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Abstract

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Embedded contact homology (ECH) capacities were defined by Hutchings and provide a family of obstructions to embeddings of four-dimensional symplectic manifolds. In Part I of this thesis, we prove that for a four-dimensional Liouville domain with all ECH capacities finite, the asymptotics of the ECH capacities recover the symplectic volume. This was joint work with Daniel Cristofaro-Gardiner and Michael Hutchings. In Part II of this thesis, we construct topological absolute gradings in Heegaard Floer homology and bordered Floer homology that satify all of the expected properties. This was joint work with Yang Huang. We also show that the isomorphism between Heegaard Floer homology and ECH preserves the absolute gradings.

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Part I

Asymptotics of ECH capacities

Chapter 1

Introduction

Define a four-dimensional *Liouville domain*[1] to be a compact symplectic four-manifold (X, ω) with oriented boundary Y such that ω is exact on X, and there exists a contact form λ on Y with $d\lambda = \omega|_Y$. In [13], a sequence of real numbers

$$0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty$$

called *ECH capacities* was defined. The definition is reviewed below in §1.2. The ECH capacities obstruct symplectic embeddings: If (X, ω) symplectically embeds into (X', ω'), then

$$c_k(X, \omega) \leq c_k(X', \omega') \tag{1.1}$$

for all k. For example, a theorem of McDuff [25], see also the survey [14], shows that ECH capacities give a sharp obstruction to symplectically embedding one four-dimensional ellipsoid into another.

The first goal of this paper is to prove the following theorem, relating the asymptotics of the ECH capacities to volume. This result was conjectured in [13] based on experimental evidence; it was proved in [13, §8] for star-shaped domains in $\mathbb{R}^4$ and some other examples.

Theorem 1.0.1. *[13, Conj. 1.12] Let (X, ω) be a four-dimensional Liouville domain such that $c_k(X, \omega) < \infty$ for all k. Then*

$$\lim_{k \to \infty} \frac{c_k(X, \omega)^2}{k} = 4 \operatorname{vol}(X, \omega).$$

Here the symplectic volume is defined by

$$\operatorname{vol}(X, \omega) = \frac{1}{2} \int_X \omega \wedge \omega.$$

In particular, when all ECH capacities are finite, the embedding obstruction (1.1) for large k recovers the obvious volume constraint $\operatorname{vol}(X, \omega) \leq \operatorname{vol}(X', \omega')$. As we review below,

[1] This definition of "Liouville domain" is slightly weaker than the usual definition, which would require that ω have a primitive λ on X which restricts to a contact form on Y.

the hypothesis that $c_k(X,\omega) < \infty$ for all k is a purely topological condition on the contact structure on the boundary; for example it holds whenever X is a star-shaped domain in $\mathbb{R}^4$.

We will obtain Theorem 1.0.1 as a corollary of the more general Theorem 1.3.1 below, which also has applications to refinements of the Weinstein conjecture in Corollary 1.3.2. To state Theorem 1.3.1, we first need to review some notions from embedded contact homology (ECH). More details about ECH may be found in [11] and the references therein.

1.1 Embedded contact homology

Let Y be a closed oriented three-manifold and let λ be a contact form on Y, meaning that $\lambda \wedge d\lambda > 0$. The contact form λ determines a contact structure $\xi = \mathrm{Ker}(\lambda)$, and the Reeb vector field R characterized by $d\lambda(R,\cdot) = 0$ and $\lambda(R) = 1$. Assume that λ is nondegenerate, meaning that all Reeb orbits are nondegenerate. Fix $\Gamma \in H_1(Y)$. The *embedded contact homology* $ECH(Y,\xi,\Gamma)$ is the homology of a chain complex over $\mathbb{Z}/2$ defined as follows.

A generator of the chain complex is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the α_i are distinct embedded Reeb orbits, the m_i are positive integers, $m_i = 1$ whenever α_i is hyperbolic, and the total homology class $\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y)$. To define the chain complex differential ∂ one chooses a generic almost complex structure J on $\mathbb{R}\times Y$ such that $J(\partial_s) = R$ where s denotes the $\mathbb{R}$ coordinate, $J(\xi) = \xi$ with $d\lambda(v, Jv) \geq 0$ for $v \in \xi$, and J is $\mathbb{R}$-invariant. Given another chain complex generator $\beta = \{(\beta_j, n_j)\}$, the differential coefficient $\langle \partial\alpha, \beta\rangle \in \mathbb{Z}/2$ is a mod 2 count of J-holomorphic curves in $\mathbb{R}\times Y$ that converge as currents to $\sum_i m_i\alpha_i$ as $s \to +\infty$ and to $\sum_j n_j\beta_j$ as $s \to -\infty$, and that have "ECH index" equal to 1. The definition of the ECH index is explained in [10]; all we need to know here is that the ECH index defines a relative $\mathbb{Z}/d$-grading on the chain complex, where d denotes the divisibility of $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ in $H^2(Y;\mathbb{Z})$ mod torsion. It is shown in [16, §7] that $\partial^2 = 0$.

One now defines $ECH(Y,\lambda,\Gamma,J)$ to be the homology of the chain complex $ECC(Y,\lambda,\Gamma,J)$. Taubes [35] proved that if Y is connected, then there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$-modules

$$ECH_*(Y,\lambda,\Gamma,J) = \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma)). \tag{1.2}$$

Here $\widehat{HM}^*$ denotes the 'from' version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [20], with $\mathbb{Z}/2$ coefficients[2], and $\mathfrak{s}_\xi$ denotes the spin-c structure determined by the oriented 2-plane field ξ, see e.g. [12, Ex. 8.2]. It follows that, whether or not Y is connected, $ECH(Y,\lambda,\Gamma,J)$ depends only on Y, ξ, and Γ, and so can be denoted by $ECH_*(Y,\xi,\Gamma)$.

There is a degree -2 map

$$U : ECH_*(Y,\xi,\Gamma) \longrightarrow ECH_{*-2}(Y,\xi,\Gamma). \tag{1.3}$$

[2]One can define ECH with integer coefficients [17, §9], and the isomorphism (1.2) also exists over $\mathbb{Z}$, as shown in [37]. However $\mathbb{Z}/2$ coefficients will suffice for this paper.

This map on homology is induced by a chain map which counts J-holomorphic curves with ECH index 2 that pass through a base point in $\mathbb{R} \times Y$. When Y is connected, the U map (1.3) does not depend on the choice of base point, and agrees under Taubes's isomorphism (1.2) with an analogous map on Seiberg-Witten Floer cohomology [38]. If Y is disconnected, then there is one U map for each component of Y.

Although ECH is a topological invariant by (1.2), it contains a distinguished class which can distinguish some contact structures. Namely, the empty set of Reeb orbits is a generator of $ECC(Y, \lambda, 0, J)$; it is a cycle by the conditions on J, and so it defines a distinguished class

$$[\emptyset] \in ECH(Y, \xi, 0), \tag{1.4}$$

called the *ECH contact invariant*. Under the isomorphism (1.2), the ECH contact invariant agrees with an analogous contact invariant in Seiberg-Witten Floer cohomology [38].

There is also a "filtered" version of ECH, which is sensitive to the contact form and not just the contact structure. If $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the chain complex $ECC(Y, \lambda, \Gamma, J)$, its *symplectic action* is defined by

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda. \tag{1.5}$$

It follows from the conditions on the almost complex structure J that if the differential coefficient $\langle \partial\alpha, \beta \rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$. Consequently, for each $L \in \mathbb{R}$, the span of those generators α with $\mathcal{A}(\alpha) < L$ is a subcomplex, which is denoted by $ECC^L(Y, \lambda, \Gamma, J)$. The homology of this subcomplex is the *filtered ECH* which is denoted by $ECH^L(Y, \lambda, \Gamma)$. Inclusion of chain complexes induces a map

$$ECH^L(Y, \lambda, \Gamma) \longrightarrow ECH(Y, \xi, \Gamma). \tag{1.6}$$

It is shown in [15, Thm. 1.3] that $ECH^L(Y, \lambda, \Gamma)$ and the map (1.6) do not depend on the almost complex structure J.

A useful way to extract invariants of the contact form out of filtered ECH is as follows. Given a nonzero class $\sigma \in ECH(Y, \xi, \Gamma)$, define

$$c_\sigma(Y, \lambda) \in \mathbb{R}$$

to be the infimum over L such that the class σ is in the image of the inclusion-induced map (1.6). So far we have been assuming that the contact form λ is nondegenerate. If λ is degenerate, one defines $c_\sigma(Y, \lambda) = \lim_{n\to\infty} c_\sigma(Y, \lambda_n)$, where $\{\lambda_n\}$ is a sequence of nondegenerate contact forms which C^0-converges to λ, cf. [13, §3.1].

1.2 ECH capacities

Let (Y, λ) be a closed contact three-manifold and assume that the ECH contact invariant (1.4) is nonzero. Given a nonnegative integer k, define $c_k(Y, \lambda)$ to be the minimum of $c_\sigma(Y, \lambda)$,

where σ ranges over classes in $ECH(Y,\xi,0)$ such that $A\sigma = [\emptyset]$ whenever A is a composition of k of the U maps associated to the components of Y. If no such class σ exists, define $c_k(Y,\lambda) = \infty$. The sequence $\{c_k(Y,\lambda)\}_{k=0,1,\ldots}$ is called the *ECH spectrum* of (Y,λ).

Now let (X,ω) be a Liouville domain with boundary Y and let λ be a contact form on Y with $d\lambda = \omega|_Y$. One then defines the ECH capacities of (X,ω) in terms of the ECH spectrum of (Y,λ) by

$$c_k(X,\omega) = c_k(Y,\lambda).$$

This definition is valid because the ECH contact invariant of (Y,λ) is nonzero by [15, Thm. 1.9]. It follows from [13, Lem. 3.9] that $c_k(X,\omega)$ does not depend on the choice of contact form λ on Y with $d\lambda = \omega|_Y$.

Theorem 1.0.1 is now a consequence of the following result about the ECH spectrum:

Theorem 1.2.1. *[13, Conj. 8.1] Let (Y,λ) be a closed contact three-manifold with nonzero ECH contact invariant. If $c_k(Y,\lambda) < \infty$ for all k, then*

$$\lim_{k\to\infty} \frac{c_k(Y,\lambda)^2}{k} = 2\,\mathrm{vol}(Y,\lambda).$$

Here the contact volume is defined by

$$\mathrm{vol}(Y,\lambda) = \int_Y \lambda \wedge d\lambda. \tag{1.7}$$

Note that the hypothesis $c_k(Y,\lambda) < \infty$ just means that the ECH contact invariant is in the image of all powers of the U map when Y is connected, or all compositions of powers of the U maps when Y is disconnected. The comparison with Seiberg-Witten theory implies that this is possible only if $c_1(\xi) \in H^2(Y;\mathbb{Z})$ is torsion; see [13, Rem. 4.4(b)].

By [13, Prop. 8.4], to prove Theorem 1.2.1 it suffices to consider the case when Y is connected. Theorem 1.2.1 in this case follows from our main theorem which we now state.

1.3 The main theorem

Recall from §1.1 that if $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y;\mathbb{Z})$ is torsion, then $ECH(Y,\xi,\Gamma)$ has a relative $\mathbb{Z}$-grading, and we can arbitrarily refine this to an absolute $\mathbb{Z}$-grading. The main theorem is now:

Theorem 1.3.1. *[13, Conj. 8.7] Let Y be a closed connected contact three-manifold with a contact form λ and let $\Gamma \in H_1(Y)$. Suppose that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion in $H^2(Y;\mathbb{Z})$, and let I be an absolute $\mathbb{Z}$-grading of $ECH(Y,\xi,\Gamma)$. Let $\{\sigma_k\}_{k\geq 1}$ be a sequence of nonzero homogeneous classes in $ECH(Y,\xi,\Gamma)$ with $\lim_{k\to\infty} I(\sigma_k) = \infty$. Then*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} = \mathrm{vol}(Y,\lambda). \tag{1.8}$$

The following application of Theorem 1.3.1 was obtained in [5]:

Corollary 1.3.2. *[5, Thm. 1.1] Every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits.*

The proof of Theorem 1.3.1 has two parts. In §2 we show that the left hand side of (1.8) (with lim replaced by lim sup) is less than or equal to the right hand side. This is actually all that is needed for Corollary 1.3.2. In §3 we show that the left hand side (with lim replaced by lim inf) is greater than or equal to the right hand side. The two arguments are independent of each other and can be read in either order. The proof of the upper bound uses ingredients from Taubes's proof of the isomorphism (1.2). The proof of the lower bound uses properties of ECH cobordism maps to reduce to the case of a sphere, where (1.8) can be checked explicitly. Part I was joint work with Daniel Cristofaro-Gardiner and Michael Hutchings.

Chapter 2

The upper bound

In this chapter we prove the upper bound half of Theorem 1.3.1:

Proposition 2.0.3. *Under the assumptions of Theorem 1.3.1,*

$$\limsup_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \le \mathrm{vol}(Y,\lambda). \tag{2.1}$$

To prove Proposition 2.0.3, we can assume without loss of generality that λ is nondegenerate. To see this, assume that (2.1) holds for nondegenerate contact forms and suppose that λ is degenerate. We can find a sequence of functions $f_1 > f_2 > \cdots > 1$, which C^0-converges to 1, such that $f_n\lambda$ is nondegenerate for each n. It follows from the monotonicity property in [13, Lem. 4.2] that

$$c_{\sigma_k}(Y,\lambda) \le c_{\sigma_k}(Y, f_n\lambda)$$

for every n and k. For each n, it follows from this and the inequality (2.1) for λ_n that

$$\limsup_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \le \mathrm{vol}(Y, f_n\lambda).$$

Since $\lim_{n\to\infty} \mathrm{vol}(Y, f_n\lambda) = \mathrm{vol}(Y,\lambda)$, we deduce the inequality (2.1) for λ.

Assume henceforth that λ is nondegenerate. In §2.1–§2.6 below we review some aspects of Taubes's proof of the isomorphism (1.2) and prove some related lemmas. In §2.7 we use these to prove Proposition 2.0.3.

2.1 Seiberg-Witten Floer cohomology

The proof of the isomorphism (1.2) involves perturbing the Seiberg-Witten equations on Y. To write down the Seiberg-Witten equations we first need to choose a Riemannian metric on Y. Let J be a generic almost complex structure on $\mathbb{R}\times Y$ as needed to define the ECH

chain complex. The almost complex structure J determines a Riemannian metric g on Y such that the Reeb vector field R has length 1 and is orthogonal to the contact planes ξ, and

$$g(v,w) = \frac{1}{2} d\lambda(v, Jw), \qquad v, w \in \xi_y. \tag{2.2}$$

Note that this metric satisfies

$$|\lambda| = 1, \qquad *d\lambda = 2\lambda. \tag{2.3}$$

One could dispense with the factors of 2 in (2.2) and (2.3), but we are keeping them for consistency with [12] and its sequels.

Let $\mathbb{S}$ denote the spin bundle for the spin-c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$. The inputs to the Seiberg-Witten equations for this spin-c structure are a connection $\mathbb{A}$ on $\det(\mathbb{S})$ and a section ψ of $\mathbb{S}$. The spin bundle $\mathbb{S}$ splits as a direct sum

$$\mathbb{S} = E \oplus (E \otimes \xi),$$

where E and $E \otimes \xi$ are, respectively, the $+i$ and $-i$ eigenspaces of Clifford multiplication by λ. Here ξ is regarded as a complex line bundle using the metric and the orientation. A connection $\mathbb{A}$ on $\det(\mathbb{S})$ is then equivalent to a (Hermitian) connection A on E via the relation $\mathbb{A} = A_0 + 2A$, where A_0 is a distinguished connection on ξ reviewed in [33, §2.1].

For a positive real number r, consider the following version of the perturbed Seiberg-Witten equations for a connection A on E and spinor ψ:

$$\begin{aligned} *F_A &= r(\langle cl(\cdot)\psi, \psi\rangle - i\lambda) + i(*d\mu + \bar{\omega}) \\ D_A\psi &= 0. \end{aligned} \tag{2.4}$$

Here cl denotes Clifford multiplcation, $\bar{\omega}$ denotes the harmonic 1-form such that $*\bar{\omega}/\pi$ represents the image of $c_1(\xi)$ in $H^2(Y;\mathbb{R})$, and μ is a generic coclosed 1-form that is L^2-orthogonal to the space of harmonic 1-forms and that has "P-norm" less than 1, see [33, §2.1].

The group of *gauge transformations* $C^\infty(Y, S^1)$ acts on the space of pairs $(\mathbb{A}, \psi)$ by $g \cdot (\mathbb{A}, \psi) = (\mathbb{A} - 2g^{-1}dg, g\psi)$. The quotient of the space of pairs $(\mathbb{A}, \psi)$ by the group of gauge transformations is called the *configuration space*. The set of solutions to (2.4) is invariant under gauge transformations. A solution to the Seiberg-Witten equations is called *reducible* if $\psi \equiv 0$ and *irreducible* otherwise. An irreducible solution is called *nondegenerate* if it is cut out transversely after modding out by gauge transformations, see [33, §3.1].

For fixed μ, when r is not in a certain discrete set, there are only finitely many irreducible solutions to (2.4) and these are all nondegenerate. In this case one can define the Seiberg-Witten Floer cohomology chain complex with $\mathbb{Z}/2$ coefficients, which we denote by $\widehat{CM}^*(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda, J, r)$. The chain complex is generated by irreducible solutions to (2.4), along with additional generators determined by the reducible solutions. The differential counts solutions to a small abstract perturbation of the four-dimensional Seiberg-Witten equations on $\mathbb{R} \times Y$. In principle the chain complex differential may depend on the choice of abstract perturbation, but since the abstract perturbation is irrelevant to the proof of Proposition 2.0.3, we will omit it from the notation.

2.2 The grading

The chain complex $\widehat{CM}^*$ has a noncanonical absolute $\mathbb{Z}$-grading defined as follows. The linearization of the equations (2.4) modulo gauge equivalence at a pair (A,ψ), not necessarily solving the equations (2.4), defines a self-adjoint Fredholm operator $\mathcal{L}_{A,\psi}$. If (A,ψ) is a nondegenerate irreducible solution to (2.4), then the operator $\mathcal{L}_{A,\psi}$ has trivial kernel, and one defines the grading $gr(A,\psi)\in\mathbb{Z}$ to be the spectral flow from $\mathcal{L}_{A,\psi}$ to a reference self-adjoint Fredholm operator $\mathcal{L}_0$ between the same spaces with trivial kernel. The grading function gr depends on the choice of reference operator; fix one below. To describe the gradings of the remaining generators, recall that the set of reducible solutions modulo gauge equivalence is a torus $\mathbb{T}$ of dimension $b_1(Y)$. As explained in [20, §35.1], one can perturb the Seiberg-Witten equations using a Morse function

$$f:\mathbb{T}\to\mathbb{R}, \tag{2.5}$$

so that the chain complex generators arising from reducibles are identified with pairs $((A,0),\phi)$, where $(A,0)$ is a critical point of f and ϕ is a suitable eigenfunction of the Dirac operator D_A. The grading of each such generator is less than or equal to $gr(A,0)$, where the latter is defined as the spectral flow to $\mathcal{L}_0$ from an appropriate perturbation of the operator $\mathcal{L}_{A,0}$.

We will need the following key result of Taubes relating the grading to the Chern-Simons functional. Fix a reference connection A_E on E. Given any other connection A on E, define the *Chern-Simons functional*

$$cs(A)=-\int_Y (A-A_E)\wedge(F_A+F_{A_E}-2i*\bar{\omega}). \tag{2.6}$$

Note that this functional is gauge invariant because the spin-c structure $\mathfrak{s}_\xi$+PD(Γ) is assumed torsion.

Proposition 2.2.1. *[33, Prop. 5.1] There exists $K>0$ such that for all r sufficiently large, if (A,ψ) is a nondegenerate irreducible solution to* (2.4)*, or a reducible solution which is a critical point of* (2.5)*, then*

$$\left|gr(A,\psi)+\frac{1}{4\pi^2}cs(A)\right|<Kr^{31/16}. \tag{2.7}$$

2.3 Energy

Given a connection A on E, define the *energy*

$$\mathcal{E}(A)=i\int_Y \lambda\wedge F_A.$$

Filtered ECH has a Seiberg-Witten analogue defined using the energy functional as follows. Given a real number L, define $\widehat{CM}^*_L$ to be the submodule of $\widehat{CM}^*$ spanned by generators

with energy less than $2\pi L$. It is shown in [33], as reviewed in [15, Lem. 2.3], that if r is sufficiently large, then all chain complex generators with energy less than $2\pi L$ are irreducible, and $\widehat{CM}^*_L$ is a subcomplex, whose homology we denote by $\widehat{HM}^*_L$. Moreover, as shown in [33] and reviewed in [15, Eq.(3.3)], if there are no ECH generators of action exactly L and if r is sufficiently large, then there is a canonical isomorphism of relatively graded chain complexes

$$ECC^L_*(Y,\lambda,\Gamma,J) \longrightarrow \widehat{CM}_L^{-*}(Y,\mathfrak{s}_{\xi,\Gamma},\lambda_1,J_1,r). \tag{2.8}$$

Here (λ_1, J_1) is an "L-flat approximation" to (λ, J), which is obtained by suitably modifying (λ, J) near the Reeb orbits of action less than L; the precise definition is reviewed in [15, §3.1] and will not be needed here.

The isomorphism (2.8) is induced by a bijection on generators; the idea is that in the L-flat case[1], if r is sufficiently large, then for every ECH generator α of action less than L, there is a corresponding irreducible solution (A,ψ) to (2.4) such that the zero set of the E component of ψ is close to the Reeb orbits in α, the curvature F_A is concentrated near these Reeb orbits, and the energy of this solution is approximately $2\pi\mathcal{A}(\alpha)$.

The isomorphism of chain complexes (2.8) induces an isomorphism on homology

$$ECH^L_*(Y,\lambda,\Gamma,J) \xrightarrow{\simeq} \widehat{HM}_L^{-*}(Y,\mathfrak{s}_{\xi,\Gamma},\lambda_1,J_1,r), \tag{2.9}$$

and inclusion of chain complexes defines a map

$$\widehat{HM}_L^{-*}(Y,\mathfrak{s}_{\xi,\Gamma},\lambda_1,J_1,r) \longrightarrow \widehat{HM}^{-*}(Y,\mathfrak{s}_{\xi,\Gamma}). \tag{2.10}$$

Composing the above two maps gives a map

$$ECH^L_*(Y,\lambda,\Gamma,J) \longrightarrow \widehat{HM}^{-*}(Y,\mathfrak{s}_{\xi,\Gamma}). \tag{2.11}$$

The isomorphism (1.2) is the direct limit over L of the maps (2.11).

2.4 Volume in Seiberg-Witten theory

The volume enters into the proof of Proposition 2.0.3 in two essential ways.

The first way is as follows. It is shown in [12, §3] that for any given grading, there are no generators arising from reducibles if r is sufficiently large. That is, given an integer j, let s_j be the supremum of all values of r such that there exists a chain complex generator with grading at least $-j$ associated to a reducible solution to (2.4). Then $s_j < \infty$ for all j.

We now give an upper bound on the number s_j in terms of the volume. Fix $0 < \delta < \frac{1}{16}$. Given a positive integer j, let r_j be the largest real number such that

$$j = \frac{1}{16\pi^2} r_j^2 \operatorname{vol}(Y,\lambda) - r_j^{2-\delta}. \tag{2.12}$$

[1]In the non-L-flat case, there may be several Seiberg-Witten solutions corresponding to the same ECH generator, and/or Seiberg-Witten solutions corresponding to sets of Reeb orbits with multiplicities which are not ECH generators because they include hyperbolic orbits with multiplicity greater than one.

Lemma 2.4.1. *If j is sufficiently large, then $s_j < r_j$.*

Proof. Observe that $(A_r^{red}, \psi) = (A_E - \frac{1}{2}ir\lambda + i\mu, 0)$ is a solution to (2.4). Moreover, every other reducible solution is given by $(A, 0)$, where $A = A_r^{red} + \alpha$ for a closed 1-form α. It follows from (2.6) that

$$cs(A) = cs(A_r^{red}) = \frac{1}{4}r^2 \operatorname{vol}(Y, \lambda) + O(r). \tag{2.13}$$

Now suppose that j is sufficiently large that Proposition 2.2.1 is applicable to $r = r_j$, fix $r > r_j$, and suppose that $gr(A, 0) \geq -j$. Then equation (2.13) contradicts Proposition 2.2.1 if r is sufficiently large, which is the case if j is sufficiently large. □

The second essential way that volume enters into the proof of Proposition 2.0.3 is via the following a priori upper bound on the energy:

Lemma 2.4.2. *There is an r-independent constant C such that any solution (A, ψ) to (2.4) satisfies*

$$\mathcal{E}(A) \leq \frac{r}{2} \operatorname{vol}(Y, \lambda) + C. \tag{2.14}$$

Proof. This follows from [33, Eq. (2.7)], which is proved using a priori estimates on solutions to the Seiberg-Witten equations. Note that there is a factor of 1/2 in (2.14) which is not present in [33, Eq. (2.7)]. The reason is that the latter uses the Riemannian volume as defined by the metric (2.3), which is half of the contact volume (1.7) which we are using. □

2.5 Max-min families

Given a connection A on E and a section ψ of $\mathbb{S}$, define a functional

$$\mathcal{F}(A, \psi) = \frac{1}{2}(cs(A) - r\mathcal{E}(A)) + e_\mu(A) + \frac{r}{2}\int_Y \langle D_A\psi, \psi\rangle d\text{vol},$$

where

$$e_\mu(A) = i\int_Y F_A \wedge \mu.$$

Since the spin-c structure $\mathfrak{s}_\xi + \text{PD}(\Gamma)$ is assumed torsion, the functional $\mathcal{F}$ is gauge invariant. The significance of the functional $\mathcal{F}$ is that the differential on the chain complex $\widehat{CM}^*$ counts solutions to abstract perturbations of the upward gradient flow equation for $\mathcal{F}$. In particular, $\mathcal{F}$ agrees with an appropriately perturbed version of the Chern-Simons-Dirac functional from [20], up to addition of an r-dependent constant, see [15, Eq. (7.2)].

A key step in Taubes's proof of the Weinstein conjecture [33] is to use a "minimax" approach to find a sequence (r_n, ψ_n, A_n), where $r_n \to \infty$ and (ψ_n, A_n) is a solution to (2.4) for $r = r_n$ with an n-independent bound on the energy. We will use a similar construction in the proof of Proposition 2.0.3.

Specifically, fix an integer j, and let s_j be the number from §2.2. Let $\hat{\sigma} \in \widehat{HM}^*(Y, \mathfrak{s}_{\xi,\Gamma})$ be a nonzero homogeneous class with grading greater than or equal to $-j$. Fix $r > s_j$ for which the chain complex $\widehat{CM}^*(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda, J, r)$ is defined. Since we are using $\mathbb{Z}/2$-coefficients, any cycle representing the class $\hat{\sigma}$ has the form $\eta = \Sigma_i(A_i, \psi_i)$, where the pairs (A_i, ψ_i) are distinct gauge equivalence classes of solutions to (2.4). Define $\mathcal{F}_{\min}(\eta) = \min_i \mathcal{F}(A_i, \psi_i)$, and $\mathcal{F}_{\hat{\sigma}}(r) = \max_{[\eta]=\hat{\sigma}} \mathcal{F}_{\min}(\eta)$. Note that $\mathcal{F}_{\hat{\sigma}}(r)$ must be finite because there are only finitely many irreducible solutions to (2.4).

The construction in [35, §4.e] shows that for any such class $\hat{\sigma}$, there exists a piecewise smooth, possibly discontinuous family of solutions $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$ to (2.4) of the same grading as $\hat{\sigma}$ defined for $r > s_j$ such that $\mathcal{F}_{\hat{\sigma}}(r) = \mathcal{F}(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$. Call the family $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))_{r>s_j}$ a *max-min family* for $\hat{\sigma}$. Given such a max-min family, define $\mathcal{E}_{\hat{\sigma}}(r) = \mathcal{E}(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$.

Lemma 2.5.1. *(a) $\mathcal{F}_{\hat{\sigma}}(r)$ is a continuous and piecewise smooth function of $r \in (s_j, \infty)$.*

(b) $\dfrac{d}{dr}\mathcal{F}_{\hat{\sigma}}(r) = -\dfrac{1}{2}\mathcal{E}_{\hat{\sigma}}(r)$.

Proof. (a) follows from [35, Prop. 4.7], and (b) follows from [33, Eq. (4.6)]. □

In particular, $\mathcal{E}_{\hat{\sigma}}(r)$ does not depend on the choice of max-min family, except for a discrete set of real numbers r.

2.6 Max-min energy and min-max symplectic action

The numbers $\mathcal{E}_{\hat{\sigma}}(r)$ from §2.5 are related to the numbers $c_\sigma(Y, \lambda)$ from §1.2 as follows:

Proposition 2.6.1. *Let σ be a nonzero homogeneous class in $ECH(Y, \xi, \Gamma)$, and let $\hat{\sigma} \in \widehat{HM}^*(Y, \mathfrak{s}_{\xi,\Gamma})$ denote the class corresponding to σ under the isomorphism* (1.2)*. Then*

$$\lim_{r\to\infty} \mathcal{E}_{\hat{\sigma}}(r) = 2\pi c_\sigma(Y, \lambda).$$

The proof of Proposition 2.6.1 requires two preliminary lemmas which will also be needed later. To state the first lemma, recall from [34, Prop. 2.8] that in the case $\Gamma = 0$, if r is sufficiently large then there is a unique (up to gauge equivalence) "trivial" solution (A_{triv}, ψ_{triv}) to (2.4) such that $1 - |\psi| < 1/2$ on all of Y. If (λ, J) is L-flat with $L > 0$, then (A_{triv}, ψ_{triv}) corresponds to the empty set of Reeb orbits under the isomorphism (2.8) with $\Gamma = 0$, see the beginning of [36, §3]. Any solution not gauge equivalent to (A_{triv}, ψ_{triv}) will be called "nontrivial". Let L_0 denote one half the minimum symplectic action of a Reeb orbit.

Lemma 2.6.2. *There exists an r-independent constant c such that if r is sufficiently large, then every nontrivial solution (A, ψ) to* (2.4) *satisfies $\mathcal{E}(A) > 2\pi L_0$ and*

$$|cs(A)| \leq cr^{2/3}\mathcal{E}(A)^{4/3}. \tag{2.15}$$

Proof. The chain complex $ECC_*^{L_0}(Y,\lambda,\Gamma,J)$ has no generators unless $\Gamma = 0$, in which case the only generator is the empty set of Reeb orbits. In particular, the pair (λ, J) is L_0-flat. By (2.8), if r is sufficiently large then every nontrivial solution (A,ψ) to (2.4) has $\mathcal{E}(A) \geq 2\pi L_0$. Given this positive lower bound on the energy, the estimate (2.15) now follows as in [33, Eq. (4.9)]. Note that it is assumed there that $E(A) \geq 1$, but the same argument works as long as there is a positive lower bound on $E(A)$. □

Now fix a positive number γ such that $\gamma < \delta/4$.

Lemma 2.6.3. *For every integer j there exists $\rho \geq 0$ such if $r \geq \rho$ and (A,ψ) is a nontrivial irreducible solution to* (2.4) *of grading $-j$, then*

$$|cs(A)| \leq r^{1-\gamma}\mathcal{E}(A). \tag{2.16}$$

Proof. Fix j. Let (A,ψ) be a nontrivial solution to (2.4) of grading $-j$ with

$$|cs(A)| > r^{1-\gamma}\mathcal{E}(A). \tag{2.17}$$

By Lemma 2.6.2, if r is sufficiently large then

$$|cs(A)| \leq cr^{2/3}\mathcal{E}(A)^{4/3}. \tag{2.18}$$

Combining (2.17) with (2.18), we conclude that $\mathcal{E}(A) \geq c^{-3}r^{1-3\gamma}$. Using (2.17) again, it follows that

$$|cs(A)| > c^{-3}r^{2-4\gamma}.$$

But this contradicts Proposition 2.2.1 when r is sufficiently large with respect to j, since $\delta > 4\gamma$. □

Proof of Proposition 2.6.1. Choose $L_0 > c_\sigma(Y,\lambda)$ and let (λ_1, J_1) be an L_0-flat approximation to (λ, J). For r large, define $f_1(r)$ to be the infimum over L such that the class $\hat{\sigma}$ is in the image of the map (2.10). We first claim that

$$\lim_{r\to\infty}\left(f_1(r) - c_\sigma(Y,\lambda)\right) = 0. \tag{2.19}$$

This holds because for every $L \leq L_0$ which is not the symplectic action of an ECH generator, in particular $L \neq c_\sigma(Y,\lambda)$, if r is sufficiently large that the isomorphism (2.9) is defined, then the class $\hat{\sigma}$ is in the image of the map (2.10) if and only if $L > c_\sigma(Y,\lambda)$.

Next define $f(r)$ for r large to be the infimum over L such that the class $\hat{\sigma}$ is in the image of the inclusion-induced map

$$\widehat{HM}^*_L(Y,\mathfrak{s}_{\xi,\Gamma},\lambda,J,r) \to \widehat{HM}^*(Y,\mathfrak{s}_{\xi,\Gamma}). \tag{2.20}$$

It follows from [15, Lem. 3.4(c)] that

$$\lim_{r\to\infty}\left(f(r) - f_1(r)\right) = 0. \tag{2.21}$$

By (2.19) and (2.21), to complete the proof of Proposition 2.6.1 it is enough to show that

$$\lim_{r\to\infty}\left(\mathcal{E}_{\hat{\sigma}}(r)-2\pi f(r)\right)=0. \tag{2.22}$$

To prepare for the proof of (2.22), assume that r is sufficiently large so that Lemma 2.6.2 is applicable and Lemma 2.6.3 is applicable to $j=-gr(\hat{\sigma})$. Also assume that r is sufficiently large so that all nontrivial Seiberg-Witten solutions in grading $gr(\hat{\sigma})$ are irreducible and have positive energy. Let (A,ψ) be a nontrivial solution in grading $gr(\hat{\sigma})$. Then

$$\mathcal{F}(A,\psi)=\frac{1}{2}(cs(A)-r\mathcal{E}(A))+e_{\mu}(A).$$

By [33, Eq. (4.2)] and Lemma 2.6.2, we have

$$|e_{\mu}(A)|\leq\kappa\mathcal{E}(A) \tag{2.23}$$

where κ is an r-independent constant. The above and Lemma 2.6.3 imply that

$$(1-r^{-\gamma}-2\kappa r^{-1})\mathcal{E}(A)\leq\frac{-2}{r}\mathcal{F}(A,\psi)\leq(1+r^{-\gamma}+2\kappa r^{-1})\mathcal{E}(A). \tag{2.24}$$

Also, it follows from the construction of the trivial solution in [34] that

$$\lim_{r\to\infty}\mathcal{E}(A_{triv})=\lim_{r\to\infty}\frac{\mathcal{F}(A_{triv},\psi_{triv})}{r}=0. \tag{2.25}$$

Now (2.22) can be deduced easily from (2.24) and (2.25). The details are as follows. Fix $\varepsilon>0$ and suppose that r is sufficiently large as in the above paragraph. By the definition of $f(r)$, the class $\hat{\sigma}$ is in the image of the map (2.20) for $L=f(r)+\varepsilon$. Also, if r is sufficiently large, then by (2.24) and (2.25), and the fact that L has an upper bound when r is large by (2.19) and (2.21), if $\eta=\sum_i(A_i,\psi_i)$ is a cycle in $\widehat{CM}_L$ representing the class $\hat{\sigma}$, then $-2\mathcal{F}(A_i,\psi_i)/r<2\pi(L+\varepsilon)$ for each i. Consequently $-2\mathcal{F}_{\hat{\sigma}}(r)/r<2\pi(L+\varepsilon)$. By (2.24) and (2.25) again, if r is sufficiently large then $\mathcal{E}_{\hat{\sigma}}(r)<2\pi(L+2\varepsilon)$, which means that $\mathcal{E}_{\hat{\sigma}}(r)<f(r)+3\varepsilon$.

By similar reasoning, if $\mathcal{E}_{\hat{\sigma}}(r)<f(r)-\varepsilon$, then if r is sufficiently large, the class $\hat{\sigma}$ is in the image of the map (2.20) for $L=f(r)-\varepsilon/2$, which contradicts the definition of $f(r)$. □

2.7 Proof of the upper bound

Proof of Proposition 2.0.3. The proof has six steps.

Step 1: Setup. If $\sigma\in ECH_*(Y,\xi,\Gamma)$ is a nonzero homogeneous class, let $\hat{\sigma}\in\widehat{HM}^*(Y,\mathfrak{s}_{\xi,\Gamma})$ denote the corresponding class in Seiberg-Witten Floer cohomology via the isomorphism (1.2). We can choose the absolute grading I on $ECH(Y,\xi,\Gamma)$ so that the Seiberg-Witten grading of $\hat{\sigma}$ is $-I(\sigma)$ for all σ. For Steps 1–5, fix such a class σ and write $j=I(\sigma)$. We

will obtain an upper bound on $c_\sigma(Y,\lambda)$ in terms of j when j is sufficiently large, see (2.37) below.

To start, we always assume that j is sufficiently large so that $j > 0$, the number r_j defined in (2.12) satisfies $r_j \geq 1$, Proposition 2.2.1 and Lemma 2.6.2 are applicable to $r \geq r_j$, Lemma 2.4.1 is applicable so that $r_j > s_j$, and the trivial solution (A_{triv}, ψ_{triv}) does not have grading $-j$.

Fix a max-min family $(A_{\hat\sigma}(r), \psi_{\hat\sigma}(r))_{r>s_j}$ for $\hat\sigma$ as in §2.5. For $r > s_j$ define

$$\begin{aligned} \mathcal{E}(r) &= \mathcal{E}_{\hat\sigma}(r) = \mathcal{E}(A_{\hat\sigma}(r)), \\ cs(r) &= cs(A_{\hat\sigma}(r)), \\ e_\mu(r) &= e_\mu(A_{\hat\sigma}(r)), \\ v(r) &= -\frac{2\mathcal{F}_{\hat\sigma}(r)}{r} = \mathcal{E}(r) - \frac{cs(r)}{r} - \frac{2e_\mu(r)}{r}. \end{aligned} \tag{2.26}$$

It follows from Lemma 2.5.1 that $v(r)$ is continuous and piecewise smooth, and

$$\frac{dv(r)}{dr} = \frac{cs(r)}{r^2} + \frac{2e_\mu(r)}{r^2}. \tag{2.27}$$

By Proposition 2.2.1 we have the key estimate

$$\left| -j + \frac{1}{4\pi^2} cs(r) \right| < K r^{2-\delta} \tag{2.28}$$

whenever $r \geq r_j$. Here we are using the fact that Lemma 2.4.1 is applicable, so that the solution $(A_{\hat\sigma}(r), \psi_{\hat\sigma}(r))$ is irreducible, so that $gr(A_{\hat\sigma}(r), \psi_{\hat\sigma}(r)) = -j$.

Define a number $\overline{r} = \overline{r}_{\hat\sigma}$ as follows. We know from Lemma 2.6.3 that if r is sufficiently large then

$$|cs(r)| \leq r^{1-\gamma} \mathcal{E}(r). \tag{2.29}$$

If (2.29) holds for all $r \geq r_j$, define $\overline{r} = r_j$. Otherwise define $\overline{r}$ to be the supremum of the set of r for which (2.29) does not hold.

Step 2. We now show that

$$\limsup_{r \geq \bar r} \mathcal{E}(r) \leq v(\bar r) g(\bar r), \tag{2.30}$$

where

$$g(r) = \exp\left(\frac{r^{-\gamma} + 2\gamma\kappa r^{-1}}{\gamma\left(1 - r^{-\gamma} - 2\kappa r^{-1}\right)} \right), \tag{2.31}$$

and κ is the constant in (2.23). Here and below we assume that j is sufficiently large so that $1 - r_j^{-\gamma} - 2\kappa r_j^{-1} > 0$.

To prove (2.30), assume that $r \geq \bar r$. Then by (2.26), (2.29), and (2.23), as in (2.24), we have

$$\mathcal{E}(r) \leq \frac{1}{1 - r^{-\gamma} - 2\kappa r^{-1}} v(r). \tag{2.32}$$

Also $v(r) > 0$, since $r \geq 1$. By (2.27), (2.29), (2.23) and (2.32) we have

$$\frac{dv(r)}{dr} \leq (r^{-1-\gamma} + 2\kappa r^{-2})\mathcal{E}(r) \leq \frac{r^{-1-\gamma} + 2\kappa r^{-2}}{1 - r^{-\gamma} - 2\kappa r^{-1}} v(r) \leq \frac{r^{-1-\gamma} + 2\kappa r^{-2}}{1 - \bar{r}^{-\gamma} - 2\kappa \bar{r}^{-1}} v(r).$$

Dividing this inequality by $v(r)$ and integrating from $\bar{r}$ to r gives

$$\begin{aligned} \ln\left(\frac{v(r)}{v(\bar{r})}\right) &\leq \frac{\bar{r}^{-\gamma} + 2\gamma\kappa\bar{r}^{-1} - r^{-\gamma} - 2\gamma\kappa r^{-1}}{\gamma\left(1 - \bar{r}^{-\gamma} - 2\kappa\bar{r}^{-1}\right)} \\ &< \frac{\bar{r}^{-\gamma} + 2\gamma\kappa\bar{r}^{-1}}{\gamma\left(1 - \bar{r}^{-\gamma} - 2\kappa\bar{r}^{-1}\right)}. \end{aligned}$$

Therefore

$$v(r) < v(\bar{r})g(\bar{r}).$$

Together with (2.32), this proves (2.30).

Step 3. We claim now that

$$v(\bar{r}) \leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) + C_0 \bar{r}^{1-\delta}. \tag{2.33}$$

Here and below, $C_0, C_1, C_2 \ldots$ denote positive constants which do not depend on $\hat{\sigma}$ or r, and which we do not need to know anything more about.

To prove (2.33), use (2.27), (2.28), (2.23), and Lemma 2.4.2 to obtain

$$\frac{dv}{dr} \leq \frac{4\pi^2(j + Kr^{2-\delta})}{r^2} + C_1 r^{-1}.$$

Integrating this inequality from r_j to $\bar{r}$ and using $j > 0$, we deduce that

$$\begin{aligned} v(\bar{r}) - v(r_j) &\leq \frac{4\pi^2 j}{r_j} - \frac{4\pi^2 j}{\bar{r}} + \frac{4\pi^2 K(\bar{r}^{1-\delta} - r_j^{1-\delta})}{1-\delta} + C_1(\ln \bar{r} - \ln r_j) \\ &\leq \frac{4\pi^2 j}{r_j} + C_2 \bar{r}^{1-\delta}. \end{aligned} \tag{2.34}$$

Also, by (2.26), (2.28), (2.23), and Lemma 2.4.2, we have

$$\begin{aligned} v(r_j) &\leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) + C + \frac{4\pi^2(-j + Kr_j^{2-\delta}) + 2\kappa(r_j \operatorname{vol}(Y, \lambda)/2 + C)}{r_j} \\ &\leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) - \frac{4\pi^2 j}{r_j} + C_3 r_j^{1-\delta}. \end{aligned} \tag{2.35}$$

Adding (2.34) and (2.35) gives (2.33).

Step 4. We claim now that if j is sufficiently large then

$$\bar{r} \leq C_4 r_j^{\frac{1}{1-2\gamma}}. \tag{2.36}$$

To prove this, by the definition of $\bar{r}$, if $\bar{r} > r_j$ then there exists a number r slightly smaller than $\bar{r}$ such that $|cs(r)| > r^{1-\gamma}\mathcal{E}(r)$. It then follows from Lemma 2.6.2 that

$$r^{1-\gamma}\mathcal{E}(r) < cr^{2/3}\mathcal{E}(r)^{4/3}.$$

Therefore

$$r^{2-4\gamma} \leq c^3 r^{1-\gamma}\mathcal{E}(r) \leq c^3|cs(r)|.$$

By (2.28) and the definition of r_j in (2.12), we have

$$c^3|cs(r)| \leq C_5 r_j^2 + C_6 r^{2-\delta}.$$

Combining the above two inequalities and using the fact that r can be arbitrarily close to $\bar{r}$, we obtain

$$\bar{r}^{2-4\gamma} \leq C_5 r_j^2 + C_6 \bar{r}^{2-\delta}.$$

Since $\delta > 4\gamma$ and $\bar{r} > r_j \to \infty$ as $j \to \infty$, if j is sufficiently large then

$$C_6 \bar{r}^{2-\delta} \leq \frac{1}{2}\bar{r}^{2-4\gamma}.$$

Combining the above two inequalities proves (2.36).

Assume henceforth that j is sufficiently large so that (2.36) holds.

Step 5. We claim now that

$$c_\sigma(Y,\lambda) \leq \frac{1}{4\pi} r_j \operatorname{vol}(Y,\lambda) g(\bar{r}) + C_7 r_j^{1-\nu}, \tag{2.37}$$

where $\nu = 1 - \frac{1-\delta}{1-2\gamma} > 0$.

To prove (2.37), insert (2.36) into (2.33) to obtain

$$v(\bar{r}) \leq \frac{1}{2} r_j \operatorname{vol}(Y,\lambda) + C_8 r_j^{1-\nu}.$$

The above inequality and (2.30) imply that

$$\begin{aligned}\limsup_{r\to\infty} \mathcal{E}(r) &\leq \left(\frac{1}{2} r_j \operatorname{vol}(Y,\lambda) + C_8 r_j^{1-\nu}\right) g(\bar{r}) \\ &\leq \frac{1}{2} r_j \operatorname{vol}(Y,\lambda) g(\bar{r}) + C_9 r_j^{1-\nu}.\end{aligned}$$

It follows from this and Proposition 2.6.1 that (2.37) holds.

Step 6. We now complete the proof of Proposition 2.0.3 by applying (2.37) to the sequence $\{\sigma_k\}$ and taking the limit as $k \to \infty$.

Let $j_k = I(\sigma_k)$ and $\bar{r}_k = \bar{r}_{\hat{\sigma}_k}$. It then follows from (2.37) and the definition of the numbers r_{j_k} in (2.12) that for every k sufficiently large,

$$\begin{aligned} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} &\leq \frac{(16\pi^2)^{-1} r_{j_k}^2 \operatorname{vol}(Y,\lambda)^2 g(\bar{r}_k)^2 + C_{10} r_{j_k}^{2-\nu}}{(16\pi^2)^{-1} r_{j_k}^2 \operatorname{vol}(Y,\lambda) - r_{j_k}^{2-\delta}} \\ &= \frac{\operatorname{vol}(Y,\lambda) g(\bar{r}_k)^2 + C_{11} r_{j_k}^{-\nu}}{1 - C_{12} r_{j_k}^{-\delta}}. \end{aligned} \tag{2.38}$$

By hypothesis, as $k \to \infty$ we have $j_k \to \infty$, and hence $\bar{r}_k > r_{j_k} \to \infty$. It then follows from (2.31) that $\lim_{k\to\infty} g(\bar{r}_k) = 1$. Putting all this into the above inequality proves (2.1). □

Chapter 3

The lower bound

In this last section we prove the following proposition, which is the lower bound half of Theorem 1.3.1:

Proposition 3.0.1. *Under the assumptions of Theorem 1.3.1,*

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \geq \mathrm{vol}(Y,\lambda). \tag{3.1}$$

In §3.1 we review some aspects of ECH cobordism maps, and in §3.2 we use these to prove Proposition 3.0.1.

3.1 ECH cobordism maps

Let (Y_+,λ_+) and (Y_-,λ_-) be closed oriented three-manifolds, not necessarily connected, with nondegenerate contact forms. Following [13], define a "weakly exact symplectic cobordism" from (Y_+,λ_+) to (Y_-,λ_-) to be a compact symplectic four-manifold (X,ω) with boundary $\partial X = Y_+ - Y_-$, such that the symplectic form ω is exact on X, and $\omega|_{Y_\pm} = d\lambda_\pm$.

It is shown in [13, Thm. 2.3], by a slight modification of [15, Thm. 1.9], that a weakly exact symplectic cobordism as above induces a map

$$\Phi^L(X,\omega) : ECH^L(Y_+,\lambda_+,0) \longrightarrow ECH^L(Y_-,\lambda_-,0)$$

for each $L \in \mathbb{R}$, defined by counting solutions to the Seiberg-Witten equations, perturbed using ω, on a "completion" of X.

More generally, let $A \in H_2(X,\partial X)$, and write $\partial A = \Gamma_+ - \Gamma_-$ where $\Gamma_\pm \in H_1(Y_\pm)$. Suppose that ω has a primitive on X which agrees with $\lambda_\pm$ on each component of $Y_\pm$ for which the corresponding component of $\Gamma_\pm$ is nonzero. Then the same argument constructs a map

$$\Phi^L(X,\omega,A) : ECH^L(Y_+,\lambda_+,\Gamma_+) \longrightarrow ECH^L(Y_-,\lambda_-,\Gamma_-), \tag{3.2}$$

defined by counting solutions to the Seiberg-Witten equations in the spin-c structure corresponding to A. As in [13, Thm. 2.3(a)], there is a well-defined direct limit map

$$\Phi(X,\omega,A) = \lim_{L\to\infty} \Phi^L(X,\omega,A) : ECH(Y_+,\xi_+,\Gamma_+) \longrightarrow ECH(Y_-,\xi_-,\Gamma_-), \tag{3.3}$$

where $\xi_\pm = \mathrm{Ker}(\lambda_\pm)$.

The relevance of the map (3.3) for Proposition 3.0.1 is that given a class $\sigma_+ \in ECH(Y_+,\xi_+,\Gamma_+)$, if $\sigma_- = \Phi(X,\omega,A)\sigma_+$, then

$$c_{\sigma_+}(Y_+,\lambda_+) \geq c_{\sigma_-}(Y_-,\lambda_-). \tag{3.4}$$

The inequality (3.4) follows directly from (3.3) and the definition of $c_{\sigma_\pm}$ in §1.1, cf. [13, Lem. 4.2]. Here we interpret $c_\sigma = -\infty$ if $\sigma = 0$. By a limiting argument as in [13, Prop. 3.6], the inequality (3.4) also holds if the contact forms $\lambda_\pm$ are allowed to be degenerate.

The map (3.2) is a special case of the construction in [9] of maps on ECH induced by general strong symplectic cobordisms. Without the assumption on the primitive of ω, these maps can shift the symplectic action filtration, but the limiting map (3.3) is still defined.

For computations we will need four properties of the map (3.3). First, if $X = ([a,b]\times Y, d(e^s\lambda))$ is a trivial cobordism from $(Y,e^b\lambda)$ to $(Y,e^a\lambda)$, where s denotes the $[a,b]$ coordinate, then

$$\Phi(X,\omega,[a,b]\times\Gamma) = \mathrm{id}_{ECH(Y,\xi,\Gamma)}. \tag{3.5}$$

This follows for example from [15, Cor. 5.8].

Second, suppose that (X,ω) is the composition of strong symplectic cobordisms (X_+,ω_+) from (Y_+,λ_+) to (Y_0,λ_0) and (X_-,ω_-) from (Y_0,λ_0) to (Y_-,λ_-). Let $\Gamma_0 \in H_1(Y_0)$ and let $A_\pm \in H_2(X_\pm,\partial_\pm X_\pm)$ be classes with $\partial A_+ = \Gamma_+ - \Gamma_0$ and $\partial A_- = \Gamma_0 - \Gamma_-$. Then

$$\Phi(X_-,\omega_-,A_-)\circ\Phi(X_+,\omega_+,A_+) = \sum_{A|_{X_\pm}=A_\pm} \Phi(X,\omega,A). \tag{3.6}$$

This is proved the same way as the composition property in [15, Thm. 1.9].

Third, if X is connected and $Y_\pm$ are both nonempty, then

$$\Phi(X,\omega,A)\circ U_+ = U_-\circ\Phi(X,\omega,A), \tag{3.7}$$

where $U_\pm$ can be the U map associated to any of the components of $Y_\pm$. This is proved as in [13, Thm. 2.3(d)].

Fourth, since we are using coefficients in the field $\mathbb{Z}/2$, it follows from the definitions that the ECH of a disjoint union is given by the tensor product

$$ECH((Y,\xi)\sqcup(Y',\xi'),\Gamma\oplus\Gamma') = ECH(Y,\xi,\Gamma)\otimes ECH(Y',\xi',\Gamma'). \tag{3.8}$$

If (X,ω) is a strong symplectic cobordism from (Y_+,λ_+) to (Y_-,λ_-), and if (X',ω') is a strong symplectic cobordism from (Y'_+,λ'_+) to (Y'_-,λ'_-), then it follows from the construction of the cobordism map that the disjoint union of the cobordisms induces the tensor product of the cobordism maps:

$$\Phi((X,\omega)\sqcup(X',\omega'),A\oplus A') = \Phi(X,\omega,A)\otimes\Phi(X',\omega',A'). \tag{3.9}$$

3.2 Proof of the lower bound

Proof of Proposition 3.0.1. The proof has four steps.

Step 1. We can assume without loss of generality that

$$U\sigma_{k+1} = \sigma_k \tag{3.10}$$

for each $k \geq 1$. To see this, note that by the isomorphism (1.2) of ECH with Seiberg-Witten Floer cohomology, together with properties of the latter proved in [20, Lemmas 22.3.3, 33.3.9], we know that if the grading $*$ is sufficiently large, then $ECH_*(Y,\xi,\Gamma)$ is finitely generated and

$$U : ECH_*(Y,\xi,\Gamma) \longrightarrow ECH_{*-2}(Y,\xi,\Gamma)$$

is an isomorphism. Hence there is a finite collection of sequences satisfying (3.10) such that every nonzero homogeneous class in $ECH(Y,\xi,\Gamma)$ of sufficiently large grading is contained in one of these sequences (recall that we are using $\mathbb{Z}/2$ coefficients). Thus it is enough to prove (3.1) for a sequence satisfying (3.10). Furthermore, in this case (3.1) is equivalent to

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{k} \geq 2\,\mathrm{vol}(Y,\lambda). \tag{3.11}$$

Step 2. When (Y,λ) is the boundary of a Liouville domain, the lower bound (3.11) was proved for a particular sequence $\{\sigma_k\}$ satisfying (3.10) in [13, Prop. 8.6(a)]. We now set up a modified version of this argument.

Fix $a > 0$ and consider the symplectic manifold

$$([-a,0]\times Y, \omega = d(e^s\lambda))$$

where s denotes the $[-a,0]$ coordinate. The idea is that if a is large, then $([-a,0]\times Y,\omega)$ is "almost" a Liouville domain whose boundary is (Y,λ).

Fix $\varepsilon > 0$. We adopt the notation that if $r > 0$, then $B(r)$ denotes the closed ball

$$B(r) = \{z \in \mathbb{C}^2 \mid \pi|z|^2 \leq r\}.$$

Choose disjoint symplectic embeddings

$$\{\varphi_i : B(r_i) \to [-a,0]\times Y\}_{i=1,\dots,N}$$

such that $([-a,0]\times Y)\setminus \sqcup_i \varphi_i(B(r_i))$ has symplectic volume less than ε. Let

$$X = ([-a,0]\times Y) \setminus \bigsqcup_{i=1}^N \mathrm{int}(\varphi_i(B(r_i))).$$

Then (X,ω) is a weakly exact symplectic cobordism from (Y,λ) to $(Y,e^{-a}\lambda)\sqcup\bigsqcup_{i=1}^N \partial B(r_i)$. Here we can take the contact form on $B(r_i)$ to be the restriction of the 1-form $\frac{1}{2}\sum_{k=1}^2(x_k dy_k - y_k dx_k)$ on $\mathbb{R}^4$; we omit this from the notation. Note that there is a canonical isomorphism

$$H_2(X,\partial X) = H_1(Y).$$

The symplectic form ω on X has a primitive $e^s\lambda$ which restricts to the contact forms on the convex boundary (Y,λ) and on the component $(Y,e^{-a}\lambda)$ of the concave boundary. Hence, as explained in §3.1, we have a well-defined map

$$\Phi = \Phi(X,\omega,\Gamma) : ECH(Y,\xi,\Gamma) \longrightarrow ECH\left((Y,\xi) \sqcup \coprod_{i=1}^{n} \partial B(r_i), (\Gamma, 0, \dots, 0)\right) \tag{3.12}$$

which satisfies (3.4). By (3.8), the target of this map is

$$ECH\left((Y,\xi) \sqcup \coprod_{i=1}^{n} \partial B(r_i), (\Gamma, 0, \dots, 0)\right) = ECH(Y,\xi,\Gamma) \otimes \bigotimes_{i=1}^{n} ECH(\partial B(r_i)).$$

Let U_0 denote the U map on the left hand side associated to the component Y, and let U_i denote the U map on the left hand side associated to the component $\partial B(r_i)$. Note that U_0 or U_i acts on the right hand side as the tensor product of the U map on the appropriate factor with the identity on the other factors. By (3.7) we have

$$\Phi(U_0\sigma) = U_i\Phi(\sigma) \tag{3.13}$$

for all $\sigma \in ECH(Y,\xi,\Gamma)$ and for all $i = 0,\dots,N$.

Step 3. We now give an explicit formula for the cobordism map Φ in (3.12).

Recall that $ECH(\partial B(r_i))$ has a basis $\{\zeta_k\}_{k\geq 0}$ where $\zeta_0 = [\emptyset]$ and $U_i\zeta_{k+1} = \zeta_k$. This follows either from the computation of the Seiberg-Witten Floer homology of S^3 in [20], or from direct calculations in ECH, most of which are explained in [18, Ex. 4.2]. We can now state the formula for Φ:

Lemma 3.2.1. *For any class $\sigma \in ECH(Y,\xi,\Gamma)$, we have*

$$\Phi(\sigma) = \sum_{k\geq 0} \sum_{k_1+\dots+k_N=k} U_0^k\sigma \otimes \zeta_{k_1} \otimes \cdots \otimes \zeta_{k_N}.$$

Note that the sum on the right is finite because the map U_0 decreases symplectic action.

Proof of Lemma 3.2.1. Given σ, we can expand $\Phi(\sigma)$ as

$$\Phi(\sigma) = \sum_{k_1,\dots,k_N\geq 0} \sigma_{k_1,\dots,k_N} \otimes \zeta_{k_1} \otimes \cdots \otimes \zeta_{k_N} \tag{3.14}$$

where $\sigma_{k_1,\dots,k_N} \in ECH(Y,\xi,\Gamma)$. We need to show that

$$\sigma_{k_1,\dots,k_N} = U_0^{k_1+\cdots+k_N}\sigma. \tag{3.15}$$

We will prove by induction on $k = k_1 + \cdots + k_N$ that equation (3.15) holds for all σ.

To prove (3.15) when $k = 0$, let X' denote the disjoint union of the trivial cobordism $([-a-1,a]\times Y, d(e^s\lambda))$ and the balls $B(r_i)$. Then the composition $X'\circ X$ is the trivial

cobordism $([-a-1,0]\times Y, d(e^s\lambda))$ from (Y,e^λ) to $(Y,e^{-a-1}\lambda)$. Now each ball $B(r_i)$ induces a cobordism map

$$\Phi_{B(r_i)} : ECH(\partial B(r_i)) \longrightarrow \mathbb{Z}/2$$

as in (3.3). By (3.9) and (3.5) we have

$$\Phi(X',\Gamma) = \mathrm{id}_{ECH(Y,\xi,\Gamma)} \otimes \Phi_{B(r_1)} \otimes \cdots \otimes \Phi_{B(r_N)}.$$

It then follows from (3.5) and the composition property (3.6) that

$$\begin{aligned}\sigma &= (\Phi(X',\Gamma)\circ\Phi)(\sigma)) \\ &= \sum_{k_1,\ldots,k_N\geq 0} \sigma_{k_1,\ldots,k_N} \prod_{i=1}^N \Phi_{B(r_i)}(\zeta_{k_i}).\end{aligned}$$

Now $\Phi_{B(r_i)}$ sends ζ_0 to 1 by [13, Thm. 2.3(b)], and ζ_m to 0 for all $m>0$ by grading considerations (the corresponding moduli space of Seiberg-Witten solutions in the completed cobordism has dimension $2m$). Therefore $\sigma = \sigma_{0,\ldots,0}$ as desired.

Next let $k>0$ and suppose that (3.15) holds for smaller values of k. To prove (3.15), we can assume without loss of generality that $k_1>0$. Applying U_1 to equation (3.14) and then using equation (3.13) with $i=1$, we obtain

$$\sigma_{k_1,\ldots,k_N} = (U_0\sigma)_{k_1-1,k_2,\ldots,k_N}.$$

By inductive hypothesis,

$$(U_0\sigma)_{k_1-1,k_2,\ldots,k_N} = U_0^{k-1}(U_0\sigma).$$

The above two equations imply (3.15), completing the proof of Lemma 3.2.1. □

Step 4. We now complete the proof of Proposition 3.0.1. Let $\{\sigma_k\}_{k\geq 1}$ be a sequence in $ECH(Y,\xi,\Gamma)$ satisfying (3.10). By (3.4) we have

$$c_{\sigma_k}(Y,\lambda) \geq c_{\Phi(\sigma_k)}\left((Y,e^{-a}\lambda)\sqcup\coprod_{i=1}^N \partial B(r_i)\right).$$

By Lemma 3.2.1 and [13, Eq. (5.6)], we have

$$\begin{aligned}c_{\Phi(\sigma_k)}&\left((Y,e^{-a}\lambda)\sqcup\coprod_{i=1}^N \partial B(r_i)\right) = \\ &\max_{U^{k'}\sigma_k\neq 0}\ \max_{k_1+\cdots+k_N=k'}\left(c_{U_0^{k'}\sigma_k}(Y,e^{-a}\lambda) + \sum_{i=1}^N c_{\zeta_{k_i}}(\partial B(r_i))\right).\end{aligned}$$

Since $U^{k-1}\sigma_k = \sigma_1 \neq 0$, it follows from the above equation and inequality that

$$c_{\sigma_k}(Y,\lambda) \geq \max_{k_1+\cdots+k_N=k-1} \sum_{i=1}^N c_{\zeta_{k_i}}(\partial B(r_i)). \tag{3.16}$$

Now recall from [13] that Theorem 1.3.1 holds for $B(r)$. In detail, we know from [13, Cor. 1.3] that

$$c_{\zeta_k}(\partial B(r)) = dr$$

where d is the unique nonnegative integer such that

$$\frac{d^2+d}{2} \le k \le \frac{d^2+3d}{2}.$$

Consequently,

$$\lim_{k\to\infty} \frac{c_{\zeta_k}(\partial B(r))^2}{k} = 2r^2 = 4\,\mathrm{vol}(B(r)). \tag{3.17}$$

It follows from (3.16) and (3.17) and the elementary calculation in [13, Prop. 8.4] that

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{k} \ge 4\sum_{i=1}^{N} \mathrm{vol}(B(r_i)). \tag{3.18}$$

By the construction in Step 2,

$$\begin{aligned} \sum_{i=1}^{N} \mathrm{vol}(B(r_i)) &\ge \mathrm{vol}([-a,0]\times Y, d(e^s\lambda)) - \varepsilon \\ &= \frac{1-e^{-a}}{2}\,\mathrm{vol}(Y,\lambda) - \varepsilon. \end{aligned} \tag{3.19}$$

Since $a > 0$ can be arbitrarily large and $\varepsilon > 0$ can be arbitrarily small, (3.18) and (3.19) imply (3.11). This completes the proof of Proposition 3.0.1. □

Part II

Absolute gradings in Floer homologies

Chapter 4

Introduction

4.1 Heegaard Floer homology

For a closed oriented 3-manifold Y, Ozsváth and Szabó [28] defined a collection of invariants of Y, the Heegaard Floer homology groups $HF^\circ(Y)$, where $HF^\circ(Y)$ denotes either $\widehat{HF}(Y)$, $HF^+(Y)$, $HF^-(Y)$, or $HF^\infty(Y)$. They showed that $HF^\circ(Y)$ splits into a direct sum by Spin^c structures

$$HF^\circ(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} HF^\circ(Y,\mathfrak{s}).$$

For each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, they also defined a relative grading on $HF^\circ(Y,\mathfrak{s})$, that takes values in $\mathbb{Z}/d(c_1(\mathfrak{s}))$, where $d(c_1(\mathfrak{s}))$ is the divisibility of $c_1(\mathfrak{s}) \in H^2(Y;\mathbb{Z})$, i.e. $d(c_1(\mathfrak{s}))\mathbb{Z} = \langle c_1(\mathfrak{s}), H_2(Y)\rangle$.

Moreover given a 4-dimensional compact oriented cobordism $W : Y_0 \to Y_1$, i.e. $\partial W = -Y_0 \cup Y_1$ as oriented manifolds, and given a Spin^c structure $\mathfrak{t}$ on W, there is a natural map $F_{W,\mathfrak{t}} : HF^\circ(Y_0, \mathfrak{t}|_{Y_0}) \to HF^\circ(Y_1, \mathfrak{t}|_{Y_1})$ defined by Ozsváth-Szabó [31].

It has been shown that Heegaard Floer homology is isomorphic to two other homology theories: Seiberg-Witten Floer homology [20] and embedded contact homology (ECH) [11]. For a proof of the existence of these isomorphisms, see [1, 21, 35]. It is known that both ECH [10] and Seiberg-Witten Floer homology [20] are absolutely graded by homotopy classes of oriented 2-plane fields, but no such absolute grading had been defined for Heegaard Floer homology. In this paper, we construct such an absolute grading for Heegaard Floer homology, which is compatible with the relative grading and cobordism maps discussed above.

We will now fix some notation that will be used in this paper. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram of Y. Here Σ is a genus g surface, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)$ are collections of disjoint circles on Σ and the basepoint z is a point on Σ in the complement of $\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g$. We also require that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are linearly independent sets in $H_1(Y)$ and that α_i and β_j intersect transversely for every i and j. We consider the tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ in the symmetric product $\mathrm{Sym}^g(\Sigma)$. Recall that the Heegaard Floer chain complex $\widehat{CF}(Y)$ is the free abelian group generated by the intersection

points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. If $\mathbf{x}$ and $\mathbf{y}$ are intersection points in the same Spin^c structure, we denote by $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ their relative grading, as defined in [28].

We denote by $\mathcal{P}(Y)$ the set of homotopy classes of oriented 2-plane fields on Y. Each homotopy class of oriented 2-plane fields belongs to a Spin^c structure, as we will explain in §5.1. Therefore $\mathcal{P}(Y)$ splits by Spin^c structures as

$$\mathcal{P}(Y) = \coprod_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \mathcal{P}(Y, \mathfrak{s}).$$

It turns out that $\mathcal{P}(Y, \mathfrak{s})$ is an affine space over $\mathbb{Z}/d(c_1(\mathfrak{s}))$. For each Spin^c structure $\mathfrak{s}$, we will construct an absolute grading $\widetilde{\mathrm{gr}}$ on $\widehat{CF}(Y, \mathfrak{s})$ with values in $\mathcal{P}(Y, \mathfrak{s})$.

For a contact structure ξ on Y, Ozsváth-Szabó [29] defined the contact invariant $c(\xi) \in \widehat{HF}(-Y)$. In [28], Ozsváth-Szabó showed that a Heegaard move induces an isomorphism on Heegaard Floer homology.

Consider a compact oriented cobordism $W : Y_0 \to Y_1$. Let ξ_0 and ξ_1 be oriented 2-plane fields on Y_0 and Y_1 respectively. We say that $\xi_0 \sim_W \xi_1$ if there exists an almost complex structure J on W such that $[\xi_0] = [TY_0 \cap J(TY_0)]$ and $[\xi_1] = [TY_1 \cap J(TY_1)]$ as homotopy classes of oriented 2-plane fields.

We can now state the first main theorem of Part II.

Theorem 4.1.1. *For every Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ of Y, there exists a canonical function $\widetilde{gr} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathcal{P}(Y)$ such that:*

(a) If $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are in the same Spin^c structure $\mathfrak{s}$, then $\widetilde{gr}(\mathbf{x})$ and $\widetilde{gr}(\mathbf{y})$ belong to $\mathcal{P}(Y, \mathfrak{s})$ and $\widetilde{gr}(\mathbf{x}) - \widetilde{gr}(\mathbf{y}) = gr(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}/d(c_1(\mathfrak{s}))$. In particular, $\widetilde{gr}$ extends to the set of homogeneous elements of $\widehat{CF}(Y)$.

(b) Let ξ be a contact structure on Y, and let $c(\xi) \in \widehat{HF}(-Y)$ be the contact invariant. Then $\widetilde{gr}(c(\xi)) = [\xi]$ as homotopy classes of oriented 2-plane fields.

(c) This absolute grading is invariant under the isomorphisms induced by Heegaard moves and hence it induces an absolute grading on $\widehat{HF}(Y)$ which is independent of the Heegaard diagram.

(d) Let $W : Y_0 \to Y_1$ be a compact, oriented cobordism, and let $\mathfrak{t}$ be a Spin^c structure on W. Then the induced map $F_{W,\mathfrak{t}} : \widehat{HF}(Y_0, \mathfrak{t}|_{Y_0}) \to \widehat{HF}(Y_1, \mathfrak{t}|_{Y_1})$ respects the grading in the sense that $\widetilde{gr}(\mathbf{x}) \sim_W \widetilde{gr}(\mathbf{y})$ for any homogeneous element $\mathbf{x} \in \widehat{HF}(Y_0, \mathfrak{t}|_{Y_0})$ and any $\mathbf{y} \in \widehat{HF}(Y_1, \mathfrak{t}|_{Y_1})$, which is a homogeneous summand of $F_{W,\mathfrak{t}}(\mathbf{x})$.

Remark 4.1.2. Theorem 4.1.1(a) implies that we have the following decomposition by degrees.

$$\widehat{CF}(Y; \mathfrak{s}) = \bigoplus_{\rho \in \mathcal{P}(Y, \mathfrak{s})} \widehat{CF}_\rho(Y; \mathfrak{s}). \tag{4.1}$$

Here $\widehat{CF}_\rho(Y; \mathfrak{s})$ is the $\mathbb{Z}$-module generated by all $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\widetilde{\mathrm{gr}}(\mathbf{x}) = \rho$.

Remark 4.1.3. The generators of $HF^\infty(Y)$ are of the form $[\mathbf{x}, i]$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \in \mathbb{Z}$. We recall that $\mathbb{Z}$ acts on $\mathcal{P}(Y)$, since $\mathcal{P}(Y, \mathfrak{s})$ is an affine space over $\mathbb{Z}/d(c_1(\mathfrak{s}))$. So we can define an absolute grading on $HF^\infty(Y)$, and hence on $HF^-(Y)$ and $HF^+(Y)$, by $\widetilde{\mathrm{gr}}([\mathbf{x}, i]) = \widetilde{\mathrm{gr}}(\mathbf{x}) + 2i$, for a homogeneous element $\mathbf{x}$. It is easy to see that Theorem 4.1.1 implies that (a),(c) and (d) also hold for $HF^\infty(Y)$, $HF^-(Y)$ and $HF^+(Y)$.

Remark 4.1.4. Using the absolute grading function $\widetilde{\mathrm{gr}}$ constructed in Theorem 4.1.1, one can recover the absolute $\mathbb{Q}$-grading for $HF^\circ(Y, \mathfrak{s})$ defined by Ozsváth-Szabó when $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ is a torsion class. See Corollary 5.4.3 for details.

We can also generalize the absolute grading function $\widetilde{gr}$ to the twisted Heegaard Floer homology groups defined by Ozsváth-Szabó [27]. Recall that the twisted Heegaard Floer homology group $\underline{HF}(Y, \mathfrak{s})$ is the homology of the twisted Heegaard Floer chain complex $CF(Y; \mathfrak{s}) \otimes \mathbb{Z}[H^1(Y; \mathbb{Z})]$, where the (infinity version) differential is defined by

$$\underline{\partial}^\infty[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \Big(\sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} \#\mathcal{M}(\phi) e^{A(\phi)} [\mathbf{y}, i - n_z(\phi)] \Big)$$

where $A : \pi_2(\mathbf{x}, \mathbf{y}) \to H^1(Y; \mathbb{Z})$ is a surjective, additive assignment. See [27] for more details. Now we define the twisted absolute grading function by simply ignoring the twisted coefficient as follows:

$$\begin{aligned} \widetilde{gr}_{tw} : \mathbb{Z}[H^1(Y; \mathbb{Z})](\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \to &\quad \mathcal{P}(Y) \\ e^\xi \mathbf{x} \mapsto &\quad \widetilde{gr}(\mathbf{x}), \end{aligned} \tag{4.2}$$

where $\xi \in H^1(Y; \mathbb{Z})$ and we write $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ multiplicatively.[1] Using an obvious twisted version of Theorem 4.1.1(b), we will prove the following corollaries in Section 5.3.

Let $\mathcal{F}_Y$ denote the set of homotopy classes (as 2-plane fields) of contact structures on Y which are weakly fillable.

Corollary 4.1.5 (Kronheimer-Mrowka [19])**.** *$\mathcal{F}_Y$ is finite.*

Corollary 4.1.6. *If Y is an L-space, then $|\mathcal{F}_Y| \leq |H_1(Y; \mathbb{Z})|$.*

Corollary 4.1.7 (Lisca [24])**.** *If Y admits a metric of constant positive curvature, then $|\mathcal{F}_Y| \leq |H_1(Y; \mathbb{Z})|$.*

Remark 4.1.8. Corollary 4.1.5 and Corollary 4.1.7 are previously proved using the relationship between Seiberg-Witten theory and contact topology.

[1] The twisted absolute grading defined here does not refine the relative $\mathbb{Z}$-grading within each Spin^c structure defined in [27]. A slightly more sophisticated construction of the twisted grading is needed to recover the relative $\mathbb{Z}$-grading. But since we do not need this refinement in this paper, we do not include the details here.

Remark 4.1.9. In fact the assertion in Corollary 4.1.5 holds for the set of homotopy classes of 2-plane fields which support a tight contact structure by the work of Colin-Giroux-Honda [4]. But our result does not imply this generalization. In particular we do not have an upper bound on $|\mathcal{F}(Y)|$ for tight contact structures.

4.2 Bordered Floer homology

We now briefly review the construction of bordered Heegaard Floer homology, following [23]. Consider a compact oriented 3-manifold Y with non-empty connected boundary. A parametrization of ∂Y is an orientation preserving diffeomorphism $\phi : \partial Y \to F$, where F is a closed oriented surface with a prescribed handle decomposition. According to [23], one can associate to F a differential graded algebra $\mathcal{A}(F)$. See § 6.1 for the precise definition of $\mathcal{A}(F)$. Then one defines the so-called *type A* and *type D* modules of Y, denoted by $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$. The type A module $\widehat{CFA}(Y)$ is a right A^∞-module over $\mathcal{A}(F)$. That means that there exist maps

$$m_l : \widehat{CFA}(Y) \otimes \mathcal{A}(F)^{\otimes(l-1)} \to \widehat{CFA}(Y),$$

satisfying the A^∞-relations, see e.g. [23, Eq. (2.6)]. Here the tensor product is taken over an appropriate ring, as we will review in §6.2. The type D module $\widehat{CFD}(Y)$ is a left differential module over $\mathcal{A}(-F)$, that is there exists a map $\partial : \widehat{CFD}(Y) \to \widehat{CFD}(Y)$, which squares to 0 and which satisfies the Leibniz rule with respect to the left action of $\mathcal{A}(-F)$. It is also shown in [23] that if Y_1 and Y_2 are compact 3-manifolds such that $\partial Y_1 = -\partial Y_2$, then there is a homotopy equivalence

$$\Phi : \widehat{CFA}(Y_1) \,\widetilde{\otimes}\, \widehat{CFD}(Y_2) \to \widehat{CF}(Y_1 \cup_F Y_2). \tag{4.3}$$

Here $\widetilde{\otimes}$ denotes the derived tensor product. For a closed oriented 3-manifold Y, we denote by $\mathrm{Vect}(Y)$ the set of homotopy classes of non-vanishing vector fields on Y. The goal of this paper is to prove the following theorems.

Theorem 4.2.1. *Given a parameterized surface F as above, there exist a groupoid $G(F)$, with a $\mathbb{Z}$-action denoted by λ^n for a given $n \in \mathbb{Z}$, and a grading function* gr with values on $G(F)$ satisfying the following conditions:

1. If a, b are two composable generators of $\mathcal{A}(F)$, then $\mathrm{gr}(a \cdot b) = \mathrm{gr}(a) \cdot \mathrm{gr}(b)$.
2. If a is a generator of $\mathcal{A}(F)$, then $\mathrm{gr}(\partial a) = \lambda^{-1}\mathrm{gr}(a)$.

Remark 4.2.2. It turns out that $G(F)$ is by construction a set of co-oriented plane fields on $F \times [0,1]$ modulo homotopy.[2] The multiplication rule is by the obvious stacking of plane fields when the boundary condition matches.

[2]By choosing a Riemannian metric on a 3-manifold, we can identify the set of nonvanishing vector fields with the set of co-oriented plane fields, modulo homotopy, by taking the orthogonal complement.

Theorem 4.2.3. *For any compact 3-manifold Y with boundary F, there exist a set $S(Y)$, admitting a right action by $G(F)$ and a left action by $G(-F)$, and a grading* gr on $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ with values on $S(Y)$ such that

(a) If x is a generator of $\widehat{CFA}(Y)$ and $a_1, \dots, a_l$ are generators of $\mathcal{A}(F)$ such that $m_{l+1}(x; a_1, \dots, a_l) \neq 0$, then

$$\mathrm{gr}(m_{l+1}(x; a_1, \dots, a_l)) = \lambda^{l-1}\mathrm{gr}(x) \cdot \mathrm{gr}(a_1) \dots \mathrm{gr}(a_l).$$

(b) If x is a generator of $\widehat{CFD}(Y)$, then $\mathrm{gr}(\partial x) = \lambda^{-1}\mathrm{gr}(x)$.

Theorem 4.2.4. *Let Y_1 and Y_2 be compact 3-manifolds such that $\partial Y_1 = -\partial Y_2$. Then there exist a set $S(Y_1) \otimes S(Y_2)$ and a map $\Psi : S(Y_1) \otimes S(Y_2) \to \mathcal{P}(Y)$ such that*

$$\widetilde{\mathrm{gr}}(\Phi(a \otimes b)) = \Psi(\mathrm{gr}(a) \otimes \mathrm{gr}(b))$$

for any generators a in $\widehat{CFA}(Y_1)$ and b in $\widehat{CFD}(Y_2)$. Here $\widetilde{\mathrm{gr}}$ denotes the absolute grading in Heegaard Floer homology.

Remark 4.2.5. Using essentially the same constructions that we will work out on this paper, Theorem 4.2.1 can be generalized to any surface F, not necessarily with connected boundary, using the generalized strands algebra defined by Zarev [39]. Both Theorems 4.2.3 and 4.2.4 can be generalized to the bimodules $\widehat{CFDD}$, $\widehat{CFDA}$, $\widehat{CFAA}$ constructed in [23], as well as the setting of bordered sutured Floer homology [39], in which case $F \subset \partial Y$, where the inclusion can be strict. The main difference in the construction in the latter case is that one needs to fix a nonvanishing vector field in $\partial Y \setminus F$, similarly to how Spin^c structures are assigned to generators in [39].

Part II is organized as follows: In §5.1, we construct the absolute grading on $\widehat{CF}$ and in §5.2, we prove that it refines the relative grading defined in [28]. That proves part (a) of the Theorem 4.1.1. In §5.3, we compute the absolute grading of the contact invariant and show that it is the homotopy class of the contact structure, which proves part (b) of the Theorem 4.1.1. This fact is known, by construction, for the absolute grading in ECH [10]. In §5.4, we prove Theorem 4.1.1(d) at the chain level, showing that $\widetilde{\mathrm{gr}}$ is natural under cobordism maps, as stated in Theorem 5.4.1. This was shown for Seiberg-Witten Floer homology by Kronheimer-Mrowka [20]. In §5.5, we prove that $\widetilde{\mathrm{gr}}$ is preserved under Heegaard moves, see Theorem 5.5.1. That means that the decomposition (4.1) is preserved under Heegaard moves and therefore it also holds in the homology level. That implies that Theorem 4.1.1(d) also holds in homology.
In §6.1, we review the definition of the strand algebra $\mathcal{A}(F)$ associated to a parameterized closed surface F following [23]. Then we construct the groupoid $G(F)$ in which the grading on $\mathcal{A}(F)$ takes value, and give the proof for Theorem 4.2.1. We finish this section by comparing our geometric grading on $\mathcal{A}(F)$ with the previously constructed grading in [23]. In §6.2, we

construct the "left-$G(-F)$ and right-$G(F)$ bimodule" $S(Y)$ in which the grading on $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ takes value. Some variations of the standard Pontryagin-Thom construction are made in this section which enable us to compute the relative gradings needed for the proof of Theorem 4.2.3. The proof of Theorem 4.2.4 is provided in §6.3. This was joint work with Yang Huang. In Chapter 7, we prove that te isomorphism between Heegaard Floer homology and ECH constructed by Colin-Ghiggini-Honda preserves the absolute grading.

Chapter 5

The absolute grading on Heegaard Floer homology

5.1 The construction

Let Y be an oriented closed 3-manifold and let $\mathcal{P}(Y)$ denote the set of homotopy classes of oriented 2-plane fields on Y. Let us first recall that there is a surjection $\psi : \mathcal{P}(Y) \to \mathrm{Spin}^c(Y)$. Also, for a fixed Spin^c structure $\mathfrak{s}$, we can endow $\psi^{-1}(\mathfrak{s}) = \mathcal{P}(Y, \mathfrak{s})$ with the structure of an affine space over $\mathbb{Z}/d(c_1(\mathfrak{s}))$, where $d(c_1(\mathfrak{s}))$ is the divisibility of the first Chern class of $\mathfrak{s}$. So, given $\xi, \eta \in \mathcal{P}(Y)$ mapping to the same Spin^c structure $\mathfrak{s}$, there is a well-defined difference $\xi - \eta$. One way of seeing this affine space structure is by using the Pontryagin-Thom construction, as follows. Each $\xi \in \mathcal{P}(Y)$ corresponds to a unique homotopy class of nonvanishing vector fields, which we denote by $[v_\xi]$. Fixing a representative v_ξ and a trivialization of TY, and after a normalization, we can think of v_ξ as a map $Y \to S^2$. The preimage of a regular value of this map gives a link and the preimage of the tangent plane to this regular point under the derivative map determines a framing of this link. We recall that two framed links $L_O, L_1 \subset Y$ are called framed cobordant, if there exists a framed surface $S \subset Y \times [0,1]$, whose boundary is $-L_O \times \{0\} \cup L_1 \times \{1\}$ and such that the framing restricted to the boundary coincides with the initial framings on L_0 and L_1. It follows from Pontryagin-Thom theory that two nonvanishing vector fields are homotopic if and only if the respective framed links are framed cobordant. If ξ, η map to the same Spin^c structure, then the respective links are cobordant and the difference of framings is $\xi - \eta \in \mathbb{Z}/d(c_1(\mathfrak{s}))$. The sign convention we are using here is that a left-handed twist increases a framing by $+1$.

Now let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram representing Y, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)$. Recall that the generators of $\widehat{CF}(Y)$ are the intersection points of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\mathrm{Sym}^g(\Sigma)$. Our goal in this section is to construct a canonical map $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathcal{P}(Y)$ that refines the relative grading, which we denote by gr, and the map that assigns a Spin^c structure to a generator, which we denote by $\mathfrak{s}_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$. For the definitions of these maps, see [28].

Theorem 5.1.1. *There is a canonical map* $\widetilde{gr} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathcal{P}(Y)$*, such that if* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ *are such that* $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y}) = \mathfrak{s}$*, then*

$$\widetilde{gr}(\mathbf{x}) - \widetilde{gr}(\mathbf{y}) = gr(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}/d(c_1(\mathfrak{s})).$$

We fix a self-indexing Morse function $f : Y \to \mathbb{R}$ compatible with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then $\mathbf{x}$ corresponds to g points $x_1, \ldots, x_g$ on Σ, which give rise to flow lines $\gamma_{x_1}, \ldots, \gamma_{x_g}$ connecting the index 1 critical points to the index 2 critical points. The basepoint z determines a flow line γ_0 from the index 0 critical point to the index 3 critical point. We can choose a gradient-like vector field v, tubular neighborhoods $N(\gamma_{x_i})$ of γ_{x_i} and diffeomorphisms $N(\gamma_{x_i}) \cong B^3$ such that, under these diffeomorphisms, $v_{|N(\gamma_{x_i})} : B^3 \to \mathbb{R}^3$ is given by $v(x, y, z) = (x, -y, 1 - 2z^2)$, for $i \neq 0$ and $v_{|N(\gamma_0)} : B^3 \to \mathbb{R}^3$ is given by $v(x, y, z) = (2xz, 2yz, 1 - 2z^2)$. Figure 5.1(a) shows two cross-sections of $v_{|N(\gamma_{x_i})}$, for $i \neq 0$. Figure 5.1(b) shows $v_{|N(\gamma_0)}$ on any plane passing through the origin containing the z-axis. Outside the union of the neighborhoods $N(\gamma_{x_i})$, v is a nonvanishing vector field. We will define a nonvanishing continuous vector field $w_{\mathbf{x}}$ on Y that coincides with v in the complement of the neighborhoods $N(\gamma_{x_i})$.

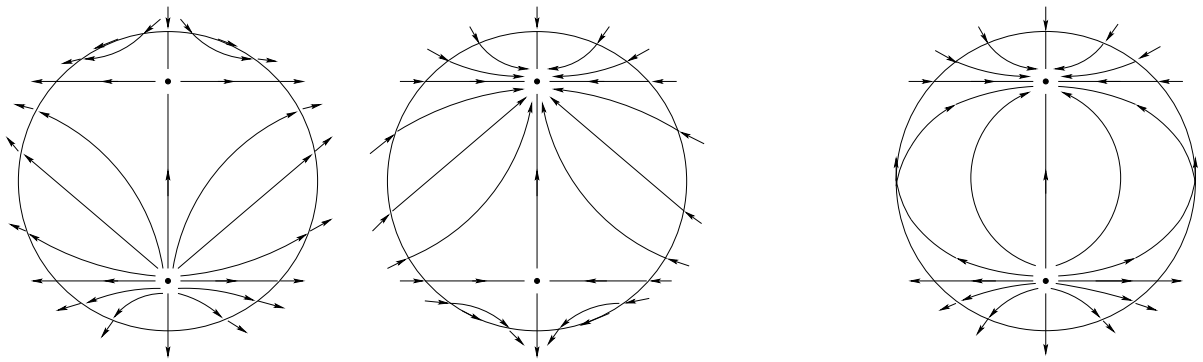


Figure 5.1:

For $i \neq 0$, on $\partial N(\gamma_{x_i}) \cong \partial B^3$, we note that

$$v(x, y, z) = (x, -y, 1 - 2z^2) = (x, -y, 2x^2 + 2y^2 - 1).$$

We define $w_{\mathbf{x}} = (x, -y, 2x^2 + 2y^2 - 1)$ in $N(\gamma_i)$, see Fig 5.2(a). This is a nonzero vector field in $N(\gamma_{x_i})$ that coincides with v on $\partial N(\gamma_{x_i})$. Also, on $\partial N(\gamma_0)$, we see that

$$v(x, y, z) = (-2xz, -2yz, 1 - 2z^2) = (-2xz, -2yz, 2x^2 + 2y^2 - 1).$$

This new vector field is still zero on the circle $C = \{(x, y, z) | x^2 + y^2 = 1/2, z = 0\}$. A vertical section of it in B^3 is shown in Figure 5.2(b).So we define $w_{\mathbf{x}}$ in $N(\gamma_0)$ by

$$w_{\mathbf{x}}(x, y, z) = (-2xz, -2yz, 2x^2 + 2y^2 - 1) + \phi(x, y, z)(y, -x, 0),$$

where ϕ is a bump function around C (i.e. $\phi = 1$ on C and $\phi = 0$ in the complement of a small neighborhood of C). Therefore $w_{\mathbf{x}}$ is a nonvanishing vector field on Y that equals v outside the union of the neighborhoods $N(\gamma_{x_i})$. We can perturb $w_{\mathbf{x}}$ to a smooth vector field. Finally we define $\widetilde{\mathrm{gr}}(\mathbf{x})$ to be the homotopy class of the orthogonal complement of $w_{\mathbf{x}}$.

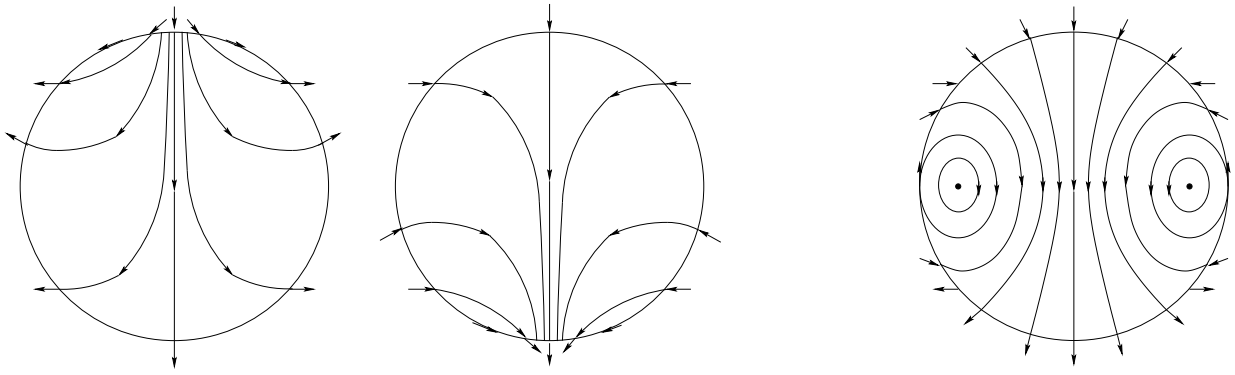


Figure 5.2:

Remark 5.1.2. We could use the gradient vector field itself instead of some other gradient-like vector field to define the absolute grading, but it would be harder to write down the formulas for the canonical modification of the gradient vector field in the neighborhoods of the flow lines. Nevertheless, we would obtain the same homotopy class.

5.2 The relative grading

This subsection is dedicated to proving that the absolute grading refines the relative grading. Given two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ such that $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$, there exists a Whitney disk $A \in \pi_2(x, y)$, as proven in [28]. This means that A is a homotopy class of maps $\varphi : D^2 \subset \mathbb{C} \to \mathrm{Sym}^g(\Sigma)$ taking i to $\mathbf{x}$, $-i$ to $\mathbf{y}$, the semicircle with positive real part to $\mathbb{T}_\beta$ and the one with negative real part to $\mathbb{T}_\alpha$. Let $D_1, \ldots, D_n$ denote the closures of the connected components of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. We write $D(A) = \sum_{k=1}^n a_k D_k$, where a_k is the multiplicity of φ on each D_k. We can choose a Whitney disk A so that $a_k \geq 0$ for every k.

We will now construct surfaces $F_1 \supset \cdots \supset F_m$, whose union projects to $\sum_{k=1}^n a_k D_k = D(A)$ on Σ. We take a_k copies of each D_k and we glue them along their boundaries in the following way: we construct F_1 by gluing one copy of each D_k with $a_k > 0$. Then we construct F_2 by gluing one copy of each D_k such that $a_k - 1 > 0$. Inductively we construct surfaces $F_1, \ldots, F_m$, where $m = \max a_k$. So the union of the surfaces F_l can be identified with $D(A)$. (Similar constructions can be found in [22, 28, 32]).

The Euler measure of a surface with corners S, denoted by $e(S)$, is defined to be $\chi(S) - \frac{p}{4} + \frac{q}{4}$, where p is the number of convex corners of S and q is the number of concave corners of S. If $w \in \alpha_i \cap \beta_j$, for some i, j, then a small neighborhood of w, when intersected with the complement of the union of the α and the β curves, gives rise to four regions. We define $n_w(D_k)$ to be $1/4$ times the number of those regions contained in D_k. We extend n_w linearly to the $\mathbb{Z}$-module generated by the domains D_k. Now we define $n_{\mathbf{x}}$ to be the sum of all n_{x_i}, for $i = 1, \ldots, g$. For example, a convex corner x_i of F_l contributes to $n_{\mathbf{x}}(F_l)$ with $1/4$ and a concave corner x_i with $3/4$. Similarly we define $n_{\mathbf{y}}$. By Lipshitz [22], the Maslov index of the Whitney disk A, denoted by $\mu(A)$, is given by

$$\mu(A) = \mathrm{ind}(A) = e(D(A)) + n_{\mathbf{x}}(D(A)) + n_{\mathbf{y}}(D(A)) = \sum_{l=1}^{m} \Big(e(F_l) + n_{\mathbf{x}}(F_l) + n_{\mathbf{y}}(F_l) \Big).$$

For each D_k, we define $n_z(D_k)$ to be 0 if $z \notin D_k$ and 1 if $z \in D_k$, and we extend n_z linearly to sums of D_k. The relative grading was defined by Ozsváth-Szabó [28] to be

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(A) - 2n_z(D(A)) \in \mathbb{Z}/d,$$

where d is the divisibility of $c_1(\mathfrak{s}(\mathbf{x}))$. So we need to show that

$$\widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y}) = \sum_{l=1}^{m} \Big(e(F_l) + n_{\mathbf{x}}(F_l) + n_{\mathbf{y}}(F_l) - 2n_z(F_l) \Big) \in \mathbb{Z}/d.$$

Step 1: We first assume that $m = 1$ and that $n_z(F_1) = 0$. Recall that a corner x_i is called degenerate if $x_i = y_j$ for some j. We also assume that there are no degenerate corners.

We will now choose a convenient trivialization of TY in order to apply the Pontryagin-Thom construction. Let f be a self-indexing Morse function f, which is compatible with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Let $F := F_1$. Let p_i be the index 1 critical point corresponding to α_i and q_j the index 2 critical point corresponding to β_j. Each edge of the boundary of F is part of an α_i or a β_j. So each edge of ∂F determines a surface by flowing downwards or upwards towards a p_i or q_j, respectively, and, by adding p_i and q_j, we get a compact surface with corners. This surface has typically three corners unless it corresponds to an edge starting at a boundary degenerate corner in which case, this edge is actually a circle and the surface corresponding to it is a disk. We call A_i and B_j the surfaces corresponding to the edges contained in α_i and β_j, respectively. We note that the flow we consider here is the one generated by a gradient-like vector field v compatible with the Morse function f.

Let C be the union of F and the surfaces A_i and B_j. We will first choose a trivialization of TY on C. We start by defining a unit vector field E_1, which is tangent to F. The orientation of Σ induces an orientation on F. We set E_1 to be the positive unit tangent vector along ∂F, with respect to its boundary orientation, outside a small neighborhood of the corners. At a neighborhood of a corner, we define E_1 on ∂F by keeping it tangent to F and rotating it by the smallest possible angle. That means that once we start rotating,

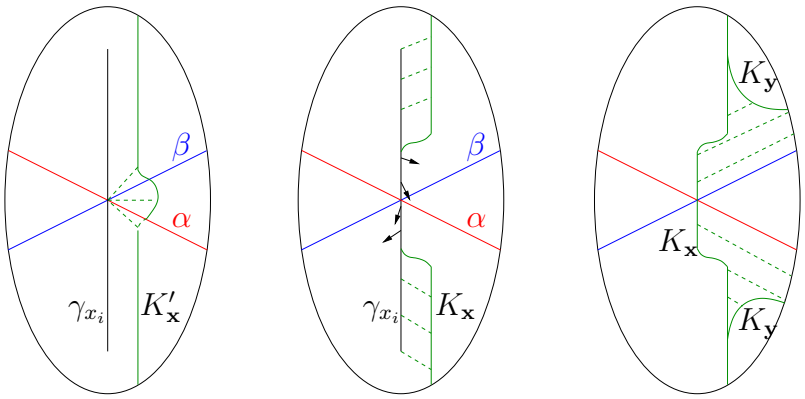
E_1 will not be tangent to ∂F at any point. In other words, each connected component of the set of points of ∂F at which E_1 is not tangent to ∂F contains exactly one corner of F. We also have to choose a corner to rotate an extra $2\pi\chi(F)$ clockwise. That allows us to extend E_1 to F. We now define E_1 on each A_i and B_j to be an extension of E_1 on ∂F such that it is tangent to A_i and B_j everywhere outside small neighborhoods of the corners x_i and y_j and such that it is always transverse to the flow lines γ_{x_i} and γ_{y_j}. In particular E_1 is tangent to A_i near p_i and to B_j near q_j. Near the corners x_i and y_j, we require E_1 to never be tangent to A_i and B_j, similarly to how we defined E_1 on F. We define E_3 on F to be the positive normal vector field to F, and we extend it to A_i and B_j so that $\{E_1, E_3\}$ is an oriented orthonormal frame on the respective tangent spaces, except maybe outside a small neighborhood of ∂F. In this neighborhood, we require that each connected component of the set of points where E_3 is not tangent to A_i or B_j intersects F. Now we take E_2 to be the unit vector field on C orthogonal to E_1 and E_3 such that $\{E_1, E_2, E_3\}$ is an oriented basis of TY. So mapping E_i to $e_i \in \mathbb{R}^3$, we get a trivialization of TY along C. We extend this trivialization to a neighborhood of C in such a way that E_1 and E_3 are still tangent to the corresponding unstable and stable surfaces near the critical points p_i and q_j and that e_1 is a regular value of $w_{\mathbf{x}}$ and $w_{\mathbf{y}}$ when seen as maps $Y \to S^2$. Now, since there are no degenerate points, C does not contain an α or β curve. Therefore there is no obstruction to extending this trivialization to all of Y. So we choose one of those extensions.

Now we define $K'_{\mathbf{x}} = w_{\mathbf{x}}^{-1}(e_1)$ and $K'_{\mathbf{y}} = w_{\mathbf{y}}^{-1}(e_1)$ as framed links. We note that inside neighborhoods of the flow lines γ_{x_i} and γ_{y_i}, these are one stranded braids contained in the corresponding unstable or stable surface, except that near each corner of F, this braid rotates around the respective flow line as much as E_1 restricted to this flow line does, but in the opposite direction. This is shown in Figure 5.3(a). It follows from the way that we chose the trivialization on C that $K'_{\mathbf{x}}$ and $K'_{\mathbf{y}}$ do not intersect C outside of those neighborhoods.

We can isotope $K'_{\mathbf{x}}$ in neighborhoods of each γ_{x_i} in the following way. Near each corner, this link is rotating around γ_{x_i}. We isotope a neighborhood of this part of the link to the segment of the flow line about which it is rotating fixing the endpoints. Outside of this neighborhood of the corner, but still inside the neighborhood of the flow line, the link is contained in the corresponding unstable or stable surface. We will call this new link $K_{\mathbf{x}}$. We can think of the framing of a link as a unit normal vector field to the link. So the framing on $K_{\mathbf{x}}$ induced from this isotopy can be seen by a vector field that is normal to the stable and unstable surfaces away from the corners and rotates with respect to the stable surface as much as $K'_{\mathbf{x}}$ rotates about the flow line, as seen in Figure 5.3(b). We denote this framing by $\tau_{\mathbf{x}}$. We note that once we fix which of the two unit normal vector fields to the stable surface we choose, the unit normal vector field to the unstable surface is determined.

We can do the same for $K'_{\mathbf{y}}$ and define $K_{\mathbf{y}}$ with framing denoted by $\eta_{\mathbf{y}}$. Figure 5.3(c) shows a picture of both $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$ at a neighborhood of a flow line γ_{x_i}. Now we modify C in the following way. For each edge of F, we substitute the corresponding A_i or B_j by the region on the unstable or stable surface bounded by the corresponding edge of F and the segments of $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$, see Figure 5.3(c). We smooth the edges of this surface and denote by $\tilde{C}$ this smooth surface with boundary, which has cusps. We note that $\tilde{C}$ gives rise to a

cobordism $S \subset Y \times [0,1]$ between $K_{\mathbf{x}} \times \{0\}$ and $K_{\mathbf{y}} \times \{1\}$ that is trivial where $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$ coincide.

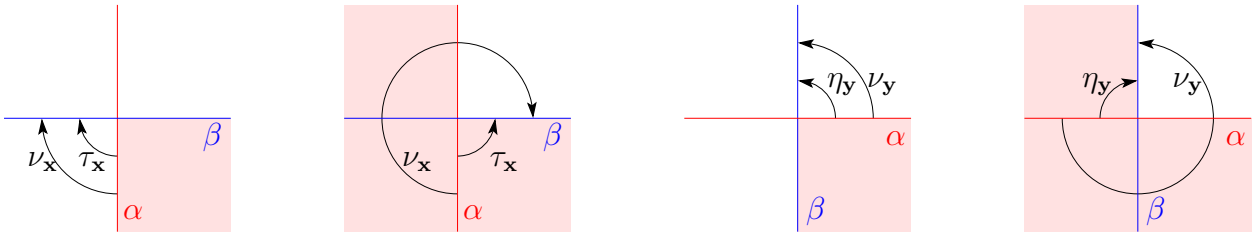


(a) (b) (c)

Figure 5.3:

If we are given a link cobordism between two links and a framing of one, then it induces a framing of the other. So $\tau_{\mathbf{x}}$ induces a framing $\tau_{\mathbf{y}}$ of $K_{\mathbf{y}}$. The Pontryagin-Thom construction tells us that $\widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y})$ equals $\tau_{\mathbf{y}} - \eta_{\mathbf{y}}$. We will now compute this difference. Since $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$ coincide as framed links outside of $\tilde{C}$, we only need to do this calculation in a neighborhood of $\tilde{C}$. To do so, we take a normal vector field N to $\tilde{C}$ and extend it arbitrarily to $K_{\mathbf{x}} \cap K_{\mathbf{y}}$. So N gives rise to a framing of S, which we call ν. We denote by $\nu_{\mathbf{x}}$ and $\nu_{\mathbf{y}}$ the restrictions of ν to $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$, resp. We will compute the difference between the framings by first comparing them with ν and then using the fact that

$$\tau_{\mathbf{y}} - \eta_{\mathbf{y}} = (\tau_{\mathbf{y}} - \nu_{\mathbf{y}}) - (\eta_{\mathbf{y}} - \nu_{\mathbf{y}}) = (\tau_{\mathbf{x}} - \nu_{\mathbf{x}}) - (\eta_{\mathbf{y}} - \nu_{\mathbf{y}}).$$



convex x_i concave x_i convex y_j concave y_j

Figure 5.4:

We will look at a neighborhood of the corners of F. In fact we only need to compute how many times $\tau_{\mathbf{x}}$ rotates with respect to $\nu_{\mathbf{x}}$, where $K_{\mathbf{x}}$ coincides with each γ_{x_i} and similarly

for $\eta_{\mathbf{y}}$. We call a nondegenerate corner of F *convex*[1] if it is a corner of some $D_k \subset F$ for only one k and *concave*[1] if it is a corner of some $D_k \subset F$ for three values of k. For convex vertices, the difference is 0 for both an x_i and a y_j. For concave vertices, it is $+1$ for an x_i and -1 for a y_j, as shown in Figure 5.4. In this picture, the orientation of the link is pointing down, so a counterclockwise turn counts as a $+1$, since that is a left-handed twist. At the distinguished corner, we rotated E_1 by an additional $2\pi\chi(F)$ clockwise. If this is an x_i it accounts for $\chi(F)$ in $\tau_{\mathbf{x}} - \nu_{\mathbf{x}}$ and if it is a y_j, it accounts for $-\chi(F)$ in $\eta_{\mathbf{y}} - \nu_{\mathbf{y}}$. So $\tau_{\mathbf{y}} - \eta_{\mathbf{y}} = \chi(F) + q$, where q is the number of concave corners.

Now if we denote by p the number of convex corners, by Lipshitz's formula,

$$\begin{aligned} \mathrm{ind}(F) &= e(F) + n_{\mathbf{x}}(F) + n_{\mathbf{y}}(F) \\ &= \chi(F) - \tfrac{1}{4}p + \tfrac{1}{4}q + \tfrac{1}{4}p + \tfrac{3}{4}q \\ &= \chi(F) + q = \tau_{\mathbf{y}} - \eta_{\mathbf{y}}. \end{aligned}$$

Since $n_z(F) = 0$, we conclude that $\widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y}) = \tau_{\mathbf{y}} - \eta_{\mathbf{y}} = \mu(A) = \mathrm{gr}(\mathbf{x}, \mathbf{y})$.

Step 2: We will now prove a technical lemma that will be useful in the general case.

Given two links K_1 and K_2 in Y that belong to the same homology class, let S be an immersed cobordism between them. That means that S is an immersed oriented compact surface in $Y \times [0,1]$ that is embedded near its boundary and such that $\partial S = K_1 \times \{1\} \cup (-K_2) \times \{0\}$. Since an immersed surface also has a normal bundle, we can ask whether framings of K_1 and K_2 extend to a framing of S. So given a framing of K_1, the surface S induces a framing of K_2. The induced framing of K_2 depends heavily on S. In fact, if we denote the signed number of self-intersections of S by $\delta(S)$, we have the following lemma. Here we orient $Y \times [0,1]$ by declaring that $\{\partial_t, E_1, E_2, E_3\}$ is an oriented basis, where $\{E_1, E_2, E_3\}$ is an oriented basis for TY and t is the coordinate function on $[0,1]$.

Lemma 5.2.1. *Let K_1 and K_2 be links in Y that belong to the same homology class and let S and S' be immersed cobordisms between them, which are in the same relative homology class. Given a framing of K_1, let ζ_S and $\zeta_{S'}$ be the framings induced on K_2 by S and S', respectively. Then $\zeta_S - \zeta_{S'} = 2(\delta(S) - \delta(S'))$.*

To prove that, we will use another lemma, which is a standard result in Differential Topology.

Lemma 5.2.2. *Let Σ be a closed oriented surface immersed into a closed oriented 4-manifold X. Let $e(N_\Sigma)$ be the Euler class ot the normal bundle of Σ with the orientation induced by the orientation of X. Then*

$$[\Sigma] \cdot [\Sigma] = e(N_\Sigma) + 2\delta(\Sigma).$$

Proof of Lemma 5.2.1. We are given $S, S' \subset Y \times [0,1]$ such that $\partial S' = \partial S = K_1 \times \{1\} \cup (-K_2 \times \{0\})$ and such that $S' - S$ vanishes in $H_2(Y \times [0,1])$. Now we take two copies of $Y \times [0,1]$, switch the orientation of one of them and glue along their common boundaries.

[1]Some authors use the adjectives acute and obtuse to denote convex and concave, respectively.

We can think of this as $Y \times [-1,1]$ with the obvious identification of $Y \times \{-1\}$ and $Y \times \{1\}$, which gives us $Y \times S^1$. We can also glue $S \subset Y \times [0,1]$ to $-S' \subset Y \times [-1,0]$ and we get a closed surface that we call Σ. Now we can assume that in $Y \times [-\varepsilon, \varepsilon]$, the surface Σ is $K_2 \times [-\varepsilon, \varepsilon]$, for ε small. We use S to get a framing on $K_2 \subset Y \times \{\varepsilon\}$ and S' to get a framing on $K_2 \subset Y \times \{-\varepsilon\}$. These are exactly ζ_S and $\zeta_{S'}$, respectively. It follows that the relative Euler class of the normal bundle of Σ restricted to $K_2 \times [-\varepsilon, \varepsilon]$ given these two framings is $\zeta_{S'} - \zeta_S$. Therefore $e(N_\Sigma) = \zeta_{S'} - \zeta_S$. Now, if we think of S, S' and Σ as chains in $Y \times S^1$, we can write $\Sigma = S - S'$. So $\Sigma - (K_1 \times S^1)$ vanishes in $H_2(Y \times S^1)$. Hence

$$[\Sigma] \cdot [\Sigma] = [K_1 \times S^1] \cdot [K_1 \times S^1] = 0.$$

Therefore, by Lemma 5.2.2,

$$\zeta_S - \zeta_{S'} = 2\delta(\Sigma) = 2(\delta(S) - \delta(S')).$$

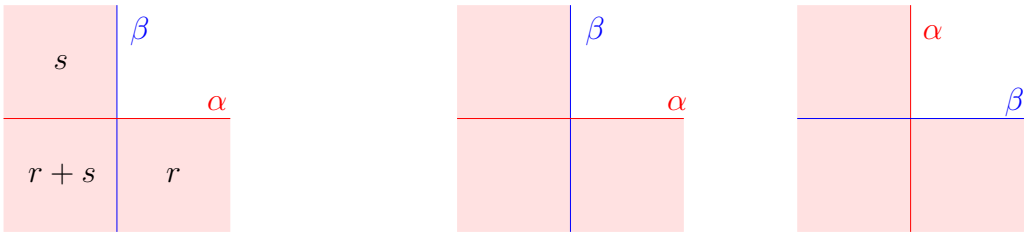
□

Step 3: We now proceed to the general case. We had written $D(\varphi)$ as a union of surfaces $F_l \subset \Sigma$, which can be seen as 2-chains in Σ. We need to show that

$$\widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y}) = \sum_{l=1}^{m} \Big(e(F_l) + n_{\mathbf{x}}(F_l) + n_{\mathbf{y}}(F_l) - 2n_z(F_l) \Big).$$

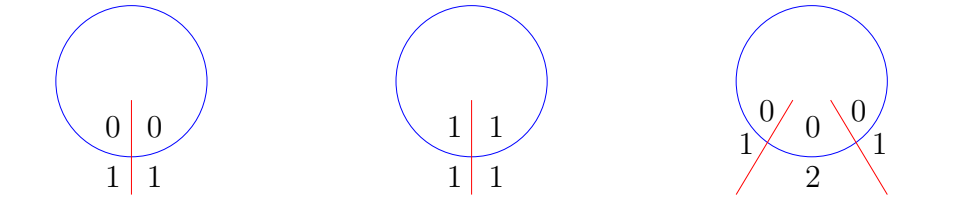
Let γ_a be the projection to Σ of the image of $\partial D^2 \cap \{z; \mathrm{Re}(z) \leq 0\}$ under φ and γ_b be the projection of the image of $\partial D^2 \cap \{z; \mathrm{Re}(z) \geq 0\}$. Then $\gamma_a - \gamma_b = \partial D(A) = \sum_l \partial F_l$. We observe that the a corner of F_l can either be an x_i, a y_j or neither. If it is neither of the two, then the interiors of γ_a and γ_b intersect at that point. We call this point an auxiliary corner and denote each of them by w_k for some k. Now fix and auxiliary corner w_k. Let r be the multiplicity of γ_a and s be the multiplicity of γ_b in a neighborhood of w_k and assume $r < s$, see Figure 5.5(a). We might also have an extra t to the multiplicity of all the four regions. But that will not affect the calculations. So, for simplicity, we can assume that $t = 0$. We get a convex corner for r of the F_l's and a concave one for r of the F_l's. For $(s - r)$ of the F_l's, this point lies on the boundary and is not a corner. We denote by γ_{w_k} the flow line passing through w_k. We say that w_k is positive if it behaves as a convex x_i (i.e γ_{w_k} is positively oriented) and as a concave y_j (i.e γ_{w_k} is negatively oriented), and that w_k is negative if the opposite happens, as shown in Figure 5.5(b).

The orientations on γ_a and $-\gamma_b$ give rise to an orientation of ∂F_l. That is also the orientation induced from Σ, since $A \geq 0$. Now we need to define $\{E_1, E_2, E_3\}$. We want to define E_1 on F_l in the same way as we did when we had only one F_l. But we have to be more careful since we may have α and β curves contained on the surface F_l. This can happen in three different ways: there is a boundary degenerate corner, an interior degenerate corner or a pair of nondegenerate corners that are on ∂F_l but are not corners of ∂F_l for some l. Figure 5.6 shows an example of each of those case.



(a) positive w_k (b) negative w_k

Figure 5.5:



boundary degenerate interior degenerate pair of nondegenerate

Figure 5.6:

For each F_l, we can define C_l, just as we did to define C in Step 1, except that when one of the edges of F_l is a circle, we will attach a disk to it, not a triangular surface. We will first define E_1 on F_m. For each edge of F_m that is not a circle, we define E_1 to be the positive unit tangent vector to ∂F_m outside neighborhoods of the corners. Along an edge that is a circle, we define E_1 to be any vector field whose rotation number along this circle is 0. We note that nondegenerate corners along this circle, e.g. Figure 5.6, cannot happen for F_m. If we have an α or β circle contained in the interior of F_m, then we define E_1 along this circle such that its rotation number is 0. In a neighborhood of each corner including the auxiliary ones, we rotate E_1 as least as possible, as we did in Step 1. We also need to choose some nondegenerate corners, i.e. not auxiliary corners, to rotate a total of $\chi(F_m) + d(F_m)$, where $d(F_m)$ denotes the number of boundary degenerate corners of F_m. After doing that, we can now extend E_1 to a vector field on F_m. Now we extend it to the triangular surfaces belonging to C_m just as we did in Step 1. For each circle on ∂F_m, we extend E_1 to the attaching disk by requiring that it is tangent to the surface $f^{-1}(t)$, for every $3/2 \leq t \leq 2$, if the circle is a β_j and for every $1 \leq t \leq 3/2$ if the circle is an α_i. We note that E_1 is not tangent to this disk at any point except for the corresponding critical point, i.e when $t = 1$ or 2, and on Σ.

Now we want to extend E_1 to $F_{m-1} \supset F_m$. We first define E_1 on ∂F_{m-1}. We can do it the same way as we did for ∂F_m except near the intersection of ∂F_{m-1} and F_m, where E_1 is already defined. This can only happen in two cases. The first one is when they intersect at an auxiliary corner. In this case we just rotate E_1 along ∂F_{m-1} as least as possible, so that it coincides with E_1 at the corner. The second case is when there is a circle in F_{m-1}

that contains two nondegenerate corners. In this case, E_1 is already defined in the segment connecting the two nondegenerate corners. So we extend it to all of this circle in such a way that its rotation number is 0. After doing that, we can extend E_1 to C_{m-1} just as we did for C_m. Proceeding by induction, we define E_1 on C_l, for $l = m, m-1, \ldots, 1$.

We can define E_3 on C_l as we did before, but when we have a circle on ∂C_l, we extend E_3 to the corresponding disk by requiring that E_3 is normal to $f^{-1}(t)$ for every t. Now we define E_2 such that $\{E_1, E_2, E_3\}$ is an orthonormal basis for TY along C_l for all l.

For every α or β circle contained in F_1, either we have attached the corresponding disk to it in some C_l or it contains an interior degenerate corner, in which case, we have also required that the rotation number of E_1 along this circle is 0. So in the latter case, we can extend E_1 and E_3 as we did when the circle was in the boundary. Now, there is no obstruction to extending the orthonormal frame $\{E_1, E_2, E_3\}$ to all of Y and, as before, that determines a trivialization by sending E_i to $e_i \in \mathbb{R}^3$.

Again, we take $K'_{\mathbf{x}} = w_{\mathbf{x}}^{-1}(e_1)$ and $K'_{\mathbf{y}} = w_{\mathbf{y}}^{-1}(e_1)$. We can isotope them the same way as before to get $K_{\mathbf{x}}$ and $K_{\mathbf{y}}$ so that they contain segments of γ_{x_i} and γ_{y_i} near the respective corners. We also define the surfaces $\tilde{C}_l$ in the same fashion as we did in Step 1. Now, to compute the difference of their framings, we will use several immersed cobordisms. We start from $K_{\mathbf{y}}$. We use $\tilde{C}_1$ to define an immersed cobordism. This cobordism exchanges segments of the flow lines γ_{y_j} corresponding to corners y_j of F_1 with segments of some γ_{x_i} corresponding to corners x_i of F_1 and possibly segments of some γ_{w_k}, corresponding to concave auxiliary corners w_k. The next step is to use $\tilde{C}_2$ to construct an immersed cobordism which exchanges segments of some γ_{y_i} by segments of some γ_{x_i}, possibly involves auxiliary corners and keeps the rest of the link fixed. We can continue this construction inductively and define immersed cobordisms for $\tilde{C}_1, \ldots, \tilde{C}_m$. Every time we obtain a γ_{w_k}, it will first appear as a concave corner and later as a convex corner. If w_k is positively oriented, then it will appear as a positive concave angle and a negative convex angle, which means that they just cancel, when we stack the immersed cobordisms. If w_k is negatively oriented, then it will appear as a negative concave corner first and as a positive convex corner later. In this case, we add trivial cobordisms to the immersed cobordisms where the segment of γ_{w_k} appears and to all of the ones in between. After stacking all those, the auxiliary corners cancel and we obtain an immersed cobordism from $K_{\mathbf{y}}$ to $K_{\mathbf{x}}$. Similarly to the case when we had only one F_l, we conclude that the difference of the framings using the cobordism induced by $\tilde{C}_l$ is $\chi(F_l) + d(F_l) + q(F_l)$ for each l, where $q(F_l)$ is the number of concave corners of F_l, not counting the auxiliary corners. Moreover for each auxiliary corner w_k, the difference of framings is $+1$ if w_k is positive, and -1 if w_k is negative. So using this immersed cobordism from $K_{\mathbf{y}}$ to $K_{\mathbf{x}}$, the difference between the framings is $\sum_{l=1}^{m} \Big(\chi(F_l) + d(F_l) + q(F_l)\Big)$ plus the signed count of the auxiliary corners.

We know that there is an embedded link cobordism from $K_{\mathbf{y}}$ to $K_{\mathbf{x}}$ in the same relative homology class as the immersed cobordism we were considering. So, by Lemma 5.2.1, $\tau_{\mathbf{y}} - \eta_{\mathbf{y}}$ equals the difference obtained using the immersed cobordism minus twice the signed number of self-intersections of the immersed cobordism, since the self-intersection number of an

embedded cobordism is 0. We now need to consider three cases.

(i) There are boundary degenerate corners or a pair of nondegenerate corners on an α or β curve contained in some ∂F_l.

(ii) There are interior degenerate corners

(iii) There are nondegenerate corners in the interior of some F_l.

(iii) The basepoint z in in the interior of F_1.

In case (i), self-intersections could exist if $K_{\mathbf{x}}$ or $K_{\mathbf{y}}$ intersects C_l for l such that C_l contains the disk we attach to the corresponding α or β circle. Let x_i and y_j be the corresponding corners. Then C_l divides $N(\gamma_{x_i})$ in two disconnected components and we can see that $K_{\mathbf{x}}$ enters and exits $N(\gamma_{x_i})$ in the same component. Similarly for y_j. Therefore the signed number of intersections with C_l is 0. In this case, $n_{x_i} + n_{y_j} = 1$. But this $+1$ appears in the difference of framings when we added $d(F_l)$ turns to E_1 near a nondegenerate corner.

In case (ii), let $x_i = y_j$ be the interior degenerate corner. So, $n_{x_i} + n_{y_j} = 2$. Also, $K_{\mathbf{x}} = K_{\mathbf{y}}$ in $N(\gamma_{x_i})$. Also, $K_{\mathbf{x}}$ intersects C_l negatively at only one point. Therefore, by Lemma 5.2.1, we have two add $+2$ to the difference of the framings.

In case (iii), since $F_i \supset F_j$, for $i < j$, and the cobordism corresponding to $\tilde{C}_i$ is taken before the one corresponding to $\tilde{C}_j$, only the nondegenerate y_j's which are in the interior of an F_j correspond to intersections. So, by Lemma 5.2.1, we have to add twice the number of interior nondegenerate y_j's. On the other hand, if we had built our immersed cobordisms in the opposite order, i.e. starting with F_m and going all the way to F_1, then we would get the same result, except that we would be counting twice the number of interior nondegenerate corners x_i, but in this case the sign of the auxiliary corners are switched. Since the two calculations have to coincide, it follows that the number of interior nondegenerate corners x_i plus the number of positive auxiliary corners equals the number of interior nondegenerate corners y_j plus the number of negative auxiliary corners. So twice the number of interior nondegenerate x_i's plus the signed count of the auxiliary corners equals the total number of interior nondegenerate corners. That is exactly what we were missing to get the full $n_{\mathbf{x}}(F_l)$ and $n_{\mathbf{y}}(F_l)$. Therefore, combining cases (i),(ii) and (iii), we conclude that the difference of the framings is $\sum_{l=1}^{m} \Big(e(F_l) + n_{\mathbf{x}}(F_l) + n_{\mathbf{y}}(F_l)\Big)$, which is equal to $\mu(A)$.

In case (iv), then $K_{\mathbf{x}} = K_{\mathbf{y}}$ near γ_z. If $K_{\mathbf{x}}$ intersects F_l, then it does so positively. Hence, by Lemma 5.2.1, we get an extra $-2\sum_l n_z(F_l)$ in the difference of framings. Therefore

$$\widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y}) = \tau_{\mathbf{y}} - \eta_{\mathbf{y}} = \mu(A) - 2n_z(A) = \mathrm{gr}(\mathbf{x}, \mathbf{y}).$$

5.3 The absolute grading of the contact invariant

In [29], Oszváth-Szabó defined the contact class $c(\xi) \in \widehat{HF}(-Y)$ for a contact 3-manifold (Y, ξ), and they showed that it is an invariant of ξ. Later, Honda-Kazez-Matić [8] gave an

alternative definition of $c(\xi)$ using an open book decomposition adapted to ξ. In this section, we compute the absolute grading of the contact invariant $c(\xi)$.

Contact topology and open book decompositions

Let Y be a closed oriented 3-manifold. A contact structure ξ is a maximally non-integrable co-oriented 2-plane field, i.e. there exists a 1-form λ such that $\lambda \wedge d\lambda > 0$ and $\xi = ker\lambda$. We call such λ a *contact form* of ξ. The *Reeb vector field* R_λ associated with λ is the unique vector field which satisfies (i) $R_\lambda \lrcorner d\lambda = 0$, (ii) $R_\lambda \lrcorner \lambda = 1$. Although the dynamics of R_λ depend heavily on the choice of λ, its homotopy class is an invariant of ξ. In fact, two contact structures are homotopic if and only if their associated Reeb vector fields are homotopic.

Now recall that an *open book decomposition* of Y is a pair (S, h), where S is a compact, oriented surface of genus g with boundary, $h : S \to S$ is a diffeomorphism which is the identity on ∂S, and Y is homeomorphic to $(S \times [0,1])/\sim$. The equivalence relation $\sim$ is defined by $(x, 1) \sim (h(x), 0)$ for $x \in S$ and $(y, t) \sim (y, t')$ for $y \in \partial S$ and $t, t' \in [0, 1]$. Given a contact structure ξ on Y, an open book (S, h) is *adapted to* ξ if there exists a contact form λ for ξ such that R_λ is positively transverse to $int(S)$ and positively tangent to ∂S.

Fix an adapted open book (S, h) of (Y, λ). Following [8], let $\{a_1, \cdots, a_{2g}\}$ be a set of pairwise disjoint, properly embedded arcs on S such that $S \setminus \bigcup_{i=1}^{2g} a_i$ is a single polygon. We call $\{a_1, \cdots, a_{2g}\}$ a *basis* for S. Next let b_i be an arc which is isotopic to a_i by a small isotopy so that the following hold:

1. The endpoints of a_i are isotoped along ∂S, in the direction given by the boundary orientation of S.

2. a_i and b_i intersect transversely in one point x_i in the interior of S.

3. If we orient a_i, and b_i is given the induced orientation from the isotopy, then the sign of the intersection $a_i \cap b_i$ is $+1$.

See Figure 5.7.

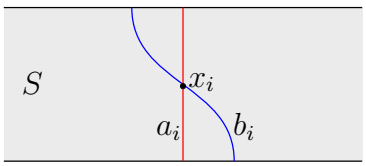


Figure 5.7: The arcs a_i and b_i on S.

Observe that (S, h) naturally induces a Heegaard splitting of Y by letting $H_1 = (S \times [0, 1/2])/\sim$ and $H_2 = (S \times [1/2, 1])/\sim$. This gives a Heegaard decomposition of Y of

genus $2g$ with Heegaard surface $\Sigma = \partial H_1 = -\partial H_2$. By choosing a basis $\{a_1, \cdots, a_{2g}\}$ for S and following the constructions above, we obtain two collections of simple closed curves $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_{2g}\}$ and $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_{2g}\}$ on Σ, where $\alpha_i = \partial(a_i \times [0, 1/2])$ and $\beta_i = \partial(b_i \times [1/2, 1])$ for $i = 1, \cdots, 2g$. Then one can properly place the basepoint z and reverse the orientation of Y to obtain a weakly admissible Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ for $-Y$. It is observed in [8] that $\mathbf{x} = (x_1, \cdots, x_{2g}) \in \widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ defines a cycle, where $x_i = a_i \cap b_i \in \alpha_i \cap \beta_i$, $i = 1, \cdots, 2g$.

Theorem 5.3.1 (Honda-Kazez-Matić [8])**.** *The class* $[\mathbf{x}] \in \widehat{HF}(-Y)$ *represented by* $\mathbf{x} \in \widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ *from above is an invariant of* ξ *and it is equal to* $c(\xi)$ *defined in [29].*

Remark 5.3.2. In light of Theorem 5.3.1, in order to prove Theorem 4.1.1(b), it suffices to show

$$\widetilde{\mathrm{gr}}(\mathbf{x}) = [\xi] \tag{5.1}$$

as homotopy classes of oriented 2-plane fields.

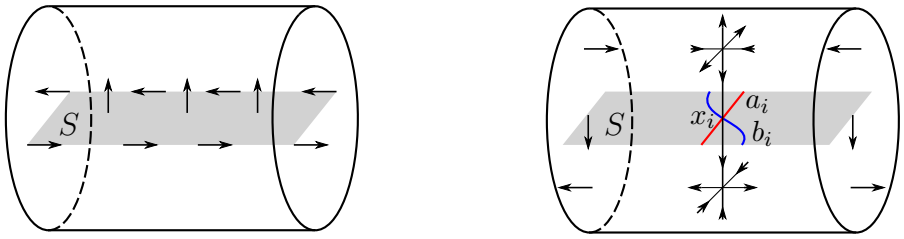
Proof of Theorem 4.1.1(b)

Throughout this section, we fix a contact form λ and an adapted open book decomposition (S, h) of (Y, λ). Note that the contact invariant is presented as an intersection point $\mathbf{x}$ in $\widehat{CF}(-Y)$. The plan is to use the Pontryagin-Thom construction to show that the vector field constructed in§5.1 to define $\widetilde{\mathrm{gr}}(\mathbf{x})$ is homotopic to the Reeb vector field R_λ.

Proof of Theorem 4.1.1(b). Let f be a Morse function adapted to our special Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where $\Sigma = (S \times \{0\}) \cup (S \times \{1/2\})$. Note that one needs to reverse the orientation of Y to define $[\mathbf{x}] = c(\xi)$. Equivalently, we shall consider, for the rest of the proof, the same Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, but with the downward gradient vector field $-\nabla f$. All the constructions of the absolute grading function carry over by simply reversing the direction of all vector fields. Let $v_{\mathbf{x}}$ be a nonvanishing vector field, which is a modification of $-\nabla f$, as defined in §5.1. In particular, the homotopy class of the orthogonal complement of $v_{\mathbf{x}}$ equals $\widetilde{\mathrm{gr}}(\mathbf{x})$. Let $\tilde{S} \subset int(S)$ be a closed subsurface such that S deformation retracts onto $\tilde{S}$, and assume that h is supported in $\tilde{S} \times \{1\}$. It is easy to see that $-\nabla f$ is homotopic to R_λ by linear interpolation in a small neighborhood $N(\tilde{S} \times \{1\})$ of $\tilde{S} \times \{1\}$ in M because they are both positively transverse to $\tilde{S} \times \{1\}$. Let $H = Y \setminus N(\tilde{S} \times \{1\})$ be the genus $2g$ handlebody[2]. So it suffices to show that $v_{\mathbf{x}}|_H$ is homotopic to $R_\lambda|_H$ relative to ∂H.

To do so, consider a closed collar neighborhood $a_i \times [-1, 1] \subset S \times \{1/2\}$ of a_i on the middle page such that it contains b_i in the interior, for $i = 1, \cdots, 2g$. Let $B_i = (a_i \times [-1, 1] \times [0, 1]) \cap H \subset H$ be a 3-ball (with corners) in H, which contains a_i and b_i in the interior. See Figure 5.8 for pictures of the vector fields $R_\lambda|_{B_i}$ and $-\nabla f|_{B_i}$.

[2]In fact H is a handlebody with corners, but this is irrelevant here because we are considering continuous vector fields.



(a) (b)

Figure 5.8: (a) The Reeb vector field R_λ restricted to B_i. (b) The downward gradient vector field $-\nabla f$ restricted to B_i.

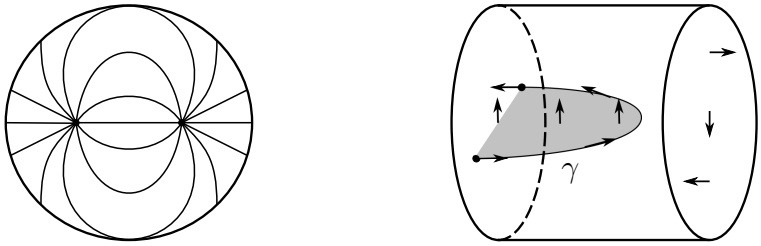
Claim: There exists a non-singular vector field R'_λ on H, homotopic to R_λ relative to ∂H, such that (i) $R'_\lambda|_{\partial B_i} = v_{\mathbf{x}}|_{\partial B_i}$, (ii) $R'_\lambda|_{B_i}$ is homotopic to $v_{\mathbf{x}}|_{B_i}$ relative to ∂B_i, for $i = 1, \cdots, 2g$.

Proof of Claim. Let $D_l = (a_i \times \{-1\} \times [0,1]) \cap H$ and $D_r = (a_i \times \{1\} \times [0,1]) \cap H$ be the left and right disk boundaries of B_i, respectively. Observe that $R_\lambda = v_{\mathbf{x}}$ on $\partial B_i \setminus (D_l \cup D_r)$ by construction. We shall consider a collar neighborhood $N(D_l) = (a_i \times [-1-\delta, -1+\delta] \times [0,1]) \cap H$ of D_l for some small $\delta > 0$, and homotope R_λ to R'_λ with the desired properties within $N(D_l)$. Note that the same construction can be carried over to a collar neighborhood of D_r.

We construct a model vector field V_l on $D^2 \times [-1,1]$ in steps. First let $\mathcal{F}_0$ be a singular foliation on D^2 which has two elliptic singularities as depicted in Figure 5.9(a). Let $\gamma \subset D^2 \times [-1,0]$ be a properly embedded, boundary parallel arc such that $\partial\gamma$ is exactly the union of the two singularities of $\mathcal{F}_0$ on $D^2 \times \{-1\}$. Then there exists a foliation $\mathcal{F}$ by disks on $D^2 \times [-1,0]$ such that for any leaf F of $\mathcal{F}$, we have $\partial F \cap int(D^2 \times [-1,0]) = \gamma$, and $\partial F \cap (D^2 \times \{-1\})$ is a leaf of $\mathcal{F}_0$. Let V'_l be a non-singular vector field on $D^2 \times [-1,0]$ such that it is positively tangent to γ and positively transverse to the interior of all leaves of $\mathcal{F}$ as depicted in Figure 5.9(b). Up to homotopy, we can assume that $V'_l|_{D^2 \times \{0\}} = v_{\mathbf{x}}|_{D_l}$ as vector fields on a disk. By fixing a trivialization of the tangent bundle $T(D^2 \times [-1,1])$ using the standard embedding $D^2 \times [-1,1] \subset \mathbb{R}^3$, we define the vector field V_l on $D^2 \times [-1,1]$ by

$$V_l(x,t) = \begin{cases} V'_l(x,t) & \text{if } -1 \le t \le 0, \\ V'_l(x,-t) & \text{if } 0 \le t \le 1. \end{cases}$$

where $x \in D^2$ is any point. Identify $D^2 \times [-1,1]$ with $N(D_l)$ by rescaling in the $[-1,1]$-direction such that D_l is identified with $D^2 \times \{0\}$, $N(D_l) \setminus B_i$ is identified with $D^2 \times [-1,0]$, and $N(D_l) \cap B_i$ is identified with $D^2 \times [0,1]$. It is easy to see that $R_\lambda|_{N(D_l)}$ is homotopic to V_l as vector fields on $N(D_l)$ relative to the boundary. Similarly, one can define a non-singular vector field V_r on $N(D_r)$ such that $R_\lambda|_{N(D_r)}$ is homotopic to V_r as vector fields on $N(D_r)$ relative to the boundary. By applying the above homotopy, which is supported in $N(D_l) \cup N(D_r)$, to R_λ, and repeat this process for every B_i, $i = 1, \cdots, 2g$, we obtain a new non-singular vector field R'_λ. Observe that R'_λ satisfies condition (i) by construction.



(a) (b)

Figure 5.9: (a) The singular foliation on D^2. (b) The vector field V_l' on a leaf of $\mathcal{F}$ in $D^2 \times [-1, 0]$.

To show that R'_λ satisfies condition (ii), we use the Pontryagin-Thom construction. Trivialize the tangent bundle TB_i by embedding $B_i \subset \mathbb{R}^3$ such that D_l (or D_r) is parallel to the xz-plane, and the $[-1, 1]$-direction is parallel to the y-axis. Consider the associated Gauss maps $G_{v_\mathbf{x}}|_{B_i} : B_i \to S^2$ and $G_{R'_\lambda}|_{B_i} : B_i \to S^2$. Without loss of generality, we assume that $G_{v_\mathbf{x}}|_{B_i}$ and $G_{R'_\lambda}|_{B_i}$ are smooth, and $p = (0, 1, 0) \in S^2$ is a common regular value. Let $p' = (\varepsilon, \sqrt{1-\varepsilon^2}, 0) \in S^2$ be a nearby common regular value which keeps track of the framing, where $\varepsilon > 0$ is small. It is now a straightforward computation that the Pontryagin submanifolds $G^{-1}_{v_\mathbf{x}}(p)$ and $G^{-1}_{R'_\lambda}(p)$ are both framed cobordant to the framed arc depicted in Figure 5.10 relative to the boundary. Hence $R'_\lambda|_{B_i}$ is homotopic to $v_\mathbf{x}|_{B_i}$ relative to ∂B_i, for all $i = 1, \cdots, 2g$. This finishes the proof of the claim. □

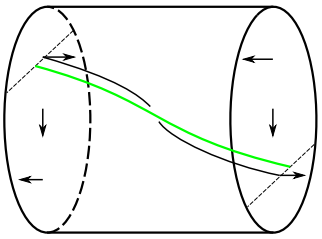


Figure 5.10: A framed arc in B_i, where the framing is indicated by the green arc.

It remains to show that R'_λ is homotopic to $v_\mathbf{x}$ on $\overline{H \setminus (\bigcup_{i=1}^{2g} B_i)}$ relative to the boundary. Let (D^2, id) be the trivial open book of S^3, and $\tilde{D} \subset int(D^2)$ be a slightly smaller disk. Let $\tilde{H}$ denote $\overline{H \setminus (\bigcup_{i=1}^{2g} B_i)}$ and observe that it is naturally identified with $(D^2 \times [0, 1] \setminus ((\tilde{D} \times [0, \varepsilon)) \cup (\tilde{D} \times (1-\varepsilon, 1])))/\sim$ by construction. On the one hand, it is easy to see that $R'_\lambda|_{\tilde{H}}$ is homotopic to the restriction of the Reeb vector field compatible with the open book (D^2, id). On the other hand, note that $\tilde{H}$ is nothing but a neighborhood of the gradient trajectory which connects the index 0 critical point to the index 3 critical point. Hence it

follows immediately from our construction of $\widetilde{\mathrm{gr}}(\mathbf{x})$ that $v_{\mathbf{x}}|_{\tilde{H}}$ is also homotopic to the Reeb vector field compatible with (D^2, id). This finishes the proof of Theorem 4.1.1(b). □

Now we compute the twisted absolute grading of the twisted contact invariant defined in [26]. Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be the generator in $\widehat{CF}(-Y)$, which defines the usual contact invariant as before. Let $\mathbb{Z}[H^1(Y;\mathbb{Z})]^\times$ denote the set of invertible elements in $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. First recall that the twisted contact invariant $\underline{c}(\xi)$ associated with the contact structure ξ is defined by

$$\underline{c}(\xi) = [u \cdot \mathbf{x}] \in \widehat{\underline{HF}}(-Y)/\mathbb{Z}[H^1(Y;\mathbb{Z})]^\times$$

where $u \in \mathbb{Z}[H^1(Y;\mathbb{Z})]^\times$. Although $\underline{c}(\xi)$ is only well-defined up to a unit in $\mathbb{Z}[H^1(Y;\mathbb{Z})]$, the twisted absolute grading $\widetilde{gr}_{tw}(\underline{c}(\xi))$ defined by (4.2) still makes sense. The following result is immediate.

Corollary 5.3.3. *If ξ is a contact structure on Y, then $\widetilde{gr}_{tw}(\underline{c}(\xi)) = [\xi] \in \mathcal{P}(Y)$.*

Proof. This follows immediately from (4.2) and Theorem 4.1.1(b). □

Now we are ready to prove the corollaries given in Section 1.

Proof of Corollary 4.1.5. If (Y,ξ) is strongly fillable, then $c(\xi) \neq 0 \in \widehat{HF}(-Y)$ according to [29]. Since $\widehat{HF}(-Y)$ is a finitely generated Abelian group, there can be only finitely many absolute gradings, i.e., homotopy classes of 2-plane fields, that support strongly fillable contact structures.

Now if (Y,ξ) is weakly fillable, then $\underline{c}(\xi) \neq 0 \in \widehat{\underline{HF}}(-Y)/\mathbb{Z}[H^1(Y;\mathbb{Z})]^\times$ according to [26]. Since $\widehat{\underline{HF}}(-Y)$ is finitely generated as a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ module, the same argument as above together with Corollary 5.3.3 implies that there can be only finitely many homotopy classes of 2-plane fields in Y that support weakly fillable contact structures. □

Proof of Corollary 4.1.6. By definition if Y is an L-space, then $\widehat{HF}(-Y)$ is a free Abelian group of rank $|H_1(Y;\mathbb{Z})|$. Therefore there are at most $|H_1(Y;\mathbb{Z})|$-many homotopy classes of 2-plane fields that support strongly fillable contact structures. To get the same result for weakly fillable contact structures, it suffices to observe that since Y is a rational homology sphere by assumption, we have

$$\widehat{\underline{HF}}(-Y) \simeq \widehat{HF}(-Y) \otimes \mathbb{Z}[H^1(Y;\mathbb{Z})].$$

Hence $\widehat{\underline{HF}}(-Y)$ is a free $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ module of rank $|H_1(Y;\mathbb{Z})|$, and therefore the conclusion follows as before. □

Proof of Corollary 4.1.7. It suffices to note that according to [30], if Y admits a metric of constant positive curvature, then Y is an L-space. □

5.4 4-dimensional cobordism and absolute $\mathbb{Q}$-grading

Let W be a connected compact oriented 4-dimensional cobordism between two connected oriented 3-manifolds Y_0 and Y_1 such that $\partial W = -Y_0 \cup Y_1$. Fixing a Spin^c structure $\mathfrak{t}$ on W, Ozsváth-Szabó [31] constructed a map $F_{W,\mathfrak{s}} : HF^\circ(Y_0, \mathfrak{t}|_{Y_0}) \to HF^\circ(Y_1, \mathfrak{t}|_{Y_1})$ between Heegaard Floer homology groups by choosing a handle decomposition of W, and counting holomorphic triangles. It turns out that $F_{W,\mathfrak{t}}$ is an invariant of W, i.e., it is independent of the choice of a handle decomposition of W. Throughout this section we fix a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for Y_0 and a handle decomposition of W. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ be the associated Heegaard diagram for Y_1 as constructed in [31]. We consider the associated chain map $F_{W,\mathfrak{t}} : \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}|_{Y_0}) \to \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathfrak{t}|_{Y_1})$.

Observe that $F_{W,\mathfrak{t}} : \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}|_{Y_0}) \to \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathfrak{t}|_{Y_1})$ is a linear map between graded vector spaces. However, according to Theorem 4.1.1(a), $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}|_{Y_i})$ is graded by the set of homotopy classes of oriented 2-plane fields $\mathcal{P}(Y_i)$, $i = 0, 1$, so it is not possible to define an integer degree of $F_{W,\mathfrak{t}}$. There is a weaker notion which is applicable here. Namely, let $W : Y_0 \to Y_1$ be a cobordism and ξ_i be an oriented 2-plane field on Y_i, for $i = 0, 1$. We say $\xi_0 \sim_W \xi_1$ if and only if there exists an almost complex structure J on W such that $[\xi_i] = [TY_i \cap J(TY_i)]$, for $i = 0, 1$, as homotopy classes of oriented 2-plane fields.

The main goal of this section is to prove Theorem 4.1.1(d) on the chain level, which we formalize in the following theorem for the reader's convenience.

Theorem 5.4.1. *Let $W : Y_0 \to Y_1$ be a compact oriented cobordism with a fixed handle decomposition, $\mathfrak{t} \in \text{Spin}^c(W)$ a Spin^c structure on W, and $F_{W,\mathfrak{t}} : \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}|_{Y_0}) \to \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathfrak{t}|_{Y_1})$ the associated cobordism map as discussed above. Then $\widetilde{gr}(\mathbf{x}) \sim_W \widetilde{gr}(\mathbf{y})$ for any homogeneous generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}|_{Y_0})$, and any homogeneous summand $\mathbf{y}$ of $F_{W,\mathfrak{t}}(\mathbf{x})$.*

Before we give the proof of Theorem 5.4.1, we take a step back and look at the Heegaard Floer homology $HF^\circ(Y, \mathfrak{s})$ for a torsion Spin^c structure $\mathfrak{s}$. By [31], there is an absolute $\mathbb{Q}$-grading of $HF^\circ(Y, \mathfrak{s})$ which lifts the relative $\mathbb{Z}$-grading. We shall see that our construction indeed generalizes their absolute $\mathbb{Q}$-grading. To do so, recall the following construction due to R. Gompf [7]. Let ξ be an oriented 2-plane field on a closed, oriented 3-manifold Y. Then there exists a compact, almost complex 4-manifold (X, J) whose *almost-complex boundary* is (Y, ξ), i.e. $Y = \partial X$ (as oriented manifolds) and $\xi = TY \cap J(TY)$ with the complex orientation. If $c_1(\xi)$ is a torsion class, then let $\theta(\xi) = (PD\ c_1(X))^2 - 2\chi(X) - 3\sigma(X) \in \mathbb{Q}$, where χ is the Euler characteristic and σ is the signature. Observe that $\theta(\xi)$ is independent of the choice of the capping almost complex 4-manifold (X, J) because the quantity $(PD\ c_1(X))^2 - 2\chi(X) - 3\sigma(X)$ vanishes for a closed X.

Let $\mathfrak{s} \in \text{Spin}^c(Y)$ be a Spin^c structure such that $c_1(\mathfrak{s})$ is a torsion class, and let $\mathfrak{U}$ be the set of homogeneous elements in $\widehat{CF}(Y, \mathfrak{s})$. We define an absolute grading function $\widetilde{\text{gr}}_0 : \mathfrak{U} \to \mathbb{Q}$ by $\widetilde{\text{gr}}_0(\mathbf{x}) = (2 + \theta(\widetilde{\text{gr}}(\mathbf{x})))/4 \in \mathbb{Q}$ for any $\mathbf{x} \in \mathfrak{U}$. This induces an absolute grading function

on $CF^\infty(Y,\mathfrak{s})$ by $\widetilde{\mathrm{gr}}_0([\mathbf{x},i]) = 2i + \widetilde{\mathrm{gr}}_0(\mathbf{x})$, and hence on the sub- and quotient-complexes $CF^-(Y,\mathfrak{s})$ and $CF^+(Y,\mathfrak{s})$.

For reader's convenience, we recall the following theorem/definition of the absolute $\mathbb{Q}$-grading due to Ozsváth-Szabó [31].

Theorem 5.4.2 (Ozsváth-Szabó)**.** *There exists an absolute grading function $\overline{gr} : \mathfrak{U} \to \mathbb{Q}$ satisfying the following properties:*

1. *The homogeneous elements of least grading in $\widehat{HF}(S^3,\mathfrak{s}_0)$ have absolute grading zero.*
2. *The absolute grading lifts the relative grading, in the sense that if $\mathbf{x},\mathbf{y} \in \mathfrak{U}$, then $\overline{gr}(\mathbf{x},\mathbf{y}) = \overline{gr}(\mathbf{x}) - \overline{gr}(\mathbf{y})$.*
3. *If W is a cobordism from Y_0 to Y_1 endowed with a Spin^c structure $\mathfrak{t}$ whose restriction to Y_i is torsion for $i = 0, 1$, then*

$$\overline{gr}(F_{W,\mathfrak{t}}(\mathbf{x})) - \overline{gr}(\mathbf{x}) = \frac{(PD\ c_1(\mathfrak{t}))^2 - 2\chi(W) - 3\sigma(W)}{4}$$

for any $\mathbf{x} \in \mathfrak{U}$.

We have the following corollary:

Corollary 5.4.3. *The function $\widetilde{gr}_0$ described above defines an absolute $\mathbb{Q}$-grading for $HF^\circ(Y,\mathfrak{s})$, which coincides with the absolute $\mathbb{Q}$-grading $\overline{gr}$ defined above.*

Proof. We use the Pontryagin-Thom construction. By fixing a trivialization of TY, the homotopy classes of oriented 2-plane fields on Y are 1-1 correspondent to the framed cobordism classes of framed links in Y. The first assertion of the corollary follows from Theorem 4.1.1(a) and the observation that adding a right-handed full twist to ξ is equivalent to decreasing $\theta(\xi)$ by 4.

It follows from the proof of Theorem 5.4.1 that if $\mathfrak{t}$ be a Spin^c structure on W whose restriction to Y_i is torsion, for $i = 0, 1$, then $F_{W,\mathfrak{t}}(\mathbf{x})$ is homogeneous for every homogeneous element $\mathbf{x} \in \mathfrak{U}$. Since we have shown in Theorem 5.1.1 that our absolute grading $\widetilde{\mathrm{gr}}$ refines the relative grading, in order to show that $\widetilde{\mathrm{gr}}_0$ coincides with the absolute $\mathbb{Q}$-grading defined in [31], it suffices to verify the following two conditions:

1. (Normalization) For the standard contact 3-sphere (S^3,ξ_{std}), $\widetilde{\mathrm{gr}}_0(c(\xi_{std})) = 0$.
2. (Cobordism formula) Let $W : Y_0 \to Y_1$ be a cobordism, and $\mathfrak{t}$ be a Spin^c structure on W whose restriction to Y_i is torsion, $i = 0, 1$. Then

$$\widetilde{\mathrm{gr}}_0(F_{W,\mathfrak{t}}(\mathbf{x})) - \widetilde{\mathrm{gr}}_0(\mathbf{x}) = \frac{(PD\ c_1(\mathfrak{t}))^2 - 2\chi(W) - 3\sigma(W)}{4}$$

for any homogeneous $\mathbf{x} \in \mathfrak{U}$.

To prove (1), note that it follows from the fact that (S^3, ξ_{std}) is the almost complex boundary of the standard unit 4-ball $B^4 \subset \mathbb{C}^2$.

To prove (2), let (X, J) be an almost complex 4-manifold with almost complex boundary $(Y_0, \widetilde{\mathrm{gr}}(\mathbf{x}))$. By Theorem 5.4.1, there exists an almost complex structure J' on W such that both $\widetilde{\mathrm{gr}}(\mathbf{x})$ and $\widetilde{\mathrm{gr}}(F_{W,\mathfrak{t}}(\mathbf{x}))$ are J'-invariant with the complex orientation. We obtain a new almost complex 4-manifold with almost complex boundary $(X \cup_{Y_0} W, \widetilde{\mathrm{gr}}(F_{W,\mathfrak{t}}(\mathbf{x})))$ by gluing (X, J) and (W, J') along Y_0. Recall the following theorem on the signature of 4-manifolds due to Novikov:

Theorem 5.4.4 (Novikov)**.** *Let M be an oriented 4-manifold obtained by gluing two 4-manifolds M_1 and M_2 along some components of their boundaries. Then the signature is additive:*

$$\sigma(M) = \sigma(M_1) + \sigma(M_2).$$

We therefore calculate as follows:

$$\begin{aligned}
\widetilde{\mathrm{gr}}_0(F_{W,\mathfrak{t}}(\mathbf{x})) - \widetilde{\mathrm{gr}}_0(\mathbf{x}) &= \frac{\theta(\widetilde{\mathrm{gr}}(F_{W,\mathfrak{t}}(\mathbf{x}))) - \theta(\widetilde{\mathrm{gr}}(\mathbf{x}))}{4} \\
&= \frac{(PD\ c_1(W, J'))^2 - 2\chi(W) - 3\sigma(W)}{4} \\
&= \frac{(PD\ c_1(\mathfrak{t}))^2 - 2\chi(W) - 3\sigma(W)}{4},
\end{aligned}$$

This finishes the proof of the second assertion of the corollary. □

The proof of Theorem 5.4.1 occupies the rest of this section. We shall follow the construction of $F_{W,\mathfrak{t}}$ given in [31].

Proof of Theorem 5.4.1. We fix a handle decomposition of W, and study the 2-handle attachments and 1- and 3-handle attachments in W separately.

CASE 1. Suppose W is given by 2-handle attachments along a framed link $L \subset Y_0$. Let Δ denote the two-simplex, with vertices $v_\alpha, v_\beta, v_\gamma$ labeled clockwise, and let e_i denote the edge v_j to v_k, where $\{i, j, k\} = \{\alpha, \beta, \gamma\}$. Recall that given a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, one can associate to it a 4-manifold

$$W_{\alpha,\beta,\gamma} = \frac{(\Delta \times \Sigma) \coprod (e_\alpha \times U_\alpha) \coprod (e_\beta \times U_\beta) \coprod (e_\gamma \times U_\gamma)}{(e_\alpha \times \Sigma) \sim (e_\alpha \times \partial U_\alpha), (e_\beta \times \Sigma) \sim (e_\beta \times \partial U_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial U_\gamma)} \tag{5.2}$$

where U_α (resp. U_β, U_γ) is the handlebody determined by the $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$) curves. Let $Y_{\alpha,\beta} = U_\alpha \cup U_\beta$, $Y_{\beta,\gamma} = U_\beta \cap U_\gamma$, and $Y_{\alpha,\gamma} = U_\alpha \cup U_\gamma$ be the 3-manifolds obtained by gluing the α-, β- and γ-handlebodies along Σ in pairs. After smoothing the corners, we have

$$\partial W_{\alpha,\beta,\gamma} = -Y_{\alpha,\beta} - Y_{\beta,\gamma} + Y_{\alpha,\gamma}$$

as oriented manifolds. See Figure 5.11.

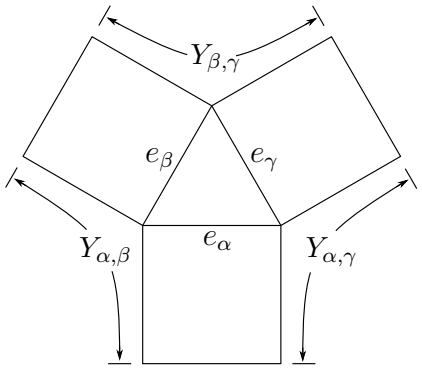


Figure 5.11: The 4-manifold $W_{\alpha,\beta,\gamma}$ associated with a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$.

According to [31], if W is obtained by attaching 2-handles along a framed link L, then there exists a triple Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ such that $Y_{\alpha,\beta} = Y_0$, $Y_{\beta,\gamma} = \#^n(S^1 \times S^2)$ for some $n \geq 1$, and $Y_{\alpha,\gamma} = Y_1$. Moreover, after filling in the boundary component $Y_{\beta,\gamma}$ by the boundary connected sum $\#_b^n(S^1 \times B^3)$, we obtain the original cobordism W. Fix a Spin^c structure $\mathfrak{t}$ on W with $\mathfrak{s}_i = \mathfrak{t}|_{Y_i}$, $i = 0, 1$. Let $\Theta \in \widehat{CF}(\#^n(S^1 \times S^2))$ be the top dimensional generator and let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. By definition, the image of $\mathbf{x}$ under the cobordism map $F_{W,\mathfrak{t}} : \widehat{CF}(Y_0, \mathfrak{s}_0) \to \widehat{CF}(Y_1, \mathfrak{s}_1)$ is a linear combination of the generators $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ with coefficients being the count of Maslov index 0 holomorphic triangles connecting $\mathbf{x}$, Θ and $\mathbf{y}$. Let $\mathbf{y}$ be a generator appearing in $F_{W,\mathfrak{t}}$ with a nonzero coefficient. We prove the following claim.

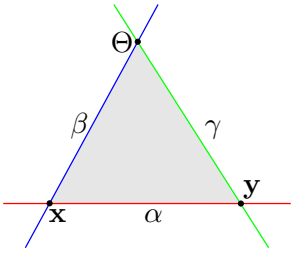
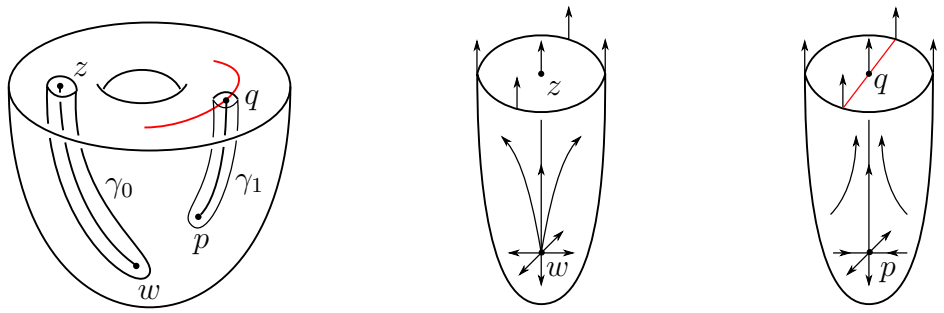


Figure 5.12: A holomorphic triangle on Σ which connects $\mathbf{x}$, Θ, and $\mathbf{y}$.

Claim: There exists an almost complex structure J on $W_{\alpha,\beta,\gamma}$ such that $\widetilde{\mathrm{gr}}(\mathbf{x}) \in \mathcal{P}(Y_0)$, $\widetilde{\mathrm{gr}}(\Theta) \in \mathcal{P}(\#^n(S^1 \times S^2))$, and $\widetilde{\mathrm{gr}}(\mathbf{y}) \in \mathcal{P}(Y_1)$ are all J-invariant with the complex orientation.

Proof of Claim. We first assume that $\mathbf{y}$ is the intersection point as shown in Figure 5.12, which is connected to $\mathbf{x}$ and Θ by the obvious (embedded) holomorphic triangle. We begin by constructing a 2-plane field on $e_\alpha \times U_\alpha$, and note that the same construction carries over to $e_\beta \times U_\beta$ and $e_\gamma \times U_\gamma$.

For simplicity of notations, we assume $g(\Sigma) = 1$, so, for instance, $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is just one point instead of a g-tuple of points. The same argument applies to Heegaard surfaces of arbitrary genus without difficulty. Let V_α be the gradient flow on U_α compatible with the α-curve so that it is pointing out along ∂U_α. Let $p \in U_\alpha$ be the index 1 critical point of V_α and $w \in U_\alpha$ be the index 0 critical point of V_α. Identify the edge $e_\alpha \subset \Delta$ with the subarc of the α-curve from $\mathbf{x}$ to $\mathbf{y}$, which is an edge of the holomorphic triangle, such that v_γ is identified with $\mathbf{x}$ and v_β is identified with $\mathbf{y}$. Abusing notations, we shall not distinguish a point on e_α and the corresponding point on the α-curve under the above identification. For any $q \in e_\alpha$, let γ_0 and γ_1 be the gradient trajectories which connect w to z and p to q respectively. Let $N(\gamma_i)$ be a tubular neighborhood of γ_i as depicted in Figure 5.13, for $i = 0, 1$. By restricting the construction of the absolute grading in Section 5.1 to U_α, we obtain a non-vanishing vector field $V'_{\alpha,q}$ on U_α which depends on the choice of $q \in e_\alpha$ as depicted in Figure 5.14. Thus we have constructed a 2-plane field $\xi_\alpha(q, x) = (V'_{\alpha,q}(x))^{\perp_3}$ on $e_\alpha \times U_\alpha$, for any $q \in e_\alpha$ and $x \in U_\alpha$. Here $\perp_3$ denotes taking the orthogonal complement of $V'_{\alpha,q}$ within TU_α.



(a) (b) (c)

Figure 5.13: (a) The α-handlebody U_α and tubular neighborhoods of the gradient trajectories γ_0 and γ_1. (b) The gradient vector field $V_\alpha|_{N(\gamma_0)}$ in $N(\gamma_0)$. (c) The gradient vector field $V_\alpha|_{N(\gamma_1)}$ in $N(\gamma_1)$.

Similarly one constructs 2-plane fields ξ_β and ξ_γ on $e_\beta \times U_\beta$ and $e_\gamma \times U_\gamma$, respectively. However, note that the boundary component $Y_{\alpha,\beta} = (v_\gamma \times U_\alpha) \cup (v_\gamma \times U_\beta)$ of $W_{\alpha,\beta,\gamma}$ is a 3-manifold with corners, and the 2-plane fields ξ_α and ξ_β do not agree along $v_\gamma \times \Sigma$ because they are tangent to the α- and β-handlebodies which intersect each other in an angle. To smooth the corners, we replace the triangle Δ in (5.2) with a hexagon H with right corners and attach α, β, and γ handles accordingly as depicted in Figure 5.15. In this way we obtain a smooth cobordism which we still denote by $W_{\alpha,\beta,\gamma} : Y_0 \coprod (S^1 \times S^2) \to Y_1$, where $Y_0 = (v_\gamma \times U_\alpha) \cup ([0,1] \times \Sigma) \cup (v_\gamma \times U_\beta)$, $Y_1 = (v_\beta \times U_\alpha) \cup ([0,1] \times \Sigma) \cup (v_\beta \times U_\gamma)$, and $S^1 \times S^2 = (v_\alpha \times U_\beta) \cup ([0,1] \times \Sigma) \cup (v_\alpha \times U_\gamma)$ are smooth 3-manifolds. We construct a 2-plane field ξ on $(e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma) \cup \partial W_{\alpha,\beta,\gamma}$ by extending ξ_α, ξ_β, and ξ_γ to the three copies of $[0,1] \times \Sigma$ such that it is translation invariant in the $[0,1]$-direction on each copy.

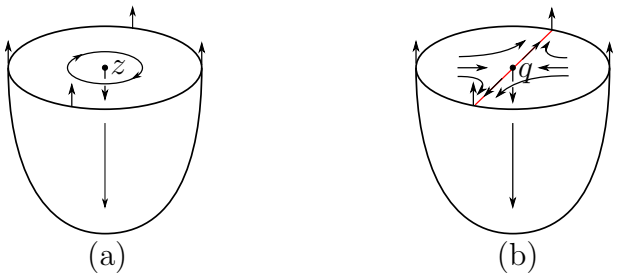


Figure 5.14: (a) The non-vanishing vector field $V'_{\alpha,q}$ restricted to $N(\gamma_0)$. (b) The non-vanishing vector field $V'_{\alpha,q}$ restricted to $N(\gamma_1)$.

By construction, it is easy to see that $\xi|_{Y_0} \simeq \widetilde{\mathrm{gr}}(\mathbf{x})$, $\xi|_{S^1\times S^2} \simeq \widetilde{\mathrm{gr}}(\Theta)$, and $\xi|_{Y_1} \simeq \widetilde{\mathrm{gr}}(\mathbf{y})$.

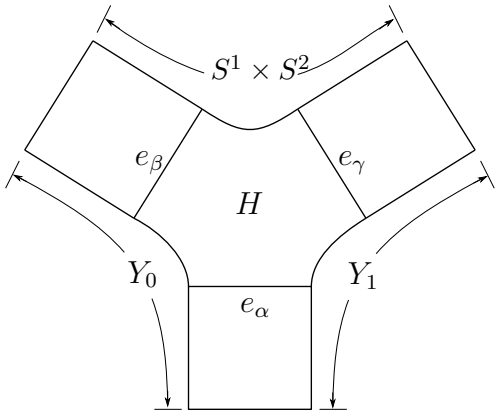


Figure 5.15: The smooth cobordism $W_{\alpha,\beta,\gamma} : Y_0 \coprod (S^1 \times S^2) \to Y_1$.

Let $D_1 \subset \Sigma$ be a closed neighborhood of z, and $D_2 \subset \Sigma$ be a closed neighborhood of the holomorphic triangle so that the non-vanishing vector field $V'_{i,q}$ is transverse to $T\Sigma$ along $\Sigma \setminus (D_1 \cup D_2)$ for any $i \in \{\alpha, \beta, \gamma\}$, $q \in \partial\Delta$. We extend ξ to the metric closure of $H \times (\Sigma \setminus (D_1 \cup D_2))$ by letting $\xi(x, y) = T_y\Sigma$ for any $x \in H$, and $y \in \Sigma \setminus (D_1 \cup D_2)$. We construct an almost complex structure J on a subset of $W_{\alpha,\beta,\gamma}$ by asking ξ and $\xi^{\perp_4}$ to be complex line bundles, where $\perp_4$ denotes taking the orthogonal complement in $TW_{\alpha,\beta,\gamma}$. In fact J is defined everywhere on $W_{\alpha,\beta,\gamma}$ except finitely many 4-balls (with corners), namely, $H \times D_1$ and $H \times D_2$. To extend J to the whole $W_{\alpha,\beta,\gamma}$, we round the corners of $\partial(H \times D_i)$, $i = 1, 2$, in two steps.

Step 1. To round the corners of $\partial H \times D_1$ and $\partial H \times D_2$ near each vertex of H, we first construct a local model for corner-rounding as follows.

Let (x_1, y_1, x_2, y_2) be coordinates on $\mathbb{R}^2 \times \mathbb{R}^2$ with the Euclidean metric. Consider a

non-singular vector field

$$v(x_1, y_1, x_2, y_2) = f(x_2, y_2)\frac{\partial}{\partial y_1} + g(x_2, y_2)\frac{\partial}{\partial x_2} + h(x_2, y_2)\frac{\partial}{\partial y_2}$$

on $\mathbb{R}^2 \times \mathbb{R}^2$, namely, f, g and h cannot be simultaneously zero. Observe that v is everywhere tangent to $\mathbb{R}^3 \simeq \{(x_1, y_1, x_2, y_2) \mid x_1 = \text{constant}\}$. Define $v^{\perp_3}$ to be the pointwise orthogonal complement to v inside $\mathbb{R}^3 \simeq \{(x_1, y_1, x_2, y_2) \mid x_1 = \text{constant}\}$. Let J be an almost complex structure on $\mathbb{R}^2 \times \mathbb{R}^2$ which preserves the metric and satisfies:

- $J(\frac{\partial}{\partial x_1}) = \frac{v}{||v||}$,
- $J(v^{\perp_3}) = v^{\perp_3}$.

Let $\mathcal{L} = \{(x_1, 0) \mid x_1 \geq 0\} \cup \{(0, y_1) \mid y_1 \geq 0\} \subset \mathbb{R}^2$ be a L-shaped broken line with a corner at the origin. We round the corner of $\mathcal{L}$ by considering

$$\mathcal{L}_r = \{(x_1, 0) \mid x_1 \geq 1\} \cup \{(0, y_1) \mid y_1 \geq 1\} \cup \{(x_1-1)^2+(y_1-1)^2 = 1 \mid 0 \leq x_1 \leq 1, 0 \leq y_1 \leq 1\}.$$

Consider the smooth submanifold $\bar{\mathcal{L}} = \mathcal{L}_r \times \mathbb{R}^2$ in $\mathbb{R}^2 \times \mathbb{R}^2$. We compute the complex line distribution $T\bar{\mathcal{L}} \cap J(T\bar{\mathcal{L}})$ on $T\bar{\mathcal{L}}$ with respect to J. To do so, identify $\bar{\mathcal{L}}$ with $(-\infty, \infty) \times \mathbb{R}^2$ such that $\{(0, y_1) \mid y_1 \geq 1\}$ is identified with $(-\infty, 0] \times \mathbb{R}^2$, $\{(x_1, 0) \mid x_1 \geq 1\}$ is identified with $[1, \infty) \times \mathbb{R}^2$, and $\{(x_1 - 1)^2 + (y_1 - 1)^2 = 1 \mid 0 \leq x_1 \leq 1, 0 \leq y_1 \leq 1\}$ is identified with $[0, 1] \times \mathbb{R}^2$. Let $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the clockwise rotation about the x-axis by $\chi(t)\pi/2$, where (x, y, z) are coordinates on $\mathbb{R}^3$ and

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Lemma 5.4.5. *The 2-plane field $T\bar{\mathcal{L}} \cap J(T\bar{\mathcal{L}})$ on $\bar{\mathcal{L}} \simeq (-\infty, \infty) \times \mathbb{R}^2$ is the orthogonal complement of the non-singular vector field $\mu(t, x_2, y_2) = \phi_t(v(x_2, y_2))$.*

Proof of Lemma 5.4.5. We first compute $J(\frac{\partial}{\partial y_1})$ as follows. Note that

$$v^{\perp_3} = \begin{cases} span\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\} & \text{if } g = h = 0, \\ span\{g\frac{\partial}{\partial y_2} - h\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1} - \frac{fg}{\lambda^2}\frac{\partial}{\partial x_2} - \frac{fh}{\lambda^2}\frac{\partial}{\partial y_2}\} & \text{otherwise.} \end{cases}$$

where $\lambda = \sqrt{g^2 + h^2}$. Since we assume that J preserves the Euclidean metric, we have

$$\begin{cases} J(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial y_2} & \text{if } g = h = 0, \\ J(g\frac{\partial}{\partial y_2} - h\frac{\partial}{\partial x_2}) = \frac{\lambda^2}{\sqrt{f^2+\lambda^2}}(\frac{\partial}{\partial y_1} - \frac{fg}{\lambda^2}\frac{\partial}{\partial x_2} - \frac{fh}{\lambda^2}\frac{\partial}{\partial y_2}) & \text{otherwise.} \end{cases} \tag{5.3}$$

It follows from (5.3) and the equation $J(\frac{\partial}{\partial x_1}) = \frac{v}{||v||}$ that

$$J\Big(\frac{\partial}{\partial y_1}\Big) = \frac{1}{\sqrt{f^2+\lambda^2}}\Big(-f\frac{\partial}{\partial x_1} - g\frac{\partial}{\partial y_2} + h\frac{\partial}{\partial x_2}\Big).$$

It is easy to see that $T\bar{\mathcal{L}} \cap J(T\bar{\mathcal{L}})$ restricted to $\{t\} \times \mathbb{R}^2$, $t \geq 1$, is the orthogonal complement of $J(\frac{\partial}{\partial y_1}) = \mu(1, \cdot)$ up to positive rescaling within $T\bar{\mathcal{L}}$. Moreover observe that $T\bar{\mathcal{L}} \cap J(T\bar{\mathcal{L}})$ restricted to $\{t\} \times \mathbb{R}^2$, for $0 \leq t \leq 1$, is the orthogonal complement of $J(t\frac{\partial}{\partial y_1} + (1-t)\frac{\partial}{\partial x_1})$, which is exactly $\mu(t, \cdot)$ up to positive rescaling. □

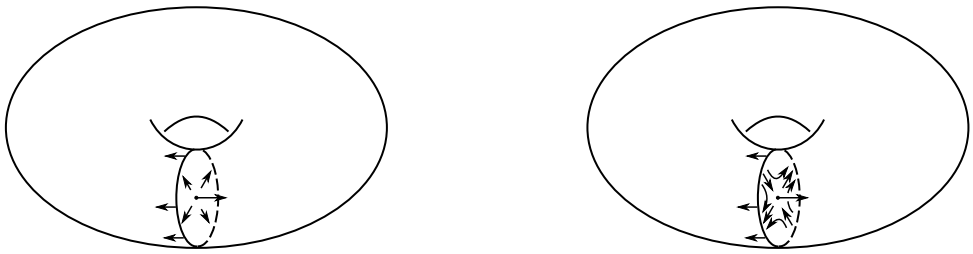
Without loss of generality, let q be a vertex of H whose adjacent edges are e_α and $[0, 1]$, where $[0, 1]$ is an edge of H connecting α- and β-handlebodies. Take a small neighborhood $N(q)$ of q in H. Identify $N(q)$ with a small neighborhood of the origin in $\mathbb{R}^2$ restricted to the first quadrant such that $e_\alpha \cup [0, 1]$ is identified with $\mathcal{L}$. We can further assume that J is defined on $N(q) \times D_i$ by taking $N(q)$ sufficiently small, and that it is invariant under translation in any direction tangent to $N(q)$. Hence we can apply Lemma 5.4.5 to compute the complex line distribution on $\mathcal{L}_r \times D_i \subset N(q) \times D_i$, $i = 1, 2$, with respect to J. By rounding all the corners of H and applying Lemma 5.4.5, we conclude that:

1. The complex line distribution $T(\partial H \times D_1) \cap JT(\partial H \times D_1)$ on $\partial H \times D_1$ is, up to homotopy relative to the boundary, the orthogonal complement of the non-singular vector field v_1, where $v_1|_{\{p\}\times D_1}$ is shown on Figure 5.16(a). In particular v_1 is defined to be invariant in the direction of ∂H.

2. Let $\theta \in [0, 2\pi)$ be the coordinate on ∂H with the boundary orientation and $\psi : \partial H \times D_2 \to \partial H \times D_2$ be a diffeomorphism defined by $\psi(\theta, z) = (\theta, e^{i\theta}z)$. The complex line distribution $T(\partial H \times D_2) \cap JT(\partial H \times D_2)$ on $\partial H \times D_2$ is, up to homotopy relative to the boundary, the orthogonal complement of the non-singular vector field $v_2 = \psi_*(v_2')$, where v_2' is invariant in the direction of ∂H and its restriction to $p \times D_2$, $p \in \partial H$, is shown on Figure 5.16(b).

Step 2. Now we round the corners of $\partial(H \times D_i) = (\partial H \times D_i) \cup (H \times \partial D_i)$, which is the union of two solid tori meeting each other orthogonally. Note that the 2-plane field $T(H \times \partial D_i) \cap JT(H \times \partial D_i)$ on $H \times \partial D_i$ is everywhere tangent to H by our choice of $D_i \subset \Sigma$, for $i = 1, 2$. Abusing notations, we still denote by $\partial(H \times D_i)$ the smooth 3-sphere obtained by rounding the corners in the standard way. Let ξ_i denote $T(\partial(H \times D_i)) \cap JT(\partial(H \times D_i))$, for $i = 1, 2$. So ξ_1 and ξ_2 are oriented 2-plane fields. Using the Pontryagin-Thom construction, we see that ξ_1 is homotopic to the negative standard contact structure on S^3, while ξ_2 is homotopic to the positive standard contact structure on S^3. Embed $H \times D_i = B^4 \subset \mathbb{C}^2$ such that H and D_i are contained in orthogonal complex planes respectively. Let

$$J_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad J_0' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$



$\partial H \times D_1$ (a) $\partial H \times D_2$ (b)

Figure 5.16:

be complex structures on $\mathbb{C}^2$. Then it is standard to check that $\xi_1 \simeq TS^3 \cap J_0' TS^3$ and $\xi_2 \simeq TS^3 \cap J_0 TS^3$ as oriented 2-plane fields, where $S^3 = \partial B^4 \subset \mathbb{C}^2$. Hence we can extend J to the whole $W_{\alpha,\beta,\gamma}$ satisfying all the desired properties.

Now we turn to the general case. Let $\mathbf{y}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ be another intersection point in $F_{W,\mathfrak{t}}$, i.e. there exists a holomorphic triangle $\psi' \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}')$ such that the Maslov index $\mu(\psi') = 0$. Let $\mathbf{y} \in F_{W,\mathfrak{t}}(\mathbf{x})$ be the intersection point as shown in Figure 5.12 and $\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$ be the obvious holomorphic triangle of Maslov index $\mu(\psi) = 0$. Since ψ and ψ' induces the same Spin^c structure $\mathfrak{t}$ on W, we have $\psi' = \psi + \phi_1 + \phi_2 + \phi_3$ for $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{x})$, $\phi_2 \in \pi_2(\Theta, \Theta)$, and $\phi_3 \in \pi_2(\mathbf{y}, \mathbf{y}')$. This implies

$$\mu(\psi') = \mu(\psi) + \mu(\phi_1) + \mu(\phi_2) + \mu(\phi_3).$$

Therefore

$$\mu(\phi_1) - 2n_z(\phi_1) = -(\mu(\phi_3) - 2n_z(\phi_3)),$$

because $\mu(\psi) = \mu(\psi') = n_z(\psi) = n_z(\psi') = \mu(\phi_2) - 2n_z(\phi_2) = 0$. Since we have shown that there exists an almost complex structure J on $W_{\alpha,\beta,\gamma}$ such that $\widetilde{\mathrm{gr}}(\mathbf{x}) \in \mathcal{P}(Y_0)$, $\widetilde{\mathrm{gr}}(\mathbf{y}) \in \mathcal{P}(Y_1)$ and $\widetilde{\mathrm{gr}}(\Theta) \in \mathcal{P}(\#^n(S^1 \times S^2))$ are all J-invariant with the complex orientation, it is easy to show that there exists another almost complex structure J' on $W_{\alpha,\beta,\gamma}$ such that $\widetilde{\mathrm{gr}}(\mathbf{x}) + \mu(\phi_1) - 2n_z(\phi_1)$, $\widetilde{\mathrm{gr}}(\mathbf{y}) - (\mu(\phi_3) - 2n_z(\phi_3))$, and $\widetilde{\mathrm{gr}}(\Theta)$ are all J'-invariant with the complex orientation. Here we are using the $\mathbb{Z}$-action as explained in Remark 4.1.3. Now it remains to observe that $\widetilde{\mathrm{gr}}(\mathbf{x}) = \widetilde{\mathrm{gr}}(\mathbf{x}) + \mu(\phi_1) - 2n_z(\phi_1) \in \mathcal{P}(Y_0)$ since $\mu(\phi_1) - 2n_z(\phi_1)$ is an integral multiple of the divisibility of $c_1(\widetilde{\mathrm{gr}}(\mathbf{x})) \in H^2(Y_0; \mathbb{Z})$, and that

$$\widetilde{\mathrm{gr}}(\mathbf{y}') = \widetilde{\mathrm{gr}}(\mathbf{y}) - \mathrm{gr}(\mathbf{y}, \mathbf{y}') = \widetilde{\mathrm{gr}}(\mathbf{y}) - (\mu(\phi_3) - 2n_z(\phi_3)).$$

□

It remains to show that J can be extended to W. Recall that $W = W_{\alpha,\beta,\gamma} \cup \#_b^n(S^1 \times B^3)$. We need to show that there exists an almost complex structure on $\#_b^n(S^1 \times B^3)$ such that its restriction to $\#^n(S^1 \times S^2) = \partial(\#_b^n(S^1 \times B^3))$ coincides with $J|_{\#^n(S^1 \times S^2)}$. Note that

$[\Theta] \in \widehat{HF}(-\#^n(S^1 \times S^2))$ defines the contact invariant of the standard contact structure on $\#^n(S^1 \times S^2)$, which is holomorphically fillable. Hence the conclusion follows immediately from Theorem 4.1.1(b). We finish the proof of Case 1.

Case 2. Suppose W is given by attaching 1- and 3-handles. By duality, it suffices to consider the case that W consists of 1-handle attachments. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram of Y_0 and $(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, z_0)$ a standard Heegaard diagram of $\#^n(S^1 \times S^2)$. We obtain a Heegaard diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z') = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)\#(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, z_0)$ of Y_1. There is an associated map between the Heegaard Floer homology groups

$$F_{W,\mathfrak{t}} : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, \mathfrak{t}|_{Y_0}) \to \widehat{CF}(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z', \mathfrak{t}|_{Y_1})$$

which is induced by $F_{W,\mathfrak{t}}(\mathbf{x}) = \mathbf{x} \otimes \Theta$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a generator in the Spin^c structure $\mathfrak{t}|_{Y_0}$, and $\Theta \in \widehat{CF}(\#^n(S^1 \times S^2))$ is the top dimensional generator. Now the existence of an almost complex structure J on W with desired properties follows from Theorem 4.1.1(b) and the fact that the standard contact structure on $\#^n(S^1 \times S^2)$ is fillable by $(\#^n_b(S^1 \times B^3), J')$ for some almost complex structure J'. So Case 2 is also proved. □

5.5 The invariance under Heegaard moves

Our aim for this section is to show that the absolute grading is an invariant of the 3-manifold. That means that if we have two different Heegaard diagrams for the same 3-manifold, then the absolute grading is preserved under the isomorphism between the Floer homologies defined in [28]. It is shown in [28] that any two Heegaard diagrams for the same manifold differ by a sequence of Heegaard moves, i.e. isotopies, handleslides, stabilizations and destabilizations. Every Heegaard move gives rise to a chain map between the Floer complexes, which induces an isomorphism in homology. It is easy to see that these chain maps take homogeneous elements to homogeneous elements. We will show the following theorem.

Theorem 5.5.1. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram for Y and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ a Heegaard diagram obtained by a Heegaard move from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Let $\Gamma : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{CF}(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ be the chain map defined in [28]. If $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, then $\widetilde{gr}(\mathbf{x}) = \widetilde{gr}(\Gamma(\mathbf{x}))$.*

Remark 5.5.2. Theorem 5.5.1 gives the invariance we wanted and implies that the following decomposition is independent of the Heegaard diagram.

$$\widehat{HF}(Y; \mathfrak{s}) = \bigoplus_{\rho \in \mathcal{P}(Y, \mathfrak{s})} \widehat{HF}_\rho(Y; \mathfrak{s}),$$

To prove Theorem 5.5.1, we will consider each type of Heegaard move at a time.

Isotopies

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram for Y and let $\boldsymbol{\alpha}'$ be given by moving α_1 to α_1' by a Hamiltonian isotopy without passing through z. Then there is a continuation map $\Gamma : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, z)$ defined by counting Maslov index 0 holomorphic disks with dynamic boundary conditions, as defined in [28]. If this isotopy does not create or destroy intersections between α and β curves, then it corresponds to isotoping the Morse function without introducing or removing any critical point. In this case it is clear that Γ is an isomorphism and that it preserves the absolute grading.

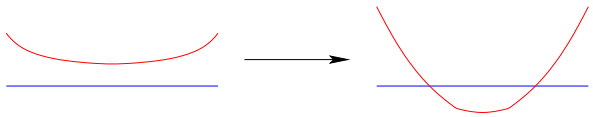


Figure 5.17:

A finger move is a Hamiltonian isotopy that creates a canceling pair of intersections, as shown in Figure 5.17. We only need to show that Γ is invariant when the isotopy introduces or eliminates one finger move and the general isotopy invariance follows from that. First assume that α_1' is obtained from α_1 by introducing one finger move. Let $\mathbf{x} = (x_1, \ldots, x_g) \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$, for some permutation σ. Then x_1 is moved to a point $x_1' \in \alpha_1' \cap \beta_{\sigma(1)}$. We note that x_1' is never one of the two new intersection points. It is easy to see an index 0 holomorphic disk from x_1 to x_1', which is actually just a flow line along $\beta_{\sigma(1)}$. So if we take $\mathbf{x}' = (x_1', x_2, \ldots, x_g)$, then $\mathbf{x}'$ is one of the terms in $\Gamma(\mathbf{x})$. It is easy to see that $\widetilde{\mathrm{gr}}(\mathbf{x}) = \widetilde{\mathrm{gr}}(\mathbf{x}')$. Therefore Γ preserves the absolute grading. Now we assume that α_1' is obtained from α_1 by eliminating a finger move. It remains to see what happens when x_1 is one of the two points that disappears. So we assume that x_1 is one of those two points, such that $\mathbf{x} = (x_1, \ldots, x_g) \in \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. If $\Gamma(\mathbf{x}) = 0$, then there is nothing to prove. Assume that $\Gamma(\mathbf{x}) \neq 0$. So we can take a term $\mathbf{x}'$ in $\Gamma(\mathbf{x})$. Then since we only isotoped α_1, none of the points x_i, for $i > 1$, have moved. So we can write $\mathbf{x}' = (x_1', x_2, \ldots, x_g)$, where $x_1' \in \alpha_1' \cap \beta_{\sigma(1)}$. That means that there exists a Maslov index 0 holomorphic disk φ from x_1 to x_1'. Now undoing this isotopy and introducing the finger move again, x_1' corresponds to an intersection $x_1'' \in \alpha_1 \cap \beta_{\sigma(1)}$ and there is a Maslov index zero holomorphic disk ψ from x_1' to x_1''. We now observe that the composition $\varphi * \psi$ is homotopic to a Whitney disk from x_1 to x_1'' with stationary boundary conditions, i.e. there exists a Whitney disk from x_1 to x_1'' with its boundary mapping to $\alpha_1 \cup \beta_{\sigma(1)}$. Therefore there is an index zero Whitney disk from x_1 to x_1''. So, since the absolute grading refines the relative grading in $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, it follows that $\widetilde{\mathrm{gr}}(\mathbf{x}) = \widetilde{\mathrm{gr}}(\mathbf{x}'')$, where $\mathbf{x}'' = (x_1'', x_2, \ldots, x_g)$, and hence $\widetilde{\mathrm{gr}}(\mathbf{x}) = \widetilde{\mathrm{gr}}(\mathbf{x}')$. That implies that Γ preserves the absolute grading when a finger move is undone.

Handleslides

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram for Y and let β_1' be the closed curve obtained by handlesliding β_1 over β_2. Now we define $\boldsymbol{\beta'} = (\beta_1', \beta_2, \ldots, \beta_g)$. This handleslide gives rise to a trivial cobordism $W = Y \times [0,1]$, which can also be obtained from the Heegaard triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta'})$ by attaching g copies of $S^1 \times D^3$, as explained in [28]. Let $F_W : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta'}, z)$ be the induced chain map. Then, it follows from Theorem 4.1.1(c) that $\widetilde{\mathrm{gr}}(\mathbf{x}) \sim_W \widetilde{\mathrm{gr}}(F_W(\mathbf{x}))$. That means that there exists an almost-complex structure J on W such that $[T(Y \times \{0\}) \cap J(T(Y \times \{0\}))] = \widetilde{\mathrm{gr}}(\mathbf{x})$ and $[T(Y \times \{1\}) \cap J(T(Y \times \{1\}))] = \widetilde{\mathrm{gr}}(F_W(\mathbf{x}))$. Now let $\xi_t = T(Y \times \{t\}) \cap J(T(Y \times \{t\}))$, for $0 \leq t \leq 1$. Under the canonical identification $Y \simeq Y \times \{t\}$, $\{\xi_t\}$ gives a homotopy between $T(Y \times \{0\}) \cap J(T(Y \times \{0\}))$ and $T(Y \times \{1\}) \cap J(T(Y \times \{1\}))$. Therefore $\widetilde{\mathrm{gr}}(\mathbf{x}) = \widetilde{\mathrm{gr}}(F_W(\mathbf{x}))$.

Stabilization

Given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ we stabilize it by taking the connected sum with a two-torus and introducing a new pair of α and β curves in this two-torus that intersect at exactly one point. This is equivalent to taking the connect sum of Y with an S^3, that is endowed with the standard genus one Heegaard decomposition. We can write $(\Sigma', \boldsymbol{\alpha'}, \boldsymbol{\beta'}, z')$ for the Heegaard diagram of the stabilization. Here $\Sigma' = \Sigma \# E$, for a two-torus E, $\boldsymbol{\alpha'} = (\alpha_1, \ldots, \alpha_g, \alpha_{g+1})$, $\boldsymbol{\beta'} = (\beta_1, \ldots, \beta_g, \beta_{g+1})$ and $z' \in \Sigma'$ is naturally associated with z, assuming that the connected sum removes a ball from Σ that does not contain z. Let w be the unique point in $\alpha_{g+1} \cap \beta_{g+1}$. It is clear that $\Gamma : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{CF}(\Sigma', \boldsymbol{\alpha'}, \boldsymbol{\beta'}, z')$, which takes $(x_1, \ldots, x_g)$ to $(x_1, \ldots, x_g, w)$, is an isomorphism. Is is also shown in [28] that this map gives rise to an isomorphism in homology. We need to show that the absolute grading is invariant under Γ. Let $\mathbf{x} = (x_1, \ldots, x_g) \in \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. In the definition of $\widetilde{\mathrm{gr}}(\mathbf{x})$ we modify a gradient-like vector field in neighborhoods of the flow lines γ_{x_i} and γ_0 to get a nonzero vector field. We can write

$$Y \# S^3 = (Y \setminus B_\varepsilon) \cup_\phi (S^3 \setminus B_R),$$

where B_ε is a small ball, B_R is a large ball and $\phi : \partial B_\varepsilon \to \partial B_R$ is a diffeomorphism. We can see the same neighborhoods $N(\gamma_{x_i}) \subset Y$ and $N(\gamma_0) \subset Y$ in $Y \# S^3$. Now we take a gradient-like vector field v for a Morse function compatible with $(\Sigma', \boldsymbol{\alpha'}, \boldsymbol{\beta}, z')$. The definition of $\widetilde{\mathrm{gr}}(\Gamma(\mathbf{x}))$ clearly implies that the vector field $w_{\Gamma(\mathbf{x})}$ is homotopic to $w_{\mathbf{x}}$ in $Y \setminus B_\varepsilon$. So it remains to show that $w_{\mathbf{x}}$ and $w_{\Gamma(\mathbf{x})}$ are also homotopic in $S^3 \setminus B_R$. We can think of $S^3 \setminus B_R$ as a small ball B_δ in R^3, where $w_{\mathbf{x}}$ is very close to being constant with respect to the standard trivialization. We note that v has only two critical points in B_δ. It is easy to homotope $w_{\mathbf{x}}$ in a neighborhood of B_δ so that it coincides with v on ∂B_δ. It is also easy to see that after we modify v in $N(\gamma_{x_{g+1}})$, the vector field we obtain is homotopic to $w_{\mathbf{x}}$ in B_δ. That concludes the proof of Theorem 5.5.1.

Chapter 6

The absolute grading on bordered Floer homology

6.1 The grading on the algebra

In this section, we construct the grading on the algebra $\mathcal{A}(\mathcal{Z})$. This grading takes values in a certain groupoid $G(\mathcal{Z})$. Before defining $G(\mathcal{Z})$ and the grading, we will quickly review the construction of $\mathcal{A}(\mathcal{Z})$. For a more thorough exposition, see [23].

The construction of the algebra $\mathcal{A}(\mathcal{Z})$

The *strand algebra* $\mathcal{A}(\mathcal{Z})$ is defined as a subalgebra of $\mathcal{A}(4k)$. As a $\mathbb{Z}/2$-vector space, $\mathcal{A}(4k)$ is generated by partial permutations (S,T,ϕ), where S and T are subsets of $\{1,\dots,4k\}$ containing the same number of elements and $\phi : S \to T$ is a bijection such that $\phi(i) \geq i$ for every $i \in S$. We can represent (S,T,ϕ) by a diagram with $4k$ points on the left and on the right and with strands connecting the set S on the left with the set T on the right. This diagram is required to have the smallest possible number of crossings. Each crossing corresponds to what is called an *inversion*, i.e. a pair of points $i,j \in \{1,\dots,4k\}$ with $i<j$ and $\phi(i)>\phi(j)$. It follows from this definition that the strands either go up or stay horizontal if we read from left to right. The product of (S,T,ϕ) with (S',T',ϕ') is defined to be $(S,T',\phi'\circ\phi)$ provided that $T=S'$ and that the number of inversions of the diagram for $(S,T',\phi'\circ\phi)$ equals the sum of the number of inversions of the diagrams for (S,T,ϕ) with (S',T',ϕ'). Otherwise, the product is set to be 0. For each subset S, one can define an idempotent element $I(S)=(S,S,\mathbb{I}_S)$. One can also define a differential on $\mathcal{A}(4k)$ as follows. For a generator a of $\mathcal{A}(4k)$, let ∂a be the sum over all ways to smooth one crossing of a, where we require all the terms of this sum to have exactly one less intersection than a. In other words, if smoothing one crossings decreases the number of inversions by more than 1, we set that term to zero.

We denote by $[2k]$ the set $\{1,\dots,2k\}$. A *pointed matched circle* $\mathcal{Z}$ is a quadruple $(Z,\mathbf{a},M,z)$ consisting of an oriented circle Z, a set of $4k$ points $\mathbf{a}$ in Z, a two-to-one function

$M : \mathbf{a} \to [2k]$ and a basepoint $z \in Z \setminus \mathbf{a}$. We also require that 0-surgery on Z along the pairs of points that are matched by M yields to a single circle. A pointed matched circle gives rise to a surface $F(\mathcal{Z})$ of genus k, which we often denote by F. The surface F is obtained by starting with a disk whose oriented boundary is Z, attaching 1-handles along all the pairs matched by M and attaching a 0-handle to the boundary circle. We observe that we can find a self-indexing Morse function $f : F \to [0, 2]$ such that $Z = f^{-1}(3/2)$ and $\mathbf{a}$ is the intersection between Z and the ascending manifolds from the index one critical points. We can identify $[2k]$ with the set of index one critical points $\{p_1, \dots, p_{2k}\}$.

By a *Reeb chord* ρ, we mean an oriented arc on $Z \setminus z$, with the same orientation as Z, whose boundary lies in $\mathbf{a}$. We denote by ρ^- the initial endpoint of ρ and by ρ^+ its final endpoint. We write $\rho = [\rho^-, \rho^+]$. A set $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_m\}$ of Reeb chords is said to be *consistent* if both sets $\boldsymbol{\rho}^- := \{\rho_1^-, \dots, \rho_m^-\}$ and $\boldsymbol{\rho}^+ := \{\rho_1^+, \dots, \rho_m^+\}$ have exactly m elements. A consistent set of Reeb chords $\boldsymbol{\rho}$ gives rise to an element $a_0(\boldsymbol{\rho})$ in $\mathcal{A}(4k)$ given by

$$a_0(\boldsymbol{\rho}) = \sum_{\substack{S \subset \{0,\dots,4k\} \\ S \cap (\boldsymbol{\rho}^- \cup \boldsymbol{\rho}^+) = \emptyset}} (S \cup \boldsymbol{\rho}^-, S \cup \boldsymbol{\rho}^+, \phi_S)$$

where $\phi_S|_S = \mathbb{I}$ and $\phi_S(\rho_i^-) = \rho_i^+$ for every i. Now, for every $\mathbf{s} \subset [2k]$, we can define the following idempotent

$$I(\mathbf{s}) := \sum_{\substack{S \subset \{0,\dots,4k\} \\ M \text{ maps } S \text{ bijectively to } \mathbf{s}}} I(S).$$

We let $\mathcal{I}(\mathcal{Z})$ be the ring of idempotents, which is defined to be the algebra generated by the elements $I(\mathbf{s})$ for $\mathbf{s} \in [2k]$. The unit of this algebra is

$$\mathbf{I} := \sum_{\mathbf{s} \subset [2k]} I(\mathbf{s}).$$

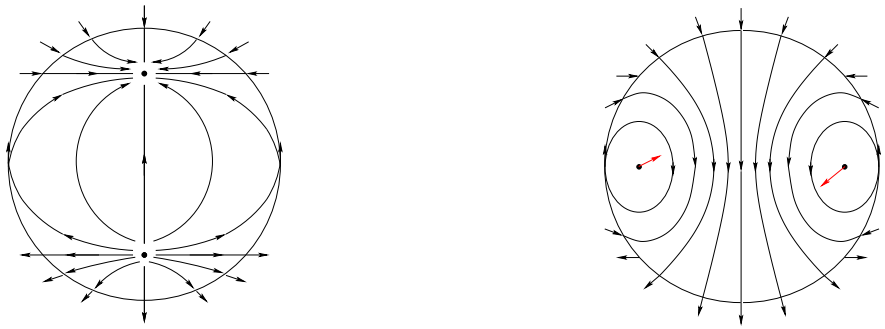
We now define the algebra $\mathcal{A}(\mathcal{Z})$ to be the subalgebra of $\mathcal{A}(4k)$ generated by $\mathcal{I}(\mathcal{Z})$ and by the elements $a(\boldsymbol{\rho}) := \mathbf{I} a_0(\boldsymbol{\rho}) \mathbf{I}$, for every consistent set of Reeb chords $\boldsymbol{\rho}$. The algebra $\mathcal{A}(\mathcal{Z})$ is generated as a $\mathbb{Z}/2$-vector space by elements of the form $I(\mathbf{s})a(\boldsymbol{\rho})$. We note that if $I(\mathbf{s})a(\boldsymbol{\rho}) \neq 0$, then $M|_{\boldsymbol{\rho}^-}$ and $M|_{\boldsymbol{\rho}^+}$ are injective, $M(\boldsymbol{\rho}^-) \subset \mathbf{s}$ and $(\mathbf{s} \setminus M(\boldsymbol{\rho}^-)) \cap M(\boldsymbol{\rho}^+) = \emptyset$.

Recall the three different ways that two Reeb chords can intersect. A pair of Reeb chords $\{\rho_1, \rho_2\}$ is said to be *interleaved* if $\rho_i^- < \rho_j^- < \rho_i^+ < \rho_j^+$ for $i = j + 1$ or $i = j - 1$, and *nested* if $\rho_i^- < \rho_j^- < \rho_j^+ < \rho_i^+$ for $i = j + 1$ or $i = j - 1$. The Reeb chords ρ_1 and ρ_2 are said to *abut* if $\rho_1^+ = \rho_2^-$. In this case, one defines their *join* to be $\rho_1 \uplus \rho_2 := [\rho_1^-, \rho_2^+]$.

The groupoid $G(\mathcal{Z})$

Let $F = F(\mathcal{Z})$. We consider the bundle $TF \oplus \underline{\mathbb{R}} \to F$, where $\underline{\mathbb{R}}$ is the trivial real line bundle. We interpret this bundle as the pullback of the tangent bundle of a three-manifold in which F is embedded, so we call sections of this bundle vector fields on F. We will now construct

a vector field $v_0' : F \to TF \oplus \underline{\mathbb{R}}$. Let f be a self-indexing Morse function compatible with $\mathcal{Z}$ as above. Consider its gradient vector field ∇f and modify it to first eliminate the index zero and index two critical points as follows. Let γ be the flow line passing through the basepoint z, which connects the index zero critical point to the index two critical point. Let $N(\gamma)$ denote a neighborhood of γ. Figure 6.1(a) illustrates ∇f restricted to $N(\gamma)$. We now define a nonvanishing vector field on $N(\gamma)$, which coincides with ∇f on $\partial N(\gamma)$, as shown in Figure 6.1(b). This picture determines the desired vector field up to homotopy relative to the boundary. Let v_0' denote the vector field given by this this construction in $N(\gamma)$ and by ∇f in the complement of $N(\gamma)$.



(a) (b)

Figure 6.1: (a) The gradient vector field ∇f in a neighborhood of the flow line passing through z. (b) The non-vanishing vector field in the same neighborhood after modification. The red arrow on the left is pointing into the page and the arrow on the right is pointing out.

Note that each subset $\mathbf{s} \subset [2k]$ corresponds to a set of index one critical points of f, under the identification $[2k] = \{p_1, \ldots, p_{2k}\}$. We denote by $\bar{\mathbf{s}}$ the subset $[2k] \setminus \mathbf{s}$. For $\mathbf{s} \subset [2k]$, let $\phi_{\mathbf{s}}$ be a bump function, which equals 1 at each point of $\mathbf{s}$ and 0 outside of small neighborhoods of each point of $\mathbf{s}$. We denote by $|\mathbf{s}|$ the cardinality of the set $\mathbf{s}$.

Definition 6.1.1. For each $\mathbf{s} \in [2k]$, we define $v_{\mathbf{s}} : F \to TF \oplus \underline{\mathbb{R}}$ to be the vector field given by

$$v_{\mathbf{s}} = v_0' + \phi_{\mathbf{s}} \frac{\partial}{\partial t} - \phi_{\bar{\mathbf{s}}} \frac{\partial}{\partial t}.$$

Here t denotes the $\mathbb{R}$-coordinate.

We can now define the grading set $G(\mathcal{Z})$.

Definition 6.1.2. For $\mathbf{s}, \mathbf{t} \in [2k]$, such that $|\mathbf{s}| = |\mathbf{t}|$, we define $G(\mathbf{s}, \mathbf{t})$ to be set of the homotopy class of nonvanishing vector fields on $F \times [0,1]$ that restrict to $v_{\mathbf{s}}$ on $F \times \{0\}$ and to $v_{\mathbf{t}}$ in $F \times \{1\}$. We define $G(\mathcal{Z})$ to be the disjoint union of $G(\mathbf{s}, \mathbf{t})$ for all $\mathbf{s}, \mathbf{t} \subset [2k]$ such that $|\mathbf{s}| = |\mathbf{t}|$.

Given vector fields v, w on $F \times [0,1]$ such that $v|_{F\times\{1\}} = w|_{F\times\{0\}}$, we can take their concatenation $v \cdot w$, which we see as a vector field on $F \times [0,1]$. So given $[v] \in G(\mathbf{s}, \mathbf{t})$ and $[w] \in G(\mathbf{t}, \mathbf{u})$, we define their composition by $[v] \cdot [w] := [v \cdot w] \in G(\mathbf{s}, \mathbf{u})$. We now recall the definition of a groupoid.

Definition 6.1.3. A groupoid is the set of morphisms of a small category[1] in which every morphism is invertible.

We observe that $G(\mathcal{Z})$ is a groupoid, coming from a category whose objects are the vector fields $v_{\mathbf{s}}$ for $\mathbf{s} \subset [2k]$. The groupoid $G(\mathcal{Z})$ admits a $\mathbb{Z}$-action, defined as follows. We will denote the action of an integer $n \in \mathbb{Z}$ by λ^n. First observe that, since $\pi_3(S^2) \simeq \mathbb{Z}$, there is a $\mathbb{Z}$-action on the set of homotopy classes of nonvanishing vector fields on a ball B^3 relative to its boundary. Our sign convention is such that the Hopf map $S^3 \to S^2$ acts on B^3 by λ^{-1}. Let $[v] \in G(\mathcal{Z})$ and fix a ball B in the interior of $F \times [0,1]$. For $n \in \mathbb{Z}$, we define $\lambda^n \cdot [v]$ to be the relative homotopy class of the vector field obtained by the acting on $v|_B$ by λ^n and keeping v constant outside B. We observe that

$$\lambda^n \cdot ([v] \cdot [w]) = (\lambda^n \cdot [v]) \cdot [w] = [v] \cdot (\lambda^n \cdot [w]).$$

A $G(\mathcal{Z})$-grading on $\mathcal{A}(\mathcal{Z})$

Recall that the strand algebra $\mathcal{A}(\mathcal{Z})$ is generated as a $\mathbb{Z}/2$ vector field by all the elements of the form $I(\mathbf{s})a(\boldsymbol{\rho})$, where $\mathbf{s} \subset [2k]$ and $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_m\}$ is a consistent set of Reeb chords. For every element $I(\mathbf{s})a(\boldsymbol{\rho}) \neq 0$, we will define its grading $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})) \in G(\mathbf{s}, \mathbf{t})$. where $\mathbf{t} = M(\boldsymbol{\rho}^+) \cup (\mathbf{s} \setminus M(\boldsymbol{\rho}^-))$.

For a general $\mathbf{s} \subset [2k]$, in order to draw a picture of $v_{\mathbf{s}} : F \to TF \oplus \underline{\mathbb{R}}$ away from the index 0 and 2 critical points, we will project it to a vector field on TF and decorate the zeros of this vector field using the following convention: an index one critical point p is decorated with " $+$ " if $v_{\mathbf{s}} = \partial_t$ at p, and with " $-$ " if $v_{\mathbf{s}} = -\partial_t$ at p.

We will define the grading function gr by steps as follows.

Step 1. Assume that $\boldsymbol{\rho}$ consists of a single Reeb orbit ρ, such that $M(\rho^-) \neq M(\rho^+)$. We now construct $\mathrm{gr}(I(\mathbf{s})a(\rho))$.

We will define a vector field $v_{(\mathbf{s},\rho)}$ on $F \times [0,1]$ such that $[v_{(\mathbf{s},\rho)}] \in G(\mathbf{s}, \mathbf{t})$. Recall that we are identifying a point in $[2k]$ with its corresponding index one critical point. Let $p_i = M(\rho^-)$ and $p_j = M(\rho^+)$. So $\mathbf{t} = \{p_j\} \cup (\mathbf{s} \setminus \{p_i\})$. It follows from our construction in §6.1 that $v_{\mathbf{s}}$ and $v_{\mathbf{t}}$ only differ in small neighborhoods of p_i and p_j.

Let $\hat{\rho}$ be the arc from p_i to p_j consisting of three pieces: the gradient trajectory from p_i to ρ^-, the Reeb chord ρ and the gradient trajectory from p_j to ρ^+, as shown in Figure 6.2.

Let $N(\hat{\rho}) \subset F$ be a tubular neighborhood of $\hat{\rho}$. The vector field $v_{\mathbf{s}}$ restricted to $N(\hat{\rho})$ is depicted in Figure 6.3.

[1] A small category is a category where the objects form a set.

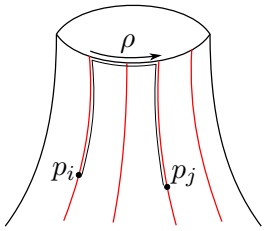


Figure 6.2: Reeb chord ρ.

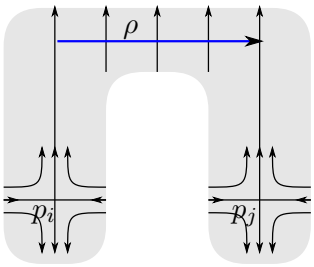


Figure 6.3: The neighborhood $N(\hat{\rho})$ of $\hat{\rho}$.

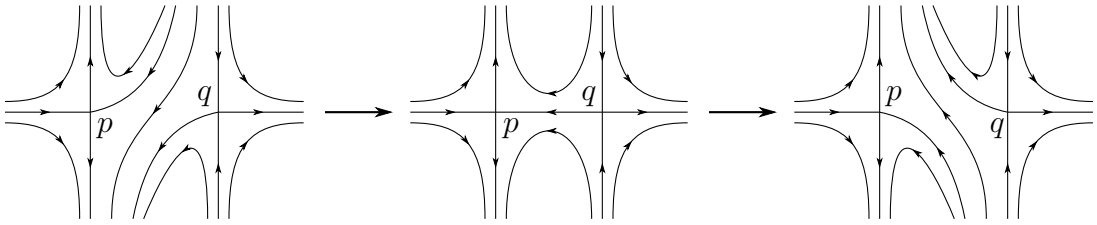
Define $v_{(\mathbf{s},\rho)}$ on $F\times\{0\}$ and $F\times\{1\}$ by setting it equal to $v_{\mathbf{s}}$ and $v_{\mathbf{t}}$, respectively. Since $v_{\mathbf{s}} = v_{\mathbf{t}}$ on the complement of $N(\hat{\rho})$, we can extend $v_{(\mathbf{s},\rho)}$ on $(F\setminus N(\hat{\rho}))\times[0,1]$, by requiring it to be invariant in the $[0,1]$-direction. The embedding $N(\hat{\rho})\subset\mathbb{R}^2$, as shown in Figure 6.3, gives rise to a trivialization of $TF|_{N(\hat{\rho})}$ and, therefore, we obtain a trivialization of $T(F\times[0,1])|_{N(\hat{\rho})\times[0,1]}$. We observe that, under the identification given by this trivialization, $v_{\mathbf{s}}|^{-1}_{N(\hat{\rho})}(0,0,1) = p_i$ and $v_{\mathbf{t}}|^{-1}_{N(\hat{\rho})}(0,0,1) = p_j$. The points p_i and p_j are framed codimension two submanifolds of $N(\hat{\rho})$. By the relative Pontryagin-Thom construction, in order to define a nonvanishing vector field on $N(\hat{\rho})\times[0,1]$ with the given boundary condition, it is enough to choose a framed 1-manifold, whose intersection with the boundary is $\{p_i\}\times\{0\}\cup\{p_j\}\times\{1\}$ with the given framing. We choose a framed 1-manifold as follows. Let $\gamma : [0,1]\to F$ be a smoothing of $\hat{\rho}$ such that $\gamma(0) = p_i$ and $\gamma(1) = p_j$. Let $\tilde{\gamma} : [0,1]\to F\times[0,1]$ be the arc defined by $\tilde{\gamma}(t) = (\gamma(t), t)$. Since $F\times\{t\}$ is always transverse to $\tilde{\gamma}$, the embedding $N(\hat{\rho})\subset\mathbb{R}^2$ gives a canonical framing on $\tilde{\gamma}$. Now, using this framed 1-manifold, the Pontryagin-Thom construction allows us to extend $v_{(\mathbf{s},\rho)}$ to the interior of $N(\hat{\rho})\times[0,1]$. We note that $v_{(\mathbf{s},\rho)}|_{N(\hat{\rho})\times[0,1]}$ is well-defined up to homotopy relative to the boundary. We now define $\mathrm{gr}(I(\mathbf{s})a(\rho))$ to be the homotopy class of $v_{(\mathbf{s},\rho)}$, which is an element of $G(\mathbf{s},\mathbf{t})$.

It will be useful later to have a more concrete description of $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$. To do so, we view a vector field on $F\times[0,1]$ as a smooth one-parameter family of nonvanishing sections $F\to TF\times\underline{\mathbb{R}}$, indexed by $t\in[0,1]$. We will, in fact, define a family of such sections $\{v^t_{(\mathbf{s},\rho)}\}_{t\in[0,1]}$. This family can be explicitly defined by a composition of three bifurcations and necessary isotopies, which we now describe.

Consider the following model situation: Let Ξ^0 be a singular vector field on the unit disk $D \subset \mathbb{R}^2$ with two saddle points p, q as depicted in Figure 6.4(a). Then there exists a 1-parameter family of vector fields Ξ^t, for $0 \leq t \leq 1$, such that

- each Ξ^t has only two saddle points which are p and q, and Ξ^t is constant near ∂D as t goes from 0 to 1,
- for exactly one t, say $t = 1/2$, the vector field Ξ^t has a saddle-saddle connection from q to p.

See Figure 6.4 for a pictorial illustration of Ξ^t. We call the one-parameter family $\{\Xi^t\}_{t\in[0,1]}$, a *bifurcation*. Notice that in the situation of Figure 6.4, we decided to fix the unstable trajectories of p and the stable trajectories of q throughout the homotopy, however, we could instead fix the stable trajectories of p and the unstable trajectories of q throughout the homotopy to define another similar one-parameter family of vector fields with the same boundary condition, which we also call a bifurcation.



(a) (b) (c)

Figure 6.4: A bifurcation.

We can now define $v^t_{(\mathbf{s},\rho)}$ to be constant and equal to $v_{\mathbf{s}}$ in the complement of $N(\hat{\rho}) \times [0, 1]$. In $N(\hat{\rho}) \times [0, 1]$, we define $v^t_{(\mathbf{s},\rho)}$ via a composition of bifurcations and isotopies, as shown in Figure 6.5.

The family $\{v^t_{(\mathbf{s},\rho)}\}$ gives rise to a vector field on $F\times[0, 1]$, which we still denote $\{v^t_{(\mathbf{s},\rho)}\}$. As before, the embedding $N(\hat{\rho}) \subset \mathbb{R}^2$, as in Figure 6.3, induces a trivialization of $T(N(\hat{\rho})\times[0, 1])$. We observe that, in $N(\hat{\rho}) \times [0, 1]$, the framed arc $(\{v^t_{(\mathbf{s},\rho)}\})^{-1}(0, 0, 1)$ is isotopic, and hence cobordant, to the arc $\tilde{\gamma}$. Moreover their framings coincide under the isotopy. Therefore, by the Pontryagin-Thom construction, this $\{v^t_{(\mathbf{s},\rho)}\}$ is a representative of homotopy class $\mathrm{gr}(I(\mathbf{s})a(\rho))$.

Step 2. Now assume that $\boldsymbol{\rho}$ still consists of only one Reeb orbit ρ, but $M(\rho^-) = M(\rho^+)$.

Let $p = M(\rho^-) = M(\rho^+)$ and let $\hat{\rho}'$ be the union of ρ and the flow lines connecting p to $\rho^- \cup \rho^+$. We construct a one-parameter family $\{\Theta^t\}_{t\in[0,1]}$ of vector fields on F as follows. Set $\Theta^0 = v_{\mathbf{s}}$ and $\Theta^t \equiv \Theta^0$, for $t \in [0, 1]$, outside $N(\hat{\rho}')$. Fix a small $\varepsilon > 0$. For $t \in [0, \varepsilon]$, define Θ^t in $N(\hat{\rho}')$ to be the homotopy which creates an extra pair of singular points near p

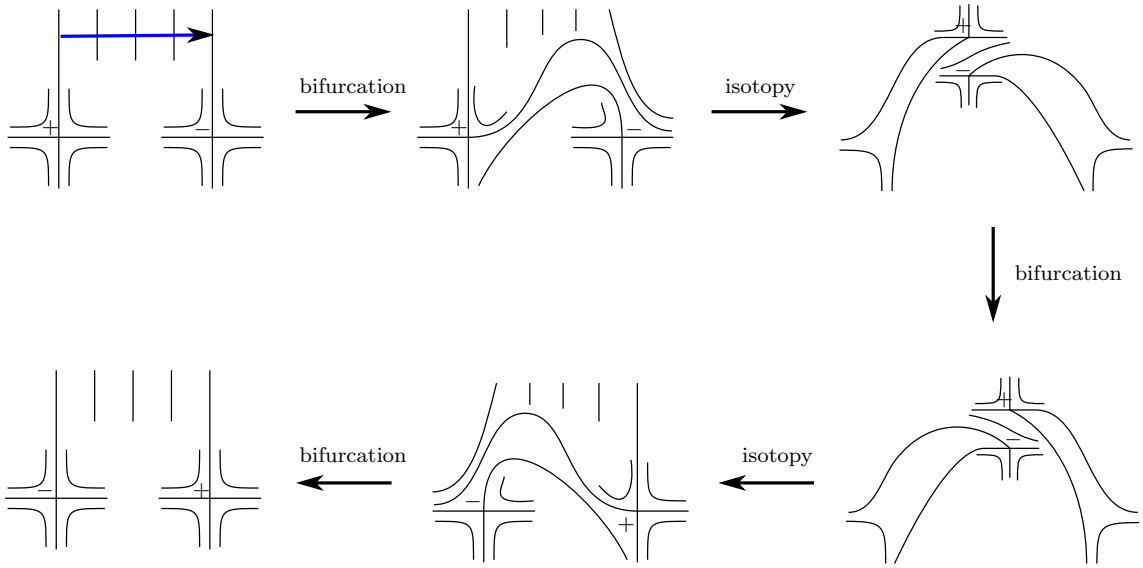


Figure 6.5: A sequence of three bifurcations which defines the grading of ρ. Arrows are omitted for simplicity of the picture (cf. Figure 6.3).

decorated with negative signs, along the unstable trajectories of p as depicted in Figure 6.6. More precisely, under the projection to TF, we create a pair of canceling critical points μ of index one and ν of index two, lying on the flow line of ∇f connecting p to ρ^+. Consider the (broken) arc $\hat{\rho}$ from p to ν, which is the union of the trajectory from p to ρ^-, ρ and the trajectory from ν to ρ^+. Now we can repeat the method from Step 1 for $\Theta^\varepsilon|_{N(\hat{\rho})}$ and obtain a homotopy $\Theta^t|_{N(\hat{\rho})}$ for $t \in [\varepsilon, 1-\varepsilon]$, which exchanges the signs of the index one critical points. We define Θ^t in $N(\hat{\rho}') \setminus N(\hat{\rho})$ to be constant and equal to Θ^ε. For $t \in [1-\varepsilon, 1]$, let $\Theta^t|_{N(\hat{\rho})}$ be the homotopy which cancels the extra pair of "negative" singular points. The family $\{\Theta^t\}_{t\in[0,1]}$ gives rise to a vector field, which is again denoted by $v_{(\mathbf{s},\rho)}$. Finally $\mathrm{gr}(I(\mathbf{s})a(\rho))$ is defined to be the homotopy class of $v_{(\mathbf{s},\rho)}$.

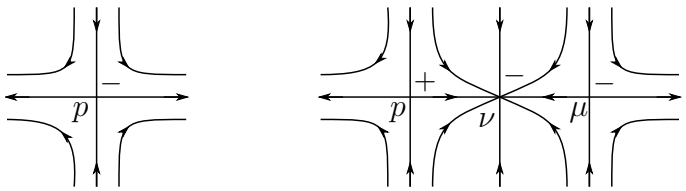


Figure 6.6: Creating a canceling pair of critical points with negative sign.

Step 3. The general case.

Suppose $\boldsymbol{\rho} = \{\rho_1, \cdots, \rho_l\}$. Note that the choice of a basepoint z and an orientation on Z induce an ordering on $\mathbf{a}$: if we start from z and follow the positive orientation on Z,

then $a_i < a_j$ if and only if we meet a_i before a_j, where $a_i, a_j \in \mathbf{a}$. We define a ordering on $\boldsymbol{\rho}$ by setting $\rho_i < \rho_j$ whenever $\rho_i^+ > \rho_j^+$ in $\mathbf{a}$. Up to re-ordering, we may assume that $\rho_1 < \rho_2 < \cdots < \rho_l$. We want to define a relative homotopy class $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})) \in G(v_{\mathbf{s}}, v_{\mathbf{t}})$, where $\mathbf{t} = M(\boldsymbol{\rho}^+) \cup (\mathbf{s} \setminus M(\boldsymbol{\rho}^-))$. First, for every point in $(M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+)) \setminus (\boldsymbol{\rho}^- \cap \boldsymbol{\rho}^+)$, we create a pair of canceling "negative" singular points, as follows. If $p = M(\rho_i^+) = M(\rho_j^-)$, then we create a pair of "negative" singular points on the flow line connecting p to ρ_i^+, as in Step 2. This construction gives rise to a vector field v_ε in $F \times [0,1]$ similar to $\{\Theta|_t\}_{t\in[0,\varepsilon]}$ from Step 2. We also consider the vector field $v_{-\varepsilon}$, which corresponds to canceling the "negative" singular points added to $v_{\mathbf{t}}$. Now consider the arcs $\hat{\rho}_i$ associated to ρ_i as before, namely, $\hat{\rho}_i$ is the union of ρ_i with the gradient trajectories connecting index one critical points to $\rho_1^- \cup \rho_1^+$. Note that $\hat{\rho}_i$ always connects a "positive" saddle to a "negative" saddle. Now we can define $v^1_{(\mathbf{s},\boldsymbol{\rho})}$ to be the homotopy class of the vector field supported on $N(\hat{\rho}_1)$ defined in Step 1. We repeat the same procedure for $\rho_2, \ldots, \rho_l$, such that for every $i \geq 2$, the vector field $v^i_{(\mathbf{s},\boldsymbol{\rho})}$ corresponding to ρ_i is supported in $N(\hat{\rho}_i)$. In particular,

$$v^{i-1}_{(\mathbf{s},\boldsymbol{\rho})}|_{F\times\{1\}} = v^i_{(\mathbf{s},\boldsymbol{\rho})}|_{F\times\{0\}}.$$

Let $v_{(\mathbf{s},\boldsymbol{\rho})}$ be the concatenation

$$v_{(\mathbf{s},\boldsymbol{\rho})} := v_\varepsilon \cdot v^1_{(\mathbf{s},\boldsymbol{\rho})} \ldots v^l_{(\mathbf{s},\boldsymbol{\rho})} \cdot v_{-\varepsilon}. \tag{6.1}$$

Finally, we define $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$ to be the relative homotopy class of $v_{(\mathbf{s},\boldsymbol{\rho})}$, which is an element of $G(\mathbf{s},\mathbf{t})$.

The properties of the grading on $\mathcal{A}(\mathcal{Z})$

We now show that the grading we constructed in the previous subsection satisfies the desired properties.

Proposition 6.1.4. *The grading function* $\mathrm{gr} : \mathcal{A}(\mathcal{Z}) \to \mathcal{G}(\mathcal{Z})$ *constructed above defines a grading on the* dg *algebra* $\mathcal{A}(\mathcal{Z})$*, i.e., it satisfies the following:*

- *For any two sets of Reeb chords* $\boldsymbol{\rho}, \boldsymbol{\sigma}$*, if* $I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})a(\boldsymbol{\sigma}) \neq 0$*, then*

$$\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})) \cdot \mathrm{gr}(I(\mathbf{t})a(\boldsymbol{\sigma})) = \mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})a(\boldsymbol{\sigma})),$$

- *For any* $\boldsymbol{\rho}$*, if* $\partial(I(\mathbf{s})a(\boldsymbol{\rho})) \neq 0$*, then*

$$\mathrm{gr}(\partial(I(\mathbf{s})a(\boldsymbol{\rho}))) = \lambda^{-1} \cdot \mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})).$$

Proof. For each index one critical point $p_i \in [2k]$, we denote by h_i the core of the corresponding 1-handle and by σ_i the Reeb chord connecting the two points in $h_i \cap Z$. We also

denote by $N(z)$ and $N(p_i)$ small neighborhoods of z and p_i. Let $\mathfrak{N}$ be a small neighborhood of

$$(Z \setminus N(z)) \cup \bigcup_{i=1}^{2k} (h_i \setminus N(p_i)).$$

We can choose an orientation-preserving embedding $\mathfrak{N} \hookrightarrow \mathbb{R}^2$ such that $Z \cap \mathfrak{N}$ is parallel to the horizontal vector ∂_x and its orientation is positive with respect to ∂_x, and such that $h_i \cap \mathfrak{N}$ is parallel to the vertical vector ∂_y, see Figure 6.7(a). Let $\bar{\mathfrak{N}} := \mathfrak{N} \cup \bigcup_i N(p_i)$. So we can extend the embedding from above to an immersion $\bar{\mathfrak{N}} \looparrowright \mathbb{R}^2$ such that $h_i \cap N(p_i)$ maps to a half-circle, see Figure 6.7(b). This immersion induces a trivialization of $TF|_{\bar{\mathfrak{N}}}$. We obtain a trivialization of $(TF \oplus \underline{\mathbb{R}})|_{\bar{\mathfrak{N}}}$, which induces a $[0,1]$-invariant trivialization τ of $T(F \times [0,1])$ over $\bar{\mathfrak{N}}$.

(a) (b)

Figure 6.7:

Fix a generator $I(\mathbf{s})a(\boldsymbol{\rho}) \in \mathcal{A}(\mathcal{Z})$. We write $\boldsymbol{\rho} = \{\rho_1, \ldots, \rho_l\}$ and we order the Reeb chords as in Step 3 above. We now define a one-manifold $Q_{(\mathbf{s},\boldsymbol{\rho})}$ in $\bar{\mathfrak{N}} \times [0,1]$. Let $\gamma_i \subset \bar{\mathfrak{N}}$ be a smoothing of the union[2] of ρ_i with the gradient flow trajectories connecting $M(\rho_i^-) \cup M(\rho_i^+)$ to $\rho_i^- \cup \rho_i^+$. We can perturb the arcs γ_i on F such that the interior of every two of these arcs intersect, at most, at one point. This happens precisely for an interleaved pair of Reeb chords. We define arcs $\tilde{\gamma}_i$ on $F \times [0,1]$ by $\tilde{\gamma}_i(t) = (\gamma(t), t)$. Now if $\gamma_i(t) = \gamma_j(t)$ for some $t \in (0,1)$, for $i < j$, we perturb $\tilde{\gamma}_i$ near t so that $\tilde{\gamma}_i(t) < \tilde{\gamma}_j(t)$. Hence the arcs $\tilde{\gamma}_i$ are all pairwise disjoint. So we can define $Q_{(\mathbf{s},\boldsymbol{\rho})}$ to be the union of $\tilde{\gamma}_i$ for all i and the constant arcs $p \times [0,1]$, where $p \in \mathbf{s} \setminus M(\boldsymbol{\rho}^-)$. We observe that the $Q_{(\mathbf{s},\boldsymbol{\rho})} \cap \mathfrak{N}$ can be represented by the strand diagram corresponding to $\boldsymbol{\rho}$. Here if ρ_i and ρ_j intersect and $i < j$, then ρ_i goes under ρ_j. See Figure 6.8 for examples of $Q_{(\mathbf{s},\boldsymbol{\rho})} \cap \mathfrak{N}$.

(a) (b) (c)

Figure 6.8: (a) An abutting pair. (b) A nested pair. (c) An interleaved pair.

We shall use the Pontryagin-Thom construction to prove both assertions of the proposition. For the first one, let $I(\mathbf{s})a(\boldsymbol{\rho})$ and $I(\mathbf{t})a(\boldsymbol{\sigma})$ be generators of $\mathcal{A}(F)$ whose product

[2]If $M(\rho_i^+) \neq M(\rho_j^-)$ for every j or $\rho_i^+ = \rho_j^-$ for some j, then this union is just $\hat{\rho}_i$. Otherwise, we need to add a small segment connecting $M(\rho_i^+)$ to the "negative" index one critical point corresponding to ρ_i^+.

is nonzero. Recall that the *join* $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ is obtained from the union $\boldsymbol{\rho} \cup \boldsymbol{\sigma}$ where for every abutting pair (ρ, σ) with $\rho \in \boldsymbol{\rho}$ and $\sigma \in \boldsymbol{\sigma}$ is substituted by $\rho \uplus \sigma$. It follows that if $I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})a(\boldsymbol{\sigma}) \neq 0$, then

$$I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})a(\boldsymbol{\sigma}) = I(\mathbf{s})a(\boldsymbol{\rho})a(\boldsymbol{\sigma}) = I(\mathbf{s})a(\boldsymbol{\rho} \uplus \boldsymbol{\sigma}). \tag{6.2}$$

Let $v_{(\mathbf{s},\boldsymbol{\rho})}$, $v_{(\mathbf{t},\boldsymbol{\sigma})}$ and $v_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}$ be the vector fields as in the construction of the grading, whose homotopy classes are $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$, $\mathrm{gr}(I(\mathbf{t})a(\boldsymbol{\sigma}))$ and $\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}\uplus\sigma)$, respectively. We want to show that the product $v_{(\mathbf{s},\boldsymbol{\rho})} \cdot v_{(\mathbf{t},\boldsymbol{\sigma})}$ is homotopic to $v_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}$. We clearly only need to look at the restriction of these vector fields to $\bar{\mathfrak{N}} \times [0,1]$. Using the trivialization τ we can see the restriction of each of these vector fields as maps $\bar{\mathfrak{N}} \to S^2$. Up to a small perturbation of the vector fields, we can assume that $\mathbf{n} = (0,0,1) \in S^2$ is a regular value of all of these three maps. Then,

$$\begin{aligned} v^{-1}_{(\mathbf{s},\boldsymbol{\rho})}(\mathbf{n}) &\simeq Q_{(\mathbf{s},\boldsymbol{\rho})}, \\ v^{-1}_{(\mathbf{t},\sigma)}(\mathbf{n}) &\simeq Q_{(\mathbf{t},\boldsymbol{\sigma})}, \\ v^{-1}_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}(\mathbf{n}) &\simeq Q_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}. \end{aligned}$$

Here the symbol $\simeq$ denotes relative framed cobordism. The framing on each of these one-manifolds is trivial in $\mathfrak{N} \times [0,1]$ and has a standard form near every p_i[3]. We now want to concatenate $Q_{(\mathbf{s},\boldsymbol{\rho})}$ and $Q_{(\mathbf{t},\boldsymbol{\sigma})}$.

Write $\boldsymbol{\rho} = \{\rho_1, \ldots, \rho_l\}$ and $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_m\}$ with the ordering given as in Step 3 of the construction of the grading. Observe that $Q_{(\mathbf{s},\boldsymbol{\rho})}$ is isotopic to the concatenation of one-manifolds $Q_{1,1} \cdot \ldots \cdot Q_{1,l}$ in $F \times [0,l]$ where each $Q_{1,i}$ consists of the union of the arc corresponding to ρ_i and constant arcs. Note that the one-manifolds $Q_{1,i}$ all have the same number of arcs. Similarly, we can write $Q_{(\mathbf{t},\boldsymbol{\sigma})}$ as a concatenation of one-manifolds $Q_{2,1} \cdot \ldots \cdot Q_{2,m}$ in $F \times [0,m]$ corresponding to $\sigma_1, \ldots, \sigma_m$. So

$$Q_{(\mathbf{s},\boldsymbol{\rho})} \cdot Q_{(\mathbf{s},\boldsymbol{\sigma})} \cong Q_{1,1} \cdot \ldots \cdot Q_{1,l} \cdot Q_{2,1} \cdot \ldots \cdot Q_{2,m}.$$

Here $\cdot$ denotes concatenation and $\cong$ denotes isotopy relative to the boundary, where we identify $F \times [0,1] \cong (F \times [0,l]) \cdot (F \times [0,m])$. Now we want to reorder this concatenation in order to obtain the decomposition of $Q_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}$ defined as above. We will move $Q_{2,1}, \ldots, Q_{2,m}$ to the left one by one, as necessary, which we explain in what follows. We start with $Q_{2,1}$. First note that if $\rho_i \cap \sigma_j = \emptyset$, then $Q_{1,i} \cdot Q_{2,j} \cong Q_{2,j} \cdot Q_{1,i}$. So we can move $Q_{2,1}$ to the left of $Q_{1,i}$, whenever $\rho_i^+ < \sigma_1^-$. If there exists ρ_i such that ρ_i and σ_1 abut then we stop at $Q_{1,i} \cdot Q_{2,1}$, since this concatenation is isotopic to the one-manifold corresponding to $\rho_i \uplus \sigma_1$ in the decomposition of $Q_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}$. If that is the case, we can move $Q_{1,i} \cdot Q_{2,1}$ to the left of all the terms $Q_{1,i'}$ for which $\rho_{i'}^+ < \sigma_1^+$. In fact, let $Q_{1,i'}$ be such a term, i.e. $\rho_i^+ < \rho_{i'}^+ < \sigma_1$. Then $\{\rho_i \uplus \sigma_1, \rho_{i'}\}$ has to be nested, otherwise $a(\boldsymbol{\rho})a(\boldsymbol{\sigma}) = 0$. So

[3]The standard framing is a rotation by π in $N(p_i)$ either positively or negatively depending on whether $p_i = M(\rho^-)$ or $p_i = M(\rho^+)$ for the corresponding Reeb chord ρ, but it does not depend on anything else.

$Q_{1,i'} \cdot (Q_{1,i} \cdot Q_{2,1}) \cong (Q_{1,i} \cdot Q_{2,1}) \cdot Q_{1,i'}$. Now assume that there does not exist ρ_i such that ρ_i and σ_1 abut. Let ρ_i be such that $\rho_i \cap \sigma \neq \emptyset$. Then $\{\rho_i, \sigma_1\}$ has to be nested, otherwise $a(\boldsymbol{\rho})a(\boldsymbol{\sigma}) = 0$. Therefore $Q_{1,i} \cdot Q_{2,1} \cong Q_{2,1} \cdot Q_{1,i}$. We now proceed analogously with $Q_{2,2}, \ldots, Q_{2,m}$. After all these isotopies, we obtain the decomposition of $Q_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}$ by one-manifolds corresponding to the Reeb chords in $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$. Therefore we conclude that

$$Q_{1,1} \cdot \ldots \cdot Q_{1,l} \cdot Q_{2,1} \cdot \ldots \cdot Q_{2,m} \cong Q_{(\mathbf{s},\boldsymbol{\rho}\uplus\boldsymbol{\sigma})}.$$

Note that the framing on the one-manifold corresponding to an abutting pair $\{\rho_i, \sigma_j\}$ is the same as the framing on $Q_{1,i} \cdot Q_{2,j}$. Therefore, since the framings on all of these manifolds are standard, it follows that all these isotopies give rise to framed cobordisms. Therefore

$$\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))\mathrm{gr}(I(\mathbf{t})a(\boldsymbol{\sigma})) = \mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho} \uplus \boldsymbol{\sigma})).$$

The first assertion follows from (6.2).

Now let $I(\mathbf{s})a(\boldsymbol{\rho})$ be a generator of $\mathcal{A}(F)$ and let $v_{(\mathbf{s},\boldsymbol{\rho})}$ and $Q_{(\mathbf{s},\boldsymbol{\rho})}$ be as above. Recall that the differential of $I(\mathbf{s})a(\boldsymbol{\rho})$ is given by resolving one crossing of $I(\mathbf{s})a(\boldsymbol{\rho})$ at a time. We denote by $\boldsymbol{\rho}_j$ the sets of Reeb chords, such that $\partial(I(\mathbf{s})a(\boldsymbol{\rho})) = \sum_j I(\mathbf{s})a(\boldsymbol{\rho}_j)$, and let $v_{(\mathbf{s},\boldsymbol{\rho}_j)}$ and $Q_{(\mathbf{s},\boldsymbol{\rho}_j)}$ be as above. So, resolving a crossing of the projection of $Q_{(\mathbf{s},\boldsymbol{\rho})}$ onto F is equivalent to doing a 0-surgery on $Q_{(\mathbf{s},\boldsymbol{\rho})}$, leading to a one-manifold, which is isotopic to $Q_{(\mathbf{s},\boldsymbol{\rho}_j)}$ for some j. We observe that the framing on the result of the 0-surgery is one unit less[4] than the framing on $Q_{(\mathbf{s},\boldsymbol{\rho})}$. Therefore

$$\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}_j)) = \lambda^{-1}\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})).$$

Hence the differential decreases the grading by 1. □

Comparison with the grading by a non-commutative group

We now compare our topological grading constructed in §6.1 with the gradings on $\mathcal{A}(\mathcal{Z})$ defined in [23].

We first recall the definition of the non-commutative groups in which the gradings defined in [23] takes values. The group $G'(4k)$ is a $\mathbb{Z}$-central extension of $H_1(Z \setminus z, \mathbf{a})$. In order to give a more concrete definition of $G'(4k)$, we need to recall a few definitions from [23]. For a Reeb chord α in $Z \setminus z$ and $p \in Z \setminus z$, let $m(p, \alpha)$ be the average multiplicity with which α covers p, i.e. $m(p, \alpha) = 1/2$ for a boundary point, $m(p, \alpha) = 1$ for an interior point and $m(p, \alpha) = 0$, otherwise. One can extend m bilinearly to a function $m : H_1(Z \setminus z, \mathbf{a}) \times H_0(\mathbf{a}) \to \frac{1}{2}\mathbb{Z}$. For $\alpha_1, \alpha_2 \in H_1(Z \setminus z, \mathbf{a})$, one can define $L(\alpha_1, \alpha_2) = m(\partial\alpha_1, \alpha_2)$, where $\partial : H_1(Z \setminus z, \mathbf{a}) \to H_0(\mathbf{a})$ is the boundary map. Also, for $\alpha \in H_1(Z \setminus z, \mathbf{a})$, let $\varepsilon(\alpha) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})$ be $\frac{1}{4}$ times the number of parity changes in α mod 1. One can now define $G'(4k) = \{(j, \alpha) \in \frac{1}{2}\mathbb{Z} \times H_1(Z \setminus z, \mathbf{a}) | \varepsilon(\alpha) = j \,(\mathrm{mod}\ 1)\}$. The multiplication is defined by

$$(j_1, \alpha_1) \cdot (j_2, \alpha_2) = (j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

[4]Note that, using our sign conversions, the writhe of $Q_{(\mathbf{s},\boldsymbol{\rho})}$ is one unit more than the writhe of $Q_{(\mathbf{s},\boldsymbol{\rho}_j)}$ with respect to the canonical projection to F.

It follows from [23, Prop. 3.37] that this operation defines a multiplication in $G'(4k)$. For an element $g = (j, \alpha) \in G'(4k)$, the number $j \in \frac{1}{2}\mathbb{Z}$ is called the *Maslov component* of g and α is called the *Spin^c component* of g.

Given an element $a \in \mathcal{A}(4k)$, it determines a class $[a] \in H_1(Z \setminus z, \mathbf{a})$. We denote by $\mathrm{inv}(a)$ the number of inversions of a. Write $a = (S, T, \phi)$. Let $\iota(a) = \mathrm{inv}(a) - m(S, [a])$. Then one can define

$$\mathrm{gr}'(a) = (\iota(a), [a]).$$

It follows from [23, Prop. 3.39] that $\mathrm{gr}'(a) \in G'(4k)$. Moreover gr' is invariant under adding horizontal strands. Let $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_n\}$ be a set of Reeb chords and $\mathbf{s} \subset [0, 2k]$ be such that $I(\mathbf{s})a(\boldsymbol{\rho}) \neq 0$. We can see $\boldsymbol{\rho}$ as an element of $\mathcal{A}(4k)$ with no horizontal strands. So $\mathrm{gr}'(I(\mathbf{s})a(\boldsymbol{\rho})) = \mathrm{gr}'(\boldsymbol{\rho})$. Let $|\boldsymbol{\rho}|$ denote the number of elements of $\boldsymbol{\rho}$, and let $|\mathrm{ab}(\boldsymbol{\rho})|$ and $|\mathrm{int}(\boldsymbol{\rho})|$ denote the number of abutting and interleaved pairs in $\boldsymbol{\rho}$, respectively. Then a calculation shows that

$$\iota(\boldsymbol{\rho}) = -\frac{|\boldsymbol{\rho}|}{2} - \frac{|\mathrm{ab}(\boldsymbol{\rho})|}{2} - |\mathrm{int}(\boldsymbol{\rho})|. \tag{6.3}$$

By making some non-canonical choices, one can also define a *refined grading* taking values on a subgroup of $G'(4k)$, see [23, §3.3.2]. That is necessary for the gluing theorems to behave well with respect to grading. Alternatively, as suggested in [23, Rem. 10.44], one could consider a more canonical subset of $G'(4k)$, as follows. Let $M_* : H_0(\mathbf{a}) \to H_0([2k])$ denote the pushforward of the map $M : \mathbf{a} \to [2k]$. Define $G'(\mathcal{Z})$ to be the set of elements (j, α) in $G'(4k)$ such that $M_*(\partial\alpha) = \mathbf{t} - \mathbf{s}$, for $\mathbf{t}, \mathbf{s} \subset [2k]$, with $|\mathbf{t}| = |\mathbf{s}|$. We observe that $G'(\mathcal{Z})$ is a groupoid and that $\mathrm{gr}'(a) \in G'(\mathcal{Z})$ for every homogeneous element $a \in \mathcal{A}(\mathcal{Z})$. We now have the following proposition.

Proposition 6.1.5. *There exists a homomorphism $\mathcal{F} : G(\mathcal{Z}) \to G'(\mathcal{Z})$ such that $\mathcal{F}(\mathrm{gr}(a)) = \mathrm{gr}'(a)$ for every homogeneous element $a \in \mathcal{A}(\mathcal{Z})$.*

Proof. Let $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ be as in the proof of Proposition 6.1.4. Let τ be the trivialization of $T(F \times [0,1])$ in $\bar{\mathfrak{N}} \times [0,1]$ constructed in that proof. We extend this trivialization to a trivialization of $T(F \times [0,1])$ arbitrarily.

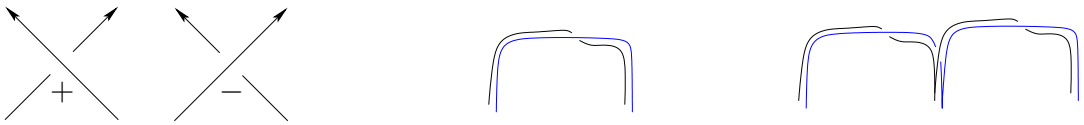
For each $\mathbf{s} \subset [2k]$, we now see $v_{\mathbf{s}}$ as a map $F \to S^2$. We can slightly perturb the vector fields $v_{\mathbf{s}}$ so that $(0,0,1)$ is a regular value of these maps. Observe that $v_{\mathbf{s}}^{-1}(0,0,1) = \mathbf{s} \cup P$, where P is a set of points in the complement of $\bar{\mathfrak{N}}$ and does not depend on $\mathbf{s}$.

Now let $[v] \in G(\mathcal{Z})$. Then $[v] \in G(\mathbf{s}, \mathbf{t})$, for some $\mathbf{s}, \mathbf{t} \subset [2k]$, such that $|\mathbf{s}| = |\mathbf{t}|$. We see the vector field v as a map $F \times [0,1] \to S^2$. We can slightly homotope v in $F \times (0,1)$ so that $(0,0,1)$ is a regular value of v. Now consider $L_v := v^{-1}(0,0,1)$. Observe that $L_v \cap (F \times \{0\}) = (\mathbf{s} \times \{0\}) \cup P$ and $L_v \cap (F \times \{1\}) = (\mathbf{t} \times \{1\}) \cup P$. Since $H_1(F)$ is generated by $h_i \cup \sigma_i \subset \bar{\mathfrak{N}} \times [0,1]$ for $i = 1, \dots, 2k$, it follows that L_v is framed homotopic to $\tilde{L}_v \cup (P \times [0,1])$ relative to the boundary, where $\tilde{L}_v$ is a framed 1-manifold cointained in $\bar{\mathfrak{N}} \times [0,1]$ and the framing on $P \times [0,1]$ is trivial. By the Pontryagin-Thom construction, we can homotope v and obtain v' such that $L_{v'} = \tilde{L}_v \cup (P \times [0,1])$. So we can assume, without loss of generality, that $L_v = \tilde{L}_v \cup (P \times [0,1])$, where $\tilde{L}_v \subset \bar{\mathfrak{N}} \times [0,1]$ and the framing on

$P \times [0,1]$ is trivial. Now, observe that $\mathfrak{N}$ deformation retracts to $\mathfrak{N} \cap (Z \cup \bigcup_i h_i)$. Projecting $\tilde{L}_v \cap (\mathfrak{N} \times [0,1])$ to F and using the deformation retraction from above, we obtain an element in $H_1(Z \setminus z, \mathbf{a})$. We define $\mathcal{F}_{\text{sp}}([v])$ to be this relative homology class in $H_1(Z \setminus z, \mathbf{a})$. Note that $M_*(\partial \mathcal{F}_{\text{sp}}([v])) = \mathbf{t} - \mathbf{s}$.

In order to define the Maslov component $\mathcal{F}_m([v])$, we compute the framing on $\tilde{L}_v$. Up to homotoping v, we can assume that, everytime $\tilde{L}_v$ intersects $N(p_i) \times \{t\}$ for some $t \in (0,1)$, the vector field v has the standard form in $N(p_i) \times [0,1]$, given by $v_{\{p_i\}}|_{N(p_i)}$. In order to see the framing on $\tilde{L}_v$, we let $K_v = v^{-1}(\delta, 0, \sqrt{1-\delta^2})$, for a small $\delta > 0$, such that $(\delta, 0, \sqrt{1-\delta^2})$ is a regular value of v. We define $\mathcal{F}_m([v])$ to be one half times the algebraic count of intersections of the projections of $L_v \cap \mathfrak{N}$ and $K_v \cap \mathfrak{N}$ to $\mathfrak{N}$, where the signs are as in Figure 6.9(a). We observe that $\varepsilon(\mathcal{F}_{\text{sp}}([v])) = \mathcal{F}_m([v]) \pmod 1$. So we can define

$$\mathcal{F}([v]) = (\mathcal{F}_m([v]), \mathcal{F}_{\text{sp}}([v])) \in G'(4k).$$



(a) (b) (c)

Figure 6.9:

To prove that $\mathcal{F}$ is a homomorphism, we need to show that $\mathcal{F}([v] \cdot [w]) = \mathcal{F}([v]) \cdot \mathcal{F}([w])$. We first observe that the Spin^c component of $\mathcal{F}([v] \cdot [w])$ is $\mathcal{F}_{\text{sp}}([v]) + \mathcal{F}_{\text{sp}}([w])$. Moreover, the count of intersections of the composition, which gives the Maslov component of $\mathcal{F}([v] \cdot [w])$, is the sum of the intersections of each piece plus the intersections between the pieces. We observe that one half times the number of intersections between the two pieces equals $L(\mathcal{F}_{\text{sp}}([v]), \mathcal{F}_{\text{sp}}([w]))$. Therefore $\mathcal{F}_m([v] \cdot [w])$ is the Maslov component of $\mathcal{F}([v]) \cdot \mathcal{F}([w])$. We also observe that $\mathcal{F}([v]^{-1}) = \mathcal{F}([v])^{-1}$, since taking the inverse of a vector field v is equivalent to switching the signs of the intersections of the projection of L_v to F.

It remains to show that for a generator $I(\mathbf{s})a(\boldsymbol{\rho})$ of $\mathcal{A}(\mathcal{Z})$, we have $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))) = \text{gr}'(\boldsymbol{\rho})$. We first order $\rho_1, \ldots, \rho_n$ as in Step 3 of §6.1. Let v be the vector field constructed in §6.1 whose relative homotopy class is $\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$. Let L_v and K_v be as above. The 1-manifold $L_v \cap (\mathfrak{N} \times [0,1])$ is the union of arcs $\tilde{\gamma}_i$, one for each Reeb chord ρ_i. Up to a relative isotopy of L_v, we can assume that the projection of $L_v \cap (\mathfrak{N} \times [0,1])$ has minimal number of intersections, i.e. there is no relative isotopy of L_v that decreases the number of intersections. It follows from the ordering of the Reeb chords that if the projections of $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ intersect for $i < j$, then the pair $\{\rho_i, \rho_j\}$ is interleaved and this is a negative intersection. Now we note that the arc $K_v \cap (\mathfrak{N} \times [0,1])$ does not rotate around $\tilde{\gamma}_i$, since the framing of L_v is trivial in $\mathfrak{N} \times [0,1]$ with respect to the trivialization. That implies the projection to $\mathfrak{N}$ has one negative intersection corresponding to each $\tilde{\rho}_i$ as in Figure 6.9(b). So for each Reeb chord ρ_i, we get a contribution of $-1/2$ to the Maslov component of $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$. Moreover, each interleaved pair gives rise to two negative intersections of the projections of $\tilde{L}_v$ and K_v.

So each interleaved pair contributes to the Maslov component of $\mathcal{F}(\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$ by -1. Finally if ρ_i and ρ_j abut, then we get an extra negative intersection, see Figure 6.9(c). So an abutting pair contributes by $-1/2$ to the Maslov component of $\mathcal{F}(\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$. Therefore, using (6.3), we conclude that

$$\mathcal{F}_m(\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))) = \iota(\boldsymbol{\rho}).$$

Hence $\mathcal{F}(\mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))) = \mathrm{gr}'(\boldsymbol{\rho})$. □

6.2 Grading on the modules

Let Y be an oriented connected compact 3-manifold with connected boundary. Following [23], we consider the bordered Heegaard diagram

$$\mathcal{H} = (\Sigma, \alpha_1^c, \cdots, \alpha_{g-k}^c, \alpha_1^a, \cdots, \alpha_{2k}^a, \beta_1, \cdots, \beta_g, z)$$

which is compatible with Y in the sense that the following conditions are satisfied:

- Σ is a compact oriented surface with a single boundary component.
- $(\Sigma \cup_\partial D^2, \alpha^c, \beta)$ is a Heegaard diagram for Y.
- $\alpha_1^a, \cdots, \alpha_{2k}^a$ are pairwise disjoint, embedded arcs in Σ with boundary on $\partial\Sigma$, and are disjoint from the α_i^c.
- $\Sigma \setminus (\alpha_1^c \cup \cdots \cup \alpha_{g-k}^c \cup \alpha_1^a \cup \cdots \cup \alpha_{2k}^a)$ is a disk with $2(g-k)$ holes.
- z is a point in $\partial\Sigma$, disjoint from all of the α_i^a.

We will abbreviate $\boldsymbol{\alpha}^c = \alpha_1^c \cup \cdots \cup \alpha_{g-k}^c$, $\boldsymbol{\alpha}^a = \alpha_1^a \cup \cdots \cup \alpha_{2k}^a$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^a$, and $\boldsymbol{\beta} = \beta_1 \cup \cdots \cup \beta_g$.

In this section, we explain how to define the grading on the modules $\widehat{CFA}(\mathcal{H})$ and $\widehat{CDF}(\mathcal{H})$. We start by defining the grading sets $S(\mathcal{H})$ and $\bar{S}(\mathcal{H})$.

The grading set

Let $F = \partial Y$. We recall from [23] that $\mathcal{H}$ gives rise to a pointed matched circle $\mathcal{Z} = (Z, \mathbf{a}, M, z)$, where $Z = \partial\Sigma$, $\mathbf{a} = \boldsymbol{\alpha}^c \cap Z$ and M maps both points in $\alpha_i^c \cap Z$ to $i \in [2k]$ for every i. For $\mathbf{s} \in [2k]$, we denote by $\mathrm{Vect}(Y, v_\mathbf{s})$ the set of homotopy classes of nonvanishing vector fields in Y whose restriction to F is $v_\mathbf{s}$. Since F is connected, $\mathrm{Vect}(Y, v_\mathbf{s})$ is nonempty if and only if $|\mathbf{s}| = k$. Let

$$S(\mathcal{H}) = \coprod_{|\mathbf{s}|=k} \mathrm{Vect}(Y, v_\mathbf{s}).$$

We observe that the groupoid $G(\mathcal{Z})$ acts on $S(\mathcal{H})$ on the right by concatenation. More precisely, given vector fields v and w such that $[v] \in \mathrm{Vect}(Y, v_{\mathbf{s}})$ and $[w] \in G(\mathbf{s}, \mathbf{t})$, define $[v] \cdot [w]$ as follows. Identify a collar neighborhood $N(F)$ of F in Y with $F \times [0,1]$ and take a representative $\tilde{v}$ of $[v]$ which is $[0,1]$-invariant in $N(F) \cong F \times [0,1]$. Now define $[v] \cdot [w] \in \mathrm{Vect}(Y, v_{\mathbf{t}})$ to be the relative homotopy class of the vector field which equals $\tilde{v}$ in the complement of $N(F)$ and w in $N(F) \cong F \times [0,1]$. Note that we also have a $\mathbb{Z}$-action on $S(\mathcal{H})$ just as before, which we again denote multiplicatively by λ^n on the left. We also observe that this action need not be free. In fact, let $[v] \in S(\mathcal{H})$ and denote by $v^\perp$ the orthogonal complement of v, seen as a complex line bundle. Then $\lambda^d \cdot [v] = [v]$ for every $d = \langle c_1(v^\perp), A \rangle$, for some $A \in H_2(Y)$.

Now we denote by $-\mathcal{Z}$ the pointed matched circle obtained by switching the orientation of Z, i.e. $-\mathcal{Z} = (-Z, \mathbf{a}, M, z)$. We observe that the groupoid $G(-\mathcal{Z})$ acts on $S(\mathcal{H})$ on the left, as follows. Given a vector field w in $(-F) \times [0,1]$, we define $\bar{w}$ to be the vector field in $F \times [0,1]$ given by $\bar{w}(x,t) = w(x, 1-t)$. So, given a vector field v in Y, if v and $\bar{w}$ coincide along $F \cong F \times \{1\}$, we can glue them along $F \cong F \times \{1\}$ and obtain a new vector field in Y, which we denote by $\bar{w} \cdot v$. So, given $[w] \in G(\mathbf{s}, \mathbf{t}) \subset G(-\mathcal{Z})$ and $[v] \in \mathrm{Vect}(Y, v_{\mathbf{s}})$, we can define $[v] \cdot [w]$ to be $[\bar{v} \cdot w]$.

The homotopy classes $[v], [w] \in \mathrm{Vect}(Y, v_{\mathbf{s}})$ are said to be in the same relative Spin^c structure if v is homotopic to w on the 2-skeleton relative to the boundary. We observe that there exists $n \in \mathbb{Z}$ such that $[v] = \lambda^n \cdot [w]$ if, and only if, $[v], [w] \in \mathrm{Vect}(Y, v_{\mathbf{s}})$ and v an w are in the same relative Spin^c structure.

Homotopy classes of vector fields

The goal of this section is provide a new way to compute the difference between homotopy classes of nonvanishing vector fields, based on Pontryagin-Thom construction. The construction here is inspired and very similar to the work of Dufraine [6]. Let Y be a closed oriented 3-manifold. Suppose ξ, η are nonvanishing vector fields on Y. By a C^∞-small perturbation, we can assume that the set

$$L = L_{\xi,\eta} = \{y \in Y \mid \xi(y) = -\eta(y)\}$$

is a link in Y. In the case that $[L] = 0 \in H_1(Y; \mathbb{Z})$, there exists an embedded compact surface $\Sigma \subset Y$ with $\partial\Sigma = L$. Choosing a Riemannian metric on Y, we consider the orthogonal complement $\eta^\perp$ of η, which is a co-oriented plane field on Y. Since Σ deformation retracts onto a wedge of circles, we can choose a trivialization $\tau : \eta^\perp|_\Sigma \to \Sigma \times \mathbb{R}^2$. This in turn gives a trivialization $\tilde{\tau} : TY|_\Sigma \to \Sigma \times \mathbb{R}^3$ by setting $\tilde{\tau}^*(\partial_z)$ to be equal to η, where (x, y, z) are the coordinates in $\mathbb{R}^3$. Let $N(\Sigma)$ denote a small tubular neighborhood of Σ in Y. Then τ gives rise to a trivialization $TY|_{N(\Sigma)} \cong N(\Sigma) \times \mathbb{R}^3$.

Using the above trivialization, we can see $\xi|_{N(\Sigma)}$ as a map $\xi_\tau : N(\Sigma) \to S^2 \subset \mathbb{R}^3$. It is clear from construction that $L_{\xi,\eta} = \xi_\tau^{-1}(0,0,-1) = \partial\Sigma$. Taking the pre-image of a regular value close to $(0,0,-1)$ in S^2, we get a framing on $L_{\xi,\eta}$. We represent this framing by a

number $n_{\xi,\eta}$, given by the difference from the Seifert framing. The following proposition gives a way to compute the difference between homotopy classes of nonzero vector fields. The result was essentially known by Dufraine [6] but we write down a proof here for the readers' convenience.

Proposition 6.2.1. *Given ξ, η nonvanishing vector fields on Y, ξ is homotopic to η if and only if $L_{\xi,\eta}$ is null-homologous and the framing $n_{\xi,\eta} = 0$.*

Proof. Suppose there exists a 1-parameter family of nonvanishing vector fields $\{\xi_t\}_{t\in[0,1]}$, on Y such that $\xi_0 = \xi$, $\xi_1 = \eta$. We choose a Riemannian metric on Y such that ξ_t is of unit length. Therefore we define a section $\Xi : Y \times [0,1] \to STY \times [0,1]$ by $\Xi(y,t) = (\xi_t(y), t)$ for all $y \in Y, t \in [0,1]$, where STY denotes the unit tangent bundle. We can also define a section $\mathbb{I} : Y \times [0,1] \to STY \times [0,1]$ by $\mathbb{I}(y,t) = (-\eta(x), t)$.

We observe that $L_{\xi,\eta} = \{(y,0) \in Y \times [0,1] \mid \Xi(y,0) = \mathbb{I}(y,0)\}$ and $\{(y,1) \in Y \times [0,1] \mid \Xi(y,1) = \mathbb{I}(y,1)\} = \emptyset$. By the standard transversality argument, we can assume that

$$\{(y,t) \in Y \times [0,1] \mid \Xi(y,t) = \mathbb{I}(y,t)\}$$

is an embedded surface in $Y \times [0,1]$. Therefore $[L_{\xi,\eta}] = 0 \in H_1(Y;\mathbb{Z})$.

Now everything follows from the usual Pontryagin-Thom construction. Namely, let $\Sigma \subset Y$ be a compact surface such that $\partial\Sigma = L_{\xi,\eta}$, and consider a neighborhood $N(\Sigma)$ of Σ in Y. Observe that ξ is homotopic to η on the complement of $N(\Sigma)$ by a linear homotopy, so we can assume that $\xi = \eta$ on $Y \setminus N(\Sigma)$. Since, again, $N(\Sigma)$ deformation retracts onto a wedge of circles, we can trivialize $\eta^\perp|_{N(\Sigma)}$ and therefore obtain a trivialization of $TY|_{N(\Sigma)}$ by writing $TY = \eta \oplus \eta^\perp$. The vector field ξ, under this trivialization, sends $L_{\xi,\eta}$ to $(0,0,-1) \in S^2$ as before. The Pontryagin-Thom construction asserts that ξ is homotopic to η if and only if $L_{\xi,\eta}$ is framed cobordant to the empty set. □

We obtain the following corollary.

Corollary 6.2.2. *Let ξ, η be nonvanishing vector fields on Y. Then ξ and η are in the same Spin^c structure if, and only if, $[L_{\xi,\eta}] = 0$. And if that is the case, then $[\xi] = \lambda^{n_{\xi,\eta}} \cdot [\eta]$.*

Proof. If ξ and η are in the same Spin^c structure, then there exists $m \in \mathbb{Z}$ such that $[\xi] = \lambda^m \cdot [\eta]$. Let $\tilde{\eta}$ be a nonvanishing vector field in Y given by modifying η in a very small ball, corresponding to the action of $\lambda^m \in \pi_3(S^2)$. By definition, $[\tilde{\eta}] = \lambda^m \cdot [\eta]$. So ξ and $\tilde{\eta}$ are homotopic. By Proposition 6.2.1, $[L_{\xi,\eta}] = [L_{\xi,\tilde{\eta}}] = 0$.

Conversely if $[L_{\xi,\eta}] = 0$, then, as explained above, we obtain a framing $n_{\xi,\eta}$ on $L_{\xi,\eta}$. Now we act on η by $\lambda^{n_{\xi,\eta}} \in \pi_3(S^2)$, obtaining a vector field $\tilde{\eta}$. We observe that $L_{\xi,\eta}$ is still nullhomologous and that $n_{\xi,\tilde{\eta}} = 0$. By Proposition 6.2.1, we conclude that $[\xi] = [\tilde{\eta}]$. So $[\xi] = \lambda^{n_{\xi,\eta}} \cdot [\eta]$. In particular, ξ and η are in the same Spin^c structure. Note that we also proved the second assertion. □

Remark 6.2.3. The point of our approach is that in order to compute the difference between ξ and η, it suffices to trivialize TY along a Seifert surface, which is much easier in practice.

Grading on $\widehat{CFA}(\mathcal{H})$

We start by recalling the definition of the A^∞-module $\widehat{CFA}(\mathcal{H})$ from [23]. Let $\mathfrak{G}(\mathcal{H})$ be the set of g-tuples $\mathbf{x} = \{x_1, \cdots, x_g\} \subset \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ such that there is exactly one point x_i on each β-circle and on each α-circle and there is at most one x_i on each α-arc. Then $\widehat{CFA}(\mathcal{H})$ is generated as a vector space over $\mathbb{Z}/2$ by $\mathfrak{G}(\mathcal{H})$. We also recall that given $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, there is an idempotent $I_A(\mathbf{x}) := I(o(\mathbf{x}))$, where $o(\mathbf{x}) \subset [2k]$ is the set of α-arcs containing x_i for some i. We have a right action of the ring of idempotents $\mathcal{I} := \mathcal{I}(\mathcal{Z})$ on $\widehat{CFA}(\mathcal{H})$ given by

$$\mathbf{x} \cdot I(\mathbf{s}) = \begin{cases} \mathbf{x}, & \text{if } I_A(\mathbf{x}) = I(\mathbf{s}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} := \mathcal{A}(\mathcal{Z})$. As explained in [23, Ch. 7], the A^∞-structure on $\widehat{CFA}(\mathcal{H})$ is given by maps

$$m_{l+1} : \widehat{CFA}(\mathcal{H}) \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A} \to \widehat{CFA}(\mathcal{H}).$$

Now we want to define a grading function

$$\mathrm{gr} : \mathfrak{G}(\mathcal{H}) \to S(\mathcal{H}),$$

compatible with the maps m_{l+1}. More precisely, let $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$ and let $a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_l)$ be generators of $\mathcal{A}$. If $\mathbf{x} \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_l) \neq 0$ then we can write

$$\mathbf{x} \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_l) = \mathbf{x} \otimes_{\mathcal{I}} I(\mathbf{s}_1)a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} I(\mathbf{s}_l)a(\boldsymbol{\rho}_l),$$

for some $\mathbf{s}_1, \ldots, \mathbf{s}_l \subset [0, 2k]$. Note, in particular, that $I(\mathbf{s}_1) = I_A(\mathbf{x})$. If $\mathbf{y}$ is a summand in $m_{l+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_l))$, we want gr to satisfy

$$\mathrm{gr}(\mathbf{y}) = \lambda^{l-1} \cdot \mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(I(\mathbf{s}_1)a(\boldsymbol{\rho}_1)) \ldots \mathrm{gr}(I(\mathbf{s}_l)a(\boldsymbol{\rho}_l)).$$

Recall the following definition from [23].

Definition 6.2.4. Given a compact 3-manifold Y with bordered Heegaard diagram $\mathcal{H}$, we say that a pair consisting of a Riemannian metric g on Y and a self-indexing Morse function $h : Y \to [0, 3]$ is *compatible with* $\mathcal{H}$ if

- the boundary of Y is geodesic,
- the gradient vector field $\nabla h|_{\partial Y}$ is tangent to ∂Y,
- h has a unique index 0 and a unique index 3 critical point, both of which lie on ∂Y, and are the unique index 0 and 2 critical points of $h|_{\partial Y}$, respectively,
- the index 1 critical points of $h|_{\partial Y}$ are also index 1 critical points of h,

- $h|_{\partial Y}$, viewed as a Morse function on $F = \partial Y$, is compatible with the pointed matched circle $\mathcal{Z}$.

Fix a compatible Morse function $h : Y \to [0,3]$, and consider the gradient vector field ∇h on Y. For any $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, the pair $(\mathbf{x}, z)$ determines $g+1$ gradient trajectories $\{\gamma_0, \cdots, \gamma_g\}$, where γ_0 connects the index 0 and index 3 critical points passing through z, and γ_i connects the index 1 and index 2 critical points passing through x_i. We define $\mathrm{gr}(\mathbf{x}) \in S(\mathcal{H})$ by modifying ∇h near tubular neighborhoods of the trajectories γ_i.

Let $N(\gamma_0)$ be a small neighborhood of γ_0 in Y. Let $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, x \geq 0\}$. Then $N(\gamma_0)$ is diffeomorphic to $D \times [0,\pi]/\sim$, where the equivalence relation is given by $((0,y),t) \simeq ((0,y),t')$ for every t, t', and where $(D \times \{0\}) \cup (D \times \{\pi\})/\sim$ is identified with $N(\gamma_0) \cap \partial Y$, see Figure 6.10(a). Using the above identification, the vector field ∇h restricted to $D \times \{t\}$ is depicted in Figure 6.11(a). For each $t \in [0,\pi]$, we modify ∇h in $D \times \{t\}$ as shown in Figure 6.11(d). Since these modifications coincide on $D \cap \{y = 0\}$, we get a nonvanishing vector field on $D \times [0,\pi]/\sim$. This is the restriction to the half-ball of the analogous modification, used to define the grading on Heegaard Floer homology.

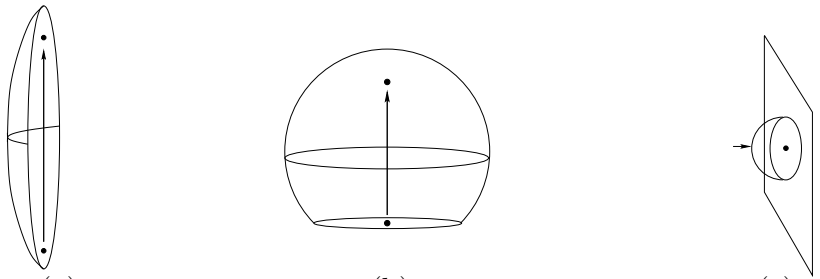
We order the flow lines $\gamma_1, \ldots, \gamma_g$ so that the index one critical points corresponding to $\gamma_1, \ldots, \gamma_k$ lie on ∂Y. For each $i = 1, \ldots, k$, let $N(\gamma_i)$ be a small neighborhood of γ_i in Y. Let $\tilde{B}$ be the intersection of the unit ball in $\mathbb{R}^3$ with $\{z \geq -1/2\}$. Then $N(\gamma_i)$ is diffeomorphic to $\tilde{B}$, see Figure 6.10(b). Let $\tilde{D} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, y \geq -1/2\}$. Each vertical cross-section of $\tilde{B}$ can be identified with $\tilde{D}$. The vector field ∇h restricted to $N(\gamma_i)$ can be viewed as an interpolation between ∇h restricted to two transverse vertical cross-sections, corresponding to the unstable manifold of the index one critical point and the stable manifold of the index two critical point. Figure 6.11(b,c) shows the restriction of ∇h to these two cross-sections. We modify ∇h on these cross-sections as in Figure 6.11(e,f). Again, this is very similar to the corresponding construction on Heegaard Floer homology. Namely, this is the restriction to $\{z \geq -1/2\}$ of the vector field defined in Chapter 5. The reader can find a formula describing this modification in Chapter 5. For each $i = k+1, \ldots, g$, the corresponding index one critical point lies in the interior of Y. So do the same modification as in Chapter 5.

We still have to eliminate the boundary index one critical points which do not belong to any γ_i. We do so by slightly perturbing ∇h in a neighborhood of each of these points so that it points to the interior of Y. Alternatively, we observe that Y is diffeomorphic to the complement of the union of small neighborhoods of each of these points. So ∇h restricted to a tubular neighborhood of the boundary of this complement gives the desired modification of ∇h, see Figure 6.10(c). Let $v_{\mathbf{x}}$ denote the vector field in Y obtained by modifying ∇h as explained above. Then we define $\mathrm{gr}(\mathbf{x})$ to be the relative homotopy class of $v_{\mathbf{x}}$. We note that $\mathrm{gr}(\mathbf{x}) \in \mathrm{Vect}(Y, v_{o(\mathbf{x})})$.

Following [23], given generators $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, we consider the relative homology group

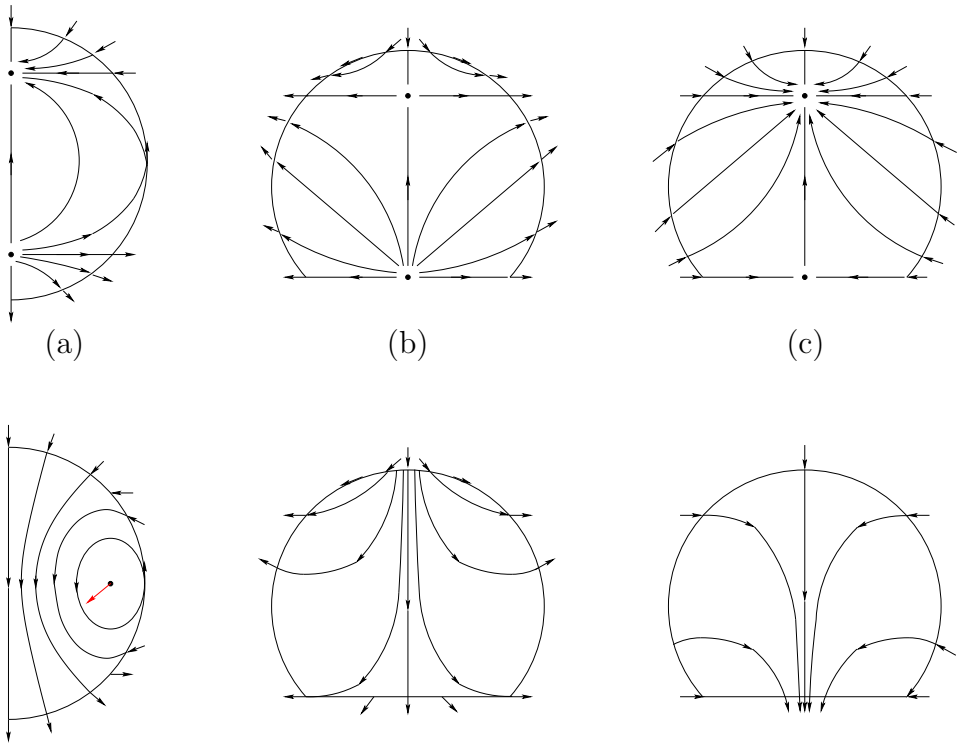
$$H_2(\Sigma \times [0,1] \times [0,1], ((S_{\boldsymbol{\alpha}} \cup S_{\boldsymbol{\beta}} \cup S_\partial) \times [0,1]) \cup G_{\mathbf{x}} \times \{0\} \cup G_{\mathbf{y}} \times \{1\}),$$

where $S_{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \{1\}$, $S_{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \{0\}$, $S_\partial = (\partial\Sigma \setminus z) \times [0,1]$, $G_{\mathbf{x}} = \mathbf{x} \times [0,1]$ and $G_{\mathbf{y}} = \mathbf{y} \times [0,1]$.



(a) (b) (c)

Figure 6.10:



(a) (b) (c)

(d) (e) (f)

Figure 6.11: Modifying ∇h to a nonvanishing vector field.

This group is usually denoted by $\pi_2(\mathbf{x}, \mathbf{y})$, following the tradition from [28].

A homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ can be interpreted as a domain in Σ. As such, one defines $e(B)$ to be the Euler measure of this domain as follows. For each positively covered region in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, we define its Euler measure to equal its Euler characteristic $\chi(B)$ plus one quarter of the number of concave corners minus the number of convex corners. We can extend this linearly to domains in Σ. One also defines $n_{\mathbf{x}}(B)$ to be one quarter of the number of components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ in B adjencent to $\mathbf{x}$, counted with multiplicity. One

defines $n_{\mathbf{y}}$ similarly. For $B \in \pi_2(\mathbf{x}, \mathbf{y})$, one defines $\partial^\partial B$ to be the piece of the boundary of B contained in $\partial\Sigma$. We think of $\partial^\partial B$ as a class in $H_1(Z \setminus \{z\}, \mathbf{a})$. Let $\overrightarrow{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_l)$ be an l-tuple of sets of Reeb chords. Recall that $a(\overrightarrow{\boldsymbol{\rho}})$ is defined to be the product $a(\overrightarrow{\boldsymbol{\rho}}) = a(\boldsymbol{\rho}_1) \dots a(\boldsymbol{\rho}_l)$ and $\iota(\overrightarrow{\boldsymbol{\rho}})$ to be the Maslov component of $\mathrm{gr}'(a(\overrightarrow{\boldsymbol{\rho}}))$. One can also define $[\overrightarrow{\boldsymbol{\rho}}] = [\boldsymbol{\rho}_1] + \dots + [\boldsymbol{\rho}_l] \in H_1(Z \setminus z, \mathbf{a})$. Now recall the definition of $\mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}})$ for $B \in \pi(\mathbf{x}, \mathbf{y})$ and $\overrightarrow{\boldsymbol{\rho}}$ satisfying $\partial^\partial B = [\overrightarrow{\boldsymbol{\rho}}]$.

$$\mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \iota(\overrightarrow{\boldsymbol{\rho}}) + l.$$

Given $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$ such that $\pi_2(\mathbf{x}, \mathbf{y})$ is nonempty[5], we now compare $\mathrm{gr}(\mathbf{x})$ and $\mathrm{gr}(\mathbf{y})$. The main result of this section is the following proposition.

Proposition 6.2.5. *Let $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\overrightarrow{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_l)$ such that $\partial^\partial B = [\overrightarrow{\boldsymbol{\rho}}]$. Then*

$$\mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(I_A(\mathbf{x}) a(\overrightarrow{\boldsymbol{\rho}})) = \lambda^{\mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}}) - l} \cdot \mathrm{gr}(\mathbf{y}). \tag{6.4}$$

Proof. Instead of doing a direct computation, we reduce the problem to the computation of relative gradings in Heegaard Floer homology, which has been done in Chapter 5. First it follows from (6.2) that

$$I_A(\mathbf{x}) a(\overrightarrow{\boldsymbol{\rho}}) = I_A(\mathbf{x}) a(\boldsymbol{\rho}_1 \uplus \dots \uplus \boldsymbol{\rho}_l).$$

Moreover, since $\iota(\overrightarrow{\boldsymbol{\rho}}) = \iota(\boldsymbol{\rho}_1 \uplus \dots \uplus \boldsymbol{\rho}_l)$, it follows that $\mathrm{ind}(B, \overline{\boldsymbol{\rho}}) - l = \mathrm{ind}(B, \boldsymbol{\rho}_1 \uplus \dots \uplus \boldsymbol{\rho}_l) - 1$. Therefore it suffices to prove (6.4) for $l = 1$. From now on, we shall assume that $\overrightarrow{\boldsymbol{\rho}} = \{\boldsymbol{\rho}\}$. We shall prove that

$$\mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(I_A(\mathbf{x}) a(\boldsymbol{\rho})) = \lambda^{e(B) + n_x(B) + n_y(B) + \iota(\boldsymbol{\rho})} \cdot \mathrm{gr}(\mathbf{y}) \tag{6.5}$$

Let (ρ, σ) be an abutting pair in $\boldsymbol{\rho}$ and let $\tilde{\boldsymbol{\rho}}$ be the set obtained from $\boldsymbol{\rho}$ by substituting the pair $\{\rho, \sigma\}$ by their join $\rho \uplus \sigma$. Since $I_A(\mathbf{x}) a(\boldsymbol{\rho})$ is a term in the differential of $I_A(\mathbf{x}) a(\tilde{\boldsymbol{\rho}})$, by Proposition 6.1.4, $\mathrm{gr}(I_A(\mathbf{x}) a(\boldsymbol{\rho})) = \lambda^{-1} \mathrm{gr}(I_A(\mathbf{x}) a(\tilde{\boldsymbol{\rho}}))$. Moreover, by (6.3), $\iota(\boldsymbol{\rho}) = \iota(\tilde{\boldsymbol{\rho}}) - 1$. So if (6.5) holds for $(B, \tilde{\boldsymbol{\rho}})$, then it also holds for $(B, \boldsymbol{\rho})$. Hence we can assume that $\boldsymbol{\rho}$ has no abutting pairs.

Now let $\{\rho, \sigma\}$ be an interleaved pair in $\boldsymbol{\rho}$ so that $\rho^- < \sigma^- < \rho^+ < \sigma^+$. We substitute the pair $\{\rho, \sigma\}$ by the nested pair $\{[\rho^-, \sigma^+], [\sigma^-, \rho^+]\}$ giving rise to a set of Reeb chords, denoted once again by $\tilde{\boldsymbol{\rho}}$. We observe that, again, $I_A(\mathbf{x}) a(\boldsymbol{\rho})$ is a term in the differential of $I_A(\mathbf{x}) a(\tilde{\boldsymbol{\rho}})$. So $\mathrm{gr}(I_A(\mathbf{x}) a(\boldsymbol{\rho})) = \lambda^{-1} \mathrm{gr}(I_A(\mathbf{x}) a(\tilde{\boldsymbol{\rho}}))$. By (6.3), $\iota(\boldsymbol{\rho}) = \iota(\tilde{\boldsymbol{\rho}}) - 1$. Therefore we can also assume that $\boldsymbol{\rho}$ has no interleaved pairs.

Now write $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_m\}$ with the ordering as in §6.1. Let Σ' be a closed surface obtained by gluing a compact surface of genus k with boundary $-Z$ to Σ along the boundary. We construct a Heegaard diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z)$ as follows. For each arc α_i^a, we glue an arc on $\Sigma' \setminus \Sigma$ to obtain a closed circle on Σ', which we denote by α_i'. We can always choose the completion of the α-arcs such that $\boldsymbol{\alpha}' = \{\alpha_1^c, \dots, \alpha_{g-k}^c, \alpha_1', \dots, \alpha_{2k}'\}$ is a set of pairwise disjoint

[5] This is equivalent to $\mathbf{x}$ and $\mathbf{y}$ being in the same Spin^c structure.

curves which are linearly independent in $H_1(\Sigma')$. Recall that $Z \setminus N(z) \subset \partial\Sigma$ is a line segment containing all Reeb chords. Now consider k translates of $Z \setminus N(z)$ on a collar neighborhood of $\partial\Sigma$ in $\Sigma' \setminus \Sigma$ with an ordering by the distance to $\partial\Sigma$. For each $i = 1, \dots, k$, we can define a circle β_i' on $\Sigma' \setminus \Sigma$ containing the i-th translate, such that these circles are pairwise disjoint and linearly independent in homology. So we let $\boldsymbol{\beta}' = \{\beta_1, \dots, \beta_g, \beta_1', \dots, \beta_k'\}$. Therefore we obtain a Heegaard diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z)$, which gives rise to a closed three-manifold containing Y, denoted by Y'.

The domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ naturally extends over Σ', as follows. Each Reeb chord ρ_i can be translated to β_i' giving rise to a segment, whose endpoints are on the α-circles corresponding to the endpoints of ρ_i. So each ρ_i gives rise to two intersection points on β_i'. We obtain a new domain B' on Σ' by taking the union of B with a domain in $\Sigma \setminus \Sigma'$ bounded by the translates of ρ_i and the corresponding α-circles, as in Figure 6.12. For the time being, let us assume that $M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+) = \emptyset$, so that the new intersection points can be added to $\mathbf{x}$ and $\mathbf{y}$, respectively, giving rise to intersection points $\mathbf{x}'$ and $\mathbf{y}'$ in $\mathbb{T}_{\boldsymbol{\alpha}'} \cap \mathbb{T}_{\boldsymbol{\beta}'}$. So $B' \in \pi_2(\mathbf{x}', \mathbf{y}')$. The case when $M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+) \neq \emptyset$ is slightly technical and will be postponed to the end of the proof.

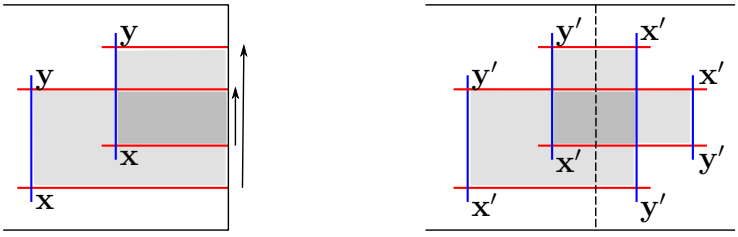


Figure 6.12: The left side is a domain on Σ. The right side is the completion of the domain on Σ'.

We recall the index formula from [22]

$$\mathrm{ind}(B') = e(B') + n_{\mathbf{x}'}(B') + n_{\mathbf{y}'}(B'). \tag{6.6}$$

We observe that $e(B) = e(B')$. The points in $\mathbf{x}'$ and $\mathbf{y}'$ are either the elements of $\mathbf{x}$ and $\mathbf{y}$ or the new convex corners on $\Sigma' \setminus \Sigma$, which are not interior corners, since we assumed that any two Reeb orbits are either disjoint or nested. Since there are $|\boldsymbol{\rho}|/2$ such corners, we conclude that

$$\mathrm{ind}(B') = e(B') + n_{\mathbf{x}'}(B') + n_{\mathbf{y}'}(B') = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\boldsymbol{\rho}|/2. \tag{6.7}$$

In Chapter 5, we defined an absolute grading function

$$\widetilde{\mathrm{gr}} : \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \to \mathcal{P}(Y'),$$

where $\mathcal{P}(Y')$ is the set of homotopy classes of nonzero vector fields on Y'. This is such that for any $B' \in \pi_2(\mathbf{x}', \mathbf{y}')$, which does not intersect the basepoint z,

$$\widetilde{\mathrm{gr}}(\mathbf{x}') = \lambda^{\mathrm{ind}(B')} \cdot \widetilde{\mathrm{gr}}(\mathbf{y}').$$

Let $\mathbf{s} = I_A(\mathbf{x})$ and $\mathbf{t} = I_A(\mathbf{y})$. Note that we can decompose Y as $Y' = Y \cup_F (F \times [0,1]) \cup_F \hat{Y}$. Following our construction of the gradings, let $v_{\mathbf{x}}$, $v_{\mathbf{y}}$ be the vector fields whose relative homotopy classes are $\mathrm{gr}(\mathbf{x})$ and $\mathrm{gr}(\mathbf{y})$, and let $v_{(\mathbf{s},\boldsymbol{\rho})}$ be the vector field defined in §6.1, such that $[v_{(\mathbf{s},\boldsymbol{\rho})}] = \mathrm{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$. Let $\mathbb{I}_{\mathbf{t}}$ denote the $[0,1]$-invariant vector field on $F \times [0,1]$, whose restriction to $F \times \{t\}$ equals $v_{\mathbf{t}}$. Then the action of $[\mathbb{I}_{\mathbf{t}}]$ on $\mathrm{Vect}(Y, v_{\mathbf{t}})$ is trivial. So $[v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}] = \mathrm{gr}(\mathbf{y})$. We now show that $v_{\mathbf{x}} \cdot v_{(\mathbf{s},\boldsymbol{\rho})}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$ are in the same Spin^c structure and we compute their difference.

Since $v_{(\mathbf{s},\boldsymbol{\rho})}$ and $\mathbb{I}_{\mathbf{t}}$ coincide on $F \times \{1\}$, we can extend $v_{\mathbf{x}} \cdot v_{(\mathbf{s},\boldsymbol{\rho})}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$ to Y' so that they coincide in $\hat{Y}$. Let X_1 and X_2 be the vector fields obtained by this extension from $v_{\mathbf{x}} \cdot v_{(\mathbf{s},\boldsymbol{\rho})}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$, respectively. We apply Proposition 6.2.1, obtaining a link denoted by $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$ defined as

$$L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} := \{y \in Y' | X_1(y) = -X_2(y)\}.$$

Since X_1 and X_2 coincide in $\hat{Y}$, the link $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$ is contained in $Y \cup (F \times [0,1))$ and it is independent of the extension of the vector fields to $\hat{Y}$.

We define $L_{\mathbf{x}',\mathbf{y}'}$ to be the link in Y given by

$$L_{\mathbf{x}',\mathbf{y}'} = \{y \in Y' | v_{\mathbf{x}'}(y) = -v_{\mathbf{y}'}(y)\}.$$

We note that $v_{\mathbf{x}'}|_Y = v_{\mathbf{x}}$ and $v_{\mathbf{y}'}|_Y = v_{\mathbf{y}}$. So the restrictions of $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$ and $L_{\mathbf{x}',\mathbf{y}'}$ to Y coincide. We observe that this is the union of the flow lines corresponding to all points in $\mathbf{x}$ and $\mathbf{y}$, up to a small isotopy in neighborhoods of the critical points. Moreover the domain B gives rise to a surface S in Y relative to the boundary, bounding $L_{\mathbf{x}',\mathbf{y}'} \cap Y$. We note that, up to a small perturbation, $S \cap \partial Y$ is the union of the arcs $\hat{\rho}_i$, as defined in §6.1.

We will now show that $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$ and $L_{\mathbf{x}',\mathbf{y}'}$ are both nullhomologous and isotopic to each other. We first look at $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1])$. Observe that $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times \{0\}) = M(\boldsymbol{\rho}^-) \cup M(\boldsymbol{\rho}^+)$ by construction. Using the bifurcation description of $v_{(\mathbf{s},\boldsymbol{\rho})}$ illustrated in Figure 6.4, we observe that $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1])$ is the union of embedded arcs, each of which is as depicted in Figure 6.13. In fact each tangency of L_F and $F \times \{t\}$ happens exactly when t corresponds to the middle of the second bifurcation for a Reeb chord. It follows that B' gives

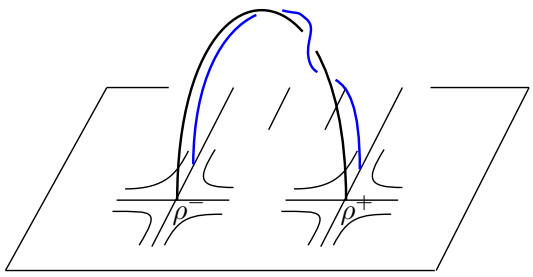


Figure 6.13: The Pontryagin submanifold $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$. The blue arc indicates the framing.

rise to a surface $S_{\boldsymbol{\rho}}$ containing S whose boundary is $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$. An example of $S_{\boldsymbol{\rho}} \cap (F \times [0,1])$

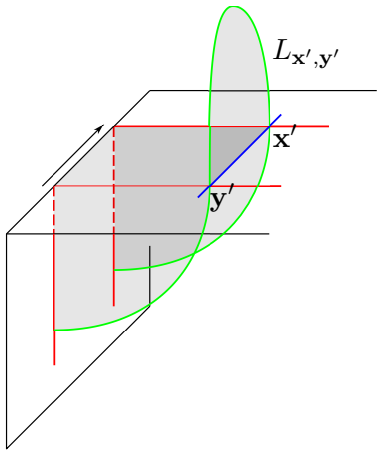


Figure 6.14: The Pontryagin submanifold $L_{\mathbf{x}',\mathbf{y}'}$ (in green).

is shown in Figure 6.13. So $[L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}] = 0 \in H_1(Y \cup (F \times [0,1]))$. Therefore $v_{\mathbf{x}} \cdot v_{(\mathbf{s},\boldsymbol{\rho})}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$ are in the same relative Spin^c structure. We will now consider the framing on $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y})}$, as in §6.2. It is clear that $S_{\boldsymbol{\rho}} \cap (F \times [0,1])$ is topologically the union of disjoint disks. Let $\mathbb{I}_{\mathbf{t}}^{\perp}$ be the oriented 2-plane field on $F \times [0,1]$ which is orthogonal to $\mathbb{I}_{\mathbf{t}}$. We can choose a trivialization of $\mathbb{I}_{\mathbf{t}}^{\perp}|_{S_{\boldsymbol{\rho}}}$ such that the vector field ∂/∂_x is everywhere tangent to $S_{\boldsymbol{\rho}}$ and points into $S_{\boldsymbol{\rho}}$ along $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1])$. We extend this trivialization to a small neighborhood $N(S_{\boldsymbol{\rho}})$ of $S_{\boldsymbol{\rho}}$ and see $v_{(\boldsymbol{\rho},\mathbf{s})}$ as a map $N(S_{\boldsymbol{\rho}}) \to S^2$. So, to compute the framing on $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1]) = v_{(\boldsymbol{\rho},\mathbf{s})}^{-1}(0,0,-1)$, we can look at $v_{(\boldsymbol{\rho},\mathbf{s})}^{-1}(\varepsilon, 0, -\sqrt{1-\varepsilon^2})$, for small $\varepsilon > 0$. For each Reeb chord, which corresponds to an arc in $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1])$, we observe that this framing is represented by a negative full-twist as depicted in Figure 6.13.

Now we observe that $L_{\mathbf{x}',\mathbf{y}'} \cap (F \times [0,1])$ is the union of the flow lines corresponding to all points in $\mathbf{x}' \setminus \mathbf{x}$ and $\mathbf{y}' \setminus \mathbf{y}$, up to a small isotopy in neighborhoods of the critical points. Note that we can isotope $L_{\mathbf{x}',\mathbf{y}'} \cap (F \times [0,1])$ to $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}} \cap (F \times [0,1])$ relative to the endpoints and, after that isotopy, $S_{\boldsymbol{\rho}}$ is a Seifert surface[6] for $L_{\mathbf{x}',\mathbf{y}'}$. We can choose a trivialization of $v_{\mathbf{y}'}^{\perp}|_{S_{\boldsymbol{\rho}}}$ such that in a neighborhood of $S_{\boldsymbol{\rho}}$, the vector field ∂_x is everywhere tangent to $S_{\boldsymbol{\rho}}$ and points into $S_{\boldsymbol{\rho}}$ along $L_{\mathbf{x}',\mathbf{y}'}$. As before, we extend this trivialization to $N(S_{\boldsymbol{\rho}})$. Again, seeing $v_{\mathbf{x}'}$ as a map $N(S_{\boldsymbol{\rho}}) \to S^2$ and taking the preimage of a regular value near $(0,0,-1)$, we observe that the framing on $L_{\mathbf{x}',\mathbf{y}'} \cap (F \times [0,1])$ is trivial.

Therefore the framing on $L_{(\mathbf{x},\boldsymbol{\rho}),\mathbf{y}}$ equals the framing on $L_{\mathbf{x}',\mathbf{y}'}$ minus the number of Reeb chords. Therefore, by (6.7) and (6.3),

$$\mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(I_A(\mathbf{x})a(\boldsymbol{\rho})) = \lambda^{\mathrm{ind}(B')-|\boldsymbol{\rho}|} \cdot \mathrm{gr}(\mathbf{y}) = \lambda^{e(B)+n_{\mathbf{y}}(B)+n_{\mathbf{y}}(B)+\iota(\boldsymbol{\rho})} \cdot \mathrm{gr}(\mathbf{y}). \tag{6.8}$$

Therefore we proved (6.5), when $M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+) = \emptyset$.

[6]Here we are using the fact that $\boldsymbol{\rho}$ does not contain any abutting or interleaved pair. Otherwise we would not be able to use the same Seifert surface for both links.

Now assume that $M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+) \neq \emptyset$. See Figure 6.15 for an example. In this case we may still construct the extended domain B' connecting $\mathbf{x}'$ and $\mathbf{y}'$ although they are not generators of $\widehat{CF}(Y')$. Nevertheless the "index" of B' is defined using the combinatorial formula (6.6). To verify (6.8) in this case, we will modify the Morse function on Y near the index one critical points corresponding to $M(\boldsymbol{\rho}^-) \cap M(\boldsymbol{\rho}^+)$. For simplicity of notations, we assume that $\boldsymbol{\rho} = \{\rho\}$ consists of one chord as depicted in left side of Figure 6.15. The general case is just an iterated application of the modification we describe below.

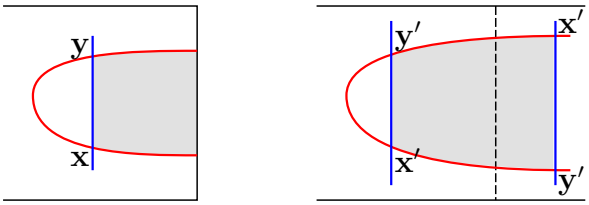


Figure 6.15: The completion of a domain where the new $\mathbf{x}'$ and $\mathbf{y}'$ are not generators.

Recall that h is a Morse function on Y such that $\nabla h|_{\partial Y}$ is compatible with the parametrization of $\partial Y = F$. Let p be the index one critical point corresponding to $M(\rho^-) = M(\rho^+)$. We construct a new Morse function h' on Y such that $h' = h$ in the complement of a neighborhood of p, and near p, $\nabla h'$ has three critical points: p, p' of index one and q of index two. See Figure 6.16. This construction should be compared with the construction of $\mathrm{gr}(I(\mathbf{s})a(\rho))$ using Figure 6.6. More precisely, assuming that the trajectories of $\nabla h'$ connects $\mathbf{x}$ to p and $\mathbf{y}$ to p' as in Figure 6.16, we define a variant $\mathrm{gr}'(\mathbf{x})$ to be equal to $\mathrm{gr}(\mathbf{x})$ away from a neighborhood of p where $h' \neq h$, and in this neighborhood the vector field is pointing out of Y near p and pointing into Y near p' and q. We also define $\mathrm{gr}'(\mathbf{y})$ analogously. It follows from the construction that $\mathrm{gr}(\mathbf{x}) \cdot \mathrm{gr}(I(\mathbf{s})a(\rho)) - \mathrm{gr}(\mathbf{y})$ is equal to $\mathrm{gr}'(\mathbf{x}) \cdot \mathrm{gr}(I(\mathbf{s})a(\rho)) - \mathrm{gr}'(\mathbf{y})$. Here by taking differences we mean taking the power of λ in (6.8). Moreover for the latter difference to be well-defined, we have to skip the steps of creating/canceling pairs of critical points when defining $\mathrm{gr}(I(\mathbf{s})a(\rho))$ in Step 2 of Section 6.1. After all, the computation of $\mathrm{gr}'(\mathbf{x}) \cdot \mathrm{gr}(I(\mathbf{s})a(\rho)) - \mathrm{gr}'(\mathbf{y})$, as well as the computation of the difference between the absolute gradings $\widetilde{\mathrm{gr}}(\mathbf{x}')$ and $\widetilde{\mathrm{gr}}(\mathbf{y}')$ where $\mathbf{x}', \mathbf{y}'$ are extended generators as before, is identical to the computation we did for the case that $M(\rho^-) \neq M(\rho^+)$. Hence we have proved the proposition. □

Theorem 4.2.3(a) is an immediate corollary of Proposition 6.2.5.

The grading on $\widehat{CFD}(\mathcal{H})$

We start by recalling the definition of the module $\widehat{CDF}(\mathcal{H})$. For $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, let $\bar{o}(\mathbf{x}) = [2k] \setminus o(\mathbf{x})$ and define $I_D(\mathbf{x}) = I([2k] \setminus o(\mathbf{x}))$. We have a left action of the set of idempotents

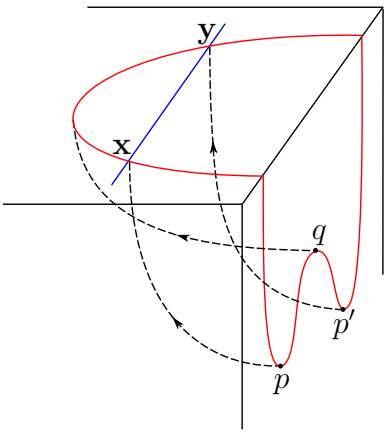


Figure 6.16: The modified Morse function h'.

$\mathcal{I}$ on $\mathfrak{G}(\mathcal{H})$ given by

$$I(\mathbf{s}) \cdot \mathbf{x} = \begin{cases} \mathbf{x}, & \text{if } I_D(\mathbf{x}) = I(\mathbf{s}), \\ 0, & \text{otherwise.} \end{cases}$$

The module $\widehat{CDF}(\mathcal{H})$ is generated over $\mathbb{Z}/2$ by the elements of the form $a \otimes \mathbf{x}$, where $a \in \mathcal{A}(-\mathcal{Z})$ and $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, and the tensor is taken over $\mathcal{I}$. Its module structure is given by the obvious left $\mathcal{A}(-\mathcal{Z})$-action.

We can define the grading gr on a generator $a(-\boldsymbol{\rho}) \otimes \mathbf{x}$ of $\widehat{CDF}(\mathcal{H})$ by

$$\mathrm{gr}(a(-\boldsymbol{\rho}) \otimes \mathbf{x}) := \mathrm{gr}(a(-\boldsymbol{\rho}) I_D(\mathbf{x})) \cdot \mathrm{gr}(\mathbf{x}).$$

The differential ∂ on $\widehat{CDF}(\mathcal{H})$ is defined in [23] by counting moduli spaces of holomorphic curves of the form $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\rho})$, where $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_n)$ is a sequence of Reeb chords. More precisely $\partial(I_D(\mathbf{x}) \otimes \mathbf{x})$ is a some of terms of the form $a(-\overrightarrow{\rho}) \otimes \mathbf{y}$, where $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\mathrm{ind}(B, \overrightarrow{\rho}) = 1$. Here $-\overrightarrow{\rho}$ denotes $(-\rho_1, \ldots, -\rho_n)$ and $a(-\overrightarrow{\rho})$ denotes the product $a(-\rho_1) \ldots a(-\rho_n)$.

Proposition 6.2.6. *Let $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\overrightarrow{\rho}$ such that $\partial^\partial B = [\overrightarrow{\rho}]$. If $a(-\overrightarrow{\rho}) \otimes \mathbf{y} \neq 0$, then*

$$\mathrm{gr}(a(-\overrightarrow{\rho}) I_D(\mathbf{y})) \cdot \mathrm{gr}(\mathbf{y}) = \lambda^{-\mathrm{ind}(B, \overrightarrow{\rho})} \mathrm{gr}(\mathbf{x}).$$

Proof. The proof is very to similar to that of Proposition 6.2.5. For the purposes of this calculation, we again group all the Reeb chords in $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_l)$ into one set $\boldsymbol{\rho}$ and assume that $\boldsymbol{\rho}$ contains no interleaved or abutting pairs, so $\mathrm{ind}(B, \overrightarrow{\rho}) - l = \mathrm{ind}(B, \boldsymbol{\rho}) - 1$. We again construct a closed manifold

$$Y' = \hat{Y} \cup_{\bar{F}} \cup \bar{F} \times [0, 1] \cup_{\bar{F}} \bar{Y}.$$

And we extend $\mathcal{H}$ to a Heegaard decomposition of Y' so that the new β-curves are translates of the Reeb chords. We again get generators $\mathbf{x}'$, $\mathbf{y}'$ of $\widehat{CF}(Y')$ and a homology class $B' \in \pi(\mathbf{x}', \mathbf{y}')$. So

$$\mathrm{ind}(B') = \mathrm{ind}(B, \boldsymbol{\rho}) - 1 + |\boldsymbol{\rho}| = \mathrm{ind}(B, \overrightarrow{\rho}) - l + l = \mathrm{ind}(B, \overrightarrow{\rho}).$$

Now the main difference in the calculation is that, when we compare the vector fields $\mathrm{gr}(a(-\overrightarrow{\rho})I_D(\mathbf{y})) \cdot \mathrm{gr}(\mathbf{y})$ and $\mathbb{I} \cdot \mathrm{gr}(\mathbf{x})$ in $(\bar{F} \times [0,1])$ where $\mathbb{I}$ is I-invariant, we obtain an arc with trivial framing for each Reeb chord. Therefore

$$\mathrm{gr}(\mathbf{x}) = \lambda^{\mathrm{ind}(B')} \cdot \mathrm{gr}(a(-\overrightarrow{\rho})I_D(\mathbf{y})) \cdot \mathrm{gr}(\mathbf{y}).$$

That implies our claim. □

We have therefore proven Theorem 4.2.3(b).

6.3 The pairing theorems

Our absolute grading is also compatible with the pairing theorems proved in [23]. More precisely, given two bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ for Y_1 and Y_2, respectively, with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$, we obtain a Heegaard diagram $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ for the closed manifold $Y := Y_1 \cup_\partial Y_2$. Let $F = \partial Y_1 = -\partial Y_2$ be the parameterized boundary.

Recall that the box tensor product $\widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$ is $\mathfrak{G}(\mathcal{H}_1) \otimes_{\mathcal{I}(\mathcal{Z})} \mathfrak{G}(\mathcal{H}_2)$ as a set. See [23, Def. 2.26] for the definition of the differential. If $\mathbf{x}_1 \in \mathfrak{G}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{G}(\mathcal{H}_2)$, such that $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$ is nonzero, then $\mathbf{x}_1$ and $\mathbf{x}_2$ must lie on complementary α-arcs. Therefore the pair $(\mathbf{x}_1, \mathbf{x}_2)$ corresponds to a generator of $\widehat{CF}(Y)$. So there is a canonical map

$$\Phi : \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2) \to \widehat{CF}(Y). \tag{6.9}$$

We recall the following theorem from [23].

Theorem 6.3.1 ([23, Thm. 1.3])**.** *The map* (6.9) *is a homotopy equivalence.*

Let $S(\mathcal{H}_1) \times_F S(\mathcal{H}_2)$ denote the set of elements of the form $([v_1], [v_2])$ with $[v_1] \in S(\mathcal{H}_1)$ and $[v_2] \in S(\mathcal{H}_2)$, such that $[v_1]$ and $[v_2]$ agree along F. Recall that $G(\mathcal{Z}_1) = G(-\mathcal{Z}_2)$ acts on $S(\mathcal{H}_1)$ on the right and on $S(\mathcal{H}_2)$ on the left. We now define $S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2)$ to be the quotient of $S(\mathcal{H}_1) \times_F S(\mathcal{H}_2)$ by the equivalence relation given by $(\xi_1 \cdot a, \xi_2) \sim (\xi_1, a \cdot \xi_2)$, where $\xi_i \in S(\mathcal{H}_i)$ for $i = 1, 2$ and $a \in G(\mathcal{Z}_1)$. Recall that the absolute grading on $\widehat{CF}(Y)$ takes value in $\mathrm{Vect}(Y)$. Now given nonvanishing vector fields v_1 in Y_1 and v_2 in Y_2, which agree along $\partial Y_1 = -\partial Y_2$, we obtain a vector field $v_1 \cdot v_2$ on Y by gluing along the boundary. Therefore we obtain a map

$$\Psi : S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2) \to \mathrm{Vect}(Y).$$

We have the following proposition.

Proposition 6.3.2. *The map Ψ is a bijection.*

Proof. To show that Ψ is surjective, let v be a nonvanishing vector field on Y and write $v = v_1 \cdot v_2$, where v_1 and v_2 are nonvanishing vector fields on Y_1 and Y_2, respectively. Now we fix a trivialization of TY, and hence a trivialization of $TY|_F$. By the Pontryagin-Thom construction, two maps $F \to S^2$ are isomorphic if, and only if, their pullback of the generator of $H^2(S^2;\mathbb{Z})$ coincide. We observe that the pullback map $\iota^* : H^2(Y_1,\mathbb{Z}) \to H^2(F)$ is trivial. Hence $v_1|_F$ is homotopic to the constant map $F \to S^2$. Now fix $\mathbf{s} \subset [0,2k]$, such that $|\mathbf{s}| = k$. Since we can extend $v_{\mathbf{s}}$ to a vector field in Y, it follows that $v_{\mathbf{s}}$ is again homotopic to the constant map. Therefore there exists a nonvanishing vector field u in $F \times [0,1]$ such that $u|_{F\times\{0\}} = v_1$ and $u|_{F\times\{1\}} = v_{\mathbf{s}}$. Let $\bar{u}$ denote the inverse of the homotopy determined by u. It follows that $v_1 \cdot u \cdot \bar{u} \cdot v_2$ is homotopic to $v_1 \cdot v_2$. So $\Psi([v_1 \cdot u] \otimes [\bar{u} \cdot v_2]) = [v_1 \cdot v_2]$. Hence Ψ is surjective.

Now let $[v_1], [w_1] \in S(\mathcal{H}_1)$ and $[v_2], [w_2] \in S(\mathcal{H}_2)$ such that $\Psi([v_1] \otimes [v_2]) = \Psi([w_1] \otimes [w_2])$. So $[v_1 \cdot v_2] = [w_1 \cdot w_2]$ as elements in $\mathrm{Vect}(Y)$. Let $H : Y \times [0,1]$ denote the homotopy from $v_1 \cdot v_2$ to $w_1 \cdot w_2$. Let u be the restriction of H to $F \times [0,1]$. So $u|_{F\times\{0\}} = v_1|_F$ and $u|_{F\times\{1\}} = w_1|_F$. We observe that $[v_1 \cdot u] = [w_1] \in S(\mathcal{H}_1)$ and that $[\bar{u} \cdot v_1] = [w_2] \in S(\mathcal{H}_2)$. So

$$[v_1] \otimes [v_2] = [v_1 \cdot u] \otimes [\bar{u} \cdot v_2] = [w_1] \otimes [w_2] \in S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2).$$

Therefore Ψ is injective. □

We can now prove that the map (6.9) preserves the absolute grading.

Theorem 6.3.3. *Given $\mathbf{x}_1 \in \mathfrak{G}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{G}(\mathcal{H}_2)$, such that $\mathbf{x}_1 \otimes \mathbf{x}_2 \neq 0$. Then*

$$\widetilde{\mathrm{gr}}(\Phi(\mathbf{x}_1 \otimes \mathbf{x}_2)) = \Psi(\mathrm{gr}(\mathbf{x}_1) \otimes \mathrm{gr}(\mathbf{x}_2)).$$

Proof. This follows immediately from our construction of the gradings in §3.3 and from the definition of the grading on Heegaard Floer homology. □

Chapter 7

The isomorphism between Heegaard Floer homology and ECH

In this chapter, we will prove that the absolute grading is preserved under the isomorphism map from Heegaard Floer homology to ECH defined by Colin-Ghiggini-Honda in [3].

We start by recalling the definition of the absolute grading on ECH from [10]. Let $\gamma = \{(\gamma_i, m_i)\}$ be an orbit set. The absolute grading $I(\gamma)$ is the homotopy class of nonvanishing vector fields obtained by modifying the Reeb vector field in disjoint neighborhoods of the Reeb orbits γ_i, as follows. For each i, choose a braid ζ_i with m_i strands around γ_i, such that the braids ζ_i belong to disjoint neighborhoods of γ_i. Let L be the union of z_i. A trivialization τ_i of ξ over each γ_i, induces a framing τ_i on each ζ_i. Let τ denote the framing on L. Now, for each component K of L, its framing induces a diffeomorphism $\phi_K : N_K \to S^1 \times D^2$ and a trivialization of TN_K, identifying $\xi = \{0\} \oplus \mathbb{R}^2$ and $R = (1, 0, 0)$. Using the previous identifications, one can define a vector field P on N_K as

$$\begin{aligned} P : S^1 \times D^2 &\to \mathbb{R} \oplus \mathbb{R}^2 \\ (t, re^{i\theta}) &= (-\cos(\pi r), \sin(\pi r)e^{-i\theta}). \end{aligned}$$

One can now construct a vector field by defining it to be given by P in each neighborhood N_K and by the Reeb vector field outside these neighborhoods. Let $P_\tau(L)$ to be be the homotopy class of this vector field. Now define

$$I(\gamma) = P_\tau(L) - \sum_i w_{\tau_i}(\zeta_i) + CZ^I_\tau(\gamma),$$

Here $w_{\tau_i}(\zeta_i)$ denotes the writhe of ζ_i with respect to τ_i and

$$CZ^I_\tau(\gamma) = \sum_i \sum_{j=1}^{m_i} CZ_\tau(\gamma_i^k),$$

where $CZ_\tau(\gamma_i^k)$ is the Conley-Zehnder index of γ_i^k with respect to τ. Hutchings showed that $I(\gamma)$ does not depend on the choice of τ or L, and that this absolute grading refines the relative grading defined by the ECH index.

The isomorphism $\widehat{HF}(-Y) \cong \widehat{ECH}(Y)$ is given by a sequence of maps that we will now recall from [3]. First, we fix an open book decomposition (S, h) for Y. That is, we choose a surface S with boundary and a diffeomorphism h of S, which is the identity near the boundary. Then $Y = (S \times [0,1])/\sim$, where $(x,1) \sim (h(x),0)$ for every $x \in S$ and $(x,t) \sim (x,t')$ for every $x \in \partial S$ and for every $t \in [0,1]$. We will assume that S and ∂S are connected. Moreover, by [3, Lemma 2.1.1], we can assume that there exists a diffeomorphism of a neighborhood of ∂S in S to $[-\varepsilon, 0] \times \partial S$ such that the monodromy h is given by $h(y,\theta) = (y, \theta - y)$ in this neighborhood. Such an open book decomposition gives rise to a Heegaard decomposition, where the two handlebodies are given by $H_1 = (S \times [0,1/2])/\sim$ and $H_2 = (S \times [1/2,1])/\sim$. The Heegaard surface is $\Sigma := S_{1/2} \cup -S_0$, where S_t denotes $S \times \{t\}$. This surface has even genus, which we denote by $2g$. We choose a set of properly embedded arcs $\mathbf{a} = \{a_1, \dots, a_{2g}\}$ of S such that $S \setminus \mathbf{a}$ is a disk. One can then let $\alpha_i = a_i^\dagger \cup a_i$, where a_i is seen as an arc in S_0 and $a_i^\dagger$ is its copy in $S_{1/2}$. One also lets $\beta_i = \beta_i^\dagger \cup h(a_i)$, where $b_i^\dagger$ is the simplest arc in $S_{1/2}$ which is isotopic to $a_i^\dagger$ and extends $h(a_i)$ to a smooth curve in Σ. For a picture, see [3, Fig. 1]. We fix z on the binding away from the curves γ_i. Hence $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ is a Heegaard diagram for $-Y$. Let f be a Morse function and X a gradient-like vector field such that the pair (f, X) is compatible with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Then $(-f, -X)$ is compatible with $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.

For each i, there are three intersections of $a_i^\dagger$ with $b_i^\dagger$ in $S_{1/2}$. Following [3], we label them x_i, x_i', x_i'', where x_i'' is the only one in the interior of $S_{1/2}$. One can define $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$ to be the subcomplex of $\widehat{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ generated by $2g$-tuples of intersection points contained in S_0. One also defines $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ to be $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))/\sim$, where two $2g$-tuples of intersection points in S_0 are equivalent if they differ by substituting x_i by x_i'. There is an induced differential on $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$. It is shown in [3, Theorem 4.9.4] that the homology of this chain complex is isomorphic to $\widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$. We also get an induced grading on $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$, as the following lemma shows.

Lemma 7.0.4. *If $\mathbf{x}$ is a $2g$-tuples of intersection points in S_0 containing x_i and $\mathbf{x}' = (\mathbf{x} \setminus \{x_i\}) \cup \{x_i'\}$, then $gr(\mathbf{x}) = gr(\mathbf{x}')$. Therefore gr is well-defined on the quotient $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$.*

Proof. We observe that $\mathbf{x}$ and $\mathbf{x}'$ are in the same Spin^c structure and that $\mathrm{gr}(\mathbf{x}, \mathbf{x}') = 0$. Therefore, since the absolute grading refines the relative grading Theorem 4.1.1,

$$\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{x}') = 0.$$

□

We now explain how to obtain $\widehat{ECH}(Y)$ using the open book decomposition. For more details, see [2]. Let $N = (S \times [0,1])/\sim$, where $(x,1) \sim (h(x),0)$, i.e. Y is obtained from N by collapsing the S^1 direction along the boundary of the pages. We choose a contact form λ on N such that the Reeb vector field R_λ is positively transverse to $S \times \{t\}$. One can assume that R_λ is parallel to ∂_t on S. Hence the torus ∂N is foliated by Reeb orbits.

After a Morse-Bott perturbation, we obtain a pair of Reeb orbits $\{e, h\}$ on ∂N. One now defines $ECC_j(N, \lambda)$ to be the chain complex generated by orbit sets whose homology class intersects $S \times \{t\}$ exactly j times. The inclusions $ECC_j(N, \lambda) \to ECC_{j+1}(N, \lambda)$ are given by the map $\gamma \mapsto e\gamma$. In [2], it is proven that

$$\widehat{ECH}(Y) \cong \lim_{j\to\infty} ECH_j(N, \lambda).$$

In order to construct the isomorphism $\widehat{HF}(-Y) \cong \widehat{ECH}(Y)$, Colin, Ghiggini and Honda first choose a stable Hamiltonian structure (λ_0, ω) on N and define a map from $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ to a periodic Floer chain complex for a stable Hamiltonian structure, as follows. Let t denote the $[0, 1]$-coordinate, $\lambda_0 := Cdt$, for a constant $C > 0$ and let ω be an area form on S. The pair (λ_0, ω) is a stable Hamiltonian structure on N. It is still defines a Reeb vector field which is parallel to ∂_t. Then one can construct periodic Floer chain complexes $PFC_j(N, \lambda_0, \omega)$ generated by orbit sets, just as ECH for a contact form. It is shown in [3, §3.1] that for if C is large enough, then we can find a smooth family of 1-forms $\{\lambda_\tau\}_{\tau\in[0,1]}$ such that λ_τ is a contact form for every $\tau > 0$ and $\lambda_1 = \lambda$. In [3, §3.6], it is shown that[1] given $j > 0$, there exists τ_0, such that for all $0 < \tau \leq \tau_0$,

$$PFC_j(N, \lambda_0) \cong ECC_j(N, \lambda_\tau).$$

The next step is the construction of a map $\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \to PFC_{2g}(N, \lambda_0)$. This map is constructed by counting certain holomorphic curves, as we now review. Let $\pi_{S^1} : N \to S^1$ denote the projection $(x, t) \mapsto t$ and let $B' := \mathbb{R} \times S^1$. Now let $\pi_{B'} : \mathbb{R} \times N \to B'$ be the map $(s, x, t) \mapsto (s, \pi_{S^1}(x, t))$ and let $B_+ := B \setminus ((0, \infty) \times (1/2, 1))$ with the corners rounded. Now define $W_+ = \pi_{B'}(B_+)$ and $\Omega_+ = ds \wedge dt + \omega$. Then (W_+, Ω_+) is a symplectic manifold with boundary and ends $S \times [0, 1/2]$ and N. The symplectic fibration $\pi_{B_+} : (W_+, \Omega_+) \to (B_+, ds \wedge dt)$ admits a symplectic connection. Now if we take a copy of $\boldsymbol{\alpha}$ on the fiber $\pi_{B_+}^{-1}(1, 1/2)$ and take its parallel transport along ∂B_+, we obtain a Lagrangian submanifold of (W_+, Ω_+), which is denoted by $L^+_{\boldsymbol{\alpha}}$. The reason for the $+$-subscript in all those objects is that one can also define simlar objects that are used to construct a map $PFC_{2g}(N, \lambda_0) \to \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$. But we will not need to use this map.

Now to each generator $\mathbf{x}$ of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$, we can associate a subset of $S \times [0, 1/2]$ given by the union of $x_i \times [0, 1/2]$, for all $x_i \in \mathbf{x}$. We will still denote this union of Reeb chords by $\mathbf{x}$. Given $\mathbf{x}$, a generator γ of $PFC_{2g}(N, \lambda_0)$ and an admissible[2] almost-complex structure J_+, one defines $\mathcal{M}_{J_+}(\mathbf{x}, \gamma)$ to be the moduli space of J_+-holomorphic maps $u : (\dot{F}, j) \to (W_+, J_+)$, where $(\dot{F}, j)$ is a Riemmann surface with boundary and punctures, both in the interior and on the boundary, satisfying the following conditions:

- $u(\partial\dot{F}) \subset L^+_{\mathbf{a}}$ and each component of $\partial\dot{F}$ is mapped to a different $L^+_{a_i}$.

[1] We must also fix a family of almost complex structures, but we will omit this from the notation, since it has no effect in our grading computations.

[2] We do note need to use the definition of admissibility. We refer the reader to [3, §5.4].

- The boundary punctures are positive[3] and the interior punctures are negative.
- At a boundary puncture, u converges to $\mathbf{x} \times [0, 1/2]$.
- At an interior puncture, u converges[4] to γ.
- The energy of u is bounded.

For a map $u : (\dot{F}, j) \to (W_+, J_+)$ in $\mathcal{M}_{J_+}(\mathbf{x}, \gamma)$, one defines its ECH-index $I(u)$ as follows. First let $\breve{W}_+$ be the compactification of W_+. We can construct $\breve{W}_+$ by taking the intersection of W_+ with $[-R, R] \times N$ for $R >> 0$. We choose an orientation of the arcs a_i and that gives rise to a trivialization τ of TS along $L^+_{\mathbf{a}} \cap \breve{W}_+$. We extend this trivialization arbitrarily along $\{R\} \times \mathbf{x} \times [0, 1/2]$ and along $\{-R\} \times \gamma$. Then $(\tau, \partial_t$ gives rise to a trivialization of $T\breve{W}_+$ along $\partial\breve{W}_+$. Let $c_1(u^*T\breve{W}_+, (\tau, \partial_t))$ denote the first Chern class of the complex bundle $u^*T\breve{W}_+$ relative to (τ, ∂_t). In other words, if we take two generic complex sections of $u^*T\breve{W}_+$ which are trivial and linearly independent on $\partial\breve{W}_+$ with respect to (τ, ∂_t), then $c_1(u^*T\breve{W}_+, (\tau, \partial_t))$ is a signed count of the points where the sections are linearly dependent. For each γ_i, the intersection $u(\dot{F}) \cap (\{-R\} \times N)$ is a braid around γ_i with m_i strands for $R >> 0$. The writhe of this braid with respect to τ is independent of R, for R sufficiently large. Then one defines $w^-_\tau(u)$ to be the sum of the writhes of all the braids corresponding to each γ_i. Now let $\mathcal{L}_0$ be a real rank one subbundle of TS along $\mathbf{x} \times [0, 1/2]$ defined as follows. At $\mathbf{x} \times \{0\}$, let $\mathcal{L}_0 = Th(\mathbf{a})$ and at $\mathbf{x} \times \{1/2\}$, let $\mathcal{L}_0 = T\mathbf{a}$ in TS. Then $\mathcal{L}_0$ is defined by rotating counterclockwise by the minimum possible amount as we travel along $\mathbf{x} \times [0, 1/2]$. Then one defines $\mu_\tau(\mathbf{x})$ to be the sum of the Maslov indices of $\mathcal{L}_0$ along each $x_i \times [0, 1/2]$. Finally, let $\delta(u) \geq 0$ denote the number of singularities[5] of u. Now define

$$I(u) = -\chi(\dot{F}) + 2c_1(u^*T\breve{W}_+, (\tau, \partial_t)) + w^-_\tau(u) + \mu_\tau(\mathbf{x}) - CZ^I_\tau(\gamma) - 2g + 2\delta(u).$$

This follows from [3, Def. 5.6.5] combined with an adjunction formula [3, Lemma 5.6.3].

Finally $\langle \Phi(\mathbf{x}), \gamma \rangle$ is defined to be the signed count of maps u in $\mathcal{M}_{J_+}(\mathbf{x}, \gamma)$ with $I(u) = 0$. It turns out that for a sufficiently generic J_+, this is well-defined and all the maps that are counted are embeddings.

In order to prove that the isomorphism between Heegaard Floer homology and ECH preserves the absolute gradings, it is enough to show the following proposition.

Proposition 7.0.5. *If $u : (\dot{F}, j) \to (W_+, J_+)$ be a map in $\mathcal{M}_{J_+}(\mathbf{x}, \gamma)$, where $\mathbf{x}$ and γ be generators of $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$ and $PFC_{2g}(N, \lambda_0, \omega)$, respectively. Assume that u is an immersion. Then*

$$gr(\mathbf{x}) - I(\gamma) = I(u).$$

[3] A positive puncture p is a point such that the s-coordinate of $u(x)$ converges to ∞ as $x \to p$. A negative puncture is defined analogously.

[4] This convergence is in the sense of currents. That is, for each interior puncture, u converges to a cylinder over a Reeb orbit γ_j with a certain multiplicity. The sum of all the multiplicities of all these cylinders corresponding to a fixed γ_j is required to be m_j.

[5] The number $\delta(u)$ is zero precisely when u is embedded.

Before proving Proposition 7.0.5, we will prove a simple lemma. Let $\mathbf{x}$ be a generator of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$. Let L be a link given by the union of the compactified gradient trajectories corresponding to each intersection point in $\mathbf{x}$ and to each x_i'. We fix a vector field V in the homotopy class $\mathrm{gr}(\mathbf{x})$. Then on the boundary of a small enough neighborhood H of L, the vector field V is positively transverse to ξ. So by a homotopy, we can assume that $V = R_\lambda$ on ∂H.

Lemma 7.0.6. *Let $\mathbf{x}$ be a generator of $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$. Let L, H and V be as in the paragraph above. Then $V|_{Y\setminus H}$ is homotopic to R_λ relative to ∂H.*

Proof. The proof of this lemma is a slight modification of the proof of Theorem 4.1.1(b). □

Proof of Proposition 7.0.5. We write $\gamma = \{(\gamma_i, m_i)\}$. We first note that by rounding the corners of $\breve{W}_+$, we obtain a trivial cobordism from N to itself, which we denote by X. Here we identify $X \simeq N \times [0,1]$. So we can see $u(\dot{F})$ as a (non-necessarilly embedded) cobordism $\breve{F}$ from $\mathbf{x}'$ to a union L of braids ζ_i around γ_i with total multiplicity m_i, where $\mathbf{x}'$ is the union of $\mathbf{x}$ with segments on the arcs a_i. Up to a small isotopy, we can assume that $\mathbf{x}' \times \{1\}$ is transverse to $S \times \{t\} \times \{1\}$.

We will now use the relative Pontryagin-Thom construction. We first choose a nonvanishing tangent vector field along the arcs a_i. That induces a trivialization of $TS|_{L^+_{\mathbf{a}}}$. We extend it arbitrarily to the Reeb chords $\mathbf{x}$. We also choose a trivialization of $TS|_L$. We can now extend this trivialization to a small neighborhoods of $\mathbf{x}'$ and L in N. We denote this trivialization of TS on these two neighborhoods by τ. Therefore (τ, ∂_t) gives rise to a trivialization of TN in these neighborhoods. Now we evoke the Pontryagin-Thom construction. Using τ, taking the preimage of $(0,0,1)$ under an appropriate choice[6] of vector fields in the homotopy classes $\mathrm{gr}(\mathbf{x})$ and $P_\tau(L)$, we obtain $\mathbf{x}'$ in $N \times \{1\}$ and L in $N \times \{0\}$. Moreover, we also get framings on $\mathbf{x}'$ and L, which we denote by ν. We claim that

$$\mathrm{gr}(\mathbf{x}) - P_\tau(L) = c_1(N\breve{F}, \nu) + c_1(TS|_{\breve{F}}) + 2\delta(\breve{F}). \tag{7.1}$$

To see that, first extend X to $N \times [0, 1+\varepsilon]$ and glue a cylinder $\hat{F} := \mathbf{x}' \times [1, 1+\varepsilon]$ to $\breve{F}$. We denote the resulting surface by $\breve{F}'$. Let τ' be a trivialization of $TS|_{\breve{F}'}$, which coincides with τ over $L \times \{0\}$. Then τ' induces a trivialization of $TS|_{\mathbf{x}' \times \{1+\varepsilon\}}$. Using τ' as above, we obtain a framing ν' on $\mathbf{x}' \times \{1+\varepsilon\} \cup L \times \{0\}$. By the Pontryagin-Thom construction for immersed surfaces

$$\mathrm{gr}(\mathbf{x}) - P_\tau(L) = c_1(N\breve{F}', \nu') + 2\delta(\breve{F}') = c_1(N\breve{F}, \nu) + (\nu'_{\mathbf{x}'} - \nu_{\mathbf{x}'}) + 2\delta(\breve{F}). \tag{7.2}$$

We note that $\nu'_{\mathbf{x}'} - \nu_{\mathbf{x}'} = -c_1(TS|_{\hat{F}}, (\tau, \tau'))$, where (τ, τ') is the trivialization of $TS|_{\partial\hat{F}}$ given by τ on $\mathbf{x}' \times \{1\}$ and by τ' in $\mathbf{x}' \times \{1+\varepsilon\}$. Moreover, since τ' is a trivialization of TS over $\breve{F}'$, it follows that $c_1(TS|_{\breve{F}'}, \tau') = 0$. So

$$c_1(TS_{\breve{F}}, \tau) = -c_1(TS|_{\hat{F}}, (\tau, \tau')) = \nu'_{\mathbf{x}'} - \nu_{\mathbf{x}'}. \tag{7.3}$$

[6]The vector field we used to define these homotopy classes gives the desired preimage, up to a small isotopy.

So combining (7.2) and (7.3), we obtain (7.1).

We claim that $c_1(N\breve{F},\nu) = c_1(N\breve{F},\tau) + \mu_\tau(\mathbf{x}) - 2g$. In fact, the difference $c_1(N\breve{F},\nu) - c_1(N\breve{F};\tau)$ is given by $(\nu^+ - \tau^+) - (\nu^- - \tau^-)$. Here ν^+ and ν^- denote the restrictions of ν to $\mathbf{x}'$ and L, respectively. One defines τ^+ and τ^- analogously. It follows from calculations in [10] that $\nu^- - \tau^- = 0$. We will now compute the difference $\nu^+ - \tau^+$. Define $(\nu^+)_x$ to be $(\nu^+)^{-1}(1,0)$ and define $(\tau^+)_x$ similarly. We note that $(\nu^+)_x = (\tau^+)_x$ in $\pi^{-1}([-\varepsilon, 1/2-\varepsilon])$ and $(\nu^+)_x = -(\tau^+)_x$ in $\pi^{-1}([1/2+\varepsilon, 1-\varepsilon])$. We also observe that, along each strand, $(\nu^+)_x$ does half a turn clockwise with respect to $(\tau^+)_x$ in $\pi^{-1}([1/2-\varepsilon, 1/2+\varepsilon])$. Now to see how many times $(\nu^+)_x$ rotates with respect to $(\tau^+)_x$ in $\pi^{-1}([-\varepsilon,\varepsilon])$, first assume that, along each strand, the projection of $(\tau^+)_x$ to S rotates a quarter turn counterclockwise as we go from $-\varepsilon$ to ε. Then $(\nu^+)_x$ does half a turn clockwise with respect to $(\tau^+)_x$. Moreover $\mu_\tau(\mathbf{x}) = 0$. Now if we change τ^+ along a strand in $\pi^{-1}([-\varepsilon,\varepsilon])$, the difference $\nu^+ - \tau^+$ will change by the same amount as $\mu_\tau(\mathbf{x})$. Therefore $\nu^+ - \tau^+ = \mu_\tau(\mathbf{x}) - 2g$. Hence

$$\mathrm{gr}(\mathbf{x}) - P_\tau(L) = \mu_\tau(\mathbf{x}) - 2g + c_1(N\breve{F},\tau) + c_1(TS|_{\breve{F}},(\tau,\partial_t)).$$

We observe that

$$\begin{aligned} c_1(TS|_{\breve{F}},\tau) = c_1(u^*TX,(\tau,\partial_t)) &= c_1(T\breve{F},\partial_t) + c_1(N\breve{F},\tau) \\ &= \chi(\breve{F}) + c_1(N\breve{F},\tau). \end{aligned}$$

So

$$\mathrm{gr}(\mathbf{x}) - P_\tau(L) = -\chi(\dot{F}) + 2c_1(u^*TX,(\tau,\partial_t)) + \mu_\tau(\mathbf{x}) - 2g + 2\delta(\breve{F}).$$

We recall $I(\gamma) = P_\tau(L) - \sum w_{\tau_i}(\zeta_i) + \mu_\tau(\gamma)$. Therefore

$$\mathrm{gr}(\mathbf{x}) - I(\mathbf{x}) = -\chi(\dot{F}) + 2c_1(u^*T\breve{W},(\tau,\partial_t)) + w_\tau^-(u) + \mu_\tau(\mathbf{x}) - CZ_\tau^I(\gamma) - 2g + 2\delta(u)$$

So the result follows, since the right hand side equals $I(u)$. □

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