p,s,n} N^{-s} \||\nabla|^s f_{\geq N}\|_{L^p_x(\mathbb{R}^n)} \\
\|f_{\leq N}\|_{L^q_x(\mathbb{R}^n)} &\lesssim_{p,q,n} N^{\frac{n}{p}-\frac{n}{q}} \|f_{\leq N}\|_{L^p_x(\mathbb{R}^n)} \\
\||\nabla|^s f_{\leq N}\|_{L^p_x(\mathbb{R}^n)} &\sim_{p,s,n} N^s \|f_{\leq N}\|_{L^p_x(\mathbb{R}^n)}.
\end{aligned}$$

Lemma 1.1.2. (Littlewood-Paley Inequality)[52] *When $1 < p < \infty$, we have the following estimate.*

$$\|f\|_{L^p_x(\mathbb{R}^n)} \sim_{p,n} \|(\sum_N f_N^2)^{1/2}\|_{L^p_x(\mathbb{R}^n)}.$$

1.2 Dispersive equations

A constant-coefficient linear dispersive PDE generally takes the form

$$\partial_t u(t,x) = Lu(t,x); \qquad u(0,x) = u_0(x) \tag{1.3}$$

where $u : \mathbb{R} \times \mathbb{R}^n \to H$ takes value in a finite dimenstional Hilbert Space H, and L is a skew-adjoing constant coefficient differential operator, taking the form

$$Lu(x) := \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha u(x)$$

where c_α are constant coefficients, $k \in \mathbb{Z}+$ is the order of the differential operator, $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^n_+$ ranges over all multi-indices with $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

$$\partial_x^\alpha := (\frac{\partial}{\partial_{x_1}})^{\alpha_1} \cdots (\frac{\partial}{\partial_{x_n}})^{\alpha_n}$$

Classically, this operator is only defined on k times continuously differentiable functions, but we can extend it to distributions and thus talk about both classical and weak distributional solutions to (1.3).

Not let us write $L = ih(D)$, where D is the frequency operator

$$D := \frac{1}{i}\nabla = (\frac{1}{i}\partial_{x_1}, \cdots, \frac{1}{i}\partial_{x_n})$$

and

$$h(\xi_1, \cdots, \xi_n) = \sum_{|\alpha| \le k} i^{|\alpha|-1} c_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

Since we assume L is skew-adjoint, we can verify that h is real-valued polynomial. We call h the *dispersion relation* of equation (1.3).

We define the free evolution operator e^{tL}

$$e^{tL} f(x) := \int_{\mathbb{R}^n} e^{ith(\xi)+ix\cdot\xi} \hat{f}(\xi) d\xi \tag{1.4}$$

The operator is initially defined for Schwartz functions, but can of course be extended to other spaces, i.e. tempered distributions.

The *fundamental solution* K_t can be viewed as the propagator e^{tL} applied to delta function.

$$K_t(x) := \int e^{i(x\cdot\xi + th(\xi))} d\xi. \tag{1.5}$$

The integral here is not absolutely convergent, but it can be interpreted as the limit of

$$K_t(x) = \lim_{\epsilon \to 0} \int e^{i(x\cdot\xi+th(\xi))} e^{-\epsilon|\xi|^2} d\xi.$$

in the sense of distribution.

And we have the solution of (1.3) as a convolution of initial data with fundamental solution.

$$u(t, x) = u_0(x) * K_t(x) = \int_{\mathbb{R}^n} u_0(x - y) K_t(y) dy. \tag{1.6}$$

The representation (1.6), together with

$$u(t, x) = e^{tL} u_0(x),$$

help us to understand the solution from different aspects. They are both useful for the proof of spacetime estimate in the next section.

We notice that for fixed frequency $\xi_0 \in \mathbb{R}^n$, the plane wave $e^{ith(\xi_0)+ix\cdot\xi_0}$ solves the equation (1.3). Thus the solution (1.4) is a superposition of plane waves, and each of them travels at velocity $-\nabla h(\xi)$, which is called *group velocity.* So different frequencies in this equation will tend to propagate at different velocities, thus dispersing the solution over time.

Now let us write down the important examples in this thesis.

Example 1.2.1. (The free Schrödinger Equation)

$$\partial_t u - i\Delta u = 0. \tag{1.7}$$

Here $u : \mathbb{R} \times \mathbb{R}^n \to H$ *is a vector field taking value in Hilbert space* H*, and* $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial_{x_j}^2}$ *is the Laplacian. The dispersion relation is* $h(\xi) = -|\xi|^2$ *and group velocity is* -2ξ*.*

Example 1.2.2. (The Airy Equation)

$$\partial_t u + \partial_x^3 u = 0. \tag{1.8}$$

Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a real scalar function. The dispersion relation is* $h(\xi) = \xi^3$ *and group velocity is* $3\xi^2$*.*

1.3 Spacetime estimates

In this section, we collect some useful estimates that are crucial to control the size of the solutions. Here we will specify our dispersive equations to be the free Schrödinger equation (1.7) and the Airy equation (1.8).

From the fundamental solution and the method of stationary phase, we get the dispersive inequalities.

$$\|e^{it\Delta}u_0(x)\|_{L_x^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}}\|u_0(x)\|_{L_x^1(\mathbb{R}^n)}. \tag{1.9}$$

$$\|e^{t\partial_x^3}u_0(x)\|_{L_x^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{3}}\|u_0(x)\|_{L_x^1(\mathbb{R})}. \tag{1.10}$$

Notice that the L_x^2 mass of the solution is conserved, so we can apply interpolation and get the following dispersive estimates.

Lemma 1.3.1. (Dispersive estimate)

$$\|e^{it\Delta}u_0(x)\|_{L_x^{p'}(\mathbb{R}^n)} \lesssim t^{-n(\frac{1}{p}-\frac{1}{2})}\|u_0(x)\|_{L_x^p(\mathbb{R}^n)}. \tag{1.11}$$

$$\|e^{t\partial_x^3}u_0(x)\|_{L_x^{p'}(\mathbb{R})} \lesssim t^{-\frac{2}{3}(\frac{1}{p}-\frac{1}{2})}\|u_0(x)\|_{L_x^p(\mathbb{R})}. \tag{1.12}$$

$1 \le p \le 2, \frac{1}{p} + \frac{1}{p'} = 1.$

Combining the above dispersive estimates with some duality argument, we obtain the well-known *Strichartz estimates.*

Theorem 1.3.2. (Strichartz estimates for Schrödinger Equation) [23, 60, 39] *Let* (q, r) *be any admissible exponents, i.e.* $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ *and* $(q, r, n) \neq (2, \infty, 2)$*. Then we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta}f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|f\|_{L_x^2(\mathbb{R}^2)}. \tag{1.13}$$

Theorem 1.3.3. (Strichartz estimates for Airy Equation) [43, 56] *Let (q,r) be Strichartz pair*

$$\frac{2}{q}+\frac{1}{r}=\frac{1}{2}, \quad 4\le q\le\infty. \tag{1.14}$$

Then we have

$$\|e^{t\partial_x^3}f\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R})}\lesssim \||D|^{-\frac{1}{q}}f\|_{L_x^2(\mathbb{R})}. \tag{1.15}$$

Also, we have the local smoothing and maximal function estimates. For a unit vector $\mathbf{e}\in\mathbb{S}^{n-1}$, we denote by $H_{\mathbf{e}}$ its orthogonal complement in $\mathbb{R}^n$ with the induced measure. Define the lateral spaces $L_{\mathbf{e}}^{p,q}$ with norms

$$\|f\|_{L_{\mathbf{e}}^{p,q}}=\left[\int_{\mathbb{R}}\left[\int_{H_{\mathbf{e}}\times\mathbb{R}}|f(xe+x',t)|^q dx'dt\right]^{\frac{p}{q}}dx\right]^{\frac{1}{p}},$$

with the usual modifications when $p=\infty$ or $q=\infty$.

Define the operator $P_{N,\mathbf{e}}$ by the Fourier multiplier $\xi\to\psi_N(\xi\cdot\mathbf{e})$, where ψ_N is the same function we used to define frequency projection P_N in section 1.

Now we can state the local smoothing and maximal function estimates for Schrödinger Equation.

Theorem 1.3.4. [31, 32] *Let $f\in L^2(\mathbb{R}^n)$, $N\in 2^{\mathbb{Z}}, N\ge 1$, and $\mathbf{e}\in\mathbb{S}^{n-1}$. Then we have the local smoothing estimate*

$$\|e^{it\Delta}P_{N,\mathbf{e}}f\|_{L_{\mathbf{e}}^{\infty,2}}\lesssim N^{-\frac{n}{2}}\|f\|_{L^2}. \tag{1.16}$$

In addition, if $n\ge 3$, we have the maximal function estimate

$$\|e^{it\Delta}P_Nf\|_{L_{\mathbf{e}}^{2,\infty}}\lesssim N^{\frac{n-1}{2}}\|f\|_{L^2}. \tag{1.17}$$

Remark 1.3.5. In dimension 2, the maximal function estimate fails, but only with a logarithmic loss.

We have similar estimates for Airy Equation.

Theorem 1.3.6. [43, 56] *The following estimates hold true.*

$$\|\partial_x e^{t\partial_x^3}f\|_{L_x^\infty L_t^2}\lesssim\|f\|_{L^2}, \tag{1.18}$$

$$\|\partial_x^{-\frac{1}{4}}e^{t\partial_x^3}f\|_{L_x^4L_t^\infty}\lesssim\|f\|_{L^2}. \tag{1.19}$$

1.4 $X^{s,b}$ spaces

If we take spacetime Fourier transform on equation (1.3), we get

$$i\tau\tilde{u}(\tau,\xi) = ih(\xi)\tilde{u}(\tau,\xi)$$

and thus we get

$$(\tau - h(\xi))\tilde{u}(\tau,\xi) = 0.$$

So $\tilde{u}(\tau,\xi)$ is supported in the hypersurface $\{(\tau,\xi) : \tau = h(\xi)\}$.

Now we are working on solutions in local time interval, and also nonlinear dispersive equation

$$\partial_t u = Lu + \mathcal{N}(u), \tag{1.20}$$

The solution does not lie in the characteristic hypersurface anymore, but we can still expect $\widetilde{\eta(t)u}$ concentrates near the hypersurface. This motivates the definition of $X^{s,b}$ spaces.

Let us define the modulation $\sigma = |\tau - h(\xi)|$. As in our definition for frequency projection (1.2), we can also define the modulation projection operator Q_M by localizing the modulation to dyadic region M.

$$\widetilde{Q_M f}(x,t) = \psi_M(\tau - h(\xi))f(\tau,\xi).$$

Now we can define the $X^{s,b}$ spaces introduced by Bourgain[5, 6].

Definition 1.4.1. The space $X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)$ or sometimes abbreviated by $X^{s,b}$ is defined as the closure of Schwartz functions $S_{t,x}(\mathbb{R}\times\mathbb{R}^n)$ under the norm.

$$\begin{aligned}\|u\|^2_{X^{s,b}_{\tau=h(\xi)}} &= \int |\hat{u}(\tau,\xi)|^2(1+|\xi|^2)^s(1+|\tau-h(\xi)|^2)^b d\tau d\xi \\ &= \sum_{N,M\in 2^{\mathbb{Z}^+}} N^{2s}M^{2b}\|P_N Q_M u\|^2_{L^2_{x,t}}.\end{aligned}$$

Remark 1.4.2. The spaces are only adapted to local in time solutions.

Here, we list some interesting properties of $X^{s,b}$ spaces. The proof can be found in standard literature [56, 25].

1.*Nesting and Duality*

We have the trivial nesting $X^{s',b'} \subset X^{s,b}$ whenever $s' < s, b' < b$. And by Parseval's identity and Cauchy-Schwarz we have the duality

$$\left(X^{s,b}_{\tau=h(\xi)}\right)^* = X^{-s,-b}_{\tau=-h(-\xi)} \tag{1.21}$$

Also notice the following fact

$$\|\bar{u}\|_{X^{s,b}_{\tau=-h(-\xi)}} = \|u\|_{X^{s,b}_{\tau=h(\xi)}} \tag{1.22}$$

2.*The free solution lies in* $X^{s,b}$ locally

Proposition 1.4.3. *Let $f \in H^s_x(\mathbb{R}^n)$ for some $s \in \mathbb{R}$, then for any Schwartz cutoff, $\eta(t) \in S_(\mathbb{R})$, we have*

$$\|\eta(t)e^{tL}f\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|f\|_{H^s_x(\mathbb{R}^n)}.$$

3.*Any estimate for free solution would extends to functions in* $X^{s,b}$, $b > \frac{1}{2}$.

Proposition 1.4.4. *Suppose that $b > \frac{1}{2}$, and let Y be a Banach space on $\mathbb{R} \times \mathbb{R}^n$, for which the following estimate holds*

$$\|e^{it\tau_0}e^{tL}f\|_Y \lesssim \|f\|_{H^s_x(\mathbb{R}^n)}$$

for all $f \in H^s_x(\mathbb{R}^n)$ and any $s, \tau_0 \in \mathbb{R}$. Then we have

$$\|u\|_Y \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

4.$X^{s,b}$ *is stable with regard to time localization.*

Proposition 1.4.5. *Let $\eta(t)$ be a Schwartz time cutoff. Then we have*

$$\|\eta(t)u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{\eta,b} \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)}.$$

5.*Energy estimate holds true for functions in* $X^{s,b}$.

Proposition 1.4.6. *For the linear equation*

$$\partial_t u = Lu + f \tag{1.23}$$

suppose u is a smooth solution, then we have the following energy estimate holds true for any $s \in \mathbb{R}, b > \frac{1}{2}$ and any Schwartz time cutoff $\eta(t)$.

$$\|\eta(t)u\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{\eta,b} \|u(x,0)\|_{H^s_x(\mathbb{R}^n)} + \|f\|_{X^{s,b-1}_{\tau=h(\xi)}}$$

Remark 1.4.7. The above Propositions 1.4.4 and 1.4.6 fails at endpoint $b = \frac{1}{2}$ logarithmically in certain regions, in which case, we should seek for alternatives. Two of the candidates are the $X^{s,\frac{1}{2},1}$ and $X^{s,\frac{1}{2},\infty}$ spaces defined via norms

$$\|u\|_{\dot{X}^{s,\frac{1}{2},1}} = \sum_{\vartheta\in 2^{\mathbb{Z}}} \left(\int_{|\tau-h(\xi)|=\vartheta} |\tilde{u}(\tau,\xi)|^2|\xi|^{2s}|\tau-h(\xi)|\, d\xi\, d\tau \right)^{\frac{1}{2}}$$

and

$$\|u\|_{\dot{X}^{s,\frac{1}{2},\infty}} = \sup_{\vartheta\in 2^{\mathbb{Z}}} \left(\int_{|\tau-h(\xi)|=\vartheta} |\tilde{u}(\tau,\xi)|^2|\xi|^{2s}|\tau-h(\xi)|\, d\xi\, d\tau \right)^{\frac{1}{2}}.$$

A better candidate is the U^2, V^2 spaces defined in the next section.

1.5 U^p, V^p spaces

In this section we discuss the function spaces of U^p, V^p type, which were first introduced by Tataru in unpublished work on wave map problem. It has been used to obtain critical results in different problems [45, 29, 27, 28] as a useful replacement of $X^{s,b}$ spaces in limiting cases.

Throughout this section let H be a separable Hilbert space over $\mathbb{C}$. Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K \le \infty$ of the real line. If $t_K = \infty$ and $v : \mathbb{R} \to H$, then we adopt the convention that $v(t_K) := 0$. Let $\chi_I : \mathbb{R} \to \mathbb{R}$ denote the (sharp) characteristic function of a set $I \subset \mathbb{R}$.

Definition 1.5.1. Let $1 \le p < \infty$. For any $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset H$ with $\sum_{k=0}^{K-1} \|\phi_k\|_H^p = 1$, we call the function $a : \mathbb{R} \to H$ defined by

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)} \phi_{k-1}$$

a U^p-atom. We define the atomic space $U^p(\mathbb{R}, H)$ as the set of all functions $u : \mathbb{R} \to H$ admitting a representation

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \ \text{ for } U^p\text{-atoms } a_j,\ \{\lambda_j\} \in \ell^1$$

and endow it with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| :\ u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j \text{ a } U^p\text{-atom} \right\}. \tag{1.24}$$

Remark 1.5.2. The spaces $U^p(\mathbb{R}, H)$ are Banach spaces and we observe that $U^p(\mathbb{R}, H) \hookrightarrow L^\infty(\mathbb{R}; H)$. Every $u \in U^p(\mathbb{R}, H)$ is right-continuous and u tends to 0 as $t \to -\infty$.

Definition 1.5.3. Let $1 \le p < \infty$.

1. We define $V^p(\mathbb{R}, H)$ as the space of all functions $v : \mathbb{R} \to H$ such that $v(\infty) := \lim_{t\to\infty} v(t) = 0$ and $v(-\infty)$ exists and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_H^p \right)^{\frac{1}{p}} \tag{1.25}$$

   is finite.

2. Likewise, let $V^p_{rc}(\mathbb{R}, H)$ denote the closed subspace of all right-continuous functions $v : \mathbb{R} \to H$ such that $\lim_{t\to-\infty} v(t) = 0$.

Remark 1.5.4. The spaces $V^p(\mathbb{R}, H)$, $V^p_{rc}(\mathbb{R}, H)$ are Banach spaces and the space $V^p_{rc}(\mathbb{R}, H)$ inherits its norm from $V^p(\mathbb{R}, H)$. The requirement of functions being right continuous in $V^p_{rc}(\mathbb{R}, H)$ guarantees that we can identify functions as distributions.

We now introduce U^p, V^p-type spaces that are adapted to our equation (1.3).

Definition 1.5.5. For $s \in \mathbb{R}$, let $U^p_L H$ (resp. $V^p_L H$) be the space of all functions $u : \mathbb{R} \to H$ such that $t \mapsto e^{-tL}u(t)$ is in $U^p(\mathbb{R}, H)$ (resp. $V^p(\mathbb{R}, H)$), with respective norms

$$\|u\|_{U^p_L H} = \|e^{-tL}u\|_{U^p(\mathbb{R},H)}, \qquad \|u\|_{V^p_L H} = \|e^{-tL}u\|_{V^p(\mathbb{R},H)}. \tag{1.26}$$

The same definition also extends to $V^p_{rc,L}H$.

We simplify our notation by omitting H unless necessary.
As we did for $X^{s,b}$ spaces, we also list some of the key features for U^p, V^p spaces.

1.*Duality and Embedding*
As in [27, Proposition 2.10], we define the following paring. Take $u \in U^p, v \in V^{p'}$, with (p, p') dual exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. And a partition $\mathrm{t} = \{t_k\}_{k=0}^{K-1} \in \mathcal{Z}$, we define

$$B_{\mathrm{t}}(u, v) := \sum_{k=1}^{K} \langle u(t_{k-1}), v(t_k) - v(t_{k-1})\rangle_H$$

Then there is an unique extension $B(u, v) : U^p \times V^{p'} : (u, v) \to B(u, v)$, such that for all $\epsilon > 0$, we can find a partition $\mathrm{t} \in \mathcal{Z}$, such that any finer partition $\mathrm{t} \subset \mathrm{t}'$

$$|B(u, v) - B_{\mathrm{t}'}(u, v)| < \epsilon$$

And

$$|B(u, v)| < \|u\|_{U^p}\|v\|_{V^{p'}}$$

It was further shown that $B(u, v)$ takes the following representation for functions $u \in U^p, v \in V^{p'}, 1 < p < \infty$

$$B(u, v) = -\int_{\mathbb{R}} \langle u'(t), v(t)\rangle \, dt$$

Proposition 1.5.6. [27] *Let $1 < p < \infty$, we have the duality*

$$(U^p)^* = V^{p'}$$

under the pairing $B(u, v)$, in the sense that $T : V^{p'} \to (U^p)^*, T(v) := B(, v)$ is an isometric isomorphism.

Hence we also get duality

$$(U^p_L)^* = V^{p'}_L$$

Proposition 1.5.7. [45] *We have the following embedding among the U^p, V^p spaces.*

$$U^p(\mathbb{R}, H) \hookrightarrow V^p_{rc}(\mathbb{R}, H) \hookrightarrow U^q(\mathbb{R}, H) \hookrightarrow L^\infty(\mathbb{R}; H), 1 \le p < q < \infty$$

Same is true if we replace U^p, V^P_{rc} by $U^p_L, V^P_{rc,L}$

Proposition 1.5.8. [45, 27] *We have the following embedding connects U^2_L, $V^2_{rc,L}$ spaces with refined $X^{s,b}$ spaces.*

$$\dot{X}^{s,\frac{1}{2},1} \subset U^2_L \subset V^2_{rc,L} \subset \dot{X}^{s,\frac{1}{2},\infty}. \tag{1.27}$$

It is worth observing that for functions at fixed modulation, the U^2_L and V^2_L norms are equivalent from this lemma.

2.*The free solution lies in U^p spaces and U^p, V^p are stable with time truncation.*

The first claim follows immediately from the atomic structure of U^p, and the fact that each atom is a piecewise free solution.

The second one is from the fact that time truncation does not increase the norm for atoms in U^p space, and reduces variation for V^p functions.

3.*The estimates for free solution extend to functions in U^p.*

We recall the following lemma, which is in the same spirit as Proposition 1.4.4. Notice here we can not take any arbitrary Banach space Y, we need the space satisfy triangle inequality with respect to time truncation.

Lemma 1.5.9. [27, Proposition 2.19] *Let $T : H \times \cdots \times H \to Y = L^p_t \tilde{Y}$ be an n-linear operator. $\tilde{Y}$ is Banach space of functions on $\mathbb{R}^n$. Assume the following bound is true for free solutions*

$$\|T(e^{tL}\phi_1, \cdots, e^{tL}\phi_m)\|_Y \lesssim \prod_{i=1}^m \|\phi_i\|_H$$

Then we can extend T to functions in U^p_L and get

$$\|T(u_1, \cdots, u_m)\|_Y \lesssim \prod_{i=1}^m \|u_i\|_{U^p_L}.$$

Remark 1.5.10. In the practical problems, we have $Y = L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)$ or $Y = L^q_x(\mathbb{R}; L^p_{t,x'}(\mathbb{R}\times\mathbb{R}^{n-1}))$. In fact we have better extension: we can extend the operator T to functions on U^r_L, $r = \min(p, q)$.

4.*The linear estimate*

Lemma 1.5.11. *For the equation*

$$\partial_t u = Lu + f; \quad u(x, 0) = u_0,$$

we have the linear estimate

$$\|u\|_{u_L^p[0,T)} \lesssim \|u(x,0)\|_H + \sup_{v \in V_L^{p'}, \|v\|_{V_L^{p'}}=1} |\int_0^T \langle f, v\rangle_H \, dt|$$

Proof. we write down the solution in the Duhamel form

$$u(t,x) = e^{tL}u_0 + \int_0^t e^{(t-s)L} f(s) ds$$

Now let's apply a time cutoff $1_{[0,T)}$ to both side, notice that linear solution is controlled in U_L^p norm, we get

$$\|u(t,x)\|_{u_L^p[0,T)} \lesssim \|u(x,0)\|_H + \|1_{[0,T)} \int_0^t e^{(t-s)L} f(s) ds\|_{u_L^p[0,T)}$$

For $\mathcal{I}(f) = 1_{[0,T)} \int_0^t e^{(t-s)L} f(s) ds$, we can extend it continuously by $\int_0^T e^{(T-s)L} f(s) ds$, hence by duality, we get

$$\|\mathcal{I}(f)\|_{u_L^p[0,T)} \leq \sup_{v \in V^{p'}, \|v\|_{V^{p'}=1}} \int \left\langle \partial_t (e^{-tL}\mathcal{I}(f)), v \right\rangle dt \leq \sup_{v \in V^{p'}, \|v\|_{V^{p'}=1}} \int \left\langle f, e^{tL} v \right\rangle_H dt$$

Just replace $e^{tL}v$ with another $\tilde{v} \in V_L^p$, the claim follows. □

5.*Modulation projections*

Let us list the estimates concerning modulation projection.

Proposition 1.5.12. [27, Corollary 2.18] *We specify $H = L^2(\mathbb{R}^n)$ here, hence $u : \mathbb{R}_t \to \mathbb{R}_x^n$. We have*

$$\|Q_M u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_L^2}$$
$$\|Q_{\geq M} u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_L^2}.$$

Chapter 2

A-priori bounds for KdV equation below $H^{-\frac{3}{4}}$

2.1 Introduction

In this chapter, we consider the Korteweg-de Vries (KdV) equation,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = 0, \quad u : \mathbb{R} \times [0,T] \to \mathbb{R}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{2.1}$$

The equation is invariant respect to the scaling law

$$u(t,x) \to \lambda^2 u(\lambda^3 t, \lambda x),$$

which implies the scale invariance for initial data in $\dot{H}^{-\frac{3}{2}}(\mathbb{R})$. It has been shown to be locally well-posed (LWP) in H^s for $s > -\frac{3}{4}$ by Kenig, Ponce and Vega [40] using a bilinear estimate. They constructed solution on a time interval $[0, \delta]$, with δ depending on $\|u_0\|_{H^s(\mathbb{R})}$. Later, the result was extended to global well-posedness (GWP) for $s > -\frac{3}{4}$ by Colliander, Keel, Staffilani, Takaoka and Tao [13] using the I-method and almost conserved quantities. See also the references [4], [11], [38], [22], [43], [6], [42] for earlier results, and [10], [26], [44] for local and global results at the endpoint $s = -\frac{3}{4}$.

In [51], Nakanishi, Takaoka and Tsutsumi showed that the essential bilinear estimate fails if $s < -\frac{3}{4}$. In fact, Christ, Colliander and Tao [10] proved a weak form of illposedness of the $\mathbb{R}$-valued KdV equation for $s < -\frac{3}{4}$. Precisely, they showed that the solution map fails to be uniformly continuous. See [41] for the corresponding result for the $\mathbb{C}$-valued KdV equation.

On the other hand, the same question was posed in the periodic setting ($u : \mathbb{T} \times [0,T] \to \mathbb{R}$), where for $s \geq -1/2$, we have the results of LWP[40] and GWP[13]. Also, Kappeler and Topalov [36], using the inverse scattering method [21], proved GWP for inital data in $H^\beta(\mathbb{T}), \beta \geq -1$ in the sense that the solution map is C^0 globally in time. Their proof depends heavily on the complete integrability of the KdV equation. Interested readers are also referred to the work of Lax and Levermore [47], Deift and Zhou [14], [15]. There they used inverse scattering and Riemann-Hilbert methods to study the semiclassical limit of the completely integrable equations.

Concerning the KdV problem with initial data in $H^{-1}(\mathbb{R})$, there has been several results recently. In [49], Molinet showed that the solution map can not be continuously extended in $H^s(\mathbb{R})$ when $s < -1$. In [37], Kappeler, Perry, Shubin and Topalov showed that given certain assumptions on the initial data $u_0 \in H^{-1}$, there exists a global weak solution to the KdV equation. Buckmaster and Koch [8] proved the existence of weak solutions to KdV equation with H^{-1} initial data. The approach in [37] and [8] both use the Miura transformation to link the KdV equation to the mKdV equation, and the proofs involve the study of Muria map, and the existence of weak L^2 solutions to mKdV or mKdV around a soliton.

In addition, there is an interesting result by Molinet and Ribaud [50] on the initial-value problem for KdV-Burgers equation.

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) - \partial_x^2 u = 0, & t \in \mathbb{R}_+,\ x \in \mathbb{R} \text{ or } \mathbb{T}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{2.2}$$

They showed that (2.2) is GWP in the space $H^s(\mathbb{R})$ for $s \geq -1$, and ill-posed when $s < -1$ in the sense that the corresponding solution map is not C^2. This is a bit surprising since the initial-value problem for the Burgers equation

$$\begin{cases} \partial_t u + \partial_x(u^2) - \partial_x^2 u = 0, & t \in \mathbb{R}_+,\ x \in \mathbb{R}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{2.3}$$

is known to be LWP in the space $H^s(\mathbb{R})$ for $s \geq -\frac{1}{2}$, and is ill-posed in $H^s(\mathbb{R})$ for $s < -\frac{1}{2}$, see references [2] and [17]. Notice that the critical result for Burgers equation (2.3) agrees with prediction from usual scaling arguments. While KdV-Burgers equation(2.2) has no scaling invariance, the sharp result by Molinet and Ribaud $s = -1$ is lower than $s = -\frac{3}{4}$ for KdV, and $s = -\frac{1}{2}$ for Burgers equation.

From all the results mentioned before, it seems reasonable to conjecture well-posedness of KdV equation (2.1) in $H^s(\mathbb{R})$, in the range $-1 \leq s < -\frac{3}{4}$, with some continuous but not uniform continuous dependence on the initial data.

Another related topic is one dimensional cubic Nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, & u : \mathbb{R} \times [0, T] \to \mathbb{C}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{2.4}$$

The NLS has scaling invariance for initial data in $\dot{H}^{-\frac{1}{2}}(\mathbb{R})$. It has GWP for initial data in $u_0 \in L^2$ and locally in time the solution has a uniform Lipschitz dependence on the initial data in balls. But below this scale, it has been shown that uniform dependence fails [10],[41]. Koch and Tataru [46] proved an a-priori local-in-time bounds for initial data in $H^s, s \geq -\frac{1}{4}$. Similar results were previously obtained by Koch and Tataru [45] for $s \geq -\frac{1}{6}$, and by Colliander, Christ and Tao [9] for $s > -\frac{1}{12}$. These a-priori estimates ensure that the equation is satisfied in the sense of distributions even for weak limits, and hence they also obtain existence of global weak solutions without uniqueness.

Inspired by the results above, we look at the KdV equation with initial data in H^s when $s < -\frac{3}{4}$, and prove that the solution satisfies a-prori local in time H^s bounds in terms of the H^s size of the initial data, for $s \geq -\frac{4}{5}$. The advantage here is that we performed detailed analysis about the interactions in the nonlinearity, which gives us better understanding of the real obstruction towards establishing wellposedness result in low regularity.

Our main result is as follows:

Theorem 2.1.1. (A-priori bound) *Let $s \geq -\frac{4}{5}$. For any $M > 0$ there exists time T and constant C, so that for any initial data in $H^{-\frac{3}{4}}$ satisfying*

$$\|u_0\|_{H^s} < M,$$

there exists a solution $u \in C([0,T], H^{-\frac{3}{4}})$ to the KdV equation which satisfies

$$\|u\|_{L_t^\infty H^s} \leq C\|u_0\|_{H^s}. \tag{2.5}$$

Using the uniform bound (2.5), together with the uniform bound on nonlinearity

$$\|\chi_{[-T,T]}u\|_{X^s \cap X^s_{le}} + \|\chi_{[-T,T]}\partial_x(u^2)\|_{X^s \cap X^s_{le}} \lesssim \|u_0\|_{H^s},$$

which come as a byproduct of our analysis in the previous theorem, one may also prove the existence of weak solution following a similar argument as in [9].

Theorem 2.1.2. (Existence of weak solution) *Let $s \geq -\frac{4}{5}$. For any $M > 0$ there exists time T and constant C, so that for any initial data in H^s satisfying*

$$\|u_0\|_{H^s} < M,$$

there exists a weak solution $u \in C([0,T], H^s) \cap (X^s \cap X^s_{le})$ to the KdV equation which satisfies

$$\|u\|_{L_t^\infty H^s} + \|\chi_{[-T,T]}u\|_{X^s \cap X^s_{le}} + \|\chi_{[-T,T]}\partial_x(u^2)\|_{X^s \cap X^s_{le}} \leq C\|u_0\|_{H^s}.$$

Remark 2.1.3. We can always rescale the initial data and hence just need to prove the theorems in case $M \ll 1$.

We recall the Littlewood-Paley frequency projection P_λ defined in chapter 1 (1.1). For each λ we also use a spatial partition of unity on the λ^{4s+5} scale

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x), \quad \chi_j^\lambda(x) = \chi(\lambda^{-4s-5}x - j),$$

with $\chi(x) \in C_0^\infty(-1,1)$.

In order to prove the theorem, we need Banach spaces

- X^s and X^s_{le} to measure the regularity of the solution u. The first one measures dyadic pieces of the solution on a frequency dependent timescale, and the second one measures the spatially localized size of the solution on unit time scale. They are similar to the ones used by Koch and Tataru in [46].
- The corresponding Y^s and Y^s_{le} to measure the regularity of the nonlinear term.
- Energy spaces
$$\|u\|^2_{l^2_\lambda L^\infty_t H^s} = \sum_{\lambda\geq 1} \lambda^{2s}\|u_\lambda\|^2_{L^\infty_t L^2_x},$$
and a local energy space
$$\|u\|^2_{l^2_\lambda l^\infty_j L^2_t H^{-s-\frac{3}{2}}} = \sum_{\lambda\geq 1} \lambda^{-2s-5}\sup_j \|\chi^\lambda_j \partial_x u_\lambda\|^2_{L^2_{x,t}}.$$

With the spaces above, we will prove the following three propositions.
The first one is about the linear equation.

Proposition 2.1.4. *The following energy estimates hold for* (2.1):
$$\|u\|_{X^s} \lesssim \|u\|_{l^2_\lambda L^\infty_t H^s} + \|(\partial_t + \partial_x^3)u\|_{Y^s}, \tag{2.6}$$
$$\|u\|_{X^s_{le}} \lesssim \|u\|_{l^2_\lambda l^\infty_j L^2_t H^{-s-\frac{3}{2}}} + \|(\partial_t + \partial_x^3)u\|_{Y^s_{le}}. \tag{2.7}$$

The second one controls the nonlinearity.

Proposition 2.1.5. *Let $s > -1$ and $u \in X^s \cap X^s_{le}$ be a solution to equation* (2.1), *then*
$$\|\partial_x(u^2)\|_{Y^s\cap Y^s_{le}} \lesssim \|u\|^2_{X^s\cap X^s_{le}} + \|u\|^3_{X^s\cap X^s_{le}}. \tag{2.8}$$

Finally, to close the argument we need to propagate the energy norms.

Proposition 2.1.6. *Let $s \geq -\frac{4}{5}$ and u be a solution to the* (2.1) *with*
$$\|u\|_{l^2_\lambda L^\infty_t H^s} \ll 1.$$
Then we have the bound for energy norm
$$\|u\|_{l^2_\lambda L^\infty_t H^s} \lesssim \|u_0\|_{H^s} + \sum_{k=3}^{6} \|u\|^k_{X^s\cap X^s_{le}}, \tag{2.9}$$
and respectively the local energy norm
$$\|u\|_{l^2_\lambda l^\infty_j L^2_t H^{-s-\frac{3}{2}}} \lesssim \|u_0\|_{H^s} + \sum_{k=3}^{6} \|u\|^k_{X^s\cap X^s_{le}}. \tag{2.10}$$

We organize our paper as follows: In section 2.2, we will define the spaces X^s, X^s_{le}, respectively Y^s, Y^s_{le}, and establish the linear mapping properties in Proposition 2.1.4. In section 2.3 we discuss the linear and bilinear Strichartz estimates for free solutions, and collect some useful estimates related to our spaces. In section 2.4 we control the nonlinearity as in Proposition 2.1.5. In sections 2.5, 2.6 we use a variation of the I-method to construct a quasi-conserved energy functional and compute its behavior along the flow, thus proving Proposition 2.1.6.

Now we end this section by showing that the three propositions imply Theorem 2.1.1.

Proof. Since $u_0 \in H^{-\frac{3}{4}}$, we can solve the equation iteratively to get a solution up to time 1, which implies that $u \in l^2_\lambda L^\infty_t H^s$ and also that $u \in X^s \cap X^s_{le}$, because the space we use has the nesting property $X^{s_1} \subset X^{s_2}, s_1 < s_2$, same for $l^2_\lambda L^\infty_t H^s$ and X^s_{le}.

Then we use a continuity argument. Suppose ϵ is a small constant and $\|u_0\|_{H^s(\mathbb{R})} < \epsilon$. Take a small δ, so that $\epsilon \ll \delta \ll 1$, denote

$$A = \{T \in [0,1];\ \|u\|_{l^2_\lambda L^\infty_t H^s([0,T]\times\mathbb{R})} \leq 2\delta, \quad \|u\|_{X^s \cap X^s_{le}([0,T]\times\mathbb{R})} \leq 2\delta\}$$

and we just need to prove $A = [0,1]$. Clearly A is not empty and $0 \in A$. We need to prove that it is closed and open.

From definition in the next section, we can see that the norms used in A are continuous with respect to T, so A is closed.

Secondly, if $T \in A$, we have by proposition 2.1.6

$$\|u\|_{l^2_\lambda L^\infty_t H^s([0,T]\times\mathbb{R})} \lesssim \epsilon + \delta^3,$$

and by proposition 2.1.4 and 2.1.5, we have

$$\|u\|_{X^s \cap X^s_{le}([0,T]\times\mathbb{R})} \lesssim \epsilon + \delta^2 + \delta^3.$$

So by taking ϵ and δ sufficiently small, we can conclude that

$$\|u\|_{l^2_\lambda L^\infty_t H^s([0,T]\times\mathbb{R})} \leq \delta, \quad \|u\|_{X^s \cap X^s_{le}([0,T]\times\mathbb{R})} \leq \delta.$$

Since the norms are continuous with respect to T, it follows that a neighborhood of T is in A. Hence we proved Theorem 2.1.1. □

2.2 Function spaces

The idea here follows the work of Koch and Tataru [45][46]. We begin with some heuristic argument: If the initial data in (2.1) has norm $\|u_0\|_{H^{-\frac{3}{4}}} \leq 1$, then the equation can be solved iteratively up to time 1. Now when taking the same problem with initial data $u_0 \in H^s, s < -\frac{3}{4}$, localized at frequency λ, the initial data will have norm $\|u\|_{H^{-\frac{3}{4}}} \leq \lambda^{-s-\frac{3}{4}}$. Now if we rescale it to have $H^{-\frac{3}{4}}$ norm 1, we see that the evolution will still be described by linear dynamics on time intervals of size λ^{4s+3}. So we decompose our solution into frequency pieces $u = \sum_{\lambda\geq 1} u_\lambda$ and measure each piece uniformly in size λ^{4s+3} time intervals.

Another important idea is to look at waves of frequency λ travelling with speed λ^2, so for time λ^{4s+3}, it travels in spatial region of size λ^{4s+5}. So we also decompose the space into a grid of size λ^{4s+5} by using the partition of unity

$$1 = \sum_{j\in\mathbb{Z}} \chi_j^\lambda(x).$$

$\chi_j^\lambda(x)$ is defined as before, and it's easy to see that the spatial scales increase with λ.

Denote $\eta_I(t)$ as sharp time cutoff with respect to any time interval I. Let I_λ be a time interval of size λ^{4s+3}, then we use $\eta_\lambda(t)$ or η_λ as a simplified notation for $\eta_{I_\lambda}(t)$. And $\chi^\lambda(x)$ is the smooth space cutoff with respect to spatial intervals of size λ^{4s+5} as before.

Define $|D|^\alpha$ to be the multiplier operator with Fourier multiplier $|\xi|^\alpha$. We use the convention that $f \in |D|^{-s}X \Leftrightarrow \|f\|^2 = \sum \lambda^{2s}\|f_\lambda\|_X^2 < \infty$ in our definitions.

Definition 2.2.1. The spaces we use contain the following elements:

(i) Given an interval $I = [t_0, t_1]$, we define the space

$$\|\phi\|^2_{X^{0,1}[I]} = \|\phi(t_0)\|^2_{L^2} + |I|\|(\partial_t + \partial_x^3)\phi\|^2_{L^2[I]},$$

$$\|\phi\|^2_{X^1[I]} = \sum_\lambda \lambda^{2s}\|\phi_\lambda\|^2_{X^{0,1}[I]}.$$

$X^1[I]$ is used to control the low modulation part of the solution in a classical space, which is extendable on the real line.

(ii) We use sums of spaces, i.e. $\|u\|_{A+B} = \inf\{\|u_1\|_A + \|u_2\|_B, u = u_1 + u_2\}$ to define

$$Z = (X^{-3-4s,2s+2}_{\tau=\xi^3} + |D|^{-2s-2}X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}) \cap |D|L^\infty_{t,x}.$$

Z will always be used for very high modulations ($\geq |\xi|^3$), i.e. in what are called the elliptic region.

(iii) The space S is defined by putting high and low modulation in different spaces.

$$\|u_\lambda\|_S = \lambda^{3s+\frac{3}{2s}+\frac{11}{2}}\|Q_{\sigma\leq\lambda^{4+\frac{3}{2s}}}u_\lambda\|_{L^2_{x,t}} + \|Q_{\lambda^{4+\frac{3}{2s}}\leq\sigma\leq\frac{1}{10}\lambda^3}u_\lambda\|_{X^{-s,1+s}_{\tau=\xi^3}} + \|Q_{\geq\frac{1}{10}\lambda^3}u_\lambda\|_Z.$$

The good thing here is space S is stable with respect to sharp time truncations, the L^2 structure deals with the tails when multiplying by a time-interval cutoff.
In particular, we have

$$\|\eta_\lambda(t)u_\lambda\|_S \lesssim \|u_\lambda\|_S.$$

(iv) Let $X_\lambda[I] = X^1[I] + S[I]$. Now we can define X^s norm in a time interval I by measuring the dyadic parts of u on small frequency-dependent time scales

$$\|u\|^2_{X^s[I]} = \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset I} \|\eta_J(t) u_\lambda\|^2_{X_\lambda[J]},$$

X^s_{le} measures the spatially localized size of the solution on the unit time scale

$$\|u\|^2_{X^s_{le}[I]} = \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset I} \|\chi^\lambda_j(x) \eta_J(t) u_\lambda\|^2_{X_\lambda[J]}.$$

(v) Correspondingly, we have the space Y^s and Y^s_{le}

$$\|u\|^2_{Y^s[I]} = \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset I} \|\eta_J(t) u_\lambda\|^2_{Y_\lambda[J]},$$

$$\|u\|^2_{Y^s_{le}[I]} = \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset I} \|\chi^\lambda_j(x) \eta_J(t) u_\lambda\|^2_{Y_\lambda[J]}.$$

Here

$$Y_\lambda[I] = |D_x|^{-s} |I|^{-\frac{1}{2}} L^2 + DS[I],$$

$DS = \{f = (\partial_t + \partial_x^3)u;\ u \in S\}$ with the induced norm and $DS[I] = \{f|_I, f \in DS\}$.

Through our paper, we will mostly drop the interval I in the notation if $I = [0, 1]$.

Remark 2.2.2. We look at each of the spaces in detail.

1. $X^1[I]$ is not stable with respect to sharp time truncation as it would cause jumps at both ends. Also in order to talk about modulation, we need to extend functions so that they are defined on the real line. To fix the problem, we define

$$\|\phi\|^2_{X^{0,1}_I} = \|\phi(t_0)\|^2_{L^2} + |I| \|(\partial_t + \partial_x^3)\phi\|^2_{L^2_{t,x}},$$

$$\|\phi\|^2_{X^1_I} = \sum_\lambda \lambda^{2s} \|\phi_\lambda\|^2_{X^{0,1}_I}.$$

Now take any function $u \in X^1[I]$, denote $u_E = \theta(t)\widetilde{u}$, where $\widetilde{u}$ is the extension of u by free solutions with matching data at both ends and $\theta(t)$ is a smooth cutoff on a neighborhood of I. Clearly, $\|u\|_{X^1[I]} = \|u_E\|_{X^1_I}$, and when we talk about function $u \in X^1[I]$, we always mean u_E.

While $S[I]$ is stable with sharp time cutoff, $DS[I]$ is not. We can extend functions in $S[I]$ by 0 outside the interval. And from the definition, functions in $DS[I]$ always come from interval restriction of functions in DS, which are defined on the real line.

2. The space $X^1[I]$ is compatible with solutions to the homogeneous equation. Namely for any smooth time cutoff $\eta(t)$, we can prove

$$\|\eta(t)e^{t\partial_x^3}u_0\|_{X^1[I]} \lesssim \|u_0\|_{H^s},$$

It is also compatible with energy estimates

$$\|u\|_{L_t^\infty(I;H^s)} \lesssim \|u\|_{X^1[I]}.$$

3. We will ignore the subscript notation $\tau = \xi^3$ in the $X^{s,b}_{\tau=\xi^3}$ space except for the special curve $\tau = \frac{1}{4}\xi^3$ which arises when two high frequency wave interact and generate an almost equally high frequency.

4. Since we are using sums of spaces, it is interesting to compare the norms of these spaces. We note the following facts by Bernstein inequality.

$$\begin{cases} \|u_\lambda\|_{X_\lambda[I_\lambda]} \approx \|u_\lambda\|_{X^1[I_\lambda]}, & \text{when } |\tau - \xi^3| \lesssim \lambda^{4+\frac{3}{2s}}, \\ \|u_\lambda\|_Z \approx \|u_\lambda\|_{|D|^{-2s-2}X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3} \cap |D|L^\infty_{t,x}}, & \text{when } |\tau - \frac{1}{4}\xi^3| \leq \frac{1}{10}\lambda^3, \\ \|u_\lambda\|_Z \approx \|u_\lambda\|_{X^{-3-4s,2s+2}\cap|D|L^\infty_{t,x}} \approx \|u_\lambda\|_{X^{-s,1+s}}, & \text{when } |\tau - \xi^3| \approx \frac{1}{10}\lambda^3. \end{cases}$$

The X^1 and S norm balance at modulation $|\tau - \xi^3| \approx \lambda^{4+\frac{3}{2s}}$, which is also where we split S into the L^2 structure and $X^{-s,1+s}$. Hence whenever we split into an X^1 and an S part, we always assume the S part have modulation larger than $\lambda^{4+\frac{3}{2s}}$ (which is larger than λ^2). The same applies for $|D|^{-s}|I|^{-\frac{1}{2}}L^2$ and DS.

The third equality is because when modulation is around $\frac{1}{10}\lambda^3$, the Z norm is in fact $X^{-3-4s,2s+2} \cap |D|L^\infty_{t,x}$. Using Bernstein, we can see that it matches with $X^{-s,1+s}$.

Now let us prove Proposition 2.1.4.

Proof. It suffices to prove the Proposition for a fixed dyadic frequency λ. We restrict our attention to time interval $J = [a, b]$ with size λ^{4s+3}, and we need to prove that

$$\|u_\lambda\|_{X_\lambda[J]} \lesssim \|u_\lambda\|_{L_t^\infty H^s} + \|f_\lambda\|_{Y_\lambda[J]}, \qquad (\partial_t + \partial_x^3)u_\lambda = f_\lambda. \tag{2.11}$$

We now split f_λ into two components

$$f_\lambda = f_{1,\lambda} + f_{2,\lambda}, \quad f_{1,\lambda} \in L^2, \quad f_{2,\lambda} \in DS.$$

Pick v_λ such that $(\partial_t + \partial_x^3)v_\lambda = f_{2,\lambda}$, $\|f_{2,\lambda}\|_{DS} = \|v_\lambda\|_S$. (or $(v_\lambda^i)_1^\infty$ with $\|v_\lambda^i\|_S \to \|f_2\|_{DS}$.)

Then we have $(\partial_t + \partial_x^3)(u_\lambda - v_\lambda) = f_{1,\lambda}$.

Notice the fact that, for any function ϕ and time interval $I = [t_0, t_1]$

$$\|\phi_\lambda\|_{X^1[I]} \approx \lambda^s|I|^{-\frac{1}{2}}\|\phi_\lambda\|_{L^2_{t,x}[I]} + \lambda^s|I|^{\frac{1}{2}}\|(\partial_t + \partial_x^3)\phi_\lambda\|_{L^2_{t,x}[I]}.$$

So we get

$$\begin{aligned}\|u_\lambda\|_{X_\lambda[J]} &\lesssim \|u_\lambda - v_\lambda\|_{X^1[J]} + \|v_\lambda\|_{S[J]} \\ &\lesssim \lambda^s |J|^{-\frac{1}{2}} \|u_\lambda - v_\lambda\|_{L^2_{t,x}[J]} + \|f_{1,\lambda}\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2[J]} + \|f_{2,\lambda}\|_{DS[J]} \\ &\lesssim \|u_\lambda\|_{L^\infty_t H^s} + \|f_{1,\lambda}\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2[J]} + \|f_{2,\lambda}\|_{DS[J]}.\end{aligned}$$

Here we used the fact

$$\lambda^s |J|^{-\frac{1}{2}} \|v_\lambda\|_{L^2_{t,x}[J]} \lesssim \|v_\lambda\|_{S[J]},$$

which can be checked easily.

For the second estimate about local energy space, we can still localize to fixed frequency, and need to show that

$$\sup_j \sum_{J\subset I}^{|J|=\lambda^{4s+3}} \|\chi_j^\lambda u_\lambda\|^2_{X_\lambda[J]} \lesssim \sup_j \sum_{J\subset I}^{|J|=\lambda^{4s+3}} (\lambda^{-2s-5} \|\chi_j^\lambda \partial_x u_\lambda\|^2_{L^2_{t,x}[J]} + \|\chi_j^\lambda f_\lambda\|^2_{Y_\lambda[J]}) \tag{2.12}$$

To prove the estimate, let us consider the inhomogeneous problem on interval $J = [a, b]$ of size $|J| = \lambda^{4s+3}$,

$$(\partial_t + \partial_x^3) u_\lambda^k = P_\lambda \chi_k^\lambda f_\lambda, \quad u_\lambda^k(a) = \chi_k^\lambda u_{0,\lambda}$$

and prove that

$$\|\chi_j^\lambda u_\lambda^k\|_{X_\lambda[J]} \lesssim \langle j-k\rangle^{-N} (\lambda^s |J|^{-\frac{1}{2}} \|\chi_k^\lambda u_\lambda^k\|_{L^2_{t,x}} + \|\chi_k^\lambda f_\lambda\|_{Y_\lambda[J]}). \tag{2.13}$$

When $j \approx k$, it is essentially the same as (2.11). Notice in the process of proving (2.11), we get

$$\|u_\lambda\|_{X_\lambda[J]} \lesssim \lambda^s |J|^{-\frac{1}{2}} \|u_\lambda\|_{L^2_{t,x}[J]} + \|f_\lambda\|_{Y_\lambda[J]}.$$

When $|j-k| \gg 1$, it follows from the rapid decay estimate on the kernel K_{jk} of $\chi_j^\lambda e^{t\partial_x^3} P_\lambda \chi_k^\lambda$:

$$|K_{jk}(t,x,y)| \lesssim \lambda^{-N} \langle j-k\rangle^{-N}, \quad |t| \le \lambda^{4s+3}.$$

Since $u_\lambda = \sum_k u_\lambda^k$, so we sum up k in (2.13), and get

$$\|\chi_j^\lambda(x) u_\lambda\|_{X_\lambda[J]} \lesssim \sum_k \langle j-k\rangle^{-N} (\lambda^s |J|^{-\frac{1}{2}} \|\chi_k^\lambda u_\lambda\|_{L^2_{t,x}[J]} + \|\chi_k^\lambda(x) f_\lambda\|_{Y_\lambda[J]}),$$

which is equivalent to (2.12). □

2.3 Linear and bilinear estimate

In this section, we look at solutions to the Airy equation (1.8), which satisfy the Strichartz, local smoothing and maximal function estimates, see theorem 1.3.3 1.3.6.

Once we have estimates for linear equation, we can extend it to X^1.

Corollary 2.3.1. *Let* (q, r) *be a Strichartz pair as in relation* (1.14). *Then we have*

$$\|\eta_I(t)u_\lambda\|_{L_t^q L_x^r} \lesssim \lambda^{-\frac{1}{q}-s}\|u_\lambda\|_{X^1[I]}, \tag{2.14}$$

Also, the following smoothing estimate and maximal function estimate hold

$$\|\eta_I(t)u_\lambda\|_{L_x^\infty L_t^2} \lesssim \lambda^{-1-s}\|u_\lambda\|_{X^1[I]}, \tag{2.15}$$

$$\|\eta_I(t)u_\lambda\|_{L_x^4 L_t^\infty} \lesssim \lambda^{\frac{1}{4}-s}\|u_\lambda\|_{X^1[I]}, \tag{2.16}$$

Proof. The results follow by expanding u_λ via Duhamel's formula.

If $(\partial_t + \partial_x^3)u_\lambda = f$, then

$$u_\lambda = e^{-t\partial_x^3}u_\lambda(t_0) + \int_{t_0}^t e^{-(t-s)\partial_x^3}f(s)ds.$$

From Strichartz estimate, and its dual form - the inhomogeneous Strichartz estimate, see Theorem 2.3 in Tao [56] section 2.3, and we get

$$\begin{aligned}\|\eta_I(t)u_\lambda\|_{L_t^q L_x^r} &\lesssim \|\eta_I(t)e^{-t\partial_x^3}u_\lambda(t_0)\|_{L_t^q L_x^r} + \lambda^{-\frac{1}{q}}\|\eta_I(t)f\|_{L_t^1 L_x^2} \\ &\lesssim \lambda^{-\frac{1}{q}-s}\|u_\lambda\|_{X^1[I]}.\end{aligned}$$

We can prove the local smoothing and maximal estimate in the same way. □

We will also need the bilinear estimate as in [24].

Proposition 2.3.2. *Let* $I_\pm^s$ *be defined by its Fourier transform in the space variable:*

$$\mathcal{F}_x I_\pm^s(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi} |\xi_1 \pm \xi_2|^s \hat{f}(\xi_1)\hat{g}(\xi_2)d\xi_1.$$

Assume u, v *be two solutions to the Airy equation with initial data* u_0, v_0. *Then we have the bilinear estimate*

$$\|I_+^{\frac{1}{2}}I_-^{\frac{1}{2}}(u,v)\|_{L_{x,t}^2} \lesssim \|u_0\|_{L_x^2}\|v_0\|_{L_x^2}. \tag{2.17}$$

Proof. For a solution to the Airy equation, we can write down its Fourier transform,

$$\widetilde{u} = \delta(\tau - \xi^3)\hat{u}_0, \qquad \widetilde{v} = \delta(\tau - \xi^3)\hat{v}_0.$$

Then

$$I_+^{\frac{1}{2}}\widetilde{I_-^{\frac{1}{2}}(u,v)}(\tau,\xi) = \int_{\xi_1+\xi_2=\xi} |\xi_1+\xi_2|^{\frac{1}{2}}|\xi_1-\xi_2|^{\frac{1}{2}}\hat{u}_0(\xi_1)\hat{v}_0(\xi_2)\delta(\tau-\xi_1^3-\xi_2^3)\,d\xi_1.$$

Let us make change of variable $\xi_1+\xi_2=\xi, \tau-\xi_1^3-\xi_2^3=\eta$.
With τ, ξ fixed, we have

$$d\xi_1 = \frac{1}{3|\xi_1+\xi_2||\xi_1-\xi_2|}d\eta,$$

hence we get

$$I_+^{\frac{1}{2}}\widetilde{I_-^{\frac{1}{2}}(u,v)}(\tau,\xi) = \frac{1}{3|\xi_1+\xi_2|^{\frac{1}{2}}|\xi_1-\xi_2|^{\frac{1}{2}}}\hat{u}_0(\xi_1)\hat{v}_0(\xi_2).$$

Now ξ_1, ξ_2 are solutions to

$$\xi_1+\xi_2=\xi, \quad \xi_1^3+\xi_2^3=\tau,$$

So we have

$$d\tau d\xi = 3|\xi_1^2-\xi_2^2|d\xi_1 d\xi_2,$$

and it follows

$$\|I_+^{\frac{1}{2}}I_-^{\frac{1}{2}}(u,v)\|_{L^2_{x,t}} \lesssim \|u_0\|_{L^2_x}\|v_0\|_{L^2_x}.$$

□

Remark 2.3.3. Propostion 2.3.2 gives us the usual L^2 estimate on product of two free solutions whenever they have frequency separation, i.e. $|\xi_1 \pm \xi_2| \neq 0$, $\xi_1 \in supp\,\hat{u}$, $\xi_2 \in supp\,\hat{v}$.

It is very useful especially when we localize the solutions into dyadic frequency pieces, then the operators $I_\pm^s$ can be simply replaced by scaler multiplication. We have the following cases:

When $|\xi_1| \approx \mu, |\xi_2| \approx \lambda, \mu \gg \lambda$, then we get

$$\|uv\|_{L^2_{t,x}} \lesssim \mu^{-1}\|u_0\|_{L^2}\|v_0\|_{L^2}. \tag{2.18}$$

When $|\xi_1| \approx |\xi_2| \approx \lambda$, and ξ_1, ξ_2 have opposite sign, so the output has frequency $|\xi_1+\xi_2| \approx \alpha \lesssim \lambda$, then we get

$$\|uv\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\|u_0\|_{L^2}\|v_0\|_{L^2}. \tag{2.19}$$

In case $|\xi_1| \approx |\xi_2| \approx \lambda$, but ξ_1, ξ_2 have same sign, the output lies close to a new curve $\tau = \frac{1}{4}\xi^3$. Following the idea in [40], we have the following Proposition.

Proposition 2.3.4. *Assume u,v are two smooth solutions to the Airy equation with initial data u_0, v_0, localized at frequencies about the comparable size and also the same sign, and I be an interval of size less than 1, then we have the following estimate*

$$\|\eta_I(t)uv\|_{X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}} \lesssim \|u_0\|_{L^2_x}\|v_0\|_{L^2_x}. \tag{2.20}$$

Proof. The proof is essentially the same as Proposition 2.3.2. There we first take two frequency really close, but have small separation, i.e. $|\xi_1 - \xi_2| \geq \epsilon$, so that all the calculation are still true, and we get the estimate (2.17). Notice that

$$\xi_1 + \xi_2 = \xi, \quad \xi_1^3 + \xi_2^3 = \tau.$$

So we solve for ξ_1, ξ_2 and get $|(\tau - \frac{1}{4}\xi^3)\xi|^{\frac{1}{2}} = \frac{3}{4}|(\xi_1+\xi_2)(\xi_1-\xi_2)|$, which is exactly the multiplier we have in the space $\dot{X}^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}$. Then we take the limit as $\epsilon \to 0$, and the norm converges as long as we are considering smooth functions. So we get

$$\|uv\|_{\dot{X}^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}} \lesssim \|u_0\|_{L^2_x}\|v_0\|_{L^2_x}. \tag{2.21}$$

To pass to nonhomogeneous space, notice the following estimate

$$\|\eta_I(t)f\|_{L^2_{t,x}} \lesssim |I|^{\frac{1}{4}}\|f\|_{L^4_t L^2_x} \lesssim \|f\|_{\dot{X}^{0,\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}}.$$

The last inequality is by Sobolev embedding. □

In Proposition 2.3.6, we will extend these estimates (2.18) (2.19) from free solutions to functions in X^1.

Now we list some L^p estimates, which are mostly straightforward.

Proposition 2.3.5. *When $-1 \leq s \leq -\frac{3}{4}$, we have the following estimates.*

$$\|\eta_I(t)Q_\sigma u_\lambda\|_{L^2_{x,t}} \lesssim \sigma^{-1}\lambda^{-s}|I|^{-\frac{1}{2}}\|u_\lambda\|_{X^1}, \tag{2.22}$$

$$\|Q_{\lambda^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \lambda^3} u_\lambda\|_{L^2_{x,t}} \lesssim \lambda^{-3s-\frac{3}{2s}-\frac{11}{2}}\|u_\lambda\|_{X^{-s,1+s}} \lesssim \lambda^{-2-s}\|u_\lambda\|_{X^{-s,1+s}}, \tag{2.23}$$

$$\|Q_{\lambda^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \lambda^3} u_\lambda\|_{L^\infty_{x,t}} \lesssim \lambda^{-2s-1}\|u_\lambda\|_{X^{-s,1+s}}, \tag{2.24}$$

$$\|u_\lambda\|_{L^3_{x,t}} \lesssim \lambda^{-\frac{1}{3}-2s-2}\|u_\lambda\|_{|D|^{-2s-2}X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}}, \tag{2.25}$$

$$\|Q_{\gtrsim\lambda^3} u_\lambda\|_{L^2_{t,x}} \lesssim \lambda^{-2s-3}\|u_\lambda\|_{X^{-3-4s,2s+2}}, \tag{2.26}$$

$$\|Q_{\gtrsim\lambda^3} u_\lambda\|_{L^q_{t,x}} \lesssim \lambda^{1-\frac{4(s+2)}{q}}\|u_\lambda\|_{X^{-3-4s,2s+2}\cap|D|L^\infty_{t,x}}, \ 2 \leq q \leq p, \tag{2.27}$$

$$\|Q_{\gtrsim\lambda^3}u_\lambda\|_{L^3_{t,x}} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\|u_\lambda\|_Z, \tag{2.28}$$

$$\|Q_{\gtrsim\lambda^3}u_\lambda\|_{L^6_{t,x}} \lesssim \lambda^{-\frac{2}{3}(s+1)+\frac{1}{3}}\|u_\lambda\|_Z. \tag{2.29}$$

Proof. The proofs are mostly simple. (2.22) is by definition combined with the size of the interval. (2.23) (2.24)(2.26) are consequences of Bernstein inequality. (2.27) is by interpolating the L^2 estimate with L^p.

The only nontrivial one is (2.25), Similar to [57], we look at the operator $S(\sigma)$ defined by multiplier $e^{\sigma^2}\Gamma(\sigma)\frac{1}{(\tau-\frac{1}{4}\xi^3\pm i0)^\sigma}$, where $\Gamma(\sigma)$ is the complex valued Gamma-function. We claim that

$$S(0+iy)P_\lambda : L^2_{t,x} \to L^2_{x,t},$$
$$S(\frac{3}{2}+iy)P_\lambda : L^1_{x,t} \to L^\infty_{t,x}.$$

Let us prove the second one by computing its Fourier inversion,

$$\mathcal{F}^{-1}\frac{\theta_\lambda(\xi)}{(\tau-\frac{1}{4}\xi^3\pm i0)^{\frac{3}{2}+iy}} = \mathcal{F}^{-1}(\tau\pm i0)^{-\frac{3}{2}-iy}\cdot\mathcal{F}^{-1}[\theta_\lambda(\xi)\delta_{\tau=\frac{1}{4}\xi^3}].$$

$\theta_\lambda(\xi)$ is some smooth bump function around $\xi=\lambda$, which we used to define P_λ.

From direct computation, we have

$$\|e^{(\frac{3}{2}+iy)^2}\Gamma(\frac{3}{2}+iy)\mathcal{F}^{-1}(\tau\pm i0)^{-\frac{3}{2}-iy}\|_{L^\infty} \lesssim t^{\frac{1}{2}},$$

and by stationary phase we get

$$\|\mathcal{F}^{-1}[\theta_\lambda(\xi)\delta_{\tau=\frac{1}{4}\xi^3}]\|_{L^\infty} = \|\int\theta_\lambda(\xi)e^{ix\xi+i\frac{1}{4}t\xi^3}\,d\xi\|_{L^\infty} \lesssim (t\lambda)^{-\frac{1}{2}}.$$

Combining them together, we get

$$\|S(\frac{3}{2}+iy)P_\lambda\|_{L^1_{x,t}\to L^\infty_{t,x}} \lesssim \lambda^{-\frac{1}{2}}.$$

Also notice the trivial bound

$$\|S(0+iy)P_\lambda\|_{L^2_{t,x}\to L^2_{x,t}} \lesssim C.$$

We interpolate to get

$$\|S(\frac{1}{2}+iy)P_\lambda\|_{L^{\frac{3}{2}}_{x,t}\to L^3_{x,t}} \lesssim \lambda^{-\frac{1}{6}}.$$

Define the operator T by multiplier $\frac{1}{(\tau-\frac{1}{4}\xi^3\pm i0)^{\frac{1}{4}}}$, and $S(\frac{1}{2})=cTT^*$, $c=e^{\frac{1}{4}}\Gamma(\frac{1}{2})$. So by the TT^* argument [58] [56], we have $\|TP_\lambda\|_{L^2_{t,x}\to L^3_{x,t}} \lesssim \lambda^{-\frac{1}{12}}$.
Hence we get

$$\|u_\lambda\|_{L^3_{x,t}} \lesssim \lambda^{-\frac{1}{3}-2s-2}\|u_\lambda\|_{|D|^{-2s-2}X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}}.$$

If we take $q = 3$ in (2.27), combining with (2.25) we get (2.28).

If we take $q = 6$ in (2.27), also compare with

$$\|Q_{\sigma\approx\lambda^3} u_\lambda\|_{L^6_{t,x}} \lesssim (\lambda\sigma)^{\frac{1}{6}} \|u_\lambda\|_{L^3_{t,x}} \lesssim \lambda^{-2s-2+\frac{1}{3}} \|u_\lambda\|_{|D|^{-2s-2} X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}},$$

we get (2.29). From Remark 2.2.2(4), we only put pieces in $|D|^{-2s-2} X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}$ norm when it lies close to the special curve, and in that case its modulation is close to λ^3. □

Also, let us collect some bilinear estimates that will be very useful in the next section.

Proposition 2.3.6. *For $\mu \gg \lambda \geq \alpha$, as before $\eta_\lambda(t)$ is the sharp cutoff on time interval I_λ of size $|I_\lambda| = \lambda^{4s+3}$. We have the following estimates:*

$$\|\eta_\mu(t) u_\mu v_\lambda\|_{L^2_{t,x}} \lesssim \mu^{-1-s} \lambda^{-s} \|u_\mu\|_{X^1[I_\mu]} \|v_\lambda\|_{X^1[I_\lambda]}, \tag{2.30}$$

$$\|\eta_\mu(t) P_{\approx\lambda}(u_\mu v_\mu)\|_{L^2_{t,x}} \lesssim \mu^{-\frac{1}{2}-2s} \lambda^{-\frac{1}{2}} \|u_\mu\|_{X^1[I_\mu]} \|v_\lambda\|_{X^1[I_\mu]}, \tag{2.31}$$

$$\|\eta_\lambda(t) u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \max\{\lambda^{-\frac{1}{3}-2s-2} \alpha^{-\frac{1}{6}-s}, \lambda^{-2-s} \alpha^{\frac{1}{2}-s}\} \|u_\lambda\|_{S[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]}. \tag{2.32}$$

Proof. For (2.30) and (2.31), we expand u, v via Duhamel's formula, and apply the bilinear estimates (2.18) (2.19) repeatedly. See [12] Lemma 3.4 for a similar proof.

For (2.32), we still break u_λ by the size of modulation, and see that the worst estimate comes when $u_\lambda \in |D|^{-2s-2} X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3} \cap |D| L^\infty_{t,x}$. Then we use L^3 for u_λ, and L^6 for v_α.

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \|\eta_\lambda u_\lambda\|_{L^2_{t,x}} \|\eta_\alpha v_\alpha\|_{L^\infty_{t,x}} \lesssim \lambda^{-2s-3} \alpha^{\frac{1}{2}-s} \|u_\lambda\|_{X^{-3-4s,2s+2}[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]},$$

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \|\eta_\lambda u_\lambda\|_{L^3_{t,x}} \|\eta_\alpha v_\alpha\|_{L^6_{t,x}} \lesssim \lambda^{-\frac{1}{3}-2s-2} \alpha^{-\frac{1}{6}-s} \|u_\lambda\|_{|D|^{-2s-2} X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]}.$$

By comparing the coefficients in the estimates above, we get

$$\|\eta_\lambda(t) u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{3}-2s-2} \alpha^{-\frac{1}{6}-s} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]}. \tag{2.33}$$

If we also consider the case of $u_\lambda \in X^{-s,1+s}$,

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \|\eta_\lambda u_\lambda\|_{L^2_{t,x}} \|\eta_\alpha v_\alpha\|_{L^\infty_{t,x}} \lesssim \lambda^{-2-s} \alpha^{\frac{1}{2}-s} \|u_\lambda\|_{X^{-s,1+s}[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]} \tag{2.34}$$

and compare the coefficients, we get (2.32). □

Remark 2.3.7. We don't have a good L^2 estimate on the product of two pieces both in S. But we will still list here some of the cases, which are manageable.

When $u_\lambda, v_\alpha \in X^{-s,1+s}$, bound u_λ in $L^2_{t,x}$, and v_α in $L^\infty_{t,x}$.

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \lambda^{-2-s}\alpha^{-2s-1}\|u_\lambda\|_{X^{-s,1+s}[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}. \tag{2.35}$$

When $u_\lambda, v_\alpha \in Z$, bound u_λ in L^3, and v_α in L^6, we get

$$\|\eta_\lambda(t) u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]}. \tag{2.36}$$

When $u_\lambda \in Z, v_\alpha \in X^{-s,1+s}$, bound u_λ in L^3, and v_α in L^6 which comes from Bernstein together with L^2 bound, we get

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\max\{\alpha^{-s-1}, \alpha^{-\frac{5}{3}-2s}\}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}. \tag{2.37}$$

The above three inequalities imply that

$$\|\eta_\lambda(t) u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}}\|u_\lambda\|_{S[I_\lambda]}\|v_\alpha\|_{S[I_\alpha]}. \tag{2.38}$$

is true except for the case $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$, which corresponds to case the high-frequency low modulation interacting with low-frequency high modulation.

To estimate the case $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$, use L^2 on u_λ, L^∞ on v_α, and we get

$$\begin{aligned}\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} &\lesssim \|\eta_\lambda u_\lambda\|_{L^2_{t,x}}\|\eta_\lambda v_\alpha\|_{L^\infty_{t,x}} \\ &\lesssim \lambda^{-2-s}\alpha\|u_\lambda\|_{X^{-s,1+s}[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]}.\end{aligned} \tag{2.39}$$

The bound here is worse than the one in (2.38).

2.4 Estimating the nonlinearity

The goal of this part is to estimate the nonlinearity as in Proposition 2.1.5. Since functions in $X^s \cap X^s_{le}$ have different piece measured differently, we show that the estimate

$$\|\partial_x(uv)\|_{Y^s\cap Y^s_{le}} \lesssim \|u\|_{X^s\cap X^s_{le}}\|v\|_{X^s\cap X^s_{le}} \tag{2.40}$$

is almost true except for one special case.

Let us expand the estimate (2.40), the energy norm takes the form

$$\begin{aligned}&\sum_{\lambda\geq 1}\sup_{|J|=\lambda^{4s+3}, J\subset[0,1]}\|\eta_J(t)P_\lambda(\partial_x(uv))\|^2_{Y_\lambda[J]} \\ \lesssim\ &\sum_{\lambda\geq 1}\sup_{|J|=\lambda^{4s+3}, J\subset[0,1]}\|\eta_J(t)\lambda\sum_{\alpha\ll\lambda}P_\lambda(u_\lambda v_\alpha)\|^2_{Y_\lambda[J]} \\ +\ &\sum_{\lambda\geq 1}\sup_{|J|=\lambda^{4s+3}, J\subset[0,1]}\|\eta_J(t)\lambda\sum_{\mu\gtrsim\lambda}P_\lambda(u_\mu v_\mu)\|^2_{Y_\lambda[J]}.\end{aligned}$$

We can do same expansion for the local energy norm.

In the case of high-low frequency interaction, our goal would be to prove

$$\|\lambda\eta_J(t)P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]} \lesssim C\|u_\lambda\|_{X_\lambda[J]}\|v_\alpha\|_{X_\alpha[K]}. \tag{2.41}$$

Here $C = C(\lambda,\alpha) \lesssim 1$, and K is a time interval with size α^{4s+3}, so that $J \subset K$.

Now given (2.41), we get bound for energy norm in the case of high-low interaction

$$\begin{aligned}
&\sum_{\lambda\geq 1}\sup_{|J|=\lambda^{4s+3},J\subset[0,1]}\|\eta_J(t)\lambda\sum_{\alpha\ll\lambda}P_\lambda(u_\lambda v_\alpha)\|^2_{Y_\lambda[J]}\\
\lesssim\;&\sum_{\lambda\geq 1}C(\lambda,\alpha)^2\sup_{|J|=\lambda^{4s+3},J\subset[0,1]}\|u_\lambda\|^2_{X_\lambda[J]}\sum_{\alpha\ll\lambda}\|v_\alpha\|^2_{X^s}\\
\lesssim\;&\|u\|^2_{X^s}\|v\|^2_{X^s}.
\end{aligned}$$

And we can prove a spatial localized version of (2.41) in exactly the same way.

$$\|\lambda\chi^\lambda_j(x)\eta_J(t)P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]} \lesssim C\|\chi^\lambda_j(x)u_\lambda\|_{X_\lambda[J]}\|v_\alpha\|_{X_\alpha[K]}. \tag{2.42}$$

Then we also get bound for local energy norm in the case of high-low interaction

$$\begin{aligned}
&\sum_{\lambda\geq 1}\sup_j\sum_{|J|=\lambda^{4s+3},J\subset[0,1]}\|\chi^\lambda_j(x)\eta_J(t)\lambda\sum_{\alpha\ll\lambda}P_\lambda(u_\lambda v_\alpha)\|^2_{Y_\lambda[J]}\\
\lesssim\;&\sum_{\lambda\geq 1}C(\lambda,\alpha)^2\sup_j\sum_{|J|=\lambda^{4s+3},J\subset[0,1]}\|\chi^\lambda_j(x)u_\lambda\|^2_{X_\lambda[J]}\sum_{\alpha\ll\lambda}\|v_\alpha\|^2_{X^s}\\
\lesssim\;&\|u\|^2_{X^s_{le}}\|v\|^2_{X^s}.
\end{aligned}$$

One remark here is that we secretly turn the summation of α from l^1 to l^2 summation, which is not true in general. Luckily, in our proof for (2.41), the bound $C(\lambda,\alpha)$ mostly involves negative power of α or λ, which makes the summation valid. The only case worth attention is case 1.1(b), where we illuminate the α summation in detail.

In the case of high-high frequency interaction, we need to measure each u_μ on smaller time interval $I_\mu \subset J$, of size $|I_\mu| = \mu^{4s+3}$.

We will prove the estimate

$$\|\lambda\eta_J P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]} \lesssim C\sup_{I_\mu\subset J}\|u_\mu\|_{X_\mu[I_\mu]}\|v_\mu\|_{X^s\cap X^s_{le}}, \tag{2.43}$$

and its corresponding spatial localized version

$$\|\lambda\eta_J\chi^\lambda_j(x)P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]} \lesssim C\sup_{I_\mu\subset J}\|\chi^\mu_{k(j)}u_\mu\|_{X_\mu[I_\mu]}\|v_\mu\|_{X^s\cap X^s_{le}}. \tag{2.44}$$

Here $\chi^\mu_{k(j)}(x)$ is a chosen spatial cutoff so that $\chi^\lambda_j(x) \leq \chi^\mu_{k(j)}(x)$ (we might need two adjacent spatial cutoffs), $C = C(\lambda,\mu) \lesssim 1$.

Given (2.43), we get the bound for energy norm in the case of high-high interaction

$$
\begin{aligned}
&\sum_{\lambda\geq 1}\sup_{J\subset[0,1]}^{|J|=\lambda^{4s+3}}\|\lambda\eta_J(t)\sum_{\mu\gtrsim\lambda}P_\lambda(u_\mu v_\mu)\|^2_{Y_\lambda[J]}\\
\lesssim\ &\sum_{\lambda\geq 1}C(\lambda,\mu)^2\left(\sum_{\mu\gtrsim\lambda}\sup_{I_\mu\subset[0,1]}^{|I_\mu|=\mu^{4s+3}}\|u_\mu\|^2_{X_\mu[I_\mu]}\right)\sum_{\mu\gtrsim\lambda}\|v_\mu\|^2_{X^s\cap X^s_{le}}\\
\lesssim\ &\|u_\mu\|^2_{X^s}\|v_\mu\|^2_{X^s\cap X^s_{le}}.
\end{aligned}
$$

And with (2.44), we can bound the local energy norm in the case of high-high interaction

$$
\begin{aligned}
&\sum_{\lambda\geq 1}\sup_j\sum_{|J|=\lambda^{4s+3},J\subset[0,1]}\|\chi_j^\lambda(x)\eta_J(t)\lambda\sum_{\mu\gtrsim\lambda}P_\lambda(u_\mu v_\mu)\|^2_{Y_\lambda[J]}\\
\lesssim\ &\sum_{\lambda\geq 1}C(\lambda,\mu)^2\left(\sum_{\mu\gtrsim\lambda}\sup_{k(j)}\sum_{|I_\mu|=\mu^{4s+3},I_\mu\subset[0,1]}\|\chi^\mu_{k(j)}(x)u_\mu\|^2_{X_\mu[I_\mu]}\right)\sum_{\mu\gtrsim\lambda}\|v_\mu\|^2_{X^s\cap X^s_{le}}\\
\lesssim\ &\|u_\mu\|^2_{X^s_{le}}\|v_\mu\|^2_{X^s\cap X^s_{le}}.
\end{aligned}
$$

In both of the estimates, we need change the order of λ, μ summation. Luckily the bound $C(\lambda,\mu)$ in (2.43) (2.44) will help us to perform the λ summation.

Since the proofs for (2.42) (2.44) are essentially the same as (2.41) (2.43). We will discard the spatial cutoff in our proofs unless needed.

Remark 2.4.1. To be more precise, for spatial localization, instead of writing a function as $u_\lambda=\sum_j\chi_j^\lambda(x)u_\lambda$, we need to decompose each function as

$$
u_\lambda=\sum_j u_{\lambda,j},\qquad u_{\lambda,j}=P_\lambda(\chi_j^\lambda u_\lambda).
$$

In this way, we preserve the frequency localization while blurring the spatial localization. But thanks to the fast decay property of the kernel of $\chi_k^\lambda(x)P_\lambda\chi_j^\lambda(x)$, we have

$$
|\chi_k^\lambda u_{\lambda,j}|\lesssim|k-j|^{-N}\lambda^{-N}\|\chi_j^\lambda u_\lambda\|_{L^\infty_tL^2_x},\qquad |k-j|\gg 1.
$$

So the difference of the two decompositions is really negligible. Similar reasoning applies when we interchange the modulation localization and time localization.

Before getting into detail, notice that $\widetilde{uv}(\tau,\xi)=\widetilde{u}(\tau_1,\xi_1)*\widetilde{v}(\tau_2,\xi_2)$, so we have

$$
\tau=\tau_1+\tau_2,\quad \xi=\xi_1+\xi_2,
$$

and the resonance identity

$$
\tau-\xi^3=(\tau_1-\xi_1^3)+(\tau_2-\xi_2^3)-3\xi\xi_1\xi_2. \tag{2.45}
$$

Also, the following high modulation relation is quite useful in our proof.

$$
\sigma_m=\max(|\tau-\xi^3|,|\tau_1-\xi_1^3|,|\tau_2-\xi_2^3|)\gtrsim|\xi\xi_1\xi_2|. \tag{2.46}
$$

This relation forces high modulation either on the input or on output, which gives a gain.

Estimate for $X^1 \times X^1$.

When $u, v \in X^1$, we break them into dyadic pieces and discuss the problem in different cases. As pointed out in Remark 2.2.2(1), for function $u_\lambda \in X^1[I_\lambda]$, $|I_\lambda| = \lambda^{4s+3}$, we think of it as its extension $u_{\lambda,E}$, which is defined on the whole real time line and still supported on neighborhood of I_λ.

Case 1.1: High-Low frequency interaction. Suppose $\lambda \gg \alpha$, then the output frequency is λ. From (2.46), let $M = \lambda^2\alpha$, then

$$\lambda\eta_\lambda u_\alpha v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim M}, Q_{\ll M}\}} \lambda\eta_\lambda Q_1[Q_2 u_\alpha Q_3 v_\lambda].$$

Clearly in each term, at least one of Q_i must be $Q_{\gtrsim M}$.

Case 1.1(a): When high modulation comes from input, simply bound that piece in L^2 and the other in L^∞. Combining with Bernstein inequality, we get

$$\begin{aligned}
\|\lambda\eta_\lambda Q_1[Q_2 u_\alpha Q_3 v_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \|\lambda\eta_\lambda Q_1[Q_2 u_\alpha Q_3 v_\lambda]\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2[I_\lambda]} \\
&\lesssim \lambda\alpha^{-s}M^{-1}(\alpha^{\frac{1}{2}} + \lambda^{\frac{1}{2}})\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]} \\
&\lesssim \lambda^{-\frac{1}{2}}\alpha^{-1-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]}.
\end{aligned}$$

For $s \geq -1$ we can sum up with respect to α then λ.

Case 1.1(b): If none of Q_2, Q_3 have high modulation, this forces $Q_1 = Q_{\approx M}$. Depends on the size of M, we bound the output in different spaces ($|D|^{-s}|I|^{-\frac{1}{2}}L^2$ or $X^{-s,s}$). Using the bilinear estimate (2.30), we have

$$\begin{aligned}
\|\lambda\eta_\lambda Q_{\approx M \lesssim \lambda^{4+\frac{3}{2s}}}[Q_2 u_\alpha Q_3 u_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\lambda^{-1-s}\alpha^{-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]} \\
&\lesssim \|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]},
\end{aligned}$$

$$\begin{aligned}
\|\lambda\eta_\lambda Q_{\approx M \geq \lambda^{4+\frac{3}{2s}}}[Q_2 u_\alpha Q_3 u_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1-s}M^s\lambda^{-1-s}\alpha^{-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]} \\
&\lesssim \|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^1[I_\lambda]}.
\end{aligned}$$

Remark 2.4.2. We need to be careful with α summation in above estimates. For the first one we use factor α^s to turn l^1 summation to l^2. A careful way of doing the second one is to write the modulation as a multiple of $\lambda^2\alpha$, and use the l^2 summability of modulation.

$$\begin{aligned}
\|\sum_{\alpha\ll\lambda}\sum_\theta \lambda\eta_\lambda Q_{(\lambda^2\alpha)\theta}(u_\lambda v_\alpha)\|_{Y_\lambda}^2 &\lesssim (\sum_\theta \|\sum_{\alpha\ll\lambda}\lambda\eta_\lambda Q_{(\lambda^2\alpha)\theta}(u_\lambda v_\alpha)\|_{DS})^2 \\
&\lesssim \{\sum_\theta(\sum_{\alpha\ll\lambda}\|\lambda\eta_\lambda Q_{(\lambda^2\alpha)\theta}(u_\lambda v_\alpha)\|_{DS}^2)^{1/2}\}^2 \\
&\lesssim \{\sum_\theta(\sum_{\alpha\ll\lambda}\theta^{2s}\|u_\lambda\|_{X^1[I_\lambda]}^2\|v_\alpha\|_{X^1[I_\alpha]}^2)^{1/2}\}^2 \\
&\lesssim (\sum_\theta \theta^s)^{\frac{1}{2}}\|u_\lambda\|_{X^1[I_\lambda]}^2\|v\|_{X^s}^2.
\end{aligned}$$

In second inequality, since the modulation is different, we do have the l^2 summation.

Case 1.2: High-High frequency interaction with low frequency output, $\lambda \ll \mu$. Here we need to cut the interval I_λ into finer scale so that u_μ is measured on smaller intervals I_μ.

$$u_\mu = \sum_i u_\mu^i, \quad u_\mu^i \in X^1[I_\mu^i], \quad \cup I_\mu^i = I_\lambda.$$

Then the output has the expression

$$\lambda \eta_\lambda u_\mu v_\mu = \sum_i \sum_{Q_j \in \{Q_{\gtrsim \lambda\mu^2}, Q_{\ll \lambda\mu^2}\}} \lambda Q_1[Q_2 u_\mu^i Q_3 v_\mu^i].$$

Case 1.2(a): When $Q_1 = Q_{\gtrsim \lambda\mu^2}$, we place the output in $DZ[I_\lambda]$, by using (2.31)

$$\|Q_2 u_\mu^i Q_3 v_\mu^i\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \mu^{-2s} \|u_\mu\|_{X^1[I_\mu^i]} \|v_\mu\|_{X^1[I_\mu^i]}$$

and the almost orthogonality of the product $\lambda Q_\sigma(u_\mu^i v_\mu^i)$ with $\lambda Q_\sigma(u_\mu^j v_\mu^j)$, we get

$$\begin{aligned} \|\sum_i \lambda Q_{\sigma \gtrsim \lambda\mu^2}[Q_2 u_\mu^i Q_3 v_\mu^i]\|_{X^{-3-4s,2s+1}[I_\lambda]} &\lesssim \lambda^{2-4s} (\sigma_{\gtrsim \lambda\mu^2})^{2s+1} |\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}} \|u_\mu^i v_\mu^i\|_{L^2_{t,x}} \\ &\lesssim \sup_{I_\mu^i \subset I_\lambda} \|u_\mu\|^2_{X^1[I_\mu^i]} \|v_\mu\|^2_{X^s}. \end{aligned}$$

The DZ norm also has the L^p component. Here because the modulation is high, we can interchange interval and modulation cutoff and have l^p summation of the intervals. Using Strichartz estimates (2.14) and Bernstein inequality on the product, we get

$$\begin{aligned} &\|\sum_i \lambda Q_{\sigma \gtrsim \lambda\mu^2}[Q_2 u_\mu^i Q_3 v_\mu^i]\|_{|D_t - D_x^3||D| L^\infty_{t,x}[I_\lambda]} \\ \lesssim\ & \sup_{I_\mu^i} \|\lambda Q_{\sigma \gtrsim \lambda\mu^2}[Q_2 u_\mu^i Q_3 v_\mu^i]\|_{|D_t - D_x^3||D| L^\infty_{t,x}[I_\mu^i]} \\ \lesssim\ & \sup_{I_\mu^i} \frac{\lambda}{\lambda\mu^2} \|u_\mu\|_{L_t^\infty L_x^2[I_\mu^i]} \|v_\mu\|_{L_t^\infty L_x^2[I_\mu^i]} \\ \lesssim\ & \mu^{-2(s+1)} \sup_{I_\mu \subset I_\lambda} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

Because of the summation on λ here, we have only $s > -1$ in Proposition 2.1.5, but not at the endpoint $s = -1$.

Case 1.2(b): When input has high modulation, we use the local energy space to get good control of the interval summation.

Before that, let us state a useful lemma:

Lemma 2.4.3. *Suppose* $-1 \leq s \leq -\frac{3}{4}$, $0 \leq k \leq \frac{1}{2}$, *then we have*

$$\|Q_{\sigma \gtrsim \lambda^3} f_\lambda\|_{DZ[I_\lambda]} \lesssim \sup_{\sigma \gtrsim \lambda^3} \lambda^{2s+3k} \sigma^{-k} \|Q_\sigma f_\lambda\|_{L^2_{t,x}[I_\lambda]},$$

$$\|f_\lambda\|_{Y_\lambda[I_\lambda]} \lesssim \sup_\sigma \lambda^{3s+\frac{3}{2}+3k} \sigma^{-k} \|Q_\sigma f_\lambda\|_{L^2_{t,x}[I_\lambda]}.$$

Proof. From the definition of $Y_\lambda[I_\lambda]$, we just need to bound different modulation in suitable spaces, and compare the bounds with the ones in our lemma. The DZ norm also has L^p component, we use Bernstein to turn L^p into L^2 norm. □

Remark 2.4.4. These estimates are very crude. When applying on the nonlinearity, we might need to do modulation analysis, or use better interval summation in some cases, e.g. case 1.2(a). But when one of the inputs has high modulation, a simple L^2 estimate saves us from tedious case by case calculation.

Let us first bound the spatial localized output in L^2.

$$\begin{aligned}
& \|\lambda \eta_\lambda \chi_j^\lambda(x) Q_\sigma [\sum_i (Q_{\gtrsim \lambda\mu^2} u_\mu^i)(Q_3 v_\mu^i)]\|^2_{L^2_{t,x}[I_\lambda]} \\
\lesssim\ & \sigma \|\lambda \eta_\lambda \chi_j^\lambda(x) [\sum_i (Q_{\gtrsim \lambda\mu^2} u_\mu^i)(Q_3 v_\mu^i)]\|^2_{L^2_x L^1_t} \\
\lesssim\ & \lambda^2 \sigma \sum_i \|\chi_j^\lambda(x) Q_{\gtrsim \lambda\mu^2} u_\mu^i\|^2_{L^2_{t,x}[I_\mu^i]} \sum_i \|\chi_j^\lambda(x) Q_3 v_\mu^i\|^2_{L^\infty_x L^2_t[I_\mu^i]} \\
\lesssim\ & \lambda^2 \sigma |\frac{I_\lambda}{I_\mu}| \sup_i \|\chi_j^\lambda(x) Q_{\gtrsim \lambda\mu^2} u_\mu^i\|^2_{L^2_{t,x}[I_\mu^i]} \sup_j \sum_i \|\chi_j^\lambda(x) Q_3 v_\mu^i\|^2_{L^\infty_x L^2_t[I_\mu^i]} \\
\lesssim\ & \sigma \lambda^{4s+3} \mu^{-12s-12} \sup_{I_\mu} \|\chi_{k(j)}^\mu(x) u_\mu\|^2_{X^1[I_\mu]} \|v_\mu\|^2_{X^s_{le}}.
\end{aligned}$$

To get same estimate without the spatial localization, we need to sum up j

$$\begin{aligned}
& \sum_j \|\lambda \eta_\lambda \chi_j^\lambda(x) Q_\sigma [\sum_i (Q_{\gtrsim \lambda\mu^2} u_\mu^i)(Q_3 v_\mu^i)]\|^2_{L^2_{t,x}[I_\lambda]} \\
\lesssim\ & \lambda^2 \sigma \sum_{i,j} \|\chi_j^\lambda(x) Q_{\gtrsim \lambda\mu^2} u_\mu^i\|^2_{L^2_{t,x}[I_\mu^i]} \sup_j \sum_i \|\chi_j^\lambda(x) Q_3 v_\mu^i\|^2_{L^\infty_x L^2_t[I_\mu^i]} \\
\lesssim\ & \lambda^2 \sigma |\frac{I_\lambda}{I_\mu}| \sup_i \|Q_{\gtrsim \lambda\mu^2} u_\mu^i\|^2_{L^2_{t,x}[I_\mu^i]} \sup_j \sum_i \|\chi_j^\lambda(x) Q_3 v_\mu^i\|^2_{L^\infty_x L^2_t[I_\mu^i]} \\
\lesssim\ & \sigma \lambda^{4s+3} \mu^{-12s-12} \sup_{I_\mu} \|u_\mu\|^2_{X^1[I_\mu]} \|v_\mu\|^2_{X^s_{le}}.
\end{aligned}$$

By Lemma 2.4.3 $k = \frac{1}{2}$, we have the following estimate with or without spatial localization.

$$\|\lambda \eta_\lambda [\sum_i Q_{\gtrsim \lambda\mu^2} u_\mu^i Q_3 v_\mu^i]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{5s+9/2} \mu^{-6s-6} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s_{le}}.$$

We can sum up frequency λ and μ when $-1 \le s \le -\frac{3}{4}$.

Remark 2.4.5. The estimates above demonstrate how we can use local energy norm to get good interval summations, especially in the case when time truncation blur the output modulation too much.

Case 1.3: High-High frequency interaction giving out the output of the same size. Now high modulation (2.46) means λ^3.

$$\lambda\eta_\lambda u_\lambda v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim\lambda^3}, Q_{\ll\lambda^3}\}} \lambda\eta_\lambda Q_1[Q_2 u_\lambda Q_3 v_\lambda].$$

Case 1.3(a): When high modulation comes from input, we estimate the output in $|D|^{-s}|I|^{-\frac{1}{2}}L^2$

$$\begin{aligned} \|\lambda\eta_\lambda Q_1[Q_{\sigma\gtrsim\lambda^3}u_\lambda Q_3 v_\lambda]\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\|Q_{\sigma\gtrsim\lambda^3}(\eta_\lambda u_\lambda)\|_{L^2_{t,x}}\|\eta_\lambda v_\lambda\|_{L^\infty_{t,x}} \\ &\lesssim \lambda^{-\frac{3}{2}-s}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\lambda\|_{X^1[I_\lambda]}. \end{aligned}$$

Case 1.3(b): When inputs have low modulation, this forces the output to have modulation approximately λ^3. In fact, the output has Fourier support lying closer to another curve $\tau = \frac{1}{4}\xi^3$. To give a good bound in this case, we want to prove

$$\|\lambda P_\lambda(u_\lambda v_\lambda)\|_{|\partial_t+\partial_x^3|^{-1}|D|^{-2s-2}X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}} \lesssim \|u_\lambda\|_{X^1[I_\lambda]}\|v_\lambda\|_{X^1[I_\lambda]}. \tag{2.47}$$

To do this, let us use the space $\dot{X}^{s,\frac{1}{2},1}$ defined in remark 1.4.7 and claim the embedding inequality

$$\|u_\lambda\|_{\dot{X}^{s,\frac{1}{2},1}} \lesssim \|u_\lambda\|_{X^1[I_\lambda]}, \tag{2.48}$$

which is proved by looking at the extension $u_{\lambda,E}$, and definitions of both norms.

Now for functions in $\dot{X}^{s,\frac{1}{2},1}$, we use foliation. The idea is same as in Chapter 2.6 Lemma 2.9 in Tao [56]. From Fourier inversion, we have

$$u_\lambda(t,x) = \frac{1}{(2\pi)^2}\int\int \tilde{u}_\lambda(\tau,\xi)e^{it\tau+ix\xi}\,d\tau\,d\xi.$$

Then if we write $\tau_0 = \tau - \xi^3$, we will have the foliation

$$u_\lambda(t,x) = \frac{1}{2\pi}\int e^{it\tau_0}e^{t\partial_x^3}f_{\tau_0}\,d\tau_0,$$

where

$$e^{t\partial_x^3}f_{\tau_0} = \frac{1}{2\pi}\int \tilde{u}_\lambda(\tau_0+\xi^3,\xi)e^{it\xi^3+ix\xi}\,d\xi,$$

and f_{τ_0} has frequency about size λ, modulation about size τ_0.

Similarly we write down $v_\lambda = u_\lambda(t,x) = \frac{1}{2\pi}\int e^{it\tau_0}e^{t\partial_x^3}g_{\tau_0'}\,d\tau_0'$.

Now using (2.21) and Minkowski inequality

$$\begin{aligned}
\|\lambda P_\lambda(u_\lambda v_\lambda)\|_{|(\partial_t+\partial_x^3)|^{-1}|D|^{-2s-2}\dot{X}^{\frac14,\frac14}_{\tau=\frac14\xi^3}} &\lesssim \frac{\lambda^{2s+3}}{\lambda^3}\sum_{\tau_0,\tau_0'}\iint \|e^{t\partial_x^3}f_{\tau_0}e^{t\partial_x^3}g_{\tau_0'}\|_{\dot{X}^{\frac14,\frac14}_{\tau=\frac14\xi^3}}\,d\tau_0\,d\tau_0' \\
&\lesssim \lambda^{2s}\sum_{\tau_0,\tau_0'}\iint \|f_{\tau_0}\|_{L_x^2}\|g_{\tau_0'}\|_{L_x^2}\,d\tau_0\,d\tau_0' \\
&\lesssim \|u_\lambda\|_{\dot{X}^{s,\frac12,1}}\|v_\lambda\|_{\dot{X}^{s,\frac12,1}}.
\end{aligned}$$

With the time cutoff we can pass to nonhomogeneous space, as in Proposition 2.3.4. Combining with the embedding (2.48), we proved (2.47).

Estimate for $S \times S$.

When $u, v \in S$, we still need to consider different frequency interaction. Notice that because of Remark 2.2.2(4), we only consider pieces that have relatively high modulation: $|\tau - \xi^3| \gtrsim |\xi|^{4+\frac{3}{2s}}$

Case 2.1: High low frequency interaction. The nonlinearity look like $\lambda\eta_\lambda u_\lambda v_\alpha$, $\lambda \gg \alpha$. As discussed in Remark 2.3.7, we don't have a good bilinear estimate, but (2.38) breaks down only for one case.

Case 2.1.1: If $u_\lambda, v_\alpha \in X^{-s,1+s}$, or $u_\lambda \in Z, v_\alpha \in X^{-s,1+s}$, or $u_\lambda, v_\alpha \in Z$, we can still use the L^2 estimate (2.38) and Lemma 2.4.3 with $k = 0$ to get

$$\begin{aligned}
\|\lambda\eta_\lambda u_\lambda v_\alpha\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{3s+\frac32}\lambda^{1-\frac43(s+1)-\frac13}\alpha^{-\frac23(s+1)+\frac13}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]} \\
&\lesssim \lambda^{\frac16+s+\frac23(s+1)}\alpha^{-\frac23(s+1)+\frac13}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]}.
\end{aligned}$$

Notice that the exponents add up to $-\frac32 - s < 0$, we can still sum up frequencies.

Case 2.1.2: Now if $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$, where (2.38) failed. We use L^2 on u_λ and $L_t^3L_x^\infty$ on v_α, still by Bernstein,

$$\|\lambda Q_\sigma(\eta_\lambda u_\lambda v_\alpha)\|_{L^2_{t,x}} \lesssim \lambda\alpha^{\frac13}\sigma^{\frac13}\lambda^{-2-s}\alpha^{-\frac43(s+1)-\frac13}\|u_\lambda\|_{X^{-s,1+s}[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]},$$

so we from Lemma 2.4.3, we get

$$\|\lambda\eta_\lambda u_\lambda v_\alpha\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{2s+\frac32}\alpha^{-\frac43(s+1)}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.$$

And we can still sum up the frequencies.

Case 2.2: High-high frequency interaction giving out equal or lower frequency, $\lambda \lesssim \mu$. When $\lambda \ll \mu$, we cut up intervals as in case (1.2). When $\lambda \approx \mu$, this procedure degenerate.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \lambda u_\mu^i v_\mu^i, \qquad u_\mu^i, v_\mu^i \in X_\mu[I_\mu^i]$$

Here we don't have a good L^2 bound on the product, so we need to do modulation analysis again to get better control. Also, all the estimates here have the corresponding version with spatial localization, the proofs are exactly the same.

Case 2.2.1: $X^{-s,1+s} \times X^{-s,1+s}$, both u_μ^i,v_μ^i have modulation $\mu^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \mu^3$. we use Berstein inequality for frequency on product, for modulation on any one of input. And we have l^2 summation of the small intervals.

$$\begin{aligned} \|\sum_i \lambda(Q_\sigma u_\mu^i)v_\mu^i\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}|\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}} \sup_i \|(Q_\sigma u_\mu^i)v_\mu^i\|_{L^2_{t,x}[I_\mu^i]} \\ &\lesssim \lambda^{1+s}|I_\lambda||I_\mu|^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\sigma^{\frac{1}{2}} \sup_i \|u_\mu^i\|_{L^2_{t,x}}\|v_\mu^i\|_{L^2_{t,x}} \\ &\lesssim \lambda^{5s+\frac{9}{2}}\mu^{-5s-5} \sup_{I_\mu} \|u_\mu\|_{S[I_\mu]}\|v_\mu\|_{X^s}. \end{aligned}$$

Case 2.2.2: $X^{-s,1+s} \times Z$, suppose v_μ has modulation $\sigma_m \gtrsim \mu^3$. By modulation analysis (2.46), this forces another high modulation on the output.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \lambda Q_{\approx\sigma_m}[u_\mu^i(Q_{\sigma_m}v_\mu^i)]$$

We comment that when $\sigma_m \approx \mu^3$, there is chance high modulation can also fall on u_μ. But in that case, from Prop 2.2.2(4), the norm Z and $X^{-s,1+s}$ match with each other. So it is essentially the same as in the following case 2.2.3.

We use L^2 (2.23) on u_μ , and L^p for v_μ, together with Bernstein.

$$\begin{aligned} &\|\sum_i \lambda Q_{\sigma_m}[u_\mu^i(Q_{\sigma_m}v_\mu^i)]\|_{X^{-3-4s,2s+1}[I_\lambda]} \\ \lesssim\ & \lambda^{-2-4s}\sigma_m^{2s+1}|\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}} \sup_i \|u_\mu^i(Q_{\sigma_m}v_\mu^i)\|_{L^2_{t,x}[I_\mu]} \\ \lesssim\ & \lambda^{-2-4s}\sigma_m^{2s+1}|\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}}(\lambda\sigma_m)^{\frac{1}{p}} \sup_i \|u_\mu^i\|_{L^2_{t,x}[I_\mu^i]}\|Q_{\sigma_m}v_\mu^i)\|_{L^p_{t,x}[I_\mu^i]} \\ \lesssim\ & \lambda^{-2s-\frac{1}{2}+\frac{1}{p}}\mu^{3s+\frac{1}{2}+\frac{4s+3}{p}} \sup_{I_\mu} \|u_\mu\|_{S[I_\mu]}\|v_\mu\|_{X^s}. \end{aligned}$$

We also need to bound the L^p component, here we exchange the interval cutoff with modulation factor and have l^p summation.

$$\begin{aligned} &\|\sum_i \lambda Q_{\approx\sigma}(u_\mu^i Q_\sigma v_\mu^i)\|_{|D_t-D_x^3||D|L^\infty_{t,x}[I_\lambda]} \\ \lesssim\ & \sigma^{-1}\sup_i \|u_\mu^i\|_{L^\infty_{t,x}}\|Q_\sigma v_\mu^i\|_{L^p_{x,t}[I_\mu^i]} \\ \lesssim\ & \sigma^{-1}\mu^{-2s-1}\mu \sup_i \|u_\mu^i\|_{X^{-s,1+s}[I_\mu^i]}\|v_\mu^i\|_{|D|L^\infty_{t,x}[I_\mu^i]} \\ \lesssim\ & \mu^{-2s-3}\sup_{I_\mu} \|u_\mu\|_{S[I_\mu]}\|v_\mu\|_{X^s}. \end{aligned}$$

In both case, we can sum up frequency when $-1 \leq s \leq -\frac{3}{4}$.
Case 2.2.3: $Z \times Z$. When u_μ, v_μ both have high modulation, we put them in L^3 (2.28).

We begin with the L^2 estimate

$$\begin{aligned}\|\sum_i \lambda Q_\sigma(u^i_\mu v^i_\mu)\|_{L^2_{t,x}[I_\lambda]} &\lesssim \lambda(\lambda\sigma)^{\frac{1}{6}}\|\sum_i Q_\sigma(u^i_\mu v^i_\mu)\|_{L^{\frac{3}{2}}_{t,x}[I_\lambda]} \\ &\lesssim \lambda(\lambda\sigma)^{\frac{1}{6}}|\frac{I_\lambda}{I_\mu}|^{\frac{2}{3}}\sup_i \|u^i_\mu\|_{L^3_{t,x}[I^i_\mu]}\|u^i_\mu\|_{L^3_{t,x}[I^i_\mu]} \\ &\lesssim \sigma^{\frac{1}{6}}\lambda^{\frac{19}{6}+\frac{8s}{3}}\mu^{-\frac{16}{3}(s+1)}\sup_{I_\mu}\|u_\mu\|_{Z[I_\mu]}\|v_\mu\|_{X^s}.\end{aligned}$$

Here notice we used $l^{\frac{3}{2}}$ summation of the intervals.

From Lemma 2.4.3, we get

$$\|\sum_i \lambda u^i_\mu v^i_\mu\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{-\frac{1}{2}+\frac{17(s+1)}{3}}\mu^{-\frac{16}{3}(s+1)}\sup_{I_\mu}\|u_\mu\|_{Z[I_\mu]}\|v_\mu\|_{X^s}.$$

To see we can sum up frequency, notice exponent for μ is negative and all the exponents add up to $-\frac{1}{2}+\frac{1}{3}(s+1) < 0$.

Estimate for $X^1 \times S$.

Suppose $u \in X^1$, $v \in S$. This includes the most dedicate case, i.e. low frequency high modulation piece interact with high frequency low modulation, where we can not prove the bilinear estimate (2.40). Instead we have to reiterate the equation and turn the bilinear estimate to trilinear. Let us work on high-high frequency interaction first.
Case 3.1: High-high frequency interaction giving out equal or lower frequency, $\lambda \lesssim \mu$. Same as before, we need to cut into smaller intervals if $\lambda \ll \mu$, and this procedure degenerate if $\lambda \approx \mu$.
Case 3.1.1: $X^1 \times X^{-s,1+s}$, by (2.46) we must have modulation $\sigma \gtrsim \lambda\mu^2$ in some term.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \sum_{Q_j \in \{Q_{\gtrsim\lambda\mu^2}, Q_{\ll\lambda\mu^2}\}} \lambda Q_1[(Q_2 u^i_\mu)(Q_3 v^i_\mu)].$$

Case 3.1.1(a): When high modulation is on output, i.e. $Q_1 = Q_{\sigma\gtrsim\lambda\mu^2}$. Using $L^\infty_t L^2_x$ on u^i_μ, $L^2_{t,x}$ on v^i_μ, together with Bernstein on the product, we get,

$$\begin{aligned}\|\lambda\sum_i Q_\sigma[(Q_2 u^i_\mu)(Q_3 v^i_\mu)]\|_{L^2_{t,x}[I_\lambda]} &\lesssim \lambda^{\frac{3}{2}}|\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}}\sup_i\|u^i_\mu\|_{L^\infty_t L^2_x[I^i_\mu]}\|v^i_\mu\|_{L^2_{t,x}[I^i_\mu]} \\ &\lesssim \lambda^{2s+3}\mu^{-4s-\frac{7}{2}}\sup_{I_\mu}\|u_\mu\|_{X^1[I_\mu]}\|v_\mu\|_{X^s}.\end{aligned}$$

Using the fact that output has high modulation and Lemma 2.4.3 with $k=\frac{1}{2}$, we get

$$\|\lambda\sum_i Q_{\sigma\gtrsim\lambda\mu^2}(u^i_\mu v^i_\mu)\|_{DZ[I_\lambda]} \lesssim \lambda^{4s+4}\mu^{-4s-\frac{9}{2}}\sup_{I_\mu}\|u_\mu\|_{X^1[I_\mu]}\|v_\mu\|_{X^s}.$$

Case 3.1.1(b): High modulation on u_μ, $Q_2 = Q_{\gtrsim \lambda\mu^2}$. We put them both in $L^2_{t,x}$.

$$\begin{aligned} & \|\sum_i \lambda Q_\sigma[(Q_{\gtrsim\lambda\mu^2} u^i_\mu) Q_3 v^i_\mu]\|_{L^2_{t,x}[I_\lambda]} \\ \lesssim\ & \lambda(\lambda\sigma)^{\frac{1}{2}} \|\sum_i Q_\sigma[(Q_{\gtrsim\lambda\mu^2} u^i_\mu) Q_3 v^i_\mu]\|_{L^1_{t,x}[I_\lambda]} \\ \lesssim\ & \lambda(\lambda\sigma)^{\frac{1}{2}} |\frac{I_\lambda}{I_\mu}| \sup_i \|Q_{\gtrsim\lambda\mu^2} u^i_\mu\|_{L^2_{t,x}[I^i_\mu]} \|Q_3 v^i_\mu\|_{L^2_{t,x}[I^i_\mu]} \\ \lesssim\ & \sigma^{\frac{1}{2}} \lambda^{4s+\frac{7}{2}} \mu^{-8s-\frac{17}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

hence we have

$$\|\sum_i \lambda(Q_{\gtrsim\lambda\mu^2} u^i_\mu) Q_3 v^i_\mu\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{7s+\frac{13}{2}} \mu^{-8s-\frac{17}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.$$

Case 3.1.1(c): High modulation comes from input $Q_3 = Q_{\gtrsim\lambda\mu^2}$. We use local smoothing (2.15) on u_μ, and $L^2_{t,x}$ on v_μ.

$$\begin{aligned} & \|\sum_i \lambda Q_\sigma[Q_2 u^i_\mu (Q_{\gtrsim\lambda\mu^2} v^i_\mu)]\|_{L^2_{t,x}[I_\lambda]} \\ \lesssim\ & \lambda\sigma^{\frac{1}{2}} \|\sum_i Q_\sigma[Q_2 u^i_\mu (Q_{\gtrsim\lambda\mu^2} v^i_\mu)]\|_{L^2_x L^1_t[I_\lambda]} \\ \lesssim\ & \lambda\sigma^{\frac{1}{2}} |\frac{I_\lambda}{I_\mu}| \sup_i \|Q_2 u^i_\mu\|_{L^\infty_x L^2_t[I^i_\mu]} \|Q_{\gtrsim\lambda\mu^2} v^i_\mu\|_{L^2_{t,x}[I^i_\mu]} \\ \lesssim\ & \sigma^{\frac{1}{2}} \lambda^{3s+3} \mu^{-6s-6} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

Hence we have

$$\|\sum_i \lambda Q_2 u^i_\mu (Q_{\gtrsim\lambda\mu^2} v^i_\mu)\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{6s+6} \mu^{-6s-6} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{S[I_\mu]}.$$

Case 3.1.2: $X^1 \times Z$. This forces high modulation $\sigma_m \gtrsim \mu^3$ also on the output.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \lambda Q_{\approx\sigma_m}[u^i_\mu (Q_{\sigma_m} v^i_\mu)]$$

We still bound the output in L^2 by using L^6 on u_μ, L^3 (2.28) on v_μ.

$$\begin{aligned} & \|\sum_i \lambda Q_\sigma[u^i_\mu (Q_{\sigma_m} v^i_\mu)]\|_{L^2_{t,x}[I_\lambda]} \\ \lesssim\ & \lambda |\frac{I_\lambda}{I_\mu}|^{\frac{1}{2}} \sup_i \|u^i_\mu\|_{L^6_{t,x}[I^i_\mu]} \|Q_{\sigma_m} v^i_\mu\|_{L^3_{t,x}[I^i_\mu]} \\ \lesssim\ & \lambda^{2s+\frac{5}{2}} \mu^{-2s-\frac{3}{2}} \mu^{-\frac{1}{6}-s} \mu^{-\frac{4}{3}(s+1)-\frac{1}{3}} \sup_i \|u^i_\mu\|_{X^1[I^i_\mu]} \|v^i_\mu\|_{S[I^i_\mu]} \\ \lesssim\ & \lambda^{2s+\frac{5}{2}} \mu^{-5s-4+\frac{2}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

From Lemma 2.4.3 with $k = \frac{1}{2}$, we get

$$\|\sum_i \lambda Q_{\sigma_m}[u_\mu^i(Q_{\sigma_m} v_\mu^i)]\|_{DZ[I_\lambda]} \lesssim \lambda^{4s+4}\mu^{-5s-\frac{11}{2}+\frac{2}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]}\|v_\mu\|_{X^s}.$$

Case 3.2: High low frequency interaction. $u_\alpha \in X^1, v_\lambda \in S$, $\lambda \gg \alpha$. The bilinear estimate (2.32) is not good enough, so we have to break into more cases.
Case 3.2.1: $u_\alpha \in X^1, v_\lambda \in X^{-s,1+s}$. Because of high modulation relation (2.46), we have

$$\lambda\eta_\lambda u_\alpha v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim \lambda^2\alpha}, Q_{\ll \lambda^2\alpha}\}} \lambda\eta_\lambda Q_1[(Q_2 u_\alpha)(Q_3 v_\lambda)].$$

Case 3.2.1(a): High modulation on u_α. $Q_2 = Q_{\sigma \gtrsim \lambda^2\alpha}$. Put u_α in L^2, v_λ in L^∞ (2.24).

$$\|\lambda\eta_\lambda Q_1[(Q_{\sigma\gtrsim\lambda^2\alpha}u_\alpha)(Q_3 v_\lambda)]\|_{L^2_{t,x}[I_\lambda]} \lesssim \lambda^{-4s-\frac{7}{2}}\alpha^{-1-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]},$$

so from Lemma 2.4.3, we get

$$\|\lambda\eta_\lambda[(Q_{\sigma\gtrsim\lambda^2\alpha}u_\alpha)(Q_3 v_\lambda)]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{-s-2}\alpha^{-1-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]}.$$

Case 3.2.1(b): High modulation on v_λ. $Q_3 = Q_{\sigma\gtrsim\lambda^2\alpha}$. Put u_α in L^∞, v_λ in L^2.

$$\|\lambda\eta_\lambda Q_1[(Q_2 u_\alpha)(Q_{\sigma\gtrsim\lambda^2\alpha}v_\lambda)]\|_{L^2_{t,x}[I_\lambda]} \lesssim \lambda^{-1-s}\alpha^{-\frac{1}{2}-2s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]},$$

so we get

$$\|\lambda\eta_\lambda[(Q_2 u_\alpha)(Q_{\sigma\gtrsim\lambda^2\alpha}v_\lambda)]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{2s+\frac{1}{2}}\alpha^{-\frac{1}{2}-2s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]}.$$

Case 3.2.1(c): When none of u_α, v_λ have high modulation, this forces the output to be approximately $\lambda^2\alpha$. $Q_1 = Q_{\sigma\approx\lambda^2\alpha}$, put u_α in L^∞, v_λ in L^2.

When $\lambda^2\alpha \lesssim \lambda^{4+\frac{3}{2s}}$, i.e. $\alpha \lesssim \lambda^{2+\frac{3}{2s}}$ we have

$$\begin{aligned} &\|\lambda\eta_\lambda Q_{\sigma\approx\lambda^2\alpha}[(Q_2 u_\alpha)(Q_3 v_\lambda)]\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \\ \lesssim\ & \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\alpha^{\frac{1}{2}-s}\lambda^{-2-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]} \\ \lesssim\ & \alpha^{\frac{1}{2}-s}\lambda^{2s+\frac{1}{2}}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]}, \end{aligned}$$

notice we have $\alpha^{\frac{1}{2}-s}\lambda^{2s+\frac{1}{2}} \lesssim \lambda^{\frac{3}{4s}}$, which is good for summation.

When $\lambda^2\alpha \gtrsim \lambda^{4+\frac{3}{2s}}$, we have

$$\begin{aligned} &\|\lambda\eta_\lambda Q_{\sigma\approx\lambda^2\alpha}[(Q_2 u_\alpha)(Q_3 v_\lambda)]\|_{X^{-s,s}} \\ \lesssim\ & \lambda^{1-s}\sigma^s\alpha^{\frac{1}{2}-s}\lambda^{-2-s}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]} \\ \lesssim\ & \alpha^{\frac{1}{2}}\lambda^{-1}\|u_\alpha\|_{X^1[I_\alpha]}\|v_\lambda\|_{X^{-s,1+s}[I_\lambda]}. \end{aligned}$$

Case 3.2.2: $u_\alpha \in X^1, v_\lambda \in Z$. Here the bilinear estimate (2.33) is good enough.

$$\begin{aligned}\|\lambda\eta_\lambda u_\alpha v_\lambda\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2[I_\lambda]} &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\|\eta_\lambda u_\alpha v_\lambda\|_{L^2_{t,x}[I_\lambda]} \\ &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\lambda^{-\frac{1}{3}-2s-2}\alpha^{-\frac{1}{6}-s}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]} \\ &\lesssim \lambda^{\frac{1}{6}+s}\alpha^{-\frac{1}{6}-s}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]}.\end{aligned}$$

Case 3.3: High low frequency interaction. $u_\lambda \in X^1, v_\alpha \in S$, $\lambda \gg \alpha$.
Case 3.3.1: $u_\lambda \in X^1$, $v_\alpha \in X^{-s,1+s}$. Without going into modulation analysis, we use $L^\infty_x L^2_t$ on u_λ, and $L^2_x L^\infty_t$ on v_α, together with Bernstein and notice the modulation on v_α is small.

$$\begin{aligned}\|\lambda\eta_\lambda u_\lambda v_\alpha\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\|v_\lambda\|_{L^\infty_x L^2_t[I_\lambda]}\|v_\alpha\|_{L^2_x L^\infty_t[I_\alpha]} \\ &\lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\lambda^{-1-s}\alpha^{-\frac{3}{2}-2s}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]} \\ &\lesssim \lambda^{2s+\frac{3}{2}}\alpha^{-\frac{3}{2}-2s}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.\end{aligned}$$

Case 3.3.2: $u_\lambda \in X^1$, $v_\alpha \in Z$. Here we can not prove any bilinear estimate if high modulation fall on v_α, so we need the following lemma to reiterate the equation.

Lemma 2.4.6. *(Reiterate the equation) Let w be a solution to KdV equation* (2.1)*. Then we can write its high modulation part as*

$$Q_{\sigma\gtrsim\alpha^3}w_\alpha = M_1 + M_2 + R,$$

where M_1, M_2, R are as follows:

- *M_1 is the output of two higher frequency-low modulation interaction,*
$$M_1 = \sum_{\alpha\lesssim\beta_1\approx\beta_2}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_\sigma(w_{\beta_1}w_{\beta_2}), \quad w_{\beta_1}, w_{\beta_2} \in X^1$$
where w_{β_1}, w_{β_2} all have very low modulation $|\tau-\xi^3| \lesssim |\xi|^{4+\frac{3}{2s}}$.

- *M_2 is the output of the high frequency-low modulation piece interact with low frequency-high modulation piece.*
$$M_2 = \sum_{\sigma\gtrsim\alpha^3,\gamma\ll\beta\approx\alpha}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_\sigma(w_\beta w_\gamma), \quad w_\beta \in X^1, w_\gamma \in Z.$$
w_β has modulation $|\tau-\xi^3| \lesssim |\xi|^{4+\frac{3}{2s}}$, w_γ has high modulation $|\tau-\xi^3| \gtrsim |\xi|^3$.

- *R is the remainder, which comes from interaction of all other cases*
$$R = \sum_{\sigma\gtrsim\alpha^3,\beta,\gamma}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_\sigma(w_\beta w_\gamma).$$

For R, we have the estimate

$$\|\eta_\alpha(t)R_\alpha\|_{\alpha^{-2s-\frac{3}{2}}L^2_x L^\infty_t} \lesssim \|w\|^2_{X^s\cap X^s_{le}[I_\alpha]}. \tag{2.49}$$

The decomposition above is true modulo ± sign on each term.

Proof. : If we apply frequency and modulation projection on the equation, we get

$$(\partial_t + \partial_x^3) P_\alpha Q_\sigma w = -P_\alpha Q_\sigma \partial_x (w^2).$$

Hence modulo ± sign we have

$$P_\alpha Q_\sigma w = (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (w^2).$$

Here we decompose w into dyadic pieces, $P_\alpha Q_\sigma w = (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (w_\beta w_\gamma)$. Now we first break each w_λ into sum of functions supported on time scale $|\lambda|^{4s+3}$. Next, for each $w_\lambda \in X^s \cap X^s_{le}[I_\lambda]$, let us decompose it as $w_\lambda = w_{\lambda,1} + w_{\lambda,2}$, $w_{\lambda,1} \in X^1$, $w_{\lambda,2} \in S$. Then we can just take u_β, v_γ to represent $w_{\beta,i}$, $w_{\gamma,j}$, $i, j \in \{1, 2\}$.

We will prove that except for the two cases in M_1 and M_2, we have the estimate (2.49). We list the estimates of all cases below, which are similar to what we have done before. Notice the modulation is always larger than α^3 in the summation.

Case 1: $\beta \approx \alpha \gg \gamma$.

(1) $u_\alpha, v_\gamma \in X^1$, use Bernstein and bilinear estimate (2.30)

$$\begin{aligned} \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (u_\alpha v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} &\lesssim \frac{\alpha^{2s+\frac{5}{2}}}{\sigma^{\frac{1}{2}}} \alpha^{-1-s} \gamma^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]} \\ &\lesssim \alpha^{-1+s} \gamma^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]}. \end{aligned}$$

(2) $u_\alpha \in X^1, v_\gamma \in S$ we only deal with $v_\gamma \in X^{-s,1+s}$. And leave $v_\gamma \in Z$ term into M_2. Notice here u_α must have high modulation σ (2.46). Put L^2 on u_α, L^∞ on v_γ

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma ((Q_\sigma u_\alpha) v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}} \sigma^{-\frac{1}{2}} \sigma^{-1} |I_\alpha|^{-\frac{1}{2}} \alpha^{-s} \gamma^{-2s-1} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]} \\ \lesssim\ & \alpha^{-s-\frac{7}{2}} \gamma^{-2s-1} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}. \end{aligned}$$

(3) $u_\alpha \in S, v_\gamma \in X^1$, use the bilinear estimate (2.32)

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (u_\alpha v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}} \sigma^{-\frac{1}{2}} \max\{\alpha^{-\frac{1}{3}-2s-2} \gamma^{-\frac{1}{6}-s}, \alpha^{-2-s} \gamma^{\frac{1}{2}-s}\} \|u_\alpha\|_{S[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]} \\ \lesssim\ & \max\{\alpha^{-\frac{4}{3}} \gamma^{-\frac{1}{6}-s}, \alpha^{s-1} \gamma^{\frac{1}{2}-s}\} \|u_\alpha\|_{S[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]}. \end{aligned}$$

(4) $u_\alpha, v_\gamma \in S$, we consider several cases:

If $u_\alpha, v_\gamma \in X^{-s,1+s}$, or $u_\alpha \in Z, v_\gamma \in X^{-s,1+s}$, or $u_\alpha, v_\gamma \in Z$, then we have the bilinear estimate

(2.38). So we have

$$\begin{aligned}
&\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma ((Q_\sigma u_\alpha) v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\
\lesssim\ & \frac{\alpha^{2s+\frac{5}{2}}}{\sigma^{\frac{1}{2}}} \alpha^{-\frac{4}{3}(s+1)-\frac{1}{3}} \gamma^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]} \\
\lesssim\ & \alpha^{-\frac{4}{3}+\frac{2}{3}(s+1)} \gamma^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}.
\end{aligned}$$

Notice the exponents add up to -1.

If $u_\alpha \in X^{-s,1+s}, v_\gamma \in Z$, use L^2 on u_α, L^p on v_γ

$$\begin{aligned}
&\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (u_\alpha v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\
\lesssim\ & \alpha^{2s+\frac{5}{2}} \sigma^{-1} \sigma^{\frac{1}{2}} \alpha^{-2-s} \gamma^1 \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{Z[I_\gamma]} \\
\lesssim\ & \alpha^{s-1} \gamma \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}.
\end{aligned}$$

The exponents add up to $s < 0$.

Case 2: $\beta \approx \gamma \gtrsim \alpha$. This part is every similar to the estimates in Case 1.2, 2.2 and 3.1. We still need to decompose u_β into sums of functions that are supported on the μ^{4s+3} time scale. $u_\beta = \sum_i u_\beta^i, \;\; u_\beta^i \in X_\beta[I_\beta^i]$

(1) $u_\beta, v_\beta \in X^1$, when one of input e.g. u_β has high modulation $Q_{\gtrsim \alpha\beta^2}$, estimate u_β in L^2, and v_β in $L_x^\infty L_t^2$. Here because we want to use Bernstein, but also want to have better summation of time intervals. So we need to use local energy space X_{le}^s similarly as in case 1.2(b).

$$\begin{aligned}
&\sum_j \|\chi_j^\alpha(x) \eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma ((Q_{\sigma_m} u_\beta) v_\beta)\|^2_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\
\lesssim\ & \sum_j \alpha^{4s+5} \|\chi_j^\alpha(x) \sum_\sigma P_\alpha Q_\sigma ((Q_{\sigma_m} u_\beta) v_\beta)\|^2_{L_x^2 L_t^1[I_\alpha]} \\
\lesssim\ & \sum_j \alpha^{4s+5} \|\chi_j^\alpha(x) Q_{\sigma_m} u_\beta\|^2_{L_{t,x}^2[I_\alpha]} \|\chi_j^\alpha(x) v_\beta\|^2_{L_x^\infty L_t^2[I_\alpha]} \\
\lesssim\ & \alpha^{4s+5} \sum_{i,j} \|\chi_j^\alpha(x) Q_{\sigma_m} u_\beta^i\|^2_{L_{t,x}^2[I_\beta^i]} \sup_j \sum_i \|\chi_j^\alpha(x) v_\beta^i\|^2_{L_x^\infty L_t^2[I_\beta^i]} \\
\lesssim\ & \alpha^{4s+5} |\frac{I_\alpha}{I_\beta}| \sup_i \|Q_{\sigma_m} u_\beta^i\|_{L_{t,x}^2[I_\beta^i]} \sup_j \sum_i \|\chi_j^\alpha(x) v_\beta^i\|^2_{L_x^\infty L_t^2[I_\beta^i]} \\
\lesssim\ & \alpha^{8s+6} \beta^{-12s-12} \|u_\beta\|^2_{X_\beta[I_\beta]} \|v_\beta\|^2_{X_{le}^s}.
\end{aligned}$$

Remark 2.4.7. In these estimates, we need to sum up all the modulations larger than α^3. It is fine as long as there is a negative factor of σ through the estimate. But in the one above, we need be more careful. Split the problem into $\sigma \approx \alpha^3$, and $\sigma \gg \alpha$.

When $\sigma \approx \alpha^3$, we can sum up modulation easily.

When $\sigma \gg \alpha^3$, we prove $(\partial_t + \partial_x^3)^{-1} : L_x^2 L_t^1 \to L_x^2 L_t^\infty$ is bounded operator which is done by looking at the symbol $\frac{1}{\tau - \xi^3} = \frac{1}{\tau} + \frac{\xi^3}{\tau(\tau - \xi^3)} \approx \frac{1}{\tau}$. And $\partial_t^{-1} : L_x^2 L_t^1 \to L_x^2 L_t^\infty$ is bounded if it acts on functions which vanish at ∞.

(2) $u_\beta \in X^1, v_\beta \in S$, we also split it into two cases:
(a) When $u_\beta \in X^1, v_\beta \in X^{-s,1+s}$. Now if the output modulation $\sigma \gtrsim \alpha\beta^2$, use L^∞ on u_β, and L^2 on v_β,

$$\begin{aligned} &\|\eta_\alpha \sum_{\sigma \gtrsim \alpha\beta^2} (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \frac{\alpha^{2s+\frac{5}{2}}}{\sigma^{\frac{1}{2}}} |\frac{I_\alpha}{I_\beta}|^{\frac{1}{2}} \beta^{\frac{1}{2}-s} \beta^{-2-s} \sup_i \|u_\beta^i\|_{X^1[I_\beta]^i} \|v_\beta^i\|_{X^{-s,1+s}[I_\beta^i]} \\ \lesssim\ & \alpha^{4s+\frac{7}{2}} \beta^{-4s-4} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s} \end{aligned}$$

And if $\sigma \ll \alpha\beta^2$, we use $L_x^\infty L_t^2$ on u_β, L^2 on v_β. We still play the trick: using local energy space to get l^2 summation of the intervals.

$$\begin{aligned} &\|\eta_\alpha \sum_{\sigma \ll \alpha\beta^2} (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}} |\frac{I_\alpha}{I_\beta}|^{\frac{1}{2}} \beta^{-1-s} \beta^{-2-s} \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{X_{le}^s} \\ \lesssim\ & \alpha^{4s+4} \beta^{-4s-\frac{9}{2}} \|u_\beta\|_{X^s} \|v_\beta\|_{X_{le}^s}. \end{aligned}$$

The point here is we can sum up the modulation $\alpha^3 \lesssim \sigma \ll \alpha\beta^2$, which give us at most $\log\beta$ loss. But we are fine because of the negative power on β. We will do a similar thing whenever we want to be careful with modulation summation, hence we will ignore it.
(b) When $u_\beta \in X^1, v_\beta \in Z$. This force high modulation $\sigma_m \gtrsim \beta^3$ on u_β, or on output.

When σ_m is on u_β, use L^2 on u_β, L^∞ on v_β.

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \frac{\alpha^{2s+\frac{5}{2}}}{\sigma} \sigma^{\frac{1}{2}} |\frac{I_\alpha}{I_\beta}|^{\frac{1}{2}} \sigma_m^{-1} |I_\beta|^{-\frac{1}{2}} \beta^{-s} \beta \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{Z[I_\beta^i]} \\ \lesssim\ & \alpha^{4s+\frac{5}{2}} \beta^{-5s-5} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

When σ_m is on output, simply put L^6 on u_β, and L^3 on v_β.

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i Q_{\sigma_m} v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \frac{\alpha^{2s+\frac{5}{2}}}{\sigma_m} \sigma_m^{\frac{1}{2}} |\frac{I_\alpha}{I_\beta}|^{\frac{1}{2}} \beta^{-\frac{1}{6}-s} \beta^{-\frac{4}{3}(s+1)-\frac{1}{3}} \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{Z[I_\beta^i]} \\ \lesssim\ & \alpha^{4s+\frac{7}{2}} \beta^{-5s-5-\frac{2}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(3) $u_\beta, v_\beta \in S$. We still break into cases.
(a) $u_\beta, v_\beta \in X^{-s,1+s}$, use L^2 on both, and l^1 summation of interval is good enough.

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+3} |\frac{I_\alpha}{I_\beta}| \sup_i \|u_\beta^i\|_{L^2_{t,x}[I_\beta^i]} \|v_\beta^i\|_{L^2_{t,x}[I_\beta^i]} \\ \lesssim\ & \alpha^{6s+6} \beta^{-6s-7} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(b) $u_\beta \in X^{-s,1+s}, v_\beta \in Z$, L^2 on u_β, L^3 on v_β, with a l^1 summation of interval.

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}+\frac{1}{3}} \sigma^{-\frac{1}{6}} |\frac{I_\alpha}{I_\beta}| \sup_i \|u_\beta^i\|_{L^2_{t,x}[I_\beta^i]} \|v_\beta^i\|_{L^3_{t,x}[I_\beta^i]} \\ \lesssim\ & \alpha^{6s+\frac{16}{3}} \beta^{-7s-7-\frac{1}{3}+\frac{2}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(c) $u_\beta, v_\beta \in Z$, Here we are a bit careful about interval cut off, using the $l^{\frac{3}{2}}$ summation.

$$\begin{aligned} &\|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}+\frac{1}{6}} \sigma^{-\frac{1}{3}} \|\sum_i P_\alpha Q_\sigma (u_\beta^i v_\beta^i)\|_{L^{\frac{3}{2}}_{t,x}[I_\alpha]} \\ \lesssim\ & \alpha^{2s+\frac{5}{2}+\frac{1}{6}} \sigma^{-\frac{1}{3}} |\frac{I_\alpha}{I_\beta}|^{\frac{2}{3}} \sup_i \|u_\beta^i\|_{L^3_{t,x}[I_\beta^i]} \|v_\beta^i\|_{L^3_{t,x}[I_\beta^i]} \\ \lesssim\ & \alpha^{2s+1+\frac{8s+8}{3}} \beta^{-\frac{16}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

□

Now we use this lemma to finish our estimate of Case 3.3, $u_\lambda \in X^1, v_\alpha \in Z$.

$$\lambda u_\lambda v_\alpha = \lambda u_\lambda (M_{1\alpha} + M_{2\alpha} + R_\alpha).$$

Step 1: Let us do R_α first, using the estimate for R_α in the lemma.

$$\begin{aligned} \|\lambda \eta_\lambda u_\lambda R_\alpha\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2} &\lesssim \lambda^{3s+\frac{5}{2}} \|\eta_\lambda u_\lambda\|_{L_x^\infty L_t^2} \|\eta_\lambda R_\alpha\|_{L_x^2 L_t^\infty} \\ &\lesssim \lambda^{3s+\frac{5}{2}} \lambda^{-1-s} \alpha^{-2s-\frac{3}{2}} \|\eta_\lambda u_\lambda\|_{X^1} \|\eta_\lambda R_\alpha\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ &\lesssim (\frac{\alpha}{\lambda})^{-2s-\frac{3}{2}} \|u_\lambda\|_{X^s[I_\lambda]} \|v\|^2_{X^s \cap X^s_{le}}. \end{aligned}$$

Step 2: Feed M_1 into the bilinear term, we divide it into two terms.

$$\lambda u_\lambda \sum_{\sigma \approx \alpha\beta^2, \alpha \lesssim \beta \ll \lambda} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha\beta^2} (v_\beta v_\beta) + \lambda u_\lambda \sum_{\sigma \approx \alpha\beta^2, \alpha \lesssim \lambda \lesssim \beta} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha\beta^2} (v_\beta v_\beta)$$

The first term, we will bilinear estimate for $u_\beta v_\lambda$, also here for fixed β, $P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)$ is almost althogonal to each other, so we can sum up α

$$\begin{aligned}
&\|\lambda\eta_\lambda u_\lambda \sum_{\sigma\approx\alpha\beta^2,\alpha\lesssim\beta\ll\lambda}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_{\sigma\approx\alpha\beta^2}(v_\beta v_\beta)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2}\\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\sum_{\alpha\lesssim\beta\ll\lambda}\frac{1}{\beta^2}\|\eta_\lambda u_\lambda P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)\|_{L^2_{t,x}}\\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\sum_{\alpha\lesssim\beta\ll\lambda}\frac{1}{\beta^2}\|\eta_\lambda u_\lambda u_\beta\|_{L^2_{t,x}}\|\eta_\lambda v_\beta\|_{L^\infty_{t,x}}\\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\sum_{\alpha\lesssim\beta\ll\lambda}\frac{1}{\beta^2}\lambda^{-1-s}\alpha^{\frac{1}{2}}\beta^{-2s}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\beta\|_{X^1[I_\beta]}\|v_\beta\|_{X^1[I_\beta]}\\
\lesssim\ & (\frac{\lambda}{\beta})^{2s+\frac{3}{2}}\|u_\lambda\|_{X^s[I_\lambda]}\|v_\beta\|_{X^s[I_\beta]}\|v_\beta\|_{X^s[I_\beta]}.
\end{aligned}$$

Here we actually used the fact that, when fix α, the two v_β's can be decomposed to functions with $\hat{v}_\beta$ supported on size α interval, so we used bernstein to get

$$\|v_\beta\|_{L^\infty_{t,x}}\lesssim\alpha^{\frac{1}{2}}\|v_\beta\|_{L^\infty_t L^2_x}.$$

So for $s\le-\frac{3}{4}$, we can sum up β.

For the second term, we will use at least l^4 interval summation (or better if we use local energy space). The good thing is that for β fixed, then $P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)$ are almost orthogonal to each other in both space and time, so we can sum up α and then ignore it. Also because u_β is measured on the smallest time scale, we still need to cut the interval.

$$\begin{aligned}
&\|\lambda\eta_\lambda u_\lambda \sum_{\sigma\approx\alpha\beta^2,\alpha\lesssim\lambda\lesssim\beta}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_{\sigma\approx\alpha\beta^2}(v_\beta v_\beta)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2}\\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\|\eta_\lambda u_\lambda\|_{L^4_tL^\infty_x}\|\eta_\lambda\sum_{\alpha\lesssim\lambda\lesssim\beta}\frac{1}{\beta^2}P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)\|_{L^4_tL^2_x}\\
\lesssim\ & \lambda^{3s+\frac{5}{2}-\frac{1}{4}-s}\|\eta_\lambda u_\lambda\|_{X^1}\sum_{\lambda\lesssim\beta}\|\eta_\lambda\sum_{\alpha\lesssim\lambda}P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)\|_{L^4_tL^2_x}\\
\lesssim\ & \lambda^{2s+\frac{9}{4}}\|u_\lambda\|_{X^s}\sum_{\lambda\lesssim\beta}|\frac{I_\lambda}{I_\beta}|^{\frac{1}{4}}\|\eta_\beta(v_\beta v_\beta)\|_{L^4_tL^2_x}\\
\lesssim\ & \lambda^{2s+\frac{9}{4}}\|u_\lambda\|_{X^s}\sum_{\lambda\lesssim\beta}|\frac{I_\lambda}{I_\beta}|^{\frac{1}{4}}\|\eta_\beta v_\beta\|_{L^8_tL^4_x}\|\eta_\beta v_\beta\|_{L^8_tL^4_x}\\
\lesssim\ & (\frac{\lambda}{\beta})^{3s+3}\|u_\lambda\|_{X^s}\|v_\beta\|^2_{X^s}
\end{aligned}$$

So we combine the two cases together and get

$$\|\lambda u_\lambda\sum_{\alpha\ll\lambda}M_{1\alpha}\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2}\lesssim\|u_\lambda\|_{X^s}\|v\|^2_{X^s}.$$

Step 3: Now we feed in the term M_2, We want to use local energy norm, so let us cut up the space using $\chi_j^\alpha(x)$.

$$\begin{aligned}
&\|\lambda\eta_\lambda\chi_j^\alpha(x)u_\lambda \sum_{\gamma\ll\alpha\lesssim\lambda,\sigma\gtrsim\alpha^3}(\partial_t+\partial_x^3)^{-1}\alpha P_\alpha Q_\sigma(v_\alpha v_\gamma)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\sum_{\gamma\ll\alpha\lesssim\lambda,\sigma\gtrsim\alpha^3}\frac{\alpha}{\sigma}\|\eta_\lambda u_\lambda\|_{L_x^\infty L_t^2}\|\eta_\lambda\chi_j^\alpha(x)P_\alpha Q_\sigma(v_\alpha v_\gamma)\|_{L_x^2L_t^\infty} \\
\lesssim\ & \lambda^{3s+\frac{5}{2}}\lambda^{-1-s}\|u_\lambda\|_{X^s}\sum_{\gamma\ll\alpha\lesssim\lambda,\sigma\gtrsim\alpha^3}\frac{\alpha}{\sigma}\|\eta_\lambda\chi_j^\alpha(x)P_\alpha Q_\sigma(v_\alpha v_\gamma)\|_{L_x^2L_t^\infty} \\
\lesssim\ & \sum_{\gamma\ll\alpha\lesssim\lambda,\sigma\gtrsim\alpha^3}\lambda^{2s+\frac{3}{2}}\|u_\lambda\|_{X^s}\alpha\sigma^{-1}\|\eta_\lambda\chi_j^\alpha(x)v_\alpha\|_{L_x^2L_t^\infty}\|\eta_\lambda\chi^\alpha(x)v_\gamma\|_{L_{t,x}^\infty} \\
\lesssim\ & \sum_{\gamma\ll\alpha\lesssim\lambda,\sigma\gtrsim\alpha^3}\lambda^{2s+\frac{3}{2}}\alpha^{s+\frac{5}{2}}\sigma^{-1}\gamma\|u_\lambda\|_{X^s}\|\chi_j^\alpha(x)v_\alpha\|_{X^1[I_\alpha]}\|v_\gamma\|_{X^s} \\
\lesssim\ & \lambda^{2s+\frac{3}{2}}\alpha^{s+\frac{1}{2}}\|u_\lambda\|_{X^s}\|\|\chi_j^\alpha(x)v_\alpha\|_{X^1[I_\alpha]}\|\|v\|_{X^s}
\end{aligned}$$

We can also square sum up the spatial cutoff in the estimate above, and get

$$\|\lambda\eta_\lambda u_\lambda M_2\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \lesssim \|u_\lambda\|_{X^s}\|v\|^2_{X^s\cap X^s_{le}}.$$

In the proof we used the estimate

$$\|\eta_\lambda\chi_j^\alpha(x)v_\alpha\|_{L_x^2L_t^\infty}\lesssim\|\chi_j^\alpha(x)\|_{L_x^4}\|\eta_\lambda\chi_j^\alpha(x)v_\alpha\|_{L_x^4L_t^\infty}\lesssim\alpha^{s+\frac{3}{2}}\|u_\alpha\|_{X^1[I_\alpha]}.$$

Actually we also have $L_x^2L_t^\infty$ maximal function estimate [43] on small time interval.

We end this section with two bilinear estimates, as a companion to Proposition 2.3.6. The proof is essentially repeating what we did preivously.

Proposition 2.4.8. *For $\lambda\gg\alpha$ we have the following estimates*

$$\|\eta_\lambda u_\lambda(Q_{\sigma\gtrsim\alpha^3}v_\alpha)\|_{L^2_{t,x}}\lesssim\lambda^{-3s-\frac{5}{2}}\|u_\lambda\|_{X^1[I_\lambda]}\|v\|^2_{X^s\cap X^s_{le}}, \tag{2.50}$$

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}}\lesssim\max(\lambda^{-1-s}\alpha^{-s},\lambda^{-3s-\frac{5}{2}})\|u_\lambda\|_{X^s\cap X^s_{le}}(\|v\|_{X^s\cap X^s_{le}}+\|v\|^2_{X^s\cap X^s_{le}}), \tag{2.51}$$

$$\|\eta_\lambda(Q_{\sigma\gtrsim\lambda^3}u_\lambda)v_\lambda\|_{L^2_{t,x}}\lesssim\lambda^{-3s-\frac{5}{2}}\|u_\lambda\|_{X^s\cap X^s_{le}}\|v_\lambda\|_{X^s\cap X^s_{le}}. \tag{2.52}$$

Proof. For (2.50), we reiterate the equation, and notice in all the proofs we did, we are proving a L^2 estimate of the product, with weight $\lambda^{3s+\frac{5}{2}}$.

For (2.51), we compare the estimate in the following cases

If $u_\lambda, v_\alpha\in X^1$, we have (2.30); If $u_\lambda\in S, v_\alpha\in X^1$, we have (2.32).

If $u_\lambda \in X^1, v_\alpha \in S$, we have (2.50) and

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L^2_{t,x}} \lesssim \|\eta_\lambda u_\lambda\|_{L^\infty_x L^2_t} \|\eta_\lambda v_\alpha\|_{L^2_x L^\infty_t} \lesssim \lambda^{-1-s}\alpha^{-2s-\frac{3}{2}}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.$$

If $u_\lambda, v_\alpha \in S$, we have (2.38) except for $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$. But notice that the estimate (2.39) is larger than $\lambda^{-1-s}\alpha^{-s}$.

Hence we can sum up the estimates to get (2.51).

The proof of (2.52) is carried out in the same way as all the detailed analysis before. We discuss cases of $u_\lambda \in X^1$ or $X^{-s,1+s}$ or Z, and be a bit careful when $Q_{\sigma \gtrsim \lambda^3} u_\lambda \in X^{-s,1+s}$ or $|D|^{-2s-2} X^{\frac{1}{4},\frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}$. □

2.5 Energy conservation

In this section, we aim to study the conservation of H^s energy, this part of calculation follows similar as in [13] and [45].

Given a positive multiplier a, we set

$$E_2(u) =< a(D)u, u > .$$

We want to take the symbol $a(\xi) = (1+\xi^2)^s$, but as in [45], [46],we will allow a slightly larger class of symbols.

Definition 2.5.1. a) Let $s \in \mathbb{R}, \epsilon > 0$. Then S^s_ϵ is the class of spherically symmetric symbols with the following properties:
(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(1+\xi^2)^{-\frac{\alpha}{2}}.$$

(ii) decay at infinity,

$$s \le \frac{\ln a(\xi)}{\ln(1+\xi^2)} \le s+\epsilon, \quad s-\epsilon \le \frac{\ln a(\xi)}{\ln(1+\xi^2)} \le s+\epsilon.$$

b) If a satisfies (i) and (ii) then we say that d is dominated by a, written as $d \in S(a)$, if

$$|\partial^\alpha d| \lesssim a(\xi)(1+\xi^2)^{-\frac{\alpha}{2}},$$

with constant depending only on a.

Definition 2.5.2. (a) A k-multiplier generates a k-linear functional or k-form acting on k functions $u_1, \cdots, u_k$

$$\Lambda_k(m; u_1, \cdots, u_k) = \int_{\xi_1+\cdots+\xi_k=0} m(\xi_1, \cdots, \xi_k)\widehat{u}_1(\xi_1)\cdots\widehat{u}_k(\xi_k).$$

We will write $\Lambda_k(m)$ for $\Lambda_k(m; u, \cdots, u)$.
(b) The symmetrization of a k-multiplier m is the multiplier

$$[m]_{sym}(\xi) = \frac{1}{n!} \sum_{\sigma \in S_k} m(\sigma(\xi)).$$

We have the following computation [13].

Proposition 2.5.3. *Suppose u satisfies the KdV equation* (2.1) *and m is a symmetric k-multiplier. Then*

$$\frac{d}{dt}\Lambda_k(m) = \Lambda_k(m\Delta_k) - i\frac{k}{2}\Lambda_{k+1}(m(\xi_1, \cdots, \cdots \xi_{k-1}, \xi_k + \xi_{k+1})(\xi_k + \xi_{k+1})),$$

where

$$\Delta_k = i(\xi_1^3 + \cdots + \xi_k^3).$$

Symbol calculation of modified energy

Here we construct modified energy, following the calculation in [13].
We first compute the derivative of E_2 along the flow

$$\frac{d}{dt}E_2(u) = \Lambda_3(M_3).$$

Easy to see that $M_3 = c\sum_{i=1}^3 (a(\xi_i)\xi_i)$, we will ignore the constant.
Now we form modified energy

$$E_3(u) = E_2(u) + \Lambda_3(\sigma_3),$$

and we aim to choose the symmetric 3-multiplier σ_3 to achieve a cancellation.

$$\frac{d}{dt}E_3(u) = \Lambda_3(M_3) + \Lambda_3(\sigma_3\Delta_3) + \Lambda_4(-i\frac{3}{2}\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)).$$

So if we take

$$\sigma_3 = -\frac{M_3}{\Delta_3},$$

we get

$$\frac{d}{dt}E_3(u) = \Lambda_4(M_4), \qquad M_4 = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}.$$

Similarly, we can define $E_4(u) = E_3(u) + \Lambda_4(\sigma_4)$, $\sigma_4 = -\frac{M_4}{\Delta_4}$,

$$\frac{d}{dt}E_4(u) = \Lambda_5(M_5),$$

then we have

$$M_5 = -2i[\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)]_{sym}.$$

This process can be continued to have further corrections, but we will stop here, since higher corrections are harder to estimate.

Bounds for multipliers

In order to estimate the derivative of modified energy, we need to have good bounds for M_i and σ_i. Also now M_i is defined only on the diagonal $\xi_1 + \cdots \xi_k = 0$, but in order to separate variables, we want to extend it off diagonal, this is useful when we prove local energy decay later on.

Proposition 2.5.4. *Assume that $a \in S_\epsilon^s$ and $d \in S(a)$, then there exist functions b and c such that*

$$\sum_{i=1}^{3} a(\xi_i)\xi_i = b(\xi_1, \xi_2, \xi_3)(\xi_1^3 + \xi_2^3 + \xi_3^3) + c(\xi_1, \xi_2, \xi_3)(\xi_1 + \xi_2 + \xi_3).$$

And on each dyadic region $\{\xi_1 \sim \alpha,\ \xi_2 \sim \lambda,\ \xi_3 \sim \mu,\ \alpha \leq \lambda \leq \mu\}$, we have the regularity conditions

$$\partial_1^{s_1}\partial_2^{s_2}\partial_3^{s_3} b(\xi_1, \xi_2, \xi_3) \lesssim a(\alpha)\lambda^{-1}\mu^{-1}\alpha^{-s_1}\lambda^{-s_2}\mu^{-s_3},$$
$$\partial_1^{s_1}\partial_2^{s_2}\partial_3^{s_3} c(\xi_1, \xi_2, \xi_3) \lesssim a(\alpha)\lambda^{-1}\mu\alpha^{-s_1}\lambda^{-s_2}\mu^{-s_3}.$$

Proof. Since

$$\xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3 + (\xi_1 + \xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 - \xi_1\xi_3).$$

Let's construct

$$b = \frac{\sum_{i=1}^{3} a(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3},$$
$$c = -b(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 - \xi_1\xi_3).$$

Notice that $a(x)x$ is a decreasing function for x, then the estimates are straightforward. □

Bound for M_3 and σ_3

We have $M_3 = \sum_{i=1}^{3} a(\xi_i)\xi_i$, $\sigma_3 = \frac{M_3}{\Delta_3}$ modulo a constant.

Proposition 2.5.5. *On the set*

$$\Omega = \{\xi_1 + \xi_2 + \xi_3 = 0\} \cap \{\xi_1 \sim \alpha, \xi_2 \sim \xi_3 \approx \lambda \geq \alpha\}$$

we have

$$|M_3(\xi_1, \xi_2, \xi_3)| \lesssim a(\alpha)\alpha,$$
$$|\sigma_3(\xi_1, \xi_2, \xi_3)| \lesssim \frac{a(\alpha)}{\lambda^2}.$$

Proof. If $\alpha \approx \lambda$, no need to do any proof. In case $\alpha \ll \lambda$, using the fact a is spherical symmetric,

$$\sum_{i=1}^{3} a(\xi_i)\xi_i = a(\xi_1)\xi_1 - a(\xi_2)\xi_1 - a(\xi_2)\xi_3 + a(\xi_3)\xi_3$$

and we have $|a(\xi_3)\xi_3 - a(\xi_2)\xi_3| \lesssim |a'(\xi_3)\xi_1\xi_3| \lesssim |a(\xi_3)\xi_1|$. So the estimate for M_3 become obvious. Using the fact that $\Delta_3 = 3\xi_1\xi_2\xi_3$ on set Ω, we get bounds for σ_3. □

From this we can prove that $E_3(u)$ is bounded by $E_2(u)$.

Proposition 2.5.6. *We have the fact that*

$$|\Lambda_3(\sigma_3)| \lesssim |E_2(u)|^{\frac{3}{2}}. \tag{2.53}$$

Proof. We can expand the trilinear expression in dyadic frequency band $\{\lambda, \lambda, \alpha \leq \lambda\}$. Then using the estimate for σ_3, we can bound $|\Lambda_3(\sigma_3)|$ by

$$\begin{aligned} a(\alpha)\lambda^{-2}\int u_\lambda u_\lambda u_\alpha dx &\lesssim a(\alpha)\lambda^{-2}\alpha^{\frac{1}{2}}\|u_\lambda\|_{L^2}\|u_\lambda\|_{L^2}\|u_\alpha\|_{L^2} \\ &\lesssim (a(\alpha)\alpha)^{\frac{1}{2}}(a(\lambda)\lambda^2)^{-1}E_2(u_\lambda)E_2(u_\alpha)^{\frac{1}{2}}. \end{aligned}$$

We can sum up the frequencies and get (2.53). □

Bound for M_4 and σ_4

Recall that

$$M_4 = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}$$

We adopt the calculation done in [13] (Notice, our $a(\xi)$ corresponds to $m^2(\xi)$, Δ_k corresponds to α_k in their paper), we have the following formula for M_4

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{-1}{108}\frac{\Delta_4}{\xi_1\xi_2\xi_3\xi_4}[a(\xi_1) + \cdots + a(\xi_4) - a(\xi_{12}) - a(\xi_{13}) - a(\xi_{14})] \tag{2.54}$$

$$+\frac{1}{36}[\frac{a(\xi_1)}{\xi_1} + \cdots + \frac{a(\xi_4)}{\xi_4}].$$

Here we used the notation $\xi_{jk} = \xi_j + \xi_k$, and

$$\Delta_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = 3(\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4) = 3\xi_{12}\xi_{13}\xi_{14}. \tag{2.55}$$

Proposition 2.5.7. *We have the estimate for M_4*

$$|M_4| \lesssim \frac{\Delta_4 a(\min(|\xi_i|, |\xi_{jk}|))}{|\xi_1\xi_2\xi_3\xi_4|}. \tag{2.56}$$

Proof. The proof repeats the argument of Lemma 4.4 in [13]. We can also deduce it from our next proposition. □

We have bounds on σ_4 immediately from Proposition 2.5.7. But in order to do correction, we need improve it slightly.

Proposition 2.5.8.

$$|\sigma_4| \lesssim \frac{a(\min(|\xi_i|, |\xi_{jk}|))}{|\xi_1\xi_2\xi_3\xi_4|}, \qquad |\Lambda_4(\sigma_4)| \lesssim |E_2(u)|^2. \tag{2.57}$$

Proof. We look at $\Lambda_4(\sigma_4)$, expand it into dyadic frequency components, since ξ_i are symmetric, we can assume $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$

(1) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \lambda\}, \mu \gg \lambda$. Then we have $\min(\xi_i, \xi_{ij}) = \xi_{12} \lesssim \lambda$ and $|\sigma_4| \lesssim \frac{a(\xi_{12})}{\lambda^2\mu^2}$. In this case, we can bound $\Lambda_4(\sigma_4)$ by

$$\begin{aligned} & a(\xi_{12})\lambda^{-2}\mu^{-2}\int u_\mu u_\mu u_\lambda u_\lambda dx \\ \lesssim\ & a(\xi_{12})|\xi_{34}|\lambda^{-2}\mu^{-2}\|u_\mu\|_{L^2}\|u_\mu\|_{L^2}\|u_\lambda\|_{L^2}\|u_\lambda\|_{L^2} \\ \lesssim\ & a(\xi_{12})|\xi_{12}|(a(\mu)\mu^2)^{-1}(a(\lambda)\lambda^2)^{-1}E_2(u_\mu)E_2(u_\lambda). \end{aligned}$$

Here notice that $a(x)x$ is bounded and we can sum up the frequencies.

(2) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \alpha\}, \mu \gg \lambda \gg \alpha$. In this case, we have $\min(\xi_i, \xi_{ij}) = \xi_4$, but we need attention with the estimate here. In fact, with the expression for M_4(2.54), we can separate the expression of σ_4 into two parts.

One term looks like

$$-\frac{1}{108}\frac{1}{\xi_1\xi_2\xi_3\xi_4}[a(\xi_1) + a(\xi_2) - a(\xi_{13}) - a(\xi_{14})] + \frac{1}{36\Delta_4}[\frac{a(\xi_1)}{\xi_1} + \frac{a(\xi_2)}{\xi_2}]$$

and it is bounded by $\frac{a(\mu)}{\alpha\lambda\mu^2}$.
And the other term looks like(if we ignore the constant $-\frac{1}{108}$),

$$\begin{aligned} & \frac{a(\xi_3) + a(\xi_4) - a(\xi_{12})}{\xi_1\xi_2\xi_3\xi_4} - \frac{1}{\xi_{12}\xi_{13}\xi_{14}}[\frac{a(\xi_3)}{\xi_3} + \frac{a(\xi_4)}{\xi_4}] \\ = \ & \frac{a(\xi_3)\xi_3\xi_4\xi_{12} + a(\xi_3)\xi_1\xi_2\xi_3 + a(\xi_4)\xi_3\xi_4\xi_{12} + a(\xi_4)\xi_1\xi_2\xi_4 - a(\xi_{12})\xi_{12}\xi_{13}\xi_{14}}{\xi_1\xi_2\xi_3\xi_4\xi_{12}\xi_{13}\xi_{14}} \end{aligned}$$

So it is bounded by $\frac{a(\alpha)}{\lambda^2\mu^2}$. Now we can bound $\Lambda_4(\sigma_4)$ by

$$\begin{aligned} & a(\alpha)\lambda^{-2}\mu^{-2}\int u_\mu u_\mu u_\lambda u_\alpha dx \\ \lesssim\ & a(\alpha)\lambda^{-2}\mu^{-2}\lambda^{\frac{1}{2}}\alpha^{\frac{1}{2}}\|u_\mu\|_{L^2}\|u_\mu\|_{L^2}\|u_\lambda\|_{L^2}\|u_\alpha\|_{L^2} \\ \lesssim\ & (a(\alpha)\alpha)^{\frac{1}{2}}(a(\lambda)\lambda^3)^{-\frac{1}{2}}(a(\mu)\mu^2)^{-1}E_2(u_\mu)E_2(u_\lambda)^{\frac{1}{2}}E_2(u_\alpha)^{\frac{1}{2}}. \end{aligned}$$

(3) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \lambda\}, \mu \gg \lambda$. Here $\min(\xi_i, \xi_{ij}) = \lambda$, we can do same estimate as in previous case and get $|\sigma_4| \lesssim \frac{a(\lambda)}{\mu^4}$, we need bound the expression

$$\begin{aligned} & a(\lambda)\mu^{-4} \int u_\mu u_\mu u_\mu u_\lambda dx \\ \lesssim \ & a(\lambda)\mu^{-4}\lambda^{\frac{1}{2}}\mu^{\frac{1}{2}} \|u_\mu\|_{L^2}\|u_\mu\|_{L^2}\|u_\lambda\|_{L^2}\|u_\alpha\|_{L^2} \\ \lesssim \ & (a(\lambda)\lambda)^{\frac{1}{2}}(a(\mu)\mu^2)^{-1}(a(\mu)\mu^3)^{-\frac{1}{2}} E_2(u_\mu)^{\frac{3}{2}} E_2(u_\lambda)^{\frac{1}{2}}. \end{aligned}$$

(4) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \mu\}, \min(\xi_i, \xi_{ij}) = \xi_{ij}$. For convenience, suppose it is ξ_{12}, then we have $|\sigma_4| \lesssim \frac{a(\xi_{12})}{\mu^4}$. And we can bound $\Lambda_4(\sigma_4)$ by

$$a(\xi_{12})\mu^{-4} \int u_\mu u_\mu u_\mu u_\mu dx \lesssim a(\xi_{12})|\xi_{12}|(a(\mu)\mu^2)^{-2} E_2(u_\mu)^2.$$

In all the cases above, we can sum up the frequency and get (2.57). □

Remark 2.5.9. From the estimate in the proof, we see that actually we have slightly better bound for M_4 than Proposition 2.5.7 in the following two cases

1. $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \alpha\}, \alpha \ll \lambda \ll \mu$, $|M_4| \lesssim \frac{a(\alpha)}{\lambda}$,

2. $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \lambda\}, \lambda \ll \mu$, $|M_4| \lesssim \frac{a(\lambda)}{\mu}$.

Proposition 2.5.10. *We have the error estimate when* $s \geq -\frac{4}{5}$

$$|\int_0^1 \Lambda_4(M_4)dt| \lesssim \|u\|^4_{X^s \cap X^s_{le}}(1 + \|u\|_{X^s \cap X^s_{le}} + \|u\|^2_{X^s \cap X^s_{le}})$$

Proof. As before, we expand the error term $\Lambda_4(M_4)$ in the dyadic frequency component and discuss in each cases. Since $u \in X^s \cap X^s_{le}$, we still decompose each piece as $u_\lambda = u_{\lambda,1} + u_{\lambda,2}$, $u_{\lambda,1} \in X^1[I_\lambda]$, $u_{\lambda,2} \in S[I_\lambda]$. We abuse the notation and still use u_λ to represent any of them. We assume $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$.

One thing to notice the the high modulation relation. Since

$$\int_0^1 \Lambda_4(M_4)dt = \int_\Sigma M_4 \widetilde{u}_1 \widetilde{u}_2 \widetilde{u}_3 \widetilde{u}_4 \, d\xi \, d\tau.$$

$$\Sigma = \{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0\}.$$

We have

$$(\tau_1 - \xi_1^3) + (\tau_2 - \xi_2^3) + (\tau_3 - \xi_3^3) + (\tau_4 - \xi_4^3) = -\Delta_4 = -3\xi_{12}\xi_{13}\xi_{14}. \tag{2.58}$$

Hence we get the high modulation

$$\sigma_M = \max\{|\tau_1-\xi_1^3|, |\tau_2-\xi_2^3|, |\tau_3-\xi_3^3|, |\tau_4-\xi_4^3|\} \gtrsim |\xi_{12}\xi_{13}\xi_{14}|. \tag{2.59}$$

(1) $\{\xi_1,\xi_2,\xi_3,\xi_4\} = \{\mu,\mu,\lambda,\lambda\}, \mu \gg \lambda$. Then we have $\min(\xi_i,\xi_{ij}) = \xi_{12} \lesssim \lambda$ and $|M_4| \lesssim |\frac{a(\xi_{12})\xi_{12}}{\lambda^2}|$, also notice function $a(x)x$ is bounded. Let us use the crude bilinear estimate (2.51), and also we need cut the time interval $[0,1]$ into smaller scale of size μ^{4s+3}.

$$\begin{aligned} &\int_0^1 \Lambda_4(M_4)dt \\ \lesssim\ & |\frac{a(\xi_{12})\xi_{12}}{\lambda^2}|(\max\{\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}}\})^2\mu^{-4s-3}\|u_\mu\|^2_{X^s\cap X^s_{le}}(\|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}})^2 \\ \lesssim\ & \max\{\mu^{-6s-5}\lambda^{-2s-2}, \mu^{-10s-8}\lambda^{-2}\}\|u_\mu\|^2_{X^s\cap X^s_{le}}\sum_{k=2}^4\|u\|^k_{X^s\cap X^s_{le}}. \end{aligned}$$

It is summable when $s \geq -\frac{4}{5}$.
(2) $\{\xi_1,\xi_2,\xi_3,\xi_4\} = \{\mu,\mu,\lambda,\alpha\}, \mu \gg \lambda \gg \alpha$. $|M_4| \lesssim \frac{a(\alpha)}{\lambda}$. We estimate it in exactly the same way as (1).

$$\begin{aligned} &\int_0^1 \Lambda_4(M_4)dt \\ \lesssim\ & \frac{a(\alpha)}{\lambda}\max\{\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}}\}\max\{\mu^{-1-s}\alpha^{-s}, \mu^{-3s-\frac{5}{2}}\}\mu^{-4s-3} \\ & \times\|u_\mu\|^2_{X^s\cap X^s_{le}}(\|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}})^2. \end{aligned}$$

By computing the exponents, we can sum up the frequencies when when $s \geq -\frac{4}{5}$.
(3) $\{\xi_1,\xi_2,\xi_3,\xi_4\} = \{\mu,\mu,\mu,\lambda\}, \mu \gg \lambda$, here $\min(\xi_i,\xi_{ij}) = \lambda$, $\sigma_m \gtrsim \mu^3$.
Case 1 When at least one of u_μ have high modulation, here we cut the interval to size μ^{4s+3} and use bilinear on $(Q_{\sigma_m}u_\mu)u_\mu$ (2.52) and $u_\mu u_\lambda$, we see that we get the bound

$$\begin{aligned} &\int_0^1\int_{\mathbb{R}} \frac{a(\lambda)}{\mu}(Q_{\sigma_m}u_\mu)u_\mu u_\mu u_\lambda dxdt \\ \lesssim\ & \frac{a(\lambda)}{\mu}\mu^{-3s-\frac{5}{2}}\max(\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}})\mu^{-4s-3}\|u_\mu\|^3_{X^s\cap X^s_{le}}(\|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}}) \\ \lesssim\ & \mu^{-10s-9}\|u_\mu\|^3_{X^s\cap X^s_{le}}(\|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}}). \end{aligned}$$

So it is summable for $s \geq -\frac{9}{10}$
Case 2 When the high modulation fall on u_λ, this is the hard case, we use the L^2 on $Q_{\sigma_m}u_\lambda$, and L^2 on the product $u_\mu u_\mu u_\mu$.

$$\|\eta_\lambda Q_{\sigma\gtrsim\mu^3}u_\lambda\|_{L^2_{t,x}} \lesssim \lambda^{-3s-\frac{3}{2}}\mu^{-3}\|u_\lambda\|_{X_\lambda[I_\lambda]}, \tag{2.60}$$

$$\|\eta_\lambda Q_{\sigma \gtrsim \mu^3} u_\lambda\|_{L^2_{t,x}} \lesssim \lambda^{3+4s}\mu^{-6s-6}\|u_\lambda\|_{X^{-3-4s,2s+2}[I_\lambda]}, \tag{2.61}$$

$$\|\eta_\mu u_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-\frac{1}{2}-3s}\|u_\mu\|^3_{X^s}. \tag{2.62}$$

The third one is proved by discussing $u_\mu \in X^1$ or S, and notice that none of them has high modulation. Then we get

$$\begin{aligned} &\int_0^1\int_{\mathbb{R}} \frac{a(\lambda)}{\mu} u_\mu u_\mu u_\mu Q_{\sigma\gtrsim\mu^3} u_\lambda dtdx \\ \lesssim\ & \frac{a(\lambda)}{\mu} \max\{\lambda^{-3s-\frac{3}{2}}\mu^{-3}, \lambda^{3+4s}\mu^{-6s-6}\}\mu^{-\frac{1}{2}-3s}\mu^{-4s-3}\|u_\mu\|^3_{X^s\cap X^s_{le}}\|v_\lambda\|_{X^s\cap X^s_{le}} \end{aligned}$$

And we can sum up frequencies when $s \geq -\frac{21}{26}$.
(4) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \mu\}$ here we need to discuss the size of ξ_{ij}.

$$\xi_{12} + \xi_{13} + \xi_{14} = 2\xi_1$$

so at least one of them is of size μ
Case 1: When $\xi_{ij} \gtrsim \mu$, then we have $|M_4| \lesssim \frac{a(\mu)}{\mu}$, and we have the high modulation factor $\sigma_m \gtrsim \mu^3$, so we use bilinear on $(Q_{\sigma_m} u_\mu)u_\mu$, and L^2 for each of $u_\mu u_\mu$.

Notice the (8,4) is Strichartz pair and using the size of interval we get

$$\|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{\frac{1}{2}-s}\|u_\mu\|^2_{X^1_\mu[I_\mu]}. \tag{2.63}$$

From (2.32) we have

$$\|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-\frac{3}{2}-2s}\|u_\mu\|_{X^1_\mu[I_\mu]}\|u_\mu\|_{S[I_\mu]}. \tag{2.64}$$

From (2.38) and (2.39) we get

$$\|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-1-s}\|u_\mu\|_{S[I_\mu]}\|u_\mu\|_{S[I_\mu]}. \tag{2.65}$$

$$\begin{aligned} &\int_0^1\int_{\mathbb{R}} \frac{a(\mu)}{\mu}(Q_{\sigma_m} u_\mu) u_\mu u_\mu u_\mu dxdt \\ \lesssim\ & \frac{a(\mu)}{\mu}\mu^{-3s-\frac{5}{2}}\mu^{\frac{1}{2}-s}\mu^{-4s-3}\|u_\mu\|^4_{X^s} \\ \lesssim\ & a(\mu)\mu^{-8s-6}\|u_\mu\|^4_{X^s} \end{aligned}$$

so it is summable when $s \geq -1$.
Case 2: When two of ξ_{ij} is big, one is small, let's assume $\xi_{13} \ll \mu, \xi_{12}, \xi_{14} \gtrsim \mu$, we have $|M_4| \lesssim |\frac{a(\xi_{13})\xi_{13}}{\mu^2}|$. Then we can easily calculate that

$$(\xi_1 - \xi_2) + (\xi_1 - \xi_4) - (\xi_1 + \xi_3) = 2\xi_1$$

since $\xi_{13} \ll \mu$, we must have at least one of $\xi_1 - \xi_2$ or $\xi_1 - \xi_4$ be of size μ, with out loss of generality, we assume $|\xi_1 - \xi_2| \gtrsim \mu$, so we have separation of frequency, i.e

$$|\xi_1 - \xi_2| \approx \mu, |\xi_1 + \xi_2| \approx \mu$$

and we can also prove that

$$|\xi_3 + \xi_4| = |\xi_1 + \xi_2| \approx \mu, |\xi_3 - \xi_4| = |\xi_3 + \xi_1 - (\xi_1 + \xi_4)| \approx \mu$$

Now we have the bilinear estimate of two u_μ's which have frequency separation.

$$\|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-1-2s}\|u_\mu\|_{X^1[I_\mu]}. \tag{2.66}$$

Together with (2.64) and (2.65), we get

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}} |\frac{a(\xi_{12})\xi_{12}}{\mu^2}| u_\mu u_\mu u_\mu u_\mu dxdt \\ \lesssim\ & |\frac{a(\xi_{12})\xi_{12}}{\mu^2}|(\mu^{-1-2s})^2\mu^{-4s-3}\|u_\mu\|^4_{X^s} \\ \lesssim\ & \mu^{-8s-7}\|u_\mu\|^4_{X^s}. \end{aligned}$$

so we can sum up for $s \geq -\frac{7}{8}$.

Case 3: When one of ξ_{1j} is big, the other two small. We can assume $\xi_{12} \leq \xi_{13} \ll \mu$, $\xi_{14} \gtrsim \mu$. In this case, we don't have frequency separation. $|M_4| \lesssim |\frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3}|$.

But we still have (2.31), so together with (2.64) and (2.65), we get

$$\begin{aligned} &\int_0^1 \int \mathbb{R} |\frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3}| u_\mu u_\mu u_\mu u_\mu dxdt \\ \lesssim\ & |\frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3}|((|\xi_{13}|^{-\frac{1}{2}}\mu^{-\frac{1}{2}})^2\mu^{-4s}\mu^{-4s-3}\|u_\mu\|^4_{X^s} \\ \lesssim\ & \mu^{-8s-7}\|u_\mu\|^4_{X^s} \end{aligned}$$

so it is summable when $s \geq -\frac{7}{8}$. □

2.6 Local energy decay

Let $\chi(x)$ be a positive, rapidly decaying function, with Fourier transform supported in [-1, 1]. Let a be as in the previous section. We define the indefinite quadratic form

$$\widetilde{E}_2(u) = \sum_\lambda \frac{1}{2}\int(\phi_\lambda a(D) + a(D)\phi_\lambda)u_\lambda u_\lambda dx. \tag{2.67}$$

Here ϕ_λ is an odd smooth function whose derivative has the form $\phi'_\lambda(x) = \psi_\lambda(x)^2$, $\psi_\lambda(x) = \lambda^{-2s-\frac{5}{2}}\chi(\frac{x}{\lambda^{4s+5}})$. We will abuse the notation a bit, and (2.67)

$$\widetilde{E}_2(u) = \frac{1}{2}\int (\phi a(D) + a(D)\phi)uudx, \tag{2.68}$$

with the understanding that it is really defined on each dyadic pieces, and $\phi = \phi_\lambda$ on each piece.

Then we have the calculation

$$\frac{d}{dt}\widetilde{E}_2(u) = \tilde{R}_2(u) + \tilde{R}_3(u), \tag{2.69}$$

where

$$\tilde{R}_2(u) = \langle (a(D)\phi_x + \phi_x a(D))u_x, u_x\rangle + \langle (a(D)\phi_{xxx} + \phi_{xxx}a(D))u, u\rangle,$$
$$\tilde{R}_3(u) = cRe\langle (a(D)\phi + \phi a(D))u, (u^2)_x\rangle.$$

We will see in the following propositions that $\tilde{R}_2$ can be used to measure local energy.

Proposition 2.6.1. *Let $a \in S^s_\epsilon$, ϕ defined as above, then we have the fixed time bound*

$$|\widetilde{E}_2(u)| \lesssim E_2(u),$$
$$|\langle (a(D)\phi_{xxx} + \phi_{xxx}a(D))u, u\rangle| \lesssim E_2(u).$$

Proof. Since ϕ and ϕ_{xxx} are bounded and its fourier transform has compact support,

$$|\langle a(D)\phi u, u\rangle| = |\langle (a(D)^{1/2}\phi a(D)^{-1/2})a(D)^{1/2}u, a(D)^{1/2}u\rangle| \lesssim E_2(u).$$

Other terms are proved similarly. □

Proposition 2.6.2. *We can use R_2 to bound the local energy*

$$\|\psi a(D)^{\frac{1}{2}}Du\|^2_{L^2_x} \lesssim \tilde{R}_2(u) + cE_2(u). \tag{2.70}$$

Proof.

$$\langle (a(D)\phi_x + \phi_x a(D))u_x, u_x\rangle = 2\|\psi(a(D)^{\frac{1}{2}}D)u\|^2_{L^2_x} + \langle r^w(x, D)u, u\rangle.$$

Here

$$r^w(x, D) = [a(D)^{1/2}, [a(D)^{1/2}, \psi^2]],$$

so its symbol r satisfy the estimate

$$\partial^\alpha_x\partial^\beta_\xi r(x, \xi) \lesssim \langle x\rangle^{-N}(1 + \xi)^{-\frac{\beta}{2}}a(\xi).$$

Hence

$$|\langle r^w(x, D)u, u\rangle| \lesssim E_2(u).$$

Combine with previous proposition and the formula for $\tilde{R}_2$, we get estimate (2.70). □

Integrating (2.69) and (2.70) on time interval $[0,1]$, together with Proposition 2.6.1 we get

$$\int_0^1 \|\psi a(D)^{\frac{1}{2}} Du\|_{L_x^2}^2 dt \lesssim \|u\|_{l_\lambda^2 L_t^\infty H^s}^2 + |\int_0^1 \tilde{R}_3(u)dt|. \tag{2.71}$$

Next, we can rewrite $\tilde{R}_3$ in the Fourier space. Notice that original definition of (2.67) is on dyadic pieces, so $\tilde{R}_3$ takes the following form

$$\tilde{R}_3(u) = 2\int_{\mathbb{R}} \phi(x)e^{ix\xi} \int_{P_\xi} (a(\xi_1-\xi)+a(\xi_1))\chi(\xi)(\xi_{23})\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx,$$

$$P_\xi = \{\xi_1+\xi_2+\xi_3=\xi\}.$$

Here ϕ is actually ϕ_λ, $\chi(\xi)$ is the multiplier used to define projection P_λ.

Now we can symmetrize it, using the notation $A(\xi_i) = (a(\xi_i-\xi)+a(\xi_i))\chi(\xi_i)$

$$\begin{aligned}\tilde{R}_3(u) &= \int_{\mathbb{R}} \phi(x)e^{ix\xi} \int_{P_\xi} (\sum_{i=1}^{3} A(\xi_i))\xi\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx \\ &- \int_{\mathbb{R}} \phi(x)e^{ix\xi} \int_{P_\xi} (\sum_{i=1}^{3} A(\xi_i)\xi_i)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx.\end{aligned}$$

To better estimate it, we use proposition 2.5.4, and rewrite

$$\sum_{i=1}^{3} A(\xi_i)\xi_i = B(\xi_1,\xi_2,\xi_3)(\xi_1^3+\xi_2^3+\xi_3^3) + C(\xi_1,\xi_2,\xi_3)(\xi_1+\xi_2+\xi_3). \tag{2.72}$$

So we split $\tilde{R}_3$ into

$$\tilde{R}_3(u) = \tilde{R}_{good,3} + \tilde{R}_{bad,3},$$

where $\tilde{R}_{good,3}$ and $\tilde{R}_{bad,3}$ take the following form,

$$\tilde{R}_{good,3} = \int_{\mathbb{R}} \phi(x)e^{ix\xi} \int_{P_\xi} (\sum_{i=1}^{3} A(\xi_i) - C)\xi\ \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx,$$

$$\tilde{R}_{bad,3} = -\int_{\mathbb{R}} \phi(x)e^{ix\xi} \int_{P_\xi} (B(\xi_1,\xi_2,\xi_3)(\xi_1^3+\xi_2^3+\xi_3^3))\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx.$$

Proposition 2.6.3. *Let a, ϕ as before, then we have the estimate*

$$|\int_0^1 \tilde{R}_{good,3}(u)dt| \lesssim \sum_{k=3,4} \|u\|_{X^s \cap X_{le}^s}^k.$$

Proof. As in proposition 2.5.4, we have

$$C = -\frac{\sum A(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3}(\xi_1^2+\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3-\xi_2\xi_3).$$

Let's look at one term of $\sum A(\xi) - C$,

$$\begin{aligned} & A(\xi_1) + \frac{A(\xi_1)\xi_1}{3\xi_1\xi_2\xi_3}(\xi_1^2+\xi_2^2+\xi_3^2-\xi_1\xi_2-\xi_1\xi_3-\xi_2\xi_3) \\ = & \frac{A(\xi_1)}{3\xi_2\xi_3}[(\xi_1+\xi_2+\xi_3)^2 - 3\xi_1(\xi_1+\xi_2+\xi_3)+3\xi_1^2]. \end{aligned}$$

So on P_ξ, we have

$$\begin{aligned} & \sum A(\xi) - C \\ = & \frac{\sum A(\xi_i)\xi_i\xi^2 - 3\sum A(\xi_i)\xi_i^2\xi + 3\sum A(\xi_i)\xi_i^3}{3\xi_1\xi_2\xi_3}. \end{aligned}$$

When we feed it to the integral, we can do integration by parts to trade ξ for derivative of ϕ

$$\begin{aligned} \tilde{R}_{good,3} &= -i\int_{\mathbb{R}}\phi_{xxx}(x)e^{ix\xi}\int_{P_\xi}\frac{\sum A(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx \\ &+ \int_{\mathbb{R}}\phi_{xx}(x)e^{ix\xi}\int_{P_\xi}\frac{\sum A(\xi_i)\xi_i^2}{\xi_1\xi_2\xi_3}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx \\ &+ i\int_{\mathbb{R}}\phi_{x}(x)e^{ix\xi}\int_{P_\xi}\frac{\sum A(\xi_i)\xi_i^3}{\xi_1\xi_2\xi_3}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx. \end{aligned}$$

Let's decompose the region into dyadic region $\{\alpha,\lambda,\lambda\}$, $\alpha \le \lambda$ and we can estimate the symbols, using the fact $a \in S^s_\epsilon$, the the proof is similar to proposition 2.5.5.

$$|\frac{\sum A(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3}| \lesssim \frac{a(\alpha)}{\lambda^2}, \quad |\frac{\sum A(\xi_i)\xi_i^2}{3\xi_1\xi_2\xi_3}| \lesssim \frac{a(\lambda)}{\alpha}, \quad |\frac{\sum A(\xi_i)\xi_i^3}{3\xi_1\xi_2\xi_3}| \lesssim \frac{a(\lambda)\lambda}{\alpha}.$$

The three terms in $\tilde{R}_{good,3}$ are similar, so we only do the third term, since that has the worst bound. Denote it as III

$$|\int_0^1 III| \lesssim \frac{a(\lambda)\lambda}{\alpha}\int_0^1\int_{\mathbb{R}}\phi_x(x)u_\lambda u_\lambda u_\alpha dxdt.$$

Case 1. $\alpha \ll \lambda$, put L^2 on one of u_λ, and bilinear estimate on $u_\lambda u_\alpha$ (2.51) , also notice ϕ_x is fast decaying on spatial scale λ^{4s+5}, so we can use local energy norm to avoid interval summation. (Based on our computation below, we can even perform interval summation with no difficulty.)

$$\begin{aligned}
|\int_0^1 III| &\lesssim \frac{a(\lambda)\lambda}{\alpha}\sum_{I_\lambda}\int_{I_\lambda}\int_{\mathbb{R}}\phi_x(x)u_\lambda u_\lambda u_\alpha dxdt \\
&\lesssim \sum_{I_\lambda}\frac{a(\lambda)\lambda}{\alpha}\lambda^{-4s-5}\|\eta_\lambda\chi^\lambda(x)u_\lambda\|_{L^2_{t,x}}\|\eta_\lambda\chi^\lambda(x)u_\lambda u_\alpha\|_{L^2_{t,x}} \\
&\lesssim \frac{a(\lambda)\lambda^{-4s-4}}{\alpha}\lambda^{\frac{3}{2}+s}\max\{\lambda^{-3s-\frac{5}{2}},\lambda^{-1-s}\alpha^{-s}\}\sum_{I_\lambda}\|\eta_\lambda\chi^\lambda(x)u_\lambda\|^2_{X_\lambda}(\|u\|_{X^s}+\|u\|^2_{X^s}) \\
&\lesssim \lambda^{-4s-5}\max\{\alpha^{-1},\lambda^{2s+\frac{3}{2}}\alpha^{-1-s}\}\|u_\lambda\|^2_{X^s_{le}}(\|u\|_{X^s}+\|u\|^2_{X^s}).
\end{aligned}$$

Case 2. $\alpha\approx\lambda$, notice we have high modulation $\sigma_m\gtrsim\lambda^3$. Then bound $(Q_{\sigma_m}u_\lambda)u_\lambda$ in L^2 (2.52), and the other one in L^2.

$$\begin{aligned}
|\int_0^1 III| &\lesssim a(\lambda)\sum_{I_\lambda}\int_{I_\lambda}\int_{\mathbb{R}}\phi_x(x)(Q_{\sigma_m}u_\lambda)u_\lambda u_\lambda dxdt \\
&\lesssim a(\lambda)\lambda^{-3s-\frac{5}{2}}\lambda^{\frac{3}{2}+s}\lambda^{-4s-5}\|u_\lambda\|^3_{X^s\cap X^s_{le}} \\
&\lesssim \lambda^{-4s-6}\|u_\lambda\|^3_{X^s\cap X^s_{le}}.
\end{aligned}$$

□

For the part $\tilde{R}_{bad,3}$ we can not estimate it directly, so we will add some correction as we did before. Take

$$\tilde{E}_3(u)=\tilde{E}_2(u)+\Lambda_B(u),$$

$$\Lambda_B(u)=-i\int_{\mathbb{R}}\phi(x)e^{ix\xi}\int_{P_\xi}B(\xi_1,\xi_2,\xi_3)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)d\xi_i d\xi dx.$$

Notice (2.72), then we have

$$\frac{d}{dt}\tilde{E}_3(u)=\tilde{R}_2(u)+\tilde{R}_{good,3}+\tilde{R}_4(u).$$

Here

$$\tilde{R}_4(u)=-\int_{\mathbb{R}}\phi(x)e^{ix\xi}\int_{P_\xi}[B(\xi_1,\xi_2,\xi_{34})\xi_{34}]_{sym}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)d\xi_i d\xi dx.$$

Here we need to do two things, show $|\tilde{E}_3(u)|\lesssim E_2^{\frac{3}{2}}(u)$, which is same as proposition 2.6.1 and 2.5.6 and 2.6.2 . And estimate the error, which repeats the proof of proposition 2.5.10 using the fact that ϕ is bounded, $\hat{\phi}$ has compact support, so they does no change to the proof, in fact, since we have the spatial localization, we have the privilege of omitting the interval summation by control them in local energy space.

Proposition 2.6.4. *With a, ϕ as before, when $s \geq -\frac{4}{5}$, we have the error estimate*

$$|\int_0^1 \tilde{R}_4(u)dt| \lesssim \|u\|^4_{X^s\cap X^s_{le}}(1+\|u\|_{X^s\cap X^s_{le}}+\|u\|^2_{X^s\cap X^s_{le}}).$$

Combining all the propositions in this section, we get the local energy bound.

Lemma 2.6.5. *The solution to the KdV equation (2.1) satisfy the following bound*

$$\begin{aligned}&\sum_\lambda \lambda^{-2s-5}\sup_j \|\chi_j^\lambda \partial_x u_\lambda\|^2_{L^2_{x,t}}\\ \lesssim\ &\sup_t \|u(t)\|^2_{H^s}(1+\|u(t)\|_{H^s})+\|u\|^3_{X^s\cap X^s_{le}}+\|u\|^4_{X^s\cap X^s_{le}}(1+\|u\|_{X^s\cap X^s_{le}}+\|u\|^2_{X^s\cap X^s_{le}}).\end{aligned}$$

2.7 Finishing the proof

To finish the whole argument, we need to pick suitable symbol $a(\xi)$ in the previous two sections. As in [45], we pick slow varying sequence.

$$\beta^0_\lambda = \frac{\lambda^{2s}\|u_{0\lambda}\|^2_{L^2}}{\|u_0\|^2_{H^s}},$$

$$\beta_\lambda = \sum_\mu 2^{-\frac{\epsilon}{2}|\log\lambda-\log\mu|}\beta^0_\mu.$$

These β_λ satisfy the following property

(i) $\lambda^{2s}\|u_{0\lambda}\|^2_{L^2} \lesssim \beta_\lambda \|u_0\|^2_{H^s}$,

(ii) $\sum \beta_\lambda \lesssim 1$,

(iii) β_λ is slow varying in the sense that

$$|\log_2\beta_\lambda - \log_2\beta_\mu| \lesssim \frac{\epsilon}{2}|\log_2\lambda-\log_2\mu|. \tag{2.73}$$

Now if we take $a_\lambda = \lambda^{2s}\max(1, \beta^{-1}_{\lambda_0}2^{-\epsilon|\log_2\lambda-\log_2\lambda_0|})$, and correspondingly we take

$$a(\xi)\approx a_\lambda, \quad |\xi|\approx\lambda$$

Then from the slow varying property (2.73), we get

$$\sum_\lambda a_\lambda\|u_{0\lambda}\|^2_{L^2_x} \lesssim \sum_\lambda \lambda^{2s}\|u_{0\lambda}\|^2_{L^2_x} + 2^{-\epsilon|\log_2\lambda-\log_2\lambda_0|}\lambda^{2s}\beta^{-1}_{\lambda_0}\|u_{0\lambda}\|^2_{L^2_x} \lesssim \|u_0\|^2_{H^s}.$$

Assume that $\|u\|_{l^2_\lambda L^\infty_t H^s} \ll 1$, which implies $\sup_t E_2(u(t)) \ll 1$. Recall that

$$\frac{d}{dt}(E_2(u) + \Lambda_3(\sigma_3)) = \Lambda_4(M_4),$$

so from Proposition 2.5.6 and 2.5.10, we get

$$(\sum_\lambda a(\lambda)\|u_\lambda(t)\|^2_{L^2_x})^{\frac{1}{2}} \lesssim \|u_0\|_{H^s} + \|u\|^4_{X^s\cap X^s_{le}}(1 + \|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}}).$$

At fixed frequency $\lambda = \lambda_0$, we get

$$\sup_t \lambda_0^s\|u_{\lambda_0}(t)\|_{L^2} \lesssim \beta^{\frac{1}{2}}_{\lambda_0}(\|u_0\|_{H^s} + \|u\|^4_{X^s\cap X^s_{le}}(1 + \|u\|_{X^s\cap X^s_{le}} + \|u\|^2_{X^s\cap X^s_{le}})).$$

From the property of β_λ, we can sum up λ_0, and get (2.9).

Together with the previous section, we can prove the local energy bound in exactly the same way, so we conclude the proof of proposition 2.1.6.

Chapter 3

Local wellposedness of Chern-Simons-Schrödinger

3.1 Introduction

We consider the initial value problem for Chern-Simons-Schrödinger system

$$\begin{cases} D_t\phi & = iD_\ell D_\ell \phi + ig|\phi|^2\phi \\ \partial_t A_1 - \partial_1 A_t & = -\mathrm{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2 - \partial_2 A_t & = \mathrm{Im}(\bar{\phi} D_1 \phi) \\ \partial_1 A_2 - \partial_2 A_1 & = -\frac{1}{2}|\phi|^2. \end{cases} \tag{3.1}$$

Here ϕ is a complex-valued function and the connection coefficients A_α are real-valued functions. The covariant derivatives D_α are defined in terms of the gauge potential A_α via

$$D_\alpha := \partial_\alpha + iA_\alpha. \tag{3.2}$$

Regarding indices, we use $\alpha = 0$ for the time variable t and $\alpha = 1, 2$ for the spatial variables x_1, x_2. When we wish to exclude the time variable in a certain expression, we switch from Greek indices to Roman. Repeated indices are assumed to be summed. We will discuss initial conditions in §3.2.

The System (3.1) is a basic model of Chern-Simons dynamics [33, 18, 19, 34]. For further physical motivation for studying (3.1), see [35, 48, 59].

Local wellposedness in H^2 is established in [3]. Also given are conditions ensuring finite-time blowup. With a regularization argument, [3] demonstrates global existence in H^1 for small L^2 data.

The system (3.1) is Galilean-invariant and has conserved *charge*

$$M(\phi) := \int_{\mathbb{R}^2} |\phi|^2 dx$$

and *energy*

$$E(\phi) := \int_{\mathbb{R}^2} |D_x\phi|^2 - \frac{g}{2}|\phi|^4 dx.$$

Notice that the scaling symmetry

$$\phi(t,x) \to \lambda\phi(\lambda^2 t, \lambda x); \quad \phi_0(x) \to \lambda\phi_0(\lambda x), \quad \lambda > 0$$

preserves the charge of the initial data $M(\phi_0)$, L^2_x is the critical space for equation (3.1).

We use J_i to denote

$$J_i := \mathrm{Im}(\bar{\phi} D_i \phi).$$

By expanding the first equation in (3.1) using (3.2), we have the following nonlinear Schrödinger equation for ϕ:

$$(i\partial_t + \Delta)\phi = -i\partial_\ell(A_\ell\phi) - iA_\ell\partial_\ell\phi + (A_t + A_x^2)\phi - g|\phi|^2\phi. \tag{3.3}$$

We establish local wellposedness for (3.1) in spaces in the full subcritical range.

3.2 Gauge selection

The Chern-Simons-Schrödinger system exhibits gauge freedom in that (3.1) is invariant with respect to the transformations

$$\phi \mapsto e^{i\theta}\phi \qquad A_\alpha \mapsto A_\alpha + \partial_\alpha\theta$$

for real-valued functions $\theta(t,x)$. Therefore in order for (3.1) to be well-posed, a gauge must be selected.

A classical choice is the *Coulomb gauge*, which is derived by imposing the constraint $\nabla \cdot A_x = 0$. In low dimension, however, the Coulomb gauge has unfavorable high$\times$high $\to$ low interactions, leading us to search for a suitable replacement. In the $d = 2$ setting of wave maps into hyperbolic space, where a similar difficulty arises, Tao [54] introduced the caloric gauge as an alternative to the Coulomb gauge. See [55] for an application of the caloric gauge to wave maps ($d = 2$, large data) and [1] for an application to Schrödinger maps ($d \geq 2$, small data). We refer the reader to [56, Chapter 6] for a lengthier discussion and a comparison of various gauges. Both wave maps and Schrödinger maps are geometric evolution equations, and in such settings the function ϕ takes values not in $\mathbb{C}$, but rather more generally in some (suitable) manifold M. A gauge system arises when considering evolution equations at the level of the tangent bundle $\phi^* TM$, where ϕ^* denotes the pullback, and the caloric gauge construction is closely tied to the presence of this underlying manifold M.

In this article we adopt from [16] a different variation of the Coulomb gauge called the *parabolic gauge*. We shall also refer to it as the *heat gauge*. The defining condition of the heat gauge is

$$\nabla \cdot A_x = A_t. \tag{3.4}$$

Differentiating in the x_1 and x_2 directions the second and third equations (respectively) in (3.1) yields

$$\begin{cases} \partial_t\partial_1 A_1 - \partial_1^2 A_t &= -\partial_1 \mathrm{Im}(\bar{\phi} D_2 \phi) \\ \partial_t\partial_2 A_2 - \partial_2^2 A_t &= \partial_2 \mathrm{Im}(\bar{\phi} D_1 \phi). \end{cases}$$

Adding these, we get

$$\partial_t(\nabla \cdot A_x) - \Delta A_t = -\partial_1 J_2 + \partial_2 J_1,$$

which, in view of the heat gauge condition (3.4), implies that A_t evolves according to the nonlinear heat equation

$$(\partial_t - \Delta) A_t = -\partial_1 J_2 + \partial_2 J_1. \tag{3.5}$$

Similarly, we obtain (coupled) parabolic evolution equations for A_1 and A_2:

$$\begin{cases} (\partial_t - \Delta) A_1 &= -J_2 - \frac{1}{2}\partial_2 |\phi|^2 \\ (\partial_t - \Delta) A_2 &= J_1 + \frac{1}{2}\partial_1 |\phi|^2. \end{cases} \tag{3.6}$$

It remains to determine initial conditions for (3.5), (3.56). Since the heat gauge is dynamic, we have an additional degree of freedom. We impose $A_t(0) = \nabla \cdot A_x(0) = 0$. To see that such a choice is consistent with (3.1), observe that $\nabla \cdot A_x(0) = 0$ coupled with the fourth equation of (3.1) yields the system

$$\begin{cases} \partial_1 A_1(t=0) + \partial_2 A_2(t=0) = 0 \\ \partial_1 A_2(t=0) - \partial_2 A_1(t=0) = -\frac{1}{2}|\phi_0|^2, \end{cases} \tag{3.7}$$

which in turn implies

$$\begin{cases} \Delta A_1(t=0) &= \frac{1}{2}\partial_2 |\phi_0|^2 \\ \Delta A_2(t=0) &= -\frac{1}{2}\partial_1 |\phi_0|^2. \end{cases} \tag{3.8}$$

Substituting (3.8) into (3.56) yields

$$\begin{cases} \partial_t A_1(t=0) &= -\mathrm{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2(t=0) &= \mathrm{Im}(\bar{\phi} D_1 \phi), \end{cases}$$

which is exactly what we obtain directly from the second and third equations of (3.1) at $t = 0$ with the choice $A_t(t = 0) \equiv 0$.

So having imposed an additional equation in order to fix a gauge, we study the initial value problem for the system

$$\begin{cases} D_t \phi &= i D_\ell D_\ell \phi + ig|\phi|^2 \phi \\ \partial_t A_1 - \partial_1 A_t &= -\mathrm{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2 - \partial_2 A_t &= \mathrm{Im}(\bar{\phi} D_1 \phi) \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2}|\phi|^2 \\ A_t &= \nabla \cdot A_x \end{cases} \tag{3.9}$$

with initial data

$$\begin{cases} \phi(0,x) &= \phi_0(x) \\ A_t(0,x) &= 0 \\ A_1(0,x) &= \frac{1}{2}\Delta^{-1}\partial_2|\phi_0|^2(x) \\ A_2(0,x) &= -\frac{1}{2}\Delta^{-1}\partial_1|\phi_0|^2(x). \end{cases} \tag{3.10}$$

Our main result is the following.

Theorem 3.2.1. For initial data $\phi_0 \in H^s(\mathbb{R}^2), s > 0$, there is a positive time T depending only on $\|\phi_0\|_{H^s}$ such that (3.9) with initial data (3.10) has a unique solution $\phi(t,x) \in C([0,T], H^s(\mathbb{R}^2))$. In addition, $\phi_0 \to \phi$ is Lipschitz continuous from $H^s(\mathbb{R}^2)$ to $C([0,T], H^s(\mathbb{R}^2))$.

3.3 Reductions with heat gauge

Let us define H^{-1} as Fourier multiplier operator

$$H^{-1}f := \int \frac{1}{i\tau + |\xi|^2} e^{i(t\tau + x\cdot\xi)} \tilde{f}(\tau,\xi) d\tau d\xi. \tag{3.11}$$

When we apply it on initial data, which is functions with only spatial variable, we have

$$H^{-1}(f(x)\delta_{t=0}) = 1_{t\geq 0} e^{t\Delta} f(x).$$

We also define $H^{-\frac{1}{2}}$ in the same way

$$H^{-\frac{1}{2}}f := \int \frac{1}{(i\tau + |\xi|^2)^{\frac{1}{2}}} e^{i(t\tau + x\cdot\xi)} \tilde{f}(\tau,\xi) d\tau d\xi. \tag{3.12}$$

Here we use principal square root of the complex function $i\tau + |\xi|^2$ by taking the positive real axis as the branch cut.

All these operators apply only to functions on positive time intervals.

Using (3.5), we can rewrite A_t as

$$A_t = -H^{-1}((Q_{12}(\bar{\phi},\phi))) - H^{-1}(\partial_1(A_2|\phi|^2)) + H^{-1}(\partial_2(A_1|\phi|^2)), \tag{3.13}$$

where $Q_{12}(\phi,\bar{\phi}) = \mathrm{Im}(\partial_1\phi\partial_2\bar{\phi} - \partial_2\phi\partial_1\bar{\phi})$.

Similarly, by (3.56) and initial condition (3.8), we can rewrite A_x as follows:

$$\begin{aligned} A_1 &= H^{-1}A_1(0) - H^{-1}[\mathrm{Re}(\bar{\phi}\partial_2\phi) + \mathrm{Im}(\bar{\phi}\partial_2\phi)] - H^{-1}(A_2|\phi|^2) \\ A_2 &= H^{-1}A_2(0) + H^{-1}[\mathrm{Re}(\bar{\phi}\partial_1\phi) + \mathrm{Im}(\bar{\phi}\partial_1\phi)] + H^{-1}(A_1|\phi|^2). \end{aligned} \tag{3.14}$$

Here

$$A_1(0) = \frac{1}{2}\Delta^{-1}\partial_2|\phi_0|^2, \quad A_2(0) = -\frac{1}{2}\Delta^{-1}\partial_1|\phi_0|^2.$$

Now we rewrite (3.3) as

$$(i\partial_t + \Delta)\phi = -2iA_\ell\partial_\ell\phi - i\partial_\ell A_\ell\phi + A_t\phi + A_x^2\phi - g|\phi|^2\phi.$$

Using (3.13), (3.14), we can conclude the following schematic representations of the nonlinear terms.

(1) For $A_j(\partial_j\phi)$, we have

$$(H^{-1}Q_{12})(\bar{\phi},\phi,\phi) + H^{-1}(A_1|\phi|^2)\partial_2\phi - H^{-1}(A_2|\phi|^2)\partial_1\phi + I_1$$

where $(H^{-1}Q_{12})(\bar{\phi},\phi,\phi) = H^{-1}(\bar{\phi}\partial_1\phi)\partial_2\phi - H^{-1}(\bar{\phi}\partial_2\phi)\partial_1\phi$.

Here I_1 comes from initial data with the following form.

$$I_1 = H^{-1}A_x(0)\partial_x\phi = (H^{-1}\Delta^{-1}\partial_1|\phi_0|^2\partial_2\phi - H^{-1}\Delta^{-1}\partial_2|\phi_0|^2\partial_1\phi)$$

(2) For $(\partial_j A_j)\phi$ and $A_t\phi$:

$$H^{-1}(Q_{12}(\bar{\phi},\phi))\phi + H^{-1}(\partial_x(A_x|\phi|^2))\phi$$

(3) For $A_x^2\phi$:

$$H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}(\bar{\phi}\partial_x\phi)\phi + H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}(A_x|\phi|^2)\phi + H^{-1}(A_x|\phi|^2)H^{-1}(A_x|\phi|^2)\phi + I_2$$

$$I_2 = [H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}A_x(0) + H^{-1}(A_x|\phi|^2)H^{-1}A_x(0) + H^{-1}A_x(0)H^{-1}A_x(0)]\phi.$$

3.4 Function spaces

In this section we define the function spaces we need for our problem. The starting point is the $U^2_\Delta H^s(\mathbb{R}^2)$, $V^2_\Delta H^s(\mathbb{R}^2)$ as define in chapter 1. But it is not good enough due to large log loss when we try to estimate a product of two functions with separated frequencies.

So we make two modifications for the U^2, V^2 spaces: first, weaken norm for the high modulation part; second, apply U, V norm to functions localized at cubes on dyadic shell.

For functions at frequency λ, we introduce a minor variation of the U^2_Δ and V^2_Δ spaces, which we respectively denote by U^2_λ, V^2_λ. We define these spaces in terms of the following norms:

$$\|u_\lambda\|^2_{U^2_\lambda} = \|Q_{\le\lambda^2}u_\lambda\|^2_{U^2_\Delta} + \sum_{|I|=\lambda^{-2},|J|=\lambda^{-1}\times\lambda^{-1}} \|\chi_I(t)\chi_J(x)Q_{\ge\lambda^2}u_\lambda\|^2_{U^2},$$

$$\|u_\lambda\|^2_{V^2_\lambda} = \|Q_{\le\lambda^2}u_\lambda\|^2_{V^2_\Delta} + \sum_{|I|=\lambda^{-2},|J|=\lambda^{-1}\times\lambda^{-1}} \|\chi_I(t)\chi_J(x)Q_{\ge\lambda^2}u_\lambda\|^2_{V^2}.$$

Here $\chi_I(t), \chi_J(x)$ denote sharp time and spatial cutoffs. These modifications, first introduced in [45], allow us to replace a logarithm of the high frequency by a logarithm of the low frequency in bilinear estimates on products of U^2_Δ, V^2_Δ functions. This gain is essential for our argument.

For each lattice point $z \in \mathbb{Z}^2$, let C_z denote the cube $C_z := z + [0,1)^2$. The collection of all such cubes yields a disjoint partition of $\mathbb{R}^2$: $\cup_{z\in\mathbb{Z}^2} C_z = \mathbb{R}^2$. Define P_{C_z} as the Fourier multiplier with symbol χ_{C_z}, where χ_{C_z} denotes the characteristic function of C_z. So $P_{C_z}u_\lambda$ means further localize a function u_λ with frequency support contained in the λ dyadic shell to a cube within that shell.

Now we are ready to define the basic function spaces we shall need in this chapter.

Definition 3.4.1. Let $s \in \mathbb{R}$ be given.

1. Define X^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{R}^2)$ such that $P_{C_z}u_\lambda \in U^2_\lambda(\mathbb{R}, H^s(\mathbb{R}^2))$ for every $z \in \mathbb{Z}^2$ and
$$\|u\|_{X^s} := \left(\sum_{\lambda\geq 1} \sum_{z\in\mathbb{Z}^d} \|P_{C_z}u\|^2_{U^2_\lambda(\mathbb{R},H^s)} \right)^{\frac{1}{2}} < +\infty.$$

2. Define Y^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{R}^2)$ such that $P_{C_z}u_\lambda \in V^2_\lambda(\mathbb{R}, H^s(\mathbb{R}^2))$ for every $z \in \mathbb{Z}^d$ and
$$\|u\|_{Y^s} := \left(\sum_{\lambda\geq 1} \sum_{z\in\mathbb{Z}^2} \|P_{C_z}u_\lambda\|^2_{V^2_\lambda(\mathbb{R},H^s)} \right)^{\frac{1}{2}} < +\infty.$$

As usual, for a time interval $I \subset \mathbb{R}$, we also consider the restriction spaces $X^s(I)$, etc.

The following corollary shows that the X^s and Y^s spaces are well adapted to localizations finer than the usual dyadic one.

Corollary 3.4.2. *Let $\{S_k\}$ be a partition of $\mathbb{R}^d$ into measurable sets S_k with the property*
$$\sup_{z\in\mathbb{Z}^d} \#\{k : C_z \cap S_k \neq \emptyset\} < +\infty.$$
Then
$$\sum_k \|P_{S_k}u_\lambda\|^2_{U^2_\lambda H^s} \lesssim \|u_\lambda\|^2_{X^s}.$$
The same holds if we replace U^2_λ by V^2_λ, and X^s by Y^s.

We will still show the following two facts as we care for most function spaces:

1. *Linear solution lies in the space*

Proposition 3.4.3. *Let $s \geq 0$, $0 < T \leq \infty$ and $\phi \in H^s(\mathbb{R}^2)$. Then, for the linear solution $u(t) := e^{it\Delta}\phi$, we have for $t \geq 0$ that $u \in X^s([0,T))$ and*
$$\|u\|_{X^s([0,T))} \leq \|\phi\|_{H^s}. \tag{3.15}$$

2. *Linear estimate for the Duhamel term*
Let $f \in L^1_{loc}([0,\infty); L^2(\mathbb{R}^2))$ and define

$$\mathcal{I}(f)(t) := \int_0^t e^{i(t-s)\Delta} f(s) ds \qquad t \geq 0, \tag{3.16}$$

as well as $\mathcal{I}(f)(t) := 0$ for $t < 0$. We have the following linear estimate for the Duhamel term.

Proposition 3.4.4. *Let $s \geq 0$ and $T > 0$. For $f \in L^1([0,T); H^s(\mathbb{R}^2))$ we have $\mathcal{I}(f) \in X^s([0,T))$ and*

$$\|\mathcal{I}(f)\|_{X^s([0,T))} \leq \sup \int_0^T \int_M f(t,x)\overline{v(t,x)} dx dt,$$

where the supremum is taken over all $v \in Y^{-s}([0,T))$ with $\|v\|_{Y^{-s}} = 1$.

We also record a useful interpolation property of the spaces U^p and V^p (resp. U^p_Δ, V^p_Δ) (cf. [27, Proposition 2.20]).

Lemma 3.4.1. Let $q_1, q_2 > 2$, E be a Banach space and

$$T : U^{q_1} \times U^{q_2} \to E$$

a bounded bilinear operator with $\|T(u_1,u_2)\|_E \leq C \prod_{j=1}^2 \|u_j\|_{U^{q_j}}$. In addition, assume that there exists $C_2 \in (0, C]$ such that the estimate $\|T(u_1,u_2)\|_E \leq C_2 \prod_{j=1}^2 \|u_j\|_{U^2}$ holds true. Then T satisfies the estimate

$$\|T(u_1,u_2)\|_E \lesssim C_2 (\ln \frac{C}{C_2} + 1)^2 \prod_{j=1}^2 \|u_j\|_{V^2}, \quad u_j \in V^2_{rc},\ j = 1,2.$$

Proof. The proof is the same as that in [28, Lemma 2.4]. For fixed u_2, let $T_1 u := T(u, u_2)$. Then we have that

$$\|T_1 u\|_E \leq D_1 \|u\|_{U^{q_1}} \quad \text{and} \quad \|T_1 u\|_E \leq D_1' \|u\|_{U^2}.$$

Here $D_1 = C\|u_2\|_{U^{q_2}}, D_1' = C_2 \|u_2\|_{U^2}$.
From the fact that $\|u_2\|_{U^{q_j}} \leq \|u_2\|_{U^2}$ and [27, Proposition 2.20], we obtain

$$\|T(u_1,u_2)\|_E = \|T_1 u_1\|_E \lesssim C_2 (\ln \frac{C}{C_2} + 1) \|u_1\|_{V^2} \|u_2\|_{U^2}. \tag{3.17}$$

Then we can repeat the argument by fixing u_1, using estimate (3.17), and

$$\|T(u_1,u_2)\|_E \leq C \prod_{j=1}^2 \|u_j\|_{U^{q_j}} \leq C \|u_1\|_{V^2} \|u_2\|_{U^{q_j}}.$$

□

Remark 3.4.2. In our analysis, we usually have the bilinear estimate proved by finer technics, and also by L^4-Strichartz estimate, which we can interpolate to get a log loss for product of functions in U^2_Δ, V^2_Δ. Unfortunately, sometimes this log loss is too big. That's where U^2_λ, and V^2_λ come into play, see lemma 3.5.4.

3.5 Linear and Bilinear estimates

Linear estimate

From chapter 1, we have the Strichartz (1.13) and local smoothing estimate (1.16) for free Schrödinger equation. Using [27, Proposition 2.19], we can extend the local smoothing estimate and Strichartz estimates to general U^p_Δ functions:

$$\|e^{it\Delta}P_{N,\mathbf{e}}f\|_{L^{\infty,2}_{\mathbf{e}}} \lesssim N^{-\frac{1}{2}}\|f\|_{U^2_\Delta}, \tag{3.18}$$

$$\|e^{it\Delta}f\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|f\|_{U^p_\Delta}. \tag{3.19}$$

Here (q,r) is any admissible pair of exponents and $p := \min(q,r)$.

Bilinear Estimates for free waves

We introduce an improved bilinear Strichartz estimate that is a slight generalization of that first shown in [7, Lemma 111].

Lemma 3.5.1 (Improved bilinear Strichartz)**.** Let $u(x,t) = e^{it\Delta}u_0(x), v(x,t) = e^{it\Delta}v_0(x)$, where $u_0, v_0 \in L^2(\mathbb{R}^2)$. Let Ω_1 denote the support of $\hat{u}_0(\xi_1)$, Ω_2 the support of $\hat{v}_0(\xi_2)$, and set $\Omega = \Omega_1 \times \Omega_2$. Assume that Ω_1 and Ω_2 are open and separated by some positive distance. Then

$$\|u\bar{v}\|_{L^2_{t,x}} \lesssim \left(\frac{\sup_{\xi,\tau}\int_{\substack{\xi=\xi_1-\xi_2\\ \tau=|\xi_1|^2-|\xi_2|^2}} \chi_\Omega(\xi_1,\xi_2)d\mathcal{H}^1(\xi_1,\xi_2)}{\operatorname{dist}(\Omega_1,\Omega_2)}\right)^{1/2} \cdot \|u_0\|_{L^2}\|v_0\|_{L^2}, \tag{3.20}$$

where $d\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure (on $\mathbb{R}^4$) and $\chi_\Omega(\xi_1,\xi_2)$ the characteristic function of Ω.

Proof. To control $\|u\bar{v}\|_{L^2_{t,x}}$, we are led by duality to estimating

$$\int_{\xi_1,\xi_2} g(\xi_1-\xi_2, |\xi_1|^2-|\xi_2|^2)\hat{u}_0(\xi_1)\bar{\hat{v}}_0(\xi_2)d\xi_1 d\xi_2.$$

We apply Cauchy-Schwarz and reduce the problem to bounding

$$G := \int_{(\xi_1,\xi_2)\in\Omega} |g(\xi_1-\xi_2, |\xi_1|^2-|\xi_2|^2)|^2 d\xi_1 d\xi_2.$$

Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $\mathbb{R}^2 \times \mathbb{R}^2 \ni (\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2) =: (\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}$. The differential corresponding to this change of coordinates is

$$df = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 2\xi_1^{(1)} & 2\xi_1^{(2)} & -2\xi_2^{(1)} & -2\xi_2^{(2)} \end{bmatrix}.$$

The size $|J_3 f|$ of the 3-dimensional Jacobian of f is defined to be the square root of the sum of the squares of the determinants of the 3×3 minors of the differential df:

$$|J_3 f| := 2\sqrt{2}\left((\xi_2^{(2)} - \xi_1^{(2)})^2 + (\xi_2^{(1)} - \xi_1^{(1)})^2 + (\xi_2^{(2)} - \xi_1^{(2)})^2 + (\xi_1^{(1)} - \xi_1^{(1)})^2\right)^{1/2}.$$

Hence

$$|J_3 f| = C|\xi_2 - \xi_1| \geq C \operatorname{dist}(\Omega_1, \Omega_2). \tag{3.21}$$

By the coarea formula (see [20, §3]),

$$\begin{aligned} G &= \int_{(\xi_1,\xi_2)\in\Omega} |g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2)|^2 d\xi_1 d\xi_2 \\ &= \int_{\xi,\tau} \int_{\substack{(\xi_1,\xi_2)\in\Omega: \\ \xi=\xi_1-\xi_2 \\ \tau=|\xi_1|^2-|\xi_2|^2}} |g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2)|^2 |J_3 f|^{-1}(\xi_1,\xi_2) d\mathcal{H}^1(\xi_1,\xi_2) d\xi d\tau \\ &\leq \int_{\xi,\tau} |g(\xi,\tau)|^2 \int_{\substack{(\xi_1,\xi_2)\in\Omega: \\ \xi=\xi_1-\xi_2 \\ \tau=|\xi_1|^2-|\xi_2|^2}} |J_3 f|^{-1}(\xi_1,\xi_2) d\mathcal{H}^1(\xi_1,\xi_2) d\xi d\tau \\ &\leq \int_{\xi,\tau} |g(\xi,\tau)|^2 d\xi d\tau \cdot \sup_{\xi,\tau} \int_{\substack{(\xi_1,\xi_2)\in\Omega: \\ \xi=\xi_1-\xi_2 \\ \tau=|\xi_1|^2-|\xi_2|^2}} |J_3 f|^{-1}(\xi_1,\xi_2) d\mathcal{H}^1(\xi_1,\xi_2). \end{aligned} \tag{3.22}$$

In view of (3.21), the right hand side of (3.22) is bounded (up to a constant) by

$$\|g\|_{L^2}^2 \cdot \operatorname{dist}(\Omega_1,\Omega_2)^{-1} \cdot \sup_{\xi,\tau} \int_{\substack{\xi=\xi_1-\xi_2 \\ \tau=|\xi_1|^2-|\xi_2|^2}} \chi_\Omega(\xi_1,\xi_2) d\mathcal{H}^1(\xi_1,\xi_2).$$

□

Let I_λ denote the frequency annulus $\{\xi \in \mathbb{R}^2 : \lambda/2 \leq |\xi| \leq 2\lambda\}$.

A straightforward application of Lemma 3.5.1 yields

Corollary 3.5.1 (Bourgain's improved bilinear Strichartz estimate [7])**.** *Let μ, λ be dyadic frequencies, $\mu \ll \lambda$. Let ϕ_μ, ψ_λ denote free waves respectively localized in frequency to I_μ and I_λ. Then*

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \|\phi_\mu(0)\|_{L_x^2} \|\psi_\lambda(0)\|_{L_x^2}. \tag{3.23}$$

Remark 3.5.2. If ϕ_μ, ψ_λ are further localized into boxes of size $\alpha \times \alpha$, we have better estimate

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\phi_\mu(0)\|_{L^2_x} \|\psi_\lambda(0)\|_{L^2_x}. \tag{3.24}$$

As a Corollary of the proof of Lemma 3.5.1, we obtain the following.

Corollary 3.5.2. *Let $u(x,t) = e^{it\Delta}u_0(x), v(x,s) = e^{is\Delta}v_0(x)$, where $u_0, v_0 \in L^2(\mathbb{R}^2)$. Let Ω_1 denote the support of $\hat{u}_0(\xi_1)$, Ω_2 the support of $\hat{v}_0(\xi_2)$. Assume that for all $\xi_1 \in \Omega_1$ and $\xi_2 \in \Omega_2$ we have*

$$|\xi_1 \wedge \xi_2| \sim \beta.$$

Then

$$\|u\bar{v}\|_{L^2_{s,t,x}} \lesssim \beta^{-1/2} \|u_0\|_{L^2} \|v_0\|_{L^2}. \tag{3.25}$$

Proof. As in the proof of Lemma 3.5.1, we use a duality argument. The key is to bound

$$\int_{(\xi_1,\xi_2)\in\Omega} |g(\xi_1 - \xi_2, |\xi_1|^2, |\xi_2|^2)| d\xi_1 d\xi_2$$

in L^2. In this setting, the proof is simpler because the change of variables f is given by $\mathbb{R}^2 \times \mathbb{R}^2 \ni (\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, |\xi_1|^2, |\xi_2|^2) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ so that $f : \mathbb{R}^4 \to \mathbb{R}^4$ and $|df| \sim |\xi_1 \wedge \xi_1|$. □

In order to achieve a gain at matched frequencies, we localize the output in both frequency and modulation, seeking to bound $P_\mu Q_\nu(\bar{\phi}_\lambda \psi_\lambda)$ in L^2. That Lemma 3.5.1 may be used efficiently, we introduce an adapted frequency-space decomposition of annuli $I_\lambda \subset \mathbb{R}^2$ that depends upon both the output frequency and modulation cutoff scales μ and ν.

Definition 3.5.3 (Frequency decomposition)**.** Suppose μ, ν, λ are dyadic frequencies satisfying $\mu \ll \lambda$ and $\nu \leq \mu\lambda$. We define a partition of I_λ into curved boxes defined as follows. First, partition I_λ into λ^2/ν annuli of equal thickness. Next, uniformly partition the annuli into λ/μ sectors of equal angle. The resulting set of curved boxes we call $\mathcal{Q} = \mathcal{Q}(\mu, \nu, \lambda)$.

We make a few remarks regarding this decomposition. The curved sides of the boxes in $\mathcal{Q}$ have length $\sim \mu$, whereas the straight sides of the boxes have length $\sim \nu/\lambda$. By adapting a suitable partition of unity to the decomposition, we have

$$f = \sum_{\substack{\mu \ll \lambda \\ \nu \leq \mu\lambda}} \sum_{R \in \mathcal{Q}(\mu,\nu,\lambda)} P_R f.$$

Note that we may extend $\mathcal{Q}(\mu, \nu, \lambda)$ to all smaller dyadic scales $\lambda' < \lambda$ in the following way: take the partition $\mathcal{Q}(\mu, \nu, \lambda')$ and cut the annuli into λ/λ' smaller annuli of equal thickness. In this way we can impose a finer scale on low frequencies.

Corollary 3.5.4. *Let μ, ν, λ be dyadic frequencies satisfying $\mu \lesssim \lambda$ and $\nu \lesssim \mu\lambda$. Let $\phi_\lambda, \psi_\lambda$ be free waves with frequency support contained in I_λ. Then*

$$\|P_\mu Q_\nu(\bar{\phi}_\lambda \psi_\lambda)\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|\phi_\lambda\|_{L^2_x} \|\psi_\lambda\|_{L^2_x}. \tag{3.26}$$

Proof. The frequency restriction P_μ applied to $P_R\bar{\phi}_\lambda P_{R'}\psi_\lambda$ restricts us to looking at the subcollection of boxes $R, R' \in \mathcal{Q}$ separated by a distance $\sim \mu$. This subcollection is further restricted by the modulation multiplier Q_ν. Let $\xi_1 \in R, \xi_2 \in R', \xi := \xi_1 - \xi_2$. We call ξ the *output frequency* and sometimes denote it by $\mathrm{Freq}_{\mathrm{out}}$. The modulation of the product $P_R\bar{\phi}_\lambda P_{R'}\psi_\lambda$ is given by

$$|\xi_1|^2 - |\xi_2|^2 - |\xi_1 - \xi_2|^2 = 2\xi_2 \cdot \xi,$$

which we call $\mathrm{Mod}_{\mathrm{out}}$ for *output modulation*. As $\xi_1 \in R$, $\xi_2 \in R'$, it holds that $|\xi_i| \sim \lambda$, $i = 1, 2$; because we apply P_μ, we also have $|\xi| = |\mathrm{Freq}_{\mathrm{out}}| \sim \mu$. Hence $\mathrm{Mod}_{\mathrm{out}} \sim 2\lambda\mu\cos\theta$, where θ is the angle between ξ_2 and ξ. Applying Q_ν restricts $\mathrm{Mod}_{\mathrm{out}}$ so that $|\mathrm{Mod}_{\mathrm{out}}| \sim \nu$, which in turn implies $|\cos\theta| \sim \nu/(\mu\lambda)$.

These restrictions motivate defining the set of *interacting pairs* $\mathcal{P} = \mathcal{P}(\mu, \nu, \lambda)$ as the collection of all pairs $(R, R') \in \mathcal{Q} \times \mathcal{Q}$ (where $\mathcal{Q} = \mathcal{Q}(\mu, \nu, \lambda)$ as in Definition 3.5.3) for which $\xi \in \{\xi_1 - \xi_2 : (\xi_1, \xi_2) \in R \times R'\}$ and $\tau \in \{|\xi_1|^2 - |\xi_2|^2 : (\xi_1, \xi_2) \in R \times R'\}$ satisfy $|\xi| \sim \mu$ and $|\tau - |\xi|^2| \sim \nu$.

Note that, for $R \in \mathcal{Q}(\mu, \nu, \lambda)$ fixed, the number p of interacting pairs $P \in \mathcal{P}(\mu, \nu, \lambda)$ containing R is $\mathrm{O}(1)$ uniformly in μ, ν, λ. This is a consequence of the restrictions $|\cos\theta| \sim \nu/(\mu\lambda)$ and $|\xi| \sim \mu$: they jointly enforce at most $O(1)$ translations of a distance $\sim \nu/\lambda$, which is precisely the scale of the short sides of the boxes. In other words, if one box in a pair is taken as fixed, then the positional uncertainty in frequency space of the remaining box induced by the cutoffs coincides with the dimensions of the box.

It remains only to show that for $(R, R') \in \mathcal{P}$ we have

$$\sup_{\xi,\tau} \int_{\substack{\xi=\xi_1-\xi_2 \\ \tau=|\xi_1|^2-|\xi_2|^2}} \chi_R(\xi_1)\chi_{R'}(\xi_2) d\mathcal{H}^1(\xi_1, \xi_2) \lesssim \frac{\nu}{\lambda}. \tag{3.27}$$

Fix $\xi \in \mathbb{R}^2, \tau \in \mathbb{R}$, $\xi \neq 0$, and consider the constraint equations

$$\begin{cases} \xi &= \xi_1 - \xi_2 \\ \tau &= |\xi_1|^2 - |\xi_2|^2. \end{cases} \tag{3.28}$$

These determine a line in $\mathbb{R}^2$:

$$\tau = -\xi \cdot (\xi - 2\xi_1).$$

Suppose this line intersects R. The angle ρ that it forms with the long side length of R satisfies $|\cos\rho| \sim \nu/(\mu\lambda)$ due to the modulation constraint (note that at the scale of these boxes, the effects of curvature can be neglected). Since the long side of R has length $\sim \mu$ and the short side length $\sim \nu/\lambda$, it follows that the total intersection length is $O(\nu/\lambda)$. □

Corollary 3.5.5. *Let μ, ν, λ be dyadic frequencies satisfying $\mu \lesssim \lambda$ and $\nu \lesssim \mu\lambda$. Let $\phi_\lambda, \psi_\lambda$ be free waves with frequency support contained in I_λ. Then*

$$\|P_\mu Q_\nu(\bar{\phi}_\lambda \psi_\lambda)\|_{L^\infty} \lesssim \frac{\mu\nu}{\lambda} \|\phi_\lambda\|_{L^2_x} \|\psi_\lambda\|_{L^2_x}.$$

Proof. In view of the proof of Corollary 3.5.4 and the L^2-orthogonality of the partition, it suffices to prove the corollary for $\bar{\phi}_\lambda$ and ψ_λ localized to boxes $R, R' \in \mathcal{P}(\mu, \nu, \lambda)$ (see the proof of Corollary 3.5.4 for the definition of $\mathcal{P}(\mu, \nu, \lambda)$). Denote these localizations by $\bar{\phi}_R$ and $\psi_{R'}$, respectively.

By Bernstein, Cauchy-Schwarz, and energy conservation,

$$\begin{aligned}
\|\bar{\phi}_R \psi_{R'}\|_{L^\infty} &\lesssim \frac{\mu\nu}{\lambda} \|\bar{\phi}_R \psi_{R'}\|_{L^\infty_t L^1_x} \\
&\lesssim \frac{\mu\nu}{\lambda} \|\bar{\phi}_R\|_{L^\infty_t L^2_x} \|\psi_{R'}\|_{L^\infty_t L^2_x} \\
&= \frac{\mu\nu}{\lambda} \|\bar{\phi}_R\|_{L^2_x} \|\psi_{R'}\|_{L^2_x}.
\end{aligned}$$

□

Extensions to U^2_λ.

We make frequent use of the proof of [27, Proposition 19], which extends $L^p_t L^q_x$ bilinear estimates for free waves to analogous estimates for U^2_Δ functions by reducing from U^2_Δ functions to U^2_Δ atoms, commuting the spatial norms with the time cutoffs, and using Hölder's inequality. The proof of [27, Proposition] also extends to local smoothing spaces.

Our first application of this proposition is in observing that (3.23) of Corollary 3.5.1 extends to U^2_Δ functions.

Corollary 3.5.6. *Let $\phi_\mu, \psi_\lambda \in U^2_\lambda$ be respectively localized in frequency to I_μ and I_λ, $\mu \ll \lambda$. Then*

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \|\phi_\mu\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda}. \tag{3.29}$$

We may similarly conclude the following.

Corollary 3.5.7. *Let $u, v \in U^2_\Delta$ be respectively localized in frequency to Ω_1 and Ω_2, where $\Omega_1, \Omega_2 \subset I_\lambda$. Assume that for all $\xi_1 \in \Omega_1$ and $\xi_1 \in \Omega_2$ we have*

$$|\xi_1 \wedge \xi_2| \sim \beta.$$

Then

$$\|u\bar{v}\|_{L^2_{s,t,x}} \lesssim \beta^{-1/2} \|u\|_{U^2_\lambda} \|v\|_{U^2_\lambda}. \tag{3.30}$$

Our next corollary is an extension of Corollary 3.5.4 to U^2_Δ functions.

Corollary 3.5.8. *Let μ, ν, λ be dyadic frequencies satisfying $\mu \lesssim \lambda$ and $\nu \lesssim \mu\lambda$. Let ϕ_λ, $\psi_\lambda \in U^2_\lambda$ have frequency support contained in I_λ. Then*

$$\|P_\mu Q_\nu(\bar{\phi}_\lambda \psi_\lambda)\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|\phi_\lambda\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda}. \tag{3.31}$$

Proof. We decompose the functions into low and high modulation pieces, writing $\phi_\lambda = Q_{\ll\nu}\phi_\lambda + Q_{\gtrsim\nu}\phi_\lambda$ and similarly for ψ_λ.

We first consider the low-low modulation interactions. The output modulation is given by

$$\mathrm{Mod}_{\mathrm{out}} = \tau_1 - \tau_2 - |\xi_1 - \xi_2|^2.$$

Due to the projections $Q_{\ll\nu}$, we also have $|\tau_j - |\xi_j|^2| \ll \nu$ for $j = 1, 2$. Therefore

$$\mathrm{Mod}_{\mathrm{out}} = |\xi_1|^2 - |\xi_2|^2 - |\xi_1 - \xi_2|^2 + O(\nu).$$

Since $\mathrm{Mod}_{\mathrm{out}} \sim \nu$, we conclude

$$|\xi_1|^2 - |\xi_2|^2 - |\xi_1 - \xi_2|^2 \sim \nu.$$

Consequently, we may localize ϕ_λ and ψ_λ to boxes lying in $\cup_{\nu' \sim \nu} \mathcal{Q}(\mu, \nu', \lambda)$. Having localized the inputs $\phi_\lambda, \psi_\lambda$ in frequency, to, say boxes R, R', we drop the frequency and modulation localizations:

$$\|P_\mu Q_\nu(P_R \bar{\phi}_\lambda P_{R'} \psi_\lambda)\|_{L^2} \leq \|P_R \bar{\phi}_\lambda P_{R'} \psi_\lambda\|_{L^2}$$

We now may invoke [27, Proposition 19] and the proof of Corollary 3.5.4 to conclude

$$\|P_R \bar{\phi}_\lambda P_{R'} \psi_\lambda\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|P_R \bar{\phi}_\lambda\|_{U^2_\lambda} \|P_{R'} \psi_\lambda\|_{U^2_\lambda},$$

which, by L^2-orthogonality of the partition, is enough to conclude

$$\|P_\mu Q_\nu(Q_{\ll\nu} \bar{\phi}_\lambda Q_{\ll\nu} \psi_\lambda)\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|Q_{\ll\nu} \phi_\lambda\|_{U^2_\lambda} \|Q_{\ll\nu} \psi_\lambda\|_{U^2_\lambda}.$$

Suppose that at least one of the functions is localized to high modulations. Without loss of generality, we place the multiplier on ψ_λ and proceed to bound $Q_{\gtrsim\nu}\psi_\lambda$ in L^2 and ϕ_λ in $L^{\infty,2}_{\mathbf{e}}$.

$$\begin{aligned}
\|P_\alpha Q_\nu(\phi_\lambda Q_{\gtrsim\nu} \psi_\lambda)\|_{L^2} &\lesssim (\alpha\nu)^{1/2} \|\phi_\lambda Q_{\gtrsim\nu} \psi_\lambda\|_{L^{2,1}_{\mathbf{e}}} \\
&\lesssim (\alpha\nu)^{1/2} \|\phi_\lambda\|_{L^{\infty,2}_{\mathbf{e}}} \|Q_{\gtrsim\nu} \psi_\lambda\|_{L^2} \\
&\lesssim (\alpha\nu)^{1/2} \nu^{-1/2} \lambda^{-1/2} \|\phi_\lambda\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda} \\
&\lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\phi_\lambda\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda}.
\end{aligned}$$

Since Fourier projections are bounded on $L^{\infty,2}_{\mathbf{e}}$, it follows that in the above argument ϕ_λ may be replaced by $Q_{\gtrsim\nu}\phi_\lambda$ (or $Q_{\ll\nu}\phi_\lambda$). □

Lemma 3.5.3. Let $Q_1, Q_2 \in \{Q_{\leq \nu_1}, Q_{\nu_2}, Q_{\geq \nu_3}, 1 : \nu_1, \nu_2, \nu_3 \text{ dyadic}\}$. Let $\phi_\mu, \psi_\lambda \in U^2_\Delta$ have respective frequency supports contained in α boxes lying in I_μ and I_λ, where $\mu \ll \lambda$. Then

$$\|Q_1\phi_\mu \cdot Q_2\psi_\lambda\|_{L^2} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\psi_\mu\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda}. \tag{3.32}$$

Proof. First consider the case where $Q_1 = 1$ and Q_2 is of the form $Q_{\leq \nu}$. As Q_2 is a Fourier multiplier with (Schwartz) symbol

$$b(\xi, \tau) := \chi((\tau - |\xi|^2)/\nu),$$

we have

$$Q_2\psi_\lambda(x,t) = (\tilde{b} * \psi_\lambda)(x,t) = \int \tilde{b}(y,s)\psi_\lambda(x-y, t-s)dyds,$$

and so it follows that the left hand side of (3.32) admits the representation

$$\|\phi_\mu(x,t) \int \tilde{b}(y,s)\psi_\lambda(x-y,t-s)dyds\|_{L^2_{x,t}}.$$

Suppose we freeze y, s and consider

$$\|\phi_\mu(x,t)\tilde{\psi}_\lambda(x-y,t-s)\|_{L^2_{x,t}}.$$

By replacing ϕ_μ and the translated ψ_λ with atoms, we obtain by Lemma 3.5.1 and the fact that the U^2_Δ spaces are translation invariant that

$$\|\phi_\mu(x,t)\tilde{\psi}_\lambda(x-y,t-s)\|_{L^2_{x,t}} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\psi_\mu\|_{U^2_\lambda} \|\psi_\lambda\|_{U^2_\lambda}.$$

Since $\tilde{b}(x,t)$ is integrable with bound independent of ν, (3.32) follows in this special case.

This argument clearly generalizes to $Q_1, Q_2 \in \{Q_{\leq \nu_1}, Q_{\nu_2}, 1 : \nu_1, \nu_2 \text{ dyadic}\}$. In order to accommodate $Q_{\geq \nu}$, we apply the above argument to $1 - Q_{\geq \nu}$ and use the triangle inequality. □

Bilinear estimate on product of U^2_λ, V^2_λ.

By using the L^4 Strichartz estimate (1.13), we obtain the bound

$$\|\phi_\lambda \phi_\mu\|_{L^2_{x,t}} \lesssim \|\phi_\lambda\|_{U^4_\Delta} \|\phi_\mu\|_{U^4_\Delta},$$

which, together with 3.5.6 (which holds true for U^2_Δ functions) and Lemma 3.4.1, implies

$$\|\phi_\lambda \phi_\mu\|_{L^2_{x,t}} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \log \lambda \|\phi_\lambda\|_{U^2_\Delta} \|\phi_\mu\|_{V^2_\Delta},$$

which is not sufficient to close our bootstrap. We therefore use the augmented counterparts of U^2_Δ, V^2_Δ, namely U^2_λ, V^2_λ.

Lemma 3.5.4. Let $\mu \ll \lambda$. Then

$$\|\phi_\lambda \phi_\mu\|_{L^2([0,1]\times\mathbb{R}^2)} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \log \mu \|\phi_\lambda\|_{U^2_\lambda} \|\phi_\mu\|_{V^2_\mu}. \tag{3.33}$$

Proof. The proof here imitates [45, Proposition 3.7]. We split the V^2 function into low and high modulation components:

$$\phi_\mu = Q_{\leq \mu^2}\phi_\lambda + Q_{\geq \mu^2}\phi_\mu.$$

For the low modulation part, we apply (3.29) and use the observation that U^2_Δ and V^2_Δ are equivalent on each dyadic region. Thanks to the time truncation, we have $O(\mu)$-many dyadic regions, leading to the $\log \mu$ loss in the estimate.

For the high modulation component, we localize further to rectangles R of size $\mu^{-2} \times \mu^{-1} \times \mu^{-1}$. By using Bernstein, we have

$$\|Q_{\geq \mu^2}\phi_\mu\|_{L^\infty} \lesssim \mu^{1/2} \|\phi_\mu\|_{V^2}$$

And by using the size of interval and local smoothing estimate, we have

$$\|\phi_\lambda\|_{L^2(R)} \lesssim \mu^{-1/2}\lambda^{-1/2} \|\phi_\lambda\|_{U^2_\lambda}$$

So we get

$$\begin{aligned} \|\phi_\lambda \phi_\mu\|^2_{L^2[0,1]} &\lesssim \sum_{|I|=\mu^{-1}, |J|=\mu^{-1}\times\mu^{-1}} \|\chi_I(t)\chi_J(x)\phi_\lambda\|^2_{L^2} \sup_{I,J} \|\chi_I(t)\chi_J(x)\phi_\mu\|^2_{L^\infty} \\ &\lesssim \|\phi_\lambda\|^2_{U^2_\lambda} \|\phi_\mu\|^2_{V^2_\mu}. \end{aligned}$$

□

Remark 3.5.5. When we have the high frequency term $\phi_\lambda \in V^2_\lambda$ paired with a low frequency term $\phi_\mu \in U^2_\mu$, we apply a Galilean transform to swap the frequencies. This allows us to then use the same proof as above.

Remark 3.5.6. The same method holds when we want to have product of two functions supported at small box.

3.6 Perturbative analysis

In this section we decompose the various terms of the nonlinearity into main terms and error terms. At the end of this section, we show how to dispense with the error terms. In the next sections, we look at the more challenging main terms.

Reductions

In section 3.3, we listed all schematic representation of the nonlinearity. To estimate it, we pair up the nonlinear terms with $\bar{\psi}$, and bound the them in $L^1_{t,x}$.

Before doing that we recall $A_j(\partial_j\phi)\bar{\psi}$ has the representation

$$(H^{-1}Q_{12})(\bar{\phi},\phi,\phi)\bar{\psi} + H^{-1}(A_1|\phi|^2)\partial_2\phi\bar{\psi} - H^{-1}(A_2|\phi|^2)\partial_1\phi\bar{\psi} + I_1$$

We take one more step in the last two terms, putting H^{-1} on $\partial_x\phi\bar{\psi}$ and expanding A_x again using (3.14). Thus $A_j(\partial_j\phi)\bar{\psi}$ admits the following schematic representation:

$$\begin{aligned}&(H^{-1}Q_{12})(\bar{\phi},\phi,\phi)\bar{\psi} + H^{-1}(\bar{\psi}\partial_x\phi)H^{-1}(\bar{\phi}\partial_x\phi)|\phi|^2 + H^{-1}(\bar{\psi}\partial_x\phi)H^{-1}(A_x|\phi|^2)|\phi|^2\\ &+ H^{-1}(\bar{\psi}\partial_x\phi)H^{-1}A_x(0)|\phi|^2 + I_1.\end{aligned}$$

Together with other schematic representation we get for $(\partial_jA_j)\phi\bar{\psi}$, $A_t\phi\bar{\psi}$ and $A_x^2\phi\bar{\psi}$ in section 3.3.

We therefore seek to control in $L^1_{t,x}$ the fourth-order terms

$$\begin{aligned}\mathrm{Main}_{4,1} &:= (H^{-1}Q_{12})(\bar{\phi},\phi,\phi)\bar{\psi},\\ \mathrm{Main}_{4,2} &:= H^{-1}(Q_{12}(\bar{\phi},\phi))\phi\bar{\psi},\\ \mathrm{Main}_{4,3} &:= |\phi|^2\phi\bar{\psi},\end{aligned} \tag{3.34}$$

the sixth-order terms

$$\begin{aligned}\mathrm{Main}_{6,1} &:= H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}(\bar{\psi}\partial_x\phi)|\phi|^2,\\ \mathrm{Main}_{6,2} &:= H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}(\bar{\phi}\partial_x\phi)\phi\bar{\psi},\end{aligned} \tag{3.35}$$

and the error terms.

$$\begin{aligned}\mathrm{Err}_1 &:= H^{-1}(\bar{\psi}\partial_x\phi)H^{-1}(A_x|\phi|^2)|\phi|^2,\\ \mathrm{Err}_2 &:= H^{-1}(\partial_x(A_x|\phi|^2))\phi\bar{\psi},\\ \mathrm{Err}_3 &:= H^{-1}(A_x|\phi|^2)H^{-1}(A_x|\phi|^2)\phi\bar{\psi}\\ \mathrm{Err}_4 &:= H^{-1}(\bar{\psi}\partial_x\phi)H^{-1}A_x(0)|\phi|^2\\ \mathrm{Err}_5 &:= (H^{-1}\Delta^{-1}\partial_1|\phi_0|^2\partial_2\phi - H^{-1}\Delta^{-1}\partial_2|\phi_0|^2\partial_1\phi)\bar{\psi}\\ \mathrm{Err}_6 &:= \big(H^{-1}(\bar{\phi}\partial_x\phi)H^{-1}A_x(0) + H^{-1}(A_x|\phi|^2)H^{-1}A_x(0)+\\ &\quad H^{-1}A_x(0)H^{-1}A_x(0)\big)\,\phi\bar{\psi}.\end{aligned} \tag{3.36}$$

Estimates for the initial data

For initial data $\phi_0 \in H^s, s > 0$, we have $|\phi_0|^2 \in W^{s,1}$.

Now let us split the initial data into low and high frequency parts.

$$|\phi_0|^2 = f_1 + f_2, \quad f_1 = P_1|\phi_0|^2, \quad f_2 = P_{>1}|\phi_0|^2$$

For the high frequency part, we use Bernstein's inequality.

$$\begin{aligned}
\|H^{-1}\Delta^{-1}\partial_j f_2\|_{L^4_{t,x}[0,T)} &\lesssim \sum_{\lambda>1} \|e^{-t\lambda^2}\frac{1}{\lambda} f_{2,\lambda}\|_{L^4_{t,x}[0,T)} \\
&\lesssim \sum_{\lambda>1} (\lambda^{-2})^{\frac{1}{4}}\lambda^{-1}\|f_{2,\lambda}\|_{L^4_x} \\
&\lesssim \sum_{\lambda>1} \|f_{2,\lambda}\|_{L^1_x} \\
&\lesssim \sum_{\lambda>1} \lambda^{-s}\|f_{2,\lambda}\|_{W^{s,1}} \lesssim \|f_2\|_{W^{s,1}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|H^{-1}\Delta^{-1}\partial_j f_2\|_{L^{2,6}_{\mathbf{e}}[0,T)} &\lesssim \sum_{\lambda>1} \|e^{-t\lambda^2}\frac{1}{\lambda} f_{2,\lambda}\|_{L^{2,6}_{\mathbf{e}}[0,T)} \\
&\lesssim \sum_{\lambda>1} (\lambda^{-2})^{\frac{1}{6}}\lambda^{-1}\lambda^{\frac{1}{2}+\frac{5}{6}}\|f_{2,\lambda}\|_{L^1_x} \\
&\lesssim \|f_2\|_{W^{s,1}}.
\end{aligned}$$

For the low frequency part,

$$\Delta^{-1}\partial_j f_1 = K_j(x) * |\phi_0|^2, K_j(x) = \mathcal{F}^{-1}(\psi(|\xi|)\frac{\xi_j}{|\xi|^2})$$

Here $\psi(\xi)$ is the same bump function used to define the Littlewood-Payley projection P_1.

For $K_j(x)$, notice the following two facts:

(1) $|K_j(x)| \lesssim 1$. This is because

$$|K_j(x)| \lesssim \int \frac{|\xi_j|}{|\xi|^2}\psi(|\xi|)|\xi|d|\xi| \lesssim \int \psi(r)dr.$$

(2) $K_j(x) \sim \frac{x_j}{|x|^2}$ as $x \to \infty$. In radial coordinates, $(\xi_1, \xi_2) = (r\cos\theta, r\sin\theta)$, and we can assume ξ_j to be ξ_1. We write

$$\begin{aligned}
K_j(x) &= \int_0^{2\pi}\int_0^\infty \frac{\psi(r)r\cos\theta}{r^2} e^{ir(x_1\cos\theta + x_2\sin\theta)} r dr d\theta \\
&= \int_0^{2\pi}\int_0^\infty \psi(r)\cos\theta e^{i|x|r(\frac{x_1}{|x|}\cos\theta+\frac{x_2}{|x|}\sin\theta)} dr d\theta.
\end{aligned}$$

By the method of stationary phase, we get the asymptotic $\frac{x_j}{|x|^2}$.

With this, we have

$$\|\Delta^{-1}\partial_j f_1\|_{L^4_x} \lesssim \|K_j(x)\|_{L^4_x}\|f_1\|_{L^1_x} \lesssim \|f_1\|_{L^1_x}$$

$$\|\Delta^{-1}\partial_j f_1\|_{L^{2,6}_{\mathbf{e},x}} \lesssim \|K_j(x)\|_{L^{2,6}_{\mathbf{e},x}} \|f_1\|_{L^1_x} \lesssim \|f_1\|_{L^1_x}.$$

Here

$$\|u\|_{L^{p,q}_{\mathbf{e},x}} := \left[\int_{\mathbb{R}} \left[\int_{H_{\mathbf{e}}} |f(xe+x')|^q dx'\right]^{\frac{p}{q}} dx\right]^{\frac{1}{p}},$$

and we use the fact that $|K(x)| \lesssim (x\cdot \mathbf{e})^{-\frac{2}{3}} \left(x\cdot \mathbf{e}^{\perp}\right)^{-\frac{1}{3}}$ as $x\to\infty$.

On a finite time interval, the symbol of H^{-1} is bounded, and so we get $H^{-1}\Delta^{-1}\partial_j f_1 \in L^4_{x,t}[0,T)$ and $L^{2,6}_{\mathbf{e}}[0,T)$.

To summarize, we proved $H^{-1}A(0) \in L^4_{x,t}[0,T) \cap L^{2,6}_{\mathbf{e}}[0,T)$.

Showing that A_x is equal to $H^{-1}(\phi\partial\phi)$ up to a small error

The main purpose here is to get a space-time bound for $H^{-1}(\phi\partial\phi)$. The space $L^4_{t,x}$ would be a candidate; however, $H^{-1}(\phi\partial\phi)$ in general fails to belong to this space, leading us to search for a suitable replacement.

Lemma 3.6.1. In the representation

$$A_x = H^{-1}A_x(0) + H^{-1}(\bar\phi\partial_x\phi) + H^{-1}(A_x|\phi|^2),$$

we have the following bounds for the main term $H^{-1}(\bar\phi\partial_x\phi)$:

$$\begin{aligned} &\|H^{-1}P_{\lambda_3}(\bar\phi_{\lambda_1}\partial_x\phi_{\lambda_2})\|_{L^{\infty,3}_{\mathbf{e}}} \lesssim \|\phi_{\lambda_1}\|_{U^2_\Delta}\|\phi_{\lambda_2}\|_{U^2_\Delta}, \text{ for } \lambda_1\sim\lambda_2\gg\lambda_3 \\ &\|H^{-1}P_{\lambda_3}(\bar\phi_{\lambda_1}\partial_x\phi_{\lambda_2})\|_{H^{-\frac{1}{2}}L^2_{x,t}} \lesssim \|\phi_{\lambda_1}\|_{U^2_\Delta}\|\phi_{\lambda_2}\|_{U^2_\Delta}, \text{ for } \lambda_3\sim\max(\lambda_1,\lambda_2). \end{aligned}$$

And $H^{-1}(A_x|\phi|^2)$ in $H^{-\frac{1}{2}}L^2_{x,t}[0,T)$.

Remark 3.6.2. Notice the following Sobolev embeddings hold:

$$H^{-1}(L^{2,6/5}_{\mathbf{e}}) \hookrightarrow H^{-\frac{1}{2}}L^2_{t,x}, \tag{3.37}$$

$$H^{-\frac{1}{2}}L^2_{t,x} \hookrightarrow L^4_{t,x}, \tag{3.38}$$

$$H^{-\frac{1}{2}}L^2_{t,x} \hookrightarrow L^{2,6}_{\mathbf{e}}. \tag{3.39}$$

So $H^{-1}(A_x|\phi|^2) \in L^4_{t,x} \cap L^{2,6}_{\mathbf{e}}$.

Proof. In the high × high → low case, $H^{-1}(\bar\phi\partial_x\phi)$ has the form $\lambda H^{-1}P_\alpha(\phi_\lambda\phi_\lambda)$, $\alpha\ll\lambda$.

For ϕ_λ supported on the annuli I_λ, we cut the annuli into 8 equal pieces, and decompose each ϕ_λ as $\phi_\lambda = \sum_{k=1}^8 \phi_\lambda^{(k)}$, where $\phi_\lambda^{(k)}$ has Fourier support in one of the eight pieces of the annulus. Now for any two $\phi_\lambda^{(k)}, \phi_\lambda^{(j)}$, we can take a direction $\mathbf{e}$ so that angle between any $\pm\xi$ in the union of the supports and $\mathbf{e}$ is bounded from below by some uniform constant. Hence we can apply the local smoothing estimate (3.18) to obtain

$$\|\phi_\lambda^{(k)}\|_{L^{\infty,2}_{\mathbf{e}}} \lesssim \lambda^{-\frac{1}{2}}\|\phi_\lambda^{(k)}\|_{U^2_\Delta}.$$

The same holds for $\phi^{(j)}$ by our choice of $\mathbf{e}$.

So we get

$$\|\lambda P_\alpha(\bar{\phi}_\lambda^{(k)}\phi_\lambda^{(j)})\|_{L_{\mathbf{e}}^{\infty,1}} \lesssim \|\phi_\lambda^{(k)}\|_{U_\Delta^2}\|\phi_\lambda^{(j)}\|_{U_\Delta^2}$$

and hence by Sobolev embedding that $\lambda H^{-1}P_\alpha(\phi_\lambda\phi_\lambda) \in L_{\mathbf{e}}^{\infty,3}$.

We now consider the case where the output frequency is comparable to the input, e.g., high×low → high. In this case, $H^{-1}(\bar{\phi}\partial_x\phi)$ takes the form $\alpha H^{-1}P_\lambda(\bar{\phi}_\lambda\phi_\alpha)$ or $\alpha H^{-1}P_\lambda(\phi_\lambda\bar{\phi}_\alpha)$, $\alpha \lesssim \lambda$, depending upon where the derivative lies. We apply Strichartz estimate (3.19) to obtain

$$\|\lambda H^{-\frac{1}{2}}P_\lambda(\phi_\lambda\bar{\phi}_\alpha)\|_{L_{x,t}^2} \lesssim \|\phi_\lambda\|_{L_{x,t}^4}\|\phi_\alpha\|_{L_{x,t}^4} \lesssim \|\phi_\lambda\|_{U_\Delta^4}\|\phi_\alpha\|_{U_\Delta^4}. \tag{3.40}$$

The desired bound for $H^{-1}(\bar{\phi}\partial\phi)$ is then obtained by using the embedding $U_\Delta^4 \hookrightarrow U_\Delta^2$.

Finally, in order to estimate $H^{-1}(A_x|\phi|^2)$, set

$$B_1 = H^{-1}(A_2|\phi|^2) \quad \text{and} \quad B_2 = H^{-1}(A_1|\phi|^2).$$

Using (3.14), we may rewrite this as

$$\begin{aligned} B_1 &= H^{-1}(B_2|\phi|^2) + H^{-1}\left(H^{-1}(\phi\partial_1\phi)|\phi|^2\right) + H^{-1}\left(H^{-1}A_2(0)|\phi|^2\right) \\ B_2 &= -H^{-1}(B_1|\phi|^2) - H^{-1}\left(H^{-1}(\phi\partial_2\phi)|\phi|^2)\right) + H^{-1}\left(H^{-1}A_1(0)|\phi|^2\right). \end{aligned}$$

From our previous discussion, $H^{-1}(\bar{\phi}\partial\phi) \in L_{\mathbf{e}}^{\infty,3}$ or $H^{-\frac{1}{2}}L_{x,t}^2$, and so by embeddings (3.38) and (3.39) we can conclude that it also lies in $L_{\mathbf{e}}^{\infty,3}$ or $L_{x,t}^4 \cap L_{\mathbf{e}}^{2,6}$. Also by virtue of the Strichartz estimate (3.19), we have $\phi \in L_{x,t}^4$. By Sobolev embedding, we conclude

$$H^{-1}\left(H^{-1}(\phi\partial_1\phi)|\phi|^2\right) \in H^{-\frac{1}{2}}L_{x,t}^2.$$

Similarly, because $H^{-1}A_x(0) \in L_{x,t}^4[0,T)$, we have

$$H^{-1}\left(H^{-1}A_x(0)|\phi|^2\right) \in H^{-\frac{1}{2}}L_{x,t}^2[0,T).$$

We can therefore apply a fixed point argument to (B_1, B_2) in $H^{-\frac{1}{2}}L_{x,t}^2[0,T) \times H^{-\frac{1}{2}}L_{x,t}^2[0,T)$ for T small. □

Controlling the "error" terms of equation (3.3)

Let us look at Err_5 term first, which needs a bit care

$$\int \left[H^{-1}\Delta^{-1}\partial_1|\phi_0|^2\partial_2\phi\cdot\bar{\psi} - H^{-1}\Delta^{-1}\partial_2|\phi_0|^2\partial_1\phi\cdot\bar{\psi}\right].$$

By applying integration by parts, we can move the $H^{-1}\partial_x$ onto $\partial_x\phi\bar{\psi}$. Hence we we can modulo signs rewrite the integrand as

$$H^{-1}(\partial_2\phi\partial_1\bar{\psi} - \partial_1\phi\partial_2\bar{\psi})\Delta^{-1}(\phi_0\bar{\phi}_0). \tag{3.41}$$

Let us denote the respective frequencies on $\phi, \psi, |\phi_0|^2$ by ξ_1, ξ_2, ξ_3. Then we have $\xi_1+\xi_2+\xi_3 = 0$.

Now if $\xi_1 \approx \xi_3 \gtrsim \xi_2$, suppose $\xi_1, \xi_2 \sim \lambda$, $\xi_2 \sim \alpha$. Then the symbol of $H^{-1}\partial_x$ and Δ^{-1} gives $\frac{\alpha}{\lambda^3}$. Using L^∞ on $\phi\bar{\psi}$ together with Bernstein inequality, and L^1 on $|\phi_0|^2$, we get the bound $\frac{\alpha^3}{\lambda^3}$ for Err_5 in this case.

If $\xi_1 \approx \xi_2 \sim \lambda$, $\xi_3 \sim \alpha$, $\lambda \gg \alpha$. We assume the modulation is on $\phi\bar{\psi}$ is ν. Because we can break the support of ϕ, ψ into $\alpha \times \alpha$ size boxes, the null form give us $\lambda^2 \frac{\alpha}{\lambda}$. So by applying L^∞ bound on $\phi\bar{\psi}$ (a version of Cor 3.5.5 for U^2, V^2 function), and $L^1_{t,x}$. We get

$$\frac{\alpha\lambda}{\nu}\frac{1}{\alpha^2}\frac{\alpha\nu}{\lambda} \lesssim 1.$$

Now based upon the preceding discussion of A_x, we can bound the other error terms (3.36) in $L^1_{x,t}$.

We can control Err_1 because $H^{-1}(\bar{\psi}\partial_x\phi) \in L^{\infty,3}_{\mathbf{e}}$ or $L^4_{x,t}$, $H^{-1}(A_x|\phi|^2) \in H^{-\frac{1}{2}}L^2_{x,t} \hookrightarrow L^4_{x,t} \cap L^{2,6}_{\mathbf{e}}$, and $\phi \in L^4_{x,t}$.

Control on Err_2 follows from $H^{-1}(\partial_x(A_x|\phi|^2)) \in L^2_{x,t}$ and $\phi, \psi \in L^4_{x,t}$.

Control on Err_3 comes from $H^{-1}(A_x|\phi|^2), \phi, \psi \in L^4_{x,t}$.

Control on Err_4 comes from $H^{-1}(\bar{\psi}\partial_x\phi) \in L^{\infty,3}_{\mathbf{e}}$ or $L^4_{x,t}$, $H^{-1}A(0) \in L^4_{x,t} \cap L^{2,6}_{\mathbf{e}}$ and $\phi \in L^4_{x,t}$.

Control on Err_6 comes from $H^{-1}(A_x|\phi|^2), \phi, \psi \in L^4_{x,t}$, $H^{-1}(\bar{\phi}\partial_x\phi) \in L^{\infty,3}_{\mathbf{e}}$ or $L^4_{x,t}$, and $H^{-1}A(0) \in L^4_{x,t} \cap L^{2,6}_{\mathbf{e}}$.

3.7 Quadrilinear bounds

In this section, we focus on controlling the main quadrilinear terms of the nonlinearity. Initially we assume that each input function is a free wave. After proving the desired bounds under this assumption, we then go on to show how to extend these bounds to general U^2_Δ functions.

Quadrilinear terms

Without loss of generality, we work only with the most difficult quadrilinear term to control, namely

$$\int \left(H^{-1}(\bar{\phi}\partial_1\phi)\bar{\phi}\partial_2\phi - H^{-1}(\bar{\phi}\partial_2\phi)\bar{\phi}\partial_1\phi\right),$$

as analogous arguments can be used to bound all of the other quadrilinear terms. The integrand is the same as the first term appearing in (3.34).

Suppose that each ϕ is a free wave, and that the four respective input frequencies are λ_j, $j = 1, 2, 3, 4$. We distinguish two cases according to whether the two input frequencies of H^{-1} are balanced or unbalanced. We say that a pair $\phi_{\lambda_{2j-1}}\phi_{\lambda_{2j}}$ is *balanced* if $\lambda_{2j-1} \sim \lambda_{2j}$ and *unbalanced* otherwise.

The frequencies of the four interacting free solutions must satisfy

$$\begin{cases} \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 &= 0 \\ \xi_1 - \xi_2 + \xi_3 - \xi_4 &= 0. \end{cases} \tag{3.42}$$

Factoring the first equation in (3.42), we get

$$(\xi_1 - \xi_2) \cdot (\xi_1 + \xi_2) + (\xi_3 - \xi_4) \cdot (\xi_3 + \xi_4) = 0.$$

Making substitutions from the second equation in (3.42), we obtain

$$\begin{cases} (\xi_1 - \xi_2) \cdot (\xi_1 + \xi_2 - \xi_3 - \xi_4) &= 0 \\ (\xi_3 - \xi_4) \cdot (-\xi_1 - \xi_2 + \xi_3 + \xi_4) &= 0. \end{cases} \tag{3.43}$$

Using in (3.43) the second equation in (3.42), we obtain the (not all independent) constraints

$$\begin{cases} (\xi_1 - \xi_2) \cdot (\xi_2 - \xi_3) &= 0 \\ (\xi_1 - \xi_2) \cdot (\xi_1 - \xi_4) &= 0 \\ (\xi_3 - \xi_4) \cdot (\xi_4 - \xi_1) &= 0 \\ (\xi_3 - \xi_4) \cdot (\xi_3 - \xi_2) &= 0. \end{cases} \tag{3.44}$$

In particular, the restrictions (3.44) imply that $(\xi_1, \xi_2, \xi_3, \xi_4)$ must form a rectangle.

Let (μ_1, λ_1) denote one pair of input frequencies and (μ_2, λ_2) the other. We assume without loss of generality that $\mu_j \leq \lambda_j$, $j = 1, 2$. Because of the second equation in (3.42),

$$\lambda_1/2 \leq |\xi_1 - \xi_2| = |\xi_3 - \xi_4| \leq 2\lambda_2,$$

which implies $\lambda_2 \gtrsim \lambda_1$. By symmetry $\lambda_1 \sim \lambda_2$. So without loss of generality we can replace λ_1 and λ_2 with λ.

We make two general remarks. The first concerns pairs of frequencies (μ_j, λ).

Remark 3.7.1. If the output frequency is comparable to the input frequencies, then refined bilinear estimates are not necessary, and it suffices to place the corresponding waves in $L^4_{t,x}$. For the sake of exposition we explicitly treat the cases where the output frequency is much lower than the input frequencies.

The second concerns modulation cutoffs.

Remark 3.7.2. If $\nu < \mu^2$, then throughout we replace Q_ν by $Q_{<\mu^2}$. For the sake of exposition, we do not explicitly point out each time this is done.

Unbalanced case

It suffices to consider the case where the derivatives fall on the higher frequency λ terms. The goal is to control

$$\left| \int H^{-1}(\bar{\phi}_{\mu_1} \partial_1 \phi_\lambda) \bar{\phi}_{\mu_2} \partial_2 \phi_\lambda \right|. \tag{3.45}$$

Without loss of generality, we suppose that each wave is normalized to 1 in L^2_x. To each pair $\bar{\phi}_{\mu_j}\phi_\lambda$ we apply bilinear estimate (3.23), obtaining a combined bound of $(\mu_1\mu_2)^{1/2}/\lambda$. The two derivatives in (3.45) are multipliers whose contribution is bounded by λ^2, while H^{-1} is a multiplier controlled here by λ^{-2}. Therefore (3.45) is bounded by $O(\mu_1^{1/2}\mu_2^{1/2}/\lambda)$, which we sum in μ_j over $\mu_j \ll \lambda$ to obtain a bound of $O(1)$.

Balanced case

Here we suppose that one pair of inputs is at frequency $\sim \lambda$ and that the other is at $\sim \mu$. Without loss of generality, we can always assume that H^{-1} takes the λ-frequency inputs. Our aim is to control

$$\left| \int H^{-1}\left(P_\alpha Q_\nu(\bar{\phi}_\lambda \partial_1 \phi_\lambda)\right) \bar{\phi}_{\mu_1}\partial_2\phi_{\mu_2} - \int H^{-1}\left(P_\alpha Q_\nu(\bar{\phi}_\lambda \partial_2 \phi_\lambda)\right) \bar{\phi}_{\mu_1}\partial_1\phi_{\mu_2} \right|. \tag{3.46}$$

We use the orthogonal partition $\mathcal{Q}(\alpha,\nu,\lambda)$ of I_λ and I_μ. Let R_1, R_2, R_3, R_4 be boxes belonging to this partition, where R_1, R_2 are α-separated at frequency λ and R_3, R_4 are α-separated at frequency μ. By the L^2-orthogonality of the partition, it suffices to estimate

$$\left| \int H^{-1}(\bar{\phi}_{R_1}\partial_1\phi_{R_2})\bar{\phi}_{R_3}\partial_2\phi_{R_4} - \int H^{-1}(\bar{\phi}_{R_1}\partial_2\phi_{R_2})\bar{\phi}_{R_3}\partial_1\phi_{R_4} \right|. \tag{3.47}$$

We may normalize so that each ϕ_{R_i} has an L^2_x norm of 1.

We now split into two subcases.

Subcase I:

Suppose that there is $\beta \gtrsim \alpha\lambda$ such that $|\xi_2 \wedge \xi_4| \sim \beta$ for all $\xi_2 \in R_2$ and $\xi_4 \in R_4$.

Let $b(y,s)$ denote kernel of $P_\alpha Q_\nu$ so that $\tilde{b}(\xi,\tau) = \chi(\xi/\alpha)\chi((\tau - \xi^2)/\nu)$.

Our goal is to control

$$\left| \int b(y,s)H^{-1}(\bar{\phi}_{R_1}\partial_1\phi_{R_2})(x-y,t-s)(\bar{\phi}_{R_3}\partial_2\phi_{R_4})(x,t) \right.$$
$$\left. - b(y,s)H^{-1}(\bar{\phi}_{R_1}\partial_2\phi_{R_2})(x-y,t-s)\bar{\phi}_{R_3}\partial_1\phi_{R_4}(x,t)dsdtdxdy \right|.$$

This is bounded by

$$\int \|H^{-1}(\bar{\phi}_{R_1}\partial_1\phi_{R_2})\bar{\phi}_{R_3}\partial_2\phi_{R_4} - H^{-1}(\bar{\phi}_{R_1}\partial_2\phi_{R_2})\bar{\phi}_{R_3}\partial_1\phi_{R_4}\|_{L^1_{t,s,x}} \sup_s |b(y,s)|dy.$$

The first term is translation invariant, and so we may obtain a bound on

$$\|H^{-1}(\bar{\phi}_{R_1}\partial_1\phi_{R_2})\bar{\phi}_{R_3}\partial_2\phi_{R_4} - H^{-1}(\bar{\phi}_{R_1}\partial_2\phi_{R_2})\bar{\phi}_{R_3}\partial_1\phi_{R_4}\|_{L^1_{t,s,x}} \tag{3.48}$$

uniform in y. In particular, we have a contribution of order $O(\nu^{-1})$ from H^{-1}, a contribution of β from the null form, and from two applications of (3.25), a contribution of β^{-1}. Taking the

L^2 normalizations into account, we have that (3.48) is $O(\nu^{-1})$. Finally, with the observation that

$$\int \sup_s |b(y,s)| dy \lesssim \nu,$$

we conclude that (3.47) is $O(1)$ in this subcase.

Subcase II:

Suppose now that $|\xi_2 \wedge \xi_4| \lesssim \alpha\lambda$ for all $\xi_2 \in R_2$ and $\xi_4 \in R_4$. Together H^{-1} and the null form contribute $\alpha\lambda/\nu$, and so we can apply (3.26) twice to conclude that term (3.47) is $O(1)$.

Extending quadrilinear bounds to U^2_λ functions.

We split into two principal cases. Throughout we take advantage of the fact that we can always renormalize functions to 1 in U^2_Δ.

Unbalanced case

Here our goal is to control

$$\left| \int H^{-1}(\bar{\phi}_{\mu_1} \partial_1 \phi_{\lambda_1}) \bar{\phi}_{\mu_2} \partial_2 \phi_{\lambda_2} \right|.$$

We conclude $\lambda_1 \sim \lambda_2$ as in the free case since the second equation of (3.42) holds for any four interacting frequencies. Therefore we need only replicate the free-wave bound on (3.45). This, however, is achieved simply by repeating the argument in the free case, but replacing (3.23) with (3.29).

Balanced case

We assume without loss of generality that $\mu_1 \sim \lambda_1 \gtrsim \lambda_2$ and $\mu_2 \lesssim \lambda_2$. Without loss of generality, we may always place the Fourier multipliers on the two large balanced frequencies. Relabeling frequencies, we therefore have as our goal to control

$$\left| \int H^{-1} \left(P_\alpha Q_\nu (\bar{\phi}_\lambda \partial_1 \phi_\lambda) \right) \bar{\phi}_{\mu_1} \partial_2 \phi_{\mu_2} - \int H^{-1} \left(P_\alpha Q_\nu (\bar{\phi}_\lambda \partial_2 \phi_\lambda) \right) \bar{\phi}_{\mu_1} \partial_1 \phi_{\mu_2} \right|. \tag{3.49}$$

As in the proof of Corollary 3.5.8, we decompose each function ϕ into low and high modulation pieces, writing $\phi = Q_{\ll \nu}\phi + Q_{\gtrsim}\phi_\lambda$.

Balanced low modulation inputs

We suppose that both ϕ_λ terms are restricted in modulation via $Q_{\ll \nu}$. Following the proof of Corollary 3.5.8, we conclude that each ϕ_λ is localized to a pair of boxes R_1, R_2 lying in $\cup_{\nu' \sim \nu} \mathcal{Q}(\alpha, \nu', \lambda)$. Since $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$, we likewise conclude that for fixed R_1, R_2, we may restrict ξ_3 and ξ_4 to boxes in $\cup_{\nu' \sim \nu} \mathcal{Q}(\alpha, \nu', \lambda)$ at frequency $\mu_1 \sim \mu_2$ (if μ_1 and μ_2 are not comparable, then there is no output to control).

There are two subcases: $\nu \lesssim \alpha\lambda$ and $\nu \gtrsim \alpha\lambda$.

When $\nu \lesssim \alpha\lambda$, we proceed as in the free balanced case up to the point of the L^2 estimates, where we invoke (3.29), (3.30), and (3.31) as opposed to (3.23), (3.25), and (3.26).

If $\nu \gtrsim \alpha\lambda$ and $\mu \ll \lambda$, then we apply bilinear estimates (3.29). The derivatives and null form are then $O(\mu\lambda/\nu)$ while from (3.29) we have a contribution of $O(\alpha\lambda)$. Therefore (3.49) is $O(\alpha\mu/\nu)$, where $\alpha\mu/\nu \lesssim 1$.

Suppose now that instead it is ϕ_{μ_1} and ϕ_{μ_2} that are restricted in modulation via $Q_{\ll\nu}$. We can then shift the Fourier multipliers in (3.49) onto these terms and again proceed as in the proof of Corollary 3.5.8 to localize in frequency. Here, however, we use the finer partition $\cup_{\nu'\sim\nu}\mathcal{Q}(\alpha,\nu',\lambda)$ at all scales under consideration. Once we localize to boxes in this partition, we can carry out the same argument as above.

Alternating high modulation inputs

Here we suppose that at least one of the ϕ_λ terms and at least one of the ϕ_{μ_j} terms are restricted in modulation by $Q_{\gtrsim\nu}$.

We localize in frequency to boxes R_j, $j = 1, 2, 3, 4$, of order $\alpha \times \alpha$ in size.

The two high modulation factors can be used to localize to a ν^{-1} time interval. Once this is done we discard the modulation information. Then we simply reduce to the free case by first reducing to atoms.

Remark 3.7.3. We remark here that in our argument we use bilinear estimates of U^2_λ and V^2_λ functions. If they are at different frequencies, then we get a log loss in the lower frequency. If these two are at same frequency, then we apply bilinear estimates if all of the terms are at same frequency and cross bilinear estimates (with a log loss on lower frequency) otherwise.

3.8 Sextilinear bounds

In this section, we focus on controlling the main sextilinear terms of the nonlinearity. Initially we assume that each input function is a free wave. After proving desired bounds under this assumption, we then go on to show how to extend the bounds to general U^2_Δ functions.

Sextilinear terms

The basic expression is

$$\int H^{-1}(\phi\partial\phi)H^{-1}(\phi\partial\phi)(\phi\bar{\phi}), \tag{3.50}$$

with integrand as in (3.35).

Here we are thinking of each ϕ as denoting a free wave. As in the quadrilinear case, we localize each ϕ to a frequency annulus:

$$\int H^{-1}(\phi_{\lambda_1}\partial\phi_{\lambda_2})H^{-1}(\phi_{\lambda_3}\partial\phi_{\lambda_4})(\phi_{\lambda_5}\bar{\phi}_{\lambda_6})$$

Throughout, Remarks 3.7.1 and 3.7.2 are assumed to be in forcer.

The possible inputs are summarized as follows, in nondecreasing order of difficulty.

λ_1, λ_2	λ_3, λ_4	λ_5, λ_6
unbalanced	unbalanced	either
balanced	unbalanced	either
unbalanced	balanced	either
balanced	balanced	unbalanced
balanced	balanced	balanced

Unbalanced-unbalanced-either

First consider the unbalanced-unbalanced-either case. Lemma 3.6.1, Sobolev embedding (3.38), and the $(q, r) = (4, 4)$ Strichartz estimate (1.13) allow us to put each of the three terms in L^4.

Balanced-unbalanced-either; unbalanced-balanced-either

Second, consider the balanced-unbalanced-either case or its symmetric counterpart unbalanced-balanced-either. Here we put the labeled balanced term in $L_{\mathbf{e}}^{\infty,3}$, the labeled unbalanced term in $L_{\mathbf{e}}^{2,6}$, and the remaining ϕ^2 term in L^2. This we can achieve thanks to Lemma 3.6.1, Sobolev embedding (3.39), and the $(q, r) = (4, 4)$ Strichartz estimate (1.13).

Preparation for the remaining cases

In the third and fourth cases we are led to further refining our frequency and modulation restrictions. We do so by applying frequency and modulation cutoffs on pairs of interacting waves. Note that in both the third and fourth cases the first two pairs of waves are balanced, meaning $\lambda_1 \sim \lambda_2$ and $\lambda_3 \sim \lambda_4$; for the purposes of estimates it suffices to treat these comparable frequencies as though they were equal. Relabeling accordingly, we consider

$$\int H^{-1}\left[P_{\mu_1} Q_{\nu_1}(\phi_{\lambda_1} \partial \phi_{\lambda_1})\right] H^{-1}\left[P_{\mu_2} Q_{\nu_2}(\phi_{\lambda_2} \partial \phi_{\lambda_2})\right] P_{\mu_3} Q_{\nu_3}(\phi_{\lambda_3} \bar{\phi}_{\lambda_4}).$$

Here the last two frequencies do not share a label because we admit the possibility that they are unbalanced.

Note that

$$\mu_1 + \mu_2 + \mu_3 = 0. \tag{3.51}$$

As a consequence,

$$|\mu_i| \sim |\mu_j| \gtrsim |\mu_k| \tag{3.52}$$

for some permutation (i, j, k) of $(1, 2, 3)$. We denote the larger magnitude scale by μ_{hi} and the smaller one by μ_{lo}.

Next we address the modulation constraints. A priori, 0 is the lower bound on each ν_i. However, the smallest scale that we actually need to consider is μ_i^2. The reason for this is twofold. Firstly, at lower modulations the symbol of H^{-1} is dominated by the μ_i term and not the ν_i term. Secondly, by dyadic summation, the bilinear estimates from §3.5 at modulation $Q_{<\mu_i^2}$ are controlled by those at modulation $Q_{\mu_i^2}$. Therefore in the following we assume that each ν_i is dyadic, satisfies $\mu_i^2 \leq \nu_i^2$, and that any of the following estimates that hold with a multiplier $Q_{\mu_i^2}$ also hold with that multiplier replaced by $Q_{<\mu_i^2}$.

A convenient consequence of the modulation constraints

$$\mu_i^2 \leq \nu_i \tag{3.53}$$

is that the two largest modulations are comparable in size. To see that this is so, let $n_1 = |\xi_1|^2 - |\xi_2|^2 - |\xi_1 - \xi_2|^2$ and similarly for n_2, n_3, so that $|n_i| = \nu_i$ for each i. Arrange each set in increasing order. Of course, $\nu_{\min} = |n_{\min}|, \nu_{\text{mid}} = |n_{\text{mid}}|, \nu_{\max} = |n_{\max}|$. We have

$$n_{\min} + n_{\text{mid}} + n_{\max} = -\mu_1^2 - \mu_2^2 - \mu_3^2. \tag{3.54}$$

Hence

$$|n_{\min} + n_{\text{mid}} + n_{\max}| \sim \mu_{\text{hi}}^2.$$

Suppose now that $\nu_{\max} \sim \mu_{\text{hi}}^2$. As a consequence of (3.53) and (3.52), we have that $\mu_{\text{hi}}^2 \lesssim \nu_{\text{mid}}$. Therefore $\mu_{\text{hi}}^2 \lesssim \nu_{\text{mid}} \lesssim \nu_{\max} \sim \mu_{\text{hi}}^2$, which implies $\nu_{\max} \sim \nu_{\text{mid}}$. Suppose now on the other hand that $\nu_{\max} \gg \mu_{\text{hi}}^2$. Then the fact that the left hand side of (3.54) is $O(\mu_{\text{hi}}^2)$ combined with the trivial bound $\nu_{\min} \leq \nu_{\text{mid}}$ forces $\nu_{\max} \sim \nu_{\text{mid}}$.

We therefore let $\nu_{\text{hi}} \sim \nu_{\max} \sim \nu_{\text{mid}}$ and $\nu_{\text{lo}} \sim \nu_{\min}$.

A simple consequence of (3.53) is $\mu_{\text{hi}}^2 \lesssim \nu_{\text{hi}}$.

Balanced-balanced-unbalanced

Now we are ready to analyze the balanced-balanced-unbalanced case.

Suppose that $Q_{\nu_{\text{lo}}}$ is paired with an H^{-1}; without loss of generality, let $|\nu_2| \sim \nu_{\text{lo}}$. Then we apply the L^∞ bound to this term and use (3.23) on the remaining pairs to obtain an upper bound of

$$\frac{\lambda_1}{\nu_{\text{hi}}} \frac{\lambda_2}{\nu_{\text{lo}}} \frac{\mu_2 \nu_{\text{lo}}}{\lambda_2} \frac{\mu_{\text{hi}}}{\max\{\lambda_1, \lambda_3, \lambda_4\}} \lesssim \frac{\mu_{\text{hi}}^2}{\nu_{\text{hi}}} \lesssim 1.$$

Suppose now on the other hand that $Q_{\nu_{\text{lo}}}$ is not paired with an H^{-1}. Then we apply L^∞ to one of the terms inside H^{-1} and use (3.23) on the remaining terms:

$$\frac{\lambda_1 \lambda_2}{\nu_{\text{hi}}^2} \frac{\mu_{\text{hi}} \nu_{\text{hi}}}{\lambda_2} \frac{\mu_{\text{hi}}}{\max\{\lambda_1, \lambda_3, \lambda_4\}} \lesssim \frac{\mu_{\text{hi}}^2}{\nu_{\text{hi}}} \lesssim 1.$$

Balanced-balanced-balanced

What remains is the balanced-balanced-balanced case. Again relabeling, we have only to consider

$$\int H^{-1} \left[P_{\mu_1} Q_{\nu_1} (\phi_{\lambda_1} \partial \phi_{\lambda_1}) \right] H^{-1} \left[P_{\mu_2} Q_{\nu_2} (\phi_{\lambda_2} \partial \phi_{\lambda_2}) \right] P_{\mu_3} Q_{\nu_3} (\phi_{\lambda_3} \bar{\phi}_{\lambda_3}).$$

The following combinations of multipliers are exhaustive:

- $H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}, H^{-1} P_{\mu_{\text{lo}}} Q_{\nu_{\text{lo}}}, P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}$
- $H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}, H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}, P_{\mu_{\text{lo}}} Q_{\nu_{\text{lo}}}$
- $H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}, H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{lo}}}, P_{\mu_{\text{lo}}} Q_{\nu_{\text{hi}}}$
- $H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}, H^{-1} P_{\mu_{\text{lo}}} Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}} Q_{\nu_{\text{lo}}}$
- $H^{-1} P_{\mu_{\text{hi}}} Q_{\nu_{\text{lo}}}, H^{-1} P_{\mu_{\text{lo}}} Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}} Q_{\nu_{\text{hi}}}$.

In addition to the above, we should also be cognizant of the following possible relationships between the input frequencies:

- $\lambda_{\text{min}} \ll \lambda_{\text{mid}} \ll \lambda_{\text{max}}$
- $\lambda_{\text{min}} \sim \lambda_{\text{mid}} \ll \lambda_{\text{max}}$
- $\lambda_{\text{min}} \ll \lambda_{\text{mid}} \sim \lambda_{\text{max}}$
- $\lambda_{\text{min}} \sim \lambda_{\text{mid}} \sim \lambda_{\text{max}}$.

The matched cases: $P_{\mu_{\text{lo}}}$ paired with $Q_{\nu_{\text{lo}}}$

Suppose that the multipliers $H^{-1}, P_{\mu_{\text{lo}}}, Q_{\nu_{\text{lo}}}$ appear together so that they are applied to the same pair of inputs, i.e., we have the combination

$$H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{lo}}}, P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}.$$

If ν_{lo} appears inside an H^{-1}, then we apply L^∞ to that term and L^2 estimates on the remaining terms. Without loss of generality, we suppose that $\nu_2 = \nu_{\text{lo}}$.

$$\frac{\lambda_1}{\nu_{\text{hi}}}\frac{\lambda_2}{\nu_{\text{lo}}}\frac{\mu_{\text{lo}}\nu_{\text{lo}}}{\lambda_2}\frac{\nu_{\text{hi}}}{\mu_{\text{hi}}\max\{\lambda_1,\lambda_3\}} \lesssim \frac{\mu_{\text{lo}}}{\mu_{\text{hi}}} \lesssim 1.$$

Note that here in applying (3.23) we pick up the smaller of the two μ's and the larger of the two λ's.

If we can't use (3.23) (when $\lambda_1 \sim \lambda_3$), then we apply bilinear estimates inside each H^{-1}. (Cross-bilinear gives us a better estimate, but we do not need it.)

Now suppose that we have the combination

$$H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, P_{\mu_{\text{lo}}}Q_{\nu_{\text{lo}}}.$$

First suppose that $\lambda_3 \neq \lambda_{\text{min}}$. Then we apply L^∞ to the ν_{lo} term:

$$\frac{\lambda_1\lambda_2}{\nu_{\text{hi}}^2}\frac{\mu_{\text{lo}}\nu_{\text{lo}}}{\lambda_3}\frac{\nu_{\text{hi}}}{\mu_{\text{hi}}\max\{\lambda_1,\lambda_2\}} \lesssim \frac{\mu_{\text{lo}}}{\mu_{\text{hi}}}\frac{\nu_{\text{lo}}}{\nu_{\text{hi}}} \lesssim 1.$$

Next suppose that $\lambda_3 = \lambda_{\text{min}}$. If $\lambda_{\text{mid}} \sim \lambda_{\text{min}}$, then the same strategy as above works. So we need only consider $\lambda_3 = \lambda_{\text{min}} \ll \lambda_{\text{mid}}$. In this case we apply the L^∞ estimate to the λ_{max} (or λ_{mid}) term and use (3.23) on the remaining:

$$\frac{\lambda_1\lambda_2}{\nu_{\text{hi}}^2}\frac{\mu_{\text{hi}}\nu_{\text{hi}}}{\lambda_{\text{max}}}\frac{\mu_{\text{lo}}}{\lambda_{\text{mid}}} \lesssim \frac{\mu_{\text{lo}}\mu_{\text{hi}}}{\nu_{\text{hi}}} \lesssim 1.$$

The mismatched cases: $P_{\mu_{\text{hi}}}$ paired with $Q_{\nu_{\text{lo}}}$

It therefore only remains to consider the cases where ν_{lo} pairs with μ_{hi}. That is, we only have the following combinations left to consider:

- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}, P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}$
- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}$
- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}, H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}$

The next lemma encapsulates a simple observation that will prove helpful in all of the remaining cases.

Lemma 3.8.1. Suppose that $Q_{\nu_{\text{lo}}}$ does not pair with $P_{\mu_{\text{lo}}}$ Then

$$\mu_{\text{lo}}^2 \ll \mu_{\text{hi}}^2 \lesssim \nu_{\text{lo}} \ll \nu_{\text{hi}}. \tag{3.55}$$

Proof. If $\mu_{\text{lo}} \sim \mu_{\text{hi}}$, then necessarily ν_{lo} pairs with μ_{lo}, and hence $\mu_{\text{lo}} \ll \mu_{\text{hi}}$. Similarly, if $\nu_{\text{lo}} \sim \nu_{\text{hi}}$, then again we effectively have ν_{lo} pairing with μ_{lo}. Hence $\nu_{\text{lo}} \ll \nu_{\text{hi}}$.

Note that by assumption it is always the case that $\nu_{\text{lo}} \gtrsim \mu_{\text{lo}}^2$. However, if ν_{lo} does not pair with μ_{lo}, then it pairs with μ_{hi}, which implies $\nu_{\text{lo}} \gtrsim \mu_{\text{hi}}^2$ □

We may therefore assume throughout the remainder of this subsection that we are in the regime dictated by (3.55).

We split into two principal cases.

μ_{hi} **inputs are at widely separated frequency scales**

Subcase I:

Suppose first that the we have to consider $H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}$. Suppose its input is λ_2 and that λ_1 and λ_3 are widely separated. Then we put L^∞ on that term (the μ_{lo} term) and get

$$\frac{\lambda_1\lambda_2}{\nu_{\text{hi}}\nu_?}\frac{\mu_{\text{lo}}\nu_{\text{hi}}}{\lambda_2}\frac{\mu_{\text{hi}}}{\max\{\lambda_1,\lambda_3\}} \lesssim \frac{\mu_{\text{lo}}\mu_{\text{hi}}}{\nu_{\text{lo}}} \lesssim 1$$

by (3.55).

This eliminates cases

$$\begin{matrix} H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}} & H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}} & P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}} \\ H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}} & H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}} & P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}} \end{matrix}$$

(except when $\lambda_1 \sim \lambda_3$).

Subcase II:

So suppose now that we are in case

$$H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}, P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}.$$

If $\lambda_3 \gtrsim \min\{\lambda_1, \lambda_2\}$, then we apply L^∞ to the μ_{lo} term to get

$$\frac{\lambda_1}{\nu_{\text{hi}}}\frac{\lambda_2}{\nu_{\text{lo}}}\frac{\mu_{\text{lo}}\nu_{\text{hi}}}{\lambda_3}\frac{\mu_{\text{hi}}}{\max\{\lambda_1,\lambda_2\}} \lesssim \frac{\mu_{\text{lo}}\mu_{\text{hi}}}{\nu_{\text{lo}}} \lesssim 1.$$

So suppose $\lambda_3 = \lambda_{\min} \ll \lambda_{\text{mid}}$. Then we apply L^∞ to the ν_{lo} term:

$$\frac{\lambda_1}{\nu_{\text{hi}}}\frac{\lambda_2}{\nu_{\text{lo}}}\frac{\mu_{\text{hi}}\nu_{\text{lo}}}{\lambda_2}\frac{\mu_{\text{lo}}}{\lambda_1} \lesssim \frac{\mu_{\text{lo}}\mu_{\text{hi}}}{\nu_{\text{hi}}} \lesssim 1.$$

Therefore we may assume that the two μ_{hi} terms have comparable input frequencies.

μ_{hi} inputs are at comparable frequency scales

We assume $\lambda_1 \sim \lambda_3$. If λ_1 and λ_2 are widely separated, then we apply L^∞ to the ν_{lo} term and apply cross-bilinear to get

$$\frac{\lambda_1\lambda_2}{\nu_{\text{hi}}\nu_{\text{lo}}}\frac{\mu_{\text{hi}}\nu_{\text{lo}}}{\lambda_1}\frac{\mu_{\text{lo}}}{\max\{\lambda_1,\lambda_2\}} \lesssim 1.$$

Therefore we have reduced to the case where $\lambda_1 \sim \lambda_2 \sim \lambda_3$, i.e., all frequencies are comparable to a single frequency λ.

To to summarize, we have reduced to the case where all input frequencies are of size $\sim \lambda$ and where a $P_{\mu_{\text{hi}}}$ is paired with $Q_{\nu_{\text{lo}}}$. Recall we are in the regime given by (3.55).

The following are the possible configurations:

- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}, P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}$
- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}, H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}$
- $H^{-1}P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}, H^{-1}P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}, P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}$

The derivative and modulation contributions together will be

$$\frac{\lambda^2}{\nu_{\text{hi}}\nu_{\text{lo}}} \quad \text{or} \quad \frac{\lambda^2}{\nu_{\text{hi}}^2}.$$

Once these are taken into account, we may place the three outputs $P_{\mu_{\text{hi}}}Q_{\nu_{\text{lo}}}$, $P_{\mu_{\text{lo}}}Q_{\nu_{\text{hi}}}$, $P_{\mu_{\text{hi}}}Q_{\nu_{\text{hi}}}$ on equal footing. We cut everything down to the finer scale using the orthogonality of the three functions coming from product, so that everything is localized to size $\mu_{\text{lo}} \times \frac{\mu_{\text{lo}}^2}{\lambda} \times \nu_{\text{lo}}$. In this case, we can use (3.23) on each term, together with Bernstein from L^∞ to L^2. We obtain

$$\frac{\lambda^2}{\nu_{\text{hi}}\nu_{\text{lo}}}\left(\frac{\nu_{\text{lo}}}{\mu_{\text{hi}}\lambda}\right)^{\frac{1}{2}}\left(\frac{\nu_{\text{hi}}}{\mu_{\text{lo}}\lambda}\right)^{\frac{1}{2}}\left(\frac{\nu_{\text{hi}}}{\mu_{\text{hi}}\lambda}\right)^{\frac{1}{2}}\left(\frac{\nu_{\text{lo}}\mu_{\text{lo}}^3}{\lambda}\right)^{\frac{1}{2}} \lesssim \frac{\mu_{\text{lo}}}{\mu_{\text{hi}}} \lesssim 1.$$

Extending sextilinear bounds to U^2_λ functions.

Notice that in our argument, we basically used L^2 and L^∞ estimate on product of free solutions. Thanks to 3.5.6 and 3.5.8, we can extend these estimates to U^2_λ functions.

We only need to be careful about extending estimate to a product of U^2_λ V^2_λ functions with certain logarithmical loss.

Similarly to Remark 3.7.3 at the end of previous section, we remark here that all of our estimates in this section immediately generalize from free solutions to U^2_λ functions.

3.9 Lipschitz dependence

Now let us look at the difference of two solutions ϕ, ϕ' corresponding to data ϕ_0, ϕ_0' and prove the following lemma.

Lemma 3.9.1. Let $\phi \in X^s$, $\mathcal{N}$ is defined as the nonliearity in the right hand side of equation *(3.3)*. Assume $\|\phi\|_{X^s} \le a \le 1$, then

$$\|\mathcal{N}\|_{X^s} \lesssim a^3$$

Also assume the same holds for ϕ', and $\mathcal{N}'$ defined in the same manner. Set $b = \|\phi - \phi'\|_{X^s} \le 2a$, $c = \|\phi_0 - \phi_0'\|_{H^s}$, then

$$\|\mathcal{N} - \mathcal{N}'\|_{X^s} \lesssim a^2 b + a^2 c$$

Proof. The first estimate on $\mathcal{N}$ is basically what we have proved before. To do the second one, we basically need to bound $\|A_x - A_x'\|$ in terms of $\phi - \phi'$ and $\phi_0 - \phi_0'$. (This is the different part than Coulomb gauge, where we don't need the difference of initial data.)

$$\begin{cases} (\partial_t - \Delta)A_1 & = -J_2 - \frac{1}{2}\partial_2|\phi|^2 \\ (\partial_t - \Delta)A_2 & = J_1 + \frac{1}{2}\partial_1|\phi|^2. \end{cases} \tag{3.56}$$

So we can write down the difference equation for A, which roughly looks like

$$(\partial_t - \Delta)(A - A') = (\bar{\phi} - \bar{\phi}')\phi + \bar{\phi}'(\phi - \phi') + (A - A')|\phi|^2 + A'[(\bar{\phi} - \bar{\phi}')\phi + \bar{\phi}'(\phi - \phi')]$$

$$(A - A')(0) = (\bar{\phi}_0 - \bar{\phi}_0')\phi_0 + \bar{\phi}_0'(\phi_0 - \phi_0')$$

Because ϕ, ϕ' satisfy the same linear estimates, so we can conclude all our estimate for ϕ, A, would just apply to the difference equation. Hence we get

$$\|\mathcal{N} - \mathcal{N}'\|_{X^s} \lesssim a^2 b + a^2 c.$$

□

From this lemma, we apply fixed point argument for equation (3.3) in $X^s[0,T)$ for T small enough, to get unique solution with Lipschitz dependence on the initial data.

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