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Isoperimetry and noise sensitivity in Gaussian space

by

Joseph Oliver Neeman

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

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Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Elchanan Mossel, Chair
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Joseph Oliver Neeman

Abstract

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Professor Elchanan Mossel, Chair

We study two kinds of extremal subsets of Gaussian space: sets which minimize the surface area, and sets which minimize the noise sensitivity. For both problems, affine half-spaces are extremal: in the case of surface area, this was discovered independently by Borell and by Sudakov and Tsirelson in the 1970s; for noise sensitivity, it is due to Borell in the 1980s. We give a self-contained treatment of these two subjects, using semigroup methods. For Gaussian surface area, these methods were developed by Bakry and Ledoux, but their application to noise sensitivity is new.

Compared to other approaches to these two problems, the semigroup method has the advantage of giving accurate characterizations of the extremal and near-extremal sets. We review the Carlen-Kerce argument showing that (up to null sets) half-spaces are the unique minimizers of Gaussian surface area. We then give an analogous argument for noise sensitivity, proving that half-spaces are also the unique minimizers of noise sensitivity. Unlike the case of Gaussian isoperimetry, not even a partial characterization of the minimizers of noise sensitivity was previously known. After characterizing the extremal sets, we study near-extremal sets. For both surface area and noise sensitivity, we show that near-extremal sets must be close to half-spaces. Our bounds are dimension-independent, but they are not sharp.

Finally, we discuss some applications of noise sensitivity: in economics, we characterize the extremal voting methods in Kalai's quantitative version of Arrow's impossibility theorem. In computer science, we characterize the optimal rounding methods in Goemans and Williamson's semidefinite relaxation of the Max-Cut problem.

Contents

Contents	i
1 Gaussian isoperimetry	1
1.1 The Gaussian isoperimetric inequality	1
1.2 Bobkov’s inequality	4
1.3 The Ornstein-Uhlenbeck semigroup	5
1.4 A semigroup proof of Bobkov’s inequality	12
1.5 Deforming Ehrhard sets	13
1.6 The equality cases in Bobkov’s inequality	16
2 Gaussian noise stability	17
2.1 Gaussian noise stability	17
2.2 The connection to Gaussian isoperimetry	19
2.3 A functional version of Borell’s inequality	23
2.4 A semigroup proof of Borell’s inequality	24
2.5 The equality cases in Borell’s inequality	26
2.6 The exchangeable Gaussians inequality	26
3 Robust Gaussian isoperimetry	33
3.1 Approximation for large t	35
3.2 Approximation for small t	43
4 Robust Gaussian noise stability	48
4.1 Approximation for large t	52
4.2 Approximation for small t	57
4.3 Optimal dependence on ρ	64
5 Applications of Gaussian noise stability	74
5.1 Boolean noise stability and Arrow’s theorem	74
5.2 Spherical noise stability and Max-Cut rounding	85
References	94

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Table of notation

γ_n	the standard Gaussian measure on \mathbb{R}^n , with density $(2\pi)^{-n/2}e^{- x ^2/2}$
A_r	the r -enlargement of the set $A \subset \mathbb{R}^n$: $A_r = \{x \in \mathbb{R}^n : d(x, A) < r\}$
γ_n^+	the Minkowski content associated to γ_n :

$$\gamma_n^+(A) = \liminf_{r \rightarrow 0^+} \frac{\gamma_n(A_r) - \gamma_n(A)}{r}$$

ϕ	the density of γ_1 : $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
Φ	the cumulative distribution function of ϕ : $\Phi(x) = \int_{-\infty}^x \phi(y) dy$
I	the isoperimetric profile of (\mathbb{R}^n, γ_n) : $I(x) = \phi(\Phi^{-1}(x))$
P_t	the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma_n(y)$$

L	the infinitesimal generator of P_t :
-----	--

$$Lf = \Delta f - \langle x, \nabla f \rangle = \left. \frac{d}{dt} \right|_{t=0} P_t f$$

H_α	the normalized Hermite polynomial with multiindex $\alpha \in \{0, 1, \dots, \}^n$
$\text{Hess}(f)$	the Hessian matrix of the function f
$\ A\ _F$	the Frobenius norm of the matrix A : $\ A\ _F^2 = \text{tr}(A^T A)$
L_t	$\frac{e^{-t}}{\sqrt{1 - e^{-2t}}}$
Pr_ρ	the probability distribution under which $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$ is distributed as a mean-zero Gaussian vector with covariance matrix $\begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}$
J_ρ	the function $[0, 1]^2 \rightarrow [0, 1]$ defined by $J_\rho(x, y) = \text{Pr}_\rho(X_1 \leq \Phi^{-1}(x), Y_1 \leq \Phi^{-1}(y))$
f_t, g_t	$P_t f, P_t g$
v_t, w_t	$\Phi^{-1} \circ f_t, \Phi^{-1} \circ g_t$ (when the range of f and g lies in $[0, 1]$)
$m(f)$	$\mathbb{E}f(1 - \mathbb{E}f)$
δ	the deficit in a context-dependent inequality: in Chapter 3,

$$\delta(f) = \mathbb{E}\sqrt{I(f) + |\nabla f|^2} - I(\mathbb{E}f),$$

while in Chapter 4,

$$\delta(f, g) = J_\rho(\mathbb{E}f, \mathbb{E}g) - \mathbb{E}_\rho J_\rho(f(X), g(Y))$$

Chapter 1

Gaussian isoperimetry

An isoperimetric inequality is a lower bound on the size of a set's boundary in terms of its volume. The oldest and most famous isoperimetric inequality is the 2-dimensional Euclidean isoperimetric inequality, which says that a bounded subset of \mathbb{R}^2 has a longer boundary than the disk of the same volume [44]. Although not proven rigorously until the late 19th century, this fact was known to the ancient Greeks, and possibly also to Queen Dido of Carthage almost three thousand years ago; for the history of the isoperimetric inequality, see, for example, section 2.2 of [44]).

Since the notions of volume and surface area are not confined to Euclidean space, one can study isoperimetric problems in different spaces; perhaps the first example of this is Lévy's proof that spherical caps have minimal surface area among all subsets of the sphere [37] with a given volume. Here, we will take a slightly different direction: back in Euclidean space, we consider the isoperimetric problem under the Gaussian measure instead of the Lebesgue measure.

The standard Gaussian measure on \mathbb{R}^n is the probability measure on \mathbb{R}^n whose density is $(2\pi)^{-n/2}e^{-|x|^2/2}$, where $|\cdot|$ denotes the Euclidean norm. We will denote this measure by γ_n . It is a central object in probability and we will assume some familiarity with its basic properties.

1.1 The Gaussian isoperimetric inequality

Gaussian surface area

For $r > 0$ and a set $A \subset \mathbb{R}^n$, let A_r denote the enlargement $\{x \in \mathbb{R}^n : d(x, A) < r\}$ where $d(x, A) = \inf\{|x - y| : y \in A\}$. We define the Gaussian surface area of A (denoted $\gamma_n^+(A)$) by Minkowski's formula [18]

$$\gamma_n^+(A) = \liminf_{r \rightarrow 0^+} \frac{\gamma_n(A_r) - \gamma_n(A)}{r}.$$

The Gaussian isoperimetric inequality – the main object of study for this chapter – is a

lower bound on $\gamma_n^+(A)$ in terms of $\gamma_n(A)$. In order to state it, define the functions

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ \Phi(x) &= \int_{-\infty}^x \phi(y) dy.\end{aligned}$$

Recall that ϕ is the density of γ_1 and so $\Phi(b) = \gamma_1((-\infty, b])$. Note that Φ is a strictly increasing function that maps \mathbb{R} to $(0, 1)$ and so it has a well-defined inverse $\Phi^{-1} : (0, 1) \rightarrow \mathbb{R}$. For notational convenience, we will extend Φ^{-1} to $[0, 1]$ by declaring that $\Phi^{-1}(0) = -\infty$ and $\Phi^{-1}(1) = \infty$. Similarly, we set $\phi(\pm\infty) = 0$.

The Gaussian isoperimetric inequality is a sharp lower bound on the Gaussian surface area of A in terms of its volume. It was proved independently, and with essentially the same method, by Borell [10] and by Sudakov and Tsirelson [47]. Since then, several other proofs have appeared, including a symmetrization proof by Ehrhard [16] and a proof, by Bobkov [6], using a tensorized inequality on the discrete cube.

Theorem 1.1 (Gaussian isoperimetric inequality). *For any measurable $A \subset \mathbb{R}^n$,*

$$\gamma_n^+(A) \geq \phi(\Phi^{-1}(\gamma_n(A))). \quad (1.1)$$

Following the usual notation in the literature, we will define the function $I(x) = \phi(\Phi^{-1}(x))$ so that (1.1) may be written as $\gamma_n^+(A) \geq I(\gamma_n(A))$. The function I is known as the “isoperimetric profile” of γ_n . Note that we have so far defined I on the interval $(0, 1)$, but it may be extended continuously to $[0, 1]$ by setting $I(0) = I(1) = 0$.

As with any other isoperimetric inequality, the inequality (1.1) is only half of the story; the other half comes from the equality cases. In the Gaussian case, equality is attained for half-spaces (i.e. sets of the form $\{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$). This is quite straightforward to check: first of all, the rotational invariance of γ_n implies that it suffices to consider only half-spaces of the form $A = \{x \in \mathbb{R}^n : x_1 \leq b\}$. Since γ_n is a product measure, we have $\gamma_n(A) = \gamma_1((-\infty, b]) = \Phi(b)$, and $\gamma_n^+(A) = \gamma_1^+((-\infty, b]) = \phi(b)$. Hence $\gamma_n^+(A) = \phi(b) = \phi(\Phi^{-1}(\Phi(b))) = \phi(\Phi^{-1}(\gamma_n(A)))$ and so equality is attained in (1.1).

The isoperimetric inequality (1.1) may therefore be restated as follows: for any measurable $A \subset \mathbb{R}^n$, the Gaussian surface area of A is at least as large as the Gaussian surface area of a half-space with the same volume as A . This restatement can be compared to the analogous – and perhaps more familiar – statement in Euclidean space: for all measurable $A \subset \mathbb{R}^n$, the (Lebesgue) surface area of A is at least as large as the (Lebesgue) surface area of a Euclidean ball with the same (Lebesgue) volume as A .

While it was easy to check that half-spaces achieve equality in (1.1), checking that half-spaces are the *only* sets that achieve equality is non-trivial. For sufficiently nice sets (namely, those which are the closures of their interior), this was proved by Ehrhard [15]; the general case was only solved in 2001 by Carlen and Kerce [11]:

Theorem 1.2. *If $A \subset \mathbb{R}^n$ is a measurable set with $\gamma_n^+(A) = I(\gamma_n(A))$ then A is a half-space (up to a set of measure zero).*

Let us also mention a non-infinitesimal version of Theorem 1.1.

Corollary 1.3. *For any measurable $A \subset \mathbb{R}^n$ and any $r > 0$,*

$$\gamma_n(A_r) \geq \Phi(\Phi^{-1}(\gamma_n(A)) + r).$$

Note that for the half-space $A = \{x : x_1 \leq a\}$ (which has measure $\Phi(a)$), we have $A_r = \{x : x_1 \leq a+r\}$ (which has measure $\Phi(a+r)$). Hence, half-spaces attain equality in Corollary 1.3.

Proof. Consider the function $F(s) = \Phi^{-1}(\gamma_n(A_r))$. Our goal is then to show that $F(r) \geq F(0) + r$; applying Φ to both sides of this inequality will then prove the claim. Now, the inverse function formula yields $\frac{d\Phi^{-1}(x)}{dx} = \frac{1}{\phi(\Phi^{-1}(x))} = \frac{1}{I(x)}$. Hence (by the chain rule),

$$\liminf_{h \downarrow 0} \frac{F(s+h) - F(s)}{h} = \frac{1}{I(\gamma_n(A_s))} \liminf_{h \downarrow 0} \frac{\gamma_n(A_{r+h}) - \gamma_n(A_r)}{h} = \frac{\gamma_n^+(A_r)}{I(\gamma_n(A_r))}.$$

By Theorem 1.1, the right hand side is at least 1. Then, by the Lebesgue differentiation theorem,

$$F(r) - F(0) \geq \int_0^r \liminf_{h \downarrow 0} \frac{F(s+h) - F(s)}{h} ds \geq r. \quad \square$$

We will prove Theorems 1.1 and 1.2 later in this chapter, but first we will give a simple application. This application serves a dual purpose: on the one hand, it provides a probabilistic view of Theorem 1.1 which is perhaps not obvious from (1.1) because it may not be clear how to interpret γ_n^+ probabilistically. On the other hand, our application is genuinely useful consequence of isoperimetry, and it has been widely applied in probability theory. Indeed, we will use it ourselves in Chapters 3 and 4.

Theorem 1.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an L -Lipschitz function and let M be its median with respect to γ_n . Then for any $t \geq 0$,*

$$\gamma_n(\{x : f(x) \geq M + t\}) \leq \Phi(-t/L) \leq e^{-\frac{t^2}{2L^2}}.$$

In other words, every Lipschitz function on \mathbb{R}^n is concentrated around its median. This turns out to be a very useful feature of Gaussian measures (and some other measures too), with applications in geometry and probability. A general discussion of concentration, with applications to various areas in mathematics, can be found in the book by Ledoux [35].

Proof of Theorem 1.4. Let $A = \{x : f(x) \leq M\}$; by the definition of the median, $\gamma_n(A) = 1/2$. Since f is L -Lipschitz, $|f(x) - M| \leq Ld(x, A)$ and so $A_{t/L} \subset \{x : f(x) \leq M + t\}$. By Corollary 1.3, and since $\Phi^{-1}(\gamma_n(A)) = \Phi^{-1}(1/2) = 0$,

$$\gamma_n(\{x : f(x) \leq M + t\}) \geq \gamma_n(A_{t/L}) \geq \Phi(t/L).$$

This proves the first claimed inequality; the second is the usual approximation of Gaussian tails. \square

1.2 Bobkov's inequality

Like many other inequalities about sets, the Gaussian isoperimetric inequality has an equivalent functional version. Although the functional version was already known to Ehrhard in the 1980s [16], it was brought to prominence more recently by Bobkov [7] who used it to give a new proof of the Gaussian isoperimetric inequality.

For the sake of concision, we will switch to a more probabilistic notation: let X be a random variable distributed according to γ_n and denote the expectation operator by \mathbb{E} ; that is, $\mathbb{E}f(X) = \int_{\mathbb{R}^n} f(x) d\gamma_n(x)$. We will sometimes omit the X and just write $\mathbb{E}f$.

Theorem 1.5 (Bobkov's inequality). *For any smooth $f : \mathbb{R}^n \rightarrow [0, 1]$,*

$$I(\mathbb{E}f) \leq \mathbb{E}\sqrt{I^2(f(X)) + |\nabla f(X)|^2}. \quad (1.2)$$

Before proceeding further, let us show how Theorems 1.1 and 1.5 are equivalent. One direction is easy to see intuitively: for any set A , define $f(x) = 1_A(x)$ (i.e., $f(x) = 1$ if $x \in A$ and 0 otherwise). Since $I(0) = I(1) = 0$, (1.2) reduces to $I(\mathbb{E}f) \leq \mathbb{E}|\nabla f|$. Now, $I(\mathbb{E}f) = \phi(\Phi^{-1}(\gamma_n(A)))$ and $\mathbb{E}|\nabla f|$ is intuitively the boundary measure of A because $|\nabla f|$ is a δ -function on the boundary of A . Hence, we recover Theorem 1.1 from Theorem 1.5.

In order to make this argument rigorous, we need a better way of relating $\mathbb{E}|\nabla f|$ to $\gamma_n^+(A)$. To do this, we will replace 1_A by a suitable smooth version of it. Fix some $\epsilon > 0$ and let $f_\epsilon(x)$ be a smooth function such that $f_\epsilon(x) = 1$ for $x \in A$, $f_\epsilon(x) = 0$ for $x \notin A_\epsilon$, and $|\nabla f_\epsilon| \leq 1 + 1/\epsilon$. (Such a function may be constructed explicitly, essentially by convolving the function $x \mapsto \max\{0, \min\{1, d(x, A)/\epsilon\}\}$ with a smooth function of small support; for more details, see Appendix C of [17].) Applying (1.2) to f_ϵ , we obtain

$$I(\mathbb{E}f_\epsilon) \leq \mathbb{E}\sqrt{I^2(f_\epsilon) + |\nabla f_\epsilon|^2} \leq \mathbb{E}I(f_\epsilon) + \mathbb{E}|\nabla f_\epsilon|.$$

Now, $I(f_\epsilon) = 0$ if $f_\epsilon \in \{0, 1\}$ and so $I(f_\epsilon)$ is supported on $A_\epsilon \setminus A$. Moreover, $I(f_\epsilon)$ is bounded pointwise by $(2\pi)^{-1/2} \leq 1$, and hence $\mathbb{E}I(f_\epsilon) \leq \gamma_n(A_\epsilon) - \gamma_n(A)$. Similarly, $|\nabla f_\epsilon|$ is bounded pointwise by $1 + 1/\epsilon$ and supported on $A_\epsilon \setminus A$; hence $\mathbb{E}|\nabla f_\epsilon| \leq (1 + 1/\epsilon)(\gamma_n(A_\epsilon) - \gamma_n(A))$. Thus we obtain

$$I(\mathbb{E}f_\epsilon) \leq \mathbb{E}I(f_\epsilon) + \mathbb{E}|\nabla f_\epsilon| \leq \left(2 + \frac{1}{\epsilon}\right)(\gamma_n(A_\epsilon) - \gamma_n(A)). \quad (1.3)$$

Let us now compare $I(\mathbb{E}f_\epsilon)$ to $I(\gamma_n(A))$. Recall that $\gamma_n(A) \leq \mathbb{E}f_\epsilon \leq \gamma_n(A_\epsilon)$. We may assume that $\gamma_n(A_\epsilon) \rightarrow \gamma_n(A)$ as $\epsilon \rightarrow 0$ (if not, then (1.1) is vacuous because $\gamma_n^+(A) = \infty$). Hence, $\mathbb{E}f_\epsilon \rightarrow \gamma_n(A)$. Since I is continuous, $I(\mathbb{E}f_\epsilon) \rightarrow I(\gamma_n(A))$. Then by taking the liminf as $\epsilon \rightarrow 0$ on both sides of (1.3), we recover (1.1).

The other implication (i.e. that (1.1) implies (1.2)) is less obvious, but quite important because it uses a dimension-raising trick that we will see again later. For a smooth function $f : \mathbb{R}^n \rightarrow [0, 1]$, define $A_f \subset \mathbb{R}^{n+1}$ by $A_f = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \leq \Phi^{-1}(f(x))\}$. By Fubini's

theorem,

$$\begin{aligned}\gamma_{n+1}(A_f) &= \int_{\mathbb{R}^n} \gamma_1(\{x_{n+1} : (x, x_{n+1}) \in A_f\}) d\gamma_n(x) \\ &= \int_{\mathbb{R}^n} \gamma_1((-\infty, \Phi^{-1}(f(x))]) d\gamma_n(x) \\ &= \int_{\mathbb{R}^n} f(x) d\gamma_n(x).\end{aligned}$$

On the other hand, one can easily check that for a set $A \subset \mathbb{R}^{n+1}$ with smooth boundary, $\gamma_{n+1}^+(A)$ is simply equal to the integral of the Gaussian density over ∂A . Let $\phi_{n+1}(y) = (2\pi)^{-(n+1)/2} e^{-|y|^2/2}$ be the density of γ_{n+1} , and recall that if ∂A is the graph of a smooth function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ then the volume element of ∂A at $(x, g(x))$ is $\sqrt{1 + |\nabla g(x)|^2} dx$. Applying this to ∂A_f (which is the graph of $\Phi^{-1} \circ f$),

$$\gamma_{n+1}^+(A_f) = \int_{\mathbb{R}^n} \phi_{n+1}(x, \Phi^{-1}(f(x))) \sqrt{1 + |\nabla(\Phi^{-1} \circ f)(x)|^2} dx.$$

Now, $\phi_{n+1}(x, x_{n+1}) = \phi_n(x)\phi(x_{n+1})$. Also, $\nabla(\Phi^{-1} \circ f) = \nabla f / (I \circ f)$ by the chain rule. Hence,

$$\begin{aligned}\gamma_{n+1}^+(A_f) &= \int_{\mathbb{R}^n} \phi(\Phi^{-1}(f(x))) \sqrt{1 + |\nabla f(x)|^2 / I(f(x))^2} d\gamma_n(x) \\ &= \int_{\mathbb{R}^n} \sqrt{I^2(f(x)) + |\nabla f(x)|^2} d\gamma_n(x).\end{aligned}$$

To summarize, we have shown that $\gamma_n(A_f) = \mathbb{E}f$ and $\gamma_{n+1}^+(A_f) = \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2}$. Applying (1.1) to A_f , we recover (1.2), and so we see that the Gaussian isoperimetric inequality implies Bobkov's inequality.

Using the same dimension-raising trick, we can apply Theorem 1.2 to find the equality cases for Theorem 1.5: f attains equality in Theorem 1.5 if and only if $A = \{(x, x_{n+1}) : x_{n+1} \leq \Phi^{-1}(f(x))\}$ is a half-space. This is of course equivalent to asking that $\Phi^{-1} \circ f$ be a linear function; hence we see that equality is attained in (1.2) exactly for functions of the form $f(x) = \Phi(\langle a, x - b \rangle)$. That such functions achieve equality can also be checked directly.

Theorem 1.6. *If $f : \mathbb{R}^n \rightarrow [0, 1]$ is a smooth function with $I(\mathbb{E}f) = \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2}$ then there exist $a, b \in \mathbb{R}^n$ such that $f(x) = \Phi(\langle a, x - b \rangle)$ for all $x \in \mathbb{R}^n$.*

1.3 The Ornstein-Uhlenbeck semigroup

Our study of the Gaussian isoperimetric inequality revolves around the proof due to Bakry and Ledoux [3]. Although several other proofs are known, the Bakry-Ledoux proof is the only one we know which gives detailed information on the equality and near-equality cases of (1.1). Their proof also uses a particularly nice idea which we will apply in other ways

later on. The main tool in this proof is the Ornstein-Uhlenbeck operator semigroup: for any $t \geq 0$, define the operator P_t by

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma_n(y). \quad (1.4)$$

This definition makes sense for any $f \in L_1(\gamma_n)$, and one may easily check that for any $t > 0$, the range of P_t consists of smooth functions.

Let us also note some trivial – but useful – properties of P_t :

1. $\{P_t : t \geq 0\}$ is a semigroup; that is, $P_t \circ P_s = P_{t+s}$
2. P_t is positivity-preserving: if $f \geq 0$ then $P_t f \geq 0$
3. $P_t 1 = 1$
4. if $f \in L_p(\gamma_n)$ for $1 \leq p < \infty$ then $P_t f \rightarrow f$ in $L_p(\gamma_n)$ as $t \rightarrow 0$
5. if $f \in L_p(\gamma_n)$ for $1 \leq p < \infty$ then $P_t f \rightarrow \mathbb{E}f$ in $L_p(\gamma_n)$ as $t \rightarrow \infty$
6. for any smooth f , $\nabla P_t f = e^{-t} P_t \nabla f$
7. for any convex $\psi : \mathbb{R} \rightarrow \mathbb{R}$, $\psi(P_t f) \leq P_t(\psi \circ f)$ pointwise
8. for any $f, g \in L_2(\gamma_n)$, $P_t(fg) \leq \sqrt{P_t(f^2)P_t(g^2)}$.

The last two properties follow from Jensen's inequality and the Cauchy-Schwarz inequality, because for any fixed $x \in \mathbb{R}^n$, $(P_t f)(x)$ is simply the integral of f against some probability measure.

Define the operator L by

$$(Lf)(x) = \sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_i^2}(x) - x_i \frac{\partial f}{\partial x_i}(x) \right) = \Delta f - \langle x, \nabla f \rangle.$$

Then L defined on $C^\infty(\mathbb{R}^n)$ and one can easily compute that for $f \in C^\infty(\mathbb{R}^n)$, $\frac{d}{dt} \Big|_{t=0} P_t f = Lf$. By the semigroup property, we also have $\frac{dP_t f}{dt} = LP_t f = P_t Lf$ for all $t \geq 0$. This operator L is known as the *generator* of P_t , which is often written as $P_t = e^{tL}$, meaning that $P_t f = \sum_{k \geq 0} \frac{L^k f}{k!}$ whenever the right hand side makes sense.

In probabilistic language, we say that P_t is the Markov operator of the Ornstein-Uhlenbeck process and that L is its generator. We will not dwell on the general theory of such objects except to mention that there is one, and that it is the subject of an excellent survey article by Ledoux [36]. For our purposes, it suffices to note two properties of L . The first is an integration by parts formula, which can be checked using Stokes' formula:

$$\int_{\mathbb{R}^n} f(x) Lg(x) d\gamma_n(x) = - \int_{\mathbb{R}^n} \langle \nabla f(x), \nabla g(x) \rangle d\gamma_n(x). \quad (1.5)$$

The second property is a sort of chain rule that can be checked by differentiation: for any smooth $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}$ and f_1, \dots, f_k mapping \mathbb{R}^n into \mathbb{R} ,

$$L\Psi(f_1, \dots, f_k) = \sum_{i,j=1}^k \frac{\partial^2 \Psi}{\partial x_i \partial x_j}(f_1, \dots, f_k) \langle \nabla f_i, \nabla f_j \rangle + \sum_{i=1}^k \frac{\partial \Psi}{\partial x_i}(f_1, \dots, f_k) Lf_i. \quad (1.6)$$

For the Ornstein-Uhlenbeck process, (1.6) can be checked directly. However, (1.6) is useful more generally. In fact, for a general collection of operators $\{P_t : t \geq 0\}$, we say that P_t is a *Markov semigroup* if it satisfies properties 1-4 above. Defining $L = \left. \frac{dP_t}{dt} \right|_{t=0}$ (on some domain where the right hand side makes sense), and the quadratic form Γ by $\Gamma(f, g) = L(fg) - gLf - fLg$, we then say that P_t is a *diffusion semigroup* if (1.6) holds with $\langle \nabla f_i, \nabla f_j \rangle$ replaced by $\Gamma(f_i, f_j)$. Much of the material from this chapter extends to more general Markov semigroups, although generalizing the results in Chapters 2 through 4 remains an open problem. For the rest of this thesis, we will not mention more general Markov semigroups; for more information on them, see the survey article by Ledoux [36].

From (1.6), the chain rule, and the definition of L , we derive the following useful formula: abbreviating $P_{t-s}f_1, \dots, P_{t-s}f_k$ by $P_{t-s}f$,

$$\begin{aligned} \frac{d}{ds} P_s \Psi(P_{t-s}f) &= LP_s \Psi(P_{t-s}f) - \sum_{i=1}^k P_s \left(\frac{\partial \Psi}{\partial x_i}(P_{t-s}f) LP_{t-s}f_i \right) \\ &= P_s \left(L\Psi(P_{t-s}f) - \sum_{i=1}^k \frac{\partial \Psi}{\partial x_i}(P_{t-s}f) LP_{t-s}f_i \right) \\ &= P_s \sum_{i,j=1}^k \langle \nabla P_{t-s}f_i, \nabla P_{t-s}f_j \rangle \frac{\partial^2 \Psi}{\partial x_i \partial x_j}(P_{t-s}f). \end{aligned} \quad (1.7)$$

This formula is useful because, in conjunction with the fundamental theorem of calculus, it may be used to derive bounds on $P_t \Psi(f_1, \dots, f_k) - \Psi(P_t f_1, \dots, P_t f_k)$. As we will see, there are many important inequalities that may be written as bounds on $P_t \Psi(f_1, \dots, f_k) - \Psi(P_t f_1, \dots, P_t f_k)$ for an appropriate Ψ .

Poincaré's inequality

Let us consider two simple examples to demonstrate the utility of (1.7). These examples serve a dual purpose: besides giving a flavor of more complicated derivations to come, the classical inequalities that we prove here will also be useful to us later. First, consider $\Psi(x) = x^2$. In this case, (1.7) reduces to

$$\frac{d}{ds} P_s (P_{t-s}f)^2 = 2P_s |\nabla P_{t-s}f|^2 = 2e^{2(s-t)} P_s |P_{t-s} \nabla f|^2, \quad (1.8)$$

where the second equality follows because $\nabla P_{t-s}f = e^{s-t} P_{t-s} \nabla f$, by property 6 above. Now we will consider two different ways to obtain inequalities from (1.8). Since $|\cdot|^2$ is convex, we

may apply Jensen's inequality (property 7 of P_t) in one of two directions: either by pushing P_s into $|\cdot|^2$ or by pulling P_{t-s} out of $|\cdot|^2$. That is,

$$\begin{aligned}\frac{d}{ds}P_s(P_{t-s}f)^2 &\leq 2e^{2(s-t)}P_sP_{t-s}|\nabla f|^2 = 2e^{2(s-t)}P_t|\nabla f|^2 \\ \frac{d}{ds}P_s(P_{t-s}f)^2 &\geq 2e^{2(s-t)}|P_sP_{t-s}\nabla f|^2 = 2e^{2s}|\nabla P_t f|^2.\end{aligned}$$

If we integrate these two inequalities from zero up to t , we obtain

$$P_t f^2 - (P_t f)^2 \leq (1 - e^{-2t})P_t|\nabla f|^2 \quad (1.9)$$

$$P_t f^2 - (P_t f)^2 \geq (e^{2t} - 1)|\nabla P_t f|^2. \quad (1.10)$$

Note that if we send $t \rightarrow \infty$ in (1.9), then we obtain the classical Poincaré inequality for Gaussian measures: $\text{Var}(f) = \mathbb{E}f^2 - (\mathbb{E}f)^2 \leq \mathbb{E}|\nabla f|^2$. On the other hand, (1.10) becomes $\text{Var}(f) \geq \mathbb{E}|\nabla f|^2$ in the same limit. Note that both of these inequalities become equalities for linear functions.

The log-Sobolev inequality

Next, we will consider (1.7) in the case that $\Psi(x) = x \log x$ and f is a strictly positive function. Then (1.7) reduces to

$$\frac{d}{ds}P_s\Psi(P_{t-s}f) = P_s\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f} = e^{2(s-t)}P_s\frac{|P_{t-s}\nabla f|^2}{P_{t-s}f}, \quad (1.11)$$

where the second equality follows because $\nabla P_{t-s}f = e^{s-t}P_{t-s}\nabla f$. To bound the right hand side, note that the Cauchy-Schwarz inequality implies that $(P_\tau g)^2 \leq P_\tau h P_\tau \frac{g^2}{h}$ for any positive g and h , and any $\tau > 0$. In particular, if we apply this with $g = |\nabla f|$, $h = f$, and $\tau = t-s$ then we obtain

$$|P_{t-s}\nabla f|^2 \leq (P_{t-s}|\nabla f|)^2 \leq (P_{t-s}f)P_{t-s}\frac{|\nabla f|^2}{f},$$

where the first inequality follows from the convexity of $|\cdot|$. Applying this to (1.11) yields

$$\frac{d}{ds}P_s\Psi(P_{t-s}f) \leq e^{2(s-t)}P_sP_{t-s}\frac{|\nabla f|^2}{f} = e^{2(s-t)}P_t\frac{|\nabla f|^2}{f}.$$

As we did for Poincaré's inequality, we can integrate out s . This time, we obtain $P_t(f \log f) - (P_t f) \log(P_t f) \leq \frac{1-e^{-2t}}{2}P_t\frac{|\nabla f|^2}{f}$. Setting $f = g^2$ and taking $t \rightarrow \infty$, we recover Gross' logarithmic Sobolev inequality [22]

$$\mathbb{E}(g^2 \log g^2) - (\mathbb{E}g^2) \log(\mathbb{E}g^2) \leq 2\mathbb{E}|\nabla g|^2.$$

As for Poincaré's inequality, we can obtain a reversal of the log-Sobolev inequality by applying the Cauchy-Schwarz inequality to P_s (instead of P_{t-s}) on the right hand side of (1.11).

Specifically, the inequality $(P_\tau g)^2 \leq P_\tau h P_\tau \frac{g^2}{h}$ applied with $g = |P_{t-s} \nabla f|$, $h = P_{t-s} f$, and $\tau = s$ yields

$$|P_t \nabla f|^2 \leq (P_s |P_{t-s} \nabla f|)^2 \leq (P_t f) P_s \frac{|P_{t-s} \nabla f|^2}{P_{t-s} f};$$

and hence

$$\frac{|P_t \nabla f|^2}{P_t f} \leq P_s \frac{|P_{t-s} \nabla f|^2}{P_{t-s} f}.$$

Applied to (1.11), this gives

$$\frac{d}{ds} P_s \Psi(P_{t-s} f) \geq e^{2s} \frac{|\nabla P_t f|^2}{P_t f}.$$

Integrating out s , we have

$$(P_t f)(P_t f \log f) - (P_t f)^2 \log P_t f \geq \frac{e^{2t} - 1}{2} |\nabla P_t f|^2. \quad (1.12)$$

This is a useful upper bound on $|\nabla P_t f|$, as we will see in Chapter 3. It is stronger than the reverse-Poincaré inequality (1.10) because it gives a better bound when $P_t f$ is close to zero.

A sharper bound on $|\nabla P_t f|$

Having demonstrated the utility of (1.7) with some simple derivations of classical inequalities, we turn next to a more recent inequality of Bakry and Ledoux [3].

Theorem 1.7. *For any measurable function $f : \mathbb{R}^n \rightarrow [0, 1]$ and any $t > 0$,*

$$|\nabla P_t f| \leq \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} I(P_t f).$$

Equivalently,

$$|\nabla(\Phi^{-1} \circ P_t f)| \leq \frac{e^{-t}}{\sqrt{1 - e^{-2t}}}.$$

The two inequalities in Theorem 1.7 are equivalent by the chain rule. They are sharp (as we will show in Section 1.5) and will be particularly useful to us when we discuss a robust version of Theorem 1.1 in Chapter 3. We remark that since $I(x) \sim \sqrt{2 \log(1/x)}$ as $x \rightarrow 0$, the reverse log-Sobolev inequality (1.12) implies Theorem 1.7, but with a worse constant. Indeed, if $f : \mathbb{R}^n \rightarrow [0, 1]$ then $P_t f \log f < 0$, and so (1.12) implies that

$$|\nabla P_t f| \leq 2 \frac{e^{-t}}{\sqrt{e^{-2t} - 1}} P_t f \sqrt{\log P_t f} \leq C \frac{e^{-t}}{\sqrt{e^{-2t} - 1}} I(P_t f)$$

for some universal constant C . This argument cannot, however, recover the sharp constant $C = 1$.

Proof of Theorem 1.7. Set $\Psi(x) = I(x)$. Then $\Psi'(x) = -\Phi^{-1}(x)$ and $\Psi''(x) = -1/I(x)$. By (1.7),

$$P_t I(f) - I(P_t f) = - \int_0^t P_s \frac{|\nabla P_{t-s} f|^2}{I(P_{t-s} f)} ds. \quad (1.13)$$

We apply the Cauchy-Schwarz inequality in the same way as we did for the log-Sobolev inequality:

$$\frac{|P_s \nabla g|^2}{P_s I(g)} \leq P_s \frac{|\nabla g|^2}{I(g)}.$$

Applying this to (1.13) with $g = P_{t-s} f$,

$$\begin{aligned} P_t I(f) - I(P_t f) &\leq - \int_0^t \frac{|P_s \nabla P_{t-s} f|^2}{P_s I(P_{t-s} f)} ds \\ &= - \int_0^t \frac{|\nabla P_t f|^2}{P_s I(P_{t-s} f)} e^{2s} ds \\ &\leq - \frac{|\nabla P_t f|^2}{I(P_t f)} \int_0^t e^{2s} ds, \end{aligned}$$

where the last inequality follows because I is concave and so $P_s I(P_{t-s} f) \leq I(P_t f)$. Since $P_t I(f)$ is non-negative, we can remove that term from the left hand side. Then because $\int_0^t e^{2s} ds = e^{2t} - 1$, we obtain

$$(e^{2t} - 1) \frac{|\nabla P_t f|^2}{I(P_t f)} \leq I(P_t f),$$

which may be rearranged into the claimed inequality. \square

Hermite polynomials

Hermite polynomials are a useful tool in Gaussian analysis. In order to remain self-contained, we will recall their definition and prove the properties that we will use. Most importantly, we will show that the Hermite polynomials form an orthogonal basis of (\mathbb{R}^n, γ_n) on which the semigroup P_t acts diagonally. A more detailed treatment of Hermite polynomials may be found in [48]. For $k \geq 0$, define the degree- k polynomial $H_k(x)$ by

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}.$$

Equivalently, H_k may be defined by the generating function

$$e^{xt-t^2/2} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{\sqrt{k!}}.$$

The equivalence of these two definitions may be seen by taking the Taylor expansion of the function $z \mapsto e^{-z^2/2}$ around the point x . Note that our normalization is non-standard; in

some other sources, the k th Hermite polynomial is taken to be what we have called $\sqrt{k!}H_k$, while in physics another convention altogether is used. We have chosen our convention so that the set $\{H_k : k \geq 0\}$ is an orthonormal basis of $L_2(\gamma_1)$:

Proposition 1.8. *For $k, \ell \geq 0$,*

$$\int_{\mathbb{R}} H_k(x)H_\ell(x) d\gamma_1(x) = \delta_{k\ell}.$$

Moreover, $\{H_k : k \geq 0\}$ spans $L_2(\gamma_1)$.

Proof. The first statement comes from calculating the integral $\int e^{xt-t^2/2}e^{xs-s^2/2}d\gamma_1(x)$ in two ways: by completing the square and by expanding the generating function definition of H_k . The second statement may be proven by applying the Stone-Weierstrass theorem to a large interval $[-R, R]$. \square

By tensorizing the basis $\{H_k\}$ of $L_2(\gamma_1)$, we construct an orthonormal basis of $L_2(\gamma_n)$. Namely, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1, \dots\}^n$, define the function $H_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$H_\alpha(x) = \prod_{i=1}^n H_{\alpha_i}(x_i).$$

From Proposition 1.8, we see immediately that $\{H_\alpha\}$ are an orthonormal basis of (\mathbb{R}^n, γ_n) .

One remarkable property of the Ornstein-Uhlenbeck semigroup is that it acts diagonally on the Hermite basis. This can be easily seen from the generating function characterization of the Hermite polynomials: indeed, for any fixed $t > 0$, if $f_t(x) = e^{xt-t^2/2}$ then

$$(P_s f_t)(x) = \exp(e^{-s}xt - t^2/2) \int_{-\infty}^{\infty} \exp(\sqrt{1-e^{-2s}}yt) d\gamma_1(y) = \exp(xe^{-s}t - e^{-2s}t^2/2) = f_{e^{-s}t}(x).$$

Writing both f_t and $f_{e^{-s}t}$ in terms of Hermite polynomials, we have

$$\sum_{k \geq 0} \frac{t^k}{k!} P_s H_k = P_s f_t = f_{e^{-s}t} = \sum_{k \geq 0} \frac{t^k e^{-sk}}{k!} H_k,$$

and so we conclude that $P_s H_k = e^{-sk} H_k$. This may be easily extended to the multi-dimensional Hermite functions, where it yields

$$P_s H_\alpha = e^{-s|\alpha|} H_\alpha. \tag{1.14}$$

Since $\{H_\alpha : \alpha \in \{0, 1, \dots\}^n\}$ span $L_2(\gamma_n)$, it follows that P_t is injective. Certain other properties of P_t (which we will not need) also follow from (1.14), such as the fact that P_t is trace-class.

1.4 A semigroup proof of Bobkov's inequality

The semigroup method that we explored in the previous section can be extended to prove Bobkov's inequality (1.2). The proof that we will give is due to Bakry and Ledoux [3], who give it in a rather more general framework. Actually, we will prove a slight strengthening of Bobkov's inequality, namely that for all $t \geq 0$ and all smooth $f : \mathbb{R}^n \rightarrow [0, 1]$, the following inequality holds pointwise:

$$\sqrt{I^2(P_t f) + |\nabla P_t f|^2} \leq P_t \sqrt{I^2(f) + |\nabla f|^2}. \quad (1.15)$$

Sending $t \rightarrow \infty$ in (1.15) recovers (1.2).

We intend to follow the same approach to (1.15) as we did for the Poincaré and log-Sobolev inequalities. However, there is a minor obstacle: because $\nabla P_s f \neq P_s \nabla f$, the quantity $\sqrt{I^2(P_s f) + |\nabla P_s f|^2}$ cannot be written in the form $\Psi(P_s g_1, \dots, P_s g_k)$, and so we cannot apply (1.7) directly. This obstacle is only minor; we simply re-derive a modified version of (1.7) involving ∇f . Since $\nabla P_s f = e^{-s} P_s \nabla f$, we have

$$\frac{d}{ds} \nabla P_s f = L e^{-s} P_s \nabla f - e^{-s} P_s \nabla f = L \nabla P_s f - \nabla P_s f.$$

Hence, for any smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ (which we write as $\Psi(x, y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$),

$$\begin{aligned} & \frac{d}{ds} P_s \Psi \left(P_{t-s} f, \frac{\partial P_{t-s} f}{\partial x_1}, \dots, \frac{\partial P_{t-s} f}{\partial x_n} \right) \\ &= P_s \left(L \Psi - \frac{\partial \Psi}{\partial x} L P_{t-s} f + \sum_{i=1}^n \frac{\partial \Psi}{\partial y_i} \frac{d}{ds} \frac{\partial P_{t-s} f}{\partial x_i} \right) \\ &= P_s \left(L \Psi - \frac{\partial \Psi}{\partial x} L P_{t-s} f - \sum_{i=1}^n \frac{\partial \Psi}{\partial y_i} L \frac{\partial P_{t-s} f}{\partial x_i} + \sum_{i=1}^n \frac{\partial \Psi}{\partial y_i} \frac{\partial P_{t-s} f}{\partial x_i} \right) \\ &= P_s \left(\frac{\partial^2 \Psi}{\partial x^2} |\nabla P_{t-s} f|^2 + 2 \sum_{i=1}^n \frac{\partial^2 \Psi}{\partial x \partial y_i} \langle \nabla P_{t-s} f, \nabla \frac{\partial P_{t-s} f}{\partial x_i} \rangle \right. \\ & \quad \left. + \sum_{i,j=1}^n \frac{\partial^2 \Psi}{\partial y_i \partial y_j} \langle \nabla \frac{\partial P_{t-s} f}{\partial x_i}, \nabla \frac{\partial P_{t-s} f}{\partial x_j} \rangle + \sum_{i=1}^n \frac{\partial \Psi}{\partial y_i} \frac{\partial P_{t-s} f}{\partial x_i} \right) \end{aligned} \quad (1.16)$$

(Everywhere that a derivative of Ψ appears, we mean for it to be evaluated at the point $(P_{t-s} f, \frac{\partial P_{t-s} f}{\partial x_1}, \dots, \frac{\partial P_{t-s} f}{\partial x_n})$.) Besides the fact that we have named one of Ψ 's arguments x and the rest y , (1.16) differs from (1.7) only by the addition of the last term, $\sum \frac{\partial \Psi}{\partial y_i} \frac{\partial P_{t-s} f}{\partial x_i}$, which appeared when we differentiated the commutation relation $\nabla P_s = e^{-s} P_s \nabla$.

Now let us apply (1.16) to prove (1.15): for $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$, let $\Psi(x, y) = \sqrt{I^2(x) + |y|^2}$.

Elementary calculus yields

$$\begin{aligned}\frac{\partial \Psi}{\partial y_i} &= \frac{y_i}{\Psi} \\ \frac{\partial^2 \Psi}{\partial x^2} &= \frac{(I')^2 \Psi^2 - \Psi^2 - (II')^2}{\Psi^3} \\ \frac{\partial^2 \Psi}{\partial x \partial y_i} &= -\frac{y_i I I'}{\Psi^3} \\ \frac{\partial^2 \Psi}{\partial y_i \partial y_j} &= \frac{\delta_{ij} \Psi^2 - y_i y_j}{\Psi^3}.\end{aligned}$$

If we set, for brevity, $v = \nabla P_{t-s} f$ and $w_i = \nabla \frac{\partial f_{t-s}}{\partial x_i}$ then (1.16) becomes

$$\begin{aligned}\frac{d}{ds} P_s \Psi \left(P_{t-s} f, \frac{\partial P_{t-s} f}{\partial x_1}, \dots, \frac{\partial P_{t-s} f}{\partial x_n} \right) \\ = P_s \left(|v|^2 \frac{\partial^2 \Psi}{\partial x^2} + 2 \sum_{i=1}^n \frac{\partial^2 \Psi}{\partial x \partial y_i} \langle w_i, v \rangle + \sum_{i,j=1}^n \frac{\partial^2 \Psi}{\partial y_i \partial y_j} \langle w_i, w_j \rangle + \sum_{i=1}^n \frac{\partial \Psi}{\partial y_i} v_i \right).\end{aligned}$$

For our specific function Ψ , we arrive (after canceling some terms) at the expression

$$\begin{aligned}P_s \frac{1}{\Psi^3} \left(|v|^4 (I')^2 - 2II' \sum_i v_i \langle w_i, v \rangle + (I^2 + |v|^2) \sum_i |w_i|^2 - \sum_{i,j} \langle w_i, w_j \rangle v_i v_j \right) \\ = P_s \frac{1}{\Psi^3} \left(|v|^4 (I')^2 - 2II' v^T H_f v + (I^2 + |v|^2) \|H_f\|_F^2 - |H_f v|^2 \right) \\ = P_s \frac{1}{\Psi^3} \left(\|IH_f - I' v v^T\|_F^2 + |v|^2 \|H_f\|_F^2 - |H_f v|^2 \right).\end{aligned}\tag{1.17}$$

where H_f denotes the Hessian matrix of $P_{t-s} f$ and $\|\cdot\|_F$ is the Frobenius norm $\|A\|_F^2 = \text{tr}(A^T A)$. Since $\|H_f\|_F |v| \geq |H_f v|$, the last line of (1.17) is non-negative pointwise, thus proving that $\frac{d}{ds} P_s \Psi \left(P_{t-s} f, \frac{\partial P_{t-s} f}{\partial x_1}, \dots, \frac{\partial P_{t-s} f}{\partial x_n} \right) \geq 0$. Recalling the definition of Ψ , we have proved (1.15) and hence also Bobkov's inequality (1.2).

1.5 Deforming Ehrhard sets

In a few pages, we proved Bobkov's inequality using only calculus and some linear algebra. However, we would not blame the reader for being somewhat bewildered by the proof: although the original inequality was motivated geometrically, the proof seems to be simply a calculation. To give a geometric intuition to the proof, recall our discussion following Theorem 1.5 where we showed that the Gaussian isoperimetric inequality in \mathbb{R}^{n+1} implies Bobkov's inequality in \mathbb{R}^n . We did so by associating to each $f : \mathbb{R}^n \rightarrow [0, 1]$ the set $A_f = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \leq \Phi^{-1}(f(x))\}$ and noting that $\mathbb{E} \sqrt{I^2(f) + |\nabla f|^2} = \gamma_{n+1}^+(A_f)$ and $\mathbb{E} f = \gamma_{n+1}(A_f)$.

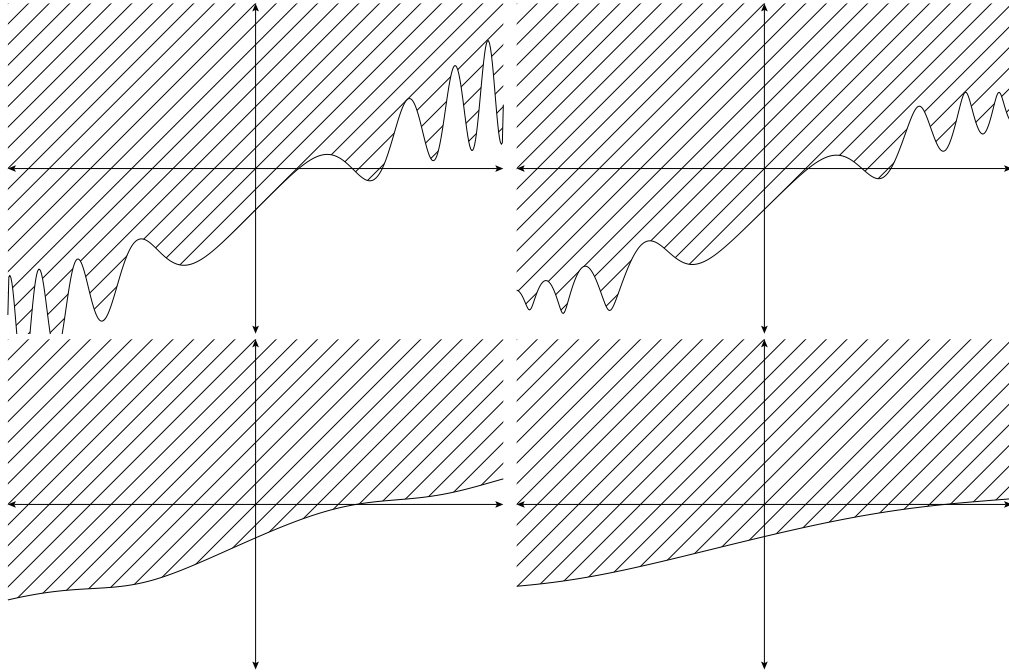


Figure 1.1: The deformation of an Ehrhard set under the semigroup P_t . From left to right, top to bottom, $t = 0, 0.01, 0.05, 0.5$, and 1

We will call $A \subset \mathbb{R}^{n+1}$ an *Ehrhard set* if it can be written as the subgraph of some function $g : \mathbb{R}^n \rightarrow \mathbb{R}$; that is, $A = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \leq g(x)\}$. For an Ehrhard set A , define the function $f_A : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f_A(x) = \Phi(\sup\{y \in \mathbb{R} : (x, y) \in A\})$. Note that the correspondence $A \mapsto f_A$ is the inverse of the correspondence $f \mapsto A_f$ that we described in the previous paragraph. In particular, this implies that $\mathbb{E}\sqrt{I^2(f_A) + |\nabla f_A|^2} = \gamma_{n+1}^+(A)$ and $\mathbb{E}f_A = \gamma_{n+1}(A)$.

For any $t \geq 0$, define $A(t) = A_{P_t f_A}$. Then $A(0) = A$ and (since $P_t f \rightarrow \mathbb{E}f$ as $t \rightarrow \infty$) $A(\infty) = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \leq \Phi^{-1}(\gamma_{n+1}(A))\}$. Note also that because $\mathbb{E}P_t f = \mathbb{E}f$ for all functions f , $\gamma_{n+1}(A(t)) = \gamma_{n+1}(A(0))$ for all t . Since $A(\infty)$ is always a half-space, we may view $A(t)$ as a measure-preserving, continuous deformation from an arbitrary Ehrhard set to a half-space (see Figure 1.1). Now, (1.15) implies that the Gaussian surface area of $A(t)$ is non-increasing in t . Thus we arrive at a geometric interpretation of the Bakry-Ledoux semigroup proof: we have constructed a continuous deformation on Ehrhard sets which decreases the surface area, preserves the measure, and ultimately turns everything into a half-space. The page or so of calculus that was involved amounted to checking that the surface area is non-increasing under this deformation.

We conclude this section with a short but confidence-inspiring calculation: if we believe that half-spaces minimize Gaussian surface area and that $\gamma_{n+1}^+(A(t))$ is non-increasing in t , then our deformation had better preserve half-spaces: if A is a half-space then $A(t)$ should be one also. We can check this with a direct calculation. If an Ehrhard set A is a half-space

then there are two possibilities: A is subgraph of a linear function, or $A = A' \times \mathbb{R}$ where $A' \subset \mathbb{R}^n$ is a half-space. In the first case, $f_A(x) = \Phi(\langle a, x - b \rangle)$ for some $a, b \in \mathbb{R}^n$; in the second case, $f_A(x) = 1_{\{\langle a, x \rangle \geq b\}}$. To show that $A(t)$ is a half-space, it suffices to find some $a', b' \in \mathbb{R}^n$ such that $P_t f_A = \Phi(\langle a', x - b' \rangle)$.

Lemma 1.9. *If $f(x) = \Phi(\langle a, x - b \rangle)$ then*

$$(P_t f)(x) = \Phi\left(\frac{\langle a, e^{-t}x - b \rangle}{\sqrt{1 + |a|^2(1 - e^{-2t})}}\right).$$

If $f(x) = 1_{\{\langle a, x - b \rangle \geq 0\}}$ then

$$(P_t f)(x) = \Phi\left(\frac{\langle a, e^{-t}x - b \rangle}{|a|\sqrt{1 - e^{-2t}}}\right).$$

Note that besides showing that half-spaces are preserved under deformation by P_t , Lemma 1.9 also provides the promised example that proves the sharpness of Theorem 1.7: if $f(x) = 1_{\{\langle a, x - b \rangle \geq 0\}}$ with $|a| = 1$ then by Lemma 1.9, $(\Phi^{-1} \circ P_t f)(x) = (1 - e^{-2t})^{-1/2} \langle a, e^{-t}x - b \rangle$, and so $|\nabla(\Phi^{-1} \circ P_t f)|$ is equal to $e^{-t}/\sqrt{1 - e^{-2t}}$ everywhere.

Proof of Lemma 1.9. Since γ_n is a product measure, it suffices to consider the case $n = 1$: suppose that $f(x) = \Phi(a(x - b))$. By the definition of P_t and Φ , we write out

$$\begin{aligned} (P_t f)(x) &= \int_{-\infty}^{\infty} \Phi(a(e^{-t}x + \sqrt{1 - e^{-2t}}y - b))\phi(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{a(e^{-t}x + \sqrt{1 - e^{-2t}}y - b)} \phi(z)\phi(y) dz dy. \end{aligned}$$

Now, the function $(z, y) \mapsto \phi(z)\phi(y)$ is rotationally invariant, so we may introduce the orthogonal change of variables

$$\begin{aligned} w &= \frac{z - a\sqrt{1 - e^{-2t}}y}{\sqrt{1 + a^2(1 - e^{-2t})}} \\ v &= \frac{y + a\sqrt{1 - e^{-2t}}z}{\sqrt{1 + a^2(1 - e^{-2t})}}. \end{aligned}$$

Under this change of variables,

$$\left\{ (y, z) \in \mathbb{R}^2 : z \leq a(e^{-t}x + \sqrt{1 - e^{-2t}}y) - b \right\} = \left\{ (v, w) \in \mathbb{R}^2 : w \leq \frac{a(e^{-t}x - b)}{\sqrt{1 + a^2(1 - e^{-2t})}} \right\}.$$

Hence,

$$(P_t f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{a(e^{-t}x - b)}{\sqrt{1 + a^2(1 - e^{-2t})}}} \phi(w)\phi(v) dw dv = \Phi\left(\frac{ae^{-t}x - ab}{\sqrt{1 + a^2(1 - e^{-2t})}}\right).$$

This proves the first claim of the Lemma; the second follows from the first by taking the limit $|a| \rightarrow \infty$. \square

1.6 The equality cases in Bobkov's inequality

Carlen and Kerce [11] noticed that the semigroup proof of Bobkov's inequality reveals the equality cases with relatively little additional effort. Let $g = \Phi^{-1} \circ f_{t-s}$; by differentiating twice, one checks that $IH_f - I'vv^T = I^2H_g$, where H_f is the Hessian of f , H_g is the Hessian of g , and $v = \nabla f_{t-s}$. Recognizing this term in (1.17), we have

$$\frac{d}{ds} P_s \Psi \left(P_{t-s} f, \frac{\partial P_{t-s} f}{\partial x_1}, \dots, \frac{\partial P_{t-s} f}{\partial x_n} \right) \geq P_s \frac{I^4 \|H_g\|_F^2}{\Psi^3}$$

and hence

$$P_t \sqrt{I^2(f) + |\nabla f|^2} - \sqrt{I^2(P_t f) + |\nabla P_t f|^2} \geq \int_0^t P_{t-s} \frac{I^4(P_s f) \|\text{Hess}(\Phi^{-1} \circ P_s f)\|_F^2}{(I^2(P_s f) + |\nabla P_s f|^2)^{3/2}} ds. \quad (1.18)$$

Now, every term in the integral is strictly positive except for $\|\text{Hess}(\Phi^{-1} \circ P_s f)\|_F^2$. If the left hand side is identically zero, we must then have $\text{Hess}(\Phi^{-1} \circ P_s f) \equiv 0$ for almost every s . In particular, there is some $s > 0$ such that $\text{Hess}(\Phi^{-1} \circ P_s f) \equiv 0$. Since $P_s f$ is smooth, it follows that $\Phi^{-1} \circ P_s f$ is a linear function. That is, there exist $a', b' \in \mathbb{R}^n$ such that $(P_s f)(x) = \Phi(\langle a', x - b' \rangle)$. We would like to argue that f takes the same form.

Lemma 1.10. *Suppose that $(P_s f)(x) = \Phi(\langle a', x - b' \rangle)$ for some $s \geq 0$, $a', b' \in \mathbb{R}^n$, and $f \in L_2(\gamma_n)$. Then there exist $a, b \in \mathbb{R}^n$ such that either $|a'| < \frac{e^{-s}}{\sqrt{1-e^{-2s}}}$ and $f(x) = \Phi(\langle a, x - b \rangle)$ or $|a'| = \frac{e^{-s}}{\sqrt{1-e^{-2s}}}$ and $f(x) = 1_{\{\langle a, x - b \rangle \geq 0\}}$.*

Under the additional assumption that f is smooth, only the first case of Lemma 1.10 is possible. Thus, Lemma 1.10 and the argument preceding it establish Theorem 1.6.

Proof of Lemma 1.10. Let $L_s = \frac{e^{-s}}{\sqrt{1-e^{-2s}}}$. Since $\nabla(\Phi^{-1} \circ P_s f) = a'$, Theorem 1.7 implies that $|a'| \leq L_s$. Suppose first that $|a'| = L_s$. By Lemma 1.9, as g ranges over the set $\{1_{\{\langle a, x - b \rangle \geq 0\}} : a, b \in \mathbb{R}^n\}$, $P_s g$ ranges over the set $\{\Phi(\langle a', x - b' \rangle) : a', b' \in \mathbb{R}^n, |a'| = L_s\}$. Since $P_s f$ belongs, by assumption, to this latter set, there must be some $a, b \in \mathbb{R}^n$ such that $P_s 1_{\{\langle a, -b \rangle \geq 0\}} = P_s f$. But P_s is injective (by (1.14)), and so $f = 1_{\{\langle a, \cdot - b \rangle\}}$.

The case $|a'| < L_s$ is similar. By Lemma 1.9, as g ranges over the set $\{\Phi(\langle a, x - b \rangle \geq 0) : a, b \in \mathbb{R}^n\}$, $P_s g$ ranges over the set $\{\Phi(\langle a', x - b' \rangle) : a', b' \in \mathbb{R}^n, |a'| < L_s\}$. In particular, if $|a'| < L_s$ then there exists some $a, b \in \mathbb{R}^n$ such that $\Phi(\langle a', \cdot - b' \rangle) = P_s f$. Since P_s is injective, $f = \Phi(\langle a, \cdot - b \rangle)$. \square

Chapter 2

Gaussian noise stability

2.1 Gaussian noise stability

Our next topic is a generalization of isoperimetry, although it may not appear that way at first. Fix a parameter $0 < \rho < 1$ and suppose that $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$ is a mean-zero Gaussian random vector with covariance matrix $\begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}$. In other words, if Z is a standard $2n$ -dimensional Gaussian vector then

$$(X, Y) \stackrel{d}{=} \begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}^{1/2} Z.$$

We will use the notation \Pr_ρ to refer to the distribution of (X, Y) .

For a set $A \subset \mathbb{R}^n$, we define the *noise stability* of A to be $\Pr_\rho(X \in A, Y \in A)$, while the *noise sensitivity* of A is $\Pr_\rho(X \in A, Y \notin A)$. It is natural to ask for an upper bound on the noise stability of A in terms of $\gamma_n(A)$. Of course, there is always the trivial bound $\Pr_\rho(X \in A, Y \in A) \leq \Pr_\rho(X \in A) = \gamma_n(A)$ but this is only sharp for $\rho = 1$ or $\gamma_n(A) \in \{0, 1\}$.

Remarkably, Borell [9] gave an upper bound on noise stability which is sharp for any $0 < \rho < 1$. In particular, he showed that among all sets with a given Gaussian measure, half-spaces are the most noise stable. Since the set $\{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(\gamma_n(A))\}$ is a half-space with the same measure as A , we may write Borell's result as an inequality:

$$\Pr_\rho(X \in A, Y \in A) \leq \Pr_\rho(X_1 \leq \Phi^{-1}(\gamma_n(A)), Y_1 \leq \Phi^{-1}(\gamma_n(A))). \quad (2.1)$$

As we will show in the next section, (2.1) is closely linked with the Gaussian isoperimetric inequality, and so it is no coincidence that half-spaces feature prominently in both inequalities. Note that by subtracting both sides of (2.1) from $\gamma_n(A)$, one obtains an inequality for noise sensitivity:

$$\Pr_\rho(X \in A, Y \notin A) \geq \Pr_\rho(X_1 \leq \Phi^{-1}(\gamma_n(A)), Y_1 \geq \Phi^{-1}(\gamma_n(A))).$$

That is, among all sets with given Gaussian measure, half-spaces are the least noise sensitive. Since noise stability and noise sensitivity can be related in this way, it is enough to study one of them.

The inequality (2.1) may be extended to a two-set version, which states that if we want to maximize $\Pr_\rho(X \in A, Y \in B)$ over all pairs of sets $A, B \subset \mathbb{R}^n$ with prescribed volumes, then we should choose A and B to be parallel half-spaces of the required volumes. By “parallel half-spaces,” we mean that there exist $a, b, d \in \mathbb{R}^n$ such that $A = \{x : \langle a, x - b \rangle \leq 0\}$ and $B = \{x : \langle a, x - d \rangle \leq 0\}$. By rotational invariance, we may choose the pair $\{x : x_1 \leq a\}$ and $\{x : x_1 \leq b\}$ (for $a, b \in \mathbb{R}$) as a canonical pair of parallel half-spaces. With this canonical pair of half-spaces, the two-set version of (2.1) may be written in the same inequality form as the one-set version:

Theorem 2.1. *For any measurable $A, B \subset \mathbb{R}^n$ and all $0 < \rho < 1$,*

$$\Pr_\rho(X \in A, Y \in B) \leq \Pr_\rho(X_1 \leq \Phi^{-1}(\gamma_n(A)), Y_1 \leq \Phi^{-1}(\gamma_n(B))). \quad (2.2)$$

Subtracting both sides of (2.2) from $\gamma_n(B)$ and replacing B with its complement, we also have

$$\Pr_\rho(X \in A, Y \in B) \geq \Pr_\rho(X_1 \leq \Phi^{-1}(\gamma_n(A)), Y_1 \geq \Phi^{-1}(\gamma_n(B))). \quad (2.3)$$

That is, (2.2) and (2.3) say that parallel half-spaces maximize the noise stability and anti-parallel half-spaces minimize the noise stability, where “anti-parallel half-spaces” means a pair of the form $\{\langle a, x - b \rangle \leq 0\}$, $\{\langle a, x - d \rangle \geq 0\}$.

Given that we wrote the inequality (2.2) with certain equality cases in mind, it may not be surprising that these are the only equality cases. That is, equality is attained in (2.2) only when A and B are parallel half-spaces. What might be more surprising is that this characterization of equality cases is new, having first appeared in a recent paper with Mossel [41]. Indeed, it seems that the earlier proofs of Theorem 2.1 [9, 25, 30] are not as well suited to the study of equality cases as the proof that we will present here.

Theorem 2.2. *If $A, B \subset \mathbb{R}^n$ are measurable sets and there exists $0 < \rho < 1$ such that*

$$\Pr_\rho(X \in A, Y \in B) = \Pr_\rho(X_1 \leq \Phi^{-1}(\gamma_n(A)), Y_1 \leq \Phi^{-1}(\gamma_n(B))),$$

then there exist $a, b, d \in \mathbb{R}^n$ such that (up to sets of measure zero) $A = \{x : \langle a, x - b \rangle \leq 0\}$ and $B = \{x : \langle a, x - d \rangle \leq 0\}$.

We assumed above that $0 < \rho < 1$, and we will mostly maintain that assumption throughout. However, let us briefly mention some other possible values of ρ : if $-1 < \rho < 0$ then the inequality in Theorem 2.1 is reversed: for any measurable A, B and all $-1 < \rho < 0$,

$$\Pr_\rho(X \in A, Y \in B) \geq \Pr_\rho(X_1 \geq \Phi^{-1}(\gamma_n(A)), Y_1 \leq \Phi^{-1}(\gamma_n(B))). \quad (2.4)$$

This follows by applying Theorem 2.1 to the pair $(X, -Y)$, which has correlation $-\rho \in (0, 1)$. The same argument shows that the inequality in (2.3) also reverses when $-1 < \rho < 0$. Hence, when $-1 < \rho < 0$, parallel half-spaces are the least noise stable pairs of sets, and anti-parallel half-spaces are the most noise stable pairs of sets.

Finally, we consider the cases $\rho \in \{-1, 0, 1\}$: these cases are degenerate and can be studied by elementary considerations. When $\rho = 0$, all sets attain equality in Theorem 2.1 because both sides of (2.2) are equal to $\gamma_n(A)\gamma_n(B)$. When $\rho = 1$, X and Y are almost surely equal, and so the left hand side of (2.2) equals $\gamma_n(A \cap B)$ while the right hand side equals $\min\{\gamma_n(A), \gamma_n(B)\}$; hence, the inequality (2.2) still holds, with equality whenever $A \subset B$ or $B \subset A$. The case $\rho = -1$ is similar, but with B replaced by $-B$. Since the matrix $\begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}$ is non-negative only for $-1 \leq \rho \leq 1$, these cases exhaust all possible values of ρ .

2.2 The connection to Gaussian isoperimetry

Ledoux [34] showed that by taking the limit as $\rho \rightarrow 1$, (2.1) recovers the Gaussian isoperimetric inequality, thus motivating our earlier statement that noise stability is a generalization of isoperimetry. We will reproduce Ledoux's argument shortly, but first let us motivate why one might expect it to be true. Since $\Pr_\rho(X \in A, Y \in A) = \gamma_n(A) - \Pr_\rho(X \in A, Y \notin A)$, maximizing the noise stability is equivalent to minimizing the noise sensitivity. If ρ is very close to 1, then X and Y are very near each other with high probability (to be precise, $X - Y$ is a Gaussian vector with covariance $2(1 - \rho)I_n$). Thus, if the boundary of A is locally flat and $1 - \rho$ is small, then $\Pr_\rho(Y \notin A | X \in A)$ essentially depends only on the distance between X and ∂A ; moreover, $\Pr_\rho(Y \notin A | X \in A)$ is negligible unless X is very close to ∂A . Hence, minimizing the noise sensitivity is similar to minimizing $\Pr_\rho(d(X, \partial A) < \sqrt{1 - \rho})$, which is like minimizing the Gaussian surface area of A .

To prove rigorously the relationship between noise sensitivity and isoperimetry, we turn to a functional inequality and its semigroup proof.

Theorem 2.3. *For smooth functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ with $g \geq 0$,*

$$\mathbb{E}gP_t f - \mathbb{E}g f \leq \frac{\|g\|_\infty}{\sqrt{2\pi}} \arccos(e^{-t}) \mathbb{E}|\nabla f|,$$

where $\|g\|_\infty$ denotes $\sup\{|g(x)| : x \in \mathbb{R}^n\}$.

Proof. We will make use of the integration by parts formula $\mathbb{E}gLf = -\mathbb{E}\langle \nabla g, \nabla f \rangle$, which may be verified by direct computation. Writing out $P_t f - f = \int_0^t \frac{d}{ds} P_s f ds = \int_0^t LP_s f ds$, we have

$$\begin{aligned} \mathbb{E}gP_t f - \mathbb{E}g f &= \int_0^t \mathbb{E}gLP_s f ds \\ &= - \int_0^t \mathbb{E}\langle \nabla g, \nabla P_s f \rangle ds \\ &= - \int_0^t \mathbb{E}\langle \nabla P_s g, \nabla f \rangle ds, \end{aligned} \tag{2.5}$$

where the last equality follows because P_s is a self-adjoint operator and $\nabla P_s = e^{-s} P_s \nabla$. Expanding the definition of P_s ,

$$\begin{aligned} (\nabla P_s g)(x) &= e^{-s} \int_{\mathbb{R}^n} (\nabla g)(e^{-s}x + \sqrt{1 - e^{-2s}}y) d\gamma_n(y) \\ &= \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \int_{\mathbb{R}^n} yg(e^{-s}x + \sqrt{1 - e^{-2s}}y) d\gamma_n(y), \end{aligned}$$

where the second equality follows from integration by parts, because the gradient of the Gaussian density is $-yd\gamma_n(y)$. Note that the last line above may be equivalently written in probabilistic notation as $\frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \mathbb{E}Zg(e^{-s}x + \sqrt{1 - e^{-2s}}Z)$. Plugging our formula for $\nabla P_s g$ back into (2.5),

$$\mathbb{E}gP_t f - \mathbb{E}g f = - \int_0^t \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \mathbb{E} \left(\langle Z_2, \nabla f(Z_1) \rangle g(e^{-s}Z_1 + \sqrt{1 - e^{-2s}}Z_2) \right) ds, \quad (2.6)$$

where the expectation is over independent Gaussian vectors Z_1 and Z_2 . Consider the inner term for a moment. Since g is non-negative,

$$\mathbb{E} \left(\langle Z_2, \nabla f(Z_1) \rangle g(e^{-s}Z_1 + \sqrt{1 - e^{-2s}}Z_2) \right) \geq \|g\|_\infty \mathbb{E} \min\{0, \langle Z_2, \nabla f(Z_1) \rangle\}. \quad (2.7)$$

Now, for any $a \in \mathbb{R}^n$, $\mathbb{E}\langle Z_2, a \rangle = 0$ and $\mathbb{E}|\langle Z_2, a \rangle| = |a|\sqrt{2/\pi}$. Hence,

$$\mathbb{E} \min\{0, \langle Z_2, a \rangle\} = -\frac{1}{2} \mathbb{E}|\langle Z_2, a \rangle| = -\frac{1}{\sqrt{2\pi}} |a|.$$

Applying this conditionally on Z_1 , we have $\mathbb{E} \min\{0, \langle Z_2, \nabla f(Z_1) \rangle\} \geq -\frac{1}{\sqrt{2\pi}} \mathbb{E}|\nabla f|$, which we plug into (2.7) to obtain

$$\mathbb{E}gP_t f - \mathbb{E}g f \leq \frac{\|g\|_\infty}{\sqrt{2\pi}} \int_0^t \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \mathbb{E}|\nabla f| ds,$$

and the conclusion follows by computing the integral via a trigonometric substitution. \square

As we did following the statement of Theorem 1.5, we may approximate the indicator function of $A \subset \mathbb{R}^n$ by a smooth function $f : \mathbb{R}^n \rightarrow [0, 1]$ such that $\mathbb{E}|\nabla f|$ is bounded by $\gamma_n^+(A)$. Setting $g = f$ and $e^{-t} = \rho$ in Theorem 2.3, we obtain an inequality on sets.

Corollary 2.4. *For every measurable $A \subset \mathbb{R}^n$,*

$$Pr_\rho(X \in A, Y \notin A) \leq \frac{1}{\sqrt{2\pi}} \arccos(\rho) \gamma_n^+(A).$$

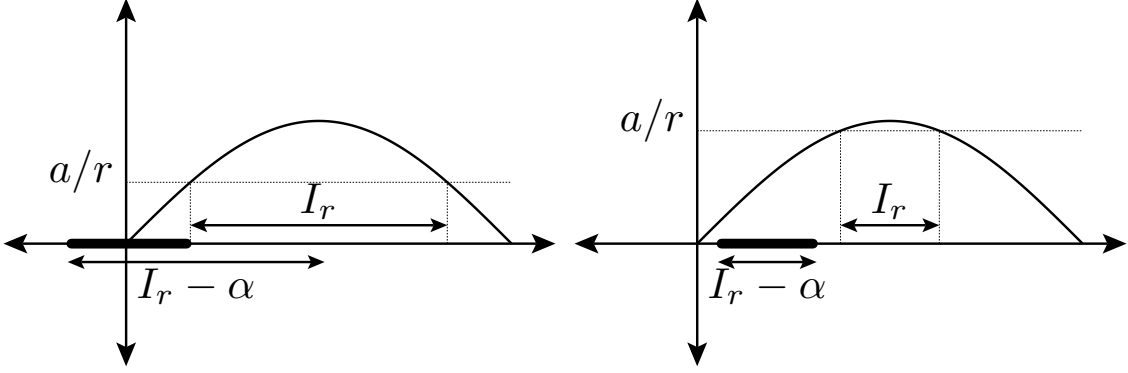


Figure 2.1: Two plots of $\sin(x)$ showing the intervals I_r and $I_r - \alpha$ for different values of a/r . In the left image, a/r is small, and so I_r and $I_r - \alpha$ overlap. In the right image, a/r is large, and so I_r and $I_r - \alpha$ are disjoint. The interval in bold is $(I_r - \alpha) \setminus I_r$; its width is either α or $|I_r|$.

We can check the sharpness of this inequality by testing it on a half-space of measure $1/2$: let $A = \{x \in \mathbb{R}^n : x_1 \leq 0\}$. For this A , $\gamma_n^+(A) = \phi(0) = \frac{1}{\sqrt{2\pi}}$. On the other hand, $\Pr_\rho(X \in A, Y \notin A) = \Pr_\rho(X_1 \leq 0, Y_1 \geq 0)$ which, after a change of variables, is the same as $\Pr(Z_1 \leq 0, Z_2 \geq -\rho Z_1 / \sqrt{1 - \rho^2})$ for independent Gaussian variables Z_1 and Z_2 . This last quantity is just the Gaussian area of a wedge in \mathbb{R}^2 which subtends an angle of $\arccos(\rho)$. Since the Gaussian measure is rotationally invariant and has total mass 1, the Gaussian area of the wedge is $\arccos(\rho)/(2\pi)$; hence $\Pr_\rho(X \in A, Y \notin A) = \arccos(\rho)/(2\pi)$ and so equality is attained in Corollary 2.4.

For a half-space A with a different measure and a fixed $0 < \rho < 1$, equality is not attained in Corollary 2.4. However, in the limit as $\rho \rightarrow 1$, Corollary 2.4 is sharp for any half-space; we can check this by computing in polar coordinates: setting $\alpha = \arccos(\rho)$, we have $\rho \sin(\theta) + \sqrt{1 - \rho^2} \cos(\theta) = \cos(\alpha) \sin(\theta) + \sin(\alpha) \cos(\theta) = \cos(\theta + \alpha)$ and so

$$\begin{aligned} \Pr_\rho(X_1 \leq a, Y_1 \geq a) &= \int_{\mathbb{R}^2} 1_{\{x \leq a\}} 1_{\{\rho x + \sqrt{1 - \rho^2} y \geq a\}} \phi(x) \phi(y) dx dy \\ &= \frac{1}{2\pi} \int_0^\infty r e^{-r^2/2} \int_0^{2\pi} 1_{\{r \sin(\theta) \leq a\}} 1_{\{r \sin(\theta + \alpha) \geq a\}} d\theta dr. \end{aligned}$$

Some care needs to be taken with the inner integral because there are a few possible cases. Suppose without loss of generality that $a > 0$. If $a/r > 1$ then the inner integral is zero because the set $\{r \sin(\theta + \alpha) \geq a\}$ is empty. If $a/r < 1$ then the set $\{\theta \in [0, 2\pi] : r \sin(\theta) \geq a\}$ is an interval, I_r say, and the inner integral above is exactly the length of $(I_r - \alpha) \setminus I_r$. In the case that $(I_r - \alpha)$ and I_r intersect, the length of $(I_r - \alpha) \setminus I_r$ is α ; in the case that they do not intersect, the length is simply the length of I_r , which is $\pi - 2 \arcsin(a/r)$. (See Figure 2.1 for an illustration of these two cases.)

Putting these cases together,

$$\int_0^{2\pi} 1_{\{r \sin(\theta) \leq a\}} 1_{\{r \sin(\theta + \alpha) \geq a\}} d\theta = 1_{\{a < r\}} \min\{\alpha, \pi - 2 \arcsin(a/r)\}.$$

Integrating this over r (and noting that $\alpha = \pi - 2 \arcsin(a/r)$ when $a/r = \sin((\pi - \alpha)/2) = \cos(\alpha/2)$),

$$\begin{aligned} 2\pi \Pr_\rho(X_1 \leq a, Y_1 \geq a) &= \int_a^\infty \min\{\alpha, \pi - 2 \arcsin(a/r)\} r e^{-r^2/2} dr \\ &= \int_a^\infty \alpha r e^{-r^2/2} dr - \int_a^{a/\cos(\alpha/2)} r e^{-r^2/2} (\alpha - \pi + 2 \arcsin(a/r)) dr \\ &= \alpha e^{-a^2/2} - \int_a^{a/\cos(\alpha/2)} r e^{-r^2/2} (\alpha - \pi + 2 \arcsin(a/r)) dr. \end{aligned}$$

When $\rho \rightarrow 1$ then $\alpha = \arccos(\rho) \rightarrow 0$. We observe that in the last line above, the second term is dominated by the first as $\alpha \rightarrow 0$. Indeed, $\cos(\alpha/2) \sim 1 - \alpha^2/8$ as $\alpha \rightarrow 0$ and so $a/\cos(\alpha/2) \sim a + a\alpha^2/8$. In particular, the integral in the last equation above runs over an interval whose length is of order α^2 . Since the integrand is bounded, it follows that the integral term is of order at most α^2 , and so it is dominated by the first term when $\alpha \rightarrow 0$.

Recalling that $\alpha = \arccos(\rho)$, we have shown that

$$\lim_{\rho \rightarrow 1} \frac{\Pr_\rho(X_1 \leq a, Y_1 \geq a)}{\arccos(\rho)} = \frac{1}{2\pi} e^{-a^2/2} = \frac{\phi(a)}{\sqrt{2\pi}}. \quad (2.8)$$

Since $\gamma_n^+(\{x : x_1 \leq a\}) = \phi(a)$, we see that Corollary 2.4 is asymptotically sharp for the set $A = \{x : x_1 \leq a\}$ as $\rho \rightarrow 1$. This observation allows us to show why Theorem 2.1 generalizes Theorem 1.1: take a general set $A \subset \mathbb{R}^n$ and let $B = \{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(\gamma_n(A))\}$ so that B is a half-space with the same volume as A . Since Corollary 2.4 holds for any $0 < \rho < 1$, it remains true in the limit:

$$\limsup_{\rho \rightarrow 1} \frac{\sqrt{2\pi} \Pr_\rho(X \in A, Y \notin A)}{\arccos(\rho)} \leq \gamma_n^+(A).$$

By Theorem 2.1 and since $\gamma_n(A) = \gamma_n(B)$, $\Pr_\rho(X \in B, Y \notin B) \leq \Pr_\rho(X \in A, Y \notin A)$ for every $0 < \rho < 1$. Hence,

$$\limsup_{\rho \rightarrow 1} \frac{\sqrt{2\pi} \Pr_\rho(X \in B, Y \notin B)}{\arccos(\rho)} \leq \limsup_{\rho \rightarrow 1} \frac{\sqrt{2\pi} \Pr_\rho(X \in A, Y \notin A)}{\arccos(\rho)} \leq \gamma_n^+(A).$$

By (2.8), the left hand side is equal to $\phi(\Phi^{-1}(\gamma_n(B))) = I(\gamma_n(A))$ and so we recover Theorem 1.1.

2.3 A functional version of Borell's inequality

In Chapter 1, we saw that the Gaussian isoperimetric inequality can be equivalently written in a functional way (Bobkov's inequality); the crucial point was a correspondence between functions $f : \mathbb{R}^n \rightarrow [0, 1]$ and certain sets $A_f \subset \mathbb{R}^{n+1}$ given by $A_f = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \leq \Phi^{-1}(f(x))\}$. Under this correspondence, Bobkov's functional of f is exactly the surface area of A_f . If we consider the noise stability of A_f instead of the surface area, what functional will this correspond to on f ? By Fubini's theorem,

$$\Pr_\rho((X, X_{n+1}) \in A_f, (Y, Y_{n+1}) \in A_g) = \mathbb{E}_\rho(\Pr_\rho(X_{n+1} \leq \Phi^{-1}(f(X)), Y_{n+1} \leq \Phi^{-1}(g(Y)) \mid X, Y)).$$

Let us give a name to the inner quantity: define $J_\rho(x, y) = \Pr_\rho(X_1 \leq \Phi^{-1}(x), Y_1 \leq \Phi^{-1}(y))$; then

$$\Pr_\rho((X, X_{n+1}) \in A_f, (Y, Y_{n+1}) \in A_g) = \mathbb{E}_\rho J_\rho(f(X), g(Y)).$$

On the other hand, $J_\rho(a, b)$ is by definition the noise stability of a pair of parallel half-spaces with volumes a and b ; since $\gamma_{n+1}(A_f) = \mathbb{E}f$ and $\gamma_{n+1}(A_g) = \mathbb{E}g$, Borell's inequality implies the following functional inequality:

Theorem 2.5. *For any measurable functions $f, g : \mathbb{R}^n \rightarrow [0, 1]$ and any $0 < \rho < 1$,*

$$\mathbb{E}_\rho J_\rho(f(X), g(Y)) \geq J_\rho(\mathbb{E}f, \mathbb{E}g), \quad (2.9)$$

where $J_\rho(x, y) = \Pr_\rho(X_1 \leq \Phi^{-1}(x), Y_1 \leq \Phi^{-1}(y))$.

In order to complete the connection between Theorem 2.5 and the set version (Theorem 2.1), we need to show that Theorem 2.5 implies Theorem 2.1. This follows trivially, because $J_\rho(0, 0) = J_\rho(0, 1) = J_\rho(1, 0) = 0$ and $J_\rho(1, 1) = 1$ and so when plugging in $f = 1_A$ and $g = 1_B$ we have $J_\rho(1_A(X), 1_B(Y)) = 1_{\{X \in A, Y \in B\}}$. Hence $\mathbb{E}_\rho J_\rho(1_A(X), 1_B(Y)) = \Pr_\rho(X \in A, Y \in B)$ and so we recover Theorem 2.1. We should point out that this correspondence between Theorem 2.1 and Theorem 2.5 is actually simpler than the analogous correspondence for the Gaussian isoperimetric inequality, because in that case gradients were involved and so some approximation arguments were required. One technical attraction of Gaussian noise stability over isoperimetry is that such smoothness issues no longer arise.

After moving from a set inequality to a functional inequality we need to reconsider the equality cases. Recall that A_f is a half-space if and only if either f is the indicator of a half-space or f takes the form $\Phi(\langle a, x - b \rangle)$. Hence the following statement is equivalent to Theorem 2.2:

Theorem 2.6. *For any measurable functions $f, g : \mathbb{R}^n \rightarrow [0, 1]$, if there exists $0 < \rho < 1$ such that $\mathbb{E}_\rho J_\rho(f(X), g(Y)) = J_\rho(\mathbb{E}f, \mathbb{E}g)$ then there exist $a, b, d \in \mathbb{R}^n$ such that either*

$$\begin{aligned} f(x) &= \Phi(\langle a, x - b \rangle) \text{ a.s.} \\ g(x) &= \Phi(\langle a, x - d \rangle) \text{ a.s.} \end{aligned}$$

or

$$\begin{aligned} f(x) &= 1_{\{(a,x-b) \geq 0\}} \text{ a.s.} \\ g(x) &= 1_{\{(a,x-d) \geq 0\}} \text{ a.s.} \end{aligned}$$

2.4 A semigroup proof of Borell's inequality

Recall the Ornstein-Uhlenbeck semigroup P_t that we introduced in Section 1.3. We used it in Section 1.4 to Bobkov's inequality by showing that $P_s \sqrt{I^2(P_{t-s}f) + |\nabla P_{t-s}f|^2}$ is non-decreasing in s . One nice feature of the functional inequality (2.9) is that the same approach works. In [40] we showed, with Mossel, that $\mathbb{E}_\rho J_\rho(P_t f(X), P_t f(Y))$ is non-decreasing in t ; in this section we will reproduce that argument. One advantage of the this new approach is that it brings the equality and near-equality cases of (2.9) within range: in the next section, we will characterize the equality cases (i.e., prove Theorem 2.6), and we will study the question of almost-equality in Chapter 4.

Consider the quantity

$$R_t = \mathbb{E}_\rho J_\rho(P_t f(X), P_t g(Y)). \quad (2.10)$$

Recall that $P_t f \rightarrow f$ as $t \rightarrow 0$ and $P_t f \rightarrow \mathbb{E}f$ as $t \rightarrow \infty$. Hence, R_t converges to the left hand side of (2.9) as $t \rightarrow 0$; as $t \rightarrow \infty$, R_t converges to the right hand side of (2.9). We will prove Theorem 2.5 by showing that $\frac{dR_t}{dt} \leq 0$ for all $t > 0$.

For brevity, define $f_t = P_t f$, $g_t = P_t g$, $v_t = \Phi^{-1} \circ f_t$, and $w_t = \Phi^{-1} \circ g_t$. Let $K_\rho(x, y) = \Pr_\rho(X \leq x, Y \leq y)$ so that $J_\rho(f_t(x), g_t(y)) = K_\rho(v_t(x), w_t(y))$ (we will tend to drop the explicit mention of ρ in J and K from now on unless we need to emphasize it).

Lemma 2.7.

$$\begin{aligned} \frac{\partial K(x, y)}{\partial x} &= \phi(x) \Phi\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right) \\ \frac{\partial K(x, y)}{\partial y} &= \phi(y) \Phi\left(\frac{x - \rho y}{\sqrt{1 - \rho^2}}\right). \end{aligned}$$

Proof. Note that $(X, Y) \stackrel{d}{=} (X, \rho X + \sqrt{1 - \rho^2} Z)$, where Z is a standard Gaussian vector that is independent from X and Y . Then $\Pr_\rho(X_1 \leq x, Y_1 \leq y) = \Pr(X_1 \leq x, Z_1 \leq \frac{y - \rho X_1}{\sqrt{1 - \rho^2}})$, and so

$$K(x, y) = \int_{-\infty}^x \int_{-\infty}^{\frac{y - \rho s}{\sqrt{1 - \rho^2}}} \phi(s) \phi(t) dt ds.$$

Differentiating in x ,

$$\frac{\partial K(x, y)}{\partial x} = \int_{-\infty}^{\frac{y - \rho x}{\sqrt{1 - \rho^2}}} \phi(x) \phi(t) dt = \phi(x) \Phi\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right).$$

This proves the first claim. The second follows because $K(x, y)$ is symmetric in x and y . \square

With Lemma 2.7 in hand, differentiating R_t is not difficult.

Lemma 2.8.

$$\frac{dR_t}{dt} = \frac{\rho}{2\pi\sqrt{1-\rho^2}} \mathbb{E}_\rho \left(\exp \left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)} \right) |\nabla v_t - \nabla w_t|^2 \right).$$

Before we prove Lemma 2.8, note that it immediately implies Theorem 2.5 because the right hand side in Lemma 2.8 is clearly non-negative.

Proof. By the chain rule,

$$\frac{dR_t}{dt} = \mathbb{E}_\rho \left(K_x(v_t(X), w_t(Y)) \frac{dv_t(X)}{dt} \right) + \mathbb{E}_\rho \left(K_y(v_t(X), w_t(Y)) \frac{dw_t(X)}{dt} \right). \quad (2.11)$$

Now, the chain rule applied to $v_t = \Phi^{-1} \circ f_t$ implies that $\frac{dv_t}{dt} = \frac{L f_t}{\phi(v_t)}$. Hence, the first term of (2.11) is

$$\mathbb{E}_\rho \left(\frac{K_x(v_t(X), w_t(Y))}{\phi(v_t(X))} L f_t(X) \right) = \mathbb{E}_\rho \Phi \left(\frac{w_t(Y) - \rho v_t(X)}{\sqrt{1-\rho^2}} \right) L f_t(X), \quad (2.12)$$

where we have used Lemma 2.7. Now we will integrate by parts; we will do so carefully because it is easy to misplace a factor of ρ : since X and Y have covariance ρI_n , we may write the expectation in integral form as

$$\begin{aligned} (2.12) &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi \left(\frac{w_t(\rho x + \sqrt{1-\rho^2}z) - \rho v_t(x)}{\sqrt{1-\rho^2}} \right) L f_t(x) d\gamma_n(x) d\gamma_n(z) \\ &= -\frac{\rho}{\sqrt{1-\rho^2}} \iint \phi \left(\frac{w_t(\rho x + \sqrt{1-\rho^2}z) - \rho v_t(x)}{\sqrt{1-\rho^2}} \right) \langle \nabla w_t - \nabla v_t, \nabla f_t \rangle d\gamma_n(x) d\gamma_n(z) \\ &= -\frac{\rho}{\sqrt{1-\rho^2}} \mathbb{E}_\rho \phi \left(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}} \right) \langle \nabla w_t - \nabla v_t, \nabla f_t \rangle \\ &= \frac{\rho}{\sqrt{1-\rho^2}} \mathbb{E}_\rho \phi \left(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}} \right) \phi(v_t) \langle \nabla v_t - \nabla w_t, \nabla v_t \rangle, \end{aligned} \quad (2.13)$$

where we have written, for brevity, v_t and w_t instead of $v_t(X)$ and $w_t(Y)$. (The purpose of turning the expectation into an integral and back again was to be clear about why the factor ρ appears in front of ∇w_t when we integrated the inner integral by parts.) Since K is symmetric in its arguments, there is a similar computation for the second term of (2.11):

$$\mathbb{E} \left(K_y(v_t(X), w_t(Y)) \frac{dw_t(X)}{dt} \right) = -\frac{\rho}{\sqrt{1-\rho^2}} \mathbb{E}_\rho \phi \left(\frac{v_t - \rho w_t}{\sqrt{1-\rho^2}} \right) \phi(w_t) \langle \nabla v_t - \nabla w_t, \nabla w_t \rangle. \quad (2.14)$$

Note that the terms involving ϕ in (2.13) and (2.14) are actually the same:

$$\phi \left(\frac{w_t - \rho v_t}{\sqrt{1-\rho^2}} \right) \phi(v_t) = \phi \left(\frac{v_t - \rho w_t}{\sqrt{1-\rho^2}} \right) \phi(w_t) = \frac{1}{2\pi} \exp \left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)} \right);$$

hence, we can plug (2.13) and (2.14) into (2.11) to obtain

$$\frac{dR_t}{dt} = \frac{\rho}{2\pi\sqrt{1-\rho^2}} \mathbb{E} \left(\exp \left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)} \right) \middle| |\nabla v_t - \nabla w_t|^2 \right). \quad \square$$

2.5 The equality cases in Borell's inequality

Lemma 2.8 allows us to analyze the equality case (Theorem 2.6), with very little additional effort. The method we will pursue here is analogous to that of Carlen and Kerce [11], which we presented in Section 1.6 to analyze the equality case in the standard Gaussian isoperimetric problem. Suppose equality is attained in (2.9). Since $\frac{dR_t}{dt}$ is non-negative, it must be zero for almost every $t > 0$. In particular, we may fix some $t > 0$ such that $\frac{dR_t}{dt} = 0$. Note that everything in Lemma 2.8 is strictly positive, except for the term $|\nabla v_t(X) - \nabla w_t(Y)|^2$, which can be zero. Therefore, $\frac{dR_t}{dt} = 0$ implies that $\nabla v_t(X) = \nabla w_t(Y)$ almost surely. Since the conditional distribution of Y given X is fully supported, ∇v_t and ∇w_t must be almost surely equal to some constant $a' \in \mathbb{R}^n$. Moreover, v_t and w_t are smooth functions (because f_t, g_t and Φ^{-1} are smooth); hence, $v_t(x) = \langle a', x - b' \rangle$ and $w_t(x) = \langle a', x - d' \rangle$ for some $b', d' \in \mathbb{R}^n$, and so

$$\begin{aligned} f_t(x) &= \Phi(\langle a', x - b' \rangle) \\ g_t(x) &= \Phi(\langle a', x - d' \rangle). \end{aligned}$$

Theorem 2.6 follows by applying Lemma 1.10 to f and g separately.

2.6 The exchangeable Gaussians inequality

In this section, we will develop a more general point of view regarding the computation we carried out in the last section. As the main application of this generalization, we re-derive the ‘‘exchangeable Gaussians theorem’’ of Isaksson and Mossel [25].

Theorem 2.9. *Let $(X^{(1)}, \dots, X^{(k)})$ be a mean zero Gaussian vector in $\mathbb{R}^n \times \dots \times \mathbb{R}^n \cong \mathbb{R}^{kn}$ with covariance matrix*

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} \otimes I_n$$

for some $0 < \rho < 1$. Then for any sets A_1, \dots, A_k ,

$$\Pr(X^{(i)} \in A_i \text{ for all } i) \leq \Pr(X_1^{(i)} \in A_i) \leq \Phi^{-1}(\gamma_n(A_i)) \text{ for all } i. \quad (2.15)$$

Note that this generalizes Borell's inequality, which is recovered in the case $k = 2$. One advantage of our proof over the original [25] is that we obtain a characterization of the equality cases:

Theorem 2.10. *If the sets $A_1, \dots, A_k \subset \mathbb{R}^n$ achieve equality in (2.15) then there exist $a, b_1, \dots, b_k \in \mathbb{R}^n$ such that for all i , $A_i = \{x \in \mathbb{R}^n : \langle a, x - b_i \rangle \geq 0\}$ up to a set of measure zero.*

Our main tool in Chapter 1 was (1.7). Note that while (1.7) dealt with a standard Gaussian vector, in this chapter we have been discussing correlated Gaussian vectors. This is only a superficial difference, since any correlated Gaussian vector can be constructed from a standard Gaussian vector by a linear transformation.

Proposition 2.11. *Suppose f_1, \dots, f_k are measurable functions $\mathbb{R}^n \rightarrow [0, 1]$ and $M = (m_{ij}) \geq 0$ is a $k \times k$ matrix with $m_{ii} = 1$. If $\Psi : [0, 1]^k \rightarrow \mathbb{R}$ satisfies $M \odot \text{Hess}(\Psi) \leq 0$ then*

$$\mathbb{E}\Psi(f_1(X^{(1)}), \dots, f_k(X^{(k)})) \leq \Psi(\mathbb{E}f_1(X^{(1)}), \dots, \mathbb{E}f_k(X^{(k)})), \quad (2.16)$$

where \odot denotes the element-wise (or Hadamard) product, and the expectation is with respect to a kn -dimensional Gaussian vector $(X^{(1)}, \dots, X^{(k)})$ with mean zero and covariance $M \otimes I_n$.

In order to write (1.7) for non-standard Gaussian vectors, we introduce some new notation: for any $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and any $n \times m$ matrix M , denote the function $f \circ M : \mathbb{R}^m \rightarrow \mathbb{R}$ by f^M .

Proof. Let $Q = (q_{ij})$ be the positive definite square root of M , and for $i = 1, \dots, k$, let Q_i be the $n \times kn$ matrix $(q_{i1}I_n \cdots q_{ik}I_n)$. Let Z be a standard Gaussian vector in \mathbb{R}^{kn} , and note that $QZ = (Q_1Z, \dots, Q_kZ)$ is a kn -dimensional Gaussian vector with mean 0 and covariance $M \otimes I_n$. We consider the quantity

$$F(s, t, z) = P_s \Psi(P_{t-s}f_1^{Q_1}(z_1), \dots, P_{t-s}f_k^{Q_k}(z_k))$$

for $z = (z_1, \dots, z_k) \in \mathbb{R}^{kn}$. Since $m_{ii} = 1$, we have $Q_i^T Q_i = I_n$ and so

$$\begin{aligned} (P_t f_i^{Q_i})(x) &= \int_{\mathbb{R}^n} f_i(e^{-t}Q_i x + \sqrt{1 - e^{-2t}}Q_i y) d\gamma_{kn}(y) \\ &= \int_{\mathbb{R}^n} f_i(e^{-t}Q_i x + \sqrt{1 - e^{-2t}}y) d\gamma_n(y) \\ &= (P_t f_i)^{Q_i}(x). \end{aligned}$$

Hence, $\nabla P_{t-s}f_i^{Q_i} = \nabla(P_{t-s}f_i)^{Q_i} = Q_i^T(\nabla P_{t-s}f_i)^{Q_i}$ and so

$$\langle \nabla P_{t-s}f_i^{Q_i}, \nabla P_{t-s}f_j^{Q_j} \rangle = m_{ij} \langle (\nabla P_{t-s}f_i)^{Q_i}, (\nabla P_{t-s}f_j)^{Q_j} \rangle.$$

Thus, by (1.7),

$$\frac{\partial F(s, t, z)}{\partial s} = P_s \sum_{i,j=1}^k \langle (\nabla P_{t-s}f_i)^{Q_i}, (\nabla P_{t-s}f_j)^{Q_j} \rangle m_{ij} \frac{\partial^2 \Psi}{\partial x_i \partial x_j} (P_{t-s}f_1^{Q_1}, \dots, P_{t-s}f_k^{Q_k}). \quad (2.17)$$

If the matrix $M \odot \text{Hess}(\Psi)$ is non-positive definite, then $\frac{\partial F(s, t, z)}{\partial s} \leq 0$ for every s, t and z . In particular, $\lim_{t \rightarrow \infty} F(t, t, Z) \leq \lim_{t \rightarrow \infty} F(0, t, Z)$. But since (Q_1Z, \dots, Q_kZ) are distributed as $(X^{(1)}, \dots, X^{(k)})$, $\mathbb{E}F(t, t, Z)$ converges to the left hand side of (2.16) and $\mathbb{E}F(0, t, Z)$ converges to the right hand side. \square

On attraction of this proof is that we obtain an explicit expression for the deficit in (2.16). As for the Gaussian isoperimetric inequality, where the explicit formula (1.17) allowed us to analyze the equality cases of Bobkov's inequality, the expression we obtain here will allow us to analyze the equality cases in Theorem 2.5:

Corollary 2.12. *Equality is attained in (2.16) if and only if for every $s > 0$ and every $x = (x_1, \dots, x_k)$ in the support of $(X^{(1)}, \dots, X^{(n)})$, $(\nabla P_s f_1(x_1), \dots, \nabla P_s f_k(x_k))$ is in the kernel of the matrix*

$$\left(m_{ij} \frac{\partial^2 \Psi}{\partial x_i \partial x_j} (P_s f_1(x_1), \dots, P_s f_k(x_k)) \right)_{i,j=1}^k \otimes I_n.$$

Proof. With $F(s, t, x)$ defined as in the previous proof, we have

$$\mathbb{E} \Psi(f_1(X^{(1)}), \dots, f_k(X^{(k)})) - \Psi(\mathbb{E} f_1, \dots, \mathbb{E} f_k) = \lim_{t \rightarrow \infty} \int_0^t \mathbb{E} \frac{\partial F(s, t, X)}{\partial s} ds.$$

Since (as we showed in the previous proof) $\frac{\partial F(s, t, z)}{\partial s} \leq 0$ for all s, t , and z , it follows that for equality to be attained in (2.16), we must have $\mathbb{E} \frac{\partial F(s, t, Z)}{\partial s} = 0$ for all s, t , and z . Now, the right hand side of (2.17) may be written as

$$\frac{\partial F(s, t, z)}{\partial s} = v^T \left(\left(m_{ij} \frac{\partial^2 \Psi}{\partial x_i \partial x_j} (P_s f_1(x_1), \dots, P_s f_k(x_k)) \right)_{i,j=1}^k \otimes I_n \right) v$$

where $v^T = ((\nabla P_{t-s} f_1(x_1))^T \dots (\nabla P_{t-s} f_k(x_k))^T)$ and $x_i = Q_i z_i$. Since the quadratic form above is non-negative definite, $\frac{\partial F(s, t, z)}{\partial s} = 0$ only if v belongs to its kernel. \square

The Hessian of J_ρ and Borell's inequality

Like (1.7), Proposition 2.11 is interesting mainly because it has interesting consequences. Here, we consider our simplest application of Proposition 2.11: the case $M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $\Psi = J_\rho$ which corresponds to Theorem 2.5. (We have already proved Theorem 2.5, but we will give a slightly different view of the proof here.)

Using Lemma 2.7 and the formula $\frac{d\Phi^{-1}}{dx} = \frac{1}{\phi(\Phi^{-1}(x))}$, we apply the chain rule to obtain the first derivatives of J :

$$\begin{aligned} \frac{\partial J(x, y)}{\partial x} &= \Phi \left(\frac{\Phi^{-1}(y) - \rho \Phi^{-1}(x)}{\sqrt{1 - \rho^2}} \right) \\ \frac{\partial J(x, y)}{\partial y} &= \Phi \left(\frac{\Phi^{-1}(x) - \rho \Phi^{-1}(y)}{\sqrt{1 - \rho^2}} \right) \end{aligned}$$

From there, the second derivatives follow by applying the chain rule again:

$$\begin{aligned}\frac{\partial^2 J(x, y)}{\partial x^2} &= -\frac{\rho}{I(x)\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(y) - \rho\Phi^{-1}(x)}{\sqrt{1-\rho^2}}\right) = -\frac{\rho\psi(x, y)}{2\pi I^2(x)\sqrt{1-\rho^2}} \\ \frac{\partial^2 J(x, y)}{\partial y^2} &= -\frac{\rho}{I(y)\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(x) - \rho\Phi^{-1}(y)}{\sqrt{1-\rho^2}}\right) = -\frac{\rho\psi(x, y)}{2\pi I^2(y)\sqrt{1-\rho^2}} \\ \frac{\partial^2 J(x, y)}{\partial x\partial y} &= \frac{1}{I(y)\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(y) - \rho\Phi^{-1}(x)}{\sqrt{1-\rho^2}}\right) = \frac{\psi(x, y)}{2\pi I(x)I(y)\sqrt{1-\rho^2}},\end{aligned}$$

where

$$\psi(x, y) = \exp\left(-\frac{\Phi^{-2}(y) - 2\rho\Phi^{-1}(x)\Phi^{-1}(y) + \Phi^{-2}(x)}{2(1-\rho^2)}\right) \quad (2.18)$$

and the second equality in each line follows by expanding out the definition of ϕ . In the end, we obtain the expression

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \odot \text{Hess}(J_\rho) = \frac{\rho\psi(x, y)}{2\pi\sqrt{1-\rho^2}} \begin{pmatrix} 1/I(x) & 0 \\ 0 & 1/I(y) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/I(x) & 0 \\ 0 & 1/I(y) \end{pmatrix}. \quad (2.19)$$

Since the right hand side is non-positive, this and Proposition 2.11 give another proof of Theorem 2.5.

Proof of the exchangeable Gaussians inequality

The exchangeable Gaussians inequality may be proved with the same logic, but slightly more complicated calculus, than the proof of Borell's inequality that we gave in the previous section. First, to state the functional version of Theorem 2.9, we define

$$K_k(x_1, \dots, x_k; \rho) = \Pr_\rho(X_1^{(i)} \leq x_i \text{ for all } i = 1, \dots, k),$$

where \Pr_ρ means the probability with respect to an kn -dimensional mean zero Gaussian vector $(X^{(1)}, \dots, X^{(k)})$ with covariance

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} \otimes I_n.$$

If we define $J_{\rho,k}(x_1, \dots, x_k) = K_k(\Phi^{-1}(x_1), \dots, \Phi^{-1}(x_k); \rho)$, then we have a functional version of Theorem 2.9:

Theorem 2.13. *For any measurable functions $f_1, \dots, f_k : \mathbb{R}^n \rightarrow [0, 1]$ and any $0 < \rho < 1$,*

$$\mathbb{E}J_{\rho,k}(f_1(X^{(1)}), \dots, f_k(X^{(k)})) \leq J_{\rho,k}(\mathbb{E}f_1, \dots, \mathbb{E}f_k)$$

With Proposition 2.11 in hand, the main work in establishing Theorem 2.13 is to compute the Hessian of $J_{\rho,k}$.

Lemma 2.14.

$$\frac{\partial K_k(x_1, \dots, x_k; \rho)}{\partial x_i} = \phi(x_i) K_{k-1} \left(\frac{x_1 - \rho x_i}{\sqrt{1 - \rho^2}}, \dots, \hat{i}, \dots, \frac{x_k - \rho x_i}{\sqrt{1 - \rho^2}}, \frac{\rho}{1 + \rho} \right), \quad (2.20)$$

where \hat{i} denotes the absence of the i th term.

Since $K_1(x) = \Pr(X_1 \leq x) = \Phi(x)$, Lemma 2.14 is a straightforward generalization of Lemma 2.7.

Proof. Let $M_{k,\rho}$ be the $k \times k$ matrix with 1 on the diagonal and ρ off the diagonal. Let $Z = (Z_1, \dots, Z_k)$ be a mean-zero Gaussian vector with covariance $M_{k,\rho}$; then $K_k(x_1, \dots, x_k; \rho) = \Pr(Z_j \leq z_j \text{ for all } j)$. The standard Schur complement formula for conditional distributions of Gaussian vectors shows that the conditional distribution of (Z_2, \dots, Z_k) given $Z_1 = x_1$ has mean $(\rho \dots \rho)^T x_1$ and covariance

$$M_{k-1,\rho} - (\rho \dots \rho) \begin{pmatrix} \rho \\ \vdots \\ \rho \end{pmatrix} = (1 - \rho^2) M_{k-1,\rho/(1+\rho)}.$$

Decomposing the probability

$$K_k(x_1, \dots, x_k; \rho) = \Pr(Z_1 \leq x_1, \dots, Z_k \leq x_k) = \Pr(Z_1 \leq x_1) \Pr(Z_2 \leq x_2, \dots, Z_k \leq x_k \mid Z_1 \leq x_1),$$

we have

$$\begin{aligned} \frac{\partial K_k(x_1, \dots, x_k; \rho)}{\partial x_1} &= \phi(x_1) \Pr(Z_2 \leq x_2, \dots, Z_k \leq x_k \mid Z_1 = x_1) \\ &= \phi(x_1) \Pr \left(\frac{Z_2 - \rho x_1}{\sqrt{1 - \rho^2}} \leq \frac{x_2 - \rho x_1}{\sqrt{1 - \rho^2}}, \dots, \frac{Z_k - \rho x_1}{\sqrt{1 - \rho^2}} \leq \frac{x_k - \rho x_1}{\sqrt{1 - \rho^2}} \mid Z_1 = x_1 \right). \end{aligned}$$

But from what we said above, the conditional distribution of $(Z_2 - \rho x_1, \dots, Z_k - \rho x_1) / \sqrt{1 - \rho^2}$ given $Z_1 = x_1$ has mean zero and covariance $M_{k-1,\rho/(1+\rho)}$. Hence, the definition of K implies that

$$\frac{\partial K_k(x_1, \dots, x_k; \rho)}{\partial x_1} = \phi(x_1) K_{k-1} \left(\frac{x_2 - \rho x_1}{\sqrt{1 - \rho^2}}, \dots, \frac{x_k - \rho x_1}{\sqrt{1 - \rho^2}}, \frac{\rho}{1 + \rho} \right).$$

This proves the claim for $i = 1$ which, by symmetry, is enough. \square

By Lemma 2.14 and the chain rule, we have the first derivatives of J :

$$\frac{\partial J_{\rho,k}}{\partial x_i} = K_{k-1} \left(\frac{\Phi^{-1}(x_1) - \rho \Phi^{-1}(x_i)}{\sqrt{1 - \rho^2}}, \dots, \hat{i}, \dots, \frac{\Phi^{-1}(x_k) - \rho \Phi^{-1}(x_i)}{\sqrt{1 - \rho^2}}, \frac{\rho}{1 + \rho} \right). \quad (2.21)$$

To calculate the second derivatives of J , we apply the chain rule to (2.21) and then use Lemma 2.14 to compute the derivative of K_{k-1} :

$$\frac{\partial^2 J_{\rho,k}}{\partial x_i \partial x_j} = \frac{1}{I(x_j) \sqrt{1-\rho^2}} \phi \left(\frac{\Phi^{-1}(x_j) - \rho \Phi^{-1}(x_i)}{\sqrt{1-\rho^2}} \right) K_{k-2} \left(\frac{\Phi^{-1}(x_1) - \frac{\rho}{1+\rho} (\Phi^{-1}(x_i) + \Phi^{-1}(x_j))}{\sqrt{(1-\rho^2)(1-(\rho/(1+\rho))^2)}}, \dots, \hat{i}, \hat{j}, \dots, \frac{\Phi^{-1}(x_k) - \frac{\rho}{1+\rho} (\Phi^{-1}(x_i) + \Phi^{-1}(x_j))}{\sqrt{(1-\rho^2)(1-(\rho/(1+\rho))^2)}} \right).$$

Let p_{ij} denote the K_{k-2} term in the equation above; by expanding the definition of ϕ and recalling the function ψ from (2.18), we have

$$\frac{\partial^2 J_{\rho,k}}{\partial x_i \partial x_j} = \frac{\psi(x_i, x_j)}{2\pi I(x_i) I(x_j) \sqrt{1-\rho^2}} p_{ij}.$$

The repeated second derivatives $\frac{\partial^2 J_{\rho,k}}{\partial x_i^2}$ are similar. There are only two differences: in (2.20), every argument to K_{k-1} contains an x_i term (whereas only the j th term contained an x_j term), and each x_i term comes with a $-\rho$ factor. Thus,

$$\frac{\partial^2 J_{\rho,k}}{\partial x_i^2} = -\frac{\rho}{2\pi I^2(x_i) \sqrt{1-\rho^2}} \sum_{j \neq i} \psi(x_i, x_j) p_{ij}.$$

Therefore,

$$M_{k,\rho} \odot \text{Hess}(J_{\rho,k}) = \frac{\rho}{2\pi \sqrt{1-\rho^2}} \mathcal{I}(x) \begin{pmatrix} -\sum_{j \neq 1} q_{1j} & q_{12} & \cdots & q_{1k} \\ q_{21} & -\sum_{j \neq 2} q_{2j} & \cdots & q_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ q_{k1} & q_{k2} & \cdots & -\sum_{j \neq k} q_{kj} \end{pmatrix} \mathcal{I}(x) \quad (2.22)$$

where $q_{ij} = \psi(x_i, x_j) p_{ij}$ and $\mathcal{I}(x)$ is the diagonal matrix with $1/I(x_i)$ as the i, i entry. It is now easy to see that $M_{k,\rho} \odot \text{Hess}(J_{\rho,k})$ is non-positive: for any $v \in \mathbb{R}^k$,

$$v^T \begin{pmatrix} -\sum_{j \neq 1} q_{1j} & q_{12} & \cdots & q_{1k} \\ q_{21} & -\sum_{j \neq 2} q_{2j} & \cdots & q_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ q_{k1} & q_{k2} & \cdots & -\sum_{j \neq k} q_{kj} \end{pmatrix} v = -\sum_{i \neq j} q_{ij} (v_i - v_j)^2 \leq 0.$$

Since $\mathcal{I} \geq 0$, it follows from (2.22) that $M_{k,\rho} \odot \text{Hess}(J_{\rho,k}) \leq 0$. By Proposition 2.11, this proves Theorem 2.13.

From here, the equality cases of Theorem 2.13 may be easily obtained. By Corollary 2.12, the quantity of interest is

$$((\nabla P_t f_1)^T \cdots (\nabla P_t f_k)^T) \left((M_{k,\rho} \odot \text{Hess}(J_{\rho,k})) \otimes I_n \right) \begin{pmatrix} \nabla P_t f_1 \\ \vdots \\ \nabla P_t f_k \end{pmatrix}, \quad (2.23)$$

which for all $t > 0$ must be equal to zero pointwise in order to have equality in Theorem 2.13. Writing $(v_t^1, \dots, v_t^k) = (\Phi^{-1} \circ P_t f_1, \dots, \Phi^{-1} \circ P_t f_k)$, we have $\nabla v_t^i = \nabla(P_t f_i)/I(P_t f_i)$ and so by (2.22),

$$(2.23) = \frac{\rho}{2\pi\sqrt{1-\rho^2}} ((\nabla v_t^1)^T \dots (\nabla v_t^k)^T) (Q \otimes I_n) \begin{pmatrix} \nabla v_t^1 \\ \vdots \\ \nabla v_t^k \end{pmatrix},$$

where Q is the $k \times k$ matrix with q_{ij} off the diagonal and $-\sum_{j \neq i} q_{ij}$ as the i th diagonal entry. Thus

$$(2.23) = -\frac{\rho}{2\pi\sqrt{1-\rho^2}} \sum_{i \neq j} q_{ij} |\nabla v_t^i - \nabla v_t^j|^2.$$

Now, q_{ij} is strictly positive pointwise, because $q_{ij} = \psi(P_t f_i, P_t f_j) p_{ij}$ where ψ is the exponential of some quantity and p_{ij} is K_{k-2} of some quantity. Since \exp and K_{k-2} are strictly positive functions, q_{ij} is strictly positive and so (2.23) can only be zero if for all $i \neq j$ and all $t > 0$, ∇v_t^i and ∇v_t^j are equal to the same constant. We have seen this condition twice already – in the equality cases for the Gaussian isoperimetric inequality and in the equality cases of Borell's inequality – and it implies that there are $a, b_1, \dots, b_k \in \mathbb{R}^n$ such that either $f_i(x) = \Phi(\langle a, x - b_i \rangle)$ for all i , or $f_i(x) = 1_{\{\langle a, x - b_i \rangle \geq 0\}}$ for all i . In particular, we have characterized the equality cases of Theorem 2.13:

Theorem 2.15. *For any measurable functions $f_1, \dots, f_k : \mathbb{R}^n \rightarrow [0, 1]$, if there exists $0 < \rho < 1$ such that equality holds in Theorem 2.13 then there exist $a, b_1, \dots, b_k \in \mathbb{R}^n$ such that either*

$$f_i(x) = \Phi(\langle a, x - b_i \rangle) \text{ a.s. for every } i$$

or

$$f_i(x) = 1_{\{\langle a, x - b_i \rangle \geq 0\}} \text{ a.s. for every } i.$$

Of course, this immediately implies Theorem 2.10.

Chapter 3

Robust Gaussian isoperimetry

For this chapter, we return to the isoperimetric inequality of Chapter 1 or, more precisely, to Bobkov's inequality: for all smooth $f : \mathbb{R}^n \rightarrow [0, 1]$,

$$\mathbb{E}\sqrt{I^2(f) + |\nabla f|^2} \geq I(\mathbb{E}f). \quad (3.1)$$

In Chapter 1, we proved this inequality and studied its equality cases: recall that equality is attained for a smooth function f if and only if f takes the form $f(x) = \Phi(\langle a, x - b \rangle)$ (if we relax the smoothness condition then there is also the limiting case $f(x) = 1_{\{\langle a, x - b \rangle \geq 0\}}$).

After studying the equality cases of an inequality, the next level of refinement is to consider the cases of almost equality. Specifically, suppose there is some small δ such that

$$\mathbb{E}\sqrt{I^2(f) + |\nabla f|^2} \leq I(\mathbb{E}f) + \delta.$$

Does this imply that f is close to one of the equality cases? If this is true (and it is) then we say that the equality cases of (3.1) are *robust*.

The study of robust isoperimetric inequalities goes back to Bonnesen [8], who studied a robust version of the Euclidean isoperimetric inequality in \mathbb{R}^2 ; thus, robust isoperimetric inequalities in Euclidean space are sometimes known as Bonnesen-type inequalities. In the Gaussian case, robust isoperimetric inequalities have appeared only recently: the first result comes from Cianchi et al. [13] in 2011.

Theorem 3.1. *If $A \subset \mathbb{R}^n$ satisfies $I(\gamma_n(A)) \geq \gamma_n^+(A) - \delta$ then there is a half-space B such that*

$$\gamma_n(A \Delta B) \leq C(n, \gamma_n(A))\sqrt{\delta},$$

where $C(n, r)$ is some function of n and r and Δ denotes the symmetric difference.

The dependence on δ in Theorem 3.1 is sharp, but the dependence on n is certainly not: indeed, the existence of such a $C(n, r)$ was shown using compactness arguments and so no upper bounds on $C(n, r)$ are even known. This is somewhat unsatisfying, since properties of Gaussian space are often dimension-independent. The isoperimetric inequality itself is an example of this phenomenon, as the Gaussian isoperimetric function I does not depend on the dimension.

Dimension-free robustness for the isoperimetric inequality

The main result of this chapter is a robust isoperimetric inequality that is stronger in one sense, but weaker in another sense, than Theorem 3.1. In particular, the dependence on n is optimal (i.e. there is none), but the dependence on δ is far from it. For a smooth function $f : \mathbb{R}^n \rightarrow [0, 1]$, define

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2} - I(\mathbb{E}f).$$

Theorem 3.2. *There exists a universal constant $C > 0$ such that for all smooth $f : \mathbb{R}^n \rightarrow [0, 1]$, there exist $a, b \in \mathbb{R}^n$ such that*

$$\mathbb{E}(f(X) - \Phi(\langle a, X - b \rangle))^2 \leq C \frac{1}{\sqrt{\log(1/\delta(f))}}.$$

The most interesting special case of Theorem 3.2 is when f is the indicator function of some set. Such an f is not smooth, of course, but the arguments of Section 1.2 show that it can be approximated by smooth functions. Thus we obtain a robustness result for the Gaussian isoperimetric inequality: for $A \subset \mathbb{R}^n$, define

$$\delta(A) = \gamma_n^+(A) - I(\gamma_n(A)).$$

Corollary 3.3. *There exists an absolute constant C such that for any measurable set $A \subset \mathbb{R}^n$ there exists an affine half-space B such that*

$$\gamma_n(A \Delta B) \leq C \frac{1}{\sqrt{\log(1/\delta(A))}}.$$

The proof outline

Our starting point for the proof Theorem 3.2 is (1.18). Taking $t \rightarrow \infty$ and setting $v_t = \Phi^{-1} \circ P_t f$, (1.18) becomes

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2} - I(\mathbb{E}f) \geq \int_0^\infty \mathbb{E} \frac{\phi(v_t) \|\text{Hess}(v_t)\|_F^2}{(1 + |\nabla v_t|^2)^{3/2}} dt. \quad (3.2)$$

From here, the proof proceeds in two main steps. In the first step, we argue that for every $t > 0$, v_t must be close to a linear function, and so f_t must be close to a function of the form $\Phi(\langle a, x - b \rangle)$. This step makes use of tools from analysis, in particular the Hölder and reverse-Hölder inequalities, and some smoothness properties of the semigroup P_t (most importantly, Theorem 1.7).

The second step of the proof is to show that if f_t is close to a function of the form $\Phi(\langle a, x - b \rangle)$ then f is also close to a function of the same form. This is unfortunately not true for general functions f , because P_t does not have a bounded inverse. Using a spectral argument, we show that P_t has a bounded inverse on functions that we care about.

One remark on notation: since this chapter will generally not be concerned with the exact value of universal constants, we will often use C and c for generic universal constants whose value may change from line to line.

3.1 Approximation for large t

This section is devoted to the proof of Proposition 3.4, which accomplishes the first step of the proof outline we gave above, showing that v_t can be approximated by an affine function for some $t \in [1, 2]$.

Proposition 3.4. *There is a universal constant $C > 0$ such that for any measurable $f : \mathbb{R}^n \rightarrow [0, 1]$ there exist $t \in [1, 2]$ and $a, b \in \mathbb{R}^n$ such that $|a| \leq L_t$ and*

$$\mathbb{E}(v_t(X) - \langle a, X - b \rangle)^2 \leq C \frac{\delta^{1/4}(f)}{m(f)^C},$$

where $v_t = \Phi^{-1} \circ (P_t f)$ and $m(f) = (\mathbb{E}f)(1 - \mathbb{E}f)$.

We remark that the restriction $t \in [1, 2]$ is not essential; the proof that we give for Proposition 3.4 may be carried out in greater generality for any $t > 0$; however, the constants in Proposition 3.4 will then depend on t .

A second-order Poincaré inequality

In proving the equality cases of Bobkov's inequality, we observed that if $\|\text{Hess}(v_t)\|_F^2$ vanishes then v_t must be a linear function. The first step towards Proposition 3.4 is a quantitative version of this observation. To that end, recall Poincaré's inequality (of which we gave a semigroup proof in Section 1.3)

$$\mathbb{E}f^2 - (\mathbb{E}f)^2 \leq \mathbb{E}|\nabla f|^2. \quad (3.3)$$

If we apply (3.3) to the partial derivatives of $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we obtain

$$\mathbb{E} \left(\frac{\partial f}{\partial x_i} \right)^2 - \left(\mathbb{E} \frac{\partial f}{\partial x_i} \right)^2 \leq \mathbb{E} \left(\frac{\partial f}{\partial x_i} \right)^2 \leq \sum_{j=1}^n \mathbb{E} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2.$$

Summing over i yields $\mathbb{E}|\nabla f|^2 - |\nabla \mathbb{E}f|^2 \leq \mathbb{E}\|\text{Hess}(f)\|_F^2$, and this can be combined with (3.3) to obtain a second-order version of Poincaré's inequality:

$$\mathbb{E}(f - \mathbb{E}f - \langle X, \mathbb{E}\nabla f \rangle)^2 = \mathbb{E}f^2 - (\mathbb{E}f)^2 - |\nabla \mathbb{E}f|^2 \leq \mathbb{E}\|\text{Hess}(f)\|_F^2, \quad (3.4)$$

where the first equality follows because integration by parts implies that $\mathbb{E}Xf(X) = \mathbb{E}\nabla f$; hence the orthogonal projection of f onto the span of linear functions is $\langle X, \mathbb{E}\nabla f \rangle$. In particular, (3.4) implies that functions f with $\mathbb{E}\|\text{Hess}(f)\|_F^2$ small are close to linear. This puts us on our way towards the first step in the proof of Theorem 3.2. Indeed, if we could remove the $\phi(v_t)(1 + |\nabla v_t|^2)^{-3/2}$ term from the right hand side of (3.2), we would be done already. The removal of this nuisance term is the topic of the next section.

The reverse-Hölder inequality

We are interested in a lower bound on $\mathbb{E}\phi(v_t)\|\text{Hess}(v_t)\|_F^2(1+|\nabla v_t|^2)^{-3/2}$. Since Theorem 1.7 gives a pointwise upper bound on $|\nabla v_s|$, it remains to find a lower bound on $\mathbb{E}\phi(v_t)\|\text{Hess}(v_t)\|_F^2$.

Proposition 3.5. *There is a universal constant $C > 0$ such that for every $t > 0$ and every $p < 1$ satisfying $p/(1-p) < L_t^{-2}/2$,*

$$\mathbb{E}(\phi(v_t)\|\text{Hess}(v_t)\|_F^2) \geq C^{(p-1)/p} e^{\frac{p-1}{p}N_t^2} (\mathbb{E}\|\text{Hess}(v_t)\|_F^{2p})^{1/p}$$

where N_t is a median of v_t .

The first step in the proof of Proposition 3.5 is the reverse-Hölder inequality, which is classical but perhaps not widely known: for any $p < 1$ and any positive functions f and g ,

$$\mathbb{E}fg \geq (\mathbb{E}f^p)^{1/p} (\mathbb{E}g^{p/(p-1)})^{(p-1)/p}. \quad (3.5)$$

The reverse-Hölder inequality actually follows from the usual Hölder inequality applied to the functions $(fg)^{1/p}$ and $g^{-1/p}$.

The other ingredient in the proof of Proposition 3.5 is an application of the isoperimetric inequality itself: Theorem 1.4. Indeed, Theorem 1.4 implies the following bound for an arbitrary Lipschitz function:

Lemma 3.6. *If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -Lipschitz with median M then for any $\lambda < 1$,*

$$\mathbb{E}e^{\frac{\lambda}{2L^2}f^2(X)} \leq \frac{2}{\sqrt{1-\lambda}} e^{\frac{\lambda}{2(1-\lambda)}M^2}.$$

In particular, recall from Theorem 1.4 that v_t is L_t -Lipschitz where $L_t = (e^{2t} - 1)^{-1/2}$. Thus if N_t is a median of v_t then for all $\beta < L_t^{-2}/2$,

$$\mathbb{E}e^{\frac{\beta}{2}v_t^2} \leq 2\sqrt{2}e^{N_t^2}. \quad (3.6)$$

Proof of Lemma 3.6. By Theorem 1.4, $(f-M)1_{\{f \geq M\}}$ is stochastically dominated by $LX_1 1_{\{X_1 \geq 0\}}$. On the other hand, Theorem 1.4 applied to $-f$ implies that $(M-f)1_{\{f \leq M\}}$ is stochastically dominated by $L|X_1|1_{\{X_1 \leq 0\}}$. Hence, $|f-M|$ is stochastically dominated by $L|X_1|$. Since the function $x \mapsto e^{\lambda(x+|M|)^2/2}$ is increasing on $[0, \infty)$,

$$\mathbb{E}e^{\lambda f^2/(2L^2)} \leq \mathbb{E}e^{\lambda(|f-M|+|M|)^2/(2L^2)} \leq \mathbb{E}e^{\lambda(|X_1|+|M|)^2/2} \leq 2\mathbb{E}e^{\lambda(X_1+|M|)^2/2},$$

where the last inequality holds because the distribution of X_1 is symmetric, and $e^{\lambda(|x+a|)^2/2} \leq e^{\lambda(x+a)^2/2} + e^{\lambda(-x+a)^2/2}$. By completing the square in the integral, one checks that

$$\mathbb{E}e^{\lambda(X_1+M)^2/2} = \frac{1}{\sqrt{1-\lambda}} e^{\frac{\lambda}{2(1-\lambda)}M^2}. \quad \square$$

Now we turn to the proof of Proposition 3.5:

Proof of Proposition 3.5. Applying (3.5) shows that for every $p < 1$,

$$\mathbb{E}\phi(v_t)\|\text{Hess}(v_t)\|_F^2 \geq \left(\mathbb{E}\|\text{Hess}(v_t)\|_F^{2p}\right)^{1/p} \left(\mathbb{E}\phi(v_t)^{p/(p-1)}\right)^{(p-1)/p}. \quad (3.7)$$

Since the first term of (3.7) is already in the form we are looking for, we turn to the second: expanding the definition of ϕ , we have

$$\left(\mathbb{E}\phi(v_t)^{p/(p-1)}\right)^{(p-1)/p} = \frac{1}{\sqrt{2\pi}} \left(\mathbb{E}\exp\left(\frac{p}{2(1-p)}v_t^2\right)\right)^{(p-1)/p}.$$

If $p/(1-p) < L_t^{-2}/2$ then we can apply (3.6) to bound the expectation:

$$\left(\mathbb{E}\exp\left(\frac{p}{2(1-p)}v_t^2\right)\right)^{(p-1)/p} \leq C e^{N_t^2}.$$

Now raise both sides to the power $(p-1)/p$; since $p < 1$, the inequality reverses and so

$$\left(\mathbb{E}\phi(v_t)^{p/(p-1)}\right)^{(p-1)/p} \geq C^{(p-1)/p} e^{\frac{p-1}{p}N_t^2}.$$

Plugging this into (3.7) completes the proof. \square

The mean and median of $P_t f$

Note that the right hand side of Proposition 3.5 depends on the median N_t of $v_t = \Phi^{-1} \circ P_t f$. In this section, we will relate N_t to a known quantity, namely $\mathbb{E}f$. First, note that since Φ is a monotone function, if $M_t = \Phi(N_t)$ is a median of $P_t f$. In particular, it is enough to relate the mean and median of $P_t f$ to one another.

It is generally true that for any L -Lipschitz function of a Gaussian random variable, the mean and the median are within CL of each other. This relationship is not helpful here, however, since we are concerned with functions $f : \mathbb{R}^n \rightarrow [0, 1]$ for which the mean and median are automatically between zero and one. In particular, a bound on the additive distance between M_t and $\mathbb{E}f$ is relatively uninformative when M_t is very small. To see what sort of relationship is appropriate, consider the indicator of a half-space: $f(x) = 1_{\{x \geq b\}}$ for some $b > 0$. Since P_t preserves expectations, $\mathbb{E}P_t f = \mathbb{E}f = \Phi(-b)$ for all $t > 0$. Since $P_t f$ is monotone for every t , the median of $P_t f$ is $(P_t f)(0) = \Phi(-b/\sqrt{1-e^{-2t}})$ (by Lemma 1.9). When b is large, $\Phi(-b) \approx e^{-b^2/2}$; under this approximation, we have $\mathbb{E}f \approx M_t^{1/(1-e^{-2t})}$.

For general functions f , we prove something which is slightly weaker than what holds in the example above. Recall that $L_t = (e^{2t} - 1)^{-1/2}$.

Lemma 3.7. *If M_t is a median of $P_t f$, then*

$$\mathbb{E}f \leq 2M_t \left(\frac{1}{1+L_t}\right)^2.$$

Proof of Lemma 3.7. Let $f_t = P_t f$ and $g_t = \sqrt{\log(1/f_t)}$; take N_t to be a median of g_t and let $M_t = e^{-N_t^2}$, so that M_t is a median of f_t . For any $\alpha < 1$,

$$\Pr(f_t \geq M_t^{\alpha^2}) = \Pr(g_t \leq \alpha \sqrt{\log(1/M_t)}) = \Pr(g_t \leq \alpha N_t).$$

Assume without loss of generality that f takes values strictly between 0 and 1 (if not, we instead consider the function $x \mapsto \max\{\epsilon, \min\{1 - \epsilon, f(x)\}\}$ for arbitrarily small ϵ). Since $f \in (0, 1)$ implies $P_t f \log f < 0$, the reverse log-Sobolev inequality (1.12) implies that

$$|\nabla f_t|^2 \leq 2L_t^2 f_t^2 \log \frac{1}{f_t}.$$

Since $\nabla g_t = -\nabla f_t / (2f_t \sqrt{\log(1/f_t)})$, this is equivalent to the inequality $|\nabla g_t|^2 \leq L_t/2$. In other words, g_t is $\frac{1}{\sqrt{2}}L_t$ -Lipschitz. By Theorem 1.4,

$$\Pr(f_t \geq M_t^{\alpha^2}) = \Pr(g_t \leq \alpha N_t) \leq \exp\left(-\frac{(1-\alpha)^2 N_t^2}{L_t^2}\right) = M_t^{\frac{(1-\alpha)^2}{L_t^2}}.$$

Setting $\alpha = \frac{1}{1+L_t}$, we have $\frac{(1-\alpha)^2}{L_t^2} = \alpha^2$. Thus, $\Pr(f_t \geq M_t^{\alpha^2}) \leq M_t^{\alpha^2}$. Since $f_t \leq 1$, Markov's inequality implies that $\mathbb{E}f_t \leq 2M_t^{\alpha^2}$. \square

The example $f(x) = 1_{\{x \geq b\}}$ shows that Lemma 3.7 is sharp up to the factor 2 and a constant factor in the exponent. We remark that a sharper result may be obtained by considering $g_t = \Phi^{-1} \circ f_t$ instead of $g_t = \sqrt{\log(1/f_t)}$ and applying Theorem 1.7 instead of the reverse log-Sobolev inequality. In particular, one can show that

$$\mathbb{E}f \leq 2\Phi\left(\frac{\Phi^{-1}(M_t)}{\sqrt{1-e^{-2t}}}\right),$$

which is sharp up to the factor 2. Since it is more convenient to work with exponentials than with Φ and Φ^{-1} , and since we are not so concerned with constant factors, we will continue to work with Lemma 3.7.

We remark that the main idea in the proof of Lemma 3.7 is that the reverse log-Sobolev inequality may be interpreted to say that $\sqrt{\log(1/f_t)}$ is Lipschitz. This fact was previously noted by Hino [24], and was also used recently by Ledoux [32].

Since medians commute with monotone functions, Lemma 3.7 may be used to relate $\mathbb{E}f$ to a median of v_t . This is in fact the main form of Lemma 3.7 that we will use, since medians of v_t will arise several times in what follows.

Corollary 3.8. *If N_t is a median of $v_t = \Phi^{-1} \circ P_t f$, then*

$$|N_t| \leq 2(1+L_t)\sqrt{\log(1/m(f))},$$

where $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$.

Proof. Suppose first that $N_t \leq 0$, and let $M_t = \Phi(N_t)$; then M_t is a median of $f_t = P_t f$. Since $\Phi(x) \leq e^{-x^2/2}$ for all $x \leq 0$, we have $M_t \leq e^{-N_t^2/2}$, which implies that $|N_t| \leq \sqrt{2 \log(1/M_t)}$. Now, taking the logarithm in Lemma 3.7 yields

$$\log \frac{1}{M_t} \leq (1 + L_t)^2 \log \frac{1}{\mathbb{E}f} + \log 2 \leq (1 + L_t)^2 \log \frac{1}{m(f)} + \log 2 \leq 2(1 + L_t)^2 \log \frac{1}{m(f)},$$

where the last inequality holds because $m(f) = \mathbb{E}f(1 - \mathbb{E}f) \leq \frac{1}{4}$ and so $\log(1/m(f)) \geq \log 2$. Comparing this to N_t , we have

$$|N_t| \leq \sqrt{2 \log(1/M_t)} \leq 2(1 + L_t) \sqrt{\log(1/m(f))}.$$

Although we have only proved the above for $N_t \leq 0$, we see immediately that it holds unconditionally because both sides are unchanged if we replace f by $1 - f$ (which changes the sign of N_t). \square

Second-derivative estimates

There is one more ingredient in the proof of Proposition 3.4: in order to connect Proposition 3.5 with the second-order Poincaré inequality (3.4), we require an upper bound on $\|\text{Hess}(v_t)\|_F$. With such a bound, we can apply Hölder's inequality to lower bound $\mathbb{E}\|\text{Hess}(v_t)\|_F^{2p}$ (with $p < 1$) in terms of $\mathbb{E}\|\text{Hess}(v_t)\|_F^2$. We will obtain the needed bound on $\|\text{Hess}(v_t)\|_F$ by pushing the reverse log-Sobolev inequality (1.12) to the second order. We are grateful to Michel Ledoux for suggesting this approach, since our original proof was rather longer.

Lemma 3.9. *There is a constant $C > 0$ such that for any $q \geq 2$ and any $t > 0$,*

$$(\mathbb{E}\|\text{Hess}(v_t)\|_F^q)^{1/q} \leq C(L_t^2 + L_t)(\sqrt{\log(1/m(f))} + L_t\sqrt{q}),$$

where $L_t = (e^{2t} - 1)^{-1/2}$.

Lemma 3.9 arises from integrating out a pointwise bound on $\|\text{Hess}(f_t)\|_F$. This pointwise bound may be of independent interest, since it is essentially a second-order version of the reverse log-Sobolev inequality (specialized to functions taking values in $[0, 1]$). Although second-order Sobolev inequalities on \mathbb{R}^n with the Lebesgue measure have appeared in the literature [23], we could not find any existing work on second-order log-Sobolev inequalities (in either direction) for Gaussian space.

Proposition 3.10. *For any smooth $f : \mathbb{R}^n \rightarrow [0, 1]$ and any $t > 0$,*

$$\|\text{Hess}(f_t)\|_F^2 \leq 8f_t^2 L_t^4 \left(2 \log \frac{1}{f_t} + \log^2 \frac{1}{f_t} \right)$$

where $f_t = P_t f$. In particular, if $v_t = \Phi^{-1} \circ f_t$ then there is a universal constant $C > 0$ such that

$$\|\text{Hess}(v_t)\|_F^2 \leq C L_t^2 (L_t^2 + L_t^2 v_t^2 + v_t^2).$$

Proof. We begin with (1.11):

$$\frac{d}{ds} P_s f_{t-s} \log f_{t-s} = P_s \frac{|\nabla f_{t-s}|^2}{f_{t-s}}. \quad (3.8)$$

Now rather than applying the Cauchy-Schwarz inequality to the right-hand side of (1.11), we apply the semigroup technique again. Specifically, we apply (1.16) with $\Psi(x, y) = |y|^2/x$. For this function, we have

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x^2} &= \frac{2|y|^2}{x^3} & \frac{\partial^2 \Psi}{\partial x \partial y_i} &= -2 \frac{y_i}{x} \\ \frac{\partial^2 \Psi}{\partial y_i \partial y_j} &= \frac{2}{x} \delta_{ij} & \frac{\partial \Psi}{\partial y_i} &= \frac{2y_i}{x}. \end{aligned}$$

After applying (1.16) and rearranging, we obtain

$$\frac{d}{ds} P_s \frac{|P_{t-s} \nabla f|^2}{P_{t-s} f} = P_s \frac{2}{f_{t-s}} \left(\left\| \text{Hess}(f_{t-s}) - \frac{(\nabla f_{t-s})(\nabla f_{t-s})^T}{f_{t-s}} \right\|_F^2 + |\nabla f_{t-s}|^2 \right). \quad (3.9)$$

Next, we apply the Cauchy-Schwarz inequality to the right hand side of (3.9). In particular, we apply the inequality $(P_s X)^2 \leq P_s X P_s \frac{X^2}{Y}$ twice, with $Y = f_{t-s}$ and X equal to $\| \text{Hess}(f_{t-s}) - (\nabla f_{t-s})(\nabla f_{t-s})^T / f_{t-s} \|_F$ and $|\nabla f_{t-s}|$ respectively. Thus we obtain

$$\begin{aligned} (3.9) &\geq \frac{2}{f_t} \left(P_s \left(\left\| \text{Hess}(f_{t-s}) - \frac{(\nabla f_{t-s})(\nabla f_{t-s})^T}{f_{t-s}} \right\|_F \right)^2 + (P_s |\nabla f_{t-s}|)^2 \right) \\ &\geq \frac{2}{f_t} \left(\left\| P_s \text{Hess}(f_{t-s}) - P_s \frac{(\nabla f_{t-s})(\nabla f_{t-s})^T}{f_{t-s}} \right\|_F^2 + |P_s \nabla f_{t-s}|^2 \right) \\ &= \frac{2}{f_t} \left(e^{4s} \left\| \text{Hess}(f_t) - \frac{(\nabla f_t)(\nabla f_t)^T}{f_{t-s}} \right\|_F^2 + e^{2s} |\nabla f_t|^2 \right). \end{aligned}$$

Applying the triangle inequality to the Frobenius norm and removing the non-negative $|\nabla f_t|^2$ term,

$$(3.9) \geq \frac{e^{4s}}{f_t} \left(\left\| \text{Hess}(f_t) \right\|_F^2 - 2 \frac{|\nabla f_t|^4}{f_t^2} \right).$$

To complete the second-order part of the proof, we integrate (3.9) from 0 to t :

$$\begin{aligned} P_t \frac{|\nabla f|^2}{f} &\geq P_t \frac{|\nabla f|^2}{f} - \frac{|\nabla P_t f|^2}{P_t f} \\ &\geq \frac{\int_0^t e^{4s} dt}{f_t} \left(\left\| \text{Hess}(f_t) \right\|_F^2 - 2 \frac{|\nabla f_t|^4}{f_t^2} \right) \\ &= \frac{e^{4t} - 1}{4f_t} \left(\left\| \text{Hess}(f_t) \right\|_F^2 - 2 \frac{|\nabla f_t|^4}{f_t^2} \right). \end{aligned} \quad (3.10)$$

Now we return to (3.8). Applying the second-order bound (3.10) with t replaced by s and f replaced by f_{t-s} ,

$$\begin{aligned} \frac{d}{ds} P_s f_{t-s} \log f_{t-s} &\geq \frac{e^{4s} - 1}{4f_t} \left(\|\text{Hess}(f_t)\|_F^2 - 2 \frac{|\nabla f_t|^4}{f_t^2} \right) \\ &\geq \frac{e^{4s} - 1}{4} \left(\frac{\|\text{Hess}(f_t)\|_F^2}{f_t} - \frac{8}{(e^{2t} - 1)^2} f_t \log^2 \frac{1}{f_t} \right), \end{aligned} \quad (3.11)$$

where the second inequality follows from the reverse log-Sobolev inequality (1.12) which, when $f_t \in (0, 1)$, implies that $|\nabla f_t|^2 \leq \frac{2}{e^{2t}-1} f_t^2 \log \frac{1}{f_t}$. Integrating (3.11) from 0 to t gives

$$\begin{aligned} f_t \log \frac{1}{f_t} &\geq P_t(f \log f) - f_t \log f_t \\ &\geq \frac{e^{4t} - 4t - 1}{16} \left(\frac{\|\text{Hess}(f_t)\|_F^2}{f_t} - \frac{8}{(e^{2t} - 1)^2} f_t \log^2 \frac{1}{f_t} \right) \\ &\geq \frac{(e^{2t} - 1)^2}{16f_t} \|\text{Hess}(f_t)\|_F^2 - \frac{1}{2} f_t \log^2 \frac{1}{f_t}, \end{aligned}$$

where the second inequality follows because $e^{4t} - 4t - 1 - (e^{2t} - 1)^2 = 2(e^{2t} - 2t - 1) \geq 0$ and so $(e^{2t} - 1)^2 \leq e^{4t} - 4t - 1$. This may then be rearranged to yield the first claim of the proposition.

For the second claim, recall that (by the chain rule)

$$\text{Hess}(v_t) = \frac{\text{Hess}(f_t)}{I(f_t)} + v_t(\nabla v_t)(\nabla v_t)^T. \quad (3.12)$$

Recall also that $I(x) \sim x\sqrt{2\log(1/x)}$ as $x \rightarrow 0$, and hence there is a universal constant C such that $x\sqrt{\log(1/x)} \leq CI(x)$ for all $x \in (0, 1/2]$. Hence, (3.12) implies that whenever $f_t \leq 1/2$,

$$\begin{aligned} \|\text{Hess}(v_t)\|_F^2 &\leq 2 \frac{\|\text{Hess}(f_t)\|_F^2}{I^2(f_t)} + 2v_t^2 |\nabla v_t|^2 \\ &\leq CL_t^4 \left(1 + \log \frac{1}{f_t} \right) + 2L_t^2 v_t^2 \end{aligned}$$

where the second inequality follows from applying the first claim of the proposition to the term involving the Hessian, and Theorem 1.7 to the term involving $|\nabla v_t|$. Since $\Phi^{-1}(x) \sim -\sqrt{2\log(1/x)}$ as $x \rightarrow 0$, there is a universal constant C such that $\log(1/f_t) \leq Cv_t^2$ whenever $f_t \leq 1/2$; hence,

$$\|\text{Hess}(v_t)\|_F^2 \leq CL_t^4 (1 + v_t^2) + 2L_t^2 v_t^2 \quad (3.13)$$

whenever $f_t \leq 1/2$. But by applying the preceding argument to $1 - f_t$ instead of f_t (which changes v_t only by a sign), we see that the formula (3.13) is valid at all points, and not just those for which $f_t \leq 1/2$. \square

Proof of Lemma 3.9. As we mentioned before, Lemma 3.9 follows by integrating out the bound of Proposition 3.10. Indeed, take $q \geq 1$; then the triangle inequality implies that

$$\left(\mathbb{E}\|\text{Hess}(v_t)\|_F^{2q}\right)^{1/q} \leq CL_t^2 \left(L_t^2 + (L_t^2 + 1)\right) \left(\mathbb{E}|v_t|^{2q}\right)^{1/q}. \quad (3.14)$$

Now, Theorem 1.7 implies that v_t is L_t -Lipschitz. Hence (by Theorem 1.4) if N_t is a median of v_t then $|v_t - N_t|/L_t$ is stochastically dominated by the absolute value of a Gaussian variable. In particular,

$$\left(\mathbb{E}|v_t|^{2q}\right)^{1/q} \leq 2N_t^2 + 2\left(\mathbb{E}|v_t - N_t|^{2q}\right)^{1/q} \leq 2N_t^2 + CL_t^2 q.$$

Plugging this back into (3.14) and taking square roots,

$$\begin{aligned} \left(\mathbb{E}\|\text{Hess}(v_t)\|_F^{2q}\right)^{1/(2q)} &\leq CL_t(L_t + (L_t + 1)N_t + (L_t^2 + L_t)\sqrt{q}) \\ &\leq CL_t(L_t + (L_t + 1)\sqrt{\log(1/m(f))} + (L_t^2 + L_t)\sqrt{q}) \\ &\leq CL_t(L_t + 1)(\sqrt{\log(1/m(f))} + L_t\sqrt{q}), \end{aligned}$$

where the second inequality follows from Corollary 3.8 and the last inequality follows because $\sqrt{\log(1/m(f))}$ is bounded away from zero. The lemma follows after replacing $2q$ by q . \square

Proof of Proposition 3.4

With all of the ingredients laid out, the proof of Proposition 3.4 follows easily.

Proof of Proposition 3.4. Fix $\tau > 0$ large enough so that $L_\tau^{-2} > 2$. By (3.2) and Theorem 1.7,

$$\begin{aligned} \delta(f) &\geq \int_0^\infty \mathbb{E} \frac{\phi(v_s) \|\text{Hess}(v_s)\|_F^2}{(1 + |\nabla v_s|^2)^{3/2}} ds \\ &\geq \int_\tau^{\tau+1} \mathbb{E} \frac{\phi(v_s) \|\text{Hess}(v_s)\|_F^2}{(1 + L_s^2)^{3/2}} ds \\ &\geq c \int_\tau^{\tau+1} \mathbb{E} \phi(v_s) \|\text{Hess}(v_s)\|_F^2 ds \end{aligned}$$

In particular, there exists $t \in [\tau, \tau + 1]$ such that

$$\delta(f) \geq c \mathbb{E} \phi(v_t) \|\text{Hess}(v_t)\|_F^2$$

Take $p = \frac{1}{2}$ in Proposition 3.5; since $L_\tau^{-2}/2 > 1$, p satisfies the condition of Proposition 3.4. Hence,

$$\delta(f) \geq ce^{-N_t^2} \mathbb{E} (\|\text{Hess}(v_t)\|_F)^2 \geq cm(f)^{2(1+L_t)^2} \mathbb{E} (\|\text{Hess}(v_t)\|_F)^2 \geq cm(f)^C \mathbb{E} (\|\text{Hess}(v_t)\|_F)^2,$$

where the second inequality follows from Corollary 3.8. Now, the Cauchy-Schwarz inequality implies that for any random variable $X \geq 0$,

$$\mathbb{E}X^2 = \mathbb{E}X^{1/2}X^{3/2} \leq (\mathbb{E}X)^{1/2}(\mathbb{E}X^3)^{1/2}.$$

With $X = \|\text{Hess}(v_t)\|_F$, this implies

$$\begin{aligned} \delta(f) &\geq cm(f)^C \frac{(\mathbb{E}\|\text{Hess}(v_t)\|_F^2)^4}{(\mathbb{E}\|\text{Hess}(v_t)\|_F^3)^2} \\ &\geq cm(f)^C \frac{(\mathbb{E}\|\text{Hess}(v_t)\|_F^2)^4}{(\sqrt{\log(1/m(f))})^6} \\ &\geq cm(f)^{C'} (\mathbb{E}\|\text{Hess}(v_t)\|_F^2)^4. \end{aligned}$$

where the second line follows from Lemma 3.9 with $q = 3$ and L_t bounded by a universal constant. The final step is to invoke (3.4): taking $a = \mathbb{E}v_t$ and $b = \mathbb{E}\nabla v_t$, we have

$$\mathbb{E}(v_t(X) - \langle a, X - b \rangle)^2 \leq \mathbb{E}\|\text{Hess}(v_t)\|_F^2 \leq C \frac{\delta^{1/4}}{m^C}.$$

Note that $|v_t| \leq L_t$ pointwise, and hence $|a| \leq L_t$. □

3.2 Approximation for small t

Proposition 3.4 shows that if f achieves almost-equality in (1.2) then v_t – for some t not too large – can be well approximated by a linear function. Since Φ is a contraction, this implies that $P_t f$ may be well approximated by a function of the form $\Phi(\langle a, x - b \rangle)$. The goal of this section is to complete the proof of Theorem 3.2 by showing that f itself can be approximated by a function of the same form. This will be accomplished mainly with spectral techniques, by expanding f in the Hermite basis.

Let $g_t(x) = \Phi(\langle a, x - b \rangle)$, where a and b satisfy the conclusion of Proposition 3.4. In particular, $|a| \leq L_t$ and so by Lemma 1.10, g_t is in the range of P_t and $P_t^{-1}g_t$ is either the indicator of a half-space or Φ composed with a linear function. Let $g = P_t^{-1}g_t$. Then Proposition 3.4 implies that

$$\mathbb{E}(P_t(f - g))^2 = \mathbb{E}(f_t - g_t)^2 \leq C \frac{\delta(f)^2}{m(f)^C},$$

and our task is to prove that $\mathbb{E}(f - g)^2$ is small. In other words, setting $h = f - g$, we want to bound $\mathbb{E}h^2$ in terms of $\mathbb{E}(P_t h)^2$. For a general function h , this is an impossible task. To see why, consider $h_k(x) = \text{sgn}(\sin(kx))$. Then $\mathbb{E}h_k^2 = 1$ for all k , but for any $t > 0$, $P_t h_k \rightarrow 0$ as $k \rightarrow \infty$. Hence $\mathbb{E}(P_t h_k)^2 \rightarrow 0$, and so $\mathbb{E}h_k^2$ cannot be bounded in terms of $\mathbb{E}(P_t h_k)^2$.

The key to bounding $\mathbb{E}(f - g)^2$ in terms of $\mathbb{E}(P_t(f - g))^2$ is to exploit some extra information that we have on $f - g$. In particular, we have assumed that f almost minimizes Bobkov's functional $\mathbb{E}\sqrt{I^2(f) + |\nabla f|^2}$. In particular,

$$\mathbb{E}|\nabla f| \leq \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2} \leq I(\mathbb{E}f) + \delta(f).$$

If we assume that $\delta(f) \leq 1$ (if not, then Theorem 3.2 is meaningless anyway), then $\mathbb{E}|\nabla f| \leq 2$. This will allow us to exploit Theorem 2.3 in order to obtain bounds on the Hermite expansion of f . Since P_t acts diagonally on the Hermite basis, these bounds will allow us to bound $\mathbb{E}(f - g)^2$ in terms of $\mathbb{E}(P_t(f - g))^2$.

We should remark that for non-negative functions h , reverse hypercontractive inequalities can be used to bound $\mathbb{E}h^2$ in terms of $\mathbb{E}(P_t h)^2$. The restriction $h \geq 0$ prevents the positive and negative parts of h from canceling out under P_t , rendering examples like $h_k(x) = \text{sgn} \sin(kx)$ impossible. For our application, however, we must consider functions that take both positive and negative values.

Smoothness and the Hermite expansion

Recall from Chapter 1 that the Hermite polynomials H_α form an orthonormal basis of (\mathbb{R}^n, γ_n) . Recall moreover that P_t acts diagonally on this basis by

$$P_t H_\alpha = e^{-|\alpha|t} H_\alpha \tag{3.15}$$

By Theorem 2.3, we observe that if $\mathbb{E}|\nabla h|$ is bounded, then its Hermite coefficients decay quickly.

Lemma 3.11. *For any smooth $h : \mathbb{R}^n \rightarrow [-1, 1]$ is a smooth function and $h = \sum_\alpha b_\alpha H_\alpha$. Then*

$$\sum_{|\alpha| \geq N} b_\alpha^2 \leq CN^{-1/2} \mathbb{E}|\nabla h|$$

for any $N \in \{1, 2, \dots\}$, where C is a universal constant.

Proof. By (3.15), $P_t h = \sum_\alpha e^{-|\alpha|t} b_\alpha H_\alpha$, and so $\mathbb{E}h P_t h = \sum_\alpha e^{-|\alpha|t} b_\alpha^2$. Hence,

$$\sum_\alpha (1 - e^{-|\alpha|t}) b_\alpha^2 = \mathbb{E}h(h - P_t h) \leq C\sqrt{t} \mathbb{E}|\nabla h|,$$

where the inequality follows from Theorem 2.3 and because $\arccos(e^{-t}) \leq C\sqrt{1 - e^{-t}} \leq C\sqrt{t}$. If $|\alpha| \geq 1/t$ then $e^{-|\alpha|t} \leq 1/e$; hence

$$(1 - 1/e) \sum_{|\alpha| \geq 1/t} b_\alpha^2 \leq \sum_\alpha (1 - e^{-|\alpha|t}) b_\alpha^2 \leq C\sqrt{t} \mathbb{E}|\nabla h|.$$

Now set $t = \frac{1}{N}$. □

Since we know how the semigroup P_t acts on the Hermite basis and we know how the Hermite coefficients of nice functions are distributed, we are in a position to bound $\mathbb{E}h^2$ in terms of $\mathbb{E}(P_t h)^2$. Essentially, Lemma 3.11 tells us that the high coefficients don't contribute much to $\mathbb{E}h^2$, while (3.15) tells us that the low coefficients contributing to $\mathbb{E}h^2$ also contribute to $\mathbb{E}(P_t h)^2$.

Lemma 3.12. *For any smooth $h : \mathbb{R}^n \rightarrow [-1, 1]$ and any $t \geq 1$,*

$$\mathbb{E}h^2 \leq C(1 + \mathbb{E}|\nabla h|) \sqrt{\frac{t}{\log(1/\mathbb{E}(P_t h)^2)}}.$$

Proof. Expand $h = \sum_{\alpha} b_{\alpha} H_{\alpha}$ and let $\epsilon = \mathbb{E}(P_t h)^2$. Then (3.15) implies that

$$\epsilon = \mathbb{E}(P_t h)^2 = \sum_{\alpha} e^{-2t|\alpha|} b_{\alpha}^2.$$

On the other hand, Lemma 3.11 implies that

$$\begin{aligned} \mathbb{E}h^2 &= \sum_{\alpha} b_{\alpha}^2 \\ &\leq e^{2t(N-1)} \sum_{|\alpha| \leq N-1} b_{\alpha}^2 e^{-2t|\alpha|} + \sum_{|\alpha| \geq N} b_{\alpha}^2 \\ &\leq e^{2t(N-1)} \epsilon + CN^{-1/2} K, \end{aligned} \tag{3.16}$$

where $K = \mathbb{E}|\nabla h|$.

Now we choose N to optimize (3.16). Let $\beta = \frac{1}{2t} \log \frac{1}{\epsilon}$ and set $N = \lceil \beta - \frac{1}{4t} \log \beta \rceil$. Since $\beta > \log \beta$ and $t \geq 1$, $N \geq \beta/2$ (and in particular, N is a positive integer). Moreover, $N - 1 \leq \beta - \frac{1}{4t} \log \beta$ and so (since $e^{2t\beta} = 1/\epsilon$) $e^{2t(N-1)} \epsilon \leq \beta^{-1/2}$. Plugging these bounds on N back into (3.16) yields

$$\mathbb{E}h^2 \leq \beta^{-1/2} + CK\beta^{-1/2} \leq C(1 + K) \sqrt{\frac{t}{\log(1/\epsilon)}}. \quad \square$$

Proof of Theorem 3.2

Finally, we are ready to prove Theorem 3.2. As we discussed at the beginning of the section, we may assume that $\delta = \delta(f) \leq 1$, which implies that $\mathbb{E}|\nabla f| \leq 2$. We may also assume that $m(f) \geq \log^{-1/2}(1/\delta)$: if not, then either $\mathbb{E}f \leq 2\log^{-1/2}(1/\delta)$ or $(1 - \mathbb{E}f) \leq 2\log^{-1/2}(1/\delta)$. In the first case, f may be approximated well by the zero function, which in turn may be approximated by functions of the form $\Phi(\langle a, x - b \rangle)$. Specifically, for any $a, b \in \mathbb{R}^n$ with $\langle a, b \rangle > 0$, $\Phi(\langle a, x - sb \rangle) \rightarrow 0$ as $s \rightarrow \infty$ and so

$$\lim_{s \rightarrow \infty} \mathbb{E}(f(X) - \Phi(\langle a, X - sb \rangle))^2 = \mathbb{E}f^2 \leq \mathbb{E}f \leq \frac{2}{\sqrt{\log(1/\delta)}}.$$

That is, if $\mathbb{E}f \leq 2\log^{-1/2}(1/\delta)$ then the conclusion of Theorem 3.2 holds trivially. A similar argument (but with the zero function replaced by the constant function 1) holds when $(1 - \mathbb{E}f) \leq 2\log^{-1/2}(1/\delta)$. Thus, we may assume that $m(f) \geq \log^{-1/2}(1/\delta)$.

As in the discussion at the beginning of the section, take $t \in [1, 2]$ and $a, b \in \mathbb{R}^n$ satisfying the conclusion of Proposition 3.4. Let $g_t(x) = \Phi(\langle a, x - b \rangle)$ and $g = P_t^{-1}g_t$ (which exists, recall, because $|a| \leq L_t$).

By Proposition 3.4 and because Φ is a contraction,

$$\mathbb{E}(g_t - f_t)^2 \leq \mathbb{E}(\langle a, X - b \rangle - v_t)^2 \leq C \frac{\delta(f)^{1/4}}{m(f)^C} \leq \delta^{C'},$$

where the last inequality follows from the assumption that $m(f) \leq \log^{-1/2}(1/\delta)$. Set $h = g - f$. Since g is Φ composed with a linear function, $\mathbb{E}|\nabla g| \leq \phi(0) \leq 1$ and hence $\mathbb{E}|\nabla h| \leq \mathbb{E}|\nabla g| + \mathbb{E}|\nabla f| \leq 3$. By Lemma 3.12,

$$\mathbb{E}(g - f)^2 \leq \frac{C}{\sqrt{\log(1/\delta^{C'})}} \leq \frac{C}{\sqrt{\log(1/\delta)}}.$$

This completes the proof of Theorem 3.2.

Open Problems

There are two natural open problems that our work leaves unresolved. The first problem asks for a sharp dependence on both δ and n simultaneously. In particular, we know of no obstacle to having a dimension-independent rate of $\sqrt{\delta}$:

Conjecture 3.13. *There is a universal constant $C > 0$ such that for every $A \subset \mathbb{R}^n$, there exists a half-space B with*

$$\gamma_n(A \Delta B) \leq C\sqrt{\delta}.$$

The second open problem asks for a generalization of our arguments to other semigroups. Although we have not discussed it here, the Bakry-Ledoux semigroup proof of Bobkov's inequality generalizes to certain non-Gaussian measures. For example (and this is still not the most general case), consider a probability measure μ on \mathbb{R}^n with density $e^{-V(x)}$ for some function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $\text{Hess}(V) \geq I_n$. Bakry and Ledoux showed that such measures also satisfy the Gaussian isoperimetric inequality: for any $A \subset \mathbb{R}^n$, $I(\mu(A)) \leq \mu^+(A)$. We could therefore ask about equality and near-equality cases of this inequality. A fairly straightforward generalization of the Carlen-Kerce argument shows that if equality is attained then A is a half-space and there is a one-dimensional subspace $E \subset \mathbb{R}^n$ such that the marginal of μ on E is Gaussian. Our proof of Theorem 3.2 seems somewhat more difficult to generalize, but it may be possible. Actually, our original proof of Theorem 3.2 in [40] contained many computations that were specific to Gaussian measures; thanks to improvements suggested by Ledoux, most of the Gaussian-specific parts have been removed from the proof presented here. However, some work remains to be done.

Conjecture 3.14. *Let μ be a probability measure on \mathbb{R}^n with a density of the form $e^{-V(x)}$ for some V satisfying $\text{Hess}(V) \geq I_n$. If $\mu^+(A) \leq I(\mu(A)) + \delta$ then there exist $a, b \in \mathbb{R}^n$ such that*

$$\mu(A \Delta \{x \in \mathbb{R}^n : \langle a, x - b \rangle \geq 0\}) \leq C\sqrt{\delta}.$$

Moreover, the marginal of μ on $\text{span}(a)$ is close to a Gaussian measure.

We deliberately left some vagueness in the phrase “close to a Gaussian measure,” because we are not sure what the right notion of closeness is. We note that the one dimensional version of this conjecture was studied by de Castro [12], who established the first part (that A must be close to a half-space). (Although in the one-dimensional case, the correct dependence on δ is $\delta/\sqrt{\log(1/\delta)}$ instead of $\sqrt{\delta}$; this was noted already by Cianchi et al. [13])

Chapter 4

Robust Gaussian noise stability

In Chapter 3, we investigated the near-equality cases in the Gaussian isoperimetric inequality from Chapter 1. In this chapter, we will consider the analogous problem for the notion of noise sensitivity that was introduced in Chapter 2. To that end, recall that \Pr_ρ denotes the distribution under which $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$ is distributed as a mean-zero Gaussian vector with covariance $\begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}$, and recall the definition of J_ρ from Section 2.3. By Theorem 2.5, $\mathbb{E}_\rho J_\rho(f(X), g(Y)) \leq J_\rho(\mathbb{E}f, \mathbb{E}g)$ for any $f, g : \mathbb{R}^n \rightarrow [0, 1]$; we also showed that equality is attained only when $f(x) = \Phi(\langle a, x - b \rangle)$ and $g(x) = \Phi(\langle a, x - d \rangle)$ for some $a, b, d \in \mathbb{R}^n$. The main result of this chapter is that if equality is almost attained in Theorem 2.5, then f and g are almost of this form. To this end, define

$$\delta(f, g) = J(\mathbb{E}f, \mathbb{E}g) - \mathbb{E}_\rho J(f(X), g(Y)). \quad (4.1)$$

Theorem 4.1. *For any $0 < \rho < 1$, there exists $C(\rho) < \infty$ such that for any $f, g : \mathbb{R}^n \rightarrow [0, 1]$ there exist $a, b, d \in \mathbb{R}^n$ such that*

$$\begin{aligned} \mathbb{E}|f(X) - \Phi(\langle a, X - b \rangle)| &\leq C(\rho)m^{-C(\rho)}\delta^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}} \\ \mathbb{E}|g(X) - \Phi(\langle a, X - d \rangle)| &\leq C(\rho)m^{-C(\rho)}\delta^{\frac{1}{4}\frac{(1-\rho)(1-\rho^2)}{1+3\rho}}, \end{aligned}$$

where $m = \mathbb{E}f(1 - \mathbb{E}f)\mathbb{E}g(1 - \mathbb{E}g)$.

The main difference between Theorem 4.1 and Theorem 3.2 (besides the fact that one deals with noise stability and one with isoperimetry) is that the rate in Theorem 4.1 is polynomial in δ ; we do not, however, believe that the exponent of δ is optimal. We should mention that a more careful tracking of constants in our proof would improve the exponent of δ slightly. However, this improvement would not bring the exponent above $\frac{1}{4}$ and it would not prevent the exponent from approaching zero as $\rho \rightarrow 1$.

Although Theorem 4.1 is stated only for $0 < \rho < 1$, the same result for $-1 < \rho < 0$ follows from certain symmetries. Indeed, one can easily check from the definition of J that

$J(x, y; \rho) = x - J(x, 1 - y; -\rho)$. Taking expectations,

$$\begin{aligned}\mathbb{E}_\rho J(f(X), g(Y); \rho) &= \mathbb{E}f - \mathbb{E}_\rho J(f(X), 1 - g(Y); -\rho) \\ &= \mathbb{E}f - \mathbb{E}_{-\rho} J(f(X), 1 - g(-Y); -\rho).\end{aligned}$$

Now, suppose that $-1 < \rho < 0$ and that f, g almost attain equality in Theorem 2.5:

$$\mathbb{E}_\rho J(f(X), g(Y); \rho) \leq J(\mathbb{E}f, \mathbb{E}g; \rho) + \delta.$$

Setting $\tilde{g}(y) = 1 - g(-y)$, this implies that

$$\mathbb{E}_{-\rho} J(f(X), \tilde{g}(Y); -\rho) \geq J(\mathbb{E}f, \mathbb{E}\tilde{g}; -\rho) - \delta.$$

Since $0 < -\rho < 1$, we can apply Theorem 4.1 to f and \tilde{g} to conclude that f and \tilde{g} are close to the equality cases of Theorem 2.6, and it follows that f and g are also close to one of these equality cases. Therefore, we will concentrate for the rest of this chapter on the case $0 < \rho < 1$.

Optimal dependence on ρ in the case $f = g$

The dependence on ρ in Theorem 4.1 is particularly interesting as $\rho \rightarrow 1$, since it is in that limit that Borell's inequality recovers the Gaussian isoperimetric inequality (as we showed in Section 2.2). As it is stated, however, Theorem 4.1 does not recover a robust version of the Gaussian isoperimetric inequality because of its poor dependence on ρ as $\rho \rightarrow 1$. In particular, as $\rho \rightarrow 1$, the exponent of δ tends to zero and the constant $C(\rho)$ tends to infinity.

It turns out that a poor dependence on ρ is necessary in some sense. To see this, take $n = 1$, $A = [2, \infty)$ and $B = [-1, 0] \cup [1, \infty)$. If $B' = [0, \infty)$ then B' is a half-space with the same measure as B ; hence,

$$\delta(A, B) = \Pr_\rho(X \in A, Y \in B') - \Pr(X \in A, Y \in B) \leq \Pr(X \in A, Y \notin B).$$

Now, if $X \in A$ and $Y \notin B$ then $X - Y \geq 1$. But $X - Y$ is a mean-zero Gaussian variable with variance $2(1 - \rho)$, and so

$$\delta(A, B) \leq \Pr_\rho(X - Y \geq 1) \leq e^{-c/(1-\rho)^2}.$$

On the other hand, the distance between B and the nearest half-space is some fixed constant. Hence, either the exponent of δ must decay like $(1 - \rho)^2$ as $\rho \rightarrow 1$, or the constant in front of δ must grow like $e^{c/(1-\rho)^2}$.

We can, however, obtain much a much better dependence on ρ if we restrict to the case $f = g$. In this case, it turns out that $\delta(f, f)$ grows only like $(1 - \rho)^{-1/2}$ as $\rho \rightarrow 1$, which is exactly the right rate for recovering the Gaussian isoperimetric inequality.

Theorem 4.2. *For every $\epsilon > 0$, there is a $\rho_0 < 1$ and a $C(\epsilon)$ such that for any $\rho_0 < \rho < 1$ and any $f : \mathbb{R}^n \rightarrow [0, 1]$ with $\mathbb{E}f = 1/2$, there exists $a \in \mathbb{R}^n$ such that*

$$\mathbb{E}|f(X) - \Phi(\langle a, X \rangle)| \leq C(\epsilon) \left(\frac{\delta(f, f)}{\sqrt{1-\rho}} \right)^{\frac{1}{4}-\epsilon}.$$

The requirements $\mathbb{E}f = 1/2$ and $\rho_0 < \rho < 1$ are there for technical reasons, and we do not believe that they are necessary (see Conjecture 4.22).

Since $\sqrt{1-\rho} \sim \arccos(\rho)$ as $\rho \rightarrow 1$, we may combine Theorem 4.2 with Theorem 2.3 to obtain a polynomial-rate, dimension-free robustness result for the Gaussian isoperimetric inequality:

Corollary 4.3. *For every $\epsilon > 0$, there is a $C(\epsilon) < \infty$ such that for every set $A \subset \mathbb{R}^n$ such that $\gamma_n(A) = 1/2$ and A has Gaussian surface area less than $\frac{1}{\sqrt{2\pi}} + \delta$, there is a half-space B such that*

$$\mathbb{P}(A \Delta B) \leq C(\epsilon) \delta^{1/4-\epsilon}.$$

Actually, one can prove Corollary 4.3 directly from Theorem 4.1, because in the case $\gamma_n(A) = 1/2$, one doesn't need to take $\rho \rightarrow 1$ in order to relate Gaussian noise sensitivity to Gaussian surface area. Indeed, recall that when $\gamma_n(A) = 1/2$, Corollary 2.4 achieves equality for every ρ . Nevertheless, we will give a proof based on Theorem 4.2. The advantage of this proof is that if Theorem 4.2 is proved without the assumption $\gamma_n(A) = 1/2$, then this proof will automatically extend to that case.

Proof. There is some potential for confusion, because we have been using δ both for the gap in the isoperimetric inequality and for the gap in Borell's inequality. Just for the duration of this proof, let δ be the isoperimetric deficit and let $\delta' = \delta'(\rho)$ be the gap in Borell's inequality. Then $\gamma_n^+(A) \leq I(\gamma_n(A)) + \delta$ by the definition of δ . By Corollary 2.4,

$$\Pr_\rho(X \in A, Y \notin A) \leq \frac{\arccos \rho}{\sqrt{2\pi}} (I(\gamma_n(A)) + \delta)$$

and so if A' is a half-space with $\gamma_n(A') = \gamma_n(A)$ then

$$\frac{\delta'}{\arccos \rho} = \frac{\Pr_\rho(X \in A, Y \notin A) - \Pr_\rho(X \in A', Y \notin A')}{\arccos \rho} \leq \frac{I(\gamma_n(A)) + \delta}{\sqrt{2\pi}} - \frac{\Pr_\rho(X \in A', Y \notin A')}{\arccos \rho}.$$

By (2.8), when we take the limit as $\rho \rightarrow 1$, the $I(\gamma_n(A))$ term on the right hand side cancels with the $\Pr_\rho(X \in A', Y \notin A')$ term (under the assumption $\gamma_n(A) = 1/2$, those two terms are actually equal for every ρ , and so there is no need to take the limit). Hence,

$$\limsup_{\rho \rightarrow 1} \frac{\delta'}{\sqrt{1-\rho}} \leq C \limsup_{\rho \rightarrow 1} \frac{\delta'}{\arccos \rho} \leq C' \delta.$$

In particular, we may take ρ large enough so that $\delta'/\sqrt{1-\rho} \leq 2C'\delta$ and then apply Theorem 4.2. \square

Note that Corollary 4.3 does not strictly improve Corollary 3.3 because of the restriction $\gamma_n(A) = 1/2$. However, a resolution of the technical issues that we alluded to earlier would remove this restriction, and would therefore go some way towards answering Conjecture 3.13.

On highly correlated functions

Let us mention one more corollary of Theorem 4.2. We have used $\mathbb{E}_\rho J(f(X), f(Y))$ as a functional generalization of $\Pr_\rho(X \in A, Y \in A)$. However, $\mathbb{E}_\rho f(X)f(Y)$ is another commonly used functional generalization of $\mathbb{P}_\rho(X \in A, Y \in A)$ which appeared, for example, in [33]. Since $xy \leq J(x, y)$ for $0 < \rho < 1$, we see immediately that Theorem 2.5 holds when the left hand side is replaced by $\mathbb{E}_\rho f(X)f(Y)$. The equality case, however, turns out to be different: whereas equality in Theorem 2.5 holds for $f(x) = \Phi(\langle a, x - b \rangle)$, there is equality in

$$\mathbb{E}_\rho f(X)f(Y) \leq J_\rho(\mathbb{E}f, \mathbb{E}f) \quad (4.2)$$

only when f is the indicator of a half-space. Moreover, a robustness result for (4.2) follows fairly easily from Theorems 4.1 and 4.2. Note that $J_\rho(\frac{1}{2}, \frac{1}{2}) = \Pr_\rho(X_1 \leq 0, Y_1 \leq 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho)$.

Corollary 4.4. *For any $0 < \rho < 1$, there is a constant $C(\rho) < \infty$ such that if $f : \mathbb{R}^n \rightarrow [0, 1]$ satisfies $\mathbb{E}f = 1/2$ and*

$$\mathbb{E}f(X)f(Y) \geq \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho) - \delta$$

then there is a half-space B such that

$$\mathbb{E}|f(X) - 1_B(X)| \leq C(\rho)\delta^c,$$

where $c > 0$ is a universal constant.

We could also state a two-function version of Corollary 4.4, at the cost of making the exponent depend on ρ .

Proof outline

Our proof of Theorem 4.1 has the same general outline as the proof of Theorem 3.2: take $R_t = \mathbb{E}_\rho J(P_t f(X), P_t g(Y))$ and consider the formula

$$\frac{dR_t}{dt} = \frac{\rho}{2\pi\sqrt{1-\rho^2}} \mathbb{E}_\rho \exp\left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right) |\nabla v_t - \nabla w_t|^2 \quad (4.3)$$

from Lemma 2.8, where $v_t = \Phi^{-1} \circ P_t f$ and $w_t = \Phi^{-1} \circ P_t g$. In the first step of the proof, we argue that for every $t > 0$, v_t and w_t must be close to linear functions, and so f_t and g_t must be close to functions of the form $\Phi(\langle a, x - b \rangle)$. This step has a similar proof as the analogous step in the proof of Theorem 3.2: we split the expectation in (4.3) using the reverse Hölder

inequality. We control one of the resulting terms using concentration and the smoothness of P_t , and we use a type of Poincaré inequality to relate the other term to the distance between v_t and a linear function.

The second step of the proof is to show that if f_t is close to a function of the form $\Phi(\langle a, x - b \rangle)$ then f is also close to a function of the same form. We cannot simply reuse the argument of Section 3.2 for this step, because the crucial assumption (that $\mathbb{E}|\nabla f|$ is bounded) may not hold. and spectral information. Here, we will use a geometric argument to say that if $h = 1_A - 1_B$ where B is a half-space, then $\mathbb{E}|h|$ can be bounded in terms of $\mathbb{E}|P_t h|$. This improved argument is essentially the reason that the rates in Theorem 4.1 are polynomial, while the rates in Theorem 3.2 were logarithmic.

4.1 Approximation for large t

This section is the analogue of Section 3.1, but for the noise sensitivity problem instead of the isoperimetric inequality. In this section, we will show that v_t is close to linear for $t > 0$. Recall the definition of δ from (4.1), and recall that $L_t = (e^{2t} - 1)^{-1/2}$.

Proposition 4.5. *For any $0 < \rho < 1$, and for any $t > 0$, there exists $C(t, \rho)$ such that for any f, g and for any $0 < \alpha < 1$, there exist $a, b, d \in \mathbb{R}^n$ with $|a| \leq L_t$ such that*

$$\begin{aligned} \mathbb{E}(f_t(X) - \Phi(\langle a, X - b \rangle))^2 + \mathbb{E}(g_t(X) - \Phi(\langle a, X - b \rangle))^2 \\ \leq C(t, \rho)(m(f)m(g))^{-C(t, \rho)} \left(\frac{\delta}{\alpha}\right)^{\frac{1}{1+4L_t^2/(1-\rho)} \frac{1}{1+\alpha}} \end{aligned}$$

where $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$.

Let us observe – and this will be important when we apply Proposition 4.5 – that by Lemma 1.10, $|a| \leq L_t$ implies that $\Phi(\langle a, \cdot \rangle - b)$ can be written in the form $P_{t+s}1_B$ for some $s > 0$ and some half-space B .

The main goal of this section is to prove Proposition 4.5. The proof proceeds according to the following steps:

- First, using a Poincaré-like inequality (Proposition 4.6) we show that if $\mathbb{E}_\rho|\nabla v(X) - \nabla w(Y)|^2$ is small then v and w are close to linear functions (with the same slope).
- In Proposition 4.8, we use the reverse Hölder inequality and some concentration properties to show that if $\frac{dR_t}{dt}$ is small, then $\mathbb{E}_\rho|\nabla v_t(X) - \nabla w_t(Y)|^{2p}$ must be small for some $p < 1$.
- Using Theorem 1.7, we argue that if $\mathbb{E}_\rho|\nabla v_t(X) - \nabla w_t(Y)|^{2p}$ is small then $\mathbb{E}_\rho|\nabla v_t(X) - \nabla w_t(Y)|^2$ is also small. Thus, we can apply the Poincaré inequality mentioned in the first bullet point, and so we obtain linear approximations for v_t and w_t .

This proof outline should appear somewhat familiar, because it is similar to the outline of Proposition 3.4's proof.

A Poincaré-like inequality

Recall that we proved the equality case by arguing that if $\frac{dR_t}{dt} = 0$ then $|\nabla v_t(X) - \nabla w_t(Y)|$ is identically zero, so ∇v_t and ∇w_t must be constant and thus v_t and w_t must be linear. The first step towards a robustness result is to show that if $|\nabla v_t(X) - \nabla w_t(Y)|$ is small, then v_t and w_t must be almost linear, and with the same slope.

Proposition 4.6. *For any smooth functions $f, g \in L_2(\gamma_n)$, if we set $a = \frac{1}{2}(\mathbb{E}\nabla f + \mathbb{E}\nabla g)$ then for any $0 < \rho < 1$,*

$$\mathbb{E}(f(X) - \langle X, a \rangle - \mathbb{E}f)^2 + \mathbb{E}(g(X) - \langle X, a \rangle - \mathbb{E}g)^2 \leq \frac{\mathbb{E}_\rho |\nabla f(X) - \nabla g(Y)|^2}{1 - \rho}.$$

By testing Proposition 4.6 against quadratic polynomials, we see that it is sharp up to a factor of 2. In fact, Proposition 4.6 may be sharpened by a factor of 2 with a slightly more complicated argument [41]. The proof we give here is simpler, though, and it generalizes more easily. The two ingredients are Poincaré's inequality (1.9) and a fairly standard correlation bound:

Lemma 4.7. *For any $f, g \in L_2(\gamma_n)$,*

$$\text{Cov}_\rho(f(X), g(Y)) \leq \rho \sqrt{\text{Var}(f) \text{Var}(g)} \leq \rho \frac{\text{Var}(f) + \text{Var}(g)}{2}. \quad (4.4)$$

Hence,

$$\mathbb{E}(f(X) - g(X))^2 \geq (\mathbb{E}f - \mathbb{E}g)^2 + (1 - \rho)(\text{Var}(f) + \text{Var}(g)) \geq (1 - \rho)(\mathbb{E}f^2 + \mathbb{E}g^2) - 2\mathbb{E}f\mathbb{E}g. \quad (4.5)$$

The property (4.4) is sufficiently useful to have its own name: a pair of random variables (X, Y) are said to have *Rényi correlation* at most ρ if for every pair of functions f, g , $\text{Cov}(f(X), g(Y)) \leq \rho \sqrt{\text{Var}(f) \text{Var}(g)}$. Under this terminology, Lemma 4.7 simply says that ρ -correlated Gaussians have Rényi correlation ρ .

Proof. Suppose without loss of generality that $\mathbb{E}f = \mathbb{E}g = 0$. If we expand $f = \sum_\alpha f_\alpha H_\alpha$ and $g = \sum_\alpha g_\alpha H_\alpha$ in the Hermite basis, then

$$\mathbb{E}f(X)P_t g(X) = \sum_\alpha e^{-t|\alpha|} f_\alpha g_\alpha \leq \sum_\alpha e^{-t|\alpha|} |f_\alpha g_\alpha|.$$

Now, $\mathbb{E}f = \mathbb{E}g = 0$ implies that $f_0 = g_0 = 0$. Hence

$$\sum_\alpha e^{-t|\alpha|} |f_\alpha g_\alpha| \leq e^{-t} \sum_\alpha |f_\alpha g_\alpha| \leq e^{-t} \left(\sum_\alpha f_\alpha^2 \sum_\alpha g_\alpha^2 \right)^{1/2} = e^{-t} \sqrt{\mathbb{E}f^2 \mathbb{E}g^2}.$$

Finally, observe that with $t = \log(1/\rho)$, $\mathbb{E}f(X)P_t g(X) = \mathbb{E}_\rho f(X)g(Y)$. This proves the first inequality in (4.4); the second inequality follows because the geometric mean is larger than the arithmetic mean. The proof of (4.5) then follows by expanding the square and applying (4.4). \square

Proof of Proposition 4.6. Assume without loss of generality that $\mathbb{E}f = \mathbb{E}g = 0$. We begin with the left hand side. Integrating by parts, one checks that $\mathbb{E}Xf(X) = \mathbb{E}\nabla f$. Hence,

$$\mathbb{E}(f(X) - \langle X, a \rangle)^2 = \mathbb{E}f^2 + |a|^2 - 2\langle a, \mathbb{E}\nabla f \rangle.$$

Adding to this the corresponding equation for g , we have

$$\begin{aligned} \mathbb{E}(f(X) - \langle X, a \rangle)^2 + \mathbb{E}(g(X) - \langle X, a \rangle)^2 &= \mathbb{E}f^2 + \mathbb{E}g^2 + 2|a|^2 - 2\langle a, \mathbb{E}\nabla f + \mathbb{E}\nabla g \rangle \\ &= \mathbb{E}f^2 + \mathbb{E}g^2 - 2|a|^2. \end{aligned} \quad (4.6)$$

On the other hand (4.5) implies that for $h_1, h_2 \in L_2(\gamma_n)$,

$$\frac{\mathbb{E}(h_1(X) - h_2(Y))^2}{1 - \rho} \geq (\mathbb{E}h_1 - \mathbb{E}h_2)^2 + \text{Var}(h_1) + \text{Var}(h_2) = \mathbb{E}h_1^2 + \mathbb{E}h_2^2 - 2\mathbb{E}h_1\mathbb{E}h_2.$$

Applying this to the partial derivatives of f and g , we have

$$\frac{\mathbb{E}_\rho |\nabla f(X) - \nabla g(Y)|^2}{1 - \rho} \geq \mathbb{E}|\nabla f|^2 + \mathbb{E}|\nabla g|^2 - 2\langle \mathbb{E}\nabla f, \mathbb{E}\nabla g \rangle. \quad (4.7)$$

Since $\mathbb{E}f = \mathbb{E}g = 0$, Poincaré's inequality implies that $\mathbb{E}f^2 \leq \mathbb{E}|\nabla f|^2$ and $\mathbb{E}g^2 \leq \mathbb{E}|\nabla g|^2$, while the Cauchy-Schwarz inequality implies that $|a|^2 = |(\mathbb{E}\nabla f + \mathbb{E}\nabla g)/2|^2 \geq \langle \mathbb{E}\nabla f, \mathbb{E}\nabla g \rangle$. Hence, the right hand side of (4.6) is at most the right hand side of (4.7). \square

The reverse-Hölder inequality

Recall the formula for $\frac{dR_t}{dt}$ given in Lemma 2.8. In this section, we will use the reverse-Hölder inequality to split this formula into an exponential term and a term depending on $|\nabla v_t(X) - \nabla w_t(X)|$. We will then use the smoothness of v_t and w_t to bound the exponential term, with the following result:

Proposition 4.8. *For any $0 < \rho < 1$ and any $t > 0$, there is a $c(t, \rho) > 0$ such that for any $r \leq \frac{1}{1+4L_t^2/(1-\rho)}$ and for any f and g ,*

$$\frac{dR_t}{dt} \geq c(t, \rho) m^{2\frac{L_t^2(1+L_t)}{1-\rho}} (\mathbb{E}|\nabla v_t(X) - \nabla w_t(Y)|^{2r})^{1/r}.$$

The three ingredients in the proof of Proposition 4.8 are the same as the ingredients in the proof of the isoperimetric analogue, Proposition 3.5: we use the reverse-Hölder inequality (3.5) to split the expectation, a Lipschitz concentration bound (Lemma 3.6) to control the exponential term, and Lemma 3.7 to relate the mean and median of $P_t f$.

Proof of Proposition 4.8. We begin by applying the reverse-Hölder inequality (3.5) to the equation in Lemma 2.8: for any $r < 1$ and $\beta = (1-r)/r$,

$$\frac{dR_t}{dt} \geq \frac{\rho}{2\pi\sqrt{1-\rho^2}} \left(\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right) \right)^{-1/\beta} \left(\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^{2r} \right)^{1/r}. \quad (4.8)$$

Let us first consider the exponential term in (4.8). Since $2|v_t w_t| \leq v_t^2 + w_t^2$, we have

$$\begin{aligned} \mathbb{E}_\rho \exp\left(\beta \frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right) &\leq \mathbb{E}_\rho \exp\left(\beta \frac{v_t^2 + w_t^2}{2(1-\rho)}\right) \\ &\leq \left(\mathbb{E} \exp\left(\beta \frac{v_t^2}{1-\rho}\right) \mathbb{E} \exp\left(\beta \frac{w_t^2}{1-\rho}\right)\right)^{1/2}, \end{aligned} \quad (4.9)$$

where we used the Cauchy-Schwarz inequality in the last line. Recall from Theorem 1.7 that v_t and w_t are both L_t -Lipschitz. Thus, we can apply Lemma 3.6 with $f = v_t$ and $\lambda = 2\beta L_t^2/(1-\rho)$; we see that if $\lambda = 2\beta L_t^2/(1-\rho) \leq \frac{1}{2}$, then

$$\mathbb{E} \exp\left(\beta \frac{v_t^2}{1-\rho}\right) \leq C e^{\lambda M_t^2},$$

where M_t is a median of v_t . Applying the same argument to w_t and plugging the result into (4.9), we have

$$\mathbb{E}_\rho \exp\left(\beta \frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right) \leq C e^{\lambda(M_t^2 + N_t^2)},$$

where N_t is a median of w_t . Going back to (4.8), we have

$$\frac{dR_t}{dt} \geq \frac{c\rho}{\sqrt{1-\rho^2}} e^{-\frac{\lambda}{\beta}(M_t^2 + N_t^2)} \left(\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^{2r}\right)^{1/r}, \quad (4.10)$$

provided that $\lambda = 2\beta L_t^2/(1-\rho) \leq \frac{1}{2}$. The equivalent condition on β is $\beta \leq \frac{1}{4}(1-\rho)/L_t^2$, and so the equivalent condition on $r = \beta/(1+\beta)$ is $r \leq \frac{1}{1+4L_t^2/(1-\rho)}$. Finally, we invoke Corollary 3.8 to show that

$$\exp\left(-\frac{\lambda}{\beta} M_t^2\right) = \exp\left(-\frac{2L_t^2 M_t^2}{1-\rho}\right) \geq (c\mathbb{E}f(1-\mathbb{E}f))^2 \frac{L_t^2(1+L_t)^2}{1-\rho}$$

(and similarly for g and N_t). Plugging this into (4.10) completes the proof. \square

Proof of Proposition 4.5

We are now prepared to prove Proposition 4.5 by combining Proposition 4.8 with Theorem 1.7 and Proposition 4.6. Besides combining these three results, there is a small technical obstacle: we know only that the integral of $\frac{dR_t}{dt}$ is small; we don't know anything about $\frac{dR_t}{dt}$ at specific values of t . So instead of showing that v_t is close to linear for every t , we will show that for every t , there is a nearby t^* such that v_{t^*} is close to linear. By ensuring that t^* is close to t , we will then be able to argue that v_t is also close to linear.

Proof of Proposition 4.5. For any $0 < r < 1$, Theorem 1.7 implies that

$$\left(\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^{2r}\right)^{1/r} \geq \left(\frac{\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^2}{\|\nabla v_t - \nabla w_t\|_\infty^{2-2r}}\right)^{1/r} \geq \frac{\left(\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^2\right)^{1/r}}{(2L_t)^{2(1-r)/r}}.$$

By Proposition 4.6 applied to v_t and w_t , if we set $a = \frac{1}{2}(\mathbb{E}\nabla v_t + \mathbb{E}\nabla w_t)$ and we define $\epsilon(v_t) = \mathbb{E}(v_t(X) - \langle X, a \rangle - \mathbb{E}v)^2$ (and similarly for $\epsilon(w_t)$), then

$$(\epsilon(v_t) + \epsilon(w_t))^{1/r} \leq \frac{(2L_t)^{2(1-r)/r}}{1-\rho} (\mathbb{E}_\rho |\nabla v_t - \nabla w_t|^{2r})^{1/r}.$$

Now we plug this into Proposition 4.8 to obtain

$$(\epsilon(v_t) + \epsilon(w_t))^{1/r} \leq C(t, \rho) m^{-C(t, \rho)} \frac{dR_t}{dt}. \quad (4.11)$$

Recall that $\delta(f, g) = \int_0^\infty \frac{dR_s}{ds} ds$. In particular,

$$\alpha t \min_{t \leq s \leq t(1+\alpha)} \left. \frac{dR_t}{dt} \right|_s \leq \int_t^{t(1+\alpha)} \frac{dR_s}{ds} ds \leq \delta(f, g)$$

and so there is some $s \in [t, t(1+\alpha)]$ such that $\left. \frac{dR_t}{dt} \right|_s \leq \frac{\delta}{\alpha t}$. If we apply this to (4.11) with t replaced by s and with $r = \frac{1}{1+4L_t^2/(1-\rho)} \leq \frac{1}{1+4L_s^2/(1-\rho)}$, we obtain

$$\epsilon(v_s) + \epsilon(w_s) \leq C(t, \rho) m^{-rC(t, \rho)} \left(\frac{\delta}{\alpha} \right)^r = C(t, \rho) m^{-C'(t, \rho)} \left(\frac{\delta}{\alpha} \right)^r.$$

Since Φ is Lipschitz, if we denote $\mathbb{E}(f_s - \Phi(\langle X, a \rangle - \mathbb{E}v_s))^2$ by $\epsilon(f_s)$ (and similarly for g_s), then we have

$$\epsilon(f_s) + \epsilon(g_s) \leq C(t, \rho) m^{-C(t, \rho)} \left(\frac{\delta}{\alpha} \right)^r. \quad (4.12)$$

So far, we have shown that f_s and g_s are close to functions of the desired form; next, we turn our attention to f_t and g_t . Define $\epsilon(f_t) = \mathbb{E}(f_t - P_{s-t}^{-1}\Phi(\langle X, a \rangle - \mathbb{E}v_s))^2$ and similarly for $\epsilon(g_t)$. By an interpolation inequality, we can compare $\epsilon(f_s)$ and $\epsilon(f_t)$. We will prove the lemma after this proof is complete.

Lemma 4.9. *For any $t < s$ and any $h \in L_2(\mathbb{R}^n, \gamma_n)$,*

$$\mathbb{E}(P_t h)^2 \leq \left(\mathbb{E}(P_s h)^2 \right)^{t/s} \left(\mathbb{E}h^2 \right)^{1-t/s}.$$

To complete the proof of Proposition 4.5, apply Lemma 4.9 with $h = f - P_s^{-1}\Phi(\langle X, a \rangle - \mathbb{E}v_s)$ (note that $P_s^{-1}\Phi(\langle X, a \rangle - \mathbb{E}v_s)$ exists by Lemma 1.10, because $|a| \leq L_s$). Since $\mathbb{E}h^2 \leq \sup |h| \leq 1$ and $s \leq (1+\alpha)t$, we see that

$$\epsilon(f_t) = \mathbb{E}(P_t h)^2 \leq \left(\mathbb{E}(P_s h)^2 \right)^{t/s} \leq \epsilon(f_s)^{1/(1+\alpha)}.$$

Applying this (and the equivalent inequality for g) to (4.12), we have

$$\epsilon(f_t) + \epsilon(g_t) \leq C(t, \rho)^{\frac{1}{1+\alpha}} m^{-C(t, \rho)/(1+\alpha)} \left(\frac{\delta}{\alpha} \right)^{\frac{r}{1+\alpha}}.$$

Since $\alpha < 1$, $\frac{1}{2} \leq \frac{1}{1+\alpha} \leq 1$ and so we can absorb the power $\frac{1}{1+\alpha}$ into the constant $C(t, \rho)$. \square

Proof of Lemma 4.9. Expand $P_s h$ in the Hermite basis as $P_s h = \sum b_\alpha H_\alpha$. Then

$$\begin{aligned}\mathbb{E}(P_s h)^2 &= \sum b_\alpha^2 \\ \mathbb{E}(P_t h)^2 &= \sum b_\alpha^2 e^{2(s-t)|\alpha|} \\ \mathbb{E}h^2 &= \sum b_\alpha^2 e^{2s|\alpha|}.\end{aligned}$$

By Hölder's inequality applied with the exponents s/t and $s/(s-t)$,

$$\begin{aligned}\mathbb{E}(P_t h)^2 &= \sum b_\alpha^{(s-t)/s} e^{2(s-t)|\alpha|} b_\alpha^{t/s} \\ &\leq \left(\sum b_\alpha^2 e^{2s|\alpha|} \right)^{(s-t)/s} \left(\sum b_\alpha^2 \right)^{t/s} \\ &= (\mathbb{E}h^2)^{(s-t)/s} (\mathbb{E}(P_s h)^2)^{t/s}.\end{aligned}$$

□

4.2 Approximation for small t

After Proposition 4.5, the second step in proving Theorem 4.1 is to show that if $P_t f$ and $P_t g$ are close to the equality cases of Theorem 2.5 then f and g are also. This step marks the main difference between this proof and the proof of Theorem 3.2. Indeed, in Chapter 3 we used a spectral argument. That spectral argument was responsible for the logarithmically slow rates (in δ) in Theorem 3.2, and it also required some smoothness properties of f (namely, a bound on $\mathbb{E}|\nabla f|$) that we can no longer assume. Here, we use a different argument that gives polynomial rates. The argument here will need the function f to take values only in $\{0, 1\}$. Thus, we will first establish Theorem 4.1 for sets; having done so, it is not difficult to extend it to functions using the correspondence, described in Section 1.2, between functions $\mathbb{R}^n \rightarrow \mathbb{R}$ and Ehrhard subsets of \mathbb{R}^{n+1} .

The main goal of our argument is to bound $\mathbb{E}|h|$ from above in terms of $\mathbb{E}|P_t h|$, for some function h . In Section 3.2, we brought up the example $h_k(x) = \text{sgn}(\sin(kx))$ to show why this is not possible for general h . The example of h_k is problematic because there is a lot of cancellation in $P_t h$. The essence of this section is that for the functions h we are interested in, there is a *geometric* reason – as opposed to a reason based on smoothness, as in Chapter 3 – which disallows too much cancellation. Indeed, we are interested in functions h of the form $1_A - 1_B$ where B is a half-space. The negative part of such a function is supported on B , while the positive part is supported on B^c . As we will see, this fact allows us to bound the amount of cancellation that occurs, and thus bound $\mathbb{E}|h|$ in terms of $\mathbb{E}|P_t h|$:

Proposition 4.10. *Let $B \subset \mathbb{R}^n$ be a half-space and $A \subset \mathbb{R}^n$ be any other set. There is an absolute constant C such that for any $t > 0$,*

$$\gamma_n(A\Delta B) \leq C \max \left\{ \mathbb{E}|P_t 1_A - P_t 1_B|, (e^{2t} - 1)^{1/4} \sqrt{\mathbb{E}|P_t 1_A - P_t 1_B|} \right\},$$

The main idea in Proposition 4.10 is in the following lemma, which states that if a non-negative function is supported on a half-space then P_t will push strictly less than half of its mass onto the complementary half-space.

Lemma 4.11. *There is a constant $c > 0$ such that for any $b \in \mathbb{R}$, if $f : \mathbb{R}^n \rightarrow [0, 1]$ is supported on $\{x_1 \leq b\}$ then for any $t > 0$,*

$$\mathbb{E}(P_t f)1_{\{X_1 \geq e^{-tb}\}} \leq \max \left\{ \frac{1}{2} \mathbb{E}f - c \frac{(\mathbb{E}f)^2}{\sqrt{e^{2t} - 1}}, \frac{3}{8} \mathbb{E}f \right\}.$$

Proof. Because P_t is self-adjoint,

$$\mathbb{E}(P_t f)1_{\{X_1 \geq e^{-tb}\}} = \mathbb{E}f P_t 1_{\{X_1 \geq e^{-tb}\}} = \mathbb{E}f \Phi \left(\frac{X_1 - b}{\sqrt{e^{2t} - 1}} \right).$$

Now, the set $\{b - \mathbb{E}f \leq x_1 \leq b\}$ has measure at most $\phi(0)\mathbb{E}f$. In particular, $\mathbb{E}f 1_{\{b - \mathbb{E}f \leq x_1 \leq b\}} \leq \phi(0)\mathbb{E}f \leq \frac{1}{2}\mathbb{E}f$.

Let $A = \{x_1 \leq b - \mathbb{E}f\}$ and $B = \{b - \mathbb{E}f \leq x_1 \leq b\}$ and recall that f is supported on $\{x_1 \leq b\}$, so that $f = f(1_A + 1_B)$. Now,

$$\Phi \left(\frac{x_1 - b}{\sqrt{e^{2t} - 1}} \right) \leq \begin{cases} \Phi \left(-\frac{\mathbb{E}f}{\sqrt{e^{2t} - 1}} \right) & x \in A \\ \frac{1}{2} & x \in B \end{cases}$$

and so

$$\begin{aligned} \mathbb{E}f \Phi \left(\frac{X_1 - b}{\sqrt{e^{2t} - 1}} \right) &= \mathbb{E}1_A f \Phi \left(\frac{X_1 - b}{\sqrt{e^{2t} - 1}} \right) + \mathbb{E}1_B f \Phi \left(\frac{X_1 - b}{\sqrt{e^{2t} - 1}} \right) \\ &\leq \Phi \left(-\frac{\mathbb{E}f}{\sqrt{e^{2t} - 1}} \right) \mathbb{E}1_A f + \frac{1}{2} \mathbb{E}1_B f \\ &= \frac{1}{2} \mathbb{E}f - \left(\frac{1}{2} - \Phi \left(-\frac{\mathbb{E}f}{\sqrt{e^{2t} - 1}} \right) \right) \mathbb{E}f 1_A. \end{aligned} \quad (4.13)$$

There is a constant $c > 0$ such that for all $x \geq 0$, $\Phi(-x) \leq \max\{\frac{1}{2} - cx, \frac{1}{4}\}$. Applying this with $x = \frac{\mathbb{E}f}{\sqrt{e^{2t} - 1}}$, we have

$$(4.13) \leq \frac{1}{2} \mathbb{E}f - \mathbb{E}f 1_A \min \left\{ c \frac{\mathbb{E}f}{\sqrt{e^{2t} - 1}}, \frac{1}{4} \right\} \leq \max \left\{ \frac{1}{2} \mathbb{E}f - c \frac{(\mathbb{E}f)^2}{\sqrt{e^{2t} - 1}}, \frac{3}{8} \mathbb{E}f \right\}$$

where in the last inequality, we recalled that $\mathbb{E}f 1_A \geq \frac{1}{2} \mathbb{E}f$. \square

Proof of Proposition 4.10. Without loss of generality, B is the half-space $\{x_1 \leq b\}$. Let f be the positive part of $1_A - 1_B$ and let g be the negative part, so that $\gamma(A \Delta B) = \mathbb{E}f + \mathbb{E}g$. Note that f is supported on B^c and g is supported on B .

Without loss of generality, $\mathbb{E}f \geq \mathbb{E}g$; Lemma 4.11 implies that if $\mathbb{E}f \leq C\sqrt{e^{2t} - 1}$ then

$$2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \leq \mathbb{E}f + \mathbb{E}g - c \frac{(\mathbb{E}f + \mathbb{E}g)^2}{\sqrt{e^{2t} - 1}}. \quad (4.14)$$

On the other hand, if $\mathbb{E}f \geq C\sqrt{e^{2t} - 1}$ then

$$2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \leq \frac{3}{4}\mathbb{E}f + \mathbb{E}g \leq \frac{7}{8}(\mathbb{E}f + \mathbb{E}g). \quad (4.15)$$

Thus,

$$\begin{aligned} \mathbb{E}|P_t f - P_t g| &= \mathbb{E}P_t f + \mathbb{E}P_t g - 2\mathbb{E}\min\{P_t f, P_t g\} \\ &= \mathbb{E}f + \mathbb{E}g - 2\mathbb{E}\min\{P_t f, P_t g\} \\ &\geq \mathbb{E}f + \mathbb{E}g - 2\mathbb{E}(1_B P_t f + 1_{B^c} P_t g) \\ &\geq \min\left\{c\frac{(\mathbb{E}f + \mathbb{E}g)^2}{\sqrt{e^{2t} - 1}}, \frac{\mathbb{E}f + \mathbb{E}g}{8}\right\}, \end{aligned}$$

Where we have applied (4.14) and (4.15) in the last inequality. Now there are two cases, depending on which term in the minimum is smaller: if the first term is smaller then

$$\mathbb{E}f + \mathbb{E}g \leq C(e^{2t} - 1)^{1/4}\sqrt{\mathbb{E}|P_t f - P_t g|};$$

otherwise, the second term in the minimum is smaller and

$$\mathbb{E}f + \mathbb{E}g \leq 8\mathbb{E}|P_t f - P_t g|.$$

In either case,

$$\gamma(A\Delta B) = \mathbb{E}f + \mathbb{E}g \leq C \max\left\{\mathbb{E}|P_t f - P_t g|, (e^{2t} - 1)^{1/4}\sqrt{\mathbb{E}|P_t f - P_t g|}\right\},$$

as claimed. \square

Synchronizing the ts

Let A be an arbitrary set. If we knew that there was a half-space B such that $\mathbb{E}(P_t 1_A - P_t 1_B)^2$ was small, then Proposition 4.10 would imply that $\gamma_n(A\Delta B)$ is small. Now, Proposition 4.5 and Lemma 1.10 imply only that that $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is small for some half-space B and some $s \geq 0$. In this section, we will argue that s must be small. Now, this is not necessarily the case for arbitrary sets A ; in fact, for any $s > 0$ one can find A such that $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is arbitrarily small. However, our sets A are not arbitrary: they are sets which are almost optimally noise stable for correlation ρ . In particular, if $e^{-t} = \rho$ then $\mathbb{E}1_A P_t 1_A$ is close to $\mathbb{E}1_B P_t 1_B$.

Using this extra information, the proof of robustness proceeds as follows: since $\mathbb{E}1_A P_t 1_A$ is close to $\mathbb{E}1_B P_t 1_B$ and $P_t 1_A$ is close to $P_{t+s} 1_B$, we will show that $\mathbb{E}1_B P_{t+s} 1_B$ is close to $\mathbb{E}1_B P_t 1_B$. But we know all about B : it is a half-space. Therefore, we can find explicit and accurate estimates for $\mathbb{E}1_B P_{t+s} 1_B$ and $\mathbb{E}1_B P_t 1_B$ in terms of t , s and $\gamma_n(B)$; using them, we can conclude that s is small. Now, if s is small then we can show (again, using explicit estimates) that $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ is small. Since $\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2$ is small (this was our starting point, remember), we can apply the triangle inequality to conclude that $\mathbb{E}(P_t 1_A - P_t 1_B)^2$ is small. Finally, we can apply Proposition 4.10 to show that $\mathbb{E}|1_A - 1_B|$ is small.

Proposition 4.12. *For every t , there is a $C(t)$ such that the following holds. For sets $A, A' \subset \mathbb{R}^n$, suppose that $B, B' \subset \mathbb{R}^n$ are parallel half-spaces with $\gamma(A) = \gamma(B)$, $\gamma(A') = \gamma(B')$. If there exist $s, \epsilon_1, \epsilon_2 > 0$ such that*

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \leq \epsilon_1^2$$

and

$$\mathbb{E} 1_A P_t 1_{A'} \geq \mathbb{E} 1_B P_t 1_{B'} - \epsilon_2$$

then

$$\left(\mathbb{E}(P_t 1_A - P_t 1_B)^2\right)^{1/2} \leq C(t) \frac{\epsilon_1 + \epsilon_2}{(I(\gamma(A))I(\gamma(A')))^{C(t)}},$$

where $I(x) = \phi(\Phi^{-1}(x))$.

Rather than prove Proposition 4.12 all at once, we have split the part relating $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ and $\mathbb{E} 1_B(P_t 1_{B'} - P_{t+s} 1_{B'})$ into a separate lemma.

Lemma 4.13. *For every t there is a $C(t)$ such that for any parallel half-spaces B and B' , and for every $s > 0$,*

$$\left(\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2\right)^{1/2} \leq C(t) \frac{\mathbb{E} 1_B(P_t 1_{B'} - P_{t+s} 1_{B'})}{(I(\gamma(B))I(\gamma(B')))^{C(t)}}.$$

Proof. First of all, one can easily check through integration by parts that for a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_b^\infty \phi(x)(Lf)(x) dx = -f'(b)\phi(b). \quad (4.16)$$

By rotating B and B' , we can assume that $B = \{x_1 \leq a\}$ and $B' = \{x_1 \leq b\}$. Let $F_{ab}(t) = \mathbb{E} 1_B P_t 1_{B'} = \int_a^\infty \phi(x) \Phi\left(\frac{e^{-t}x-b}{\sqrt{1-e^{-2t}}}\right) dx$ and consider its derivative: by (4.16), if $L_t = (e^{2t} - 1)^{-1/2}$ then

$$\begin{aligned} F'_{ab}(t) &= \int_a^\infty \phi(x) L \Phi\left(\frac{e^{-t}x-b}{\sqrt{1-e^{-2t}}}\right) dx \\ &= -L_t \phi(a) \phi\left(\frac{e^{-t}a-b}{\sqrt{1-e^{-2t}}}\right) \\ &= -\frac{L_t}{2\pi} \exp\left(-\frac{a^2+b^2-2e^{-t}ab}{2(1-e^{-2t})}\right) \\ &\leq -\frac{L_t}{2\pi} \exp\left(-\frac{a^2+b^2}{1-e^{-2t}}\right). \end{aligned}$$

Now, L_t is decreasing in t and $\exp(-x/(1-e^{-2t}))$ is increasing in t . In particular, for any $\tau \in [t, t+s]$,

$$F'_{ab}(\tau) \leq -\frac{L_{t+s}}{2\pi} \exp\left(-\frac{a^2+b^2}{1-e^{-2t}}\right).$$

Hence,

$$F_{ab}(t) - F_{ab}(t+s) \geq -s \max_{s \leq \tau \leq t} F'_{ab}(\tau) \geq \frac{sL_{t+s}}{2\pi} \exp\left(-\frac{a^2+b^2}{1-e^{-2t}}\right). \quad (4.17)$$

If s is large, this is a poor bound because sL_{t+s} decreases exponentially in s . However, when $s \geq 1$ we can instead use

$$F_{ab}(t) - F_{ab}(t+s) \geq F_{ab}(t) - F_{ab}(t+1) \geq \frac{L_{t+1}}{2\pi} \exp\left(-\frac{a^2+b^2}{1-e^{-2t}}\right). \quad (4.18)$$

Equations (4.17) and (4.18) show that if $\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})$ is small then s must be small. The next step, therefore, is to control $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ in terms of s . Now,

$$\begin{aligned} \mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 &= \mathbb{E}((P_t 1_B)^2 + (P_{t+s} 1_B)^2 - 2(P_t 1_B)(P_{t+s} 1_B)) \\ &= \mathbb{E}1_B(P_{2t} 1_B + P_{2(t+s)} 1_B - 2P_{2t+s} 1_B) \\ &= (F_{aa}(2t) - F_{aa}(2t+s)) - (F_{aa}(2t+s) - F_{aa}(2t+2s)) \\ &\leq s(F'_{aa}(2t) - F'_{aa}(2t+2s)), \end{aligned} \quad (4.19)$$

where the inequality follows because

$$F'_{aa}(t) = -\frac{L_t}{2\pi} \exp\left(-\frac{(1-e^{-t})a^2}{1-e^{-2t}}\right) = -\frac{L_t}{2\pi} \exp\left(-\frac{a^2}{1+e^{-t}}\right)$$

and so F'_{aa} is an increasing function. To control the right hand side of (4.19), we go to the second derivative of F :

$$F''(t) = \frac{e^{2t}}{2\pi(e^{2t}-1)^{3/2}} \exp\left(-\frac{a^2}{1+e^{-t}}\right) + \frac{1}{2\pi\sqrt{e^{2t}-1}} \frac{a^2 e^{-t}}{(1+e^{-t})^2} \exp\left(-\frac{a^2}{1+e^{-t}}\right)$$

This is decreasing in t ; hence

$$\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 \leq s(F'(2t) - F'(2t+2s)) \leq 2s^2 F''(2t). \quad (4.20)$$

We will now complete the proof by combining our upper bound on $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2$ with our lower bounds on $\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})$. First, assume that $s \leq 1$. Then $L_{t+s} \geq L_{t+1}$ and so (4.17) plus (4.20) implies that

$$\begin{aligned} (\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2)^{1/2} &\leq 2\pi \exp\left(\frac{a^2+b^2}{1-e^{-2t}}\right) \frac{\sqrt{2F''(2t)}}{L_{t+1}} \mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'}) \\ &= 2\pi^{1-\frac{2}{1-e^{-2t}}} \frac{\sqrt{2F''(2t)}}{L_{t+1}} \frac{\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})}{(I(\gamma(B))I(\gamma(B')))^{\frac{2}{1-e^{-2t}}}}. \end{aligned}$$

If we take $C(t) \geq \max\{\sqrt{2F''(2t)}/L_{t+1}, 2/(1-e^{-2t})\}$ then the Lemma holds in this case. On the other hand, if $s > 1$ then (4.18) implies that

$$\frac{2\pi^{1-\frac{2}{1-e^{-2t}}}}{L_{t+1}} \frac{\mathbb{E}1_B(P_t 1_{B'} - P_{t+s} 1_{B'})}{(I(\gamma(B))I(\gamma(B')))^{\frac{2}{1-e^{-2t}}}} \geq 1.$$

Since $\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2 \leq 1$ trivially, the Lemma holds in this case provided that

$$C(t) \geq \max\{1/L_{t+1}, 2/(1 - e^{-2t})\}. \quad \square$$

Proof of Proposition 4.12. By the Cauchy-Schwarz inequality,

$$\mathbb{E}1_A P_t 1_A \leq \mathbb{E}1_A P_{t+s} 1_B + \sqrt{\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2} \leq \mathbb{E}1_A P_{t+s} 1_B + \epsilon_1.$$

Moreover, $\mathbb{E}1_A P_{t+s} 1_B \leq \mathbb{E}1_B P_{t+s} 1_B$ since B is a super-level set of $P_{t+s} 1_B$ with the same volume as A . Thus,

$$\begin{aligned} \mathbb{E}1_B P_t 1_B - \epsilon_2 &\leq \mathbb{E}1_A P_t 1_A \\ &\leq \mathbb{E}1_A P_{t+s} 1_B + \epsilon_1 \\ &\leq \mathbb{E}1_B P_{t+s} 1_B + \epsilon_1. \end{aligned}$$

By Lemma 4.13,

$$\left(\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2\right)^{1/2} \leq C(t) \mathbb{E}1_B (P_t 1_B - P_{t+s} 1_B) \leq C(t)(\epsilon_1 + \epsilon_2)$$

Finally, the triangle inequality gives

$$\begin{aligned} \left(\mathbb{E}(P_t 1_A - P_t 1_B)^2\right)^{1/2} &\leq \left(\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2\right)^{1/2} + \left(\mathbb{E}(P_t 1_B - P_{t+s} 1_B)^2\right)^{1/2} \\ &\leq \epsilon_1 + C(t)(\epsilon_1 + \epsilon_2). \end{aligned}$$

Of course, 1 can be absorbed into the constant $C(t)$. □

Proof of Theorem 4.1

First, define t by $e^{-t} = \rho$. We then have $L_t^2 = \frac{\rho^2}{1-\rho^2}$ and so the exponent of δ in Proposition 4.5 becomes

$$\frac{1}{1 + 4 \frac{\rho^2}{(1-\rho^2)(1-\rho)}} \cdot \frac{1}{1 + \alpha} = \frac{(1-\rho^2)(1-\rho)}{1-\rho + 3\rho^2 + \rho^3} \cdot \frac{1}{1 + \alpha}. \quad (4.21)$$

Of course, we can define $\alpha > 0$ (depending on ρ) so that (4.21) is

$$\eta := \frac{(1-\rho^2)(1-\rho)}{1+3\rho}.$$

Now suppose that $f = 1_A$ and $g = 1_{A'}$ for some $A, A' \subset \mathbb{R}^n$. Proposition 4.5 implies that there are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $|a| \leq L_t$ and

$$\mathbb{E}\left(\left(P_t 1_A\right)(X) - \Phi(\langle a, X \rangle - b)\right)^2 \leq C(\rho) m^{c(\rho)} \delta^\eta.$$

Since $|a| \leq L_t$, Lemma 1.10 implies that we can find some $s > 0$ and a half-space B such that $\Phi(\langle a, x \rangle - b) = (P_{t+s} 1_B)(x)$; then

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \leq C(\rho) m^{c(\rho)} \delta^\eta. \quad (4.22)$$

At this point, it isn't clear that $\gamma(A) = \gamma(B)$; however, we can ensure this by modifying B slightly:

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \geq (\mathbb{E}P_t 1_A - \mathbb{E}P_{t+s} 1_B)^2 = (\gamma(A) - \gamma(B))^2.$$

Therefore let \tilde{B} be a translation of B so that $\gamma(\tilde{B}) = \gamma(A)$. By the triangle inequality,

$$\begin{aligned} (\mathbb{E}(P_t 1_A - P_{t+s} 1_{\tilde{B}})^2)^{1/2} &\leq (\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2)^{1/2} + (\mathbb{E}(P_{t+s} 1_B - P_{t+s} 1_{\tilde{B}})^2)^{1/2} \\ &\leq (\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2)^{1/2} + |\gamma(B) - \gamma(\tilde{B})|^{1/2} \\ &\leq 2(\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2)^{1/2}. \end{aligned}$$

By replacing B with \tilde{B} , we can assume in (4.22) that $\gamma(A) = \gamma(B)$ (at the cost of increasing $C(\rho)$ by a factor of 2).

Now we apply Proposition 4.12 with $\epsilon_1^2 = C(\rho)m^{c(\rho)}\delta^\eta$ and $\epsilon_2 = \delta$. The conclusion of Proposition 4.12 leaves us with

$$\begin{aligned} (\mathbb{E}(P_t 1_A - P_t 1_B)^2)^{1/2} &\leq C(\rho)m^{c(\rho)}(\epsilon_1 + \epsilon_2) \\ &\leq C(\rho)m^{c(\rho)}\delta^{\eta/2}. \end{aligned}$$

where we have absorbed the constant $C(t)$ from Proposition 4.10 into $C(\rho)$ and $c(\rho)$. Since $\mathbb{E}|X| \leq (\mathbb{E}X^2)^{1/2}$ for any random variable X , we may apply Proposition 4.10:

$$\begin{aligned} \gamma(A\Delta B) &\leq C(\rho)\sqrt{\mathbb{E}|P_t 1_A - P_t 1_B|} \\ &\leq C(\rho)(\mathbb{E}(P_t 1_A - P_t 1_B)^2)^{1/4} \\ &\leq C(\rho)m^{c(\rho)}\delta^{\eta/4}. \end{aligned}$$

By applying the same argument to A' and B' , this establishes Theorem 4.1 in the case that f and g are indicator functions.

To extend the result to other functions, note that $\mathbb{E}J(f(X), g(Y)) = \mathbb{E}J(1_A(\tilde{X}), 1_{A'}(\tilde{Y}))$ where \tilde{X} and \tilde{Y} are ρ -correlated Gaussian vectors in \mathbb{R}^{n+1} , and

$$\begin{aligned} A &= \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \Phi^{-1}(f(x))\} \\ A' &= \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \Phi^{-1}(g(x))\}. \end{aligned}$$

Moreover, $\mathbb{E}f = \gamma_{n+1}(A)$ and $\mathbb{E}g = \gamma_{n+1}(A')$. Applying Theorem 4.1 for indicator functions in dimension $n+1$, we find a half-space B so that

$$\gamma_{n+1}(A\Delta B) \leq C(\rho)m^{c(\rho)}\delta^{\eta/4}. \quad (4.23)$$

By slightly perturbing B , we can assume that it does not take the form $\{x_i \geq b\}$ for any $1 \leq i \leq n$; in particular, this means that we can write B in the form

$$B = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \langle a, x \rangle - b\}.$$

for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. But then

$$\gamma_{n+1}(A\Delta B) = \mathbb{E}|f(X) - \Phi(\langle a, X \rangle - b)|;$$

combined with (4.23), this completes the proof.

4.3 Optimal dependence on ρ

In this section, we will prove Theorem 4.2. To do so we need to improve the dependence on ρ that appeared in Theorem 4.1. Before we begin, let us list the places where the dependence on ρ can be improved:

1. In Proposition 4.8, we needed to control

$$\mathbb{E}_\rho \exp \left(\beta \frac{v_t^2(X) + w_t^2(Y) - 2\rho v_t(X)w_t(Y)}{2(1 - \rho^2)} \right).$$

Of course, the denominator of the exponent blows up as $\rho \rightarrow 1$. However, if $v_t = w_t$ then the numerator goes to zero (in law, at least) at the same rate. In this case, therefore, we are able to bound the above expectation by an expression not depending on ρ .

2. In the proof of Proposition 4.5, we used an L_∞ bound on $|\nabla v_t|$ and $|\nabla w_t|$ to show that for some $r < 1$,

$$\mathbb{E}_\rho (|\nabla v_t(X) - \nabla w_t(Y)|^2)^{1/r} \leq C(t) \mathbb{E}_\rho (|\nabla v_t(X) - \nabla w_t(Y)|^{2r})^{1/r}.$$

This inequality is not sharp in its ρ -dependence because when $v_t = w_t$, the left hand side shrinks like $(1 - \rho)^{1/r}$ as $\rho \rightarrow 1$, while the right hand side shrinks like $1 - \rho$. We can get the right ρ -dependence by using an L_p bound on $|\nabla v_t(X) - \nabla v_t(Y)|$ when applying Hölder's inequality, instead of an L_∞ bound.

3. In applying Proposition 4.12, we were forced to take $e^{-t} = \rho$. Since most of our bounds have a (necessary) dependence on t , this causes a dependence on ρ which is not optimal. To get around this, we will use the subadditivity property of Kane [28], and Kindler and O'Donnell [30] to show that we can actually choose certain values of t such that e^{-t} is much smaller than ρ . In particular, we can take t to be quite large even when ρ is close to 1.

Once we have incorporated the first two improvements, we will obtain a better version of Proposition 4.5:

Proposition 4.14. *For any $\alpha, t > 0$, there is a constant $C(t, \alpha)$ such that for any $f : \mathbb{R}^n \rightarrow [0, 1]$, there exist $a \in \mathbb{R}^n, b \in \mathbb{R}$ with $|a| \leq L_t$ such that*

$$\mathbb{E} (f_t(X) - \Phi(\langle X, a \rangle - b))^2 \leq C(t, \alpha) m^{\frac{4L_t^2(1+L_t)^2}{1+8L_t^2} - \alpha} \left(\frac{\delta}{\rho \sqrt{1 - \rho}} \right)^{\frac{1}{1+8L_t^2} - \alpha}.$$

where $L_t = (e^{2t} - 1)^{-1/2}$, $\delta(f) = \mathbb{E}_\rho J(f(X), f(Y)) - J(\mathbb{E}f, \mathbb{E}f)$, and $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$.

Moreover, this statement holds with a $C(t, \alpha)$ which, for any fixed α , is decreasing in t .

Once we have incorporated the third improvement above, we will use the arguments of Section 4.2 to prove Theorem 4.2.

A better bound on the auxiliary term

First, we will tackle item 1 above. Our improved bound leads to a version of Proposition 4.8 with the correct dependence on ρ .

Proposition 4.15. *Let $L_t = (e^{2t} - 1)^{-1/2}$. There are constants $0 < c, C < \infty$ such that for any $t > 0$, if $r \leq \frac{1}{1+8L_t^2}$ then*

$$\frac{dR_t}{dt} \geq \frac{\rho}{\sqrt{1-\rho^2}} (cm(f))^{4L_t^2(1+L_t)^2} \left(\mathbb{E} |\nabla v_t(X) - \nabla v_t(Y)|^{2r} \right)^{1/r}$$

where $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$.

To obtain this improvement, we note that for a Lipschitz function v , $(v(X) - v(Y))/\sqrt{1-\rho}$ satisfies a Gaussian tail bound that does not depend on ρ :

Lemma 4.16. *If $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -Lipschitz then*

$$\Pr_\rho \left(v(X) - v(Y) \geq Ls\sqrt{2(1-\rho)} \right) \leq 1 - \Phi(s).$$

In particular, if $4\beta L^2 < 1$ then

$$\mathbb{E}_\rho \exp \left(\beta \frac{(v(X) - v(Y))^2}{(1-\rho)} \right) \leq \frac{1}{\sqrt{1-4\beta L^2}}.$$

Proof. Let $Z_1 = \frac{X+Y}{2}$ and $Z_2 = \frac{X-Y}{2}$, so that $\mathbb{E}Z_1^2 = \frac{1+\rho}{2}$ and $\mathbb{E}Z_2^2 = \frac{1-\rho}{2}$. Now we condition on Z_1 : the function $v(Z_1 + Z_2) - v(Z_1 - Z_2)$ is $2L$ -Lipschitz in Z_2 and has conditional median zero (because it is odd in Z_2); thus

$$\Pr_\rho \left(v(Z_1 + Z_2) - v(Z_1 - Z_2) \geq Ls\sqrt{2(1-\rho)} \mid Z_1 \right) \leq 1 - \Phi(s).$$

Now integrate out Z_1 to prove the first claim.

Proving the second claim from the first one is a standard calculation. □

Next, we use the estimate of Lemma 4.16 to prove a bound on

$$\mathbb{E}_\rho \exp \left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X)v_t(Y)}{2(1-\rho^2)} \right)$$

that is better than the one from (4.9) which was used to derive Proposition 4.8.

Lemma 4.17. *There is a constant C such that for any $t > 0$, and for any $\beta > 0$ with $8\beta L_t^2 \leq 1$,*

$$\mathbb{E}_\rho \exp \left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X)v_t(Y)}{2(1-\rho^2)} \right) \leq 4e^{2\beta L_t^2 N_t^2},$$

where N_t is a median of v_t .

Proof. We begin with the Cauchy-Schwarz inequality:

$$\begin{aligned}
& \mathbb{E}_\rho \exp\left(\beta \frac{v_t^2(X) + v_t^2(Y) - 2\rho v_t(X)v_t(Y)}{2(1-\rho^2)}\right) \\
&= \mathbb{E}_\rho \exp\left(\beta \frac{(v_t(X) - v_t(Y))^2}{2(1-\rho^2)}\right) \exp\left(\beta \frac{v_t(X)v_t(Y)}{1+\rho}\right) \\
&\leq \left(\mathbb{E}_\rho \exp\left(2\beta \frac{(v_t(X) - v_t(Y))^2}{2(1-\rho^2)}\right)\right)^{1/2} \left(\mathbb{E} \exp\left(2\beta \frac{v_t(X)^2}{1+\rho}\right)\right)^{1/2}. \tag{4.24}
\end{aligned}$$

Now, recall from Theorem 1.7 that v_t is L_t -Lipschitz. In particular, Lemma 4.16 implies that if $8\beta L_t^2 \leq 1$ then the first term of (4.24) is at most $\sqrt{2}$. Finally, Lemma 3.6 (with $\lambda = 4L_t^2\beta$) implies that the second term of (4.24) is bounded by $2\sqrt{2}e^{2\beta L_t^2 N_t^2}$. \square

Proof of Proposition 4.15. Take $\beta \leq L_t^{-2}/8$ and let $r = \frac{\beta}{1+\beta}$ so that $\frac{1}{r} - \frac{1}{\beta} = 1$. Beginning from (4.8) in the proof of Proposition 4.8, we can apply Lemma 4.17 to obtain

$$\frac{dR_t}{dt} \geq c \frac{\rho}{\sqrt{1-\rho^2}} e^{-2L_t^2 N_t^2} \left(\mathbb{E}_\rho |\nabla v_t(X) - \nabla v_t(Y)|^{2r}\right)^{1/r},$$

and we conclude by applying Corollary 3.8, which implies that

$$e^{-2L_t^2 N_t^2} \geq (cm(f))^{4L_t^2(1+L_t)^2}. \quad \square$$

Higher moments of $|\nabla v_t(X) - \nabla v_t(Y)|$

Here, we will carry out the second step of the plan outlined at the beginning of Section 4.3. The main result is an upper bound on arbitrary moments of $|\nabla v_t(X) - \nabla v_t(Y)|$.

Proposition 4.18. *There is a constant C such that for any $t > 0$ and any $1 \leq q < \infty$,*

$$\left(\mathbb{E}_\rho |\nabla v_t(X) - \nabla v_t(Y)|^q\right)^{1/q} \leq C(L_t^2 + L_t) \sqrt{q(1-\rho)} \left(\sqrt{\log(1/m(f))} + L_t \sqrt{q}\right).$$

If we fix q and t , then the bound of Proposition 4.18 has the right dependence on ρ (this can be tested by setting v_t to be a quadratic function). In particular, we will use Proposition 4.18 instead of the uniform bound $|\nabla v_t| \leq L_t$, which does not improve as $\rho \rightarrow 1$.

The main tool for the proof of Proposition 4.18 is a result of Pinelis [46], which will allow us to relate moments of $|\nabla v_t(X) - \nabla v_t(Y)|$ to moments of $\|\text{Hess}(v_t)\|_F$.

Theorem 4.19. *Let Z and Z' be independent standard Gaussian vectors on \mathbb{R}^n . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a C^1 function with $\mathbb{E}f = 0$, and let Df be the $n \times k$ matrix of its partial derivatives. For any convex function $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}$,*

$$\mathbb{E}\Psi(f(Z)) \leq \mathbb{E}\Psi\left(\frac{\pi}{2} Df(Z) \cdot Z'\right).$$

Proof. Define $Z_\theta = Z' \cos \theta + Z \sin \theta$. Then $Z_0 = Z'$, $Z_{\pi/2} = Z$ and for every θ , Z_θ is a standard Gaussian vector on \mathbb{R}^n . By the fundamental theorem of calculus,

$$f(Z') - f(Z) = \int_0^{\pi/2} \frac{d}{d\theta} f(Z_\theta) d\theta = \int_0^{\pi/2} Df(Z_\theta) \cdot Z_{\theta+\pi/2} d\theta.$$

By Jensen's inequality applied to the normalized Lebesgue measure on $[0, \pi/2]$,

$$\Psi(f(Z') - f(Z)) \leq \frac{2}{\pi} \int_0^{\pi/2} \Psi\left(\frac{\pi}{2} Df(Z_\theta) \cdot Z_{\theta+\pi/2}\right) d\theta.$$

Note that for any θ , the pair $(Z_\theta, Z_{\theta+\pi/2})$ is equal in distribution to (Z, Z') . Hence, taking expectations of the above inequality,

$$\mathbb{E}\Psi(f(Z') - f(Z)) \leq \mathbb{E}\Psi\left(\frac{\pi}{2} Df(Z) \cdot Z'\right).$$

Since $\mathbb{E}f = 0$, Jensen's inequality applied conditioned on Z implies that $\mathbb{E}\Psi(f(Z') - f(Z)) \geq \mathbb{E}\Psi(f(Z))$. \square

It can be checked directly that for any fixed matrix A and for a standard Gaussian vector Z' , $\mathbb{E}|AZ'|^q \leq (C\sqrt{q})^{q/2} \|A\|_F$. By specializing Theorem 4.19 to the case $\Psi(x) = |x|^q$, we obtain the following corollary:

Corollary 4.20. *For any $1 \leq q < \infty$ and any smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$(\mathbb{E}|f(Z) - f(Z')|^q)^{1/q} \leq C\sqrt{q}(\mathbb{E}\|Df\|_F^q)^{1/q},$$

where C is a universal constant.

Proof. Define $h : \mathbb{R}^{2n} \rightarrow \mathbb{R}^k$ by $h(x, y) = f(x) - f(y)$. Let Z and Z' be independent standard Gaussian vectors in \mathbb{R}^n , and set $W = (Z, Z')$ so that W is a standard Gaussian in \mathbb{R}^{2n} . Applying Theorem 4.19 to h with $\Psi(x) = |x|^q$, we obtain

$$\mathbb{E}|h(W)|^q \leq C^q \mathbb{E}|Dh(W) \cdot W'|^q \leq (C\sqrt{q})^q \mathbb{E}\|Dh(W)\|_F^q,$$

where W' is an independent copy of W , and the last inequality came from applying the inequality $\mathbb{E}|AW'|^q \leq (C\sqrt{q})^{q/2} \|A\|_F$ conditioned on W . Finally, note that $\mathbb{E}|h(W)|^q = \mathbb{E}|f(Z) - f(Z')|^q$ and

$$\mathbb{E}\|Dh(W)\|_F^q = \mathbb{E}\|Df(Z) - Df(Z')\|_F^q \leq 2^q \mathbb{E}\|Df\|_F^q. \quad \square$$

To go from Corollary 4.20 to Proposition 4.18 takes two steps. The first is to decompose the ρ -correlated Gaussians X and Y in terms of independent Gaussians and apply Proposition 4.18 on the result; this step is responsible for achieving the correct dependence on ρ . The second step is to apply Lemma 3.9, the bound on $\mathbb{E}\|\text{Hess}(v_t)\|_F^q$ from the previous chapter.

Proof of Proposition 4.18. Let Z, Z_1 and Z_2 be independent standard Gaussians on \mathbb{R}^n ; set $X = \sqrt{\rho}Z + \sqrt{1-\rho}Z_1$ and $Y = \sqrt{\rho}Z + \sqrt{1-\rho}Z_2$ so that X and Y are standard Gaussians with correlation ρ . Conditioned on Z , define the function

$$h(x) = \nabla v_t(\sqrt{Z} + \sqrt{1-\rho}x),$$

so that $h(Z_1) = \nabla v_t(X)$ and $h(Z_2) = \nabla v_t(Y)$. Note that

$$(Dh)(x) = \sqrt{1-\rho} \text{Hess}(v_t)(\sqrt{\rho}Z + \sqrt{1-\rho}x);$$

thus Corollary 4.20 (conditioned on Z) implies that

$$\mathbb{E}(|\nabla v_t(X) - \nabla v_t(Y)|^q | Z) \leq (C\sqrt{q(1-\rho)})^q \mathbb{E}(\|\text{Hess}(v_t)(X)\|_F^q | Z).$$

Integrating out Z and raising both sides to the power $1/q$, we have

$$(\mathbb{E}|\nabla v_t(X) - \nabla v_t(Y)|^q)^{1/p} \leq C\sqrt{q(1-\rho)} (\mathbb{E}\|\text{Hess}(v_t)\|_F^q)^{1/q}.$$

We conclude by applying Lemma 3.9 to the right hand side. \square

With the first two steps of our outline complete, we are ready to prove Proposition 4.14. This proof is much like the proof of Proposition 4.5, except that it uses Propositions 4.15 and 4.18 in the appropriate places.

Proof of Proposition 4.14. For any non-negative random variable Z and any $0 < \alpha < 2$, $0 < r < 1$, Hölder's inequality applied with $p = 2r/\gamma$ implies that

$$\mathbb{E}Z^2 = \mathbb{E}Z^\gamma Z^{2-\gamma} \leq (\mathbb{E}Z^{2r})^{\gamma/(2r)} (\mathbb{E}Z^{2r(2-\gamma)/(2r-\gamma)})^{(2r-\gamma)/(2r)}.$$

In particular, if we set $q = 2r(2-\gamma)/(2r-\gamma)$ then we obtain

$$(\mathbb{E}Z^{2r})^{1/r} \geq \left(\frac{\mathbb{E}Z^2}{(\mathbb{E}Z^q)^{(2-\gamma)/q}} \right)^{2/\gamma}. \quad (4.25)$$

Now, set $Z = |\nabla v_t(X) - \nabla v_t(Y)|$, $a = \mathbb{E}\nabla v_t$ and $\epsilon(v_t) = \mathbb{E}(v_t(X) - \langle X, a \rangle - \mathbb{E}v_t)^2$. Then Proposition 4.6 implies that $\mathbb{E}Z^2 \geq 2(1-\rho)\epsilon(v_t)$, while Proposition 4.18 implies that

$$(\mathbb{E}Z^q)^{1/q} \leq C(L_t^2 + L_t)\sqrt{q(1-\rho)} \left(\sqrt{\log(1/m(f))} + L_t\sqrt{q} \right);$$

putting these together with (4.25), we have

$$\begin{aligned} (\mathbb{E}Z^{2r})^{1/r} &\leq \left(\frac{2(1-\rho)\epsilon(v_t)}{\left(C(L_t^2 + L_t)\sqrt{q(1-\rho)}(\sqrt{\log(1/m(f))} + L_t\sqrt{q}) \right)^{2-\gamma}} \right)^{2/\gamma} \\ &= c\sqrt{1-\rho} \left(\frac{\epsilon(v_t)}{\left((L_t^2 + L_t)\sqrt{q}(\sqrt{\log(1/m(f))} + L_t\sqrt{q}) \right)^{2-\gamma}} \right)^{2/\gamma}. \end{aligned}$$

Now define $\eta = 8L_t^2/(1 + 8L_t^2)$ and choose $r = 1 - \eta$ (so as to satisfy the hypothesis of Proposition 4.8). If we then define $\gamma = 2r - \alpha\eta = 2 - (2 + \alpha)\eta$ for some $0 < \alpha < 1$, we will find that $q = 2r \frac{2+\alpha}{\alpha} \leq 6/\alpha$. In particular, the last displayed quantity is at least

$$(1 - \rho)(c\alpha)^{(2-\gamma)/\gamma} \frac{\epsilon(v_t)^{2/\gamma}}{\left((L_t^3 + 1)\sqrt{\log(1/m(f))}\right)^{(2-\gamma)/\gamma}}$$

Since $(L_t^3 + 1)^{(2-\gamma)/\gamma}$ depends only on t , we can put this all together (going back to (4.25)) to obtain

$$\begin{aligned} (\mathbb{E}|\nabla v_t(X) - \nabla v_t(Y)|^{2r})^{1/r} &\geq c(t, \alpha)(1 - \rho) \frac{\epsilon(v_t)^{2/\gamma}}{\log^{C(t)}(1/m(f))} \\ &= c(t, \alpha)(1 - \rho) \frac{\epsilon(v_t)^{\frac{1+8L_t^2}{1-4\alpha L_t^2}}}{\log^{C(t)}(1/m(f))}. \end{aligned}$$

Combined with Proposition 4.15, this implies

$$\begin{aligned} \frac{dR_t}{dt} &\geq c(t)\rho\sqrt{1-\rho} \frac{m(f)^{4L_t^2(1+L_t)^2}}{\log^{C(t)}(1/m(f))} \epsilon(v_t)^{\frac{1+8L_t^2}{1-4\alpha L_t^2}} \\ &\geq c(t, \alpha)\rho\sqrt{1-\rho} m(f)^{4L_t^2(1+L_t)^2+\alpha} \epsilon(v_t)^{\frac{1+8L_t^2}{1-4\alpha L_t^2}}, \end{aligned} \quad (4.26)$$

where the last line follows because for every $\alpha > 0$ and every C , there is a $C'(\alpha)$ such that for every $x \leq \frac{1}{4}$, $\log^C(1/x) \leq C'(\alpha)x^{-\alpha}$. Now, with (4.26) as an analogue of (4.11), we complete the proof by following that of Proposition 4.5. Let us reiterate the main steps: recalling that $\delta = \int_0^\infty \frac{dR_s}{ds} ds$, we see that for any $\alpha, t > 0$, there is some $s \in [t, t(1 + \alpha)]$ so that $\frac{dR_t}{dt}\Big|_s \leq \frac{\delta}{\alpha t}$. By (4.26) applied with $t = s$, we have

$$\epsilon(v_s) \leq C(t, \alpha)m^{\frac{4L_t^2(1+L_t)^2(1-4\alpha L_t^2)}{1+8L_t^2}-\alpha} \left(\frac{\delta}{\rho\sqrt{1-\rho}}\right)^{\frac{1-4\alpha L_t^2}{1+8L_t^2}}.$$

Now, note that Φ is a contraction, and so Lemma 4.9 implies that

$$\mathbb{E}(f_t(X) - P_{s-t}^{-1}\Phi(\langle X, \mathbb{E}\nabla v_s \rangle - \mathbb{E}v_s))^2 \leq C(t, \alpha)m^{\frac{4L_t^2(1+L_t)^2(1-4\alpha L_t^2)}{1+8L_t^2}-\alpha} \left(\frac{\delta}{\rho\sqrt{1-\rho}}\right)^{\frac{1-4\alpha L_t^2}{1+8L_t^2}-\alpha}.$$

By changing α and adjusting $C(t, \alpha)$ accordingly, we can put this inequality into the form that was claimed in the proposition.

Finally, recall that $|\mathbb{E}\nabla v_s| \leq L_s$ by Theorem 1.7, and so $P_{s-t}^{-1}\Phi(\langle X, \mathbb{E}\nabla v_s \rangle - \mathbb{E}v_s)$ can be written in the form $\Phi(\langle X, a \rangle - b)$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ with $|a| \leq L_t$. \square

On the monotonicity of δ with respect to ρ

The final step in the proof of Theorem 4.2 is to improve the application of Lemma 4.13. Assuming, for now, that f is the indicator function of a set A , the hypothesis of Theorem 4.2 tells us if $e^{-t} = \rho$ then $\mathbb{E}1_A P_t 1_A$ is almost as large as possible; that is, it is almost as large as $\mathbb{E}1_B P_t 1_B$ where B is a half-space of probability $\mathbb{P}(A)$. This assumption allows us to apply Lemma 4.13, but only with $t = \log(1/\rho)$. In particular, this means that we will need to use this value of t in Proposition 4.14, which implies a poor dependence on ρ in our final answer.

To avoid all these difficulties, we will follow Kane [28] and Kindler and O’Donnell [30] to show if $\mathbb{E}1_A P_t 1_A$ is almost as large as possible for $t = \log(1/\rho)$, then it is also large for certain values of t that are larger.

Proposition 4.21. *Suppose $A \subset \mathbb{R}^n$ has $\mathbb{P}(A) = 1/2$. If $\theta = \cos(k \arccos \rho)$ for some $k \in \mathbb{N}$, and*

$$J(1/2, 1/2; \rho) - \mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho) \leq \delta$$

then

$$J(1/2, 1/2; \theta) - \mathbb{E}_\theta J(1_A(X), 1_A(Y); \theta) \leq k\delta$$

Proof. Let Z and Z' be independent standard Gaussians on \mathbb{R}^n and define $Z_\theta = Z' \cos \theta + Z \sin \theta$. Note that for any θ and any $j \in \mathbb{N}$, $Z_{(j+1)\theta}$ and $Z_{j\theta}$ have correlation $\cos \theta$. In particular, if $\theta = \arccos(\rho)$, then the union bound implies that

$$\begin{aligned} \mathbb{P}_\theta(X \in A, Y \notin A) &= \Pr(Z(0) \in A, Z(k\theta) \notin A) \\ &\leq \sum_{j=0}^{k-1} \Pr(Z_{j\theta} \in A, Z_{(j+1)\theta} \notin A) \\ &= k \Pr_\rho(X \in A, Y \notin A). \end{aligned} \tag{4.27}$$

The remarkable thing about this inequality is that it becomes equality when A is a half-space of measure $1/2$, because in this case, $\Pr_\rho(X \in A, Y \notin A) = \frac{1}{2\pi} \arccos(\rho)$.

Recall that $\mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho) = \Pr_\rho(X \in A, Y \in A)$. Thus, the hypothesis of the proposition can be rewritten as

$$\left(\frac{1}{2} - \frac{1}{2\pi} \arccos(\rho) \right) - \left(\gamma_n(A) - \Pr_\rho(X \in A, Y \notin A) \right) \leq \delta,$$

which rearranges to read

$$\Pr_\rho(X \in A, Y \notin A) \leq \delta + \frac{1}{2\pi} \arccos \rho.$$

By (4.27), this implies that

$$\Pr_\theta(X \in A, Y \notin A) \leq k\delta + \frac{1}{2\pi} \arccos \theta,$$

which can then be rearranged to yield the conclusion of the proposition. \square

Let us point out two deficiencies in Proposition 4.21: the requirement that $\mathbb{P}(A) = 1/2$ and that k be an integer. The first of these deficiencies is responsible for the assumption $\mathbb{E}f = \frac{1}{2}$ in Theorem 4.2, and the second one prevents us from obtaining a better constant in the exponent of δ . Both of these restrictions come from the subadditivity condition (4.27), which only makes sense for an integer k , and only achieves equality for a half-space of volume $\frac{1}{2}$. But beyond the fact that our proof fails, we have no reason not to believe that some version of Proposition 4.21 is true without these restrictions. In particular, we make the following conjecture:

Conjecture 4.22. *There is a function $k(\rho, a)$ such that*

- *for any fixed $a \in (0, 1)$, $k(\rho, a) \sim \sqrt{1 - \rho}$ as $\rho \rightarrow 1$;*
- *for any fixed $a \in (0, 1)$, $k(\rho, a) \sim \rho$ as $\rho \rightarrow 0$; and*
- *for any $a \in (0, 1)$ and any $A \subset \mathbb{R}^n$ the quantity*

$$\frac{J(a, a; \rho) - \mathbb{E}_\rho J(1_A(X), 1_A(Y); \rho)}{k(\rho, a)}$$

is increasing in ρ .

If this conjecture were true, it would tell us that sets which are almost optimal for some ρ are also almost optimal for smaller ρ , where the function $k(\rho, a)$ quantifies the almost optimality. Note that the other direction is certainly false: sets which are almost optimal for some ρ need not be almost optimal for larger ρ . A simple example is the set $(\infty, -\epsilon] \cup [0, \epsilon]$, which is almost optimal for fixed ρ as $\epsilon \rightarrow 0$, but far from optimal if $\sqrt{1 - \rho} \ll \epsilon$.

In any case, let us move on to the proof of Theorem 4.2. If the conjecture is true, then the following proof will directly benefit from the improvement.

Proof of Theorem 4.2. We will prove the theorem when f is the indicator function of a set A . The extension to general f follows from the same argument that was made in the proof of Theorem 4.1.

Fix $\epsilon > 0$. If ρ_0 is close enough to 1 then for every $\rho_0 < \rho < 1$, there is a $k \in \mathbb{N}$ such that $k \arccos(\rho) \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2} - \frac{\epsilon}{2}]$. In particular, this means that $\cos(k \arccos(\rho)) \in [c_1(\epsilon), c_2(\epsilon)]$, where $c_1(\epsilon)$ and $c_2(\epsilon)$ converge to zero as $\epsilon \rightarrow 0$. Moreover, this k must satisfy

$$k \leq \frac{C(\epsilon)}{\arccos(\rho)} \leq \frac{C(\epsilon)}{\sqrt{1 - \rho}}.$$

Now let $\theta = \cos(k \arccos(\rho))$. By Proposition 4.21, A satisfies

$$J(1/2, 1/2; \theta) - \mathbb{E}_\theta J(1_A(X), 1_A(Y); \theta) \leq C(\epsilon) \frac{\delta}{\sqrt{1 - \rho}}.$$

Now we will apply Proposition 4.14 with ρ replaced by θ and $t = \log(1/\theta)$. Since $\theta \leq c_2(\epsilon)$, it follows that $L_t = \theta/\sqrt{1-\theta^2} \leq c_3(\epsilon)$ (where $c_3(\epsilon) \rightarrow 0$ with ϵ). Thus, the conclusion of Proposition 4.14 gives us $a \in \mathbb{R}^n$, $b \in R$ such that

$$\begin{aligned} \mathbb{E}((P_t 1_A)(X) - \Phi(\langle X, a \rangle - b))^2 &\leq C \left(\frac{\delta}{\theta \sqrt{(1-\theta)(1-\rho)}} \right)^{1-c_4(\epsilon)} \\ &\leq C(\epsilon) \left(\frac{\delta}{\sqrt{1-\rho}} \right)^{1-c_4(\epsilon)}. \end{aligned} \quad (4.28)$$

Now we apply the same small- t argument as in Theorem 4.1: Lemma 3.1 implies that there is some $s > 0$ and a half-space B such that

$$\mathbb{E}(P_t 1_A - P_{t+s} 1_B)^2 \leq C(\epsilon) (\delta/\sqrt{1-\rho})^{1-c_4(\epsilon)}$$

and we can assume, at the cost of increasing $C(\epsilon)$, that $\mathbb{P}(B) = \mathbb{P}(A)$. Then Proposition 4.12 implies that

$$\mathbb{E}(P_t 1_A - P_t 1_B)^2 \leq C(\epsilon) (\delta/\sqrt{1-\rho})^{1-c_4(\epsilon)},$$

and we apply Proposition 4.10 (recalling that t is bounded above and below by constants depending on ϵ) to conclude that

$$\mathbb{P}(A \Delta B) \leq C(\epsilon) (\delta/\sqrt{1-\rho})^{1/4-c_4(\epsilon)/4}.$$

Recall that $c_4(\epsilon)$ is some quantity tending to zero with ϵ . Therefore, we can derive the claim of the theorem from the equation above by modifying $C(\epsilon)$. \square

Finally, we will prove Corollary 4.4.

Proof of Corollary 4.4. Since $xy \leq J(x, y)$, the hypothesis of Corollary 4.4 implies that

$$\mathbb{E}J(f(X), f(Y)) \geq \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho) - \delta.$$

Now, consider Theorem 4.2 with $\epsilon = 1/8$. If $\rho > \rho_0$ then apply it; if not, apply Theorem 4.1. In either case, the conclusion is that there is some $a \in \mathbb{R}^n$ such that

$$\mathbb{E}|f(X) - \Phi(\langle X, a \rangle)| \leq C(\rho)\delta^c.$$

Setting $g(X) = \Phi(\langle X, a \rangle)$, Hölder's inequality implies that

$$\begin{aligned} |\mathbb{E}g(X)g(Y) - \mathbb{E}f(X)f(Y)| &= |\mathbb{E}(g(X) - f(X))g(Y) + \mathbb{E}f(X)(g(Y) - f(Y))| \\ &\leq 2\mathbb{E}|f - g|. \end{aligned}$$

In particular,

$$\mathbb{E}g(X)g(Y) \geq \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho) - \delta - C(\rho)\delta^c. \quad (4.29)$$

But the left hand side can be computed exactly: if $|a| = (e^{2t} - 1)^{-1/2}$ and $A = \{x \in \mathbb{R}^n : x_1 \leq 0\}$ then

$$\begin{aligned} \mathbb{E}g(X)g(Y) &= \mathbb{E}P_t 1_A(X)P_t 1_A(Y) \\ &= \mathbb{E}1_A(X)P_{2t-\log(\rho)} 1_A(X) \\ &= \frac{1}{4} + \frac{1}{2\pi} \arcsin(e^{-2t}\rho) \\ &\leq \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho) - \frac{1}{2\pi}\rho(1 - e^{-2t}), \end{aligned}$$

where the last line used the fact that the derivative of \arcsin is at least 1. Combining this with (4.29), we have

$$1 - e^{-2t} \leq C(\rho)\delta^c \quad (4.30)$$

On the other hand,

$$\mathbb{E}|g - 1_A| = 2(1/2 - \mathbb{E}g 1_A) = \frac{1}{2} - \frac{1}{\pi} \arcsin(e^{-t}) \leq \sqrt{1 - e^{-2t}},$$

which combines with (4.30) to prove that $\mathbb{E}|g - 1_A| \leq C(\rho)\delta^c$. Applying the triangle inequality, we conclude that

$$\mathbb{E}|f - 1_A| \leq \mathbb{E}|f - g| + \mathbb{E}|g - 1_A| \leq C(\rho)\delta^c. \quad \square$$

Chapter 5

Applications of Gaussian noise stability

In this chapter, we will survey some applications of Gaussian noise sensitivity in computer science and economics. We will begin with boolean noise stability and the “majority is stablest” theorem of Mossel, O’Donnell and Oleszkiewicz [42]. The majority is stablest theorem has applications in computational complexity and in quantitative social choice. We will briefly discuss one of these applications, and then go on to prove a robust version of the majority is stablest theorem. From boolean noise stability, we will move to spherical noise stability. Since high-dimensional Gaussian vectors behave in many ways like uniformly random points on a sphere, we can translate our results on robust Gaussian noise stability into statements on sufficiently high dimensional spheres. This has algorithmic implications for Goemans and Williamson’s Max-Cut algorithm.

Since the material in this chapter is fairly wide-ranging, we will not make any particular effort to remain self-contained. In particular, we will quote without proof the invariance principle of Mossel et al. [39, 42] and some classical properties of spherical harmonics.

5.1 Boolean noise stability and Arrow’s theorem

Consider the boolean hypercube $\{-1, 1\}^n$ and let $\xi = (\xi_1, \dots, \xi_n)$ be a uniformly random element of it. Then the ξ_i are independent and $\Pr(\xi_i = 1) = \Pr(\xi_i = -1) = 1/2$. There is a natural correlation structure on the boolean hypercube: for some parameter $\rho \in [-1, 1]$, define $\sigma = (\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$ by letting each σ_i be an independent random variable on $\{-1, 1\}$ with $\mathbb{E}\sigma_i = 0$ and $\mathbb{E}\sigma_i\xi_i = \rho$. Write \Pr_ρ for the joint distribution of σ and ξ .

For a set $A \subset \{-1, 1\}^n$, define the noise stability of A to be $\Pr_\rho(\xi \in A, \sigma \in A)$. As in the Gaussian case, we may consider the problem of choosing A to maximize the noise stability. On the boolean hypercube, however, this problem turns out to be more complicated than on \mathbb{R}^n with the Gaussian measure. In particular, the boolean hypercube is not rotationally invariant, so different half-spaces have different properties. For example, the half-space

$\{x \in \{-1, 1\}^n : x_1 \leq 0\}$ has measure $1/2$ and noise stability $(1 + \rho)/4$, while the half-space $\{x \in \{-1, 1\}^n : \sum x_i \leq 0\}$ has measure $1/2$ (at least, for odd n), and by the central limit theorem its noise stability converges to $J_\rho(1/2, 1/2) = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho)$. Since \arcsin is concave on $[0, 1]$, the subcube $\{x \in \{-1, 1\}^n : x_1 \leq 0\}$ is more noise stable than the Hamming ball $\{x \in \{-1, 1\}^n : \sum x_i \leq 0\}$.

It turns out to be profitable to consider a more restricted problem that excludes examples like the subcube. For a function $f : \{-1, 1\}^n \rightarrow [0, 1]$, we define the influence of the i th coordinate by

$$\text{Inf}_i(f) = \mathbb{E} \text{Var}(f(\xi_1, \dots, \xi_n) \mid \xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n).$$

In particular, if the range of f is $\{0, 1\}$ then $\text{Inf}_i(f)$ is the probability that changing the i th coordinate will change the output of f . The influence can also be written in terms of the Fourier expansion:

$$\text{Inf}_i(f) = \sum_{S \ni i} |S| \hat{f}_S^2. \quad (5.1)$$

Note that if $f(x) = x_1$ then $\text{Inf}_1(f) = 1$, while $\text{Inf}_i(f) = 0$ for $i \neq 1$. On the other hand, if $f(x) = \text{sgn}(\sum x_i)$ (we will call this example the “majority function” for obvious reasons) then $\text{Inf}_i(f) = O(n^{-1/2})$ for every i . In what follows, we will restrict ourselves to functions in which every variable has low influence. Intuitively, such functions do not depend much on any single variable. Our example above showed that the majority function is not the most noise stable zero-mean function, because the function $f(x) = x_1$ is more noise stable. Remarkably, if we restrict to functions of low influence then the majority function *is* – in an appropriate asymptotic sense – the most noise stable zero-mean function.

Denote the noise stability of $f : \{-1, 1\}^n \rightarrow [0, 1]$ by

$$\mathbb{S}_\rho(f) = \mathbb{E}_\rho f(\xi) f(\sigma).$$

Theorem 5.1. *For every $\delta > 0$, there is a $\tau > 0$ such that the following holds: suppose that $f : \{-1, 1\}^n \rightarrow [0, 1]$ is a function with $\text{Inf}_i(f) \leq \tau$ for every i . Then for every $0 < \rho < 1$,*

$$\mathbb{S}_\rho(f) \leq J_\rho(\mathbb{E}f, \mathbb{E}f) + C(\rho)\delta. \quad (5.2)$$

If, moreover, there is some $0 < \rho < 1$ such that

$$\mathbb{S}_\rho(f) \geq J_\rho(\mathbb{E}f, \mathbb{E}f) - \delta \quad (5.3)$$

then there exist $a, b \in \mathbb{R}^n$ such that

$$\mathbb{E}|f(\xi) - 1_{\{a, \xi - b\} \geq 0}| \leq C(\rho)\delta^{c(\rho)}, \quad (5.4)$$

where $0 < c(\rho), C(\rho) < \infty$ are constants depending only on ρ .

If we set $a_n = \frac{1}{\sqrt{n}}(1, \dots, 1)$ and $b_n = \Phi^{-1}(\mathbb{E}f)a_n$, then the central limit theorem implies that $\mathbb{E}1_{\{a_n, \xi - b_n\} \leq 0} \rightarrow \mathbb{E}f$ and $\mathbb{S}_\rho(1_{\{a_n, \xi - b_n\} \leq 0}) \rightarrow J(\mathbb{E}f, \mathbb{E}f; \rho)$. In the case $\mathbb{E}f = \frac{1}{2}$ and $b_n = 0$, (5.2) says, therefore, that no low-influence function can be much more noise stable

than the simple majority function, while (5.4) says that any low-influence function which is close to optimal must be a perturbation of some weighted majority function. We should mention that (5.2) is due to Mossel et al. [42], while (5.4) is our contribution.

We remark that Theorem 5.1 is not stated in the most general form that we can prove. In particular, we could state a two-function version of Theorem 5.1 or a version that uses the functional $\mathbb{E}_\rho J_\rho(f(\xi), f(\sigma))$ in place of $\mathbb{E}_\rho f(\xi)f(\sigma)$. These variations, however, are proved in essentially the same way, namely by combining the ideas from [42] with the appropriate Gaussian robustness result. In order to avoid repetition, therefore, we will only state and prove one version.

Arrow's theorem

Before proving Theorem 5.1, let us discuss an application: in economics, Arrow's theorem [1] says that any non-dictatorial election system between three candidates which satisfies two natural properties (namely, the "independence of irrelevant alternatives" and "neutrality") has a chance of producing a non-rational outcome. (By non-rational outcome, we mean that there are three candidates, A , B and C say, such that candidate A is preferred to candidate B , B is preferred to C and C is preferred to A .) Kalai [26, 27] showed that if the election system is such that each voter has only a small influence on the outcome, then the probability of a non-rational outcome is substantial. From Theorem 5.1, we will see that the simple majority system minimizes the chance of a non-rational outcome, and that an election system with an almost-minimal chance of non-rationality must be close to a weighted majority system.

In order to state the result precisely, we need to introduce some notation from social choice theory. For $k \geq 2$, let $L(k)$ be the set of linear orderings on $\{1, \dots, k\}$. A *social choice function for n voters on k candidates* is a function $(L(k))^n \rightarrow L(k)$. Intuitively, each voter provides a ranking on the candidates, and the social choice function aggregates these rankings into a single global ranking.

We say that a social choice function is a *dictatorship* if there is some $i \in \{1, \dots, n\}$ such that $f(r_1, \dots, r_n)$ depends only on r_i . That is, a dictatorship listens only to the opinion of the i th voter. Such a social choice function is generally considered undesirable in the theory of social choice.

For an ordering $r = (r(1) > r(2) > \dots > r(k)) \in L(k)$, and a permutation $\pi : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$, write $\pi(r)$ for the order $(\pi(r(1)) > \dots > \pi(r(k)))$. We say that a social choice function f is *neutral* if for all permutations π and all n -tuples $(r_1, \dots, r_n) \in (L(k))^n$, we have $\pi(f(r_1, \dots, r_n)) = f(\pi(r_1), \dots, \pi(r_n))$. In other words, f has no built-in preference for any candidates: if we rename the candidates then we only rename the outcome.

Next, we define the *independence of irrelevant alternatives* (IIA) property. Intuitively, a social choice function satisfies IIA if for any pair of candidates $a, b \in \{1, \dots, k\}$, the final ordering of a and b depends only the voters' preferences between a and b . More precisely, if $r_1, r'_1, \dots, r_n, r'_n \in L(k)$ are such that for every i , r_i ranks a above b if and only if r'_i ranks a above b , then $f(r_1, \dots, r_n)$ ranks a above b if and only if $f(r'_1, \dots, r'_n)$ ranks a above b . This

assumption can be motivated in several ways. For example, Arrow motivates it by arguing that if a candidate dies after the votes are cast, then the fact that they merely appeared on the ballot should not affect the relative ranking of the other candidates. Another motivation is that the IIA property rules out the possibility of “spoiler” third-party candidates.

An ideal social choice function would not be a dictatorship, it would be neutral, and independent alternatives would be irrelevant. Arrow showed, however, that this ideal is impossible:

Theorem 5.2. *Suppose f is a social choice function on $n \geq 3$ voters and $k \geq 3$ alternatives which is neutral and satisfies IIA. Then f is a dictatorship.*

We can get a different view of Arrow’s theorem by considering *generalized social choice functions*. A generalized social choice function is not constrained to take values in $L(k)$; instead of producing a linear order, it produces a collection of pairwise preferences. That is, a generalized social choice function is a map from $L(k)^n$ to the set of total, asymmetric binary relations on $\{1, \dots, k\}$. Since every linear order is a total, asymmetric binary relation, every social choice function is a generalized social choice function. But when $k \geq 3$, not every generalized social choice function is a social choice function.

The definitions of neutrality, dictatorship, and IIA extend naturally to generalized social choice functions. Suppose, then, that f is a generalized social choice function which is neutral and satisfies IIA. By Arrow’s theorem, either f is a dictatorship or it fails to be a social choice function: there exists some voting profile r_1, \dots, r_n such that $f(r_1, \dots, r_n)$ is not a linear order. Kalai [26, 27] considered a quantitative version of this claim: if f is far from a dictatorship, must there be many voting profiles that result in something that is not a total order? Equivalently, if r_1, \dots, r_n are chosen independently and uniformly at random from $L(k)$, could it be that $f(r_1, \dots, r_n)$ is a total order with high probability? Note that Theorem 5.2 only asserts the existence of a single voting profile that fails to produce a linear order. It still leaves open the possibility that we could construct a generalized social choice function that is far from a dictatorship, satisfies IIA, and for which only one out of $(k!)^n$ voting profiles fails to produce a linear order. Such a generalized social choice function would still be quite attractive to social choice theorists.

Unfortunately, no such function exists. Moreover, the answer is closely related to Theorem 5.1. To explain why, let us point out a different characterization of the IIA property: for a total binary relation R and for $a, b \in \{1, \dots, k\}$, let $R^{a>b} \in \{-1, 1\}$ be 1 if and only if aRb . Then f satisfies IIA if and only if for every $a, b \in \{1, \dots, k\}$ there exists an antisymmetric function $g_{a,b} : \{-1, 1\}^n \rightarrow \{-1, 1\}$ such that $f(r_1, \dots, r_n)^{a>b} = g_{a,b}(r_1^{a>b}, \dots, r_n^{a>b})$. This is simply another way of saying that the ordering f puts on a and b depends only on the ordering that the voters put on a and b . Under the additional assumption that f is neutral, the function $g_{a,b}$ must be the same for all $a, b \in \{1, \dots, k\}$.

Now suppose without loss of generality that $k = 3$ (under IIA, any generalized social choice function on $k \geq 3$ candidates induces a generalized social choice function on any subset of the candidates). The following two observations allow us to reduce our question on generalized social choice functions to a question on the noise sensitivity of boolean functions:

1. If r is a uniformly random element of $L(3)$ and a, b, c are distinct elements of $\{1, 2, 3\}$ then $r^{a>b}$ and $r^{b>c}$ satisfy $\mathbb{E}r^{a>b} = \mathbb{E}r^{b>c} = 0$ and $\mathbb{E}r^{a>b}r^{b>c} = -1/3$.
2. A total, symmetric binary relation R on $\{a, b, c\}$ is a linear order if and only if $R^{a>b}, R^{b>c}$, and $R^{c>a}$ are not all equal. Equivalently,

$$\frac{1 + R^{a>b}R^{b>c} + R^{b>c}R^{c>a} + R^{c>a}R^{a>b}}{4} = \begin{cases} 0 & \text{if } R \text{ is a linear order} \\ 1 & \text{otherwise.} \end{cases}$$

Hence, if f is any neutral generalized social choice function satisfying IIA, $g : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is the associated pairwise function, and r is a uniformly random element of $L(k)^n$, then

$$\begin{aligned} \Pr(f(r) \notin L(3)) &= \mathbb{E} \frac{1 + f^{a>b}(r)f^{b>c}(r) + f^{b>c}(r)f^{c>a}(r) + f^{c>a}(r)f^{a>b}(r)}{4} \\ &= \mathbb{E} \frac{1 + g(r^{a>b})g(r^{b>c}) + g(r^{b>c})g(r^{c>a}) + g(r^{c>a})g(r^{a>b})}{4} \\ &= \frac{1}{4} + \frac{3}{4}\mathbb{S}_{-1/3}(g). \end{aligned}$$

Hence, we obtain the following theorem as an immediate corollary of Theorem 5.1.

Theorem 5.3. *For any $\epsilon > 0$, there exists $\tau > 0$ such that the following holds. Let f be a neutral generalized social choice function on 3 candidates that satisfies IIA. If for all dictator functions g , $\Pr(f = g) \leq \tau$, then*

$$\Pr(f \notin L(3)) \geq \frac{1}{4} - \frac{3}{2\pi} \arcsin(1/3) - \epsilon. \quad (5.5)$$

Moreover, if

$$\Pr(f \notin L(3)) \leq \frac{1}{4} - \frac{3}{2\pi} \arcsin(1/3) + \epsilon$$

then there exists $v \in \mathbb{R}^n$ such that

$$\mathbb{E}|g(\sigma) - \text{sgn}(\langle v, \sigma \rangle)| \leq C\epsilon^c, \quad (5.6)$$

where C and c are positive universal constants.

Since $\frac{1}{4} - \frac{3}{2\pi} \arcsin(1/3) \approx 8.77\%$, (5.5) implies that every IIA and neutral generalized social choice function which is far from dictatorial has a reasonable chance of producing a non-linear ranking as its outcome. Moreover, (5.6) implies that essentially the only functions which get close to this bound are those that use a weighted majority vote to decide between each pair of candidates.

Social choice functions are often assumed to satisfy an additional constraint, *anonymity*, which requires a social choice function to be invariant under permutations of voters. Under

this additional assumption (5.6) implies that g must actually be close to a simple (i.e., unweighted) majority function. Remarkably, the generalized social choice function which makes pairwise decisions by the simple majority vote was the first such function to be mathematically studied: in 1785, the Marquis de Condorcet [14] observed that pairwise majority voting could result in a relation that is not a linear order; this phenomenon is widely known as “Condorcet’s paradox.” From this historical viewpoint, Theorem 5.3 is somewhat surprising since it says that in some sense, Condorcet’s voting method is the *best* way to obtain a linear order on the candidates.

The proof of Theorem 5.1

We begin the proof of Theorem 5.1 by recalling some Fourier-theoretic properties of $\{-1, 1\}^n$. For more background on the Fourier analysis of boolean functions, see the lecture notes by O’Donnell [45]. For $S \subset [n]$, define $\chi_S : \{-1, 1\}^n \rightarrow \{-1, 1\}$ by $\chi_S(x) = \prod_{i \in S} x_i$. Then $\{\chi_S : S \subset [n]\}$ form an orthonormal basis of $L_2(\{-1, 1\}^n)$. We will write \hat{f}_S for the coefficients of f in this basis; that is,

$$f(x) = \sum_{S \subset [n]} \hat{f}_S \chi_S(x). \quad (5.7)$$

Recall that Pr_ρ denotes the distribution on $\{-1, 1\}^n \times \{-1, 1\}^n$ under which $(\xi_i, \sigma_i)_{i=1}^n$ are independent, $\mathbb{E}_\rho \xi_i = \mathbb{E}_\rho \sigma_i = 0$, and $\mathbb{E}_\rho \xi_i \sigma_i = 1$. Define the Bonami-Beckner semigroup Q_t by

$$(Q_t f)(\xi) = \mathbb{E}_{e^{-t}}(f(\sigma) \mid \xi).$$

In terms of the Fourier expansion, one can check that

$$Q_t f = \sum_{S \subset [n]} e^{-t|S|} \hat{f}_S \chi_S. \quad (5.8)$$

Also, Q_t is a self-adjoint operator, and it satisfies

$$\mathbb{E}_\rho f(\xi) g(\sigma) = \mathbb{E} f(\xi) (Q_{\log(1/\rho)} g)(\xi) = \mathbb{E} g(\xi) (Q_{\log(1/\rho)} f)(\xi). \quad (5.9)$$

The invariance principle

Note that any function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ can be extended to a multilinear function $\mathbb{R}^n \rightarrow \mathbb{R}$ through the Fourier expansion (5.7): since $\chi_S(x)$ is defined for all $x \in \mathbb{R}^n$, we may define $g(x)$ for $x \in \mathbb{R}^n$ by $g(x) = \sum_S \hat{f}_S \chi_S(x)$. We will say that g is the *multilinear extension* of f ; note that g and f agree on $\{-1, 1\}^n$, thereby justifying the term “extension.” A word of caution: we will sometimes define functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ by formulas that make sense on all of \mathbb{R}^n (for example, $f(x) = 1_{\{(a, x-b) \geq 0\}}$). In such a case, the multilinear extension of f is not the same as the function $1_{\{(a, x-b) \geq 0\}} : \mathbb{R}^n \rightarrow \mathbb{R}$.

Let us remark on some well-known and important properties of multilinear polynomials. First of all, let $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$ be a mean-zero Gaussian vector with covariance matrix

$\begin{pmatrix} I_n & \rho I_n \\ \rho I_n & I_n \end{pmatrix}$, as it was in the last chapter. Since $\mathbb{E}\xi_i = \mathbb{E}X_i = 0$, $\mathbb{E}\xi_i^2 = \mathbb{E}X_i^2 = 1$, and $\mathbb{E}_\rho \xi_i \sigma_i = \mathbb{E}_\rho X_i Y_i = \rho$, it is trivial to check that for multilinear functions f and g ,

$$\begin{aligned} \mathbb{E}f(\xi) &= \mathbb{E}f(X) \\ \mathbb{E}f^2(\xi) &= \mathbb{E}f^2(X) \\ \mathbb{E}_\rho f(\xi)g(\sigma) &= \mathbb{E}_\rho f(X)g(Y). \end{aligned} \tag{5.10}$$

Recall the Ornstein-Uhlenbeck semigroup from Chapter 1; by (1.14) and (5.8), it follows that if f is a multilinear polynomial then for any $t > 0$, $Q_t f$ and $P_t f$ are equal (as multilinear polynomials). In particular, there is no ambiguity in using the notation f_t for both $P_t f$ and $Q_t f$.

Despite these similarities, $f(X)$ and $f(\xi)$ can have very different distributions in general (for example, if $f(x) = x_1$). The main technical result of [42] is that when f has low influence and $t > 0$, then $f_t(X)$ and $f_t(\xi)$ have similar distributions. We will quote a rather weaker statement than the one proved in [42], which will nevertheless be sufficient for our purposes. In particular, we will only need to know that if $g(\xi)$ takes values in $[0, 1]$, then $g(X)$ mostly takes values in $[0, 1]$. Before stating the theorem from [42], let us introduce some notation: for a function f taking values in \mathbb{R} , let \bar{f} be its truncation which takes values in $[0, 1]$:

$$\bar{f}(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } 0 \leq f(x) \leq 1 \\ 1 & \text{if } 1 < f(x). \end{cases}$$

Theorem 5.4. *Suppose f is a multilinear polynomial such that $f(\xi) \in [0, 1]$ for all $\xi \in \{-1, 1\}^n$. If f satisfies $\max_i \text{Inf}_i(f) \leq \tau$ then for any $\eta > 0$,*

$$\mathbb{E}(f_\eta(X) - \bar{f}_\eta(X))^2 \leq C\tau^{c\eta} \tag{5.11}$$

We will now use Theorem 5.4 to prove Theorem 5.1. First, (5.11) and the triangle inequality imply that for any $0 < \rho' < 1$,

$$\mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) \leq \mathbb{E}_{\rho'} \bar{f}_\eta(X) \bar{f}_\eta(Y) + C\tau^{c\eta}. \tag{5.12}$$

By (5.10) and (5.9),

$$\mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) = \mathbb{E}_{\rho'} f_\eta(\xi) f_\eta(\sigma) = \mathbb{E}_{e^{-2\eta\rho'}} f(\xi) f(\sigma). \tag{5.13}$$

Now set $\rho' = e^{2\eta\rho}$ (assuming that η is small enough so that $e^{2\eta\rho} < 1$). By (5.13) and (5.12),

$$\mathbb{E}_\rho f(\xi) f(\sigma) = \mathbb{E}_{\rho'} f_\eta(X) f_\eta(Y) \leq \mathbb{E}_{\rho'} \bar{f}_\eta(X) \bar{f}_\eta(Y) + C\tau^{c\eta}. \tag{5.14}$$

Applying Theorem 2.5 to \bar{f}_η , we see that $\mathbb{E}_\rho f(\xi) f(\sigma) \leq J_{\rho'}(\mathbb{E}\bar{f}_\eta, \mathbb{E}\bar{f}_\eta) + C\tau^{c\eta}$. Now, Theorem 5.4 implies that $|\mathbb{E}\bar{f}_\eta - \mathbb{E}f| \leq C\tau^{c\eta}$, and the derivatives of $J_\rho(x, x)$ in both x and ρ can be bounded by a constant depending only on ρ ; hence,

$$J_{\rho'}(\mathbb{E}\bar{f}_\eta, \mathbb{E}\bar{f}_\eta) \leq J_\rho(\mathbb{E}f, \mathbb{E}f) + C(\rho)(|\rho - \rho'| + |\mathbb{E}\bar{f}_\eta - \mathbb{E}f|) \leq J_\rho(\mathbb{E}f, \mathbb{E}f) + C(\rho)(\eta + C\tau^{c\eta}).$$

Plugging this into (5.14), we have $\mathbb{E}_\rho f(\xi)f(\sigma) \leq J_\rho(\mathbb{E}f, \mathbb{E}f)$, which proves (5.2).

Next, we prove (5.4). Under our assumption that $\mathbb{E}_\rho f(\xi)f(\sigma) \geq J_\rho(\mathbb{E}f, \mathbb{E}f) - \delta$, (5.14) implies that

$$\begin{aligned} \mathbb{E}_{\rho'} \overline{f_\eta}(X) \overline{f_\eta}(Y) &\geq J_\rho(\mathbb{E}f, \mathbb{E}f) - C\tau^{c\eta} - \delta \\ &\geq J_\rho(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) - C\tau^{c\eta} - \delta \\ &\geq J_{\rho'}(\mathbb{E}\overline{f_\eta}, \mathbb{E}\overline{f_\eta}) - C(\rho)\eta - C\tau^{c\eta} - \delta, \end{aligned}$$

where the second inequality follows because $|\mathbb{E}f - \mathbb{E}\overline{f_\eta}| \leq C\tau^{c\eta}$ and $\frac{\partial J(x,y;\rho)}{\partial x}$ is bounded. Applying Theorem 4.1 (with ρ' in place of ρ) to $\overline{f_\eta}$, we see that there are $a, b \in \mathbb{R}^n$ such that

$$\mathbb{E}(\overline{f_\eta}(X) - 1_{\{(a, X-b) \geq 0\}})^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

By (5.11) and the triangle inequality, we may replace $\overline{f_\eta}$ by f_η :

$$\mathbb{E}(f_\eta(X) - 1_{\{(a, X-b) \geq 0\}})^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}. \quad (5.15)$$

The next step is to pull (5.15) back to the discrete cube by replacing X with ξ on the left hand side of (5.15). We will do this using Theorem 5.4. As a prerequisite, we need to show that $1_{\{(a, x-b) \geq 0\}}$ has small influences; this is essentially the same as saying that a is well-spread:

Lemma 5.5. *There is an $a \in \mathbb{R}^n$ satisfying (5.15) with $\sum a_i^2 = 1$ and $\max_i |a_i| \leq C\tau^c$.*

Once we have shown that $1_{\{(a, x-b) \geq 0\}}$ has small influences, we can use Theorem 5.4 to show that the multilinear extension of $1_{\{(a, x-b) \geq 0\}}$ is close to $1_{\{(a, x-b) \geq 0\}}$:

Lemma 5.6. *Let $g^{a,b}$ be the multilinear extension of the function $x \mapsto 1_{\{(a, x-b) \geq 0\}}$. If $\sum_i a_i^2 = 1$ and $\max_i |a_i| \leq \tau$ then for any $\eta > 0$,*

$$\mathbb{E}(g_\eta^{a,b}(X) - 1_{\{(a, X-b) \geq 0\}})^2 \leq C(\eta + \tau^{c\eta}).$$

From Lemma 5.6 and the triangle inequality, we conclude from (5.15) that

$$\mathbb{E}(f_\eta(X) - g_\eta^{a,b}(X))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

Since $f_\eta - g_\eta^{a,b}$ is a multilinear polynomial, its second moment remains unchanged when X is replaced by ξ :

$$\mathbb{E}(f_\eta(\xi) - g_\eta^{a,b}(\xi))^2 = \mathbb{E}(f_\eta(X) - g_\eta^{a,b}(X))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}.$$

Now, $g^{a,b}$ is the indicator of a half-space on the cube; thus, $\mathbb{E}(g_\eta^{a,b}(\xi) - g^{a,b}(\xi))^2 \leq C\eta^c$ (see, for example, [5]). Applying this and the triangle inequality, we have

$$\mathbb{E}(f_\eta(\xi) - g^{a,b}(\xi))^2 \leq C(\rho)(\eta + \tau^{c\eta} + \delta)^{c(\rho)}. \quad (5.16)$$

The last piece is to replace f_η by f . We do this with a simple lemma which shows that for any function f , if f_η is close to some indicator function then f is also close to the same indicator function.

Lemma 5.7. For any functions $f : \{-1, 1\}^n \rightarrow [0, 1]$ and $g : \{-1, 1\}^n \rightarrow \{0, 1\}$ and any $\eta > 0$,

$$\mathbb{E}(f(\xi) - g(\xi))^2 \leq C\sqrt{\mathbb{E}(f_\eta(\xi) - g(\xi))^2}.$$

Applying Lemma 5.7 to (5.16), we obtain

$$\mathbb{E}(f(\xi) - g^{a,b}(\xi))^2 \leq C(\rho)(\eta + \tau^{cn} + \delta)^{c(\rho)}.$$

By choosing τ and η small enough compared to δ , the proof of Theorem 5.1 is complete, modulo the proofs of Lemmas 5.5, 5.6 and 5.7. We will prove them in the coming section.

Gaussian and boolean half-spaces

Here we will prove the lemmas of the previous section. Before doing so, let us observe that $\mathbb{E}X_i 1_{\{\langle a, X-b \rangle \geq 0\}}$ is proportional to a_i , a fact which has already been noted by Matulef et al. [38]:

Lemma 5.8.

$$\mathbb{E}X_i 1_{\{\langle a, X-b \rangle \geq 0\}} = a_i \phi(\langle a, b \rangle).$$

Proof. Let $e_i \in \mathbb{R}^n$ be the vector with 1 in position i and 0 elsewhere. We may write $e_i = a_i a + a^\perp$, where a^\perp is some element of \mathbb{R}^n which is orthogonal to a . Note that $\langle X, a^\perp \rangle$ is independent of $\langle X, a \rangle$ and so $\mathbb{E}\langle X, a^\perp \rangle 1_{\{\langle a, X-b \rangle \geq 0\}} = 0$. Hence,

$$\mathbb{E}X_i 1_{\{\langle a, X-b \rangle \geq 0\}} = \mathbb{E}\langle a_i a + a^\perp, X \rangle 1_{\{\langle a, X-b \rangle \geq 0\}} = a_i \mathbb{E}\langle a, X \rangle 1_{\{\langle a, X-b \rangle \geq 0\}} = a_i \mathbb{E}X_1 1_{\{X_1 \geq \langle a, b \rangle\}},$$

where the last equality follows because, by the rotational invariance of the Gaussian measure, $\langle a, X \rangle$ has the same distribution as X_1 . Finally, integration by parts shows that $\mathbb{E}X_1 1_{\{X_1 \geq \langle a, b \rangle\}} = \phi(\langle a, b \rangle)$. \square

Next, we prove Lemma 5.5. The point is that if a half-space is close to a low-influence function f then that half-space must also have low influences. We can then perturb the half-space to have even lower influences without increasing its distance to f by much.

Proof of Lemma 5.5. Suppose that f has influences bounded by τ , and that

$$\mathbb{E}(f(X) - 1_{\{\langle a, X-b \rangle \geq 0\}})^2 \leq \gamma, \tag{5.17}$$

where $\gamma = C(\rho)(\eta + \tau^{cn} + \epsilon)^c$. We will show that there is some \tilde{a} such that $\sum_i \tilde{a}_i^2 = 1$, $\max_i |\tilde{a}_i| \leq C\tau^c$, and

$$\mathbb{E}(f(X) - 1_{\{\langle \tilde{a}, X-b \rangle \geq 0\}})^2 \leq C\gamma^c. \tag{5.18}$$

When applied to the function f_η , this will imply the claim of Lemma 5.5.

Since X_1, \dots, X_n are orthonormal,

$$\begin{aligned} \mathbb{E}(f(X) - 1_{\{(a, X-b) \geq 0\}})^2 &\geq \sum_{i=1}^n \left(\mathbb{E}X_i f(X) - \mathbb{E}X_i 1_{\{(a, X-b) \geq 0\}} \right)^2 \\ &= \sum_{i=1}^n \left(\hat{f}_{\{i\}} - a_i \phi(\langle a, b \rangle) \right)^2, \end{aligned} \quad (5.19)$$

where the equality used Lemma 5.8. Define $\kappa_{a,b} = \phi(\langle a, b \rangle)$, and note from (5.1) that since the influences of f are bounded by τ , $|\hat{f}_{\{i\}}| \leq \sqrt{\tau}$ for every i . Hence for any i with $|a_i| \kappa_{a,b} \geq 2\sqrt{\tau}$, we have $(\hat{f}_{\{i\}} - a_i \kappa_{a,b})^2 \geq a_i^2 \kappa_{a,b}^2 / 4$. Combining this with (5.17) and (5.19),

$$\gamma \geq \mathbb{E}(f(X) - 1_{\{(a, X-b) \geq 0\}})^2 \geq \frac{\kappa_{a,b}^2}{4} \sum_{\{i: |a_i| \kappa_{a,b} \geq 2\sqrt{\tau}\}} a_i^2. \quad (5.20)$$

We now consider two cases, depending on whether $\kappa_{a,b}$ is large or small. First, suppose that $\kappa_{a,b} \leq \gamma^{1/3}$; suppose also, without loss of generality, that $\langle a, b \rangle \leq 0$ (if not, replace f by $1 - f$). Then $\kappa_{a,b} = \phi(\langle a, b \rangle) \geq \Phi(\langle a, b \rangle) = \mathbb{E}1_{\{(a, X-b) \geq 0\}}$; on the other hand, (5.17) implies that $(\mathbb{E}f - \mathbb{E}1_{\{(a, X-b) \geq 0\}})^2 \leq \mathbb{E}(f - 1_{\{(a, X-b) \geq 0\}})^2 \leq \gamma$ and so

$$\mathbb{E}f \leq \sqrt{\gamma} + \mathbb{E}1_{\{(a, X-b) \geq 0\}} \leq \sqrt{\gamma} + \kappa_{a,b} \leq 2\gamma^{1/3}.$$

Since f takes values in $[0, 1]$, it follows that $\mathbb{E}f^2 \leq C\gamma^c$; in particular, *any* half-space with small enough measure will satisfy (5.18).

Now suppose that $\kappa_{a,b} \geq \gamma^{1/3}$ (which is in turn larger than $\tau^{1/3}$ by definition); then (5.20) implies that

$$\sum_{\{i: |a_i| \geq 2\tau^{1/6}\}} a_i^2 \leq \sum_{\{i: |a_i| \kappa_{a,b} \geq 2\sqrt{\tau}\}} a_i^2 \leq 4\gamma^{1/3}.$$

If we define \bar{a} to be the truncated version of a (i.e. $\bar{a}_i = a_i$ if $|a_i| < 2\tau^{1/6}$ and $\bar{a}_i = 0$ otherwise), then this implies that $|a - \bar{a}|^2 \leq 4\gamma^{1/3}$. Since $|a| = 1$, it then follows from the triangle inequality that $|\bar{a}| \geq 1 - 2\gamma^{1/6}$. Set $\tilde{a} = \bar{a}/|\bar{a}|$. If γ is small enough so that $1 - 2\gamma^{1/6} \leq 1/2$ then

$$\max_i |\tilde{a}_i| = \frac{1}{|\bar{a}|} \max_i |\bar{a}_i| \leq \frac{2\tau^{1/6}}{1 - 2\gamma^{1/6}} \leq 4\tau^{1/6}$$

and

$$|a - \tilde{a}| \leq |a - \bar{a}| + |\bar{a} - \tilde{a}| \leq 2\gamma^{1/6} + \frac{1 - |\bar{a}|}{|\bar{a}|} \leq 8\gamma^{1/6}.$$

By the triangle inequality, \tilde{a} satisfies (5.18). \square

Next, we will prove Lemma 5.6: if $g^{a,b}$ is the multilinear extension of a low-influence half-space, then $g^{a,b}$ is close to a half-space. Observe that this is very much not the case for general half-spaces: the multilinear extension of $1_{\{x_1 \geq 0\}}$ is x_1 , which is not close, in $L_2(\mathbb{R}^n, \gamma_n)$, to any half-space.

The main idea of the proof is to study the quantity $\overline{\mathbb{E}g^{a,b}(X)} \langle a, X - b \rangle$. By showing that this is almost as large as $\mathbb{E}1_{\{(a, X-b) \geq 0\}} \langle a, X - b \rangle$, we show that $g^{a,b}(X)$ is close to $1_{\{(a, X-b) \geq 0\}}$.

Proof of Lemma 5.6. Suppose without loss of generality that $\langle a, b \rangle \geq 0$. Let $h(x) = \langle a, x - b \rangle$ and let g be the multilinear extension of $1_{\{h \geq 0\}}$. First of all, the Berry-Esseen [20] theorem implies that for any $t \in \mathbb{R}$, $|\Pr(\langle a, x \rangle \geq t) - \Pr(\langle a, X \rangle \geq t)| \leq \tau$. By the formula $\mathbb{E}Z = \int_0^\infty \Pr(Z \geq t) dt$ for a non-negative random variable Z , we have

$$\begin{aligned}
\mathbb{E}g(\xi)h(\xi) &= \mathbb{E}h(\xi)1_{\{h(\xi) \geq 0\}} \\
&= \int_0^\infty \Pr(\langle a, \xi - b \rangle \geq t) dt \\
&= \int_{\langle a, b \rangle}^\infty \Pr(\langle a, \xi \rangle \geq t) dt \\
&\geq \int_{\langle a, b \rangle}^M \Pr(\langle a, \xi \rangle \geq t) dt \\
&\geq \int_{\langle a, b \rangle}^M \Pr(X_1 \geq t) dt - M\tau \\
&\geq \int_{\langle a, b \rangle}^\infty \Pr(X_1 \geq t) dt - M\tau - Ce^{-M^2/2} \\
&= \mathbb{E}h(X)1_{\{h(X) \geq 0\}} - M\tau - Ce^{-M^2/2}.
\end{aligned}$$

Choosing $M = \sqrt{\log(1/\tau)}$, we have

$$\mathbb{E}g(\xi)h(\xi) \geq \mathbb{E}h(X)1_{\{h(X) \geq 0\}} - C\tau^c. \quad (5.21)$$

Now, h is linear and so $h_t = e^{-t}h$; since Q_η is self-adjoint, Theorem 5.4 implies that

$$\begin{aligned}
\mathbb{E}g(\xi)h(\xi) &= e^\eta \mathbb{E}g(\xi)h_\eta(\xi) \\
&= e^\eta \mathbb{E}g_\eta(\xi)h(\xi) \\
&= e^\eta \mathbb{E}g_\eta(X)h(X) \\
&\leq e^\eta \mathbb{E}\overline{g}_\eta(X)h(X) + Ce^\eta(\eta + \tau^{c\eta}) \\
&\leq \mathbb{E}\overline{g}_\eta(X)h(X) + C(\eta + \tau^{c\eta}),
\end{aligned}$$

where the last inequality assumes that $\eta < 1$ (if not then the lemma is trivial anyway), and uses the fact that $\mathbb{E}\overline{g}_\eta(X)h(X)$ is bounded by a universal constant. Combining this with (5.21),

$$\mathbb{E}h(X)1_{\{h(X) \geq 0\}} \leq \mathbb{E}\overline{g}_\eta(X)h(X) + C(\eta + \tau^{c\eta}). \quad (5.22)$$

Now, let $m(X) = 1_{\{\langle a, X - b \rangle \geq 0\}} - \overline{g}_\eta(X)$ and take $\epsilon = \mathbb{E}|m|$. Note that because $\overline{g}_\eta \in [0, 1]$, when $m \neq 0$ then m and h have the same sign; in particular, $m(x)h(x) \geq 0$. Let $A = \{x : \langle a, X - b \rangle \in [-\epsilon/2, \epsilon/2]\}$. Then $\Pr(A) \leq \epsilon/2$, and since $|m| \leq 1$ pointwise we must have $\mathbb{E}|m|1_{A^c} \geq \mathbb{E}|m| - \Pr(A) \geq \epsilon/2$. But on A^c we have $|h(x)| \geq \epsilon/2$; since $m(x)h(x) \geq 0$,

$$\mathbb{E}m(X)h(X) \geq \mathbb{E}m(X)h(X)1_{\{X \in A^c\}} \geq \frac{\epsilon}{2} \mathbb{E}|m|1_{A^c} \geq \frac{\epsilon^2}{4}.$$

Applying this to (5.22) yields $\epsilon \leq C(\eta + \tau^{c\eta})^c$. So if we recall the definition of ϵ , then we see that

$$\mathbb{E}|1_{\{(a, X-b) \geq 0\}} - \overline{g_\eta}(X)| \leq C(\eta e^{2\eta} + \tau^{c\eta})^c.$$

By changing the constant c , we may replace $\mathbb{E}|\cdot|$ with $\mathbb{E}(\cdot)^2$; by (5.11) and the triangle inequality, we may replace $\overline{g_\eta}$ by g_η . This completes the proof of the lemma. Note that the only reason for proving this lemma with g_η instead of g was for extra convenience when applying it; the statement of the lemma is also true with g instead of g_η . \square

The only remaining piece is Lemma 5.7.

Proof of Lemma 5.7. Suppose $f : \{-1, 1\}^n \rightarrow [-1, 1]$ and $g : \{-1, 1\}^n \rightarrow \{-1, 1\}$. This does not exactly correspond to the statement of the lemma, but it will be more convenient for the proof; we can recover the statement of the lemma by replacing f by $\frac{1+f}{2}$ and g by $\frac{1+g}{2}$.

Let $\epsilon = \mathbb{E}(f_\eta(\xi) - g(\xi))^2$. Since g takes values in $\{-1, 1\}$, we have $\mathbb{E}g^2 = 1$. Then the triangle inequality implies that $(\mathbb{E}g^2)^{1/2} \leq (\mathbb{E}f_\eta^2)^{1/2} + \sqrt{\epsilon}$; squaring both sides, we have

$$\mathbb{E}f_\eta^2 \geq \mathbb{E}g^2 - 2\mathbb{E}(f_\eta^2)^{1/2}\sqrt{\epsilon} \geq 1 - 3\sqrt{\epsilon}.$$

Since $\mathbb{E}f^2 \leq 1$, we have

$$\begin{aligned} \mathbb{E}(f - f_\eta)^2 &= \sum_{S \subset [n]} \hat{f}_S^2 (1 - e^{-\eta|S|})^2 \\ &\leq \sum_{S \subset [n]} \hat{f}_S^2 (1 - e^{-\eta|S|}) \\ &= \mathbb{E}f^2 - \mathbb{E}f_\eta^2 \\ &\leq 3\sqrt{\epsilon}. \end{aligned}$$

It then follows by the triangle inequality that $\mathbb{E}(f - g)^2 \leq C\sqrt{\epsilon}$. \square

5.2 Spherical noise stability and Max-Cut rounding

The well-known similarity between a Gaussian vector and a uniformly random vector on a high-dimensional sphere suggests that there might be a spherical analogue of our Gaussian noise sensitivity result. The correlation structure on the sphere that is most useful for our purposes is the uniform measure over all pairs of points (x, y) whose inner product $\langle x, y \rangle$ is exactly ρ . Under this model of noise, we can use robust Gaussian noise sensitivity to show, asymptotically in the dimension, robustness for spherical noise sensitivity. This uses the theory of spherical harmonics and has applications to rounding semidefinite programs (in particular, the Goemans-Williamson algorithm for Max-Cut). Our proof uses and generalizes the work of Klartag and Regev [31], in which a related problem was studied in the context of one-way communication complexity.

Our spherical noise stability result mostly follows from Theorem 4.1, by replacing X and Y by $X/|X|$ and $Y/|Y|$. When n is large, these renormalized Gaussian vectors are uniformly distributed on the sphere and their inner product is tightly concentrated around ρ . The fact that their inner product is not *exactly* ρ causes some difficulty, particularly because Q_ρ is actually orthogonal to the joint distribution of two normalized Gaussians. Working through this difficulty with some properties of spherical harmonics, we obtain the following spherical analogue of Theorem 4.1:

Theorem 5.9. *Let $0 < \rho < 1$ and write Q_ρ for the measure of (X, Y) on the sphere S^{n-1} where the pair (X, Y) is uniformly distributed in*

$$\{(x, y) \in S^{n-1} \times S^{n-1} : \langle x, y \rangle = \rho\}.$$

For measurable $A_1, A_2 \subset S^{n-1}$, define

$$\delta = \delta(A_1, A_2) = Q_\rho(X \in B_1, Y \in B_2) - Q_\rho(X \in A_1, Y \in A_2),$$

where B_1 and B_2 are parallel spherical caps with the same volumes as A_1 and A_2 respectively. Define also

$$m(A_1, A_2) = p(1-p)q(1-q)$$

where $p = \Pr(X \in A_1)$ and $q = \Pr(Y \in A_2)$.

For any $A_1, A_2 \subset S^{n-1}$, there exist parallel spherical caps B_1 and B_2 such that

$$Q(A_1 \Delta B_1) \leq C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4} \frac{(1-\rho)(1-\rho^2)}{1+3\rho}}$$

$$Q(A_2 \Delta B_2) \leq C(\rho) m^{-C(\rho)} \delta_*^{\frac{1}{4} \frac{(1-\rho)(1-\rho^2)}{1+3\rho}}.$$

where $\delta_* = \max(\delta, n^{-1/2} \log n)$.

The case $\rho = 0$ of the above theorem is related to work by Klartag and Regev [31]. In this case one expects that X and Y should behave as independent random variables on S^{n-1} and that therefore for *all* A_1, A_2 , $Q_0(X \in A_1, Y \in A_2)$ should be close to $Q(X \in A_1)Q(Y \in A_2)$. Indeed the main technical statement of Klartag and Regev (Theorem 5.2) says that for every two sets,

$$|Q_0(X \in A_1, Y \in A_2) - Q(X \in A_1)Q(Y \in A_2)| \leq \frac{C}{n}.$$

In other words the results of Klartag and Regev show that in the case $\rho = 0$, a uniform orthogonal pair (X, Y) on the sphere behaves like a pair of independent random variables up to an error of order n^{-1} , while our results show that for $0 < \rho < 1$, (X, Y) that are ρ correlated behave like Gaussians with the same correlation.

That spherical caps minimize the quantity $Q_\rho(X \in A_1, Y \in A_2)$ over all sets A_1 and A_2 with some prescribed volumes is originally due to Baernstein and Taylor [2], while a similar result for a different noise model is due to Beckner [4]. Their results do not follow from ours because of the dependence on n in Theorem 5.9, and so one could ask for a sharper version of Theorem 5.9 that does imply these earlier results. One obstacle is that we do not know a proof of Beckner's inequality that gives control of the deficit.

Rounding the Goemans-Williamson algorithm

Let $G = (V, E)$ be a graph and recall that the Max-Cut problem is to find a set $A \subset V$ such that the number of edges between A and $V \setminus A$ is maximal. It is of course equivalent to look for a function $f : V \rightarrow \{-1, 1\}$ such that $\sum_{(u,v) \in E} |f(u) - f(v)|^2$ is maximal. Goemans' and Williamson's breakthrough was to realize that this combinatorial optimization problem can be efficiently solved if we relax the range $\{-1, 1\}$ to S^{n-1} . Let us say, therefore, that an embedding f of a graph $G = (V, E)$ into the sphere S^{n-1} is *optimal* if

$$\sum_{(u,v) \in E} |f(u) - f(v)|^2$$

is maximal. An oblivious rounding procedure is a (possibly random) function $R : S^{n-1} \rightarrow \{-1, 1\}$ (we call it "oblivious" because it does not look at the graph G). We will then denote by $\text{Cut}(G, R)$ the expected value of the cut produced by rounding the worst possible optimal spherical embedding of G :

$$\text{Cut}(G, R) = \frac{1}{2} \min_f \mathbb{E} \sum_{(u,v) \in E} |R(f(u)) - R(f(v))|,$$

where the minimum is over all optimal embeddings f . If MaxCut denotes the maximum cut in G , then Goemans and Williamson [21] showed that when $R(x) = \text{sgn}(\langle X, x \rangle)$ for a standard Gaussian vector X , then for every graph G ,

$$\text{Cut}(G, R) \geq \text{MaxCut}(G) \min_{\theta} \alpha_{\theta},$$

where $\alpha_{\theta} = \frac{2}{\pi} \frac{\theta}{1 - \cos \theta}$. In the other direction, Feige and Schechtman [19] showed that for every oblivious rounding scheme R and every $\epsilon > 0$, there is a graph G such that

$$\text{Cut}(G, R) \leq \text{MaxCut}(G) \left(\epsilon + \min_{\theta} \alpha_{\theta} \right).$$

In other words, no rounding scheme is better than the half-space rounding scheme. Using Theorem 4.1, we can go further:

Theorem 5.10. *Suppose R is a rounding scheme on S^{n-1} such that for every graph G with n vertices,*

$$\text{Cut}(G, R) \geq \text{MaxCut}(G) \left(\min_{\theta} \alpha_{\theta} - \epsilon \right).$$

Then there is a hyperplane rounding scheme \tilde{R} such that

$$\mathbb{E} |R(Y) - \tilde{R}(Y)| \leq C \epsilon_{\star}^c,$$

where Y is a uniform (independent of R and \tilde{R}) random vector on S^{n-1} , C and c are absolute constants, and $\epsilon_{\star} = \max\{\epsilon, n^{-1/2} \log n\}$.

In other words, any rounding scheme that is almost optimal is essentially the same as rounding by a random half-space.

Proof of Theorem 5.9

To make the connection between Theorem 4.1 and Theorem 5.9, we define, for a subset $A \subset S^{n-1}$, the *radial extension* $\bar{A} \subset \mathbb{R}^n$ by

$$\bar{A} = \{x \in \mathbb{R}^n : x \neq 0 \text{ and } \frac{x}{|x|} \in A\}$$

From the spherical symmetry of the Gaussian distribution it immediately follows that $\gamma_n(\bar{A}) = Q(A)$. The proof of Theorem 5.9 crucially relies on the fact that $Q_\rho(A_1, A_2)$ is close to $\text{Pr}_\rho(\bar{A}_1, \bar{A}_2)$ in high dimensions. More explicitly it uses the following lemmas:

Lemma 5.11. *For any half-space $H = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$ there is a spherical cap $B = \{x \in S^{n-1} : \langle a, x \rangle \leq b'\}$ such that $\gamma_n(\bar{B}) = \gamma_n(H)$ and*

$$\gamma_n(\bar{B} \Delta H) \leq Cn^{-1/2} \log n.$$

Lemma 5.12. *For any two sets $A_1, A_2 \subset S^{n-1}$ and any $\rho \in [-1 + \epsilon, 1 - \epsilon]$ it holds that*

$$|Q_\rho(A_1, A_2) - \text{Pr}_\rho(\bar{A}_1, \bar{A}_2)| \leq C(\epsilon)n^{-1/2} \log n.$$

Given Lemmas 5.12 and 5.11, the proof of Theorem 5.9 is an easy corollary of Theorem 4.1:

Proof of Theorem 5.9. Define $\delta_* = \delta(\bar{A}_1, \bar{A}_2)$. Let H_1, H_2 be parallel half-spaces with $\gamma_n(H_i) = \gamma_n(\bar{A}_i)$, and let B_1, B_2 be the corresponding caps whose existence is guaranteed by Lemma 5.11. Then

$$\begin{aligned} \delta_* &= \delta(\bar{A}_1, \bar{A}_2) \\ &= \text{Pr}_\rho(X \in H_1, Y \in H_2) - \text{Pr}_\rho(X \in \bar{A}_1, Y \in \bar{A}_2) \\ &\leq \text{Pr}_\rho(X \in \bar{B}_1, Y \in \bar{B}_2) - \text{Pr}_\rho(X \in \bar{A}_1, Y \in \bar{A}_2) + O(n^{-1/2} \log n) \\ &\leq Q_\rho(X \in B_1, Y \in B_2) - Q_\rho(X \in A_1, Y \in A_2) + O(n^{-1/2} \log n) \\ &= \delta(A_1, A_2) + O(n^{-1/2} \log n), \end{aligned}$$

where the first inequality follows from Lemma 5.11 and the second follows from Lemma 5.12.

From Theorem 4.1 it follows that there are parallel half-spaces H_1 and H_2 with $\gamma_n(H_i) = \gamma_n(\bar{A}_i)$ satisfying

$$\gamma_n(\bar{A}_i \Delta H_i) \leq C(\rho)m^{-C(\rho)} \delta_*^{\frac{1}{4} \frac{(1-\rho)(1-\rho^2)}{1+3\rho}}.$$

By Lemma 5.11, there are parallel caps B_1 and B_2 such that

$$Q(A_i \Delta B_i) = \gamma_n(\bar{A}_i \Delta \bar{B}_i) \leq C(\rho)m^{-C(\rho)} \delta_*^{\frac{1}{4} \frac{(1-\rho)(1-\rho^2)}{1+3\rho}}. \quad \square$$

The proof of Lemma 5.11 is quite simple, so we present it first:

Proof of Lemma 5.11. Let $H = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$, and suppose without loss of generality that $b \geq 0$. For any $\epsilon > 0$, define

$$\begin{aligned} H_\epsilon^+ &= \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b(1 + \epsilon)\} \\ H_\epsilon^- &= \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b(1 - \epsilon)\}. \end{aligned}$$

Note that $\gamma_n(H_\epsilon^+ \setminus H_\epsilon^-) \leq C\epsilon$.

Now define $B = \{x \in S^{n-1} : \langle x, a \rangle \leq b/\sqrt{n}\}$. Then $\bar{B} = \{x \in \mathbb{R}^n : \langle x, a \rangle \leq b|x|/\sqrt{n}\}$, and so

$$\begin{aligned} \gamma_n(\bar{B} \setminus H_\epsilon^+) &= \gamma_n((1 + \epsilon)b \leq \langle X, a \rangle \leq b|X|/\sqrt{n}) \\ &\leq \gamma_n(|X| \geq (1 + \epsilon)\sqrt{n}) \\ &\leq Ce^{-c\epsilon^2 n}, \end{aligned}$$

where the last line follows from standard concentration inequalities (Bernstein's inequalities, for example). Similarly,

$$\gamma_n(H_\epsilon^- \setminus \bar{B}) \leq \gamma_n(|X| \leq (1 - \epsilon)\sqrt{n}) \leq Ce^{-c\epsilon^2 n}.$$

Since $H_\epsilon^- \subset H \subset H_\epsilon^+$ and $\gamma_n(H_\epsilon^+ \setminus H_\epsilon^-) \leq C\epsilon$, it follows that

$$\gamma_n(H \Delta \bar{B}) \leq C\epsilon + Ce^{-c\epsilon^2 n}.$$

By choosing $\epsilon = Cn^{-1/2} \log n$, we have

$$\gamma_n(H \Delta \bar{B}) \leq Cn^{-1/2} \log n. \tag{5.23}$$

Now, the lemma claimed that we could ensure $\gamma_n(\bar{B}) = \gamma_n(H)$. Since the volume of the cap $B' := \{\langle a, x \rangle \leq b'|x|\}$ is continuous and strictly increasing in b' , we may define b' to be the unique real number such that $\gamma_n(\bar{B}') = \gamma_n(H)$. Now, either $B \subset B'$ or $B' \subset B$; hence $\gamma_n(\bar{B} \Delta \bar{B}') = |\gamma_n(\bar{B}) - \gamma_n(\bar{B}')|$. On the other hand, (5.23) implies that

$$|\gamma_n(\bar{B}) - \gamma_n(\bar{B}')| = |\gamma_n(\bar{B}) - \gamma_n(H)| \leq Cn^{-1/2} \log n,$$

and so the triangle inequality leaves us with

$$\gamma_n(H \Delta \bar{B}') \leq \gamma_n(H \Delta \bar{B}) + \gamma_n(B \Delta \bar{B}') \leq Cn^{-1/2} \log n. \quad \square$$

We defer the proof of Lemma 5.12 until the next section, since this proof requires an introduction to spherical harmonics.

Spherical harmonics and Lemma 5.12

We will try to give an introduction to spherical harmonics which is as brief as possible, while still containing enough material for us to explain the proof of Lemma 5.12 adequately. A slightly less brief introduction is contained in [31]; for a full treatment, see [43].

Let \mathcal{S}_k be the linear space consisting of harmonic, homogeneous, degree- k polynomials. We will think of \mathcal{S}_k as a subspace of $L_2(S^{n-1}, Q)$; then $\{\mathcal{S}_k : k \geq 0\}$ spans $L_2(S^{n-1}, Q)$. One can easily check that \mathcal{S}_k is invariant under rotations. Hence it is a representation of $SO(n)$. It turns out, moreover, that \mathcal{S}_k is an irreducible representation of $SO(n)$; combined with Schur's lemma, this leads to the following important property:

Lemma 5.13. *If $T : L_2(S^{n-1}) \rightarrow L_2(S^{n-1})$ commutes with rotations then $\{\mathcal{S}_k : k \geq 0\}$ are the eigenspaces of T .*

In particular, we will apply Lemma 5.13 to the operators T_ρ defined by $(T_\rho f)(X) = \mathbb{E}(f(Y)|X)$, where $(X, Y) \sim Q_\rho$. In other words, $(T_\rho f)(x)$ is the average of f over the set $\{y \in S^{n-1} : \langle x, y \rangle = \rho\}$. Clearly, T_ρ commutes with rotations; hence Lemma 5.13 implies that $\{\mathcal{S}_k : k \geq 0\}$ are the eigenspaces of T_ρ . In particular, there exist $\{\mu_k(\rho) : k \geq 0\}$ such that $T_\rho f = \mu_k(\rho)f$ for all $f \in \mathcal{S}_k$. Moreover, to compute $\mu_k(\rho)$, it is enough to compute $T_\rho f$ for a single $f \in \mathcal{S}_k$. For this task, the Gegenbauer polynomials provide good candidates: define

$$G_k(t) = \mathbb{E}(t + iW_1\sqrt{1-t^2})^k,$$

where the expectation is over $W = (W_1, \dots, W_{n-1})$ distributed uniformly on the sphere S^{n-2} . Define $f_k(x) = G_k(x_1)$; it turns out that $f_k \in \mathcal{S}_k$; on the other hand, one can easily check that $f_k(e_1) = 1$, while $(T_\rho f_k)(e_1) = G_k(\rho)$. From the discussion above, it then follows that

$$\mu_k(\rho) = \mathbb{E}(\rho + iW_1\sqrt{1-\rho^2})^k.$$

With this explicit formula, we can show that $\mu_k(\rho)$ is continuous in ρ :

Lemma 5.14. *For any $\epsilon > 0$ there exists $C(\epsilon)$ such that if $\rho, \eta \in [-1 + \epsilon, 1 - \epsilon]$ then*

$$|\mu_k(\rho) - \mu_k(\eta)| \leq C(\epsilon)(|\rho - \eta| + n^{-1/2}).$$

We will leave the proof of Lemma 5.14 to the end. Instead, let us show how it can be used to prove that $Q_\rho(X \in A_1, Y \in A_2)$ is continuous in ρ .

Lemma 5.15. *For any $\epsilon > 0$ there exists $C(\epsilon)$ such that if $\rho, \eta \in [-1 + \epsilon, 1 - \epsilon]$ then*

$$|Q_\rho(X \in A_1, Y \in A_2) - Q_\eta(X \in A_1, Y \in A_2)| \leq C(\epsilon)Q^{1/2}(A_1)Q^{1/2}(A_2)(|\rho - \eta| + n^{-1/2}).$$

Proof. Take $f, g \in L_2(S^{n-1}, Q)$ and write $f = \sum_{k=0}^{\infty} f_k$ where $f_k \in \mathcal{S}_k$. Then

$$|\mathbb{E}gT_\rho f - \mathbb{E}gT_\eta f| \leq \|T_\rho f - T_\eta f\|_2 \|g\|_2$$

(where $\|f\|_2$ denotes $\sqrt{\mathbb{E}f^2}$) and

$$\|T_\rho f - T_\eta f\|_2^2 = \sum_{k=0}^{\infty} (\mu_k(\rho) - \mu_k(\eta))^2 \|f_k\|_2^2$$

By Lemma 5.14, we have

$$\|T_\rho f - T_\eta f\|_2 \leq C(\epsilon)(|\rho - \eta| + n^{-1/2})\|f\|_2,$$

and therefore

$$|\mathbb{E}gT_\rho f - \mathbb{E}gT_\eta f| \leq C(\epsilon)\|f\|_2\|g\|_2(|\rho - \eta| + n^{-1/2}).$$

Note that if $f = 1_{A_1}$ and $g = 1_{A_2}$ then $\mathbb{E}gT_\rho f = Q_\rho(X \in A_1, Y \in A_2)$, while $\|f\|_2 = Q(A_1)^{1/2}$. \square

The proof of Lemma 5.12 is straightforward once we know Lemma 5.15. As we have already mentioned, normalized Gaussian vectors from \Pr_ρ have a joint distribution that is similar to Q_ρ , except that their inner products are close to ρ instead of being exactly ρ . But Lemma 5.15 implies that a small difference in ρ doesn't affect the noise sensitivity by much.

Proof of Lemma 5.12. Let X, Y be distributed according to \Pr_ρ . Then

$$\Pr_\rho(X \in \bar{A}_1, Y \in \bar{A}_2) = \Pr_\rho\left(\frac{X}{|X|} \in A_1, \frac{Y}{|Y|} \in A_2\right),$$

Note that conditioned on $|X|, |Y|$ and $\langle X, Y \rangle$, the variables $X/|X|, Y/|Y|$ are distributed according to Q_r , where $r = \langle X, Y \rangle / (|X||Y|)$. Now with probability $1 - \frac{1}{n^2}$ it holds that

$$|X|^2, |Y|^2 \in n \pm Cn^{1/2} \log n, \quad \langle X, Y \rangle \in \rho n \pm Cn^{1/2} \log n.$$

On this event, we have

$$r = \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle \in \rho \pm Cn^{-1/2} \log n.$$

Using Lemma 5.15 we get that

$$\Pr_\rho(X \in \bar{A}_1, Y \in \bar{A}_2) \leq Q_\rho(X \in A_1, Y \in A_2) + C(\epsilon)n^{-1/2} \log n.$$

A similar argument yields a bound in the other direction and concludes the proof. \square

Our final task is the proof of Lemma 5.14:

Proof of Lemma 5.14. Define $Z_\rho = \rho + iW_1\sqrt{1-\rho^2}$ (recalling that $W = (W_1, \dots, W_{n-1})$ is uniformly distributed on S^{n-2}) so that $\mu_k(\rho) = \mathbb{E}Z_\rho^k$. Note that if $|W_1| \leq \frac{1}{2}$ (which happens with probability at least $1 - \exp(-cn)$) then

$$|Z_\rho| = \rho^2 + W_1(1 - \rho^2) \leq \frac{1 + \rho^2}{2} \leq 1 - \frac{\epsilon}{2} \leq \exp(-c\epsilon).$$

Now,

$$\begin{aligned}\mu_k(\rho) - \mu_k(\eta) &= \mathbb{E}(Z_\rho^k - Z_\eta^k) \\ &= \mathbb{E}(Z_\rho - Z_\eta) \sum_{j=1}^{k-1} Z_\rho^j Z_\eta^{k-1-j}.\end{aligned}\tag{5.24}$$

If $|W_1| \leq \frac{1}{2}$ then $|Z_\rho^j Z_\eta^{k-1-j}| \leq \exp(-c\epsilon k)$ and so

$$\left| \sum_j Z_\rho^j Z_\eta^{k-1-j} \right| \leq k \exp(-c\epsilon k) \leq C(\epsilon)$$

Applying this to (5.24), we have

$$\begin{aligned}|\mu_k(\rho) - \mu_k(\eta)| &= \mathbb{E}(Z_\rho^k - Z_\eta^k) \mathbf{1}_{\{|W_1| \geq 1/2\}} + \mathbb{E} \mathbf{1}_{\{|W_1| < 1/2\}} (Z_\rho - Z_\eta) \sum_{j=1}^{k-1} Z_\rho^j Z_\eta^{k-1-j} \\ &\leq 2\gamma_n(|W_1| \geq 1/2) + C(\epsilon) \mathbb{E}|Z_\rho - Z_\eta| \\ &\leq \exp(-cn) + C(\epsilon)|\rho - \eta|,\end{aligned}$$

where $\mathbb{E}|Z_\rho - Z_\eta| \leq C(\epsilon)|\rho - \eta|$ because $|\sqrt{1 - \rho^2} - \sqrt{1 - \eta^2}| \leq C(\epsilon)|\rho - \eta|$. \square

Spherical noise and Max-Cut

In this section, we will outline how robust noise sensitivity on the sphere (Theorem 5.9) implies that half-space rounding for the Goemans-Williamson algorithm is robustly optimal (Theorem 5.10). The key for making this connection is Karloff's family of graphs [29]: for any $n, d \in \mathbb{N}$, let $G_{n,d} = (V_{n,d}, E_{n,d})$ be the graph whose vertices are the $\binom{n}{n/2}$ balanced elements of $\{-n^{-1/2}, n^{-1/2}\}^n$, and with an edge between u and v if $\langle u, v \rangle = d/n$. Karloff showed that if $d \leq n/24$ then the optimal cut of $G_{n,d}$ has value $|E_{n,d}|(1 - d/n)$. Moreover, the identity embedding (and any rotation of it) is an optimal embedding of $G_{n,d}$ into S^{n-1} . In these embeddings, every angle between two connected vertices is d/n ; hence, it is easy to calculate the expected value of a rounding scheme:

Lemma 5.16. *Let (X, Y) be distributed according to $Q_{d/n}$. For any rounding scheme R ,*

$$\text{Cut}(G_{n,d}, R) \leq \frac{|E_{n,d}|}{2} \mathbb{E}|R(X) - R(Y)|,$$

where the expectation is with respect to X, Y and R .

Proof. Recall that

$$\begin{aligned}\text{Cut}(G, R) &= \frac{1}{2} \min_f \mathbb{E}_R \sum_{(u,v) \in E} |R(f(u)) - R(f(v))| \\ &\leq \frac{1}{2} \mathbb{E}_R \mathbb{E}_f \sum_{(u,v) \in E} |R(f(u)) - R(f(v))|,\end{aligned}$$

where the expectation is taken over all rotations f . But if f is a uniformly random rotation then for every $(u, v) \in E_{n,d}$, the pair $(f(u), f(v))$ is equal in distribution to the pair (X, Y) (and both pairs are independent of R). \square

Theorem 5.10 follows fairly easily from Lemma 5.16, Theorem 5.9, and the fact that $\text{MaxCut}(G_{n,d}) = |E_{n,d}|(1 - d/n)$. Indeed, choose n and d such that $|d/n - \cos^{-1} \theta^*| \leq n^{-1}$, where $\theta^* \approx 2.33$ minimizes α_θ , and suppose there is a rounding scheme R such that

$$\text{Cut}(G_{n,d}, R) \geq \text{MaxCut}(G_{n,d})(\alpha_{\theta^*} - \epsilon).$$

Let $\theta = \cos(d/n)$; since α_θ is continuous in θ , it follows that $|\alpha_\theta - \alpha_{\theta^*}| \leq \frac{C}{n}$. Taking $\epsilon_\star = \max\{\epsilon, n^{-1/2} \log n\}$, we have $|\alpha_\theta - \alpha_{\theta^*}| \leq C\epsilon_\star$ and so

$$\begin{aligned} \text{Cut}(G_{n,d}, R) &\geq \text{MaxCut}(G_{n,d})(\alpha_\theta - C\epsilon_\star) \\ &= |E_{n,d}|(1 - \cos \theta)(\alpha_\theta - C\epsilon_\star) \\ &= \frac{2}{\pi} \theta |E_{n,d}|(1 - C\epsilon_\star). \end{aligned}$$

By Lemma 5.16, $\frac{1}{2} \mathbb{E}|R(X) - R(Y)| \geq \frac{2}{\pi} \theta(1 - C\epsilon_\star)$. If we define the (random) subset $A_R \subset S^{n-1}$ by $A_R = \{x : R(x) = 1\}$, and set $\rho = \cos \theta$ then

$$Q(A_R) - \mathbb{S}_\rho(A_R) = \frac{1}{2} \mathbb{E}(|R(X) - R(Y)| | R)$$

Taking expectations,

$$\mathbb{E}(Q(A_R) - \mathbb{S}_\rho(A_R)) = \frac{1}{2} \mathbb{E}|R(X) - R(Y)| \geq \frac{2}{\pi} \arccos \rho - C\epsilon_\star. \quad (5.25)$$

Let δ_R be the random deficit $\delta_R = \frac{2}{\pi} \arccos \rho - (Q(A_R) - \mathbb{S}_\rho(A_R))$, so that (5.25) implies $\mathbb{E}\delta_R \leq C\epsilon_\star$. Take η_R to be the distance from A_R to the nearest hemisphere: $\eta_R = \min\{Q(A_R \Delta B) : B \text{ is a hemisphere}\}$ and let B_R be a hemisphere that achieves the minimum (which is attained because the set of hemispheres is compact with respect to the distance $d(A, B) = Q(A \Delta B)$). Recall that $\theta \approx \theta^* \approx 2.33$ and so $\rho = \cos \theta < 0$; by the same symmetries discussed following Theorem 4.1, Theorem 5.9 applies for $\rho < 0$, but with the deficit inequality reversed. Hence, $\eta_R \leq C \max\{\delta_R, n^{-1/2} \log n\}^c$. Taking expectations,

$$\mathbb{E}\eta_R \leq C \mathbb{E} \max\{\delta_R, n^{-1/2} \log n\}^c \leq C \max\{\mathbb{E}\delta_R, n^{-1/2} \log n\}^c = C'\epsilon_\star^c.$$

Consider the rounding scheme $\tilde{R}(y)$ which is 1 when $y \in B_R$ and -1 otherwise. Then $\mathbb{E}(|R(Y) - \tilde{R}(Y)| | R) = 2\eta_R$, and so the displayed equation above implies that

$$\mathbb{E}|R(Y) - \tilde{R}(Y)| \leq C\epsilon_\star^c.$$

Since \tilde{R} is a hyperplane rounding scheme, this completes the proof of Theorem 5.10.

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