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by

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Abstract

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In this thesis we address several questions involving quantum groups, quantum cluster algebras, and integrable systems, and provide some novel examples of the very useful interplay between these subjects. In the Chapter 2, we introduce the classical reflection equation (CRE), and give a construction of integrable Hamiltonian systems on G/K, where G is a quasitriangular Poisson Lie group and K is a Lie subgroup arising as the fixed point set of a group automorphism σ of G satisfying the CRE. As an application, we provide a detailed treatment of the algebraic integrability of the XXZ spin chain with reflecting boundary conditions.

In Chapter 3, we study doubles of Hopf algebras and dual pairs of quantum moment maps. For any Hopf algebra A, we construct a natural generalization of the (quantized) Grothendieck-Springer resolution; the standard resolution corresponds to taking A a quantum Borel subalgebra. In this latter case, we apply the general construction to yield an algebra embedding of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ into a quantum torus algebra which is a central extension of the quantum coordinate ring of the reduced big double Bruhat cell in the corresponding simply-connected group G.

Chapter 4 gives an alternative geometric description of this quantum torus embedding. Namely, we construct an embedding of $U_q(\mathfrak{sl}_n)$ into a quantum cluster chart on a quantum character variety associated to a marked punctured disk. We obtain a description of the coproduct of $U_q(\mathfrak{sl}_n)$ in terms of a quantum character variety associated to the marked twice punctured disk, and express the action of the R-matrix in terms of a mapping class group element corresponding to the half-Dehn twist rotating one puncture about the other. As a consequence, we realize the algebra automorphism of $U_q(\mathfrak{sl}_n)^{\otimes 2}$ given by conjugation by the R-matrix as an explicit sequence of cluster mutations, and derive a refined factorization of the R-matrix into quantum dilogarithms of cluster monomials.

We conclude by mentioning some applications of our cluster realization of quantum groups to the decomposition of tensor products of positive representations, and the construction of a modular functor from quantum higher Teichmuller theory.

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Chapter 1

Introduction

In the present thesis we address several problems involving integrable systems, quantum groups, and cluster algebras. The central strategy we pursue is to exploit the rich and fruitful relationship between geometric objects such as moduli spaces and integrable systems, corresponding algebraic ones, including Poisson-Lie groups, Hopf algebras, and quantum groups. We begin in Chapter 2 by developing the theory of Poisson homogeneous spaces of quasitriangular Poisson-Lie groups, which turns out to be governed by an algebraic structure called the *classical reflection equation*. From solutions of this reflection equation, we explain how one can construct a general class of integrable Hamiltonian systems whose time evolution is given by a Lax equation, and whose trajectories can be computed by solving a certain factorization problem of Iwasawa type in the underlying Poisson-Lie group. This class of integrable systems turns out to often admit a physical interpretation as models with integrable reflecting boundary conditions. An example of particular interest, which is related to a homogeneous space for the the loop group LSL_2, is the XXZ spin chain with open boundaries, and we apply our formalism to give a complete treatment of the algebraic integrability of this model.

In Chapter 3, we pass from classical to quantum and turn our attention to the theory of doubles of Hopf algebras. Motivated by the Poisson geometry of the Grothendieck-Springer simultaneous resolution of a complex semisimple Lie group, we explain how to construct an analog of the quantization of that resolution starting from any Hopf algebra. The Grothendieck-Springer resolution is recovered when the underlying Hopf algebra is the quantum Borel subalgebra in the corresponding quantum group $U_q(\mathfrak{g})$. The key to the construction is the notion of a commuting dual pair of quantum moment maps. As an interesting by product, we also obtain non-trivial homomorphisms of certain reflection equation algebras, whose defining relations are given by the quantum analog of the reflection equation studied in Chapter 2. In the case of the Grothendieck-Springer resolution, we are able to combine our general algebraic construction with certain properties of the R-matrix and quantum Weyl group of $U_q(\mathfrak{g})$ in order to obtain an algebra embedding of $U_q(\mathfrak{g})$ into a *quantum torus algebra*; that is, an associative algebra generated by variables $X_1^{\pm 1}, \ldots, X_l^{\pm 1}$ subject to the q-commutativity relations $X_iX_j = q^{b_{ij}}X_jX_i$ for some $b_{ij} \in \mathbb{Z}$. This embedding can be regarded as a q-deformation of the familiar realization of $U(g)$ in terms of the Weyl algebra of differential operators on the big cell of the flag variety.

Our goal in Chapter 4 is to provide an alternative geometric description of the quantum torus embedding of $U_q(\mathfrak{sl}_n)$, using the tools of higher Teichmüller theory and quantum cluster algebras. More precisely, we construct an embedding of $U_q(\mathfrak{sl}_n)$ into a quantum cluster chart on the moduli space of framed PGL_n-local systems on a disk $\widehat{S}$ with a single puncture p, and with two marked points x_1, x_2 on its boundary. Our embedding has the property that for each Chevalley generator of $U_q(\mathfrak{sl}_n)$, there is a cluster mutation-equivalent to $\mathcal{D}_n$ in which the image of that generator is a cluster monomial. We also solve the problem of describing the coproduct and R-matrix of $U_q(\mathfrak{sl}_n)$ in geometric terms, this time involving the moduli space of framed local systems on a twice punctured disk. Conjugation by the R-matrix is realized as the element of the cluster mapping class group corresponding to the half-Dehn twist rotating one puncture about the other.

We conclude in Chapter 5 by commenting on some exciting applications of our cluster realization of quantum groups to the decomposition of tensor products of positive representations, and the construction of a modular functor from quantum higher Teichmüller theory.

1.1 Introduction to Chapter 2

A large and well-studied class of integrable Hamiltonian systems consists of those whose phase space can be realized as a Poisson submanifold of a quasitriangular Poisson-Lie group G. In this situation, the conjugation invariant functions $I_G \subset C(G)$ form a Poisson commutative subalgebra, and particular integrable systems arise by restricting these functions to symplectic leaves in G.

In this Chapter we construct integrable systems on Poisson homogeneous spaces of the form G/K, where (G, r) is a quasitriangular Poisson Lie group and K is a Lie subgroup of G which arises as the fixed point set of a Lie group automorphism $\sigma : G \to G$. In this setting, the condition for G/K to inherit a Poisson structure from G is equivalent to the requirement that the quantity

$$C_\sigma(r) = (\sigma \otimes \sigma)(r) + r - (\sigma \otimes 1 + 1 \otimes \sigma)(r)$$

be a Lie(K)-invariant in $\mathfrak{g} \otimes \mathfrak{g}$. In the special case $C_\sigma(r) = 0$, we say that (r, σ) is a solution of the *classical reflection equation* (CRE). In this case, we construct a *classical reflection monodromy matrix* $\mathcal{T}$ with the property that the classical reflection transfer matrices obtained by taking the trace of $\mathcal{T}$ in finite dimensional representations of G form a Poisson commuting family of functions in $C(G/K) \subset C(G)$. These functions are no longer Ad_G-invariant, but are instead bi-invariant under the action of $K \times K$ on G by left and right translations.

The motivation for our construction comes from the quantum spin chains with reflecting boundary conditions introduced by Sklyanin [57]. It is known [7] that these quantum integrable systems are closely related to coideal subalgebras in the quantum affine algebras

$U_q(\widehat{\mathfrak{g}})$. When $\mathfrak{g}$ is a finite dimensional simple Lie algebra, coideal subalgebras in $U_q(\mathfrak{g})$ have been studied by many authors [44],[38],[18],[60] and may be regarded as quantizations of the classical symmetric spaces G/K. Of particular relevance is the work of Belliard, Crampé and Regelskis [11],[12], who introduced the notion of *Manin triple twists* to understand the semiclassical limit of coideal subalgebras.

We will show that the semiclassical limit of Sklyanin's quantum reflection equation coincides with the CRE for an appropriate choice of group G and automorphism σ. We shall also explain how to derive local Hamiltonians for the corresponding homogeneous classical spin chain, the XXZ spin chain with reflecting boundaries.

Next, we proceed to a give a detailed analysis of the reflection XXZ chain. Namely, we solve the following three fundamental problems

- Separation of variables on the system's phase space
- Integration of the system's equations of motion
- Construction of the system's action-angle variables.

Crucial to our analysis is the so-called *algebraic integrability* of the system: the tori on which the flows of the reflection Hamiltonians are linearized are in fact abelian varieties, arising as Jacobians of the spectral curves of the reflection monodromy matrix. For general background on the notion algebraic integrability and examples of its applications, we refer the reader to the books and surveys [65],[64], [24].

Let us also say a word about the broader physical and mathematical context in which the classical spin chains appear. On the one hand, they may be regarded as lattice versions of Landau-Lifshitz type continuum models, see [48] and [58], [14]. On the other, they can also be obtained as infinite spin or mean-field limits of lattice models in quantum statistical physics, see [32]. We hope that the analysis presented here can also prove useful in the study of the continuum and quantum systems related to the classical XXZ chain with reflecting boundaries.

1.2 Introduction to Chapter 3

The Grothendieck-Springer simultaneous resolution of a complex simple Lie group G plays a central role in the geometric representation theory. Recall that if $B \subset G$ is a Borel subgroup in G, and we write $\mathfrak{g}, \mathfrak{b}$ for the Lie algebras of G, B respectively, then the Grothendieck-Springer resolution is the following map of Poisson varieties:

$$G \times_B \mathfrak{b} \longrightarrow \mathfrak{g}, \qquad (g, x)B \longmapsto gxg^{-1}. \tag{1.2.1}$$

Indeed, the Poisson map (1.2.1) admits a quantization, yielding an embedding of the enveloping algebra $U(\mathfrak{g})$ into the ring of global differential operators on the principal affine space G/N.

It was shown in [49] that both sides of the multiplicative Grothendieck-Springer resolution

$$G \times_B B \longrightarrow G, \qquad (g,b)B \longmapsto gbg^{-1} \tag{1.2.2}$$

admit natural, nontrivial Poisson structures such that the resolution map is Poisson. In [25], we showed that the resolution (1.2.2) can be also quantized, this time to yield an embedding of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ into a certain ring of quantum differential operators on G/N.

One remarkable property of $U_q(\mathfrak{g})$ is that it can be realized as a quotient of the Drinfeld double $D(U_q(\mathfrak{b}))$ of a quantum Borel subalgebra $U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$. In this note, we observe that an analog of the quantization of the resolution (1.2.2) exists under mild conditions for the Drinfeld double $D(A)$ of a topological Hopf algebra A. The key to the construction of this quantization is the existence of a pair $\mu_L, \mu_R \colon D(A) \to H(D(A)^{*,op})$ of commuting quantum moment maps from $D(A)$ to the Heisenberg double of a certain Hopf algebra $D(A)^{*,op}$ opposite dual to $D(A)$. In this general setting, the role of quantum differential operators on G/N is played by the quantum Hamiltonian reduction of $H(D(A)^{*,op})$ by $\mu_L(A)$, and the resolution map is given by the residual quantum moment map $\mu_R \colon D(A) \to H(D(A)^{*,op})//\mu_L(A)$.

Although a similar construction has appeared before in the context of the quantum Beilinson-Bernstein theorem, we believe that the following results are new. First, we show that the quantum Hamiltonian reduction $H(D(A)^{*,op})//\mu_L(A)$ is isomorphic to the Heisenberg double $H(A)$. Recall [43] that the Heisenberg double $H(A)$ of a finite-dimensional Hopf algebra A is isomorphic to the algebra of its endomorphisms $\mathrm{End}(A)$. Thus, the natural action of $D(A)$ on A yields a homomorphism $D(A) \to H(A)$. We show that it coincides with the map $\mu_R \colon D(A) \to H(A)$ when A is finite dimsensional. Second, we provide an explicit Faddeev-Reshetikhin-Takhtajan type presentation of the map μ_R in terms of universal R-matrices, which leads to a homomorphism between certain reflection equation algebras.

Having explained the general theory, we proceed to apply it in the important special case of the quantized enveloping algebra $U_q(\mathfrak{g})$, and as an application obtain an algebra embedding of $U_q(\mathfrak{g})$ into a quantum torus algebra.

1.3 Introduction to Chapter 4

In [23], an intriguing realization of the quantum group $U_q(\mathfrak{sl}_2)$ and the Drinfeld double of its Borel subalgebra was presented in terms of a quantum torus algebra $\mathfrak{D}$. Explicitly, the algebra $\mathfrak{D}$ has generators $\{w_1, w_2, w_3, w_4\}$, with the relations

$$w_i w_{i+1} = q^{-2} w_{i+1} w_i, \qquad \text{and} \qquad w_i w_{i+2} = w_{i+2} w_i \tag{1.3.1}$$

where $i \in \mathbb{Z}/4\mathbb{Z}$. In terms of the standard generators E, F, K, K' of the Drinfeld double (see Section 3 for the definitions), the embedding proposed in [23] takes the form

$$\begin{aligned} E &\mapsto \mathbf{i}(w_1 + w_2), \qquad & K &\mapsto q w_2 w_3, \\ F &\mapsto \mathbf{i}(w_3 + w_4), \qquad & K' &\mapsto q w_4 w_1, \end{aligned} \tag{1.3.2}$$

where $\mathbf{i} = \sqrt{-1}$.

The embedding (1.3.2) has some striking properties. Firstly, as proposed in [23], one can use the Weyl-type relations (1.3.1) to define a modular double of $U_q(\mathfrak{sl}_2)$ compatible with the regime $|q| = 1$. Next, the image of the quasi R-matrix under this embedding admits a remarkable factorization into the product of four quantum dilogarithms:

$$\bar{\mathcal{R}} = \Psi^q(w_1 \otimes w_3)\,\Psi^q(w_1 \otimes w_4)\,\Psi^q(w_2 \otimes w_3)\,\Psi^q(w_2 \otimes w_4)\,. \tag{1.3.3}$$

These properties have been exploited in [10], [9], [2] to define and study a new continuous braided monoidal category of 'principal series' representations of $U_q(\mathfrak{sl}_2)$.

On the other hand, factorizations of the $U_q(\mathfrak{sl}_2)$ R-matrix of the form (1.3.3) have also appeared in quantum Teichmuller theory. In [37], the action of the R-matrix is identified, up to a simple permutation, with an element of the mapping class group of the twice punctured disc. The mapping class group element in question corresponds to the half-Dehn twist rotating one puncture about the other. After triangulating the punctured disc, this transformation can be decomposed into a sequence of four flips of the triangulation, as shown in Figure 4.8. One is thus led to interpret each dilogarithm in the factorization (1.3.3) as corresponding to a flip of a triangulation. In [36], this observation was used to re-derive Kashaev's knot invariant.

In this chapter, we explain how to generalize Faddeev's embedding (1.3.2) to the case of the quantum group $U_q(\mathfrak{sl}_{n+1})$ using the language of quantum cluster algebras. The key to our construction is the quantum cluster structure associated to moduli spaces of PGL_{n+1}-local systems on a decorated, marked surface, see [63]. Cluster charts on these varieties are obtained from an ideal triangulation of the surface by 'gluing' certain simpler cluster charts associated to each triangle. In the case of moduli spaces of PGL_{n+1}-local systems, a flip of a triangulation can be realized as sequences of $\binom{n+2}{3}$ cluster mutations.

Taking a particular cluster chart on the moduli space associated to a triangulation of the punctured disk (defined precisely in Section 2), we obtain by this gluing procedure a quiver and a corresponding quantum torus algebra $\mathfrak{D}_n$. Our first main result, Theorem 4.4.4, is to describe an explicit embedding of $U_q(\mathfrak{sl}_{n+1})$ into $\mathfrak{D}_n$. Our embedding has the property that each Chevalley generator E_i, F_i of $U_q(\mathfrak{sl}_{n+1})$ is a cluster monomial in some cluster torus mutation equivalent to $\mathfrak{D}_n$. In the simplest case $n = 1$, our result reproduces Faddeev's realization (1.3.2) of $U_q(\mathfrak{sl}_2)$ in terms of the quantum torus $\mathfrak{D}_1$ associated to the cyclic quiver with four nodes (see Figure 4.4). Moreover, our cluster embedding turns out to be compatible with the action of $U_q(\mathfrak{sl}_{n+1})$ in its *positive representations* [30], which are higher rank generalizations of the principal series representations of $U_q(\mathfrak{sl}_2)$.

Next, we turn to the problem of describing the coproduct and R-matrix of $U_q(\mathfrak{sl}_{n+1})$ in terms of our embedding. We formulate this description in terms of a quantum cluster chart $\mathcal{Z}_n$ from another quantum cluster variety, this time corresponding to a quiver built from a triangulation of the twice punctured disk. As we explain in Remark 4.6.4, the coproduct admits a simple description in terms of the cluster variables of $\mathcal{Z}_n$.

Finally, we prove in Theorem 4.6.1 that the automorphism $P \circ \mathrm{Ad}_R$ of $U_q(\mathfrak{sl}_{n+1})^{\otimes 2}$, given by conjugation by the R-matrix followed by the flip of tensor factors, restricts to $\mathcal{Z}_n$ and coincides with a cluster transformation given by the composite of the half-Dehn twist and a certain permutation. In the course of the proof, we obtain (Theorem 4.7.4) a refined factorization of R with $4\binom{n+2}{3}$ quantum dilogarithm factors, one for each mutation required to achieve the half-Dehn twist realized as a sequence of four flips. In the case of $U_q(\mathfrak{sl}_2)$, when each flip can be achieved by a single cluster mutation, we again recover Faddeev's factorization (1.3.3).

Chapter 2

Integrable systems with reflecting boundary conditions

2.1 Lie bialgebras and Poisson-Lie groups

In this section we collect some standard definitions and facts about Poisson-Lie groups and their Lie bialgebras that we shall use throughout this Chapter. For further details, see for example [17, 22]. Recall that a *Lie bialgebra* is a Lie algebra $\mathfrak{g}$ together with a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ satisfying the following two conditions:

1. The dual mapping $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$
2. The map δ satisfies the 1-cocycle condition

$$\delta([X, Y]) = X \cdot \delta(Y) - Y \cdot \delta(X)$$

where the $\mathfrak{g}$ acts on $\mathfrak{g} \wedge \mathfrak{g}$ by the exterior square of the adjoint representation.

We shall focus on Lie bialgebras for which the 1-cocycle δ is actually a coboundary. This means that there exists an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta(X) = X \cdot r$. One checks [17] that the induced bracket on $\mathfrak{g}^*$ will be skew and satisfy the Jacobi identity if and only if the symmetric part $J = \frac{1}{2}(r + r_{21})$ of r as well as the quantity

$$[[r, r]] := [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}]$$

are $\mathfrak{g}$-invariants in $\mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ respectively. In the case that $[[r, r]] = 0$, we say r is a solution of the *classical Yang-Baxter equation*, and that the Lie bialgebra $\mathfrak{g}$ is *quasitriangular.* If $\mathfrak{g}$ is a quasitriangular Lie bialgebra and the symmetric bilinear form on $\mathfrak{g}^*$ defined by J is nondegenerate, we say that $\mathfrak{g}$ is *factorizable.*

A *Poisson-Lie group* is Lie group equipped with a Poisson structure such that the group multiplication is a Poisson map. As is well known [17], the category of Lie bialgebras is equivalent to the category of connected, simply connected Poisson-Lie groups. The Lie bialgebra corresponding to a given Poisson-Lie group is called its *tangent Lie bialgebra*. We say that a Poisson-Lie group G is quasitriangular (resp. factorizable) if its tangent Lie bialgebra is.

The Poisson bracket on a coboundary Poisson-Lie group (G, r) may be described quite explicitly. If $\{X_s\}$ is a basis for $\mathfrak{g}$, let us expand

$$r = \sum_{s,t} r^{s,t} X_s \otimes X_t$$

Then if $f_1, f_2 \in \mathbb{C}[G]$, we have

$$\{f_1, f_2\} = \sum_{s,t} r^{s,t} \left(X_s^L[f_1] X_t^L[f_2] - X_s^R[f_1] X_t^R[f_2] \right) \tag{2.1.1}$$

where $Y^{L,R}$ denote left/right derivatives with respect to $Y \in \mathfrak{g}$:

$$Y^L[f](g) = \frac{d}{dt}\bigg|_{t=0} f(e^{tY} g), \quad Y^R[f](g) = \frac{d}{dt}\bigg|_{t=0} f(g e^{tY})$$

2.2 Poisson homogeneous spaces and the classical reflection equation

Let $\mathfrak{g}$ be a Lie bialgebra, and let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra in $\mathfrak{g}$. We say that $\mathfrak{k}$ is a *coideal Lie subalgebra* in $\mathfrak{g}$ if

$$\delta(\mathfrak{k}) \subset \mathfrak{g} \otimes \mathfrak{k} + \mathfrak{k} \otimes \mathfrak{g}$$

Our interest in coideal Lie subalgebras stems from the following fact, whose straightforward proof may be found in [22].

Proposition 2.2.1. *Let $K \subset G$ be a closed Lie subgroup in G. Then the homogeneous space G/K inherits a unique Poisson structure from G such that the natural projection $\pi : G \to G/K$ is Poisson if and only if $\mathfrak{k} = \mathrm{Lie}(K)$ is a coideal Lie subalgebra in $\mathfrak{g}$.*

Note that this condition is weaker than the condition that the subgroup K be a Poisson submanifold of G, for which we require $\eta(k) \subset T_k K^{\otimes 2} \subset T_k G^{\otimes 2}$ for all $k \in K$.

It is also worthing noting that Proposition 2.2.1 may also be stated in a dual form at the level of function algebras; in this formulation, which goes back to [54], the subalgebra $\mathbb{C}[G/K]$ of right K-invariant functions on G is a Poisson subalgebra in $\mathbb{C}[G]$ if and only if the annihilator $\mathfrak{k}^\perp \subset \mathfrak{g}^*$ of $\mathfrak{k}$ is a Lie subalgebra in the dual Lie bialgebra $\mathfrak{g}^*$.

Suppose now that $(\mathfrak{g}, r)$ is a coboundary Lie bialgebra, and that $\sigma : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism. Then the fixed point set

$$\mathfrak{k} = \mathfrak{g}^\sigma = \{x \in \mathfrak{g} \big| \sigma(x) = x\}$$

is a Lie subalgebra of $\mathfrak{g}$. Our first goal is to characterize when $\mathfrak{k}$ is a coideal Lie subalgebra. For this purpose, we shall introduce the quantity

$$C_\sigma(r) = (\sigma \otimes \sigma)(r) + r - (\sigma \otimes 1 + 1 \otimes \sigma)(r) \tag{2.2.1}$$

Note that if $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is any $\mathfrak{g}$-invariant, then $C_\sigma(\Omega)$ is a $\mathfrak{k}$-invariant. In particular, we have that $C_\sigma(J)$ is a $\mathfrak{k}$-invariant, where $J = \frac{1}{2}(r + r_{21})$ denotes the symmetric part of r.

Theorem 2.2.2. *Let $(\mathfrak{g}, r)$ be the coboundary Lie bialgebra determined by an element $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then the Lie subalgebra $\mathfrak{k} = \mathfrak{g}^\sigma$ is a coideal Lie subalgebra in $(\mathfrak{g}, r)$ if and only if the quantity $C_\sigma(r)$ is a $\mathfrak{k}$-invariant in $\mathfrak{g} \otimes \mathfrak{g}$.*

Proof. The subspace $\mathfrak{g} \otimes \mathfrak{k} + \mathfrak{k} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ is the kernel of the vector space endomorphism $A = (\sigma - 1) \otimes (\sigma - 1)$. So $\mathfrak{k}$ is a coideal Lie subalgebra if and only if $A \circ \delta(\mathfrak{k}) = 0$. Let us expand the r-matrix as $r = \sum_i a_i \otimes b_i$ for some $a_i, b_i \in \mathfrak{g}$, and take $x \in \mathfrak{k}$. Then we compute

$$\begin{aligned}
A \circ \delta(x) &= A([x, a_i] \otimes b_i + a_i \otimes [x, b_i]) \\
&= \sigma[x, a_i] \otimes \sigma b_i + [x, a_i] \otimes b_i - \sigma[x, a_i] \otimes b_i - [x, a_i] \otimes \sigma b_i \\
&+ \sigma a_i \otimes \sigma[x, b_i] + a_i \otimes [x, b_i] - \sigma a_i \otimes [x, b_i] - a_i \otimes \sigma[x, b_i] \\
&= [x, \sigma a_i] \otimes \sigma b_i + [x, a_i] \otimes b_i - [x, \sigma a_i] \otimes b_i - [x, a_i] \otimes \sigma b_i \\
&+ \sigma a_i \otimes [x, \sigma b_i] + a_i \otimes [x, b_i] - \sigma a_i \otimes [x, b_i] - a_i \otimes [x, \sigma b_i] \\
&= [x, (\sigma \otimes \sigma)r] + [x, r] - [x, (\sigma \otimes 1)r] - [x, (1 \otimes \sigma)r] \\
&= [x, C_\sigma(r)]
\end{aligned}$$

This is zero if and only if $C_\sigma(r)$ is a $\mathfrak{k}$-invariant. □

Corollary 2.2.3. *If $C_\sigma(r)$ is a $\mathfrak{k}$-invariant, the corresponding homogeneous space G/K inherits the structure of a Poisson manifold.*

The simplest way to satisfy this condition is to demand $C_\sigma(r) = 0$. In this case, we say that the pair (r, σ) is a solution of the *classical reflection equation*

$$(\sigma \otimes \sigma)(r) + r - (\sigma \otimes 1 + 1 \otimes \sigma)(r) = 0 \tag{2.2.2}$$

Additionally, it is important to note that, because of the $\mathfrak{g}$-invariance of the symmetric part J of the r-matrix, the condition that $C_\sigma(r)$ be a $\mathfrak{k}$-invariant will also be satisfied if we assume that the skew part $\hat{r}$ of the r-matrix satisfies the reflection equation $C_\sigma(\hat{r}) = 0$.

We will mostly be interested in the case when the automorphism σ is an involution, i.e. $\sigma^2 = \mathrm{id}$. Then if $\mathfrak{p}$ is the (-1)-eigenspace of σ, we have the decomposition of $\mathfrak{k}$-modules $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If we decompose the r-matrix as

$$r = r_{\mathfrak{k}\mathfrak{k}} + r_{\mathfrak{k}\mathfrak{p}} + r_{\mathfrak{p}\mathfrak{k}} + r_{\mathfrak{p}\mathfrak{p}} \tag{2.2.3}$$

we find $C_\sigma(r) = 4r_{\mathfrak{p}\mathfrak{p}}$. Hence, we have

Proposition 2.2.4. *If σ is an involution, $\mathfrak{k}$ is a coideal Lie subalgebra if and only if $r_{\mathfrak{p}\mathfrak{p}}$ is $\mathfrak{k}$-invariant. The pair (r, σ) is a solution of the CRE if and only if $r_{\mathfrak{p}\mathfrak{p}} = 0$.*

Remark 2.2.5. In this work, we only consider Poisson structures on G/K with the property that the projection $G \to G/K$ is a Poisson map. In [20], Drinfeld classifies Poisson structures on G/K compatible with the Poisson structure on G in the sense that the mapping $G \times G/K \to G/K$ is Poisson. Such Poisson structures are shown to correspond to Lagrangian subalgebras in the double $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$, those under consideration in the present work being given by Lagrangians $L_{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$. It would be interesting to try to construct integrable systems on the more general class of Poisson homogeneous spaces classified in [20].

Remark 2.2.6. Let us conclude this section by commenting on the relation between our construction and the one outlined in [11] in terms of Manin triple twists. Suppose we have a quasitriangular Lie bialgebra $(\mathfrak{g}, r)$, with the corresponding Manin triple $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, and suppose also that we have a Lie algebra involution ϕ of $\mathfrak{g}$. We may then attempt to extend ϕ to an anti-invariant Manin triple twist simply by declaring that σ acts on $\mathfrak{g}^*$ by

$$(\phi(\xi), X) = -\langle \xi, \phi(X) \rangle$$

where the round brackets to refer to the invariant symmetric bilinear form on $\mathfrak{d}$ and the angle brackets to refer to the canonical pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. What must be checked is that this extension respects the Lie algebra structure on $\mathfrak{d}$. In order for this to hold, we require that $(\phi \otimes \phi)(r) + r = 0$. Note, by applying $\phi \otimes 1$, that this condition also implies $(\phi \otimes 1 + 1 \otimes \phi)(r) = 0$. Therefore, the Manin triple twist constructed in this fashion gives a solution of the classical reflection equation. The solutions constructed in this fashion have the special property that both 'sides' of the reflection equation vanish in their own right.

2.3 Construction of integrable systems

Quasitriangular Poisson-Lie groups play a prominent role in the theory of classical integrable systems because of the following simple consequence of formula (2.1.1) for the Poisson bracket on G.

Proposition 2.3.1. *[48] If (G, r) is a quasitriangular Poisson-Lie group, then the subspace $I_G \subset C(G)$ of conjugation-invariant functions is a Poisson commutative subalgebra.*

Restricting this Poisson commutative subalgebra of functions to symplectic leaves of appropriate dimension in G, it is often possible to obtain classical integrable systems. Examples of integrable systems that can be derived in this framework include the Coxeter-Toda lattice [61], its affine counterpart [67], and the classical XXZ spin chain with periodic boundaries, see the survey [47] and references therein. We will now explain how to construct integrable systems on the Poisson homogeneous spaces G/K described in the previous section.

In order to describe K-invariant functions on G explicitly, we introduce the *classical reflection monodromy matrix*

$$\mathcal{T}(g) = g\sigma(g)^{-1} \tag{2.3.1}$$

Observe that if $k \in K = G^\sigma$, we have $\mathcal{T}(gk) = \mathcal{T}(g)$. Hence matrix elements $\mathcal{T}^V$ of $\mathcal{T}$ in any finite dimensional representation V are elements of the ring of K-invariant functions $C(G/K)$. Taking the trace, we obtain the *reflection transfer matrix*

$$\tau^V(g) = tr_V \; \mathcal{T}^V(g) \tag{2.3.2}$$

In contrast to those arising from the standard construction of integrable systems on G, the reflection transfer matrices $\tau^V(g)$ are not in general Ad_G-invariant. On the other hand, observe that $\tau^V(kg) = \tau^V(g)$, so the reflection transfer matrices lie in $\mathbb{C}(K\backslash G/K)$, the subalgebra of K-bi-invariant functions on G.

Theorem 2.3.2. *Suppose (r,σ) is a solution of the classical reflection equation. Then the subalgebra $\mathbb{C}(K\backslash G/K)$ is Poisson commutative. In particular, for any pair of finite dimensional representations V, W of G, the reflection transfer matrices τ^V, τ^W satisfy*

$$\{\tau^V, \tau^W\} = 0$$

Proof. If $H \in \mathbb{C}(K\backslash G/K)$ is a bi-invariant function, then for all $X \in \mathfrak{k}$ we have $X^{L,R}[H] = 0$. Hence the result follows from the decomposition (5) of the r-matrix. □

Poisson commutativity of the reflection transfer matrices also follows immediately from the following expression for the Poisson brackets of matrix elements of the reflection monodromy matrix:

Proposition 2.3.3. *Let r be the r-matrix of a quasitriangular Lie bialgebra, and let $\hat{r}$ denote its skew part. If (r,σ) is a solution of the classical reflection equation, matrix elements of the reflection monodromy matrix satisfy*

$$\{\mathcal{T}_1 \otimes \mathcal{T}_2\} = [r, \mathcal{T}_1\mathcal{T}_2] + \mathcal{T}_1[\mathcal{T}_2, \sigma_1 r] + \mathcal{T}_2[\mathcal{T}_1, \sigma_2 r] \tag{2.3.3}$$

Similarly, if $(\hat{r},\sigma)$ is a solution of the classical reflection equation, then matrix elements of the reflection monodromy matrix satisfy

$$\{\mathcal{T}_1 \otimes \mathcal{T}_2\} = [\hat{r}, \mathcal{T}_1\mathcal{T}_2] + \mathcal{T}_1[\mathcal{T}_2, \sigma_1 \hat{r}] + \mathcal{T}_2[\mathcal{T}_1, \sigma_2 \hat{r}] \tag{2.3.4}$$

Proof. Recall [17] that the matrix elements $\rho(g)$ of g in a finite dimensional representation have the Poisson brackets

$$\begin{aligned}\{\rho_1 \otimes \rho_2\} &= [r_{12}, \rho_1\rho_2] \\ &= [\hat{r}_{12}, \rho_1\rho_2]\end{aligned}$$

Now since σ is a group automorphism, we have

$$\{\rho_1 \circ \sigma, \rho_2\} = [\sigma_1 r, (\rho_1 \circ \sigma)\rho_2]$$

et cetera. Moreover, since $\mathcal{T}(g) = \rho(g) \cdot (\rho^{-1} \circ \sigma)(g)$, applying the Leibniz rule yields

$$\begin{aligned}\{\mathcal{T}_1 \otimes \mathcal{T}_2\} &= \rho_1\{(\rho_1^{-1} \circ \sigma), \rho_2\}(\rho_2^{-1} \circ \sigma) + \rho_1\rho_2\{\rho_1^{-1} \circ \sigma, \rho_2^{-1} \circ \sigma\} \\ &\quad + \{\rho_1, \rho_2\}(\rho_1^{-1} \circ \sigma)(\rho_2^{-1} \circ \sigma) + \rho_2\{\rho_1, \rho_2^{-1} \circ \sigma\}(\rho_1^{-1} \circ \sigma) \\ &= r\mathcal{T}_1\mathcal{T}_2 + \mathcal{T}_1\mathcal{T}_2(\sigma_1\sigma_2 r) - \mathcal{T}_1(\sigma_1 r)\mathcal{T}_2 - \mathcal{T}_2(\sigma_2 r)\mathcal{T}_1 \\ &\quad + \rho_1\rho_2\left(\sigma_1 r + \sigma_2 r - r - \sigma_1\sigma_2 r\right)(\rho_2^{-1} \circ \sigma)(\rho_1^{-1} \circ \sigma)\end{aligned}$$

This expression may be rewritten as

$$\begin{aligned}\{\mathcal{T}_1 \otimes \mathcal{T}_2\} &= [r, \mathcal{T}_1\mathcal{T}_2] + \mathcal{T}_1[\mathcal{T}_2, \sigma_1 r] + \mathcal{T}_2[\mathcal{T}_1, \sigma_2 r] \\ &\quad + \mathcal{T}_1\mathcal{T}_2 C_\sigma(r) - \rho_1\rho_2 C_\sigma(r)(\rho_2^{-1} \circ \sigma)(\rho_1^{-1} \circ \sigma)\end{aligned} \tag{2.3.5}$$

Applying the CRE $C_\sigma(r) = 0$, one arrives at formula (2.3.4). Of course, the case in which it is the skew part $\hat{r}$ that satisfies the CRE is treated identically. □

Let us conclude this section by observing that the commutativity of the reflection transfer matrices continues to hold under the weaker assumption that $C_\sigma(r)$ is a (possibly nonzero) $\mathfrak{k}$-invariant.

Proposition 2.3.4. *Suppose σ is an involution, and (r, σ) satisfies the condition (2.2.1): i.e. $C_\sigma(r)$ is a $\mathfrak{k}$-invariant. Then the reflection transfer matrices form a Poisson commutative subalgebra.*

Proof. As in the previous section, write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces of σ. Recall [29] that we may take neighborhoods $V_\mathfrak{k}, V_\mathfrak{p}$ in $\mathfrak{k}, \mathfrak{p}$ such that the map $V_\mathfrak{p} \times V_\mathfrak{k} \to G,\ (z, y) \mapsto \exp(z)\exp(y)$ is a diffeomorphism onto an open neighborhood U of the identity in G. For $g = \exp(z)\exp(y) =: pk \in U$, the K-invariance of $C_\sigma(r)$ implies

$$tr_{V\otimes W}\ (g \otimes g)C_\sigma(r)(\sigma(g^{-1}) \otimes \sigma(g^{-1})) = tr_{V\otimes W}(p^2 \otimes p^2)C_\sigma(r)$$

Since $g\sigma(g^{-1}) = p^2$, the terms involving $C_\sigma(r)$ in (2.3.5) cancel and we obtain the result.

□

2.4 Twisting

The construction of the previous section also admits a twisted version, which we shall now describe. Let σ be an involution on $(\mathfrak{g}, r)$ which is a solution of the classical reflection equation $C_\sigma(r) = 0$, and suppose that $\varphi_\pm$ are two automorphisms of $\mathfrak{g}$. As usual, we denote the Lie algebra of fixed points of σ by $\mathfrak{k}$, and the corresponding Lie group by K. The automorphisms $\varphi_\pm$ allow us to define twisted left and right actions of K on G:

$$k \triangleright g = \varphi_+(k)g, \quad g \triangleleft k = g\varphi_-(k) \tag{2.4.1}$$

Proposition 2.4.1. *Suppose that the involutions $\sigma_\pm = \varphi_\pm \circ \sigma \circ \varphi_\pm^{-1}$ are again solutions of $C_{\sigma_\pm}(r) = 0$. Then the subalgebra $\mathbb{C}(K_+\backslash G/K_-)$ of twisted K-bi-invariant functions on G is Poisson commutative.*

Proof. Since $C_{\sigma_\pm}(r) = 0$, we have $r_{\mathfrak{p}_\pm\mathfrak{p}_\pm} = 0$ in the corresponding decompositions (5) of r. The $\sigma_\pm$ fixed-subalgebras $\mathfrak{k}_\pm$ are related to $\mathfrak{k}$ by $\mathfrak{k}_\pm = \varphi_\pm(\mathfrak{k})$. But f is a twisted K-bi-invariant function we have

$$\varphi_+(X)^L f = 0 = \varphi_-(X)^R f$$

for all $X \in \mathfrak{k}$, and from formula (2.1.1) for the Poisson bracket the result follows. □

Of particular interest to us is when the the automorphisms $\varphi_\pm$ are of the form $\mathrm{Ad}_{h_\pm}$ for some $h_\pm \in G$. In this case, we may form the (right) twisted monodromy matrix

$$\mathcal{T}(g) = gh_-\sigma(h_-^{-1}g^{-1}) \tag{2.4.2}$$

whose matrix elements in any finite dimensional representation V are invariant under twisted right action of K. Then the twisted transfer matrices

$$\tau^V = \mathrm{tr}_V\left(\mathcal{T}(g)\sigma(h_+)h_+^{-1}\right) \tag{2.4.3}$$

are twisted bi-invariant functions on G. Putting $K_+ = \sigma(h_+)h_+^{-1}$ and $K_- = h_-\sigma(h_-)^{-1}$, we may write the twisted transfer matrix more economically as

$$\tau^V = tr_V\left(gK_-\sigma(g^{-1})K_+\right) \tag{2.4.4}$$

2.5 Factorization dynamics

We now proceed to the description of the dynamics of the systems constructed in the previous sections. In the case of flows on factorizable Poisson-Lie groups generated by Ad_G-invariant Hamiltonians, it is well-known that the dynamics is governed by the solution of a certain factorization problem in G. The first statement of this result in the Poisson-Lie context can be found in [54]; for further details see [48]. In the present case, we will show that the solution of the reflection dynamics generated by K-bi-invariant Hamiltonians is related to a factorization problem of Iwasawa-type.

For simplicity, we shall work in the untwisted setting. Let us begin by writing down the equations of motion generated by K-bi-invariant Hamiltonians.

Given a function $H \in \mathbb{C}(G)$, its left and right gradients at a point $g \in G$ are functionals $\nabla^{\pm}H(g) \in \mathfrak{g}^*$ defined by

$$\langle \nabla^+ H(g), X \rangle = \frac{d}{dt}\bigg|_{t=0} H(e^{tX}g), \quad \langle \nabla^- H(g), X \rangle = \frac{d}{dt}\bigg|_{t=0} H(ge^{tX})$$

Let $\hat{r} = \frac{1}{2}(r - r_{21})$ be the skew part of the r-matrix, and let $J = \frac{1}{2}(r + r_{21})$ be its symmetric part. We may regard the tensors $r, J, \hat{r} \in \mathfrak{g} \otimes \mathfrak{g}$ as linear maps $\mathfrak{g}^* \to \mathfrak{g}$ by contraction in the first tensor factor.

In this section, we shall assume that we have fixed a finite dimensional representation (ρ, V) of G, and to simplify notation we shall confuse group elements $g \in G$ with their images $\rho(g) \in \mathrm{End}(V)$. Now, it follows from formula (2.1.1) that the matrix g evolves under the Hamiltonian flow of H by

$$\dot{g}(t) = \hat{r}\left(\nabla^+ H\right) g - g\hat{r}\left(\nabla^- H\right) \tag{2.5.1}$$

Proposition 2.5.1. *Suppose $(\hat{r}, \sigma)$ is a solution of the classical reflection equation, and that $H \in C(K\backslash G/K)$ is a K-bi-invariant Hamiltonian. Then the Hamiltonian flow of H takes place on $K \times K$-orbits in G, and the reflection monodromy matrix evolves in time by the Lax equation*

$$\dot{\mathcal{T}}(t) = [\hat{r}\left(\nabla^+ H\right)(t), \mathcal{T}(t)] \tag{2.5.2}$$

Proof. If $H \in \mathbb{C}(K\backslash G/K)$, then for $X \in \mathfrak{k}$, $\langle \nabla^{\pm} H, X \rangle = 0$, and so from the decomposition (5) for the skew r-matrix $\hat{r}$, it follows that $\hat{r}_{\mathfrak{pp}} = 0$ and $\hat{r}(\nabla^{\pm} H) \in \mathfrak{k}$. In view of the equation of motion (2.5.1), this proves the first part of the proposition. The equation of motion for the reflection monodromy matrix is obtained by straightforward calculation using formula (2.1.1). □

Let us now assume that (G, r) is factorizable, with r satisfying the classical Yang-Baxter equation in the form

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0 \tag{2.5.3}$$

and suppose that the *skew part* $\hat{r}$ of r satisfies $C_\sigma(\hat{r}) = 0$ for some involution σ. By the Yang-Baxter condition (2.5.3), the linear map

$$r : \mathfrak{g}^* \to \mathfrak{g}, \quad \xi \longmapsto \langle \xi \otimes 1, r \rangle \tag{2.5.4}$$

is a homomorphism of Lie algebras, so its image $\mathfrak{b} = r(\mathfrak{g}^*)$ is a Lie subalgebra in $\mathfrak{g}$. We will further assume that $\mathfrak{g}$ admits an *Iwasawa decomposition* $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{k}$. At the group level, this means that in a neighborhood of the identity, each element of g admits a unique factorization $g = bk^{-1}$ with $b \in B$, $k \in K$.

Proposition 2.5.2. *Under the above assumptions on $(G, \hat{r}, \sigma)$, the time evolution of the matrix $g(t)$ under the Hamiltonian flow of $H \in \mathbb{C}(K\backslash G/K)$ is given for sufficiently small time t by*

$$g(t) = k_+^{-1}(t) g_0 k_-(t) \tag{2.5.5}$$

where the matrices $k_\pm(t)$ are solutions of the following factorization problems in G:

$$\exp\big(tJ(\nabla^\pm H(g_0))\big) = b(t) k_\pm^{-1}(t)$$

Proof. For brevity, let us denote $Q_\pm = J(\nabla^\pm H(g_0))$. By the Iwasawa decomposition, for sufficiently small time we have a unique factorization

$$e^{tQ_\pm} = b(t) k_\pm^{-1}(t)$$

where $b(t) \in B, k_\pm(t) \in K$. Differentiating with respect to time shows that

$$b^{-1}\dot{b} - k_\pm^{-1}\dot{k_\pm} = k_\pm^{-1} Q k_\pm$$

Now set $g(t) = k_+^{-1}(t) g_0 k_-(t)$. By the bi-invariance of H and the Ad-invariance of J, we have

$$\begin{aligned} k_\pm^{-1}(t) Q k_\pm(t) &= J\left(\nabla^\pm_{g(t)} H\right) \\ &= r\left(\nabla^\pm_{g(t)} H\right) - \hat{r}\left(\nabla^\pm_{g(t)} H\right) \end{aligned}$$

But by the Iwasawa decomposition at the Lie algebra level this implies

$$k_\pm^{-1}\dot{k_\pm} = \hat{r}\left(\nabla^\pm_{g(t)} H\right)$$

and by comparison with the equation of motion (2.5.1) the result follows. □

Corollary 2.5.3. *Under the above assumptions on $(G, \hat{r}, \sigma)$, the isospectral evolution of the reflection monodromy matrix under the Hamiltonian flow of $H \in \mathbb{C}(K\backslash G/K)$ is given explicitly by*

$$\mathcal{T}(t) = k_+^{-1}(t) \mathcal{T}_0 k_+(t) \tag{2.5.6}$$

In particular, it follows that all spectral invariants of the reflection monodromy matrix are conserved quantities for the reflection Hamiltonians.

2.6 Finite dimensional examples

We will now show how our construction may be applied to split real semisimple Lie algebras. For simplicity, we will focus on the case of type A_n, when $\mathfrak{g} = sl_{n+1}(\mathbb{R})$.

Let us choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and a system of Chevalley generators $\{E_i, F_i, H_i\}$. We denote the set of positive roots by Δ^+. The standard Lie bialgebra structure on $\mathfrak{g}$ is defined by

$$\delta(H_i) = 0, \quad \delta(E_i) = E_i \wedge H_i, \quad \delta(F_i) = F_i \wedge H_i, \tag{2.6.1}$$

or equivalently by the r-matrix

$$r = \sum_{\alpha \in \Delta^+} E_\alpha \wedge F_\alpha$$

The *Cartan involution* on $\mathfrak{g}$ is the Lie algebra automorphism θ defined by

$$\theta(H_i) = -H_i, \quad \theta(E_i) = -F_i, \quad \theta(F_i) = -E_i$$

This involution gives rise to a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into its ± 1 eigenspaces known as the *Cartan decomposition.* The fixed point set $\mathfrak{k} = so_n(\mathbb{R})$ is a Lie subalgebra in $\mathfrak{g}$, the anti-fixed point set $\mathfrak{p}$ consisting of traceless symmetric matrices is a $\mathfrak{k}$-module.

Proposition 2.6.1. *The pair (r, θ) is a solution of the classical reflection equation.*

From this (or by inspection of the formulas for the cobracket), it follows that $\mathfrak{k}$ is a coideal Lie subalgebra in $\mathfrak{g}$. Note also that $\mathfrak{k}$ does *not* satisfy $\delta(\mathfrak{k}) \subset \mathfrak{k} \otimes \mathfrak{k}$, so K is not a Poisson-Lie subgroup in G.

Denote by A, B the analytic subgroups of G with Lie algebras $\mathfrak{h}, \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ respectively. By virtue (see [29]) of the Iwasawa decomposition $G = BK$, we may identify the homogeneous space G/K with B. Observe that $B \subset G$ is a Poisson submanifold.

Proposition 2.6.2. *The Poisson structure on G/K coincides with the Poisson structure on B coming from its inclusion into G.*

Proof. Let $\pi : G \to G/K \simeq B$ be the projection with respect to Iwasawa decomposition, and $\iota : G/K \simeq B \to G$ be inclusion. We must show that ι is Poisson, where $B \simeq G/K$ is equipped with the *quotient* Poisson structure. For $b \in B$, we calculate $\iota_*^{\otimes 2}\left(\eta_{G/K}(bK)\right)$. By definition of the quotient Poisson structure and the chain rule, we have

$$\begin{aligned} \iota_*^{\otimes 2}\left(\eta_{G/K}(bK)\right) &= \iota_*^{\otimes 2}\pi_*^{\otimes 2}\left(\eta_G(b)\right) \\ &= (\iota \circ \pi)_*^{\otimes 2}\left(\eta_G(b)\right) \end{aligned}$$

Since $B \subset G$ is Poisson, $\eta_G(b) \subset TB \otimes TB$. Moreover, we have $(\iota \circ \pi)|_B = \mathrm{id}_B$. Hence

$$(\iota \circ \pi)_*^{\otimes 2}\left(\eta_G(b)\right) = \eta_G(b)$$

which shows that ι is Poisson. $\square$

Remark 2.6.3. Recall [29] that we also have the *global Cartan decomposition* $G = PK$: any element of g may be uniquely factored $g = pk$, where $k \in SO_n$ and p is a symmetric positive definite matrix. This shows that we may also identify G/K with the space P of symmetric positive definite matrices. The element p in the factorization of g is essentially the reflection monodromy matrix, since it may be explicitly computed as $p^2 = gg^T = g\theta(g^{-1})$.

By the so-called KAK decomposition (see again [29]) , the reflection Hamiltonians may be regarded as functions on $K\backslash G/K \simeq A$, where A is the n-dimensional Cartan subgroup of unit determinant diagonal matrices. Thus in order to describe integrable systems on G/K, we must identify symplectic leaves of dimension $2n$. Recall [61], [67] that the double Bruhat cells $G^{u,v}$ are A-invariant Poisson subvarieties of G. Consider the $2n$-dimensional double Bruhat cell $G^{c,1} \subset B$ where c is the Coxeter element $c = s_n \cdots s_1$ in the symmetric group S_{n+1}.

Proposition 2.6.4. *The double Bruhat cell $G^{c,1}$ is mapped isomorphically under the quotient projection to a symplectic manifold $M_c \subset G/K$ of dimension $2n$, to and the restriction of $C(K\backslash G/K)$ to M_c defines an integrable system.*

Indeed, $G^{c,1}$ consists of all upper triangular matrices X with positive diagonal entries and with all entries of distance > 1 from the diagonal equal to zero. If we write

$$a_k = X_{kk}, k = 1, \ldots, n+1, \; b_k = X_{k,k+1},$$

the non-zero Poisson brackets of coordinates are given by

$$\{a_k, b_k\} = a_k b_k, \quad \{a_{k+1}, b_k\} = -a_{k+1} b_k$$

The coordinates (a_k, b_k) can be expressed in terms of canonically conjugate coordinates $\{p_k, q_k\} = 1, \; k = 1, \ldots n$ by

$$a_k = e^{q_{k-1} - q_k}, \quad b_k = e^{p_k}, \quad k = 1, \ldots, n$$

where we understand $q_0 = q_{n+1} = 0$. The reflection monodromy matrix takes the symmetric tridiagonal form

$$\mathcal{T} = \begin{bmatrix} a_1^2 + b_1^2 & b_1 a_2 & 0 & 0 & 0 & 0 \\ b_1 a_2 & a_2^2 + b_2^2 & b_2 a_3 & 0 & 0 & 0 \\ 0 & b_2 a_3 & a_3^2 + b_3^2 & b_3 a_4 & 0 & 0 \\ & & \ddots & & & \\ & & & \ddots & & \\ 0 & 0 & 0 & b_{n-1} a_n & a_n^2 + b_n^2 & b_n a_{n+1} \\ 0 & 0 & 0 & 0 & b_n a_{n+1} & a_{n+1}^2 \end{bmatrix}$$

Its trace is the quadratic local Hamiltonian

$$tr\ \mathcal{T} = \sum_{k=1}^{n+1} e^{2(q_{k-1}-q_k)} + \sum_{k=1}^{n} e^{2p_k}$$

This system essentially coincides with the open Coxeter-Toda system with phase space $G^{c,c^{-1}}/A$ [27]. Explicitly, $G^{c,c^{-1}}/A$ consists of the unit determinant tridiagonal matrices, modulo conjugation by diagonal matrices, where we equip $SL_{n+1}(\mathbb{R})$ with the Poisson structure defined by the scaled r-matrix $2r$. Then under this normalization, the map

$$M_c \to G^{c,c^{-1}}/A, \quad \mathcal{T} \mapsto [\mathcal{T}] \in SL_{n+1}/A$$

is Poisson, and carries open Coxeter-Toda Hamiltonians to reflection Hamiltonians. Hence, our construction gives a "symmetric" Lax representation of the open Coxeter-Toda system. This result is a non-linear analog of the fact that the phase space of the non-relativistic open Toda chain with its linear Poisson structure may be realized in two ways: either as lower Hessenberg matrices, or as symmetric tridiagonal matrices, see [55] and [28].

One may also restrict the reflection Hamiltonians to the $2n$-dimensional symplectic leaves of the double Bruhat cells $G^{c,c}$, although we have been unable to identify the integrable systems obtained in this fashion with those in the existing literature.

Remark 2.6.5. Although the restriction of the reflection Hamiltonians to symplectic leaves of dimension greater than $2n$ cannot yield an integrable system, the explicit solution of the equations of motion in terms of the factorization problem in G given in Section 6 remains valid on such leaves. This leads us to suspect that such systems may be *degenerately integrable*, as their Ad_G-invariant counterparts were shown to be in [46]. We leave the detailed investigation of this subject for a future work.

2.7 Classical XXZ spin chain with reflecting boundaries

We will now apply our general scheme to the case of the formal loop algebra $Lsl_2 = sl_2 \otimes \mathbb{C}[z^{\pm 1}]$. In doing so, we shall recover the semiclassical limit of Sklyanin's XXZ model with reflecting boundary conditions [57]. Let

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

be the usual root basis of $\mathfrak{g} = sl_2$. The infinite dimensional Lie algebra $L\mathfrak{g}$ has a basis $\{x[n] = x \otimes z^n \mid x \in \{E, H, F\}, n \in \mathbb{Z}\}$. It admits several pseudo-triangular Lie bialgebra structures [17], the one of interest to us being determined by the (trigonometric) r-matrix

$$r(z,w) = \frac{z^2+w^2}{z^2-w^2}\left(\frac{H\otimes H}{2}\right) + \frac{2zw}{z^2-w^2}\left(E\otimes F + F\otimes E\right) \tag{2.7.1}$$

Here we regard $Lsl_2 \otimes Lsl_2 \simeq (\mathfrak{g}\otimes\mathfrak{g})[z^{\pm1}, w^{\pm1}]$ as being embedded in the larger space $(\mathfrak{g}\otimes\mathfrak{g})(z,w)$ of $\mathfrak{g}\otimes\mathfrak{g}$-valued rational functions of z and w. The cobracket is given by the formula

$$\delta(x)(z,w) = \left(\mathrm{ad}_{x(z)}\otimes 1 + 1\otimes \mathrm{ad}_{x(w)}\right) r(z,w)$$

Setting $r_{12}(z/w) = r(z,w)$, we obtain a solution of the classical Yang-Baxter equation in $(\mathfrak{g}\otimes\mathfrak{g})(z,w)$ with multiplicative spectral parameter:

$$[r_{12}(z), r_{13}(zw)] + [r_{12}(z), r_{23}(w)] + [r_{13}(zw), r_{23}(w)] = 0$$

Observe that $r(z)$ satisfies the 'unitarity' condition

$$r_{12}(z^{-1}) = -r_{12}(z)$$

Now let $\rho : sl_2 \to \mathrm{End}(\mathbb{C}^2)$ be the vector representation of sl_2, and consider evaluation representations $\rho_z = \mathbb{C}^2\otimes\mathbb{C}[z^{\pm1}], \rho_w = \mathbb{C}^2\otimes\mathbb{C}[w^{\pm1}]$. Again, we embed $\rho_z\otimes\rho_w$ inside $(\mathbb{C}^2\otimes\mathbb{C}^2)(z,w)$. Then the image of $r(z,w)$ is

$$r(z,w) = \frac{1}{2(z^2-w^2)}\begin{bmatrix} z^2+w^2 & 0 & 0 & 0 \\ 0 & -(z^2+w^2) & 4zw & 0 \\ 0 & 4zw & -(z^2+w^2) & 0 \\ 0 & 0 & 0 & z^2+w^2 \end{bmatrix}$$

The classical monodromy matrix $T(z)$ is defined as the matrix elements of LSL_2 in the evaluation representation ρ_z. Poisson brackets of its elements are given by

$$\{T_1(z), T_2(w)\} = [r_{12}(z,w), T_1(z)T_2(w)] \tag{2.7.2}$$

The loop algebra Lsl_2 has an involution θ defined by $(\theta x)(z) = x(z^{-1})$ for $x(z)\in Lsl_2$. Conjugation by any element of the loop group LGL_2 also defines an automorphism of Lsl_2. Let us define a one-parameter family of loop group elements $h(z;\xi)$ by

$$h(z;\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \xi z^{-1} - z\xi^{-1} \end{pmatrix}$$

and consider the composite automorphism $\sigma_\xi = \mathrm{Ad}_{h(z;\xi)}\circ\theta\circ\mathrm{Ad}_{h^{-1}(z;\xi)}$. Note that setting $\xi = i$ recovers $\sigma_i = \theta$.

Proposition 2.7.1. *We have*

$$(\sigma_\xi\otimes\sigma_\xi)r(z,w) + r(z,w) - (\sigma_\xi\otimes 1 + 1\otimes\sigma_\xi)r(z,w) = 0 \tag{2.7.3}$$

so that σ_ξ defines a one-parameter family of solutions of the classical reflection equation.

Introducing

$$K(z;\xi) = h(z;\xi)h^{-1}(z^{-1};\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\xi z^{-1}-\xi^{-1}z}{\xi z - z^{-1}\xi^{-1}} \end{pmatrix}$$

equation (2.7.3) may be written more explicitly as

$$\begin{aligned} & r_{12}(z/w)K_1(z)K_2(w) + K_1(z)r_{12}(zw)K_2(w) \\ & = K_2(w)r_{12}(zw)K_1(z) + K_2(w)K_1(z)r_{12}(z/w) \end{aligned}$$

where we for brevity we have suppressed in our notation the ξ-dependence of the matrix $K(z)$.

Remark 2.7.2. This latter formula can be recognized as the semiclassical limit of Sklyanin's quantum reflection equation [57]

$$R_{12}(z_1/z_2)K_1(z_1)R_{12}(z_1z_2)K_2(z_2) = K_2(z_2)R_{12}(z_1z_2)K_1(z_1)R_{12}(z_1/z_2)$$

where the quantum R-matrix

$$R(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for} \quad b(z) = \frac{z - z^{-1}}{qz - q^{-1}z^{-1}}, \quad c(z) = \frac{q - q^{-1}}{qz - q^{-1}z^{-1}}$$

is related to $r(z)$ by

$$f(z)R(z) = 1 + hr(z) + O(h^2),\ q = e^h,\ h \to 0$$

where

$$f(z) = \frac{q^{1/2}z - q^{-1/2}z^{-1}}{z - z^{-1}}$$

By virtue of Proposition 2.7.1, we may perform the twisting outlined in section 5. If ξ_+, ξ_- are complex numbers, we shall twist on the left by $\mathrm{Ad}_{h(z;\xi_+^{-1})}$, and on the right by $\mathrm{Ad}_{h(z;\xi_-)}$. The corresponding twisted (right) reflection monodromy matrix is

$$\mathcal{T}(z) = T(z)K_-(z)T^{-1}(z^{-1})$$

A simple calculation along the lines of the proof of formula (2.3.4) shows that

$$\begin{aligned} \{\mathcal{T}_1(z) \otimes \mathcal{T}_2(w)\} = {} & [r_{12}(z/w), \mathcal{T}_1(z)\mathcal{T}_2(w)] \\ & + \mathcal{T}_1(z)r_{12}(zw)\mathcal{T}_2(w) - \mathcal{T}_2(w)r_{12}(zw)\mathcal{T}_1(z) \end{aligned}$$

which coincides with the formula given in Sklyanin's original paper [57].

In order to describe particular finite dimensional systems, we must identify symplectic leaves in LSL_2. The leaves we will consider can be described in terms of $(SL_2)^*$, the Poisson-Lie group dual to SL_2 with its standard Poisson structure. Explicitly, we have $\mathbb{C}[SL_2^*] \simeq \mathbb{C}[e, f, k^{\pm 1}]$ with the Poisson brackets

$$\{k, e\} = ke$$
$$\{k, f\} = -kf$$
$$\{e, f\} = 2(k^2 - k^{-2})$$

The function $\omega = k^2 + k^{-2} + ef$ is a Casimir element of the Poisson algebra $\mathbb{C}[SL_2^*]$, and it is thus constant on symplectic leaves. We shall parameterize its value by $\omega = t^2 + t^{-2}$. The generic level set ω_t is a two-dimensional symplectic leaf Σ_t in SL_2^*.

One can check that the Local Lax matrix

$$L(z) = \begin{pmatrix} zk - z^{-1}k^{-1} & e \\ f & zk^{-1} - z^{-1}k \end{pmatrix} \tag{2.7.4}$$

satisfies $\{L_1(z), L_2(w)\} = [r_{12}(z/w), L_1(z)L_2(w)]$. Hence the mapping

$$T(z) = L_1(z) \cdots L_N(z)$$

defines a Poisson embedding of the $2N$-dimensional symplectic manifold $\Sigma_{t_1} \times \cdots \times \Sigma_{t_N}$ into LSL_2.

Let us conclude by deriving the local Hamiltonian of the homogeneous chain where $\omega_i \equiv \omega = t^2 + t^{-2}$, using the technique explained in [47] and references therein. Note that since

$$\det L(z) = z^2 + z^{-2} - \omega$$

when $z = t^{\pm 1}$ the Lax matrix degenerates into the projector

$$L(t) = \alpha \otimes \beta^T$$

for

$$\alpha = \begin{pmatrix} 1 \\ (tk^{-1} - t^{-1}k)/e \end{pmatrix}, \quad \beta = \begin{pmatrix} tk - t^{-1}k^{-1} \\ e \end{pmatrix}$$

We also have the identity

$$L(z)L(z^{-1}) = -\det L(z)\mathrm{Id}$$

To describe the local Hamiltonian of the chain, we introduce the regularized reflection monodromy matrix

$$S(z) = \left((-1)^N \det T(z)\right) \mathcal{T}(z) = L_1(z) \cdots L_N(z) K_-(z) L_N(z) \cdots L_1(z)$$

One then computes

$$tr\ S(t)K_+(z) = (K_-(t)\alpha_N, \beta_N)(\beta_1, K_+(t)\alpha_1)\prod_{n=1}^{N-1}(\beta_n, \alpha_{n+1})(\beta_{n+1}, \alpha_n)$$

But

$$\begin{aligned}(\beta_n, \alpha_{n+1})(\beta_{n+1}, \alpha_n) &= tr\ \ (L_n(t)L_{n+1}(t)) \\ &= e_n f_{n+1} + f_{n+1}e_n + \omega(k_n k_{n+1} + k_n^{-1}k_{n+1}^{-1}) - 2(k_n k_{n+1}^{-1} + k_{n+1}k_n^{-1})\end{aligned}$$

and

$$(K_\pm(t)\alpha_n, \beta_n) = \text{const.} \times (\xi_\pm k_n - \xi_\pm^{-1}k_n^{-1})$$

Hence setting

$$\begin{aligned}H_{n,n+1} &= \log\left(e_n f_{n+1} + e_{n+1}f_n + \omega(k_n k_{n+1} + k_n^{-1}k_{n+1}^{-1}) - 2(k_n k_{n+1}^{-1} + k_{n+1}k_n^{-1})\right) \\ H_0 &= \log(\xi_+ k_1 - \xi_+^{-1}k_1^{-1}), \quad H_N = \log(\xi_- k_N - \xi_-^{-1}k_N^{-1})\end{aligned}$$

we obtain the local reflection Hamiltonian

$$\mathcal{H} = H_0 + \sum_{n=1}^{N-1} H_{n,n+1} + H_N.$$

2.8 Algebraic integrability of the XXZ model with reflecting boundaries: preliminaries

We now proceed towards our goal of establishing the algebraic integrability of the XXZ model with reflecting boundary conditions. Let us begin by making a few further general remarks on the model that will prove useful in the sequel.

Firstly, we observe that the local Lax matrix (2.7.4) satisfies satisfies

$$\det L(z) = z^2 + z^{-2} - \omega \tag{2.8.1}$$

as well as the identities

$$L(z)L(z^{-1}) = -\det L(z)\text{Id} \tag{2.8.2}$$

$$L(z^{-1})^t = -\sigma_2 L(z)\sigma_2^{-1} \tag{2.8.3}$$

$$L(-z) = -\sigma_3 L(z)\sigma_3^{-1} \tag{2.8.4}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. From the symmetries (2.8.2) of $L(z)$, it follows that the reflection monodromy matrix $\mathcal{T}(z)$ satisfies

$$\mathcal{T}(-z) = \sigma_3 \mathcal{T}(z) \sigma_3^{-1} \tag{2.8.5}$$
$$\mathcal{T}(z^{-1})^t = -\sigma_2 \mathcal{T}(z) \sigma_2^{-1} \tag{2.8.6}$$

Later, we will need the following explicit formulae for the Poisson brackets of matrix elements of $\mathcal{T}(z)$:

$$\{A(z_1), A(z_2)\} = \frac{2}{z_1 z_2 - z_1^{-1} z_2^{-1}} \Big(B(z_1)C(z_2) - C(z_1)B(z_2) \Big) \tag{2.8.7}$$

$$\{C(z_1), A(z_2)\} = \frac{2z_1}{(z_2^2 - z_1^2)(z_1^2 z_2^2 - 1)} \Big(z_1 z_2^4 C(z_1)A(z_2) - z_1^2 z_2^3 A(z_1)C(z_2) - z_1^2 z_2 D(z_1)C(z_2) + z_2^3 D(z_1)C(z_2) - z_1 C(z_1)A(z_2) + z_2 A(z_1)C(z_2) \Big) \tag{2.8.8}$$

Note that unlike in the periodic case, the functions $A(z)$ do not form a Poisson commutative family.

The *reflection transfer matrix* is the Laurent polynomial $t(z)$ defined by

$$t(z) = \frac{1}{2} \mathrm{tr}\ \mathcal{T}(z)$$

Proposition 2.8.1. ([57],[52]) *The reflection transfer matrix satisfies*

$$\{t(z_1), t(z_2)\} = 0$$

and thus its coefficients generate a Poisson commutative subalgebra in $\mathbb{C}[M_N]$.

Proof. The commutativity of the reflection transfer matrices follows immediately by taking the trace over $\mathbb{C}^2 \otimes \mathbb{C}^2$ in formula (2.3.4) for the Poisson bracket of matrix elements of the reflection monodromy matrix. □

Let us describe some properties of the transfer matrix. Firstly, by the symmetries (2.8.2) of $\mathcal{T}(z)$, we have

$$t(-z) = t(z),\ t(z^{-1}) = -t(z)$$

The transfer matrix $t(z)$ therefore a function of the variable $w = z^2$, which admits an expansion

$$t(z) = \frac{1}{2} \left(\frac{w+1}{w-1} \right) \left(P_N \left(\frac{w^N + w^{-N}}{2} \right) + P_{N-1} \left(\frac{w^{N-1} + w^{1-N}}{2} \right) + \cdots + P_0 \right) \tag{2.8.9}$$

Note also that

$$t(z) = \frac{A(z) - A(z^{-1})}{2}$$

with the function $A(z)$ taking the form

$$A(z) = \frac{Pz^{2N+1} + \cdots - P^{-1}z^{-2N-1}}{z - z^{-1}}$$

where the leading coefficient

$$P = \xi_- \prod_{j=1}^{N} k_j^2$$

is proportional to the deformed total σ^z-component of spin. The leading coefficient of the transfer matrix is

$$\begin{aligned} \frac{P_N}{2} &= \xi_- \prod_{j=1}^{N} (k_j)^2 - \xi_-^{-1} \prod_{j=1}^{N} (k_j)^{-2} \\ &= P - P^{-1} \end{aligned}$$

The following lemma, giving a linear relation between the reflection Hamiltonians, is a simple consequence of formulas (2.8.1) and (2.8.2).

Lemma 2.8.2. *The reflection transfer matrix $t(z)$ satisfies*

$$\lim_{z \to 1} (z - z^{-1}) t(z) = \sum_{j=0}^{N} P_j = \left(\xi_- - \xi_-^{-1}\right) \prod_{k=1}^{N} (\omega_k - a_k^2 - a_k^{-2})$$

We also have the following proposition, which shows that the functions $(P_1, \ldots, P_N)$ form a set of N functionally independent Hamiltonians.

Proposition 2.8.3. *For generic values of the constants ξ, ω_i, a_i, the reflection Hamiltonians $P_1, \ldots, P_N$ are functionally independent.*

Proof. Since the functional independence is an open condition, it suffices to consider the case $\xi = a_i \equiv 1$. We will prove the stronger statement that the $P_1, \ldots, P_N$ remain independent when restricted to the N-dimensional subvariety of phase space cut out by $\{f_j = 0 | j = 1, \ldots, N\}$. On this locus, the local Lax operators become upper triangular, so the reflection monodromy matrix becomes

$$t(z) = \left(\frac{z + z^{-1}}{z - z^{-1}}\right) \left(\prod_{j=1}^{N} (z k_j - z^{-1} k_j^{-1})^2 + \prod_{j=1}^{N} (z k_j^{-1} - z^{-1} k_j)^2\right)$$

Note that the reflection Hamiltonians P_j are functions of the variables $\tilde{k}_j = k_j^2$: explicitly, for $1 \leq j \leq N$, we have

$$P_j = \sum_{r_i \in \{0, \pm 1\}, r_1 + \cdots + r_k = j} \left(\prod_{i=1}^{N} (-2)^{\delta_{r_i,0}} (\tilde{k}_i^{r_i} + \tilde{k}_i^{-r_i})\right)$$

To verify their algebraic independence, it suffices to check that the Jacobian $J(\tilde{k}) = \det\left[\frac{\partial P_i}{\partial_{\tilde{k}_j}}\right]$ is not identically zero. Indeed, by counting degrees one sees that the Laurent monomial $k_1^{N-1}k_2^{N-2}\cdots k_{N-1}$ can only be obtained from the diagonal term in the expansion of the determinant $J(\tilde{k})$, where it appears with coefficient $(-2)^{N(N+1)/2}$.

□

This proposition shows that the classical XXZ spin chain with reflecting boundary conditions is an integrable system. Note that the reflection Hamiltonians can be written

$$P_k = 2^{2-\delta_{k,0}}\mathrm{Res}_{z=0}\left(\frac{z-z^{-1}}{z+z^{-1}}\right)z^{-2k-1}t(z)dz$$

Let us now write down the equations of motion generated by the P_k. For this we need to introduce some notations. Given any Laurent polynomial

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{C}[z, z^{-1}]$$

we may uniquely decompose f as

$$f = f^\sigma + f^+$$

where f_σ satisfies $f^\sigma(z) = f^\sigma(z^{-1})$ and $f^+ \in z\mathbb{C}[z]$. Let us also introduce the matrices

$$M_k^\sigma(z) = 2^{2-\delta_{k,0}}\left(\left(\frac{z-z^{-1}}{z+z^{-1}}\right)z^{-2k}\mathcal{T}(z)\right)^\sigma$$

$$M_k^+(z) = 2^{1-\delta_{k,0}}\left(\left(\frac{z-z^{-1}}{z+z^{-1}}\right)z^{-2k}\mathcal{T}(z)\right)^+$$

Taking the trace over the first space in equation (2.3.4), we find that the equations of motion take the following Lax form:

$$\frac{\partial}{\partial t_k}\mathcal{T}(z) := \{\mathcal{T}(z), P_k\} = [\mathcal{T}(z), M_k^\sigma(z)] \tag{2.8.10}$$

$$= \left[M_k^+(z), \mathcal{T}(z)\right] \tag{2.8.11}$$

We therefore obtain the following corollary, which opens the door to studying the system using the algebro-geometric techniques explained in [66],[45],[33],[48] and references therein.

Corollary 2.8.4. *The spectrum of the reflection monodromy matrix $\mathcal{T}(z)$ is preserved under the Hamiltonian flows of the reflection Hamiltonians. In particular, the coefficients of the characteristic polynomial* $\det(\zeta - \mathcal{T}(z))$ *are invariant under these flows.*

2.9 Spectral curves

Motivated by Corollary 2.8.4, we consider the invariant spectral curve

$$M \ : \ \det(\zeta - \mathcal{T}(z)) = 0 \tag{2.9.1}$$

cut out of $\mathbb{C} \times \mathbb{C}^*$ by the characteristic polynomial of the reflection monodromy matrix $\mathcal{T}(z)$. More precisely, we shall work with the compact Riemann surface obtained by adding four points at infinity, two points over $z = 0$ and another two points over $z = \infty$. In what follows, we will use the notation M to refer to this compact Riemann surface. Introducing

$$y = \zeta - t(z) \tag{2.9.2}$$

we have

$$y^2 = t(z)^2 - \det \mathcal{T}(z)$$

By (2.8.1), the coefficients of $\det \mathcal{T}(z)$ are constant on a symplectic leaf, so that all degrees of freedom for the moduli of M are in fact encoded by the transfer matrix $t(z)$ and its coefficients $\{P_j\}$.

Let us introduce the notations $\lambda = z^2 + z^{-2}$ and

$$\mathcal{Q}(z) = t(z)^2 - \det \mathcal{T}(z)$$

Lemma 2.9.1. *We have $\mathcal{Q}(z) = \mathcal{Q}_{2N}(\lambda)$ where $\mathcal{Q}_{2N}(\lambda)$ is a polynomial of degree $2N$ in λ.*

This fact has the following geometric meaning. Firstly, the curve M is a 4-fold cover of a genus $N-1$ hyperelliptic curve

$$\Gamma \ : \ y^2 - \mathcal{Q}_{2N}(\lambda) = 0$$

and a 2-fold cover of the intermediate genus $2N-1$ spectral curve

$$\Sigma \ : \ y^2 - \tilde{\mathcal{Q}}(w) = 0$$

where $w = z^2$ and $\mathcal{Q}(z) = \tilde{\mathcal{Q}}(w) = \mathcal{Q}_{2N}(\lambda)$. The projection $\pi : \Sigma \to \Gamma$ is given by $\lambda = w + w^{-1}$. Note that $\Gamma = \Sigma/\tau$, where $\tau : \Sigma \to \Sigma$ is the involution $\tau(w, y) = (w^{-1}, y)$.

We now turn to the description of the holomorphic differentials on the various spectral curves. The space $H^0(\Sigma, K)$ of holomorphic differentials on Σ has dimension $g(\Sigma) = 2N-1$. We may decompose $H^0(\Sigma, K)$ into its ± 1 eigenspaces $V_\pm$ with respect to the induced action of the involution τ. Bases may be chosen as

$$V_+ = \operatorname{span}\left\{ \omega_j^+ = \frac{(w - w^{-1})(w^j + w^{-j})}{yw} dw \ \middle| \ 0 \le j \le N-2 \right\}$$

$$V_- = \text{span}\left\{ \omega_k^- = \frac{(w^k + w^{-k})}{yw} dw \;\middle|\; 0 \le k \le N-1 \right\}$$

The subspace V_+ coincides with $\pi^* H^0(\Gamma, K)$, and its elements may be regarded as holomorphic differentials on Γ. The following basis for V_+ will prove well adapted to the description of the flows of our chosen basis of reflection Hamiltonians P_j:

$$\Omega_j = \left(\frac{w+1}{w-1} \right) \frac{(w^j + w^{-j} - 2)}{8yw} dw \ , \ 1 \le j \le N-1$$

We will also need the following differential of the third kind

$$\Omega_N = -(P + P^{-1}) \left(\frac{w+1}{w-1} \right) \frac{(w^N + w^{-N} - 2)}{2yw} dw$$

which has simple poles at the two points $\infty_\pm$ lying over $\lambda = \infty$ and is regular elsewhere. We shall label the points $\infty_\pm$ by

$$(\lambda^{-N} y)(\infty_\pm) = \pm \left(\frac{P + P^{-1}}{2} \right)$$

Observe that Ω_N is defined so as to have the normalization

$$\text{Res}_{\infty_+} \Omega_N = 1 = -\text{Res}_{\infty_-} \Omega_N$$

2.10 Separation of variables

The next step in our analysis of the model is to find a system of local Darboux coordinates on the symplectic manifold M_N. To do this, we apply Sklyanin's method of (classical) separation of variables, as explained in [59].

From the symmetries (2.8.2) of $\mathcal{T}(z)$, we have that

$$A(z^{-1}) = -D(z), \quad C(z^{-1}) = C(z) \tag{2.10.1}$$

$$C(-z) = -C(z), \quad A(-z) = A(z) \tag{2.10.2}$$

In view of the symmetries of $C(z)$, it is natural to consider

$$\tilde{C}(z) = \frac{C(z)}{z + z^{-1}}$$

which satisfies

$$\tilde{C}(z^{-1}) = \tilde{C}(z), \quad \tilde{C}(-z) = \tilde{C}(z)$$

and is therefore a function of λ. In fact, $\tilde{C}(\lambda)$ is a polynomial of degree $N-1$, and following Sklyanin [59], we may introduce coordinates $(\lambda_1, \ldots, \lambda_{N-1}, Q)$ as its zeros and asymptotic as $\lambda \to \infty$:

$$\tilde{C}(z) = Q \prod_{k=1}^{N-1} (\lambda - \lambda_k) \tag{2.10.3}$$

Note that in order to obtain a well defined set of coordinates in this fashion one must specify a locally consistent ordering of the roots λ_j. However, the angle coordinates constructed in Section 5 will turn out to be independent of this choice of ordering. Note also that the leading coefficient Q is given by

$$Q = \sum_{j=1}^{N} f_j \left((k_j/a_j) \prod_{i>j} k_i^2 \xi_- - (k_j/a_j)^{-1} \prod_{i>j} k_i^{-2} \xi_-^{-1} \right)$$

We also introduce the corresponding multi-valued w-coordinates

$$w_j + w_j^{-1} = \lambda_j$$

Observe that since when $C(z)$ vanishes the reflection monodromy matrix becomes upper triangular, the points $(w, \zeta) = (w_j, A(z_j^{\pm 1}))$ where $z_j^2 = w_j$ lie on the curve Σ, and the points $(\lambda, \zeta) = (\lambda_j, A(z_j^{\pm 1}))$ lie on the curve Γ.

Let us fix a particular branch of the equation $w + w^{-1} = \lambda$ to give us a locally defined set of functions $w_1, \ldots, w_{N-1}$. Again, the angle coordinates we construct will be independent of this choice. We may then introduce a further $(N-1)$ local coordinates

$$\zeta_k = A(w_k) \tag{2.10.4}$$

In terms of the function y defined by (2.9.2), we have

$$y_j := \zeta_j - t(z_j) = \frac{A(z_j) - D(z_j)}{2}$$

which by (2.10.1) is independent of our choice of branch of w.

We now have the following proposition, which is proved by direct calculation using formulae (2.8.7) for the Poisson brackets of reflection monodromy matrix elements.

Proposition 2.10.1. *The coordinates* $(Q, w_1, \ldots, w_{N-1}\ ;\ P, \zeta_1, \ldots, \zeta_{N-1})$ *are log-canonical: we have*

$$\{w_k, \zeta_j\} = 2\delta_{j,k} w_j \zeta_k, \quad \{Q, P\} = 2QP \tag{2.10.5}$$

and the Poisson brackets of all other pairs of coordinates are zero.

To summarize, we obtain a system of log-canonical coordinates consisting of the asymptotics Q, P of $\tilde{C}(\lambda), A(z)$ respectively, together with a degree $(N-1)$ divisor $(w, \zeta) = (w_k, \zeta_k)$ on Σ which projects onto the zero locus of the polynomial $\tilde{C}(\lambda)$.

2.11 Linearization of flows and algebraic integrability.

In this section we explain how to construct affine coordinates on the Liouville tori in M_N cut out by the reflection Hamiltonians $\{P_j\}$, with respect to which the Hamiltonian flows of the P_j correspond to linear motion with constant velocity. To do this, we will use the Hamilton-Jacobi method; for further details, see [8],[45],[33] and references therein.

The first step is to use the log-canonical coordinates constructed in the previous section to write down a local expression for a primitive α for the symplectic form on M_N. We find

$$\alpha = \frac{\log P}{2Q} dQ + \frac{1}{2} \sum_k \log(\zeta_k) \frac{dw_k}{w_k} \tag{2.11.1}$$

We must now restrict α to the level sets of the reflection Hamiltonians P_j and integrate in order to form the Hamilton-Jacobi action. The final step consists of differentiating with respect to the invariants P_j to obtain the canonically conjugate angle variables F_j. The action is given by

$$S(Q, \lambda_1, \cdots, \lambda_{N-1}, P_1, \ldots, P_N) = \frac{(\log P)(\log Q)}{2} + \frac{1}{2} \sum_{k=1}^{N-1} \int_{w_0}^{w_k} \log(\zeta) \frac{dw}{w}$$

where the integral is understood as being taken on the spectral curve Σ. We therefore find

$$F_j = \frac{\partial S}{\partial P_j} = \begin{cases} \displaystyle\sum_{k=1}^{N-1} \int_{\lambda_0}^{\lambda_k} \Omega_j^+ & 1 \le j \le N-1 \\ \displaystyle\frac{\log(Q)}{4(P+P^{-1})} - \frac{1}{4(P+P^{-1})} \sum_{k=1}^{N-1} \int_{\lambda_0}^{\lambda_k} \Omega_N & j = N \end{cases} \tag{2.11.2}$$

where we may now regard the integrals as being taken on the genus $N-1$ curve Γ. The symplectic form being written as

$$\omega = \sum_{k=1}^{N} dF_k \wedge dP_k$$

the time evolution under the reflection flows becomes linear in these coordinates:

$$F_j(t_k) = F_j(0) + t_k \delta_{jk}$$

Note that the coordinates $F_1, \ldots, F_{N-1}$ coincide with the Abel map applied to the degree $g(\Gamma) = N-1$ divisor

$$\mathcal{D} = p_1 + \cdots + p_{N-1}$$

on Γ, where we write p_j for the point $(\lambda, y) = (\lambda_j, y_j)$. Hence the reflection flow is linearized on the $\mathrm{Jac}(\Gamma)$, the Jacobian variety of Γ, which establishes the algebraic integrability of the system.

2.12 Action-angle variables

In this section we explain how to construct complex action-angle variables for the system. Let us choose a canonical basis

$$(A_1, \dots, A_{N-1}, B_1, \dots, B_{N-1})$$

for $H_1(\Gamma_0, \mathbb{Z})$, where Γ_0 is some fixed spectral curve. By Gauss-Manin, this choice of basis has a well defined propagation to a canonical homology basis for all nearby spectral curves Γ. We will also need to introduce $\gamma := A_N$, a contractible loop on Σ winding once around the point ∞_+.

In order to define the action-angle variables, we must choose a lifting of $A_1, \dots, A_N$ to homology classes $\tilde{A}_1, \dots, \tilde{A}_N$ on the curve Σ^c obtained by deleting slits between the branch points of the multi-valued function $\log(\zeta)$ on Σ. On the cut Riemann surface Σ^c, we have a well-defined meromorphic differential

$$\eta = \log(\zeta)\frac{dw}{w} \tag{2.12.1}$$

Then the action variables $J_1, \dots, J_N$ are defined as the A-periods of the differential η:

$$J_k = \oint_{A_k} \eta, \quad 1 \le k \le N \tag{2.12.2}$$

A priori, this definition of the action variables depends on our choice of lifting of the homology classes A_i. However, the following proposition shows that this dependence is of a tame nature.

Proposition 2.12.1. *Let $\{J_k\}, \{J'_k\}$ be two sets of coordinates defined by formula (2.12.2) for two different choices of sets of lifts $\{\tilde{A}_k\}, \{\tilde{A}'_k\}$ of the homology classes $\{[A_k]\} \subset H_1(\Gamma, \mathbb{Z})$, having the same winding numbers around ∞_+. Then each difference $J_k - J'_k$ is a constant function on M, and the map*

$$(P_1, \dots, P_N) \longmapsto (J_1, \dots, J_N) \tag{2.12.3}$$

is a change of coordinates.

Proof. Let us first show that (2.12.3) is a change of coordinates. For this, note that

$$\frac{\partial J_i}{\partial P_k} = \oint_{A_i} \Omega_k \tag{2.12.4}$$

Since the pairing between $H^0(\Gamma, K)$ and the span of the A-cycles is perfect, and Ω_N is the only differential of the Ω_j with nonzero residue at ∞_+, it follows that the Jacobian matrix of (2.12.3) is of full rank, which shows that (2.12.3) is a change of coordinates. Now to prove the first assertion of proposition amounts to showing that

$$\{F_k, (J_i - J_i')\} = \frac{\partial}{\partial P_k}(J_i - J_i') = 0$$

for all j, k. But since the differentials $\Omega_1, \ldots, \Omega_{N-1}$ are well defined meromorphic differentials on Γ, and by definition A_N, A_N' have the same winding number around ∞_+, we have

$$\frac{\partial J_i}{\partial P_k} = \oint_{A_i} \Omega_k = \oint_{A_i'} \Omega_k = \frac{\partial J_i'}{\partial P_k}$$

□

Let us also note that , again up to a shift by an additive constant, the action variable J_N is given by $J_N = 2\pi i \log P$.

With these results in hand we can proceed to the construction of the angle variables ω_k as the coordinates canonically conjugate to the J_k by the Hamilton-Jacobi method:

$$\omega_k = \frac{\partial S}{\partial J_k} \tag{2.12.5}$$

Note that these coordinates are independent of our choices of representative for the homology classes A_k, and the differentials $d\omega_k$ are τ-invariant and thus descend to the curve Γ. Moreover, for $1 \leq k \leq N-1$ we have

$$\begin{aligned}
\oint_{A_i} d\omega_k &= \frac{\partial}{\partial J_k} \oint_{A_i} dS \\
&= \frac{\partial}{\partial J_k} \oint_{A_i} \left(\sum_r p_r dq_r + F_r dP_r \right) \\
&= \frac{\partial}{\partial J_k} \oint_{A_i} \alpha \\
&= \delta_{ik}
\end{aligned}$$

which shows that the angle variables are indeed normalized correctly with respect to the A-cycles of Γ, and that all A-periods of the differential $d\omega_N$ vanish. Note that if $p \in \{\infty_\pm\}$, and γ_p is a contractible loop in Γ with winding number 1 around p, we also have

$$\begin{aligned}
\oint_{\gamma_p} d\omega_k &= \frac{\partial}{\partial J_k} \oint_{\gamma_p} dS \\
&= \frac{\partial}{\partial J_k} \oint_{\gamma_p} \eta \\
&= \pm \delta_{kN}
\end{aligned}$$

which shows that the differentials $d\omega_1, \ldots, d\omega_{N-1}$ are holomorphic, and that

$$\mathrm{Res}_{\infty_+} d\omega_N = 1 = -\mathrm{Res}_{\infty_-} d\omega_N.$$

2.13 Solutions in theta functions

We will now apply the geometric description of the system given in the previous sections to write explicit formulas for the flows of the reflection Hamiltonians using Riemann theta functions.

Let (A_i, B_i) be the canonical homology basis and $\{dw_j\}$ be the normalized abelian differentials constructed in the previous section. The *matrix of b-periods* corresponding to this data is the $(N-1) \times (N-1)$ symmetric matrix

$$\mathcal{B}_{jk} = \oint_{B_j} d\omega_k, \quad 1 \leq j, j \leq N-1$$

This matrix gives the rise to the model

$$\mathrm{Jac}(\Gamma) = \mathbb{C}^{N-1}/(\mathbb{Z}^{N-1} + \mathcal{B}\mathbb{Z}^{N-1})$$

for the Jacobian of Γ. Expanding $d\omega_j = \sum_k \mathcal{N}_{jk}\Omega_k$ where $\mathcal{N}_{jk} \in \mathbb{C}$, we define the normalized angle variables

$$\widetilde{F}_j = \sum_k \mathcal{N}_{jk} F_k, \quad j = 1, \ldots, N-1$$

$$\widetilde{F}_N = 4(P + P^{-1}) \left(\sum_{k=1}^{N} \mathcal{N}_{jk} F_k \right)$$

Note that $\mathcal{N}_{jN} = 0$ for $j = 1, \ldots, N-1$ and $\mathcal{N}_{NN} = 1$ so that we have

$$\widetilde{F}_N = \log Q - \sum_{k=1}^{N-1} \int_{\lambda_0}^{\lambda_k} d\omega_N$$

In these coordinates the time evolution takes the form

$$\widetilde{F}_i(t_k) = \widetilde{F}_i(0) + t_k \mathcal{N}_{ik}, \quad i = 1, \ldots, N-1 \tag{2.13.1}$$

$$\widetilde{F}_N(t_k) = \widetilde{F}_N(0) + c_k t_k \tag{2.13.2}$$

where $c_k = 4(P + P^{-1})\mathcal{N}_{Nk}$. If we define the normalized Abel map with base point p_0

$$\mathcal{A}_j(p_1 + \ldots + p_{N-1}) = \sum_{k=1}^{N-1} \int_{p_0}^{p_k} d\omega_j \tag{2.13.3}$$

we have

$$\mathcal{A}(\mathcal{D}(t)) = \mathcal{A}(\mathcal{D}(0)) + t_k U^{(k)}$$

where the velocity vector $U^{(k)}$ is given by

$$U_j^{(k)} = \mathcal{N}_{jk}$$

and

$$\widetilde{F}_N(t) = \widetilde{F}_N(0) + c_k t_k.$$

Let us now recall some background on theta functions. For a more detailed discussion of this subject, see [66] and references therein. The Riemann theta function associated to the spectral curve Γ and its matrix of b-periods $\mathcal{B}$ is the following holomorphic function on $\mathbb{C}^{N-1}$:

$$\theta(z) = \sum_{m \in \mathbb{Z}^{N-1}} e^{2\pi i (m,z) + \pi i (\mathcal{B}m, m)} \tag{2.13.4}$$

The theta function is automorphic with respect to the lattice of periods of Γ: if $n \in \mathbb{Z}^{N-1}$, we have

$$\begin{aligned} \theta(z+n) &= \theta(z) \\ \theta(z+\mathcal{B}n) &= \exp\left(-2\pi i (n,z) - \pi i (\mathcal{B}n, n)\right) \theta(z) \end{aligned} \tag{2.13.5}$$

From these formulas, it follows that the divisor Θ of $\theta(z)$ is a well defined analytic subset of the Jacobian $\mathrm{Jac}(\Gamma)$. Let us fix a so-called *odd non-singular* point $e \in \Theta \subset \mathbb{C}^{N-1}$ of the theta divisor. Then the third kind differential $\widetilde{\Omega}_N$ can be expressed in terms of the odd theta function $\theta_e(z) := \theta(z+e)$ as

$$\widetilde{\Omega}_N(p) = d \log \left(\frac{\theta_e(\int_{\infty_+}^p \omega)}{\theta_e(\int_{\infty_-}^p \omega)} \right)$$

where we use the shorthand notation

$$\int_q^p \omega = \mathcal{A}(p) - \mathcal{A}(q)$$

Hence from our formula (2.13.2) for the time evolution of $\widetilde{F}_N$, we obtain the following expression for the time evolution of the observable Q under the Hamiltonian flow of the reflection Hamiltonian P_k:

$$Q(t_k) = Q(0) e^{c_k t_k} \prod_{j=1}^{N-1} \frac{\theta_e(\int_{p_j(t_k)}^{\infty_+} \omega) \theta_e(\int_{p_j(0)}^{\infty_-} \omega)}{\theta_e(\int_{p_j(0)}^{\infty_+} \omega) \theta_e(\int_{p_j(t_k)}^{\infty_-} \omega)} \tag{2.13.6}$$

However this formula is of limited practical value, in that it requires knowledge of the points $p_k(t)$ for all times t, whereas all we know explicitly is the (linear) time evolution of $\mathcal{A}(\mathcal{D}(t))$. We may remedy this defect as follows. Let K denote the Riemann point for the based Abel map (2.13.3). Consider two non-special effective degree $g = N-1$ divisors

$$\mathcal{D} = p_1 + \cdots + p_g, \quad \mathcal{D}' = q_1 + \cdots + q_g$$

and form the meromorphic function

$$m(p) = \prod_{j=1}^{g} \frac{\theta_e(\int_{p_j}^{p} \omega)}{\theta_e(\int_{q_j}^{p} \omega)} \cdot \frac{\theta(\mathcal{A}(p) - \mathcal{A}(D') - K)}{\theta(\mathcal{A}(p) - \mathcal{A}(D) - K)}$$

which must be constant since it has no poles. We therefore obtain, for any point q on the curve,

$$\prod_{j=1}^{g} \frac{\theta_e(\int_{p_j}^{p} \omega)\theta_e(\int_{q_j}^{q} \omega)}{\theta_e(\int_{q_j}^{p} \omega)\theta_e(\int_{p_j}^{q} \omega)} = \frac{\theta(\mathcal{A}(p) - \mathcal{A}(D) - K)}{\theta(\mathcal{A}(p) - \mathcal{A}(D') - K)} \frac{\theta(\mathcal{A}(q) - \mathcal{A}(D') - K)}{\theta(\mathcal{A}(q) - \mathcal{A}(D) - K)}$$

Applying this formula to in the case $\mathcal{D} = p_1(t) + \cdots + p_g(t), \mathcal{D}' = p_1(0) + \cdots + p_g(0)$, $p = \infty_+, q = \infty_-$, we find

$$\begin{aligned} Q(t_k) &= Q(0)e^{c_k t_k} \frac{\theta(\mathcal{A}(\infty_+) - \mathcal{A}(\mathcal{D}(t)) - K)}{\theta(\mathcal{A}(\infty_+) - \mathcal{A}(\mathcal{D}(0)) - K)} \frac{\theta(\mathcal{A}(\infty_-) - \mathcal{A}(\mathcal{D}(0)) - K)}{\theta(\mathcal{A}(\infty_-) - \mathcal{A}(\mathcal{D}(t)) - K)} \\ &= Q(0)e^{c_k t_k} \frac{\theta(\mathcal{A}(\infty_+) - \mathcal{A}(\mathcal{D}(0)) - t_k U^{(k)} - K)\theta(\mathcal{A}(\infty_-) - \mathcal{A}(\mathcal{D}(0)) - K)}{\theta(\mathcal{A}(\infty_-) - \mathcal{A}(\mathcal{D}(0)) - t_k U^{(k)} - K)\theta(\mathcal{A}(\infty_+) - \mathcal{A}(\mathcal{D}(0)) - K)} \end{aligned}$$

which is an explicit formula for the time evolution of Q.

We now turn to the problem of reconstructing the full reflection monodromy matrix. For this, we introduce the following meromorphic function on Γ:

$$\rho = \frac{Q(z + z^{-1})}{P + P^{-1}} \cdot \frac{y + h(\lambda)}{C(z)} \tag{2.13.7}$$

$$= \frac{1}{P + P^{-1}} \cdot \frac{y + h(\lambda)}{(\lambda + 2) \prod_{k=1}^{N-1} (\lambda - \lambda_k)} \tag{2.13.8}$$

where we write

$$h(\lambda) = \frac{A(z) - D(z)}{2}.$$

The relevance of the function ρ to our problem is that the vector

$$\psi = \left(1, \frac{(P + P^{-1})}{Q(z + z^{-1})} \cdot \rho\right)^t$$

spans the eigenspace of $\mathcal{T}(z)$ corresponding to the given point on the spectral curve. We have the following proposition characterizing the function ρ.

Proposition 2.13.1. *The meromorphic function ρ has exactly N poles, $N - 1$ of them at the divisor $\mathcal{D}$, and one at the point $q^+ = (-2, h(-2))$ lying over $\lambda = -2$. In addition, ρ has a zero at ∞_-. Its value at ∞_+ is*

$$\rho(\infty_+) = 1$$

Proof. The assertion about the pole at q^+ follows from the identity

$$h^2(-2) = \mathcal{Q}_{2N}(-2) = \left(\frac{\xi - \xi^{-1}}{4}\right) \prod_{k=1}^{N} (\omega_k + a_k^2 + a_k^{-2})^2$$

□

By the Riemann-Roch theorem there is generically a unique such ρ, which can be written as

$$\rho = \frac{\theta_e(\int_{\infty_-}^{p} \omega)\theta(\mathcal{A}(p) - \mathcal{A}(D) - K + W)}{\theta_e(\int_{q^+}^{p} \omega)\theta(\mathcal{A}(p) - \mathcal{A}(D) - K)} \cdot \frac{\theta_e(\int_{q^+}^{\infty_+} \omega)\theta(\mathcal{A}(\infty_+) - \mathcal{A}(D) - K)}{\theta_e(\int_{\infty_-}^{\infty_+} \omega)\theta(\mathcal{A}(\infty_+) - \mathcal{A}(D) - K + W)}$$

where the vector W is defined as the vector of b-periods of the unique normalized third kind differential Ω_{∞_-,q^+} with residue 1 at ∞_- and residue -1 at q^+:

$$W_j = \oint_{B_j} \Omega_{\infty_-,q^+}$$

Hence the time evolution of ρ under the flow of the reflection Hamiltonian P_k is given by the explicit formula

$$\begin{aligned}\rho(p,t_k) =& \frac{\theta_e(\int_{\infty_-}^{p} \omega)\theta(\mathcal{A}(p) - \mathcal{A}(D(0)) - t_k U^{(k)} - K + W)}{\theta_e(\int_{q^+}^{p} \omega)\theta(\mathcal{A}(p) - \mathcal{A}(D(0)) - t_k U^{(k)} - K)} \\ &\cdot \frac{\theta_e(\int_{q^+}^{\infty_+} \omega)\theta(\mathcal{A}(\infty_+) - \mathcal{A}(D(0)) - t_k U^{(k)} - K)}{\theta_e(\int_{\infty_-}^{\infty_+} \omega)\theta(\mathcal{A}(\infty_+) - \mathcal{A}(D(0)) - t_k U^{(k)} - K + W)}\end{aligned}$$

From this we can reconstruct the eigenvector $\psi(p)$ and therefore the full reflection monodromy matrix $\mathcal{T}(z)$.

Chapter 3

Doubles of Hopf algebras and quantization of the Grothendieck-Springer resolution

3.1 Poisson geometry

Preliminaries

Recall that a Poisson-Lie group is a Lie group G with a Poisson structure such that the multiplication map $G \times G \to G$ is a morphism of Poisson varieties. Let G^* be the (connected, simply-connected) Poisson-Lie dual of G, and $\mathcal{D}(G)$ be the double of G. The Lie algebra $\mathfrak{d} = Lie(\mathcal{D}(G))$ can be written as $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. We will say that there exist local isomorphisms

$$\mathcal{D}(G) \simeq G \times G^* \simeq G^* \times G.$$

Let us consider a pair of dual bases (x_i) and (x^i) of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then the element $r \in \mathfrak{d} \wedge \mathfrak{d}$ defined by

$$r = \frac{1}{2} \sum_i (x_i, 0) \wedge (0, x^i)$$

is independent of the choice of bases. Let X^R, X^L denote respectively the right- and left-invariant tensor fields on a Lie group, taking the value $X^R(e) = X^L(e) = X$ at the identity element of the group. Then the bivectors

$$\pi_{\pm} = r^R \pm r^L$$

define a pair of Poisson structures on the Lie group $\mathcal{D}(G)$. We abbreviate the resulting Poisson manifolds by $\mathcal{D}_{\pm}(G)$. In fact, $\mathcal{D}_{-}(G)$ is a Poisson-Lie group, while $\mathcal{D}_{+}(G)$ is only a Poisson manifold.

Remark 3.1.1. As a manifold, the group $\mathcal{D}_-(G)$ is locally isomorphic to $\mathcal{D}_-(G)^*$. In general, however, this is neither an isomorphism of Lie groups, or of Poisson manifolds.

The action of a Poisson-Lie group G on a Poisson variety P is said to be Poisson, if so is the map $G \times P \to P$. Given a Poisson map $P \to G^*$, one can obtain a local Poisson action $G \times P \to P$ using the group-valued moment map. Recall that the group-valued moment map is defined (see [40]) as follows.

Definition 3.1.2. Let π be the Poisson bivector field defining the Poisson structure on the manifold P. A map $\mu\colon P \to G^*$ is said to be a moment map for the Poisson action $G \times P \to P$, if for every $X \in \mathfrak{g}$ one has

$$\mu_X = \left\langle \pi, \mu^* X^R \otimes - \right\rangle,$$

where μ_X is the vector field on P generated by the action $\mu_{\exp(tX)}$.

Remark 3.1.3. A moment map is Poisson, if exists.

Remark 3.1.4. Recall that there are open subsets of factorizable elements $G^* \cdot G$ and $G \cdot G^*$ in the double $\mathcal{D}_-(G)$. Hence we may regard G^* as a submanifold in $\mathcal{D}_-(G)/G$, and may regard the moment map μ in Definition 2.2 as taking values in $\mathcal{D}_-(G)/G$ or $G\backslash\mathcal{D}_-(G)$.

The following theorem is well-known (see e.g. [54, 40]).

Proposition 3.1.5. *Let G be a Poisson-Lie group, and $\mathcal{D}_\pm(G)$ its double with Poisson bivectors $\pi_\pm$. Then*

1. *the actions of $\mathcal{D}_-(G)$ on $\mathcal{D}_+(G)$ by left and right multiplications are Poisson;*
2. *the moment map for the Poisson action of the subgroup $G \subset \mathcal{D}_-(G)$ on $\mathcal{D}_+(G)$ by left (resp. right) multiplication is the natural projection $\mathcal{D}(G) \to \mathcal{D}(G)/G$ (resp. $\mathcal{D}(G) \to G\backslash\mathcal{D}(G)$).*

Remark 3.1.6. Let $\mathcal{P}$ be the category of Poisson-Lie groups. Consider a Poisson-Lie group G and its connected, simply-connected Poisson-Lie dual G^*. Then the assignment $G \to G^*$ defines a functor $\mathcal{P} \to \mathcal{P}^{op}$. Therefore, any Poisson-Lie subgroup $H \subset G$ induces a map $p\colon G^* \to H^*$. Now consider a Poisson action $G \times P \to P$ with the moment map μ_G. It gives rise to the Poisson action $H \times P \to P$ with the moment map $\mu_H = p \circ \mu_G$.

Double of the double construction

Now, let us start with the Poisson-Lie group $D = \mathcal{D}_-(G)$ and consider its double $\mathcal{D}(D) = \mathcal{D}(\mathcal{D}(G))$. The Lie algebra $\mathfrak{D} = Lie(\mathcal{D}(D))$ may be written as $\mathfrak{D} = \mathfrak{d} \oplus \mathfrak{d} = \mathfrak{d}_\Delta \oplus \mathfrak{d}^*$ where

$$\mathfrak{d}_\Delta = \{((x,\alpha),(x,\alpha)) \in \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{g} \oplus \mathfrak{g}^*\}$$

is the diagonal embedding of $\mathfrak{d}$ into $\mathfrak{d} \oplus \mathfrak{d}$ and

$$\mathfrak{d}^* = \{((y,0),(0,\beta)) \in \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{g} \oplus \mathfrak{g}^*\}.$$

Using the local isomorphism $\mathcal{D}(D) \simeq D_\Delta \times D^*$ we may coordinatize the moment map ν_r for the right Poisson action of $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ as

$$\nu_r \colon \mathcal{D}_+(D) \longrightarrow D^*, \qquad (dg, d\alpha) \longmapsto \alpha^{-1}g$$

for any triple of elements $g \in G$, $\alpha \in G^*$, $d \in \mathcal{D}(\mathcal{D}(G))$. Similarly, using the local isomorphism $\mathcal{D}(D) \simeq D^* \times D_\Delta$ we write the moment map ν_l for the left Poisson action of $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ as

$$\nu_l \colon \mathcal{D}_+(D) \longrightarrow D^*, \qquad (gd, \alpha d) \longmapsto g\alpha^{-1}.$$

Hamiltonian reduction

Consider the Poisson action of the subgroup $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ by left multiplications and the Poisson action of $G \subset D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ by right multiplications. Clearly, the two actions commute, because so do the left and right actions of $\mathcal{D}_-(D)$. We illustrate these actions as follows

$$D_\Delta \curvearrowright \mathcal{D}_+(D) \curvearrowleft D_\Delta \supset G.$$

By Remark 3.1.6, the moment map μ_r for the right action of G is given as

$$\mu_r \colon \mathcal{D}_+(D) \longrightarrow (G \times D)\backslash\mathcal{D}_+(D).$$

The Hamiltonian reduction of $\mathcal{D}_+(D)$ by the moment map μ_r becomes

$$\mu_r^{-1}(e)/G_\Delta = \{(dg, d) \mid d \in D, g \in G\}/G_\Delta.$$

Therefore, we can identify

$$\mu_r^{-1}(e)/G_\Delta \simeq D \times_G G,$$

where $D \times_G G$ denotes the set of G-orbits through $D \times G$ under the right action

$$(D \times G) \times G \longrightarrow D \times G, \qquad ((d,g),h) \longmapsto (dh, h^{-1}gh)$$

with $g, h \in G$ and $d \in D$.

On the other hand, since the left and right D_Δ-actions on $\mathcal{D}(D)$ commute, the variety $D \times_G G$ admits the residual D_Δ-action by left multiplication. The corresponding moment map is

$$\mu_l \colon D \times_G G \longrightarrow \mathcal{D}(D)/D_\Delta.$$

As explained in Remark 3.1.1, we may use local diffeomorphism of D with D^* to write a local expression for the map μ_l as $\mu_l((q,g)G) = qgq^{-1} \in D$.

The following Proposition follows easily from considering Poisson bivectors for the Poisson varieties under consideration.

Proposition 3.1.7. *There is a local Poisson isomorphism*

$$\mathcal{D}_+(G) \longrightarrow D \times_G G, \qquad \alpha g \longmapsto (\alpha, g)G,$$

where $g \in G$, $\alpha \in G^*$ and we identify $\mathcal{D}(G) \simeq G^* \times G$.

Under this identification, the moment map μ_l becomes

$$\mathcal{D}_+(G) \longrightarrow \mathcal{D}_-(G), \qquad \alpha g \longmapsto \alpha g \alpha^{-1}. \tag{3.1.1}$$

3.2 Reminder on Hopf algebras

To fix our notations, we will recall some standard notions from the theory of Hopf algebras. In what follows, we choose to work in the setting of topological Hopf algebras over the ring $k[[\hbar]]$ of formal power series over a ground field k. In particular, all tensor products are to be understood as completed in the $\hbar$-adic topology.

Basic notations.

Let A be a topological Hopf algebra over $\mathbb{K} := k[[\hbar]]$, with the quadruple (m, Δ, ϵ, S) denoting the multiplication, comultiplication, counit, and antipode of A respectively. We say that a pair of topological Hopf algebras A and A^* form a dual pair if there exists a non-degenerate Hopf pairing $\langle -, - \rangle \colon A \otimes A^* \to \mathbb{K}$, that is a non-degenerate pairing satisfying

1. $\langle ab, x \rangle = \langle a \otimes b, \Delta(x) \rangle$
2. $\langle a, xy \rangle = \langle \Delta(a), x \otimes y \rangle$
3. $\langle 1_A, - \rangle = \epsilon_{A^*}$ and $\langle -, 1_{A^*} \rangle = \epsilon_A$
4. $\langle S(a), x \rangle = \langle a, S(x) \rangle$

for all $a, b \in A$ and $x, y \in A^*$. In fact, condition (4) follows from the other three, see [6, Section 1.2.5, Proposition 9]. We will also use the notation A^{op} for the Hopf algebra $(A, m^{op}, \Delta, S^{-1})$, and A^{cop} for the Hopf algebra $(A, m, \Delta^{op}, S^{-1})$.

Module algebras.

The category of modules Mod_A over a Hopf algebra A has a monoidal structure determined by the coproduct $\Delta \colon A \to A \otimes A$. We say that M is an *A-module algebra* if it is an algebra object in the monoidal category Mod_A, that is

$$a \cdot 1_M = \epsilon(a) 1_M \qquad \text{and} \qquad a \cdot (mn) = (a_1 \cdot m)(a_2 \cdot n)$$

for any $a \in A$ and $m, n \in M$.

A Hopf algebra A can be naturally regarded as a module algebra over itself using the *adjoint action*

$$\mathrm{ad}\colon A\otimes A\longrightarrow A, \qquad a\otimes b\longmapsto a\triangleright b := a_1 b S a_2.$$

A Hopf algebra A^* dually paired with A can also be regarded as a module algebra over A using the *left coregular action*

$$\mathrm{coreg}\colon A\otimes A^*\longrightarrow A^*, \qquad a\otimes x\longmapsto a\rightharpoonup x := \langle a, x_2\rangle x_1.$$

There is also a *right coregular action* of A^{op} on A^*, defined by

$$a\otimes x\longmapsto x\leftharpoonup a := \langle a, x_1\rangle x_2.$$

The Drinfeld double

Suppose A, A^* is a dual pair of Hopf algebras. In what follows, we assume that the pairing $\langle\cdot,\cdot\rangle$ is such that a topological basis $\{a_i\}$ for A gives rise to a dual topological basis $\{x^i\}$ in A^* with the property that $\langle a_i, x^j\rangle = \delta_i^j$, and there is a well-defined element $\sum_i a_i\otimes x^i\in A\otimes A^*$, where as usual tensor product is completed in the $\hbar$-adic topology. For instance, this hypothesis will be satisfied whenever A and A^* are a dual pair of QUE-algebras in the sense of Drinfeld [19], and of course whenever A is finitely generated and projective over $k[[\hbar]]$.

Under the above assumption, there exists a Hopf algebra $D(A)$ called the *Drinfeld double* of A, with the following properties:

1. as a coalgebra, $D(A)\simeq (A^*)^{cop}\otimes A$;
2. the maps $a\mapsto 1\otimes a$ and $x\mapsto x\otimes 1$ are embeddings of Hopf algebras;
3. let (a_i) and (x^i) be dual bases for A and A^* respectively. Then the canonical element

$$R=\sum_i (1\otimes a_i)\otimes(x^i\otimes 1)\in D(A)^{\otimes 2},$$

called the universal R-matrix of the Drinfeld double, satisfies

$$R\Delta_D(d) = \Delta_D^{op}(d)R$$

for all $d\in D(A)$.

From the above properties one derives the following explicit formula for the multiplication in $D(A)$:

$$(x\otimes a)(y\otimes b) = \langle a_1, y_3\rangle\langle a_3, S^{-1}y_1\rangle xy_2\otimes a_2 b. \tag{3.2.1}$$

It also follows from the definition of the double, that the R-matrix is invertible, with inverse

$$R^{-1} = (S_D\otimes\mathrm{id})(R)$$

and that the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^{\otimes 3}$$

holds in the triple tensor product $D(A)^{\otimes 3}$.

Proposition 3.2.1. *If A is a Hopf algebra and $D(A)$ its Drinfeld double, the following formula equips A with the structure of a $D(A)$-module algebra:*

$$\begin{aligned} (1 \otimes a) \cdot b &= a_1 b S a_2 \\ (x \otimes 1) \cdot b &= b \leftharpoondown S^{-1}x \end{aligned} \tag{3.2.2}$$

In the action (3.2.2), the Hopf subalgebra $A \subset D(A)$ acts adjointly on A, while the Hopf subalgebra $(A^*)^{cop} \subset D(A)$ acts by its right coregular action.

The dual of the Drinfeld double.

In addition to the Drinfeld double, we will also make use of another Hopf algebra $T(A) = D(A)^*$ dually paired with $D(A)$. As an algebra, we have $T(A) \simeq A^{op} \otimes A^*$, and the pairing $\langle\langle \cdot, \cdot \rangle\rangle : D(A) \otimes T(A) \to k[[\hbar]]$ is defined by

$$\langle\langle x \otimes a, b \otimes y \rangle\rangle = \langle b, x\rangle \langle a, y\rangle. \tag{3.2.3}$$

The formula for its comultiplication can be found by dualizing (3.2.1) and reads

$$\Delta_T(a \otimes x) = \left(a_1 \otimes x^r x_1 x^t\right) \otimes \left(S^{-1}a_t a_2 a_r \otimes x_2\right) \in T(A)^{\otimes 2}.$$

Similarly, the antipode in $T(A)$ can be written as

$$S_T(a \otimes x) = a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t S(x) x^r.$$

The Heisenberg double

Given a Hopf algebra A and its module algebra M, one defines their *smash-product* $M\#A$ as an associative algebra on the vector space $M \otimes A$ with the multiplication given by

$$(m\#x)(n\#y) = m(x_1 \cdot n)\#y_2 b$$

for any elements $x, y \in A$ and $m, n \in M$. Recall [41], that the *Heisenberg double* $H(A)$ of an associative algebra A is the smash product $H(A) = A\#A^*$ with respect to the coregular action of A^* on A. Thus, the multiplication in $H(A)$ is determined by the formula

$$(a\#x)(b\#y) = a(x_1 \rightharpoonup b)\#x_2 y = \langle x_1, b_2\rangle\, ab_1\#x_2 y$$

for any $a, b \in A$ and $x, y \in A^*$. Note that one has the following inclusions of algebras

$$A \longrightarrow H(A), \qquad a \mapsto a\#1,$$
$$A^* \longrightarrow H(A), \qquad x \mapsto 1\#x.$$

By construction, the Heisenberg double $H(A)$ acts on A via

$$(a\#x) \cdot_L (b) = a(x \rightharpoonup b) = \langle x, b_2 \rangle \, ab_1 \tag{3.2.4}$$

In fact, $H(A)$ also acts on A via

$$(a\#x) \cdot_R (b) = (b \leftharpoonup S^{-1}x)Sa = \langle x, Sb_1 \rangle \, b_2 S^{-1} a \tag{3.2.5}$$

The Heisenberg double $H(A)$ has the following well-known properties:

Proposition 3.2.2. *[56] The antipode S_T of $T(A)$, when regarded as an operator $\iota\colon H(A) \to H(A)$ via*

$$\iota\colon H(A) \longrightarrow H(A), \qquad a \otimes x \longmapsto a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t S(x) x^r, \tag{3.2.6}$$

defines an algebra automorphism of $H(A)$.

Note that the automorphism ι intertwines the two actions 3.2.4, 3.2.5 of $H(A)$ on A.

Corollary 3.2.3. *One has the following inclusions of algebras*

$$A \longrightarrow H(A), \qquad a \mapsto \iota(a\#1) = a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t x^r,$$
$$A^* \longrightarrow H(A), \qquad x \mapsto \iota(1\#x) = a_r S^{-1}(a_t) \otimes x^t S(x) x^r.$$

Since the actions $(A\#1, \cdot_L)$, $(A\#1, \cdot_R)$ commute, we have

Proposition 3.2.4. *[56] The maps*

$$A \otimes A \longrightarrow H(A), \qquad a \otimes b \longmapsto (a\#1)\iota(b\#1),$$
$$A^* \otimes A^* \longrightarrow H(A), \qquad x \otimes y \longmapsto (x\#1)\iota(1\#y)$$

are homomorphisms of associative algebras.

Quantum Hamiltonian reduction

Let us briefly recall the notion of quantum Hamiltonian reduction. Suppose that A is a Hopf algebra, V is an associative algebra, $\mu\colon A \to V$ is a homomorphism of associative algebras, and I is a 2-sided ideal in A preserved by the adjoint action of A. Then, by the ad-invariance of I, the action of A on V defined by the formula

$$a \circ v = \mu(a_1) v \mu(Sa_2)$$

descends to an action of A on the V-module $V/\mu(I)$, where we abuse notation and write $\mu(I)$ for the left ideal in V generated by $\mu(I)$. The *quantum Hamiltonian reduction* $V//\mu(A)$ of V by the *quantum moment map* $\mu\colon A \to V$ at the ideal I is defined as the set of A-invariants

$$\begin{aligned} V//\mu(A) :=& (V/V\mu(I))^A \\ =& \{a \in V/V\mu(I) \mid a \circ v = \epsilon(a)v \quad \text{for all} \quad a \in A\} \end{aligned}$$

One checks that $V//\mu(A)$ inherits a well-defined associative algebra structure from that of V, such that $V//\mu(A)$ is an A-module algebra.

3.3 Construction of the quantum resolution

The double of a double

Suppose that A is a Hopf algebra, and let $D(A), T(A)$, and $H(A)$ be its Drinfeld double, dual to the Drinfeld double, and the Heisenberg double respectively. Consider the Heisenberg double

$$H(T(A)^{op}) = T(A)^{op}\#D(A)^{cop},$$

of the algebra $T(A)^{op}$. One has an algebra embedding

$$\mu_L\colon D(A) \longrightarrow H(T(A)^{op}), \qquad u \mapsto 1\#u \in H(T(A)^{op})$$

which may be regarded as the quantum moment map for the following $D(A)$-module algebra structure on $H(T(A)^{op})$:

$$u \circ_L (\phi\#v) = (u_3 \rightharpoonup \phi)\#u_2 v S^{-1}_{D(A)} u_1. \tag{3.3.1}$$

As in Corollary 3.2.3, there exists another algebra embedding defined by

$$\mu_R\colon D(A) \longrightarrow H(T(A)^{op}), \qquad u \mapsto \iota^{-1}(1\#u). \tag{3.3.2}$$

It generates the following $D(A)$-module algebra structure on $H(T(A)^{op})$:

$$u \circ_R (\phi\#v) = (\phi \leftharpoonup S^{-1}_{D(A)} u)\#v \tag{3.3.3}$$

By Proposition 3.2.4, the subalgebras $\mu_L(D(A))$ and $\mu_R(D(A))$ commute with each other in $H(T(A)^{op})$. This forces the actions (3.3.1) and (3.3.3) to commute as well.

Dual pairs of quantum moment maps.

We shall now restrict the action (3.3.1) to the Hopf subalgebra $A \subset D(A)$, and consider the quantum Hamiltonian reduction of $H(T(A)^{op})$ at the augmentation ideal $I_A = \ker(\epsilon_A)$ of A. We denote the algebra obtained as a result of the quantum Hamiltonian reduction by $H(T(A)^{op})//\mu_L(A)$.

We also have the moment map $\mu_R\colon D(A) \to H(T(A)^{op})$ given in (3.3.2), and the action (3.3.3) of $D(A)$ on $H(T(A)^{op})$ that it defines.

Proposition 3.3.1. *The action (3.3.3) of $D(A)$ on $H(T(A)^{op})$ descends to a well-defined action*

$$D(A) \times H(T(A)^{op})//\mu_L(A) \longrightarrow H(T(A)^{op})//\mu_L(A) \tag{3.3.4}$$

In turn, the map μ_R descends to a well-defined homomorphism of $D(A)$-module algebras

$$\mu_R \colon D(A) \longrightarrow H(T(A)^{op})//\mu_L(A)$$

which is a moment map for the action (3.3.4)*.*

Proof. The Proposition is a simple consequence of the fact that the subalgebras $\mu_L(D(A))$ and $\mu_R(D(A))$ commute with one another. Indeed, this commutativity implies that for all $a \in A$, $u \in D(A)$, one has

$$\begin{aligned} a \circ_L (\mu_R(u) + \mu_L(I_A)) &= a_1 \mu_R(u) S a_2 + \mu_L(I_A) \\ &= \mu_R(u) a_1 S a_2 + \mu_L(I_A) \\ &= \epsilon(a)\,(\mu_R(u) + \mu_L(I_A)) \end{aligned}$$

which shows that

$$\mu_R(u) + \mu_L(I_A) \in \big(H(T(A)^{op})/\mu_L(I_A)\big)^A =: H(T(A)^{op})//\mu_L(A).$$

It follows from the definition of the algebra structure of the quantum Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$ that $\mu_R \colon D(A) \to H(T(A)^{op})//\mu_L(A)$ is a homomorphism of algebras. Regarding this homomorphism as a quantum moment map, we obtain an action of $D(A)$ on $H(T(A)^{op})//\mu_L(A)$ which by construction descends from (3.3.3), and such that $\mu_R \colon D(A) \to H(T(A)^{op})$ is a morphism of $D(A)$-module algebras. □

$H(A)$ from quantum Hamiltonian reduction

We now examine the algebra structure of the Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$ in more detail.

Proposition 3.3.2. *There is an isomorphism of algebras*

$$\varphi : H(T(A)^{op})//\mu_L(A) \longrightarrow H(A) \tag{3.3.5}$$

Proof. Let us begin by making explicit the structure of the Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$. Firstly, note that we can identify the quotient $H(T(A)^{op})/I_A$ with the vector space $T(A)^{op} \otimes A^*$. It is easy to check that induced action of A on $T(A)^{op} \otimes A^*$ is then given by

$$a \circ_L \big((b \otimes y) \otimes x\big) = (b \otimes a_2 \rightharpoonup y) \otimes \mathrm{ad}^*_{a_1}(x)$$

where

$$\mathrm{ad}^*_a(x) = \langle a_1, x_3 \rangle \langle S^{-1} a_2, x_1 \rangle x_2.$$

Hence the algebra $H(T(A)^{op})//\mu_L(A)$ of A-invariants in $H(T(A)^{op})/I_A$ may be identified with $H(A) = A\#A^*$, as a vector space, under the map

$$\varphi\colon H(A) \longrightarrow H(T(A)^{op})//\mu_L(A), \qquad a\#x \longmapsto (a \otimes x_1 Sx_3) \otimes x_2. \tag{3.3.6}$$

Finally, we claim that the map (3.3.6) is in fact an isomorphism of algebras. Indeed, in $H(T(A)^{op})//\mu_L(A)$, one computes

$$\begin{aligned}\varphi(a\#x)\varphi(b\#y) &= ((a \otimes x_1Sx_3) \otimes x_2)\,((b \otimes y_1Sy_3) \otimes y_2)\\ &= \langle x_2, S^{-1}a_t b_2 a_r\rangle(ab_1 \otimes x^r y_1 Sy_3 x^t x_1 Sx_3) \otimes x_3 y_2\\ &= \langle x_3, b_2\rangle(ab_1 \otimes x_4 y_1 Sy_3 S^{-1}x_2 x_1 Sx_6) \otimes x_5 y_2\\ &= \langle x_1, b_2\rangle(ab_1 \otimes x_2 y_1 Sy_3 Sx_4) \otimes x_3 y_2\\ &= \varphi(\langle x_1, b_2\rangle ab_1 \otimes x_2 y)\\ &= \varphi\left((a\#x)(b\#y)\right)\end{aligned}$$

which completes the proof. □

Corollary 3.3.3. *Under the isomorphism φ defined in* (3.3.5), *the moment map*

$$\mu_R\colon D(A) \to H(T(A)^{op})//\mu_L(A) \simeq H(A)$$

takes the form

$$\mu_R\colon D(A) \longrightarrow H(A), \qquad by \longmapsto b_1 a_r Sb_2 a_t \# S^{-1}x^t S^{-1}yx^r. \tag{3.3.7}$$

Using the homomorphism μ_R, one can pull back the defining representation (3.2.4) of $H(A)$ on A to obtain a representation of $D(A)$. A straightforward computation establishes

Proposition 3.3.4. *The pullback under μ_R of the action* (3.2.4) *coincides with the representation* (3.2.2) *of $D(A)$ on A.*

Remark 3.3.5. In [39], the formula (3.3.7) is derived in the finite-dimensional setting from the action (3.2.2) of $D(A)$, together with the fact, see e.g. [43], that $H(A) \simeq \mathrm{End}(A)$ as algebras.

Example: quantized Grothendieck-Springer resolution

Suppose now that $\mathfrak{g}$ is a complex simple Lie algebra, and denote by $U_\hbar(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$, see [19, 17]. Recall that $U_\hbar(\mathfrak{g})$ may be regarded as the quantized algebra of functions on a formal neighborhood of the identity element $e \in G^*$, where G is a simple Lie group endowed with its standard Poisson structure. Let us apply our constructions to the case $A = U_\hbar(\mathfrak{b})$, where $U_\hbar(\mathfrak{b})$ is the quantum Borel subalgebra in $U_\hbar(\mathfrak{g})$. Then there is an isomorphism of algebras $D(A) \simeq U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{h})$, where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of $\mathfrak{g}$, see [19]. The restriction of the homomorphism (3.3.7) to

$U_\hbar(\mathfrak{g}) \subset D(A)$ defines a map of algebras $\Phi\colon U_\hbar(\mathfrak{g}) \to H(A)$. In [25], it was shown (in the setting of the rational form $U_q(\mathfrak{g})$) that Φ is injective, and that its image is contained in a certain subalgebra $H(A)^{\mathfrak{h}}$ of $U_\hbar(\mathfrak{h})$-invariants.

In the above setup, the semiclassical limit of the map Φ is closely related to the well-known Grothendieck-Springer resolution

$$G \times_B B \longrightarrow G, \qquad (g,b)B \longmapsto gbg^{-1}$$

where G is a complex simple Lie group, and $B \subset G$ is a Borel subgroup. More precisely, the algebra $H(A)^{\mathfrak{h}}$ can be regarded as the quantized algebra of functions on a formal neighborhood of $(e,e)B \in G \times_B B$. The Poisson geometric structure is exactly the one described in [49].

3.4 R-matrix formalism

In this section we rewrite the homomorphism (3.3.7) in terms of canonical elements of the algebras $D(A)$ and $T(A)$. As before, let

$$R = R_{12} = \sum_i a_i \otimes x^i \in D(A) \otimes D(A)$$

be the universal R-matrix of $D(A)$. In what follows we make use of elements

$$R_{21} = \sum_i x^i \otimes a_i$$

and

$$\mathcal{L} = R_{21}R_{12} \in D(A) \otimes D(A).$$

Recall [56] that the element $\mathcal{L}$ satisfies the *reflection equation*

$$\mathcal{L}_1 R_{12} \mathcal{L}_2 R_{21} = R_{12} \mathcal{L}_2 R_{21} \mathcal{L}_1 \in D(A)^{\otimes 3} \tag{3.4.1}$$

where $\mathcal{L}_1 = R_{31}R_{13}$, $\mathcal{L}_2 = R_{32}R_{23}$. Let us also introduce canonical elements $\Theta, \Omega \in D(A) \otimes H(A)$ defined by

$$\Theta = \sum_i a_i \otimes x_i \qquad \text{and} \qquad \Omega = \sum_i x^i \otimes a_i.$$

These elements satisfy the relations

$$\begin{aligned} R_{12}\Theta_1\Theta_2 &= \Theta_2\Theta_1 R_{12} \\ R_{12}\Omega_1\Omega_2 &= \Omega_2\Omega_1 R_{12} \\ R_{12}\Theta_1\Omega_2^{-1} &= \Omega_2^{-1}\Theta_1 \end{aligned} \tag{3.4.2}$$

If ι is the automorphism of $H(A)$ defined by (3.2.6), we write

$$\widetilde{\Theta} = (\mathrm{id} \otimes \iota)(\Theta) \qquad \text{and} \qquad \widetilde{\Omega} = (\mathrm{id} \otimes \iota)(\Omega).$$

The following proposition is straightforward.

Proposition 3.4.1. *Let $\mu_R\colon D(A)\to H(A)$ be the homomorphism defined by* (3.3.7)*. Then one has*

$$
\begin{aligned}
(\mathrm{id}\otimes\mu_R)\,(R_{12}) &= \widetilde{\Theta},\\
(\mathrm{id}\otimes\mu_R)\,(R_{21}) &= \Omega\widetilde{\Omega},
\end{aligned}
$$

and hence

$$(\mathrm{id}\otimes\mu_R)\,(\mathcal{L}) = \Omega\widetilde{\Omega}\widetilde{\Theta}.$$

Recall [6, Section 8.1.3, Proposition 5] that the element $u\in D(A)$ defined by

$$u = Sa_iSx^i \in D(A)$$

satisfies

$$udu^{-1} = S_D^2(d) \qquad \text{for all} \qquad d\in D(A).$$

Proposition 3.4.2. *The following identity holds in $D(A)\otimes H(A)$*

$$\Theta^{-1}\Omega^{-1} = u_1\widetilde{\Omega}\widetilde{\Theta},$$

where $u_1 = u\otimes 1\in D(A)\otimes H(A)$.

Proof. We have

$$
\begin{aligned}
\Theta^{-1}\Omega^{-1} &= \sum Sa_iSx^j\otimes x^ia_j\\
&= \sum S(a_ka_t)S(x^rx^k)\otimes(a_r\#x^t)\\
&= \sum a_tux^r\otimes(Sa_r\#Sx^t)\\
&= u_1\sum a_tx^r\otimes(Sa_r\#S^{-1}x^t).
\end{aligned}
$$

Using the formula

$$ax = \langle a_{(1)},x_{(3)}\rangle\langle S^{-1}a_{(3)},x_{(1)}\rangle x_{(2)}\otimes a_{(2)}$$

for the multiplication in the Drinfeld double $D(A)$, we arrive at

$$
\begin{aligned}
\Theta^{-1}\Omega^{-1} &= u_1\sum a_tx^r\otimes(Sa_r\#S^{-1}x^t)\\
&= u_1\sum x^pa_q\otimes\left(a_\alpha Sa_qS^{-1}a_\beta\#x^\beta S^{-1}x^px^\alpha\right)\\
&= u_1\widetilde{\Omega}\widetilde{\Theta}
\end{aligned}
$$

which completes the proof. □

Corollary 3.4.3. *One has*

$$(\mathrm{id}\otimes\mu_R)\,(\mathcal{L}) = \Omega u_1^{-1}\Theta\Omega^{-1}. \tag{3.4.3}$$

Remark 3.4.4. Since the first tensor factor in $\mathcal{L}$ runs over the basis of $D(A)$, the homomorphism μ_R is completely defined by the formula (3.4.3). The latter can be thought of as a quantization of the map (3.1.1), where u_1 is a quantum correction, invisible on the level of Poisson geometry.

Corollary 3.4.5. *The element*

$$\widehat{\mathcal{L}} = \Omega u_1^{-1}\Theta\Omega^{-1} \in D(A)\otimes H(A) \tag{3.4.4}$$

provides a solution to the reflection equation (3.4.1).

Remark 3.4.6. In fact, one can check using the relations (3.4.2) that the element

$$\widehat{\mathcal{L}}' = \Omega\Theta\Omega^{-1} \in D(A)\otimes H(A)$$

obtained from (3.4.4) by omitting u_1^{-1}, also satisfies the reflection equation (3.4.1). In general, however, the linear map $D(A)\to H(A)$ defined by $\mathcal{L}\mapsto\widehat{\mathcal{L}}'$ will fail to be a homomorphism of algebras. On the other hand, suppose that $R\in \mathrm{End}(V\otimes V)$ is a scalar solution of the Yang-Baxter equation. Then, following Faddeev-Reshetikhin-Takhtajan, one can define a *reflection equation algebra* $\mathcal{A}$ as the algebra generated by entries of $\mathcal{L}\in\mathcal{A}\otimes\mathrm{End}(V)$, subject to the defining relations (3.4.1). Similarly, one can define an algebra $\mathcal{H}$ generated by entries of the elements $\Theta,\Omega\in\mathcal{H}\otimes\mathrm{End}(V)$ subject to the relations (3.4.2). Then we get a well-defined homomorphism of algebras

$$\mathcal{A}\longrightarrow\mathcal{H},\qquad \mathcal{L}\longmapsto\Omega\Theta\Omega^{-1}.$$

3.5 Application to the quantized Grothendieck-Springer resolution

We now consider in detail the application of the general formalism developed in this chapter to the important special case of the Grothendieck-Springer resolution. We begin by recalling the definitions and various well-known properties of the quantum groups that will be used extensively in the sequel. Our conventions match those of of [35]. We refer the reader to [42, 35, 6] for further details and proofs of many of the results in this section.

Conventions

In what follows, $\mathfrak{g}$ will denote a finite-dimensional complex simple Lie algebra of rank r, equipped with a choice of Cartan subalgebra $\mathfrak{h}$ and a set of simple roots $\{\alpha_1,\ldots,\alpha_r\}$. We write P,Q for the weight and root lattices associated to the corresponding root system Π, and denote the fundamental weights by $\omega_1,\ldots,\omega_r$. Denote by $(\cdot,\cdot)$ the unique symmetric bilinear form on $\mathfrak{h}^*$ invariant under the Weyl group W, such that $(\alpha,\alpha)=2$ for all short roots

$\alpha \in \Pi$. Let $k = \mathbb{C}(q^{1/N})$ be the field of rational functions in a formal variable $q^{1/N}$, where $N \in \mathbb{N}$ is such that $\frac{1}{2}(\lambda, \mu) \in \frac{1}{N}\mathbb{Z}$ for any pair of weights $\lambda, \mu \in P$. If A is a Hopf algebra, we denote by A^{op} the Hopf algebra with the opposite multiplication to A, and denote by A^{cop} the Hopf algebra with the opposite comultiplication to A. We will use the Sweedler notation

$$\Delta(a) = \sum a_1 \otimes a_2$$

to express coproducts. Throughout the paper, all modules for the quantum group $U_q(\mathfrak{g})$ are assumed to be of type I.

Quantized enveloping algebras

The (simply-connected) quantized universal enveloping algebra $U \stackrel{\text{def}}{=} U_q(\mathfrak{g})$ is the k-algebra generated by elements

$$\{E_i, F_i, K^\lambda \mid i = 1, \dots, r, \lambda \in P\}$$

subject to the relations

$$K^\lambda E_i = q^{(\lambda,\alpha_i)} E_i K^\lambda, \qquad K^\lambda K^\mu = K^{\lambda+\mu},$$
$$K^\lambda F_i = q^{-(\lambda,\alpha_i)} F_i K^\lambda, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

together with the quantum Serre relations (see [35], p.53). In the relations above we have set $K_i \stackrel{\text{def}}{=} K^{\alpha_i}$ and $q_i = q^{(\alpha_i,\alpha_i)/2}$. The algebra U is a Hopf algebra, with the comultiplication

$$\Delta(K^\lambda) = K^\lambda \otimes K^\lambda, \qquad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

the antipode

$$S(K^\lambda) = K^{-\lambda}, \qquad S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i$$

and the counit

$$\epsilon(K^\lambda) = 1, \qquad \epsilon(E_i) = 0, \qquad \epsilon(F_i) = 0.$$

Let $U_{\geq 0}$ denote the subalgebra of U generated by all K^λ, E_i, and $U_{\leq 0}$ denote the subalgebra generated by all K^λ, F_i. We also write U_0 for the subalgebra generated by K^λ, $\lambda \in P$. The algebras $U_{\geq 0}$, $U_{\leq 0}$, U_0 are Hopf subalgebras in U. Recall that $(U_{\leq 0})^{cop}$ stands for the co-opposite Hopf algebra to $U_{\leq 0}$. There is a non-degenerate Hopf pairing

$$\langle \cdot, \cdot \rangle : U_{\geq 0} \times (U_{\leq 0})^{cop} \longrightarrow k \tag{3.5.1}$$

defined by

$$\langle K^\lambda, K^\mu \rangle = q^{-(\lambda,\mu)}, \qquad \langle K^\lambda, E_i \rangle = 0 = \langle K^\lambda, F_i \rangle, \qquad \langle E_i, F_j \rangle = -\frac{\delta_{ij}}{q_i - q_i^{-1}}.$$

Let U^+ and U^- denote the subalgebras generated by all E_i and by all F_i respectively. Then the quantum group U admits a triangular decomposition: the natural multiplication map defines an isomorphism of $\mathbb{C}(q)$-modules

$$U^+ \otimes U_0 \otimes U^- \longrightarrow U \tag{3.5.2}$$

The algebra U is graded by the root lattice Q. Indeed, setting

$$U_\nu = \{u \in U \mid K^\lambda u = q^{(\lambda,\nu)} u K^\lambda\}, \tag{3.5.3}$$

we have $U = \bigoplus_{\nu \in Q} U_\nu$. If we set $U_\nu^+ = U^+ \cap U_\nu$ and $U_\nu^- = U^- \cap U_\nu$, then the pairing (3.5.1) has the orthogonality property

$$\langle U_\nu^+, U_{-\mu}^- \rangle = 0 \quad \text{if} \quad \mu \neq \nu. \tag{3.5.4}$$

Remark 3.5.1. The Hopf algebra U can be described as a quotient of the Drinfeld double of the dual pair $(U_{\geq 0}, U_{\leq 0}^{cop})$, which in particular implies the relation

$$xy = \langle x_1, y_1 \rangle \langle x_3, Sy_3 \rangle y_2 x_2 \quad \text{for all} \quad x \in U_{\geq 0},\ y \in U_{\leq 0}. \tag{3.5.5}$$

Quantized coordinate rings.

Let G be the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}$. The quantized algebra of functions on G, which we denote by $\mathcal{O}_q[G]$, is defined to be the Hopf algebra of matrix elements of finite-dimensional U-modules. For a finite-dimensional U-module V of highest weight λ and a pair of elements $v \in V$ and $f \in V^*$ we denote the corresponding matrix element by $c^\lambda_{f,v}$, or simply by $c_{f,v}$ when it does not cause ambiguity. By construction, there is a Hopf pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathcal{O}_q[G] \otimes U \longrightarrow k \tag{3.5.6}$$

defined by evaluation of matrix elements against elements of U. Pairing (3.5.6) is non-degenerate, since no non-zero element of U acts as zero in all finite-dimensional representations [35].

The algebra $\mathcal{O}_q[G]$ is a left $U \otimes U^{cop}$ module algebra via the left and right coregular actions

$$((x \otimes y) \circ \psi)(u) = \psi(Syux) \qquad \text{where} \qquad x, u \in U,\ y \in U^{cop},\ \psi \in \mathcal{O}_q[G]. \tag{3.5.7}$$

As a $U \otimes U^{cop}$-module, $\mathcal{O}_q[G]$ admits the Peter-Weyl decomposition

$$\mathcal{O}_q[G] = \bigoplus_{\lambda \in P^+} L(\lambda)^* \otimes L(\lambda)$$

where $L(\lambda)$ is the finite-dimensional U-module of highest weight λ, and $L(\lambda)^*$ is its dual.

The algebra $\mathcal{O}_q[G]$ is graded by two copies of the weight lattice P as follows

$$\mathcal{O}_q[G] = \bigoplus_{\lambda,\mu\in P} \mathcal{O}_q[G]_{\lambda,\mu}$$

where

$$\mathcal{O}_q[G]_{\lambda,\mu} = \left\{\psi \in \mathcal{O}_q[G] \mid (K^\nu \otimes K^\rho)\psi = q^{(\mu,\nu)+(\lambda,\rho)}\psi\right\}.$$

If V is a representation of U and $v \in V$ satisfies $K^\lambda v = q^{(\lambda,\mu)} v$ for all $\lambda \in P$, we say that v is a weight vector of weight μ, and write $\mathrm{wt}(v) = \mu$. The subspace $\mathcal{O}_q[G]_{\lambda,\mu}$ is spanned by matrix elements $c_{f,v}$ with $\mathrm{wt}(f) = \lambda$, $\mathrm{wt}(v) = \mu$. Note that $S(\mathcal{O}_q[G]_{\lambda,\mu}) = \mathcal{O}_q[G]_{\mu,\lambda}$ and for $x_\nu \in U_\nu$, $\psi_{\lambda,\mu} \in \mathcal{O}_q[G]_{\lambda,\mu}$ we have

$$\psi_{\lambda,\mu}(x_\nu) \neq 0 \quad \Longrightarrow \nu + \lambda + \mu = 0$$

Moreover, if $\psi \in \mathcal{O}_q[G]_{\lambda,\mu}$ its coproduct takes the form

$$\Delta(\psi) = \sum_i \psi_{\lambda,\nu_i} \otimes \psi_{-\nu_i,\mu} \quad \text{where} \quad \psi_{\alpha,\beta} \in \mathcal{O}_q[G]_{\alpha,\beta}.$$

Quantum Weyl group

Let $\widehat{U}$ be the completion of U with respect to the weak topology generated by all matrix elements of finite-dimensional U-modules (see [34, Section 3]). As an algebra $\widehat{U}$, is isomorphic to $\prod_{\lambda\in P_+} \mathrm{End}_{\mathbb{C}(q)} L(\lambda)$. We will also regard an element $u \in \widehat{U}$ as a functional on $\mathcal{O}_q[G]$ via the evaluation pairing $\langle\langle c_{f,v}, u\rangle\rangle = f(uv)$.

Definition 3.5.2. [42] Define an element T_i of $\widehat{U}$ which acts on any weight vector v by

$$T_i(v) \quad \stackrel{\text{def}}{=} \sum_{\substack{a,b,c\geqslant 0 \\ a-b+c=(\mathrm{wt}(v),\alpha_i)}} (-1)^b q^{ac-b} F_i^{(a)} E_i^{(b)} F_i^{(c)}(v).$$

By [42, Theorem 39.4.3], the elements T_i generate an action of the braid group on any finite-dimensional U-module. The subalgebra of $\widehat{U}$ generated by U together with the T_i is often referred to as the *quantum Weyl group*, and it is known [5] to in fact be a Hopf algebra. Moreover, let w_0 be the longest element of the Weyl group, and $w = s_{i_1} \ldots s_{i_k}$ any of its reduced decompositions into simple reflections. Then the element T_{w_0} defined by

$$T_{w_0} = T_{i_1} \ldots T_{i_k} \tag{3.5.8}$$

is independent of the choice of reduced expression for w_0.

Quantum minors

We now recall the definition of certain elements of $\mathcal{O}_q[G]$ that will prove useful in the sequel. For each dominant weight $\lambda \in P^+$, we fix a highest weight vector $v_\lambda \in L(\lambda)$. Then, as in [34], we define the corresponding lowest weight vectors $v_{w_0(\lambda)} \in L(\lambda)$ by

$$T_{w_0} v_{w_0(\lambda)} = (-1)^{\langle 2\lambda, \rho^\vee \rangle} q^{-2(\lambda,\rho)} v_\lambda$$

Proposition 3.5.3. *[34, Comment 5.10] The vectors $v_\lambda, v_{w_0(\lambda)}$ satisfy*

$$T_{w_0} v_\lambda = v_{w_0(\lambda)}.$$

For each $\lambda \in P^+$, there is a unique pairing

$$\langle -, - \rangle_\lambda \colon L(-w_0(\lambda)) \otimes L(\lambda) \longrightarrow k$$

satisfying conditions

$$\langle v_{-\lambda}, v_\lambda \rangle_\lambda = 1 \qquad \text{and} \qquad \langle xw, v \rangle_\lambda = \langle w, Sxv \rangle_\lambda$$

for all $x \in U$, $v \in L(\lambda)$, and $w \in L(-w_0(\lambda))$. The following definition coincides with the one given in [1].

Definition 3.5.4. The *quantum principal minor* Δ^λ is the element of $\mathcal{O}_q[G]$ whose value on any $x \in U$ is given by

$$\Delta^\lambda(x) = \langle v_{-\lambda}, x v_\lambda \rangle_\lambda$$

Given $(u, v) \in W \times W$ we choose reduced decompositions $u = s_{i_l} \cdots s_{i_1}$ and $v = s_{j_{l'}} \cdots s_{j_1}$ and set

$$n_k = \langle s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee), \lambda \rangle, \qquad m_k = \langle s_{j_1} \cdots s_{j_{k-1}}(\alpha_{i_k}^\vee), \lambda \rangle.$$

Then the quantum minor $\Delta_{u,v}^\lambda$ is defined by

$$\Delta_{u,v}^\lambda(x) = \Delta^\lambda \left(E_{i_1}^{(n_1)} \cdots E_{i_l}^{(n_l)} x F_{j_{l'}}^{(m_{l'})} \cdots F_{j_1}^{(m_1)} \right)$$

where $a^{(n)}$ stands for the n-th q-divided power of a.

$\mathcal{O}_q[G]$ as a co-quasitriangular Hopf algebra

Write Θ_ν for the canonical element in $U_\nu^+ \otimes U_\nu^-$ with respect to the pairing 3.5.1. If V, W are two finite-dimensional representations of U, then the action of the formal sum $\Theta = \sum_{\nu \in Q} \Theta_\nu$ is well defined in the tensor product $V \otimes W$. Let $f_{V,W}$ be the operator in $V \otimes W$ defined by

$$f_{V,W}(v \otimes w) = q^{-(\mathrm{wt}(v), \mathrm{wt}(w))}(v \otimes w)$$

for any weight vectors $v, w \in V, W$. Then define R_{VW} to be the following operator in $V \otimes W$

$$R_{VW}(v \otimes w) = \Theta \circ f_{VW}$$

The operator R gives rise to a bilinear form $r\colon \mathcal{O}_q[G] \times \mathcal{O}_q[G] \longrightarrow k$ defined by

$$\begin{aligned} r(c_{f,v}, c_{g,w}) &= (f \otimes g)(R_{VW}(v \otimes w)) \\ &= \sum_{\alpha} q^{-(\mathrm{wt}(v),\mathrm{wt}(w))} f(\Theta_\alpha v) g(\Theta_{-\alpha} w). \end{aligned}$$

The form r equips $\mathcal{O}_q[G]$ with the structure of a *co-quasitriangular Hopf algebra* [6, 51]. This means that, for all triples $\phi, \psi, \rho \in \mathcal{O}_q[G]$, we have

$$r(\phi_1, \psi_1)\phi_2\psi_2 = \psi_1\phi_1 r(\phi_2, \psi_2), \tag{3.5.9}$$
$$r(\phi\psi, \rho) = r(\phi, \rho_1) r(\psi, \rho_2), \tag{3.5.10}$$
$$r(\rho, \phi\psi) = r(\rho_1, \psi) r(\rho_2, \phi). \tag{3.5.11}$$

As the following Proposition shows, the form r is closely related to the longest element T_{w_0} of the quantum Weyl group.

Proposition 3.5.5. *[5, 34] Let C be the element of $\widehat{U}$ defined by*

$$C(v) = q^{(wt(v),\rho)-(wt(v),wt(v))/2} v$$

where ρ is the half-sum of positive roots. Then setting

$$Y = CT_{w_0}, \tag{3.5.12}$$

we have the following equality in $\mathcal{O}_q[G]^* \otimes \mathcal{O}_q[G]^*$

$$r = (Y^{-1} \otimes Y^{-1})\Delta(Y). \tag{3.5.13}$$

l-operators

Let $\mathcal{O}_q[G]^*$ be the full linear dual of $\mathcal{O}_q[G]$, and define maps

$$l^{\pm},\ 'l^{\pm}\colon \mathcal{O}_q[G] \longrightarrow \mathcal{O}_q[G]^*$$

by

$$l^+(\phi) = r(\cdot, \phi), \qquad 'l^+(\phi) = r(\cdot, S^{-1}\phi),$$
$$'l^-(\phi) = r(\phi, \cdot), \qquad l^-(\phi) = r(S\phi, \cdot).$$

Lemma 3.5.6. *[51, Lemma 1.4] The maps $l^{\pm} : \mathcal{O}_q[G] \to \mathcal{O}_q[G]^*$ are anti-homomorphisms of algebras, while the maps $'l^{\pm}$ are homomorphisms of algebras. Additionally, we have*

$$l^+, 'l^+ \colon \mathcal{O}_q[G] \longrightarrow U_{\geq 0}, \qquad l^-, 'l^- \colon \mathcal{O}_q[G] \longrightarrow U_{\leq 0}$$

with explicit formulas given by

$$l^+(c_{f,v}) = \sum_\alpha f(\Theta_{-\alpha} v)\Theta_\alpha K^{-\operatorname{wt}(v)}$$
$$'l^-(c_{f,v}) = \sum_\alpha f(\Theta_{\alpha} v)\Theta_{-\alpha} K^{-\operatorname{wt}(v)}$$

We also have

Lemma 3.5.7. *Let Δ, S_U denote the coproduct and antipode in U. Then*

$$'l^{\pm} = S_U \circ l^{\pm}$$

and

$$\Delta \circ l^{\pm}(\phi) = l^{\pm}(\phi_1) \otimes l^{\pm}(\phi_2),$$
$$\Delta \circ 'l^{\pm}(\phi) = 'l^{\pm}(\phi_2) \otimes 'l^{\pm}(\phi_1).$$

Proof. These identities follow directly from the properties (3.5.10), (3.5.11) of r, together with the non-degeneracy of the Hopf pairing between U and $\mathcal{O}_q[G]$. □

We will make frequent use of the following lemma relating the universal r-form to the Hopf pairing (3.5.1).

Lemma 3.5.8. *Let $\langle \cdot, \cdot \rangle$ be the pairing* (3.5.1) *of $U_{\geq 0}$ with $U_{\leq 0}$. Then*

$$\langle l^+(\phi),\ 'l^-(\psi)\rangle = r(\psi, \phi)$$

Proof. We verify the claim for any pair of matrix elements $c_{f,v}, c_{g,w} \in \mathcal{O}_q[G]$. Let us expand $\Theta = \sum_i \Theta_{+i} \otimes \Theta_{-i}$ where $\langle \Theta_{+i}, \Theta_{-j} \rangle = \delta_{ij}$. Then using the relation

$$\langle \Theta_{+i} K^\lambda, \Theta_{-j} K^\mu \rangle = q^{-(\lambda,\mu)} \delta_{ij}$$

from [35, p. 6.13] we compute

$$\begin{aligned}\langle l^+(c_{g,w}),\ 'l^-(c_{f,v})\rangle &= \sum_{i,j} g(\Theta_{-i} w) f(\Theta_{+j} v) \langle \Theta_{+i} K^{-\operatorname{wt}(w)}, \Theta_{-j} K^{-\operatorname{wt}(v)} \rangle \\ &= \sum_i q^{-(\operatorname{wt}(v),\operatorname{wt}(w))} f(\Theta_{+i} v) g(\Theta_{-i} w) = r(c_{f,v}, c_{g,w}).\end{aligned}$$

□

The ad-integrable part of U

Consider the left (right) adjoint actions ad_l (respectively, ad_r) of U on itself defined by

$$\mathrm{ad}_l(x)(y) = x_1 y S x_2 \tag{3.5.14}$$
$$\mathrm{ad}_r(x)(y) = S x_1 y x_2 \tag{3.5.15}$$

Definition 3.5.9. The *left ad-integrable part of* U is defined as the subset

$$F_l(U) = \{x \in U \mid \dim \mathrm{ad}_l(U)x < \infty\}$$

Similarly, the *right ad-integrable part of* U is defined as the subset

$$F_r(U) = \{x \in U \mid \dim \mathrm{ad}_r(U)x < \infty\}$$

Proposition 3.5.10. *[6] The ad-integrable parts $F_l(U)$, $F_r(U)$ are subalgebras in U. Moreover, they are left and right coideals respectively:*

$$\Delta(F_l(U)) \subset U \otimes F_l(U), \qquad \Delta(F_r(U)) \subset F_r(U) \otimes U.$$

Now consider the maps

$$I \colon \mathcal{O}_q[G] \longrightarrow U_{\geq 0} \otimes U_{\leq 0}, \qquad I = (l^+ \otimes {}'l^-) \circ \Delta \tag{3.5.16}$$

and

$$J \colon \mathcal{O}_q[G] \longrightarrow U, \qquad J = m \circ I \tag{3.5.17}$$

where

$$m \colon U_{\geq 0} \otimes U_{\leq 0} \longrightarrow U, \qquad u_+ \otimes u_- \mapsto u_+ u_-$$

is the multiplication in U. Note also that the action (3.5.15) induces a coadjoint action $\mathrm{ad}_r^* \colon U \otimes \mathcal{O}_q[G] \longrightarrow \mathcal{O}_q[G]$ given by

$$\langle \mathrm{ad}_r^*(x)(\psi), y \rangle = \langle \psi, S(x_1) y x_2 \rangle, \qquad x, y \in U, \ \psi \in \mathcal{O}_q[G]. \tag{3.5.18}$$

The following theorem was proven by Joseph and Letzter in [4], building on results of Caldero [16].

Theorem 3.5.11. *[4] The map J is an injection of U-modules, with respect to the action* (3.5.14) *on U and the action* (3.5.18) *on $\mathcal{O}_q[G]$. Its image is*

$$F_l(U) = \bigoplus_{\lambda \in P^+} (\mathrm{ad}_l\, U)(K^{-2\lambda}) \tag{3.5.19}$$

Since $S(F_l(U)) = F_r(U)$, the theorem implies that the map

$$J' \stackrel{\text{def}}{=} S \circ J \colon \mathcal{O}_q[G] \longrightarrow F_r(U)$$

is also an isomorphism of U-modules. Indeed, for all $x \in U$, $\phi \in \mathcal{O}_q[G]$ we have

$$x_2 J'(\phi) S^{-1} x_1 = J'(\mathrm{ad}_r^*(S^{-2}x)\phi) \tag{3.5.20}$$

Despite being a morphism of U-modules, the map J is not a morphism of algebras. However, as explained in [51], one can equip $\mathcal{O}_q[G]$ with a twisted algebra structure so that J becomes an algebra homomorphism:

Proposition 3.5.12. *The following formula defines an associative product $\bullet_F$ in $\mathcal{O}_q[G]$*

$$\begin{aligned} \phi \bullet_F \psi &= r(\phi_1, \psi_2) r(\phi_3, S\psi_1) \phi_2 \psi_3 \\ &= r(\phi_2, \psi_3) r(\phi_3, S\psi_1) \psi_2 \phi_1 \end{aligned}$$

If we write ${}^F\mathcal{O}_q[G]$ for the algebra obtained by equipping $\mathcal{O}_q[G]$ with the product $\bullet_F$, then the map $J \colon {}^F\mathcal{O}_q[G] \longrightarrow F_l(U)$ is an isomorphism of U-module algebras.

Similarly, the map J' is an isomorphism of algebras $({}^F\mathcal{O}_q[G])^{op} \simeq F_r(U)$.

The Heisenberg double of $U_{\geq 0}$

We define the Heisenberg double of $U_{\geq 0}$ to be the smash product $\mathcal{H}_q = U_{\geq 0} \# U_{\leq 0}$ of the dual pair of Hopf algebras $U_{\geq 0}$ and $U_{\leq 0}^{cop}$ with respect to the pairing (3.5.1). The product in $\mathcal{H}_q$ can be written explicitly as

$$(a\#x)(b\#y) = \langle b_2, x_2 \rangle a b_1 \otimes x_1 y$$

Let us make a few remarks on the structure of $\mathcal{H}_q$ that will prove useful in the sequel. Consider the torus

$$\mathbb{T} = U_0 \otimes U_0 \subset \mathcal{H}_q$$

and the following three subtori

$$T_+ = U_0 \otimes 1, \qquad T_- = 1 \otimes U_0, \qquad \text{and} \qquad T_c = (1 \otimes S) \circ \Delta(U_0).$$

The Heisenberg double $\mathcal{H}_q$ has the following T_--module algebra structure

$$(1 \otimes K^\lambda) \circ (a\#x) = (1\#K^\lambda)(a\#x)(1\#K^{-\lambda}) = \langle K^\lambda, a_2 \rangle a_1 \# K^\lambda x K^{-\lambda}.$$

It also admits a T_+-module algebra structure given by

$$(K^\mu \otimes 1) \circ (a\#x) = \langle K^\mu, x_1 \rangle a \# x_2$$

Since the actions of T_+ and T_- commute, we may combine them into an action of $\mathbb{T}$ on $\mathcal{H}_q$. Using the grading (3.5.3), the restriction of this $\mathbb{T}$-action to the subalgebras T_- and T_c can be computed explicitly as

$$(1 \otimes K^\rho) \circ (x_\nu K^\lambda \# y_\alpha K^\mu) = q^{(\rho, \alpha-\lambda)} x_\nu K^\lambda \# y_\alpha K^\mu$$
$$(K^\rho \otimes K^{-\rho}) \circ (x_\nu K^\lambda \# y_\alpha K^\mu) = q^{(\rho, \lambda-\alpha-\mu)} x_\nu K^\lambda \# y_\alpha K^\mu$$

for any $x_\nu \in U^+_\nu$ and $y_\alpha \in U^-_\alpha$. Therefore, we have

Proposition 3.5.13. *The T_- invariants in $\mathcal{H}_q$ coincide with the subalgebra*

$$\mathcal{H}_q^{T_-} = \bigoplus_{\nu \in Q_+} U^+ K^{-\nu} \# U^-_{-\nu} T,$$

the T^c invariants coincide with the subalgebra

$$\mathcal{H}_q^{T_c} = \bigoplus_{\lambda \in P,\, \nu \in Q_+} U^+ K^\lambda \# U^-_{-\nu} K^{\lambda+\nu}, \tag{3.5.21}$$

and the $\mathbb{T}$-invariants coincide with the subalgebra

$$\mathcal{H}_q^{\mathbb{T}} = \bigoplus_{\lambda \in P,\, \nu \in Q_+} U^+ K^{-\nu} \# U^-_{-\nu}.$$

Note that, the subalgebra of T_- invariants $\mathcal{H}_q^{T_-}$ commute with the subalgebra $1\#T$. Hence we obtain

Corollary 3.5.14. *Multiplication in $\mathcal{H}_q$ yields an algebra isomorphism*

$$\mathcal{H}_q^{\mathbb{T}} \otimes T \longrightarrow \mathcal{H}_q^{T^-}, \qquad (a\#x) \otimes (1\#K^\rho) \longmapsto a\#xK^\rho.$$

Remark 3.5.15. The torus $\mathbb{T}(A)$ is naturally embedded into the Drinfeld double of the dual pair $(U_{\geq 0}, U_{\leq 0}^{cop})$. The action of $\mathbb{T}$ used in this section arises from the action of the Drinfeld double on the Heisenberg double considered in [41].

The following formula defines an action of $\mathcal{H}_q$ on $U_{\geq 0}$

$$(a\#x) \circ b = \langle x, b_2 \rangle a b_1 \tag{3.5.22}$$

where $a\#x \in \mathcal{H}_q$ and $b \in U_{\geq 0}$. We have the following lemma regarding the restriction of this action to the subalgebra $\mathcal{H}_q^{T_-} \subset \mathcal{H}_q$.

Lemma 3.5.16. *As $\mathcal{H}_q^{T_-}$-modules, we have*

$$U_{\geq 0} = \bigoplus_{\lambda \in P} U^+ K^\lambda$$

Proof. It suffices to check that $K^{-\nu}\#y_{-\nu} \in \mathcal{H}_q^{T_-}$ preserves U^+K^λ. This follows from the orthogonality property (3.5.4) of the pairing $\langle\cdot,\cdot\rangle$ and the fact that given $x \in U_\alpha^+$, its coproduct can be expanded as

$$\Delta(x) = \sum_\beta x_{\alpha-\beta}K^\beta \otimes x_\beta$$

where $x_\beta \in U_\beta^+$ and $x_{\alpha-\beta} \in U_{\alpha-\beta}^+$. □

3.6 Embedding $F_l(U)$ into the Heisenberg double $\mathcal{H}_q$.

In the language of l-operators, Corollary (3.3.7) specializes to the following statement in the present case.

Proposition 3.6.1. *The maps*

$$\zeta\colon {}^F\mathcal{O}_q[G] \longrightarrow \mathcal{H}_q, \qquad \phi \longmapsto l^+(S^{-1}\phi_3\phi_1)\#' l^-(S^{-1}\phi_2) \tag{3.6.1}$$

$$\widehat{\zeta}\colon F_l(U) \longrightarrow \mathcal{H}_q, \qquad \widehat{\zeta} = \zeta \circ J^{-1} \tag{3.6.2}$$

are homomorphisms of algebras.

Proposition 3.6.2. *The image of ζ is contained in the subalgebra $\mathcal{H}_q^{T_-}$ of T_- invariants.*

Proof. Suppose that $\psi \in \mathcal{O}_q[G]_{\lambda,\mu}$. Then we may expand

$$\Delta^2(\psi) = \sum_{\nu_1,\nu_2} \psi_{\lambda,\nu_1} \otimes \psi_{-\nu_1,\nu_2} \otimes \psi_{-\nu_2,\mu}$$

with $\psi_{\alpha,\beta} \in \mathcal{O}_q[G]_{\alpha,\beta}$. Note that

$$S^{-1}\psi_{-\nu_2,\mu}\psi_{\lambda,\nu_1} \in \mathcal{O}_q[G]_{\lambda+\mu,\nu_1-\nu_2} \quad \text{and} \quad S^{-1}\psi_{-\nu_1,\nu_2} \in \mathcal{O}_q[G]_{\nu_2,-\nu_1}$$

and recall that $\psi_{\alpha,\beta}(x_\rho)$ is non-zero only if $\rho+\alpha+\beta = 0$. Therefore we have

$$\begin{aligned}\zeta(\psi) &= \sum_{\nu_1,\nu_2,\alpha,\beta} (S^{-1}\psi_{-\nu_2,\mu}\psi_{\lambda,\nu_1})(\Theta_{-\alpha})\Theta_\alpha K^{-\nu_1+\nu_2}\#(S^{-1}\psi_{-\nu_1,\nu_2})(\Theta_\beta)\Theta_{-\beta}K^{-\nu_1} \\ &= \sum_{\nu_1,\beta}(S^{-1}\psi_{\nu_1-\beta,\mu}\psi_{\lambda,\nu_1})(\Theta_{-\lambda-\mu-\beta})(S^{-1}\psi_{\nu_1,\nu_1-\beta})(\Theta_\beta)\Theta_{\lambda+\mu+\beta}K^{-\beta}\#\Theta_{-\beta}K^{-\nu_1}\end{aligned}$$

which implies $\zeta(\psi) \in \mathcal{H}_q^{T_-}$. □

Recall the defining representation (3.5.22) of $\mathcal{H}_q$ on $U_{\geq 0}$. Pulling this representation back under the algebra homomorphism (3.6.2), we obtain an action of the algebra $F_l(U)$ on $U_{\geq 0}$. In studying this representation, it will be convenient to describe $U_{\geq 0}$ by means of the surjective homomorphism $l^+\colon \mathcal{O}_q[G] \to U_{\geq 0}$. The following formula is easily deduced from the formula (3.6.1) for ζ, the coquasitriangularity of r, and Lemma 3.5.8.

Lemma 3.6.3. *The action of $J(\psi) \in F_l(U)$ on $l^+(\varphi) \in U_{\geq 0}$ induced by $\widehat{\zeta}$ is given by*

$$J(\psi) \cdot l^+(\varphi) = r(S^{-1}\psi_3, \varphi_1)\ l^+(S^{-1}\psi_2\varphi_2\psi_1)$$

Since $\zeta({}^F\mathcal{O}_q[G]) \subset \mathcal{H}_q^{T_-}$, it follows from Lemma 3.5.16 that the space $U_{\geq 0}$ decomposes as an $F_l(U)$-module as

$$U_{\geq 0} = \bigoplus_{\lambda \in P} U^+ K^\lambda \tag{3.6.3}$$

We will now identify the $F_l(U)$-modules U^+K^λ. Recall the definition of the contragredient Verma module $M(\mu)^\vee$ for U. Let $\mathbb{C}_\mu$ be the one-dimensional $U_{\geq 0}$-module with basis w_μ and $U_{\geq 0}$-module structure defined by

$$a \cdot w_\mu = \langle a, K^\mu \rangle,$$

which is a slight abuse of notation for $\mu \notin P$. Regard U as a $U_{\geq 0}$ module via the action $a \cdot u = uS(a)$. Then

$$M(\mu)^\vee \stackrel{\text{def}}{=} \operatorname{Hom}_{U_{\geq 0}}(U, \mathbb{C}_\mu)$$

where $\operatorname{Hom}_{U_{\geq 0}}$ denotes the restricted (graded) Hom of $U_{\geq 0}$-modules. The action of U on $M(\mu)^\vee$ is then given by

$$(u \cdot \phi)(v) = \phi(Suv).$$

Note that because of the triangular decomposition of U, elements of $M(\mu)^\vee$ are uniquely determined by their values on $U_{\leq 0} \subset U$.

Proposition 3.6.4. *The $F_l(U)$-module U^+K^λ in the decomposition* (3.6.3) *is isomorphic to the restriction to $F_l(U)$ of the contragredient Verma module $M(\lambda/2)^\vee$.*

Proof. Given $a \in U^+K^\lambda$, define an element $\phi_a \in M(\lambda/2)^\vee$ by declaring

$$\phi_a(y) = \langle aK^{-\lambda/2}, y \rangle$$

for all $y \in U_{\leq 0}$. We claim that the map $a \mapsto \phi_a$ is an isomorphism of $F_l(U)$-modules. By the non-degeneracy of $\langle \cdot, \cdot \rangle$, it is an isomorphism of linear spaces. To show that it respects the $F_l(U)$-module structure, we compute the action of the subalgebras $U_{\geq 0}$ and $U_{\leq 0}$ on $M(\lambda/2)^\vee$.

Suppose first that $z \in U_{\leq 0}$, with $Sz \in U^-K^\rho$. Then for all $y \in U_{\leq 0}$ we have

$$(z \cdot \phi_a)(y) = \phi_a(Szy) = \langle aK^{-\lambda/2}, Szy \rangle = \langle a_1K^{-\lambda/2}, Sz \rangle \langle a_2K^{-\lambda/2}, y \rangle = q^{\frac{1}{2}(\lambda,\rho)} \langle a_1, Sz \rangle \phi_{a_2}(y)$$

At the same time, for $b \in U^+K^\rho$, we have

$$\begin{aligned}(b \cdot \phi_a)(y) = \phi_a(Sby) &= \langle Sb_3, y_3 \rangle \langle b_1, y_1 \rangle \phi_a(y_2Sb_2) = \langle Sb_3, y_3 \rangle \langle b_1, y_1 \rangle \langle Sb_2, K^{\lambda/2} \rangle \langle aK^{-\lambda/2}, y_2 \rangle \\ &= \langle b_1aK^{-\lambda/2}Sb_3, y \rangle \langle Sb_2, K^{\lambda/2} \rangle = q^{\frac{1}{2}(\lambda,\rho)} \phi_{b_1aSb_2}(y).\end{aligned}$$

Here we used formula (3.5.5) for the product in U, together with the homogeneity of the coproduct in $U_{\geq 0}$. Now given $\psi \in {}^F\mathcal{O}_q[G]_{\gamma,\mu}$, we compute the action of $J(\psi) = l^+(\psi_1)'l^-(\psi_2)$ on $\phi_a \in M(\lambda/2)^\vee$ with the help of Lemma 3.6.3. Note that in the expansion

$$\Delta(\psi) = \sum_\nu \psi_{\gamma,\nu} \otimes \psi_{-\nu,\mu}$$

we have

$$'l^-(S^{-1}\psi_{-\nu,\mu}) \in U^- K^\nu \quad \text{and} \quad l^+(\psi_{\gamma,\nu}) \in U^+ K^{-\nu}.$$

Then

$$J(\psi) \cdot \phi_a = l^+(\psi_1) \cdot \left(q^{\frac{1}{2}(\lambda,\nu)} \langle a_1, 'l^-(S^{-1}\psi_2) \rangle \phi_{a_2} \right) = \langle a_1, 'l^-(S^{-1}\psi_3) \rangle \phi_{l^+(\psi_1)a_2 l^+(S^{-1}\psi_2)}.$$

Therefore taking $a = l^+(\varphi)$, we find

$$J(\psi) \cdot \phi_a = r(S^{-1}\psi_3, \varphi_1) \phi_{l^+(S^{-1}\psi_2 \varphi_2 \psi_1)} = \phi_{\psi \cdot a}$$

which shows that the map $a \mapsto \phi_a$ intertwines the two actions of $F_l(U)$. □

Corollary 3.6.5. *The homomorphisms $\widehat{\zeta}$ and $\widehat{\xi}$ are injective.*

Proof. For any $\lambda \in P^+$, the contragredient Verma module $M(\lambda)^\vee$ contains the finite-dimensional U-module $L(\lambda)$ as a submodule. Hence the corollary follows from the fact [35, p. 5.11] that no non-zero element of U acts by zero in all finite-dimensional representations. □

As in Corollary **??**, we may extend $\widehat{\zeta}$ to obtain a homomorphism of algebras

$$\widetilde{\zeta} \colon F_l(U) \otimes_Z U_0 \longrightarrow \mathcal{H}_q^{T-}, \qquad u \otimes t \mapsto \mu(u)t. \tag{3.6.4}$$

Proposition 3.6.6. *The homomorphism $\widetilde{\zeta}$ is injective.*

Proof. Since $U_0 \simeq \mathbb{C}[P]$ we may regard $\widetilde{U} \overset{\text{def}}{=} F_l(U) \otimes_Z U_0$ as a quasi-coherent sheaf on Spec$\mathbb{C}[P]$, whose stalk at $\lambda \in \mathbb{C}[P]$ we denote by $(\widetilde{U})_\lambda$. We may similarly regard $\mathcal{H}_q^{T-}$ as a sheaf over Spec$\mathbb{C}[P]$ and denote its stalk at $\lambda \in \mathbb{C}[P]$ by $(\mathcal{H}_q^{T-})_\lambda$. Let $\widetilde{\zeta}_\lambda \colon (\widetilde{U})_\lambda \longrightarrow (\mathcal{H}_q^{T-})_\lambda$ be the induced map. Then $\ker \widetilde{\zeta}$ is a subsheaf of $\widetilde{U}$, and $\ker \widetilde{\zeta}_\lambda$ is its stalk at point λ. Thus, it is enough to show that $\ker \widetilde{\zeta}_\lambda = 0$ for any λ.

Let $\mathcal{I}_\lambda \subset \widetilde{U}$ denote the ideal generated by $\left\langle 1 \otimes K^\mu - q^{\langle \lambda,\mu \rangle} \right\rangle_{\mu \in P}$ and $\mathcal{J}_\lambda \subset \mathcal{H}_q^{T-}$ denote the ideal generated by $\left\langle 1\#K^\mu - q^{\langle \lambda,\mu \rangle} \right\rangle_{\mu \in P}$. Let U^λ be the quotient of U by the central character of the Verma module of weight λ. Note, that $\widetilde{U}/\mathcal{I}_\lambda \simeq U^\lambda$. Set $\mathcal{H}_q^\lambda \overset{\text{def}}{=} \mathcal{H}_q^{T-}/\mathcal{J}_\lambda$ and let $\widehat{\zeta}^\lambda \colon U^\lambda \longrightarrow \mathcal{H}_q^\lambda$ be the induced homomorphism. By quantum Duflo theorem, we know that U^λ acts faithfully on the Verma module $M(\lambda)$. In view of Proposition 3.6.4 and the existence of a nondegenerate pairing between a Verma module and the corresponding contragredient Verma module, we obtain $\ker \widehat{\zeta}^\lambda = 0$.

Now, let $\mathbb{C}[P]_\lambda$ denote the local ring at λ and $\mathfrak{m}_\lambda$ be its maximal ideal. Then one has

$$\mathcal{U}^\lambda = \big(\widetilde{U}\big)_\lambda / \mathfrak{m}_\lambda \big(\widetilde{U}\big)_\lambda \qquad \text{and} \qquad \mathcal{H}_q^\lambda = \big(\mathcal{H}_q^{T_-}\big)_\lambda / \mathfrak{m}_\lambda \big(\mathcal{H}_q^{T_-}\big)_\lambda,$$

so that

$$\mathfrak{m}_\lambda \ker \widetilde{\zeta}_\lambda = \ker \widetilde{\zeta}_\lambda.$$

At this point the Proposition would from Nakayama's lemma if $\ker \widetilde{\zeta}_\lambda$ were a finitely-generated $\mathbb{C}[P]_\lambda$ module. Therefore, it remains to filter $\ker \widetilde{\zeta}_\lambda$ by finitely generated submodules. There is a natural filtration on $\big(\widetilde{U}\big)_\lambda$ (by the sum of modulus of exponents in the Poincaré-Birkhoff-Witt basis), so let $\ker_n \widetilde{\zeta}_\lambda$ denote the intersection of the n-th filtered component with $\ker \widetilde{\zeta}_\lambda$. Then the submodules $\ker_n \widetilde{\zeta}_\lambda$ are finitely generated (as submodules of a finitely generated module over a Noetherian ring) and deliver the required filtration on $\ker \widetilde{\zeta}_\lambda$. □

3.7 The R-twisted quantum coordinate ring

In this section we introduce the R-twist ${}^R\mathcal{O}_q[G]$ of the quantum coordinate ring $\mathcal{O}_q[G]$, and explain its relation with the Heisenberg double $\mathcal{H}_q$.

The Heisenberg double and ${}^R\mathcal{O}_q[G]$

Proposition 3.7.1. *The following formula defines an associative product $\bullet_R$ in $\mathcal{O}_q[G]$*

$$\phi \bullet_R \psi = r(\phi_1, \psi_1)\phi_2\psi_2 \tag{3.7.1}$$

Proof. This follows straightforwardly from the co-quasitriangularity properties (3.5.9) of the universal r-form. □

Definition 3.7.2. We define ${}^R\mathcal{O}_q[G]$ to be the associative algebra with multiplication defined by (3.7.1).

Proposition 3.7.3. *The map I given by* (3.5.16) *defines an embedding of algebras*

$$I\colon {}^R\mathcal{O}_q[G] \longrightarrow \mathcal{H}_q.$$

Proof. That I is injective follows from the injectivity of the map $J = m \circ I$. To prove that I is a homomorphism of algebras, we compute

$$I(\phi \bullet_R \psi) = r(\phi_1, \psi_1) I(\phi_2\psi_2) = r(\phi_1, \psi_1) l^+(\phi_2\psi_2) \#' l^-(\phi_3\psi_3)$$

On the other hand, in $\mathcal{H}_q$ we have

$$\begin{aligned} I(\phi)\cdot I(\psi) &= \big(l^+(\phi_1)\#'l^-(\phi_2)\big)\cdot\big(l^+(\psi_1)\#'l^-(\psi_2)\big) \\ &= \big\langle l^+(\psi_1)_2, {}'l^-(\phi_2)_2\big\rangle\, l^+(\phi_1)l^+(\psi_1)_1\#'l^-(\phi_2)_1{}'l^-(\psi_2) \\ &= \big\langle l^+(\psi_2), {}'l^-(\phi_2)\big\rangle\, l^+(\phi_1)l^+(\psi_1)\#'l^-(\phi_3)'l^-(\psi_3) \\ &= r(\phi_2,\psi_2)l^+(\psi_1\phi_1)\#'l^-(\phi_3\psi_3) \\ &= r(\phi_1,\psi_1)l^+(\phi_2\psi_2)\#'l^-(\phi_3\psi_3) \\ &= I(\phi\bullet_R\psi) \end{aligned}$$

□

Proposition 3.7.4. *The image $I({}^R\mathcal{O}_q[G])\subset\mathcal{H}_q$ is contained in the subalgebra $\mathcal{H}_q^{T^c}$ of T^c-invariants.*

Proof. Suppose that $\psi\in\mathcal{O}_q[G]_{\lambda,\mu}$, and

$$\Delta(\psi)=\sum_\nu \psi_{\lambda,\nu}\otimes\psi_{-\nu,\mu}$$

Then

$$I(\psi)=\sum\psi_{\lambda,\nu}(\Theta_{-\alpha})\psi_{-\nu,\mu}(\Theta_\beta)\Theta_\alpha K^{-\nu}\#\Theta_{-\beta}K^{-\mu}$$

The only non-zero terms in the sum must have $\beta+\mu-\nu=0$, $\lambda+\nu-\alpha=0$. Hence we find

$$I(\psi)=\sum\psi_{\lambda,\mu+\beta}(\Theta_{-\lambda-\mu-\beta})\psi_{-\mu-\beta,\mu}(\Theta_\beta)\Theta_{\lambda+\mu+\beta}K^{-\mu-\beta}\#\Theta_{-\beta}K^{-\mu}\in\mathcal{H}_q^{T_c}$$

□

Although $I:{}^R\mathcal{O}_q[G]\to\mathcal{H}_q^{T_c}$ is an embedding, it is not surjective. In order to obtain an isomorphism, we must localize at certain elements of ${}^R\mathcal{O}_q[G]$. We define elements $\phi_i^\pm\in\mathcal{O}_q[G]$ by

$$\begin{aligned} \phi_i^+ &= (q_i^{-1}-q_i)^{-1}\Delta^{\omega_i}_{s_i,1}, \\ \phi_i^- &= (q_i^{-1}-q_i)^{-1}\Delta^{\omega_i}_{1,s_i}. \end{aligned}$$

Lemma 3.7.5. *The following equalities hold*

$$\begin{aligned} I(\Delta^{\omega_i}) &= K^{-\omega_i}\#K^{-\omega_i}, \\ I(\phi_i^+) &= E_iK^{-\omega_i}\#K^{-\omega_i}, \\ I(\phi_i^-) &= K^{-\omega_i}\#F_iK^{\alpha_i-\omega_i}. \end{aligned}$$

Proof. One can see that

$$\phi_i^+=(1-q_i^{-2})^{-1}\operatorname{ad}_r^*(E_i)(\Delta^{\omega_i})\qquad\text{and}\qquad\phi_i^-=(q_i^{-1}-q_i)^{-1}\operatorname{ad}_r^*(F_i)(\Delta^{\omega_i}).$$

The rest of the proof is a straightforward calculation using the U-equivariance of J. □

Proposition 3.7.6. *The algebra $\mathcal{H}_q^{T_c}$ is generated by $I\left({}^R\mathcal{O}_q[G]\right)$ together with the elements*

$$K^{\omega_i}\#K^{\omega_i} = q^{(\omega_i,\omega_i)} I\left(\Delta^{\omega_i}\right)^{-1}.$$

Hence, we have the isomorphism

$$I\colon {}^R\mathcal{O}_q[G][(\Delta^{\omega_i})^{-1}]_{i=1}^r \longrightarrow \mathcal{H}_q^{T_c} \tag{3.7.2}$$

Proof. Existence of the map and its injectivity follow from the fact that $I(\Delta^{\omega_i})$ is invertible in $\mathcal{H}_q$, with inverse given by

$$I(\Delta^{\omega_i})^{-1} = q^{-(\omega_i,\omega_i)} K^{\omega_i}\#K^{\omega_i}$$

The surjectivity follows from Lemma 3.7.5 together with the description (3.5.21) of $\mathcal{H}_q^{T_c}$. □

Set

$$\mathcal{O}_q[G^\circ] \overset{\text{def}}{=} \mathcal{O}_q[G][(\Delta^{\omega_i})^{-1}]_{i=1}^r$$

and let

$$\mathcal{O}_q[G^\circ/H] \overset{\text{def}}{=} \left\{\phi \in \mathcal{O}_q[G^\circ] \mid (K^\lambda \otimes 1)\cdot\phi = \phi \quad \text{for any} \quad \lambda \in P\right\}.$$

be the subalgebra of U_0-invariants in $\mathcal{O}_q[G^\circ]$ under the coregular action defined by (3.5.7).

Corollary 3.7.7. *The restriction of the map* (3.7.2)

$$I\colon {}^R\mathcal{O}_q[G^\circ/H] \longrightarrow \mathcal{H}_q^{\mathbb{T}}$$

is an isomorphism of algebras.

Images of the Chevalley generators under $\widehat{\zeta}$

By Lemma 3.7.5, it suffices to calculate $\zeta(\Delta^{\omega_i}), \zeta(\phi_i^{\pm})$. First, suppose that $\phi \in \mathcal{O}_q[G]_{\lambda,\mu}$ satisfies $\phi(xu) = \epsilon(x)\phi(u)$ for all $x \in U^-$, $u \in U$. Then we have $l^+(\phi) = \epsilon(\phi)K^\lambda$, so

$$\zeta(\phi) = (1\#K^\lambda)I(S^{-1}\phi)$$

which implies

$$\zeta(\Delta^{\omega_i}) = q^{-(\omega_i,\omega_i)} I(\Delta^{\omega_i})\cdot I(S^{-1}\Delta^{\omega_i})t^{\omega_i},$$
$$\zeta(\phi_i^-) = q^{-(\omega_i,\omega_i)} I(\Delta^{\omega_i})\cdot I(S^{-1}\phi_i^-)t^{\omega_i},$$

where t^λ stands for $1\#K^\lambda$.

In order to calculate $\zeta(\phi_i^+)$, suppose that $\phi \in \mathcal{O}_q[G]_{\lambda,\mu}$ satisfies $\phi(ua) = \epsilon(a)\phi(u)$ for all $a \in U^+$ and $u \in U$. Then we have $J(\phi) = l^+(\phi)K^{-\mu}$, and hence

$$\Delta_U(J(\phi)) = l^+(\phi_1)K^{-\mu} \otimes J(\phi_2), \tag{3.7.3}$$

where Δ_U denotes the comultiplication in U. In turn, this implies

$$\zeta(\phi_i^+) = q^{-(\omega_i,\omega_i)}\Big(I(\phi_i^+)\cdot I(S^{-1}\Delta^{\omega_i})t^{\omega_i} + q^{(\alpha_i-\omega_i,\omega_i)} I(\Delta^{[\alpha_i]_-})\cdot I(\Delta^{\omega_i})^{-1}\cdot I(S^{-1}\phi_i^+)t^{\omega_i-\alpha_i}\Big),$$

where $[\alpha_i]_- \in P^+$ is defined by $\alpha_i = 2\omega_i - [\alpha_i]_-$.

Corollary 3.7.8. *We have*

$$\widehat{\zeta}(K^{-2\omega_i}) = q^{-(\omega_i,\omega_i)} I(\Delta^{\omega_i}) \cdot I(S^{-1}\Delta^{\omega_i}) t^{\omega_i},$$
$$\widehat{\zeta}(F_i K^{\alpha_i-2\omega_i}) = q^{-(\omega_i,\omega_i)} q_i^{-1} I(\Delta^{\omega_i}) \cdot I(S^{-1}\phi_i^-) t^{\omega_i},$$
$$\widehat{\zeta}(E_i K^{-2\omega_i}) = q^{-(\omega_i,\omega_i)} \Big(I(\phi_i^+) \cdot I(S^{-1}\Delta^{\omega_i}) t^{\omega_i} + q^{(\alpha_i-\omega_i,\omega_i)} I(\Delta^{[\alpha_i]_-}) \cdot I(\Delta^{\omega_i})^{-1} \cdot I(S^{-1}\phi_i^+) t^{\omega_i-\alpha_i} \Big).$$

An isomorphism between ${}^R\mathcal{O}_q[G]$ and $\mathcal{O}_q[G]$

We now explain how to use the quantum Weyl group to construct an isomorphism between ${}^R\mathcal{O}_q[G]$ and $\mathcal{O}_q[G]$. Recall the element Y defined by (3.5.12). Then the identity (3.5.13) implies the following proposition.

Proposition 3.7.9. *The element Y defines an isomorphism of algebras*

$$\iota_Y \colon \mathcal{O}_q[G] \longrightarrow {}^R\mathcal{O}_q[G], \qquad \phi \longmapsto \langle Y, \phi_1 \rangle \phi_2$$

Proof. Using the relation (3.5.13), we compute

$$\begin{aligned} \iota_Y(\phi) \bullet_R \iota_Y(\psi) &= \langle Y, \phi_1 \rangle \langle Y, \psi_1 \rangle \phi_1 \bullet_R \psi_2 = \langle Y, \phi_1 \rangle \langle Y, \psi_1 \rangle r(\phi_2, \psi_2) \phi_3 \psi_3 \\ &= \langle Y, \phi_1 \rangle \langle Y, \psi_1 \rangle \langle (Y^{-1} \otimes Y^{-1}) \Delta(Y), \phi_2 \otimes \psi_2 \rangle \phi_3 \psi_3 \\ &= \langle \Delta(Y), \phi_1 \otimes \psi_1 \rangle \phi_2 \psi_2 = \langle Y, \phi_1 \psi_1 \rangle \phi_2 \psi_2 = \iota_Y(\phi\psi). \end{aligned}$$

□

Definition 3.7.10. Let θ be the Dynkin diagram automorphism such that $w_0 s_i = s_{\theta(i)} w_0$ holds for all simple reflections s_i.

Using the definition of Y one obtains the following explicit formulas for ι_Y in terms of generalized minors.

Lemma 3.7.11. *One has*

$$\begin{aligned} (\iota_Y)^{-1}(\Delta^{\omega_i}) &= q^{(\omega_i,\rho+\omega_i/2)} \Delta^{\omega_i}_{w_0,1} \\ (\iota_Y)^{-1}(S^{-1}\Delta^{\omega_i}) &= (-1)^{\langle 2\omega_i,\rho^\vee \rangle} q^{(\omega_i,\rho+\omega_i/2)} \Delta^{\omega_{\theta(i)}}_{1,w_0} \\ (q_i^{-1} - q_i)(\iota_Y)^{-1}(\phi_i^+) &= -q^{(\omega_i,\rho+\omega_i/2)} \Delta^{\omega_i}_{w_0 s_i,1} \\ (q_i^{-1} - q_i)(\iota_Y)^{-1}(S^{-1}\phi_i^+) &= (-1)^{\langle 2\omega_i,\rho^\vee \rangle + 1} q_i^{-1} q^{(\omega_i,\rho+\omega_i/2)} \Delta^{\omega_{\theta(i)}}_{1,s_i w_0} \\ (q_i^{-1} - q_i)(\iota_Y)^{-1}(S^{-1}\phi_i^-) &= (-1)^{\langle 2\omega_i,\rho^\vee \rangle} q_i q^{(\omega_i,\rho+\omega_i/2)} \Delta^{\omega_{\theta(i)}}_{s_{\theta(i)},w_0} \end{aligned}$$

Corollary 3.7.12. *The map ι_Y establishes an isomorphism between the localizations*

$$\iota_Y \colon \mathcal{O}_q[G][(\Delta^{\omega_i}_{w_0,1})^{-1}]_{i=1,\dots,r} \longrightarrow {}^R\mathcal{O}_q[G][(\Delta^{\omega_i})^{-1}]_{i=1,\dots,r}.$$

As explained in [1], the algebra $\mathcal{O}_q[G][(\Delta^{\omega_i}_{w_0,1})^{-1}]_{i=1,\dots,r}$ can be regarded as the quantum coordinate ring $\mathcal{O}_q[G^{w_0}]$ of the big open Bruhat cell $G^{w_0} = B_+ w_0 B_+ \subset G$.

3.8 Main results

Let us introduce the notation

$$\mathcal{O}_q[G^{w_0}/H] \overset{\text{def}}{=} \left\{\phi \in \mathcal{O}_q[G^{w_0}] \mid (K^\lambda \otimes 1)\cdot\phi = \phi \quad \text{for any} \quad \lambda \in P\right\}$$

for the subalgebra of U_0-invariants in $\mathcal{O}_q[G^{w_0}]$ under the coregular action defined by (3.5.7). By Corollary 3.7.7 and Corollary 3.7.12 the map

$$(\iota_Y^{-1} \circ I^{-1}) \otimes \mathrm{id}\colon \mathcal{H}_q^{\mathbb{T}} \otimes T \simeq \mathcal{H}_q^{T-} \longrightarrow \mathcal{O}_q[G^{w_0}/H] \otimes T \tag{3.8.1}$$

is an isomorphism of algebras. Combining this isomorphism with Corollary 3.5.14, we arrive at

Theorem 3.8.1. *The map* Φ *obtained by composing the homomorphism* $\widetilde{\zeta}$ *defined in* (3.6.4) *with the isomorphism* (3.8.1) *is an embedding of algebras*

$$\Phi\colon F_l(U) \otimes_Z U_0 \longrightarrow \mathcal{O}_q[G^{w_0}/H] \otimes T.$$

Remark 3.8.2. Note that by Corollary 3.7.8, in order to extend the homomorphism Φ to the entire quantum group $U_q(\mathfrak{g})$, we must localize further by inverting the products $\Delta^{\omega_i}_{w_0,1}\Delta^{\omega_{\theta(i)}}_{1,w_0}$ for all $i = 1, \cdots r$. Hence the target of the homomorphism becomes $\mathcal{O}_q[G^{w_0,w_0}/H]$, the quantum coordinate ring of the reduced big double Bruhat cell in G. In fact, we must also adjoin the square roots $\left(\Delta^{\omega_i}_{w_0,1}\Delta^{\omega_{\theta(i)}}_{1,w_0}\right)^{1/2}$ to $\mathcal{O}_q[G^{w_0,w_0}/H]$, although this poses no difficulties. This phenomenon is related to the fact that the maps $\eta_i\colon G^* \to G_*$ in (**??**), (**??**), while local diffeomorphisms, are in fact 2^r-fold coverings.

Notation 3.8.3. $\mathcal{O}'_q[G^{w_0,w_0}/H]$ denotes the algebra obtained by adjoining $\left(\Delta^{\omega_i}_{w_0,1}\Delta^{\omega_{\theta(i)}}_{1,w_0}\right)^{1/2}$ for $i = 1, \dots, r$ to $\mathcal{O}_q[G^{w_0,w_0}/H]$. Similarly, T' stands for $\mathbb{C}[P/2] \supset T$.

Corollary 3.8.4. *Let* $\chi\colon T \to \mathbb{C}$ *be a character of the torus* T. *Denote by the same letter the induced character of the center* $Z \subset U_q(\mathfrak{g})$ *coming from the embedding* $\widehat{\xi}|_Z\colon Z \hookrightarrow T$. *Then* Φ *extends to an embedding* $\Phi'\colon U_q(\mathfrak{g}) \to \mathcal{O}'_q[G^{w_0,w_0}/H] \otimes T'$ *such that the following diagram commutes*

$$\begin{array}{ccc} U_q(\mathfrak{g}) & \xrightarrow{\Phi'} & \mathcal{O}'_q[G^{w_0,w_0}/H] \otimes T' \\ \downarrow & & \downarrow {\scriptstyle \mathrm{id}\otimes\chi} \\ U_q(\mathfrak{g})/\mathcal{I}_\chi & \longrightarrow & \mathcal{O}'_q[G^{w_0,w_0}/H] \end{array}$$

where $\mathcal{I}_\chi$ *is the ideal generated by the kernel of* χ.

Corollary 3.8.5. *One has the following explicit formulas for* Φ'

$$\Phi'(K^{-2\omega_i}) = (-1)^{\langle 2\omega_i, \rho^\vee\rangle} q^{2(\omega_i,\rho)} \Delta^{\omega_i}_{w_0,1}\Delta^{\omega_{\theta(i)}}_{1,w_0} t^{\omega_i},$$

$$\Phi'(\widehat{F}_i K^{\alpha_i}) = q_i \Delta^{\omega_{\theta(i)}}_{s_{\theta(i)},w_0} \left(\Delta^{\omega_{\theta(i)}}_{1,w_0}\right)^{-1},$$

$$\Phi'(\widehat{E}_i) = -\left(\Delta^{\omega_i}_{w_0 s_i,1}\left(\Delta^{\omega_i}_{w_0,1}\right)^{-1} + q_i^{-1}\Delta^{[\alpha_i]_-}_{w_0,1}\Delta^{\omega_{\theta(i)}}_{1,w_0 s_{\theta(i)}}\left(\Delta^{\omega_{\theta(i)}}_{1,w_0}\right)^{-1}\left(\Delta^{\omega_i}_{w_0,1}\right)^{-2} t^{-\alpha_i}\right),$$

where $\widehat{E}_i = (q_i^{-1} - q_i)E_i$ *and* $\widehat{F}_i = (q_i^{-1} - q_i)F_i$.

Proof. This follows from combining Corollary 3.7.8 with Lemma 3.7.11. □

Remark 3.8.6. As mentioned in the Introduction, the Feigin homomorphisms [13] allow us to further explicitize our formulas for Φ' and compare our realization of $U_q(\mathfrak{g})$ to those of [30, 53]. While we save the full details of this comparision for the forthcoming paper [26], let us present the essential idea in the case $\mathfrak{g} = \mathfrak{sl}_n$. In [53], the quantized moduli space of decorated PGL_n-local systems on a punctured disk with two boundary marked points was used to construct an embedding of $U_q(\mathfrak{g})$ into a quantum torus. That quantum torus arises as a quantum cluster chart on the moduli space, corresponding to an ideal triangulation of the punctured disk in which two triangles are glued by two sides. In [31], it was checked that this geometric approach reproduces the embedding of [30]. On the other hand, one can consider the quantum cluster chart corresponding to a *self-folded* triangulation of the punctured disk; this quantum torus is related to the original one by an explicit sequence of $\binom{n+1}{3}$ quantum cluster mutations. Now recall that to specify a Feigin homomorphism amounts to picking a pair of reduced expressions i_1, i_2 for the longest element w_0 of the Weyl group; for an appropriate choice of i_1, i_2 one verifies that Φ' coincides with the embedding of $U_q(\mathfrak{g})$ into the quantum cluster chart corresponding to the self-folded triangulation. Thus the quantum torus realization of $U_q(\mathfrak{g})$ presented here is mutation equivalent to those of [30, 53].

We end this section with the following conjecture:

Conjecture 3.8.7. *We have an isomorphism of non-commutative fraction fields*

$$\mathrm{Frac}\,(F_l(U) \otimes_Z U_0) = \mathrm{Frac}\,(\mathcal{O}_q[G/H] \otimes T)\,.$$

In particular, $\mathrm{Frac}\,(F_l(U) \otimes_Z U_0)$ *coincides with a non-commutative fraction field of a quantum torus algebra, i.e. the quantum Gelfand-Kirillov property holds for* $F_l(U) \otimes_Z U_0$.

3.9 Example for $\mathfrak{g} = \mathfrak{sl}_2$

We conclude by providing a detailed example of our construction for the case $\mathfrak{g} = \mathfrak{sl}_2$. Let us write $E, F, K^{1/2}$ for the generators of the simply-connected form of $U_q(\mathfrak{sl}_2)$. Recall that the fundamental representation of $U_q(\mathfrak{sl}_2)$ on $\mathbb{C}^2$ is determined by

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K^{1/2} \mapsto \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

The Hopf algebra $\mathcal{O}_q(SL_2)$ is generated by the matrix coefficients of the fundamental representation. More explicitly, $\mathcal{O}_q(SL_2)$ has generators $\langle x_{11}, x_{12}, x_{21}, x_{22} \rangle$ subject to the relations

$$\begin{aligned} x_{11}x_{12} &= qx_{12}x_{11} & x_{12}x_{22} &= qx_{22}x_{12} & x_{12}x_{21} &= x_{21}x_{12} \\ x_{11}x_{21} &= qx_{21}x_{11} & x_{21}x_{22} &= qx_{22}x_{21} & [x_{11}, x_{22}] &= (q - q^{-1})x_{12}x_{21} \end{aligned}$$

as well as the quantum determinant relation

$$x_{11}x_{22} - qx_{12}x_{21} = 1.$$

The coalgebra structure of $\mathcal{O}_q(SL_2)$ is given by

$$\Delta(x_{ij}) = x_{i1} \otimes x_{1j} + x_{i2} \otimes x_{2j} \qquad \text{and} \qquad \epsilon(x_{ij}) = \delta_{ij}$$

while the antipode is given by

$$S(x_{11}) = x_{22}, \qquad S(x_{12}) = -q^{-1}x_{12}, \qquad S(x_{21}) = -qx_{21}, \qquad S(x_{22}) = x_{11}.$$

The quantum coordinate ring of the big Bruhat cell $Bw_0B \subset SL_2$ is

$$\mathcal{O}_q[SL_2^{w_0}] = \mathcal{O}_q[SL_2][x_{21}^{-1}]$$

while the quantum coordinate ring of the big double Bruhat cell $Bw_0B \cap B_-w_0B_- \subset SL_2$ is given by

$$\mathcal{O}_q[SL_2^{w_0,w_0}] = \mathcal{O}_q[SL_2][x_{12}^{-1}x_{21}^{-1}].$$

The quantum coordinate ring of the reduced big double Bruhat cell $\mathcal{O}_q[SL_2^{w_0,w_0}/H] \otimes T$ embeds into the quantum torus algebra

$$\mathcal{A} = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}, z^{\pm 1}\rangle \big/ (uv = q^2vu, zu = uz, zv = vz)$$

via the identification

$$u = -q^2x_{22}x_{21}, \qquad v = -q^{-1}x_{12}^{-1}x_{21}^{-1}, \qquad z = t.$$

As in Corollary 3.8.5, we introduce the normalized generators of $U_q(\mathfrak{sl}_2)$

$$\widehat{E} = (q^{-1} - q)E \qquad \text{and} \qquad \widehat{F} = (q^{-1} - q)F.$$

Then the values of the l-operators on the matrix coefficients x_{ij} are easily computed to be

$$\begin{aligned} l^+(x_{11}) &= K^{-1/2} & \qquad {}'l^-(x_{11}) &= K^{-1/2} \\ l^+(x_{12}) &= 0 & {}'l^-(x_{12}) &= \widehat{F}K^{1/2} \\ l^+(x_{21}) &= \widehat{E}K^{-1/2} & {}'l^-(x_{21}) &= 0 \\ l^+(x_{22}) &= K^{1/2} & {}'l^-(x_{22}) &= K^{1/2} \end{aligned}$$

It follows that the isomorphism $J\colon \mathcal{O}_q(SL_2) \longrightarrow F_l(U_q(\mathfrak{sl}_2))$ is given by

$$J(x_{11}) = K^{-1}, \qquad J(x_{12}) = q\widehat{F}, \qquad J(x_{21}) = \widehat{E}K^{-1}, \qquad J(x_{22}) = K + q\widehat{E}\widehat{F}.$$

The homomorphism $\zeta\colon \mathcal{O}_q(SL_2) \longrightarrow \mathcal{H}_q^{T-}$ takes the form

$$\begin{aligned}
\zeta(x_{11}) &= \big(1 + q^{-1}\widehat{E}K^{-1}\#\widehat{F}\big)\cdot t\\
\zeta(x_{12}) &= -q\big(K^{-1}\#\widehat{F}\big)\cdot t\\
\zeta(x_{21}) &= \big(\widehat{E}\#1 + q^{-1}\widehat{E}^2K^{-1}\#\widehat{F}\big)\cdot t - \big(\widehat{E}\#1\big)\cdot t^{-1}\\
\zeta(x_{22}) &= -q\big(\widehat{E}K^{-1}\#\widehat{F}\big)\cdot t + t^{-1}
\end{aligned}$$

where we write $t = 1\#K^{1/2} \in \mathcal{H}_q^{T-}$.

The isomorphism $\iota_Y\colon \mathcal{O}_q(SL_2) \longrightarrow {}^R\mathcal{O}_q(SL_2)$ is given by

$$\begin{aligned}
\iota_Y(x_{11}) &= -q^{-3/4}x_{21}, & \iota_Y(x_{21}) &= q^{-3/4}x_{11},\\
\iota_Y(x_{12}) &= -q^{-3/4}x_{22}, & \iota_Y(x_{22}) &= q^{-3/4}x_{12}.
\end{aligned}$$

and the isomorphism $I\colon \mathcal{O}_q^R(SL_2)[\Delta_i^{-1}]_{i=1}^r \longrightarrow \mathcal{H}_q^{T_c}$ is

$$\begin{aligned}
I(x_{11}) &= K^{-1/2}\#K^{-1/2} & I(x_{21}) &= \widehat{E}K^{-1/2}\#K^{-1/2}\\
I(x_{12}) &= K^{-1/2}\#\widehat{F}K^{1/2} & I(x_{22}) &= K^{1/2}\#K^{1/2} + \widehat{E}K^{-1/2}\#\widehat{F}K^{1/2}
\end{aligned}$$

The algebra embedding $\Phi\colon F_l(U_q(\mathfrak{sl}_2)) \longrightarrow \mathcal{O}_q[SL_2^{w_0}/H] \otimes T$ in Theorem 4.6.1 takes the form

$$\begin{aligned}
K^{-1} &\mapsto -qx_{12}x_{21}t,\\
\widehat{F} &\mapsto -q^2x_{22}x_{21}t,\\
\widehat{E}K^{-1} &\mapsto qx_{11}x_{12}t + x_{11}x_{21}^{-1}t^{-1}.
\end{aligned} \tag{3.9.1}$$

As explained in Remark 3.8.2, in order to embed $U_q(\mathfrak{sl}_2)$ we must localize further at $x_{12}x_{21}$ and adjoin $(x_{12}x_{21})^{1/2}$. Therefore let $\mathcal{A}'$ be the quantum torus algebra obtained from $\mathcal{A}$ by adjoining the elements $v^{1/2}$ and $z^{1/2}$. Then we obtain the following quantum torus algebra realization of $U_q(\mathfrak{sl}_2)$:

$$\Phi'\colon U_q(\mathfrak{sl}_2) \longrightarrow \mathcal{A}'$$

$$K^{1/2} \mapsto v^{1/2}z^{-1/2}, \qquad \widehat{F} \mapsto uz, \qquad \widehat{E} \mapsto z^{-1}u^{-1}(qv^{1/2} - q^{-1}v^{-1/2})(v^{-1/2}z - v^{1/2}z^{-1}).$$

Chapter 4

Quantum character varieties, cluster algebras and quantum groups

4.1 Quantum cluster $\mathcal{X}$-tori

In this section we recall a few basic facts about cluster $\mathcal{X}$-tori and their quantization following [63]. We shall need only skew-symmetric exchange matrices, and we incorporate this in the definition of a cluster seed.

A *seed* $\mathbf{i}$ is triple (I, I_0, ε) where I is a finite set, $I_0 \subset I$ is a subset and $\varepsilon = (\varepsilon_{ij})_{i,j \in I}$ is a skew-symmetric[1] $\mathbb{Q}$-valued matrix, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$. To a seed $\mathbf{i}$ we associate an algebraic torus $\mathcal{X}_{\mathbf{i}} = (\mathbb{C}^\times)^{|I|}$, equipped with a set of coordinates $\{X_1, \ldots, X_{|I|}\}$ and a Poisson structure defined by

$$\{X_i, X_j\} = 2\varepsilon_{ij} X_i X_j, \qquad i, j \in I.$$

We refer to the torus $\mathcal{X}_{\mathbf{i}}$ as the *cluster torus* and to the matrix ε as the *exchange matrix.* The coordinates X_i are called *cluster variables* and they are said to be *frozen* if $i \in I_0$.

Given a pair of seeds $\mathbf{i} = (I, I_0, \varepsilon)$, $\mathbf{i}' = (I', I_0', \varepsilon')$, and an element $k \in I \setminus I_0$ we say that an isomorphism $\mu_k \colon I \to I'$ is a *cluster mutation in direction* k if $\mu_k(I_0) = I_0'$ and

$$\varepsilon'_{\mu_k(i),\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leqslant 0, \\ \varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0. \end{cases} \tag{4.1.1}$$

A mutation μ_k induces an isomorphism of cluster tori $\mu_k^* \colon \mathcal{X}_{\mathbf{i}} \to \mathcal{X}_{\mathbf{i}'}$ as follows:

$$\mu_k^* X_{\mu_k(i)} = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i \left(1 + X_k^{-\operatorname{sgn}(\varepsilon_{ki})}\right)^{-\varepsilon_{ki}} & \text{if } i \neq k. \end{cases}$$

[1] in general, the matrix ε is allowed to be skew-symmetrizable.

Note that the data of a cluster seed can be conveniently encoded by a quiver with vertices $\{v_i\}$ labelled by elements of the set I and with adjacency matrix ε. The arrows $v_i \to v_j$ between a pair of frozen variables are considered to be weighted by ε_{ij}. Then the mutation μ_k of the corresponding quiver can be performed in three steps:

1. reverse all the arrows incident to the vertex k;
2. for each pair of arrows $k \to i$ and $j \to k$ draw an arrow $i \to j$;
3. delete pairs of arrows $i \to j$ and $j \to i$ going in the opposite directions.

The algebra of functions $\mathcal{O}(\mathcal{X}_{\mathbf{i}})$ admits a quantization $\mathcal{X}_{\mathbf{i}}^q$ called the *quantum torus algebra* associated to the seed $\mathbf{i}$. It is an associative algebra over $\mathbb{C}(q)$ defined by generators $X_i^{\pm 1}$, $i \in I$ subject to relations

$$X_i X_j = q^{2\varepsilon_{ji}} X_j X_i.$$

The cluster mutation in the direction k induces an automorphism μ_k^q of $\mathcal{X}_{\mathbf{i}}^q$ called the *quantum cluster mutation*, defined by

$$\mu_k^q(X_i) = \begin{cases} X_k^{-1}, & \text{if } i = k, \\ X_i \prod_{r=1}^{\varepsilon_{ki}} \left(1 + q^{2r-1} X_k^{-1}\right)^{-1}, & \text{if } i \neq k \text{ and } \varepsilon_{ki} \geqslant 0, \\ X_i \prod_{r=1}^{-\varepsilon_{ki}} \left(1 + q^{2r-1} X_k\right), & \text{if } i \neq k \text{ and } \varepsilon_{ki} \leqslant 0. \end{cases}$$

The quantum cluster mutation μ_k^q can be written as a composition of two homomorphisms, namely

$$\mu_k^q = \mu_k^\sharp \circ \mu_k'$$

where μ_k' is a monomial transformation defined by

$$X_i \longmapsto \begin{cases} X_k^{-1}, & \text{if } i = k, \\ q^{\varepsilon_{ik}\varepsilon_{ki}} X_i X_k^{\varepsilon_{ki}}, & \text{if } i \neq k \text{ and } \varepsilon_{ki} \geqslant 0, \\ X_i, & \text{if } i \neq k \text{ and } \varepsilon_{ki} \leqslant 0. \end{cases}$$

and

$$\mu_k^\sharp = \mathrm{Ad}_{\Psi^q(X_k)}$$

is a conjugation by the quantum dilogarithm function

$$\Psi^q(x) = \frac{1}{(1+qx)(1+q^3x)\dots}.$$

Mutation of the exchange matrix is incorporated into the monomial transformation μ_k'. The following lemma will prove very useful.

Lemma 4.1.1. *A sequence of mutations* $\mu^q_{i_k} \dots \mu^q_{i_1}$ *can be written as follows*

$$\mu^q_{i_k} \dots \mu^q_{i_1} = \Phi_k \circ \mathrm{M}_k$$

where

$$\Phi_k = \mathrm{Ad}_{\Psi^q\left(X_{i_1}\right)} \mathrm{Ad}_{\Psi^q\left(\mu'_{i_1}(X_{i_2})\right)} \dots \mathrm{Ad}_{\Psi^q\left(\mu'_{i_{k-1}} \dots \mu'_{i_1}(X_{i_k})\right)}$$

and

$$\mathrm{M}_k = \mu'_{i_k} \dots \mu'_{i_2} \mu'_{i_1}.$$

Proof. We shall prove the lemma by induction. Assume the statement holds for some $k = r-1$. Then

$$\mu^q_{i_r} \dots \mu^q_{i_1} = \mathrm{Ad}_{\Psi^q(\Phi_{r-1}(\mathrm{M}_{r-1}(X_{i_r})))}\, \mu'_{i_r} \Phi_{r-1} \mathrm{M}_{r-1}.$$

Now the proof follows from the fact that the homomorphisms μ'_{i_r} and Φ_{r-1} commute and the following relation:

$$\mathrm{Ad}_{\Psi^q(\Phi_{r-1}(\mathrm{M}_{r-1}(X_{i_r})))}\, \Phi_{r-1} = \Phi_{r-1}\, \mathrm{Ad}_{\Psi^q(\mathrm{M}_{r-1}(X_{i_r}))} = \Phi_r.$$

□

We conclude this section with the two properties of the quantum dilogarithm which we will use liberally throughout the paper. For any u and v such that $uv = q^{-2}vu$ we have

$$\Psi^q(u)\Psi^q(v) = \Psi^q(uv) \tag{4.1.2}$$

$$\Psi^q(v)\Psi^q(u) = \Psi^q(u)\Psi^q(qvu)\Psi^q(v) \tag{4.1.3}$$

The first equality is nothing but a q-analogue of the addition law for exponentials, while the second one is known as the *pentagon identity*.

4.2 Quantum character varieties

We now recall some elements of the theory of quantum character varieties as defined in [63]. Let $\widehat{S}$ be a decorated surface — that is, a topological surface S with boundary ∂S, equipped with a finite collection of marked points $x_1, \dots, x_r \in \partial S$ and punctures $p_1, \dots, p_s$. In [63], the moduli space $\mathcal{X}_{\widehat{S}, PGL_m}$ of PGL_m-local systems on S with reductions to Borel subgroups at each marked point x_i and each puncture p_i, was defined and shown to admit the structure of a cluster $\mathcal{X}$-variety. In particular, suppose that T is an ideal triangulation of S: recall that this means that all vertices of T are at either marked points or punctures. Then it was shown in [63] that for each such ideal triangulation, one can produce a cluster $\mathcal{X}$-chart on $\mathcal{X}_{\widehat{S}, PGL_m}$. Moreover, the Poisson algebra of functions on such a chart admits a canonical quantization, whose construction we shall now recall.

The first step is to describe the quantum cluster $\mathcal{X}$-chart associated to a single triangle. To do this, consider a triangle ABC given by the equation $x + y + z = m$, $x, y, z \geqslant 0$ and intersect it with lines $x = p$, $y = p$, and $z = p$ for all $0 < p < m$, $p \in \mathbb{Z}$. The resulting picture is called the m-triangulation of the triangle ABC. Let us now color the triangles of the m-triangulation in black and white, as in Figure 4.1 so that triangles adjacent to vertices A, B, or C are black, and two triangles sharing an edge are of different color. We shall also orient the edges of white triangles counterclockwise. Finally, we connect the vertices of the m-triangulation lying on the same side of the triangle ABC by dashed arrows in the clockwise direction. The resulting graph is shown in Figure 4.1. Note that the vertices on the boundary of ABC are depicted by squares. Throughout the text we will use square vertices for frozen variables. All dashed arrows will be of weight $\frac{1}{2}$, that is a dashed arrow $v_i \to v_j$ denotes the commutation relation $X_i X_j = q^{-1} X_j X_i$.

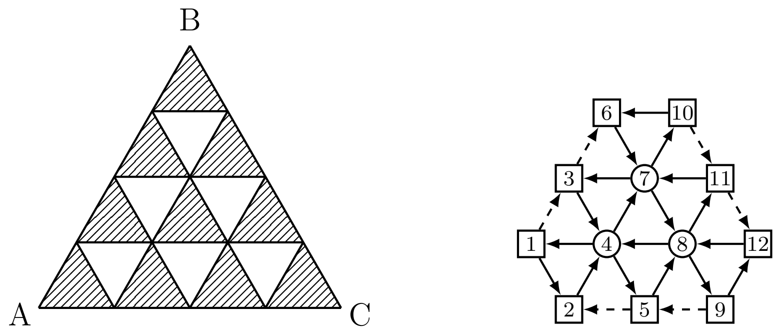

Figure 4.1: Cluster $\mathcal{X}$-coordinates on the configuration space of 3 flags and 3 lines.

Now, let us recall the procedure of *amalgamating* two quivers by a subset of frozen variables, following [62]. In simple words, amalgamation is nothing but the gluing of two quivers by a number of frozen vertices. More formally, let Q_1, Q_2 be a pair of quivers, and I_1, I_2 be certain subsets of frozen variables in Q_1, Q_2 respectively. Assuming there exists a bijection ϕ: $I_1 \to I_2$ we can amalgamate quivers Q_1 and Q_2 by the subsets I_1, I_2 along ϕ. The result is a new quiver Q constructed in the following two steps:

1. for any $i \in I_1$ identify vertices $v_i \in Q_1$ and $v_{\phi(i)} \in Q_2$ in the union $Q_1 \sqcup Q_2$;

2. for any pair $i, j \in I_1$ with an arrow $v_i \to v_j$ in Q_1 labelled by ε_{ij} and an arrow $v_{\phi(i)} \to v_{\phi(j)}$ in Q_2 labelled by $\varepsilon_{\phi(i),\phi(j)}$, label the arrow between corresponding vertices in Q by $\varepsilon_{ij} + \varepsilon_{\phi(i),\phi(j)}$

Amalgamation of a pair of quivers Q_1, Q_2 into a quiver Q induces an embedding $\mathcal{X} \to \mathcal{X}_1 \otimes \mathcal{X}_2$ of the corresponding cluster $\mathcal{X}$-tori:

$$X_i \mapsto \begin{cases} X_i \otimes 1, & \text{if } i \in Q_1 \setminus I_1, \\ 1 \otimes X_i, & \text{if } i \in Q_2 \setminus I_2, \\ X_i \otimes X_{\phi(i)}, & \text{otherwise.} \end{cases}$$

An example of amalgamation is shown in Figure 4.2. There, the left quiver is obtained by amalgamating a triangle ABC from Figure 4.1 with a similar triangle along the side BC (or more precisely, along frozen vertices 10, 11, and 12 on the edge BC). Another example is shown in Figure 4.7 where a triangle ABC is now amalgamated by 2 sides. Finally, the process of amalgamation is best shown in Figure 4.6.

As explained in [63], in order to construct the cluster $\mathcal{X}$-coordinate chart on $\mathcal{X}_{\widehat{S},PGL_m}$ corresponding to an ideal triangulation T of $\widehat{S}$, one performs the following procedure:

1. m-triangulate each of the ideal triangles in T;
2. for any pair of ideal triangles in T sharing an edge, amalgamate the corresponding pair of quivers by this edge.

In general, different ideal triangulations of an $\widehat{S}$ result in different quivers, and hence different cluster $\mathcal{X}$-tori. However, any triangulation can be transformed into any other by a sequence of *flips* that replace one diagonal in an ideal 4-gon with the other one. Each flip corresponds to the following sequence of cluster mutations that we shall recall on the example shown in Figure 4.2. There, a flip is obtained in three steps. First, mutate at vertices 10, 11, 12, second, mutate at vertices 7, 8, 14, 15, and third, mutate at vertices 4, 11, 18. Note, that the order of mutations within one step does not matter. In general, a flip in an m-triangulated 4-gon consists of $m-1$ steps. On the i-th step, one should do the following. First, inscribe an i-by-$(m-i)$ rectangle in the 4-gon, such that vertices of the rectangle coincide with boundary vertices of the m-triangulation and the side of the rectangle of length $m-i$ goes along the diagonal of a 4-gon. Second, divide the rectangle into $i(m-i)$ squares and mutate at the center of each square. As in the example, the order of mutations within a single step does not matter.

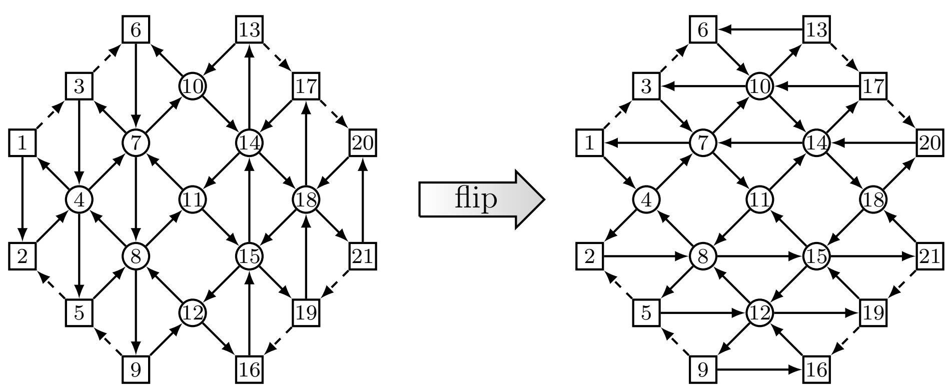

Figure 4.2: A pair of triangles amalgamated by 1 side.

4.3 Quantum groups

In what follows, we consider the Lie algebra $\mathfrak{sl}_{n+1} = \mathfrak{sl}_{n+1}(\mathbb{C})$ equipped with a pair of opposite Borel subalgebras $\mathfrak{b}_\pm$ and a Cartan subalgebra $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$. The corresponding root system Δ is equipped with a polarization $\Delta = \Delta_+ \sqcup \Delta_-$, consistent with the choice of Borel subalgebras $\mathfrak{b}_\pm$, and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\} \subset \Delta_+$. We denote by $(\cdot,\cdot)$ the unique symmetric bilinear form on $\mathfrak{h}^*$ invariant under the Weyl group W, such that $(\alpha,\alpha) = 2$ for all roots $\alpha \in \Delta$. Entries of the Cartan matrix are denoted $a_{ij} = (\alpha_i, \alpha_j)$.

Let q be a formal parameter, and consider an associative $\mathbb{C}(q)$-algebra $\mathfrak{D}_n$ generated by elements

$$\{E_i, F_i, K_i, K_i' \,|\, i = 1, \ldots, n\},$$

subject to the relations

$$\begin{aligned} K_iE_j &= q^{a_{ij}}E_jK_i, & K_i'E_j &= q^{-a_{ij}}E_jK_i', & K_iK_j &= K_jK_i, \\ K_iF_j &= q^{-a_{ij}}F_jK_i, & K_i'F_j &= q^{a_{ij}}F_jK_i', & K_iK_j' &= K_j'K_i, \end{aligned} \tag{4.3.1}$$

the relation

$$[E_i, F_j] = \delta_{ij}\left(q - q^{-1}\right)(K_i - K_i'), \tag{4.3.2}$$

and the quantum Serre relations

$$\begin{aligned} &E_i^2E_{i\pm1} - (q+q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0, \\ &F_i^2F_{i\pm1} - (q+q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0, \\ &[E_i, E_j] = [F_i, F_j] = 0 \quad \text{if} \quad |i-j| > 1. \end{aligned} \tag{4.3.3}$$

The algebra $\mathfrak{D}_n$ is a Hopf algebra, with the comultiplication

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(F_i) &= F_i \otimes K_i' + 1 \otimes F_i, & \Delta(K_i') &= K_i' \otimes K_i', \end{aligned}$$

the antipode

$$\begin{aligned} S(E_i) &= -K_i^{-1}E_i, & S(K_i) &= K_i^{-1}, \\ S(F_i) &= -F_iK_i, & S(K_i') &= (K_i')^{-1}, \end{aligned}$$

and the counit

$$\epsilon(K_i) = \epsilon(K_i') = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The quantum group $U_q(\mathfrak{sl}_{n+1})$ is defined as the quotient

$$U_q(\mathfrak{sl}_{n+1}) = \mathfrak{D}_n / \langle K_iK_i' = 1 \,|\, i = 1, \ldots, n\rangle.$$

Note that the quantum group $U_q(\mathfrak{sl}_{n+1})$ inherits a well-defined Hopf algebra structure from $\mathfrak{D}_n$. The subalgebra $U_q(\mathfrak{b}) \subset \mathfrak{D}_n$ generated by all K_i, E_i is a Hopf subalgebra in $\mathfrak{D}_n$. The algebra $U_q(\mathfrak{b})$ is isomorphic to its image under the projection onto $U_q(\mathfrak{sl}_{n+1})$ and is called the quantum Borel subalgebra of $U_q(\mathfrak{sl}_{n+1})$. Note that $\mathfrak{D}_n$ is nothing but the Drinfeld double of $U_q(\mathfrak{b})$.

Let us fix a *normal ordering* $\prec$ on Δ_+, that is a total ordering such that $\alpha \prec \alpha+\beta \prec \beta$ for any $\alpha, \beta \in \Delta_+$. We set $E_{\alpha_i} = E_i$, $F_{\alpha_i} = F_i$, and define inductively

$$E_{\alpha+\beta} = \frac{E_\alpha E_\beta - q^{-(\alpha,\beta)} E_\beta E_\alpha}{q - q^{-1}}, \tag{4.3.4}$$

$$F_{\alpha+\beta} = \frac{F_\beta F_\alpha - q^{(\alpha,\beta)} F_\alpha F_\beta}{q - q^{-1}}. \tag{4.3.5}$$

Then the set of all normally ordered monomials in K_α, K'_α, E_α, and F_α for $\alpha \in \Delta_+$ forms a Poincaré-Birkhoff-Witt (PBW) basis for $\mathfrak{D}_n$ as a $\mathbb{C}(q)$-module. In what follows, we denote

$$E_{ij} = E_{\alpha_i+\alpha_{i+1}+\ldots+\alpha_j} \qquad \text{and} \qquad F_{ij} = F_{\alpha_i+\alpha_{i+1}+\ldots+\alpha_j}.$$

Finally, let us introduce for future reference the automorphism θ of the Dynkin diagram of $U_q(\mathfrak{sl}_{n+1})$ defined by

$$\theta(i) = n+1-i, \qquad 1 \le i \le n. \tag{4.3.6}$$

4.4 An embedding of $U_q(\mathfrak{sl}_{n+1})$

Let us now explain how to embed $U_q(\mathfrak{sl}_{n+1})$ into a quantum cluster $\mathcal{X}$-chart on the quantum character variety of decorated PGL_{n+1}-local systems on an disk $\widehat{S}$ with a single puncture p, and with two marked points x_1, x_2 on its boundary.

We consider the ideal triangulation of $\widehat{S}$ in which we take the pair of triangles from Figure 4.1 and amalgamate them by two sides as in Figure 4.6. The resulting quiver is shown on Figure 4.3. Note that the vertices in the central column used to be frozen before amalgamation. We shall refer to this quiver as the $\mathcal{D}_n$*-quiver* and denote the corresponding quantum torus algebra by $\mathcal{D}_n$. The $\mathcal{D}_n$-quivers for $n = 1$, 2, and 3 are shown on Figures 4.4, 4.5, and 4.7 respectively.

Let us explain our convention for labelling the vertices of the $\mathcal{D}_n$-quiver. We denote frozen vertices in the left column by $\mathrm{V}_{i,-i}$ with $i = 1, \ldots, n$ counting South to North. Now, choose a frozen vertex $\mathrm{V}_{i,-i}$ and follow the arrows in the South-East direction until you hit one of the vertices in the central column. Each vertex along the way is labelled by $\mathrm{V}_{i,r}$, $r = -i, \ldots, 0$. Then, start from the central vertex $\mathrm{V}_{i,0}$ and follow arrows in the North-East direction labelling vertices $\mathrm{V}_{i,r}$, $r = 0, \ldots, i$, on your way until you hit a frozen vertex in the right column, which receives the label $\mathrm{V}_{i,i}$. This way we label all the vertices except for the upper half of those in the central column. Now, let us rotate the $\mathcal{D}_n$-quiver by 180°, and label the image of the vertex $\mathrm{V}_{i,r}$ by $\Lambda_{i,r}$. Now, we have labelled every vertex twice by some

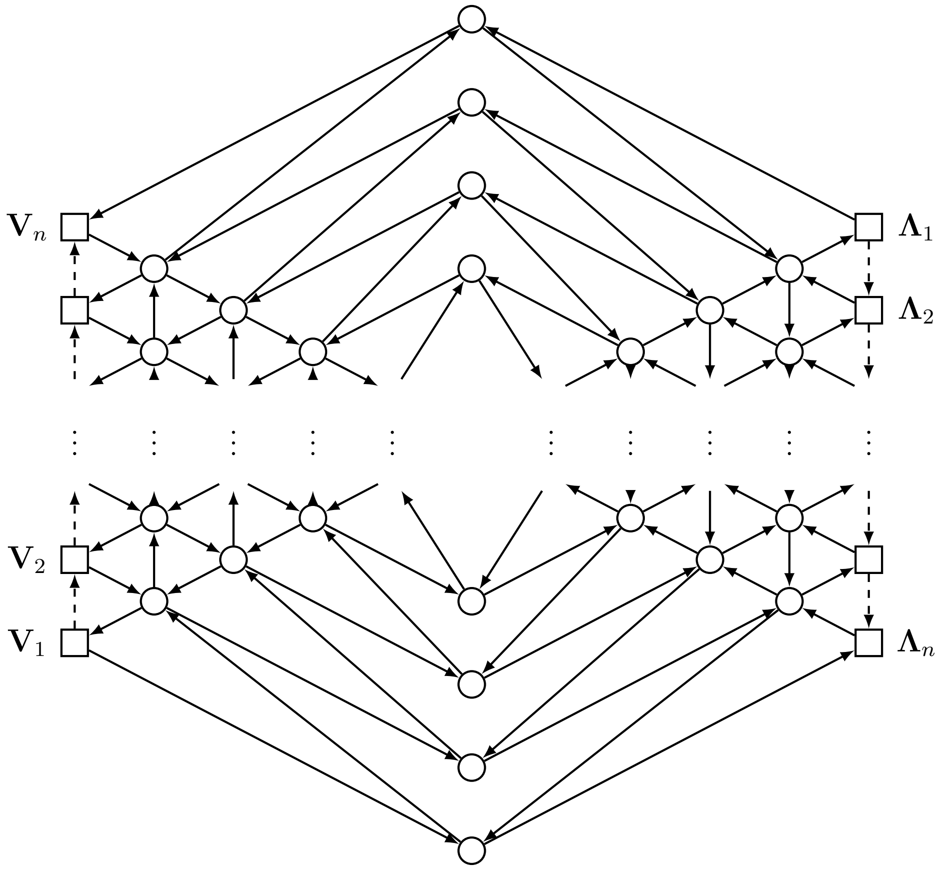

Figure 4.3: $\mathcal{D}_n$-quiver.

V and some Λ except for those in the central column. This way to label vertices, although redundant, will prove very convenient in the sequel. The following relation is easy to verify:

$$\mathrm{V}_{i,\pm r} = \Lambda_{\theta(\pm r),\mp\theta(i)}, \qquad 1 \leqslant r \leqslant i \leqslant n.$$

In the above formula, θ denotes the diagram automorphism defined in (4.3.6). Finally, we refer to the subset of vertices $\{\mathrm{V}_{i,r} \mid -i \leqslant r < i\}$ as the V_i*-path*. Similarly, the Λ_i*-path* is $\{\Lambda_{i,r} \mid -i \leqslant r < i\}$.

Example 4.4.1. Let us refer to the i-th vertex in Figure 4.4 by X_i. Then the labelling suggested above is as follows:

$$\mathrm{V}_{1,-1} = X_1, \qquad \mathrm{V}_{1,0} = X_2, \qquad \mathrm{V}_{1,1} = X_3,$$
$$\Lambda_{1,-1} = X_3, \qquad \Lambda_{1,0} = X_4, \qquad \Lambda_{1,1} = X_1.$$

Example 4.4.2. Similarly, we refer to the i-th vertex in Figure 4.5 by X_i. Then, one has

$$\begin{array}{llll} \mathrm{V}_{1,-1} = X_1, & \mathrm{V}_{1,0} = X_2, & \mathrm{V}_{1,1} = X_3, & \mathrm{V}_{2,-2} = X_4, \\ \mathrm{V}_{2,-1} = X_5, & \mathrm{V}_{2,0} = X_6, & \mathrm{V}_{2,1} = X_7, & \mathrm{V}_{2,2} = X_8, \\ \Lambda_{1,-1} = X_8, & \Lambda_{1,0} = X_9, & \Lambda_{1,1} = X_4, & \Lambda_{2,-2} = X_3, \\ \Lambda_{2,-1} = X_7, & \Lambda_{2,0} = X_{10}, & \Lambda_{2,1} = X_5, & \Lambda_{2,2} = X_1. \end{array}$$

Remark 4.4.3. As shown in [25], for any semisimple Lie algebra $\mathfrak{g}$ the algebra $U_q(\mathfrak{g})$ can be embedded into the quantized algebra of global functions on the Grothendieck-Springer resolution $G \times_B B$, where $B \subset G$ is a fixed Borel subgroup in G. On the other hand, the variety $G \times_B B$ is isomorphic to the moduli space of G-local systems on the punctured disc, equipped with reduction to a Borel subgroup at the puncture, as well as a trivialization at one marked point on the boundary. Classically, this moduli space is birational to $\mathcal{X}_{\widehat{S},G}$, and it would be interesting to understand the precise relation between the corresponding quantizations.

We now come to the first main result of the paper.

Theorem 4.4.4. *There is an embedding of algebras $\iota\colon \mathfrak{D}_n \to \mathcal{D}_n$ defined by the following assignment for $i = 1, \dots, n$:*

$$\widehat{E}_i \longmapsto \mathbf{i} \sum_{r=-i}^{i-1} q^{i+r} \mathrm{V}_{i,-i} \mathrm{V}_{i,1-i} \dots \mathrm{V}_{i,r}, \tag{4.4.1}$$

$$K_i \longmapsto q^{2i} \mathrm{V}_{i,-i} \mathrm{V}_{i,1-i} \dots \mathrm{V}_{i,i}, \tag{4.4.2}$$

$$\widehat{F}_{\theta(i)} \longmapsto \mathbf{i} \sum_{r=-i}^{i-1} q^{i+r} \Lambda_{i,-i} \Lambda_{i,1-i} \dots \Lambda_{i,r}, \tag{4.4.3}$$

$$K'_{\theta(i)} \longmapsto q^{2i} \Lambda_{i,-i} \Lambda_{i,1-i} \dots \Lambda_{i,i}. \tag{4.4.4}$$

Remark 4.4.5. The algebra embedding (4.4.1) – (4.4.4) turns out to be equivalent to the homomorphism from $U_q(\mathfrak{sl}_n)$ into an algebra of difference operators used to constuct the *positive representations* introduced in [30]. We thank I. Ip for pointing this out to us.

Remark 4.4.6. Formulas (4.4.1) and (4.4.3) can be rewritten as follows:

$$\begin{aligned} E_i &\longmapsto \mathbf{i}\, \mathrm{Ad}_{\Psi^q(\mathrm{V}_{i,i-1})} \dots \mathrm{Ad}_{\Psi^q(\mathrm{V}_{i,1-i})} \mathrm{V}_{i,-i}, \\ F_{\theta(i)} &\longmapsto \mathbf{i}\, \mathrm{Ad}_{\Psi^q(\Lambda_{i,i-1})} \dots \mathrm{Ad}_{\Psi^q(\Lambda_{i,1-i})} \Lambda_{i,-i}. \end{aligned}$$

Note, that the right hand side of the formula (4.4.1) coincides with the cluster X-variable corresponding to the vertex$\mathrm{V}_{i,-i}$, in the cluster obtained from the initial one by consecutive application of mutations at variables $\mathrm{V}_{i,r}$, where r runs from $i-1$ to $1-i$. Similarly, the right hand side of the formula (4.4.3) coincides with the cluster X-variable for vertex $\Lambda_{i,-i}$ in the cluster obtained from the initial one by consecutive application of mutations at variables $\Lambda_{i,r}$, where r runs from $i-1$ to $1-i$.

Example 4.4.7. For $n = 1$, in the notations of Figure 4.4, the embedding ι reads

$$E \mapsto \mathbf{i}X_1(1 + qX_2), \qquad K \mapsto q^2X_1X_2X_3,$$
$$F \mapsto \mathbf{i}X_3(1 + qX_4), \qquad K' \mapsto q^2X_4X_3X_2.$$

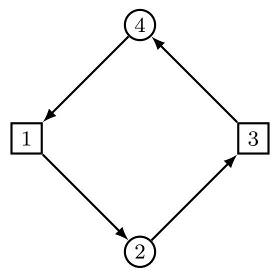

Figure 4.4: $\mathcal{D}_1$-quiver.

Example 4.4.8. For $n = 2$, in the notations of Figure 4.5, the embedding ι reads

$$E_1 \mapsto \mathbf{i}X_1(1 + qX_2), \qquad K_2 \mapsto q^4X_3X_4X_5X_6X_7,$$
$$E_2 \mapsto \mathbf{i}X_4(1 + qX_5(1 + qX_6(1 + qX_7))), \qquad K_1 \mapsto q^2X_1X_2X_3,$$
$$F_1 \mapsto \mathbf{i}X_3(1 + qX_7(1 + qX_{10}(1 + qX_5))), \qquad K_2' \mapsto q^2X_8X_9X_4,$$
$$F_2 \mapsto \mathbf{i}X_8(1 + qX_9), \qquad K_1' \mapsto q^4X_7X_4X_{10}X_5X_1.$$

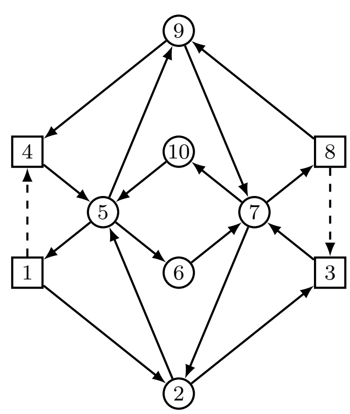

Figure 4.5: $\mathcal{D}_2$-quiver.

The proof of Theorem 4.4.4 will follow from Propositions 4.4.9 and 4.4.10 stated below.

Proposition 4.4.9. *The formulas* (4.4.1) – (4.4.4) *define a homomorphism of algebras.*

Proof. In what follows we abuse notations and denote an element of the algebra $\mathfrak{D}_n$ and its image under ι the same. For any $1 \leqslant i \leqslant n$ and $-i \leqslant r < i$, let us define

$$w_i^r = \mathbf{i}q^{i+r}\mathrm{V}_{i,-i}\dots\mathrm{V}_{i,r},$$
$$m_i^r = \mathbf{i}q^{i+r}\Lambda_{i,-i}\dots\Lambda_{i,r}.$$

Then, the formulas (4.4.1) – (4.4.4) can be rewritten as follows:

$$\widehat{E}_i = w_i^{-i} + \dots + w_i^{i-1}, \qquad K_i = -qw_i^{i-1}m_{\theta(i)}^{-\theta(i)},$$
$$\widehat{F}_{\theta(i)} = m_i^{-i} + \dots + m_i^{i-1}, \qquad K'_{\theta(i)} = -qm_i^{i-1}w_{\theta(i)}^{-\theta(i)}.$$

It is immediate from inspecting the quiver that the relations (4.3.1) hold, as well as $[E_i, E_j] = [F_i, F_j] = 0$ for $|i-j| > 1$. To verify (4.3.2) it suffices to notice that $i < \theta(j)$ implies $w_i^r m_j^s = m_j^s w_i^r$, while

$$i = \theta(j) \implies w_i^r m_j^s = \begin{cases} q^2 m_j^s w_i^r & \text{if } r = -i, s = j-1, \\ q^{-2} m_j^s w_i^r & \text{if } r = i-1, s = -j, \\ m_j^s w_i^r & \text{otherwise,} \end{cases}$$
$$i > \theta(j) \implies w_i^r m_j^s = \begin{cases} q^2 m_j^s w_i^r & \text{if } r = \pm\theta(j), s = \mp\theta(i) - 1, \\ q^{-2} m_j^s w_i^r & \text{if } s = \pm\theta(i), r = \mp\theta(j) - 1, \\ m_j^s w_i^r & \text{otherwise.} \end{cases}$$

Let us now check the Serre relation

$$E_{i+1}^2 E_i + E_i E_{i+1}^2 = (q+q^{-1}) E_{i+1} E_i E_{i+1}.$$

Suppose $-i \leqslant t \leqslant i-1$ and $-i-1 \leqslant r \leqslant i$. We write

$$t \lhd r \quad \text{if} \quad w_{i+1}^r w_i^t = q^{-1} w_i^t w_{i+1}^r,$$
$$t \rhd r \quad \text{if} \quad w_{i+1}^r w_i^t = q w_i^t w_{i+1}^r.$$

It is easy to verify that

$$t \lhd r \iff \begin{cases} t \leqslant r & \text{if } r < 0, \\ t < r & \text{if } r \geqslant 0 \end{cases} \qquad \text{and} \qquad t \rhd r \iff \begin{cases} t > r & \text{if } r < 0, \\ t \geqslant r & \text{if } r \geqslant 0. \end{cases}$$

We can now express

$$\begin{aligned}
E_{i+1}^2E_i &= \sum_{r,s,t} w_{i+1}^r w_{i+1}^s w_i^t \\
&= \sum_{t\rhd r, t\rhd s} w_{i+1}^r w_{i+1}^s w_i^t + \sum_{t\rhd r, t\lhd s} w_{i+1}^r w_{i+1}^s w_i^t \\
&+ \sum_{t\lhd r, t\rhd s} w_{i+1}^r w_{i+1}^s w_i^t + \sum_{t\lhd r, t\lhd s} w_{i+1}^r w_{i+1}^s w_i^t \\
&= q\sum_{t\rhd r, t\rhd s} w_{i+1}^r w_i^t w_{i+1}^s + q^{-1}\sum_{t\rhd r, t\lhd s} w_{i+1}^r w_i^t w_{i+1}^s \\
&+ q\sum_{t\lhd r, t\rhd s} w_{i+1}^r w_i^t w_{i+1}^s + q^{-1}\sum_{t\lhd r, t\lhd s} w_{i+1}^r w_i^t w_{i+1}^s.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
E_iE_{i+1}^2 &= q\sum_{t\lhd r, t\rhd s} w_{i+1}^r w_i^t w_{i+1}^s + q^{-1}\sum_{t\rhd r, t\rhd s} w_{i+1}^r w_i^t w_{i+1}^s \\
&+ q\sum_{t\lhd r, t\lhd s} w_{i+1}^r w_i^t w_{i+1}^s + q^{-1}\sum_{t\rhd r, t\lhd s} w_{i+1}^r w_i^t w_{i+1}^s.
\end{aligned}$$

Observe that if $t \rhd r$ and $t \lhd s$, then one necessarily has $r < s$, which in turn implies $w_{i+1}^r w_{i+1}^s = q^{-2} w_{i+1}^s w_{i+1}^r$. Similarly, if $t\lhd r$ and $t\rhd s$, it follows that $r > s$ and $w_{i+1}^r w_{i+1}^s = q^2 w_{i+1}^s w_{i+1}^r$. Hence

$$\begin{aligned}
\sum_{t\rhd r, t\lhd s} w_{i+1}^r w_i^t w_{i+1}^s &= \sum_{t\rhd r, t\lhd s} w_{i+1}^s w_i^t w_{i+1}^r, \\
\sum_{t\lhd r, t\rhd s} w_{i+1}^r w_i^t w_{i+1}^s &= \sum_{t\lhd r, t\rhd s} w_{i+1}^s w_i^t w_{i+1}^r.
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
&E_{i+1}^2E_i + E_iE_{i+1}^2 \\
&\qquad = \left(q-q^{-1}\right)\left(\sum_{t\rhd r, t\rhd s} + \sum_{t\rhd r, t\lhd s} + \sum_{t\lhd r, t\rhd s} + \sum_{t\lhd r, t\lhd s}\right) w_{i+1}^r w_i^t w_{i+1}^s \\
&\qquad = E_{i+1}E_iE_{i+1}.
\end{aligned}$$

The other nontrivial Serre relations are proved in an identical fashion. □

Proposition 4.4.10. *The homomorphism $\iota\colon \mathfrak{D}_n \to \mathcal{D}_n$ is injective.*

Proof. It will be convenient to choose a different PBW basis of $\mathfrak{D}_n$ from the one we considered in Section 4.3. Namely, for any simple root α we set $F'_\alpha = F_\alpha$, then define inductively

$$F'_{\alpha+\beta} = \frac{F'_\alpha F'_\beta - q^{-(\alpha,\beta)}F'_\beta F'_\alpha}{q-q^{-1}}.$$

By the PBW theorem, the set Mon_{PBW} of all normally ordered monomials in K_α, K'_α, E_α, and F'_α, $\alpha \in \Delta_+$, forms a basis for $\mathfrak{D}_n$ over $\mathbb{C}(q)$. Let us now fix a degree-lexigocraphic order on the set of all monomials in the quantum torus $\mathcal{D}_n$, taken with respect to any total order on the generators $\{X_i\}$. To establish injectivity of ι, it will suffice to show that there are no two PBW monomials $m_1, m_2 \in \mathrm{Mon}_{PBW}$, such that $\iota(m_1)$ and $\iota(m_2)$ have the same leading term with respect to our chosen monomial order for $\mathcal{D}_n$. Indeed, if this is true, our monomial order induces a total order on Mon_{PBW} with respect to which the map ι becomes triangular. In fact, given a monomial $\vec{X} \in \mathcal{D}_n$ that arises as the leading term of some PBW monomial, one can reconstruct the unique PBW monomial $m_{\vec{X}}$ such that the leading term of $\iota(m_{\vec{X}})$ is $\vec{X}$ as follows. In the cluster monomial $\vec{X}$, let n_{ij}, s_{ij}, e_{ij}, and w_{ij} be respectively the degrees of the cluster variables corresponding to North, South, East, and West nodes of the rhombus labelled by ij in the right triangle in Figure 4.6. Let us also declare $w_{1n} = 0$. Then the degree of E_{ij} in $m_{\vec{X}}$ is equal to $n_{ij} + s_{ij} - e_{ij} - w_{ij}$ and the degree of K_i is equal to $e_{ii} - n_{in}$.

To see this, first observe that the leading term of $\iota(m)$ for $m \in \{E_\alpha, F_\alpha, K_i, K'_i\}$ will contribute one to the power of a cluster variable in $\vec{X}$ iff the corresponding vertex of the quiver appears in $\iota(m)$. Hence, the quantity $N + S - E - W$ for any rhombus will be an integer combination of exponents of the $\{E_\alpha, F_\alpha, K_i, K'_i\}$. Now, the SE and NW edges of any rhombus in the right triangle form part of some V_r, V_{r+1}-paths respectively. Note that N appears in K'_i iff E does, and S appears in K'_i iff W does. Hence $n_{K'_i}$ contributes nothing to $N + S - E - W$. Similar arguments apply to show $n_{K'_i}$ also contributes nothing to $N + S - E - W$. Moreover, since we are in the right triangle, exactly the same arguments also apply to n_{F_α} for any root α.

Now,suppose we have a root $E_{i,j}$. Then if $i > r+1$, or $j < r$, none of the rhomus vertices appear in $\iota(E_{i,j})$. If $i \leq r$ and $r+1 \leq j$, then S appears in E_{ij} iff W does, and E appears in E_{ij} iff N does. Hence $n_{E_{ij}}$, the power of E_{ij} in a PBW monomial makes no contribution $N + S - E - W$. Similarly, if $i = r+1$, then W appears in $E_{i,j}$ iff N does, since N can never be a frozen vertex. The only remaining case is when $j = r$: in this case, we see that $E_{i,j}$ will contribute to $N + S - E - W$ iff S is the last vertex in V_r appearing in $E_{i,j}$. The formulas for the exponents of F_α, K_i, K'_i are proved by similar arguments.

Now, let n_{ij}, s_{ij}, e_{ij}, and w_{ij} denote the degrees in $m_{\vec{X}}$ of the cluster variables corresponding the North, South, East, and West nodes of the corresponding rhombus in the left triangle in Figure 4.6, where we set $e_{1n} = 0$. Then the degree of $F'_{\theta(i)\theta(j)}$ equals $n_{ij} + s_{ij} - e_{ij} - w_{ij}$ in the left triangle where we set $e_{1n} = 0$ and the degree of $K'_{\theta(i)}$ equals $w_{i,i} - s_{i,n}$. □

Corollary 4.4.11. *The homomorphism ι induces an embedding of the quantum group $U_q(\mathfrak{sl}_{n+1})$ into the quotient of the algebra $\mathcal{D}_n$ by relations*

$$q^{2n+2} \mathrm{V}_{i,-i} \dots \mathrm{V}_{i,i} \cdot \Lambda_{\theta(i),-\theta(i)} \dots \Lambda_{\theta(i),\theta(i)} = 1$$

for all $1 \leqslant i \leqslant n$.

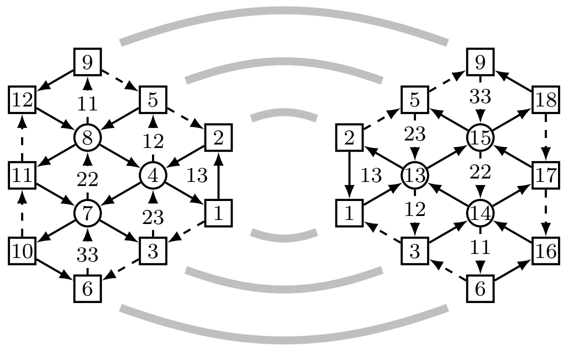

Figure 4.6: A pair of triangles amalgamated by 2 sides.

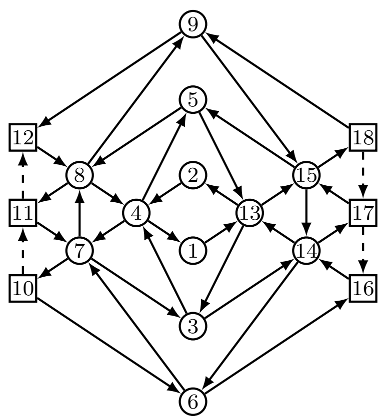

Figure 4.7: $\mathcal{D}_3$-quiver.

4.5 The Dehn twist on a twice punctured disk

In order to describe the coalgebra structure of $U_q(\mathfrak{sl}_{n+1})$, we will need to consider the moduli space $\mathcal{X}_{\widehat{S}_2,PGL_{n+1}}$ of PGL_{n+1}-local systems on $\widehat{S}_2$, a disk with *two* punctures p_1, p_2, and two marked points x_1, x_2 on its boundary. To obtain a quantum cluster chart on this moduli space, we consider the quiver corresponding to the $(n+1)$-triangulation of the left-most disk in Figure 4.8. Note that this quiver is formed by amalgamating two $\mathcal{D}_n$-quivers by one column of frozen variables, see Figure 4.3. An example of two amalgamated $\mathcal{D}_2$-quivers is shown in Figure 4.9, where one should disregard the gray arrows. We refer to the result of this amalgamation as the $\mathcal{Z}_n$-quiver and denote the corresponding quantum torus algebra by $\mathcal{Z}_n$.

Figure 4.8 shows four different ideal triangulations of a twice punctured disk with two marked points on the boundary; the arrows correspond to flips of ideal triangulations. Note that the right-most disk may be obtained from the left-most one by applying the half-Dehn twist rotating the left puncture clockwise about the right one. Hence this half-Dehn twist may be decomposed into a sequence of 4 flips. Let $\mathcal{Z}'_n$ be the quiver obtained from the $(n+1)$-triangulation of the right-most disk. It is evident from inspecting the corresponding $(n+1)$-triangulations that there exists an isomorphism σ between the $\mathcal{Z}_n$- and the $\mathcal{Z}'_n$-quivers that preserves all frozen variables. On the other hand, since there is no nontrivial automorphism of the $\mathcal{Z}_n$-quiver fixing its frozen variables, we conclude that the isomorphism σ is unique.

Let us now describe σ explicitly. Recall that each $(n+1)$-triangulated triangle contains exactly n solid oriented paths parallel to each of its sides. For example, in the 4-triangulation shown in Figure 4.1, one sees paths $1 \to 2$, $3 \to 4 \to 5$, and $6 \to 7 \to 8 \to 9$, parallel to the side BC. Now, consider the second disk in Figure 4.8, recall that the $(n+1)$-triangulation of the pair of triangles in the middle is shown in the right part of Figure 4.2. For $i = 1, \ldots, n$ we define the i-th *permutation cycle* to

- follow the i-th solid path parallel to the side a in the triangle Δ_{abc} along the orientation,
- follow the i-th solid path parallel to the side d in the triangle Δ_{bde} in the direction opposite to the orientation,
- follow the i-th solid path parallel to the side g in the triangle Δ_{efg} along the orientation,
- follow the i-th solid path parallel to the side d in the triangle Δ_{cdf} in the direction opposite to the orientation.

Now, the isomorphism σ is defined as follows: each vertex in the i-th permutation cycle is moved i steps along the cycle, frozen variables are left intact, the rest of the vertices are rotated by 180°. In Figure 4.9, the 2 cycles in the quiver $\mathcal{Z}_2$ and the rotation of vertices 9 and 11 are shown by gray arrows; the action of σ reads

$$\sigma = (2\ 7\ 15\ 17\ 13\ 4)\ (3\ 16\ 18)\ (8\ 10\ 12)\ (9\ 11),$$

where the 2nd permutation cycle breaks into (3 16 18) (8 10 12).

4.6 Cluster realization of the R-matrix

Recall that the universal R-matrix of the quantum group $U_q(\mathfrak{sl}_{n+1})$ is an element

$$\mathcal{R} \in U_q(\mathfrak{sl}_{n+1}) \widetilde{\otimes} U_q(\mathfrak{sl}_{n+1})$$

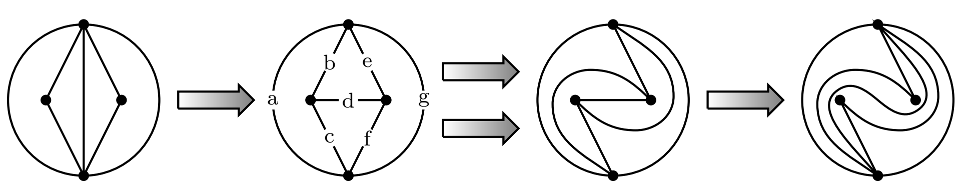

Figure 4.8: The half Dehn twist as a sequence of 4 flips.

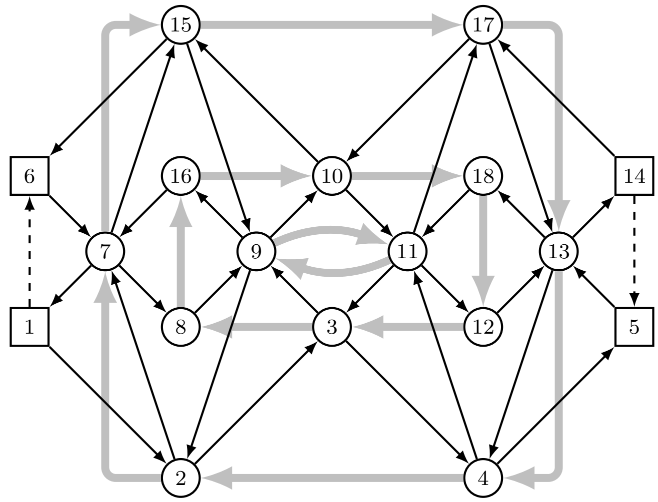

Figure 4.9: Permutation on the $\mathcal{Z}_2$-quiver.

of a certain extension of its tensor square, and gives rise to a braiding on the category of finite dimensional $U_q(\mathfrak{sl}_{n+1})$-modues. The universal R-matrix admits decomposition

$$\mathcal{R} = \bar{\mathcal{R}}\mathcal{K}.$$

where

$$\mathcal{K} = q^{\sum_{i,j} c_{ij} H_i \otimes H_j},$$

(c_{ij}) is the inverse of the Cartan matrix, and H, H' are defined from the relations

$$K = q^H \qquad \text{and} \qquad K' = q^{H'}.$$

The tensor $\bar{\mathcal{R}}$ is called the quasi R-matrix and is given by the formula

$$\bar{\mathcal{R}} = \prod_{\alpha \in \Delta_+}^{\rightarrow} \Psi^q \left(-E_\alpha \otimes F_\alpha\right), \tag{4.6.1}$$

where the product is ordered consistently with the previously chosen normal ordering $\prec$ on Δ_+.

Let $\mathrm{Ad}_{\mathcal{K}}$ and $\mathrm{Ad}_{\bar{\mathcal{R}}}$ denote the automorphisms of $\mathfrak{D}_n \otimes \mathfrak{D}_n$ that conjugate by $\mathcal{K}$ and $\bar{\mathcal{R}}$ respectively. It is clear that both $\mathrm{Ad}_{\mathcal{K}}$ and $\mathrm{Ad}_{\bar{\mathcal{R}}}$ extend to automorphisms of $\mathcal{D}_n \otimes \mathcal{D}_n$ defined in the same way. We write P for the automorphism of $\mathcal{D}_n \otimes \mathcal{D}_n$ permuting the tensor factors:

$$P(X \otimes Y) = Y \otimes X.$$

Recall the isomorphism of quivers described in the previous section. It defines a permutation of cluster variables $X_i \mapsto X_{\sigma(i)}$ which we also denote by σ with a slight abuse of notation. Note that each of the 4 flips shown in Figure 4.8 corresponds to a sequence of $\binom{n+2}{3}$ cluster mutations, as explained at the end of Section 4.2. Let

$$N = 4 \cdot \binom{n+2}{3}$$

and $\mu_N \dots \mu_1$ be the sequence of quantum cluster mutations constituting the half-Dehn twist. Now we are ready to formulate the next main result of the paper.

Theorem 4.6.1. *The composition*

$$P \circ \mathrm{Ad}_{\mathcal{R}} \colon \mathcal{D}_n \otimes \mathcal{D}_n \longrightarrow \mathcal{D}_n \otimes \mathcal{D}_n$$

restricts to the subalgebra $\mathcal{Z}_n$. Moreover, the following automorphisms of $\mathcal{Z}_n$ coincide:

$$P \circ \mathrm{Ad}_{\mathcal{R}} = \mu_N \dots \mu_1 \circ \sigma,$$

where the sequence of quantum cluster mutations $\mu_N \dots \mu_1$ constitutes the half Dehn twist.

Proof. By Lemma 4.1.1 we have

$$\mu_N \dots \mu_1 = \Phi_N \circ \mathrm{M}_N,$$

where M_N is a monomial transformation, and Φ_N is a conjugation by a sequence of N quantum dilogarithms. The result of the theorem then follows from Propositions 4.6.2 and 4.6.3 below.

□

Proposition 4.6.2. *The following automorphisms of $\mathcal{Z}_n$ coincide:*

$$P \circ \mathrm{Ad}_{\mathcal{K}} = \mathrm{M}_N \circ \sigma. \tag{4.6.2}$$

Proof. We define the $\Lambda\mathrm{V}_i$-path in the $\mathcal{Z}_n$-quiver as the concatenation of the $\Lambda_{\theta(i)}$-path in the left $\mathcal{D}_n$-quiver with the V_i-path in the right $\mathcal{D}_n$-quiver. For example, in the notations of Figure 4.9, the $\Lambda\mathrm{V}_1$-path consists of vertices 1, 7, 16, 9, 3, 4, 5. Each mutation from the sequence $\mu_N \dots \mu_1$ happens at a vertex that belongs to a certain $\Lambda\mathrm{V}_i$-path, has exactly two outgoing edges within this path, and has exactly two incoming edges from vertices that do not belong to the path. This claim can be easily verified by inspecting the $\mathcal{Z}_n$-quiver and the sequence of mutations under discussion. In turn, it implies that the monomial transformation M_N restricts to each $\Lambda\mathrm{V}$-path. The action of M_N on the $\Lambda\mathrm{V}_i$-path is shown in Figure 4.10, where

$$\begin{aligned}
Z_- &= q^{2\theta(i)} \cdot X_1 X_2 \dots X_{2\theta(i)+1} \cdot Y_1, \\
Z_0 &= q^{-2n} \cdot X_{2\theta(i)}^{-1} \dots X_2^{-1} X_1^{-1} \cdot Y_{2i}^{-1} \dots Y_2^{-1} Y_1^{-1}, \\
Z_+ &= q^{2i} \cdot X_1 \cdot Y_1 Y_2 \dots Y_{2i+1}.
\end{aligned}$$

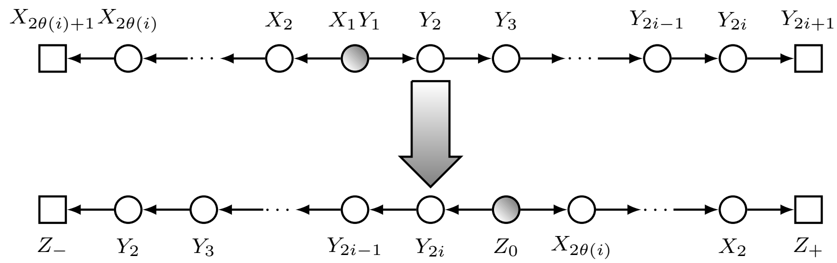

Figure 4.10: Action of M_N on the $\Lambda\mathrm{V}_i$-path.

On the other hand, it is easy to see that the automorphism $P \circ \mathrm{Ad}_{\mathcal{K}}$ acts as P on all nonfrozen variables in the product $\mathfrak{D}_n \otimes \mathfrak{D}_n$. It is a matter of a straightforward calculation to verify that $P \circ \mathrm{Ad}_{\mathcal{K}}$ acts on the frozen variables of $\mathcal{Z}_n$, and on those variables that used to be frozen before the amalgamation, as follows:

$$P(\mathrm{Ad}_{\mathcal{K}}(X_{2\theta(r)+1})) = q^{2\theta(r)} \cdot X_1X_2 \dots X_{2\theta(r)+1} \cdot Y_1,$$
$$P(\mathrm{Ad}_{\mathcal{K}}(X_1Y_1)) = q^{-2n} \cdot X_{2\theta(r)}^{-1} \dots X_2^{-1}X_1^{-1} \cdot Y_{2r}^{-1} \dots Y_2^{-1}Y_1^{-1},$$
$$P(\mathrm{Ad}_{\mathcal{K}}(Y_{2r+1})) = q^{2r} \cdot X_1 \cdot Y_1Y_2 \dots Y_{2r+1}.$$

Now we can see that under the action of $P \circ \mathrm{Ad}_{\mathcal{K}}$, the initial cluster $\mathcal{X}$ of the quiver $\mathcal{Z}_n$ is transformed into a different cluster $\mathcal{X}'$ with the underlying quiver isomorphic to $\mathcal{Z}_n$. At the same time, M_N also turns $\mathcal{X}$ into $\mathcal{X}'$, but the corresponding quiver is $\mathcal{Z}_n'$. Since there are no nontrivial automorphisms of the quiver $\mathcal{Z}_n$ fixing the frozen variables, we conclude that the permutation σ satisfies (4.6.2). □

Proposition 4.6.3. *The following automorphisms of $\mathcal{Z}_n$ coincide:*

$$\mathrm{Ad}_{P(\bar{\mathcal{R}})} = \Phi_N.$$

Proof. Consider the factorization (4.7.18) of the quasi R-matrix obtained in Theorem 4.7.4. On the other hand, we have a different factorization of the R-matrix from inspecting the sequence of flips realizing the Dehn twist along with the corresponding sequence of mutations.

The latter factorization reads

$$\begin{aligned}\bar{\mathcal{R}} =&\prod_{k=0}^{n-1}\prod_{j=\theta(k)}^{n+1}\prod_{i=1}^{\theta(k+1)}\psi\left(m_{j-i}^{i-\theta(k)}\otimes w_{i+\theta(j)}^{-i}\right)\\ &\cdot\prod_{k=0}^{n-1}\prod_{j=\theta(k)}^{n+1}\prod_{i=1}^{\theta(k+1)}\psi\left(m_{j-i}^{i-\theta(k)}\otimes w_{i+\theta(j)}^{\theta(j)}\right)\\ &\cdot\prod_{k=1}^{n}\prod_{j=k+1}^{n+1}\prod_{i=1}^{k}\psi\left(m_{i+\theta(j)}^{i-1}\otimes w_{j-i}^{k-j}\right)\\ &\cdot\prod_{k=1}^{n}\prod_{j=k+1}^{n+1}\prod_{i=1}^{k}\psi\left(m_{i+\theta(j)}^{i-1}\otimes w_{j-i}^{k-i}\right),\end{aligned} \tag{4.6.3}$$

where all three products are taken in ascending order and expanded from left to right. Now, it suffices to show that formulas (4.7.18) and (4.6.3) coincide.

Let us write $(a_1,\dots,a_N)$ for the sequence of dilogarithm arguments appearing in the factorization (4.7.18), read from left to right. Similarly, we write $(b_1,b_2,\dots,b_N)$ for the sequence of dilogarithm arguments appearing in the factorization (4.6.3), again read from left to right. It is easy to see that the underlying sets $(a_1,\dots,a_N)$ and $(b_1,\dots,b_N)$ coincide. Moreover, we claim that for every pair (b_i,b_j) with $i<j$ such that $(b_i,b_j)=(a_k,a_l)$ for some $k>l$, we have $[b_i,b_j]=0$. This follows from commutation relations

$$\begin{aligned} &w_i^r w_i^s = q^{2\,\mathrm{sgn}(r-s)} w_i^s w_i^r,\\ &w_i^r w_j^s = w_j^s w_i^r \quad \text{if} \quad |i-j|>1,\\ &w_i^r w_{i+1}^s = \begin{cases} q w_{i+1}^s w_i^r & \text{if } r\lhd s,\\ q^{-1} w_{i+1}^s w_i^r & \text{if } r\rhd s,\end{cases}\end{aligned}$$

and similar relations for variables m_i^r, all of which can be read from the $\mathcal{D}_n$-quiver. Hence one can freely re-order the dilogarithms $\psi(b_i)$ to match the order arising in (4.7.18), and the Proposition is proved. □

Remark 4.6.4. The homomorphism $(\iota\otimes\iota)\circ\Delta\colon \mathfrak{D}_n\to\mathcal{D}_n\otimes\mathcal{D}_n$ given composition of the comultiplication map with the tensor square of ι factors through the subalgebra $\mathcal{Z}_n$: we have

$$(\iota\otimes\iota)\circ\Delta\colon \mathfrak{D}_n\to\mathcal{Z}_n\subset\mathcal{D}_n\otimes\mathcal{D}_n.$$

Let us refer to the concatenation of the two V_i-paths in a pair of amalgamated $\mathcal{D}_n$-quivers as a VV_i-path. Then, the formula for $\Delta(E_i)$ is obtained by conjugating the first (frozen) variable in the VV_i-path by quantum dilogarithms with arguments running over consecutive vertices in the VV_i-path not including the last (frozen) vertex, and multiplying the result by $\mathbf{i}$. In particular, in the notations of Figure 4.9 one gets

$$\Delta(E_1) = \mathbf{i}X_1(1 + qX_2(1 + qX_3(1 + qX_4))),$$
$$\Delta(E_2) = \mathbf{i}X_6(1 + qX_7(\ldots(1 + qX_{12}(1 + qX_{13}))\ldots)).$$

The coproduct $\Delta(K_i)$ is equal to the product of all the variables along the VV_i-path multiplied by q^{4i}. Again, in the notations of Figure 4.9 one gets

$$\Delta(K_1) = q^4 X_1X_2X_3X_4X_5,$$
$$\Delta(K_2) = q^8 X_6X_7X_8X_9X_{10}X_{11}X_{12}X_{13}X_{14.}$$

Formulas for $\Delta(F_{\theta(i)})$ and $\Delta(K'_{\theta(i)})$ can be obtained from those for $\Delta(E_i)$ and $\Delta(K_i)$ via rotating the $\mathcal{Z}_n$-quiver by 180°. Similarly, one can get formulas for iterated coproducts $\Delta^k(A)$, $A \in \mathfrak{D}_n$, by amalgamating $k+1$ copies of the $\mathcal{D}_n$-quiver.

4.7 Factorization of the R-matrix

In this section, we show that the embedding (4.4.4) gives rise to the refined factorization of the R-matrix of $U_q(\mathfrak{sl}_{n+1})$ used in the proof of Theorem 4.6.1.

We begin with some preparatory lemmas and remarks. It follows from formulas (4.3.4) and (4.4.1) that for every $-i \leqslant r < i$ and $-j \leqslant s < j$ there exist unique decompositions

$$E_{i+1,j} = E^{\downarrow r_-}_{i+1,j} + E^{\downarrow r_+}_{i+1,j}, \qquad E_{i,j-1} = E^{\uparrow s_-}_{i,j-1} + E^{\uparrow s_+}_{i,j-1}$$

where the summands satisfy

$$w_i^r E^{\downarrow r_\pm}_{i+1,j} = q^{\pm 1} E^{\downarrow r_\pm}_{i+1,j} w_i^r, \qquad w_j^s E^{\uparrow s_\pm}_{i,j-1} = q^{\pm 1} E^{\uparrow s_\pm}_{i,j-1} w_j^s.$$

In a similar fashion, formulas (4.3.5) and (4.4.3) imply decompositions

$$F_{i+1,j} = F^{\downarrow r_-}_{i+1,j} + F^{\downarrow r_+}_{i+1,j}, \qquad F_{i,j-1} = F^{\uparrow s_-}_{i,j-1} + F^{\uparrow s_+}_{i,j-1}$$

where the summands are defined by

$$m^r_{\theta(i)} F^{\downarrow r_\pm}_{i+1,j} = q^{\pm 1} F^{\downarrow r_\pm}_{i+1,j} m^r_{\theta(i)}, \qquad m^s_{\theta(j)} F^{\uparrow s_\pm}_{i,j-1} = q^{\pm 1} F^{\uparrow s_\pm}_{i,j-1} m^s_{\theta(j)}.$$

It is also evident that E_{ij} and F_{ij} can be decomposed as

$$E_{ij} = -q \sum_{s=-j}^{j-1} E^{\uparrow s_+}_{i,j-1} w_j^s = q^2 \sum_{s=-r}^{r-1} E^{\uparrow s_+}_{i,r-1} w_r^s E^{\downarrow s_-}_{r+1,j} = -q \sum_{s=-i}^{i-1} w_i^s E^{\downarrow s_-}_{i+1,j}, \tag{4.7.1}$$

$$F_{ij} = \sum_{s=-\theta(j)}^{\theta(j)-1} F^{\uparrow s_+}_{i,j-1} m^s_{\theta(j)} = \sum_{s=-\theta(r)}^{\theta(r)-1} F^{\uparrow s_+}_{i,r-1} m^s_{\theta(r)} F^{\downarrow s_-}_{r+1,j} = \sum_{s=-\theta(i)}^{\theta(i)-1} m^s_{\theta(i)} F^{\downarrow s_-}_{i+1,j} \tag{4.7.2}$$

for any $i < r < j$. We say that formulas (4.7.1) show decompositions of E_{ij} *with respect to the* V_i*-,* V_r*-, and* V_j*-paths.* Similarly, formulas (4.7.2) show decompositions of F_{ij} *with respect to the* $\Lambda_{\theta(i)}$*-,* $\Lambda_{\theta(r)}$*-, and* $\Lambda_{\theta(j)}$*-paths.*

Lemma 4.7.1. *For all $a < b$, we have*

$$\left(E^{\uparrow a_+}_{i,j-1} w^a_j\right)\left(E^{\uparrow b_+}_{i,j-1} w^b_j\right) = q^{-2}\left(E^{\uparrow b_+}_{i,j-1} w^b_j\right)\left(E^{\uparrow a_+}_{i,j-1} w^a_j\right), \tag{4.7.3}$$

$$\left(F^{\uparrow a_+}_{i,j-1} m^a_{\theta(j)}\right)\left(F^{\uparrow b_+}_{i,j-1} m^b_{\theta(j)}\right) = q^{-2}\left(F^{\uparrow b_+}_{i,j-1} m^b_{\theta(j)}\right)\left(F^{\uparrow a_+}_{i,j-1} m^a_{\theta(j)}\right), \tag{4.7.4}$$

and

$$\left(E^{\uparrow a_+}_{i,j-1} w^a_j E^{\downarrow a_-}_{j+1,k}\right)\left(E^{\uparrow b_+}_{i,j-1} w^b_j E^{\downarrow b_-}_{j+1,k}\right) = q^{-2}\left(E^{\uparrow b_+}_{i,j-1} w^b_j E^{\downarrow b_-}_{j+1,k}\right)\left(E^{\uparrow a_+}_{i,j-1} w^a_j E^{\downarrow a_-}_{j+1,k}\right). \tag{4.7.5}$$

Proof. We shall only prove the first relation as the proofs of the other two are similar. First, note that since $a < b$ we have

$$w^a_j E^{\uparrow b_+}_{i,j-1} = q E^{\uparrow b_+}_{i,j-1} w^a_j \qquad \text{and} \qquad w^a_j w^b_j = q^{-2} w^b_j w^a_j,$$

therefore

$$w^a_j \left(E^{\uparrow b_+}_{i,j-1} w^b_j\right) = q^{-1}\left(E^{\uparrow b_+}_{i,j-1} w^b_j\right) w^a_j.$$

Let us set

$$E^{a\triangledown b}_{i,j-1} = E^{\uparrow a_+}_{i,j-1} - E^{\uparrow b_+}_{i,j-1}.$$

Then by definition we have

$$E^{\uparrow b_+}_{i,j-1} w^b_j = q^{-1} w^b_j E^{\uparrow b_+}_{i,j-1}$$

and it only remains to commute $E^{a\triangledown b}_{i,j-1}$ through $E^{\uparrow b_+}_{i,j-1} w^b_j$. For this, is enough to show that

$$E^{a\triangledown b}_{i,j-1} E^{\uparrow b_+}_{i,j-1} = q^{-2} E^{\uparrow b_+}_{i,j-1} E^{a\triangledown b}_{i,j-1} \tag{4.7.6}$$

since

$$E^{a\triangledown b}_{i,j-1} w^b_j = q w^b_j E^{a\triangledown b}_{i,j-1}.$$

We finish the proof by induction on j. Assume that equalities (4.7.3) and (4.7.6) hold for all $j < k$. To prove the base of induction, it is enough to note that if $j = i+1$, the relation (4.7.6) follows readily from inspecting the quiver, which in turn implies (4.7.3). In order to make the step of induction, we decompose both $E^{a\triangledown b}_{i,k-1}$ and $E^{\uparrow b_+}_{i,k-1}$ with respect to the V_{k-1}-path and apply (4.7.3) for $j = k-2$. □

Lemma 4.7.2. *For $i < j$ we have*

$$E_{i,j} E_k = \begin{cases} q^{-1} E_k E_{i,j} & \text{if } k = j, \\ q E_k E_{i,j} & \text{if } k = i, \\ E_k E_{i,j} & \text{if } i < k < j. \end{cases} \tag{4.7.7}$$

Proof. The proof follows from the decomposition (4.7.1) and considerations similar to the proof of those in the proof of Lemma 4.7.1. □

For any $i < j$ let us declare

$$F_{i,j}^{\geqslant s} = \sum_{r \geqslant s} F_{i,j-1}^{\uparrow r_+} m_{\theta(j)}^r.$$

Note that

$$F_{i,j}^{\geqslant s} = \begin{cases} F_{i,j}^{\uparrow s_+} & \text{if} \quad s < 0, \\ F_{i,j-1}^{\uparrow 0_+} m_{\theta(j)}^0 + F_{i,j}^{\uparrow s_+} & \text{if} \quad s \geqslant 0. \end{cases} \tag{4.7.8}$$

In what follows we use the following shorthand:

$$\psi(x) = \Psi^q(-x)$$

Note that the pentagon identity (4.1.3) now reads

$$\psi(v)\psi(u) = \psi(u)\psi(-quv)\psi(v) \tag{4.7.9}$$

for any u and v satisfying $vu = q^2 uv$.

Lemma 4.7.3. *We have*

$$\begin{aligned} &\psi\left(E_{i,j} \otimes F_{i,j}^{\uparrow s_+}\right) \psi\left(E_{i,j+1} \otimes F_{i,j+1}^{\geqslant s}\right) \psi\left(E_{j+1} \otimes m_{\theta(j+1)}^s\right) \\ &\quad = \psi\left(E_{j+1} \otimes m_{\theta(j+1)}^s\right) \psi\left(E_{i,j} \otimes F_{i,j}^{\uparrow s_+}\right) \psi\left(E_{i,j+1} \otimes F_{i,j+1}^{\geqslant s+1}\right). \end{aligned} \tag{4.7.10}$$

Proof. By equality (4.1.2) and Lemma 4.7.1, there exists a factorization

$$\psi\left(E_{i,j+1} \otimes F_{i,j+1}^{\geqslant s}\right) = \prod_{r \geq s} \psi\left(E_{i,j+1} \otimes F_{i,j}^{\uparrow r_+} m_{\theta(j+1)}^r\right),$$

where the product is taken in ascending order. Note that by Lemma 4.7.2, the dilogarithm $\psi\left(E_{j+1} \otimes m_{\theta(j+1)}^s\right)$ commutes with all but the left-most factor in this product. Hence the left-hand side of (4.7.10) may be re-ordered so that we have a triple of adjacent factors

$$\psi\left(E_{i,j} \otimes F_{i,j}^{\uparrow s_+}\right) \psi\left(E_{i,j+1} \otimes F_{i,j}^{\uparrow s_+} m_{\theta(j+1)}^s\right) \psi\left(E_{j+1} \otimes m_{\theta(j+1)}^s\right).$$

Now, using (4.7.1) we get similar factorizations

$$\psi\left(E_{i,j} \otimes F_{i,j}^{\uparrow s_+}\right) = \prod_{r=-i}^{i-1} \psi\left(-qE_{i,j-1}^{\uparrow r_+} w_j^r \otimes F_{i,j}^{\uparrow s_+}\right) \tag{4.7.11}$$

and

$$\psi\left(E_{i,j+1}\otimes F_{i,j}^{\uparrow s+}m^s_{\theta(j+1)}\right)=\prod_{r=-i}^{i-1}\psi\left(q^2E_{i,j-1}^{\uparrow r+}w_j^rE_{j+1}^{\downarrow r-}\otimes F_{i,j}^{\uparrow s+}m^s_{\theta(j+1)}\right), \tag{4.7.12}$$

with the products again being taken in ascending order. By Lemma 4.7.1, the rightmost factor $\psi\left(-qE_{i,j-1}^{\uparrow(i-1)+}w_j^{i-1}\otimes F_{i,j}^{\uparrow s+}\right)$ in (4.7.11) commutes with all but the rightmost factor in (4.7.12), so we can re-order again to get a triple of adjacent factors

$$\begin{aligned}\psi\left(-qE_{i,j-1}^{\uparrow(i-1)+}w_j^{i-1}\otimes F_{i,j}^{\uparrow s+}\right)\psi\left(q^2E_{i,j-1}^{\uparrow(i-1)+}w_j^{i-1}E_{j+1}^{\downarrow(i-1)-}\otimes F_{i,j}^{\uparrow s+}m^s_{\theta(j+1)}\right)\\ \cdot\psi\left(E_{j+1}\otimes m^s_{\theta(j+1)}\right).\end{aligned} \tag{4.7.13}$$

On the other hand, we can factor

$$\psi\left(E_{j+1}\otimes m^s_{\theta(j+1)}\right)=\psi\left(E_{j+1}^{\downarrow(i-1)-}\otimes m^s_{\theta(j+1)}\right)\psi\left(E_{j+1}^{\uparrow(i-1)+}\otimes m^s_{\theta(j+1)}\right), \tag{4.7.14}$$

and then apply the pentagon identity (4.7.9) to (4.7.13), yielding

$$\begin{aligned}\psi\left(E_{j+1}^{\downarrow(i-1)-}\otimes m^s_{\theta(j+1)}\right)\psi\left(-qE_{i,j-1}^{\uparrow(i-1)+}w_j^{i-1}\otimes F_{i,j}^{\uparrow s+}\right)\\ \cdot\psi\left(E_{j+1}^{\uparrow(i-1)+}\otimes m^s_{\theta(j+1)}\right).\end{aligned} \tag{4.7.15}$$

Note that the right two factors in the product (4.7.15) commute, so it can be re-expressed as

$$\psi\left(E_{j+1}\otimes m^s_{\theta(j+1)}\right)\psi\left(-qE_{i,j-1}^{\uparrow(i-1)+}w_j^{i-1}\otimes F_{i,j}^{\uparrow s+}\right). \tag{4.7.16}$$

Repeating the same procedure for each of the remaining factors in the product (4.7.11), one arrives at (4.7.10). □

Theorem 4.7.4. *The quasi R-matrix of $U_q(\mathfrak{sl}_{n+1})$ can be factored as follows:*

$$\begin{aligned}\bar{\mathcal{R}}_n=&\psi\left(E_1\otimes m_n^{-n}\right)\psi\left(E_2\otimes m_{n-1}^{1-n}\right)\cdots\psi\left(E_n\otimes m_1^{-1}\right)\\ &\cdot\psi\left(E_1\otimes m_n^{1-n}\right)\psi\left(E_2\otimes m_{n-1}^{2-n}\right)\cdots\psi\left(E_{n-1}\otimes m_2^{-1}\right)\\ &\vdots\\ &\cdot\psi\left(E_1\otimes m_n^{-2}\right)\psi\left(E_2\otimes m_{n-1}^{-1}\right)\\ &\cdot\psi\left(E_1\otimes m_n^{-1}\right)\\ &\cdot\psi\left(E_1\otimes m_n^{0}\right)\\ &\cdot\psi\left(E_2\otimes m_{n-1}^{0}\right)\psi\left(E_1\otimes m_n^{1}\right)\\ &\vdots\\ &\cdot\psi\left(E_{n-1}\otimes m_2^{0}\right)\psi\left(E_{n-2}\otimes m_3^{1}\right)\cdots\psi\left(E_1\otimes m_n^{n-2}\right)\\ &\cdot\psi\left(E_n\otimes m_1^{0}\right)\psi\left(E_{n-1}\otimes m_2^{1}\right)\cdots\psi\left(E_1\otimes m_n^{n-1}\right).\end{aligned} \tag{4.7.17}$$

Equivalently, we have

$$\begin{aligned}\bar{\mathcal{R}} =& \prod_{k=1}^{n} \prod_{j=1}^{\theta(k)} \prod_{i=-j}^{j-1} \psi\left(w_j^i \otimes m_{\theta(j)}^{k-\theta(j)-1}\right) \\ &\cdot \prod_{k=1}^{n} \prod_{j=\theta(k)}^{n} \prod_{i=-\theta(j)}^{\theta(j)-1} \psi\left(w_{\theta(j)}^i \otimes m_j^{j-\theta(k)}\right),\end{aligned} \tag{4.7.18}$$

where in the above formula, the products are taken in ascending order[2] *and expanded from left to right, that is one should first expand the formula in k, then in j, and then in i.*

Example 4.7.5. In the case of $U_q(\mathfrak{sl}_3)$, formula (4.7.18) yields a factorization of the quasi R-matrix into the following 16 factors:

$$\begin{aligned}\bar{\mathcal{R}} =&\psi\left(w_1^{-1} \otimes m_2^{-2}\right) \psi\left(w_1^{0} \otimes m_2^{-2}\right) \psi\left(w_2^{-2} \otimes m_1^{-1}\right) \psi\left(w_2^{-1} \otimes m_1^{-1}\right) \\ &\cdot\psi\left(w_2^{0} \otimes m_1^{-1}\right) \psi\left(w_2^{1} \otimes m_1^{-1}\right) \psi\left(w_1^{-1} \otimes m_2^{-1}\right) \psi\left(w_1^{0} \otimes m_2^{-1}\right) \\ &\cdot\psi\left(w_1^{-1} \otimes m_2^{0}\right) \psi\left(w_1^{0} \otimes m_2^{0}\right) \psi\left(w_2^{-2} \otimes m_1^{0}\right) \psi\left(w_2^{-1} \otimes m_1^{0}\right) \\ &\cdot\psi\left(w_2^{0} \otimes m_1^{0}\right) \psi\left(w_2^{1} \otimes m_1^{0}\right) \psi\left(w_1^{-1} \otimes m_2^{1}\right) \psi\left(w_1^{0} \otimes m_2^{1}\right).\end{aligned}$$

Proof. Choosing the normal ordering

$$\alpha_1 \prec (\alpha_1+\alpha_2) \prec (\alpha_1+\cdots+\alpha_n) \prec \alpha_2 \prec \cdots \prec (\alpha_2+\cdots+\alpha_n) \prec \cdots \prec \alpha_n$$

in the formula (4.6.1), we can write

$$\bar{\mathcal{R}}_{n+1} = \psi\left(E_1 \otimes F_1\right) \psi\left(E_{1,2} \otimes F_{1,2}\right) \cdots \psi\left(E_{1,n+1} \otimes F_{1,n+1}\right) \cdot \bar{\mathcal{R}}_n. \tag{4.7.19}$$

where we may assume by induction that $\bar{\mathcal{R}}_n$ factors as follows:

$$\begin{aligned}\bar{\mathcal{R}}_n =&\psi\left(E_2 \otimes m_n^{-n}\right) \psi\left(E_3 \otimes m_{n-1}^{1-n}\right) \cdots \psi\left(E_{n+1} \otimes m_1^{-1}\right) \\ &\vdots \\ &\cdot\psi\left(E_2 \otimes m_n^{-2}\right) \psi\left(E_3 \otimes m_{n-1}^{-1}\right) \\ &\cdot\psi\left(E_2 \otimes m_n^{-1}\right) \\ &\cdot\psi\left(E_2 \otimes m_n^{0}\right) \\ &\cdot\psi\left(E_3 \otimes m_{n-1}^{0}\right) \psi\left(E_2 \otimes m_n^{1}\right) \\ &\vdots \\ &\cdot\psi\left(E_n \otimes m_2^{0}\right) \psi\left(E_{n-1} \otimes m_3^{1}\right) \cdots \psi\left(E_2 \otimes m_n^{n-2}\right) \\ &\cdot\psi\left(E_{n+1} \otimes m_1^{0}\right) \psi\left(E_n \otimes m_2^{1}\right) \cdots \psi\left(E_2 \otimes m_n^{n-1}\right).\end{aligned} \tag{4.7.20}$$

[2] In fact, one only needs to order the product over k, for the reason that all factors with a fixed k commute. However, it is slightly easier to check that formulas (4.7.18) and (4.6.3) coincide if all three products are ordered.

By Lemma 4.7.2, we may shuffle the prefix of (4.7.19) and the first row of (4.7.20) into the following form:

$$\begin{aligned}&\psi\left(E_1\otimes F_1\right)\psi\left(E_{1,2}\otimes F_{1,2}\right)\psi\left(E_2\otimes m_n^{-n}\right)\cdots\psi\left(E_{1,n}\otimes F_{1,n}\right)\psi\left(E_n\otimes m_2^{-2}\right)\\&\cdot\psi\left(E_{1,n+1}\otimes F_{1,n+1}\right)\psi\left(E_{n+1}\otimes m_1^{-1}\right).\end{aligned}\tag{4.7.21}$$

We can then apply Lemma 4.7.3 to write

$$\begin{aligned}&\psi\left(E_1\otimes F_1\right)\psi\left(E_{1,2}\otimes F_{1,2}\right)\psi\left(E_2\otimes m_n^{-n}\right)\\&=\psi\left(E_1\otimes m_{n+1}^{-n-1}\right)\psi\left(E_1\otimes F_1^{\uparrow(-n)_+}\right)\psi\left(E_{1,2}\otimes F_{1,2}\right)\psi\left(E_2\otimes m_n^{-n}\right)\\&=\psi\left(E_1\otimes m_{n+1}^{-n-1}\right)\psi\left(E_2\otimes m_n^{-n}\right)\psi\left(E_1\otimes F_1^{\uparrow(-n)_+}\right)\psi\left(E_{1,2}\otimes F_{1,2}^{\uparrow(1-n)_+}\right).\end{aligned}$$

After repeated applications of Lemma 4.7.3, the last of these being to write

$$\begin{aligned}&\psi\left(E_{1,n}\otimes F_{1,n}^{\uparrow(-1)_+}\right)\psi\left(E_{1,n+1}\otimes F_{1,n+1}\right)\psi\left(E_{n+1}\otimes m_1^{-1}\right)\\=&\psi\left(E_{n+1}\otimes m_1^{-1}\right)\psi\left(E_{1,n}\otimes F_{1,n}^{\uparrow(-1)_+}\right)\psi\left(E_{1,n+1}\otimes F_{1,n+1}^{\geqslant 0}\right),\end{aligned}$$

we arrive at the following form of (4.7.21):

$$\begin{aligned}&\psi\left(E_1\otimes m_{n+1}^{-n-1}\right)\psi\left(E_2\otimes m_n^{-n}\right)\cdots\psi\left(E_{n+1}\otimes m_1^{-1}\right)\\&\cdot\psi\left(E_1\otimes F_1^{\uparrow(-n)_+}\right)\cdots\psi\left(E_{1,n}\otimes F_{1,n}^{\uparrow(-1)_+}\right)\psi\left(E_{1,n+1}\otimes F_{1,n+1}^{\geqslant 0}\right).\end{aligned}$$

We can now repeat this reasoning for each of the next $n-1$ rows in the product (4.7.20). This results in an expression for $\bar{\mathcal{R}}_{n+1}$ of the form

$$\begin{aligned}\bar{\mathcal{R}}_{n+1}=&\psi\left(E_1\otimes m_{n+1}^{-n-1}\right)\psi\left(E_2\otimes m_n^{-n}\right)\cdots\psi\left(E_{n+1}\otimes m_1^{-1}\right)\\&\cdot\psi\left(E_1\otimes m_{n+1}^{-n}\right)\psi\left(E_2\otimes m_n^{-n+1}\right)\cdots\psi\left(E_n\otimes m_2^{-1}\right)\\&\vdots\\&\cdot\psi\left(E_1\otimes m_{n+1}^{-2}\right)\psi\left(E_2\otimes m_n^{-1}\right)\\&\cdot\psi\left(E_1\otimes m_{n+1}^{-1}\right)\\&\cdot\psi\left(E_1\otimes m_{n+1}^{0}\right)\\&\cdot\psi\left(E_1\otimes F_1^{\uparrow 0_+}\right)\psi\left(E_{1,2}\otimes F_{1,2}^{\geqslant 0}\right)\cdots\psi\left(E_{1,n+1}\otimes F_{1,n+1}^{\geqslant 0}\right)\\&\cdot\psi\left(E_2\otimes m_n^{0}\right)\\&\cdot\psi\left(E_3\otimes m_{n-1}^{0}\right)\psi\left(E_2\otimes m_n^{1}\right)\\&\vdots\\&\cdot\psi\left(E_{n+1}\otimes m_1^{0}\right)\psi\left(E_n\otimes m_2^{1}\right)\cdots\psi\left(E_2\otimes m_n^{0}\right).\end{aligned}$$

Note that the first $n+2$ rows of factors in this product are now in the desired form. Now we need to focus on the following factor:

$$\begin{aligned}&\psi\left(E_1\otimes F_1^{\uparrow 0+}\right)\psi\left(E_{1,2}\otimes F_{1,2}^{\geqslant 0}\right)\cdots\psi\left(E_{1,n+1}\otimes F_{1,n+1}^{\geqslant 0}\right)\\ &\cdot\psi\left(E_2\otimes m_n^0\right)\\ &\cdot\psi\left(E_3\otimes m_{n-1}^0\right)\psi\left(E_2\otimes m_n^1\right)\\ &\vdots\\ &\cdot\psi\left(E_{n+1}\otimes m_1^0\right)\psi\left(E_n\otimes m_2^1\right)\cdots\psi\left(E_2\otimes m_n^0\right).\end{aligned}$$

By Lemma 4.7.2, we can reshuffle this block so that it begins with an adjacent triple of terms

$$\begin{aligned}&\psi\left(E_1\otimes F_1^{\uparrow 0+}\right)\psi\left(E_{1,2}\otimes F_{1,2}^{\geqslant 0}\right)\psi\left(E_2\otimes m_n^0\right)\\ &=\psi\left(E_2\otimes m_n^0\right)\psi\left(E_1\otimes F_1^{\uparrow 0+}\right)\Psi\left(E_{1,2}\otimes F_{1,2}^{\uparrow 0+}\right)\\ &=\psi\left(E_2\otimes m_n^0\right)\psi\left(E_1\otimes m_{n+1}^1\right)\psi\left(E_1\otimes F_1^{\uparrow 1+}\right)\psi\left(E_{1,2}\otimes F_{1,2}^{\uparrow 0+}\right),\end{aligned}$$

where we once again used Lemma 4.7.3. Note that now this recovers the correct form of row $(n+3)$ in (4.7.17), continuing in a similar fashion one arrives at the desired expression for $\bar{\mathcal{R}}$. □

4.8 Comparison with Faddeev's results

We conclude by comparing the rank 1 case of our results with Faddeev's embedding (1.3.2) as promised in the introduction. Consider the quiver in Figure 4.4. The corresponding quantum cluster $\mathcal{D}_1$ has initial variables $\langle X_1, X_2, X_3, X_4\rangle$ subject to the relations

$$X_iX_{i+1} = q^{-2}X_{i+1}X_i \quad \text{and} \quad X_iX_{i+2} = X_{i+2}X_i \quad \text{where} \quad i\in\mathbb{Z}/4\mathbb{Z}.$$

In this case, the embedding (4.4.4) takes the form

$$\begin{aligned}\widehat{E}&\mapsto \mathbf{i}X_1(1+qX_2), &\qquad K&\mapsto q^2X_1X_2X_3,\\ \widehat{F}&\mapsto \mathbf{i}X_3(1+qX_4), &\qquad K'&\mapsto q^2X_3X_4X_1,\end{aligned}$$

while our formula (4.7.17) for the universal R-matrix reads $\mathcal{R} = \bar{\mathcal{R}}\mathcal{K}$, with

$$\begin{aligned}\bar{\mathcal{R}} = \Psi^q\left(X_1\otimes X_3\right)&\Psi^q\left(qX_1\otimes X_3X_4\right)\cdot\\ &\Psi^q\left(qX_1X_2\otimes X_3\right)\Psi^q\left(q^2X_1X_2\otimes X_3X_4\right).\end{aligned} \tag{4.8.1}$$

Hence, Faddeev's formulas (1.3.2) and (1.3.3) are recovered from ours under the monomial change of variables

$$w_1\mapsto X_1,\quad w_2\mapsto qX_1X_2,\quad w_3\mapsto X_3,\quad w_4\mapsto qX_3X_4. \tag{4.8.2}$$

Chapter 5

Outlook

The results obtained in this thesis suggest several interesting directions for future research. First, the explicit formula for the images of the Chevalley generators of $U_q(\mathfrak{sl}_n)$ under the embedding described in Chapter 5 turns out to be compatible with their action under the *positive representations* of $U_q(\mathfrak{sl}_n)$ defined by Frenkel and Ip in [30]. These representations are obtained from representations of the quantum cluster torus in which the quantum cluster variables X_i act by positive, essentially self-adjoint operators, and can be regarded as the quantum analog of the celebrated totally positive part of the underlying cluster variety.

In the case of $\mathfrak{sl}_2$, a remarkable theorem of Ponsot and Teschner [9] asserts that the category of positive representations is in fact closed under tensor products in the sense of a direct integral, and thus forms a 'continuous tensor category.' In higher rank, however, the establishing the closure under tensor product has remained an important open problem. Thus, it is natural to pose

Problem 5.0.1. Using the cluster embedding constructed in Chapter 5, apply the tools of quantum Teichmüller theory to equip the positive representations of $U_q(\mathfrak{sl}_n)$ with the structure of a continuous tensor category.

In ongoing work [26] with A. Shapiro, we have developed a novel approach to Problem 5.0.1, which turns out to be closely related to the quantum analogs of classical integrable systems like the ones studied in Chapter 1.

The essential idea is that the decomposition of the tensor product $P_\lambda \otimes P_\mu$ of two positive representations arises from the isomorphism of quantum higher Teichmüller spaces corresponding to cutting along the simple closed curve γ shown in Figure 25. The right hand side of Figure 2 corresponds to a 'fiber product' of two quantum Teichmüller spaces: the diagonally acting copy of $U_q(\mathfrak{sl}_n)$ corresponding to a punctured disk with two marked points on its boundary; and the quantum Teichmüller space for a thrice-punctured sphere with two specified monodromies λ, μ. The fiber product is taken over the unspecified monodromy ν around the loop γ, which is associated to the remaining third puncture of the sphere, and parameterizes the central characters of $U_q(\mathfrak{sl}_n)$. Moreover, the quantum Teichmüller space for the thrice punctured sphere on the right corresponds to the endomorphism algebra of the

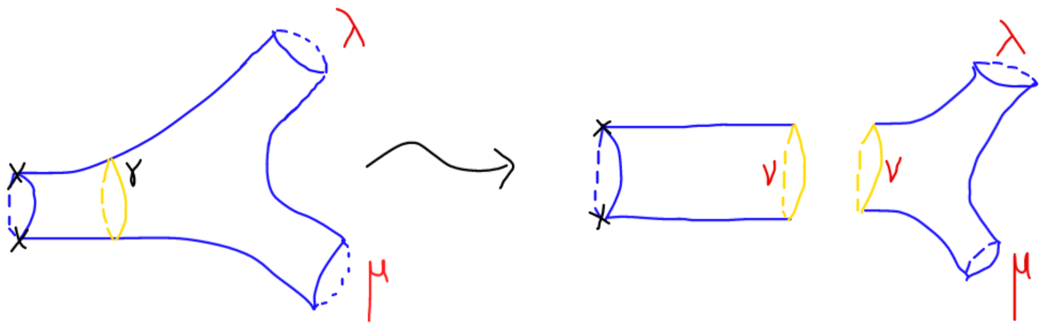

Figure 5.1: Teichmüller theory interpretation of the decomposition of the tensor product of positive representations $P_\lambda \otimes P_\mu$ into positive representations P_ν.

multiplicity space $M_{\lambda\mu}^{\nu}$. Thus Problem 5.0.1 reduces to the problem of understanding the spectrum of the commuting quantum Hamiltonians determining the monodromy ν. However, it turns out that these Hamiltonians can be identified, after a natural sequence of cluster transformations, with those of the *q-difference open Toda chain* [21]. The eigenfunctions of these operators are the celebrated *q-Whittaker functions*, which have been extensively studied [21, 50, 3, 15]. In fact, we are confident that the analytic properties of the q-Whittaker functions can be used to complete the solution of Problem 5.0.1.

Somewhat more ambitiously, one can hope to generalize our results to the infinite dimensional setting of affine Kac-Moody algebras. This would represent a major advance for the field, since the definition of positive representations of a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ has so far been unclear. So we conclude by posing the challenging

Problem 5.0.2. Generalize the results of [53] to give a quantum cluster realization of the quantum affine algebras $U_q(\widehat{\mathfrak{g}})$, and use this to define a category of positive representations of $U_q(\widehat{\mathfrak{g}})$.

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