Saturated de Rham-Witt complexes with unit-root coefficients

by

Ravi Fernando

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Professor Martin Olsson, Chair
Professor Sug Woo Shin
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Abstract

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The saturated de Rham-Witt complex, introduced by Bhatt-Lurie-Mathew, is a variant of the classical de Rham-Witt complex which provides a conceptual simplification of the classical construction and which is expected to produce better results for non-smooth varieties. In this paper, we introduce a generalization of the saturated de Rham-Witt complex which allows coefficients in a unit-root F-crystal. We define our complex by a universal property in a category of so-called de Rham-Witt modules. We prove a number of results about it, including existence, quasicoherence, and comparisons to the de Rham-Witt complex of Bhatt-Lurie-Mathew and (in the smooth case) to crystalline cohomology and the classical de Rham-Witt complex with coefficients.

Contents

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Chapter 1

Introduction

1.1 The classical de Rham-Witt complex

Suppose X is a variety over a perfect field k of characteristic p. A central object of study is the crystalline cohomology $H^*_{\text{cris}}(X/W(k))$, conceived by Grothendieck and developed by Berthelot using the crystalline site. Crystalline cohomology is a Weil cohomology theory; it serves as a characteristic–0 lift of algebraic de Rham cohomology in characteristic p, and it fills the gap among ℓ-adic étale cohomology theories at $\ell = p$.

Following ideas of Bloch and Deligne, Illusie's classic paper [12] associates to X a pro-complex $(W_r\Omega^*_X)_r$, called the *de Rham-Witt pro-complex* of X, whose inverse limit $W\Omega^*_X$ is called the *de Rham-Witt complex* of X. These are p-adic lifts of the de Rham complex $\Omega^*_{X/k} = W_1\Omega^*_X$, and when X/k is smooth, they compute crystalline cohomology in the same sense that the de Rham complex computes algebraic de Rham cohomology. More precisely, Illusie shows that these complexes are representatives of the derived pushforward of the crystalline structure sheaf to the Zariski site, and therefore that their hypercohomology recovers crystalline cohomology:

Theorem 1.1.1. *([12, II, Théorème 1.4]) Suppose X/k is smooth. Then for each $r > 0$, we have an isomorphism*

$$Ru_{X/W_r*}\mathcal{O}_{X/W_r} \xrightarrow{\sim} W_r\Omega^*_X$$

in the derived category $D(X_{\text{zar}}, W_r(k))$ of sheaves of $W_r(k)$-modules on X. Passing to hypercohomology, this induces isomorphisms

$$H^*_{\text{cris}}(X/W_r) \xrightarrow{\sim} \mathbb{H}^*(X, W_r\Omega^*_X) := R^*\Gamma(X_{\text{zar}}, W_r\Omega^*_X) \text{ and}$$
$$H^*_{\text{cris}}(X/W) \xrightarrow{\sim} \mathbb{H}^*(X, W\Omega^*_X) := R^*\Gamma(X_{\text{zar}}, W\Omega^*_X).$$

In the decades since the introduction of the de Rham-Witt complex, there has been a wealth of further work to generalize its construction, understand its structure, and study its implications for crystalline cohomology and characteristic-p algebraic geometry more broadly. To give just a few examples: we now have de Rham-Witt complexes with coefficients ([8]) and over a nontrivial base ([18]); a good understanding ([12], [13]) of the two spectral sequences computing crystalline cohomology in terms of de Rham-Witt, and proofs of the Künneth and duality formulas for crystalline cohomology via de Rham-Witt ([7], [6]).

1.2 The saturated de Rham-Witt complex

Recently, Bhatt-Lurie-Mathew ([3]) introduced a variant called the *saturated* de Rham-Witt complex. This is denoted $\mathcal{W}\Omega^*_X$, with a calligraphic $\mathcal{W}$ to distinguish it from the classical de Rham-Witt complex $W\Omega^*_X$. Although the saturated de Rham-Witt complex agrees with the classical one in good situations, it is expected to have better properties for non-smooth varieties. Moreover, its construction provides a new perspective that simplifies the classical picture in several ways.

Let us recall the general outline of Bhatt-Lurie-Mathew's approach. The construction is affine-local, and most of the work happens in the category of Dieudonné complexes, so we begin by recalling the main features of this category.

Definition 1.2.1. ([3, Definition 2.1.1]) A *Dieudonné complex* is a complex (X^*, d) of abelian groups, equipped with a graded group endomorphism F such that we have $dF = pFd$ as morphisms $X^i \to X^{i+1}$ for each i. Dieudonné complexes form a category $\mathbf{DC}$, with the obvious definition of morphisms.

The category $\mathbf{DC}$ has two important full subcategories

$$\mathbf{DC}_{\mathrm{str}} \hookrightarrow \mathbf{DC}_{\mathrm{sat}} \hookrightarrow \mathbf{DC},$$

called the categories of *strict* and *saturated* Dieudonné complexes respectively. Both inclusions have left adjoints, called the *saturation* $\mathrm{Sat} : \mathbf{DC} \to \mathbf{DC}_{\mathrm{sat}}$ and the *completion* $\mathcal{W} : \mathbf{DC}_{\mathrm{sat}} \to \mathbf{DC}_{\mathrm{str}}$. (Thus the composition $\mathcal{W}\,\mathrm{Sat} : \mathbf{DC} \to \mathbf{DC}_{\mathrm{str}}$ is left-adjoint to the inclusion $\mathbf{DC}_{\mathrm{str}} \to \mathbf{DC}$; we will call $\mathcal{W}\,\mathrm{Sat}$ the *strictification*.) Roughly speaking, a Dieudonné complex is saturated if it is p-torsionfree and F is a bijection onto its "expected image"; the saturation is a suitable colimit which forces this to be so. If X^* is saturated, it can then be endowed with a Verschiebung operator $V : X^i \to X^i$ for each i, satisfying $FV = VF = p$ among other identities. We then call a Dieudonné complex strict if it is saturated and is complete with respect to the filtration given by $(\mathrm{im}\, V^r + \mathrm{im}\, dV^r)_r$; the strictification functor is the completion.

An especially pleasant feature of Dieudonné complexes is that they provide a very clean dictionary between p-power torsion and torsion-free objects. Namely, suppose X^* is a saturated Dieudonné complex, and let $\mathcal{W}_r\, X^* = X^*/(\mathrm{im}\, V^r + \mathrm{im}\, dV^r)$. These objects form a pro-complex, where each $\mathcal{W}_r\, X^*$ is killed by p^r. The Frobenius and Verschiebung operators on X^* pass to the tower, giving it the structure of a *strict Dieudonné tower* ([3, Definition 2.6.1]). One then shows that the category **TD** of strict Dieudonné towers is equivalent to $\mathbf{DC}_{\mathrm{str}}$, where the forward equivalence is given by taking the limit and the inverse is $(\mathcal{W}_r(-))_r$.

Bhatt-Lurie-Mathew also introduces *Dieudonné algebras*, which are commutative algebra objects in **DC** satisfying a few extra hypotheses. Dieudonné algebras form a category **DA**, and the subcategory of Dieudonné algebras that are strict as Dieudonné complexes is called $\mathbf{DA}_{\mathrm{str}}$.

With this setup, it is quite simple to define the saturated de Rham-Witt complex:

Definition 1.2.2. ([3, Definition 4.1.1]) If R is an $\mathbb{F}_p$-algebra, a *saturated de Rham-Witt complex associated to* R is a strict Dieudonné algebra A^* equipped with a ring map

$$f : R \to \mathcal{W}_1 A^0 = A^0/VA^0$$

that is initial among such collections of data; that is, such that every ring map $R \to \mathcal{W}_1 B^0$ for a strict Dieudonné algebra B^* factors as $\mathcal{W}_1(g) \circ f$ for a unique morphism $g : A^* \to B^*$ in $\mathbf{DA}_{\mathrm{str}}$.

Assuming existence, this is clearly unique up to unique isomorphism, and the functor

$$\mathcal{W}\Omega^*_{(-)} : \mathbb{F}_p\text{–}\mathbf{alg} \to \mathbf{DA}_{\mathrm{str}}$$

is left-adjoint to the functor

$$\mathcal{W}_1(-)^0 = (-)^0/V(-)^0 : \mathbf{DA}_{\mathrm{str}} \to \mathbb{F}_p\text{–}\mathbf{alg}.$$

Bhatt-Lurie-Mathew provides two explicit constructions of $\mathcal{W}\Omega^*_R$: one completely general construction built from Witt vectors, and a second construction under the hypothesis that R admits a p-torsionfree lift with Frobenius. Both constructions begin with the following observation.

Proposition 1.2.3. *([3, Proposition 3.2.1]) Suppose A is a p-torsionfree ring equipped with a homomorphism $\phi : A \to A$ lifting the absolute Frobenius endomorphism of A/pA. Then there is a unique graded ring homomorphism $F : \Omega^*_A \to \Omega^*_A$ which extends the Frobenius $\phi : A \to A$ in degree 0 and satisfies the identity*

$$F(dx) = x^{p-1}dx + d\left(\frac{\phi(x) - x^p}{p}\right)$$

for all $x \in A$. Moreover, this F gives Ω^*_A the structure of a Dieudonné algebra.

Remark 1.2.4. One can easily verify that the endomorphism F of Proposition 1.2.3 satisfies the identity

$$p^n F(x \cdot dx_1 \wedge \cdots \wedge dx_n) = \phi(x) \cdot d\phi(x_1) \wedge \cdots \wedge d\phi(x_n);$$

that is, $p^n F$ agrees with the map ϕ^* which induced on Ω^n_A by the functoriality of ϕ. For this reason, we call F a *divided Frobenius*. Note however that we cannot simply define F as $\frac{\phi^*}{p^n}$ in degree n, as Ω^*_A may have p-torsion even when A does not.

The constructions are as follows:

Construction 1.2.5. ([3, Proposition 4.1.4]) Let R be an $\mathbb{F}_p$-algebra, R_{red} its reduction, and $W(R_{\mathrm{red}})$ the ring of p-typical Witt vectors. Then $W(R_{\mathrm{red}})$ is p-torsionfree and comes equipped with a Frobenius endomorphism F, so Proposition 1.2.3 gives the (naive) de Rham complex $\Omega^*_{W(R_{\mathrm{red}})} = \Omega^*_{W(R_{\mathrm{red}})/\mathbb{Z}}$ the structure of a Dieudonné algebra. Then the saturated de Rham-Witt complex functor is given by $R \mapsto \mathcal{W}\operatorname{Sat}(\Omega^*_{W(R_{\mathrm{red}})})$.

Construction 1.2.6. ([3, Variant 3.3.1 and Corollary 4.2.3]) Suppose A is a p-torsionfree ring equipped with a lift ϕ of the absolute Frobenius endomorphism of $R := A/pA$. Then the p-adically completed de Rham complex $\widehat{\Omega}^*_A$ has the structure of a Dieudonné algebra, and its strictification $\mathcal{W}\operatorname{Sat}(\widehat{\Omega}^*_A)$ is a saturated de Rham-Witt complex associated to R.

Remark 1.2.7. Implicit in Construction 1.2.5 is the fact that for any $\mathbb{F}_p$-algebra R, the natural morphism $\mathcal{W}\,\Omega^*_R \to \mathcal{W}\,\Omega^*_{R_{\mathrm{red}}}$ is an isomorphism. This follows from the definition of $\mathcal{W}\,\Omega^*_R$ in light of [3, Lemma 3.6.1] and does not require the construction.

Bhatt-Lurie-Mathew proves a number of results about the saturated de Rham-Witt complex, of which the following few are the most relevant to us.

Theorem 1.2.8. *([3, Corollary 4.4.11, Theorem 4.4.12]) For any $\mathbb{F}_p$-algebra R, we have canonical maps*[1]

$$\zeta_r : W_r\Omega^*_R \xrightarrow{\sim} \mathcal{W}_r\,\Omega^*_R \textit{ for all } r, \textit{ and}$$
$$\zeta : W\Omega^*_R \xrightarrow{\sim} \mathcal{W}\,\Omega^*_R,$$

where $W\Omega^*_R = \lim_{\leftarrow r} W_r\Omega^*_R$ is the classical de Rham-Witt complex of [12]. If R is a regular noetherian $\mathbb{F}_p$-algebra, then the maps ζ_r and ζ are isomorphisms.

Theorem 1.2.9. *([3, Theorem 5.3.7, Remark 5.2.3]) Let X be an arbitrary $\mathbb{F}_p$-scheme. The functors* $\operatorname{Spec} R \mapsto \mathcal{W}\,\Omega^*_R$ *and* $\operatorname{Spec} R \mapsto \mathcal{W}_r\,\Omega^*_R$ *define sheaves for the étale topology on X. Moreover, the latter is a quasicoherent sheaf of $W_r\mathcal{O}_X$-modules.*

We denote these sheaves $\mathcal{W}\,\Omega^*_X$ and $\mathcal{W}_r\,\Omega^*_X$, and call them the *de Rham-Witt complex* and the *de Rham-Witt pro-complex* of X respectively. Bhatt-Lurie-Mathew then shows that under reasonable hypotheses, the saturated de Rham-Witt complex of X agrees with the classical de Rham-Witt complex and computes crystalline cohomology:

[1] These maps are called γ_r and γ in the original; we have changed the notation to avoid any potential confusion later with divided power structures denoted γ.

Theorem 1.2.10. *([3, Corollary 4.4.11, Theorem 4.4.12]) Let R be an $\mathbb{F}_p$-algebra. Then there is a canonical morphism of differential graded algebras from the classical de Rham-Witt complex $W\Omega^*_R$ to the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$. If R is regular Noetherian, then this map is an isomorphism.*

Theorem 1.2.11. *([3, Theorem 10.1.1 and surrounding remarks]) Let k be a perfect field of characteristic p, and X a smooth k-scheme. Then there is a canonical isomorphism of cohomology rings*

$$H^*_{\mathrm{cris}}(X/W(k)) \simeq \mathbb{H}^*(X, \mathcal{W}\Omega^*_X).$$

Of course Theorem 1.2.11 follows from Theorems 1.2.10 and 1.1.1, but [3, §10] proves it directly as well. Ogus ([20, Corollary 5.4]) has given an alternative proof.

1.3 Summary of main results

Our goal in this paper is to generalize the approach of Bhatt-Lurie-Mathew in order to define and construct saturated de Rham-Witt complexes with coefficients in a unit-root F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$. (The necessary facts about F-crystals are collected in section 2.5.) As in [3], we work affine-locally; thus, suppose we are given a perfect field k of characteristic p, a k-algebra R, and a unit-root F-crystal $\mathcal{E}$ on $\mathrm{Cris}(\mathrm{Spec}\, R/W(k))$.

To begin, we must say what category our saturated de Rham-Witt complexes with coefficients live in. This is as follows (see Definition 4.1.3 for details):

Definition 1.3.1. A *de Rham-Witt module over* $(R, \mathcal{E})$ is a collection of the following data: a left $\mathcal{W}\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with $W_r(R)$-linear maps $\iota_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r M^0$ for each r, compatible with quotient and Frobenius maps, and compatible with connections in a suitable sense.

Within this category, our de Rham-Witt complexes are defined by a concise universal property:

Definition 1.3.2. A *saturated de Rham-Witt complex associated to* $\mathcal{E}$ *over* R is an initial object in the category of de Rham-Witt modules over $(R, \mathcal{E})$. Such an object is unique up to unique isomorphism if it exists; we will denote it by $\mathcal{W}\Omega^*_{R,\mathcal{E}}$.

Since we define our de Rham-Witt complexes by universal property, we had better prove that they exist:

Theorem 1.3.3. *(See Theorem 6.1.10.) Suppose R is a k-algebra, and $\mathcal{E}$ is a unit-root F-crystal on* $\mathrm{Cris}(\mathrm{Spec}\, R/W)$. *Then there exists a saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ associated to $(R, \mathcal{E})$.*

The proof is more complicated than in the case of the trivial crystal, since we have no analogue of Construction 1.2.5. Roughly speaking, we will prove an analogue of Construction 1.2.6 and then reduce to the case where its hypothesis is satisfied.

We also prove an analogue of Theorem 1.2.9:

Proposition 1.3.4. *(See Proposition 4.5.3.) Let X be a k-scheme, and $\mathcal{E}$ a unit-root F-crystal on* $\mathrm{Cris}(X/W(k))$. *The functors*

$$\begin{aligned}&\operatorname{Spec} R \mapsto \mathcal{W}\,\Omega^*_{R,\mathcal{E}} \text{ and}\\ &\operatorname{Spec} R \mapsto \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \text{ for } r > 0\end{aligned}$$

define sheaves for the étale topology on X. The latter is a quasicoherent sheaf of $W_r\mathcal{O}_X$-modules.

Let $\mathcal{W}\,\Omega^*_{X,\mathcal{E}}$ and $\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$ denote the sheaves produced by the proposition. Our next main result relates these sheaves to the cohomology of $\mathcal{E}$ provided that X/k is smooth:

Theorem 1.3.5. *(See Corollary 6.2.16.) If X is a smooth k-scheme and $(\mathcal{E}, \phi_{\mathcal{E}})$ is a unit-root F-crystal on* $\mathrm{Cris}(X/W(k))$*, then there are canonical isomorphisms*

$$\mathbb{H}^i(X_{\mathrm{zar}}, \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}) \simeq H^i((X/W_r)_{\mathrm{cris}}, \mathcal{E})$$

for each $r > 0$, and

$$\mathbb{H}^i(X_{\mathrm{zar}}, \mathcal{W}\,\Omega^*_{X,\mathcal{E}}) \simeq H^i((X/W)_{\mathrm{cris}}, \mathcal{E})$$

satisfying various compatibilities as described in Proposition 6.2.18.

Finally, again assuming smoothness, we compare our de Rham-Witt (pro-)complexes to the classical ones constructed in [8]. These have the form

$$\begin{aligned}W_r\Omega^*_{X,\mathcal{E}} &= \mathrm{dR}(\mathcal{E}_{W_r(X),\gamma}) \otimes_{\Omega^*_{W_r(\mathcal{O}_X),\gamma}} W_r\Omega^*_X \text{ and}\\ W\Omega^*_{X,\mathcal{E}} &= \lim_{\leftarrow r} W_r\Omega^*_{X,\mathcal{E}}.\end{aligned}$$

where $\mathrm{dR}(\mathcal{E}_{W_r(X),\gamma})$ is the PD-de Rham complex associated to $\mathcal{E}$ on $W_r(X)$ (cf. §2.7), and $(W_r\Omega^*_X)_r$ is the classical de Rham-Witt pro-complex of X. (As in the case of trivial coefficients, we use a calligraphic $\mathcal{W}$ for the saturated de Rham-Witt complex and a roman W for the classical one.)

Theorem 1.3.6. *(See Theorem 6.3.6.) Suppose X is a k-scheme and $(\mathcal{E}, \phi_{\mathcal{E}})$ is a unit-root F-crystal on* $\mathrm{Cris}(X/W(k))$*. Then we have compatible maps*

$$\begin{aligned}&W_r\Omega^*_{X,\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \text{ for each } r, \text{ and}\\ &W\Omega^*_{X,\mathcal{E}} \to \mathcal{W}\,\Omega^*_{X,\mathcal{E}},\end{aligned}$$

which are isomorphisms if X/k is smooth.

1.4 New ingredients

The central idea introduced in this work is the category $\mathrm{dRWM}_{R,\mathcal{E}}$ of de Rham-Witt modules (Definition 4.1.3). Whereas the classical treatment of de Rham-Witt complexes with coefficients, due to Étesse ([8]), is more constructive in nature, de Rham-Witt modules provide a clean setting in which to define our saturated de Rham-Witt complexes $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ by universal property: namely, $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ is defined to be an initial object of $\mathrm{dRWM}_{R,\mathcal{E}}$. This is in the spirit of Illusie's original definition ([12, I, §1])as well as that of Bhatt-Lurie-Mathew ([3, §4]): we build a category of objects that carry all the structure we want, and we ask for an initial object in this category. The actual construction is done only later, first in the special case where R admits a p-torsionfree lift with Frobenius, and later in general by reducing to this case.

A significant—and necessary—difference between our universal property and its predecessors is that while de Rham-Witt complexes with trivial coefficients carry an algebra structure (corresponding to the algebra structure of the trivial F-crystal $\mathcal{O}_{X/W}$), ours instead carries the structure of a module over $\mathcal{W}\Omega^*_R$ (corresponding to the structure of the coefficient crystal $\mathcal{E}$ as an $\mathcal{O}_{X/W}$-module). Note that this has some content even when $\mathcal{E}$ *is* the trivial crystal: $\mathcal{W}\Omega^*_R$ satisfies a universal property not only as an algebra but also as a module over itself.

Accordingly, we must make systematic use of modules in the categories $\mathbf{DC}$ and $\mathbf{DC}_{\mathrm{str}}$. We develop this theory in chapter 3. Although the symmetric monoidal structures of these categories appear already in [3], we study them in greater detail, including concrete descriptions of the resulting algebra and module objects (§3.2), an account of tensor products over a base algebra (§3.5), a discussion of the structure carried by $\mathcal{W}_r M^*$ when M^* is a module in $\mathbf{DC}_{\mathrm{str}}$ (§3.6), and a computation of the strict Dieudonné tower associated to a filtered colimit of strict Dieudonné complexes (§3.7). We also introduce an internal Hom functor on $\mathbf{DC}$ (§3.4), inspired by [7, I, §5], which may be of independent interest.

1.5 Outline

We begin in chapter 2 with some background material on Witt vectors, de Rham complexes, and crystals, which will be used throughout. In chapter 3, we develop the theory of modules in the categories $\mathbf{DC}$ and $\mathbf{DC}_{\mathrm{str}}$, as discussed above.

In chapter 4, we define the category $\mathrm{dRWM}_{R,\mathcal{E}}$ of *de Rham-Witt modules* associated to a unit-root F-crystal $\mathcal{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ on $\operatorname{Spec} R$, which houses the universal property characterizing our saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$. At this stage, we will not be able to construct $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ except in the case of the trivial crystal $\mathcal{E} = \mathcal{O}_{X/W}$ (§4.2). However, we will prove several useful formal properties about the behavior of $\mathrm{dRWM}_{R,\mathcal{E}}$ and $\mathcal{W}\Omega^*_{R,\mathcal{E}}$: functoriality (§4.3), insensitivity to nilpotent thickenings (§4.4), an étale sheaf property (§4.5), and compatibility with colimits (§4.6). Although we work on an affine k-scheme $\operatorname{Spec} R$ throughout, the étale sheaf property allows us to define a saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{X,\mathcal{E}}$ when X is not necessarily affine.

In chapter 5, we introduce our main technique for constructing $\mathcal{W}\Omega^*_{R,\mathcal{E}}$. Namely, assuming that R admits a p-torsionfree lift A with Frobenius, we introduce the category $\mathrm{dRWLM}_{A,\mathcal{E}}$ of *de Rham-Witt lift modules*. In this case, we show that the categories $\mathrm{dRWM}_{R,\mathcal{E}}$ and $\mathrm{dRWLM}_{A,\mathcal{E}}$ are equivalent. By studying the latter category, we are able to construct the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ in the Frobenius-lifted situation.

We prove our main theorems in chapter 6. Namely, in §6.1, we show that $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ always exists by reducing to the Frobenius-lifted situation. If X/k is smooth, we show in §6.2 that $\mathcal{W}\Omega^*_{X,\mathcal{E}}$ computes the cohomology of $\mathcal{E}$, and in §6.3 we compare it to the classical de Rham-Witt complex $W\Omega^*_{X,\mathcal{E}}$.

1.6 Future directions

A significant technical nuisance in this work is our lack of a rich theory of Dieudonné complexes valued in sheaves, including the tensor algebra discussed in chapter 3. Given such a theory, we could define our saturated de Rham-Witt complexes by a universal property within a category of *sheafy de Rham-Witt modules*, rather than working affine-locally and bootstrapping up to the non-affine case.

This would not only be a conceptual simplification; it would also allow for a simple alternate proof of Theorem 1.3.6. Namely, suppose X/k is smooth and $(\mathcal{E}, \phi_{\mathcal{E}})$ is a unit-root F-crystal on X, expressed as $\mathcal{L} \otimes \mathcal{O}_{X/W(k)}$ for an étale $\mathbb{Z}_p$-local system $\mathcal{L} = (\mathcal{L}_r)_r$ on X as in Construction 2.5.6. Theorem 1.3.6 would follow from an isomorphism of pro-complexes

$$(\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}})_r \simeq (\mathcal{W}_r\,\Omega^*_X \otimes_{\mathbb{Z}/p^r\mathbb{Z}} \mathcal{L}_r)_r. \tag{1.6.0.1}$$

Unfortunately, we cannot prove 1.6.0.1 affine-locally: our definition of $\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$ relies a priori on the entire tower $(\mathcal{L}_r)_r$, and there is generally no Zariski or étale cover of X that trivializes the entire tower simultaneously. Thus, we would like to embrace the idea of working with sheaves, and view both sides of the isomorphism not as sheaves valued in strict Dieudonné towers but as *strict Dieudonné towers valued in the category of abelian sheaves*.

The notion of a strict Dieudonné tower (or a Dieudonné complex) valued in sheaves does not appear in [3], as all of their constructions are affine-local. In joint work in progress with Joe Stahl ([9]), we develop a theory of Dieudonné complexes in a general complete and cocomplete abelian category $\mathcal{A}$. This theory includes categories $\mathbf{DC}_{\mathcal{A}}$, $\mathbf{DC}_{\mathrm{sat},\mathcal{A}}$, $\mathbf{DC}_{\mathrm{str},\mathcal{A}}$, and $\mathbf{TD}_{\mathcal{A}}$ of (saturated, strict) Dieudonné complexes and strict Dieudonné towers valued in $\mathcal{A}$, which recover the usual categories when $\mathcal{A} = \mathbf{Ab}$.

Generalizing the theory to this context introduces several new difficulties, both internal to the theory of Dieudonné complexes in $\mathcal{A}$ and (when $\mathcal{A}$ is the category of abelian sheaves on a site X) involving comparisons between sheafy and non-sheafy constructions. For example, we generally do not have an equivalence of categories

$$\mathbf{DC}_{\mathrm{str},X} \simeq \mathbf{TD}_X$$

between strict Dieudonné complexes and Dieudonné towers of sheaves on X (cf. [3, Proposition 2.9.1, Corollary 2.9.4]); we have functors in both directions, but the composition

$$\mathbf{TD}_X \to \mathbf{DC}_{\mathrm{str},X} \to \mathbf{TD}_X$$

is generally not isomorphic to the identity. Similarly, given M^* in $\mathbf{DC}_{\mathrm{sat},X}$, we do not know whether the "strictification" $\mathcal{W}(M^*)$ is strict (cf. [3, Proposition 2.7.5, Corollary 2.7.6]). Both of these issues can be resolved by appealing to special properties of the objects we are interested in, namely the quasicoherence property of Proposition 1.3.4.

A further challenge is to construct a symmetric monoidal structure on either $\mathbf{DC}_{\mathrm{str},X}$ or $\mathbf{TD}_X$ generalizing the strict tensor product

$$\otimes^{\mathrm{str}} : \mathbf{DC}_{\mathrm{str}} \times \mathbf{DC}_{\mathrm{str}} \to \mathbf{DC}_{\mathrm{str}}$$

of [3, Remark 7.6.4]. This is a necessary ingredient for building a category of sheafy de Rham-Witt modules; such a module should in particular be a module over $\mathcal{W}\Omega^*_X$ in $\mathbf{DC}_{\mathrm{str},X}$. It is not difficult to define a tensor product on $\mathbf{DC}_{\mathcal{A}}$ given one on $\mathcal{A}$; however, Bhatt-Lurie-Mathew's construction of $\otimes^{\mathrm{str}}$ seems to rely crucially on the fact that $\mathbb{Z}$ has cohomological dimension 1, and it does not readily generalize to the case of sheaves.

Another direction for future work is to give an account, in the spirit of [3], of de Rham-Witt complexes with coefficients in an F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$ that is not unit-root. These objects cannot be strict Dieudonné complexes, because the map

$$\alpha_F : \mathcal{W}\Omega^*_{R,\mathcal{E}} \to \eta_p \mathcal{W}\Omega^*_{R,\mathcal{E}}$$

given by $p^i F$ in degree i will not be an isomorphism. Instead, we would need a modification of the category $\mathbf{DC}_{\mathrm{str}}$ which carries some extra data allowing the Frobenius endomorphism to have smaller-than-expected image in a controlled way, in the spirit of Fontaine-Jannsen's theory of φ-gauges ([10])—but which nonetheless has a universal construction analogous to strictification.

1.7 Notation

1.7.1. Unless otherwise specified, all rings are commutative with 1. We fix a prime number p throughout. We will always let k denote a perfect field of characteristic p, which we will usually treat as fixed but will occasionally need to vary. We let $W = W(k)$ be its ring of Witt vectors, and $W_r = W_r(k) = W/p^r W$ for each $r > 0$. We will always[2] let R denote a ring of characteristic p, which will usually be a k-algebra and will sometimes have additional restrictions. We will usually let A denote a p-torsionfree lift of R, although the exact hypotheses we impose will vary.

1.7.2. We will generally designate complexes (dg-algebras, dg-modules, Dieudonné complexes, etc.) with asterisks: A^*, M^*, $\mathcal{W}\Omega^*_R$, and so on. But we will omit this when we abbreviate a de Rham complex as $\mathrm{dR}(-)$, as in Definitions 2.6.4 and 2.7.6.

1.7.3. Several kinds of Frobenius operators will make an appearance, and we will do our best to notate them all unambiguously with the limited set of suggestive names at our disposal. The absolute Frobenius endomorphism of an $\mathbb{F}_p$-algebra R will be denoted F_R or Frob_R. If A is a lift of R, then we will generally denote a lift of F_R to A as ϕ or ϕ_A. However, the Frobenius endomorphism on the ring of Witt vectors $W(R)$ will be written as F, or as σ if $R = k$ is a perfect field of characteristic p. If A is a ring with Frobenius lift ϕ, then the induced endomorphism of its de Rham complex Ω^*_A will be called ϕ or ϕ^*; in this situation, we will reserve the name F for a divided Frobenius. (This is consistent with the convention of denoting Dieudonné complexes as (M^*, d, F); cf. [3, Remark 2.1.4].) Finally, we will write F-crystals as $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a crystal and $\phi_{\mathcal{E}}$ is a Frobenius endomorphism; we will often suppress the Frobenius $\phi_{\mathcal{E}}$ from the notation. See Example 2.5.5 for more notation related to F-crystals.

1.7.4. We will also encounter a number of PD-structures (divided power structures; cf. [2, §3]). Except in the proof of Proposition 2.2.8, we will always reserve the notation $[\,]$ for PD-structures on the ideal (p) (of various rings) given by Lemma 1.7.7; other PD-structures will be called γ or δ.

1.7.5. We will borrow a number of notational conventions from [3]. For example, if M^* is a Dieudonné complex, we will let $\rho = \rho_M : M^* \to \mathcal{W}\,\mathrm{Sat}(M^*)$ denote its strictification map; that is, the initial morphism from M^* to a strict Dieudonné complex. Similarly, α_F will denote the map of complexes $M^* \to \eta_p M^*$ (or $M^* \to M^*$ if M^* is concentrated in nonnegative degrees) which is defined by $p^i F$ in degree i.

For future reference, we collect here some setup and related observations which we will frequently invoke throughout the paper.

Situation 1.7.6.

(a) (*The lifted situation*) Let R be a k-algebra and A a p-torsion-free ring with $A/pA = R$. We set $A_r = A/p^r A$ for each $r > 0$, and $\widehat{A} = \lim_r A_r$.

[2]Occasionally we will also use the letter R for certain quotient maps in towers, in accordance with [3, Definition 2.6.1]. Note also that the unsaturated Raynaud ring R^* of 3.1.1 is neither commutative nor an $\mathbb{F}_p$-algebra.

(b) (*The Frobenius-lifted situation*) In addition to the above, let $\phi : A \to A$ be a lift of the Frobenius endomorphism of R; that is, a ring homomorphism such that $\phi(x) \equiv x^p \pmod p$ for all $x \in A$. Note that ϕ induces endomorphisms of each A_r.

Lemma 1.7.7. *In the lifted situation 1.7.6, there is a unique PD-structure* $[\,]$ *on* $(p) \subset A_r$ *that is compatible with the PD-structure on* $(p) \subset \mathbb{Z}_p$ *via the map* $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z} \to A_r$. *Moreover, as* r *varies, these are compatible via the quotient maps* $A_r \to A_{r-1}$.

Proof. This follows from [2, Proposition 3.15] along with the notes preceding it. For future reference, we record the formula:

$$\begin{aligned} px^{[n]} &= x^n p^{[n]} \\ &= \frac{p^n}{n!} x^n, \end{aligned}$$

where the rational number $p^n/n!$ belongs to $p\mathbb{Z}_{(p)}$ and can be viewed as an element of A_r via $\mathbb{Z}_{(p)} \to \mathbb{Z}/p^r\mathbb{Z} \to A_r$. □

Situation 1.7.8.

(a) (*The embedded situation*) Let R be a k-algebra and A a smooth algebra over $W = W(k)$ equipped with a quotient map $A \twoheadrightarrow R$ of W-algebras, with kernel $I \subset A$ (necessarily containing p). For each $r > 0$, we set $A_r = A/p^rA$ and $I_r = I/p^rA$, and let $(B_r, \overline{I}_r, \gamma)$ denote the PD-envelope of (A_r, I_r) over $(W_r(k), (p), [\,])$. Let $(B, \overline{I}, \gamma)$ be the inverse limit of the PD-algebras $(B_r, \overline{I}_r, \gamma)$. Note that we have $B_r = B_{r+1}/p^rB_{r+1}$ and $\overline{I}_r = \overline{I}_{r+1}/p^rB_{r+1}$ by [2, Remark 3.20.8]; it follows by [21, tag `09B8`] that $B_r = B/p^rB$ and thus also $\overline{I}_r = \overline{I}/p^rB$.

(b) (*The Frobenius-embedded situation*) In addition to the above, let $\phi = \phi_A : A \to A$ be a lift of the Frobenius endomorphism of A_1. Note that ϕ is then also compatible with the Frobenius endomorphism of R, and thus preserves the ideal I. It follows that ϕ induces PD-endomorphisms $\phi_{B_r} : (B_r, \overline{I}_r, \gamma) \to (B_r, \overline{I}_r, \gamma)$ for each r by the functoriality of the PD-envelope, and thus also $\phi_B : (B, \overline{I}, \gamma) \to (B, \overline{I}, \gamma)$ by passing to the limit.

Remark 1.7.9. In the Frobenius-embedded situation, B may or may not have p-torsion, but the endomorphism ϕ_B is a lift of the absolute Frobenius of $B_1 = B/pB$. To prove this, note that the set of $b \in B_1$ such that $\phi_{B_1}(b) = b^p$ is a subring of B_1 containing A_1, so it suffices to check that it also contains the algebra generators $\gamma_n(a)$ for all $a \in I_1$ and $n > 0$. But recalling that $a^p = p\gamma_p(a) = 0$ and similarly $\gamma_n(a)^p = 0$, we can calculate:

$$\begin{aligned} \phi_{B_1}(\gamma_n(a)) &= \gamma_n(\phi_{A_1}(a)) \\ &= \gamma_n(a^p) \\ &= \gamma_n(0) \\ &= 0 \\ &= \gamma_n(a)^p. \end{aligned}$$

Chapter 2

Background on Witt vectors, de Rham complexes, and crystals

2.1 Witt vectors

2.1.1. In this section, we will recall some background information on Witt vectors, which for us will always mean p-typical Witt vectors. Although we will only be interested in the Witt vectors of $\mathbb{F}_p$-algebras, we briefly postpone specializing to this situation. We refer to [14, 17, 11], and [12, 0, §1] for proofs and further details.

Definition 2.1.2. A δ-ring (A, δ) is a ring A equipped with a set-theoretic map $\delta : A \to A$ satisfying the identities

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}, \\ \delta(xy) &= x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y), \text{ and} \\ \delta(1) &= 0, \end{aligned}$$

for all x and y in A. We let δ-**Ring** denote the category of δ-rings.

Remark 2.1.3. ([14]) Definition 2.1.2 is motivated by the following observation: if (A, δ) is a δ-ring, then δ determines a Frobenius lift by the formula

$$\phi(x) = x^p + p\delta(x)$$

If A is moreover p-torsionfree, then we can recover $\delta(x)$ as $\frac{\phi(x)-x^p}{p}$, and this defines a bijection between δ-ring structures and Frobenius lifts on A. Thus we may regard the category of δ-rings as a substitute for the category of rings equipped with Frobenius lift which behaves better in the presence of p-torsion.

2.1.4. ([14, Théorème 2]), [17, Definition 2.4.5]) The forgetful functor

$$\delta\text{-}\mathbf{Ring} \to \mathbf{Ring}$$

admits both left and right adjoints. Its right adjoint is the Witt vector functor

$$W : \mathbf{Ring} \to \delta\text{-}\mathbf{Ring},$$

which (as Joyal first observed) we may take as a *definition* of the Witt vectors. The left adjoint of the forgetful functor is constructed as a suitable quotient of a free δ-ring; we will not discuss this further.

Remark 2.1.5. ([17, Definitions 3.1.1, 3.1.3, 3.2.1]) In order to perform calculations in rings of Witt vectors, one generally relies on various "standard" coordinate systems for Witt vectors; that is, functorial maps of sets (not compatible with addition or multiplication!)

$$W(A) \to A^{\mathbb{N}}.$$

Among these are the Joyal coordinates $x \mapsto (y_n)$, the Witt coordinates $x \mapsto (x_n)$, and the ghost coordinates $x \mapsto (w_n)$; the Joyal and Witt coordinate maps are bijective but the ghost coordinate map is generally not. When coordinates are necessary, we will exclusively use Witt coordinates.

Remark 2.1.6. If R is a ring, then the δ-ring $W(R)$ carries a ring homomorphism $F : W(R) \to W(R)$ defined as in Remark 2.1.3; we call this the Witt vector Frobenius. It also carries a *Verschiebung* map $V : W(R) \to W(R)$ defined on Witt coordinates by

$$V(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

The Verschiebung map is additive and satisfies the identities

$$FV(x) = px,$$
$$V(x \cdot Fy) = V(x) \cdot y.$$

It follows from this that $\operatorname{im} V^r$ forms an ideal of $W(R)$ for each $r > 0$. We let $W_r(R)$ denote the quotient $W(R)/\operatorname{im} V^r$; the projection $W(R) \to W_r(R)$ can be identified with the projection onto the first r Witt coordinates.

2.1.7. From now on, we will only consider Witt vectors of $\mathbb{F}_p$-algebras. We will spend the remainder of this section collecting some more or less standard results on Witt vectors in this situation which we will use throughout the paper.

Lemma 2.1.8. *Suppose R is an $\mathbb{F}_p$-algebra. If $x \in W(R)$ has Witt coordinates $(x_0, x_1, \dots)$, then we have:*

$$F(x) = (x_0^p, x_1^p, \dots) \text{ and}$$
$$px = (0, x_0^p, x_1^p, \dots) = VF(x).$$

In particular, F induces a map $W_r(R) \to W_r(R)$ for each r, and $W(R)$ is p-torsionfree if and only if R is reduced.

Proof. The first statement is [17, Corollary 3.1.5]; the second follows by expanding $px = FV(x)$ and comparing it to $VF(x)$. (We warn the reader that all of these identities require the hypothesis that R is an $\mathbb{F}_p$-algebra.) $\square$

Example 2.1.9. ([11, Theorem 6.19]) If k is a perfect $\mathbb{F}_p$-algebra, then $W(k)$ is the unique p-complete p-torsionfree ring A with $A/pA = k$. In this case, the Witt vector Frobenius is an automorphism, which we will denote as $\sigma : W(k) \to W(k)$; it reduces modulo the ideal $pW(k) = VW(k)$ to the Frobenius automorphism of k.

Lemma 2.1.10. *If R is any $\mathbb{F}_p$-algebra with reduction R_{red}, then the natural map $W(R) \to W(R_{\mathrm{red}})$ is surjective, and its kernel equals the closure of the p-power torsion in $W(R)$ with respect to the standard (V-adic) topology.*

Proof. Since the Witt coordinate maps are functorial, $W(R) \to W(R_{\mathrm{red}})$ is surjective and its kernel consists of all elements whose Witt coordinates are nilpotent. We claim that this equals the closure of the p-power torsion. Recall that if $x = (x_0, x_1, \dots)$ in Witt coordinates, then we have $px = FV(x) = (0, x_0^p, x_1^p, \dots)$. It follows that the p^n-torsion elements of $W(R)$ are precisely the elements whose Witt coordinates x_i all satisfy $x_i^{p^n} = 0$. So the p-power torsion consists of elements with uniformly nilpotent Witt coordinates, and its closure consists of all elements with nilpotent Witt coordinates. $\square$

Construction 2.1.11. ([12, pp. 0, 1.4.3]) If R is an $\mathbb{F}_p$-algebra and $r > 0$, the ideals $VW(R) \subset W(R)$ and $VW_{r-1}(R) \subset W_r(R)$ carry canonical divided power structures, both given by the formula

$$\gamma_n(Vx) = \frac{p^{n-1}}{n!} V(x^n)$$

for $n > 0$. The canonical maps $W(R) \to W_r(R) \to W_{r'}(R)$ (where $r \geq r'$) are PD-morphisms, as are $W(f)$ and $W_r(f)$ whenever $f : R \to R'$ is a morphism of $\mathbb{F}_p$-algebras. Note in particular that taking $x = Fy$ in the formula above gives

$$\gamma_n(py) = \frac{p^n}{n!} y^n.$$

Lemma 2.1.12. *In the Frobenius-lifted situation 1.7.6:*

1. *There exists a unique δ-ring map $h : A \to W(R)$ that is compatible with the quotient maps to R. (Here the δ-ring structure of A is induced by its Frobenius lift ϕ as in Remark 2.1.3.)*
2. *The induced maps $h_r : A_r \to W_r(R)$ are PD-morphisms (with respect to the PD-structures of Lemma 1.7.7 and Construction 2.1.11).*
3. *The map h intertwines the Frobenius endomorphisms of A and $W(R)$, and similarly the h_r intertwine the Frobenius endomorphisms of the towers $(A_r)_r$ and $(W_r(R))_r$.*

Proof. Part (1) is immediate from 2.1.4. Part (3) follows from this, as the Frobenius lifts are determined by the δ-ring structures.

For part (2), first note that h_r maps the ideal $(p) \subset A_r$ to $(p) \subset VW_{r-1}(R) \subset W_r(R)$. Then letting a bar denote the passage from A to A_r or from $W(R)$ to $W_r(R)$, we have for all $x \in A$ that

$$\begin{aligned} h_r((p\overline{x})^{[n]}) &= h_r\left(\frac{p^n}{n!}\overline{x}^n\right) \\ &= \frac{p^n}{n!} \cdot h_r(\overline{x})^n \\ &= \gamma_n(ph_r(\overline{x})) \\ &= \gamma_n(h_r(p\overline{x})). \end{aligned}$$

□

Finally, we recall a version of the topological invariance of the étale site, as well as how to make it explicit for the case of truncated Witt vectors:

Proposition 2.1.13.

1. *If $X \to Y$ is a morphism of schemes which induces an isomorphism $X_{\mathrm{red}} \to Y_{\mathrm{red}}$, then base change defines an equivalence of categories*

$$\{\textit{étale } Y - \textit{schemes}\} \to \{\textit{étale } X - \textit{schemes}\}.$$

   Moreover, if X and Y are affine, then this equivalence identifies the categories of affine *schemes étale over X and Y.*

2. *Suppose $X = \operatorname{Spec} R$ is an affine $\mathbb{F}_p$-scheme, equipped with its natural closed embedding into $Y = \operatorname{Spec} W_r(R)$ for some $r > 0$. Then the inverse of the equivalence of part (1) sends each affine étale X-scheme* $\operatorname{Spec} S$ *to the affine étale $W_r(R)$-scheme* $\operatorname{Spec} W_r(S)$.

Proof. Part (1) follows from [21, tags `04DZ`, `054M`, 04E0]. Given this, part (2) amounts to the statement that $W_r(R) \to W_r(S)$ is étale, which is [12, 0, Proposition 1.5.8]. □

2.2 de Rham and PD-de Rham complexes of rings

2.2.1. In this section we summarize the various types of de Rham complexes which we will need, as well as the many relationships among them. We refer the reader to [2, §3] and [1, p. I] for background on PD-structures.

Definition 2.2.2. Let $B \to A$ be a morphism of rings. The *de Rham complex of A over B* is the initial object among commutative differential graded algebras (C^*, d) equipped with a morphism $A \to C^0$ such that the differential d is B-linear. It can be constructed as the exterior algebra of the module of Kähler differentials,

$$\Omega^*_{A/B} := \bigwedge\nolimits^*_A \Omega^1_{A/B},$$

equipped with the exterior derivative

$$d(a \cdot da_1 \wedge \cdots \wedge da_n) = da \wedge da_1 \wedge \cdots \wedge da_n.$$

Definition 2.2.3. Let $B \to A$ be a morphism of rings, and suppose (A, I, γ) is a PD-algebra. The *PD-de Rham complex of A/B* is the initial object among B-linear commutative differential graded algebras C^* which are equipped with a morphism $A \to C^0$ and satisfy the identity

$$d\gamma_n(x) = \gamma_{n-1}(x)dx$$

for all $n > 0$ and $x \in I$. It can be constructed as the quotient of the usual de Rham complex $\Omega^*_{A/B}$ by the dg-ideal generated by the elements $d\gamma_n(x) - \gamma_{n-1}(x)dx$:

$$\Omega^*_{A/B,\gamma} := \Omega^*_{A/B}/\langle d\gamma_n(x) - \gamma_{n-1}(x)dx\rangle.$$

Remark 2.2.4. Note that for $n > 1$, we have

$$\begin{aligned} d\left(d\gamma_n(x) - \gamma_{n-1}(x)dx\right) &= 0 - d\gamma_{n-1}(x) \wedge dx \\ &= -d\gamma_{n-1}(x) \wedge dx + \gamma_{n-2}(x)dx \wedge dx \\ &= dx \wedge \left(d\gamma_{n-1}(x) - \gamma_{n-2}(x)dx\right), \end{aligned}$$

which lies in the graded ideal generated by elements of the form $d\gamma_n(x) - \gamma_{n-1}(x)dx$. (When $n = 1$, the identity $d\gamma_n(x)\gamma_{n-1}(x)dx$ holds already in $\Omega^1_{A/B}$.) It follows that this graded ideal is closed under the differential, and thus it agrees with the dg-ideal generated by the same elements.

We first observe that in our main cases of interest, all of the "obvious" choices of base ring are interchangeable:

Lemma 2.2.5. *Let k be a perfect $\mathbb{F}_p$-algebra, and A a $W_r(k)$-algebra for some $r > 0$. Then the de Rham complexes of A over the base rings $\mathbb{Z}, \mathbb{Z}/p^r\mathbb{Z}, \mathbb{Z}_p, W(k)$, and $W_r(k)$ all agree. Moreover, if A comes equipped with a PD-structure (I, γ), then the same is true for PD-de Rham complexes.*

Proof. For any base ring B, the de Rham complex $\Omega^*_{A/B}$ is the quotient of the absolute de Rham complex $\Omega^*_{A/\mathbb{Z}}$ by the dg-ideal generated by the elements db, where $b \in B$. It follows that $\Omega^*_{A/B}$ depends only on A and the image of the map $B \to A$. In particular, the first three and last two of the specified de Rham complexes agree.

To compare the de Rham complex over $W_r(k)$ to the absolute de Rham complex, we must show that for all $b \in W_r(k)$, we have $db = 0$ already in $\Omega^*_{A/\mathbb{Z}}$. To see this, we write b in Witt coordinates as $(b_0, \ldots, b_{r-1})$, and recall ([17, Definition 3.17]) that

$$b = \sum_{i=0}^{r-1} p^i [b_i],$$

where $[b_i]$ is the multiplicative lift. Then we write each $[b_i]$ as $[b_i^{1/p^r}]^{p^r}$ by multiplicativity, where b_i^{1/p^r} is the p^r-th root in k. This gives

$$d[b_i] = p^r [b_i^{1/p^r}]^{p^r-1} d[b_i^{1/p^r}] = 0,$$

since $W_r(k)$ is killed by p^r. So we have $db = 0$.

Finally, the claim about PD-de Rham complexes follows from the claim about ordinary de Rham complexes, as the PD-compatibility condition does not involve the base ring. □

Remark 2.2.6. Given a k-algebra R, we will mostly be interested in the de Rham and PD-de Rham complexes of the following $W_r(k)$-algebras:

1. the lifts A_r of Situation 1.7.6, with the PD-structure [ ] of Lemma 1.7.7;
2. the embeddings A_r of Situation 1.7.8;
3. the PD-envelopes B_r of Situation 1.7.8, with the PD-structure γ; and
4. the truncated Witt vectors $W_r(R)$, with the PD-structure γ of Construction 2.1.11.

We will sometimes abbreviate e.g. $\Omega^*_{B_r/W_r,\gamma}$ as $\Omega^*_{B_r,\gamma}$; there is no harm in doing so, since the (PD-)de Rham complexes over W_r agree with the absolute ones (over $\mathbb{Z}$) in view of Lemma 2.2.5.

Next we note how de Rham and PD-de Rham complexes behave under surjections of rings:

Lemma 2.2.7.

1. *Let $A \to B$ be a surjection of rings. Then the natural map $\Omega^*_A \to \Omega^*_B$ is the quotient by the dg-ideal K generated by* $\ker(A \to B)$ *in degree 0.*
2. *Let $(A, I, \gamma) \to (B, J, \delta)$ be a surjection of PD-algebras with $J = \operatorname{im}(I)$. Then the natural map $\Omega^*_{A,\gamma} \to \Omega^*_{B,\delta}$ is the quotient by the dg-ideal K generated by* $\ker(A \to B)$ *in degree 0.*

Proof. We compare universal properties. Let C^* be an arbitrary commutative differential graded algebra. We have natural bijections:

$$\begin{aligned}\mathrm{Hom}_{cdga}(\Omega^*_A/K, C^*) &\simeq \{f \in \mathrm{Hom}_{cdga}(\Omega^*_A, C^*) : f(K) = 0\}\\ &\simeq \{g \in \mathrm{Hom}(A, C^0) : g(\ker(A \to B)) = 0\}\\ &\simeq \mathrm{Hom}(B, C^0)\\ &\simeq \mathrm{Hom}_{cdga}(\Omega^*_B, C^*)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{cdga}(\Omega^*_{A,\gamma}/K, C^*) &\simeq \{f \in \mathrm{Hom}_{cdga}(\Omega^*_{A,\gamma}, C^*) : f(K) = 0\}\\ &\simeq \{g \in \mathrm{Hom}(A, C^0) : dg(\gamma_n(x)) = g(\gamma_{n-1}(x))dg(x),\\ &\qquad g(\ker(A \to B)) = 0\}\\ &\simeq \{h \in \mathrm{Hom}(B, C^0) : dh(\delta_n(y)) = h(\delta_{n-1}(y))dh(y)\}\\ &\simeq \mathrm{Hom}_{cdga}(\Omega^*_{B,\delta}, C^*),\end{aligned}$$

where x ranges over $I \subset A$ and y ranges over $J \subset B$. So the Yoneda lemma implies that we have $\Omega^*_A/K \xrightarrow{\sim} \Omega^*_B$ and $\Omega^*_{A,\gamma}/K \xrightarrow{\sim} \Omega^*_{B,\delta}$. □

In Definition 2.2.3, it will sometimes be useful to impose the divided power compatibility condition only for p-th divided powers, and to quotient by the ideal generated by the given relations instead of its closure under the differential. This is sufficient in the case of $\mathbb{Z}_{(p)}$-algebras:

Proposition 2.2.8. *Suppose $B \to A$ is a morphism of rings, and (A, I, γ) is a PD-algebra. Suppose A is moreover a $\mathbb{Z}_{(p)}$-algebra (for example, a $\mathbb{Z}/p^r\mathbb{Z}$-algebra for some r, or more generally a $\mathbb{Z}_p$-algebra). Then the dg-ideal of $\Omega^*_{A/B}$ generated by the elements $d\gamma_n(x) - \gamma_{n-1}(x)dx$ with $x \in I$ and $n \geq 1$ is in fact generated as a graded ideal by the elements of this form with $n = p$.*

Proof. For notational convenience, we will write $x^{[n]}$ in place of $\gamma_n(x)$. First of all, note that we have

$$\begin{aligned}d(dx^{[p]} - x^{[p-1]}dx) &= 0 - dx^{[p-1]} \wedge dx\\ &= -d\left(\frac{x^{p-1}}{(p-1)!}\right) \wedge dx\\ &= -\frac{x^{p-2}}{(p-2)!}dx \wedge dx = 0\end{aligned}$$

since $(p-1)!$ is invertible in A. It follows that the ideal generated by these elements is already a dg-ideal. Let J denote this ideal. We will show by strong induction on m that all elements of the form $dx^{[m]} - x^{[m-1]}dx$ lie in J. This is of course tautological for $m = p$, and it holds for $m < p$ because $dx^{[m]} - x^{[m-1]}dx$ is killed by the unit $m!$ and is therefore zero. Now suppose we are given an integer $n > p$ such that the elements $dx^{[m]} - x^{[m-1]}dx$ with $m < n$ all lie in J. Let $x \in I$. We claim that $dx^{[n]} - x^{[n-1]}dx \in J$.

First consider the case where $p \nmid n$. Then $n = pa + b$, where a and b are both positive. The PD-structure axioms imply

$$x^{[n]} = \frac{x^{[pa]}x^{[b]}}{\binom{n}{b}},$$

where the denominator is prime to p. So applying the Leibniz rule and the inductive hypothesis gives:

$$\begin{aligned}
dx^{[n]} &= \binom{n}{b}^{-1} \left(x^{[pa]}dx^{[b]} + x^{[b]}dx^{[pa]}\right) \\
&\equiv \binom{n}{b}^{-1} \left(x^{[pa]}x^{[b-1]}dx + x^{[b]}x^{[pa-1]}dx\right) \\
&= \binom{n}{b}^{-1} \left(\binom{n-1}{b-1}x^{[n-1]} + \binom{n-1}{b}x^{[n-1]}\right) dx \\
&= \frac{\binom{n-1}{b-1} + \binom{n-1}{b}}{\binom{n}{b}} x^{[n-1]}dx \\
&= x^{[n-1]}dx,
\end{aligned}$$

where the congruence is modulo J. This proves the first case. Next, suppose $n = pa$, where $a > 1$. Then the PD-structure axioms give

$$x^{[n]} = \frac{(x^{[p]})^{[a]}}{\left(\frac{(pa)!}{(p!)^a a!}\right)},$$

where again the denominator is an integer that is prime to p. We now apply the inductive hypothesis with $m = a$ and then $m = p$, and rewrite using PD-structure axioms:

$$\begin{aligned}
dx^{[n]} &\equiv \left(\frac{(pa)!}{(p!)^a a!}\right)^{-1} (x^{[p]})^{[a-1]} dx^{[p]} \\
&\equiv \left(\frac{(pa)!}{(p!)^a a!}\right)^{-1} (x^{[p]})^{[a-1]} x^{[p-1]} dx \\
&= \frac{\left(\frac{(p(a-1))!}{(p!)^{a-1}(a-1)!}\right)}{\left(\frac{(pa)!}{(p!)^a a!}\right)} \cdot x^{[p(a-1)]} x^{[p-1]} dx \\
&= \frac{\left(\frac{(p(a-1))!}{(p!)^{a-1}(a-1)!}\right)}{\left(\frac{(pa)!}{(p!)^a a!}\right)} \cdot \binom{n-1}{p-1} x^{[n-1]} dx \\
&= x^{[n-1]} dx,
\end{aligned}$$

where again the congruence is modulo J. This completes the proof. □

Corollary 2.2.9. *In the situation of Proposition 2.2.8, the kernel of the quotient map $\Omega^*_{A/B} \to \Omega^*_{A/B,\gamma}$ is killed by p.*

Proof. By the proposition, the kernel is generated by elements of the form $d\gamma_p(x) - \gamma_{p-1}(x)dx$, where $x \in I$. Multiplying this by $p!$ gives $dx^p - px^{p-1}dx = 0$, so the kernel is killed by $p!$. But since $(p-1)!$ is a unit, it follows that these elements are killed even by p. □

We finish this section by discussing how de Rham and PD-de Rham complexes of truncated Witt vectors behave under étale base change:

Lemma 2.2.10. *Let $R \to S$ be an étale map of $\mathbb{F}_p$-algebras, and endow the ideals $VW_{r-1}(R) \subset W_r(R)$ and $VW_{r-1}(S) \subset W_r(S)$ with the PD-structures of Construction 2.1.11, both denoted γ. Then the graded $W_r(S)$-module maps*

$$\begin{aligned}
W_r(S) \otimes_{W_r(R)} \Omega^*_{W_r(R)} &\to \Omega^*_{W_r(S)} \text{ and} \\
W_r(S) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma} &\to \Omega^*_{W_r(S),\gamma}
\end{aligned}$$

induced by functoriality of the de Rham and PD-de Rham complexes are isomorphisms.

Proof. The statement for ordinary de Rham complexes holds because $W_r(R) \to W_r(S)$ is étale. To prove the statement for PD-de Rham complexes, it suffices to work in degree 1, since the kernels

$$\begin{aligned}
K_R &:= \ker(\Omega^*_{W_r(R)} \to \Omega^*_{W_r(R),\gamma}), \\
K_S &:= \ker(\Omega^*_{W_r(S)} \to \Omega^*_{W_r(S),\gamma})
\end{aligned}$$

are generated in degree 1 by Remark 2.2.4. Thus, by the case of ordinary de Rham complexes, it suffices to show that the natural map

$$W_r(R) \otimes_{W_r(S)} K_R \to K_S \tag{2.2.10.1}$$

is surjective. To this end, let Vy be an arbitrary element of $VW_{r-1}(S)$, and let K denote the image of (2.2.10.1). We must show that

$$d\gamma_n(Vy) - \gamma_{n-1}(Vy)dVy \in K$$

for all n. By base-changing the short exact sequence

$$0 \to VW_{r-1}(R) \hookrightarrow W_r(R) \to R \to 0$$

along the étale map $W_r(R) \to W_r(S)$, we see that

$$W_r(S) \otimes_{W_r(R)} VW_{r-1}(R) \to VW_{r-1}(S)$$

is surjective, and so Vy lies in the $W_r(S)$-span of elements Vx with $x \in W_{r-1}(R)$.

Certainly the elements $d\gamma_n(Vx) - \gamma_{n-1}(Vx)dVx$ lie in K (or even K_R) for all n. We will finish the proof by showing that the set of elements $\alpha \in VW_{r-1}(S)$ such that $d\gamma_n(\alpha) - \gamma_{n-1}(\alpha)d\alpha$ lies in K for all n is a sub-ideal of $VW_{r-1}(S)$, and thus contains Vy. Indeed, this set is closed under scalar multiplication by $s \in W_r(S)$ by the calculation

$$\begin{aligned}
d\gamma_n(s\alpha) - \gamma_{n-1}(s\alpha)d(s\alpha) &= d(s^n\gamma_n(\alpha)) - s^{n-1}\gamma_{n-1}(\alpha)d(s\alpha) \\
&= s^n d\gamma_n(\alpha) + ns^{n-1}\gamma_n(\alpha)ds - s^{n-1}\gamma_{n-1}(\alpha)(sd\alpha + \alpha ds) \\
&= s^n \left(d\gamma_n(\alpha) - \gamma_{n-1}(\alpha)d\alpha\right) + s^{n-1}(n\gamma_n(\alpha) - \alpha\gamma_{n-1}(\alpha))ds \\
&= s^n \left(d\gamma_n(\alpha) - \gamma_{n-1}(\alpha)d\alpha\right) \in K,
\end{aligned}$$

and it is closed under addition by the calculation

$$\begin{aligned}
&d\gamma_n(\alpha+\alpha') - \gamma_{n-1}(\alpha+\alpha')d(\alpha+\alpha') \\
&= \sum_{i+j=n} d(\gamma_i(\alpha)\gamma_j(\alpha')) - \left(\sum_{i+j=n-1} \gamma_i(\alpha)\gamma_j(\alpha')\right)(d\alpha + d\alpha') \\
&= \sum_{i+j=n} (\gamma_i(\alpha)d\gamma_j(\alpha') + \gamma_j(\alpha')d\gamma_i(\alpha)) - \sum_{\substack{i+j=n \\ i\neq 0}} \gamma_j(\alpha')\gamma_{i-1}(\alpha)d\alpha - \sum_{\substack{i+j=n \\ j\neq 0}} \gamma_i(\alpha)\gamma_{j-1}(\alpha')d\alpha' \\
&= \sum_{\substack{i+j=n \\ i\neq 0}} \gamma_j(\alpha')\left(d\gamma_i(\alpha) - \gamma_{i-1}(\alpha)d\alpha\right) + \sum_{\substack{i+j=n \\ j\neq 0}} \gamma_i(\alpha)\left(d\gamma_j(\alpha') - \gamma_{j-1}(\alpha')d\alpha'\right) \in K.
\end{aligned}$$

□

2.3 Completed de Rham and PD-de Rham complexes of rings

2.3.1. In this section, we will discuss the p-adically completed versions of the de Rham and PD-de Rham complexes of p-torsionfree rings. It will be convenient to unify our discussions of the lifts $(A,(p))$ of the lifted situation 1.7.6 and the completed PD-envelopes $(B,\overline{I})$ of the embedded situation 1.7.8 (when B is p-torsionfree). Accordingly, we fix the following setup throughout this section.

Situation 2.3.2.

1. (The PD-embedded situation) Let A be a p-torsionfree ring, and let I be an ideal containing p such that the quotients $I_r := I/p^rA \subset A/p^rA =: A_r$ are endowed with compatible PD-structures, all of which we (abusively) call γ. Let R be the $\mathbb{F}_p$-algebra A/I, which agrees with A_r/I_r for all r.

2. (The PD-embedded situation with Frobenius) Suppose additionally that we are given an endomorphism $\phi : A \to A$, lifting the absolute Frobenius endomorphism of A/pA, such that each induced map $\phi_r : A_r \to A_r$ is a PD-morphism $(A_r, I_r, \gamma) \to (A_r, I_r, \gamma)$. (In fact the condition $\phi(I_r) \subseteq I_r$ is automatic, as ϕ lifts the Frobenius endomorphism of A_1 and thus also that of its quotient R.)

Example 2.3.3.

1. In the lifted situation 1.7.6, the lift $(A, I = (p))$ satisfies the hypotheses of the PD-embedded situation, where $I_r \subset A_r$ carries the PD-structure $[\ ]$ of Lemma 1.7.7.

2. In the embedded situation 1.7.8, suppose the completed PD-envelope B is p-torsionfree. Then $(B,\overline{I})$ satisfies the hypotheses of the PD-embedded situation, where $\overline{I}_r \subset B_r$ carries its canonical PD-structure γ.

Both of these examples carry through when adding Frobenius endomorphisms.

Remark 2.3.4. We do *not* assume in Situation 2.3.2 that $I \subset A$ itself carries a PD-structure. For example, we may have $I = (p) \subset \mathbb{Z} = A$, where we do not have divided powers because we cannot divide by integers that are prime to p. Of course, this is not a problem in the p^r-torsion setting, as the quotients A_r inherit PD-structures from the PD-structure on $(p) \subset \mathbb{Z}_p$. On the other hand, the PD-structure can only be interpreted in terms of honest division by p in the p-torsionfree ring A. The need to switch between p-torsionfree and p-power torsion rings will make some proofs more complicated, so we invite the reader to imagine on first reading this section that A is a p-torsionfree $\mathbb{Z}_{(p)}$-algebra, which would allow us to perform all necessary divisions in A rather than passing to its quotients.

Definition 2.3.5. Let A be a p-torsionfree ring.

1. The *completed de Rham complex* of A is

$$\widehat{\Omega}^*_A = \lim_r \Omega^*_{A/p^rA}.$$

2. If $I \subset A$ is an ideal containing (p) and we have compatible PD-structures γ on each of the ideals $I/p^rA \subset A/p^rA$, then the *completed PD-de Rham complex* of A is

$$\widehat{\Omega}^*_{A,\gamma} = \lim_r \Omega^*_{A/p^rA,\gamma},$$

Remark 2.3.6. Since the de Rham complex is compatible with base change, we have $\Omega^*_{A/p^rA} = \Omega^*_A/p^r\Omega^*_A$ for each r, and thus $\widehat{\Omega}^*_A$ agrees with the p-adic completion of the usual de Rham complex Ω^*_A. However, even when A is p-adically complete, its completed de Rham complex need not agree with its naive de Rham complex. For example, when $A = \mathbb{Z}_p$, the completed de Rham complex

$$\widehat{\Omega}^*_{\mathbb{Z}_p} = \lim_r \Omega^*_{\mathbb{Z}/p^r\mathbb{Z}} = \mathbb{Z}_p$$

is concentrated in degree 0, but the naive de Rham complex $\Omega^*_{\mathbb{Z}_p}$ is unbounded.

Remark 2.3.7. The definition above will be useful later when we are given an $\mathbb{F}_p$-algebra R which admits a lift; that is, a p-torsionfree ring A such that $A/pA \simeq R$. With no assumptions on R, we will also be interested in the tower $(\Omega^*_{W_r(R),\gamma})_r$, where the topology is coarser than the p-adic topology. However, we will not work directly with its limit, as we are not able to give it the structure of a Dieudonné complex.

Proposition 2.3.8. *In Situation 2.3.2, let π_r be the quotient map $\Omega^*_{A_r} \to \Omega^*_{A_r,\gamma}$ for each r, and let $\pi = \lim(\pi_r) : \widehat{\Omega}^*_A \to \widehat{\Omega}^*_{A,\gamma}$ be their limit. Then π is surjective, and its kernel is contained in the exact p-torsion of $\widehat{\Omega}^*_A$.*

Proof. The kernel of π is contained in the exact p-torsion because the same is true of each π_r by Proposition 2.2.8. As for surjectivity, we will apply the Mittag-Leffler criterion to the kernels $K_r = \ker(\pi_r)$. Since each K_r is generated by the elements $dx^{[n]} - x^{[n-1]}dx$ for all $x \in I_r$, it follows by lifting each x to I_{r+1} that the quotient maps send K_{r+1} surjectively onto K_r. So Mittag-Leffler implies that $R^1 \lim K_r = 0$, and thus π is surjective. □

Lemma 2.3.9. *Suppose in the situation of Proposition 2.3.8 that $I = (p)$, and the quotients I_r are endowed with the PD-structures $\gamma = [\ ]$ of Lemma 1.7.7. Then the maps $\pi_r : \Omega^*_{A_r} \to \Omega^*_{A_r,\gamma}$ and $\pi : \widehat{\Omega}^*_A \to \widehat{\Omega}^*_{A,\gamma}$ are isomorphisms.*

Proof. To show that each π_r is an isomorphism, we must show that for each $y \in (p) \subset A_r$, we have $d\gamma_n(y) = \gamma_{n-1}(y)dy$ already in $\Omega^*_{A_r}$. Write y as px for some $x \in A_r$, and calculate:

$$d\gamma_n(px) = d\left(\frac{p^n}{n!}x^n\right) = \frac{p^n}{n!} \cdot nx^{n-1}dx = \frac{p^{n-1}x^{n-1}}{(n-1)!} \cdot d(px) = \gamma_{n-1}(px)d(px).$$

So each π_r is an isomorphism, and their limit π is as well. □

Corollary 2.3.10. *In the situation of Lemma 2.3.9, suppose that we are given a lift $\phi : A \to A$ of the absolute Frobenius of A/pA. Then the divided Frobenius endomorphism F on Ω^*_A (Proposition 1.2.3) passes to each of the quotients $\Omega^*_{A_r,\gamma}$, and to their limit $\widehat{\Omega}^*_{A,\gamma}$. This Frobenius makes $\widehat{\Omega}^*_{A,\gamma}$ a Dieudonné complex, whose strictification is a saturated de Rham-Witt complex of the $\mathbb{F}_p$-algebra A/pA.*

Proof. The PD-de Rham complexes agree with their non-PD analogues by Lemma 2.3.9. The claims are proved for the latter by [3, Variant 3.3.1, Corollary 4.2.3]. □

2.3.11. We will finish this section with Proposition 2.3.13, a generalization of Corollary 2.3.10 which relaxes the assumption that $I = (p)$. This is only a mild generalization: we are allowing ourselves to impose a stronger PD-compatibility on the de Rham complex of A (and thus killing some of its p-torsion), but still completing with respect to the p-adic topology. In particular, it does not apply to $\lim_r \Omega^*_{W_r(R),\gamma}$ for a typical (imperfect) $\mathbb{F}_p$-algebra R.

First we need a lemma:

Lemma 2.3.12. *Let A, I, and γ be as in Situation 2.3.2, and suppose A is endowed with a Frobenius lift ϕ. Suppose $x \in I$ is a lift of $\overline{x} \in I_r$. Then the elements x^p and $\phi(x)$ both lie in pA, and $\frac{x^p}{p}$ reduces to the element $(p-1)!\gamma_p(\overline{x}) \in A_r$.*

Proof. Since $\overline{x}^p = p\gamma_p(\overline{x}) \in pI_r$, we have $\overline{x}^p \in pA_r$ and thus $x^p \in pA$. Since ϕ is a lift of Frobenius, it follows that $\phi(x) \equiv x^p \equiv 0 \pmod{p}$. So it makes sense to speak of the elements $\frac{x^p}{p}$ and $\frac{\phi(x)}{p} \in A$.

Next we claim that $\frac{x^p}{p}$ is a representative of the divided power $(p-1)!\gamma_p(\overline{x})$. Letting x' denote the image of x in A_{r+1}, it is clear that x^p is a representative of $x'^p = p!\gamma_p(x')$, and dividing by p implies that $\frac{x^p}{p}$ represents $(p-1)!\gamma_p(x')$ up to the p-torsion in A_{r+1}, namely $p^r A_{r+1}$. This error term is killed when we pass to the quotient A_r.

□

Proposition 2.3.13. *Let A, I, and γ be as in Situation 2.3.2, and suppose A is endowed with a Frobenius lift ϕ. Then the divided Frobenius $F : \Omega^*_A \to \Omega^*_A$ of Proposition 1.2.3 induces a (unique) map $F : \Omega^*_{A_r,\gamma} \to \Omega^*_{A_r,\gamma}$ for each r, and thus also $F : \widehat{\Omega}^*_{A,\gamma} \to \widehat{\Omega}^*_{A,\gamma}$ by passage to the limit.*

Proof. It is clear (cf. [3, Variant 3.3.1]) that F passes to an endomorphism of $\Omega^*_{A_r} = \Omega^*_A/p^r\Omega^*_A$, so we must only show that this preserves the dg-ideal generated by elements of the form $d\gamma_n(\overline{x}) - \gamma_{n-1}(\overline{x})d\overline{x}$, with $\overline{x} \in I_r$ and $n \geq 1$. Since A_r is a $\mathbb{Z}/p^r\mathbb{Z}$-algebra, Proposition 2.2.8 allows us to restrict ourselves to $n = p$ and replace "dg-ideal" with "ideal". Now fix $\overline{x} \in I_r$. We must show that the elements $F(d\gamma_p(\overline{x}))$ and $F(\gamma_{p-1}(\overline{x})d\overline{x}) \in \Omega^1_{A_r}$ map to the same element of $\Omega^1_{A_r,\gamma}$.

The proof will require some calculations. Namely, letting $x \in I$ be a lift of $\overline{x}$, we claim that the elements $Fd(\frac{x^p}{p})$ and $F(x^{p-1}dx) \in \Omega^1_A$ reduce to the same element of $\Omega^1_{A_r,\gamma}$. To see why this suffices, note that the former reduces to $(p-1)!Fd(\gamma_p(\overline{x}))$ by Lemma 2.3.12, the latter reduces to $(p-1)!F(\gamma_{p-1}(\overline{x})d\overline{x})$ since $\overline{x}^{p-1} = (p-1)!\gamma_{p-1}(\overline{x})$, and we can divide by the common factor $(p-1)!$ in the $\mathbb{Z}/p^r\mathbb{Z}$-algebra A_r. To prove our claim, we first calculate in Ω^1_A using the formula of Proposition 1.2.3; all divisions in A by powers of p are justified by applying Lemma 2.3.12 to either x or $\frac{x^p}{p}$:

$$\begin{aligned}
Fd\left(\frac{x^p}{p}\right) &= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + d\left(\frac{\phi\left(\frac{x^p}{p}\right) - \left(\frac{x^p}{p}\right)^p}{p}\right) \\
&= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + d\left(\frac{\phi(x)^p}{p^2} - \frac{x^{p^2}}{p^{p+1}}\right) \\
&= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + d\left(\frac{\phi(x)^p}{p^2}\right) - d\left(\frac{x^{p^2}}{p^{p+1}}\right) \\
&= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + p^{p-2} d\left(\frac{\phi(x)}{p}\right)^p - d\left(\frac{x^{p^2}}{p^{p+1}}\right) \\
&= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + p^{p-1}\left(\frac{\phi(x)}{p}\right)^{p-1} d\left(\frac{\phi(x)}{p}\right) - d\left(\frac{x^{p^2}}{p^{p+1}}\right) \\
&= \left(\frac{x^p}{p}\right)^{p-1} d\left(\frac{x^p}{p}\right) + \phi(x)^{p-1} d\left(\frac{\phi(x)}{p}\right) - d\left(\frac{x^{p^2}}{p^{p+1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
F(x^{p-1}dx) &= \phi(x^{p-1}) \cdot F(dx) \\
&= \phi(x)^{p-1} \cdot \left(x^{p-1}dx + d\left(\frac{\phi(x) - x^p}{p}\right)\right) \\
&= \phi(x)^{p-1} \cdot \left(x^{p-1}dx + d\left(\frac{\phi(x)}{p}\right) - d\left(\frac{x^p}{p}\right)\right) \\
&= \phi(x)^{p-1}x^{p-1}dx + \phi(x)^{p-1} d\left(\frac{\phi(x)}{p}\right) - \phi(x)^{p-1} d\left(\frac{x^p}{p}\right)
\end{aligned}$$

Note that both of these expressions contain the term $\phi(x)^{p-1}d\left(\frac{\phi(x)}{p}\right)$. We claim that when we quotient down to $\Omega^1_{A_r,\gamma}$, all the remaining terms cancel. Using Lemma 2.3.12 again, we have

$$\frac{x^p}{p} \mapsto (p-1)!\gamma_p(\overline{x}) \in A_r \qquad \text{and}$$
$$\frac{x^{p^2}}{p^{p+1}} = \frac{\left(\frac{x^p}{p}\right)^p}{p} \mapsto (p-1)!\gamma_p((p-1)!\gamma_p(\overline{x})) = (p-1)!^{p+1}\gamma_p(\gamma_p(\overline{x})) \in A_r,$$

so the remaining terms of the former map to:

$$\begin{aligned}
&((p-1)!\gamma_p(\overline{x}))^{p-1} d\left((p-1)!\gamma_p(\overline{x})\right) - d\left((p-1)!^{p+1}\gamma_p(\gamma_p(\overline{x}))\right)\\
&= (p-1)!^p \gamma_p(\overline{x})^{p-1} d\gamma_p(\overline{x}) - (p-1)!^{p+1} d\gamma_p(\gamma_p(\overline{x}))\\
&= (p-1)!^{p+1}\left(\gamma_{p-1}(\gamma_p(\overline{x})) d\gamma_p(\overline{x}) - d\gamma_p(\gamma_p(\overline{x}))\right)\\
&= 0 \in \Omega^*_{A_r,\gamma}
\end{aligned}$$

by applying the PD-compatibility relation to the element $\gamma_p(\overline{x})$. The remaining terms of the latter map to

$$\begin{aligned}
&\phi(\overline{x})^{p-1}\overline{x}^{p-1} d\overline{x} - \phi(\overline{x})^{p-1} d((p-1)!\gamma_p(\overline{x}))\\
&= \phi(\overline{x})^{p-1}(p-1)!\left(\gamma_{p-1}(\overline{x}) d\overline{x} - d(\gamma_p(\overline{x}))\right)\\
&= 0 \in \Omega^*_{A_r,\gamma},
\end{aligned}$$

by applying the same PD-compatibility to the element $\overline{x}$. □

Corollary 2.3.14. *In the situation of Proposition 2.3.13, $(\widehat{\Omega}^*_{A,\gamma}, d, F)$ is a Dieudonné algebra, and its strictification is a saturated de Rham-Witt complex associated to the $\mathbb{F}_p$-algebra A/pA.*

Proof. All the Dieudonné algebra properties are inherited via the quotient map $\widehat{\Omega}^*_A \to \widehat{\Omega}^*_{A,\gamma}$. Moreover, by Proposition 2.3.8, this quotient map is surjective with p-torsion kernel, so it becomes an isomorphism after taking saturations, and in particular also after taking strictifications. The result then follows from Construction 1.2.6. □

2.4 Crystals of modules

2.4.1. In this section, we will collect some results about crystals of modules for later use. We refer to [2, §§5-6] for background on the crystalline site and crystals of $\mathcal{O}_{X/S}$-modules. In particular, we are concerned for the moment with the (small) crystalline site $\mathrm{Cris}(X/S)$ where p is locally nilpotent on S; we will soon loosen this restriction to allow p-adic bases such as $\mathrm{Spf}\, W(k)$.

We begin with some standard results about pullbacks of crystals, which will share the following setup.

Situation 2.4.2. Suppose we are given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

where S and S' are PD-base schemes. Suppose we also have objects $(U, T, \gamma) \in \mathrm{Cris}(X/S)$ and $(U', T', \gamma') \in \mathrm{Cris}(X'/S')$, and a PD-morphism $h : (U', T', \gamma') \to (U, T, \gamma)$ over g, in the sense of [2, Definition 5.6].

Remark 2.4.3. Recall that inverse image sheaves on both the Zariski and crystalline sites are built as the sheafifications of presheaf pullback functors ([21, tag `008F`], [2, p. 5.7.2]). Working in Situation 2.4.2 and denoting these presheaf pullbacks with a superscript p, we have

$$h^p(\mathcal{E}_T) = (V' \mapsto \varinjlim_{h(V') \subset V \subset T} \mathcal{E}(V)),$$

where V ranges over open subsets of T containing $h(V')$, and

$$(g^p_{\mathrm{cris}}\mathcal{E})_{T'} = (V' \mapsto \varinjlim_{j:V' \to V} \mathcal{E}(V)),$$

where j ranges over all PD-morphisms from V' to an object V of $\mathrm{Cris}(X/S)$. We get a map

$$h^p(\mathcal{E}_T) \to (g^p_{\mathrm{cris}}\mathcal{E})_{T'}$$

by including the former colimit index set into the latter. This induces an $\mathcal{O}_{T'}$-linear map

$$h^*(\mathcal{E}_T) \to (g^*_{\mathrm{cris}}\mathcal{E})_{T'}$$

upon sheafifying and tensoring over $h^{-1}(\mathcal{O}_T)$ with $\mathcal{O}_{T'}$.

We will repeatedly use the following standard fact relating pullbacks of crystals to the Zariski pullbacks of their associated Zariski sheaves:

Lemma 2.4.4. *([2, Exercise 6.5]) The sheaf $g^*_{\mathrm{cris}}\mathcal{E}$ is a crystal, and the map $h^*(\mathcal{E}_T) \to (g^*_{\mathrm{cris}}\mathcal{E})_{T'}$ above is an isomorphism.*

The isomorphisms of Lemma 2.4.4 enjoy various compatibilities; we will make particular use of the following.

Lemma 2.4.5. *Suppose in Situation 2.4.2 that we have not one but two PD-morphisms h_1 and h_2 over g, fitting into a commutative square*

$$\begin{tikzcd} T_1' \arrow[r, "h_1"] \arrow[d, "a"'] & T_1 \arrow[d, "b"] \\ T_2' \arrow[r, "h_2"] & T_2, \end{tikzcd}$$

where the left column lives in $\mathrm{Cris}(X'/S')$ *and the right column lives in* $\mathrm{Cris}(X/S)$*. Then the diagram*

$$\begin{tikzcd}
h_1^{-1}(b^*\mathcal{E}_{T_2}) \arrow[r, "\sim"] \arrow[d] & h_1^{-1}(\mathcal{E}_{T_1}) \arrow[d] \\
h_1^*(b^*\mathcal{E}_{T_2}) \arrow[r, "\sim"] \arrow[d, equal] & h_1^*(\mathcal{E}_{T_1}) \arrow[dd, "\sim"] \\
a^*(h_2^*(\mathcal{E}_{T_2})) \arrow[d, "\sim"'] & \\
a^*((g_{\mathrm{cris}}^*\mathcal{E})_{T_2'}) \arrow[r, "\sim"] & (g_{\mathrm{cris}}^*\mathcal{E})_{T_1'}
\end{tikzcd}$$

commutes, where the horizontal maps are induced by the restriction maps of the crystals $\mathcal{E}$ *and* $g_{\mathrm{cris}}^*\mathcal{E}$*, and the vertical isomorphisms are those of Lemma 2.4.4.*

Proof. The upper square commutes because $h^{-1} \to h^*$ is a map of functors. The rest of the diagram is induced by the diagram of presheaves on T_1'

$$\begin{tikzcd}
h_1^p(b^p\mathcal{E}_{T_2}) \arrow[r, "h_1^p(b^*)"] \arrow[d, equal] & h_1^p(\mathcal{E}_{T_1}) \arrow[dd] \\
a^p(h^p\mathcal{E}_{T_2}) \arrow[d] & \\
a^p((g_{\mathrm{cris}}^p\mathcal{E})_{T_2'}) \arrow[r, "a^*"] & (g_{\mathrm{cris}}^p\mathcal{E})_{T_1'}
\end{tikzcd}$$

by sheafifying everything and tensoring up to $\mathcal{O}_{T_1'}$. The top-left corner here is the presheaf on T_1' defined by

$$V' \mapsto \varinjlim_{V \supseteq b(h_1(V'))} \mathcal{E}(V),$$

the bottom-right corner is the presheaf defined by

$$V' \mapsto \varinjlim_{j: V' \to V} \mathcal{E}(V)$$

(where j ranges over all PD-morphisms from V' to an object V of $\mathrm{Cris}(X/S)$), and both maps from the former to the latter are the tautological map given by including the former colimit index set into the latter. Thus the diagram of presheaves commutes, and therefore the diagram of $\mathcal{O}_{T_2}$-modules does as well. □

2.4.6. We now allow S to be a p-adic base as in ([2, Definition 7.17]). We will be interested in the case $S = \mathrm{Spf}\, W(k)$; note that objects (U, T, δ) in $\mathrm{Cris}(X/W)$ are still required to satisfy $p^n\mathcal{O}_T = 0$ for some n.

Definition 2.4.7. ([21, tag 07IU]) A crystal $\mathcal{E}$ of $\mathcal{O}_{X/S}$-modules is *quasicoherent* (resp. *locally free*, *finite locally free*) if for every object (U, T, δ) of $\mathrm{Cris}(X/S)$, the sheaf $\mathcal{E}_T$ of $\mathcal{O}_T$-modules is quasicoherent (resp. locally free, finite locally free).

Lemma 2.4.8. *Let X be a k-scheme and $\mathcal{E}$ a finite locally free crystal of $\mathcal{O}_{X/W}$-modules. Then the system $(\mathcal{E}_{W_r(X)})_r$ of quasicoherent sheaves on $W_r(X)$ is finite locally free, in the following sense: there exist an affine open cover $(\operatorname{Spec} R_i)_i$ of X (which we identify with $\operatorname{Spec} W_r R_i \subset W_r(X)$ as topological spaces), integers n_i, and isomorphisms $\mathcal{E}_{W_r(R_i)} \simeq \mathcal{O}^{\oplus n_i}$ for all r and i, compatible with the quotient maps from $W_r(R_i)$ to $W_{r-1}(R_i)$.*

Proof. Choose an affine open cover $(\operatorname{Spec} R_i)_i \subset X$ trivializing the Zariski sheaf $\mathcal{E}_X$. For each i, choose generators $x_1, \ldots, x_{n_i}$ for the finite free R_i-module $\mathcal{E}(R_i)$. Lift each x_j arbitrarily to a compatible family of sections $\widetilde{x}_j \in \mathcal{E}(W_r R_i)$ for each r; this is possible because $\mathcal{E}(W_r R_i) \simeq \mathcal{E}(W_{r+1} R_i) \otimes_{W_{r+1} R_i} W_r R_i$ and the map $W_{r+1} R_i \to W_r R_i$ is surjective.

Since the kernel $VW_{r-1}(R_i)$ of $W_r(R_i) \to R_i$ is nilpotent, it is contained in the Jacobson radical of $W_r(R_i)$. Thus Nakayama's lemma implies that for each r, the elements $\widetilde{x}_j$ generate $\mathcal{E}(W_r R_i)$. It follows that the $\widetilde{x}_j$ are linearly independent, as $\mathcal{E}(W_r R_i)$ is a locally free $W_r(R_i)$-module of rank n_i and thus cannot be generated by fewer than n_i elements. Thus the elements $\widetilde{x}_j$ form a basis of the $W_r(R_i)$-module $\mathcal{E}(W_r R_i)$. It follows by quasicoherence that the $\widetilde{x}_j$ identify $\mathcal{E}_{\operatorname{Spec} W_r R_i}$ with $\mathcal{O}^{\oplus n_i}$. □

Remark 2.4.9. Note that although the tower of Zariski sheaves $(\mathcal{E}_{W_r X})_r$ associated to $\mathcal{E}$ is locally free, there is generally no open cover $\{U_i\}_i$ of X such that $\mathcal{E}|_{U_i}$ is free as a crystal of $\mathcal{O}_{U_i/S}$-modules for each i. In particular, the connections we will discuss later will typically not be locally trivial.

2.5 F-crystals

In this section, X will continue to denote a k-scheme, and the PD-base S will be either $\operatorname{Spec} W_r(k)$ or $\operatorname{Spf} W(k)$, with its usual PD-structure on the ideal (p). Our main objects of interest are as follows.

Definition 2.5.1. ([21, tag 07N3])

1. An *F-crystal* is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$ where $\mathcal{E}$ is a finite locally free crystal of $\mathcal{O}_{X/S}$-modules and $\phi_{\mathcal{E}} : (F_X)^*_{\mathrm{cris}} \mathcal{E} \to \mathcal{E}$ is a morphism of $\mathcal{O}_{X/S}$-modules.

2. An F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$ is *unit-root* if $\phi_{\mathcal{E}}$ is an isomorphism.

Remark 2.5.2. Let $\mathrm{Cris\acute{E}t}(X/S)$ denote the crystalline-étale site of X/S (cf. [15, Remark 2.2.4.2]), and $(X/S)_{\text{cris-ét}}$ the corresponding topos. Then restricting from $\mathrm{Cris\acute{E}t}(X/S)$ to the usual crystalline site $\mathrm{Cris}(X/S)$ defines a functor between the respective categories of

$\mathcal{O}$-modules, which is an equivalence on the full subcategories of *quasicoherent* crystals. Moreover, a quasicoherent crystal of modules on $\mathrm{Cris\acute{E}t}(X/S)$ is finite locally free (in the étale topology) if and only if its restriction to $\mathrm{Cris}(X/S)$ is (in the Zariski topology); this follows from [21, tag 05VG]. Thus, when we are interested in quasicoherent or finite locally free crystals of modules, we may freely pass between the crystalline and crystalline-étale sites.

Example 2.5.3. The "trivial" F-crystal on $\mathrm{Cris}(X/S)$ is given by the crystalline structure sheaf $\mathcal{O}_{X/S}$ equipped with the canonical map

$$(F_X)^*_{\mathrm{cris}}\mathcal{O}_{X/S} = (F_X)^{-1}_{\mathrm{cris}}\mathcal{O}_{X/S} \otimes_{(F_X)^{-1}_{\mathrm{cris}}\mathcal{O}_{X/S}} \mathcal{O}_{X/S} \xrightarrow{\sim} \mathcal{O}_{X/S}.$$

We denote this by $(\mathcal{O}_{X/S}, F)$; it is clearly a unit-root F-crystal.

For concreteness, let us write down what kind of information is carried by the Frobenius $\phi_{\mathcal{E}}$ of an F-crystal.

Remark 2.5.4. Suppose $T = (U, T, \delta)$ and $T' = (U', T', \delta')$ are objects of $\mathrm{Cris}(X/S)$ and we have a PD-morphism $h : T' \to T$ over $F_X : X \to X$. Then Lemma 2.4.4 gives an isomorphism $((F_X)^*_{\mathrm{cris}}\mathcal{E})_{T'} \simeq h^*(\mathcal{E}_T)$. In particular, for any such h, the data of an $\mathcal{O}_{X/S}$-module map

$$\phi_{\mathcal{E}} : (F_X)^*_{\mathrm{cris}}\mathcal{E} \to \mathcal{E}$$

induces an $\mathcal{O}_{T'}$-module map

$$h^*(\mathcal{E}_T) \xrightarrow{\sim} ((F_X)^*_{\mathrm{cris}}\mathcal{E})_{T'} \to \mathcal{E}_{T'}$$

which we (slightly abusively) also denote $\phi_{\mathcal{E}}$. This is an isomorphism if $(\mathcal{E}, \phi_{\mathcal{E}})$ is unit-root.

Example 2.5.5. Let R be a k-algebra, and take $X = U = U' = \operatorname{Spec} R$ in Remark 2.5.4.

1. If h is the Frobenius map $F : \operatorname{Spec} W_r(R) \to \operatorname{Spec} W_r(R)$, then $\phi_{\mathcal{E}}$ defines a map

   $$\phi_{\mathcal{E}} : F^*(\mathcal{E}_{\operatorname{Spec} W_r(R)}) \to \mathcal{E}_{\operatorname{Spec} W_r(R)}$$

   of Zariski sheaves, and thus

   $$\mathcal{E}(W_r(R)) \otimes_{W_r(R),F} W_r(R) \to \mathcal{E}(W_r(R))$$

   upon taking sections. This gives us a Frobenius-semilinear map

   $$F : \mathcal{E}(W_r(R)) \xrightarrow{\mathrm{id}\otimes 1} \mathcal{E}(W_r(R)) \otimes_{W_r(R),F} W_r(R) \to \mathcal{E}(W_r(R)).$$

   We will often instead be interested in the composition of this map with the quotient map $\mathcal{E}(W_r(R)) \to \mathcal{E}(W_{r-1}(R))$, which may of course be obtained by the same procedure starting with the Frobenius map $h = F : \operatorname{Spec} W_{r-1}(R) \to \operatorname{Spec} W_r(R)$.

2. If R admits a smooth lift with Frobenius (A, ϕ) as in the Frobenius-lifted situation 1.7.6, then we can take $h = \phi_{A_r} : \operatorname{Spec} A_r \to \operatorname{Spec} A_r$. Then $\phi_{\mathcal{E}}$ defines a map

$$\phi_{\mathcal{E}} : \phi^*(\mathcal{E}_{\operatorname{Spec} A_r}) \to \mathcal{E}_{\operatorname{Spec} A_r}$$

   of Zariski sheaves, which as before yields a Frobenius-semilinear map

$$F : \mathcal{E}(A_r) \xrightarrow{\operatorname{id} \otimes 1} \mathcal{E}(A_r) \otimes_{A_r, \phi} A_r \to \mathcal{E}(A_r).$$

   As above, we will often work with the composition of this map with the quotient map $\mathcal{E}(A_r) \to \mathcal{E}(A_{r-1})$.

3. Suppose R admits a smooth embedding with Frobenius (A, ϕ) as in the Frobenius-embedded situation 1.7.8, and let B_r denote the resulting PD-envelope for each $r > 0$. Then taking $h = \phi_{B_r} : \operatorname{Spec} B_r \to \operatorname{Spec} B_r$, $\phi_{\mathcal{E}}$ defines a map

$$\phi_{\mathcal{E}} : \phi^*(\mathcal{E}_{\operatorname{Spec} B_r}) \to \mathcal{E}_{\operatorname{Spec} B_r}$$

   of Zariski sheaves, and

$$\mathcal{E}(B_r) \otimes_{B_r, \phi_{B_r}} B_r \to \mathcal{E}(B_r)$$

   on sections. As before, this gives a Frobenius-semilinear map

$$F : \mathcal{E}(B_r) \xrightarrow{\operatorname{id} \otimes 1} \mathcal{E}(B_r) \otimes_{B_r, \phi_{B_r}} B_r \to \mathcal{E}(B_r).$$

Our main source of unit-root F-crystals—and often the only source, as we will soon see—is as follows.

Construction 2.5.6. Given an étale $\mathbb{Z}/p^r\mathbb{Z}$-local system $\mathcal{L}$ on X, we construct a corresponding unit-root F-crystal on $\operatorname{Cris}(X/W_r(k))$ as follows. The underlying crystal is the module pullback of $\mathcal{L}$ along the projection of ringed topoi

$$u_{X/W_r} : ((X/W_r)_{\text{cris-ét}}, \mathcal{O}_{X/W_r}) \to (X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

(cf. Remark 2.5.2). More concretely, this is the $\mathcal{O}_{X/W_r}$-module given by sheafifying the presheaf on $\operatorname{CrisÉt}(X/W_r(k))$

$$(U \hookrightarrow T, \delta) \mapsto \mathcal{L}(U) \otimes_{\mathbb{Z}/p^r\mathbb{Z}} \mathcal{O}_T.$$

We will denote this as $\mathcal{L} \otimes_{\mathbb{Z}/p^r\mathbb{Z}} \mathcal{O}_{X/W_r}$, or more simply $\mathcal{L} \otimes \mathcal{O}_{X/W_r}$. This is finite locally free in the étale topology because $\mathcal{L}$ is, and thus it is also finite locally free in the Zariski topology. We endow our crystal with the Frobenius operator

$$\phi = \text{“id} \otimes F\text{”} : (F_X)^*_{\text{cris}} \left(\mathcal{L} \otimes \mathcal{O}_{X/W_r}\right) = F_X^{-1}(\mathcal{L}) \otimes (F_X)^*_{\text{cris}} \left(\mathcal{O}_{X/W_r}\right) \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_{X/W_r},$$

where the equality indicated comes from commuting the pullbacks in the diagram

$$\begin{tikzcd}
((X/W_r)_{\text{cris-ét}}, \mathcal{O}_{X/W_r}) \arrow[r, "(F_X)_{\text{cris}}"] \arrow[d, "u_{X/W_r}"'] & ((X/W_r)_{\text{cris-ét}}, \mathcal{O}_{X/W_r}) \arrow[d, "u_{X/W_r}"] \\
(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}) \arrow[r, "F_X"] & (X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})
\end{tikzcd}$$

of ringed topoi. This construction is functorial in $\mathcal{L}$. Moreover, if $\mathcal{L} = (\mathcal{L}_r)_r$ is instead a $\mathbb{Z}_p$-local system on X, then the discussion above produces compatible unit-root F-crystals $\mathcal{L}_r \otimes_{\mathbb{Z}/p^r\mathbb{Z}} \mathcal{O}_{X/W_r}$ on $\mathrm{Cris}(X/W_r(k))$ for each $r > 0$, and these together define a unit-root F-crystal on $\mathrm{Cris}(X/W(k))$.

2.5.7. When X is smooth, a theorem of Katz ([16, Proposition 4.1.1]; see also [5, Theorem 2.1] for the case of F-isocrystals) shows that unit-root F-crystals on X can be classified in terms of $\mathbb{Z}_p$-local systems. We will spend the remainder of this section recording Katz's theorem in the forms which will be useful to us, parts of whose proof (while well-known to the experts) we were unable to locate in the literature.

Situation 2.5.8. Let k be a perfect field of characteristic p and $W_r(k)$ its ring of truncated Witt vectors, as usual. Suppose we are given a smooth irreducible affine scheme $\widetilde{X}$ over $W_r(k)$, equipped with a lift $\phi_{\widetilde{X}} : \widetilde{X} \to \widetilde{X}$ of the absolute Frobenius of its special fiber $X := \widetilde{X}_k$, compatible with $\sigma : W_r(k) \to W_r(k)$. Let $i : X \hookrightarrow \widetilde{X}$ denote the embedding.

Theorem 2.5.9. *(Katz) In Situation 2.5.8, we have canonical equivalences among all of the following categories, which preserve ranks and are compatible as r varies:*

1. *the category $\mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(X)$ of finite-rank $\mathbb{Z}/p^r\mathbb{Z}$-local systems on the étale site of X,*

2. *the category $\mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X})$ of finite-rank $\mathbb{Z}/p^r\mathbb{Z}$-local systems on the étale site of $\widetilde{X}$,*

3. *the category $\mathrm{Rep}_{\mathbb{Z}/p^r\mathbb{Z}}(\pi_1(\widetilde{X}, \overline{x}))$ of finite free $\mathbb{Z}/p^r\mathbb{Z}$-modules equipped with a continuous action of the étale fundamental group $\pi_1(\widetilde{X}, \overline{x})$ (for a fixed geometric point $\overline{x} \in \widetilde{X}$),*

4. *the category $\mathrm{Vect}(\widetilde{X})^F$ of finite-rank locally free sheaves H on $\widetilde{X}$ equipped with an isomorphism $F : \phi_{\widetilde{X}}^*(H) \to H$,*

5. *the category $F - \mathrm{crys}^{\mathrm{ur}}(\widetilde{X}/W_r(k))$ of unit-root F-crystals on $\mathrm{Cris}(\widetilde{X}/W_r(k))$ relative to the lift $\phi_{\widetilde{X}}$; that is, finite-rank locally free crystals of modules on $\mathrm{Cris}(\widetilde{X}/W_r(k))$ equipped with an isomorphism*

$$\phi_{\mathcal{E}} : (\phi_{\widetilde{X}})^*_{\mathrm{cris}}\mathcal{E} \to \mathcal{E},$$

and

6. *the category* $F - \mathrm{crys}^{\mathrm{ur}}(X/W_r(k))$ *of unit-root* F*-crystals on* $\mathrm{Cris}(X/W_r(k))$.

Proof. We will construct a diagram of equivalences of categories as follows, where the six categories listed above appear in clockwise order from the top left.

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(X) \arrow[r, leftrightarrow] \arrow[d] & \mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X}) \arrow[r, leftrightarrow] \arrow[d] & \mathrm{Rep}_{\mathbb{Z}/p^r\mathbb{Z}}(\pi_1(\widetilde{X}, \overline{x})) \\
F - \mathrm{crys}^{\mathrm{ur}}(X/W_r(k)) & F - \mathrm{crys}^{\mathrm{ur}}(\widetilde{X}/W_r(k)) \arrow[l] \arrow[r] & \mathrm{Vect}(\widetilde{X})^F \arrow[ul]
\end{tikzcd} \tag{2.5.9.1}$$

The equivalence (1) $\leftrightarrow$ (2) is given by pushforward and pullback along the equivalence of étale sites $\text{Ét}(X) \simeq \text{Ét}(\widetilde{X})$ induced by i; see Proposition 2.1.13. The equivalence (2) $\leftrightarrow$ (3) is the monodromy equivalence, [21, tag `0DV5`]. The functor (5) $\to$ (6) is the pullback functor i^*_{cris}; one can see that this is an equivalence by applying [2, Corollary 6.7] to both the crystals in question and their Frobenius endomorphisms.

The functor (5) $\to$ (4) associates to a unit-root F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\widetilde{X}/W_r(k)$ the finite locally free sheaf $\mathcal{E}_{\widetilde{X}}$, equipped with the isomorphism F determined by $\phi_{\mathcal{E}}$ on Zariski sheaves. (Equivalently, viewing crystals on $\mathrm{Cris}(\widetilde{X}/W_r(k))$ as modules with integrable connection, this functor is given by forgetting the connection.) The functor (1) $\to$ (6) is given by Construction 2.5.6, and (2) $\to$ (5) is analogously given by pullback along the morphism

$$u_{\widetilde{X}/W_r} : ((\widetilde{X}/W_r)_{\text{cris-ét}}, \mathcal{O}_{\widetilde{X}/W_r}) \to (\widetilde{X}_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

of ringed topoi. (It follows by commuting pullbacks in an appropriate diagram of ringed topoi that the square in the diagram (2.5.9.1) commutes up to isomorphism of functors.) The functor (4) $\to$ (2) sends a pair (H, F) to the étale sheaf given by

$$(U \to \widetilde{X}) \mapsto \{x \in H(U) : x \text{ is fixed by } H(U) \xrightarrow{\phi^*_U} \phi^*_{\widetilde{X}}(H)(U) \xrightarrow{F} H(U)\},$$

where $\phi^*_U : U \to U$ is the unique map compatible with $\phi_{\widetilde{X}}$. (As in Remark 2.5.2, we may treat the finite locally free sheaf H as living on either the Zariski or the étale site of $\widetilde{X}$.) Note that the proof of [16, Proposition 4.1.1] shows that the output of this functor is a $\mathbb{Z}/p^r\mathbb{Z}$-local system with the same rank as the vector bundle H.

The statement of Katz's theorem given in [16, Proposition 4.1.1] is that we have an equivalence of categories (3) $\to$ (4); the equivalence constructed by Katz is

$$\begin{tikzcd}
\mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X}) \arrow[d] & \mathrm{Rep}_{\mathbb{Z}/p^r\mathbb{Z}}(\pi_1(\widetilde{X}, \overline{x})) \arrow[l, "\sim"'] \\
F - \mathrm{crys}^{\mathrm{ur}}(\widetilde{X}/W_r(k)) \arrow[r] & \mathrm{Vect}(\widetilde{X})^F,
\end{tikzcd}$$

with inverse given by

$$\begin{tikzcd} \mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X}) \arrow[r, "\sim"] & \mathrm{Rep}_{\mathbb{Z}/p^r\mathbb{Z}}(\pi_1(\widetilde{X}, \overline{x})) \\ & \mathrm{Vect}(\widetilde{X})^F. \arrow[ul] \end{tikzcd}$$

(We omit the proof that Katz's functor agrees with the specified one; this amounts to checking that given $\mathcal{L}$ in $\mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X})$, Katz's functor sends the π_1-representation $\mathcal{L}_{\overline{x}}$ to the vector bundle with Frobenius $(\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}, F)$.) Thus, to complete the proof, we must show that that the functors

$$\begin{tikzcd} \mathrm{LocSys}_{\mathbb{Z}/p^r\mathbb{Z}}(\widetilde{X}) \arrow[d, "G"'] & \\ F - \mathrm{crys}^{\mathrm{ur}}(\widetilde{X}/W_r(k)) \arrow[r, "H"'] & \mathrm{Vect}(\widetilde{X})^F \arrow[ul, "K"'] \end{tikzcd}$$

are all equivalences, with an inverse of each one given by composing the other two.

Katz shows that K is an equivalence of categories with inverse $H \circ G$, so it will suffice to show that at least one of G and H is an equivalence. The proof of this consists of four steps:

1. H is essentially surjective.
2. H is "essentially injective" (that is, each isomorphism class of objects in $\mathrm{Vect}(\widetilde{X})^F$ has at most one preimage in $F - \mathrm{crys}^{\mathrm{ur}}(\widetilde{X}/W_r(k))$ up to isomorphism), so G is essentially surjective.
3. G is faithful.
4. H is faithful, so G is full.

Steps 1 and 3 follow immediately from the fact that $H \circ G$ is an equivalence of categories. For step 4, H is faithful by viewing crystals as modules with (integrable, quasi-nilpotent) connection, and viewing H as the functor that forgets the connection; it follows by a diagram chase that G is full. Step 2 is *not* immediate; we must show that every pair $(\mathcal{E}_{\widetilde{X}}, \phi_{\mathcal{E}})$ admits at most one compatible connection ∇. Suppose on the contrary that we have a vector bundle with Frobenius $(\mathcal{E}_{\widetilde{X}}, \phi_{\mathcal{E}})$ that admits two connections

$$\nabla, \nabla' : \mathcal{E}_{\widetilde{X}} \to \mathcal{E}_{\widetilde{X}} \otimes \Omega^1_{\widetilde{X}/W_n(k)}$$

compatible with $\phi_{\mathcal{E}}$. Then $A = \nabla - \nabla'$ is an $\mathcal{O}_{\widetilde{X}}$-linear map. The compatibility of ∇ and ∇' with $\phi_{\mathcal{E}}$ implies that the diagram

$$\begin{tikzcd}
\phi_{\widetilde{X}}^* \mathcal{E}_{\widetilde{X}} \arrow[r, "\phi_{\widetilde{X}}^*(A)"] \arrow[d, "\phi_{\mathcal{E}}"', "\sim"] & \phi_{\widetilde{X}}^*(\mathcal{E}_{\widetilde{X}}) \otimes \Omega^1_{\widetilde{X}/W_n(k)} \arrow[d, "\phi_{\mathcal{E}} \otimes \phi_{\widetilde{X}}^*"] \\
\mathcal{E}_{\widetilde{X}} \arrow[r, "A"] & \mathcal{E}_{\widetilde{X}} \otimes \Omega^1_{\widetilde{X}/W_n(k)}.
\end{tikzcd}$$

commutes. So we have

$$A = (\phi_{\mathcal{E}} \otimes \phi_{\widetilde{X}}^*) \circ \phi_{\widetilde{X}}^*(A) \circ \phi_{\mathcal{E}}^{-1}.$$

Since $\phi_{\widetilde{X}}^*$, the pullback of differentials along a Frobenius lift, is divisible by p, it follows that A is divisible by p. By iterating this process, we can show that A is divisible by p^n, and therefore it vanishes. □

Remark 2.5.10. Suppose X/k is smooth, and fix $r > 0$ for the moment. Then X locally admits a lift $\widetilde{X}/W_r(k)$, since we can write X locally as $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where the matrix of partial derivatives of the f_i is nonsingular and then lift the polynomials f_i to $W_r(k)$. It follows from the lifting property of smooth morphisms that $\widetilde{X}$ admits an endomorphism $\phi_{\widetilde{X}}$ which lifts the absolute Frobenius of $X = \widetilde{X}_k$ and is compatible with $\sigma : W_r(k) \to W_r(k)$, as in Situation 2.5.8. It then follows from Theorem 2.5.9 that any unit-root F-crystal on $\mathrm{Cris}(X/W_r(k))$ admits an étale trivialization.

In fact the proof of [16, Proposition 4.1.1] tells us more: given a unit-root F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\mathrm{Cris}(X/W)$, there exists an affine étale cover of X

$$(\operatorname{Spec} S_{i,1} \to \operatorname{Spec} R)_{i \in I}$$

trivializing the restriction of $(\mathcal{E}, \phi_{\mathcal{E}})$ to $\mathrm{Cris}(X/W_1(k))$, and each $\operatorname{Spec} S_{i,1}$ admits a tower of *finite* étale covers $\operatorname{Spec} S_{i,r}$ (for $r \geq 1$) trivializing the restriction of $(\mathcal{E}, \phi_{\mathcal{E}})$ to $\mathrm{Cris}(X/W_r(k))$. Thus, setting $S_{i,\infty} = \varinjlim_r S_{i,r}$, the pullback of $(\mathcal{E}, \phi_{\mathcal{E}})$ to the affine scheme $\operatorname{Spec} S_{i,\infty}$ is isomorphic to a finite direct power of the trivial crystal

$$(\mathcal{O}_{\operatorname{Spec} S_{i,\infty}/W}, F).$$

If X is quasicompact, then we can take the index set I to be finite, and then the cover

$$\coprod_i \operatorname{Spec} S_{i,\infty} \to X$$

trivializing $(\mathcal{E}, \phi_{\mathcal{E}})$ is in particular a pro-étale affine cover in the sense of [4, Definition 4.2.1].

2.6 de Rham complexes associated to a crystal on a PD-envelope

2.6.1. In this and the next two sections, we will recall two classical constructions of de Rham complexes associated to a crystal $\mathcal{E}$ of $\mathcal{O}_{X/S}$-modules, in order to point out their compatibilities. We begin with the de Rham complex associated to a PD-envelope, as in [2, Theorem 6.6]. We will start in the following p-power torsion situation, and later take sections and pass to the limit.

Situation 2.6.2. Suppose X is a scheme over a perfect field k of characteristic p. We will work over a PD-base (S, I, γ) on which p is nilpotent, and later specialize to the PD-base $(\operatorname{Spec} W_r(k), (p), [\,])$ for $r > 0$. Suppose we are given a closed embedding $X \hookrightarrow Y$ over S, where Y/S is smooth.

Then we can consider several PD-envelopes. Let $D = D_X(Y)$ be the PD-envelope of $X \hookrightarrow Y$, D_Y the PD-envelope of the diagonal morphism $Y \hookrightarrow Y \times_S Y$, and $D_X(Y \times Y)$ the PD-envelope of the diagonally embedded $X \hookrightarrow Y \times_S Y$, all over the base S. The first and last of these are objects of $\mathrm{Cris}(X/S)$, and the two projections $Y \times_S Y \rightrightarrows Y$ induce morphisms

$$p_1, p_2 : D_X(Y \times Y) \rightrightarrows D$$

in this category. For the sake of brevity, we will abbreviate the structure sheaves of the various PD-envelopes with calligraphic letters; thus $\mathcal{D} = \mathcal{O}_D, \mathcal{D}_Y = \mathcal{O}_{D_Y}$, and $\mathcal{D}_X(Y \times Y) = \mathcal{O}_{D_X(Y \times Y)}$.

Construction 2.6.3. ([2, Theorem 6.6, (i) $\to$ (iii)]) We construct the de Rham complex associated to $\mathcal{E}$ over D as follows. We begin with the diagram below, where the top maps are isomorphisms because $\mathcal{E}$ is a crystal, the middle vertical maps are isomorphisms by [2, Corollary 6.3], and ε is a $\mathcal{D}_Y$-linear isomorphism defined by going the long way around the diagram.

$$\begin{tikzcd}
p_1^*\mathcal{E}_D \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{E}_{D_X(Y\times Y)} & p_2^*\mathcal{E}_D \arrow[l, "\sim"'] \arrow[d, equal] \\
\mathcal{E}_D \otimes_{\mathcal{D}} \mathcal{D}_X(Y \times Y) & & \mathcal{D}_X(Y \times Y) \otimes_{\mathcal{D}} \mathcal{E}_D \\
\mathcal{E}_D \otimes_{\mathcal{D}} \mathcal{D} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \arrow[u, "\sim"] \arrow[d, equal] & & \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D} \otimes_{\mathcal{D}} \mathcal{E}_D \arrow[u, "\sim"'] \arrow[d, equal] \\
\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}_Y & & \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D \arrow[ll, "\sim", "\varepsilon"']
\end{tikzcd}$$

Quotienting down to the first-order PD-envelope $\mathcal{D}_Y^1$, we get an isomorphism

$$\varepsilon_1 : \mathcal{D}_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{E}_D \xrightarrow{\sim} \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}_Y^1.$$

But the first-order PD-envelope D^1_Y agrees with the usual first-order neighborhood of the diagonal, so we can replace $\mathcal{D}^1_Y$ with $\mathcal{P}^1_{Y/S} = (\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_Y)/\mathcal{J}^2$, where $\mathcal{J}$ is the ideal sheaf generated by sections of the form $1 \otimes x - x \otimes 1$. We then construct the (non-commutative) diagram

$$\begin{tikzcd} & \mathcal{E}_D \arrow[dl, "\mathrm{id} \otimes 1"'] \arrow[dr, "1 \otimes \mathrm{id}"] & \\ \mathcal{E}_D \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes \mathcal{O}_Y)/\mathcal{J}^2 & & (\mathcal{O}_Y \otimes \mathcal{O}_Y)/\mathcal{J}^2 \otimes_{\mathcal{O}_Y} \mathcal{E}_D, \arrow[ll, "\sim", "\varepsilon_1"'] \end{tikzcd}$$

and build the map

$$\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1) : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes \mathcal{O}_Y)/\mathcal{J}^2.$$

This factors through the subsheaf $\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2$, since all three maps above reduce to $\mathrm{id} : \mathcal{E}_D \to \mathcal{E}_D$ modulo $\mathcal{J}$. We define our connection to be the resulting map

$$\nabla : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2 = \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S};$$

this is indeed a connection by [2, Proposition 2.9]. As usual, we then define

$$\nabla_i : \mathcal{E}_D \otimes \Omega^i_Y \to \mathcal{E}_D \otimes \Omega^{i+1}_Y$$

for each i by

$$\nabla_i(e \otimes \omega) = \nabla(e) \otimes \omega + e \otimes d\omega,$$

where we are implicitly applying the multiplication map

$$(\mathcal{E}_D \otimes \Omega^1_Y) \otimes \Omega^i_Y \to \mathcal{E}_D \otimes \Omega^{i+1}_Y$$

to the first term. It follows from the Leibniz rule that ∇_i is well-defined and $\nabla_0 = \nabla$.

Definition 2.6.4. We call

$$\mathcal{E}_D \xrightarrow{\nabla} \mathcal{E}_D \otimes \Omega^1_Y \xrightarrow{\nabla_1} \mathcal{E}_D \otimes \Omega^2_Y \xrightarrow{\nabla_2} \cdots$$

the *de Rham complex of $\mathcal{E}$ associated to the embedding $X \hookrightarrow Y$*. (The use of the word "complex" will be justified by the theorem below.) We will denote it as $\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y})$ or $(\mathcal{E}_D \otimes \Omega^*_Y, \nabla)$, and will usually abbreviate ∇_i as ∇.

Remark 2.6.5. In the case of the structure sheaf $\mathcal{E} = \mathcal{O}_{X/S}$, we have $\mathcal{E}_D = \mathcal{D}$; we write the resulting connection as $\nabla_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes \Omega^1_{Y/S}$. Then $(\mathcal{D} \otimes \Omega^*_Y, \nabla_{\mathcal{D}})$ is a cdga when endowed with the obvious multiplication map

$$(\alpha \otimes \omega) \cdot (\alpha' \otimes \omega') = \alpha\alpha' \otimes (\omega \wedge \omega').$$

Theorem 2.6.6. *([2, Theorem 6.6]) The connection ∇ is integrable; that is, the map*

$$\nabla^2 : \mathcal{E}_D \otimes \Omega^i_Y \to \mathcal{E}_D \otimes \Omega^{i+2}_Y$$

vanishes for all i. Moreover, it has the following compatibility with the connection $\nabla_{\mathcal{D}}$: for all sections α of $\mathcal{D}$ and x of $\mathcal{E}_D$, we have

$$\nabla_{\mathcal{E}}(\alpha e) = e \otimes \nabla_{\mathcal{D}}(\alpha) + \alpha \nabla_{\mathcal{E}}(e)$$

as sections of $\mathcal{E} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} = \mathcal{E} \otimes_{\mathcal{D}} (\mathcal{D} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S})$.

Remark 2.6.7. As one can show by computing ∇_i, the compatibility condition above is equivalent to saying that the multiplication map

$$(\alpha \otimes \omega) \cdot (e \otimes \omega') = \alpha e \otimes (\omega \wedge \omega') \tag{2.6.7.1}$$

makes $(\mathcal{E}_D \otimes \Omega^*_Y, \nabla)$ a dg-module over the dg-algebra $(\mathcal{D} \otimes \Omega^*_Y, \nabla)$.

Lemma 2.6.8. *The PD-de Rham complex is functorial in $\mathcal{E}$ as a dg-module; that is, for any morphism $f : \mathcal{E} \to \mathcal{F}$ of crystals of $\mathcal{O}_{X/S}$-modules, the map*

$$f_D \otimes \mathrm{id} : \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} \to \mathcal{F}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}$$

of graded $\mathcal{D} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}$-modules is compatible with connections.

Proof. Tracing through Construction 2.6.3, every object and morphism appearing in the construction is functorial in the crystal $\mathcal{E}$. The resulting map $\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \to \mathcal{F}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$ induced by f is simply $f_D \otimes \mathrm{id}$, which is therefore compatible with ∇. □

Remark 2.6.9. In the proof above, we implicitly used the following general fact, various instantiations of which we will continue to use freely: if A^* is a dg-algebra over $\mathcal{D} \otimes \Omega^*_Y$ and $f : \mathcal{E}_D \otimes \Omega^*_Y \to A^*$ is a map of graded $\mathcal{D} \otimes \Omega^*_Y$-modules, then f is compatible with all ∇_i (and thus a map of dg-modules) if and only if it is compatible with $\nabla = \nabla_0$. The proof consists of observing that $\mathcal{E}_D \otimes \Omega^*_Y$ is generated as a graded module by its degree-0 component and then computing both $\nabla_i(e \otimes \omega)$ and $\nabla_i(f(e \otimes \omega))$ using the definition of ∇_i.

2.6.10. We next turn to the functoriality of the complex $\mathcal{E}_D \otimes \Omega^*_Y$ in the triple $(X \hookrightarrow Y/S)$. Our main case of interest is when $\mathcal{E}$ is an F-crystal and we are given a Frobenius lift ϕ_Y, which in particular is compatible with the absolute Frobenius endomorphisms F_X and F_S.

Construction 2.6.11. Suppose we are given a commutative diagram

$$\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\big\downarrow & & \big\downarrow \\
Y' & \xrightarrow{g} & Y \\
\big\downarrow & & \big\downarrow \\
S' & \longrightarrow & S,
\end{array}$$

where $X \hookrightarrow Y/S$ and $X' \hookrightarrow Y'/S'$ are as in Situation 2.6.2 and $S' \to S$ is a PD-morphism. Let D and D' denote the PD-envelopes $D_X(Y)$ and $D_{X'}(Y')$ respectively. Then for $i = 1$ and 2, we have a commutative square

$$\begin{tikzcd} D_{X'}(Y' \times Y') \arrow[r, "\widetilde{h}"] \arrow[d, "p_i'"'] & D_X(Y \times Y) \arrow[d, "p_i"] \\ D' \arrow[r, "h"] & D \end{tikzcd} \tag{2.6.11.1}$$

of PD-envelopes, where p_i and p_i' denote the projections, and the maps h and $\widetilde{h}$ are induced respectively by g and $g \times g$. Note that being PD-thickenings, the embeddings $X \hookrightarrow D$ and $X \hookrightarrow D_X(Y \times Y)$ are both homeomorphisms, and similarly for $X' \hookrightarrow D'$ and $X' \hookrightarrow D_{X'}(Y' \times Y')$. Under the resulting identifications of topological spaces, the projections p_i and p_i' are the identity maps, and $\widetilde{h}$ agrees with h.

Proposition 2.6.12. *In the situation of Construction 2.6.11, the map*

$$h^*_{\mathcal{E}} \otimes h^*_{\Omega} : h^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'}$$

is compatible with the connections on the two sides, where $h^*_{\mathcal{E}}$ denotes the pullback map

$$h^{-1}(\mathcal{E}_D) \to h^*(\mathcal{E}_D) \simeq (f^*_{\mathrm{cris}}\mathcal{E})_{D'}$$

(cf. Lemma 2.4.4) and h^*_{Ω} is the pullback map of differential forms.

Proof. Recall that the connection ∇ on $\mathcal{E}_D$ is induced by the map $\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1)$ in the following diagram of sheaves on $D_X(Y \times Y)$ (which we recall we can identify with X and D as topological spaces):

$$\begin{tikzcd} p_1^*\mathcal{E}_D \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathcal{E}_{D_X(Y \times_S Y)} & p_1^*\mathcal{E}_D \arrow[l, "\sim"'] \arrow[d, "\sim"] \\ \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \arrow[d, two heads] & & \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D \arrow[ll, "\sim", "\varepsilon"'] \arrow[d, two heads] \\ \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y & & \mathcal{D}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D \arrow[ll, "\sim", "\varepsilon_1"'] \\ & \mathcal{E}_D \arrow[ul, "\mathrm{id} \otimes 1"] \arrow[ur, "1 \otimes \mathrm{id}"'] & \end{tikzcd} \tag{2.6.12.1}$$

Similarly, the connection on $(f^*_{\mathrm{cris}}\mathcal{E})_{D'}$ is induced by the map $\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1)$ in the following diagram of sheaves on $D_{X'}(Y' \times Y')$ $(= X' = D'$ as topological spaces):

$$\begin{tikzcd}
p_1'^*(f^*_{\mathrm{cris}}\mathcal{E})_{D'} \arrow[r, "\sim"] \arrow[d, "\sim"'] & (f^*_{\mathrm{cris}}\mathcal{E})_{D_{X'}(Y' \times_{S'} Y')} & p_1'^*(f^*_{\mathrm{cris}}\mathcal{E})_{D'} \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
(f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}_{Y'} \arrow[d, two heads] & & \mathcal{D}_{Y'} \otimes_{\mathcal{O}_{Y'}} (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \arrow[ll, "\sim", "\varepsilon"'] \arrow[d, two heads] \\
(f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}^1_{Y'} & & \mathcal{D}^1_{Y'} \otimes_{\mathcal{O}_{Y'}} (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \arrow[ll, "\sim", "\varepsilon_1"'] \\
& (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \arrow[ul, "\mathrm{id} \otimes 1"] \arrow[ur, "1 \otimes \mathrm{id}"'] &
\end{tikzcd} \tag{2.6.12.2}$$

To prove the proposition, we will reconstruct the map $h^*_{\mathcal{E}} \otimes h^*_{\Omega}$ by giving pullback maps sending $\widetilde{h}^{-1}$ $(= h^{-1})$ of each object in diagram 2.6.12.1 to the corresponding object in 2.6.12.2, compatibly with the maps within each diagram. We have already seen the map

$$h^*_{\mathcal{E}} : h^{-1}(\mathcal{E}_D) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D'},$$

of the bottom objects; the pullback map

$$\widetilde{h}^{-1}(\mathcal{E}_{D_X(Y \times Y)}) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D_{X'}(Y' \times Y')}$$

is defined similarly. The maps

$$\begin{aligned}
&\widetilde{h}^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}_{Y'}, \\
&\widetilde{h}^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D) \to \mathcal{D}_{Y'} \otimes_{\mathcal{O}_{Y'}} (f^*_{\mathrm{cris}}\mathcal{E})_{D'}, \\
&\widetilde{h}^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}^1_{Y'}, \text{ and} \\
&\widetilde{h}^{-1}(\mathcal{D}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D) \to \mathcal{D}^1_{Y'} \otimes_{\mathcal{O}_{Y'}} (f^*_{\mathrm{cris}}\mathcal{E})_{D'}
\end{aligned}$$

are all defined by tensoring $h^*_{\mathcal{E}}$ with the respective pullback maps of structure sheaves. Finally, for $i = 1$ and 2, the pullback map

$$\widetilde{h}^{-1}(p_i^*\mathcal{E}_D) \to p_i'^*((f^*_{\mathrm{cris}}\mathcal{E})_{D'})$$

is defined as the composition

$$\widetilde{h}^{-1}(p_i^*\mathcal{E}_D) \to \widetilde{h}^*(p_i^*\mathcal{E}_D) = p_i'^*(h^*\mathcal{E}_D) \xrightarrow{p_i'^*(h^*_{\mathcal{E}})} p_i'^*((f^*_{\mathrm{cris}}\mathcal{E})_{D'}).$$

It is clear that these pullback maps are compatible with the diagonal and lower vertical maps in diagrams 2.6.12.1 and 2.6.12.2, and it follows by unraveling the definitions that they are also compatible with the upper vertical maps. By applying Lemma 2.4.5 to the square

2.6.11.1, it follows that they are also compatible with the top horizontal maps, and thus by construction with the isomorphisms ε and ε_1. This implies that the diagram

$$\begin{tikzcd} h^{-1}(\mathcal{E}_D) \arrow[r, "h^{-1}(\varepsilon_1\circ(1\otimes\mathrm{id})-(\mathrm{id}\otimes 1))"] \arrow[d, "h^*_{\mathcal{E}}"'] & \widetilde{h}^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y) \arrow[d, "h^*_{\mathcal{E}}\otimes\widetilde{h}^*"] \\ \mathcal{E}_{D'} \arrow[r, "\varepsilon_1\circ(1\otimes\mathrm{id})-(\mathrm{id}\otimes 1)"] & \mathcal{E}_{D'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}^1_{Y'} \end{tikzcd}$$

commutes, where $\widetilde{h}^*$ is the algebra map $\widetilde{h}^{-1}(\mathcal{D}^1_Y) \to \mathcal{D}^1_{Y'}$. Therefore the diagram

$$\begin{tikzcd} h^{-1}(\mathcal{E}_D) \arrow[r, "h^{-1}(\nabla)"] \arrow[d, "h^*_{\mathcal{E}}"'] & h^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}) \arrow[d] \\ \mathcal{E}_{D'} \arrow[r, "\nabla"] & \mathcal{E}_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^1_{Y'/S'} \end{tikzcd}$$

obtained by passing to subobjects commutes as well. But the right vertical map is precisely the degree-1 component of $h^*_{\mathcal{E}} \otimes h^*_{\Omega}$. So $h^*_{\mathcal{E}} \otimes h^*_{\Omega}$ is a map of graded modules, compatible with connections in degree 0, and thus also compatible with connections in all degrees by the definition of ∇_i. This completes the proof. $\square$

Corollary 2.6.13. *The morphism of Proposition 2.6.12 expresses $(f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'}$ as the dg-module tensor product*

$$(\mathcal{D}' \otimes \Omega^*_{Y'/S'}) \otimes_{h^{-1}(\mathcal{D}\otimes\Omega^*_{Y/S})} h^{-1}(\mathcal{E}_D \otimes \Omega^*_{Y/S}).$$

Proof. The map in question is by construction a map of graded modules over $h^{-1}(\mathcal{D}\otimes\Omega^*_{Y/S})$, and Proposition 2.6.12 says that it is in fact a map of dg-modules. Thus it factors through a map

$$(\mathcal{D}' \otimes \Omega^*_{Y'/S'}) \otimes_{h^{-1}(\mathcal{D}\otimes\Omega^*_{Y/S})} h^{-1}(\mathcal{E}_D \otimes \Omega^*_{Y/S}) \to (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'} \tag{2.6.13.1}$$

of dg-modules over $\mathcal{D}' \otimes \Omega^*_{Y'/S'}$. To check that this is an isomorphism, note that as graded modules we have

$$\begin{aligned} (\mathcal{D}' \otimes \Omega^*_{Y'/S'}) \otimes_{h^{-1}(\mathcal{D}\otimes\Omega^*_{Y/S})} h^{-1}(\mathcal{E}_D \otimes \Omega^*_{Y/S}) &\simeq (\mathcal{D}' \otimes \Omega^*_{Y'/S'}) \otimes_{h^{-1}\mathcal{D}} h^{-1}\mathcal{E}_D \\ &\simeq (\mathcal{D}' \otimes_{h^{-1}\mathcal{D}} h^{-1}\mathcal{E}_D) \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'} \\ &= h^*\mathcal{E}_D \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'} \\ &\simeq (f^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'}, \end{aligned}$$

and by comparing maps in degree 0 we see that this isomorphism agrees with 2.6.13.1. $\square$

Lemma 2.6.14. *The functorialities of* $\mathrm{dR}(\mathcal{E}_{X\hookrightarrow Y})$ *in* $\mathcal{E}$ *and in* $X \hookrightarrow Y/S$ *commute. Namely, if in the situation of Construction 2.6.11 we have two crystals* $\mathcal{E}$ *and* $\mathcal{F}$ *of* $\mathcal{O}_{X/S}$*-modules, then the diagram*

$$\begin{tikzcd} h^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) \arrow[r, "h^{-1}(f)\otimes\mathrm{id}"] \arrow[d, "h^*_{\mathcal{E}}\otimes h^*_{\Omega}"'] & h^{-1}(\mathcal{F}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) \arrow[d, "h^*_{\mathcal{F}}\otimes h^*_{\Omega}"] \\ (g^*_{\mathrm{cris}}\mathcal{E})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'} \arrow[r, "g^*_{\mathrm{cris}}(f)\otimes\mathrm{id}"] & (g^*_{\mathrm{cris}}\mathcal{F})_{D'} \otimes_{\mathcal{O}_{Y'}} \Omega^*_{Y'/S'} \end{tikzcd}$$

commutes.

Proof. It suffices to show that $h^*_{\mathcal{F}} \circ h^{-1}(f) = g^*_{\mathrm{cris}}(f) \circ h^*_{\mathcal{E}}$. But when we identify $g^*_{\mathrm{cris}}(-)_{D'}$ with $h^*((-)_D)$ via Lemma 2.4.4, this simply says that the base-change map $h^{-1}(\mathcal{E}_D) \to h^*(\mathcal{E}_D)$ is functorial in $\mathcal{E}$. □

Construction 2.6.15. Now suppose in Situation 2.6.2 that the PD-base S is

$$(\operatorname{Spec} W_r(k), (p), [\ ]),$$

and that $(\mathcal{E}, \phi_{\mathcal{E}})$ is an F-crystal on $\mathrm{Cris}(X/S)$. Let $F_S : S \to S$ be the Witt vector Frobenius, and suppose we are given an endomorphism $\phi_Y : Y \to Y$ over F_S which lifts the absolute Frobenius of Y_k and is compatible with $[\ ]$. Then applying Proposition 2.6.12 to the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, "\phi_Y"] \arrow[d] & Y \arrow[d] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

implies that the pullback map

$$\begin{aligned} h^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) &\to (F^*_{X,\mathrm{cris}}\mathcal{E})_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} \\ e \otimes \omega &\mapsto h^*(e) \otimes \phi^*_Y(\omega) \end{aligned}$$

is compatible with the connections, where $h : D \to D$ is the map induced by (F_X, ϕ_Y). Thus this is a morphism of dg-modules over $h^{-1}(\mathcal{D} \otimes \Omega^*_{Y/S})$, where this acts on the source as in Remark 2.6.7 and on the target via pullback. The functoriality of $\mathcal{E}_D \otimes \Omega^*_{Y/S}$ in the crystal $\mathcal{E}$ provides a further dg-module map

$$\phi_{\mathcal{E}} \otimes \mathrm{id} : (F^*_{X,\mathrm{cris}}\mathcal{E})_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S} \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S},$$

which is an isomorphism if $(\mathcal{E}, \phi_{\mathcal{E}})$ is unit-root. Composing these yields a map

$$\phi : h^{-1}(\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S},$$

which we call the *functorial* or *undivided Frobenius endomorphism* of $\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}$. This is a map of dg-modules over $h^{-1}(\mathcal{D} \otimes \Omega^*_{Y/S})$, where again this acts on the source in the obvious way and on the target via h^*. Note that if $(\mathcal{E}, \phi_{\mathcal{E}})$ is unit-root, then this is a dg-module base change map as in Corollary 2.6.13.

2.6.16. For the rest of this section, we will take sections of the previous constructions on affines, write down a divided Frobenius operator, and pass to the limit as $r \to \infty$. Our goal (Remark 2.6.20) is to interpret the de Rham complex of $\mathcal{E}$ as a Dieudonné complex equipped with a suitable module structure over the de Rham complex of $\mathcal{O}_{X/S}$.

Construction 2.6.17. Suppose R, A, B, ϕ_A, and ϕ_B are as in the Frobenius-embedded situation 1.7.8, and set $X = \operatorname{Spec} R$, $Y_r = \operatorname{Spec} A_r$, and $S_r = \operatorname{Spec} W_r(k)$ for each $r > 0$. Then the PD-envelope of $X \hookrightarrow Y_r$ is $D_r = \operatorname{Spec} B_r$, which carries the PD-structure γ and the Frobenius lift $\phi_{D_r} = \phi_{B_r}$ induced by F_R and ϕ_{A_r}. Applying Construction 2.6.15 yields a map

$$\phi = (\phi_{D_r})^*_{\mathcal{E}} \otimes \phi^*_{Y_r} : \phi^{-1}_{D_r}(\mathcal{E}_{D_r} \otimes_{\mathcal{O}_{Y_r}} \Omega^*_{Y_r/S}) \to \mathcal{E}_{D_r} \otimes_{\mathcal{O}_{Y_r}} \Omega^*_{Y_r/S}$$

of dg-modules over $\phi^{-1}_{D_r}(\mathcal{D}_r \otimes \Omega^*_{Y_r/S})$. Taking global sections yields the de Rham complex

$$\mathrm{dR}(\mathcal{E}(X \hookrightarrow Y_r)) := (\mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}, \nabla)$$

and an endomorphism (as complexes of abelian groups)

$$\phi = \phi^*_{\mathcal{E}} \otimes \phi^*_{A_r} : \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r} \to \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r},$$

which is a semilinear map of dg-$(B_r \otimes_{A_r} \Omega^*_{A_r})$-modules in that we have

$$\phi((b \otimes \omega) \cdot (e \otimes \omega')) = (\phi_{B_r}(b) \otimes \phi^*_{A_r}(\omega)) \cdot \phi(e \otimes \omega')$$

for all $b \in B_r, e \in \mathcal{E}(B_r)$, and $\omega, \omega' \in \Omega^*_{A_r}$. Finally, we pass to the limit as $r \to \infty$ to obtain the de Rham complex

$$\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) := \lim_r \mathrm{dR}(\mathcal{E}(X \hookrightarrow Y_r)),$$

which is a dg-module over the cdga

$$\widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet)) = \lim_r (B_r \otimes_{A_r} \Omega^*_{A_r}, \nabla),$$

and carries an undivided Frobenius endomorphism

$$\phi : \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) \to \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)),$$

which is semilinear with respect to the algebra map

$$\lim_r (\phi_{B_r} \otimes \phi^*_{A_r}) : \widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet)) \to \widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet)).$$

Construction 2.6.18. In the situation of Construction 2.6.17, we also have *divided* Frobenius maps

$$F : \mathcal{E}(B_r) \otimes_{A_r} \Omega^i_{A_r} \to \mathcal{E}(B_r) \otimes_{A_r} \Omega^i_{A_r}$$

for each r, defined by

$$F(e \otimes \omega) = \phi_{\mathcal{E}}(e) \otimes F(\omega),$$

where the latter F is the divided Frobenius given by taking the map of Proposition 1.2.3 modulo p^r. These are maps of graded groups, not of complexes; we have $p^i F = \phi$ as endomorphisms of the degree-i piece of the complex. Passing to the limit, these induce a divided Frobenius endomorphism

$$F : \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) \to \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)).$$

Proposition 2.6.19. *In the situation above:*

1. *For each i, the maps ∇F and $pF\nabla : \mathcal{E}(B_r) \otimes_{A_r} \Omega^i_{A_r} \to \mathcal{E}(B_r) \otimes_{A_r} \Omega^{i+1}_{A_r}$ coincide.*
2. *The divided Frobenius $F : \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r} \to \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}$ is a semilinear map of graded $B_r \otimes_{A_r} \Omega^*_{A_r}$-modules, in the sense that for all $b \in B_r$, $e \in \mathcal{E}(B_r)$, and $\omega, \omega' \in \Omega^*_{A_r}$, we have*
$$F((b \otimes \omega) \cdot (e \otimes \omega')) = F(b \otimes \omega) \cdot F(e \otimes \omega').$$
3. *The analogous compatibilities hold for the completed de Rham complex $\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))$ with its $\widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet))$-module structure and divided Frobenius.*

Proof. Part (3) follows from parts (1) and (2) by passage to the limit. Part (2) follows from a calculation using only the definitions and the compatibility of F with the multiplication maps $B_r \otimes \mathcal{E}(B_r) \to \mathcal{E}(B_r)$ and $\Omega^*_{A_r} \otimes \Omega^*_{A_r} \to \Omega^*_{A_r}$:

$$\begin{aligned} F((b \otimes \omega) \cdot (e \otimes \omega')) &= F(be \otimes (\omega \wedge \omega')) \\ &= \phi_{\mathcal{E}}(be) \otimes F(\omega \wedge \omega') \\ &= (\phi_{B_r}(b) \cdot \phi_{\mathcal{E}}(e)) \otimes (F(\omega) \wedge F(\omega')) \\ &= (\phi_{B_r}(b) \otimes F(\omega)) \cdot (\phi_{\mathcal{E}}(e) \otimes F(\omega')) \\ &= F(b \otimes \omega) \cdot F(e \otimes \omega'). \end{aligned}$$

As for part (1), we first consider the finite-level statement in degree $i = 0$. By Construction 2.6.17, the diagram

$$\begin{array}{ccc} \mathcal{E}(B_r) & \xrightarrow{\nabla} & \mathcal{E}(B_r) \otimes_{A_r} \Omega^1_{A_r} \\ \downarrow{\scriptstyle \phi_{\mathcal{E}}} & & \downarrow{\scriptstyle \phi_{\mathcal{E}} \otimes \phi^*_{A_r}} \\ \mathcal{E}(B_r) & \xrightarrow{\nabla} & \mathcal{E}(B_r) \otimes_{A_r} \Omega^1_{A_r}, \end{array}$$

commutes. The left vertical map here equals our F, and the right vertical map is our pF, so the identity holds in this case.

Now take $i > 0$, and choose an input element of the form $e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)$. Since $(\mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}, \nabla)$ is a dg-module over $(B_r \otimes_{A_r} \Omega^*_{A_r}, \nabla)$ and F is multiplicative, we can compute everything in terms of the differentials on $\mathcal{E}(B_r)$ and $B_r \otimes_{A_r} \Omega^*_{A_r}$:

$$
\begin{aligned}
& pF\nabla(e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)) \\
&= pF\nabla((a \cdot da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1)) \\
&= pF\left((da \wedge da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1) + (-1)^i (a \cdot da_1 \wedge \cdots \wedge da_i) \cdot \nabla(e \otimes 1)\right) \\
&= p\left((Fda \wedge Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1) + (-1)^i (Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(\nabla(e \otimes 1))\right) \\
&= (dFa \wedge Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1) + (-1)^i (Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot \nabla(F(e \otimes 1)) \\
&= \nabla\left((Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1)\right) \\
&= \nabla F\left((a \cdot da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1)\right) \\
&= \nabla F\left(e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)\right).
\end{aligned}
$$

This completes the proof. □

Remark 2.6.20. Proposition 2.6.19 tells us that $\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))$ is a Dieudonné complex, and moreover that its dg-module structure over the cdga $\widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet))$ is compatible with the divided Frobenius endomorphisms. Once we have discussed module objects in **DC** (cf. Lemma 3.2.6), this will tell us (taking $\mathcal{E} = \mathcal{O}_{X/W}$) that $\widehat{\mathrm{dR}}(\mathcal{O}_{X/W}(X \hookrightarrow Y_\bullet))$ is an algebra object in **DC**, and (for $\mathcal{E}$ arbitrary) that $\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))$ is a module over it in **DC**. In fact the former is a Dieudonné algebra, since it is concentrated in nonnegative degrees and we have $F(x) \equiv x^p \pmod{p}$ in degree 0 by Remark 1.7.9.

Remark 2.6.21. If both R/k and A/W are smooth algebras, then it follows by [2, Corollary 3.35] that the completed PD-envelope B is p-torsionfree. In fact, since the A_r-modules $\Omega^i_{A_r}$ and the B_r-module $\mathcal{E}(B_r)$ are locally free in this case, one can show that

$$\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) = \lim_r \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}$$

is p-torsionfree as well—but we will not need this.

2.7 PD-de Rham complexes associated to a crystal

2.7.1. In this section we will collect some remarks on the PD-de Rham complex associated to $\mathcal{E}$ over an object T of $\mathrm{Cris}(X/S)$, following [8, pp. II, 1.1]. We will ultimately be interested in the cases where $X = \operatorname{Spec} R$ is an affine k-scheme and T is Spec of one of the following: a lift A_r as in Situation 1.7.6, a PD-envelope B_r of a smooth embedding as in Situation 1.7.8, or the ring of truncated Witt vectors $W_r(R)$. We will first recall the constructions in sheafy language; we will restrict to the affine case and take sections in Construction 2.7.16.

Situation 2.7.2. As usual, let k be a perfect field of characteristic p, X a k-scheme, and $S = (\operatorname{Spec} W_r(k), (p), [\ ])$. Let $\mathcal{E}$ be a crystal of $\mathcal{O}_{X/S}$-modules, let $T = (U \hookrightarrow T, \gamma)$ be an object of $\mathrm{Cris}(X/S)$, and let $\mathcal{I} \subset \mathcal{O}_T$ denote the ideal sheaf cutting out U.

2.7.3. We first recall the construction of the nth-order PD-compatible PD-envelope $D^n_{T/S,\gamma}$—a slight modification of $D^n_{T/S}$ which carries divided powers on the ideal cutting out U instead of T, and which therefore defines an object in $\mathrm{Cris}(X/S)$. We will then use this to write down the PD-connection on $\mathcal{E}$.

Construction 2.7.4. ([8, pp. II, 1.1.3]) Let $\mathcal{J}$ denote the ideal sheaf cutting out the diagonal in $T \times_S T$, generated by sections of the form $x \otimes 1 - 1 \otimes x$. Let $(D_{T/S}, \overline{\mathcal{J}}, \delta)$ be the PD-envelope of $(T \times_S T, \mathcal{J})$ over S, and $D^n_{T/S}$ be the n-th order neighborhood—cut out of $D_{T/S}$ by the ideal sheaf $\overline{\mathcal{J}}^{[n+1]}$, and with PD-ideal also denoted $(\overline{\mathcal{J}}, \delta)$ by abuse of notation. This is in particular a nilpotent thickening of the diagonally embedded T. For brevity, we will write $\mathcal{D}_{T/S}$ and $\mathcal{D}^n_{T/S}$ for $\mathcal{O}_{D_{T/S}}$ and $\mathcal{O}_{D^n_{T/S}}$ respectively.

Note that the natural map

$$\mathcal{O}_T \simeq \mathcal{O}_{T \times T}/\mathcal{J} \xrightarrow{\sim} \mathcal{D}^n_{T/S}/\overline{\mathcal{J}}$$

is an isomorphism by [2, Remark 3.20.4]. The resulting quotient map $\pi : \mathcal{D}^n_{T/S} \to \mathcal{O}_T$ admits two splittings $s_1, s_2 : \mathcal{O}_T \to \mathcal{D}^n_{T/S}$ given by the compositions

$$\mathcal{O}_T \underset{1 \otimes \mathrm{id}}{\overset{\mathrm{id} \otimes 1}{\rightrightarrows}} \mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_T \longrightarrow \mathcal{D}^n_{T/S}.$$

Thus we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathcal{J}} \arrow[r] & \mathcal{D}^n_{T/S} \arrow[r, "\pi"] & \mathcal{O}_T \arrow[r] \arrow[l, bend left, "s_1, s_2"] & 0, \end{tikzcd}$$

where each splitting induces a decomposition $\mathcal{D}^n_{T/S} \simeq \overline{\mathcal{J}} \oplus \mathcal{O}_T$. Now consider the kernel $\mathcal{K}$ of the composition $\mathcal{D}^n_{T/S} \xrightarrow{\pi} \mathcal{O}_T \twoheadrightarrow \mathcal{O}_U$, which each of the two splittings identifies with $\overline{\mathcal{J}} \oplus \mathcal{I}$. Since $\mathcal{I}$ comes with the PD-structure γ, [1, I, Proposition 1.6.5] implies that for each of the splittings s_1, s_2 there is a corresponding PD-structure γ', γ'' on $\mathcal{K}$ that restricts to δ on $\overline{\mathcal{J}}$ and γ on $\mathcal{I}$. It follows by [8, II, Corollaire 1.1.2] that there is a universal quotient of the pair $(\mathcal{D}^n_{T/S}, \mathcal{K})$ with a PD-structure compatible with both γ' and γ''. We call this $\mathcal{D}^n_{T/S,\gamma}$; abusing notation again, we call its PD-ideal $(\mathcal{K}, \delta)$. We call $D^n_{T/S,\gamma}$ the corresponding closed subscheme of $D^n_{T/S}$. Note that we have $\mathcal{D}^n_{T/S,\gamma}/\mathcal{K} = \mathcal{D}^n_{T/S}/\mathcal{K} = \mathcal{O}_U$, so that $(U \hookrightarrow D^n_{T/S,\gamma}, \delta)$ is an object of $\mathrm{Cris}(X/S)$.

Construction 2.7.5. ([8, II, Proposition 1.1.5]) Let $\mathcal{E}$ be a crystal of $\mathcal{O}_{X/S}$-modules and $T = (U, T, \delta)$ an object of $\mathrm{Cris}(X/S)$. Let $n > 0$ be arbitrary for the moment. Let

$$\mathrm{pr}_1, \mathrm{pr}_2 : D^n_{T/S,\gamma} \rightrightarrows T$$

be the PD-morphisms induced by the two projections $T \times T \rightrightarrows T$. Then we construct the following diagram of $\mathcal{D}^n_{T/S,\gamma}$-modules, where the top horizontal maps are isomorphisms because $\mathcal{E}$ is a crystal, and ε is a $\mathcal{D}^n_{T/S,\gamma}$-linear isomorphism defined by going the long way around the diagram.

$$\begin{tikzcd}
\mathrm{pr}_1^* \mathcal{E}_T \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{E}_{D^n_{T/S,\gamma}} & \mathrm{pr}_2^* \mathcal{E}_T \arrow[l, "\sim"'] \arrow[d, equal] \\
\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}^n_{T/S,\gamma} & & \mathcal{D}^n_{T/S,\gamma} \otimes_{\mathcal{O}_T} \mathcal{E}_T \arrow[ll, "\sim", "\varepsilon_n"']
\end{tikzcd}$$

Taking $n = 1$ gives an isomorphism

$$\varepsilon_1 : \mathcal{D}^1_{T/S,\gamma} \otimes_{\mathcal{O}_T} \mathcal{E}_T \xrightarrow{\sim} \mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}^1_{T/S,\gamma},$$

which lets us form the noncommutative diagram

$$\begin{tikzcd}
& \mathcal{E}_T \arrow[dl, "\mathrm{id} \otimes 1"'] \arrow[dr, "1 \otimes \mathrm{id}"] & \\
\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}^1_{T/S,\gamma} & & \mathcal{D}^1_{T/S,\gamma} \otimes_{\mathcal{O}_T} \mathcal{E}_T, \arrow[ll, "\sim", "\varepsilon_1"']
\end{tikzcd}$$

of abelian sheaves on T. (Here we are identifying the underlying topological spaces of T and $D^1_{T/S,\gamma}$, and thus viewing $\mathcal{D}^1_{T/S,\gamma}$ as a sheaf of rings on T which has two $\mathcal{O}_T$-algebra structures. The maps $\mathrm{id} \otimes 1$ and $1 \otimes \mathrm{id}$ are $\mathcal{O}_T$-linear for the left and right $\mathcal{O}_T$-module structures of their targets respectively.) We then build the map

$$\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1) : \mathcal{E}_T \to \mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}^1_{T/S,\gamma}.$$

This factors through the subsheaf

$$\mathcal{E}_T \otimes_{\mathcal{O}_T} \overline{\mathcal{J}}/\overline{\mathcal{J}}^{[2]},$$

and we define our PD-connection to be the resulting map

$$\nabla : \mathcal{E}_T \to \mathcal{E}_T \otimes_{\mathcal{O}_T} \overline{\mathcal{J}}/\overline{\mathcal{J}}^{[2]} \simeq \mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^1_{T/S,\gamma},$$

where this last isomorphism is furnished by [8, II, Lemma 1.1.4]. To see that this satisfies the Leibniz rule, let α and e be sections of $\mathcal{O}_T$ and $\mathcal{E}_T$ respectively, and calculate in $\mathcal{E}_T \otimes \mathcal{D}^1_{T/S,\gamma}$:

$$\begin{aligned}
\nabla(\alpha e) &= \varepsilon_1(1 \otimes \alpha e) - \alpha e \otimes 1 \\
&= \varepsilon_1((1 \otimes \alpha) \otimes e) - (e \otimes (\alpha \otimes 1)) \\
&= (1 \otimes \alpha) \cdot \varepsilon_1(1 \otimes e) - (\alpha \otimes 1) \cdot (e \otimes 1) \\
&= (1 \otimes \alpha) \cdot \varepsilon_1(1 \otimes e) - (1 \otimes \alpha) \cdot (e \otimes 1) + d\alpha \cdot (e \otimes 1) \\
&= (1 \otimes \alpha) \cdot \nabla(e) + d\alpha \cdot (e \otimes 1) \\
&= \alpha \cdot \nabla(e) + e \otimes d\alpha.
\end{aligned}$$

As before, we then define

$$\nabla_i : \mathcal{E}_T \otimes \Omega^i_{T/S,\gamma} \to \mathcal{E}_T \otimes \Omega^{i+1}_{T/S,\gamma}$$

for each i by

$$\nabla_i(e \otimes \omega) = \nabla(e) \otimes \omega + e \otimes d\omega,$$

where we are implicitly applying the multiplication map

$$(\mathcal{E}_T \otimes \Omega^1_{T/S,\gamma}) \otimes \Omega^i_{T/S,\gamma} \to \mathcal{E}_T \otimes \Omega^{i+1}_{T/S,\gamma}$$

to the first term. Since ∇ is a PD-connection, ∇_i is well-defined for all i and we have $\nabla_0 = \nabla$.

Definition 2.7.6. We will call

$$\mathcal{E}_T \xrightarrow{\nabla} \mathcal{E}_T \otimes \Omega^1_{T/S,\gamma} \xrightarrow{\nabla_1} \mathcal{E}_T \otimes \Omega^2_{T/S,\gamma} \xrightarrow{\nabla_2} \cdots$$

the *PD-de Rham complex associated to $\mathcal{E}$ over T*, and denote it by $\mathrm{dR}(\mathcal{E}_{T,\gamma})$ or $(\mathcal{E}_T \otimes \Omega^*_{T/S,\gamma}, \nabla)$. We will usually abbreviate ∇_i as ∇.

Proposition 2.7.7.

1. *We have $\nabla^2 = 0$, so that $\mathrm{dR}(\mathcal{E}_{T,\gamma})$ is indeed a complex.*

2. *The multiplication map*
$$\omega \cdot (e \otimes \omega') = e \otimes (\omega \wedge \omega') \tag{2.7.7.1}$$
makes $\mathrm{dR}(\mathcal{E}_{T,\gamma})$ a dg-module over $\Omega^*_{T/S,\gamma}$.

Proof. Part (1) is proved in [8, Proposition 1.1.5]. For part (2), the formula clearly defines a graded module structure; we must only show that for all sections e of $\mathcal{E}_T$ and ω, ω' of $\Omega^*_{T,\gamma}$ in degrees i and j respectively, we have

$$\nabla(\omega \cdot (e \otimes \omega')) = d\omega \cdot (e \otimes \omega') + (-1)^i \omega \cdot \nabla(e \otimes \omega').$$

To prove this, we first rewrite $\nabla_i(e \otimes \omega)$ as $(-1)^i \omega \cdot \nabla(e) + e \otimes d\omega$ for convenience, using the graded commutativity of $\Omega^*_{T/S,\gamma}$. Then we calculate:

$$\begin{aligned}
\nabla(\omega \cdot (e \otimes \omega')) &= \nabla(e \otimes (\omega \wedge \omega')) \\
&= (-1)^{i+j}(\omega \wedge \omega') \cdot \nabla(e) + e \otimes d(\omega \wedge \omega') \\
&= (-1)^{i+j}(\omega \wedge \omega') \cdot \nabla(e) + e \otimes \big((d\omega \wedge \omega') + (-1)^i \omega \wedge d\omega'\big) \\
&= e \otimes (d\omega \wedge \omega') + (-1)^i \omega \cdot \big((-1)^j \omega' \cdot \nabla(e) + e \otimes d\omega'\big) \\
&= d\omega \cdot (e \otimes \omega') + (-1)^i \omega \cdot \nabla(e \otimes \omega').
\end{aligned}$$

$\square$

Remark 2.7.8. Although only the first-order PD-compatible PD-envelope $D^1_{T/S,\gamma}$ is needed in Construction 2.7.5, the proof that $\nabla^2 = 0$ requires evaluating $\mathcal{E}$ on $D^n_{T/S,\gamma}$ for $n > 1$. This amounts to the correspondence between PD-stratifications and integrable connections; cf. [2, Theorem 4.8].

Proposition 2.7.9. *The PD-de Rham complex is functorial in $\mathcal{E}$ as a dg-module; that is, for any morphism $f : \mathcal{E} \to \mathcal{F}$ of crystals, the graded $\Omega^*_{T,\gamma}$-module map*

$$f_T \otimes \mathrm{id} : \mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma} \to \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}$$

is compatible with the connections.

Proof. Tracing through Construction 2.7.5, every object and morphism appearing in the construction is functorial in the crystal $\mathcal{E}$. The resulting map $\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^1_{T,\gamma} \to \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^1_{T,\gamma}$ induced by f is simply $f_T \otimes \mathrm{id}$, which is therefore compatible with ∇. □

Next we will examine the behavior of our PD-de Rham complexes when we have a PD-morphism $h : T' \to T$ over $g : X' \to X$ as in Situation 2.4.2.

Lemma 2.7.10. *In Situation 2.4.2, the pullback map $h^{-1}(\Omega^*_T) \to \Omega^*_{T'}$ induces a map $h^{-1}(\Omega^*_{T,\gamma}) \to \Omega^*_{T',\gamma'}$ by passage to the quotient.*

Proof. Since $\Omega^*_{T,\gamma}$ is the quotient of Ω^*_T by the dg-ideal generated by sections of the form $d\gamma_n(x) - \gamma_{n-1}(x)dx$, we must only show that the pullback of every such section vanishes in $\Omega^*_{T',\gamma'}$. But since h is a PD-morphism, we have

$$\begin{aligned} h^*(d\gamma_n(x) - \gamma_{n-1}(x)dx) &= d(h^*(\gamma_n(x))) - h^*(\gamma_{n-1}(x)) \\ &= d\gamma_n(h^*(x)) - \gamma_{n-1}(h^*(x)), \end{aligned}$$

which evidently vanishes in $\Omega^*_{T',\gamma'}$. □

Proposition 2.7.11. *In Situation 2.4.2, the map*

$$h^*_{\mathcal{E}} \otimes h^*_{\Omega} : h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}) \to (g^*_{\mathrm{cris}}\mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'}$$

is compatible with the connections on the two sides, where $h^*_{\mathcal{E}}$ denotes the pullback map

$$h^{-1}(\mathcal{E}_T) \to h^*(\mathcal{E}_T) \simeq (g^*_{\mathrm{cris}}\mathcal{E})_{T'}$$

(cf. Lemma 2.4.4) and h^*_{Ω} is the pullback map of Lemma 2.7.10.

Proof. For convenience, we will write $D = (D, \overline{\mathcal{J}}, \delta)$ and $D' = (D', \overline{\mathcal{J}}', \delta')$ for the first-order PD-compatible PD-envelopes $D^1_{T/S,\gamma}$ and $D^1_{T'/S',\gamma'}$ respectively. We then have a commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{D_h} & D \\ {\scriptstyle \mathrm{pr}'_i}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\ T' & \xrightarrow{h} & T, \end{array} \tag{2.7.11.1}$$

where pr_i and pr'_i are the projections (each for $i = 1$ and 2), and D_h is the PD-morphism induced by

$$h \times h : T' \times_{S'} T' \to T \times_S T.$$

Note that we may identify the underlying topological spaces of T and T' with those of D and D' via the respective diagonal embeddings. When we do so, the projections $\mathrm{pr}_1, \mathrm{pr}_2 : D \to T$ and $\mathrm{pr}'_1, \mathrm{pr}'_2 : D' \to T'$ become the identity; moreover, D_h is identified with h.

Recall that the PD-connection ∇ on $\mathcal{E}_T$ is induced by the map $\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1)$ in the following diagram of sheaves on D ($= T$ as topological spaces):

$$\begin{tikzcd}
\mathrm{pr}_1^* \mathcal{E}_T \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{E}_D & \mathrm{pr}_2^* \mathcal{E}_T \arrow[l, "\sim"'] \arrow[d, equal] \\
\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D} & & \mathcal{D} \otimes_{\mathcal{O}_T} \mathcal{E}_T \arrow[ll, "\sim", "\varepsilon_1"'] \\
& \mathcal{E}_T \arrow[ul, "\mathrm{id} \otimes 1"] \arrow[ur, "1 \otimes \mathrm{id}"'] &
\end{tikzcd} \tag{2.7.11.2}$$

Similarly, the PD-connection on $(g_{\mathrm{cris}}^* \mathcal{E})_{T'}$ is induced by the map $\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1)$ in the following diagram of sheaves on D' ($= T'$ as topological spaces):

$$\begin{tikzcd}
\mathrm{pr}_1'^* (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \arrow[r, "\sim"] \arrow[d, equal] & (g_{\mathrm{cris}}^* \mathcal{E})_{D'} & \mathrm{pr}_2'^* (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \arrow[l, "\sim"'] \arrow[d, equal] \\
(g_{\mathrm{cris}}^* \mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \mathcal{D}' & & \mathcal{D}' \otimes_{\mathcal{O}_{T'}} (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \arrow[ll, "\sim", "\varepsilon_1"'] \\
& (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \arrow[ul, "\mathrm{id} \otimes 1"] \arrow[ur, "1 \otimes \mathrm{id}"'] &
\end{tikzcd} \tag{2.7.11.3}$$

The proof is analogous to that of Proposition 2.6.12: we will reconstruct $h_{\mathcal{E}}^* \otimes h_{\Omega}^*$ by giving pullback maps sending D_h^{-1} ($= h^{-1}$) of each object in diagram 2.7.11.2 to the corresponding object in 2.7.11.3, compatibly with the maps within each diagram. We have already seen the pullback map

$$h_{\mathcal{E}}^* : h^{-1}(\mathcal{E}_T) \to (g_{\mathrm{cris}}^* \mathcal{E})_{T'}$$

of the bottom objects, and the map

$$D_h^{-1}(\mathcal{E}_D) \to (g_{\mathrm{cris}}^* \mathcal{E})_{D'}$$

is defined analogously. The maps

$$\begin{aligned} h_{\mathcal{E}}^* \otimes D_h^* &: D_h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}) \to (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \mathcal{D}' \text{ and} \\ D_h^* \otimes h_{\mathcal{E}}^* &: D_h^{-1}(\mathcal{D} \otimes_{\mathcal{O}_T} \mathcal{E}_T) \to \mathcal{D}' \otimes_{\mathcal{O}_{T'}} (g_{\mathrm{cris}}^* \mathcal{E})_{T'} \end{aligned}$$

are defined by tensoring $h^*_{\mathcal{E}}$ with the pullback map $D^*_h : D_h^{-1}(\mathcal{D}) \to D^*_h(\mathcal{D}) = \mathcal{D}'$ of structure sheaves. Finally, the pullback map

$$D_h^{-1}(\mathrm{pr}_i^* \mathcal{E}_T) \to \mathrm{pr}_i'^*((g^*_{\mathrm{cris}}\mathcal{E})_{T'})$$

is defined as the composition

$$D_h^{-1}(\mathrm{pr}_i^* \mathcal{E}_T) \to D_h^*(\mathrm{pr}_i^* \mathcal{E}_T) = \mathrm{pr}_i'^*(h^*\mathcal{E}_T) \xrightarrow{\mathrm{pr}_i'^*(h^*_{\mathcal{E}})} \mathrm{pr}_i'^*((g^*_{\mathrm{cris}}\mathcal{E})_{T'}).$$

It is clear that the pullback maps we have constructed are compatible with the vertical and diagonal maps in diagrams 2.7.11.2 and 2.7.11.3. By applying Lemma 2.4.5 to the square 2.7.11.1 (for $i = 1$ and 2), it follows that they are also compatible with the top horizontal maps. Therefore, they are also compatible with the isomorphisms ε_1. This shows that the diagram

$$\begin{tikzcd}
h^{-1}(\mathcal{E}_T) \arrow[r, "h^{-1}(\varepsilon_1\circ(1\otimes\mathrm{id})-(\mathrm{id}\otimes 1))"] \arrow[d, "h^*_{\mathcal{E}}"'] & h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \mathcal{D}) \arrow[d, "h^*_{\mathcal{E}}\otimes D^*_h"] \\
\mathcal{E}_{T'} \arrow[r, "\varepsilon_1\circ(1\otimes\mathrm{id})-(\mathrm{id}\otimes 1)"] & \mathcal{E}_{T'} \otimes_{\mathcal{O}_{T'}} \mathcal{D}'
\end{tikzcd}$$

commutes, as does the diagram

$$\begin{tikzcd}
h^{-1}(\mathcal{E}_T) \arrow[r, "h^{-1}(\nabla)"] \arrow[d, "h^*_{\mathcal{E}}"'] & h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^1_{T,\gamma}) \arrow[d] \\
\mathcal{E}_{T'} \arrow[r, "\nabla"] & \mathcal{E}_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^1_{T',\gamma'}
\end{tikzcd}$$

obtained by passing to subobjects. But the right vertical map is precisely the degree-1 component of $h^*_{\mathcal{E}} \otimes h^*_{\Omega}$. So $h^*_{\mathcal{E}} \otimes h^*_{\Omega}$ is a map of graded modules, compatible with connections in degree 0, and thus also compatible with connections in all degrees by the definition of ∇_i. This completes the proof. □

Corollary 2.7.12. *The morphism of Proposition 2.7.11 expresses* $\mathrm{dR}((g^*_{\mathrm{cris}}\mathcal{E})_{T',\gamma'})$ *as the dg-module tensor product* $\Omega^*_{T',\gamma'} \otimes_{h^{-1}(\Omega^*_{T,\gamma})} h^{-1}(\mathrm{dR}(\mathcal{E}_{T,\gamma}))$.

Proof. The map in question is clearly a map of graded modules over $h^{-1}(\Omega^*_{T,\gamma})$, and Proposition 2.7.11 says that it is in fact a map of dg-modules. Since the target is a dg-module over $\Omega^*_{T',\gamma'}$, we get a map

$$\Omega^*_{T',\gamma'} \otimes_{h^{-1}(\Omega^*_{T,\gamma})} h^{-1}(\mathrm{dR}(\mathcal{E}_{T,\gamma})) \to \mathrm{dR}(\mathcal{E}_{T',\gamma'}) \tag{2.7.12.1}$$

of dg-$\Omega^*_{T',\gamma'}$-modules. To check that this is an isomorphism, note that as graded modules we have

$$\begin{aligned}\Omega^*_{T',\gamma'} \otimes_{h^{-1}(\Omega^*_{T,\gamma})} h^{-1}(\mathrm{dR}(\mathcal{E}_{T,\gamma})) &\simeq \Omega^*_{T',\gamma'} \otimes_{h^{-1}(\Omega^*_{T,\gamma})} \left(h^{-1}\mathcal{E}_T \otimes_{h^{-1}\mathcal{O}_T} h^{-1}(\Omega^*_{T,\gamma})\right)\\ &\simeq h^{-1}\mathcal{E}_T \otimes_{h^{-1}\mathcal{O}_T} \Omega^*_{T',\gamma'}\\ &\simeq (h^{-1}\mathcal{E}_T \otimes_{h^{-1}\mathcal{O}_T} \mathcal{O}_{T'}) \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'}\\ &\simeq h^*\mathcal{E}_T \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'}\\ &\simeq \mathcal{E}_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'}\\ &\simeq \mathrm{dR}(\mathcal{E}_{T',\gamma'}),\end{aligned}$$

and by comparing maps in degree 0 we see that this isomorphism agrees with 2.7.12.1. □

Lemma 2.7.13. *The functorialities of* $\mathrm{dR}(\mathcal{E}_{T,\gamma})$ *in* $\mathcal{E}$ *and in* T *commute. Namely, given a PD-morphism* $h : T' \to T$ *over* $g : X' \to X$ *and a morphism* $f : \mathcal{E} \to \mathcal{F}$ *of crystals of* $\mathcal{O}_{X/S}$*-modules, the diagram*

$$\begin{tikzcd} h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}) \arrow[r, "h^{-1}(f)\otimes\mathrm{id}"] \arrow[d, "h^*_{\mathcal{E}}\otimes h^*_{\Omega}"'] & h^{-1}(\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}) \arrow[d, "h^*_{\mathcal{F}}\otimes h^*_{\Omega}"] \\ (g^*_{\mathrm{cris}}\mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'} \arrow[r, "g^*_{\mathrm{cris}}(f)\otimes\mathrm{id}"] & (g^*_{\mathrm{cris}}\mathcal{F})_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'} \end{tikzcd}$$

commutes.

Proof. It suffices to show that $h^*_{\mathcal{F}}\circ h^{-1}(f) = g^*_{\mathrm{cris}}(f)\circ h^*_{\mathcal{E}}$. But when we identify $g^*_{\mathrm{cris}}(-)_{T'}$ with $h^*((-)_T)$ via Lemma 2.4.4, this simply says that the base-change map $h^{-1}(\mathcal{E}_T) \to h^*(\mathcal{E}_T)$ is functorial in $\mathcal{E}$. □

Construction 2.7.14. Now suppose $(\mathcal{E}, \phi_{\mathcal{E}})$ is an F-crystal on X/S, and suppose $h : T' \to T$ is a PD-morphism over the absolute Frobenius morphism $g = F_X : X \to X$. Then Proposition 2.7.11 gives us a map

$$h^*_{\mathcal{E}} \otimes h^*_{\Omega} : h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^1_{T,\gamma}) \to (F^*_{X,\mathrm{cris}}\mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'},$$

and Proposition 2.7.9 gives us a further map

$$\phi_{\mathcal{E}} \otimes \mathrm{id} : (F^*_{X,\mathrm{cris}}\mathcal{E})_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'} \to \mathcal{E}_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'},$$

which is an isomorphism if $(\mathcal{E}, \phi_{\mathcal{E}})$ is unit-root. Composing these yields a Frobenius pullback map

$$\phi : h^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}) \to \mathcal{E}_{T'} \otimes_{\mathcal{O}_{T'}} \Omega^*_{T',\gamma'}$$

of dg-modules over $h^{-1}(\Omega^*_{T,\gamma})$. If $(\mathcal{E}, \phi_{\mathcal{E}})$ is unit-root, then this is a dg-module base change map as in Corollary 2.7.12.

Situation 2.7.15. For the rest of this section, we will take $X = \operatorname{Spec} R$ to be an affine scheme over a perfect field k of characteristic p, and $S = \operatorname{Spf} W(k)$. Suppose R admits a p-torsionfree PD-embedding with Frobenius (A, γ, ϕ) as in Situation 2.3.2, where γ denotes the PD-structure on $I_r \subset A_r$ for each r. Take $T = T' = \operatorname{Spec} A_r$ with its PD-structure γ and the PD-morphism $h = \phi_T : T' \to T$ lying over $g = F_X$.

Construction 2.7.16. In Situation 2.7.15, we have a PD-de Rham complex

$$\mathrm{dR}(\mathcal{E}_{T,\gamma}) = (\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}, \nabla)$$

and an undivided Frobenius endomorphism

$$\phi : \phi_T^{-1}(\mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}) \to \mathcal{E}_T \otimes_{\mathcal{O}_T} \Omega^*_{T,\gamma}$$

of dg-modules over $\phi_T^{-1}(\Omega^*_{T,\gamma})$. Passing to global sections yields a dg-module

$$\mathrm{dR}(\mathcal{E}(A_r, \gamma)) := (\mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma}, \nabla)$$

over the dg-algebra $\Omega^*_{A_r,\gamma}$, and an endomorphism (as complexes of abelian groups)

$$\phi = \phi_{\mathcal{E}} \otimes \phi^*_{A_r} : \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma} \to \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma},$$

which is semilinear over $\Omega^*_{A_r,\gamma}$ in the sense that we have

$$\phi(\omega \cdot (e \otimes \omega')) = \phi^*_{A_r}(\omega) \cdot \phi(e \otimes \omega')$$

for all $e \in \mathcal{E}(A_r)$ and $\omega, \omega' \in \Omega^*_{A_r,\gamma}$. Passing to the limit, we have a dg-module

$$\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma)) := \lim_r \mathrm{dR}(\mathcal{E}(A_r, \gamma))$$

over $\widehat{\Omega}^*_{A_r,\gamma}$ and an endomorphism

$$\phi : \widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma)) \to \widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))$$

of complexes of abelian groups, which enjoys the analogous semilinearity property with respect to the algebra map

$$\lim_r(\phi^*_{A_r}) : \widehat{\Omega}^*_{A,\gamma} \to \widehat{\Omega}^*_{A,\gamma}.$$

We call this the *functorial* or *undivided Frobenius endomorphism* of $\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))$.

Construction 2.7.17. In Situation 2.7.15, for each r, we also have a divided Frobenius map

$$F : \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma} \to \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma}$$

defined by

$$F(e \otimes \omega) = \phi_{\mathcal{E}}(e) \otimes F(\omega),$$

where the latter F is the divided Frobenius map of Proposition 2.3.13. This satisfies $p^i F = \phi$ in degree i; and it is a ϕ-semilinear map of graded A_r-modules, since the same is true of both $\phi_{\mathcal{E}}$ and F. Passing to the limit as $r \to \infty$ gives a map

$$F : \widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma)) \to \widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma)),$$

which we call the *divided Frobenius endomorphism of* $\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))$.

Remark 2.7.18. It will also be useful later to work with the PD-thickening $T = \operatorname{Spec} W_r(R)$ of $\operatorname{Spec} R$. In this case, we can write down a PD-de Rham complex

$$\mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) = \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma}$$

exactly as in Construction 2.7.16. Moreover, the Frobenius morphism

$$F : \operatorname{Spec} W_r(R) \to \operatorname{Spec} W_r(R)$$

over F_R induces an undivided Frobenius morphism

$$\phi = \phi_{\mathcal{E}} \otimes F^*_{W_r(R)} : \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma} \to \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma},$$

with a suitable semilinearity property. However, in this case we have no divided Frobenius endomorphism analogous to that of Construction 2.7.17, so we will not be able to give the limit

$$\lim_r \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma}$$

the structure of a Dieudonné complex.

Proposition 2.7.19. *In Situation 2.7.15:*

1. *For each i, the maps ∇F and $pF\nabla : \mathcal{E}(A_r) \otimes_{A_r} \Omega^i_{A_r,\gamma} \to \mathcal{E}(A_r) \otimes_{A_r} \Omega^{i+1}_{A_r,\gamma}$ coincide.*
2. *The divided Frobenius $F : \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma} \to \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma}$ is a semilinear map of graded $\Omega^*_{A_r,\gamma}$-modules, in the sense that for all $e \in \mathcal{E}(A_r)$ and $\omega, \omega' \in \Omega^*_{A_r,\gamma}$, we have*
$$F(\omega \cdot (e \otimes \omega')) = F(\omega) \cdot F(e \otimes \omega').$$
3. *The analogous compatibilities hold for the completed PD-de Rham complex $\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))$ with its $\widehat{\Omega}^*_{A,\gamma}$-module structure and divided Frobenius.*

Proof. Part (2) follows from the fact that $F : \Omega^*_{A_r,\gamma} \to \Omega^*_{A_r,\gamma}$ is a ring homomorphism, and part (3) follows from parts (1) and (2) by passage to the limit. As for part (1), we first consider the finite-level statement in degree $i = 0$. By Construction 2.7.16, the diagram

$$\begin{array}{ccc} \mathcal{E}(A_r) & \xrightarrow{\nabla} & \mathcal{E}(A_r) \otimes_{A_r} \Omega^1_{A_r,\gamma} \\ \downarrow{\scriptstyle \phi_{\mathcal{E}}} & & \downarrow{\scriptstyle \phi_{\mathcal{E}} \otimes \phi^*} \\ \mathcal{E}(A_r) & \xrightarrow{\nabla} & \mathcal{E}(A_r) \otimes_{A_r} \Omega^1_{A_r,\gamma} \end{array}$$

commutes. The left vertical map here equals our F, and the right vertical map is our pF, so the identity holds in this case.

Now take $i > 0$, and choose an input element of the form $e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)$. Since $\mathrm{dR}(\mathcal{E}(A_r, \gamma))$ is a dg-module over $(\Omega^*_{A_r,\gamma}, d)$ and F is multiplicative, we can compute everything in terms of the differentials on $\mathcal{E}(A_r)$ and $\Omega^*_{A_r,\gamma}$:

$$\begin{aligned}
&pF\nabla(e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)) \\
&= pF\nabla((a \cdot da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1)) \\
&= pF\left((da \wedge da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1) + (-1)^i (a \cdot da_1 \wedge \cdots \wedge da_i) \cdot \nabla(e \otimes 1)\right) \\
&= p\left((Fda \wedge Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1) + (-1)^i (Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(\nabla(e \otimes 1))\right) \\
&= (dFa \wedge Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1) + (-1)^i (Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot \nabla(F(e \otimes 1)) \\
&= \nabla\left((Fa \cdot Fda_1 \wedge \cdots \wedge Fda_i) \cdot F(e \otimes 1)\right) \\
&= \nabla F\left((a \cdot da_1 \wedge \cdots \wedge da_i) \cdot (e \otimes 1)\right) \\
&= \nabla F\left(e \otimes (a \cdot da_1 \wedge \cdots \wedge da_i)\right).
\end{aligned}$$

This completes the proof. □

Remark 2.7.20. Once we have discussed module objects in **DC** (cf. Lemma 3.2.6), this proposition will tell us that $\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))$ is a module over $\widehat{\Omega}^*_{A,\gamma}$ in **DC**.

2.8 Comparison between de Rham complexes

2.8.1. In this section, we will compare the de Rham complexes of the previous two sections. This will generalize a result of Illusie to the case of an arbitrary crystal of modules. In the case of a unit-root F-crystal, we will recast our comparison in the language of Dieudonné complexes by taking sections on affines and comparing divided Frobenius endomorphisms.

We will first work modulo p^r, in the following situation:

Situation 2.8.2. Let k be a perfect field of characteristic p, X a k-scheme, $S = \operatorname{Spec} W_r(k)$, and $\mathcal{E}$ a crystal of $\mathcal{O}_{X/S}$-modules. Let Y be a smooth S-scheme equipped with a closed embedding $X \hookrightarrow Y$ over S, let I denote the ideal sheaf of this embedding, and let $(D = D_X(Y), \overline{I}, \gamma)$ be the PD-envelope of $X \hookrightarrow Y$ over $(S, (p), [\ ])$. As before, we will abbreviate $\mathcal{O}_D$ as $\mathcal{D}$.

2.8.3. In Situation 2.8.2, Construction 2.7.5 produces a de Rham complex $\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}$ associated to the PD-envelope D, and Construction 2.6.3 gives us a PD-de Rham complex $\mathcal{E} \otimes_{\mathcal{D}} \Omega^*_{D/S,\gamma}$ associated to the object $(X \hookrightarrow D, \gamma)$ of $\mathrm{Cris}(X/S)$. We claim that these coincide. We are not aware of a proof of this in the literature; however, we will see that it follows from the following result of Illusie, which deals with the case $\mathcal{E} = \mathcal{O}_{X/S}$.

Proposition 2.8.4. *([12, pp. 0, 3.1.6]) The cdga map*

$$\theta : (\Omega^*_{D/S,\gamma}, d) \to (\mathcal{D} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}, \nabla)$$

induced by the identity map $\mathcal{D} \to \mathcal{D}$ in degree 0 is an isomorphism.

Proposition 2.8.5. *Continuing in Situation 2.8.2, the isomorphism*

$$\mathrm{id}_{\mathcal{E}} \otimes \theta : \mathcal{E}_D \otimes_{\mathcal{D}} \Omega^*_{D/S,\gamma} \to \mathcal{E}_D \otimes_{\mathcal{D}} (\mathcal{D} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}) = \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S}$$

is compatible with the connections of Constructions 2.7.5 and 2.6.3.

Proof. As in the proofs of Propositions 2.6.12 and 2.7.11, we will map a suitable pullback of the diagram defining

$$\nabla : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$$

to the diagram defining

$$\nabla' : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{D}} \Omega^1_{D/S,\gamma}.$$

To begin, recall the first-order PD-compatible PD-envelope $D^1_{D/S,\gamma}$ of Construction 2.7.4 and the full PD-envelope $D_X(Y \times_S Y)$ of Situation 2.6.2. We have morphisms

$$\begin{tikzcd}
D^1_{D/S,\gamma} \arrow[r, "h"] \arrow[rr, bend left, "\mathrm{pr}_1"] \arrow[rr, bend right, "\mathrm{pr}_2"'] & D_X(Y \otimes_S Y) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & D
\end{tikzcd}$$

in $\mathrm{Cris}(X/S)$, where the p_i and pr_i are the projections, and h is induced by the natural map $D \times_S D \hookrightarrow Y \times_S Y$. Note that $\mathrm{pr}_i = p_i \circ h$ for $i = 1, 2$, and all the maps are compatible homeomorphisms at the level of topological spaces. In particular, we may discuss sheaves on D, $D_X(Y \times_S Y)$, and $D^1_{D/S,\gamma}$ interchangeably via the inverse image functors $p_1^{-1} = p_2^{-1}$, $\mathrm{pr}_1^{-1} = \mathrm{pr}_2^{-1}$, and h^{-1}.

The two connections in question are induced by the maps

$$\begin{aligned}
&\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1) : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y \text{ and} \\
&\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1) : \mathcal{E}_D \to \mathcal{E}_D \otimes_{\mathcal{D}} \mathcal{D}^1_{D/S,\gamma}
\end{aligned}$$

in the following diagrams:

$$\begin{tikzcd}
p_1^*\mathcal{E}_D \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{E}_{D_X(Y \times Y)} & p_2^*\mathcal{E}_D \arrow[l, "\sim"'] \arrow[d, two heads] \\
\mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y & & \mathcal{D}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D \arrow[ll, "\sim", "\varepsilon_1"'] \\
& \mathcal{E}_D \arrow[ul, "\mathrm{id} \otimes 1"] \arrow[ur, "1 \otimes \mathrm{id}"'] &
\end{tikzcd} \tag{2.8.5.1}$$

(where the left vertical map is the isomorphism $p_1^*\mathcal{E}_D \simeq \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ of [2, Corollary 6.3] followed by the quotient map, and similarly for the right vertical map) and

$$\begin{tikzcd}
\mathrm{pr}_1^*\,\mathcal{E}_D \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{E}_{D^1_{D/S,\gamma}} & \mathrm{pr}_2^*\,\mathcal{E}_D \arrow[l, "\sim"'] \arrow[d, equal] \\
\mathcal{E}_D \otimes_{\mathcal{D}} \mathcal{D}^1_{D/S,\gamma} & & \mathcal{D}^1_{D/S,\gamma} \otimes_{\mathcal{D}} \mathcal{E}_D \arrow[ll, "\sim", "\varepsilon_1"'] \\
& \mathcal{E}_D \arrow[ul, "\mathrm{id}\otimes 1"] \arrow[ur, "1\otimes\mathrm{id}"'] &
\end{tikzcd} \tag{2.8.5.2}$$

Applying Lemma 2.4.5 to the squares

$$\begin{tikzcd}
D^1_{D/S,\gamma} \arrow[r, "h"] \arrow[d, "\mathrm{pr}_i"'] & D_X(Y \times_S Y) \arrow[d, "p_i"] \\
D \arrow[r, equal] & D,
\end{tikzcd}$$

over $g = \mathrm{id}_X$ (for $i = 1$ and 2), we get a commutative diagram

$$\begin{tikzcd}
h^{-1}p_1^*\mathcal{E}_D \arrow[r, "\sim"] \arrow[d] & h^{-1}\mathcal{E}_{D_X(Y\otimes Y)} \arrow[d] & h^{-1}p_2^*\mathcal{E}_D \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{pr}_1^*\,\mathcal{E}_D \arrow[r, "\sim"] & \mathcal{E}_{D^1_{D/S,\gamma}} & \mathrm{pr}_2^*\,\mathcal{E}_D. \arrow[l, "\sim"']
\end{tikzcd} \tag{2.8.5.3}$$

This maps the top row of 2.8.5.1 to the top row of 2.8.5.2. We map $\mathcal{E}_D$ to $\mathcal{E}_D$ by the identity map. To define maps on the remaining objects, we first note that the algebra maps

$$\mathcal{D} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \xrightarrow{\sim} \mathcal{D}_X(Y \times_S Y) \xrightarrow{h^*} \mathcal{D}^1_{D/S,\gamma} \text{ and}$$
$$\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D} \xrightarrow{\sim} \mathcal{D}_X(Y \times_S Y) \xrightarrow{h^*} \mathcal{D}^1_{D/S,\gamma}$$

factor through maps

$$\nu_1 : \mathcal{D} \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y \to \mathcal{D}^1_{D/S,\gamma} \text{ and}$$
$$\nu_2 : \mathcal{D}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{D} \to \mathcal{D}^1_{D/S,\gamma}$$

respectively, since the ideal $\overline{J}_Y^{[2]} \subset \mathcal{D}_Y$ maps to $\overline{J}_{D/S,\gamma}^{[2]} = 0 \subset \mathcal{D}^1_{D/S,\gamma}$. We then form the maps

$$\mathrm{id}_{\mathcal{E}_D} \otimes \nu_1 : \mathcal{E}_D \otimes_{\mathcal{O}_Y} \mathcal{D}^1_Y \to \mathcal{E}_D \otimes_{\mathcal{D}} \mathcal{D}^1_{D/S,\gamma} \text{ and}$$
$$\nu_2 \otimes \mathrm{id}_{\mathcal{E}_D} : \mathcal{D}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_D \to \mathcal{D}^1_{D/S,\gamma} \otimes_{\mathcal{D}} \mathcal{E}_D.$$

We have now mapped all objects in 2.8.5.1 to the corresponding objects in 2.8.5.2. We have seen that these maps are compatible with the top horizontal maps in the respective diagrams; it is clear that they are compatible with the diagonal maps, and it follows from unraveling the constructions that they are compatible with the vertical maps as well. Hence they are compatible with the isomorphisms ε_1, and thus also with the maps $\varepsilon_1 \circ (1 \otimes \mathrm{id}) - (\mathrm{id} \otimes 1)$ inducing the connections. But the map of subobjects

$$\mathcal{E}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S} \to \mathcal{E}_D \otimes_{\mathcal{D}} \Omega^1_{D/S,\gamma}$$

induced by $\mathrm{id}_{\mathcal{E}_D} \otimes \nu_1$ is precisely the degree-1 component of $\mathrm{id} \otimes \theta$. So $\mathrm{id} \otimes \theta$ is a map of graded modules, compatible with connections in degree 0, and thus also compatible with connections in all degrees by the definition of ∇_i. This completes the proof. □

2.8.6. Our goal in the remainder of this section is to extract the meaning of the previous proposition in terms of Dieudonné complexes and Dieudonné algebras when we take sections and pass to the limit. We will work in the Frobenius-embedded situation 1.7.8; note that in this situation, the schemes $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} A_r$, and $D = \operatorname{Spec} B_r$ (for any $r > 0$) satisfy the hypotheses of Situation 2.8.2. We will at times also assume that the completed PD-envelope B is p-torsionfree; this is only needed in order to define a divided Frobenius on $\mathcal{E}(B_r) \otimes_{B_r} \Omega^*_{B_r,\gamma}$ using Construction 2.7.17.

Remark 2.8.7. If B is not p-torsionfree, then we still get an isomorphism of de Rham complexes

$$\mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r} \simeq \mathcal{E}(B_r) \otimes_{B_r} \Omega^*_{B_r,\gamma},$$

along with compatibility of the respective undivided Frobenius endomorphisms. The left-hand side carries the divided Frobenius of Construction 2.6.18, but we cannot apply Construction 2.7.17 to the right-hand side due to the p-torsion in B. Of course we could define a divided Frobenius on the right-hand side by transport of structure, but we will have no need for this.

We first extract an isomorphism of Dieudonné algebras from Proposition 2.8.4:

Lemma 2.8.8. *Suppose in the Frobenius-embedded situation 1.7.8 that B is p-torsionfree.*

1. *For each r, the isomorphism*

$$\theta_r : \Omega^*_{B_r,\gamma} \xrightarrow{\sim} B_r \otimes_{A_r} \Omega^*_{A_r}$$

obtained from Proposition 2.8.4 by taking sections is compatible with divided Frobenius endomorphisms.

2. *Passing to the limit,*

$$\lim_r \theta_r : \lim_r (\Omega^*_{B_r,\gamma}, d, F) \xrightarrow{\sim} \lim_r (B_r \otimes_{A_r} \Omega^*_{A_r}, \nabla, F)$$

is an isomorphism of Dieudonné algebras.

Proof. The first statement implies the second, since we already know that each θ_r is an isomorphism of dg-algebras. To prove (1), recall that both sides are graded algebras generated by elements of the form b and db where $b \in B_r$. Since both divided Frobenius endomorphisms are algebra maps, it suffices to check that θ_r is compatible with F on inputs of these forms. It is clear in degree 0, since the isomorphism

$$\theta_r^0 : \Omega^0_{B_r,\gamma} \xrightarrow{\sim} B_r \otimes_{A_r} \Omega^0_{A_r}$$

in degree 0 is the identity $B_r \to B_r$, with the divided Frobenius endomorphism ϕ_{B_r} on both sides. It is similarly clear for elements of the form

$$\Omega^*_{B_r,\gamma} \ni da \overset{\theta_r}{\leftrightarrow} 1 \otimes da \in B_r \otimes_{A_r} \Omega^*_{A_r}$$

with $a \in A_r$, since the map $\Omega^*_{A_r} \to \Omega^*_{B_r,\gamma}$ is induced by $\Omega^*_A \to \Omega^*_B$ and is therefore compatible with the divided Frobenius endomorphism of Proposition 1.2.3.

To complete the proof, recall that B_r is generated as an A_r-algebra by elements of the form $\gamma_n(x)$ where $n \geq 1$ and $x \in I_r$. It follows by the Leibniz rule that all elements $db \in \Omega^1_{B_r,\gamma}$ lie in the graded algebra generated by B_r and elements of the form $d\gamma_n(x)$. But we have by fiat that

$$d\gamma_n(x) = \gamma_{n-1}(x)dx \in \Omega^*_{B_r,\gamma},$$

so these elements are generated by the ones we have already discussed. □

Corollary 2.8.9. *1. Suppose that we are in the embedded situation 1.7.8 and $\mathcal{E}$ is a quasicoherent crystal on* Spec R. *Then for each r, the map*

$$\mathrm{id}_{\mathcal{E}} \otimes \theta_r : \mathcal{E}(B_r) \otimes_{B_r} \Omega^*_{B_r,\gamma} \to \mathcal{E}(B_r) \otimes_{B_r} (B_r \otimes_{A_r} \Omega^*_{A_r}) = \mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}$$

of Proposition 2.8.5 is an isomorphism of dg-modules, where the source and target are respectively viewed as dg-modules over $(\Omega^*_{B_r,\gamma}, d)$ and $(B_r \otimes_{A_r} \Omega^*_{A_r}, \nabla)$, and these are identified with each other via Proposition 2.8.4.

2. *In the Frobenius-embedded situation 1.7.8, if $\mathcal{E}$ is an F-crystal and B is p-torsionfree, then the isomorphisms* $\mathrm{id}_{\mathcal{E}} \otimes \theta_r$ *are compatible with divided Frobenius endomorphisms.*

3. *The analogous statements hold for the limit*

$$\lim_r (\mathrm{id}_{\mathcal{E}} \otimes \theta_r) : \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) \simeq \widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma)). \tag{2.8.9.1}$$

Proof. The map $\mathrm{id}_{\mathcal{E}} \otimes \theta_r$ is clearly an isomorphism of graded modules, as $\mathrm{id}_{\mathcal{E}}$ and θ_r are. Proposition 2.8.5 says that it is compatible with the connections, thus a morphism of dg-modules. To prove statement (2), recall from Constructions 2.7.17 and 2.6.18 that the Frobenius endomorphisms of both sides are defined by tensoring $\phi_{\mathcal{E}}$ with the divided Frobenius endomorphisms of the respective de Rham complexes, so the requisite compatibility follows from Lemma 2.8.8. Part (3) follows by passage to the limit. □

Remark 2.8.10. Once we have discussed module objects in **DC** (cf. Lemma 3.2.6), Corollary 2.8.9 will tell us that 2.8.9.1 is an isomorphism of modules in **DC** when we identify the underlying Dieudonné algebras via Lemma 2.8.8.

2.9 Completed de Rham complexes and quotients

2.9.1. The goal of this section is to prove Proposition 2.9.4—the seemingly obvious statement that quotienting the completed de Rham complex $\widehat{\mathrm{dR}}(\mathcal{E}(A,\gamma))$ by p^r recovers the r-th level of the tower, $\mathrm{dR}(\mathcal{E}(A_r,\gamma))$. This will require a few module-theoretic lemmas.

Lemma 2.9.2. *Let A be a p-torsionfree ring, $A_r = A/p^rA$, and $\widehat{A} = \lim_r A_r$. Suppose $(M_r)_r$ is an inverse system of finite projective A_r-modules such that for all $r' > r$, the given map $M_{r'} \to M_r$ identifies M_r with $M_{r'} \otimes_{A_{r'}} A_r$. Then:*

1. *$\widehat{M} := \lim_r M_r$ is a finite projective module over $\widehat{A}$.*
2. *The natural map $\widehat{M} \to M_r$ identifies M_r with $\widehat{M} \otimes_{\widehat{A}} A_r$.*

Proof. We first choose an A_1-module surjection $\pi_1 : A_1^{\oplus m} \to M_1$ for some finite m. For any r, an arbitrary lift $\pi_r : A_r^{\oplus m} \to M_r$ is surjective by Nakayama's lemma, so we may lift π_1 to a compatible family of surjections $\pi_r : A_r^{\oplus m} \to M_r$. Since M_r is a projective A_r-module for all r, each π_r admits a section $s_r : M_r \to A_r^{\oplus m}$. Note that the chosen sections s_r are not a priori compatible as r varies; our main task will be to remedy this.

Fix r for the moment, and let $\overline{s}_{r+1}$ denote the reduction of s_{r+1} modulo p^r. Since s_{r+1} is a splitting of π_{r+1}, it follows that $\overline{s}_{r+1}$ is a splitting of π_r, so $s_r - \overline{s}_{r+1}$ maps M_r to $\ker \pi_r$. Next, we observe that the natural projection $\ker \pi_{r+1} \to \ker \pi_r$ is surjective, by applying the snake lemma in the diagram below.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & p^r A_{r+1}^{\oplus m} \arrow[r, two heads] \arrow[d] & p^r M_{r+1} \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & \ker \pi_{r+1} \arrow[r] \arrow[d] & A_{r+1}^{\oplus m} \arrow[r, "\pi_{r+1}"] \arrow[d] & M_{r+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \pi_r \arrow[r] \arrow[d] & A_r^{\oplus m} \arrow[r, "\pi_r"] \arrow[d] & M_r \arrow[r] \arrow[d] & 0 \\
 & ? \arrow[r] & 0 & 0 &
\end{tikzcd}$$

Since M_{r+1} is a projective A_{r+1}-module, it follows that the composition

$$M_{r+1} \twoheadrightarrow M_r \xrightarrow{s_r - \overline{s}_{r+1}} \ker \pi_r$$

lifts to some map $u : M_{r+1} \to \ker \pi_{r+1}$. Then $s_{r+1} + u$ is another splitting of π_{r+1}, whose reduction modulo p^r is $\overline{s}_{r+1} + (s_r - \overline{s}_{r+1}) = s_r$. Thus, by repeating this construction recursively for all r, we can arrange for the splittings s_r to be compatible.

Now consider the limit of the short exact sequences

$$0 \to \ker \pi_r \to A_r^{\oplus m} \xrightarrow{\pi_r} M_r \to 0.$$

The resulting sequence

$$0 \to \lim_r(\ker \pi_r) \to \widehat{A}^{\oplus m} \xrightarrow{\widehat{\pi}} \widehat{M} \to 0$$

is short exact by applying the Mittag-Leffler criterion to the tower $(\ker \pi_r)_r$, which we saw earlier has surjective transition maps. Then $\widehat{s} = \lim_r s_r : \widehat{M} \to \widehat{A}^{\oplus m}$ is a splitting of $\widehat{\pi}$, so $\widehat{M}$ is a direct summand of $\widehat{A}^{\oplus m}$, and is thus a finite projective $\widehat{A}$-module. This proves (1).

To prove (2), note that for all n and r, tensoring the short exact sequence

$$0 \to A_n \xrightarrow{p^r} A_{n+r} \to A_r \to 0$$

with the flat module M_{n+r} yields a short exact sequence

$$0 \to M_n \xrightarrow{p^r} M_{n+r} \to M_r \to 0.$$

Passing to the limit as $n \to \infty$ gives

$$0 \to \widehat{M} \xrightarrow{p^r} \widehat{M} \to M_r \to 0,$$

since the tower $(M_n)_n$ satisfies the Mittag-Leffler criterion. $\square$

Lemma 2.9.3. *Maintaining the setup of Lemma 2.9.2, let $(N_r)_r$ be an arbitrary inverse system of A_r-modules, and set $\widehat{N} = \lim_r N_r$. The canonical map*

$$\widehat{M} \otimes_{\widehat{A}} \widehat{N} \to \lim_r (M_r \otimes_{A_r} N_r)$$

is an isomorphism.

Proof. By Lemma 2.9.2, $\widehat{M}$ is a finite projective $\widehat{A}$-module, so its dual module

$$\widehat{M}^\vee = \mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{A})$$

satisfies

$$\mathrm{Hom}_{\widehat{A}}(P, \widehat{M} \otimes Q) = \mathrm{Hom}_{\widehat{A}}(P \otimes \widehat{M}^{\vee}, Q)$$

for all $\widehat{A}$-modules P and Q. This will allow us to compare universal properties. Fix an arbitrary $\widehat{A}$-module P, and calculate:

$$\begin{aligned}
\mathrm{Hom}_{\widehat{A}}(P, \widehat{M} \otimes_{\widehat{A}} \widehat{N}) &= \mathrm{Hom}_{\widehat{A}}(P \otimes_{\widehat{A}} \widehat{M}^{\vee}, \widehat{N}) \\
&= \lim_r \mathrm{Hom}_{\widehat{A}}(P \otimes_{\widehat{A}} \widehat{M}^{\vee}, N_r) \\
&= \lim_r \mathrm{Hom}_{\widehat{A}}(P, \widehat{M} \otimes_{\widehat{A}} N_r) \\
&= \lim_r \mathrm{Hom}_{\widehat{A}}(P, M_r \otimes_{A_r} N_r) \\
&= \mathrm{Hom}_{\widehat{A}}(P, \lim_r (M_r \otimes_{A_r} N_r)).
\end{aligned}$$

The result then follows from the Yoneda lemma. □

Proposition 2.9.4. *In the situation of Construction 2.7.16, the canonical map*

$$\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))/p^r \to \mathrm{dR}(\mathcal{E}(A_r, \gamma))$$

is an isomorphism.

Proof. Note that this statement depends only on the tower of modules $(M_r)_r = (\mathcal{E}(A_r))_r$, and not on their PD-connections or Frobenius endomorphisms. Each M_r is a finite projective A_r-module, since it is the module of global sections of the finite locally free sheaf $\mathcal{E}_{\mathrm{Spec}\, A_r}$. The crystal property of $\mathcal{E}$ also gives $M_{r'} \otimes_{A_{r'}} A_r \xrightarrow{\sim} M_r$ for $r' > r$. Thus the tower $(M_r)_r$ satisfies the hypotheses of Lemma 2.9.2.

In the case $\mathcal{E} = \mathcal{O}$, our claim says that $\widehat{\Omega}^*_{A,\gamma}/p^r \xrightarrow{\sim} \Omega^*_{A_r,\gamma}$, which follows from the universal properties of the various PD-de Rham complexes: both sides are initial among commutative differential graded algebras equipped with PD-compatible maps from A_r. For $\mathcal{E}$ arbitrary, combining the above with Lemmas 2.9.3 and 2.9.2 gives:

$$\begin{aligned}
\widehat{\mathrm{dR}}(\mathcal{E}(A, \gamma))/p^r &= \left(\lim_r \left(\mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma}\right)\right)/p^r \\
&\simeq \left((\lim_r \mathcal{E}(A_r)) \otimes_{\widehat{A}} \widehat{\Omega}^*_{A,\gamma}\right)/p^r \\
&\simeq \left((\lim_r \mathcal{E}(A_r))/p^r\right) \otimes_{\widehat{A}/p^r} (\widehat{\Omega}^*_{A,\gamma}/p^r) \\
&\simeq \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma} = \mathrm{dR}(\mathcal{E}(A_r, \gamma)).
\end{aligned}$$

□

Remark 2.9.5. In the embedded situation 1.7.8 (which, we recall, requires a smooth embedding rather than merely a p-torsionfree one), we analogously have a canonical isomorphism of the classical de Rham complexes of Construction 2.6.17,

$$\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))/p^r \xrightarrow{\sim} \mathrm{dR}(\mathcal{E}(X \hookrightarrow Y_r)),$$

for example by appealing to the isomorphisms of Corollary 2.8.9. Note that we do not need to assume that the completed PD-envelope B is p-torsionfree, in view of Remark 2.8.7.

Chapter 3

Algebra with Dieudonné complexes

The de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ with coefficients in a unit-root F-crystal $\mathcal{E}$ will have the structure of a dg-module over the Dieudonné algebra $\mathcal{W}\Omega^*_R$. Accordingly, we begin by giving an account of modules in the categories $\mathbf{DC}$ and $\mathbf{DC}_{\mathrm{str}}$.

3.1 Dieudonné complexes as R^*-modules

Recall from [3, Definition 2.1.1] that a *Dieudonné complex* is a complex (M^*, d) of abelian groups equipped with a graded group endomorphism F satisfying $dF = pFd$. We begin with an elementary reinterpretation of this definition.

Definition 3.1.1. The *unsaturated Raynaud ring* is the noncommutative graded ring $R^* = \mathbb{Z}\langle d, F\rangle/(d^2, dF - pFd)$, where d lives in degree 1 and F in degree 0.

The following lemma is immediate:

Lemma 3.1.2. *The category of Dieudonné complexes is equivalent to the category of graded left modules over R^*.*

Remark 3.1.3. Concretely, R^* is concentrated in degrees 0 and 1, and has the form

$$R^* = \mathbb{Z}[F] \xrightarrow{d} \mathbb{Z}[F] \cdot d,$$

where $d(F^n) := p^n F^n \cdot d$. In particular, a general element of R^* has the form

$$r = \sum_{i \geq 0} a_i F^i + \sum_{i \geq 0} b_i F^i d,$$

where all but finitely many a_i and b_i are equal to 0. Note that the structure of R^* as a left module over itself makes it a Dieudonné complex; for r as above, we have

$$\begin{aligned} Fr &= \sum_{i\geq 0} a_i F^{i+1} + \sum_{i\geq 0} b_i F^{i+1} d \quad \text{and} \\ dr &= \sum_{i\geq 0} p^i a_i F^i d. \end{aligned}$$

Remark 3.1.4. The forgetful functor $\mathbf{DC} \to \mathrm{Gr}(\mathrm{Ab})$ can be viewed as restriction of scalars from R^* to $\mathbb{Z}[0]$, so it admits both right and left adjoints. Explicitly, these are given by the same formulas as for ordinary (commutative, non-graded) rings (see e.g. [21, tags `05DQ` and `08YP`]): the left adjoint is the extension of scalars

$$\begin{aligned} &\mathrm{Gr}(\mathrm{Ab}) \to \mathbf{DC}, \\ &\qquad M^* \mapsto R^* \otimes M^*, \text{ with } F(r\otimes m) := Fr \otimes m \text{ and } d(r\otimes m) := dr \otimes m, \end{aligned}$$

and the right adjoint is the coinduced module construction

$$\begin{aligned} &\mathrm{Gr}(\mathrm{Ab}) \to \mathbf{DC}, \\ &\qquad M^* \mapsto \mathrm{Hom}^*_{\mathrm{Gr}(\mathrm{Ab})}(R^*, M^*), \text{ with } F(f)(r) := f(rF) \text{ and } d(f)(r) := f(rd), \end{aligned}$$

where $\mathrm{Hom}^i_{\mathrm{Gr}(\mathrm{Ab})}$ denotes the set of group maps that raise the grading by i.

Corollary 3.1.5. *The forgetful functor* $\mathbf{DC} \to \mathrm{Gr}(\mathrm{Ab})$ *commutes with all limits and colimits.*

Remark 3.1.6. The Dieudonné complex R^* is free on the generator $1 \in R^0$, in the sense that for any $X^* \in \mathbf{DC}$, evaluation on 1 gives a bijection $\mathrm{Hom}_{\mathbf{DC}}(R^*, X^*) \to X^0$.

Aside 3.1.7. The term "unsaturated Raynaud ring" is justified by the fact that $\mathrm{Sat}\, R^*$ is a free $\mathbb{Z}$-module with basis

$$\{V^i, F^j, dV^i, F^j d : i \geq 0, j > 0\},$$

and so $\mathrm{Sat}\, R^* \otimes_{\mathbb{Z}} W\mathcal{O}_S$ is the usual Raynaud ring of the ringed topos $(S, \mathcal{O}_S)$ ([7, 0, equation 5.1]). It follows that $\mathcal{W}\,\mathrm{Sat}\, R^*$ (cf. [3, Example 2.5.7]) is the completed Raynaud ring of $\mathrm{Spec}\,\mathbb{F}_p$. (Note that both $\mathrm{Sat}\, R^*$ and $\mathcal{W}\,\mathrm{Sat}\, R^*$ are saturated Dieudonné complexes, so they have Verschiebung operators V and therefore contain the element $V := V(1)$.)

We can give any saturated (resp. strict) Dieudonné complex M^* the structure of a graded module over $\mathrm{Sat}\, R^*$ (resp. $\mathcal{W}\,\mathrm{Sat}\, R^*$) by the obvious formula

$$\begin{aligned} &\left(\sum a_i V^i + \sum b_j F^j + \sum c_i dV^i + \sum d_j F^j d\right) \cdot m \\ &= \sum a_i V^i(m) + \sum b_j F^j(m) + \sum c_i dV^i(m) + \sum d_j F^j d(m). \end{aligned}$$

This defines functors

$$\mathbf{DC}_{\mathrm{sat}} \to (\mathrm{Sat}\, R^*)\text{-}\mathbf{mod} \text{ and}$$
$$\mathbf{DC}_{\mathrm{str}} \to (\mathcal{W}\,\mathrm{Sat}\, R^*)\text{-}\mathbf{mod}.$$

We emphasize that these are *not* equivalences of categories. In particular, modules over $\mathrm{Sat}\, R^*$ and $\mathcal{W}\,\mathrm{Sat}\, R^*$ may have p-torsion, whereas saturated Dieudonné complexes by definition cannot. (In particular, the categories $\mathbf{DC}_{\mathrm{sat}}$ and $\mathbf{DC}_{\mathrm{str}}$ are not abelian, as multiplication-by-p maps are monomorphisms that are not kernels.) Moreover, even in the absence of p-torsion, a $\mathrm{Sat}\, R^*$-module need not be saturated; an example is $\mathbb{Z}[\sqrt{p}]$ concentrated in degree 0 with $F = V = \sqrt{p}$. Finally, even a $\mathcal{W}\,\mathrm{Sat}\, R^*$-module that is saturated as a Dieudonné complex need not be strict; an example is $\mathbb{Q}_p$ concentrated in degree 0 with $F = 1$ and $V = p$.

Nonetheless, the analogy between $(\mathrm{Sat}\, R^*)$-$\mathbf{mod}$ and $\mathbf{DC}_{\mathrm{sat}}$ (resp. $(\mathcal{W}\,\mathrm{Sat}\, R^*)$-$\mathbf{mod}$ and $\mathbf{DC}_{\mathrm{str}}$) allows us to draw inspiration from [7, p. I], which develops a Hom-tensor formalism for derived modules over the Raynaud ring and its completion. In particular, the tensor product of [3, Remark 2.1.5] (recalled below) corresponds to the $*$ product of [7, I, Definition 3.1], and its adjoint (Definition 3.4.2) was inspired by [7, I, Definition 5.1].

3.2 Modules in DC and $\mathbf{DC}_{\mathrm{str}}$

3.2.1. In this section, we recall the symmetric monoidal structures with which Bhatt-Lurie-Mathew endows the categories $\mathbf{DC}$ and $\mathbf{DC}_{\mathrm{str}}$, denoted $\otimes$ and $\otimes^{\mathrm{str}}$,[1] and discuss the resulting categories of algebra and module objects.

Definition 3.2.2. ([3, Remarks 2.1.5 and 7.6.4]) If M^* and N^* are Dieudonné complexes, then $M^* \otimes N^*$ is given by the tensor product of the underlying complexes of abelian groups, equipped with $F = F_M \otimes F_N$ and d defined by the graded Leibniz rule. The unit of this tensor product is $\mathbb{Z} = \mathbb{Z}[0]$, with Frobenius acting as the identity. The symmetry $M^* \otimes N^* \xrightarrow{\sim} N^* \otimes M^*$ is defined by

$$x \otimes y \mapsto (-1)^{|x|\cdot|y|} y \otimes x \tag{3.2.2.1}$$

when x and y are homogeneous.

The strictified tensor product $\otimes^{\mathrm{str}}$ is the unique symmetric monoidal structure on $\mathbf{DC}_{\mathrm{str}}$ making the strictification functor $\mathcal{W}\,\mathrm{Sat} : \mathbf{DC} \to \mathbf{DC}_{\mathrm{str}}$ symmetric monoidal. In particular, we have a canonical isomorphism

$$M^* \otimes^{\mathrm{str}} N^* \simeq \mathcal{W}\,\mathrm{Sat}(M^* \otimes N^*)$$

for each M^*, $N^* \in \mathbf{DC}_{\mathrm{str}}$. The unit of $\otimes^{\mathrm{str}}$ is $\mathcal{W}\,\mathrm{Sat}(\mathbb{Z}) = \mathbb{Z}_p$, again concentrated in degree 0 with trivial Frobenius.

[1] We have promoted the subscript "str" to a superscript in order to make room for a base ring later; cf. Definition 3.5.2.

Definition 3.2.3. By an *(associative) algebra in* **DC**, we mean a monoid for the monoidal structure $\otimes$; that is, a Dieudonné complex A^* equipped with a multiplication map $m : A^* \otimes A^* \to A^*$ and a unit map $1 : \mathbb{Z} \to A^*$ in **DC**, making the usual associativity and unit diagrams

$$\begin{tikzcd} A^* \otimes A^* \otimes A^* \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "m \otimes \mathrm{id}"'] & A^* \otimes A^* \arrow[d, "m"] \\ A^* \otimes A^* \arrow[r, "m"] & A^* \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{Z} \otimes A^* \arrow[r, "1 \otimes \mathrm{id}"] \arrow[dr, equal] & A^* \otimes A^* \arrow[d, "m"] & A^* \otimes \mathbb{Z} \arrow[l, "\mathrm{id} \otimes 1"'] \arrow[dl, equal] \\ & A^* & \end{tikzcd}$$

commute. We call A^* *commutative* if multiplication is compatible with the symmetry of the tensor product; this is the usual graded-commutative law. We denote the categories of such objects, with the obvious morphisms, by Alg(**DC**) and CAlg(**DC**).

Algebra objects in $\mathbf{DC}_{\mathrm{str}}$ are defined analogously, using the symmetric monoidal structure $\otimes^{\mathrm{str}} = \mathcal{W}\,\mathrm{Sat}(- \otimes -)$ and its unit $\mathbb{Z}_p = \mathcal{W}\,\mathrm{Sat}(\mathbb{Z})$.

Definition 3.2.4. For A^* in Alg(**DC**), we define a *left A^*-module in* **DC** to be a Dieudonné complex M^* equipped with a morphism $m_M : A^* \otimes M^* \to M^*$, such that the action diagrams

$$\begin{tikzcd} A^* \otimes A^* \otimes M^* \arrow[r, "m_A \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m_M"'] & A^* \otimes M^* \arrow[d, "m_M"] \\ A^* \otimes M^* \arrow[r, "m_M"] & M^* \end{tikzcd} \tag{3.2.4.1}$$

and

$$\begin{tikzcd} \mathbb{Z} \otimes M^* \arrow[rr, "1 \otimes \mathrm{id}"] \arrow[dr, equal] & & A^* \otimes M^* \arrow[dl, "m_M"] \\ & M^* & \end{tikzcd} \tag{3.2.4.2}$$

commute. For $A^* \in \mathrm{Alg}(\mathbf{DC}_{\mathrm{str}})$, we define a *left A^*-module in* $\mathbf{DC}_{\mathrm{str}}$ analogously, where the tensor products are replaced by $\otimes^{\mathrm{str}}$.

We denote the categories of such modules, with the obvious morphisms, as $A^* - \mathrm{mod}_{\mathbf{DC}}$ and $A^* - \mathrm{mod}_{\mathrm{str}}$ respectively. We can similarly define the categories of right modules $\mathrm{mod}_{\mathbf{DC}} - A^*$ and $\mathrm{mod}_{\mathrm{str}} - A^*$, and bimodules $A^* - \mathrm{mod}_{\mathbf{DC}} - B^*$ and $A^* - \mathrm{mod}_{\mathrm{str}} - B^*$; as always, the two actions on a bimodule are required to commute.

Remark 3.2.5. As noted in [3, Remark 3.1.5], a Dieudonné algebra is just a commutative algebra object in **DC** with a few extra conditions. We will have no need for these conditions except inasmuch as they are built into the universal property of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of a k-algebra R.

We have the following concrete interpretations of algebra and module objects in **DC** and $\mathbf{DC}_{\mathrm{str}}$:

Lemma 3.2.6.

(a) An algebra object in **DC** *is a graded algebra A^* equipped with graded group maps $F : A^* \to A^*$ and $d : A^* \to A^{*+1}$ satisfying the rules*

$$\begin{aligned} d^2 &= 0, \\ dF &= pFd, \\ d(ab) &= da \cdot b + (-1)^{|a|} a \cdot db, \\ F(ab) &= Fa \cdot Fb, \\ d(1) &= 0, \ \textit{and} \\ F(1) &= 1. \end{aligned}$$

(In fact the identity $d(1) = 0$ is redundant, as it follows from the graded Leibniz rule.)

(b) Given an algebra object A^* in **DC***, an object M^* in A^*-$\mathbf{mod}_{\mathbf{DC}}$ is a graded left A^*-module equipped with graded group maps $F : M^* \to M^*$ and $d : M^* \to M^{*+1}$ such that*

$$\begin{aligned} d^2 &= 0, \\ dF &= pFd, \\ d(am) &= da \cdot m + (-1)^{|a|} a \cdot dm, \ \textit{and} \\ F(am) &= Fa \cdot Fm. \end{aligned}$$

(c) Morphisms of algebra objects (resp. module objects) are simply morphisms of graded algebras (resp. graded modules) that are compatible with F and d.

(d) The category of algebra objects (resp. modules over a fixed strict algebra A^*) in $\mathbf{DC}_{\mathrm{str}}$ is isomorphic to the full subcategory of the category of algebra objects (resp. modules over A^*) in **DC** *spanned by the objects that are strict as Dieudonné complexes.*

Proof. Parts (a-c) follow immediately from unraveling the definitions. We will sketch a proof of the algebra case of part (d); the module case is completely analogous. Let A^* be a strict Dieudonné complex. First note that the maps $m : A^* \otimes A^* \to A^*$ correspond bijectively to the maps $m^{\mathrm{str}} : A^* \otimes^{\mathrm{str}} A^* \to A^*$, by factoring m as

$$A^* \otimes A^* \xrightarrow{\rho} \mathcal{W}\mathrm{Sat}(A^* \otimes A^*) = A^* \otimes^{\mathrm{str}} A^* \xrightarrow{m^{\mathrm{str}}} A^*;$$

in particular, we have $m^{\mathrm{str}} = \mathcal{W}\,\mathrm{Sat}(m)$. The same applies to the unit maps $1 : \mathbb{Z} \to A^*$ and $1^{\mathrm{str}} : \mathbb{Z}_p \to A^*$. Then the associativity diagram for m^{str} is the strictification of the associativity diagram for m, so the universal property of

$$\rho : A^* \otimes A^* \otimes A^* \to A^* \otimes^{\mathrm{str}} A^* \otimes^{\mathrm{str}} A^*$$

implies that one commutes if and only if the other does. The same argument applies to the unit diagram. Thus the algebra structures of A^* as an object of $\mathbf{DC}$ are in bijection with those of A^* as an object of $\mathbf{DC}_{\mathrm{str}}$.

If A^* and B^* are algebra objects in $\mathbf{DC}_{\mathrm{str}}$ (which we can now identify with algebra objects in $\mathbf{DC}$ that are strict as Dieudonné complexes), then a morphism $f : A^* \to B^*$ in $\mathbf{DC}_{\mathrm{str}}$ is the same as a morphism $f : A^* \to B^*$ in $\mathbf{DC}$; the same argument as above shows that f is a morphism of algebra objects in $\mathbf{DC}_{\mathrm{str}}$ if and only if it is a morphism of algebra objects in $\mathbf{DC}$. So we have given bijections of objects and of corresponding hom-sets in the two specified categories; these define an isomorphism of categories. □

Remark 3.2.7. Note that the identities above which only involve d say exactly that A^* is a differential graded algebra and M^* is a differential graded module over it. Thus we can also describe algebra and module objects in $\mathbf{DC}$ respectively as dg-algebras and dg-modules equipped with a graded group endomorphism F satisfying some properties.

We record the following lemma for later use.

Lemma 3.2.8. *If A^* is an algebra object in $\mathbf{DC}$ and M^* is a left A^*-module in $\mathbf{DC}$, then we have a bijection*

$$\mathrm{Hom}_{A^*\text{-}\mathbf{mod}_{\mathbf{DC}}}(A^*, M^*) \leftrightarrow \{x \in M^0 : Fx = x, dx = 0\}$$

given by $f \leftrightarrow f(1)$.

Proof. An A^*-module homomorphism $f : A^* \to M^*$ is a map of graded modules commuting with F and d. In particular, f must be given by the rule $f(a) = ax$, where $x = f(1) \in M^0$. We claim that this f is a map in A^*-$\mathbf{mod}_{\mathbf{DC}}$ if and only if $Fx = x$ and $dx = 0$. These conditions are necessary because we always have $F(1) = 1$ and $d(1) = 0$ in A^*. Conversely, given any $x \in M^0$ such that $Fx = x$ and $dx = 0$, $f(a) = ax$ does define a map in A^*-$\mathbf{mod}_{\mathbf{DC}}$, since it is a map of graded A^*-modules and we have the compatibilities

$$f(Fa) = Fa \cdot x = Fa \cdot Fx = F(ax) = F(f(a))$$

and (taking a homogeneous)

$$f(da) = da \cdot x = da \cdot x + (-1)^{|a|} a \cdot dx = d(ax) = d(f(a)).$$ □

Remark 3.2.9. In the situation of Lemma 3.2.8, if the module M^* is strict, then in fact the condition $Fx = x$ implies the condition $dx = 0$, since for each r we have

$$dx = dF^r x = p^r F^r dx \equiv 0 \in \mathcal{W}_r\, M^*.$$

3.3 Right and bi-Dieudonné complexes

3.3.1. We would like to construct an internal Hom functor on the category $\mathbf{DC} \simeq R^*$-**mod**. But recall that for S a noncommutative ring and M, N left modules over it, $\mathrm{Hom}_S(M, N)$ is in general only an abelian group. In order to give $\mathrm{Hom}_S(M, N)$ the structure of an S-module, we need M to be an (S, S)-bimodule.

With this in mind, we make the following definitions:

Definition 3.3.2. A *right Dieudonné complex* is a graded right R^*-module. A *bi-Dieudonné complex* is a graded R^*-bimodule. We call the resulting categories **rDC** and **biDC**, where the morphisms are maps of degree 0 respecting the R^*-module structures.

Remark 3.3.3. Equivalently, a right Dieudonné complex consists of the data of a complex of abelian groups (X^*, d) equipped with an endomorphism $F : X^n \to X^n$ for each n, such that $F(d(x)) = pd(F(x))$ for all x. By convention, we will write the d and F to the right of their arguments, so the identity above has the form $xdF = p(xFd)$. A bi-Dieudonné complex contains the data of a graded abelian group with left and right F operators $X^n \to X^n$ and left and right d operators $X^n \to X^{n+1}$ for each n, satisfying the identities

$$\begin{aligned} (dx)d &= d(xd), & \qquad (dx)F &= d(xF), \\ (Fx)d &= F(xd), & \qquad (Fx)F &= F(xF). \end{aligned}$$

We emphasize that the left and right F (resp. d) operators on a bi-Dieudonné complex must be homogeneous of degree 0 (resp. 1) with respect to a single grading.

Example 3.3.4. The ring R^* itself is a bi-Dieudonné complex, as is any graded R^*-algebra.

Lemma 3.3.5.

1. *For $X \in \mathbf{DC}$ and $Y \in \mathbf{biDC}$, the Dieudonné complex $X \otimes Y$ has the structure of a bi-Dieudonné complex, where the right structure is given by:*

$$\begin{aligned} (x \otimes y)F &= (x \otimes yF), \\ (x \otimes y)d &= (x \otimes yd). \end{aligned}$$

2. *For $X \in \mathbf{biDC}$ and $Y \in \mathbf{DC}$, the Dieudonné complex $X \otimes Y$ has the structure of a bi-Dieudonné complex, where the right structure is given by:*

$$\begin{aligned} (x \otimes y)F &= (xF \otimes y), \\ (x \otimes y)d &= (-1)^n (xd \otimes y) \text{ for } y \in Y^n. \end{aligned}$$

Proof. We will prove the first part; the second part follows from this by transporting structure across the symmetry isomorphism $X \otimes Y \simeq Y \otimes X$ of 3.2.2.1. Recall that $X \otimes Y$ already has the structure of a Dieudonné complex. The verifications of the right Dieudonné complex identities

$$\begin{aligned}(x \otimes y)d^2 &= 0\\ (x \otimes y)dF &= p(x \otimes y)Fd\end{aligned}$$

are immediate. To show that the left and right structures commute, let $r \in \{F, d\} \subset R^*$, $x \in X^m$, and $y \in Y^n$. We calculate:

$$\begin{aligned}(F(x \otimes y))r &= (Fx \otimes Fy)r\\ &= Fx \otimes Fyr\\ &= F(x \otimes yr)\\ &= F((x \otimes y)r)\end{aligned}$$

and

$$\begin{aligned}(d(x \otimes y))r &= (dx \otimes y + (-1)^m x \otimes dy)r\\ &= dx \otimes yr + (-1)^m x \otimes dyr\\ &= d(x \otimes yr)\\ &= d((x \otimes y)r).\end{aligned}$$

□

Note that the sign appearing in part (2) of the lemma is to be expected, as we are commuting the elements $d \in R^1$ and $y \in Y^n$.

3.4 Internal Hom

Following [7, I, §5], we will now describe an internal Hom functor in the category **DC**. The first step is as follows.

Lemma 3.4.1. *For $X^* \in$ **biDC** and $Y^* \in$ **DC**, the following defines a Dieudonné complex, which we call the* naive internal Hom.

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{DC}}(X^*, Y^*)^n &:= \{(f_i : X^i \to Y^{i+n})_{i \in \mathbb{Z}} : \text{ compatible with left } F, d\},\\ (Ff)(x) &:= f(xF)\\ (df)(x) &:= f(xd).\end{aligned}$$

Proof. We must first check that the functions Ff and df have the right compatibilities. For each pair $r, s \in \{F, d\}$, the calculation

$$(rf)(sx) = f(sxr) = s(f(xr)) = s((rf)(x)),$$

shows that rf is compatible with s. So $\underline{\mathrm{Hom}}_{\mathrm{DC}}(X^*, Y^*)^*$ is a graded abelian group with additive operators F and d of degree 0 and 1 respectively. Moreover, we have

$$\begin{aligned}(d^2 f)(x) &= f(xd^2) = 0 \text{ and}\\ (dFf)(x) &= f(xdF) = f(p \cdot xFd) = p \cdot (Fdf)(x),\end{aligned}$$

so it is indeed a Dieudonné complex. □

We are now ready to define our internal Hom and prove that it is right-adjoint to $\otimes$:

Definition 3.4.2. For $X^*, Y^* \in \mathbf{DC}$, we define

$$\underline{\mathrm{Hom}}^!_{\mathbf{DC}}(X^*, Y^*) = \underline{\mathrm{Hom}}_{\mathbf{DC}}(X^* \otimes R^*, Y^*)^*,$$

where $X^* \otimes R^*$ has the bi-Dieudonné complex structure of Lemma 3.3.5. Lemma 3.4.1 gives this the structure of a Dieudonné complex.

Proposition 3.4.3. *For $X^*, Y^*, Z^* \in \mathbf{DC}$, there is a natural isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathbf{DC}}(X^*, \underline{\mathrm{Hom}}^!_{\mathbf{DC}}(Y^*, Z^*)) \simeq \mathrm{Hom}_{\mathbf{DC}}(X^* \otimes Y^*, Z^*).$$

Namely, if we have $\varphi : X^* \to \underline{\mathrm{Hom}}^!_{\mathbf{DC}}(Y^*, Z^*)$ and $\varphi' : X^* \otimes Y^* \to Z^*$, we map

$$\begin{array}{rcl} \varphi & \mapsto & \varphi'(x \otimes y) := (-1)^{mn} \varphi(x)(y \otimes 1),\\ \varphi(x)(y \otimes r) := (-1)^{(k+m)n} \varphi'(rx \otimes y) & \leftarrow\!\shortmid & \varphi', \end{array}$$

where $x \in X^m, y \in Y^n$, and $r \in R^k$.

Proof. For notational convenience, we will maintain the convention $x \in X^m, y \in Y^n$, and $r \in R^k$. We first show that given φ, the resulting φ' is a morphism of Dieudonné complexes. It is a graded group homomorphism by construction, and for $x \in X^m$ and $y \in Y^n$, we have

$$\begin{aligned}\varphi'(F(x \otimes y)) &= \varphi'(Fx \otimes Fy)\\ &= (-1)^{mn} \varphi(Fx)(Fy \otimes 1)\\ &= (-1)^{mn} (F(\varphi(x)))(Fy \otimes 1)\\ &= (-1)^{mn} \varphi(x)(Fy \otimes 1F)\\ &= (-1)^{mn} \varphi(x)(Fy \otimes F1)\\ &= (-1)^{mn} \varphi(x)(F(y \otimes 1))\\ &= (-1)^{mn} F(\varphi(x)(y \otimes 1))\\ &= F(\varphi'(x \otimes y))\end{aligned}$$

and

$$\begin{aligned}
\varphi'(d(x \otimes y)) &= \varphi'(dx \otimes y + (-1)^m x \otimes dy) \\
&= (-1)^{(m+1)n}\varphi(dx)(y \otimes 1) + (-1)^{mn}\varphi(x)(dy \otimes 1) \\
&= (-1)^{(m+1)n} d(\varphi(x))(y \otimes 1) + (-1)^{mn}\varphi(x)(dy \otimes 1) \\
&= (-1)^{(m+1)n}\varphi(x)(y \otimes 1d) + (-1)^{mn}\varphi(x)(dy \otimes 1) \\
&= (-1)^{mn}\varphi(x)\big((-1)^m y \otimes d1 + dy \otimes 1\big) \\
&= (-1)^{mn}\varphi(x)(d(y \otimes 1)) \\
&= (-1)^{mn} d(\varphi(x)(y \otimes 1)) \\
&= d(\varphi'(x \otimes y)).
\end{aligned}$$

Next we show that given φ', the resulting φ lies in $\mathrm{Hom}_{\mathbf{DC}}(X^*, \underline{\mathrm{Hom}}^!_{\mathbf{DC}}(Y^*, Z^*))$. This requires a number of calculations. First, for fixed $x \in X^m$, $\varphi(x)$ is a degree-m map of graded groups, and it commutes with d and F by the computations

$$\begin{aligned}
\varphi(x)(F(y \otimes r)) &= \varphi(x)(Fy \otimes Fr) \\
&= (-1)^{(k+m)n}\varphi'(Frx \otimes Fy) \\
&= (-1)^{(k+m)n}\varphi'(F(rx \otimes y)) \\
&= F((-1)^{(k+m)n}\varphi'(rx \otimes y)) \\
&= F(\varphi(x)(y \otimes r))
\end{aligned}$$

and

$$\begin{aligned}
\varphi(x)(d(y \otimes r)) &= \varphi(x)(dy \otimes r + (-1)^n y \otimes dr) \\
&= (-1)^{(k+m)(n+1)}\varphi'(rx \otimes dy) + (-1)^{(k+1+m)n} \cdot (-1)^n \varphi'(drx \otimes y) \\
&= (-1)^{(k+m)n}\varphi'(drx \otimes y + (-1)^{k+m} rx \otimes dy) \\
&= (-1)^{(k+m)n}\varphi'(d(rx \otimes y)) \\
&= d((-1)^{(k+m)n}\varphi'(rx \otimes y)) \\
&= d(\varphi(x)(y \otimes r)).
\end{aligned}$$

So $\varphi : X^* \to \underline{\mathrm{Hom}}^!_{\mathbf{DC}}(Y^*, Z^*)$ is a map of graded groups. This map is compatible with F and d by the calculations

$$\varphi(Fx)(y \otimes r) = (-1)^{(k+m)n}\varphi'(rFx \otimes y) = \varphi(x)(y \otimes rF) = F(\varphi(x))(y \otimes r) \text{ and}$$
$$\varphi(dx)(y \otimes r) = (-1)^{(k+m+1)n}\varphi'(rdx \otimes y) = \varphi(x)(y \otimes rd) = d(\varphi(x))(y \otimes r).$$

(For the last equality in each line, recall that F and d act on

$$\underline{\mathrm{Hom}}^!_{\mathbf{DC}}(Y^*, Z^*) = \underline{\mathrm{Hom}}_{\mathbf{DC}}(Y^* \otimes R^*, Z^*)$$

by premultiplication on the right side of R^*.)

We have now shown that both $\varphi \mapsto \varphi'$ and $\varphi' \mapsto \varphi$ land in the claimed codomains. It remains to prove that they are inverses. For the $\varphi' \mapsto \varphi \mapsto \varphi'$ direction, let φ' be given and define φ as above. Unrolling the definitions gives

$$(-1)^{mn}\varphi(x)(y \otimes 1) = \varphi'(1x \otimes y) = \varphi'(x \otimes y).$$

In the other direction, let φ be given and define φ' as above. Expanding the definitions and using the R^*-linearity of φ gives

$$\begin{aligned}(-1)^{(k+m)n}\varphi'(rx \otimes y) &= (-1)^{(k+m)n} \cdot (-1)^{(k+m)n}\varphi(rx)(y \otimes 1)\\ &= r(\varphi(x))(y \otimes 1)\\ &= \varphi(x)(y \otimes r),\end{aligned}$$

completing the proof. □

Remark 3.4.4. We can view the isomorphism of Proposition 3.4.3 as a signed currying map. To explain the signs appearing in the maps $\varphi \mapsto \varphi'$ and $\varphi \leftarrow\!\shortmid \varphi'$, note that $\varphi(x)$ is a homomorphism of left modules, which can be imagined as right multiplication by some matrix. So we are in effect commuting the elements $\varphi(x)$ of degree m and $y \otimes 1$ of degree n, which requires multiplying by $(-1)^{mn}$. The sign seems to be missing in [7, I, Proposition 5.4]; it must be $(-1)^{mn}$ regardless of whether one defines $\underline{\mathrm{Hom}}^!$ using $R^* \otimes (-)$ or $(-) \otimes R^*$.

Corollary 3.4.5. *The bifunctors $\otimes$ and $\otimes^{\mathrm{str}}$ commute with colimits.*

Proof. By Proposition 3.4.3, $\otimes$ is a left adjoint, so it commutes with colimits. (In fact we don't really need Proposition 3.4.3: since both colimits and tensor products can be computed at the level of graded groups, the existence of an internal Hom functor for graded groups is sufficient.) Since $\mathcal{W}\,\mathrm{Sat}$ is also a left adjoint and $\otimes^{\mathrm{str}} = \mathcal{W}\,\mathrm{Sat}(- \otimes -)$, it follows that $\otimes^{\mathrm{str}}$ commutes with colimits as well. □

Question 3.4.6. Does $\mathbf{DC}_{\mathrm{str}}$ have an internal Hom functor? If so, is it compatible with that of $\mathbf{DC}$ via the strictification functor?

3.5 More on modules in DC and $\mathbf{DC}_{\mathrm{str}}$

3.5.1. In this entirely formal section, we describe a few natural operations on module objects in $\mathbf{DC}$ and $\mathbf{DC}_{\mathrm{str}}$: tensor product over a base algebra (including the special case of base change along a morphism of algebras), and strictification of a module in $\mathbf{DC}$.

Definition 3.5.2. Suppose A^* is an algebra object in $\mathbf{DC}$, for example a Dieudonné algebra. If M^* and N^* are respectively left and right A^*-modules, we define $M^* \otimes_{A^*} N^*$ as the coequalizer of the diagram

$$M^* \otimes A^* \otimes N^* \underset{\mathrm{id}_M \otimes m_N}{\overset{m_M \otimes \mathrm{id}_N}{\rightrightarrows}} M^* \otimes N^*$$

in $\mathbf{DC}$. Recall by Corollary 3.1.5 that this colimit can be computed at the level of the underlying graded groups. If A^*, M^*, and N^* are strict, then we define the strictified tensor product $M^* \otimes^{\mathrm{str}}_{A^*} N^*$ as the coequalizer of the analogous diagram

$$M^* \otimes^{\mathrm{str}} A^* \otimes^{\mathrm{str}} N^* \underset{\mathrm{id}_M \otimes m_N}{\overset{m_M \otimes \mathrm{id}_N}{\rightrightarrows}} M^* \otimes^{\mathrm{str}} N^*$$

in $\mathbf{DC}_{\mathrm{str}}$.

Remark 3.5.3. We can equivalently describe $M^* \otimes^{\mathrm{str}}_{A^*} N^*$ as $\mathcal{W}\,\mathrm{Sat}(M^* \otimes_{A^*} N^*)$, as $\mathcal{W}\,\mathrm{Sat}$ is a left adjoint and therefore commutes with colimits.

Construction 3.5.4. If A^* is an algebra object in $\mathbf{DC}$, we can promote $\otimes_{A^*}$ to a bifunctor

$$(\mathbf{mod}_{\mathbf{DC}}\text{-}A^*) \times (A^*\text{-}\mathbf{mod}_{\mathbf{DC}}) \to \mathbf{DC}$$

as follows. Let $f : M^* \to M'^*$ be a morphism in $\mathbf{mod}_{\mathbf{DC}}\text{-}A^*$, and $g : N^* \to N'^*$ a morphism in $A^*\text{-}\mathbf{mod}_{\mathbf{DC}}$. Then the morphism $f \otimes g : M^* \otimes N^* \to M'^* \otimes N'^*$ induces a unique morphism $f \otimes g : M^* \otimes_{A^*} N^* \to M'^* \otimes_{A^*} N'^*$ in $\mathbf{DC}$, by chasing the diagram below.

$$\begin{tikzcd}
M^* \otimes A^* \otimes N^* \arrow[r, shift left, "m_M \otimes \mathrm{id}_N"] \arrow[r, shift right, "\mathrm{id}_M \otimes m_N"'] \arrow[d, "f \otimes \mathrm{id} \otimes g"'] & M^* \otimes N^* \arrow[r] \arrow[d, "f \otimes g"] & M^* \otimes_{A^*} N^* \arrow[d, dotted, "f \otimes g"] \\
M'^* \otimes A^* \otimes N'^* \arrow[r, shift left, "m_{M'} \otimes \mathrm{id}_{N'}"] \arrow[r, shift right, "\mathrm{id}_{M'} \otimes m_{N'}"'] & M'^* \otimes N'^* \arrow[r] & M'^* \otimes_{A^*} N'^*
\end{tikzcd}$$

By chasing the strict analogue of the same diagram, we can similarly promote $\otimes^{\mathrm{str}}_{A^*}$ to a bifunctor

$$(\mathbf{mod}_{\mathrm{str}}\text{-}A^*) \times (A^*\text{-}\mathbf{mod}_{\mathrm{str}}) \to \mathbf{DC}_{\mathrm{str}}\,.$$

Lemma 3.5.5. *(Associativity of tensor product over base algebras.)*

1. *Suppose we have algebra objects A^* and B^* in $\mathbf{DC}$, along with a right A^*-module M^*, an (A^*, B^*)-bimodule N^*, and a left B^*-module P^*. Then there exists a unique isomorphism $M^* \otimes_{A^*} (N^* \otimes_{B^*} P^*) \to (M^* \otimes_{A^*} N^*) \otimes_{B^*} P^*$ in $\mathbf{DC}$ making the diagram*

$$\begin{tikzcd}
M^* \otimes (N^* \otimes P^*) \arrow[r, "\sim"] \arrow[d] & (M^* \otimes N^*) \otimes P^* \arrow[d] \\
M^* \otimes_{A^*} (N^* \otimes_{B^*} P^*) \arrow[r, "\sim"] & (M^* \otimes_{A^*} N^*) \otimes_{B^*} P^*
\end{tikzcd} \tag{3.5.5.1}$$

commute.

2. *Suppose moreover that the Dieudonné complexes A^*, B^*, M^*, N^*, and P^* are all strict. Then there exists a unique isomorphism $M^* \otimes^{\mathrm{str}}_{A^*} (N^* \otimes^{\mathrm{str}}_{B^*} P^*) \to (M^* \otimes^{\mathrm{str}}_{A^*} N^*) \otimes^{\mathrm{str}}_{B^*} P^*$ in $\mathbf{DC}_{\mathrm{str}}$ making the analogous diagram commute.*

Proof. We will prove (1); the proof of (2) is completely analogous. We claim that the vertical maps in diagram 3.5.5.1 exhibit both $M^* \otimes_{A^*} (N^* \otimes_{B^*} P^*)$ and $(M^* \otimes_{A^*} N^*) \otimes_{B^*} P^*$ as the colimit of the diagram below, where each pair of double arrows represent the two possible contractions of the various multiplicative structures. This universal property will prove both existence and uniqueness of the desired isomorphism.

$$\begin{tikzcd}
M^* \otimes A^* \otimes N^* \otimes B^* \otimes P^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M^* \otimes N^* \otimes B^* \otimes P^* \arrow[d, shift left] \arrow[d, shift right] \\
M^* \otimes A^* \otimes N^* \otimes P^* \arrow[r, shift left] \arrow[r, shift right] & M^* \otimes N^* \otimes P^*
\end{tikzcd}$$

(We have omitted parentheses since we already know that the absolute tensor product is associative. Note that the square commutes if we choose compatibly among each pair of maps.) To prove our claim, recall that a colimit along a product of two index categories agrees with the colimit along one index category of the colimits along the other. Accordingly, we may compute the colimit above as the coequalizer of the column that is constructed by taking coequalizers of each row.

By viewing each row as a base change of the diagram

$$M^* \otimes A^* \otimes N^* \rightrightarrows M^* \otimes N^*$$

and recalling that $\otimes$ commutes with colimits, we can identify the colimits of the two rows with

$$(M^* \otimes_{A^*} N^*) \otimes B^* \otimes P^* \text{ and } (M^* \otimes_{A^*} N^*) \otimes P^*$$

respectively, with the two obvious maps between them. Thus the colimit of the entire diagram is

$$\operatorname{coeq}\Big((M^* \otimes_{A^*} N^*) \otimes B^* \otimes P^* \rightrightarrows (M^* \otimes_{A^*} N^*) \otimes P^* \Big) = (M^* \otimes_{A^*} N^*) \otimes_{B^*} P^*.$$

Repeating the calculation with rows and columns exchanged shows that the colimit is also $M^* \otimes_{A^*} (N^* \otimes_{B^*} P^*)$, completing the proof. □

Remark 3.5.6. It follows from Lemma 3.5.5 that if A^* and B^* are algebra objects in $\mathbf{DC}$, M^* is a (B^*, A^*)-bimodule, and N^* is a left A^*-module, then $M^* \otimes_{A^*} N^*$ is a left B^*-module; its multiplication map is the composition

$$B^* \otimes (M^* \otimes_{A^*} N^*) \simeq (B^* \otimes M^*) \otimes_{A^*} N^* \xrightarrow{m_M \otimes \mathrm{id}} M^* \otimes_{A^*} N^*.$$

(Note that the absolute tensor product here coincides with the tensor product over the unit object $\mathbb{Z}$.) An important special case is where $M^* = B^*$ is an A^*-algebra. Of course all of this carries over to the strict setting as well.

3.5.7. We finish this section by showing that for A^* an algebra object in $\mathbf{DC}$, the strictification of an A^*-module in $\mathbf{DC}$ is a $\mathcal{W}\operatorname{Sat}(A^*)$-module in $\mathbf{DC}_{\mathrm{str}}$. A typical case of interest is $A^* = \widehat{\Omega}^*_A$, where A is a p-torsionfree ring equipped with a lift of the Frobenius endomorphism of $R = A/pA$. By Bhatt-Lurie-Mathew's Construction 1.2.6, we have $\mathcal{W}\operatorname{Sat}(\widehat{\Omega}^*_A) = \mathcal{W}\Omega^*_R$, so this lemma says that the strictification of an $\widehat{\Omega}^*_A$-module in $\mathbf{DC}$ is a $\mathcal{W}\Omega^*_R$-module in $\mathbf{DC}_{\mathrm{str}}$. The same is true of $A^* = \Omega^*_{W(R_{\mathrm{red}})}$, in view of Construction 1.2.5.

Lemma 3.5.8. *Let A^* be an algebra object in $\mathbf{DC}$ (for example, a Dieudonné algebra), and let M^* be a left A^*-module in $\mathbf{DC}$.*

1. *The strictification $\mathcal{W}\operatorname{Sat}(M^*)$ carries a unique strict left $\mathcal{W}\operatorname{Sat}(A^*)$-module structure such that $\rho : M^* \to \mathcal{W}\operatorname{Sat}(M^*)$ is a morphism in A^*-$\mathbf{mod}_{\mathbf{DC}}$.*
2. *Sending $f : M^* \to N^*$ to $\mathcal{W}\operatorname{Sat}(f) : \mathcal{W}\operatorname{Sat}(M^*) \to \mathcal{W}\operatorname{Sat}(N^*)$ makes $\mathcal{W}\operatorname{Sat}$ a functor from A^*-$\mathbf{mod}_{\mathbf{DC}}$ to $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}}$.*
3. *The functor of part (2) is left-adjoint to the forgetful functor $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}} \to A^*$-$\mathbf{mod}_{\mathbf{DC}}$.*

Proof. First recall from [3, Remark 7.6.4] that for any M^* and N^* in $\mathbf{DC}$, the map

$$M^* \otimes N^* \overset{\rho\otimes\rho}{\to} \mathcal{W}\operatorname{Sat}(M^*) \otimes \mathcal{W}\operatorname{Sat}(N^*) \overset{\rho}{\to} \mathcal{W}\operatorname{Sat}(M^*) \otimes^{\mathrm{str}} \mathcal{W}\operatorname{Sat}(N^*)$$

is a strictification map. For part (1), we are given a map $m_M : A^* \otimes M^* \to M^*$ in $\mathbf{DC}$ satisfying the action properties; i.e. such that the associativity and unit diagrams 3.2.4.1 and 3.2.4.2 commute. We are looking for a map $m' : \mathcal{W}\operatorname{Sat}(A^*) \otimes^{\mathrm{str}} \mathcal{W}\operatorname{Sat}(M^*) \to \mathcal{W}\operatorname{Sat}(M^*)$ satisfying the analogous action properties, and which makes the outer region of the diagram

$$\begin{tikzcd}
 & A^* \otimes M^* \arrow[dl, "\mathrm{id}\otimes\rho"'] \arrow[d, "\rho\otimes\rho"] \arrow[r, "m_M"] & M^* \arrow[d, "\rho"] \\
A^* \otimes \mathcal{W}\operatorname{Sat}(M)^* \arrow[r, "\rho\otimes\mathrm{id}"] & \mathcal{W}\operatorname{Sat}(A^*) \otimes^{\mathrm{str}} \mathcal{W}\operatorname{Sat}(M^*) \arrow[r, "m'"] & \mathcal{W}\operatorname{Sat}(M^*)
\end{tikzcd}$$

commute. But since the triangle on the left commutes, the outer region commutes if and only if the square does. Since the left vertical map is a strictification map, there is a unique choice of m' which makes the diagram commute, namely $m' = \mathcal{W}\operatorname{Sat}(m_M)$. This also satisfies the associativity and unit properties, since the respective diagrams still commute after applying $\mathcal{W}\operatorname{Sat}$.

For (2), functoriality is clear; we must only show that for each $f : M^* \to N^*$ in A^*-$\mathbf{mod}_{\mathbf{DC}}$, $\mathcal{W}\operatorname{Sat}(f)$ is actually a morphism in $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}}$. This follows by strictifying the commutative square

$$\begin{tikzcd}
A^* \otimes M^* \arrow[r, "m_M"] \arrow[d, "\mathrm{id}\otimes f"'] & M^* \arrow[d, "f"] \\
A^* \otimes N^* \arrow[r, "m_N"] & N^*.
\end{tikzcd}$$

For (3), we must show that if M^* is in A^*-$\mathbf{mod}_{\mathbf{DC}}$ and N^* is in $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}}$, then any morphism $f : M^* \to N^*$ in A^*-$\mathbf{mod}_{\mathbf{DC}}$ factors uniquely through a morphism

$$\mathcal{W}\operatorname{Sat}(M^*) \to N^*$$

in $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}}$. Uniqueness is clear, as there is a unique factoring map even in $\mathbf{DC}$, namely

$$\mathcal{W}\operatorname{Sat}(f) : \mathcal{W}\operatorname{Sat}(M^*) \to \mathcal{W}\operatorname{Sat}(N^*) = N^*.$$

This is in fact a morphism in $\mathcal{W}\operatorname{Sat}(A^*)$-$\mathbf{mod}_{\mathrm{str}}$ by part (2). □

3.6 Filtrations on modules in $\mathbf{DC}_{\mathrm{str}}$

3.6.1. In this section, we examine the following question: if A^* is an algebra object in $\mathbf{DC}$ (for example a Dieudonné algebra), and M^* is a strict A^*-module in $\mathbf{DC}$, then what kind of structure does $\mathcal{W}_r M^*$ have? It is only a complex of abelian groups a priori; recall that the Frobenius maps $\mathcal{W}_r M^*$ to its quotient $\mathcal{W}_{r-1} M^*$ and not to itself. We will begin with generalities, and then specialize to the case $A^* = \mathcal{W}\Omega^*_R$ that is of greatest interest to us.

Lemma 3.6.2. *Let A^* be an algebra object in $\mathbf{DC}$, and M^* a strict left A^*-module in $\mathbf{DC}$. Then for each r, we have containments:*

1. $A^* \cdot (V^r M^* + dV^r M^*) \subset V^r M^* + dV^r M^*$, *and*
2. $(V^r A^* + dV^r A^*) \cdot M^* \subset V^r M^* + dV^r M^*$ *if A^* is saturated.*

In particular, part (1) makes $\mathcal{W}_r M^*$ a dg-module over A^*, and part (2) makes it a dg-module over $\mathcal{W}_r A^*$ when A^* is saturated.

Proof. Fix elements a of A^* and x of M^*, which for convenience we take to be homogeneous. We must show that all of the elements $a \cdot V^r x$, $(V^r a) \cdot x$, $a \cdot dV^r x$, and $(dV^r a) \cdot x$ lie in $V^r M^* + dV^r M^*$. The strategy for all four will be the same: apply F^r to the given element, simplify, and eventually cancel F^r's from both sides of the equation, using the fact that F is injective on M^*.

For $a \cdot V^r x$, we have:

$$\begin{aligned} F^r(a \cdot V^r x) &= F^r(a) \cdot F^r V^r x \\ &= F^r(a) \cdot p^r x \\ &= p^r(F^r(a) \cdot x) \\ &= F^r V^r (F^r a \cdot x). \end{aligned}$$

Canceling F^r's gives $a \cdot V^r x = V^r(F^r a \cdot x) \in V^r M^*$.

For $(V^r a) \cdot x$, we have:

$$\begin{aligned} F^r((V^r a) \cdot x) &= (F^r V^r a) \cdot F^r x \\ &= p^r a \cdot F^r x \\ &= F^r V^r (a \cdot F^r x), \end{aligned}$$

so $(V^r a) \cdot x = V^r(a \cdot F^r x) \in V^r M^*$.

For $a \cdot dV^r x$, we need the graded Leibniz rule:

$$\begin{aligned} F^r(a \cdot dV^r x) &= (F^r a) \cdot (F^r dV^r x) \\ &= (F^r a) \cdot dx \\ &= (-1)^{|a|} \left(d((F^r a) \cdot x) - (dF^r a) \cdot x\right) \\ &= (-1)^{|a|} \left(F^r dV^r((F^r a) \cdot x) - (p^r F^r da) \cdot x\right) \\ &= (-1)^{|a|} \left(F^r dV^r((F^r a) \cdot x) - F^r V^r((F^r da) \cdot x)\right) \\ &= (-1)^{|a|} F^r \left(dV^r((F^r a) \cdot x) - V^r((F^r da) \cdot x)\right). \end{aligned}$$

Again we cancel F^r's and see that the right-hand side lies in $V^r M^* + dV^r M^*$.

Finally, for $(dV^r a) \cdot x$, we have:

$$\begin{aligned} F^r((dV^r a) \cdot x) &= (F^r dV^r a) \cdot (F^r x) \\ &= da \cdot F^r x \\ &= d(a \cdot F^r x) - (-1)^{|a|} a \cdot dF^r x \\ &= F^r dV^r (a \cdot F^r x) - (-1)^{|a|} a \cdot p^r F^r dx \\ &= F^r dV^r (a \cdot F^r x) - (-1)^{|a|} F^r V^r (a \cdot F^r dx) \\ &= F^r \left(dV^r(a \cdot F^r x) - (-1)^{|a|} V^r (a \cdot F^r dx)\right). \end{aligned}$$

Canceling F^r's, we again see that $(dV^r a) \cdot x \in V^r M^* + dV^r M^*$. □

3.6.3. Now let R be an $\mathbb{F}_p$-algebra and $A^* = \mathcal{W}\,\Omega^*_R$. If M^* is a strict module over A^*, then the previous lemma tells us that $\mathcal{W}_r\, M^*$ is a dg-module over $\mathcal{W}_r\, A^*$. Our next goal is to make this more concrete by specifying some de Rham complexes which can serve as convenient substitutes for the cdga $\mathcal{W}_r\, \Omega^*_R$.

Lemma 3.6.4. *Let R be an $\mathbb{F}_p$-algebra. The dg-algebra map*

$$\Omega^*_{W(R)} \to \Omega^*_{W(R_{\mathrm{red}})} \xrightarrow{\rho} \mathcal{W}\,\mathrm{Sat}(\Omega^*_{W(R_{\mathrm{red}})}) = \mathcal{W}\,\Omega^*_R \twoheadrightarrow \mathcal{W}_r\,\Omega^*_R$$

factors uniquely through the quotient map $\Omega^*_{W(R)} \to \Omega^*_{W_r(R),\gamma}$.

Proof. Uniqueness is clear, because $\Omega^*_{W(R)} \to \Omega^*_{W_r(R),\gamma}$ is surjective. For existence, recall by Definition 2.2.3 and Lemma 2.2.7 that the kernel of $\Omega^*_{W(R)} \to \Omega^*_{W_r(R),\gamma}$ is generated as a dg-ideal by elements of the forms

$$\omega_1 = d\gamma_n(x) - \gamma_{n-1}(x)dx \text{ for } x \in VW(R), \text{ and}$$
$$\omega_2 = V^r y \text{ for } y \in W(R).$$

We must show that each of these maps to zero in $\mathcal{W}_r\,\Omega^*_R$. But ω_1 maps to zero even in $\mathcal{W}\,\Omega^*_R$, since it is killed by $n!$ and $\mathcal{W}\,\Omega^*_R$ is $\mathbb{Z}$-torsionfree. As for ω_2, since the map

$$W(R) = \Omega^0_{W(R)} \to \mathcal{W}\,\Omega^0_R$$

is compatible with Verschiebung operators, we have $\rho(V^r x) = V^r \rho(x) \in \mathcal{W}\,\Omega^*_R$, which vanishes in $\mathcal{W}_r\,\Omega^*_R$. □

Corollary 3.6.5. *Let M^* be a strict $\mathcal{W}\,\Omega^*_R$-module.*

1. *The dg-$\Omega^*_{W(R)}$-module structure of M^* coming from the strictification map $\Omega^*_{W(R)} \to \mathcal{W}\,\Omega^*_R$ induces a (unique) dg-$\Omega^*_{W_r(R),\gamma}$-module structure on $\mathcal{W}_r\,M^*$ for each r.*

2. *If $f : M^* \to N^*$ is a morphism of strict $\mathcal{W}\,\Omega^*_R$-modules, then $\mathcal{W}_r(f) : \mathcal{W}_r\,M^* \to \mathcal{W}_r\,N^*$ is a morphism of dg-$\Omega^*_{W_r(R),\gamma}$-modules for each r. Thus $\mathcal{W}_r$ defines a functor*

$$\mathcal{W}\,\Omega^*_R\text{-}\mathbf{mod}_{\mathrm{str}} \to \Omega^*_{W_r(R),\gamma}\text{-}\mathbf{dgmod}.$$

Proof. For part (a), Lemma 3.6.2 shows that the dg-$\Omega^*_{W(R)}$-module structure of $\mathcal{W}_r\,M^*$ factors through $\Omega^*_{W(R)} \to \mathcal{W}\,\Omega^*_R \to \mathcal{W}_r\,\Omega^*_R$. In view of the map $\Omega^*_{W_r(R),\gamma} \to \mathcal{W}_r\,\Omega^*_R$ of Lemma 3.6.4, it also factors through $\Omega^*_{W(R)} \to \Omega^*_{W_r(R),\gamma}$.

For part (b), we simply observe that f is a morphism of dg-$\mathcal{W}\,\Omega^*_R$-modules and thus also of dg-$\Omega^*_{W(R)}$-modules, so $\mathcal{W}_r(f)$ is a morphism of dg-$\Omega^*_{W_r(R),\gamma}$-modules. □

The following two results play the same role for the lifted construction 1.2.6 of $\mathcal{W}\,\Omega^*_R$ that the previous two play for the Witt vector construction 1.2.5.

Lemma 3.6.6. *Let R, (A, ϕ), and A_r be as in the Frobenius-lifted situation 1.7.6.*

1. *The dg-algebra map*
$$\widehat{\Omega}^*_A \xrightarrow{\rho} \mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_A) = \mathcal{W}\,\Omega^*_R \twoheadrightarrow \mathcal{W}_r\,\Omega^*_R$$
factors uniquely through the PD-de Rham complex $\Omega^*_{A_r}$.

2. *The morphism $\Omega^*_{A_r} \to \mathcal{W}_r\,\Omega^*_R$ of part (1) factors as*
$$\Omega^*_{A_r} \xrightarrow{h_r} \Omega^*_{W_r(R),\gamma} \to \mathcal{W}_r\,\Omega^*_R,$$
where the first map is induced by the morphism h_r of Lemma 2.1.12 and the second was constructed in Lemma 3.6.4.

Proof. Part (1) is clear, since the target is killed by p^r and the natural map $\widehat{\Omega}^*_A \to \Omega^*_{A_r}$ is the quotient by p^r. For part (2), we must check that the bottom triangle of the diagram below commutes; this will follow from the commutativity of the upper pentagon.

$$\begin{tikzcd}
\Omega^*_A \arrow[rr] \arrow[dr, "h"] \arrow[dd] & & \widehat{\Omega}^*_A \arrow[r, "\rho"] & \mathcal{W}\,\Omega^*_R \arrow[dd] \\
& \Omega^*_{W(R)} \arrow[r] \arrow[dd] & \Omega^*_{W(R_{\mathrm{red}})} \arrow[ur, "\rho"] & \\
\Omega^*_{A_r} \arrow[rrr] \arrow[dr, "h_r"] & & & \mathcal{W}_r\,\Omega^*_R \\
& \Omega^*_{W_r(R),\gamma} \arrow[urr] & &
\end{tikzcd}$$

By the universal property of the de Rham complex, it suffices to prove that the two maps $\Omega^*_A \to \mathcal{W}\,\Omega^*_R$ agree in degree 0; that is, the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "h"] & & \widehat{A} \arrow[r, "\rho"] & \mathcal{W}\,\Omega^0_R \\
& W(R) \arrow[r] & W(R_{\mathrm{red}}) \arrow[ur, "\rho"] &
\end{tikzcd}$$

commutes. Recall from [3, Proposition 3.6.2] that we have $\mathcal{W}\,\Omega^0_R = W(\mathcal{W}_1\,\Omega^0_R)$. All of the maps above are compatible with the various Frobenius endomorphisms, and the target $\mathcal{W}\,\Omega^0_R$ is p-torsionfree, so the two maps $A \rightrightarrows \mathcal{W}\,\Omega^0_R$ are both δ-ring maps. But by construction ([3, Proposition 4.1.4, Corollary 4.2.3]), both maps pass to the same map $A/pA = R \xrightarrow{e} \mathcal{W}_1\,\Omega^0_R$ when we take quotients. It follows by the universal property of Witt vectors that the two maps agree. ◻

Corollary 3.6.7. *Let R, (A, ϕ), and A_r be as in the Frobenius-lifted situation 1.7.6, and let M^* be a strict $\mathcal{W}\,\Omega^*_R$-module.*

1. *The dg-$\widehat{\Omega}^*_A$-module structure of M^* coming from the strictification map $\widehat{\Omega}^*_A \to \mathcal{W}\,\Omega^*_R$ induces a (unique) dg-$\Omega^*_{A_r}$-module structure on $\mathcal{W}_r\,M^*$ for each r.*

2. *If $f : M^* \to N^*$ is a morphism of strict $\mathcal{W}\,\Omega^*_R$-modules, then $\mathcal{W}_r(f) : \mathcal{W}_r\,M^* \to \mathcal{W}_r\,N^*$ is a morphism of dg-$\Omega^*_{A_r}$-modules for each r. Thus $\mathcal{W}_r$ defines a functor*

$$\mathcal{W}\,\Omega^*_R\text{-}\mathbf{mod}_{\mathrm{str}} \to \Omega^*_{A_r}\text{-}\mathbf{dgmod}.$$

3. *The functor of part (2) factors as*

$$\mathcal{W}\,\Omega^*_R\text{-}\mathbf{mod}_{\mathrm{str}} \to \Omega^*_{W_r(R),\gamma}\text{-}\mathbf{dgmod} \to \Omega^*_{A_r}\text{-}\mathbf{dgmod},$$

where the first step is the functor of Corollary 3.6.5 and the second is restriction of scalars along $h_r : \Omega^*_{A_r} \to \Omega^*_{W_r(R),\gamma}$.

Proof. For part (1), simply recall that the natural map $\widehat{\Omega}^*_A \to \Omega^*_{A_r}$ is the quotient by p^r. For part (2), note that f and therefore also $\mathcal{W}_r(f)$ are maps of dg-$\widehat{\Omega}^*_A$-modules, so the latter is a map of dg-$\Omega^*_{A_r}$-modules. Part (3) follows from part (2) of Lemma 3.6.6. $\square$

Remark 3.6.8. If $R \to S$ is an étale map of $\mathbb{F}_p$-algebras, then Lemma 2.2.10 and [3, Corollary 5.3.5] respectively tell us that the natural maps

$$\Omega^*_{W_r(R),\gamma} \to \Omega^*_{W_r(S),\gamma}$$

and

$$\mathcal{W}_r\,\Omega^*_R \to \mathcal{W}_r\,\Omega^*_S$$

are base-change maps along $W_r(R) \to W_r(S)$. It follows that the natural dg-algebra map

$$\Omega^*_{W_r(S),\gamma} \otimes_{\Omega^*_{W_r(R),\gamma}} \mathcal{W}_r\,\Omega^*_R \to \mathcal{W}_r\,\Omega^*_S$$

is an isomorphism.

3.7 Aside: strict Dieudonné complexes and colimits

3.7.1. Given a diagram $(M^*_i)_i$ of strict Dieudonné complexes, there are three things we might reasonably mean by the "colimit" of M^*_i. Namely, we could mean their colimit

$$\varinjlim_{i,\mathrm{str}} M^*_i$$

in the category $\mathbf{DC}_{\mathrm{str}}$, their colimit

$$\varinjlim_{i,\mathbf{DC}} M^*_i$$

in the category $\mathbf{DC}$ (which by Corollary 3.1.5 may be computed at the level of graded abelian groups), or the colimit

$$(\varinjlim_{i} \mathcal{W}_r\,M^*_i)_r$$

of the associated strict Dieudonné towers. In this largely formal section, we will compare these notions of colimits. We begin with what holds for arbitrary colimits, and then specialize to filtered colimits.

Lemma 3.7.2. *The category $\mathbf{DC}_{\mathrm{str}}$ has all colimits, and they can be computed as the strictifications of the corresponding colimits in $\mathbf{DC}$.*

Proof. The strictification functor is a left adjoint, so it commutes with all colimits. $\square$

Lemma 3.7.3. *Filtered colimits (in* $\mathbf{DC}$*) of saturated Dieudonné complexes are saturated. In particular,* $\mathbf{DC}_{\mathrm{sat}}$ *admits filtered colimits, and they coincide with the corresponding colimits in* $\mathbf{DC}$*.*

Proof. Let $(M_i^*)_{i\in I}$ be a filtered system in $\mathbf{DC}$ with each M_i^* saturated. Then $M^* := \varinjlim_i M_i^*$ is p- and F-torsionfree because p- and F-torsion are given by finite limits of graded groups, and filtered colimits commute with these. Thus we must only show that the map

$$F : M^* \to \{x \in M^* : dx \in pM^{*+1}\}$$

is surjective. An element of the target can be represented by an element $x \in M_i^*$ such that $dx \in pM_j^*$ for some $i, j \in I$. By choosing an element $\ell \in I$ with maps $i \to \ell$, $j \to \ell$, we may instead take $x \in M_\ell^*$ and $dx \in pM_\ell^*$. Then x lies in the image of F on M_ℓ^*, and thus also on M^*. ☐

Remark 3.7.4. Strict Dieudonné complexes are *not* closed under filtered colimits in $\mathbf{DC}$. For example, the group $\mathbb{Z}_p$ concentrated in degree 0 with $F = \mathrm{id}$ forms a strict Dieudonné complex, but its sequential colimit along the multiplication-by-p map is isomorphic to $\mathbb{Q}_p$, which is not strict. Of course, the colimit of this system in $\mathbf{DC}_{\mathrm{str}}$ is $\mathcal{W}\,\mathrm{Sat}(\mathbb{Q}_p) = 0$ by Lemma 3.7.2.

Lemma 3.7.5. *If* $(M_i^*)_i$ *is a filtered diagram of saturated Dieudonné complexes, then the canonical map*

$$\mathcal{W}_r(\varinjlim_{i,\mathbf{DC}} M_i^*) \to \varinjlim_i(\mathcal{W}_r\, M_i^*)$$

is an isomorphism.

Proof. Since $\mathcal{W}_r$ is a cokernel (in the category of graded abelian groups), this follows from the fact that colimits commute with colimits. ☐

3.7.6. We finish this section by showing that filtered colimits of Dieudonné towers—or equivalently, the Dieudonné towers corresponding to filtered limits in $\mathbf{DC}_{\mathrm{str}}$ under the equivalence of categories $\mathbf{DC}_{\mathrm{str}} \simeq \mathbf{TD}$ of [3, Corollary 2.9.4]—can be computed at the level of projective systems of graded abelian groups:

Proposition 3.7.7. *If* $(M_i^*)_i$ *is a filtered diagram of strict Dieudonné complexes, then for each* r*, we have a canonical isomorphism of graded abelian groups*

$$(\mathcal{W}_r(\varinjlim_{i,\mathrm{str}} M_i^*))_r \simeq (\varinjlim_i(\mathcal{W}_r\, M_i^*))_r,$$

compatible with their respective quotient, Frobenius, and Verschiebung maps as r *varies.*

Proof. Combining Lemmas 3.7.2, 3.7.3, and 3.7.5 gives isomorphisms of complexes

$$\begin{aligned}\mathcal{W}_r(\varinjlim_{i,\mathrm{str}} M_i^*) &\simeq \mathcal{W}_r \operatorname{Sat}(\varinjlim_{i,\mathbf{DC}}(M_i^*)) \\ &= \mathcal{W}_r(\varinjlim_{i,\mathbf{DC}}(M_i^*)) \\ &\simeq \varinjlim_{i}(\mathcal{W}_r(M_i^*)).\end{aligned}$$

These isomorphisms are clearly compatible with the quotient, Frobenius, and Verschiebung maps, as all of these maps are induced by the respective endomorphisms of the various strict Dieudonné complexes. □

Remark 3.7.8. Of course, Lemma 3.7.2 and the equivalence of categories $\mathbf{DC}_{\mathrm{str}} \simeq \mathbf{TD}$ guarantee that $\mathbf{TD}$ admits *arbitrary* colimits, computed by the left-hand side of Proposition 3.7.7. But in general these need not agree with the colimits of the underlying projective systems of graded groups. For example, the coequalizer of the diagram

$$\mathbb{Z}_p \underset{0}{\overset{p}{\rightrightarrows}} \mathbb{Z}_p$$

in $\mathbf{DC}_{\mathrm{str}}$ is zero, and thus so is the coequalizer of the corresponding diagram

$$\Big((\mathbb{Z}/p^r\mathbb{Z}) \underset{0}{\overset{p}{\rightrightarrows}} (\mathbb{Z}/p^r\mathbb{Z})\Big)_r$$

in $\mathbf{TD}$. But this latter coequalizer does not vanish when computed at the level of towers of graded groups.

Chapter 4

Definition and first properties

4.1 de Rham-Witt modules over a k-algebra

4.1.1. Let k be a perfect field of characteristic p, R a k-algebra, and $\mathcal{E}$ a unit-root F-crystal on $(\operatorname{Spec} R/W)_{\mathrm{cris}}$. In this section, we will describe a certain category of modules over the saturated de Rham-Witt complex $\mathcal{W}\,\Omega^*_R$ which are endowed with some data coming from $\mathcal{E}$. Our de Rham-Witt complex $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ will be defined as the initial object of this category.

We begin with the following observation. If M^* is a strict $\mathcal{W}\,\Omega^*_R$-module, then Corollary 3.6.5 gives $\mathcal{W}_r\,M^*$ the structure of a dg-$\Omega^*_{W_r(R),\gamma}$-module, and in particular also a graded $W_r(R)$-module. We then have the following easy lemma:

Lemma 4.1.2. *Suppose we are given a $\mathcal{W}\,\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with a $W_r(R)$-linear map $\iota : \mathcal{E}(W_r(R)) \to \mathcal{W}_r\,M^0$. Then:*

(a) The map ι extends uniquely to a map of graded left $\Omega^*_{W_r(R),\gamma}$-modules (not necessarily compatible with differentials)

$$\iota^* : \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) = \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma} \to \mathcal{W}_r\,M^*.$$

(b) If we are instead given a compatible family of maps $\iota_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r\,M^0$ for all r, then the resulting maps ι_r^* are also compatible.

Proof. Part (a) is the tensor-hom adjunction for the homomorphism $W_r(R) \to \Omega^*_{W_r(R),\gamma}$ of graded rings; explicitly, the map is

$$\iota^* : \mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma} \xrightarrow{\iota\otimes\mathrm{id}} \mathcal{W}_r\,M^0 \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma} \xrightarrow{m} \mathcal{W}_r\,M^*,$$

where m is the multiplication map. Part (b) is then clear from this formula. $\square$

Definition 4.1.3. By a *de Rham-Witt module over* $(R, \mathcal{E})$ we will mean a collection of the following data: a left $\mathcal{W}\,\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with $W_r(R)$-linear maps

$$\iota_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r\, M^0$$

for each r, such that:

1. For each $r > 0$, the diagram

$$\begin{tikzcd} \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] \arrow[d, two heads] & \mathcal{W}_r\, M^0 \arrow[d, two heads] \\ \mathcal{E}(W_{r-1}(R)) \arrow[r, "\iota_{r-1}"] & \mathcal{W}_{r-1}\, M^0 \end{tikzcd} \tag{4.1.3.1}$$

commutes, where the two vertical maps are the quotient maps.

2. For each $r > 0$, the diagram

$$\begin{tikzcd} \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] \arrow[d, "F"'] & \mathcal{W}_r\, M^0 \arrow[d, "F"] \\ \mathcal{E}(W_{r-1}(R)) \arrow[r, "\iota_{r-1}"] & \mathcal{W}_{r-1}\, M^0 \end{tikzcd} \tag{4.1.3.2}$$

commutes, where the left vertical map is defined in Example 2.5.5.

3. The maps ι_r^* of Lemma 4.1.2 are maps of complexes

$$(\mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma}, \nabla) \to (\mathcal{W}_r\, M^*, d).$$

A morphism of de Rham-Witt modules over $(R, \mathcal{E})$ is a morphism $f : M^* \to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules such that $\iota_{r,N} = \mathcal{W}_r(f^0) \circ \iota_{r,M}$ for each r. We call the resulting category $\mathrm{dRWM}_{R,\mathcal{E}}$.

Remark 4.1.4. The motivation for the maps ι_r is classical: in the case of the trivial F-crystal $\mathcal{E} = \mathcal{O}_{X/W}$, they correspond to the requirement $M^0_n = W_n(M^0_1)$ of [12, Définition 1.1, V1], which subsequent authors (e.g. [3, Definition 4.4.1(3)]) have usually replaced with the data of a map $W(R) \to M^0$.

Remark 4.1.5. Of course we can equivalently demand the data of the extension ι_r^*, rather than only $\iota_r = \iota_r^0$. In this case, we demand compatibility with F only in degree 0, as the source does not generally have a divided Frobenius operator in positive degrees (cf. Remark 2.7.18)—it only has an undivided Frobenius. As for the other compatibilities, by Lemma 4.1.2, the graded maps ι_r^* are $\Omega^*_{W_r(R),\gamma}$-linear if and only if the ι_r are $W_r(R)$-linear, and the ι_r^* are compatible with quotient maps (in all degrees) if and only if the ι_r are. We have a similar statement for morphisms $f : M^* \to N^*$: if $\iota_{r,N} = \mathcal{W}_r(f) \circ \iota_{r,M}$, then the compatibility $\iota^*_{r,N} = \mathcal{W}_r(f) \circ \iota^*_{r,M}$ in all degrees follows, because a graded $\Omega^*_{W_r(R),\gamma}$-module morphism from $\mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma}$ is determined by its degree-0 component.

This allows us to formulate an alternative to Definition 4.1.3 which focuses on the maps of complexes ι^*_r rather than their degree-0 components $\iota_r = \iota^0_r$. We will allow ourselves to pass freely between the two definitions, but will use this latter definition more often in practice.

Alternative Definition 4.1.6. A *de Rham-Witt module over* $(R, \mathcal{E})$ is a collection of the following data: a left $\mathcal{W}\,\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with maps

$$\iota^*_r : \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \to \mathcal{W}_r\, M^*$$

of dg-$\Omega^*_{W_r(R),\gamma}$-modules for each r, such that the following diagrams commute for all r:

$$\begin{tikzcd} \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \arrow[r, "\iota^*_r"] \arrow[d, two heads] & \mathcal{W}_r\, M^* \arrow[d, two heads] \\ \mathrm{dR}(\mathcal{E}(W_{r-1}(R), \gamma)) \arrow[r, "\iota^*_{r-1}"] & \mathcal{W}_{r-1}\, M^* \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{E}(W_r(R)) \arrow[r, "\iota^0_r"] \arrow[d, "F"'] & \mathcal{W}_r\, M^0 \arrow[d, "F"] \\ \mathcal{E}(W_{r-1}(R)) \arrow[r, "\iota^0_{r-1}"] & \mathcal{W}_{r-1}\, M^0 \end{tikzcd}$$

A morphism of de Rham-Witt modules over $(R, \mathcal{E})$ is a morphism $f : M^* \to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules such that $\iota^*_{r,N} = \mathcal{W}_r(f) \circ \iota^*_{r,M}$ for each r.

Definition 4.1.7. We call M^* a *saturated de Rham-Witt complex associated to* $\mathcal{E}$ *over* R if it is initial among de Rham-Witt modules over $(R, \mathcal{E})$. Such an object is unique up to unique isomorphism if it exists; we will denote it $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$.

Remark 4.1.8. Unfortunately, there is in general no "obvious" construction of an initial de Rham-Witt module. A natural attempt would be to give $\lim_r \mathrm{dR}(\mathcal{E}(W_r(R), \gamma))$ the structure of a Dieudonné complex and take its strictification; we are unable to carry this out in light of Remark 2.7.18. In lieu of such a direct construction, we will first give a construction in the case of the trivial F-crystal $\mathcal{E} = \mathcal{O}$, and later construct $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ in general by reducing to the lifted situation, which we will study in chapter 5.

Remark 4.1.9. All of the de Rham-Witt modules we are interested in will be concentrated in nonnegative degrees. This property (or more generally the vanishing of M^{-1} for a strict Dieudonné complex M^*) allows us to formulate another slight variant of Definition 4.1.3.

Namely, if $M^{-1} = 0$, then we have $\mathcal{W}_r M^0 = M^0/\operatorname{im} V^r$, so $F : M^0 \to M^0$ induces an endomorphism of $\mathcal{W}_r M^0$ rather than just a map $\mathcal{W}_r M^0 \to \mathcal{W}_{r-1} M^0$. Thus we could replace condition (2) of Definition 4.1.3 with the commutativity of the diagram

$$\begin{tikzcd} \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] \arrow[d, "F"'] & \mathcal{W}_r M^0 \arrow[d, "F"] \\ \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] & \mathcal{W}_r M^0, \end{tikzcd} \tag{4.1.9.1}$$

where the left vertical map is again given by Example 2.5.5. It should come as no surprise that this makes no difference:

Lemma 4.1.10. *Suppose M^* is a strict Dieudonné complex such that $M^{-1} = 0$, and suppose M^* is equipped with maps ι_r such that* (4.1.3.1) *commutes. Then the commutativity of* (4.1.3.2) *for all r is equivalent to the commutativity of* (4.1.9.1) *for all r.*

Proof. If (4.1.9.1) and (4.1.3.1) both commute for some integer r, then combining the two squares shows that (4.1.3.2) commutes as well. Conversely, if (4.1.3.2) commutes for the integer $r + 1$, then we can factor this diagram as

$$\begin{tikzcd} \mathcal{E}(W_{r+1}(R)) \arrow[r, "\iota_{r+1}"] \arrow[d, two heads] & \mathcal{W}_{r+1} M^0 \arrow[d, two heads] \\ \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] \arrow[d, "F"'] & \mathcal{W}_r M^0 \arrow[d, "F"] \\ \mathcal{E}(W_r(R)) \arrow[r, "\iota_r"] & \mathcal{W}_r M^0, \end{tikzcd}$$

and observe that the commutativity of the outer rectangle and the top square implies that of the bottom square. □

4.2 Example: the trivial F-crystal

4.2.1. We will first construct $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ in the case of the trivial F-crystal $(\mathcal{O}_{X/S}, F)$ of Example 2.5.3, on an arbitrary affine k-scheme $X = \operatorname{Spec} R$. This will of course recover the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Bhatt-Lurie-Mathew. The point is that we have replaced the algebra structure in the universal property of [3, Definition 4.1.1] with a module structure.

Construction 4.2.2. The "trivial" de Rham-Witt module over $(R, \mathcal{O}_{\operatorname{Spec} R/W})$ is as follows. View $\mathcal{W}\Omega^*_R$ as a left module over itself, and let $\iota_r^* : \Omega^*_{W_r(R),\gamma} \to \mathcal{W}_r \Omega^0_R$ be the dg-algebra map of Lemma 3.6.4, which is in particular also a dg-$\Omega^*_{W_r(R),\gamma}$-module map.

Proposition 4.2.3. *The object $(\mathcal{W}\Omega^*_R, (\iota_r)_r)$ of Construction 4.2.2 is a saturated de Rham-Witt complex associated to the trivial F-crystal on* $\operatorname{Spec} R$.

Proof. First, we check that it is a de Rham-Witt module over $(R, \mathcal{O}_{\operatorname{Spec} R/W})$, using Alternative Definition 4.1.6. By construction, the maps ι_r^* and ι_{r-1}^* are compatible via the quotient maps. To see that ι_r^0 and ι_{r-1}^0 are compatible via Frobenius, consider the diagram:

$$\begin{tikzcd}
W(R) \arrow[rr] \arrow[dr, "F"] \arrow[dd, two heads] & & \mathcal{W}\operatorname{Sat}(\Omega^*_{W(R_{\mathrm{red}})})^0 \arrow[dr, "F"] \arrow[dd, two heads] & \\
& W(R) \arrow[rr] \arrow[dd, two heads] & & \mathcal{W}\operatorname{Sat}(\Omega^*_{W(R_{\mathrm{red}})})^0 \arrow[dd, two heads] \\
W_r(R) \arrow[rr, "\iota_r^0"] \arrow[dr, "F"] & & \mathcal{W}_r\,\Omega^0_R \arrow[dr, "F"] & \\
& W_{r-1}(R) \arrow[rr, "\iota_{r-1}^0"] & & \mathcal{W}_{r-1}\,\Omega^0_R
\end{tikzcd}$$

The top, left, and right faces of the cube commute, as do the front and back faces by the construction of ι_r^* and ι_{r-1}^*. Since the vertical maps are surjective, it follows that the bottom face commutes.

So $(\mathcal{W}\,\Omega_R^*, (\iota_r^*)_r)$ is indeed a de Rham-Witt module over $(R, \mathcal{O}_{\operatorname{Spec} R/W})$. Now let $M^* = (M^*, (\iota_{r,M}^*)_r)$ be another such de Rham-Witt module, we must show there is a unique map $\mathcal{W}\,\Omega_R^* \to M^*$ of de Rham-Witt modules. Such a map is in particular a $\mathcal{W}\,\Omega_R^*$-module morphism; by Lemma 3.2.8 and Remark 3.2.9, these are determined by $f(1)$, which may be any $x \in M^0$ such that $Fx = x$.

But in order for f to be a morphism of de Rham-Witt modules, each $\mathcal{W}_r(f)$ must intertwine $\iota_{r,M}^*$ with $\iota_{r,\mathcal{W}\Omega}^*$. Equivalently, $\mathcal{W}_r(f)$ must send $\iota_{r,\mathcal{W}\Omega}^0(1) = 1 \in \mathcal{W}_r\,\Omega_R^*$ to $\iota_{r,M}^0(1) \in \mathcal{W}_r\,M^0$. So $f(1)$ must be the element $x := (\iota_{r,M}^0(1))_r \in \lim_r(\mathcal{W}_r\,M^0) = M^0$. This element is indeed fixed by Frobenius, since we have $F(1) = 1$ in $W_r(R)$, and $\iota_{r,M}^0$ commutes with Frobenius. Thus this f is the unique morphism of de Rham-Witt modules $\mathcal{W}\,\Omega_R^* \to M^*$, as desired. $\square$

4.3 Functoriality of $\mathrm{dRWM}_{R,\mathcal{E}}$

4.3.1. Before showing that our saturated de Rham-Witt complexes exist in general, we will discuss the functoriality of the category $\mathrm{dRWM}_{R,\mathcal{E}}$, first in the F-crystal $\mathcal{E}$ and then in the pair R/k. Both of these functorialities will behave as the identity on the underlying strict Dieudonné complexes.

Construction 4.3.2. Let $f : \mathcal{E} \to \mathcal{E}'$ be a morphism of unit-root F-crystals on $\operatorname{Spec} R$. We have a functor $f^* : \mathrm{dRWM}_{R,\mathcal{E}'} \to \mathrm{dRWM}_{R,\mathcal{E}}$, defined as follows.

Given an object $(M^*, (\iota_r'^*)_r)$ in $\mathrm{dRWM}_{R,\mathcal{E}'}$, first notice that M^* already has the structure of a $\mathcal{W}\Omega_R^*$-module in $\mathbf{DC}_{\mathrm{str}}$. We endow M^* with maps $\iota_r^* : \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathcal{W}_r M^*$ defined as the composition

$$\mathrm{dR}(\mathcal{E}(W_r(R)),\gamma) \xrightarrow{f(W_r(R))\otimes\mathrm{id}} \mathrm{dR}(\mathcal{E}'(W_r(R)),\gamma) \xrightarrow{\iota_r'^*} \mathcal{W}_r M^*.$$

We first check that $f^*(M^*) := (M^*, (\iota_r^*)_r)$ is a de Rham-Witt module over $(R,\mathcal{E})$. For each r, ι_r^* is a map of dg-$\Omega^*_{W_r(R),\gamma}$-modules, because the same is true of $f(W_r(R))\otimes\mathrm{id}$ (by Proposition 2.7.9) and $\iota_r'^*$ (by assumption). The compatibilities with quotient and Frobenius maps likewise follow from those of $f(W_r(R))\otimes\mathrm{id}$ and $\iota_r'^*$. So $(M^*, (\iota_r^*)_r)$ is a de Rham-Witt module.

Now let $M^* = (M^*, (\iota_r'^*)_r)$ and $N^* = (N^*, (\eta_r'^*)_r)$ be two de Rham-Witt modules over $(R,\mathcal{E}')$, and let ι_r^* and η_r^* be the maps constructed above, so that

$$\begin{gathered} f^*(M^*) = (M^*, (\iota_r^*)_r) \text{ and} \\ f^*(N^*) = (N^*, (\eta_r^*)_r). \end{gathered}$$

Suppose we are given a morphism $g : M^* \to N^*$ in $\mathrm{dRWM}_{R,\mathcal{E}'}$. This is by definition a map of $\mathcal{W}\Omega_R^*$-modules which makes the bottom triangle of the following diagram commute for each r:

$$\begin{tikzcd}
 & \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \arrow[d] \arrow[ddl, "\iota_r^*"'] \arrow[ddr, "\eta_r^*"] & \\
 & \mathrm{dR}(\mathcal{E}'(W_r(R),\gamma)) \arrow[dl, "\iota_r'^*"] \arrow[dr, "\eta_r'^*"'] & \\
\mathcal{W}_r M^* \arrow[rr, "\mathcal{W}_r g"] & & \mathcal{W}_r N^*
\end{tikzcd}$$

This diagram lives in the category of dg-$\Omega^*_{W_r(R),\gamma}$-modules. Since the two other small triangles commute by the construction of ι_r^* and η_r^*, it follows that the large triangle commutes, which makes the map $f^*(g) := g$ a morphism from $f^*(M^*) = (M^*, (\iota_r^*)_r)$ to $f^*(N^*) = (N^*, (\eta_r^*)_r)$ in $\mathrm{dRWM}_{R,\mathcal{E}}$. We clearly have $f^*(\mathrm{id}) = \mathrm{id}$ and $f^*(g\circ h) = f^*(g)\circ f^*(h)$ for all g, h, so f^* is a functor $\mathrm{dRWM}_{R,\mathcal{E}'} \to \mathrm{dRWM}_{R,\mathcal{E}}$.

Lemma 4.3.3. *The construction above is functorial in f: given maps $\mathcal{E} \xrightarrow{f} \mathcal{E}' \xrightarrow{g} \mathcal{E}''$, we have equalities of functors*

$$\begin{gathered} (gf)^* = f^*\circ g^* : \mathrm{dRWM}_{R,\mathcal{E}''} \to \mathrm{dRWM}_{R,\mathcal{E}} \ \textit{ and} \\ \mathrm{id}^* = \mathrm{id} : \mathrm{dRWM}_{R,\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}} \end{gathered}$$

Proof. Immediate from the construction. □

4.3.4. Next we move on to functoriality in R. In order to apply this to the Frobenius endomorphism of R later (Proposition 4.3.13), it will be useful to give an account of functoriality in the pair R/k rather than only for k-algebra maps $R \to R'$. Thus we introduce the following temporary notation: $\mathrm{dRWM}_{R/k,\mathcal{E}}$ is the category of de Rham-Witt modules over $(R, \mathcal{E})$ with ground field k, and $\mathcal{W}\,\Omega^*_{R/k,\mathcal{E}}$ is its initial object (provided that this exists).

4.3.5. Note that although the ground field k makes no direct appearance in Definition 4.1.3 (thanks to our implicit use of Lemma 2.2.5), the choices of R and $(\mathcal{E}, \phi_{\mathcal{E}})$ constrain k in opposite directions: k must be small enough so that R is a k-algebra, but large enough so that $(\mathcal{E}, \phi_{\mathcal{E}})$ is defined over $W = W(k)$. Thus, in contrast to the situation in [3], choosing $k = \mathbb{F}_p$ uniformly would cause a loss of generality.

Construction 4.3.6. Suppose we are given a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r, "f"] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} k' \arrow[r] & \operatorname{Spec} k, \end{tikzcd}$$

corresponding to a ring map $f^\sharp : R \to R'$ over $k \to k'$, and let $\mathcal{E}$ be a unit-root F-crystal on $\mathrm{Cris}(\operatorname{Spec} R/W)$. We have a functor $f_* : \mathrm{dRWM}_{R'/k', f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R/k,\mathcal{E}}$, defined as follows.

Suppose we are given an object M^* in $\mathrm{dRWM}_{R'/k', f^*_{\mathrm{cris}}\mathcal{E}}$. This is by definition a strict $\mathcal{W}\,\Omega^*_{R'}$-module equipped with morphisms $\iota'^*_r : \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(R'), \gamma)) \to \mathcal{W}_r\, M^*$ for each r, satisfying various compatibilities. We make the Dieudonné complex M^* into a module over $\mathcal{W}\,\Omega^*_R$ via the map $\mathcal{W}\,\Omega^*_{f^\sharp} : \mathcal{W}\,\Omega^*_R \to \mathcal{W}\,\Omega^*_{R'}$. By Corollary 2.7.12, we have

$$\mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} \Omega^*_{W_r(R'),\gamma} \xrightarrow{\sim} \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(R'), \gamma))$$

as a dg-module. We then define $\iota^*_r : \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \to \mathcal{W}_r\, M^*$ as the composition

$$\mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \xrightarrow{\pi^*_r} \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(R'), \gamma)) \xrightarrow{\iota'^*_r} \mathcal{W}_r\, M^*,$$

where π^*_r is the base change map. All of the compatibilities of ι^*_r follow from the corresponding compatibilities of ι'^*_r and π^*_r, so that (M^*, ι^*_r) is a de Rham-Witt module over $(R, \mathcal{E})$.

Finally, suppose we are given a morphism $g : (M^*, (\iota'^*_r)_r) \to (N^*, (\eta'^*_r)_r)$ in $\mathrm{dRWM}_{R'/k', f^*_{\mathrm{cris}}\mathcal{E}}$. This is by definition a map of strict $\mathcal{W}\,\Omega^*_{R'}$-modules making the lower triangle of the diagram

$$\begin{tikzcd} & \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \arrow[d, "\pi^*_r"] \arrow[ddl, "\iota^*_r"'] \arrow[ddr, "\eta^*_r"] & \\ & \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(R'), \gamma)) \arrow[dl, "\iota'^*_r"] \arrow[dr, "\eta'^*_r"'] & \\ \mathcal{W}_r\, M^* \arrow[rr, "\mathcal{W}_r\, g"] & & \mathcal{W}_r\, N^* \end{tikzcd}$$

commute. But as in Construction 4.3.2, commutativity of the lower triangle implies that of the outer triangle, and a map of strict $\mathcal{W}\Omega^*_{R'}$-modules is in particular a map of strict $\mathcal{W}\Omega^*_R$-modules via $\mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_{R'}$. This makes $f_*(g) := g$ a morphism in $\mathrm{dRWM}_{R/k,\mathcal{E}}$. As before, the identities $f_*(\mathrm{id}) = \mathrm{id}$ and $f_*(g \circ h) = f_*(g) \circ f_*(h)$ are immediate, making $f_* : \mathrm{dRWM}_{R'/k', f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R/k,\mathcal{E}}$ a functor.

Lemma 4.3.7. *(Functoriality of* $\mathrm{dRWM}_{R/k,\mathcal{E}}$ *in* R/k*) Given maps* $\operatorname{Spec} R'' \xrightarrow{f} \operatorname{Spec} R' \xrightarrow{g} \operatorname{Spec} R$ *of affine schemes over* $\operatorname{Spec} k'' \to \operatorname{Spec} k' \to \operatorname{Spec} k$*, and a unit-root F-crystal $\mathcal{E}$ on* $\mathrm{Cris}(\operatorname{Spec} R/W)$*, we have a commutative diagram of categories*

$$\begin{tikzcd}
\mathrm{dRWM}_{R''/k'',(gf)^*_{\mathrm{cris}}\mathcal{E}} \arrow[d, "\simeq"'] \arrow[dr, "(gf)_*"] & \\
\mathrm{dRWM}_{R''/k'',f^*_{\mathrm{cris}}g^*_{\mathrm{cris}}\mathcal{E}} \arrow[r, "f_*\circ g_*"'] & \mathrm{dRWM}_{R/k,\mathcal{E}}
\end{tikzcd}$$

where the vertical equivalence is induced by the canonical isomorphism $(gf)^*_{\mathrm{cris}}\mathcal{E} \simeq f^*_{\mathrm{cris}}g^*_{\mathrm{cris}}\mathcal{E}$*. Moreover, we have* $\mathrm{id}_* = \mathrm{id} : \mathrm{dRWM}_{R,\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$*.*

Proof. Clear from the construction. □

Lemma 4.3.8. *(The functorialities in $\mathcal{E}$ and R/k commute.) Suppose we are given a map* $f : \operatorname{Spec} R' \to \operatorname{Spec} R$ *of affine schemes over* $\operatorname{Spec} k' \to \operatorname{Spec} k$ *and a map* $g : \mathcal{E} \to \mathcal{E}'$ *of unit-root F-crystals on* $\mathrm{Cris}(\operatorname{Spec} R/W)$*. Then the diagram of categories*

$$\begin{tikzcd}
\mathrm{dRWM}_{R'/k',f^*_{\mathrm{cris}}\mathcal{E}'} \arrow[r, "f^*_{\mathrm{cris}}(g)^*"] \arrow[d, "f_*"'] & \mathrm{dRWM}_{R'/k',f^*_{\mathrm{cris}}\mathcal{E}} \arrow[d, "f_*"] \\
\mathrm{dRWM}_{R/k,\mathcal{E}'} \arrow[r, "g^*"] & \mathrm{dRWM}_{R/k,\mathcal{E}}
\end{tikzcd}$$

commutes.

Proof. This follows from the constructions and the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{W}_r M^* & \mathrm{dR}(f^*_{\mathrm{cris}}(\mathcal{E}')(W_r(R'),\gamma)) \arrow[l, "\iota_r^*"'] & \mathrm{dR}(f^*_{\mathrm{cris}}(\mathcal{E})(W_r(R'),\gamma)) \arrow[l, "f^*_{\mathrm{cris}}(g)\otimes 1"'] \\
 & \mathrm{dR}(\mathcal{E}'(W_r(R),\gamma)) \arrow[u, "W_r(f)^*"] & \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)). \arrow[l, "g\otimes 1"] \arrow[u, "W_r(f)^*"']
\end{tikzcd}$$

□

Remark 4.3.9. Examining Construction 4.3.6 reveals that the category $\mathrm{dRWM}_{R/k,\mathcal{E}}$ is independent of k in the following sense: whenever $f^\sharp : R \to R'$ is an isomorphism, the functor $f_* : \mathrm{dRWM}_{R'/k', f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R/k,\mathcal{E}}$ is an isomorphism of categories, regardless of whether $k \to k'$ is an isomorphism. Thus, at this point we will resume our previous convention of omitting k from the notation $\mathrm{dRWM}_{R,\mathcal{E}}$.

Remark 4.3.10. The functorialities of the category $\mathrm{dRWM}_{R,\mathcal{E}}$ naturally lead to functorialities of its initial object, provided that the initial object exists. Namely, suppose we are given a map $f : \operatorname{Spec} R' \to \operatorname{Spec} R$ of affine schemes over $\operatorname{Spec} k' \to \operatorname{Spec} k$ and a map $g : \mathcal{E} \to \mathcal{E}'$ of unit-root F-crystals on $\mathrm{Cris}(\operatorname{Spec} R/W)$. Then assuming the relevant saturated de Rham-Witt complexes exist, there are unique maps in $\mathrm{dRWM}_{R,\mathcal{E}}$

$$\mathcal{W}\Omega^*_{R,\mathcal{E}} \to g^*(\mathcal{W}\Omega^*_{R,\mathcal{E}'}) \quad \text{and} \quad \mathcal{W}\Omega^*_{R,\mathcal{E}} \to f_*(\mathcal{W}\Omega^*_{R', f^*_{\mathrm{cris}}\mathcal{E}}),$$

which compose as expected if we are instead given maps

$$\mathcal{E} \xrightarrow{g} \mathcal{E}' \xrightarrow{g'} \mathcal{E}'' \quad \text{or} \quad \operatorname{Spec} R'' \xrightarrow{f'} \operatorname{Spec} R' \xrightarrow{f} \operatorname{Spec} R.$$

Since g^* and f_* act as the identity on the underlying strict Dieudonné complexes, we may regard these as maps

$$\mathcal{W}\Omega^*_{R,\mathcal{E}} \to \mathcal{W}\Omega^*_{R,\mathcal{E}'} \quad \text{and} \quad \mathcal{W}\Omega^*_{R,\mathcal{E}} \to \mathcal{W}\Omega^*_{R', f^*_{\mathrm{cris}}\mathcal{E}}$$

in $\mathbf{DC}_{\mathrm{str}}$.

Remark 4.3.11. We will finish this section by computing the functoriality map of Remark 4.3.10 explicitly in the case of the Frobenius morphism. Before stating this, recall that for any Dieudonné complex M^*, the map

$$\alpha_F : M^* \to M^*$$

defined by $p^i F$ in degree i is a morphism of Dieudonné complexes, thanks to the calculations

$$\begin{aligned} F(\alpha_F(x)) &= p^i F^2(x) = \alpha_F(Fx) \text{ and} \\ d(\alpha_F(x)) &= p^i dF(x) = p^{i+1} F dx = \alpha_F(dx) \end{aligned}$$

for x homogeneous of degree i. In particular, if M^* is saturated, it makes sense to write down the maps $\mathcal{W}_r(\alpha_F) : \mathcal{W}_r M^* \to \mathcal{W}_r M^*$.

We give the result first in the case of trivial coefficients, and then for a general $\mathcal{E}$:

Lemma 4.3.12. *For any $\mathbb{F}_p$-algebra R, the endomorphism*

$$\mathcal{W}\Omega^*_{F_R} : \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_R$$

induced by the absolute Frobenius $F_R : R \to R$ coincides with α_F.

Proof. First note that α_F is a map of Dieudonné algebras, by Remark 4.3.11 and multiplicativity. Now, unwinding the universal property of [3, Definition 4.1.1], the claim says that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "F_R"'] & \mathcal{W}_1\,\Omega^0_R \arrow[d, "\mathcal{W}_1(\alpha_F)"] \\ R \arrow[r] & \mathcal{W}_1\,\Omega^0_R \end{tikzcd}$$

commutes. But in fact it follows from the definition of Dieudonné algebras that $\mathcal{W}_1(\alpha_F) = F : \mathcal{W}_1\,\Omega^0_R \to \mathcal{W}_1\,\Omega^0_R$ is the Frobenius endomorphism of this $\mathbb{F}_p$-algebra, so the diagram indeed commutes. □

Proposition 4.3.13. *Let R be a k-algebra and $\mathcal{E}$ a unit-root F-crystal on* $\mathrm{Cris}(\operatorname{Spec} R/W)$*. Consider the object $(F_{\operatorname{Spec} R})_*\phi^*_{\mathcal{E}}\,\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ of* $\mathrm{dRWM}_{R,\mathcal{E}}$ *obtained from Constructions 4.3.2 and 4.3.6, and recall that we have*

$$(F_{\operatorname{Spec} R})_*\phi^*_{\mathcal{E}}\,\mathcal{W}\,\Omega^*_{R,\mathcal{E}} = \mathcal{W}\,\Omega^*_{R,\mathcal{E}}$$

as a strict Dieudonné complex. Then the unique map

$$\mathcal{W}\,\Omega^*_{R,\mathcal{E}} \to (F_{\operatorname{Spec} R})_*\phi^*_{\mathcal{E}}\,\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$$

in $\mathrm{dRWM}_{R,\mathcal{E}}$ *coincides with the map $\alpha_F : \mathcal{W}\,\Omega^*_{R,\mathcal{E}} \to \mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ of Dieudonné complexes.*

Proof. For convenience, we will abbreviate the absolute Frobenius morphism $F_{\operatorname{Spec} R}$ as f. To show that α_F is a map of de Rham-Witt modules, we must show that it is compatible with module structures and ι maps. Unraveling the given functors

$$\mathrm{dRWM}_{R,\mathcal{E}} \xrightarrow{\phi^*_{\mathcal{E}}} \mathrm{dRWM}_{R,f^*_{\mathrm{cris}}\mathcal{E}} \xrightarrow{f_*} \mathrm{dRWM}_{R,\mathcal{E}},$$

the necessary compatibility with module structures is that for all $x \in \mathcal{W}\,\Omega^*_R$ and $m \in \mathcal{W}\,\Omega^*_{R,\mathcal{E}}$, we have

$$\alpha_F(x \cdot m) = \mathcal{W}\,\Omega^*_{F_R}(x) \cdot \alpha_F(m).$$

Taking x and m homogeneous of degree i and j respectively, this equation simplifies by Lemma 4.3.12 to the known identity

$$p^{i+j}F(x \cdot m) = p^i F(x) \cdot p^j F(m).$$

As for compatibility with ι maps, the de Rham-Witt module $f_*\phi^*_{\mathcal{E}}\,\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ is equipped by construction with the maps

$$\iota_r : \mathcal{E}(W_r(R)) \xrightarrow{W_r(f)^*} f^*_{\mathrm{cris}}\mathcal{E}(W_r(R)) \xrightarrow{\phi_{\mathcal{E}}} \mathcal{E}(W_r(R)) \xrightarrow{\iota_{r,\mathcal{W}\Omega}} \mathcal{W}_r\,\Omega^0_{R,\mathcal{E}},$$

where $\iota_{r,\mathcal{W}\Omega}$ is the map which the de Rham-Witt module $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ comes equipped with, and the composition of the first two maps is the semilinear endomorphism

$$F : \mathcal{E}(W_r(R)) \to \mathcal{E}(W_r(R))$$

of Example 2.5.5. (Note in particular that $W_r(f)$ agrees with the Witt vector Frobenius $F : W_r(R) \to W_r(R)$ by Lemma 2.1.8.) Thus, to prove that α_F is compatible with the ι maps, it suffices to prove that the square

$$\begin{array}{ccc} \mathcal{E}(W_r(R)) & \xrightarrow{\iota_r} & \mathcal{W}_r\,\Omega^0_{R,\mathcal{E}} \\ \downarrow F & & \downarrow F \\ \mathcal{E}(W_r(R)) & \xrightarrow{\iota_r} & \mathcal{W}_r\,\Omega^0_{R,\mathcal{E}} \end{array}$$

commutes. But this square is exactly (4.1.9.1), and its commutativity follows from Lemma 4.1.10 since $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ is a de Rham-Witt module concentrated in nonnegative degrees. □

4.4 Insensitivity to nilpotent thickenings

Throughout this section, we fix the following setup:

Situation 4.4.1. Let R be a k-algebra, and $\mathcal{E}$ a unit-root F-crystal on $\mathrm{Cris}(\operatorname{Spec} R/W)$. Let $f : \operatorname{Spec} R_{\mathrm{red}} \hookrightarrow \operatorname{Spec} R$ be the closed embedding over k corresponding to the reduction map $f^\sharp : R \to R_{\mathrm{red}}$.

4.4.2. Recall from Remark 1.2.7 that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Bhatt-Lurie-Mathew is insensitive to nilpotent thickenings. Our present goal is to generalize this statement to saturated de Rham-Witt complexes with unit-root coefficients. The main result is as follows.

Proposition 4.4.3. *The functor $f_* : \mathrm{dRWM}_{R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$ is an equivalence of categories.*

Before we prove this, we will need a few lemmas examining the behavior of the pullback map

$$\mathcal{E}_{W_r(R)} \to W_r(f)^*(\mathcal{E}_{W_r(R)}) = (f^*_{\mathrm{cris}}\mathcal{E})_{W_r(R_{\mathrm{red}})}$$

(cf. Lemma 2.4.4):

Lemma 4.4.4. *Fix some $r > 0$. For every local section x of $\ker(\mathcal{E}_{W_r(R)} \to (f^*_{\mathrm{cris}}\mathcal{E})_{W_r(R_{\mathrm{red}})})$, there exists an $N \gg 0$ and an open covering $\{U_i\}_i$ of $\operatorname{Spec} R$ such that $x|_{W_r(U_i)}$ lies in the image of the p^N-torsion in $\mathcal{E}_{W_n(U_i)}$ for all $n \geq r$.*

Proof. As the statement is local on $\operatorname{Spec} W_r(R)$, we may assume by Lemma 2.4.8 that the entire tower of Zariski sheaves $\mathcal{E}_{W_\bullet(R)}$ is isomorphic to $\mathcal{O}^{\oplus m}_{\operatorname{Spec} W_\bullet(R)}$. We also assume without loss of generality that x is a globally defined section of $\mathcal{E}_{W_r(R)}$; that is, an element of $W_r(R)$. Finally, we assume that the rank m of $\mathcal{E}$ is 1; the general case follows by repeating the same construction in each coordinate.

With this setup, all we must prove is the following: for every $x \in \ker(W_r(R) \to W_r(R_{\mathrm{red}}))$, there exists an $N \gg 0$ such that x lies in the image of the p^N-torsion in $W_n(R)$ for all $n \geq r$. To prove this, write x as $(x_0, x_1, \ldots, x_{r-1})$ in Witt coordinates, where each $x_i \in R$ is nilpotent. Choose N so that each $x_i^{p^N} = 0$, and lift x to $(x_0, x_1, \ldots, x_{r-1}, 0, 0, \ldots) \in W_n(R)$ for each $n \geq r$. This is p^N-torsion by Lemma 2.1.8. $\square$

We will actually need the following slight strengthening of the previous lemma:

Corollary 4.4.5. *Fix $r > 0$. For every $x \in \ker(\mathcal{E}(W_r(R)) \to (f^*_{\mathrm{cris}}\mathcal{E})(W_r(R_{\mathrm{red}})))$, there exists an $N \gg 0$ such that x lies in the image of the p^N-torsion in $\mathcal{E}(W_n(R))$ for all $n \geq r$.*

Proof. We proved above that there exists an $N \gg 0$ such that for all $n \geq r$, the sheaf-theoretic image of the reduction map

$$\mathcal{E}_{W_n(R)}[p^N] \to \mathcal{E}_{W_r(R)}$$

(both interpreted as sheaves on $\operatorname{Spec} W_n(R)$, say) contains the kernel of

$$\mathcal{E}_{W_r(R)} \to W_r(f)^*(\mathcal{E}_{W_r(R)}) = (f^*_{\mathrm{cris}}\mathcal{E})_{W_r(R_{\mathrm{red}})}$$

We must now prove the analogous statement for global sections instead of sheaves, which would ordinarily be a more delicate statement. But since we are dealing with maps of quasicoherent sheaves on an affine scheme, taking global sections is an exact functor, and thus the global sections of the sheaf-theoretic image coincide with the image of the global sections. So the two statements are equivalent. $\square$

Lemma 4.4.6. *Continuing with the setup of Proposition 4.4.3, suppose $(M^*, (\iota_r)_r)$ is a de Rham-Witt module over $(R, \mathcal{E})$. Then each map $\iota_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r M^0$ factors through the surjection $\mathcal{E}(W_r(R)) \twoheadrightarrow \mathcal{E}(W_r(R)) \otimes_{W_r(R)} W_r(R_{\mathrm{red}}) \simeq (f^*_{\mathrm{cris}}\mathcal{E})(W_r(R_{\mathrm{red}}))$.*

Proof. Let x lie in the kernel of $\mathcal{E}(W_r(R)) \twoheadrightarrow (f^*_{\mathrm{cris}}\mathcal{E})(W_r(R_{\mathrm{red}}))$. We must show that $\iota_r(x) = 0$. By Corollary 4.4.5, we can lift x to a p^N-torsion element $\widetilde{x} \in \mathcal{E}(W_{r+N}(R))$. Then we have $\iota_{r+N}(\widetilde{x}) \in \mathcal{W}_{r+N} M^0[p^N]$. By axiom 7 in the definition of Dieudonné towers ([3, Definition 2.6.1]), this must reduce to 0 in $\mathcal{W}_r M^0$. So we have $\iota_r(x) = 0$. $\square$

Proof of Proposition 4.4.3. We must show that $f_* : \mathrm{dRWM}_{R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$ is essentially surjective and fully faithful. To prove it is essentially surjective, let $(M^*, (\iota^*_r)_r)$ be a de Rham-Witt module over $(R, \mathcal{E})$; we must show this is isomorphic to an object in the image of f_*. We will construct such an object by keeping the same underlying strict Dieudonné complex M^* (viewed as a $\mathcal{W}\,\Omega^*_{R_{\mathrm{red}}}$-module via the isomorphism $\mathcal{W}\,\Omega^*_R \to \mathcal{W}\,\Omega^*_{R_{\mathrm{red}}}$ of Remark 1.2.7) and factoring the ι^*_r through suitable maps $\iota'^*_r : \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathcal{W}_r\,M^*$.

To construct ι'^*_r, first observe by Lemma 4.4.6 that the map $\iota^0_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r\,M^0$ annihilates

$$\ker(W_r(R) \to W_r(R_{\mathrm{red}})) \cdot \mathcal{E}(W_r(R)).$$

It follows that ι^*_r annihilates

$$\ker(W_r(R) \to W_r(R_{\mathrm{red}})) \cdot \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)),$$

since $\mathcal{E}(W_r(R))$ generates its PD-de Rham complex as a module over $\Omega^*_{W_r(R),\gamma}$. Therefore ι^*_r factors through

$$\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} \Omega^*_{W_r(R),\gamma}/K,$$

where K is the dg-ideal generated by $\ker(W_r(R) \to W_r(R_{\mathrm{red}}))$ in degree 0. But in view of Lemma 2.2.7 and Corollary 2.7.12, we have

$$\begin{aligned}\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} \Omega^*_{W_r(R),\gamma}/K &= \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} \Omega^*_{W_r(R_{\mathrm{red}}),\gamma} \\ &\simeq \mathrm{dR}(\mathcal{E}(W_r(R_{\mathrm{red}}),\gamma)),\end{aligned}$$

so we have succeeded in factoring ι^*_r through a map

$$\mathrm{dR}(\mathcal{E}(W_r(R_{\mathrm{red}}),\gamma)) \xrightarrow{\iota'^*_r} \mathcal{W}_r\,M^*$$

as intended.

Each ι'^*_r is a map of dg-$\Omega^*_{W_r(R_{\mathrm{red}}),\gamma}$-modules by construction, and the necessary compatibilities with Frobenius (in degree 0) and quotient maps follow from the corresponding compatibilities of ι^*_r and $\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \twoheadrightarrow \mathrm{dR}(\mathcal{E}(W_r(R_{\mathrm{red}}),\gamma))$. So the strict $\mathcal{W}\,\Omega^*_{R_{\mathrm{red}}}$-module M^*, equipped with the maps ι'^*_r, is a de Rham-Witt module over $(R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E})$. We have $f_*(M^*, (\iota'^*_r)_r) = (M^*, (\iota^*_r)_r)$ by construction, proving that f_* is essentially surjective.

To prove that f_* is fully faithful, let $M^* = (M^*, (\iota'^*_r)_r)$ and $N^* = (N^*, (\eta'^*_r)_r)$ be two objects in $\mathrm{dRWM}_{R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E}}$. Let ι^*_r and η^*_r be the maps of Construction 4.3.6; that is,

$$\begin{aligned}f_*M^* &= (M^*, (\iota^*_r)_r) \text{ and} \\ f_*N^* &= (N^*, (\eta^*_r)_r)\end{aligned}$$

in $\mathrm{dRWM}_{R,\mathcal{E}}$. Recall that a homomorphism $g : M^* \to N^*$ in $\mathrm{dRWM}_{R_{\mathrm{red}},f^*_{\mathrm{cris}}\mathcal{E}}$ is a map of $\mathcal{W}\Omega^*_{R_{\mathrm{red}}}$-modules such that small triangle in the diagram

$$\begin{tikzcd}
 & \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \arrow[d, "\pi_r^*"] \arrow[ddl, "\iota_r^*"'] \arrow[ddr, "\eta_r^*"] & \\
 & \mathrm{dR}(\mathcal{E}(W_r(R_{\mathrm{red}}),\gamma)) \arrow[dl, "\iota'^*_r"] \arrow[dr, "\eta'^*_r"'] & \\
\mathcal{W}_r\, M^* \arrow[rr, "\mathcal{W}_r\, g"] & & \mathcal{W}_r\, N^*
\end{tikzcd}$$

commutes for all r. A homomorphism $g : f_*M^* \to f_*N^*$ in $\mathrm{dRWM}_{R,f^*_{\mathrm{cris}}\mathcal{E}}$ is a map of $\mathcal{W}\Omega^*_R$-modules such that the large triangle commutes. But π_r^* is surjective and the other triangles commute, so the inner triangle commutes if and only if the outer one does. Since we also know that $\mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_{R_{\mathrm{red}}}$ is an isomorphism, it follows that

$$\mathrm{Hom}_{\mathrm{dRWM}_{R_{\mathrm{red}},f^*_{\mathrm{cris}}\mathcal{E}}}(M^*,N^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{dRWM}_{R,\mathcal{E}}}(f_*M^*, f_*N^*),$$

as desired. This completes the proof. □

Corollary 4.4.7. *In the situation of Proposition 4.4.3, suppose that either $(R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E})$ or $(R,\mathcal{E})$ admits a saturated de Rham-Witt complex. Then both pairs admit saturated de Rham-Witt complexes, and the equivalence $f_* : \mathrm{dRWM}_{R_{\mathrm{red}},f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$ sends one to the other.*

Proof. This follows immediately from Proposition 4.4.3 and the definition of saturated de Rham-Witt complexes as initial de Rham-Witt modules. □

Remark 4.4.8. Since the functor f_* preserves the underlying Dieudonné complex and its module structure over $\mathcal{W}\Omega^*_R \simeq \mathcal{W}\Omega^*_{R_{\mathrm{red}}}$, and the maps ι_r^* and ι'^*_r are compatible, we can regard the two saturated de Rham-Witt complexes in Corollary 4.4.7 as being "the same".

4.5 Étale localization

We now address the question of how our saturated de Rham-Witt modules localize along étale morphisms of affine schemes. We begin with a few lemmas:

Lemma 4.5.1. *Suppose A^* is an algebra object in* **DC**, *M^* and N^* are strict right and left A^*-modules respectively, and P^* is any strict Dieudonné complex equipped with a map $f : M^* \otimes_{A^*} N^* \to P^*$ in* **DC**. *Then the map for each $r \geq 0$, the composition*

$$f_r : M^* \otimes_{A^*} N^* \to P^* \twoheadrightarrow \mathcal{W}_r(P^*)$$

factors uniquely through a map of complexes

$$\mathcal{W}_r(M^*) \otimes_{A^*} \mathcal{W}_r(N^*) \to \mathcal{W}_r(P^*).$$

Proof. This follows from the identities

$$\begin{aligned} V^r m \otimes n &= V^r(m \otimes F^r n), \\ dV^r m \otimes n &= d(V^r m \otimes n) - (-1)^{|m|} V^r b \otimes dn \\ &= dV^r(m \otimes F^r n) - (-1)^{|m|} V^r(m \otimes F^r dn), \end{aligned}$$

and similarly for $m \otimes V^r n$ and $m \otimes dV^r n$. (Recall that one can verify the first identity by applying the injective map F^r to both sides.) □

Remark 4.5.2. If A^* is also strict, then we have

$$\mathcal{W}_r(M^*) \otimes_{A^*} \mathcal{W}_r(N^*) = \mathcal{W}_r(M^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(N^*)$$

in light of Lemma 3.6.2 and the surjectivity of $A^* \to \mathcal{W}_r A^*$, so we may view Lemma 4.5.1 as providing a map

$$\mathcal{W}_r(M^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(N^*) \to \mathcal{W}_r(P^*).$$

The main result of this section is as follows:

Proposition 4.5.3. *Let $X = \operatorname{Spec} R$ be an affine k-scheme, $f : \operatorname{Spec} S \to X$ an étale neighborhood, and $\mathcal{E}$ a unit-root F-crystal on* $\mathrm{Cris}(X/W)$*. Assume $(R, \mathcal{E})$ admits a saturated de Rham-Witt complex $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$. Then:*

1. *The pair $(S, f^*_{\mathrm{cris}}\mathcal{E})$ also admits a saturated de Rham-Witt complex $\mathcal{W}\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$.*
2. *For each $r > 0$, $S \mapsto \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$ defines a quasicoherent sheaf of $W_r\mathcal{O}$-modules on the affine étale site* $\mathrm{Aff\acute{E}t}(X)$.

Remark 4.5.4. The analogue of Proposition 4.5.3 for the classical de Rham-Witt complex with coefficients ([8, Définition 1.1.7]) is much easier to prove: the natural morphisms

$$\begin{gathered} \Omega^*_{W_r(R),\gamma} \to \Omega^*_{W_r(S),\gamma}, \\ \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathrm{dR}(f^*_{\mathrm{cris}}(\mathcal{E})(W_r(S),\gamma)), \text{ and} \\ W_r\Omega^*_R \to W_r\Omega^*_S \end{gathered}$$

are all base-change maps along $W_r(R) \to W_r(S)$ by Lemma 2.2.10 and [12, I, Proposition 1.14], thus so is

$$\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} W_r\Omega^*_R \to \mathrm{dR}(f^*_{\mathrm{cris}}(\mathcal{E})(W_r(S),\gamma)) \otimes_{\Omega^*_{W_r(S),\gamma}} W_r\Omega^*_S.$$

Remark 4.5.5. The idea of the proof of Proposition 4.5.3 will be that we have isomorphisms

$$\begin{aligned} \mathcal{W}\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}} &\simeq \mathcal{W}\,\Omega^*_S \otimes^{\mathrm{str}}_{\mathcal{W}\,\Omega^*_R} \mathcal{W}\,\Omega^*_{R,\mathcal{E}}, \text{ and} \\ \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}} &\simeq W_r(S) \otimes_{W_r(R)} \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \text{ for each } r, \end{aligned}$$

where the latter visibly defines the quasicoherent sheaf

$$\widetilde{\mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}}.$$

In order to pass between the two isomorphisms above, we must show that the strictified tensor product above can be computed at finite levels. In fact, since writing down a de Rham-Witt module requires an understanding of both its underlying strict Dieudonné complex M^* and the corresponding strict Dieudonné tower $(\mathcal{W}_r\,M^*)_r$, we will need this comparison (Proposition 4.5.9) before we can prove either isomorphism above. The proof of this proposition will take up much of this section; it takes place in the setting of a V-adically étale morphism of Dieudonné algebras (cf. [3, §5.3])—of which the prototypical example is $\mathcal{W}\,\Omega^*_R \to \mathcal{W}\,\Omega^*_S$ where $R \to S$ is étale.

4.5.6. Suppose $A^* \to B^*$ is a V-adically étale morphism of strict Dieudonné algebras, suppose M^* is a strict A^*-module, and set $N^*_r = \mathcal{W}_r\,B^* \otimes_{\mathcal{W}_r\,A^*} \mathcal{W}_r\,M^*$. In the next few results, we will give this the structure of a Dieudonné tower, and then show that its limit computes the strictified tensor product $B^* \otimes^{\mathrm{str}}_{A^*} M^*$. Note first that we have natural isomorphisms of graded modules

$$\begin{aligned} N^*_r = \mathcal{W}_r\,B^* \otimes_{\mathcal{W}_r\,A^*} \mathcal{W}_r\,M^* &\simeq \mathcal{W}_r\,B^0 \otimes_{\mathcal{W}_r\,A^0} \mathcal{W}_r\,M^* \\ &\simeq W_r(B^0/VB^0) \otimes_{W_r(A^0/VA^0)} \mathcal{W}_r\,M^* \\ &\simeq W(B^0/VB^0) \otimes_{W(A^0/VA^0)} \mathcal{W}_r\,M^*, \end{aligned}$$

where the first step is the statement that $A^* \to B^*$ is V-adically étale, and the second step follows by [3, Proposition 3.6.2].

Construction 4.5.7. Suppose $A^* \to B^*$ is a V-adically étale morphism of strict Dieudonné algebras, and suppose M^* is a strict A^*-module. Then we can endow the family of complexes $(\mathcal{W}_r\,B^* \otimes_{\mathcal{W}_r\,A^*} \mathcal{W}_r\,M^*)_r$ with maps R, F, and V making it into a strict Dieudonné tower. For convenience, we set $N^*_r = \mathcal{W}_r\,B^* \otimes_{\mathcal{W}_r\,A^*} \mathcal{W}_r\,M^*$. First observe that we have natural isomorphisms of graded modules

$$\begin{aligned} N^*_r = \mathcal{W}_r\,B^* \otimes_{\mathcal{W}_r\,A^*} \mathcal{W}_r\,M^* &\simeq \mathcal{W}_r\,B^0 \otimes_{\mathcal{W}_r\,A^0} \mathcal{W}_r\,M^* \\ &\simeq W_r(B^0/VB^0) \otimes_{W_r(A^0/VA^0)} \mathcal{W}_r\,M^* \\ &\simeq W(B^0/VB^0) \otimes_{W(A^0/VA^0)} \mathcal{W}_r\,M^*, \end{aligned}$$

where the first step is the statement that $A^* \to B^*$ is V-adically étale, and the second step follows by [3, Proposition 3.6.2]. In particular, [3, Corollary 5.4.9] implies that the maps

$$\begin{aligned} &R : \mathcal{W}_{r+1}\,M^* \to \mathcal{W}_r\,M^* \to N^*_r, \\ &F : \mathcal{W}_{r+1}\,M^* \to \mathcal{W}_r\,M^* \to N^*_r, \text{ and} \\ &V : \mathcal{W}_r\,M^* \to \mathcal{W}_{r+1}\,M^* \to N^*_{r+1} \end{aligned}$$

each extend uniquely to maps

$$
\begin{aligned}
R &: N^*_{r+1} \to N^*_r, \\
F &: N^*_{r+1} \to N^*_r, \\
V &: N^*_r \to N^*_{r+1}
\end{aligned}
$$

respectively. Note that the uniqueness of the extensions implies that we have identities

$$
\begin{aligned}
R(b \otimes m) &= Rb \otimes Rm, \\
F(b \otimes m) &= Fb \otimes Fm, \text{ and} \\
V(Fb \otimes m) &= b \otimes Vm
\end{aligned}
$$

for $b \in \mathcal{W}_r\, B^0$ and $m \in \mathcal{W}_r\, M^*$, since these are true for b in the image of $\mathcal{W}_r\, A^0 \to \mathcal{W}_r\, B^0$.

Proposition 4.5.8. *The maps R, F, and V of Construction 4.5.7 endow the system $(N^*_r)_r$ with the structure of a strict Dieudonné tower.*

Proof. Properties (1), (3), (4), and (5) of [3, Definition 2.6.1] follow from the corresponding properties of $(M^*_r)_r$ and the uniqueness of the extensions of [3, Corollary 5.4.9]. Properties (2) and (7) follow from base-changing the corresponding properties of $(M^*_r)_r$ along the étale (thus flat) map $\mathcal{W}_r\, A^0 \to \mathcal{W}_r\, B^0$. Properties (6) and (8) follow from a similar argument if we are careful about semilinearity. Namely, twisting the respective exact sequences for $(M^*_r)_r$ by various powers of Frobenius gives exact sequences of $\mathcal{W}_{r+1}\, A^0$-modules

$$M^*_{r+1} \xrightarrow{F} F^*(M^*_r) \xrightarrow{d} F^*(M^{*+1}_r)/p$$

and

$$(F^{r+1})^*(M^*_1 \oplus M^{*-1}_1) \xrightarrow{(V^r, dV^r)} F^*(M^*_{r+1}) \xrightarrow{R} F^*(M^*_r),$$

where the linearity of d and dV^r follows from the calculations

$$
\begin{aligned}
d(F(a) \cdot m) &= dF(a) \cdot m + F(a) \cdot dm \\
&= pFd \cdot m + F(a) \cdot dm \\
&\equiv F(a) \cdot dm \pmod{p},
\end{aligned}
$$

and

$$
\begin{aligned}
dV^r(F^{r+1}(a) \cdot m) &= d(F(a) \cdot V^r m) \\
&= dF(a) \cdot V^r m + F(a) \cdot dV^r m \\
&= pFd(a) \cdot V^r m + F(a) \cdot dV^r m \\
&= F(a) \cdot dV^r m \in M^*_{r+1},
\end{aligned}
$$

each for $a \in \mathcal{W}_{r+1}\, A^0$. The corresponding properties of $(N^*_r)_r$ follow by base-changing along the étale map $\mathcal{W}_r\, A^0 \to \mathcal{W}_r\, B^0$; this operation commutes with Frobenius pullbacks by [3, Remark 5.4.2]. $\square$

We now claim that this strict Dieudonné tower corresponds, under the equivalence of categories of [3, Corollary 2.9.4], to the strictified tensor product $B^* \otimes^{\mathrm{str}}_{A^*} M^*$.

Proposition 4.5.9. *Suppose $A^* \to B^*$ is a V-adically étale morphism of strict Dieudonné algebras, and suppose M^* is a strict A^*-module. Then the natural B^*-linear map*

$$\rho' : B^* \otimes_{A^*} M^* \to \varprojlim (\mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*))$$

exhibits the latter as a strictification of the former. In particular, this induces isomorphisms

$$\mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*) \simeq \mathcal{W}_r(B^* \otimes^{\mathrm{str}}_{A^*} M^*)$$

of $\mathcal{W}_r\, B^*$-modules for each r.

Proof. By the universal property of strictification, the given map ρ' factors uniquely as

$$B^* \otimes_{A^*} M^* \xrightarrow{\rho} B^* \otimes^{\mathrm{str}}_{A^*} M^* \xrightarrow{g} \varprojlim (\mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*)).$$

We must show that g is an isomorphism of strict Dieudonné complexes. We will construct its inverse by working in the equivalent category of strict Dieudonné towers.

For each $r > 0$, the composite map

$$B^* \otimes_{A^*} M^* \xrightarrow{\rho} B^* \otimes^{\mathrm{str}}_{A^*} M^* \twoheadrightarrow \mathcal{W}_r(B^* \otimes^{\mathrm{str}}_{A^*} M^*)$$

factors as

$$B^* \otimes_{A^*} M^* \twoheadrightarrow \mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*) \xrightarrow{f_r} \mathcal{W}_r(B^* \otimes^{\mathrm{str}}_{A^*} M^*),$$

by Lemma 4.5.1. It follows from the uniqueness part of [3, Corollary 5.4.9] that the maps f_r are compatible with R, F, and V, so they define a map of strict Dieudonné towers

$$(f_r)_r : (\mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*))_r \to (\mathcal{W}_r(B^* \otimes^{\mathrm{str}}_{A^*} M^*))_r.$$

Under the equivalence of categories between strict Dieudonné complexes and strict Dieudonné towers, the maps g and f_r respectively correspond to maps $g_r := \mathcal{W}_r(g)$ and $f := \lim_r f_r$ as in the following diagram:

$$\begin{tikzcd}
& B^* \otimes_{A^*} M^* \arrow[dl, "\rho'"'] \arrow[dr, "\rho"] & \\
\lim_{\leftarrow n} \left(\mathcal{W}_n(B^*) \otimes_{\mathcal{W}_n(A^*)} \mathcal{W}_n(M^*)\right) \arrow[rr, bend left, "f"] \arrow[d, two heads, "\pi'_r"'] & & B^* \otimes^{\mathrm{str}}_{A^*} M^* \arrow[ll, bend left, "g"] \arrow[d, two heads, "\pi_r"] \\
\mathcal{W}_r(B^*) \otimes_{\mathcal{W}_r(A^*)} \mathcal{W}_r(M^*) \arrow[rr, bend left, "f_r"] & & \mathcal{W}_r(B^* \otimes^{\mathrm{str}}_{A^*} M^*) \arrow[ll, bend left, "g_r"]
\end{tikzcd}$$

In the diagram above, we have

$$\rho' = g \circ \rho \text{ and}$$
$$\pi_r \circ \rho = f_r \circ \pi_r' \circ \rho'$$

by construction. It follows by passing to quotients (resp. limits) that

$$\pi_r' \circ \rho' = g_r \circ \pi_r \circ \rho \text{ and}$$
$$\rho = f \circ \rho',$$

so in particular we have

$$\rho = f \circ g \circ \rho \text{ and}$$
$$\pi_r' \circ \rho' = g_r \circ f_r \circ \pi_r' \circ \rho'.$$

By the universal properties of the maps ρ and $\pi_r' \circ \rho'$, this implies that $f \circ g$ and $g_r \circ f_r$ are identity maps on the respective objects. Since $(f_r)_r$ and $(g_r)_r$ correspond to f and g under the equivalence of categories, this proves that f and g are inverses. □

Proof of Proposition 4.5.3. Set $M^* = \mathcal{W}\,\Omega^*_S \otimes^{\mathrm{str}}_{\mathcal{W}\,\Omega^*_R} \mathcal{W}\,\Omega^*_{R,\mathcal{E}}$. To prove part (1), we will endow M^* with maps $\iota^*_{r,M} : \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma)) \to \mathcal{W}_r\,M^*$ making it a saturated de Rham-Witt complex of $(S, f^*_{\mathrm{cris}}\mathcal{E})$. Note that by Proposition 4.5.9 above and [3, Corollary 5.3.5], we have isomorphisms

$$\begin{aligned}\mathcal{W}_r(M^*) &\simeq \mathcal{W}_r\,\Omega^*_S \otimes_{\mathcal{W}_r\,\Omega^*_R} \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \text{ as dg-modules, and}\\ &\simeq W_r(S) \otimes_{W_r(R)} \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \text{ as graded modules.}\end{aligned}$$

So once we prove that M^* is a saturated de Rham-Witt complex associated to $(S, f^*_{\mathrm{cris}}\mathcal{E})$, it will follow that the functor $S \mapsto \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$ on $\mathrm{Aff\acute{E}t}(X)$ is simply the quasicoherent sheaf $\widetilde{\mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}}$ of $W_r\mathcal{O}$-modules, proving part (2).

We construct the maps $\iota^*_{r,M}$ as the composition

$$\begin{aligned}\mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma)) &\simeq \Omega^*_{W_r(S),\gamma} \otimes_{\Omega^*_{W_r(R),\gamma}} \mathrm{dR}(\mathcal{E}(W_r(R),\gamma))\\ &\simeq \mathcal{W}_r\,\Omega^*_S \otimes_{\mathcal{W}_r\,\Omega^*_R} \mathrm{dR}(\mathcal{E}(W_r(R),\gamma))\\ &\to \mathcal{W}_r\,\Omega^*_S \otimes_{\mathcal{W}_r\,\Omega^*_R} \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} = \mathcal{W}_r\,M^*,\end{aligned}$$

using the isomorphisms of Corollary 2.7.12 and Remark 3.6.8 and the map

$$\iota^*_{r,R} : \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}$$

which comes as part of the data of the universal de Rham-Witt module $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$. Each map in this composition is a map of dg-$\Omega^*_{W_r(S),\gamma}$-modules, and is compatible with reduction maps and with Frobenius in degree 0. Therefore the same is true of $\iota^*_{r,M}$, and thus the maps $\iota^*_{r,M}$ make M^* a de Rham-Witt module over $(S, f^*_{\mathrm{cris}}\mathcal{E})$.

Now let $N^* = (N^*, (\iota^*_{r,N})_r)$ be an arbitrary de Rham-Witt module over $(S, f^*_{\mathrm{cris}}\mathcal{E})$. We claim that there is a unique map $M^* \to N^*$ in $\mathrm{dRWM}_{S,f^*_{\mathrm{cris}}\mathcal{E}}$. To prove this, first recall the de Rham-Witt module $f_*N^* = (N^*, (\iota^*_{r,f_*N})_r)$ of Construction 4.3.6. By the universal property of $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$, there is a unique map $g : \mathcal{W}\,\Omega^*_{R,\mathcal{E}} \to f_*N^*$ in $\mathrm{dRWM}_{R,\mathcal{E}}$. Since N^* is a strict $\mathcal{W}\,\Omega^*_S$-module, g factors uniquely as

$$\mathcal{W}\,\Omega^*_{R,\mathcal{E}} \xrightarrow{a} \mathcal{W}\,\Omega^*_S \otimes^{\mathrm{str}}_{\mathcal{W}\,\Omega^*_R} \mathcal{W}\,\Omega^*_{R,\mathcal{E}} = M^* \xrightarrow{h} N^*,$$

where $a = 1 \otimes \mathrm{id}$. So it only remains to prove that $h : M^* \to N^*$ is in fact a map in $\mathrm{dRWM}_{S,f^*_{\mathrm{cris}}\mathcal{E}}$; that is, that $\iota^*_{r,N}$ coincides with the composition

$$\mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma)) \xrightarrow{\iota^*_{r,M}} \mathcal{W}_r\, M^* \xrightarrow{\mathcal{W}_r(h)} \mathcal{W}_r\, N^*.$$

To prove this, we chase the diagram:

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \arrow[r] \arrow[d, "\iota^*_{r,R}"] \arrow[drr, bend left, "\iota^*_{r,f_*N}"] & \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma)) \arrow[d, "\iota^*_{r,M}"'] \arrow[dr, "\iota^*_{r,N}"] & \\
\mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \arrow[r, "\mathcal{W}_r(a)"'] \arrow[rr, bend right, "\mathcal{W}_r(g)"'] & \mathcal{W}_r\, M^* \arrow[r, "\mathcal{W}_r(h)"'] & \mathcal{W}_r\, N^*
\end{tikzcd}$$

All regions except for the triangle at right commute by construction. It follows that the triangle commutes on the image of $\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma))$. But this image generates $\mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S),\gamma))$ as a graded $\Omega^*_{W_r(S),\gamma}$-module, and the maps in the triangle are all maps of dg-$\Omega^*_{W_r(S),\gamma}$-modules, so the triangle commutes. □

Remark 4.5.10. As a result of Proposition 4.5.3, we can define $\mathcal{W}\,\Omega^*_{X,\mathcal{E}}$ and $\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$ as sheaves on the étale site of X even if X is not affine; namely, they are the sheaves defined on affines by

$$\begin{aligned}
\operatorname{Spec} R &\mapsto \mathcal{W}\,\Omega^*_{R,\mathcal{E}|_{\operatorname{Spec} R}} \text{ and}\\
\operatorname{Spec} R &\mapsto \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}|_{\operatorname{Spec} R}},
\end{aligned}$$

respectively. By the proposition, these both exist provided that X has an open cover by affines $\operatorname{Spec} R_i$ such that $(R_i, \mathcal{E}|_{\operatorname{Spec} R_i})$ admits a saturated de Rham-Witt complex; if so, then $\mathcal{W}_r\,\Omega^i_{X,\mathcal{E}}$ is a quasicoherent sheaf of $W_r\mathcal{O}_X$-modules for each i.

Remark 4.5.11. As discussed in §1.6, we expect that it is also possible to define $\mathcal{W}\Omega^*_{X,\mathcal{E}}$ directly by a universal property analogous to Definition 4.1.7. This will require the theory of (strict) Dieudonné complexes and Dieudonné towers valued in sheaves, which will be developed in [9].

4.6 Compatibility with colimits

4.6.1. In this section, we will show that saturated de Rham-Witt complexes with coefficients behave well with respect to filtered colimits of rings, in a certain sense. To make this precise, we introduce the following setup:

Situation 4.6.2. Fix a k-algebra R_0 and a unit-root F-crystal $\mathcal{E}_0$ on $X_0 = \operatorname{Spec} R_0$. Let $(X_i = \operatorname{Spec} R_i)_{i\in\mathcal{I}}$ be a diagram of affine R_0-schemes, indexed by a cofiltered category $\mathcal{I}$. Let X denote the limit $\lim_i X_i$, which agrees with the spectrum of $R = \varinjlim_i R_i$. Then we have morphisms of affine schemes

$$\begin{tikzcd} X \arrow[rr, "h_i"] \arrow[dr, "g"'] & & X_i \arrow[dl, "g_i"] \\ & X_0 & \end{tikzcd}$$

for each $i \in \mathcal{I}$, and

$$X_i \xrightarrow{f_\alpha} X_j$$

for each $\alpha : i \to j$ in $\mathcal{I}$. Pulling $\mathcal{E}_0$ back along the various maps yields unit-root F-crystals

$$\mathcal{E}_i := (g_i)^*_{\mathrm{cris}}\mathcal{E}_0$$

on X_i for each $i \in \mathcal{I}$, and similarly

$$\mathcal{E} := g^*_{\mathrm{cris}}\mathcal{E}_0$$

on X.

Construction 4.6.3. In Situation 4.6.2, suppose that $\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i}$ exists for each i. We wish to assemble these into a diagram indexed by $\mathcal{I}^{\mathrm{op}}$ and give their strictified colimit the structure of a de Rham-Witt module over $(R, \mathcal{E})$. The first step is to write down the transition maps $\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i} \to \mathcal{W}\Omega^*_{R_j,\mathcal{E}_j}$ for each $\alpha : j \to i$ in $\mathcal{I}$. To this end, note that the universal property of $\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i}$ induces a map

$$\pi_\alpha : \mathcal{W}\Omega^*_{R_i,\mathcal{E}_i} \to f_{\alpha *}\,\mathcal{W}\Omega^*_{R_j,\mathcal{E}_j}$$

in $\mathrm{dRWM}_{R_i,\mathcal{E}_i}$, and recall (cf. Construction 4.3.6) that the target has the same underlying Dieudonné complex as $\mathcal{W}\Omega^*_{R_j,\mathcal{E}_j}$. So we can view π_α as a map $\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i} \to \mathcal{W}\Omega^*_{R_j,\mathcal{E}_j}$ of strict Dieudonné complexes. The uniqueness part of the universal property of $\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i}$ ensures that these maps compose correctly given morphisms $\ell \xrightarrow{\beta} j \xrightarrow{\alpha} i$ in $\mathcal{I}$, so that $(\mathcal{W}\Omega^*_{R_i,\mathcal{E}_i}, \pi_\alpha)$ forms an $\mathcal{I}^{\mathrm{op}}$-shaped diagram in $\mathbf{DC}_{\mathrm{str}}$.

Let M^* denote the colimit of this system; recall from Lemma 3.7.2 that this is the strictification of the colimit in $\mathbf{DC}$. Our next task is to give M^* the structure of a de Rham-Witt module over $(R, \mathcal{E})$. Since each $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}$ is a strict $\mathcal{W}\,\Omega^*_{R_i}$-module (compatibly with transition maps), their colimit in $\mathbf{DC}$ is a module over the Dieudonné algebra colimit

$$\varinjlim_{i,\mathbf{DA}} \mathcal{W}\,\Omega^*_{R_i}.$$

Lemma 3.5.8 then makes M^* a module over the strictification

$$\mathcal{W}\,\mathrm{Sat}(\varinjlim_{i,\mathbf{DA}} \mathcal{W}\,\Omega^*_{R_i}) = \varinjlim_{i,\mathbf{DA}_{\mathrm{str}}} \mathcal{W}\,\Omega^*_{R_i}.$$

But this latter colimit equals $\mathcal{W}\,\Omega^*_R$, since $\mathcal{W}\,\Omega^*_{-}$ is a left adjoint ([3, Definition 4.1.1]) and thus commutes with arbitrary colimits. So M^* is a module over $\mathcal{W}\,\Omega^*_R$ in $\mathbf{DC}_{\mathrm{str}}$.

To finish making M^* a de Rham-Witt module, we must endow it with a map

$$\iota_r^* : \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \to \mathcal{W}_r\, M^*$$

for each $r > 0$, satisfying various compatibilities. To construct ι_r^*, we simply take the colimit of the corresponding maps

$$\iota_{r,i}^* : \mathrm{dR}(\mathcal{E}_i(W_r(R_i), \gamma)) \to \mathcal{W}_r\, \Omega^*_{R_i,\mathcal{E}_i}$$

as i varies. We identify the target of this map, $\varinjlim_i \mathcal{W}_r\, \Omega^*_{R_i,\mathcal{E}_i}$, with $\mathcal{W}_r\, M^*$ using Proposition 3.7.7; and we identify its source with $\mathrm{dR}(\mathcal{E}(W_r(R), \gamma))$ by the calculation

$$\begin{aligned}
\varinjlim_i \mathrm{dR}(\mathcal{E}_i(W_r(R_i), \gamma)) &\simeq \varinjlim_i \left(\mathrm{dR}(\mathcal{E}_0(W_r(R_0), \gamma)) \otimes_{\Omega^*_{W_r(R_0),\gamma}} \Omega^*_{W_r(R_i),\gamma}\right) \\
&\simeq \mathrm{dR}(\mathcal{E}_0(W_r(R_0), \gamma)) \otimes_{\Omega^*_{W_r(R_0),\gamma}} \varinjlim_i \left(\Omega^*_{W_r(R_i),\gamma}\right) \\
&\simeq \mathrm{dR}(\mathcal{E}_0(W_r(R_0), \gamma)) \otimes_{\Omega^*_{W_r(R_0),\gamma}} \Omega^*_{W_r(R),\gamma} \\
&\simeq \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)).
\end{aligned}$$

Since each $\iota_{r,i}^*$ is a morphism of dg-$\Omega^*_{W_r(R_i),\gamma}$-modules, their colimit is a morphism of dg-$\Omega^*_{W_r(R),\gamma}$-modules. Finally, the compatibilities of the ι_r^* with quotient and Frobenius maps follow from the corresponding compatibilities of the $\iota_{r,i}^*$.

Proposition 4.6.4. *In the situation above, if $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}$ exists for each i, then the object*

$$M^* = \varinjlim_{i,\mathbf{DC}_{\mathrm{str}}} (\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}) \in \mathrm{dRWM}_{R,\mathcal{E}}$$

of Construction 4.6.3 is a saturated de Rham-Witt complex associated to $(R, \mathcal{E})$. In particular, we have

$$\mathcal{W}_r\, \Omega^*_{R,\mathcal{E}} = \varinjlim_i \mathcal{W}_r\, \Omega^*_{R_i,\mathcal{E}_i}$$

for each r.

Proof. Let N^* be any test object in $\mathrm{dRWM}_{R,\mathcal{E}}$; we must show there exists a unique map $M^* \to N^*$ in $\mathrm{dRWM}_{R,\mathcal{E}}$. We first show existence. For each i, pushing forward along $h_i : X \to X_i$ gives an object $h_{i*}N^*$ in $\mathrm{dRWM}_{R_i,\mathcal{E}_i}$. The universal property of $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}$ then provides a unique map

$$\psi_i : \mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i} \to h_{i*}N^*$$

in $\mathrm{dRWM}_{R_i,\mathcal{E}_i}$. Forgetting the de Rham-Witt module structure for the moment, these are in particular maps of strict $\mathcal{W}\,\Omega^*_{R_i}$-modules $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i} \to N^*$, compatible as i varies. So passing to the strictified colimit gives a map

$$\psi : M^* \to N^*$$

of strict $\mathcal{W}\,\Omega^*_R$-modules. To prove that this is a map of de Rham-Witt modules over $(R, \mathcal{E})$, consider the following diagram:

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}_i(W_r(R_i),\gamma)) \arrow[r] \arrow[d, "\iota^*_{r,i}"] \arrow[drr, bend left, "\iota^*_{r,h_{i*}N}"] & \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \arrow[d, "\iota^*_{r,M}"] \arrow[dr, "\iota^*_{r,N}"] & \\
\mathcal{W}_r\,\Omega^*_{R_i,\mathcal{E}_i} \arrow[r] \arrow[rr, bend right, "\mathcal{W}_r(\psi_i)"'] & \mathcal{W}_r\,M^* \arrow[r, "\mathcal{W}_r(\psi)"] & \mathcal{W}_r\,N^*
\end{tikzcd}$$

The various ι^* maps here are the ones giving $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}$, M^*, N^*, and $h_{i*}N^*$ the structure of de Rham-Witt modules. The square on the left commutes by construction of $\iota^*_{r,M}$, and the top and bottom regions commute by the constructions of $\iota^*_{r,h_{i*}N}$ and ψ respectively. The commutativity of the outer region is precisely the statement that $\psi_i : \mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i} \to h_{i*}N^*$ is a morphism in $\mathrm{dRWM}_{R_i,\mathcal{E}_i}$. If this is the case for all i, then since

$$\mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) = \varinjlim_i \mathrm{dR}(\mathcal{E}_i(W_r(R_i),\gamma)),$$

it follows that the small triangular region commutes as well. But this is the statement that $\psi : M^* \to N^*$ is a morphism in $\mathrm{dRWM}_{R,\mathcal{E}}$.

To complete the proof, we must show that ψ is the unique map $M^* \to N^*$ in $\mathrm{dRWM}_{R,\mathcal{E}}$. But given any such ψ, we can reconstruct the diagram above, constructing ψ_i from ψ instead of the other way around. The commutativity of the small triangle implies that of the outer region, so the ψ_i are morphisms of de Rham-Witt modules. The universal property of $\mathcal{W}\,\Omega^*_{R_i,\mathcal{E}_i}$ forces these to agree with our earlier maps ψ_i, which in turn forces ψ to agree with our earlier ψ. $\square$

Chapter 5

Construction from a lift with Frobenius

5.0.1. So far, we have only been able to construct our saturated de Rham-Witt complexes $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ in the case of the trivial crystal $\mathcal{E} = \mathcal{O}$. Our goal in this chapter is to construct $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ for a general unit-root F-crystal $\mathcal{E}$, provided that R comes equipped with a lift with Frobenius.

5.0.2. Throughout this chapter, we fix R, A, $(A_r)_r$, and ϕ as in the Frobenius-lifted situation 1.7.6. As usual, let $\mathcal{E}$ be a unit-root F-crystal on $\operatorname{Spec} R$.

5.0.3. The main player in this chapter will be a variant of the category of de Rham-Witt modules, called the category $\mathrm{dRWLM}_{A,\mathcal{E}}$ of *de Rham-Witt lift modules* over $(A, \mathcal{E})$ (Definition 5.1.3), which incorporates data coming from the lifts A_r of R rather than from the truncated Witt vectors $W_r(R)$. The usefulness of this category comes from the fact that while $\mathrm{dRWM}_{R,\mathcal{E}}$ has no "obvious" initial object, $\mathrm{dRWLM}_{A,\mathcal{E}}$ does (Corollary 5.1.10), and the two categories turn out to be equivalent (Proposition 5.2.3). Thus, although the category $\mathrm{dRWM}_{R,\mathcal{E}}$ is more canonical (as it does not depend on a lift), $\mathrm{dRWLM}_{A,\mathcal{E}}$ is a useful setting in which to perform constructions. The resulting construction can be viewed as a generalization of the formula $\mathcal{W}\,\Omega^*_R = \mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_A)$ of Construction 1.2.6.

5.1 de Rham-Witt lift modules

5.1.1. Recall that for any $r > 0$, the PD-structure $[\;]$ on $(p) \subset A_r$ makes $\operatorname{Spec} A_r$ an object of $\mathrm{Cris}(\operatorname{Spec} R/W_r)$. Then Construction 2.7.16 defines a PD-de Rham complex

$$\mathrm{dR}(\mathcal{E}(A_r, [\;])) = (\mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,[\;]}, \nabla),$$

and passing to the limit gives a completed PD-de Rham complex

$$\widehat{\mathrm{dR}}(\mathcal{E}(A, [\;])) := \lim_r \mathrm{dR}(\mathcal{E}(A_r, [\;])).$$

In fact, we have $\Omega^*_{A_r,[\;]} = \Omega^*_{A_r}$ by Lemma 2.3.9 under our hypotheses. Accordingly, we will usually suppress the PD-structure, writing the finite-level de Rham complex as $\mathrm{dR}(\mathcal{E}(A_r))$ and the completed de Rham complex as $\widehat{\mathrm{dR}}(\mathcal{E}(A))$.

We begin with the following observation. If M^* is a strict $\mathcal{W}\,\Omega^*_R$-module, then Corollary 3.6.7 gives $\mathcal{W}_r\,M^*$ the structure of a dg-$\Omega^*_{A_r}$-module, so in particular it is also a graded A_r-module. We have the following easy lemma:

Lemma 5.1.2. *Suppose we are given a $\mathcal{W}\,\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with an A_r-linear map $\lambda : \mathcal{E}(A_r) \to \mathcal{W}_r\,M^0$. Then:*

(a) The map λ extends uniquely to a map of graded left $\Omega^*_{A_r}$-modules (not necessarily compatible with differentials)

$$\lambda^* : \mathrm{dR}(\mathcal{E}(A_r)) = \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r} \to \mathcal{W}_r\,M^*.$$

(b) If we are instead given a compatible family of maps $\lambda_r : \mathcal{E}(A_r) \to \mathcal{W}_r\,M^0$ for all r, then the resulting maps λ^*_r are also compatible.

Proof. Part (a) is the tensor-hom adjunction for the homomorphism $A_r \to \Omega^*_{A_r}$ of graded rings; explicitly, the map is

$$\lambda^* : \mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r} \xrightarrow{\lambda\otimes\mathrm{id}} \mathcal{W}_r\,M^0 \otimes_{A_r} \Omega^*_{A_r} \xrightarrow{m} \mathcal{W}_r\,M^*,$$

where m is the multiplication map. Part (b) is then clear from this formula. □

Definition 5.1.3. By a *de Rham-Witt lift module for* $(A, \mathcal{E})$ we will mean a collection of the following data: a left $\mathcal{W}\,\Omega^*_R$-module in $\mathbf{DC}_{\mathrm{str}}$, equipped with A_r-linear maps

$$\lambda_r : \mathcal{E}(A_r) \to \mathcal{W}_r\,M^0$$

for each $r > 0$, such that:

1. For each $r > 0$, the diagram

$$\begin{tikzcd} \mathcal{E}(A_r) \arrow[r, "\lambda_r"] \arrow[d, two heads] & \mathcal{W}_r\,M^0 \arrow[d, two heads] \\ \mathcal{E}(A_{r-1}) \arrow[r, "\lambda_{r-1}"] & \mathcal{W}_{r-1}\,M^0 \end{tikzcd}$$

   commutes, where the two vertical maps are the quotient maps.

2. For each $r > 0$, the diagram

$$\begin{tikzcd} \mathcal{E}(A_r) \arrow[r, "\lambda_r"] \arrow[d, "F"'] & \mathcal{W}_r\,M^0 \arrow[d, "F"] \\ \mathcal{E}(A_{r-1}) \arrow[r, "\lambda_{r-1}"] & \mathcal{W}_{r-1}\,M^0 \end{tikzcd}$$

   commutes, where the left vertical map is defined in Example 2.5.5.

3. The maps λ_r^* of Lemma 5.1.2 are maps of complexes

$$(\mathcal{E}(A_r)\otimes_{A_r}\Omega^*_{A_r},\nabla)\to(\mathcal{W}_r\,M^*,d).$$

A morphism of de Rham-Witt lift modules over $(A,\mathcal{E})$ is a morphism $f : M^*\to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules such that $\lambda_{r,N}=\mathcal{W}_r(f^0)\circ\lambda_{r,M}$ for each r. We call the resulting category $\mathrm{dRWLM}_{A,\mathcal{E}}$.

Remark 5.1.4. As in Remark 4.1.5, we could equivalently demand the data of the extension λ_r^*, rather than only $\lambda_r=\lambda_r^0$. The translation works exactly as before (with the result stated as Alternative Definition 5.1.6 below), except for the following. Recall that in Definition 4.1.3, it was not possible to demand that the ι_r^* be compatible with Frobenius in all degrees, as their source $\mathrm{dR}(\mathcal{E}(W_r(R),\gamma))$ carries a divided Frobenius operator only in degree 0 (i.e. the Frobenius map $\mathcal{E}(W_r(R))\to\mathcal{E}(W_{r-1}(R))$ of Example 2.5.5 divided by $p^0=1$). Here, the source of λ_r^* carries divided Frobenius endomorphisms in all degrees (Construction 2.7.17), so we could demand that λ_r^* be compatible with them in all degrees. But in fact this comes for free, as the following lemma shows.

Lemma 5.1.5. *Let $(M^*,(\lambda_r)_r)$ be a de Rham-Witt lift module over $(A,\mathcal{E})$. Then for each r, the diagram*

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}(A_r)) \arrow[r, "\lambda_r^*"] \arrow[d, "F"'] & \mathcal{W}_r\,M^* \arrow[d, "F"] \\
\mathrm{dR}(\mathcal{E}(A_{r-1})) \arrow[r, "\lambda_{r-1}^*"] & \mathcal{W}_{r-1}\,M^*,
\end{tikzcd}$$

commutes, where the left vertical map is the divided Frobenius composed with the quotient map $\mathrm{dR}(\mathcal{E}(A_r))\to\mathrm{dR}(\mathcal{E}(A_{r-1}))$.

Proof. Consider a simple tensor $e\otimes\omega\in\mathcal{E}(A_r)\otimes_{A_r}\Omega^i_{A_r}=\mathrm{dR}(\mathcal{E}(A_r))$. We calculate:

$$\begin{aligned}
\lambda_{r-1}^*(F(e\otimes\omega)) &= \lambda_{r-1}^*(\phi_{\mathcal{E}}(e)\otimes F\omega)\\
&= F\omega\cdot\lambda_{r-1}(\phi_{\mathcal{E}}(e))\\
&= F\omega\cdot F(\lambda_r(e))\\
&= F(\omega\cdot\lambda_r(e))\\
&= F(\lambda_r^*(e\otimes\omega)),
\end{aligned}$$

where the third equality comes from the given compatibility in degree 0, and the fourth follows (after lifting ω to an element of $\widehat{\Omega}^*_A$ and $\lambda_r^*(e\otimes 1)$ to an element of M^*) from the compatibility of the multiplication map $\widehat{\Omega}^*_A\otimes M^*\to M^*$ with Frobenius. $\square$

Alternative Definition 5.1.6. A *de Rham-Witt lift module over* $(A,\mathcal{E})$ is a collection of the following data: a left $\mathcal{W}\,\Omega^*_R$-module M^* in $\mathbf{DC}_{\mathrm{str}}$, equipped with maps

$$\lambda_r^* : \mathrm{dR}(\mathcal{E}(A_r))\to\mathcal{W}_r\,M^*$$

of dg-$\Omega^*_{A_r}$-modules for each r, such that the following diagrams commute for all r:

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}(A_r)) \arrow[r, "\lambda_r^*"] \arrow[d, two heads] & \mathcal{W}_r\, M^* \arrow[d, two heads] \\
\mathrm{dR}(\mathcal{E}(A_{r-1})) \arrow[r, "\lambda_{r-1}^*"] & \mathcal{W}_{r-1}\, M^*
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}(A_r)) \arrow[r, "\lambda_r^*"] \arrow[d, "F"'] & \mathcal{W}_r\, M^* \arrow[d, "F"] \\
\mathrm{dR}(\mathcal{E}(A_{r-1})) \arrow[r, "\lambda_{r-1}^*"] & \mathcal{W}_{r-1}\, M^*
\end{tikzcd}$$

A morphism of de Rham-Witt lift modules over $(A, \mathcal{E})$ is a morphism $f : M^* \to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules such that $\lambda^*_{r,N} = \mathcal{W}_r(f) \circ \lambda^*_{r,M}$ for each r.

Remark 5.1.7. Unlike the situation in Remark 4.1.8, we will be able to directly construct an initial object in $\mathrm{dRWLM}_{A,\mathcal{E}}$. Namely, recall from Remark 2.7.20 that $\widehat{\mathrm{dR}}(\mathcal{E}(A))$ is a Dieudonné complex, and moreover a $\widehat{\Omega}^*_A$-module in **DC**. By Lemma 3.5.8, it follows that $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))$ has the structure of a strict module over $\mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_A) = \mathcal{W}\,\Omega^*_R$. We claim that this can be made into a de Rham-Witt lift module, and that it is initial. In fact we will prove more (Proposition 5.1.9): the category of de Rham-Witt lift modules is *equivalent* to the category of strict $\mathcal{W}\,\Omega^*_R$-modules equipped with a $\mathcal{W}\,\Omega^*_R$-module map from $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))$; that is, the coslice category

$$\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))/\,\mathcal{W}\,\Omega^*_R\text{-}\mathbf{mod}_{\mathrm{str}}.$$

This latter category has the obvious initial object $(\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))), \mathrm{id})$.

Lemma 5.1.8. *Suppose M^* is a strict $\mathcal{W}\,\Omega^*_R$-module equipped with a $\mathcal{W}\,\Omega^*_R$-module map*

$$f : \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))) \to M^*.$$

1. *For each r, the composition*

$$\widehat{\mathrm{dR}}(\mathcal{E}(A)) \xrightarrow{\rho} \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))) \xrightarrow{f} M^* \twoheadrightarrow \mathcal{W}_r\, M^* \tag{5.1.8.1}$$

   factors uniquely as

$$\widehat{\mathrm{dR}}(\mathcal{E}(A)) \to \mathrm{dR}(\mathcal{E}(A_r)) \xrightarrow{\lambda_r^*} \mathcal{W}_r\, M^*.$$

2. *The maps λ^*_r give M^* the structure of a de Rham-Witt lift module for $(A, \mathcal{E})$.*

3. *The construction $(M^*, f) \mapsto (M^*, (\lambda^*_r)_r)$ defines a functor Θ from the coslice category $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))/\,\mathcal{W}\,\Omega^*_R\text{-}\mathbf{mod}_{\mathrm{str}}$ to $\mathrm{dRWLM}_{\mathcal{E},A}$.*

Proof. To prove (1), simply recall from Proposition 2.9.4 that the natural map $\widehat{\mathrm{dR}}(\mathcal{E}(A)) \to \mathrm{dR}(\mathcal{E}(A_r))$ is the quotient by p^r, and the target $\mathcal{W}_r\, M^*$ is killed by p^r.

For (2), to show that the λ_r^* satisfy the conditions of Alternative Definition 5.1.6, note that even the maps (5.1.8.1) are maps of complexes, compatible with the quotient and Frobenius maps on each side. It follows that the induced maps λ_r^* are as well.

Finally, for (3), suppose we are given a morphism

$$\begin{tikzcd} & \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))) \arrow[dl, "f_M"'] \arrow[dr, "f_N"] & \\ M^* \arrow[rr, "g"] & & N^* \end{tikzcd}$$

in the coslice category, and let $\lambda_{r,M}^*$ and $\lambda_{r,N}^*$ be the maps constructed in part (1). We claim that g is a morphism of de Rham-Witt lift modules from $(M^*, (\lambda_{r,M}^*)_r)$ to $(N^*, (\lambda_{r,N}^*)_r)$. All there is to prove is that $\lambda_{r,N}^* = \mathcal{W}_r(g) \circ \lambda_{r,M}^*$ for each r. This follows by chasing the diagram below, where the commutativity of the outer region follows from that of the inner regions plus the surjectivity of the indicated map on the left.

$$\begin{tikzcd} & & & M^* \arrow[r, two heads] \arrow[dd, "g"] & \mathcal{W}_r\, M^* \arrow[dd, "\mathcal{W}_r(g)"] \\ \mathrm{dR}(\mathcal{E}(A_r)) \arrow[urrrr, bend left, "\lambda_{r,M}^*"] \arrow[drrrr, bend right, "\lambda_{r,N}^*"'] & \widehat{\mathrm{dR}}(\mathcal{E}(A)) \arrow[l, two heads] \arrow[r, "\rho"] & \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))) \arrow[ur, "f_M"] \arrow[dr, "f_N"'] & & \\ & & & N^* \arrow[r, two heads] & \mathcal{W}_r\, N^* \end{tikzcd} \tag{5.1.8.2}$$

□

Proposition 5.1.9. *The functor* $\Theta : \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))/\,\mathcal{W}\,\Omega_R^*\text{-}\mathbf{mod}_{\mathrm{str}} \to \mathrm{dRWLM}_{\mathcal{E},A}$ *of Lemma 5.1.8 is an equivalence of categories.*

Proof. We will show first that it is essentially surjective and then that it is fully faithful. Suppose $(M^*, (\lambda_r^*)_r)$ is a de Rham-Witt lift module over $(A, \mathcal{E})$, as in Alternative Definition 5.1.6. In particular, the maps $\lambda_r^* : \mathrm{dR}(\mathcal{E}(A_r)) \to \mathcal{W}_r\, M^*$ are maps of dg-$\Omega_{A_r}^*$-modules, compatible with the quotient and Frobenius maps on both sides as r varies. Passing to the limit, we get a map $\lambda^* : \widehat{\mathrm{dR}}(\mathcal{E}(A)) \to M^*$ of dg-$\widehat{\Omega}^*_A$-modules, which is again compatible with F. By Lemma 3.2.6, this is a map in $\widehat{\Omega}^*_A$-$\mathbf{mod}_{\mathbf{DC}}$. Thus by Lemma 3.5.8, its strictification

$$\mathcal{W}\,\mathrm{Sat}(\lambda^*) : \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))) \to M^*$$

is a map in $\mathcal{W}\,\Omega^*_R$-$\mathbf{mod}_{\mathrm{str}}$. Thus we have constructed an object $(M^*, \mathcal{W}\,\mathrm{Sat}(\lambda^*))$ in the coslice category $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))/\,\mathcal{W}\,\Omega^*_R$-$\mathbf{mod}_{\mathrm{str}}$, and one can check that the functor Θ sends this to $(M^*, (\lambda^*_r)_r)$.

To prove that Θ is fully faithful, note that it acts as the identity on the underlying strict $\mathcal{W}\,\Omega^*_R$-modules and the morphisms thereof, and the morphisms on both sides are morphisms $M^* \to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules satisfying some extra conditions. Namely, a morphism $g : M^* \to N^*$ in $\mathcal{W}\,\Omega^*_R$-$\mathbf{mod}_{\mathrm{str}}$ is a morphism of de Rham-Witt lift modules if we have $\lambda^*_{r,N} = \mathcal{W}_r(g) \circ \lambda^*_{r,M}$ for each r, and it is a morphism in the coslice category if $f_N = g \circ f_M$, where f_M and f_N are the given maps from $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))$. Thus it suffices to show that these conditions are equivalent to each other. We showed one implication already in Lemma 5.1.8, and the reverse implication follows by chasing the same diagram (5.1.8.2) in reverse: if the outer region commutes, then the two maps $\widehat{\mathrm{dR}}(\mathcal{E}(A)) \to \mathcal{W}_r\, N^*$ agree for all r, which implies the two maps $\widehat{\mathrm{dR}}(\mathcal{E}(A)) \to N^*$ agree, and thus $f_N = g \circ f_M$ by the universal property of ρ. $\square$

Corollary 5.1.10. *The category* $\mathrm{dRWLM}_{A,\mathcal{E}}$ *admits an initial object, whose underlying strict* $\mathcal{W}\,\Omega^*_R$*-module is* $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))$.

Proof. The coslice category $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A)))/\,\mathcal{W}\,\Omega^*_R$-$\mathbf{mod}_{\mathrm{str}}$ clearly has an initial object, namely

$$(\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(A))), \mathrm{id}).$$

The equivalence Θ sends this to an initial object of $\mathrm{dRWLM}_{A,\mathcal{E}}$ with the same underlying strict $\mathcal{W}\,\Omega^*_R$-module. $\square$

5.2 Comparing $\mathrm{dRWM}_{R,\mathcal{E}}$ and $\mathrm{dRWLM}_{A,\mathcal{E}}$

We maintain the setup and notation of the previous section.

Construction 5.2.1. Let $(M^*, (\iota^*_r)_r)$ be a de Rham-Witt module for $(R, \mathcal{E})$, as in Alternative Definition 4.1.6. We can give the $\mathcal{W}\,\Omega^*_R$-module M^* the structure of a de Rham-Witt lift module over $(A, \mathcal{E})$ as follows. For each r, recall the PD-morphism

$$h_r : (\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}\, W_r(R), \gamma) \to (\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}\, A_r, [\,])$$

of Lemma 2.1.12, and let

$$\theta^*_r : \mathrm{dR}(\mathcal{E}(A_r)) \to \mathrm{dR}(\mathcal{E}(W_r(R), \gamma))$$

be the map of dg-$\Omega^*_{A_r}$-modules given by applying Proposition 2.7.11 to h_r. Then let λ^*_r be the composition

$$\lambda^*_r : \mathrm{dR}(\mathcal{E}(A_r)) \xrightarrow{\theta^*_r} \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \xrightarrow{\iota^*_r} \mathcal{W}_r\, M^*,$$

Proposition 5.2.2. *In the situation of Construction 5.2.1, $(M^*, (\lambda^*_r)_r)$ is a de Rham-Witt lift module over $(A, \mathcal{E})$. Moreover, the map sending $(M^*, (\iota^*_r)_r)$ to $(M^*, (\lambda^*_r)_r)$ and acting as the identity on morphisms is a functor $\Psi_{A,\mathcal{E}} : \mathrm{dRWM}_{R,\mathcal{E}} \to \mathrm{dRWLM}_{A,\mathcal{E}}$.*

Proof. We must show that each λ^*_r is a map of dg-$\Omega^*_{A_r}$-modules, and that they are compatible with quotient and Frobenius maps as r varies. In view of Lemmas 5.1.2 and 5.1.5, it suffices to prove these compatibilities in degree 0.

For each r, θ^*_r is a map of dg-$\Omega^*_{A_r}$-modules by construction, while ι^*_r is by definition a map of dg-$\Omega^*_{W_r(R),\gamma}$-modules, and is in particular a map of dg-$\Omega^*_{A_r}$-modules via the natural map $\Omega^*_{A_r} \to \Omega^*_{W_r(R),\gamma}$. Therefore each λ^*_r is a map of dg-$\Omega^*_{A_r}$-modules.

The maps θ^0_r are compatible with quotient and Frobenius maps, by evaluating $\mathcal{E}$ on the commutative squares

$$\begin{tikzcd} A_r \arrow[r] \arrow[d, two heads] & W_r(R) \arrow[d, two heads] \\ A_{r-1} \arrow[r] & W_{r-1}(R) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_r \arrow[r] \arrow[d, "F"'] & W_r(R) \arrow[d, "F"] \\ A_{r-1} \arrow[r] & W_{r-1}(R) \end{tikzcd}$$

of PD-thickenings. The ι^0_r are compatible with quotient and Frobenius maps by definition, so it follows that the λ^0_r are as well. This proves that $(M^*, (\lambda^*_r)_r)$ is a de Rham-Witt lift module over $(A, \mathcal{E})$.

To prove that $\Psi_{A,\mathcal{E}}$ is a functor, suppose we are given a map $f : M^* \to N^*$ which defines a morphism $(M^*, (\iota^*_{r,M})_r) \to (N^*, (\iota^*_{r,N})_r)$ in $\mathrm{dRWM}_{R,\mathcal{E}}$. We must show that the same map f defines a morphism $(M^*, (\lambda^*_{r,M})_r) \to (N^*, (\lambda^*_{r,N})_r)$ in $\mathrm{dRWLM}_{A,\mathcal{E}}$; that is, we have $\lambda^*_{r,N} = \mathcal{W}_r(f) \circ \lambda^*_{r,M}$ for each r. This follows from chasing the diagram

$$\begin{tikzcd} & & \mathcal{W}_r\, M^* \arrow[dd, "\mathcal{W}_r(f)"] \\ \mathrm{dR}(\mathcal{E}(A_r)) \arrow[r, "\theta^*_r"] \arrow[urr, bend left, "\lambda^*_{r,M}"] \arrow[drr, bend right, "\lambda^*_{r,N}"'] & \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \arrow[ur, "\iota^*_{r,M}"] \arrow[dr, "\iota^*_{r,N}"'] & \\ & & \mathcal{W}_r\, N^*, \end{tikzcd} \tag{5.2.2.1}$$

where the commutativity of all inner regions implies that of the outer region. ☐

In fact we can prove more:

Proposition 5.2.3. *The functor* $\Psi_{A,\mathcal{E}} : \mathrm{dRWM}_{R,\mathcal{E}} \to \mathrm{dRWLM}_{A,\mathcal{E}}$ *is an equivalence of categories.*

Proof. We will show that $\Psi_{A,\mathcal{E}}$ is essentially surjective and fully faithful. For essential surjectivity, suppose $(M^*, (\lambda_r^*)_r)$ is an object of $\mathrm{dRWLM}_{A,\mathcal{E}}$. In particular, each λ_r^* is a morphism $\mathrm{dR}(\mathcal{E}(A_r)) \to \mathcal{W}_r\, M^*$ of dg-$\Omega^*_{A_r}$-modules. But Corollary 3.6.7 shows that the dg-$\Omega^*_{A_r}$-module structure of $\mathcal{W}_r\, M^*$ factors through its dg-$\Omega^*_{W_r(R),\gamma}$-module structure, so we can factor λ_r^* as

$$\mathrm{dR}(\mathcal{E}(A_r)) \to \Omega^*_{W_r(R),\gamma} \otimes_{\Omega^*_{A_r}} \mathrm{dR}(\mathcal{E}(A_r)) \simeq \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \xrightarrow{\iota_r^*} \mathcal{W}_r\, M^*,$$

where the isomorphism comes from Corollary 2.7.12, and ι_r^* is a morphism of dg-$\Omega^*_{W_r(R),\gamma}$-modules. The compatibilities of the ι_r^* with quotient and Frobenius maps follow from the same compatibilities of λ_r^*. Thus $(M^*, (\iota_r^*)_r)$ is a de Rham-Witt module over $(R, \mathcal{E})$, and by construction we have $\Psi_{A,\mathcal{E}}(M^*, (\iota_r^*)_r) = (M^*, (\lambda_r^*)_r)$.

Since $\Psi_{A,\mathcal{E}}$ acts as the identity on morphisms, it is clearly faithful. To prove that it is full, we must only chase diagram 5.2.2.1 in reverse: recall that a morphism $f : M^* \to N^*$ in $\mathcal{W}\,\Omega^*_R$-$\mathbf{mod}_{\mathrm{str}}$ defines a morphism in $\mathrm{dRWM}_{R,\mathcal{E}}$ if and only if the small triangle commutes, and it defines a morphism in $\mathrm{dRWLM}_{A,\mathcal{E}}$ if and only if the outer region commutes. The universal property of

$$\Omega^*_{W_r(R),\gamma} \otimes_{\Omega^*_{A_r}} \mathrm{dR}(\mathcal{E}(A_r)) \simeq \mathrm{dR}(\mathcal{E}(W_r(R),\gamma))$$

implies that these two conditions are in fact equivalent. ☐

Corollary 5.2.4. *Let* $(M^*, (\lambda_r)_r)$ *denote the initial object*

$$\mathcal{W}\,\mathrm{Sat}(\lim_r \mathrm{dR}(\mathcal{E}(A_r)))$$

of $\mathrm{dRWLM}_{A,\mathcal{E}}$*. Then* M^* *can be endowed with maps* $(\iota_r)_r$ *making it an initial object of* $\mathrm{dRWM}_{R,\mathcal{E}}$*; that is, a saturated de Rham-Witt complex associated to* $(R, \mathcal{E})$*.*

Chapter 6

Proofs of main results

In this chapter, we prove our three main results: the existence of our saturated de Rham-Witt complexes in general, and comparisons (for X/k smooth) to crystalline cohomology and the classical de Rham-Witt complexes of [8]. The first two sections, comprising the general construction and comparison to cohomology, are inspired by [20, §5], which does the same for the saturated de Rham-Witt complex of Bhatt-Lurie-Mathew. Working with nontrivial coefficient crystals will require us to make a few changes, one of which (the use of PD-de Rham complexes rather than naive de Rham complexes in §6.2) turns out to be a significant simplification.

In the final section, which compares our de Rham-Witt complexes to the classical ones, the idea is to work on a pro-étale cover trivializing the F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$. Such a cover exists by Remark 2.5.10, and the results of sections 4.5 and 4.6 allow us to express our saturated de Rham-Witt complexes on the cover in terms of those on the base scheme.

6.1 General construction

6.1.1. Throughout this section, we will work in the Frobenius-embedded situation 1.7.8, with two modifications: we assume that R is reduced (which does no harm in view of Corollary 4.4.7), and we allow A to be an arbitrary p-torsionfree W-algebra instead of a smooth one. We let $(\mathcal{E}, \phi_{\mathcal{E}})$ be a unit-root F-crystal on $\mathrm{Cris}(\mathrm{Spec}\, R/W)$ as usual.

Additionally, let $\widetilde{B}$ be the quotient of the p-adically completed PD-envelope B by the p-adic closure of the p-power torsion ideal, and $\widetilde{B}_r = \widetilde{B}/p^r\widetilde{B}$ for each $r > 0$. Finally, although we will mostly use ring-theoretic notation, we fix for convenience the following notation for the various affine schemes of interest to us:

$$\begin{aligned} X &= \operatorname{Spec} R, \\ Y_r &= \operatorname{Spec} A_r, \\ D_r &= \operatorname{Spec} B_r, \text{ and} \\ \widetilde{D_r} &= \operatorname{Spec} \widetilde{B_r}. \end{aligned}$$

Remark 6.1.2. If R is generated as a k-algebra by elements $\{x_i\}_{i\in I}$, we may choose A to be the polynomial algebra

$$A = W[\{t_i\}_{i\in I}],$$

with the map $A \twoheadrightarrow R$ induced by $t_i \mapsto x_i$ and the Frobenius lift given by

$$\phi\left(a\prod_i t_i^{e_i}\right) = \sigma(a)\prod_i t_i^{pe_i}.$$

For future reference, we note that this relativizes as follows. Given a morphism $f : R \to R'$ lying over a map $k \to k'$ of fields, suppose R is generated over k by elements $\{x_i\}_{i\in I}$ and R' is generated over k' by elements $\{y_j\}_{j\in J}$. Then consider the diagram

$$\begin{tikzcd} W(k)[\{t_i\}_{i\in I}] \arrow[r, hook] \arrow[d, two heads] & W(k')[\{t_i\}_{i\in I}, \{u_j\}_{j\in J}] \arrow[d, two heads] \\ R \arrow[r, "f"] & R' \end{tikzcd}$$

where the left vertical map is as before, the right one sends t_i to $f(x_i)$ and u_j to y_j, and both polynomial algebras are equipped with Frobenius lifts as before.

In particular, if R is a finite-type k-algebra (as we will assume later), then we can lift it to a polynomial algebra A on finitely many generators, which is in particular smooth over W. If $R \to R'$ is a morphism of finite-type algebras over k and k' respectively, then we can lift it to a morphism of polynomial algebras A, A' on finitely many generators over $W(k)$ and $W(k')$ respectively, which are again smooth. Note also that we are free to p-adically complete the polynomial algebras A and A', in order to discuss the smooth p-adic formal schemes $\operatorname{Spf} A$ and $\operatorname{Spf} A'$.

Remark 6.1.3. The natural map $R = A/I \to B/\overline{I}$ is an isomorphism. The analogous statement $R = A_r/I_r \xrightarrow{\sim} B_r/\overline{I_r}$ for uncompleted PD-envelopes follows from [2, Remark 3.20.4], since $(p) \subset W(k)$ is principal, and the case of the completed PD-envelope follows by quotienting by p^r (cf. [2, Remark 3.20.8]) and passing to the limit.

Lemma 6.1.4. *The kernel of $B \to \widetilde{B}$ is a sub-PD-ideal of $(\overline{I}, \gamma)$. In particular, the quotient map*

$$B \twoheadrightarrow B/\overline{I} = R$$

factors through $\widetilde{B}$*, and* γ *induces PD-structures (which we will abusively also call* γ*) on the ideals* $\ker(\widetilde{B} \to R) \subset \widetilde{B}$ *and* $\ker(\widetilde{B}_r \to R) \subset \widetilde{B}_r$ *for each* r*.*

Proof. Although our hypotheses are slightly different, the proof of [20, Lemma 5.1] goes through without change. □

Remark 6.1.5. Lemma 6.1.4 shows that for each $r > 0$, we have an object $(X \hookrightarrow \widetilde{D}_r, \gamma)$ of $\mathrm{Cris}(X/W)$. Note that the PD-structure $[\,]$ of Lemma 1.7.7 also makes $\widetilde{D}_r$ into a PD-thickening of $\widetilde{D}_1$, leading to an object $\widetilde{D}_r = (\widetilde{D}_1 \hookrightarrow \widetilde{D}_r, [\,])$ of $\mathrm{Cris}(\widetilde{D}_1/W_r)$. Both of these objects will be useful for us; for future reference, we will let f denote the natural closed embedding $X \hookrightarrow \widetilde{D}_1$, and $f^\sharp : \widetilde{B}_1 \to R$ the corresponding surjective ring map.

Remark 6.1.6. We will ultimately construct our saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ by applying Corollary 5.2.4 to the p-torsionfree lifting $\widetilde{B}$ of the $\mathbb{F}_p$-algebra $\widetilde{B}_1$, and then pulling back along f. But before we can do any of this, we must specify what unit-root F-crystal on $\widetilde{D}_1$ we are working with. This is done in the following three lemmas; the goal is to find a unit-root F-crystal $(\mathcal{F}, \phi_{\mathcal{F}})$ on $\mathrm{Cris}(\widetilde{D}_1/W_r)$ such that $f^*_{\mathrm{cris}}(\mathcal{F}) \simeq \mathcal{E}$.

Lemma 6.1.7. *For every* $x \in \ker f^\sharp$*, we have* $x^p = 0$*. In particular,* $f^\sharp$ *expresses* R *as the reduction of* $\widetilde{B}_1$*.*

Proof. Since $\ker f^\sharp$ carries a PD-structure γ, we have for all $x \in \ker f^\sharp$ that $x^p = p!\gamma_p(x)$, which vanishes because $\widetilde{B}_1$ is an $\mathbb{F}_p$-algebra. So $f^\sharp$ is surjective, it kills only nilpotents, and it kills all nilpotents because R is reduced. □

Lemma 6.1.8. *There exists a unique morphism* $g : \widetilde{D}_1 \to X$ *such that* $f \circ g$ *is the absolute Frobenius endomorphism of* $\widetilde{D}_1$*. Moreover,* $g \circ f$ *is the absolute Frobenius of* X*.*

Proof. In ring-theoretic notation, this says that $\mathrm{Frob}_{\widetilde{B}_1}$ factors uniquely as

$$\widetilde{B}_1 \xrightarrow{f^\sharp} R \xrightarrow{g^\sharp} \widetilde{B}_1,$$

and moreover that $f^\sharp \circ g^\sharp = \mathrm{Frob}_R$. For the first statement, we need only observe that $f^\sharp$ is surjective and that $\mathrm{Frob}_{\widetilde{B}_1}$ annihilates $\ker f^\sharp$ by Lemma 6.1.7. For the second statement, we calculate that

$$\begin{aligned} f^\sharp \circ g^\sharp \circ f^\sharp &= f^\sharp \circ \mathrm{Frob}_{\widetilde{B}_1} \\ &= \mathrm{Frob}_R \circ f^\sharp, \end{aligned}$$

which implies $f^\sharp \circ g^\sharp = \mathrm{Frob}_R$ because $f^\sharp$ is surjective. □

Lemma 6.1.9. *Let* $\mathcal{F} = g^*_{\mathrm{cris}}\mathcal{E}$*, equipped with the Frobenius endomorphism*

$$\phi_{\mathcal{F}} = g^*_{\mathrm{cris}}\phi_{\mathcal{E}} : F^*_{X,\mathrm{cris}}\mathcal{F} \to \mathcal{F}.$$

There exists an isomorphism of F*-crystals* $f^*_{\mathrm{cris}}(\mathcal{F}, \phi_{\mathcal{F}}) \xrightarrow{\sim} (\mathcal{E}, \phi_{\mathcal{E}})$*.*

Proof. By definition, we have

$$f_{\mathrm{cris}}^*(\mathcal{F}, \phi_{\mathcal{F}}) = f_{\mathrm{cris}}^* g_{\mathrm{cris}}^*(\mathcal{E}, \phi_{\mathcal{E}}) = F_{X,\mathrm{cris}}^*(\mathcal{E}, \phi_{\mathcal{E}}),$$

so we must give an isomorphism of F-crystals

$$F_{X,\mathrm{cris}}^*(\mathcal{E}, \phi_{\mathcal{E}}) \simeq (\mathcal{E}, \phi_{\mathcal{E}}).$$

But since $\mathcal{E}$ is a unit-root F-crystal, $\phi_{\mathcal{E}}$ itself is an isomorphism of crystals $F_{X,\mathrm{cris}}^*\mathcal{E} \to \mathcal{E}$, and it is moreover an isomorphism of F-crystals because the diagram

$$\begin{tikzcd}
F_{X,\mathrm{cris}}^* F_{X,\mathrm{cris}}^* \mathcal{E} \arrow[r, "F_{X,\mathrm{cris}}^*\phi_{\mathcal{E}}"] \arrow[d, "F_{X,\mathrm{cris}}^*\phi_{\mathcal{E}}"'] & F_{X,\mathrm{cris}}^* \mathcal{E} \arrow[d, "\phi_{\mathcal{E}}"] \\
F_{X,\mathrm{cris}}^* \mathcal{E} \arrow[r, "\phi_{\mathcal{E}}"'] & \mathcal{E}
\end{tikzcd}$$

commutes. □

Theorem 6.1.10. *Suppose R is a reduced k-algebra and $(\mathcal{E}, \phi_{\mathcal{E}})$ is a unit-root F-crystal on* Cris(Spec R/W). *Then there exists a saturated de Rham-Witt complex $\mathcal{W}\Omega_{R,\mathcal{E}}^*$ associated to $(R, \mathcal{E})$. Moreover, it is isomorphic as a strict $\mathcal{W}\Omega_R^*$-module to the object*

$$\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(\widetilde{B})))$$

of Corollary 5.1.10, viewed as a $\mathcal{W}\Omega_R^*$-module via the isomorphism $\mathcal{W}\Omega_{\widetilde{B}_1}^* \simeq \mathcal{W}\Omega_R^*$ of Remark 1.2.7.

Proof. Putting together Lemmas 6.1.9 and 6.1.7 and Propositions 4.4.3 and 5.2.3, we have equivalences of categories

$$\mathrm{dRWM}_{R,\mathcal{E}} \simeq \mathrm{dRWM}_{R, f_{\mathrm{cris}}^*\mathcal{F}} \xrightarrow{\sim} \mathrm{dRWM}_{\widetilde{B}_1,\mathcal{F}} \xrightarrow{\sim} \mathrm{dRWLM}_{\widetilde{B},\mathcal{F}},$$

each preserving the underlying strict $\mathcal{W}\Omega_R^*$-modules. The object we have specified is an initial object in the category on the right, so it corresponds to an initial object of the category on the left with the same underlying strict $\mathcal{W}\Omega_R^*$-module. □

Corollary 6.1.11. *Let R be an arbitrary k-algebra and $(\mathcal{E}, \phi_{\mathcal{E}})$ a unit-root F-crystal on* Cris(Spec R/W). *Then there exists a saturated de Rham-Witt complex $\mathcal{W}\Omega_{R,\mathcal{E}}^*$ associated to $(R, \mathcal{E})$.*

Proof. Combine Theorem 6.1.10 with Corollary 4.4.7. □

6.2 Comparison to crystalline cohomology

6.2.1. Our goal in this section is to compare the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{X,\mathcal{E}}$ of Remark 4.5.10 to the cohomology of the F-crystal $\mathcal{E}$. To prove this, we will first need a slightly different recipe for $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ when R is a smooth k-algebra, in terms of a de Rham complex of $\mathcal{E}$ rather than our auxiliary F-crystal $\mathcal{F}$.

6.2.2. We maintain the setup and notation of 6.1.1, and assume moreover that R is a smooth (and in particular finite-type) k-algebra. In light of Remark 6.1.2, we can and do take A to be smooth over W. It follows by Remark 2.6.21 that the completed PD-envelope B is p-torsionfree, so that $\widetilde{B} = B$.

Remark 6.2.3. The comparison will pass through three de Rham complexes associated to the F-crystals $\mathcal{E}$ and $\mathcal{F}$ (with $\mathcal{E} \simeq f^*_{\text{cris}}\mathcal{F}$), which we now recall. First, we have already discussed the completed PD-de Rham complex

$$\widehat{\mathrm{dR}}(\mathcal{F}(B,[\ ])) = \lim_r \left(\mathcal{F}(B_r) \otimes_{B_r} \Omega^*_{B_r,[\ ]}\right)$$

associated to $\mathcal{F}$ on the formal PD-thickening $(D_1 \hookrightarrow D_r, [\ ])_r$ of $D_1 = \operatorname{Spec} B_1$. (The reader may recall that we have $\Omega^*_{B_r,[\ ]} = \Omega^*_{B_r}$ by Lemma 2.3.9; however, we will leave the PD-structure in the notation for emphasis.) Closely related to this (cf. Remark 6.1.5) is

$$\widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma)) = \lim_r \left(\mathcal{E}(B_r) \otimes_{B_r} \Omega^*_{B_r,\gamma}\right),$$

which is associated to $\mathcal{E}$ on the formal PD-thickening $(X \hookrightarrow D_r, \gamma)_r$ of $X = \operatorname{Spec} R$. Third, recall from Definition 2.6.17 the completed de Rham complex

$$\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) = \lim_r \left(\mathcal{E}(B_r) \otimes_{A_r} \Omega^*_{A_r}\right),$$

which is constructed from the PD-envelopes of the smooth embeddings $X \hookrightarrow Y_r$.

By Remarks 2.7.20 and 2.6.20, these are modules in **DC** over $\widehat{\Omega}^*_{B,[\ ]}$, $\widehat{\Omega}^*_{B,\gamma}$, and $\lim_r(B_r \otimes_{A_r} \Omega^*_{A_r})$ respectively. They are related in several ways: by Lemma 2.8.8 and Remark 2.8.10, we have an isomorphism between the latter two Dieudonné algebras and an isomorphism of the corresponding modules when we identify the algebras. Moreover, the compatibility of the PD-structures γ (on I) and $[\ ]$ (on $(p) \subset I$) gives us a quotient map

$$\widehat{\Omega}^*_{B,[\ ]} \to \widehat{\Omega}^*_{B,\gamma},$$

which has p-torsion kernel by Proposition 2.3.8. This carries over to our chosen coefficients as follows.

Construction 6.2.4. Consider the PD-morphism

$$h = \mathrm{id}_{D_r} : (X \hookrightarrow D_r, \gamma) \to (D_1 \hookrightarrow D_r, [\ ])$$

over $f : X \hookrightarrow D_1$. Corollary 2.7.12 associates to this an isomorphism

$$\Omega^*_{B_r,\gamma} \otimes_{\Omega^*_{B_r,[\ ]}} \mathrm{dR}(\mathcal{F}(B_r,[\ ])) \simeq \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{F}(B_r,\gamma)) \simeq \mathrm{dR}(\mathcal{E}(B_r,\gamma)).$$

of dg-$\Omega^*_{B_r,\gamma}$-modules. Let

$$\pi_r : \mathrm{dR}(\mathcal{F}(B_r,[\ ])) \to \mathrm{dR}(\mathcal{E}(B_r,\gamma))$$

denote the resulting base change map, and

$$\pi : \widehat{\mathrm{dR}}(\mathcal{F}(B,[\ ])) \to \widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma))$$

the map of limits.

Lemma 6.2.5. *The map π is a morphism of $\widehat{\Omega}^*_{B,[\ ]}$-modules in* **DC**.

Proof. It is clearly a map of dg-modules over $\widehat{\Omega}^*_{B,[\ ]}$, so in light of Lemma 3.2.6, it suffices to prove that it is compatible with F. But the Frobenius endomorphisms of $\mathrm{dR}(\mathcal{F}(B_r,[\ ]))$ and $\mathrm{dR}(\mathcal{E}(B_r,\gamma))$ are defined by $\phi_{\mathcal{F}} \otimes F$ and $\phi_{\mathcal{E}} \otimes F$, respectively, where each F is the divided Frobenius endomorphism of Proposition 2.3.13. The maps of tensor factors

$$\begin{gathered} \mathcal{F}(B_r) \to \mathcal{E}(B_r) \text{ and} \\ \Omega^*_{B_r,[\ ]} \to \Omega^*_{B_r,\gamma} \end{gathered}$$

are both compatible with Frobenius, so π_r and thus π are as well. □

Lemma 6.2.6. *The strictification of π is an isomorphism*

$$\mathcal{W}\,\mathrm{Sat}(\pi) : \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\ ]))) \xrightarrow{\sim} \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma)))$$

of strict modules over $\mathcal{W}\,\Omega^*_{B_1} = \mathcal{W}\,\Omega^*_R$.

Proof. It follows from Lemma 3.5.8 that $\mathcal{W}\,\mathrm{Sat}(\pi)$ is a morphism of strict modules over $\mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_{B,[\ ]})$. But recall that $\mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_{B,[\ ]})$ is isomorphic to $\mathcal{W}\,\Omega^*_{B_1}$ by Corollary 2.3.14, and thus also to $\mathcal{W}\,\Omega^*_R$ by Remark 1.2.7. To show that $\mathcal{W}\,\mathrm{Sat}(\pi)$ is an isomorphism, first note that each π_r is surjective with exact p-torsion kernel, since this is true of the quotient map

$$\Omega^*_{B_r,[\ ]} \to \Omega^*_{B_r,\gamma}$$

(cf. Corollary 2.2.9) and we are tensoring this with the flat B_r-module $\mathcal{F}(B_r)$. It follows by passage to the limit (and a Mittag-Leffler calculation analogous to that of Proposition 2.3.8) that π itself is surjective with exact p-torsion kernel. Thus $\mathrm{Sat}(\pi)$ is an isomorphism, as is $\mathcal{W}\,\mathrm{Sat}(\pi)$. □

Corollary 6.2.7. *There exist maps $\iota^*_r : \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \to \mathcal{W}_r\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)))$ making $\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)))$ a saturated de Rham-Witt complex of $\mathcal{E}$ over R.*

Proof. We showed in Theorem 6.1.10 that $\mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,])))$ is a saturated de Rham-Witt complex, so we can simply transport this structure across the isomorphisms

$$\mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,]))) \simeq \mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma))) \simeq \mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))),$$

where the former is Lemma 6.2.6 and the latter comes from strictifying the isomorphism of Remark 2.8.10. □

Lemma 6.2.8. *The strictification map*

$$\rho : \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) \to \mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)))$$

and the induced map

$$\rho_r : \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))/p^r = \mathrm{dR}(\mathcal{E}(X \hookrightarrow Y_r)) \to \mathcal{W}_r\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)))$$

are quasi-isomorphisms.

Proof. This is analogous to [20, Corollary 5.4], and the proof goes through without change, except that we must replace the reference to [2, p. 8.20] with its unit-root generalization, [19, Corollary 7.3.6]. (Note that [20, Theorem 1.8] applies to the quasi-isomorphisms at both finite and infinite level.) □

Proposition 6.2.9. *The morphisms of Corollary 6.2.7 and Lemma 6.2.8 are functorial in the data* $(\operatorname{Spec} R \hookrightarrow (\operatorname{Spec} A_r/W_r(k))_r, \phi_A)$*. That is, suppose we are given a commutative diagram*

$$\begin{array}{ccccc} X' & \hookrightarrow & (Y'_r)_r & \longrightarrow & \operatorname{Spec} W_r(k') \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & (Y_r)_r & \longrightarrow & \operatorname{Spec} W_r(k), \end{array}$$

where each row is as in 6.2.2 and the maps $Y'_r \to Y_r$ are compatible with Frobenius lifts. Suppose $\mathcal{E}$ is a unit-root F-crystal on $\mathrm{Cris}(X/W(k))$ *and $\mathcal{E}'$ is its pullback to* $\mathrm{Cris}(X'/W(k'))$*. Then the resulting diagrams*

$$\begin{array}{ccccc} \mathrm{dR}(\mathcal{E}(X \hookrightarrow Y_r)) & \xrightarrow{\rho_r} & \mathcal{W}_r\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) & \xrightarrow{\sim} & \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{dR}(\mathcal{E}'(X' \hookrightarrow Y'_r)) & \xrightarrow{\rho'_r} & \mathcal{W}_r\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}'(X' \hookrightarrow Y'_\bullet))) & \xrightarrow{\sim} & \mathcal{W}_r\,\Omega^*_{R',\mathcal{E}'} \end{array}$$

and

$$\begin{array}{ccccc} \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) & \xrightarrow{\rho} & \mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))) & \xrightarrow{\sim} & \mathcal{W}\,\Omega^*_{R,\mathcal{E}} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathrm{dR}}(\mathcal{E}'(X' \hookrightarrow Y'_\bullet)) & \xrightarrow{\rho'} & \mathcal{W}\operatorname{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}'(X' \hookrightarrow Y'_\bullet))) & \xrightarrow{\sim} & \mathcal{W}\,\Omega^*_{R',\mathcal{E}'} \end{array}$$

commute, where the rightmost vertical maps are given by Remark 4.3.10.

Proof. We will prove the infinite-level statement, which immediately implies the mod-p^r statement. The compatibility of the left-hand square follows from the functoriality of the strictification map. As for the right-hand square, this is a tedious exercise in unraveling the proofs of Corollary 6.2.7 and its precursor Theorem 6.1.10, which we will describe now. By the universal property of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R,\mathcal{E}}$, it suffices to prove that the given map

$$\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet))) \to \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{E}'(X' \hookrightarrow Y'_\bullet))) \tag{6.2.9.1}$$

is a morphism in $\mathrm{dRWM}_{R,\mathcal{E}}$ once we use Corollary 6.2.7, Theorem 6.1.10, and Construction 4.3.6 to give its source and target this structure.

We are given maps $R \to R'$, $A_r \to A'_r$, and $W_r(k) \to W_r(k')$ of rings, which induce maps $h_r^\sharp : B_r \to B'_r$ of PD-envelopes for each r. Applying Lemma 6.1.8 to both PD-envelopes leads to a commutative diagram of rings

$$\begin{tikzcd}
B_1 \arrow[r, "f^\sharp"] \arrow[d, "h_1^\sharp"] \arrow[rr, bend left, "\mathrm{Frob}_{B_1}"] & R \arrow[r, "g^\sharp"] \arrow[d] & B_1 \arrow[d, "h_1^\sharp"] \\
B'_1 \arrow[r, "f'^\sharp"] \arrow[rr, bend right, "\mathrm{Frob}_{B'_1}"'] & R' \arrow[r, "g'^\sharp"] & B'_1,
\end{tikzcd}$$

from which we see that $\mathcal{F}' = g'^*_{\mathrm{cris}}\mathcal{E}'$ is the pullback of $\mathcal{F} = g^*_{\mathrm{cris}}\mathcal{E}$ along $h_1 : \operatorname{Spec} B'_1 \to \operatorname{Spec} B_1$. This gives us a commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,])) \arrow[r, "\pi"] \arrow[d] & \widehat{\mathrm{dR}}(\mathcal{E}(B,\gamma)) \arrow[r, "\sim"] \arrow[d] & \widehat{\mathrm{dR}}(\mathcal{E}(X \hookrightarrow Y_\bullet)) \arrow[d] \\
\widehat{\mathrm{dR}}(\mathcal{F}'(B',[\,])) \arrow[r, "\pi'"] & \widehat{\mathrm{dR}}(\mathcal{E}'(B',\gamma)) \arrow[r, "\sim"] & \widehat{\mathrm{dR}}(\mathcal{E}'(X' \hookrightarrow Y'_\bullet))
\end{tikzcd}$$

relating the maps of Remark 6.2.3 and Construction 6.2.4, where the vertical morphisms are the natural pullback maps. As π and π' become isomorphisms after applying $\mathcal{W}\,\mathrm{Sat}$, we may thus identify the given map (6.2.9.1) with the map

$$\mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,])) \to \mathcal{W}\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}'(B',[\,])). \tag{6.2.9.2}$$

By Corollary 5.1.10, these two objects are initial objects of $\mathrm{dRWLM}_{B,\mathcal{F}}$ and $\mathrm{dRWLM}_{B',\mathcal{F}'}$ respectively when endowed with the obvious maps

$$\lambda_r^* : \mathrm{dR}(\mathcal{F}(B_r,[\,])) \to \mathcal{W}_r\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,]))$$
$$\lambda_r'^* : \mathrm{dR}(\mathcal{F}'(B'_r,[\,])) \to \mathcal{W}_r\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}'(B',[\,])).$$

Next, the proof of Theorem 6.1.10 consists of turning these objects into initial objects of the categories $\mathrm{dRWM}_{R,\mathcal{E}}$ and $\mathrm{dRWM}_{R',\mathcal{E}'}$ by passing through various equivalences of categories. Unwinding these functors, this amounts to factoring λ_r^* and $\lambda_r'^*$ (uniquely) as the compositions across the rows of the following diagram:

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{F}(B_r,[\,])) \arrow[r] \arrow[d] & \mathrm{dR}(\mathcal{F}(W_r(B_1),[\,])) \arrow[r] \arrow[d] & \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \arrow[r, "\iota_r^*"] \arrow[d] & \mathcal{W}_r\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}(B,[\,]))) \arrow[d] \\
\mathrm{dR}(\mathcal{F}'(B_r',[\,])) \arrow[r] & \mathrm{dR}(\mathcal{F}'(W_r(B_1'),[\,])) \arrow[r] & \mathrm{dR}(\mathcal{E}'(W_r(R'),\gamma)) \arrow[r, "\iota_r'^*"] & \mathcal{W}_r\,\mathrm{Sat}(\widehat{\mathrm{dR}}(\mathcal{F}'(B',[\,])))
\end{tikzcd}$$

The two leftmost squares and the outer rectangle of this diagram commute, so the uniqueness of the factorization implies that the rightmost square commutes as well. This is precisely the compatibility of ι maps that we require in order for (6.2.9.2) to be a morphism in $\mathrm{dRWM}_{R,\mathcal{E}}$, where we have implicitly applied the pushforward functor of Construction 4.3.6 to its target. This map is also evidently compatible with module structures, so it is indeed a morphism of de Rham-Witt modules as claimed. □

6.2.10. We now leave the affine setting and assume only that X/k is smooth and comes equipped with a closed embedding into a smooth p-adic formal scheme $\mathcal{Y}/\operatorname{Spf} W(k)$ which is equipped with a Frobenius lift $\phi_{\mathcal{Y}}$ (over $\sigma : W(k) \to W(k)$) of $\mathcal{Y} \times_{\operatorname{Spf} W} \operatorname{Spec} k$. (By Remark 6.1.2, such an embedding exists for any smooth *affine* X/k, and this relativizes for morphisms $X' \to X$ of smooth affine schemes over $\operatorname{Spec} k' \to \operatorname{Spec} k$.) For each $r > 0$, we let Y_r denote the scheme $\mathcal{Y} \times_{\operatorname{Spf} W} \operatorname{Spec} W_r(k)$. Note that locally on $\mathcal{Y}$, any such embedding has the form

$$\operatorname{Spec} R \hookrightarrow \operatorname{Spf} A,$$

where R is a smooth k-algebra and A is a smooth and p-adically complete W-algebra equipped with a Frobenius lift ϕ_A. (In fact there exists a basis of affine open subschemes $\operatorname{Spec} R \subset X$ admitting such an embedding: given an open neighborhood U of $x \subset X$, one can choose an open formal subscheme $V \subset \mathcal{Y}$ whose preimage equals U, and one can then localize further within V.) Similarly, given a map

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d] & \mathcal{Y}' \arrow[r] \arrow[d] & \operatorname{Spf} W' \arrow[d] \\
X \arrow[r, hook] & \mathcal{Y} \arrow[r] & \operatorname{Spf} W
\end{tikzcd}$$

of two such embeddings (which we always assume is compatible with Frobenius lifts), we can localize first on $\mathcal{Y}$ and then on $\mathcal{Y}'$ to put it in the form

$$\begin{tikzcd}
\operatorname{Spec} R' \arrow[r, hook] \arrow[d] & \operatorname{Spf} A' \arrow[r] \arrow[d] & \operatorname{Spf} W' \arrow[d] \\
\operatorname{Spec} R \arrow[r, hook] & \operatorname{Spf} A \arrow[r] & \operatorname{Spf} W.
\end{tikzcd}$$

Proposition 6.2.11. *Suppose $X \hookrightarrow \mathcal{Y}/W$ is as in 6.2.10, and let $\mathcal{E}$ be a unit-root F-crystal on* $\mathrm{Cris}(X/W)$*. Then we have quasi-isomorphisms*

$$\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) \to \mathcal{W}_r\, \Omega^*_{X,\mathcal{E}}$$
$$\widehat{\mathrm{dR}}(\mathcal{E}_{X \hookrightarrow \mathcal{Y}}) \to \mathcal{W}\, \Omega^*_{X,\mathcal{E}}$$

of Zariski sheaves (of W_r-modules and W-modules respectively) on X. These are both functorial in the data $(X \hookrightarrow \mathcal{Y}/W, \phi_{\mathcal{Y}})$ in the sense that given compatible morphisms $f : X' \to X$ and $g : \mathcal{Y}' \to \mathcal{Y}$ over $\mathrm{Spf}\, W' \to \mathrm{Spf}\, W$*, the diagram*

$$\begin{tikzcd}
\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) \arrow[r] \arrow[d] & \mathcal{W}_r\Omega^*_{X,\mathcal{E}} \arrow[d] \\
f_* \,\mathrm{dR}((f^*_{\mathrm{cris}}\mathcal{E})_{X' \hookrightarrow Y'_r}) \arrow[r] & f_*(\mathcal{W}_r\Omega^*_{X', f^*_{\mathrm{cris}}\mathcal{E}})
\end{tikzcd}$$

commutes for each r, and similarly when passing to the limit.

Proof. To construct the isomorphism

$$\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) \simeq \mathcal{W}_r\, \Omega^*_{X,\mathcal{E}},$$

we work on affine open subschemes $\mathrm{Spec}\, R \subset X$ on which $X \hookrightarrow \mathcal{Y}$ restricts to an affine embedding $\mathrm{Spec}\, R \hookrightarrow \mathrm{Spf}\, A$. On each such open subscheme, Corollary 6.2.7 and Lemma 6.2.8 define a quasi-isomorphism

$$\mathrm{dR}(\mathcal{E}(\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}\, A_r)) \to \mathcal{W}_r\, \Omega^*_{R,\mathcal{E}|_{\mathrm{Spec}\, R}}.$$

By Proposition 6.2.9, these glue to a map of complexes of sheaves

$$\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) \to \mathcal{W}_r\, \Omega^*_{X,\mathcal{E}},$$

which we claim is also a quasi-isomorphism. Indeed, the cone of this map is a complex of sheaves which is exact when evaluated on a basis of affines in X; thus it is exact on stalks and therefore also at the level of sheaves. This yields the isomorphism at finite levels. We then obtain the infinite-level isomorphism by applying $R\lim_r$. Note here that $(\mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}))_r$ and $(\mathcal{W}_r\, \Omega^*_{X,\mathcal{E}})_r$ are towers of quasicoherent sheaves with surjective transition maps, so they satisfy

$$R\lim_r \mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) = \lim_r \mathrm{dR}(\mathcal{E}_{X \hookrightarrow Y_r}) = \widehat{\mathrm{dR}}(\mathcal{E}_{X \hookrightarrow \mathcal{Y}}) \text{ and}$$
$$R\lim_r \mathcal{W}_r\, \Omega^*_{X,\mathcal{E}} = \lim_r \mathcal{W}_r\, \Omega^*_{X,\mathcal{E}} = \mathcal{W}\, \Omega^*_{X,\mathcal{E}}$$

by [21, tag `0BKS`]. Functoriality in the data $(X \hookrightarrow \mathcal{Y}/W, \phi_{\mathcal{Y}})$ is guaranteed by Proposition 6.2.9. ∎

6.2.12. In order to prove Proposition 6.2.11 without assuming the existence of a smooth embedding with Frobenius, we will need the method of cohomological descent; cf. e.g. [1, pp. V, 3.4]. Our setup is as follows.

Construction 6.2.13. Let X be a smooth k-scheme and $(\mathcal{E}, \phi_{\mathcal{E}})$ a unit-root F-crystal on X. Choose an affine open cover $(U_i)_{i\in I}$ of X. For each i, fix a smooth formal embedding $U_i \hookrightarrow \mathcal{Y}_i/W$ with Frobenius lift, as in Remark 6.1.2. For every nonempty finite subset $J \subset I$, we get a smooth formal embedding

$$U_J := \bigcap_{i\in J} U_i \hookrightarrow \prod_{i\in J} \mathcal{Y}_i =: \mathcal{Y}_J, \tag{6.2.13.1}$$

where the product is taken over $\operatorname{Spf} W$ and $\mathcal{Y}_J$ carries a Frobenius lift defined componentwise. Then we can assemble the open subschemes U_J into a hypercover $X_\bullet$ of X, with

$$X_n = \coprod_{|J|=n} U_J,$$

and (6.2.13.1) defines a closed embedding of the semi-simplicial scheme $X_\bullet$ into a smooth formal semi-simplicial scheme $\mathcal{Y}_\bullet$ equipped with Frobenius lift.

Remark 6.2.14. To prevent any confusion between the simplicial embedding $X_\bullet \hookrightarrow \mathcal{Y}_\bullet$ of Construction 6.2.13 and the p-adic towers of lifts $X \hookrightarrow Y_\bullet$ of Construction 2.6.17, we will refrain from using the latter notation for the remainder of this section. When necessary, we will denote the p-adic tower associated to the semi-simplicial formal scheme $\mathcal{Y}_\bullet$ as $(\mathcal{Y}_{\bullet,r})_r$.

Theorem 6.2.15. *Suppose X is a smooth k-scheme and $\mathcal{E}$ is a unit-root F-crystal on* $\mathrm{Cris}(X/W)$*. Then there are canonical isomorphisms*

$$Ru_{X/W_r*}\mathcal{E} \simeq \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$$

in $D(X, W_r)$ for each $r > 0$, and

$$Ru_{X/W*}\mathcal{E} \simeq \mathcal{W}\,\Omega^*_{X,\mathcal{E}}$$

in $D(X, W)$.

Proof. We will prove the finite-level statement; the infinite-level statement follows as in Proposition 6.2.11, using the isomorphism

$$Ru_{X/W*}\mathcal{E} \simeq R\lim_r Ru_{X/W_r*}\mathcal{E}$$

of [2, p. 7.22.2]. Choose an affine open cover $\{U_i\}$, and let $X_\bullet$ be the resulting hypercover, equipped with a smooth formal embedding $X_\bullet \hookrightarrow \mathcal{Y}_\bullet$ with Frobenius lift as in Construction

6.2.13. Viewing $\mathcal{E}|_{X_\bullet}$ as a sheaf in the topos $(X_\bullet/W_r)_{\mathrm{cris}}$ (cf. [1, pp. V, 3.4.1]), the embedding $X_\bullet \hookrightarrow \mathcal{Y}_\bullet$ induces a quasi-isomorphism

$$\mathrm{dR}(\mathcal{E}_{X_\bullet \hookrightarrow \mathcal{Y}_{\bullet,r}}) \to \mathcal{W}_r\,\Omega^*_{X_\bullet,\mathcal{E}|X_\bullet}$$

of sheaves of W_r-modules on $X_\bullet$ by Proposition 6.2.11. This becomes an isomorphism in the derived category $D(X_\bullet, W_r)$. From [2, Theorem 7.1(2)], we also have an isomorphism

$$Ru_{X_\bullet/W_r*}\mathcal{E}|_{X_\bullet} \simeq \mathrm{dR}(\mathcal{E}_{X_\bullet \hookrightarrow \mathcal{Y}_{\bullet,r}})$$

in $D(X_\bullet, W_r)$; composing this with the above yields an isomorphism

$$Ru_{X_\bullet/W_r*}\mathcal{E}|_{X_\bullet} \simeq \mathcal{W}_r\,\Omega^*_{X_\bullet,\mathcal{E}|X_\bullet} \tag{6.2.15.1}$$

in $D(X_\bullet, W_r)$. Now let π_{zar} and π_{cris} denote the canonical morphisms of topoi

$$\begin{aligned} \pi_{\mathrm{zar}} &: X_{\bullet,\mathrm{zar}} \to X_{\mathrm{zar}} \\ \pi_{\mathrm{cris}} &: (X_\bullet/W_r)_{\mathrm{cris}} \to (X/W_r)_{\mathrm{cris}}, \end{aligned}$$

whose pullback functors are the usual pullbacks of sheaves. Combining (6.2.15.1) with the theorem of cohomological descent ([1, V, Théorème 3.4.8]), we obtain isomorphisms

$$\begin{aligned} Ru_{X/W_r*}\mathcal{E} &\simeq Ru_{X/W_r*}R\pi_{\mathrm{cris}\,*}(\pi^*_{\mathrm{cris}}\mathcal{E}) \\ &\simeq R\pi_{\mathrm{zar}\,*}Ru_{X_\bullet/W_r*}(\mathcal{E}|_{X_\bullet}) \\ &\simeq R\pi_{\mathrm{zar}\,*}\,\mathcal{W}_r\,\Omega^*_{X_\bullet,\mathcal{E}|X_\bullet} \\ &\simeq R\pi_{\mathrm{zar}\,*}\pi^*_{\mathrm{zar}}\,\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \\ &\simeq \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \end{aligned}$$

in $D(X_{\mathrm{zar}}, W_r)$.

We claim that this isomorphism does not depend on the choice of the open cover $\{U_i\}$ and the embeddings with Frobenius $(U_i \hookrightarrow \mathcal{Y}_i/W, \phi_{\mathcal{Y}_i})$. Indeed, if we are given two choices $\{U_i \hookrightarrow \mathcal{Y}_i/W, \phi_{\mathcal{Y}_i}\}$ and $\{U'_j \hookrightarrow \mathcal{Y}'_j/W, \phi_{\mathcal{Y}'_j}\}$ of such data with one factoring through the other, the functorialities of Proposition 6.2.11 and [2, Theorem 7.1(2)] ensure that the resulting isomorphisms agree. In general, given only $(U_i \hookrightarrow \mathcal{Y}_i/W, \phi_{\mathcal{Y}_i})$ and $\{U'_j \hookrightarrow \mathcal{Y}'_j/W, \phi_{\mathcal{Y}'_j}\}$, we may reduce to the case above by forming a common refinement of the open covers and taking the product of the embeddings and Frobenius lifts. □

Corollary 6.2.16. *Under the same hypotheses as Theorem 6.2.15, there are canonical isomorphisms*

$$\mathbb{H}^i(X_{\mathrm{zar}}, \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}) \simeq H^i((X/W_r)_{\mathrm{cris}}, \mathcal{E})$$

for each $r > 0$, and

$$\mathbb{H}^i(X_{\mathrm{zar}}, \mathcal{W}\,\Omega^*_{X,\mathcal{E}}) \simeq H^i((X/W)_{\mathrm{cris}}, \mathcal{E}).$$

Proof. Apply $R^i\Gamma(X_{\mathrm{zar}}, -)$ to each part of Theorem 6.2.15. $\square$

Remark 6.2.17. The isomorphisms of Theorem 6.2.15 enjoy various compatibilities. We will state them at the level of $Ru_{X/W_r*}\mathcal{E}$; of course, all of the compatibilities readily pass to the limit and to cohomology.

Proposition 6.2.18. *Fix r, and let*

$$\tau_{X,\mathcal{E}} : Ru_{X/W_r*}\mathcal{E} \simeq \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$$

denote the isomorphism of Theorem 6.2.15 in $D(X, W_r)$. Then:

1. *The isomorphism*

$$\tau_{X,\mathcal{O}} : Ru_{X/W_r*}\mathcal{O}_{X/S} \simeq \mathcal{W}_r\,\Omega^*_{X,\mathcal{O}_{X/S}} = \mathcal{W}_r\,\Omega^*_X$$

   is compatible with multiplicative structures.

2. *$\tau_{X,\mathcal{E}}$ is compatible with the multiplicative structures of the two sides over $Ru_{X/W_r*}\mathcal{O}_{X/W}$ resp. $\mathcal{W}_r\,\Omega^*_X$ (via the isomorphism $\tau_{X,\mathcal{O}}$).*

3. *$\tau_{X,\mathcal{E}}$ is functorial in the F-crystal $\mathcal{E}$.*

4. *$\tau_{X,\mathcal{E}}$ is functorial in X in the sense that given a morphism $f : X' \to X$ over* $\operatorname{Spec} k' \to \operatorname{Spec} k$*, the square*

$$\begin{CD} Ru_{X/W_r*}\mathcal{E} @>{\sim}>{\tau_{X,\mathcal{E}}}> \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \\ @VVV @VVV \\ Rf_*Ru_{X'/W_r*}(f^*_{\mathrm{cris}}\mathcal{E}) @>{\sim}>{Rf_*(\tau_{X',f^*_{\mathrm{cris}}\mathcal{E}})}> Rf_*\,\mathcal{W}_r\,\Omega^*_{X',f^*_{\mathrm{cris}}\mathcal{E}} \end{CD}$$

   commutes, where the left vertical arrow is the functoriality map of [12, 0, 3.2.5(a)], and the right one is the derived version of the morphism defined on affines by Remark 4.3.10.

5. *The diagram*

$$\begin{CD} Ru_{X/W_r*}\mathcal{E} @>{\sim}>{\tau_{X,\mathcal{E}}}> \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \\ @VVV @. \\ R(F_X)_*Ru_{X/W_r*}(F_X)^*_{\mathrm{cris}}\mathcal{E} @. @VV{\mathcal{W}_r(\alpha_F)}V \\ @V{\phi_{\mathcal{E}}}V{\sim}V @. \\ R(F_X)_*Ru_{X/W_r*}\mathcal{E} @>{\sim}>{R(F_X)_*\tau_{X,\mathcal{E}}}> R(F_X)_*\,\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \end{CD}$$

   commutes, where the upper left vertical morphism is the functoriality map of [12, 0, 3.2.5(a)] and α_F is the map defined by p^iF in degree i.

Proof. The first three compatibilities follow from stepping through all of the various quasi-isomorphisms we composed, for any choice of open cover and embeddings with Frobenius. Statement (4) follows from choosing affine open covers and embeddings $\{(U_i \hookrightarrow \mathcal{Y}_i/W, \phi_{\mathcal{Y}_i})\}$ of X and $\{(U'_j \hookrightarrow \mathcal{Y}'_j/W', \phi_{\mathcal{Y}'_j})\}$ of X' as in 6.2.10, and then tracing through the proof of Theorem 6.2.15 and using the functorialities of Proposition 6.2.11 and [2, Theorem 7.1(2)].

As for (5), by applying part (4) to the absolute Frobenius morphism $F_X : X \to X$ (over F_k) and part (3) to $\phi_{\mathcal{E}}$, we see that the diagram

$$\begin{tikzcd}
Ru_{X/W_r*}\mathcal{E} \arrow[r, "\sim", "\tau_{X,\mathcal{E}}"'] \arrow[d] & \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \arrow[d] \\
R(F_X)_*Ru_{X/W_r*}(F_X)^*_{\mathrm{cris}}\mathcal{E} \arrow[r, "\sim", "R(F_X)_*\tau_{X,(F_X)^*_{\mathrm{cris}}\mathcal{E}}"'] \arrow[d, "\phi_{\mathcal{E}}"', "\sim"] & R(F_X)_*\,\mathcal{W}_r\,\Omega^*_{X,(F_X)^*_{\mathrm{cris}}\mathcal{E}} \arrow[d] \\
R(F_X)_*Ru_{X/W_r*}\mathcal{E} \arrow[r, "\sim", "R(F_X)_*\tau_{X,\mathcal{E}}"'] & R(F_X)_*\,\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}
\end{tikzcd}$$

commutes, where the right-hand vertical arrows are induced by applying Remark 4.3.10 to F_X and $\phi_{\mathcal{E}}$ respectively. On affine opens $\operatorname{Spec} R \subset X$, the composition of these two morphisms is given by $\mathcal{W}_r$ of the de Rham-Witt module map

$$\mathcal{W}\,\Omega^*_{R,\mathcal{E}} \to (F_{\operatorname{Spec} R})_*\phi^*_{\mathcal{E}}\,\mathcal{W}\,\Omega^*_{R,\mathcal{E}},$$

which equals α_F by Proposition 4.3.13. $\square$

6.3 Comparison to the classical de Rham-Witt complex with coefficients

6.3.1. Fix, as usual, a k-algebra R and a unit-root F-crystal $\mathcal{E}$ on $\operatorname{Spec} R$. Let

$$W\Omega^*_R = \lim_{\leftarrow r} W_r\Omega^*_R$$

denote the classical de Rham-Witt complex of [12, I, Définition 1.4], and

$$W\Omega^*_{R,\mathcal{E}} = \lim_{\leftarrow r} W_r\Omega^*_{R,\mathcal{E}}$$

the classical de Rham-Witt complex with coefficients, where

$$W_r\Omega^*_{R,\mathcal{E}} = \mathrm{dR}(\mathcal{E}(W_r(R),\gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} W_r\Omega^*_R$$

([8, Définition 1.1.7]). Recall from Theorem 1.2.8 that Bhatt-Lurie-Mathew constructs maps

$$\begin{aligned}
\zeta_r &: W_r\Omega^*_R \to \mathcal{W}_r\,\Omega^*_R \text{ for each } r, \text{ and} \\
\zeta &: W\Omega^*_R \to \mathcal{W}\,\Omega^*_R,
\end{aligned}$$

for any R, which are isomorphisms if R is regular Noetherian. Our goal in this section is to prove an analogous result in the case of unit-root coefficients. Namely, given a unit-root F-crystal $\mathcal{E} = (\mathcal{E}, \phi_{\mathcal{E}})$ on $\mathrm{Cris}(\operatorname{Spec} R/W(k))$, we will construct maps

$$\begin{aligned} \zeta_{r,\mathcal{E}} &: W_r\Omega^*_{R,\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \text{ for each } r, \text{ and} \\ \zeta_{\mathcal{E}} &: W\Omega^*_{R,\mathcal{E}} \to \mathcal{W}\,\Omega^*_{R,\mathcal{E}}, \end{aligned}$$

and we will show that these maps are isomorphisms if R is a smooth k-algebra.

Remark 6.3.2. In view of Étesse's comparison ([8, II, Théorème 2.1.1]) between $Ru_{X/W_r*}\mathcal{E}$ and the classical de Rham-Witt complex $W_r\Omega^*_{X,\mathcal{E}}$, this gives an alternative proof of Theorem 6.2.15.

Construction 6.3.3. Given a k-algebra R and a unit-root F-crystal $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\operatorname{Spec} R$, the map

$$\iota_r^* : \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}$$

of dg-$\Omega^*_{W_r(R),\gamma}$-modules induces a map

$$\zeta_{r,\mathcal{E}} : W_r\Omega^*_{R,\mathcal{E}} := \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} W_r\Omega^*_R \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}$$

of dg-$W_r\Omega^*_R$-modules, where the target is a dg-$W_r\Omega^*_R$-module by Lemma 3.6.2 and the comparison map ζ_r. These maps are compatible as r varies, so passing to the limit yields a map

$$\zeta_{\mathcal{E}} : W\Omega^*_{R,\mathcal{E}} \to \mathcal{W}\,\Omega^*_{R,\mathcal{E}}$$

of dg-$W\Omega^*_R$-modules.

Lemma 6.3.4. *Suppose we have an étale map $f : \operatorname{Spec} S \to \operatorname{Spec} R$ and a unit-root F-crystal $\mathcal{E}$ on $\mathrm{Cris}(\operatorname{Spec} R/W)$. Then the map*

$$\zeta_{r,f^*_{\mathrm{cris}}\mathcal{E}} : W_r\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$$

is the base change of

$$\zeta_{r,\mathcal{E}} : W_r\Omega^*_{R,\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}$$

along $W_r(R) \to W_r(S)$, where we have identified the source and target objects using Remark 4.5.4 and Proposition 4.5.3 respectively.

Proof. We must show that the right-hand square of the diagram

$$\begin{tikzcd} \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \arrow[r, two heads] \arrow[d] & \mathrm{dR}(\mathcal{E}(W_r(R), \gamma)) \otimes_{\Omega^*_{W_r(R),\gamma}} W_r\Omega^*_R \arrow[r, "\zeta_{r,\mathcal{E}}"] \arrow[d] & \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}} \arrow[d] \\ \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S), \gamma)) \arrow[r, two heads] & \mathrm{dR}(f^*_{\mathrm{cris}}\mathcal{E}(W_r(S), \gamma)) \otimes_{\Omega^*_{W_r(S),\gamma}} W_r\Omega^*_S \arrow[r, "\zeta_{r,f^*_{\mathrm{cris}}\mathcal{E}}"] & \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}, \end{tikzcd}$$

commutes, where the vertical maps are induced by Proposition 2.7.11 and Remark 4.3.10, and are all base change maps along $W_r(R) \to W_r(S)$. In fact the right vertical map defines a morphism

$$\mathcal{W}\,\Omega^*_{R,\mathcal{E}} \to f_* \,\mathcal{W}\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$$

in $\mathrm{dRWM}_{R,\mathcal{E}}$; that is, the maps $\iota^*_{r,\mathcal{E}}$ are compatible with $\iota^*_{r,f^*_{\mathrm{cris}}\mathcal{E}}$. This is precisely the commutativity of the outer rectangle, which implies that of the square in question. □

Lemma 6.3.5. *For the trivial F-crystal $(\mathcal{O}, F)$, we have equalities of comparison maps*

$$\zeta_{r,\mathcal{O}} = \zeta_r : W_r\Omega^*_R \to \mathcal{W}_r\,\Omega^*_R \text{ and}$$
$$\zeta_{\mathcal{O}} = \zeta : W\Omega^*_R \to \mathcal{W}\,\Omega^*_R,$$

where we have identified $W\Omega^*_{R,\mathcal{O}}$ with $W\Omega^*_R$ and $\mathcal{W}\,\Omega^*_{R,\mathcal{O}}$ with $\mathcal{W}\,\Omega^*_R$, using Proposition 4.2.3 for the latter.

Proof. We will prove the finite-level statement, which of course implies the statement on limits. By the construction of $\zeta_{r,\mathcal{O}}$, it suffices to prove that the diagram

$$\begin{tikzcd} \Omega^*_{W_r(R),\gamma} \arrow[d, two heads] \arrow[dr] & \\ W_r\Omega^*_R \arrow[r, "\zeta_r"] & \mathcal{W}_r\,\Omega^*_R \end{tikzcd}$$

of differential graded algebras commutes. But this diagram commutes on the image of $W_r(R) = \Omega^0_{W_r(R),\gamma}$, which generates the PD-de Rham complex as a differential graded algebra. □

Theorem 6.3.6. *Let R be a smooth k-algebra and let $(\mathcal{E}, \phi_{\mathcal{E}})$ a unit-root F-crystal on* $\mathrm{Cris}(\mathrm{Spec}\, R/W(k))$*. Then the comparison maps $\zeta_{r,\mathcal{E}}$ and $\zeta_{\mathcal{E}}$ are isomorphisms.*

Proof. Remark 2.5.10 allows us to choose an étale cover $\mathrm{Spec}\, S_1 \to R$ and finite étale covers

$$\cdots \to \mathrm{Spec}\, S_2 \to \mathrm{Spec}\, S_1$$

such that for each n, the restriction of $(\mathcal{E}, \phi_{\mathcal{E}})$ to $\mathrm{Cris}(\mathrm{Spec}\, R/W_n)$ is trivialized on $\mathrm{Spec}\, S_n$. Let S_∞ denote the colimit of the R-algebras S_n, and let $f_n : \mathrm{Spec}\, S_n \to \mathrm{Spec}\, R$ and $f_\infty : \mathrm{Spec}\, S_\infty \to \mathrm{Spec}\, R$ denote the various covers.

For each r and n, Lemma 6.3.4 tells us that the maps

$$\zeta_{r,f^*_{n,\mathrm{cris}}\mathcal{E}} : W_r\Omega^*_{S_n,f^*_{n,\mathrm{cris}}\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{S_n,f^*_{n,\mathrm{cris}}\mathcal{E}}$$

are given by base-changing

$$\zeta_{r,\mathcal{E}} : W_r\Omega^*_{R,\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{R,\mathcal{E}}$$

along $W_r(R) \to W_r(S_n)$. Passing to colimits as $n \to \infty$ and using Proposition 4.6.4, it follows that

$$\zeta_{r,f^*_{\infty,\mathrm{cris}}\mathcal{E}} : W_r\Omega^*_{S_\infty,f^*_{\infty,\mathrm{cris}}\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{S_\infty,f^*_{\infty,\mathrm{cris}}\mathcal{E}}$$

is the base-change of $\zeta_{r,\mathcal{E}}$ along $W_r(R) \to W_r(S_\infty)$.

Since the F-crystal $f^*_{\infty,\mathrm{cris}}\mathcal{E}$ is trivial, it follows from Lemma 6.3.5 and taking finite direct sums that $\zeta_{r,f^*_{\infty,\mathrm{cris}}\mathcal{E}}$ is an isomorphism. But the pro-étale cover $\operatorname{Spec} S_\infty \to \operatorname{Spec} R$ is faithfully flat, so faithfully flat descent tells us that if

$$\zeta_{r,f^*_{\infty,\mathrm{cris}}\mathcal{E}} = \zeta_{r,\mathcal{E}} \otimes_{W_r(R)} W_r(S_\infty)$$

is an isomorphism, then so is $\zeta_{r,\mathcal{E}}$. It follows by passage to the limit that $\zeta_{\mathcal{E}}$ is an isomorphism as well. □

Corollary 6.3.7. *Let X be a k-scheme and $(\mathcal{E}, \phi_{\mathcal{E}})$ a unit-root F-crystal on* $\mathrm{Cris}(X/W)$. *Then we have compatible maps*

$$\begin{aligned} &W_r\Omega^*_{X,\mathcal{E}} \to \mathcal{W}_r\,\Omega^*_{X,\mathcal{E}} \text{ for each } r, \text{ and} \\ &W\Omega^*_{X,\mathcal{E}} \to \mathcal{W}\,\Omega^*_{X,\mathcal{E}}, \end{aligned}$$

which are isomorphisms if X/k is smooth.

Proof. The maps are defined on affines by Construction 6.3.3, and they glue by Lemma 6.3.4. If X/k is smooth, then they are isomorphisms by working affine-locally and applying Theorem 6.3.6. □

Remark 6.3.8. Since the classical de Rham-Witt complex $W\Omega^*_{X,\mathcal{E}}$ is defined without reference to a Frobenius endomorphism on $\mathcal{E}$, it follows from Corollary 6.3.7 that $\mathcal{W}\,\Omega^*_{X,\mathcal{E}}$ is determined as a complex (and even as a dg-$\mathcal{W}\,\Omega^*_X$-module) by the crystal $\mathcal{E}$ without its Frobenius endomorphism. Moreover, the finite-level comparison implies that $\mathcal{W}_r\,\Omega^*_{X,\mathcal{E}}$ is determined by the restriction of $\mathcal{E}$ to $\mathrm{Cris}(X/W_r)$. We are not aware of a direct proof of these observations, or whether they hold true for X/k not smooth.

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