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MONOIDAL STRUCTURE IN MIRROR SYMMETRY AND NONCOMMUTATIVE GEOMETRY

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MONOIDAL STRUCTURE IN MIRROR SYMMETRY AND NONCOMMUTATIVE GEOMETRY

by

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Abstract

Monoidal Structure in Mirror Symmetry and Noncommutative Geometry

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Professor Alan Weinstein, Chair

In the thesis, we initial first steps in understanding Quantum mirror symmetry and noncommutative compactification of moduli spaces of tori. To obtain a global invariant of noncommutative torus bundle, we study the monodromy of Gauss-Manin connection on periodic cyclic homology groups of Heisenberg group. Then the global monodromy map is developed, and provides a criterion to detect when a noncommutative torus bundle is dequantizable. A process to construct the dequantizing Poisson manifold is given, when the dequantization criterion is satisfied. Naively, it seems that the Morita theory for noncommutative torus bundles can be developed naturally as in Rieffel theory for rotation algebras. However, this assumption turns out to be wrong; the Morita class of a non-dequantizable noncommutative torus bundle is not a classical object in the category of C^*-algebras, and we call them C^*-stacks. Even with the extended notion of C^*-stacks, the Morita theory is still incompatible with noncommutative torus bundles with the infinite Poisson limit. There is no rotation algebra that can be used to compactify the moduli space of rotation algebras, even though we know that the "infinite Poisson rotation algebra" is strongly Morita equivalent to the classical torus. This subtlety is completely solved by a new mathematical structure hidden behind the quantization of constant Dirac structures on the tori.

We develop a new theory of quantum spaces called spatial structure to give a better understanding of quantum spaces and torus fibrations. We clarify some examples of spatial algebras and construct a monoidal structure from a given spatial structure. Using Hilsum-Skandalis maps between groupoids, we find that a groupoid presentation of a C^*- algebra implies a monoidal structure on the category of representations. We decompose the spatial product of the cyclic modules over the rotation algebras as an example, and propose a conjecture that a quantum mirror symmetry lies behind the spatial structure and the Hopfish structure in the sense of Tang and Weinstein.

to Ha, Anan and my parents

Contents

Contents ii

List of Figures v

List of Tables vi

1 Introduction 1

2 Fields of noncommutative two-tori and Dequantization 8

2.1 Noncommutative two-tori 9

2.1.1 Definitions of Noncommutative Two Tori 9

2.1.2 Automorphisms of Noncommutative two-torus 12

2.2 Morita Equivalence of Rings and Algebras 15

2.3 Continuous Fields of C^*-Algebras 19

2.3.1 Definitions of Continuous Fields 19

2.3.2 Strict Deformation Quantization and Continuous Fields 23

2.4 Moduli Stack 25

2.4.1 A review of moduli stacks 26

2.4.2 Homotopically Trivial Family of Noncommutative two-tori 30

2.4.3 Relationship with Echterhoff approach 43

2.4.4 Statement of the monodromy theorem 47

2.5 DG Algebras 49

2.6 Periodic Cyclic Homology 54

2.6.1 Concrete Approach 54

2.6.2 Abstract approach 56

2.6.3 Periodic cyclic homology 59

2.7 Kontsevich formulation and formality 61

2.7.1 Semi-classical side . . . 61
2.7.2 Algebraic side . . . 62
2.7.3 Formality for the Hochschild cochains . . . 63
2.7.4 Twisting Procedure and tangent cohomology . . . 64
2.8 Character Map and Monodromy Theorem . . . 67
2.8.1 Gauss-Manin connection . . . 67
2.8.2 Main proof for the Formal Case . . . 71
2.8.3 Proof for the smooth case: . . . 75
2.9 A global invariant . . . 76
2.9.1 Monodromy map . . . 76
2.9.2 Dequantization . . . 80
2.9.3 Morita equivalence . . . 83
2.9.4 Bundle of C^*-Stacks and compactification of the moduli spaces . . . 87

3 Spatial Structures **94**
3.1 Noncommutative Algebra Motivation . . . 94
3.2 Background on C^*- functor . . . 98
3.2.1 C^*-correspondence . . . 98
3.2.2 Groupoid Correspondence . . . 98
3.2.3 C^*-Functor . . . 100
3.3 Spatial Structure . . . 102
3.3.1 Definition and examples . . . 102
3.3.2 Spatial structure from groupoids . . . 104
3.3.3 Spatial structure and Morita equivalence . . . 107
3.3.4 Monoidal category . . . 109
3.3.5 Crossed Product of Spatial Algebras . . . 113
3.4 Monoidal Structure For Rotation Algebras . . . 116
3.4.1 Case $q_1 \neq 0$ and $q_2 \neq 0$: . . . 118
3.4.2 Only one of q_1 or $q_2 = 0$: . . . 120
3.4.3 Case $q_1 = q_2 = 0$: . . . 120
3.5 Spatial structures and continuous fields . . . 123

A Appendix **127**
A.1 Groupoids and Stacks . . . 127
A.2 $KK-$category . . . 128
A.3 Torus bundle . . . 131

Bibliography **133**

Index **142**

List of Figures

2.4.1 The Rieffel Projection 36
2.4.2 Moduli space of zigzags 41
2.4.3 Constant noncommutative bundle 42
2.4.4 Zigzag of the discrete Heisenberg group. 42
2.4.5 The zigzag of roof functions 43
2.4.6 Trivial monodromy noncommutative two-tori bundles 46
2.6.1 The product of Hochschild cochains 57
2.9.1 Decomposition of monodromy 77
2.9.2 Normalized monodromy 78
2.9.3 S and T 80
2.9.4 Cut and paste 80
2.9.5 Compactification of the moduli space of noncommutative two-tori 89
2.9.6 Moduli space of quantization of the Dirac structures 92

3.3.1 Spatial product 113
3.4.1 Spatial monoidal product 123

List of Tables

1.0.1 Triangle of areas 3

2.0.1 Main approach 9
2.6.1 Noncommutative Geometry Correspondence 61

3.1.1 Stack to Category Correspondence 95
3.1.2 Monoidal structures and Quantization 97

List of Symbols

$\mathbb{T}_\theta$	Noncommutative torus
$\Phi_{\lambda,\mu}$	Action of classical torus
τ	Canonical invariant trace on noncommutative tori
$\Gamma(E)$	space of sections of the vector bundle E
$\mathbb{T}_\theta^\infty$	Smooth noncommutative torus
$\langle .,. \rangle_A$	Rieffel inner product
$ZM(A)$	center of multiplier algebra of A
$\{.,.\}$	Poisson bracket
(A, G, ϕ)	C^*-dynamical system
$A \rtimes G$	Full crossed product
$A \rtimes_r G$	Reduced crossed product
$Prim(A)$	Primitive spectrum
$*_\hbar$	Star product
$MC(L)$	Goldman-Millson groupoid
$HH^\bullet(A)$	Hochschild cohomology of algebra A
$HH_\bullet(A)$	Hochschild homology of algebra A
$CC^\alpha_{p,q,n}$	Coefficent of Cyclic module

$Ind_A^{A \rtimes G}\Delta$	Induced spatial algebra
$T^{\alpha}_{p,q}$	Cyclic module
$(A_i, \Delta^{A_i}, \epsilon^{A_i})$	Spatial algebra
(E, ρ, F)	KK-cocycle
$KK_{\bullet}(A, B))$	KK-group
$HP_{\bullet}$	periodic cyclic homology
$PC_{\bullet}(A)$	periodic cyclic complex

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Chapter 1

Introduction

It is widely believed in super-string theory that to a Calabi-Yau manifold (X, g, J, ω), we can associate two kinds of quantum field theories (QFTs), an A-model and a B-model. The A-model is based on the symplectic structure ω and the B-model is based on the complex structure of the same Calabi-Yau manifold X.

The construction of physical models from Calabi-Yau manifolds is not 1-1, as there are many known examples of different Calabi-Yau manifolds with isomorphic QFTs. In fact, for a large class of Calabi-Yau manifolds, there is a relation between them, called mirror symmetry, and those which are mirror-symmetric to each other are called mirror pairs.

Mathematically, Kontsevich in [43] made a conjecture that mirror symmetry for a pair of Calabi-Yau manifolds is an equivalence between the derived category of quasi-coherent sheaves on one Calabi-Yau manifold and the derived Fukaya category of the other, and he calls it Homological Mirror Symmetry. Roughly speaking, it is an equivalence between categories of D-branes of A-models and B-models, and can be viewed as a correspondence from complex algebraic geometry to symplectic geometry. In [84], any Calabi-Yau manifold is conjectured to form a Special Lagrangian Torus Fibration (SLAG), and the Mirror Symmetry in this case turns out to be T-duality between torus fibrations.

However, Mirror Symmetry is not just an equivalence between the triangulated categories, but is expected to be an equivalence between monoidal categories, like the way Fourier transform intertwines the normal product and the convolution product. In [85], the monoidal structure of the Donaldson category is constructed from the addition along the torus direction. Subotic

[85] proved, with direct calculations, that the products of the elements of the Donaldson categories, that is, in the case of two-dimensional tori with a complexified symplectic structure, the monoidal Fukaya category with the addition along the torus direction is equivalent to the monoidal category of coherent sheaves on elliptic curves with the tensor product. Therefore, Mirror Symmetry is conjectured to intertwine the monoidal Fukaya structure and tensor product structure of the corresponding torus fibrations.

From a physical point of view, mirror symmetry is an isomorphism between supersymetric (SUSY) sigma models with the symplectic and complex manifolds as the targets. However, a symplectic manifold is also a Poisson manifold, and the quantization of the sigma models with the Poisson target is known to have interesting relationship with the deformation quantization of Poisson manifolds as in [13].

Given a Poisson manifold (P, π), the deformation quantization process associates to it a noncommutative algebra $A_\hbar$ depending on a parameter $\hbar$ such that A_0 is isomorphic to $C^\infty(P)$ and

$$\lim_{\hbar \mapsto 0} \frac{f *_\hbar g - g *_\hbar f}{i\hbar} = \{f, g\}_\pi.$$

Depending on the topological structure and the nature of the convergence, there are many kinds of deformation quantization known up to now (among them are the formal deformation quantization of [27, 44] for formal algebras, and the strict deformation quantization by Rieffel [74] for C^*-algebras).

The rotation algebra in some sense is viewed as the deformation quantization of the symplectic torus as in [77, 74], and also viewed as the groupoid C^*- algebra [70] of the transformation groupoid $\mathbb{Z}$ acting on S^1 by translation. In [3], the group structure on S^1 is composed with the Morita functor to obtain a new mathematical monoidal structure on the category of Modules of noncommutative two-tori, called a Hopfish algebra. The authors of [3] also compute the monoidal product between cyclic modules as an example.

The same story also appears in Geometric Langlands program, where Mirror symmetry is realized as the Langlands duality via the work [41] of Kapustin and Witten. The Langlands duality can be thought of as the composition of the Mirror symmetry for Hitchin fibrations and the quantization of the Hitchin systems by D-modules, as in the lecture by Donagi and Pantev [22]. Therefore, it should be interesting to study the analog of the Langlands correspondence from the viewpoint of C^*- algebras.

We can summarize these correspondences in the following diagram 1. 1, on the next page.

$$
\begin{tikzcd}
& \text{Noncommutative Algebra} & \\
\begin{array}{c}\text{Symplectic}\\ \text{Geometry}\end{array} \arrow[ur, "\text{Quantization}"] \arrow[rr, leftrightarrow, "\text{Mirror Symmetry}"'] & & \begin{array}{c}\text{Complex}\\ \text{Geometry}\end{array} \arrow[ul, "C^{*}-\text{ Langland duality}"']
\end{tikzcd}
$$

Table 1.0.1: Triangle of areas

Noncommutative geometry of the quantum tori arises naturally when we compactify the M-theory over a torus by the work by Connes, Douglas and Schwarz [15]. The three authors show that deforming a classical torus to a quantum torus corresponds to turning on a constant 3-form background, and the $SL(2,\mathbb{Z})-$ symmetry of noncommutative tori predicted in noncommutative geometry is equivalent to T-duality of the noncommutative Yang-Mills theory.

Later, in the famous paper [80], Seiberg and Witten use the idea to study the quantization of open super-string theories in the presence of a B-field, and show that the noncommutative Yang-Mills theory is low level limit of super-string theory. Therefore, we hope that studying the Morita equivalence of the bundle of noncommutative two-tori may help us understand the mirror symmetry in super-string theory and M-theory.

Content of the thesis

In this thesis, we study the connections between the monoidal structures on the above mathematical objects, which are the preparatory steps to understanding Mirror Symmetry. The remaining steps will appear in forthcoming papers.

The material is organized as:

In chapter two, we study the Morita equivalence between noncommutative two-torus fibrations, which can be viewed as the quantization of torus bundles with respect to Poisson structures, or by turning on magnetic fields as mentioned in [53]. It is known that Morita equivalent algebras have equivalent categories of representations, so Morita equivalence between noncommutative two-torus bundles may be viewed as an isomorphism between their representation categories. We classify the bundle up to a Morita equivalence and study the bundle of periodic cyclic homology with the tool of the Gauss-Manin connection. The monodromy of some noncommutative bundles (including group C^*-algebras of discrete Heisenberg group) is also studied, and will be a condition for a noncommutative two-torus bundle to be a quantization of a symplectic torus fibration. We develop the notion of bundles of C^*-algebras with Morita bimodules playing the role of gluing functions, which is analogous to the notion of the bundles of stacks. And finally, to understand the Morita equivalence of the bundles, it is necessary to use the Dirac structure to compactify the moduli space of noncommutative two-tori, as well as to introduce a new mathematical structure to detect the difference between various kinds of noncommutative geometries.

In chapter three, we develop a new theory of quantum spaces. In Noncommutative Geometry, it has been widely believed that a noncommutative space is determined by its category of representations. Equivalently, a quantum space (in the sense of Alain Connes [14]) is "defined" as a Morita equivalence class of C^*-algebras. However, motivated in part from Mirror Symmetry as well as from classical Harmonic Analysis (different groupoids with isomorphic C^*-algebras), we propose that more information is needed to specify the quantum spaces. The information is the rigorous mathematical formulation of the notion of "points" to remember the base manifold of the groupoid. The structure is represented in terms of monoidal structures of representation categories, which we name "spatial structure". After defining the structure, we clarify some examples of spatial algebras and develop a rigorous way to construct a monoidal structure on the category of modules. Using Hilsum-Skandalis maps between groupoids, we show that groupoid presentations of C^*-algebras implies monoidal structures on the categories of representations. For the classical case, *i.e.* commutative C^*-algebras of functions on locally compact spaces, the spatial structure reduces to familiar tensor products of sheaves. Finally, we decompose the spatial product of the cyclic modules of the rotation algebras like the way Hopfish algebra entering the picture [3]. The result we obtain, after a twisting with an automorphism, coincides with the Hopfish monoidal structure obtained from the addition of the circle, which forms a new motivation for understanding Mirror Symmetry at the quantum level.

The main results on the thesis are:

- 1-Theorem 2.4.5: Classification of homotopically trivial bundles of noncommutative two-tori up to an isomorphism.
- 2-Theorem 2.4.13: Compute the monodromy of the periodic cyclic homology of the C^*-algebra of the discrete Heisenberg groups.
- 3-Theorem 2.9.1: Construction of the monodromy map for all noncommutative two-torus bundles.
- 4-Theorem 2.9.3: Giving a condition when the semi-classical bundle exists. A construction is given when the monodromy map is trivial.
- 5-Theorem 2.9.10: Classification of Morita equivalence of noncommutative two-torus bundle over contractible spaces.
- 6-Theorem 2.9.12: For the noncommutative two-torus bundles with nontrivial monodromy, the Morita equivalence degenerates to isomorphic relation.
- 7-Definition 2.9.13 Introduction of bundles of C^*-stacks, and giving a complete classification.
- 8-Theorem 2.9.16: Construction of the spatial C^*-stack of deformation quantization of Dirac structure. A special case is spatial structure of quantum two-torus.
- 9-Examples 2.9.18: Computation the spatial products for the two tori.
- 10-Theorem 2.9.15: Show that we can not naively compactify the moduli space of noncommutative two-tori by just adding a classical torus and describe the structure of the moduli space and the infinity spatial stack singularity.
- 11-Definition 3.3.1: Give definition of spatial structure.
- 12-Theorem 3.3.6: Show that any locally compact groupoids with a Haar system can be associated with a spatial structure.
- 13-Theorem 3.3.10: Show that the algebras of Morita equivalent groupoids are spatial Morita equivalent.

- 14-Theorem 3.3.14 and example 3.3.15 computation of spatial monoidal product for the spaces and Abelian groups.

- 15-Theorem 3.3.24: Show that we can use the induction functor and give a construction of the crossed product of a spatial algebra by a locally compact group.

- 16-Theorem: 2.9.14 Compute the spatial product for the noncommutative two-torus coming from cross product of $\mathbb{Z}$ on S^1.

- 17-Theorem 3.5.1: Compute the natural spatial structure arises from the C^*-algebras of the discrete Heisenberg group.

Relationship with the work of Echterhoff [25], Hannabus and Mathai [34]

The noncommutative two-torus bundle in our sense is the actual torus bundle, not in the sense of stacks. In fact, a noncommutative two-torus bundle in the sense of [34, 25] is defined to be any bundle such that the crossed product with the action of a classical torus is **Morita equivalent** to a bundle of algebras of compact operators [34]. On the other hand, the tensor product of a noncommutative two-torus bundle with a twisted bundle of algebras of compact operators is still a noncommutative two-torus bundle in their view. Therefore, their definition requires a Douady-Dixmier class in $H^3(X, \mathbb{Z})$, which controls the twisted bundles of algebras of compact operators.

For the same reasons, their objects are in fact only defined in terms of Morita equivalent classes of algebras. Then, modulo twisting by Douady-Dixmier classes, Theorem 3.2 in [34] gives an equivalent definition to our work. In fact, most of their work was known earlier by Echterhoff, Nest and Oyono-Oyono in [25]. Although they share the idea of using the strict deformation quantization of tori to deal with the bundles, Poisson torus bundles as well as the dequantization condition has never been constructed and mentioned. For this reason, we have to go through all of the Gauss-Manin connections.

Future plan

As mentioned earlier, the thesis is a part of a series of works with many steps to understand the Mirror Symmetry conjecture, but many of them have been deferred to future papers due to limitations in the space and time (not the four dimensional space-time). In fact, we also have dequantized the above ideas to develop monoidal and spatial structures of generalized complex spaces and studying the mirror symmetry between these structures. On the other hand, in another work, we also proved that the quantization of the Subotic monoidal products for the symplectic torus bundles is isomorphic to the Hopfish structure invented by Tang, Weinstein and Zhu in [90] and developed in [3]. We also built a mathematical framework for the convergence of the elliptic curves to the noncommutative two-tori. Very recently, we show that the isomorphic relation of the moduli spaces of coisotropic sub-groupoids of the groupoids integrating Dirac tori is in $1-1$ correspondence with the Morita equivalent of the quantum tori.

All of the above fit in one picture, where there is a big mathematical structure lying behind everything and a big symmetry group controls Morita equivalence of noncommutative algebras, mirror symmetry of Generalized complex monoidal spaces and they make sense even in the level of stacks. It corresponds to the stories happening in M-theory, super-string theory, and noncommutative Yang-Mill theory.

We hope that it is possible to understand these concepts better in the future.

Chapter 2

Fields of noncommutative two-tori and Dequantization

In general, an open set of a moduli space is determined by the groupoids of continuous fields of objects with some kinds of the isomorphisms between them. It is also widely believed among the experts [51, 52] that the moduli space of noncommutative two-tori will be isomorphic to the noncommutative boundary of the noncommutative compactification of the moduli space of the enhanced elliptic curves $M_{1,1}$.

Therefore, to understand the noncommutative compactification of the moduli space of elliptic curves, we need to restrict our study to two main objects: the bundle of noncommutative two-tori and the bundle of elliptic curves with the J-invariant passing through the real line.

In this chapter, we study the bundles of noncommutative two-tori up to a Morita equivalence and the relationship with their semi-classical limit. The main technique that we use is the Gauss-Manin connections on the bundles of periodic cyclic homologies.

The main theorems in this chapter are Theorem 2.4.5 on the classification of homotopically trivial bundles of noncommutative two-tori, the monodromy theorem 2.4.13, the conditions for the existence of the dequantization bundle 2.9.3,and Theorem 2.9.16 on the Dirac structure and spatial singularity.

We expect that our work may help understanding the missing T-duality in the work of Mathai and Rosenberg [54].

$$\begin{tikzcd}
\textbf{Torus bundles} \arrow[d, "Mathai"', "missing\, T-duality"] \\
\textbf{Noncommutative torus bundles} \arrow[d, "GM\, monodromy+"', "local\, Morita\, functor"] \\
\textbf{Bundles of } C^{*}-\textbf{stacks} \arrow[d, "Global"', "Morita\, functor"] \\
\textbf{Quantization of Dirac bundles to spatial algebras.}
\end{tikzcd}$$

Table 2.0.1: Main approach

2.1 Noncommutative two-tori

In this section, we recall noncommutative two-torus algebras. Most of the material is taken from [17].

2.1.1 Definitions of Noncommutative Two Tori

A noncommutative two torus algebra is defined to be the universal $C^{*}-$algebra generated by two unitary operators U and V satisfying the relationship

$$UV = e^{2\pi i\theta}VU, \tag{2.1.1}$$

where θ is a real constant. By abuse of language, we simply call it a noncommutative two-torus. In the thesis, we use the notation $\mathbb{T}_\theta$ for the noncommutative two-tori.

There are many equivalent ways to understand noncommutative two-tori. The most natural way is to use the canonical model where the torus acts on a Hilbert space. Let H be $L^2(\mathbb{R}/\mathbb{Z}, dx)$ and consider two unitary operators, the multiplication by $e^{2\pi ix}$ and the rotation by θ. Namely, let $f \in H$, which is identified with a periodic function on $\mathbb{R}$ of period 1, the actions of two unitary operators U and V are given as

$$U(f)(x) = e^{2\pi ix}.f(x)$$

and

$$Vf(x) = f(x-\theta)$$

acting on H. A simple calculation shows that Equation 2.1.1 is satisfied.

A noncommutative two-torus can also be thought of as the groupoid C^*-algebra of a translation groupoid. Namely, let $S^1 \rtimes_\lambda \mathbb{Z}$ be the groupoid of translation of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ by λ. Do notice that the notation here is quite different, because we want to keep the consistency in Theorem 3.4.2 with [3]. The source and target maps are

$$\beta(t, n) = t$$

$$\alpha(t, n) = t + n.\lambda.$$

If the two arrows (t_1, n_1) and (t_2, n_2) are composable, *i.e.* $\beta(t_1, n_1) = \alpha(t_2, n_2)$, that is $t_1 = t_2 + n_2\lambda$, then the product of the two arrows is $(t_2, n_1 + n_2)$.

Fixing the Haar measure on S^1, and the counting measure on the fibers of the groupoid, the convolution product on the $C_c(S^1 \rtimes \mathbb{Z})$ is defined by

$$a * b(t, n) \quad = \quad \sum_{k\in\mathbb{Z}} a(t + k\lambda, n - k)b(t, k) \tag{2.1.2}$$

and the involution is defined by $a(g) = \overline{a(g^{-1})}$. For the special class of functions $a_{m,j} \in C(S^1 \times \mathbb{Z})$ given by

$$a_{m,j}(t, k) = e^{imt}\delta_{jk} \tag{2.1.3}$$

the convolution 2.1.2 is equivalent to

$$a_{m_1,j_1} * a_{m_2,j_2} = e^{im_1j_2\lambda}a_{m_1+m_2,j_1+j_2} \tag{2.1.4}$$

and the involution is equivalent to

$$a^*_{mj} = e^{i\lambda mj}a_{-m.-j} \tag{2.1.5}$$

If we write $U = a_{10}$ and $V = a_{01}$, then U and V are unitary elements and we have

$$UV = a_{10}a_{01} = e^{i\lambda}a_{11} = a_{01} * a_{10} = e^{i\lambda}VU.$$

With the substitution $\lambda = 2\pi\theta$, we obtain Equation 2.1.1. Therefore, the vector space $\langle a_{m,j}\rangle_{m,j}$ is generated by U and V, and is dense in the C^*-closure of $C_c(S^1 \rtimes \mathbb{Z})$. This gives us the second description of noncommutative two-tori.

Alternatively, from the viewpoint of noncommutative dynamical systems as in [17], it is equivalent to think of the noncommutative two-tori as the crossed product of $\mathbb{Z}$ with $C(\mathbb{R}/\mathbb{Z})$. To be precise, let V be the automorphism of $C(\mathbb{R}/Z)$ generated by the translation $f(x) \mapsto f(x-\theta)$. Then there is an action of $\mathbb{Z}$ on $C(S^1)$ given by $n \mapsto V^n$. Then the noncommutative two-torus is isomorphic to the crossed product $C(\mathbb{R}/Z) \rtimes \mathbb{Z}$ corresponding to this action, which gives us the third definition.

Now let G be a discrete group. By a 2-cocycle on G, we mean a function $\sigma : G \times G \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ such that

$$\sigma(g,h)\sigma(gh,k) = \sigma(g,hk)\sigma(h,k) \text{ for all } g,h,k \in G.$$

We assume that σ is normalized, *i.e.*

$$\sigma(g,e) = \sigma(e,g) = 1 \text{ for all } g \in G.$$

We define the twisted product for $C_c(G)$ as

$$(\sum_g a_g g).(\sum_h b_h h) = \sum_{g,h} a_g b_h.\sigma(g,h)$$

and the involution as

$$a_g^* = \sigma(g,g^{-1})a_{g^{-1}}.$$

Then the completion under the enveloping C^*- norm is called the full twisted group C^*-algebra and denoted by $C^*(G,\sigma)$. For the case with $G = \mathbb{Z}^2$ and $\sigma((m,n)(p,q)) = e^{2\pi i(mq-np)\theta}$, the twisted group C^*-algebra is also isomorphic to the noncommutative two-torus, and this constitutes the fourth definition [92]. Twisted groupoid C^*-algebras can also be developed via cocycles on groupoids [70].

The fifth way to think of a noncommutative two-torus came from a remark by Weinstein about the relation between operator algebras and Poisson geometry. The rotation algebras were first recognized in [76] as the deformation quantization of the classical torus under the direction of a constant Poisson structure $\theta\partial_x \wedge \partial_y$. In fact, in [89], under a suitable change of variable and Fourier transform in the $\mathbb{Z}$ direction, the crossed product 2.1.2 is equivalent to

$$f *_\theta g(q_1,q_2) = \frac{1}{(\pi\theta)^2}\int_{\mathbb{R}^4} e^{\frac{2\pi\sqrt{-1}}{\theta}(u_2v_1-u_1v_2)}.f(q_1+u_1,q_2+u_2).g(q_1+v_1,q_2+v_2)dudv \tag{2.1.6}$$

When θ goes to zero, then $f *_\theta g$ goes to $f.g$, the usual product of functions. From this observation, Rieffel [76, 74] has developed a theory of strict deformation quantization in 2.3.2, where the commutative algebra is the algebra of functions on a Poisson manifold, and the deformed algebra is some dense subalgebra of the C^*-algebra. Normally, the subalgebra is the algebra of smooth elements defined by the action of the classical torus, and in this case it is the algebra of fast decaying series in U and V given by

$$\mathbb{T}_\theta^\infty = \left\{ \sum a_{m,j} U^m V^j \mid \sup_{(m,j)\in\mathbb{Z}^2} \mid a_{m,j} \mid (1+ \mid m \mid + \mid j \mid)^p < \infty \text{ for all } p \in \mathbb{Z} \right\} \tag{2.1.7}$$

with the same multiplication 2.1.4 and involution 2.1.5. We call the subalgebra the **smooth noncommutative two-torus**. The definition of strict deformation quantization can be found in 2.3.2. This approach builds a bridge between the differential geometry of Poisson manifolds and the noncommutative geometry of C^*-algebras. From the viewpoint of Alain Connes [14], a noncommutative two-torus algebra is the algebra of continuous functions on a virtual object called a noncommutative two-torus.

We would like to remark that there are different hidden structures corresponding to the different definitions of a noncommutative two-torus. These spatial algebra structures will be developed in chapter 3.

2.1.2 Automorphisms of Noncommutative two-torus

It is well known that two noncommutative two tori are isomorphic if $\theta = \pm\theta' + n$ for some $n \in \mathbb{Z}$. The isomorphism between $\mathbb{T}_\theta$ and $\mathbb{T}_{-\theta}$ is realized by the flip $U \leftrightarrow V$, and the isomorphism $\mathbb{T}_\theta \cong \mathbb{T}_{\theta+1}$ is built in the definition 2.1.1. However, the full classification of noncommutative 2-tori up to an isomorphism is highly nontrivial, which is proved by Rieffel [71].

To study the structure of bundles of noncommutative two-tori, it is necessary to understand the automorphism group of a single noncommutative two-torus. There are some well known classes of automorphisms of a noncommutative two-torus.

The first class consists of the canonical action of the classical torus on the quantum torus. For any constants λ, μ on the unit circle ($|\lambda| = |\mu| = 1$), the pair $(\lambda U, \mu V)$ also satisfies the condition 2.1.1. Therefore, the operators defined as

$$\Phi_{\lambda,\mu}(\sum_{m,n} a_{m,n}.U^m.V^n) = \sum_{m,n} a_{m,n}.(\lambda U)^m.(\mu V)^n \tag{2.1.8}$$

can be extended from the subalgebra of polynomials in variables (U, V) to a C^*-automorphisms of $\mathbb{T}_t$. Consequently, $\Phi_{\lambda,\mu}$ gives an action of the classical torus $\mathbb{T}^2$ on the noncommutative two-torus $\mathbb{T}_t$. It is important that this action induces a smooth structure on the C^*-algebra so that we can do differential geometry on it. Notice that the automorphisms of this kind are isotopic to the identity.

The second class of automorphisms is generated by changes of basis. There is an action of the group $SL(2,\mathbb{Z})$ on $\mathbb{T}_\theta$ given by

$$U^m V^n \mapsto U^{m_1} V^{n_1} \tag{2.1.9}$$

where $g.\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$, which plays the role of the outer automorphism groups in the classical level. In the later part of this chapter, we will see that the automorphism induces an action of $SL(2,\mathbb{Z})$ on the odd Periodic Cyclic homology group, and induces the trivial action on the even parts.

In [23], Elliot and Rørdam proved that for irrational θ, there is an exact sequence

$$1 \longrightarrow \overline{Inn(\mathbb{T}_\theta)} \longrightarrow Aut(\mathbb{T}_\theta) \longrightarrow Aut(K(\mathbb{T}_\theta)) \longrightarrow 1.$$

Here $Inn(\mathbb{T}_\theta)$ is the inner automorphism group, which is generated by the unitary elements, and $\overline{Inn(\mathbb{T}_\theta)}$ is its closure, *i.e.* the group of approximately inner automorphisms. An automorphism of $K_\bullet(\mathbb{T}_\theta)$ is required to preserve the order structure of the K_0 group, so it must be the identity on $K_0(\mathbb{T}_\theta) \cong \mathbb{Z}+\theta\mathbb{Z}$. It follows that the outer automorphism group of the noncommutative two-torus is isomorphic to

$$Aut(K(\mathbb{T}_\theta)) \cong Aut(K_1(\mathbb{T}_\theta)) \cong Aut(\mathbb{Z}^2) \cong GL(2,\mathbb{Z}).$$

It is also helpful to see that in this case, the group of translations of the classical torus is the limit of the inner automorphisms of the noncommutative two-torus. But we do not know if the exact sequence is split; only a partial lifting by the action of $SL(2,\mathbb{Z})$ on the lattice $\mathbb{Z}^2$ of unitary generators in Equation 2.1.9 is known [4]. Notice that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps T_θ into $T_{-\theta}$, and is not qualified as an automorphism of T_θ.

For rational rotation algebras, Stacey in [82] showed that

$$1 \longrightarrow Inn(\mathbb{T}_\theta) \longrightarrow Aut(\mathbb{T}_\theta) \longrightarrow Homeo(\mathbb{T}^2) \longrightarrow 1$$

for $\theta = \frac{1}{2}$ and

$$1 \longrightarrow Inn(\mathbb{T}_\theta) \longrightarrow Aut(\mathbb{T}_\theta) \longrightarrow Homeo^+(\mathbb{T}^2) \longrightarrow 1$$

for all rational $\theta \in [0,1) \setminus \frac{1}{2}$. The key difference between the case $\theta = \frac{1}{2}$ and $\theta \neq \frac{1}{2}$ is that the noncommutative two-tori $\mathbb{T}_\theta$ and $\mathbb{T}_{1-\theta}$ coincide, which means the flip operator $U \longleftrightarrow V$ is an automorphism of the torus $\mathbb{T}_{\frac{1}{2}}$. We also notice that the inner automorphism groups of rational rotation algebras are very small compared to those of the irrational rotation algebras. This is the reason why the structures of $Aut(\mathbb{T}_\theta)$ for rational θ by the work of Stacey [82]are quite different from the work by Elliot and Rørdam[23].

For the **smooth** noncommutative two-tori, 2.1.7, the automorphism group must preserve the smooth structure, *i.e.* must preserve the subalgebra of fast decaying series in U and V. Thus, $Aut(\mathbb{T}_\theta^\infty)$ is smaller, and is proved by Elliott in[24] to be $(PU(\mathbb{T}_\theta^\infty)^0) \rtimes (\mathbb{T}^2 \rtimes SL(2,\mathbb{Z}))$ for the numbers θ which are not Liouville numbers. Namely, any automorphism of the smooth torus can be decomposed into a product of a limit of inner automorphisms which are conjugation by unitary elements, a translation by the classical torus and a $SL(2,\mathbb{Z})$ base-changing. For the Liouville numbers, there are more automorphisms than those of the Elliott type as showed by Kodaka in [42].

To classify the noncommutative two-tori, it is necessary to understand integration on them. We define the **trace** of the noncommutative two-torus $\mathbb{T}_t$ as the integration over the action of the classical torus $\mathbb{T}^2$

Definition 2.1.1.

$$\tau(a) = \int_{\mathbb{T}^2} \Phi_g(a)dg \tag{2.1.10}$$

which can be computed as

$$\tau(\sum_{m,n} a_{m,n} U^m V^n) = a_{0,0} \tag{2.1.11}$$

for $a \in A_t$. Furthermore, if t is irrational, the conditions $\tau(AB) = \tau(BA)$ and $\tau(I) = 1$ turn out to determine the trace uniquely, by [17], Proposition VI. 1. 3.

2.2 Morita Equivalence of Rings and Algebras

To understand a mathematical object, it is very useful to understand how it acts on others. In a very broad class of categories, a classical mathematical object can be uniquely determined by the representation theory of suitable function algebras on it. For example, in algebraic geometry, an affine algebraic variety can be viewed as the spectrum of its coordinate ring, which is of finite type, nonnilpotent algebra. In Gelfand's theory, a locally compact Hausdorff space can be identified with the spectrum of a commutative C^*-algebra. In measure theory, a commutative W^*-algebra is nothing but the algebra of bounded Borel functions on a measure space. Thus, the theories of these kinds of algebras are equivalent to the theories of affine algebraic varieties, locally compact spaces and measure spaces, respectively.

The maximum spectrum of a commutative algebra consists of equivalence classes of irreducible representations which are one-dimensional. Hence, in the above categories, two commutative algebras having isomorphic representation theories are always isomorphic.

For the classes of noncommutative algebras most of whose representations are not one-dimensional, however, the concept of "isomorphism equivalence" is too strong. Many noncommutative algebras are obviously non-isomorphic, but still sharing the same representation theories. For example, the finite dimensional matrix algebras over a commutative ring R and the ring R itself have the same representations, but one is commutative while the other is not. Consequently, it is no use talking about isomorphisms between function algebras when we only care about their representations.

What we need here is a weaker concept which still respects the representation theory of algebras. The most natural one is the equivalence between the categories of modules.

Definition 2.2.1. [87, 57] Two unital rings A and B are called **Morita equivalent** if and only if their categories of left modules are isomorphic.

Generally, there is a natural way to establish an additive functor from the category of $A-$modules to the category of $B-$modules. Let ${}_BX_A$ be a $(B, A)-$bimodule, *i.e.* an abelian group associated with an action of A on the right, and an action of B on the left, such that they are commuting with each other. Then Morita proved in [57] that the $(B, A)-$bimodule X forms a functor $P \mapsto {}_BX_A \underset{A}{\otimes} P$ from the category of A-modules to the category of

B-module.

The other way is also true, in [87] it is shown that any right exact additive functor between categories of modules of algebras is equivalent to the functor induced by a bimodule. Hence, to define a relation of having the isomorphic representation theories between two rings, it is possible to use a pair of bimodules which are inverses to each other.

Theorem 2.2.2. *[87] Two rings A and B are Morita equivalent if there exist bimodules ${}_AP_B$ and ${}_BQ_A$ such that ${}_AP_B \otimes_B Q_A \cong_A A_A$ and ${}_BQ_A \otimes_A P_B \cong_B B_B$.*

Morita equivalence can also be defined in a broader context, such as for groupoids, C^*-algebras, W^*- algebras, Poisson manifolds, etc.. For C^*-algebras we consider the category of Hermitian modules over it, defined as follows.

Definition 2.2.3. Let A be a C^*-algebra (possibly non-unital). By a left **Hermitian module** M over A, we mean a Hilbert space M together with:

1. A left $*-$representation ϕ of A on M,
$$\phi(a.b) = \phi(a).\phi(b), \tag{2.2.1}$$
and
$$\phi(a^*) = \phi(a)^*.$$
The representation ϕ is **non-degenerate**, *i.e.* $\phi(A)M$ is dense in M.

The notion of right Hermitian modules is obtained if we replace the left representations in Equation 2.2.1 by right ones, *i.e.* those satisfying the conditions $\phi(b.a) = \phi(a).\phi(b)$ and $\phi(a^*) = \phi(a)^*$.

Naturally, we take the representation theory of a C^*-algebra A as the theory of the left A-Hermitian modules. The collection of the left Hermitian A-modules together with the intertwining operators forms a category, which will be denoted by $Mod - A$.

In the passage from a C^*-algebra A to the category $Mod - A$ of Hermitian modules over A, many properties of the algebra A have been lost. Generally, it is only possible to recover the C^*- algebra A from $Mod - A$ with some additional conditions. Rieffel in [72] has proved that the categories of the Hermitian modules only determine the enveloping W^*- algebras of the C^*-algebras. Because there are many non-isomorphic C^*-algebras admitting the same W^*-enveloping one, the equivalence relation of the representation categories is much weaker (or somehow more interesting) than the isomorphic relation.

Given a category $Mod - A$, how much information can we obtain about the algebra itself? The answer lies in the theory of Morita equivalence for C^*-algebras developed by Rieffel in a series of papers (beginning with [72] or the reference in [93], [69]). In the light of the Theorem 2.2.2, the equivalence of the categories was also realized by the tensor product with a bimodule but in this case has been modified to be compatible with the C^*-algebra setting.

The problem is to define an inner product on the tensor products $M \otimes_A P$ of Hermitian modules. The way to define the structure was found by Rieffel in terms of an A-valued inner product on the bimodule P.

Definition 2.2.4. [74] Let A be a C^*-algebra, possibly without identity. By a (right) **Hilbert C^*-module** (or Hilbert A-module) M on A, we mean a right $A-$module structure on M, together with

1. An A-valued inner product $\langle ., . \rangle_A : M \times M \to A$, such that $\langle m, n \rangle_A a = \langle m, na \rangle_A$ and linear on second variable.
2. $\langle ., . \rangle$ is positive, *i.e.* $\langle m, m \rangle \geq 0$ in A, and $\langle m, m \rangle = 0$ if and only if $m = 0$.
3. $\langle m, n \rangle_A^* = \langle n, m \rangle_A$.
4. M is complete for the norm from the $A-$valued inner product $||m||_M = ||\langle m, m \rangle_A||_A^{\frac{1}{2}}$.

If the image of $\langle ., . \rangle_A$ is dense in A under the C^*-norm, we call M a **full C^*-module**. The analogous definition is for the left C^*- modules. With this structure, the inner product on the tensor $M_A \otimes_A P$ is determined as

$$\langle m_1 \otimes p_1, m_2 \otimes p_2 \rangle_{\mathbb{C}} = \langle p_1, \langle m_1, m_2 \rangle_A p_2 \rangle_{\mathbb{C}} \tag{2.2.2}$$

and called it **Rieffel tensor product**. Therefore, it is possible to put a Hilbert-module structure on one side of a bimodule to obtain a tensor functor between the categories of Hermitian modules.

Definition 2.2.5. Let A and B be C^*-algebras. By a $B-A$ **correspondence** structure, we mean a right C^*-module M over A, which is also a left non-degenerate B-module by means of a $*-$ homomorphism of B into $End_A(M)$.

Given a $B - A$ correspondence M, we can define an additive functor between categories of left Hermitian modules $\otimes_M : A - Mod \to B - Mod$ as ${}_AP \mapsto {}_BM_A \underset{A}{\otimes} P$ with the inner product 2.2.2. Conversely, the additive functor $M \otimes_A$ also determines the bimodule M uniquely, up to an isomorphism, as proved by Rieffel in [72].

In order to talk about equivalences between the categories of Hermitian modules, we need to define the "inverted Rieffel tensor product" or "inverted Rieffel functor". In terms of bimodules, we have the following definition.

Definition 2.2.6. [72] We say that two C^*-algebras A and B are **strongly C^*-Morita equivalent** if there exists a $(B, A)-$bimodule M, equipped with an $A-$valued and a B-valued inner product structures, with respect to which M is a B-A correspondence and A-B correspondence at the same time such that

1. $\langle m, n\rangle_B p = m\langle n, p\rangle_A$ for any $m, n, p \in M$,
2. M is both a full $C^* - A-$module and $C^* - B-$module.

Sometimes people call M a Morita functor from A to B. If M is a right A-module, then the conjugate space $\overline{M}$ is a left A-module with the action

$$a\overline{m} := m.a^*.$$

Therefore, given a Morita $(B, A)-$bimodule M, it is easy to construct a Morita $(A, B)-$bimodule $\overline{M}$. We can check without difficulty that $M \otimes_A \overline{M} \cong B$ as $(B, B)-$bimodules and $\overline{M} \otimes_B M \cong A$ as $(A, A)-$bimodules.

If A and B are C^*- algebras, then in [72] the Rieffel tensor product of a $(B, A)-$ correspondence M induces a morphism $P \mapsto M \otimes_A P$ from the category $A - Mod$ to the category $B - Mod$ [72]. Rieffel also proved [72] that C^*-algebras form a category, with the objects being C^*-algebras, the morphisms being C^*- correspondence, and the composition of morphisms P from A to B and Q from B to C given by the full tensor product of bimodules ${}_CQ \otimes_B P_A$. The new inner products are defined by

$$_C\langle q_1 \otimes p_1, q_2 \otimes p_2\rangle =_C \langle q_1, q_2 ._B\langle p_1, p_2\rangle\rangle \tag{2.2.3}$$

Then Morita equivalence is exactly an equivalent relation. People call it a **Rieffel category.**

Example 2.2.7. There are many examples of strongly Morita equivalence in the literature, we refer the reader to [92, 71, 74, 75] for more details. We only list here some of them for quick reference.

1. Any C^*-algebra A is strongly Morita equivalent to $A \otimes K$, where K is the C^*-algebra of compact operators [72].

2. Let P be any locally compact space, together with two free and wandering actions of locally compact groups H, K on P. By wandering actions, we mean that for any compact set $Q \subset P$, the following sets

$$\{h \in H \mid hQ \cap Q \neq \emptyset\}$$

$$\{k \in K \mid kQ \cap Q \neq \emptyset\}$$

   is precompact in H and K. Then $C_0(P/H) \rtimes K$ and $C_0(P/K) \rtimes H$ are strongly Morita equivalent. The result is known by Rieffel [74] and Phil Green.

3. Let E be any vector bundle over a compact space X. Then the algebras $C(X)$ and $End_{C(X)}(\Gamma(E))$ are strongly Morita equivalent. [92]

4. Let L be any constant Dirac structure on a torus. Then for any transverse section M of the foliation generated by $L \cap T\mathbb{T}^n$, there exists a strict deformation quantization of the Dirac structure L, denoted by $A_{L,M}$. Furthermore, any different choices of M lead to strong Morita equivalent C^*-algebras, by Weinstein and Tang in [91].

2.3 Continuous Fields of C^*-Algebras

2.3.1 Definitions of Continuous Fields

In this section, we define the notion of the continuous fields of C^*-algebras over a parameter space, with the main references being [28] and [26].

First, we describe the natural bundle structure of any C^*- algebra. Let A be a C^*-algebra. By a primitive ideal I_π, we mean the kernel of any irreducible $*-$representation π of A [14]. Let denote by $Prim(A)$ the space of all the primitive ideals of A together with the Jacobson topology. It is known from [18] that $Prim(A)$ is a locally compact space and satisfying the T_0-axiom for separability [14].

For any $P \in Prim(A)$, denote by $a(P)$ the image of $a \in A$ under the projection from A onto A/P. Then, by the Dauns-Hofmann theorem in [64], there exists an isomorphism between $C^b(Prim(A))$ and the center of the multiplier algebra $ZM(A)$ of A as follows: if $f \in C^b(Prim(A))$, and $a \in A$, then $f.a$ is the unique element of A satisfying $f.a(P) = f(P).a(P)$ for all $P \in Prim(A)$ and any element of $ZM(A)$ is of this form. In other words, any C^*- algebra can be viewed as an upper semi-continuous bundle over its center $Prim(A)$. Therefore, we have the following definition.

Definition 2.3.1. Let X be a locally compact Hausdorff space. By a $C_0(X)-C^*-$**algebra** in the sense of [25, 28], we mean a C^*- algebra A together with an injective $*-$ homomorphism $\Phi_A : C_0(X) \to ZM(A)$, which is non-degenerate in the sense that

$$\Phi_A(C_0(X)).A := span\{\Phi_A(f)a : f \in C_0(X) \quad \text{and} \ a \in A\}$$

is dense in A. By abuse of language, we say that A is a C^*-**bundle** over X. The map from $spec(ZM(A))$ into X is called the **structure map**.

Given a $C_0(X)-$algebra, the fiber of A over a point $x \in X$ is defined as

$$A_x = A/I_x$$

where I_x an ideal of elements generated by the functions vanishing at x

$$I_x = \{\Phi(f).a \mid a \in A(X) \, \text{and} \ f \in C_0(X) \text{ such that } f(x) = 0\}$$

In this case, there exists a continuous map σ_X from $Prim(A)$ into X, or $Prim(A)$ is fibered over X if and only if σ_X is onto X. We notice that no condition on the triviality of the bundle is required.

A routine check shows that for $a \in A$, the map $x \mapsto ||a(x)||_{A_x}$ is only upper semi-continuous. Therefore any C^*-algebra A is naturally an upper semi-continuous bundle over its primitive spectrum $Prim(A)$ with identity structure maps.

If the map $x \mapsto ||a(x)||_{A_x}$ is continuous, *i.e.* σ_X is open, then A is the section algebra of a continuous field of C^*-algebras over X [65], which is defined as follows.

Definition 2.3.2. [28, 65, 64] Let X be a locally compact Hausdorff space and we associate to any $x \in X$ a C^*- algebra A_x. By a **continuous field of C^*- algebras A_x,** we mean a $*-$subalgebra A of $\coprod_{x\in X} A_x$, containing sectional elements $a(x)$, *i.e.* mapping $x \in X$ to $a(x) \in A_x$, and satisfying the following conditions

1. The subalgebra is closed under point-wise multiplication with elements of $C_0(X)$.
2. For any $x \in X$, the set $\{a(x)\}$ is dense in A_t under the C^*- norm of A_x
3. For any section $a(x)$ of the field, the function $x \to \|a(x)\|_{A_x}$ is continuous.
4. The algebra A is complete under the C^*- norm $\|a\| = sup_{x\in X}\|a(x)\|_{A_x}$.

The space X is called the **base space,** A_x the **fibers**, and A the **section algebra** of the C^*-bundle $\{A_x\}_x$.

The existence of such a field is not automatically, since the continuity between the nearby fiber algebras in terms of the functions $x \mapsto \|a(x)\|$ is an additional requirement that we put on the bundle and the nature of the definition 2.3.2 is a way to point out directly the relationship between near-by fiber algebras. Normally, to guarantee that the fibers does not jump arbitrarily, people often require these fibers C^*-algebras act continuously on a fixed Hilbert space H.

Certainly, in Definition 2.3.2 we say "a" subalgebra due to the fact that in general the relationship between fibers is not automatically satisfied. The upper semi-continuity is mostly a built-in structure of the fields, but the lower semi-continuity often comes from fixing a Hilbert space that fiber algebras acting on. Before dealing with the continuity of the fields, we recall the definition of reduced groupoid C^*-algebras for the necessity of the lower semi-continuity.

Definition 2.3.3. [17] A C^*-**dynamical system** consists of a C^*-algebra A together with a homomorphism α of a locally compact group G into $Aut(A)$. By a **co-variant representation** of (A, G, ϕ), we mean a pair (π, U) where π is a $*-$representation of A on a Hilbert space H, and $s \to U_s$ is a unitary representation of G on the same space such that

$$U_g\pi(a)U_g^* = \pi(\phi_g(a))$$

for all $a \in A$ and $g \in G$.

The algebra AG is the space of all continuous compactly supported $A-$valued functions on G with the multiplication defined by

$$a * b(t) = \int_G a(u)\phi_u(b(u^{-1}t))du.$$

and

$$f^*(t) = \Delta(t)^{-1}\phi_t(f(t^{-1})^*)$$

with Δ the modular function of the Haar measure. Then a co-variant representation of (A, G, α) is equivalent to a $*-$representation of AG [17].

Definition 2.3.4. [17] We define the **full crossed product** of $A \rtimes_\alpha G$ as the enveloping C^*-algebra of AG, *i.e.* the C^*-norm is defined by $||f|| = \underset{\sigma}{sup}||\sigma(f)||$, where σ run over all the $*-$representation of AG.

We can construct a smaller class of representations of the dynamical system as follows. Let π be a representation of A on a Hilbert space H, and let $L^2(G, H)$ be the square summable functions on G with values over H, such that

$$||f|| = (\int_G |f(g)|^2 dg)^{\frac{1}{2}}$$

is bounded. And the co-variant representation of (A, G, ϕ) on the Hilbert space $L^2(G, H)$ is defined by

$$(\tilde{\pi}(a)f)(g) = \pi(\phi_g^{-1}(a))f(g),$$

$$\Lambda_h f(g) = f(h^{-1}g).$$

Then $(\tilde{\pi}, \Lambda)$ can be extended to a $*-$representation of AG on $L^2(G, H)$. The norm $||f|| = \underset{\sigma}{sup}||\sigma(f)||$ where σ runs on the class of representation is called the **reduced crossed product** $A \rtimes_{\phi,r} G$ **[17].** Let notice that if $A = \mathbb{C}$, we obtain the notion reduced group C^*-algebra.

Summarize, given a C^*-dynamical system (A, G, ϕ), we can construct two C^*-algebras, the full and the reduced one. In many cases, the algebras coincide, for example when G is amenable [18]. Then, we can talk about the Rieffel theorem on continuity of the bundle.

Theorem 2.3.5. *[73] Let X be a locally compact space, and for each $x \in X$, let ϕ_x be an action of the locally compact group G on a fixed C^*-algebra A such that for each $g \in G$ and $a \in A$, the function $x \mapsto \phi_g^x(a)$ is norm continuous on X. For each $f \in C_c(G, A)$ let $||f||_x$ and $||f||_x^r$ denote the norms of f in $A \rtimes_{\phi^x} G$ and $A \rtimes_{\phi^x,r} G$* Then

a) For each f, the function $x \mapsto ||f||_x$ is upper semi-continuous.

b) For each f, the function $x \mapsto ||f||_x^r$ is lower semi-continuous.

c) If $A \rtimes_{\phi^x} G = A \rtimes_{\phi^x,r} G$ for all x, for example when G is amenable, then the function $x \mapsto ||f||_x = ||f||_x^r$ is continuous. Furthermore, in the case,

$A \rtimes_{\alpha_x} G \cong A \rtimes_{\alpha_x, r} G$ is a continuous field of C^*-algebras and the algebra of sections is isomorphic to $C_\infty(X, A) \rtimes_\alpha G$.

Normally, we just deal with much less general situations, where α is assumed to be a real and continuous function with at most finitely many 2.3.5extreme points over the real line or the circle. The fields we are dealing with now come from the action Φ of $\mathbb{Z}$ on $C_0(\mathbb{R} \times S^1)$, given by

$$\Phi_n.f(t,x) = f(t, x + \theta(t)) \text{ for } \theta \in C_b(\mathbb{R})$$

which is a special case for the theorem 2.3.5.

Recently, when trying do deal with strict deformation quantization from the viewpoints of groupoids, Landsman [45] has generalized the theorem to the case of continuous fields of groupoids. We will come back to the theorem in the later part of the thesis.

2.3.2 Strict Deformation Quantization and Continuous Fields

By a Poisson algebra, we mean a commutative algebra A, together with a bi-differential operator called Poisson bracket

$$\{.,.\} : A \otimes A \to A,$$

satisfying the Leibniz rule. Then a Poisson manifold (M, π) is a manifold M such that $C^\infty(M)$ is a Poisson algebra. The bracket of a Poisson structure is determined by a 2-vector fields π as

$$\{f, g\} := \pi(df, dg).$$

The bi-vector field must satisfy the condition $[\pi, \pi] = 0$, which is equivalent to the Jacobi identity.

Definition 2.3.6. A **Formal Deformation Quantization** [**10, 27**] of a commutative Poisson algebra $C^\infty(M)$ is a $*-$product in

$$C^\infty(M)[[\hbar]] \otimes C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$$

such that:

1. $f *_0 g = f.g$,

2. $f *_\hbar g - g *_\hbar f = i\hbar\{f, g\}$,

3. $*_\hbar$ is associative.

By an abuse of language, we say that the Poisson manifold M is quantized by a $*-$product.

It is important that the $*-$product is not a actual product of functions, but a product of formal power series. Many functional analysis conditions need to be satisfied to make sure that the product converges.

In the following, we recall the Rieffel theory of strict deformation quantization, *i.e.* the above under the C^*-algebra setting. It is important that in the deformation quantization process, we cannot deform the whole C^*-algebras of continuous functions $C_0(M)$, but only some subalgebra $\mathcal{A}$ of smooth functions $C_0^\infty(M)$ or Schwartz functions, *i.e.* like in the classical Moyal-Weyl theory. [78] Let $\mathcal{A}$ be a dense subalgebra of a C^*-algebra A, equipped with a Poisson bracket. By a **strict deformation quantization** of $\mathcal{A}$ in the direction of $\{,\}$, we will mean an open interval I of real numbers containing 0, together with associative products $*_\hbar$ for $\hbar \in I$, an involution $^{*_\hbar}$, and a C^*-norm $||.||_\hbar$ (for $*_\hbar$ and $^{*_\hbar}$ on A, which for $\hbar = 0$ are the original point-wise product, complex conjugation involution, and supremum norm on $\mathcal{A}$, such that:

1. The corresponding field of C^*-algebras, with continuity structure given by the elements of $\mathcal{A}$ as constant fields, is a continuous field of C^*-algebras.

2. For every $f \in A$, the function $\hbar \mapsto ||f||_\hbar$ is continuous.

3. For every $f, g \in A$

$$||(f *_\hbar g - g *_\hbar f)/i\hbar - \{f, g\}\,||_\hbar$$

converge to 0 as $\hbar$ goes to 0.

If we let $\overline{A_\hbar}$ be the C^*-completion of A for $||.||_\hbar$, then the condition (1) mean that $\left\{\bar{A}_\hbar\right\}$ forms a continuous field of C^*-algebras. We call $\mathcal{A}$ a smooth structure of the C^*-algebra A. In many case the smooth structure comes from an action of a local Lie group G.

Definition 2.3.7. Let G be a (local) Lie group, acting on a C^*-algebra A. An element $a \in A$ is a **smooth element** if the map $g \mapsto g.a$ from G to A is smooth.

Example 2.3.8. [77] (Noncommutative two-tori) Let $\mathbb{T}^n$ be an ordinary n-torus, and θ be a real skew symmetric $n \times n$ matrix. Let A be the C^*-algebra $C(\mathbb{T}^n)$, with an action of $\mathbb{T}^n$ on $C(\mathbb{T}^n)$ induced by the translation. The algebra of smooth elements is isomorphic to $C^\infty(\mathbb{T}^n)$.

The Fourier transform carries $C^\infty(\mathbb{T}^n)$ to $S(\mathbb{Z}^n)$ and takes the Poisson bracket to another bracket on $S(\mathbb{Z}^n)$

$$\{\phi, \psi\}(p) = -4\pi^2 \sum_q \phi(q)\psi(p-q)\gamma(q, p-q),$$

where γ is the skew bi-linear form on $\mathbb{Z}^n$ defined by

$$\gamma(p, q) = \sum \theta_{jk} p_j q_k.$$

For any $\hbar \in \mathbb{R}$ we define a skew-symmetric bi-character $\sigma_\hbar$ on $\mathbb{Z}^n$ by

$$\sigma_\hbar = e^{-\pi i \hbar \gamma(p,q)}$$

and define a product $*_\hbar$ on $S(\mathbb{Z}^n)$ by

$$\phi *_\hbar \psi(p) = \sum_q \phi(q)\psi(p-q)\sigma_\hbar(q, p-q). \tag{2.3.1}$$

The involution is given by $\phi^*(p) = \overline{\phi(-p)}$. The norm $||.||_\hbar$ is the operator norm on $S(\mathbb{Z}^n)$ acting on the Hilbert space $L^2(\mathbb{Z}^n)$ by the same formula $*_\hbar$. Then the Fourier transform will take these structures back to $C^\infty(\mathbb{T}^n)$, which we have a strict deformation quantization, and the fiber algebras are recognized exactly as the noncommutative two-tori [74].

2.4 Moduli Stack

In this section, we study bundles of the rotation algebras $\{A_{\theta(t)}\}$, which are isomorphic to the cross product of $\mathbb{Z}$ and $C(\mathbb{R} \times S^1)$, given by an action Φ of $\mathbb{Z}$ on $C(\mathbb{R} \times S^1)$

$$(\Phi(n).f)(t, x) = f(t, e^{2\pi n i \theta(t)} x)$$

where þ is a continuous real valued function on $\mathbb{R}$. The graph of θ is called a **J-curve.**

In this chapter, we use the notation $\{A_{\theta(t)}\}$ for a bundle of noncommutative two-tori, $A_{\theta(t)}$ for a fiber of the family, which is a single noncommutative two-torus. In the very special case where $\theta(t) = t$ for all t, we use the notation $\{A_t\}_t$ for the bundle of tori.

Our main task here is to determine when two fields of algebras $A_{\theta(t)}$ and $A_{\psi(t)}$ are Morita equivalent. In order to do so, we develop an invariant theory for the fields of noncommutative algebras.

2.4.1 A review of moduli stacks

Recall from algebraic topology that for any group G, any principal $G-$ bundle E over M can be obtained by pulling back from a special $G-$ bundle, EG over BG, called the universal bundle. The continuous map f is called the classifying map, and the space BG is called the classifying space.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\pi"] & EG \arrow[d, "\pi"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

However, the space (EG, BG, π) is usually inaccessible. That means the classifying space can be any contractible space which admits a free and effective action of G and is only determined up to a homotopy. Furthermore, the construction of BG uses an action on a contractible space of the universal free group containing G, which yields a hardly accessible space. Thus, the classifying spaces do not always exist in the category of finite dimensional manifolds, algebraic schemes, or separable locally compact spaces.

Under usual circumstances, they are not directly touchable even if we know that they exist. It is possible to define classifying spaces axiomatically and theoretically, but that does not shed much light on their structure.

In algebraic geometry, the subtleties concerning nonexistence of classifying spaces in the the category of algebraic schemes are dealt with using a clever idea of Mumford. In [62], without explicitly mentioning groupoids or moduli spaces, Mumford develops most of the important machinery for the area and computes the homology of the moduli stack $M_{1,1}$ (for a modern treatment, see[35]). Instead of viewing BG as an algebraic scheme, he thinks of it as a pseudo-functor $BG(*) = hom(*, BG)$ from the category of topological spaces into the category of groupoids. More precisely, to any topological space X, we associate $BG(X)$ the category of all $G-$ bundles over X together with $G-$ isomorphisms between them. Naturally, $BG(X)$ forms a groupoid.

$$BG \longleftrightarrow \{BG(*) : X \mapsto Hom(X, BG)\} \longleftrightarrow \{G - \text{bundle M over X}\}$$

Therefore, BG is a "functor" that associates to any topological space X a groupoid $BG(X)$. Given a map $g : Y \to X$, we can pull the groupoid $BG(X)$ back to $BG(Y)$ via a functor g^* induced by g. Because we are dealing with functors, it is necessary to lessen the equalities to isomorphisms. For example we require $(g.h)^* \cong g^* \circ h^*$ instead of $(g.h)^* = g^* \circ h^*$. This is what we mean by a pseudo functor.

What we obtain from BG is a pseudo$-$functor $X \mapsto BG(X)$ satisfying some technical properties, and we call it a **category fibered in groupoids, or a moduli stack.** The key distinction between a stack and a space is that a stack remembers the automorphisms of points at the stacky points. We can cut the bundle of objects at the fibers corresponding to the stacky points and twisting it with an automorphism of the stacky point to obtain a new bundle. Thus, through this twisting process, there are more bundles near the stacky points than the normal points. Depending on the kind of maps between topological spaces $g : Y \to X$, we obtain different topologies on the functor BG; and it is the starting point for many kinds of cohomologies. For more details, see [59].

We approach the problem using the ideas concerning moduli stacks of curves in algebraic geometry to study the analog between noncommutative two-tori and elliptic curves. The moduli space of rotation algebras up to some kind of isomorphisms can be described as a fibered category NCT in groupoids, *i.e.* a stack A.1. Over a locally compact space X, we define $NCT(X)$ to be the groupoid with the continuous bundle of rotation algebras over X as the objects, and the isomorphisms as arrows. The family of all groupoids $NCT(X)$ plays the role of open set for the moduli space of non-commutative two-tori, and the condition on pulling back maps $g : Y \to X$ induces different kinds of topologies on the moduli space. For example, if g is etale, we call it etale topology, and the associated cohomology is called etale cohomology.

Regarding "the universal bundle of objects", Mumford [35] introduced the theory of moduli spaces to simulate the non-existence of the universal bundle of elliptic curves.

We recall from [56] that a J-invariant is a function classifying the elliptic curves $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, such that for the curve $y^2 = 4x^3 - g_2 x - g_3$,

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-4},$$

$$g_3 = 140 \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-6}$$

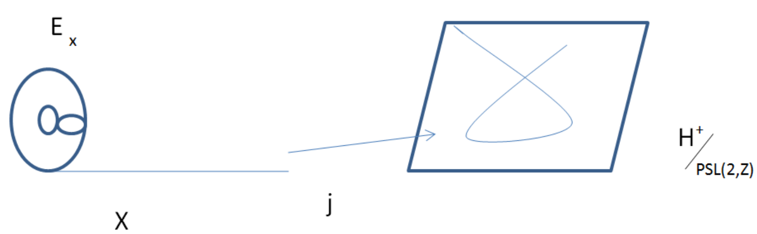
the J-invariant is

$$J(\tau) = 1728 \frac{g_2^3}{\Delta}$$

where

$$\Delta = g_2^3 - 27 g_3^2.$$

Thus, J is a function from the upper half plane to $\mathbb{C}$, which is invariant under the modular group $PSL(2, \mathbb{Z})$. Two elliptic curves E_τ and $E_{\tau'}$ are isomorphic if and only if $J(\tau) = J(\tau')$, therefore the J-invariant completely classifies the elliptic curves E_τ.



A bundle of elliptic curves E over X always induces a continuous map J from X into the upper half plane, such that $J(x)$ is the J-invariant of the elliptic curve fiber E_x over the point x.

However, there are some special points in $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$ where the modular group action is not free. This happens when the elliptic curves have nontrivial automorphisms (for example $Aut(\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ and $Aut(\mathbb{C}/\mathbb{Z} \oplus e^{\frac{\pi}{3}i}\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}$, with $J-$ invariants equal to 0 or 1024 respectively, see [56]). By twisting the elliptic curve bundles with this extra automorphism, we can construct two different twisted bundles of elliptic curves over the same non-simply connected base with the same J-images. Therefore, the twisted one cannot be obtained by pulling back from the canonical bundle on the upper half plane, or equivalently, the universal family does not exist. In another sense, the two special points are more stacky than the others.

Therefore, the moduli stack of elliptic curves could never be a topological space/scheme, but it should be an algebraic stack. The open sets for the topology of the moduli space are given by the families of elliptic curves, and the morphisms between families are required to be etale. The topology near the stacky points is described by twisted families of elliptic curves passing through them.

In most applications, there is a map π from a topological space/ scheme/ algebraic space, etc. X onto the stack $\mathcal{M}$ so that we can compare $\mathcal{M}$ and X. Therefore, a groupoid can be constructed $\begin{tikzcd} X \times_{\mathcal{M}} X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ X \end{tikzcd}$ from the stack $\mathcal{M}$, and called a presentation of stack. Because for different choices of X, the presentation groupoids are Morita equivalent, people usually say **"A stack is Morita equivalence class of groupoids".** For a more elementary exposition, we refer to [1] or to [35] for complete treatment.

The analogous question one can ask is whether any fiber bundle of noncommutative two-tori can be obtained from something like BG or $M_{1,1}$, with the push forward operator of C^*-algebras playing the role of pulling back of spaces.

$$\begin{tikzcd} A(X) & EA \arrow[l, "f^*"'] \\ C(X) \arrow[u, "\pi"] & C(BG) \arrow[l, "f^*"] \arrow[u, "\pi"'] \end{tikzcd}$$

Since a noncommutative two-torus is the limit of a family of elliptic curves $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ (see [51, 52] or our forthcoming paper for a mathematical framework of the convergence), when the J-invariant τ converges to the real line, the same story works out. Furthermore, from the viewpoint of moduli spaces, it is not natural to compactify the bundle over $\mathbb{R}$ to get one over S^1 by identifying the noncommutative two-tori at the point θ and $\theta + 1$. This is because to choose the isomorphisms between the noncommutative two-tori A_θ and $A_{\theta+1}$ is not canonical and unique, so this leads to the twisted bundles.

As a first impression, any continuous field of noncommutative two-tori can be pushed forward from the group C^*-algebra of the discrete Heisenberg group $H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z})$. From the dual point of view, the fiber bundle can be pulled back from a special bundle of noncommutative two-tori over S^1 because any fiber of A is isomorphic to only one fiber of $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}))$.

However, the construction of the universal bundle fails because of the nontrivial automorphisms of the noncommutative two-tori, both at the C^*-algebra level and at the periodic cyclic homology level.

In summary, although the universal bundle of noncommutative two-tori does not exist as a C^*-algebra, it could exist as a topological stack. The topology on the stack is induced locally by pulling back the natural topology on the Heisenberg group of noncommutative two-torus fibration $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}))$. The same story with the elliptic curves also appears here, when we push forward the functor of C^*- algebras instead of pulling back elliptic curves. Because there are many nontrivial automorphisms of noncommutative two-tori in 2.1.2, many bundles of noncommutative two-torus bundle exists and can not be pushed forward from a single universal C^*- algebra. The details will be discussed in the next subsection.

2.4.2 Homotopically Trivial Family of Noncommutative two-tori

Before proceeding with the definition of homotopically trivial families, we clarify a motivating example.

Lemma 2.4.1. *Let $H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})$ be the Heisenberg group*

$$H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \text{ where } a \text{ and } b \text{ integers, } c \in \mathbb{R} \right\}.$$

Then $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$ is isomorphic to a bundle of noncommutative two-tori fiber.

Proof. We notice that $H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}) \cong \langle U,V,W \mid VU = UVW, UW = WU, VW = WV\rangle$ is embedded canonically inside the group $H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})$. Furthermore, $H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})$ is isomorphic to the semi-direct product $(\mathbb{Z}\times\mathbb{R}) \rtimes \mathbb{Z}$, with the action of $\mathbb{Z}$ on $\mathbb{Z}\times\mathbb{R}$ given as

$$\Phi(n).(k,x) = (k, nk + x) \tag{2.4.1}$$

for $n, k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Therefore

$$\begin{aligned} C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})) &\cong C^*((\mathbb{Z}\times\mathbb{R}) \rtimes_\Phi \mathbb{Z}) \\ &\cong C^*(\mathbb{Z}\times\mathbb{R}) \rtimes_\psi \mathbb{Z} \\ &\cong C_0(S^1 \times \hat{\mathbb{R}}) \rtimes_\phi \mathbb{Z}. \end{aligned}$$

where $\hat{\mathbb{R}}$ is the unitary dual of $\mathbb{R}$, ψ is the action of $\mathbb{Z}$ on $C^*(\mathbb{Z}\times\mathbb{R})$ associated with Φ, and ϕ is the image of ψ under the Fourier-Gelfand transform of $C^*(\mathbb{Z}\times\mathbb{R})$.

To compute the action ϕ of $\mathbb{Z}$ on $C_0(S^1 \times \hat{\mathbb{R}})$, we take a typical element $\delta_k \otimes f(x) \in C^*(\mathbb{Z}\times\mathbb{R})$. The action of the element n of $\mathbb{Z}$ on the element yields $\delta_k \otimes f(x+nk)$, by Equation 2.4.1. The Fourier-Gelfand transform is

$$\begin{aligned} \mathcal{F}(\delta_k \otimes f(x-nk))(u,v) &= e^{2\pi iku}.\int_{-\infty}^{\infty} e^{2\pi ivx} f(x+nk)dx \\ &= e^{2\pi iku}.e^{-2\pi inkv}\int_{-\infty}^{\infty} e^{2\pi ivx} f(x)dx \\ &= e^{2\pi ik(u-nv)}\mathcal{F}(f)(v) \end{aligned}$$

As a result, $\phi(n) : e^{2\pi iku} \otimes \mathcal{F}(f)(v) = e^{2\pi ik(u-nv)}\mathcal{F}(f)(v)$. Here, δ_k is the Dirac delta function, $x \in \mathbb{R}$,$v \in \hat{\mathbb{R}}$. Then, the action gives

$$\phi(n) : (u,v) \mapsto (u+nv, v).$$

In summary, the C^*-algebra of the group $H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})$ is isomorphic to the cross product $C_0(S^1\times\hat{\mathbb{R}})\rtimes\mathbb{Z}$. Decomposing the system into a bundle over the dual real line $\hat{\mathbb{R}}$, the fiber algebras are isomorphic to the group C^*-algebras of the dynamical system $e^{2\pi iku} \mapsto e^{2\pi ik(u+nv)}$, *i.e.* the rotation of S^1 by v, which are just $\mathbb{T}^2_v$.

Why is the field continuous? We need to verify that the condition in Theorem 2.3.5 is satisfied for $G = \mathbb{Z}$, $A = C(S^1)$ and $X = S^1$. It is easy to verify that the the variation of the group action on a fixed element $v \mapsto f(x - nv)$ is continuous, for fixed $n \in \mathbb{Z}$ and $f \in C(S^1)$. On the other hand $C(S^1) \rtimes \mathbb{Z} \cong C(S^1) \rtimes_r \mathbb{Z}$ because $\mathbb{Z}$ is amenable. Then we apply Theorem 2.3.5 and the rest follows. QED. □

Let $H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z})$ be the Heisenberg group over $\mathbb{Z}$. Then as proved in [38], $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}))$ is isomorphic to a bundle of noncommutative two-tori over

S^1. The fiber over any point $\theta \in S^1$ of $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}))$ is isomorphic to $\mathbb{T}_\theta$. Therefore, the set of the fibers of the bundle $H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{Z})$ contains all of the isomorphism classes of noncommutative two-tori.

Motivated by the lemma 2.4.1 on the Heisenberg group, we obtain the following definition.

Definition 2.4.2. Let X be a locally compact space. By a **homotopically trivial family** of noncommutative two-tori over X, we mean a C^*-algebra $A = C_0(X) \otimes_{C_0(\mathbb{R})} C_0(S^1 \times \mathbb{R}) \rtimes_\phi \mathbb{Z}$ obtained by pushing out [67] by the following commutative diagram

$$\begin{tikzcd} A & C_0(S^1 \times \mathbb{R}) \rtimes_\phi \mathbb{Z} \arrow[l] \\ C_0(X) \arrow[u] & C_0(\mathbb{R}) \arrow[l, "\theta^*"'] \arrow[u, "i^*"'] \end{tikzcd}$$

Here, $\theta^* : C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \to C_0(X)$ is induced from a continuous and proper map $\theta : X \to \mathbb{R}$ and $i^* : C_0(\mathbb{R}) \cong C^*(\mathbb{R}) \to C_0(S^1 \times \mathbb{R}) \rtimes_\phi \mathbb{Z}$ is the canonical embedding.

By definition, there is always a built-in action of the classical torus $\mathbb{T}^2$ on any homotopically trivial family by acting on the right of the bundle of noncommutative two tori $C_0(S^1 \times \mathbb{R}) \rtimes_\phi \mathbb{Z}$.

Roughly speaking, the homotopically trivial families are those obtained by pushing forward from the "fake universal bundle" $C^*(\mathbb{H}_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$. We notice that there always exists a map from X to S^1, by composing θ with the projection from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$, called the classifying map, analogous to the J-invariant of the elliptic curves. Sometimes, by abuse of language, we call it J-invariant of noncommutative two-tori, and the image is called the J-curve.

Remark 2.4.3. We notice that, a homotopically trivial noncommutative two torus bundle always admits a built-in pair of unitary generators. Let $\{A_{\theta(t)}\}_t$ be a homotopically trivial family of rotation algebras. Then there exist two unitary sections $\{U_t\}$ and $\{V_t\}$ such that $U_t V_t = V_t U_t e^{2\pi i \theta(t)}$. As any homotopically trivial family of NCTs can be pushed forward from the universal bundle $\{A_\theta\} \cong C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$, it is enough to make clear the remark for the bundle $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$.

With the notation as in Lemma 2.4.1, we can choose the unitary sections canonically from the generators U and V of the discrete Heisenberg group.

Let I_θ be the ideal of $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$ generated by $W-\theta$, then the sections are given by

$$U \mapsto U_\theta = U + I_\theta \subseteq A_\theta \cong C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z})/I_\theta$$

$$V \mapsto V_\theta = V + I_\theta \subseteq A_\theta \cong C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z})/I_\theta.$$

Clearly, the relation $UV = VUW$ in $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$ implies that $U_\theta V_\theta = V_\theta U_\theta e^{2\pi i\theta}$.

From this remark, we imply that

Claim 2.4.4. Any homotopically trivial family of noncommutative two-tori is principal, in the sense that there exists an action of the classical torus on the noncommutative two-torus fibers. In fact, we can verify easily that, the fiber over $x \in X$ is isomorphic to $A_{\theta(x)} \cong C^*(U_\theta, V_\theta \mid U_\theta V_\theta = V_\theta U_\theta e^{2\pi i\theta}.)$ by Remark 2.4.3. Finally, the action of the classical torus is given by Equation 2.1.8.

Theorem 2.4.5 (Classification of homotopically trivial noncommutative two-torus bundle)**.** *Any isomorphism between two homotopically trivial families of noncommutative two-tori $\Theta_{\mathbb{T}^2} : A_{\theta(t)} \to A_{\tilde{\theta}(t)}$, and commuting with the action of the classical torus $\mathbb{T}^2$, can be decomposed into a homeomorphism of the base space X and an isomorphism commuting with $C_0(X)-$actions. Furthermore, the $C_0(X)-$isomorphism is given by:*

$-$The automorphism of the bundle given by $A_{\theta(x)} \mapsto A_{\theta(x)+1}$, which corresponds to the integral translation of the J-curves.

$-$The automorphism given by $A_{\theta(x)} \mapsto A_{n-\theta(x)}$ which corresponds to the reflection via the half-integral lattice $\theta(x) \to n - \theta(x)$ for $n \in \mathbb{Z}$ of the J-curves.

Proof:

Our approach is to modify Rieffel's classification of rotation algebras in [71].

Step 1: Reducing the $\mathbb{T}^2-$isomorphism to fiber-wise isomorphisms. Let $a \in \{A_\theta\}$ be an element of the bundle. Let $\tau : A_\theta \to C_0(\mathbb{R})$ be the canonical $\mathbb{T}^2-$invariant tracial state in 2.1.2. Then, τ is the expectation of the fields on the subalgebra $C_0(X)$, which projects the fiber-wise noncommutative two-tori to $\mathbb{C}$. Let i be the canonical embedding of $C_0(X)$ into A_θ. Then it is easy to see that $\tau \circ i = Id$.

$$\begin{tikzcd} A_\theta \arrow[r, "\Theta"] & A_\psi \\ C_0(X) \arrow[u, "i_\theta"] \arrow[r, "\Psi"] & C_0(X) \arrow[u, "i_\psi"'] \end{tikzcd}$$

With the same notation, we assume that $\Theta : A_{\theta(x)} \longrightarrow A_{\psi(x)}$ is an isomorphism between two C^*-algebras. Let

$$\Psi = \tau_\psi \circ \Theta \circ i_\theta.$$

Although $\tau_{\psi(x)}$ is just an expectation, not a C^*-homomorphism from $A_{\psi(x)}$ to $C_0(X)$, but the restriction of τ_β to the image of $\Theta \circ i$ (which is just the center of $A_{\psi(x)}$ generically) is a C^*- morphism. We can also check that $\Psi^{-1} = \tau_\theta \circ \Theta^{-1} \circ i_\psi$.

Thus, Ψ is a C^*-algebra automorphism of $\mathbb{C}_0(X)$, which induces a homeomorphism $\widehat{\Psi}$ on $X = Spec(C_0(X))$.

Without loss of generality, we assume that Θ preserves the fiber-algebras by considering $\Theta\circ\Psi^{-1}$ instead of Θ. Equivalently, by composing with a homeomorphism of the base, we rearrange the fibers to the right positions over the same base. It is obvious that any action of the classical torus commutes with a homeomorphism of the base.

$$\begin{aligned} T_{\lambda,\mu}(\Psi(a_{mn}(t)U_t^m V_t^n)) &= a_{mn}(\Psi^{-1}(t))(\lambda U_{\Psi^{-1}(t)})^m(\mu V_{\Psi^{-1}(t)})^n \\ &= \Psi(T_{\lambda,\mu}(a_{mn}(t)U_t^m V_t^n)). \end{aligned}$$

Therefore, the composition of Ψ^{-1} with Θ commute with the $\mathbb{T}^2$-action.

Then we only need to show that any isomorphism between two fiber-wise isomorphic noncommutative two torus bundles commuting with $\mathbb{T}^2-$action is a composition of translations and reflections of $J-$ curves.

It is easy to see that Θ is an isomorphism between two continuous fields and that Θ also induces an isomorphism between two ideals $A_\theta m_x \subset A_\theta$ and $A_\psi m_{\hat{\Psi}^{-1}(x)} \subset A_\psi$. where

$$m_x = \{f \in C_b(X) \mid f(x) = 0\}.$$

Taking the quotient, we obtain a family of isomorphisms of the fibers $\Theta_x : A_{\theta(x)} \to A_{\psi(x)}$.

In summary of step 1, we reduce the problem to the case of a $\mathbb{T}^2-$automorphism Θ of one noncommutative torus bundle and the rest is to study the structure of Θ.

Step 2: Review of Rieffel's proof. It is also shown in [71] that two noncommutative two-tori A_θ and A_ψ are isomorphic if and only if $\theta = \pm\psi$ modulo $\mathbb{Z}$. The main proof consists of four steps:

First: Rieffel projections The main idea for the proof is understanding the behavior of the positive cone $(K_0^+(A_\theta), K_0(A_\psi))$. Therefore, it is necessary to find generators for the K_0-group.

We view A_θ as the cross product $C(S^1) \rtimes_\theta \mathbb{Z}$, where U is $e^{2\pi it}$ and V is the translation by θ. For any $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta) \cap (0,1)$, Rieffel constructs a projection P with the trace $\tau(P) = \alpha$, now called a Rieffel projection. First, a projection of trace θ is constructed, then he applies the same procedure for the sub-algebra $A_{n\theta} = \langle U^n, V\rangle$ of A_θ to obtain the projection of trace $n\theta$. If P is a projection of trace α, then $I - P$ is also a projection of trace $1-\alpha$, then it is safe to construct a projection of trace θ for $0 \leq \theta \leq 1/2$.

Now we recall the Rieffel procedure to construct the projection in the form $P_\theta = M_{g(t)}V_\theta + M_{f(t)} + M_{\overline{g(t+\theta)}}.V_\theta^*$ for $f(t), g(t) \in C(S^1)$. Here, $^-$denotes complex conjugation and M_f denotes multiplication by f and t take values in $[0,1]$. The condition $P_\theta^2 = P_\theta$ is equivalent to

$$\begin{cases} g(t)g(t-\theta) & = 0 \\ g(t)(1-f(t)-f(t-\theta)) & = 0 \\ f(t)-f^2(t) & = |g(t)|^2 + |g(t-\alpha)|^2 \end{cases} \tag{2.4.2}$$

where $f, g \in C(S^1)$. Any functions f, g over S^1 satisfying 2.4.2 give a Rieffel projection $P_\theta = M_g V_\theta + M_f + M_{\overline{g(t+\theta)}}.V_\theta^*$ with $\tau(P_\theta) = \theta$.

Equations 2.4.2 can be solved explicitly. Assume that $0 < \theta < 1/2$. Pick any positive ϵ provided that $\theta + \epsilon < 1/2$. The formula he obtains is

$$f(t) = \begin{cases} \epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\ 1 & \text{for } \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta+\epsilon-t) & \text{for } \theta \leq t \leq \theta+\epsilon \\ 0 & \text{for } \theta+\epsilon \leq t \leq 1 \end{cases}$$

$$g(t) = \begin{cases} \sqrt{f(t)-f(t)^2} & \text{for } \theta \leq t \leq \theta+\epsilon \\ 0 & \text{for otherwise.} \end{cases} .$$

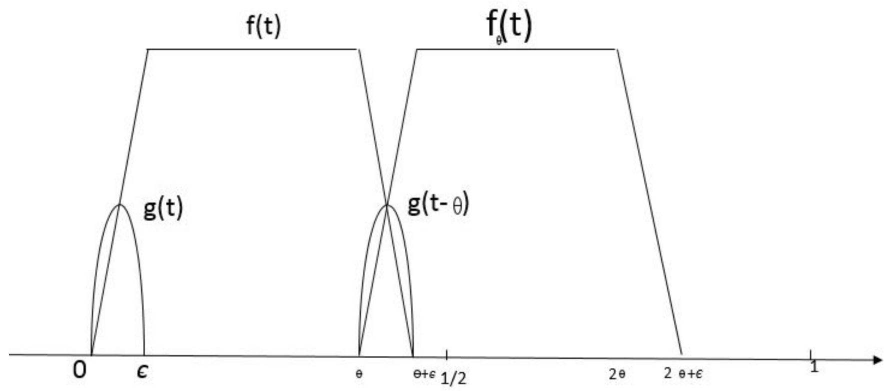


Figure 2.4.1: The Rieffel Projection

Second: Embedding into AF algebras After constructing the Rieffel projection P_θ of trace θ, it is necessary to compute the positive cone of the K_0- group. Because the trace map is positive, the positive cone always has positive trace. To compute the trace map, it is natural to move to simpler objects where the K_0-groups are fully understood.

By an $AF-$ algebra (also called Effros-Shen algebra), we mean any C^*- algebra which is isomorphic to the direct limit of finite dimensional matrix algebras. Normally, the limiting process can be described by a Bratteli diagram. Because finite dimensional matrix algebras are Morita equivalent to $\mathbb{C}$, the K_0- group of the $AF-$algebras, which are the limit of the K_0- groups of the finite dimensional ones, can be reduced to the limits of the direct sums of copies of $\mathbb{Z}$.

It is now important to construct an embedding of the irrational rotation algebra A_θ into an $AF-$algebra $\mathfrak{U}_\theta$, so that the $K-$ theory is manageable. The result can be found in the important and creative paper by Pimsner and Voiculescu [68]. Assume that

$$\theta = \lim_{n\to\infty}[a_0, a_1, ...a_n] = \lim_{n\to\infty} a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_{3} + \frac{1}{... + \frac{1}{a_n}}}}}$$

and $\frac{p_n}{q_n} = [a_0, ..a_n]$. It is known that $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$, $p_1 = a_0$, $q_1 = 1$.

Then we define $\mathfrak{U}_\theta = \overline{\cup_{n\geq 1} U_n}$ for $\mathfrak{U}_n = M_{q_n} \oplus M_{q_{n-1}}$, where the embeddings are given by $\phi_n = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$, *i.e.* a_n copies of M_{q_n} and one copy of $M_{q_{n-1}}$ are embedded into $M_{q_{n+1}}$, and one copy of $M_{q_{n-1}}$ inside $\mathfrak{U}_{n-1}$ is embedded into $\mathfrak{U}_n$. The following diagram describes the $AF-$algebra $\mathfrak{U}_\theta$.

$$\begin{tikzcd}
M_{q_n} \arrow[r, "a_n"] \arrow[dr] & M_{q_{n+1}} \arrow[r, "a_{n+1}"] \arrow[dr] & M_{q_{n+2}} \arrow[r, "a_{n+2}"] \arrow[dr] & Mat_{q_{n+3}} \cdots \\
M_{q_{n-1}} \arrow[ur] & M_{q_n} \arrow[ur] & M_{q_{n+1}} \arrow[ur] & Mat_{q_{n+2}}
\end{tikzcd}$$

We refer the reader who is not familiar with $AF-$algebra to [17]. Because $\frac{p_n}{q_n}$ approximates θ, we can approximate U and V by

$$U_n e_k^n = e_{k+1}^n, \text{and } V_n{=}e^{2\pi i \frac{p_n}{q_n}} e_k^n$$

for $\{e_k^n\}$ the standard basis in $L^2(\mathbb{Z}/\mathbb{Z}_{q_n})$. The construction of the embedding is technical and it involves finding a clever way of embedding the unitary operators U_n and V_n in a way that $\{U_n\}_n$ and $\{V_n\}_n$ are Cauchy sequences of unitary operators converging to our desired U and V.

Third: K-theory As the finite dimensional matrix algebra is Morita equivalent to a direct sum of copies of $\mathbb{C}$, the K_0- group of the $AF-$algebra is the limit of the K_0- group of the finite dimensional ones, which can be reduced to the limit of the direct sums of $\mathbb{Z}$. Therefore, the K_0^+- cone can be computed.

Furthermore, the canonical trace of the direct sums of matrix algebras can be extended uniquely to a trace σ on the $AF-$algebra. Then, it is possible to show that in our case, $K_0(\mathfrak{U}_\theta) \cong \mathbb{Z}^2$ and the trace of the positive cone $K_0^+(\mathfrak{U}_\theta)$ of the K_0- group is $\mathbb{Z} + \mathbb{Z}\theta$.

Since we can embed A_θ into $\mathfrak{U}_\theta$ by a $*-$homomorphism ρ, there is an injection of the order groups

$$\rho_* : (K_0(A_\theta), K_0^+(A_\theta)) \to (K_0(\mathfrak{U}_\theta), K_0^+(\mathfrak{U}_\theta))$$

which commutes with the canonical traces τ on A_θ and σ on $\mathfrak{U}_\theta$. Consequently, the canonical trace of $\rho_*(K_0(A_\theta))$ is contained inside $\mathbb{Z} + \theta\mathbb{Z}$.

For the <u>rational</u> θ, although we can not embed A_θ into $AF-$algebras, it is still possible to show that $A_{\frac{p}{q}}$ is isomorphic to a twisted matrix algebra over a classical torus [82, 14, 92]. And with the presentation, we can compute the trace of the K_0-group easily, and obtain the same result.

By Step 1 2.4.2, the map $\sigma \circ \rho$ from $K_0(A_\theta))$ to $\mathbb{Z} + \theta\mathbb{Z}$ is surjective, so it is an isomorphism. Hence two isomorphic rotation algebras have the same $\tau(K_0(A_\theta)) = \sigma \circ \rho(K_0(A_\theta)))$, which means

$$\mathbb{Z} + \mathbb{Z}\theta = \mathbb{Z} + \mathbb{Z}\psi.$$

If at least one of θ and ψ is irrational, then this happens if and only if $\theta = \pm\psi(mod\ \mathbb{Z})$. If θ and ψ are rational, then applying the argument in the paper [96] by Yin, we obtain the same result.

The converse part of Rieffel theorem is trivial.

Step 3: Modifying Rieffel's proof for the bundles. The same technique is also applied here, except for some small modifications.

First We modify the construction by Rieffel to continuous fields, by using the action of the classical torus on every fiber to find the image of fiber-wise Rieffel projections under the invariant trace. When t varies, the images form a curve in $X \times [0, 1]$ which coincides with the $J-$curve. We use that to classify the homotopically trivial bundles of NCTs.

Second

There is no nontrivial projection of A_θ for θ in $\mathbb{Z}$ although the K_0 groups of all the A_θ are isomorphic. The reason is that the construction of Rieffel projections in [71] collapses because of the singularity, the system of equations above has no solution for θ integers.

In the figure 2.4.1, the piecewise linear function $f(t)$ converges to the $\chi([0])$, which is clearly discontinuous. It corresponds to the fact that the classical torus is connected, so all the projections are trivial. The other way to see this is by solving Equation 2.4.2, which is quite obvious.

In fact, the generators of the K_0-group are projections in the algebra $Mat_n(A_\theta)$. Because there is no projection for $n = 1$ and $\alpha \in \mathbb{Z}$, we need to find them in $Mat_n(A_\theta)$ for $n \geqq 2$. In fact, the generators of K_0-group lie inside the bundle of 2 by 2 matrix algebras over the circle, but we do not know how to extend this projection to a continuous bundle of Rieffel projections for all nearby non-integers θ.

Third

For $\alpha = 1/2$, the formula of Rieffel projection does not work. The action of $\mathbb{Z}$ on $C(S^1)$ takes $f(e^{2\pi it})$ to $f(e^{2\pi i(t+1/2)}) = f(-e^{2\pi it})$, which is isomorphic to a twisted matrix algebra over the circle. Therefore, the existence of a projection of trace $\frac{1}{2}$ is an easy exercise in C^*- algebra, and should be wellknown in the literature. We could not find any reference, so a simple proof will be listed here.

Apply the result in page 396 in [82] to the case $p = 1, q = 2$, we obtain

$$A_{\frac{1}{2}} = \left\{ f \in C(\mathbb{R}^2, M_2) : \; f(\lambda + m, \mu + n) = Ad(W_1^n W_2^m) f(\lambda, \mu) \right\} \qquad \forall\, (\lambda, \mu) \in \mathbb{R}^2 \text{ and all } (m, n) \in \mathbb{Z}^2$$

where Ad is the adjoin operator $Ad(g)H = gHg^{-1}$, $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = SW_1S^{-1}$ for $S = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Therefore, a typical element of $A_{\frac{1}{2}}$ is

$$f(\lambda, \mu) = Ad(\begin{pmatrix} e^{2\pi i \mu} & 0 \\ 0 & e^{2\pi i \frac{\mu}{2}} \end{pmatrix})(Ad(S \begin{pmatrix} e^{2\pi i \lambda} & 0 \\ 0 & e^{2\pi i \frac{\lambda}{2}} \end{pmatrix} S^{-1}))g(\lambda, \mu) \quad (2.4.3)$$

for any periodic function $g \in C(\mathbb{R}^2, M_2)$, *i.e.* $g(\lambda, \mu + 1) = g(\lambda + 1, \mu) = g(\lambda, \mu)$. At this point, we can take $g(\lambda, \mu)$ as the constant projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which yields a projection in formula 2.4.3. Because the trace is normalized, $\tau f(\lambda, \mu) = \frac{1}{2}$.

Fourth The Rieffel projection $P_{\theta(x_0)}$ for a single value $\theta(x_0) \in (0, 1/2)$ can be extended to a projection P_θ of a bundle of noncommutative two-tori over an open set $U \subset X$ containing x_0.

From Remark 2.4.3, there exist two unitary sections U_θ and V_θ satisfying the condition $U_\theta V_\theta = V_\theta U_\theta e^{2\pi i \theta}$.

With the assumption that local extensions of unitary operators exist, we can easily construct a local extension for the Rieffel projections for all $\theta \notin \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}$ from the two unitary sections. We just take $\epsilon = (\frac{1}{2} - \theta)/2$ and just let θ vary in the formula of $P_\theta = M_{g_\theta} V_\theta + M_{f_\theta} + M_{\overline{g_\theta(t+\theta)}} . V_\theta^*$ in 2.4.2.

Fifth The construction of the imbedding of noncommutative two-tori into the AF algebras is not continuous, because the Bratelli diagram technique was based on continued fractional expansion of θ, which is not continuous when θ varies.

However, we just use the point-wise imbedding to compute the image of the K_0- group under the canonical trace and it does not depend on the imbedding into the $AF-$algebras. By the third step, we can always extend the Rieffel projections continuously to an open neighborhood, so the J-curves must be continuous (except for integral points). Even in the case $\theta \in \frac{1}{2} + \mathbb{Z}$ where the Rieffel projection can not be extended, it is still possible to show that the J-curve is continuous.

Thus, the images of the Rieffel projections (together with the unit forming the generators for the K_0-group) form continuous curves, which vary nicely and continuously until θ reaches the values in $\mathbb{Z}$.

It is also known that $A_\theta \cong A_{-\theta} \cong A_{1-\theta}$ and $A_\theta \cong A_{\theta+1}$ in Equation 2.1.2. Define the reflecting function $g(t) = \frac{1}{2} - |\theta(t) - [\theta(t)] - \frac{1}{2}|$.

$$g(t) = \begin{cases} \{\theta(t)\} \text{ if } \theta(t) \in \mathbb{Z} + [0, 1/2] \\ \{1 - \theta(t)\} \text{ if } \theta(t) \in \mathbb{Z} + [1/2, 1] \end{cases} \tag{2.4.4}$$

where $\{\theta\} = \theta - [\theta]$ denotes the fractional part. The process from the function $\theta(t)$ to $g(t)$ can be viewed as a continuous retraction from $\mathbb{R}$ to [0,1/2].

Therefore up to a homeomorphism of the base space, the bundles of non-commutative two-tori are classified up to a $\mathbb{T}^2-$isomorphism and a homeomorphism of the base, by the graph of the function $g(t) = \frac{1}{2} - |\theta(t) - [\theta(t)] - \frac{1}{2}|$.

Special case for connected open subset of the real line

In this case, we have a better description of the continuous fields in terms of zigzags.

Let $\mathcal{M}$ be the moduli space of the fields whose boundary of the critical sets and the set with the boundary value 0 or 1/2 are isolated. We want to find a description of the field which is more intuitive and manageable.

By a zigzag, we mean a piecewise linear curve on the plane, such that the projection to the real line is injective. Normally, given a graph of a function defined over a closed interval, it is possible to linearize it to obtain a new function sharing the same extreme points, but being linear piecewise. The example will be given in figure 2.4.2.

To obtain the description in terms of zigzags, we fix the extreme points and boundary points (with values 0 and $\frac{1}{2}$) of the folded J-curves, and linearize the rest. The **extreme points** are those points that the folded J-curves get the maximum and minimum. This case also contains the case that the folded functions are constant locally. By the **boundary points**, we mean the points that the folded functions get the values 0 or $\frac{1}{2}$. If the sets of boundary and extreme points have non-empty interiors, the folded functions will be linearized to constant functions.

Up to a homeomorphism of the base, for the case that the functions have non-isolated set of end points of the critical and boundary value sets, we can assume the distance between the end points are constant (normally 1) and the graph $g(t)$ is path-wise linear. On the open intervals (a, ∞) or $(-\infty, a)$, if the folded function $g(t)$ is monotone, we compactify the zigzag to obtain a finite one, by adding the limiting fibers $A_{\lim\limits_{t\to\pm\infty} g(t)}$ to the bundle. There may be constant segments corresponding to the non-empty interior sets of extreme and boundary points, and we will normalize them to be of length 1 (see example 2.4.8). Then up to a C^*- isomorphism, $\mathcal{M}$ is isomorphic to the moduli space of zigzags with values in $[0, \frac{1}{2}]$, and steps 1.

Thus, in the homeomorphic class of graphs, we can choose a representative in terms of integral zigzags, like in the zigzag picture 2.4.2. We notice that the zigzag may have the constant segments.

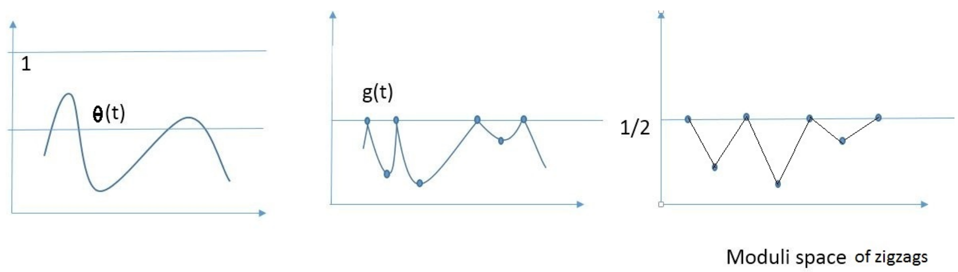


Figure 2.4.2: Moduli space of zigzags

Correspondingly, we can claim that the graphs of $g(t)$ can be classified up to homeomorphisms of the base space by its extreme values and the points $g(t)$ reaching the boundary value 0 and 1/2, which correspond 1-1 to an integral zigzag. The rest follows. QED.

Example 2.4.6. The first example is the constant noncommutative torus bundle, with all $A_{\theta(t)} \cong A_{\psi_0} \forall t \in \mathbb{R}$. The zigzag is just a closed line of length 1.

The second example is the Heisenberg bundle.

Example 2.4.7. The zigzag corresponding to the field $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$ is the following zigzag

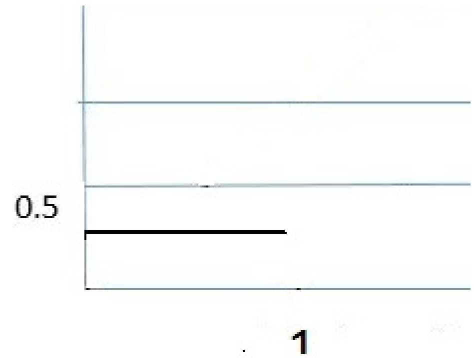


Figure 2.4.3: Constant noncommutative bundle

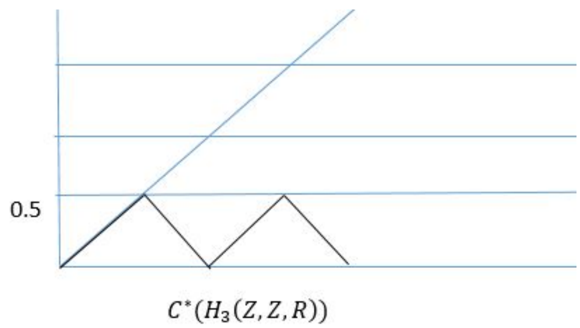


Figure 2.4.4: Zigzag of the discrete Heisenberg group.

For the bundles with non-isolated extreme points, it is enough to describe the example.

Example 2.4.8. For the bundle with $0 < \theta < \epsilon + \theta < \frac{1}{2}$

$$\alpha(t) = \begin{cases} \epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\ 1 & \text{for } \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta + \epsilon - t) & \text{for } \theta \leq t \leq \theta + \epsilon \\ 0 & \text{for } \theta + \epsilon \leq t \leq 1 \\ 0 & \text{other wise.} \end{cases}$$

Then the zigzag can be described by the following diagram, where the linear part on $[0, \epsilon]$ corresponds to two steps of the zigzag. The infinite parts are compactified into two steps of length 1, so the zigzag is finite.

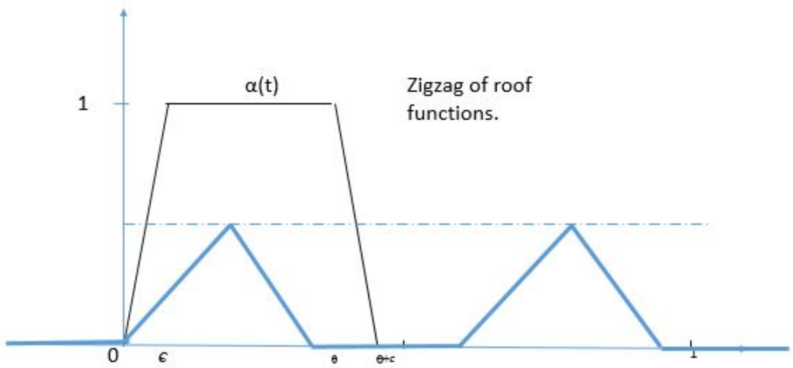


Figure 2.4.5: The zigzag of roof functions

The computation is trivial.

We will extend the result to a more general case, $i.e.$ homotopically nontrivial families, keeping in mind the example of the noncommutative two-torus bundle $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}))$ over S^1. The point is that we can not fold the whole J-curve of a non-simply connected bundle.

2.4.3 Relationship with Echterhoff approach

In [25], starting with a remark that $\mathbb{T}^2_\theta \rtimes \mathbb{T}^2 \cong K(L^2(\mathbb{R}^2/\mathbb{Z}^2))$ the algebra of compact operators on the Hilbert space $L^2(\mathbb{R}^2/\mathbb{T}^2)$, Echterhoff, Nest and Oyono-Oyono defined a noncommutative principal T-bundle over X to be a separated C^*-bundle $A(X)$ together with an action of the classical torus $\mathbb{T}^2$ such that there exists a Morita equivalence $A(X) \rtimes_\theta \mathbb{T}^2 \cong C_0(X, K(L^2(\mathbb{R}^2/\mathbb{Z}^2)))$.

On the other hand, the noncommutative principal $\mathbb{T}^2-$bundle is Morita equivalent to the crossed product $C_0(X,K) \rtimes \mathbb{Z}^2$ of the $\mathbb{Z}^2$ dual-action on the compact operator algebra over X. This bundle of actions in turn can be classified by a pair $([q : Y \to X], f)$, consisting the isomorphism class of a principal $\mathbb{T}^2-$ bundle over X and a map $f \in C_b(X, H^2(\mathbb{Z}^2, \mathbb{T}))$ as in [26]. The map q classifies classical torus bundles and f stands for the noncommutative two-torus parameter.

Hence, the work of the authors show that the principal noncommutative two-torus bundles (which is classified by the pairs (q, f) up to Morita equivalence), coincide exactly with the fields of noncommutative two-tori in our sense modulo a Morita equivalence.

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r] & Y \arrow[d, "q"] \\ f \arrow[u, "2-cocyle"] & X \end{tikzcd}$$

The authors Echterhoff, Nest and Oyono-Oyono [25] also showed that all the noncommutative two-tori are KK-equivalent and as a result, $K_0(\mathbb{T}^2_\theta) \cong K_0(C(\mathbb{T}^2))$ and $K_1(\mathbb{T}^2_\theta) \cong K_1(C(\mathbb{T}^2))$. The authors also construct a bundle of K-theories on the continuous fields and showed that the monodromy of the K-theory bundle of $C^*(H_3)$ is of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under the action of $\pi_1(S^1)$. In this case, because the K-groups are discrete, it is unnecessary to introduce any flat connection on the bundle.

However, the K-theory is not compatible with the study of the bundles, because it is very hard to find a way to extend a projection in a single algebra to its deformed one. When the J-invariant passes on integral points, a singularity happens, and the phenomenon is known as the quantum Hall Effect [53]. In addition, we don't know how to relate the K-theory of quantum algebras to their semi-classical limits, since the K-theory for Poisson manifolds by Ginzburg [31] is not very well compatible to the K-theory of their noncommutative deformation.

The key requirement in the setting of Echterhoff is the existence of an action of the classical tori on the families. Although in our situation of homotopically trivial families $C_0(X) \otimes_{C_0(\mathbb{R})} C^*(\mathbb{H}_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$, a torus action like that always occurs naturally on the fiber algebras of $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$, we should lessen the requirement to deal with the twisted bundles. If we twist the classical torus bundle by the action of $SL(2, \mathbb{Z})$, a global $\mathbb{T}^2-$symmetry may not exist. To deal with the local-global issues, the tradition approach is to glue the local principal bundles together to obtain a new kind of object.

In the rest of the chapter, we will classify these objects and clarify the relationship with Poisson geometry. Instead of using the definition of Echterhoff in [25], we proceed with another approach, which may help us to understand the moduli space.

Definition 2.4.9. Let A be a C^*-algebra fibered over a locally compact space X. A is called a ***noncommutative two-torus bundle*** if there exists a cover $\{U_i\}_{i\in I}$ of X such that $A(U_i) = C(U_i) \otimes_{C(X)} A$ are homotopically trivial family of noncommutative two-tori in definition 2.4.2, and there exists a family of gluing isomorphisms

$$\Phi_{ij} : A(U_i) \otimes_{C(X)} C(U_j) \to A(U_j) \otimes_{C(X)} C(U_i)$$

of C^*-algebras over $C(U_{ij})$ such that the gluing condition

$$\Phi_{ij}\Phi_{jk}\Phi_{ki} = Id$$

is satisfied for all i, j, k.

In this definition, it is worth remarking that the gluing factor is built in the structure of the bundle $[q : Y \to X]$, and the local classifying maps of local families $A(U_i)$ are just the local forms of the map $f \in C_b(X, H^2(\mathbb{Z}^2, \mathbb{T}))$. Note that the noncommutative two-torus bundle is different from the one by Echterhoff, Nest and Oyono-Oyono, meaning that we only require the existence of a local action of the classical tori.

It also may happen that for any two bundles $\left\{A_{\theta_1(t)}\right\}$ over U_1 and $\left\{A_{\theta_2(t)}\right\}$ over U_2, such that the restriction to $U_1 \bigcap U_2$ of two noncommutative fields are isomorphic, we can glue two bundles into one single field on $U_1 \cup U_2$ with a jumping and noncontinuous J-invariant. The issue is whether it is possible to determine a global $J-$invariant for the whole bundle. For example, we can glue $\{A_0\}_{t\in[0,1]}$ with $\{A_1\}_{t\in[\frac{1}{2},\frac{3}{2}]}$ via the isomorphisms $A_{[0,t]} \cong A_{[1,1+t]}$ and $A_{[1-t,1]} \cong A_{[3/2,1/2-t]}$ for $0 < t < 1/2$ to obtain a new continuous bundle A, but in this case, there is no global choice of the J-invariant.

Example 2.4.10. The group C^*-algebra of the discrete Heisenberg groups over $\mathbb{Z}$ is a noncommutative two-torus bundle over S^1, but not a homotopically trivial family due to Theorem 2.4.13. .

The second example is the mapping class groups.

Example 2.4.11. Let $\phi \in SL(2, \mathbb{Z})$ be an automorphism of the classical torus, coming from automorphism of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Then there exists an action of $\mathbb{Z}$ on $\mathbb{R} \times \mathbb{T}^2$ given as $\phi^n(t, x) = (t - n, \phi^n(x))$. Then the crossed product $C_0(\mathbb{R} \times \mathbb{T}^2) \rtimes_\phi \mathbb{Z}$ is a noncommutative two-torus bundle.

The next one is isomorphic fiber-wise to the bundle in example 1, but non-isomorphic globally.

Example 2.4.12. Let $\left\{A_{\theta(t)}\right\}$ be bundles of noncommutative two-tori with α varying in $[0, \frac{1}{2})$. We glue a copy of the bundles over $[0, \frac{1}{2})$ and another one on $[\frac{1}{2}, 1]$ via the isomorphism $A_\theta \cong A_{1-\theta}$ (by exchanging two generators) at the point 0 and 1 to obtain a bundle over the circle S^1. This bundle is obtained by pulling back the map $\theta(t)$ from S^1 to $\mathbb{R}$, $t \mapsto \frac{1}{2} - |\frac{1}{2} - t|$ for $0 \leq t \leq 1$. We denote it by B_t.

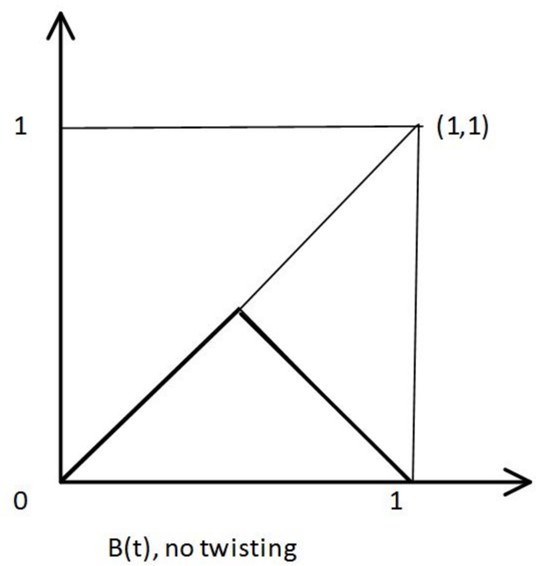


Figure 2.4.6: Trivial monodromy noncommutative two-tori bundles

After this thesis was completed, we were informed that Hannabus and Mathai in [34, 33] also constructed (**non-principal**) noncommutative torus bundles. Starting from a torus bundle $\xi : \mathbb{T}^n \to E \to X$, they constructed a torus bundle over the universal cover $\eta : \pi_1(X) \to \hat{X} \to X$ by taking the fiber product $\mathbb{T}^n \to \eta^* E \to \hat{X}$. Because $\pi_1(\hat{X})$ is trivial, then the torus bundle is a principal (see [39], also in Appendix A.3.2). With the global action of the classical torus on the fibers, the general machinery of Rieffel quantization [77, 78] is applied and yields a universal cover quantum algebra $A_{\hat{X}}$. Then noncommutative torus bundle is realized as a fixed-point subalgebra of the universal one, under the action of the fundamental group $\pi_1(X)$.

In our approach, noncommutative torus bundle is defined abstractly, like the way a manifold is constructed from the charts. The class of principle torus bundles then appear naturally as those ones with the trivial odd Gauss-Manin connection. Therefore, the two approaches complement each other.

2.4.4 Statement of the monodromy theorem

In the previous section, we have seen that there exists a noncommutative bundle without any global J-invariant. It is necessary to find an invariant explaining this phenomena.

It turns out that the existence of a global $J-$invariant is closely related to the Poisson structure of the torus bundle.

Fix a standard non-degenerated Poisson structure ω on the two dimensional torus such that the volume of the torus is 1. Then in [74], a comment made by Weinstein and Rieffel [76] is that rotation algebra $A_\hbar$ is the deformation quantization of the classical torus with respect to the standard Poisson structure $\pi = \hbar \partial_x \wedge \partial_y$. Starting from a bundle of non-degenerate Poisson tori, we can apply the general construction of Kontsevich [44] or the strict deformation quantization [77] of Rieffel to obtain quantum tori.

Because the Poisson manifold is fibered over some base X, it is very natural to see that the quantum algebra is fibered over the same base by the Kontsevich theorem [81, 44] $A/[A,A] \cong A_0/\{A_0, A_0\}_\pi$.

Therefore, the bundle of rotation algebras over the real line $A = C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$ is isomorphic to the deformation quantization bundle of a single classical symplectic torus $(\mathbb{T}^2, \pi)$, *i.e.* a family of algebras $A_\hbar$ depending on a parameter $\hbar$. However, depending on the twisting factor gluing the fibers at the point θ and $\theta + 1$ together, many interesting phenomenons may happen. There exists noncommutative two-torus bundles not coming from the quantization of any Poisson manifold, if we twist the quantization of a Poisson torus bundle by some isomorphism.

But a semi-classical counter-part still exists locally. If we localize the noncommutative two-torus bundle to an open and contractible set, it is not difficult to build a Poisson manifold corresponding to it just by pulling back from the universal Poisson bundle via the classifying map f. Certainly, the Poisson bundle that we obtain is not unique.

The main event happens when we try globalize the dequantization functor. There exists many Poisson structures on the same manifold with the isomorphic quantized algebra, and all of the Poisson structures differ an integral Poisson structure (see [74]). If the extension of the dequantization functor via two different paths leads to different Poisson fiber F, the dequantization functor may not exist. Therefore, we need to find an invariant of the quantum bundle that determine the ability to obtain the dequantized Poisson manifold.

There are some properties the invariant must satisfy. First, the invariant should be of a global nature because it controls the gluing factor of the Poisson and noncommutative two-tori. Second, it should be closely related to the invariants of the Poisson structure on the semi-classical fibers (deRham, Poisson cohomology,...) and the invariant of the $*-$product structures on the quantum bundle. Third, it should be a part of the quantization process because it relates the Poisson manifolds on the one hand and noncommutative algebras on the other hand.

The next theorem is the main example for our study and plays the role of the building block for the construction of the invariant.

Theorem 2.4.13. ***(Monodromy Theorem)*** *Let $\{A_\theta\}_\theta \cong C^*(H_3)$ be the group C^*-algebra of discrete Heisenberg group, represented as a fiber bundle of rotation algebras over S^1. Then, there exists a basis of $HP(A_\theta)$ such that the monodromy of the base under the monodromy of Gauss-Manin connection on $HP_{even}-$bundle is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

The Gauss-Manin connection is a connection on the bundle of periodic cyclic homology of a bundle of deformed algebras. For the definition, we refer to [30, 94]

Our proof is based on the computation of the Gauss-Manin connection for the periodic cyclic homology of the noncommutative fibers. At the very beginning, we had computed the monodromy map by direct computation of the cyclic chains, the calculation is technical and boring. However, we found that it is possible to overcome the complicated calculation with the star product by using the twisted version of the Shoikhet-Dolgushev morphism to map the $HP_\bullet-$group to the Poisson cohomology groups. More than simply out of convenience for the calculations, it is possible to generalize the technique to work with any noncommutative fibers, not just the two-tori. Furthermore, it builds a direct link between quantum invariants of noncommutative fibrations and Poisson invariants of Poisson manifolds.

In order to do so, in the next section, we recall necessary background of the proof, *i.e.* the periodic cyclic homology group, Kontsevich quantization functor and the character map.

2.5 DG Algebras

In this section, we review some material [44, 20, 21] about DGLAs.

Definition 2.5.1. [44] A **Differential Graded Lie Algebra** (DGLA) is a graded algebra $A^\bullet = \oplus_{k=0}^{\infty} A_k$ equipped with a differential operator d of degree 1, a Lie bracket $[.,.]$ such that

1. $[.,.] : A_k \otimes A_l \to A_{k+l}$. $\qquad d : A_k \to A_{k+1}$ for $k \geq 0$.
2. $d^2 = 0, \qquad d[\gamma_1, \gamma_2] = [d\gamma_1, \gamma_2] + (-1)^{|\gamma_1|}[\gamma_1, d\gamma_2]$,
3. $[\gamma_2, \gamma_1] = -(-1)^{|\gamma_1|.|\gamma_2|}[\gamma_1, \gamma_2]$
$[\gamma_1, [\gamma_2, \gamma_3]] + (-1)^{|\gamma_3|.(|\gamma_1|+|\gamma_2|)}[\gamma_3, [\gamma_1, \gamma_2]] + (-1)^{|\gamma_3|.(|\gamma_1|+|\gamma_2|)}[\gamma_2, [\gamma_3, \gamma_1]] = 0$.

In these formulas, $|\gamma|$ denotes the degree of the homogeneous element γ.

Definition 2.5.2. [20] A morphism between DGLAs $\mu : L \to \tilde{L}$ is a **quasi-isomorphism** if μ induces an isomorphism on the corresponding cohomology groups. We denote this by $\mu : L \tilde{\to} \tilde{L}$. Two DGLAs L and $\tilde{L}$ are **quasi-isomorphic** if they can be connected by a sequence of quasi-isomorphisms. A DGLA is called **formal** if it is quasi-isomorphic to its cohomology DGLA $H^\bullet(L)$.

Remark 2.5.3. Although the DGLA (L, d) and its cohomology DGLA $(H^\bullet(L), 0)$ have the same cohomology groups, they need not to be quasi-isomorphic. Of course, we can always map $H^\bullet(L)$ into L, *i.e.* choosing representatives of the cohomology classes via a map of vector spaces $F : H^\bullet(L) \to L$, but F is not a Lie algebra morphism. However, we can choose $F_2 : \wedge^2 H^\bullet(L) \to L$ of degree 1 such that

$$F_1([a, b]) - [F_1(a), F_1(b)] \;=\; d(F_2(a, b)).$$

But the failure of associativity of F_2 leads us to define $F_3 : \wedge^3 H^\bullet(L) \to L$ and so on. What we obtain after the process is the notion of $L_\infty-$algebras.

Definition 2.5.4. [81] An $L_\infty-$**algebra** is a $\mathbb{Z}-$ graded vector space L associated with a collection of linear maps $Q_n : \wedge^n L \to L[2-n]$, satisfying the relations

$$\sum_{i_1<..<i_p, j_1<..<j_q,\ p+q=k} \pm Q_{q+1}(Q_p(x_{i_1} \wedge x_{i_2} \wedge ...x_{i_p}) \wedge x_{j_1} \wedge .. \wedge x_{j_p}) = 0$$

for each $k \geqq 2$ and homogeneous x_s. The sign is delicate, and in this thesis we follow the Koszul rule.

Remark 2.5.5. The first relation, for $k = 2$, is $Q_1^2 = 0$, *i.e.* Q_1 is a differential operator on L.

The second relation, for $k = 3$, is that the product Q_2 is compatible with the differential Q_1.

The third relation, is that the skew-symmetric product Q_2 obeys the Jacobi identity modulo Q_3.

If $Q_n = 0$ for $n \geq 3$, we obtain the notion of 2.5.1 DGLA.

Another equivalent definition of $L_\infty-$ algebra is in terms of homological vector fields, as in [20]. Let $L = \oplus_k L^k$ be a $\mathbb{Z}-$graded vector space. On L, we define the structure of a coassociative cocommutative algebra without counit $C(L)$ cofreely cogenerated by L, with shifted parity. The vector space $C(L)$ is isomorphic to $\bigwedge^\bullet L$, where the sign convention is

$$\gamma_1 \wedge \gamma_2 \quad = \quad -(-1)^{|\gamma_1|.|\gamma_2|}\gamma_2 \wedge \gamma_1,$$

and the comultiplication

$$\Delta : C(L) \quad \to \quad C(L) \wedge C(L),$$

is defined by the formulas

$$\Delta(\gamma_1) \quad = \quad 0, \tag{2.5.1}$$

$$\Delta(\gamma_1 \wedge ... \wedge \gamma_n) \quad = \quad \sum_{k=1}^{n-1} \sum_{\epsilon \in Sh(k,n-k)} \pm \gamma_{\epsilon(1)} \wedge ... \wedge \gamma_{\epsilon(k)} \bigotimes \gamma_{\epsilon(k+1)} \wedge ... \wedge \gamma_{\epsilon(n)}$$

where $\gamma_1, ..., \gamma_n$ are homogeneous elements of L.

Definition 2.5.6. [20] A graded vector space L is an $L_\infty-$**algebra** if the coassociative cocommutative algebra without counit $C(L)$ cofreely cogenerated by L, with shifted parity, is equipped with a coderivation Q of degree 1 such that $Q^2 = 0$.

Expanding the coderivation Q to a family of maps $Q_n : \wedge^n L \to L$, it is easy to see that this definition is equivalent to Definition 2.5.4. In the same light, an $L_\infty-$morphism F between $L_\infty-$algebras is a homomorphism of the cocommutative coassociative algebras $C(L_i)$ commuting with Δ_i and Q_i, *i.e.* equivalent to a family of maps

$$F_n : \wedge^n L \to L[1-n]$$

satisfying the equations (for $n > 0$)

$$F(\gamma_1 \wedge \ldots \wedge \gamma_n) = \sum_{p \geqq 1} \sum_{k_1+k_2+\ldots+k_p=n} \sum_{\epsilon \in Sh(k_1,k_2,\ldots k_p)} \pm F_{k_1}(\gamma_{\epsilon(1)} \wedge \ldots \wedge \gamma_{\epsilon(k_1)}) \bigwedge \ldots$$
$$\ldots \bigwedge F_{k_p}(\gamma_{\epsilon(n-k_p+1)} \wedge \ldots \wedge \gamma_{\epsilon(n)}),$$

This definition boils down to the usual definition of morphism between DGLAs if the $n-$components vanishes for all $n > 2$.

Definition 2.5.7. [19, 20] Let L be an $L_\infty-$algebra. Then a graded vector space M, is endowed with a structure of an $L_\infty-$**module** over L if the cofreely cogenerated comodule $C(L) \otimes M$ over the coalgebra $C(L)$ is endowed with a 2-nilpotent coderivation ϕ of degree 1. We denote the $L_\infty-$structures by $L \xrightarrow{L_{mod}} M$.

Unfolding the definition, the comodule structure is equivalent to the coaction

$$a : C(L) \otimes M \to C(L) \otimes C(L) \otimes M$$

where

$$a(\gamma_1 \wedge \ldots \wedge \gamma_n \otimes v) = \sum_{k=1}^{n-1} \sum_{\epsilon \in Sh(k,n-k)} \pm \gamma_{\epsilon(1)} \wedge \ldots \wedge \gamma_{\epsilon(k)} \bigotimes \gamma_{\epsilon(k+1)} \wedge \ldots \wedge \gamma_{\epsilon(n)} +$$
$$+ \gamma_1 \wedge \ldots \wedge \gamma_n \otimes v$$

which satisfies the equation $(I \otimes a)a(X) = (\Delta \otimes I)a(X)$ for any $X \in C(L) \otimes M$. Then the coderivation structure ϕ can be expanded to a map from $C(L) \otimes M$ to itself, satisfying

$$\begin{aligned}
\phi(\gamma_1 \wedge ... \wedge \gamma_n \otimes v) &= \phi_n(\gamma_1, .., \gamma_n) + \\
&+ \sum_{k=1}^{n-1} \sum_{\epsilon \in Sh(k,n-k)} \pm \gamma_{\epsilon(1)} \wedge ... \wedge \gamma_{\epsilon(k)} \phi_{n-k}(\gamma_{\epsilon(k+1)}, ..., \gamma_{\epsilon(n)}, v) + \\
&+ (-1)^{k_1+k_2+..+k_n} \gamma_1 \wedge ... \wedge \gamma_n \otimes \phi_0(v) + \\
&+ \sum_{k=1}^{n} \sum_{\epsilon \in Sh(k,n-k)} \pm Q_k(\gamma_{\epsilon(1)}, .., \gamma_{\epsilon(n)}) \otimes \gamma_{\epsilon(k+1)} \wedge .. \wedge \gamma_{\epsilon(n)} \otimes v,
\end{aligned}$$

where $\gamma_i \in L^{k_i}$, $v \in M$, Q_k's are the $L_\infty-$algebra structure maps on L, and ϕ_n are arbitrary polylinear antisymmetric graded maps

$$\phi_n : \wedge^n L \otimes M \quad \rightarrow \quad M[1-n].$$

It is easy to check that $\phi^2 = 0$ is equivalent to

$$\sum_{p+q=k} \sum_{i_1<i_2<..<i_p, j_1<..<i<j_q} \pm \phi_p(x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_p} \wedge \phi_q(x_{j_1} \wedge x_{j_2} \wedge ... x_{j_q} \otimes m)) +$$
$$\sum_{p+q=k} \sum_{i_1<i_2<..<i_p, j_1<..<j_q} \pm \phi_{q+1}(Q_p(x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_p}) \wedge x_{j_1} \wedge x_{j_2} \wedge ... x_{j_q} \otimes m) = 0. \tag{2.5.2}$$

Sometimes, this formula is used to define the $L_\infty-$modules directly.

Remark 2.5.8. For $k = 0$, Equation 2.5.2 gives $\phi_0^2 = 0$, *i.e.* ϕ_0 is a differential on M. For $k = 1$, we get that $\phi_1 : L \otimes M \to M$ is a map of complexes,

$$\phi_0(\phi_1(x \otimes m)) = \phi_1(dx \otimes m) \pm \phi_1(x \otimes \phi_o m).$$

Definition 2.5.9. [20] Let L be an $L_\infty-$algebra and (M, ϕ_M), and (N, ϕ_N) be $L_\infty-$modules over L. Then a morphism f from the comodule $C(L) \otimes M$ to the comodule $C(L) \otimes N$ compatible with the coderivations ϕ_M and ϕ_N

$$f(\phi^M(X)) = \phi^N(f(X)) \quad \forall X \in C(L) \otimes M$$

is called an $L_\infty-$**morphism** between the $L_\infty-$modules (M, ϕ^M) and (N, ϕ^N).

Sometimes, we also meet the following definition of $L_\infty-$morphisms in the literature.

Definition 2.5.10. [20]An $L_\infty-$ **morphism** f between two $L_\infty-$modules M and N over L then can be defined as a family of linear maps

$$f_n : \wedge^n L \otimes M \to N[-n]$$

such that

$$\begin{aligned} f(\gamma_1 \wedge .. \wedge \gamma_n \otimes v) &= f_n(\gamma_1 \wedge .. \wedge \gamma_n, v) + \\ & \sum_{k=1}^{n-1} \sum_{\epsilon \in Sh(k,n-k)} \pm \gamma_{\epsilon(1)} \wedge .. \wedge \gamma_{\epsilon(k)} \otimes f_{n-k}(\gamma_{\epsilon(k+1)}, .., \gamma_{\epsilon(n)}, v) \\ & + \gamma_1 \wedge .. \wedge \gamma_n \otimes f_0(v) \end{aligned} \quad (2.5.3)$$

which is compatible with the differential, that is

$$\begin{aligned} &\phi_0^N f_n(\gamma_1, .., \gamma_n, v) - (-1)^n f_n(Q_1\gamma_1, \gamma_2, ..., \gamma_n) - ... \\ &\quad ... - (-1)^{k_1+k_2+...+k_n+n} f_n(\gamma_1, ..., \gamma_n, \phi_0^M v) = \\ &\sum_{p=0}^{n-1} \sum_{\epsilon \in Sh(p,n-p)} \pm f_p(\gamma_{\epsilon(1)}, .., \gamma_{\epsilon(p)}, \phi_{n-p}^M(\gamma_{\epsilon(p+1)}, .., \gamma_{\epsilon(n)}, v) \\ &- \sum_{p=1}^{n} \sum_{\epsilon \in Sh(p,n-p)} \pm \phi_p^N(\gamma_{\epsilon(1)}, .., \gamma_{\epsilon(p)}, f_{n-p}(\gamma_{\epsilon(p+1)}, .., \gamma_{\epsilon(n)}, v) \\ &+ \sum_{p=2}^{n} \sum_{\epsilon \in Sh(p,n-p)} \pm f_{n-p+1}(Q_p(\gamma_{\epsilon(1)}, .., \gamma_{\epsilon(p)}), \gamma_{\epsilon(p+1)}, .., \gamma_{\epsilon(n)}, v) \end{aligned} \quad (2.5.4)$$

for $\gamma_i \in L^{k_i}$.

Let $\hbar$ be a formal parameter. We extend the structure of DGLA from L to the space of formal power series $L[[\hbar]]$ canonically.

Definition 2.5.11. [20] A **Maurer-Cartan** (MC) element of a DGLA L is an element α of $L^1[[\hbar]]$ satisfying the equation

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \quad (2.5.5)$$

Maurer-Cartan elements generate the infinitesimal deformations of $L_\infty-$algebras. Two deformations of an $L_\infty-$algebra are called equivalent if there exists an isotropy relation between them. On infinitesimal level, because $[L_0, L_n] \subset L_n$, there exists an action of the group $exp(\hbar L_0[[\hbar]])$ on the space of Maurer-Cartan elements, which is analog to the action of the group of inner automorphisms

$$\exp(\xi)\alpha \quad = \quad \exp(ad(\xi))\alpha + \frac{e^{ad(\xi)} - 1}{ad(\xi)}\partial\xi \text{ for } \xi \in L_0, \alpha \in L$$

and preserving the Maurer-Cartan equation 2.5.5. The space of all Maurer-Cartan elements of an $L_\infty-$algebra, together with the above action forms an object called the **Goldman-Millson groupoid** [32], and denoted by $MC(L)$. In the literature, it is also called the **Deligne groupoid**. Any $L_\infty-$ morphism between $L_\infty-$algebras induces a morphism between their Goldman-Millson groupoids, *i.e.* between the moduli spaces of infinitesimal deformations. Furthermore,

Theorem 2.5.12. *[29] Let $\mu : L \to \tilde{L}$ be a quasi-morphism of DGLAs, then $\mu_* : MC(L) \to MC(\tilde{L})$ induces an isomorphism between the coarse moduli spaces $\pi_0(MC(L))$ and $\pi_0(MC(\tilde{L}))$.*

Although from now all the $L_\infty-$algebras will be DGLAs, we still consider the general $L_\infty-$morphisms between DGLAs.

2.6 Periodic Cyclic Homology

There are many ways to understand Periodic cyclic homology. The first one is the concrete approach, suitable for computation and the second is the abstract one. The main reference is [81]....

Note that on the chain complexes and on the spaces of differential forms, we use the negative grading.

2.6.1 Concrete Approach

Let A be any algebra, and let $C_\bullet(A, A)= \bar{A}^{\otimes -\bullet} \otimes A$ be the Hochschild chain complex with the differential operator b. It is concentrated in non-negative degree. Here $\bar{A} = A/k1$. For any element $a_0 \otimes a_1 \otimes ... \otimes a_n \in C_{-n}(A)$, b is defined as

$$\begin{aligned} b(a_0 \otimes a_1 \otimes ... \otimes a_n) =& a_0.a_1 \otimes a_2 \otimes ... \otimes a_n - a_0 \otimes a_1 a_2 \otimes ... \otimes a_n \\ &+ .. + (-1)^{n-1} a_0 \otimes a_1 \otimes ... \otimes a_{n-1}a_n + \\ &+ (-1)^n a_n a_0 \otimes a_1 \otimes a_2 \otimes ... \otimes a_{n-1}. \end{aligned}$$

It is easy to verify that $b^2 = 0$, and we obtain a differential complex

$$\ldots \longrightarrow \bar{A} \otimes \bar{A} \otimes \bar{A} \otimes A \xrightarrow{b} \bar{A} \otimes \bar{A} \otimes A \xrightarrow{b} \bar{A} \otimes A \xrightarrow{b} A.$$

The cohomology of the complex $(C_\bullet(A), b)$ is called **Hochschild homology** of the algebra A, denoted by $HH_\bullet(A)$

On the dual side, let $C^\bullet(A) = Hom(\bar{A}^{\otimes \bullet}, A)$ be the Hochschild cochain complex. The differential operator b acting naturally on $C^\bullet(A)$ is defined by

$$\begin{aligned} b\phi(u \otimes a_1 \otimes \ldots \otimes a_n) = a_0.\phi(a_1 \otimes a_2 \otimes \ldots \otimes a_n) - \phi(a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_n) + \\ + \phi(a_0 \otimes a_1 a_2 \otimes \ldots \otimes a_n) + .. + (-1)^n \phi(a_0 \otimes \ldots \otimes a_{n-1} a_n) + \\ + (-1)^{n+1} \phi(a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}).a_n \end{aligned}$$

for any $\phi \in C^\bullet(A)$, and $a_i \in A$. It is also easy to verify that $b^2 = 0$, and the cohomology of $(C^\bullet(A), b)$ is called **Hochschild cohomology** of A, denoted by $HH^\bullet(A)$. It is helpful to recall that $HH_\bullet$ and $HH^\bullet$ can be viewed as the generalization of **differential forms** and **vector fields.**

We want to keep the degree of $C_\bullet(A)$ concentrated in negative degree because the degree of b should be the same for both chains and cochains. The second point is to emphasize the difference between chains (negative) and cochains (positive).

Theorem 2.6.1. *(Hochschild-Kostant-Rosenberg, 1961, [14]) Let X be a smooth manifold (or a smooth algebraic variety, and $A = C^\infty(X)$ (or $O(X)$). Then we have*

$$HH_{-i}(A) : \; = \; H^{-i}(C_{-i}(A, A), b) \cong \Omega^i(X).$$

and

Theorem 2.6.2. *(Hochschild-Kostant-Rosenberg, 1961, [14]) Let X be a smooth manifold (or a smooth algebraic variety, and $A = C^\infty(X)$ (or $O(X)$). Then we have*

$$HH^i(A) : \; = \; H^i(C^i(A, A), b) \cong \Gamma(X, \wedge^i TX). \tag{2.6.1}$$

By the Hochschild-Kostant-Rosenberg theorem, if A is the algebra of smooth functions on a smooth manifold M, $HH^\bullet(A) \cong PolVect(M)$ and

$HH_\bullet(A) \cong \Omega^{-\bullet}(M)$. Therefore, it is natural to develop the familiar operators like Lie derivative, exterior differential, internal multiplication... in differential geometry into the realm of noncommutative algebras.

There are two levels that the operators can lie in. The first one is the homology/cohomology level, which is just the direct analog of the commutative case, *i.e.* these operators acting on the Hochschild cohomology and homology groups $HH^\bullet(A)$ and $HH_\bullet(A)$. However, it is very difficult to deal directly with these operators continuously when the algebra A is deformed.

The second way is to work on the level of differential chain/cochain complexes, *i.e.* passing through a quasi-isomorphism representative. The relationship between these familiar operators defined on this level should reduce to the one on the homology level, after passing to homology.

2.6.2 Abstract approach

It is desirable to use the abstract approach to the situation. First of all, we develop the terminology that works out for any vector space A, and then study how the algebraic structure enters the picture.

Let A be any vector space. Denote by $C_\bullet(A)= \bar{A}^{\otimes -\bullet} \otimes A$ and $C^\bullet(A)= Hom(\bar{A}^{\otimes \bullet}, A)$ the Hochschild chain and cochain spaces. Naturally, there exists a pairing of degree zero between $C_\bullet(A)$ and $C^\bullet(A)$ with the values on A, defined by the evaluation map.

For $\phi, \psi \in C^{\bullet+1}(A)$ with the degrees shifted by 1, the **Gerstenhaber bracket** is defined [30]by

$$[\phi,\psi] \;=\; \phi \bullet \psi - (-1)^{pq}\psi \bullet \phi \quad \phi \in C^{p+1}(A),\;\; \psi \in C^{q+1}(A), \tag{2.6.2}$$

where the composition product can be defined as

$$\phi \bullet \psi(a_1, a_2, ..., a_{p+q+1}) \;=\; \sum_{i=0}^{p+1} (-1)^{(i-1)q} \phi(a_1, .., a_{i-1}, \psi(a_i, ...a_{i+q}), ..., a_{p+q+1})$$

As a result, $C^{\bullet+1}(A)$ is a graded Lie algebra. Pictorially, we think of the product $\phi \bullet \psi$ as

On the dual side, $CC_\bullet(A)$ is graded module over the Lie algebra $CC^{\bullet+1}(A)$ where the module structure is the generalization of the **Lie derivative** of a vector field on differential forms. More precisely, for any $\phi \in C^{\bullet+1}(A)$, and

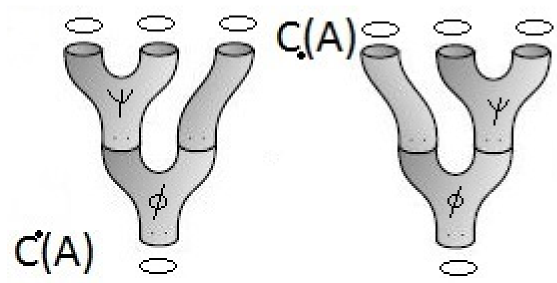


Figure 2.6.1: The product of Hochschild cochains

$a = (a_0, a_1, ...a_n) \in C_{-n}(A)$, the Lie action is given [30] by

$$L_\phi(a_0, a_1, ..., a_n) = (-1)^p \sum_{i=0}^{n-p} (-1)^{ip}(a_0, a_1, ..., a_{i-1}, \phi(a_i, ..., a_{i+p}), ..., a_n) + \\ + \sum_{i=n-p+1}^{n} (-1)^{np}(\phi(a_i, ..., a_n, a_0, ..., a_{p-n+i-1}), ..., a_{i-1}). \tag{2.6.3}$$

where we can show that

$$[L_\phi, L_\psi] \;=\; L_{[\phi,\psi]} \tag{2.6.4}$$

There also exists another differential $B : C_\bullet(A) \to C_{\bullet-1}(A)$, [30]called the Rinehart-Connes differential acting on $C_\bullet(A)$. This operator on the chain level plays the role of the exterior differential in differential geometry.

$$B(a_0, a_1, .., a_n) = \sum_{i=0}^{n} (-1)^{in}(1, a_i, a_{i+1}, ..., a_n, a_0, ...a_{i-1}) \tag{2.6.5}$$

and $B : C^{\bullet+1}(A) \to C^\bullet(A)$

$$B\phi(a_0, ..., a_n) = \sum_{i=0}^{n} (-1)^{in}\phi(1, a_i, a_{i+1}, ..., a_n, a_0, ..., a_{i-1}). \tag{2.6.6}$$

It is an easy exercise to verify that $B^2 = 0$. We notice that the Connes differential is compatible with the Lie derivative L in the sense that [30]

$$BL_\phi - (-1)^{deg(\phi)+1}L_\phi B \;=\; 0$$

Lie derivative and algebraic product μ. It is fruitful to see that $[.,.]$, B and L are natural operators in the category of vector spaces, and they are independent of the algebraic product of A. If an associative product $\mu \in C^2(A)$ is given, we can determine many other structures of the chain and cochain complexes basing on these tautological operators.

For example, it is known that [19] the familiar Hochschild differential b on $C_\bullet(A)$ is just the Lie derivative L_μ, and Hochschild differential b on $C^\bullet(A)$ is nothing but $[\mu, .]$. The vanishing of b^2 on these complexes is equivalent to the associativity of μ, and it allows us to build the homology $HH_\bullet(A)$ and cohomology $HH^\bullet(A)$ groups.

Summarize, $(C^{\bullet+1}(A), b, [.,.])$ is a Differential Graded Lie Algebra (DGLA) in the sense of 2.5.1 and $C_\bullet(A)$ is a DG Lie module over $C^{\bullet+1}(A)$.

Just like in the classical case, we can show that

$$b \circ L_\phi - (-1)^p L_\phi \circ b \quad = \quad L_{b\phi} = L_{[\mu,\phi]} \quad \text{for } \phi \in C^{p+1}(A).$$

which passes these structures to the cohomology level (cite [30]). Consequently, the Lie derivative operator L induces the action of the graded Lie algebra $HH^{\bullet+1}(A)$ on the homology $HH_\bullet(A)$ of $C_\bullet(A)$ via the generalized Lie derivative. In the case of algebra of smooth functions on a smooth manifold, it reduces to the usual Lie derivative via the Hochschild-Kostant-Rosenberg theorem 2.6.1.

Interior multiplication [30] Motivated in parts by the internal multiplication $i_\xi \omega$ between poly-vector fields and differential forms, it is natural to introduce the **internal multiplication** (the **contraction**) I between a chain and a cochain

$$I_\phi(a_0, a_1, ..., a_n) = (a_0\phi(a_1, a_2, ...a_p), a_{p+1}, ..., a_n). \tag{2.6.7}$$

for $\phi \in C^p(A)$, $a = (a_0, a_1, ..., a_n) \in C_n(A)$. See [29]

Naively, we expect that the homotopy formula $L_\phi = B \circ I_\phi - (-1)^p I_\phi \circ B$ in differential geometry also holds, but it turns out that the answer is yes only in the cohomological level, and "almost yes" in the chain/cochain complex level. It means that we need to introduce a new term that vanishes when passing through cohomology group.

Lemma 2.6.3. *[30] Let $H_\phi : C_{-n}(A) \to C_{-n+p-2}(A)$ be the map*

$$H_\phi(a_0, a_1, ..., a_n) = \sum_{i=0}^{n-p} \sum_{j=0}^{n-p-i} (-1)^{i(n-p+1)+j(p+1)}. \\ (1, a_{n+1-i}, ..., a_n, a_0, a_1, ..., a_j, \phi(a_{j+1}, ..., a_{j+p}), ..., a_{n-i}). \tag{2.6.8}$$

Then we can show that H_ϕ commutes with B, $[b, I_\phi] = I_{b\phi}$ and

$$L_\phi \quad = \quad [B, I_\phi] - H_{b\phi} + [b, H_\phi], \tag{2.6.9}$$

as elements of $End_k(C_\bullet(A))$. Furthermore,

$$[b + B, I_\phi + H_\phi] = L_\phi + I_{b\phi} + H_{b\phi}. \tag{2.6.10}$$

We also can verify that on the chain complex level some important properties are satisfied

$$[b, I_\phi] = I_{b\phi} \tag{2.6.11}$$
$$[B, H_\phi] = 0 \tag{2.6.12}$$

2.6.3 Periodic cyclic homology

Lemma 2.6.4. *[14] Given an algebra A. Then the operators B, b fit in a double complex, and satisfying condition $b^2 = B^2 = Bb + bB = 0$*

$$\begin{tikzcd}
 & ... & ... & & \\
 & A^{\otimes 4} \arrow[r, "b"] & A^{\otimes 3} \arrow[r, "b"] & A \otimes A \arrow[r, "b"] & A \\
... & A^{\otimes 3} \arrow[r, "b"] \arrow[u, "B"] & A \otimes A \arrow[r, "b"] \arrow[u, "B"] & A \arrow[u, "B"] & \\
 & A \otimes A \arrow[r, "b"] \arrow[u, "B"] & A \arrow[u, "B"] & & \\
... & A \arrow[u, "B"] & & &
\end{tikzcd} \tag{2.6.13}$$

If we extend the double complex on the left and the up-side directions by repeating the rows and columns, we obtain an infinite double complex. Taking direct sums along the diagonal direction, we obtain the the complexes

$$\cdots \underset{uB}{\overset{b}{\rightleftarrows}} PC_{odd}(A) \underset{uB}{\overset{b}{\rightleftarrows}} PC_{even}(A) \underset{uB}{\overset{b}{\rightleftarrows}} PC_{odd}(A) \qquad \cdots$$

where $PC_{odd}(A) = A \oplus A^{\otimes 3} \oplus A^{\otimes 5} \oplus \ldots$ and $PC_{even} = \mathbb{C} \oplus A^{\otimes 2} \oplus A^{\otimes 4} \oplus A^{\otimes 6} \oplus \ldots$.

It is easy to verify that $b + B$ is a differential acting on the complex $PC_\bullet(A) = PC_{odd}(A) \oplus PC_{even}(A)$

$$\xrightarrow{b+B} PC_{odd}(A) \xrightarrow{b+B} PC_{even}(A) \xrightarrow{b+B} PC_{odd}(A) \xrightarrow{b+B} \ldots$$

The cohomology of the complex is called **periodic cyclic homology** of A, and denoted by $HP_\bullet(A)$.

Theorem 2.6.5. *[14] Let M be a compact manifold. Then $HP_\bullet(C^\infty(M)) \cong H^{odd}_{dR}(M) \oplus H^{even}_{dR}(M)$.*

The meaning of the theorem is that $HP_\bullet$ group is a noncommutative generalization of the deRham cohomology. In fact, as we will see below, periodic cyclic homology is invariant under infinitesimal noncommutative deformation.

Example 2.6.6. Recall from [63] that

$$HP_k(C^\infty(\mathbb{T}_\theta)) \cong HP_k(C^\infty(S^1 \times S^1)) \cong H^k(S^1 \times S^1).$$

Therefore

$$HP_{even}(\mathbb{T}_\theta) = HP_{odd}(\mathbb{T}_\theta) = \mathbb{Z}^2.$$

By [63], for the smooth tori, $HP_1(\mathbb{T}_\theta)$ is generated by $< u^{-1}du >$ and $< v^{-1}dv >$ and $HP_2(\mathbb{T}_\theta)$ is generated by $< 1 >$ and $< u^{-1}v^{-1}(e^{2\pi i\theta}du.dv - dv.du) >$.

We summarize the above terminologies in the following table.

We believe that the table should be well known in somewhere, and anyone in the areas knows, but we do not know any reference.

Classical level	Cohomology level	Complex level
poly-vector field	$HH^\bullet(A)$	$(C^\bullet(A), b)$
differential form	$HH_\bullet(A)$	$(C_\bullet(A), b)$
exterior differential d	d	$D = b + uB$
Lie derivative L_ξ	L_ϕ action of $HH^\bullet$ on $HH_\bullet$	L_ϕ in Equation 2.6.3
Internal multiplication i_ξ	I_ϕ	I_ϕ in Equation 2.6.7
	I_ϕ well-defined	$[b, I_\phi] = I_{b\phi}$
Homotopy formula	$L_\phi = B \circ I_\phi - (-1)^p I_\phi \circ B$	Equation 2.6.10
DeRham cohomology	$HP_\bullet$	BB-complex 2.6.13
$\cup$ and $[.,.]$	Leibniz rule	unknown

Table 2.6.1: Noncommutative Geometry Correspondence

2.7 Kontsevich formulation and formality

Because our study deals with morphisms between cohomology groups of deformed algebras, we review some background on deformation theory in the sense of Kontsevich. We will try to avoid technical details of the construction of the quantization map, and refer to the original paper [44].

2.7.1 Semi-classical side

Let M be any smooth manifold, and let $\Gamma(T^{\bullet+1}M) = \Gamma(M, \wedge^{\bullet+1}TM)$ be the space of poly-vector fields associated with Schouten Bracket $[.,.]$ (special case of equation 2.6.2) and zero differential. It is known that any 2-vector field $\pi \in \Gamma(T^{\bullet+1}M)$ is a Poisson structure if it satisfies the Maurer-Cartan equation $[\pi, \pi] = 0$.

Because the differential is trivial, we can rewrite it under the form

$$d\pi + \frac{1}{2}[\pi, \pi] \;=\; 0 \qquad \text{(Maurer-Cartan equation)} \qquad (2.7.1)$$

We would like to study the moduli space $Poiss(M)$ of Poisson structures on M, modulo gauge transformations.

2.7.2 Algebraic side

Let A be the algebra of functions on M with usual product. We would like to define a new associative product on $A[[\hbar]]$ depending on a parameter $\hbar$

$$f *_\hbar g \quad = \quad f.g + \hbar.\gamma(f,g).$$

The associativity is equivalent to the equation

$$d\gamma + \frac{1}{2}[\gamma,\gamma] \quad = \quad 0 \qquad \text{(Maurer-Cartan equation)} \tag{2.7.2}$$

where $[.,.]$ is the Gerstenhaber bracket. It is known that we can γ is equivalent to a skew-symmetric product, modulo a gauge transformation. We would like to study the moduli space $Assoc(M)$ of associative product on A, modulo gauge transformation.

Quantization and Formality for Hochschild Chains Starting from a $*_\hbar-$product, it is possible to recover the Poisson structure by taking semi classical limit, also called dequantization

$$\{f,g\} = lim_{\hbar\to 0}\frac{f *_\hbar g - g *_\hbar f}{\hbar}$$

The philosophy of Quantization is that an inversion map should exist, and in many good enough case, it should be an isomorphism.

$$Poiss(M) \longleftrightarrow Assoc(M)$$

$$\text{MC Eq (2.7.1) for } \Gamma(T^{\bullet+1}M) \longleftrightarrow \text{MC Eq (2.7.2) for } C^{\bullet+1}(A) \tag{2.7.3}$$

By the Hochschild-Kostant-Rosenberg theorem 2.6.1, the cohomology of the complex $(C^{\bullet+1}(A), b)$ is isomorphic to the cohomology of the complex $\Gamma(T^{\bullet+1}M)$ with zero differential as vector spaces. The evident map is quite simple and canonical

$$U_1^{(0)} : \xi_0 \wedge \xi_1 \wedge \ldots \wedge \xi_n \to \left[(f_0 \otimes f_1 .. \otimes f_n) \mapsto \frac{1}{(n+1)!}\sum_\sigma sgn(\sigma)\prod_{i=0}^{n} \xi_{\sigma_i}(f_i)). \right]$$

However, this canonical map is not a DGLA morphism, so it does not preserve the Maurer-Cartan equations 2.7.1 and 2.7.2. As a result, it could not provide a correspondence between the Poisson structures on M and the associative product structures on $C^\infty(M)$.

To establish the correspondence, In [44], Kontsevich proved the Formality Conjecture by constructing a DGLA quasi-morphism U from $(\Gamma(T^{\bullet+1}M), 0, [.,.])$ to $(C^\bullet(A), d, [.,.])$. Being a DGLA morphism, U preserves the Lie brackets and differentials on both chain complex and poly-vector field sides, therefore, it induces a bijection between the solutions of the two Maurer-Cartan equations 2.7.1 and 2.7.2.

$$DGLA : (\Gamma(T^{\bullet+1}M), [.,.], 0) \xrightarrow[Kontsevich]{U} (C^\bullet(A), [.,.], b)$$

Therefore, there is a 1-1 correspondence induced from U between:

1. Poisson structures π, *i.e.* 2-vector fields satisfying the Maurer-Cartan equation $[\pi, \pi] = 0$

2. Deformation quantization $\gamma = U^\pi$ of the associative product satisfying the Maurer-Cartan equation on the other side

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \tag{2.7.4}$$

   Equivalently, Poisson structure can be quantized canonically with the quantization product, $f *_\hbar g = f.g + U^\pi(f \otimes g)$.

In other senses, the Kontsevich morphism plays the role of the isomorphism between two moduli spaces of solutions of Maurer-Cartan equation. The formula for the Kontsevich morphism can be found in [44].

2.7.3 Formality for the Hochschild cochains

$$\begin{tikzcd}
DGLA : (\Gamma(T^{\bullet+1}M), [.,.], 0) \arrow[r, "U", "Kontsevich"'] \arrow[d, "L_{mod}"] & (C^{\bullet+1}(A), [.,.], b) \arrow[d, "L_{mod}"] \\
(\Omega^{-\bullet}(M)[[u]], ud) & (C_\bullet(A)[[u]], b + uB) \arrow[l, "V-L_\infty modules"', "Shoikhet-Dolgushev"]
\end{tikzcd}$$

In the diagram, the vertical arrows with L_{mod} mean that the lower is a $L_\infty-$module structure of the upper, as explain in the definition 2.5.7. It is important to notice that they are not morphisms.

Equivalently, it means that $C_\bullet(A)$ is a DGLA module over $C^\bullet(A)$, so it is a DGLA module over $(\Gamma(T^{\bullet+1}M), [.,.], 0)$ via the composition with the Kontsevich map U. On the other hand, $\Omega^{-\bullet}(M)$ is a DGLA module over $(\Gamma(T^{\bullet+1}M), [.,.], 0)$ via the classical Lie derivative. The question is, how to compare these DGLA modules over $(\Gamma(T^{\bullet+1}M), [.,.], 0)$.

Shoikhet in [81] has constructed a map V of $L_\infty-$modules over $\Gamma(T^{\bullet+1}M)$ from $C_\bullet(A)$ to $\Omega^{-\bullet}(M)$ for the linear case $M = \mathbb{R}^n$. Dolgushev [19] later generalized it to general smooth manifolds using the Gelfand-Fuchs technique.

We notice that the key difference between Kontsevich deformation map and Shoikhet-Dolgushev maps is that the first one is the quantization from classical side to the noncommutative side, and the second one is the dequantization from the noncommutative side back to the classical side. It is not surprising phenomenal because the first component of the Shoikhet-Dolgushev map is just the familiar morphism in the Hochschild-Kostant-Rosenberg theorem 2.6.1.

$$V_0(a_0 \otimes a_1 \otimes .... \otimes a_n) \quad = \quad \frac{1}{n!} a_0 da_1 da_2..da_n.$$

Roughly speaking, the Kontsevich and Shoikhet-Dolgushev maps are improved versions of Hochschild-Kostant-Rosenberg for higher degrees.

2.7.4 Twisting Procedure and tangent cohomology

The twisting procedure is from the paper by Quillen, and then is used by [95] in the realm of algebraic geometry. The twisting procedure also appears in the work of Kontsevich [44], section 8, when the author discusses the compatibility with the cup product of the tangent space of the super moduli space.

The main idea is that instead of $L_\infty-$morphisms between $L_\infty-$modules, it is possible to study their deformation by a Maurer-Cartan element. Therefore, the twisting procedure can be viewed as the study of objects built out of the tangent spaces of the deformation functor. We remind the reader the twisting procedure from [19, 20, 21].

Algebras

Given a Maurer-Cartan element $\pi \in \hbar L[[\hbar]]$ of a DGLA algebra L, it is possible to modify the DGLA structure by adding a twisting component $[\pi, .]$ to the differential d. The new DGLA $(L, d + [\pi, .])$ is called the **twisted DGLA** by π. The more general twisting process can be done with $L_\infty-$algebras, where the homological vector field is twisted by

$$Q^\pi(X) = e^{-\pi\wedge}Q(e^{\pi\wedge}X).$$

Modules

Given M a DGLA module over L. Then, the twisting procedure can be carried out for modules (see proposition 2, page 38, [20]). For any $L_\infty-$module (M, ϕ), we define the tangent complex M^π as the graded space $M^\bullet = M \otimes C(L)$ together with the new differential $\phi^\pi = e^{-\pi\wedge}.\phi.e^{\pi\wedge}$ where $\phi_n : C(L) \otimes M \to M$ the module differential structure in 2.5.7. But being away from the abstract nonsense, we can compute that the formula for the differential 2.5.7 is deformed to

$$\begin{aligned}
\phi^\pi_n(\gamma_1 \wedge ... \wedge \gamma_n \otimes v) &= \phi_n(\gamma_1 \wedge ... \wedge \gamma_n \otimes v) + \phi_{n+1}(\pi, \gamma_1 \wedge ... \wedge \gamma_n \otimes v) \\
&\quad + ... + \frac{1}{p!}\phi_p(\pi, \pi, ..., \pi, \gamma_1 \wedge ... \wedge \gamma_n \otimes v) + ...
\end{aligned}$$

and the morphism f between $L_\infty-$modules 2.5.10 is deformed to

$$\begin{aligned}
f^\pi_n(\gamma_1 \wedge ... \wedge \gamma_n \otimes v) &= f_n(\gamma_1 \wedge ... \wedge \gamma_n \otimes v) + f_{n+1}(\pi, \gamma_1 \wedge ... \wedge \gamma_n \otimes v) \\
&\quad + ... + \frac{1}{p!}f_{n+p}(\pi, \pi, ..., \pi, \gamma_1 \wedge ... \wedge \gamma_n \otimes v) + ...
\end{aligned}$$

It is very simple to verify that the equation guarantees $\partial + [\alpha, .]$ is an actual differential, *i.e.* $(\partial + [\pi, .])^2 = 0$ due to the Maurer-Cartan equation 2.5.5. We denote the new DGLA

$$(L[[\hbar]], d_\pi, [., .])$$

by L^α and called it the **twisted-DGLA**. A module over the twisted DGLA M^π is called the twisted $L_\infty-$module.

It is obvious that every morphism of DGLA $U : L \to \tilde{L}$ can be extended uniquely to $U^\pi : L^\pi \to \tilde{L}^{U(\pi)}$. Furthermore, the twisting procedure preserves the quasi-isomorphisms.

Proposition 2.7.1. *[20] If π a Maurer-Cartan element in L, and $U : L \to \tilde{L}$ is a quasi-isomorphism between DGLA algebras, then so is $U^\pi : L^\pi \to \tilde{L}^{U(\pi)}$ where $U^\pi = \sum_{n=0}^{\infty} \frac{U_n^\pi}{n!}$ and*

$$U_n^\pi(a_1, ..., a_n) \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!} U_{k+n}(a_1, a_2, .., a_n, \pi, \pi, ..., \pi)$$

If $V : M \to \tilde{M}$ is a quasi-isomorphism between $L_\infty-$ modules over L, with n-Taylor components V_k, then

$$\begin{tikzcd} L^\pi \arrow[r, "="] \arrow[d, "L_{mod}"] & L^\pi \arrow[d, "L_{mod}"] \\ M \arrow[r, "V^\pi"] & \tilde{M}^\pi \end{tikzcd}$$

is also a quasi-isomorphism between $L_\infty-$modules over L^π, where $V^\pi = \sum_{n=0}^{\infty} \frac{V_n^\pi}{n!}$, and

$$V_n^\pi(b_1, b_2, ...b_n, m) \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!} V_{k+n}(\pi, \pi, ..., \pi, b_1, b_2, ..., b_n, m).$$

We notice that U_0^π is just the constant map, with the value equal to the image of π under the Kontsevich morphism U. Therefore, abusing use of language, we use U_0^π for U^π.

We are interested in the application of the above proposition to the formality conjecture, *i.e.* the map U is the Kontsevich morphism. The twisted U^π is an $\hbar-$linear $L_\infty-$ quasi-isomorphism from $\Gamma(T^{\bullet+1}M)[[\hbar]]$ with the Poisson differential $d_\pi = [\pi, .]$ to $(C^{\bullet+1}(M)[[\hbar]], b_\hbar)$ with the new Hochschild differential $b_* = b + L_{U_0^\pi}$ calculated with Kontsevich $*-$product U_0^π.

Recall that $PC_\bullet(A) = PC_{odd} \oplus PC_{even}$, the complex for Periodic cyclic homology. Then, V^π is $L_\infty-$ morphism of modules over DGLA $(\Gamma(T^{\bullet+1}M), d_\pi)$ from $(PC_\bullet(A)[[\hbar]], b + uB)$ to $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]]), ((u)), L_\pi + ud)$. See[11, 95] for more details.

$$\begin{tikzcd} DGLA : (\Gamma(T^{\bullet+1}M), [.,.], d_\pi) \arrow[r, "U^\pi", "\text{Kontsevich}"'] \arrow[d, "L_{mod}"'] & (C^\bullet(A), [.,.], b + L_{U_0(\pi)}) \arrow[d, "L_{mod}"] \\ (\Omega^{-\bullet}(M)[[u]], L_\pi + ud) & (C_\bullet(A)[[u]], b + uB) \arrow[l, "V^\pi - L_\infty modules"', "\text{S-D}"] \end{tikzcd} \tag{2.7.5}$$

where S-D stands for Shoikhet-Dolgushev. We remind that the vertical arrows is the $L_\infty-$modules structure in 2.5.7. The parameter u is formal, and will be put to be 1 for our purposes, although studying the variation of u leads to interesting problems in mirror symmetry [40].

If u is non-zero, because $L_\pi = d \circ i_\pi - i_\pi \circ d$, the map $\alpha \mapsto e^{i_\pi/u}.\alpha$ induces an isomorphism between two differential complexes $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]]), ((u)), L_\pi + ud)$ and $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]]((u)), ud)$, since

$$\begin{tikzcd}
(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]]), ((u)), L_\pi + ud) \arrow[d, "e^{i_\pi/u}"] & (PC_\bullet(A)[[\hbar]], b_* + uB)] \arrow[l, "V_0^\pi"'] \arrow[dl, "\tilde{V_0^\pi}"] \\
(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]]((u)), ud) &
\end{tikzcd}$$

Therefore in the cohomology level, we obtain an isomorphism $\tilde{V_0^\pi} = e^{i_\pi/u}.V_0^\pi$ from the periodic cyclic homology $PH_\bullet(A_\hbar)$ [46] of deformed algebras to the DeRham cohomology $H^{-\bullet}(M, \mathbb{R}[[\hbar]])((u))$, which is a morphisms of modules over Poisson Cohomology $HP^\bullet(A_\hbar)$, called ***the character map*** [11].

2.8 Character Map and Monodromy Theorem

In this section, we summarize the techniques introduced above to prove the Main Theorem 2.4.13

2.8.1 Gauss-Manin connection

Classically the Gauss-Manin connection was defined in the algebro-geometric setting by Manin [50] as a flat connection ∇ on the bundle of cohomology groups. The cohomology groups are the deRham cohomology $H^\bullet(E_s)$ of the fibers of a family of algebraic varieties over some base S and the connection is determined by a differential equation. The problem of developing the Gauss-Manin connection to noncommutative geometry setting was solved by Getzler [30] in the level of periodic cyclic homology.

We recall some necessary backgrounds from [11]. Let $\{A_t\}$ be a family of algebras over the base S. Assume that the algebras $\{A_t\}$ can be identified to A_0 as vector spaces. The products in the fiber algebras $\{A_t\}$ has the form

$\mu+\gamma(t)$ where μ is the product in A_0 and $\gamma(t)$ is a curve on the moduli space of solutions of the Maurer-Cartan equation [12]

$$b\gamma(t)+\frac{1}{2}[\gamma(t),\gamma(t)]=0.$$

where b is the Hochschild differential 2.7.2.

Theorem 2.8.1. *[11] Let $\gamma(t)$ be the deformation of the deformed product $\mu+\gamma(t)$ of the fiber algebras A_t from A_0. Then there exists a **Gauss-Manin connection** ∇ on the fiber bundle $PC_\bullet(A_t)=(\oplus A_t^{\otimes odd})\oplus(\otimes A_t^{\otimes even})$ which commutes with the $b+uB$ differential. The connection defines a differential equation of parallel transportation as follows*

$$\frac{d}{dt}c(t)+\frac{1}{u}\hat{I}_{(\dot{\gamma}(t))}c(t)=0 \tag{2.8.1}$$

where $c(t)\in PC_\bullet(A_t)$, and $c(0)$ is a cyclic cycle. Here, $\hat{I}_{\dot{\gamma}(t)}=I_{\dot{\gamma}}+uH_{\dot{\gamma}}$ is defined in Equations 2.6.6 and 2.6.7 The $\nabla-$parallel solution $c(t)$ is cyclic cycle for all t, and its homology class in $PH_\bullet(A_t)$ depends only on the class of $c(0)$.

Because the differential $b+uB$ on the complex $PC_\bullet$ commutes with the Gauss-Manin connection, the connection on the $PC_\bullet-$ complex induces a connection on the periodic cyclic homology group $HP_\bullet$.

Given a Gauss-Manin connection, it is unknown in general whether or not it is integrable. Equivalently, we do not know if the differential equation of parallel transportation has a unique solution.

The key remark about the theorem is that the Gauss-Manin connection ∇ on the level of $PC_\bullet-$ complexes in the sense of Getzler is not flat, or equivalently it depends on the path connecting two points. However, it is possible to check that [30] its curvature ∇^2 is homotopic to zero, and so the connection induces a flat connection on the level of periodic cyclic homology $PH_\bullet(A_t)$.

Tsygan in [86] constructed another flat connection ∇_{GM} on the the level of complexes $(PC_\bullet(A_t),b+B)$. This connection coincides with the one constructed by Getzler in the level of periodic cyclic homology, but it is much more complicated, compared to 2.8.1. Because two connections yield the same monodromy in the level of homology, it is sufficient to use Getzler's connection for computations.

Recently, Yashinski [94] has developed a functional analytic condition that the Gauss-Manin connection is integrable. The condition is mostly the smoothness of the bundle, which is always satisfied in our case. In this circumstance, the author even showed that the solution converges. We refer the reader to his work for more details.

We will use smooth tori from now on, with the smooth structure defined by the action of the classical torus. The reason for this is that smooth tori are compatible with both Rieffel quantization and periodic cyclic homology, in the sense that $HP_\bullet(C^\infty(\mathbb{T}^2)) \cong \mathbb{C}^2 + \mathbb{C}^2$. Meanwhile, the periodic cyclic homology can detect even the local neighborhoods of any points due to $HP_\bullet(C(\mathbb{T}^2)) \cong C(\mathbb{T}^2)$, which is too much for our purpose. It is also impossible to deform the algebra of all continuous functions, because the Poisson structure does not make sense in non-smooth context.

In order to compute the monodromy for the periodic cyclic homology groups $HP_\bullet(A_t)$, we compute the monodromy for its generators $PC_\bullet(A_t)$ by using equation 2.8.1, and then pass it down to the $HP_\bullet-$ groups.

We fix a basis $< x, y >$ for the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, then $U^{-1}dU = dx$ and $V^{-1}dV = dy$ are $\mathbb{T}^2-$invariant homology classes of the noncommutative two-tori. It is known in [63] that they from a basis for the group $HP_\bullet(\mathbb{T}_{\theta(t)})$.

We apply equation 2.8.1 to compute the star product and the Gauss-Manin connection for the case $\{\mathbb{T}_{\theta(t)}\} \cong C^*(H_3)$. To avoid technical obstructions, we compute the monodromy of the curve $c(t)$ starting from the commutative algebra A_0.

Instead of using the formulation $\mathbb{T}_t = C^*(U, V \mid UU^* = VV^* = Id, UV = VU.e^{2\pi it})$ with the notation from 2.4.1, we think of the $*-$product as an element of $Hom(A \otimes A, A) \cong A \otimes A^* \otimes A^*$.

$$\begin{aligned}
\mu + \gamma(t) &= \sum_{x,y,p,q} e^{-2\pi ityp} U^{x+p} V^{y+q} \otimes (U^x V^y)^* \otimes (U^p V^q)^* \\
&= \sum_{x,y,p,q} e^{-2\pi ityp} W_{x,y,p,q}.
\end{aligned}$$

where $W_{x,y,p,q}$ stands for

$$U^{x+p} V^{y+q} \otimes (U^x V^y)^* \otimes (U^p V^q)^*$$

and $*$ stands for the dual basis. We note that it is equivalent to Equation 2.3.1 of $*-$product on $S(\mathbb{Z}^n)$.

Thus, we obtain the variation of the $*_t$ product structure

$$\dot{\gamma}(t) = -\sum_{x,y,p,q} 2\pi iyp.e^{-2\pi ityp} W_{x,y,p,q}. \tag{2.8.2}$$

In order to obtain the monodromy, we plug the variation 2.8.2 into the differential equation 2.8.1. The Gauss-Manin connection as in the form of 2.8.1 consists of two parts, I_ϕ 2.6.7 and uH_ϕ 2.6.8. Notice that $I_{\dot{\gamma}}(a_0, a_1, ..., a_n)$ and $uH_{\dot{\gamma}}(a_0, a_1, ..., a_n)$ vanish for $n = 0$ and $n = 1$ for the obvious reason, *i.e.* checking the degree. For $n = 2$, we can verify that

$$I_{\dot{\gamma}(t)}(a \otimes b \otimes c) = a *_t \dot{\gamma}(b \otimes c),$$

$$H_{\dot{\gamma}}(a \otimes b \otimes c) = e \otimes \dot{\gamma}(b \otimes c) \otimes a,$$

and

$$\begin{aligned} H_\phi(a_0, a_1, ..., a_n) = \sum_{i=0}^{0}\sum_{j=0}^{0}(-1)^{i+j}. \\ (1, a_{n+1-i}, ..., a_n, a_0, a_1, ..., a_j, \phi(a_{j+1}, ..., a_{j+2})...., a_{2-i}). \end{aligned} \tag{2.8.3}$$

To summarize, the equation for the Gauss-Manin connection for bundle of noncommutative two-tori can be described as

$$\begin{aligned} \hat{I}_{\dot{\gamma}}[(U^xV^y \otimes (U^pV^q) \otimes (U^mV^n)] = -2\pi iqme^{-2\pi it(qm+y(p+m))}U^{x+p+m}V^{y+q+n} \\ - 2\pi iqme^{-2\pi itqm}[e] \otimes U^{p+m}V^{q+n} \otimes U^xV^y. \end{aligned} \tag{2.8.4}$$

We come up with an equation for the monodromy of the generators of the $PH_\bullet$ groups in Equation 2.6.6. Because of the vanishing of I and H for $n = 0$ and $n = 1$, the monodromy acts trivially for $H^0(\mathbb{T}^2)$, $H^1(\mathbb{T}^2)$.
For $n = 2$, we obtain

$$\hat{I}_{\dot{\gamma}}[U^{-1}V^{-1}(e^{2\pi it}dUdV - dVdU)] = -2\pi i - 2\pi ie^{-2\pi it}[e] \otimes dUV \otimes d(U^{-1}V^{-1})$$

To finish Lemma 2.4.13, we only need to show that the monodromy is nontrivial for $H^2(\mathbb{T}^2) \cong \mathbb{C}$. The proof is a direct albeit lengthy and technical calculation of the Hochschild cochains, *i.e.* by solving the differential equation of the monodromy.

2.8.2 Main proof for the Formal Case

However, Cattaneo, Felder and Willwacher provide submitted a preprint [11], which contains a lemma which help us shorten the calculation. The enlightening key is to use the twisted quasi-morphism map V^π that we reviewed in the previous sections.

Lemma 2.8.2. *([11]) Let π be a Poisson vector field on M, $A = C^\infty(M)$ and $\theta \in \Gamma(M, \wedge^p TM)$ such that $[\pi, \theta] = 0$. The twisted Shoikhet-Dolgushev quasi-morphism V^π 2.7.5 induces a map of complex*

$$\hat{i}_\theta \circ V_0^\pi(.) - V_0^\pi \circ \hat{I}_{U_1^\pi(\theta)}(.) + uV_1^\pi(\theta)(.) \qquad .$$

from $(PC_\bullet(A), b_* + uB)$ to $(\Omega^{-\bullet}(M, \mathbb{R}) \otimes \mathbb{C}((u)), L_\pi + ud)$. Furthermore, this map is trivial homotopically, so it induces a zero map on cohomology.

$$\begin{tikzcd}
(\Gamma(M, \wedge^{\bullet+1}TM), [\pi, .]) \arrow[r, "U_\bullet^\pi", "\theta \mapsto U_1^\pi(\theta)"'] \arrow[d, "\hat{i}_\theta"] & (C^\bullet(A), b + L_{U_0^\pi}) \arrow[d, "\hat{I}_{U_1^\pi(\theta)}"] \\
(\Omega^{-\bullet}(M)((u)), L_\pi + ud) & (PC_\bullet(A), b_* + uB) \arrow[l, "V_\bullet^\pi"', "V_1^\pi(\theta)"]
\end{tikzcd}$$

Where the vertical arrows are DGLA modules actions. The formula for $V_1^\pi(\theta; a_0, a_1, ...a_n)$ is

$$\sum_{m \geqq 0} \frac{\hbar^m}{m!} \sum_{\Gamma \in G(m+1,n)} (\int_{\pi_1^{-1}(U_1) \subset C_{m+1,n}} \omega_\Gamma) D_\Gamma(\theta, \pi, \pi, ...; a_0, a_1, ..., a_n)$$

The sum is taken over all the Shoikhet graphs, and D_Γ a differential operator defined for the graph Γ.

Because we do not need the precise formula for the Shoikhet-Dolgushev map, we refer the reader to the [81]. We apply the lemma to the case $\pi(t) = t.\pi$, and $\theta = \pi$. Notice that it can be proved for any smooth curve on the moduli space of MC elements $\pi(t)$ as long as $[\pi(t), \theta(t)] = 0$, for $\theta(t) = \dot{\pi}(t)$. Let $\gamma(t)$ be the quantization of $\pi(t)$, *i.e.* $\gamma(t) = U(\pi(t))$.

On the quantum side, there exists the Getzler-Gauss-Manin connection $d + \frac{1}{u}\hat{I}_{\dot{\gamma}(t)}$ on the $(PC_\bullet(A_t), d_\mu + L_{\gamma(t)})$ complex.

On the classical side, there exists another connection $d + \frac{1}{u}\hat{i}_{\dot{\pi}(t)}$ on the twisted complex $(\Omega^{-\bullet}(M), L_{\pi(t)} + ud)$. Here the connection on the classical level is

$$\hat{i}_{\dot{\pi}(t)} = i_{\dot{\pi}(t)} + \frac{u}{2} dL_{\dot{\pi}(t)} = i_{\dot{\pi}(t)} + \frac{u}{2} di_{\dot{\pi}(t)} d.$$

Lemma 2.8.3. *[11]Modulo trivial homology classes, $V^{\pi(t)}$ intertwines the Gauss-Manin connection $d + \frac{1}{u}\hat{I}_{U_1^{\pi(t)}(\dot{\pi}(t))}$ on $(PC_\bullet(A_t), b_* + B)$ and the connection $d + \frac{1}{u}\hat{i}_{\dot{\pi}(t)}$ on the trivial vector bundle on the Maurer-Cartan variety of Poisson structures with fibers $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])((u)), L_{\pi(t)} + ud)$. Furthermore, $\tilde{V}^{\pi(t)} = e^{\hat{i}_{\pi(t)}} \circ V^{\pi(t)}$ maps parallel sections on $HP_\bullet-$ bundle to constant section on $H^\bullet(M)-$bundle.*

Proof. The proof is from [11]. By the definition of the Shoikhet morphism

$$\frac{d}{dt} V_0^{\pi(t)} + \frac{1}{u}\hat{i}_{\dot{\pi}(t)} \circ V_0^{\pi(t)} = V_1^{\pi(t)}(\dot{\pi}(t)) + \frac{1}{u}\hat{i}_{\pi} \circ V_0^{\pi}.$$

By the lemma 2.8.2

$$\frac{d}{dt} V_0^{\pi(t)} + \frac{1}{u}\hat{i}_{\dot{\pi}(t)} \circ V_0^{\pi(t)} = \frac{1}{u}[V_0^{\pi(t)} \circ \hat{I}^{\pi(t)}_{U_1(\dot{\pi}(t))} + \Big\{(L_{\pi(t)} + ud) \circ X^{\pi(t)}(\dot{\pi}(t)) + X^{\pi}(\dot{\pi}(t)) \circ (b_*(t) + uB)\Big\}]$$

Applying this equation of differential operators on the cyclic cycle $c(t)$ satisfying the Gauss-Manin equation of parallel transportation, we obtain the equation

$$\begin{aligned} \frac{d}{dt}(e^{\hat{i}_{\pi/u}} V_0^{\pi(t)} c(t)) &= (\frac{d}{dt} + \frac{1}{u}\hat{i}_{\dot{\pi}(t)})(V_0^{\pi(t)} c(t)) \\ &= \left[\frac{1}{u} L_{\pi(t)} + d\right] X^{\pi(t)}(\dot{\pi}(t)) c(t) \\ &= d\left[e^{\hat{i}_{\pi(t)}/u} . X^{\pi(t)}(\dot{\pi}(t) c(t))\right] \end{aligned}$$

Thus, $(\frac{d}{dt} + \frac{1}{u}\hat{i}_{\dot{\pi}(t)})(V_0^{\pi(t)} c(t)) = \frac{d}{dt}(e^{\hat{i}_{\pi}/u} V_0^{\pi} c(t))$ is a boundary, and after passing to the cohomology level, it induces a zero homology class. To summarize, the quasi-isomorphism $V_0^{\pi(t)}$ intertwines the Gauss-Manin connection between Periodic cyclic homology bundle and $d + \frac{1}{u}\hat{i}_{\dot{\pi}(t)}$ connection on the $(\Omega(M), d + L_{\pi(t)})$ bundle. □

The meaning of this lemma is that we can trivialize the bundle to the commutative case, modulo a twisting isomorphism. For our case, $\dot{\pi}(t) = \pi$, and then

$$(\frac{d}{dt} + \frac{1}{u}\hat{i}_{\dot{\pi}(t)})(V_0^{\pi(t)}c(t)) = \frac{d}{dt}(e^{\hat{i}_{\pi}/u}V_0^{\pi(t)}c(t))$$

The problem boils down to studying the twisting factor $d + \hat{i}_{\dot{\pi}(t)} = d + i_{\dot{\pi}(t)} + \frac{u}{2}dL_{\dot{\pi}(t)} = d + i_{\dot{\pi}(t)} + \frac{u}{2}di_{\dot{\pi}(t)}d$ acting on the differential complexes $(\Omega^{-\bullet}(M), d + L_{\pi})$. Because the Gauss-Manin Connection commutes with B and b, the parallel transportation moves a cycle to a cycle and a boundary into a boundary.

$$\begin{tikzcd}
(C^\bullet(A_t[[\hbar]],*_t),b+uB) \arrow[r, "Module"] & (PC_\bullet(A_t[[\hbar]]),b_*+uB,\nabla_{GM}) \arrow[d, "V_0^\pi"] \arrow[dd, bend left, "\tilde{V}_0^\pi"] \\
(\Gamma(T^{\bullet+1}M),0) \arrow[u, "U"] \arrow[ur] \arrow[r] \arrow[dr] & (\Omega^{-\bullet}(M,\mathbb{R}[[\hbar]])((u)), L_{\pi(t)}+ud, \nabla=d+\frac{1}{u}\hat{i}_{\dot{\pi}(t)}) \arrow[d, "e^{\hat{i}_\pi/u}"] \\
 & (\Omega^{-\bullet}(M)[[\hbar]]((u)), ud, \nabla = d)
\end{tikzcd}$$

Computation of the monodromy of parallel sections

Applying the above commutative diagram to the bundle of noncommutative two-tori $A_\hbar = \langle U, V \mid UV = VUe^{2\pi i\hbar}\rangle$, we obtain the following differential equation for the horizontal sections on $(\Omega^{-\bullet}(M), d + L_{\pi(t)})$

$$\dot{c}(t) + \frac{1}{u}(i_\pi + \frac{u}{2}di_\pi d)c(t) = 0. \tag{2.8.5}$$

which induce constant sections on $(\Omega^{-\bullet}(M), d)$, modulo trivial cohomology classes. However, solving the differential equation on the infinite dimensional space $\oplus A^{\otimes\bullet}$ from 2.8.1 is quite complicated and we choose to pull back the parallel sections on the deRham cohomology bundle $(\Omega^{-\bullet}(M), d)$ back to the $HP_\bullet$ bundle.

Remark 2.8.4. The quasi-isomorphism $V^{\pi(t)}\circ(\tilde{V}^{\pi(t)})^{-1}{=}e^{\hat{i}_\pi/u}$ from $(\Omega^{-\bullet}(M), d)$ to $(\Omega^{-\bullet}(M), d + L_{t.\pi})$ can be read

$$dx \wedge dy \mapsto e^{it\partial_x\wedge\partial_y}dx \wedge dy = dx \wedge dy + t1$$

$$dx \mapsto e^{it\partial_x \wedge \partial_y} dx = dx$$
$$dy \mapsto e^{it\partial_x \wedge \partial_y} dy = dy$$
$$1 \mapsto e^{it\partial_x \wedge \partial_y} 1 = 1.$$

which induces an isomorphism between cohomology groups. The differential equation 2.8.5 is always integrable, since the differential operator $\partial_x \wedge \partial_y$ is nilpotent.

It is important to notice that two noncommutative tori A_t and A_{t+n} are always isomorphic, but the complexes $(\Omega^{-\bullet}(M), d + L_{t.\pi})$ and $(\Omega^{-\bullet}(M), d + L_{(t+n).\pi})$ are different. Therefore, to construct a local bundle of semi-classical complex $(\Omega^{-\bullet}(M), d + L_{t\pi})$, we need to choose a representative among the family of Poisson structure $(t + n)\pi$. If t is shifted by n, we obtain another family of quasi-isomorphisms $V^{\pi(t+n)}$ from $(PC_\bullet(A_t), b_* + B, \nabla_{GM})$ into $(\Omega^{-\bullet}(M), d + L_{(t+n).\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}}, \nabla)$.

$$\begin{tikzcd}
A_t \arrow[r, "\cong"] \arrow[d] & A_{t+1} \arrow[d] \\
(\Omega^{-\bullet}(M), d + L_{t\pi}) \arrow[r] \arrow[d] & (\Omega^{-\bullet}(M), d + L_{(t+1)\pi}) \arrow[d] \\
\Omega^{-\bullet}(M) \arrow[r, "="] & \Omega^{-\bullet}(M)
\end{tikzcd}$$

Thus, $V_{(t+1)\pi} \circ V_\pi^{-1}$ induces a map $e^{\hat{i}_{(t+1)\pi/u}} \circ [e^{\hat{i}_{t\pi/u}}]^{-1}$ on $(\Omega^{-\bullet}(M), d)$. By Remark 2.8.4, this morphism maps a parallel section to another parallel section.

The bundle $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}))$ is isomorphic to the bundle $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$ by gluing the fibers A_θ and $A_{\theta+1}$ together. Therefore, the monodromy of the Gauss-Manin connection on $H^\bullet(M) \cong HP^\bullet(A_0)$ is equivalent to the operator $e^{\hat{i}_{\pi/u}}$. We fix the basis

$$e^1_{odd} = (\hat{V}^\pi_0)^{-1}(dx)$$
$$e^2_{odd} = (\hat{V}^\pi_0)^{-1}(dy)$$
$$e^1_{even} = (\hat{V}^\pi_0)^{-1}(1)$$
$$e^1_{even} = (\hat{V}^\pi_0)^{-1}(dx \wedge dy)$$

By Remark 2.8.4 the monodromy acts on $HP_{even} = HC_0 \oplus HC_2 \cong \mathbb{C}^2$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Theorem 2.4.13 is proved for formal noncommutative two-torus bundle.

2.8.3 Proof for the smooth case:

$$\begin{array}{c} (PC_\bullet(\mathcal{A}_t), b_* + uB), \nabla_{GM}, \\ \Big\downarrow{\scriptstyle i} \\ (PC_\bullet(A_t[[\hbar]]), b_* + uB), \nabla_{GM} \\ \Big\downarrow{\scriptstyle V^\pi} \\ \Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])((u)), L_{\pi(t)} \end{array}$$

Recall that the multiplication for the the smooth noncommutative two-torus 2.3.1

$$\phi *_\hbar \psi(p) = \sum_q \phi(q)\psi(p-q)e^{-\pi i\hbar\gamma(q,p-q)} \ \forall \phi, \psi \in S(\mathbb{Z}^2)$$

coincides with the one for the formal noncommutative two-torus. Therefore, we obtain an imbedding of the $PC_\bullet$−complexes of bundles of smooth tori with Gauss-Manin connection into the ones of formal noncommutative two-tori.

Because the Shoikhet-Dolgushev morphism intertwines the connections, the composition $V^\pi \circ i$ does as well. By [94], the Gauss-Manin connection is integrable, so the parallel transportation exists as well. Remark that we can choose the basis for the periodic cyclic homology in both of smooth and formal cases by the finite cyclic chains

$$\left\{1, U^{-1}dU, V^{-1}dV, V^{-1}U^{-1}dUdV - U^{-1}V^{-1}dVdU\right\}.$$

On the other hand, the restriction of the Gauss-Manin connection of the formal tori to the class of fast decaying series coincides with the one in the sense of [94].

Thus, the monodromy functors for the smooth and formal cases are the same for the cyclic chains. Together with the proof for the formal case, it implies that the images of the monodromies for the elements in $PC_\bullet(\mathcal{A}_t)$ coincides with the ones for elements in $\Omega^{-\bullet}(M)[[\hbar]]$. The theorem is proved.

Remark 2.8.5. We are told via a personal correspondence with a secret expert that this case is quite simple, in the sense that the flat connection of the bundle of noncommutative two-tori leads naturally to the deformation quantization. In fact, in this case, the roles of t and $\hbar$ are almost the same.

Example 2.8.6. However, the monodromy of the bundle in Example 2.4.12 is trivial, because the parallel transportation on the intervals $[\frac{1}{2}, 1]$ and $[0, 1/2]$ is inverted to each other. The same technique is also applied, except for the fact that the monodromy map is $V_{t\pi} \circ [V_{(t+\frac{1}{2})\pi}]^{-1} \circ [V_{(t+\frac{1}{2})\pi}] \circ V_{\pi}^{-1} = Id$. Therefore, two noncommutative two-torus bundles in 2.4.12 and 2.4.13 are non-isomorphic.

Remark 2.8.7. Note that in [25] the same result has been obtained using $KK-$theory method, but it is not clear to us how the Rieffel projections behave when α passes over integer values nor the relationship with the semi-classical limit.

The critical point left in the understanding of the Shoikhet-Dolgushev morphism $\tilde{V}_0^{\pi}$ is that we do not know exactly which basis in $HP_{\bullet}(A_t)$ corresponds to the standard one $\langle 1, dx, dy, dx \wedge dy \rangle$. It is possible to do calculation, but the computation is quite complicated. Even in the case of a trivial Poisson structure $\pi = 0$, in [11] the map is quite complicated and can be computed by

$$\tilde{V}_0^{\pi=0}(c) \quad = \quad \hat{A}_u(c).C_0(c)$$

with $c \in PC_{\bullet}(A)$, $\hat{A}_u(M) = \sum_n u^n \hat{A}_{2n}(M)$ the components of the A-roof genus of M, and C_0 the Connes maps

$$C_0(a_0, a_1, ..., a_n) = a_0 da_1 \wedge ....da_n.$$

2.9 A global invariant

2.9.1 Monodromy map

We generalize the monodromy theorem 2.4.13 to the general case in order to obtain a global invariant of the noncommutative two-torus bundles.

Theorem 2.9.1. *Let $\left\{A_{\theta(m)}\right\}$ be a smooth family of noncommutative two-tori over a connected smooth compact manifold M. We assume that there* *exists at least* $m_0 \in M$ *such that* $A_{\theta(m_0)} \cong C(\mathbb{T}^2)$, *i.e.* $\theta(m_0) \in \mathbb{Z}$. *Then, there exists a homomorphism* Mon *from the fundamental group* $\pi_1(M)$ *of* M *to* $GL(2, \mathbb{Z})$.

Proof. We notice that the noncommutative two-torus bundles having no classical torus fiber can be classified in the same fashion. However, because they have no twisting factor, the bundles can be pulled back from the universal bundle $C^*(H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{R}))$. The classification then can be obtained from Theorem 2.4.5.

We construct the monodromy maps $Mon : \pi_1(M) \to GL(2, \mathbb{Z})$ by using the monodromy theorem 2.4.13. Let $\phi(.) : S^1 \to M$ be a smooth path on M, which represents an element $[\phi(.)]$ of $\pi_1(M)$. In the following construction, we assume that all the curves always start from a classical torus, *i.e.* $A_{\theta(0)} \cong A_{\theta(1)} \cong C(\mathbb{T}^2)$.

Pushing forward $A_{\theta(t)}$ through ϕ, we obtain a noncommutative fiber bundle $A_{\theta(\phi(t))}$ over S^1. Because $\mathbb{T}_\theta \cong \mathbb{T}_{\theta+1,}$ this bundle over S^1 can be decomposed into the composition of fundamental blocks, each of which is a smaller sub-bundle over sub-intervals of S^1. The principal rule for the decomposition is that all of the fibers of any fundamental block are assumed to be non-isomorphic to the classical torus, except for the start and the end points. If they are not, it is possible to decompose the block into smaller ones.

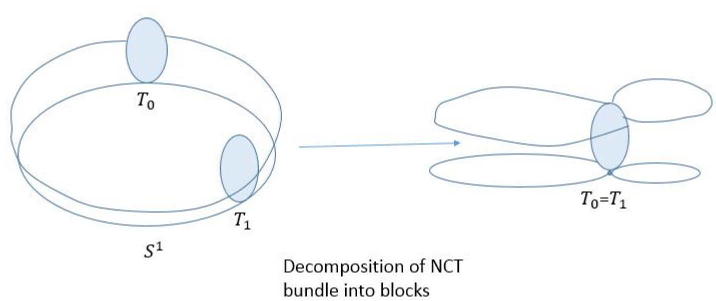


Figure 2.9.1: Decomposition of monodromy

In Theorem 2.4.5, we classified all homotopically trivial noncommutative two-torus bundles. Because any noncommutative two-torus bundle over S^1 is isomorphic to a bundle over $[0, 1]$ with identified fibers lying over 0 and 1, it is possible to identify the moduli space of fundamental blocks with classical end points

$$\left\{ \left[A_{\theta((\phi(t))} \right]_{S^1} \mid A_{\theta(1)} = C(\mathbb{T}^2) \right\}$$

and the moduli space of pairs of noncommutative two-torus bundles $\{A_{\theta((\phi(t))}\}$ over $[0,1]$ with the classical end points and a monodromy map in $GL(2,\mathbb{Z})$

$$\left\{\left[A_{\theta((\phi(t))}, g\right]_{t\in[0,1]} \mid \lim_{t\to\{0,1\}} \theta((\phi(t)) = 0, g \in End(HP_{even}(\mathbb{T}_\theta)) \cong GL_2(\mathbb{Z})\right\},$$

where g is the monodromy map along the Gauss-Manin connection. Then, we normalize the latter among the homotopy class of $\theta(t)$ to obtain the linear ones with the same end points.

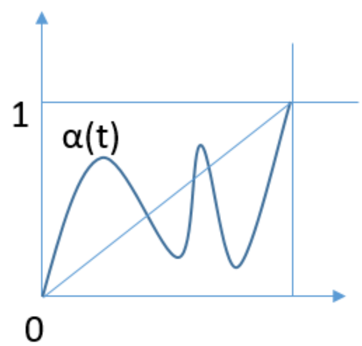


Figure 2.9.2: Normalized monodromy

We notice that because the Gauss-Manin connection is flat at the homology level, the monodromy map does not depend on the paths among their homotopy classes. Then the monodromy of the curve ϕ is obtained by taking the product of all the monodromies of the fundamental block components.

We want to clarify that the monodromy takes values in $GL(2,\mathbb{Z})$, not in $SL(2,\mathbb{Z})$. The reason is that the monodromy map for general bundle $\{[A_{\theta(\phi(t))}, g]\}$ over S^1 may be different from the one in Theorem 2.4.13. There is no twisting element for the noncommutative torus bundle associated with the group C^*- algebra of the discrete Heisenberg group. But in general, this case may happen. By the classification in [25], a principal noncommutative two-torus bundle can be found in terms of a classical torus fibration together with a strict deformation quantization of every fiber. Therefore, we need to know the action of the twisting factor of the classical torus fibers on the monodromy.

Notably, by subsection 2.1.2, the isomorphism between two smooth noncommutative tori is $(PU(\mathbb{T}^\infty_\theta)^0) \rtimes (\mathbb{T}^2 \rtimes SL(2,\mathbb{Z}))$ a product of: □

1. A translation by the classical torus $\mathbb{T}^2$.
2. Morphism Φ_{twist} inside $SL(2,\mathbb{Z})$, which is also an automorphism of A_θ.
$$\Phi_{twist} : U \mapsto U^a V^b, V \mapsto U^c V^d$$
such that $ad - bc = 1$,
3. Possibly a flip Φ_{flip} exchanging U and V, which induces an isomorphism between A_θ and $A_{-\theta}$.

It is obvious that twisting by modular group Φ_{twist} preserves the second periodic cyclic homology group and the volume form $dx \wedge dy$ of the torus. Thus, Φ_{twist} plays no role in the monodromy map on the $HP_{even}(A) \cong H^2(\mathbb{T}^2)$. The same is for the translation and the inner automorphisms.

But the flip $U \mapsto V$, $V \mapsto U$ induces an isomorphism Φ_{flip} from A_θ to $A_{-\theta}$. In the odd homology level, the map induces $dx \mapsto dy$ and $dy \mapsto dx$, which acts on $HP_{odd}(A_\theta)$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the even homology group, the twisting maps $dx \wedge dy$ into $dy \wedge dx = -dx \wedge dy$, which acts on $HP_{even}(A_\theta)$ by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In summary, the twisting factor of the fibers by a volume preserving isomorphism results in twisting of the $HP_{odd}-$group. The monodromy maps constructed above and the flip result in twisting of the $HP_{even}-$group. Then, there exists a monodromy map from $\pi_1(M)$ to $GL(2,\mathbb{Z})$. The theorem is proved.

In short, the problem of classification of noncommutative two-torus bundles boils down to the classification of homotopically trivial bundles (which has been solved), the classification of $\mathbb{T}^2-$torus bundle, which is known, and the classification of monodromy maps.

Remark 2.9.2. Because the group of all the monodromies is generated by the one for the fundamental blocks, and the maps $T : \theta \mapsto \theta + 1$ and $S : \theta \mapsto -\theta$ does not commute, the monodromy map may not factor through the first homology group $H_1(M)$. More precisely, we can compute that $STS^{-1} = T^{-1}$.

For example, the fundamental group of the plane minus two points is isomorphic to $\mathbb{Z} * \mathbb{Z} \cong \langle a, b \rangle$, and the monodromy map is different from its abelianization $\mathbb{Z}^2$. Up to now, we still do not know a **natural** construction of such a bundle over $CP^1 \setminus \{there\, points\}$ with such a monodromy, although it

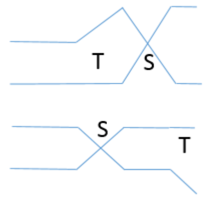


Figure 2.9.3: S and T

is possible to build one using the cut, paste and glue operations of bundles. We describe one construction here, and hope to get a better one in the future. Given the complex plane $\mathbb{C}$, to every $z \in \mathbb{C}$ we associate a trivial noncommutative two-torus $\mathbb{T}^2_{arg(z)}$. We deform the bundle a bit, so that the fibers over a very narrow strip $(-i\epsilon, i\epsilon) \times [1, \infty)$ are classical tori. Deleting a real half line $[1, \infty)$ from the complex plane, and then gluing the local open half disks together by the flip $U \leftrightarrow V$.

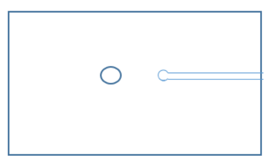


Figure 2.9.4: Cut and paste

2.9.2 Dequantization

We shall now see how the monodromy map affects the behavior of the Poisson bundles.

Theorem 2.9.3. *Let $\left\{A_{\theta(m)}\right\}$ be a smooth noncommutative two-torus bundle over a smooth manifold M. Then there exists a Poisson manifold P fibered over M, which corresponds to $A_{\theta(m)}$ only if the monodromy map is trivial.*

Proof. The proof is deduced directly from the above. Assume that there exists a Poisson manifold P dequantizing $A_{\theta(m)}$, with nontrivial monodromy maps. We choose a curve γ on M such that the monodromy map $Mon(\gamma)$ is nontrivial, then $\{A_{\theta(\gamma(t))}\}$ is a family of noncommutative two-tori over S^1

which quantize a Poisson bundle over S^1. We decompose the path into the fundamental blocks, and apply Theorem 2.4.13. The contradiction follows from the fact that there is no Poisson torus bundle quantizing to $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{Z}))$.

Construction of the Dequantization bundle

Given a noncommutative torus bundle $\left\{A_{\theta(m)}\right\}$ over a manifold M, we will construct the dequantizing Poisson manifold. The process is to build the torus fibration, then build a family of global Poisson structures from local ones.

We start with local principal families pulled back from the universal Poisson bundle $(\mathbb{R}\times\mathbb{T}^2, t.\partial_x\wedge\partial_y)$ via the Gauss-Manin connection. Any noncommutative two-torus bundle in our senses can be decomposed into noncommutative two-torus bundles over homotopically trivial open sets in the sense of Echterhoff [25]. Namely, there exist open sets $\{U_i\}_i$ covering the spectrum of the center of the noncommutative two-torus bundles, such that on each U_i, the bundle can be localized to be isomorphic to $C(U_i)\otimes_{C_0(\mathbb{R})}C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$. Although the global torus action is not given in the first place, the tensor products $A_i = C(U_i)\otimes_{C_0(\mathbb{R})} C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})$ already posses a natural one by the action on the right hand side.

Applying Echterhoff's theorem [25] to the principal bundles A_i, we obtain a family of torus bundles $q_i : \mathbb{T}^2_i \to Y_i \to U_i$ with fibers $\mathbb{T}^2$ equipped with 2-cycles σ_i. Over a single open set U_i of the open cover $\{U_i\}_i$, a noncommutative two-torus bundle defines a family of Poisson structures $(\theta_i + n_i).\partial_x \wedge \partial_y$, up to a constant $n_i \in \mathbb{Z}$.

The family of quantum gluing functions

$$\phi_{ij} : C(U_i)\otimes_{C_0(\mathbb{R})} C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R})) \to C(U_j)\otimes_{C_0(\mathbb{R})} C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$$

consist of the isomorphism of the bundle of tori. On the subbundle of smooth noncommutative tori, the automorphism group is given by $(PU(\mathbb{T}^\infty_\theta)^0) \rtimes (\mathbb{T}^2 \rtimes SL(2,\mathbb{Z}))$. However, the inner automorphism group leads to isomorphic noncommutative torus bundles, and it remains to take care of $\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})$. Therefore, a torus bundle can be obtained by gluing the trivial torus bundles $\{C(U_i)\times\mathbb{T}^2, \phi_{ij}\}$ together with the isomorphism given by the translation of the classical torus and $SL(2,\mathbb{Z})$ automorphisms of the $\mathbb{Z}^2$ lattice in $\mathbb{R}^2$. Obviously, both of these transformations preserve Poisson structure.

The only thing left is the construction of the Poisson structure from those on open sets. The condition to glue the Poisson structure is obtained by the triviality of the monodromy map. The Poisson torus bundle is then determined up to an integral Poisson structure. QED. □

Conjecture 2.9.4. *We think that if the monodromies of the odd homology classes are trivial, then the noncommutative torus bundle is principal.*

Under the stronger condition that there exist global unitary sections $\{U_t\}$ and $\{V_t\}$ satisfying the condition $U_tV_t = V_tU_te^{2\pi i\theta(t)}$, then clearly a global action of the classical torus exists by subsection 2.1.2. Furthermore, we can verify that $\langle U_t^{-1}dU_t, V_t^{-1}dV_t\rangle$ are generators of the odd $HP-$ group, and parallel under Gauss-Manin connection.

Remark 2.9.5. Given a bundle of Poisson tori $(\mathbb{T}^2, f(t)\partial_x \wedge \partial_y)$, fibered over a manifold M. Bursztyn proved in [8] that the pairing between the cohomology classes defined by the corresponding symplectic structure (for $f(t) \neq 0$) and the homology group $H_2(\mathbb{T}^2, \mathbb{Z}) \cong H^2(C(\mathbb{T}^2), \mathbb{Z})$ form a lattice inside the line bundle over M.

$$\langle f(t)^{-1}dx \wedge dy, H_2(\mathbb{T}^2)\rangle = f(t)^{-1}.\mathbb{Z}$$

Therefore, we obtain a map from $H_2(\mathbb{T}^2)$ to $C^\infty(M)$. Taking the derivative of this map, we obtain a map from $H_2(\mathbb{T}^2)$ to T^*M.

If $M \cong \mathbb{R}$, we obtain a lattice like the one in the picture 2.4.2 and the graph is $\frac{-f'(t)}{f(t)^2}\mathbb{Z}$. Obviously, the variation lattice in the sense of Dazord relates to the derivative of the parallel transportation in the Gauss-Manin connection. Therefore, we obtain the proof for the proposition.

Proposition 2.9.6. *Let P be a Poisson manifold corresponding to a quantum torus fibration. Then, the integration of the Dazord variation lattice along any closed curves is trivial.*

Example 2.9.7. The Poisson manifold for the group C^*-algebra of the discrete Heisenberg group does not exist. However, if we remove any single noncommutative two-torus fiber A_θ, then the Poisson manifold exists, which is isomorphic to $(0,1) \times T^2, (t-\theta).\partial_x \wedge \partial_y)$. The variation lattice is constant 1, which lead to $\int_{(0,1)} 1dt = 1 \neq 0$.

Remark 2.9.8. For any noncommutative two-torus bundle $\{A_{\theta(t)}\}_t$, there always exists a Poisson manifold P quantizing locally to $\{A_{\theta(t)}\}_t$.

The construction was obtained by folding the noncommutative two-torus bundle through a process like the one in Theorem 2.4.2 so that all the monodromy maps are trivial. Hence, by Theorem 2.9.3, the dequantized Poisson bundle exists.

2.9.3 Morita equivalence

We have known that the monodromy of a noncommutative two-torus bundle acting on the $HP-$group is somehow related to the element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of the mapping class group $GL(2,\mathbb{Z})$ of the two-torus. One can raise the following naive question "where does the element $S = (\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}) \in GL(2,\mathbb{Z})$ go?"

The answer lies in the Morita equivalence of the bundles, where the gluing isomorphisms are replaced by Morita functors. In order to study the Morita functor between families, we need a relationship between the fiber-wise and global Morita equivalence.

Lemma 2.9.9. *Let M be any Morita bimodule between two continuous fields of C^*- algebras. Then, if one of these C^*-algebras is continuously fibered over a locally compact Hausdorff space T, the other one can also be fibered over the same base. Furthermore, the Morita bimodule can be decomposed into a bundle of Morita bimodules between the fiber algebras over T.*

Proof. If A and B are Morita equivalent via a bimodule X, $B \cong End_A(X)$, the center of A and center of $End_A(X)$ are isomorphic. The correspondence here is the map $a \mapsto X_a$, where X_a is the left multiplication by a. Thus, the spectrums of their centers are homeomorphic as locally compact spaces. Denote the spectrum by T, then the following diagram commutes; here all the arrows are embeddings of C^*-algebras.

$$\begin{tikzcd} C_0(T) \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ End_A(X) \arrow[r, hook] & End(X) \end{tikzcd}$$

Let $t \in T$ and m_t be the ideal generated by functions in $C_0(T)$ vanishing at t. Then, let $A_t = A/A.m_t$, $B_t = B/B.m_t$, and denote by X_t the fiber $X/m_t.X$. Then there exists a structure of $A_t - B_t$ bimodules over X_t. We can verify quite easily that X_t are A_t and B_t-full modules with the norm induced from X. Then A_t and B_t are Morita equivalent, via the bimodule X_t. The lemma is proved. $\square$

Another way to view the lemma is that the center of a C^*- algebra can be identified with the center of the category of its Hermitian modules (the collection of natural transformations from the identity functor of the category to itself) by the result of Rieffel [72]. Two Morita equivalent C^*-algebras have isomorphic categories of Hermitian modules, hence the quotient categories are isomorphic to each other when we localize at the same point.

Theorem 2.9.10. *Let $A_{\theta(t)}$ be a homotopically trivial family of noncommutative two-tori over a **contractible** locally compact space T. For any $g \in GL(2,\mathbb{Z})$, such that $g\theta(t)$ is well-defined, the bundles of noncommutative two-tori $A_{\theta(t)}$ and $A_{g\theta(t)}$ are Morita equivalent.*

By contrast, any Morita equivalence between two homotopically trivial bundles can be determined by an element g of $GL(2,\mathbb{Z})$.

Proof. **Part I**

Because the elements of $GL(2,\mathbb{Z})$ are the composition of finite translations $\theta \mapsto \theta + 1$ and the inversion $\theta \mapsto \frac{1}{\theta}$, whereas the translation is just the isomorphism $A_\theta \cong A_{\theta+1}$, we only need to show that A_θ is Morita equivalent to $A_{\frac{1}{\theta}}$.

We build the bimodule directly with the techniques owed to Rieffel in [72]. For K and H locally compact groups acting freely and wanderingly on the left and the right of a locally compact space X, two C^*-algebras $C_0(K\backslash X) \rtimes H$ and $C_0(X/H) \rtimes K$ are strongly Morita equivalent (also see Example 2). Apply Rieffel's theorem to $X = \mathbb{R} \times \mathbb{R}$, $K = \theta(t).\mathbb{Z}$ and $H = \frac{1}{\theta(t)}\mathbb{Z}$ and it follows that $C(S^1) \rtimes_{\theta(t)} \mathbb{Z} \cong A_{\theta(t)}$ and $C(S^1) \rtimes_{\frac{1}{\theta(t)}} \mathbb{Z} \cong A_{\frac{1}{\theta(t)}}$ are strongly Morita Equivalence.

Part II

The converse is more complicated. Apply lemma 2.9.9, then the Morita bimodules between bundles can be decomposed into a bundle of Morita bimodules between fiber algebras. By breaking the bundle into pieces fibered over open subsets of the base, we can assume that any bundle can be obtained by pulling back from the universal bundle $C^*(H_3(\mathbb{Z},\mathbb{Z},\mathbb{R}))$. Apply Rieffel's theorem [71], for any single value $t \in \mathbb{R}$, if $A_{\theta(t)}$ and $A_{\psi(t)}$ are Morita equivalent, then $\theta(t)$ and $\psi(t)$ are $GL(2,\mathbb{Z})-$related.

Because the Rieffel projections in $A_{\theta(t)}$ and $A_{\psi(t)}$ can be extended continuously to an open neighborhood I of t as long as $\theta(I)$ and $\psi(I)$ do not contain integral points, the images of the projections under the canonical trace must coincide for all the values $t' \in I$. Therefore, due to the continuity of the images, a common element $g \in GL(2,\mathbb{Z})$ can be chose for the whole interval I.

The only thing left is to prove the lemma for the small interval containing integral points. However, we can compose the Morita bimodule with any other Morita functor in $GL(2,\mathbb{Z})$ to move away from the integer points and the same process applied. The theorem is proved. $\square$

As for the nature of the theorem, the Morita equivalence between continuous bundles equals the Morita equivalence for every fiber. Another way to prove the theorem is using the result on the invertible $KK-$elements by Dadarlat [16], where the author showed that element $\sigma \in KK(A,B)$ is invertible if and only if for all $x \in X$, $\sigma_x \in KK(A_x, B_x)$ is also $KK-$invertible. Then, Theorem 2.9.10 is obtained for σ a Morita equivalent bimodule.

In the construction of the Morita bimodules between C^*-bundles, we need to choose an element $t \in \mathbb{R}$ so that the equivalence can be realized as a transformation $t \mapsto \frac{at+b}{ct+d}$. Choosing a specific value of t among the different ones boils down to finding a Poisson torus in the class of tori $\{\mathbb{T}^2, (t+n)\partial_x \wedge \partial_y\}$ admitting the same quantization. Certainly, when we work with a single noncommutative two-torus, or even a homotopically trivial family of them, the different choices of t or the different choices of the Poisson structures do not matter, there are always a lot of Poisson structures around. However, for the bundles admitting no semi-classical limit, *i.e.* no Poisson structures for us to choose, it is predictable that there is a subtlety.

Lemma 2.9.11. *The choice of $g \in GL(2,\mathbb{Z})$ in Theorem 2.9.10 is unique. Equivalently, if there exists $g' \in GL(2,\mathbb{Z})$, such that g' also plays the role of the Morita bimodule between two local bundles with fixed semi-classical counter parts, then $g = g'$.*

The proof is easy. The equation $\frac{a\theta+b}{c\theta+d} = \frac{A\theta+B}{C\theta+D}$ for all θ in an open interval implies a system of equations, which is equivalent to $a = A, b = B, c = C, d = D$. The computation is trivial, and will be omitted.

Theorem 2.9.12. *Let $A(X)$ be a continuous field of NCTs, such that the algebra $A(X)$ admits no semi-classical limit. Then, the Morita equivalent class of $A(X)$ reduces to isomorphism class.*

Proof. Let $\gamma : S^1 \to X$ be a curve in X, such that $\Phi([\gamma]) \in SL(2,\mathbb{Z})$ is not the identity. Then, if $A(X)$ and $B(X)$ are strongly Morita equivalent with the bimodule M, $A(X) \otimes_{C_0(X)} C(S^1)$ and $B(X) \otimes_{C_0(X)} C(S^1)$ are also strongly

Morita equivalent with the bimodule $M \otimes_{C_0(X)} C(S^1)$. The problem reduces to the case $X \cong S^1$, where we can apply the computation for the Heisenberg group.

Because the local Morita equivalence can be realized as the action of an element of $GL(2,\mathbb{Z})$ by Theorem 2.9.10, we only need to study the ability to extend the Morita functor to the whole circle.

Assume that the extension is possible. For any θ, the gluing condition requires that $A_{g(\theta+1)}$ and $A_{g(\theta)}$ for all θ in an open neighborhood. Therefore,

$$g(\theta+1) = \pm g(\theta) + n \tag{2.9.1}$$

Or $g(\theta+2) = g(\theta)+2n$, for some $n \in \mathbb{Z}$. Equivalently, we obtain the equation

$$\frac{a(\theta+2)+b}{c(\theta+2)+d} = \frac{a\theta+b}{c\theta+d} + 2n$$

which implies

$$\pm 2 = n(c\theta+d)^2 + n(c\theta+d)c$$

or

$$\pm 1 = n[d^2]$$

$$\begin{cases} c & = 0 \\ nd^2 & = \pm 1 \end{cases}$$

because $c = 0$, we obtain $g(\theta) = \pm\theta + b$.

For $nd^2 = 1$, then $n = 1$ and $g(\theta) = \theta + b$, which is just the translation.

For $nd^2 = -1$ then $n = -1$ and $g(\theta) = -\theta + b$, which is just the composition of a translation and a flip $U \longleftrightarrow V$.

The theorem is proved. □

The proof can be shortened by noticing that any element of $SL(2,\mathbb{Z})$ can be decomposed into the product of a series of translations $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and at most one inversion $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The translation preserves Equation 2.9.1, but the inversion could never. The theorem is then immediate.

2.9.4 Bundle of C^*-Stacks and compactification of the moduli spaces

We will now investigate how the inversion $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of the Morita functor behaves in the $HP-$groups. A quick computation at of the classical fibers reveals that the map S acts by

$$1 \mapsto dx \wedge dy$$

$$dx \wedge dy \mapsto -1$$

on the group $HP_{even}(C^\infty(\mathbb{T}^2))$. This map clearly cannot be induced by any classical automorphism of the classical torus. In the later part, we will see that it plays the role of a generator for group $SO(n, n, \mathbb{Z})$ for higher dimensional noncommutative tori.

Note that the class of continuous fields of noncommutative two-tori is not closed under Morita equivalence. Only a subclass of those fields admitting semi-classical limit is closed under Morita equivalence.

There are two approaches to the problem: (1) we can choose to work with only those with semi-classical counters and Morita equivalence, or (2) we must extend the category of noncommutative two-torus fibrations to new objects so that the category becomes closed under the Morita functor.

The second approach leads to a new mathematical object obtained by gluing together bundles of C^*- algebras over open sets $\{U_\alpha\}_\alpha$ by continuous families of Morita bimodules fibered over $U_\alpha \cap U_\beta$ with the usual compatible conditions. Because Morita equivalence of C^*-algebras is an isomorphism between the categories of Hermitian modules, the object that we described may be understood as a continuous bundle of categories of modules.

Definition 2.9.13. Let A_i be families of bundles of C^*-algebras fibered over open subsets U_i of a topological space X, together with a family ${}_iM_j$ of Morita bimodules between $A_i \mid_{U_j \cap U_j}$, satisfying the conditions

$${}_iM_j \otimes_{A_j\, j} M_i \cong A_i$$

as $A_i - A_i$ bimodules and

$${}_iM_j \otimes_{A_j\, j} M_k \cong_i M_k$$

as $A_i - A_j$ bimodules. We call the family $\{U_i, A_{i,i} M_j\}_{i,j}$ a presentation of a **bundle of C^*-stacks.**

Between two families $\{U_i, A_{i,i} M_j\}_{i,j}$ and $\{U_i, B_{i,i} N_j\}_{i,j}$ we can define a C^*-correspondence from $\{U_i, A_{i,i} M_j\}_{i,j}$ to $\{U_i, B_{i,i} N_j\}_{i,j}$ by continuous families of C^*-correspondences $\{_{B_i}P_{iA_i}\}_i$ from A_i to B_i, such that the diagram commutes

$$\begin{tikzcd} A_i \arrow[r, "_iM_j"] \arrow[d, "P_i"] & A_j \arrow[d, "P_j"] \\ B_i \arrow[r, "_iN_j"] & B_j \end{tikzcd}$$

If two families are defined with different systems of open sets, obviously we choose a finer one. It is easy to check that the relation constitutes an equivalence relation and we call the equivalence class of families (A_i, M_i) **a bundle of C^*-stack.**

From the viewpoint of Artin stacks, two Morita equivalent groupoids are presentations of the same stack. If we think of a C^*-algebra as a C^*-algebra of groupoid, then the object that we define is a C^*- analog of the bundle of stacks.

We would like to mention the following theorem.

Theorem 2.9.14. *(Classification of C^*-stacks of the noncommutative two-tori) The category of C^*-stacks of noncommutative two-tori NCT is equivalent to the category of $\mathbb{T}^2-$bundles associated with a section s_i of trivial line bundle, and $GL(2,\mathbb{Z})-$ morphisms as gluing morphisms.*

By $GL(2,\mathbb{Z})-$morphisms, we mean elements $g_{ij} \in GL(2,\mathbb{Z})$ such that $g_{ij}(s_i) = s_j$, and satisfying the conditions $g_{ii} = Id$, $g_{ij} \circ g_{ji} = Id$, $g_{ij} \circ g_{jk} \circ g_{ki} = Id$ as elements in $GL(2,\mathbb{Z})$.

Proof. The proof is immediate for the values that are well defined under $GL(2,\mathbb{Z})$ morphisms. Given a presentation $\{U_i, A_{i,i} M_j\}_{i,j}$ of a C^*-stack, it is easy to construct a family of $\mathbb{T}^2-$bundles over U_i, together with family of J-curves on U_i. By Theorem 2.9.10, we obtain $g_{ij} \in GL(2,\mathbb{Z})$ for any $U_i \cap U_j$, and it is easy to see that the conditions are satisfied.

On the reverse side of the theorem, applying the same technique in Theorem 2.9.1, we can construct the presentation for the C^*-stack locally. Because the generators of HP_{odd} can be dealt with by the automorphisms of noncommutative two-tori, the only thing left is the HP_{even}. By Theorem 2.9.10, the rest follow by gluing the continuation of generators of HP_{even} via Gauss-Manin connections. QED. □

We ask what happens if the $GL(2,\mathbb{Z})-$ action is not defined? Although one may guess that there is no continuous noncommutative two-torus bundle associated to the Morita equivalence, it is nice to see how it happens. It is possible to compactify the real line $\mathbb{R}$ to obtain the projective line $\mathbb{R}P^1$ by adding the point $\{\infty\}$ so that the group $GL(2,\mathbb{Z})$ is well-defined everywhere. Equivalently, we need to find a fiber corresponding to the infinity point, which we will denote by A_∞.

Because the unknown A_∞ should be Morita equivalent to A_n via the Morita equivalence $\theta \mapsto \frac{1}{\theta-n}$, naively the fiber that we add should be $C(\mathbb{T}^2)$. But this is not quite right because of the singularity.

Theorem 2.9.15. *Let $\{A_{\theta(t)}\}$ be a continuous field of noncommutative two-tori defined over $(0,1]$, such that $\theta(t)=\frac{1}{t}$ for $t\in(0,1]$. Then, there is no continuous field of NCTs over $[0,1]$ extending $\left\{A_{\theta(t)}\right\}$.*

Proof. Assume that there exists $\left\{B_{\alpha(t)}\right\}_{[0,1]}$ such that $B_{\theta(t)}=A_{\theta(t)}$. By Definition 2.4.2, locally around the zero point, there exist a continuous function $f:[0,1]\to\mathbb{R}$ such that $f(t)=\frac{1}{t}$. However, it never happens for any continuous function f.

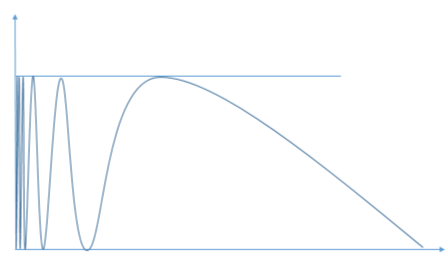


Figure 2.9.5: Compactification of the moduli space of noncommutative two-tori

We can proceed with another approach as follows. By Definition 2.3.2 and applying the technique exactly like in the lemma 2.4.3, there exists two unitary sections $U_{\theta(t)}$ and $V_{\theta(t)}$ such that

$$U_{\theta(t)}V_{\theta(t)}U_{\theta(t)}^{-1}V_{\theta(t)}^{-1}=e^{\frac{2\pi i}{t}}Id. \tag{2.9.2}$$

Choose a sequence $t_k = \frac{1}{k+\theta}$, then equation 2.9.2 implies

$$U_{\theta(t_k)}V_{\theta(t_k)}U^{-1}_{\theta(t_k)}V^{-1}_{\theta(t_k)} = e^{2\pi i(k+\theta)}Id.$$

Therefore, on the fiber B_0 contains a copy of noncommutative two-tori A_θ for any θ. This contradicts with the structure of the subalgebras of noncommutative two-tori as described in [66].

In summary, there is no noncommutative two-tori to compactify the bundle. $\square$

On the semi-classical side, when α tents to infinity, the Poisson structures corresponding to the noncommutative two-tori converge to infinity, or the symplectic structures reduce to zero. Therefore, in order to work with continuous families, we need to pass to the class of continuous fields of Dirac structures.

Recall that a Dirac structure on a manifold M is a maximal isotropic subbundle $L \subset TM \oplus T^*M$, satisfying some conditions (for details see [8]). For the two tori, the Poisson structures are given by subbundles $\langle dx + \theta\partial_y, dy - \theta\partial_x\rangle$, and when θ tends to infinity, the subbundles converge to $\langle \partial_x, \partial_y\rangle$, which is a Dirac structure.

In the paper [91]by Weinstein and Tang, the authors show that given a Dirac structure, it is possible to construct a deformation quantization up to a choice of a transverse manifold. Firstly, any Dirac structure determines a foliation F, which is given by the intersection $L \cap TM$. Choosing a transverse manifold P of the foliation, then on P it induces a Poisson structure π_L (by restricting L to P) and a groupoid structure G over P by intersecting with F on P. Finally, we quantize the Poisson manifold P to obtain a quantum algebra A_P, and the cross product $G \ltimes A_P$ is called the quantization of the Dirac structure.

Applying the machinery to the case of two tori, then:

1-For the Poisson tori with $L = \langle dx + \theta\partial_y, dy - \theta\partial_x\rangle$, the foliation consists of just points, so P contains all of $\mathbb{T}^2$. Therefore, $A_p = \mathbb{T}_\theta$ and the group action is trivial. As a result, the quantization is just the noncommutative two-tori.

2-For the Dirac tori with $L = \langle \partial_y, \partial_x\rangle = TM$ the foliation consists whole $\mathbb{T}^2$. Therefore, P is just a point, and there is a trivial action of $\mathbb{Z}^2$ on P, given by the action of $\mathbb{R}^2$ on $\mathbb{T}^2$. Therefore, $A_p = \mathbb{Z}^2 \ltimes C^*\{pt\}$, which follows that the quantization algebra is $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$.

It is very important to see that although the quantization algebra of the infinity Poisson tori is Morita equivalent to the classical tori, *i.e.* the fiber $\mathbb{T}_0^2$, by Theorem 2.9.15 we cannot just add the classical tori to the bundle. The reason is that there are more structures on the bundles that we can detect than just C^*-algebras. The new structure is developed in chapter 3, called spatial structures.

Assuming that the reader has read chapter 3, we claim that:

Theorem 2.9.16. *Let L be a Dirac structure on the tori $\mathbb{T}^2$. Then on any quantization of the Dirac structure, there exist a spatial structure.*

Proof. We proceed exactly as in the construction by Weinstein and Tang [91]. Let P be any transverse section of the foliation F, then (P, π_L) forms a Poisson torus. We even prove for tori of any dimension.

$$\begin{tikzcd}
\mathbb{T}^2 \times \mathbb{T}^2 \arrow[r, bend left] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{T}^2 \arrow[dl, "\Delta"] \arrow[dr] & \mathbb{T}^2 . \arrow[l, bend right] \arrow[d, shift left] \arrow[d, shift right] \\
\mathbb{T}^2 \times \mathbb{T}^2 & & \mathbb{T}^2 \arrow[ll, dashed]
\end{tikzcd}$$

Therefore, $\mathbb{T}^n$ is embedded into $\mathbb{T}^n \times \mathbb{T}^n$ as diagonal. Then, the bimodule structure is constructed by quantizing these Poisson tori in Example 3.3.4.

Let $\Delta : A_\theta \otimes A_\theta \to A_\theta$ be the quantization of the diagonal map, obtained by restriction of the star product $*$ to the diagonal. Then, the $(A_\theta, A_\theta \otimes A_\theta)$ bimodule structure is A_θ, and the inner product is given by equation 3.3.1.

By [91], there exists an action of $\mathbb{Z}^k$ on P, preserving the Poisson structure, with k the dimension of the leaves. Therefore, it induces an action of $\mathbb{Z}^k$ on A_θ. By the construction, it is easy to see that it induces an action of $\mathbb{Z}^k$ on the bimodule A_θ and satisfying the conditions of Theorem 3.3.24. By the theorem, there exists a spatial structure on $A_\theta \rtimes \mathbb{Z}^k$, given by $Ind_{A_\theta}^{A_\theta \rtimes \mathbb{Z}^k} A_\theta \cong A_\theta \rtimes \mathbb{Z}^k$. □

For the Poisson two tori

Example 2.9.17. For the Poisson tori with $L = \langle dx + \theta\partial_y, dy - \theta\partial_x \rangle$, an easy computation shows that the bimodule for the transverse quantum tori are given as $\Delta = \{A_\theta\}$. Because there is no group action, it is also the final spatial structure.

For the zero symplectic two torus

Example 2.9.18. For the Dirac tori with $L = \langle \partial_y, \partial_x \rangle = TM$ the foliation consists of only one leaf $\mathbb{T}^2$. The bimodule for the transverse quantum torus is given by $\Delta = C^*(\{pt\}) \cong \mathbb{C}$. Easy computations give us the spatial structure $Ind_A^{A \rtimes G} \Delta \cong \mathbb{C} \rtimes \mathbb{Z}^2 \times \mathbb{Z}^2 \cong C(\mathbb{T}^4)$. This spatial structure is not isomorphic to the spatial structure of the classical tori, which is isomorphic to $C(\mathbb{T}^2)$.

We can also see that two spatial structure is not isomorphic by decomposing the spatial product of modules like in Examples 3.3.18 and 3.3.15. The first one is isomorphic to the intersection of points, and the second one is isomorphic to the group multiplication.

On the two torus, the moduli space of the constant Dirac structures is isomorphic to $\mathbb{R}P^1$, which is the real line added with the infinity point of zero symplectic structure. Because the Morita equivalence does not preserve spatial structure, so although quantization of $SO(n, n, \mathbb{Z})-$related Dirac tori are Morita equivalent, they are not equivalent in the sense of spatial structures. The very critical example is the infinity Poisson torus and the zero Poisson torus.

Another viewpoint is that continuous fields of noncommutative two tori mostly come from a continuous fields of groupoids of translations on the circle 2.3.5. To compactify the bundle at infinity in a compatible way with groupoid structure of other fibers, we need to add a groupoid of infinity translation, which does not exist.

Summarizing the above, it is possible to see that the infinity fiber corresponding to a different spatial stack which is more spatial than the others.

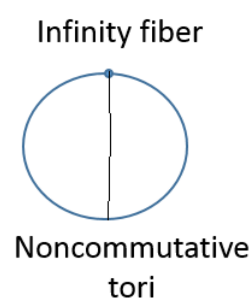


Figure 2.9.6: Moduli space of quantization of the Dirac structures

Despite the fact that there is no way to add a noncommutative two-tori to compactify the bundle, it is still possible to add a special point to the moduli space. Then, the topology of the moduli space of quantization of the constant Dirac structure on the two torus is defined by:

1-Classes of open sets that are noncommutative two-torus bundles over any bounded open sets of the real line.

2-Classes of open sets that are noncommutative two-torus bundles over any open set of the real line $[a, \pm\infty)$, added with a spatial C^*-algebra of quantization of infinity Poisson structures. The infinity quantum torus corresponds to the infinite zigzags.

Chapter 3

Spatial Structures

In this chapter, we develop a theory of spatial structure to understand the quantization of Mirror Symmetry. It should be the dual of the Hopfish structure, like the way A-models and B-models are related, which turns out to provide a new understanding of the noncommutative torus bundles.

3.1 Noncommutative Algebra Motivation

It is widely believed in noncommutative geometry that a quantum space can be represented by the category of representations of a noncommutative algebra. However, we think that additional structure is needed, even when we study very basic noncommutative objects.

In the previous chapter, we have developed the theory of continuous bundles of C^*-Module categories, which is the C^*-analog of bundles of stacks presenting noncommutative torus moduli space functor. The main motivation is that a stack can be represented by Morita equivalent groupoids, and Groupoid C^*-algebras [45, 70] of Morita equivalent groupoids are Morita equivalent C^*-algebras. Thus, a stack is associated to an isomorphism class of categories of representations. See diagram 3.1 on "Stack to Category Correspondence" on the next page.

However, the correspondence is not one to one, *i.e.* there are many non-Morita equivalent groupoids with the same groupoid C^*- algebra. For example, $\begin{array}{c}\mathbb{Z}\\ \downdownarrows\\ \{pt\}\end{array}$ and $\begin{array}{c}S^1\\ \downdownarrows\\ S^1\end{array}$ have isomorphic groupoid C^*-algebras $C^*(\mathbb{Z}) \cong C_\infty(S^1)$

via the Fourier transform but clearly they are not isomorphic groupoids, even up to a Morita equivalence. Therefore, there should be another structure on C^*-algebra level reflecting the information that we lost when passing through the Groupoid C^*-algebra functor. Because two groupoids have non-isomorphic bases $\{pt\}$ and S^1, or are even worse not Morita equivalent, the mathematical structure that we are seeking should reflect this critical difference.

$$
\begin{tikzcd}
\text{[Stack]} \arrow[d] \\
\text{[Equivalent Classes of Groupoids]} \arrow[d, "C^*-functor"'] \\
\text{[Equivalent Classes of } C^*\text{-algebras]} \arrow[d, "Morita\ Equivalent"'] \\
\text{[Category of Hermitian-Modules]}
\end{tikzcd}
$$

Table 3.1.1: Stack to Category Correspondence

The main motivation of the theory is from algebraic geometry. Recall that a scheme is separable if and only if its diagonal map $\Delta : X \to X \times X$ is a closed immersion [36]. For example, the double-origin affine line is not separable because the double point is quite sensitive to the diagonal map. Equivalently, the diagonal map "doubles" the singularity of the space, hence the topology is easier to be detected.

However, dealing with the diagonal maps leads us to the question "Is groupoid C^*- algebra functor co-variant or contra-variant?". If the stacks are classifying spaces, a group homomorphism $f : G \to H$ induces $f^* : C^*(G) \to C^*(H)$. But the groupoids are just locally compact spaces then a continuous map $f : X \to Y$ will be equivalent to $f^* : C^*([Y//Y]) \to C^*([X//X])$, which is a wrong-way morphism. The answer to the question leads to the generalization of the morphisms by bimodules as in [83, 6, 58], or by correspondences in W^*- algebra framework as in [14] or in C^*-framework as in [47].

The second motivation for our study comes from the quantization of mirror symmetry. Bressler and Soibelman in [5] have conjectured that the category of holonomic modules over quantized algebras is "deformed" from Fukaya category of symplectic manifolds, where the Lagrangian submanifolds play the role of the characteristic varieties of the holonomic modules. The authors also conjectured that the morphisms between derived categories of modules over quantized algebras can be realized as bimodules lying over Lagrangian submanifolds of the product of semi-classical symplectic manifolds. Therefore, we hope that any monoidal structure in the Fukaya category can be quantized to a monoidal bimodule of quantized algebras.

On the other hand, Subotic [85] has proved that for the case of torus fibrations, the monoidal structure of Fukaya category coming from a symplectic groupoid is isomorphic to the tensor structure of the derived category of quasi-coherent sheaves of the mirror. Therefore, the mirror symmetry of a monoidal structure in the complex category, (for example tensor product of coherent sheaves over bundles of Abelian varieties) should be a monoidal structure in the symplectic category. Even if the monoidal structure in the complex category is just a spatial structure of a complex space, the monoidal structure in the symplectic category is still interesting enough. Composing the quantization process and mirror symmetry together, we obtain the monoidal product on the category of modules of the quantized algebras.

We should also be aware of the relationship between this direction and the Geometric Langlands Program (GLP), where the mirror symmetry is realized as the Langlands duality via the work of Kapustin and Witten [41]. The Langlands duality can be thought of as the composition of the mirror symmetry for Hitchin fibrations and the quantization of the Hitchin systems by D-modules, as in the lecture by Donagi and Pantev [22]. Therefore, it should be interesting to study the analog of the Langlands correspondence and the monoidal structure from the viewpoint of the theory of C^*- algebras, which is the analysis analog of theory of D-modules.

In the semi-classical level, our approach is to explain a phenomenon that there are many complex groups, but there is no complex groupoid with the totally real base space; there are many symplectic groupoids but there is no symplectic group. Is it co-incident or there is a reason lying behind.

All of these structures depend on the base spaces of some groupoids, which correspond to spatial structures in the language of C^*-algebras that we develop in this chapter.

Recently, in [91] the deformation quantization algebras of $O(n, n, \mathbb{Z})-$ Dirac structures are proved to be strongly Morita equivalent, although the geometries of the Dirac structures are quite different. They not only have

$$
\begin{tikzcd}
\text{Complex Category} \arrow[r, leftrightarrow, "MS"] \arrow[d] & \text{Symplectic} \arrow[r, "Q"] \arrow[d] & \text{Quantized Algebras} \arrow[d] \\
\mathrm{Dcoh}^b \arrow[r, leftrightarrow] \arrow[d] & \text{Fukaya Category} \arrow[r, "\text{B\&S}"] \arrow[d] & \text{Holonomic Modules} \arrow[d] \\
\text{Monoidal} \arrow[r, leftrightarrow] & \text{Monoidal} \arrow[r] & \text{Spatial structure}
\end{tikzcd}
$$

Table 3.1.2: Monoidal structures and Quantization

non-isomorphic invariants (*e.g.* leaf space, Poisson cohomology...) but also of different kinds of geometries, among these are foliation, symplectic, Poisson, presymplectic... structures. Interestingly, all of them correspond to the same C^*-stack. Without difficulty, it is possible to guess that there should be a new mathematical structure to detect the different geometries. Furthermore, choosing different transverse manifolds in a Dirac torus [91] leads us to different choices of C^*-algebras among a Morita equivalent class, the mathematical structure should be defined on the level of C^*-stacks. Because noncommutative geometry of noncommutative tori is the toroidal compactification of M-theory [80], Mirror Symmetry exchanges different kinds of geometries. And since Morita equivalence in the sense of Schwarz [79] is related to duality in M-theory, we hope that this structure should help understanding the quantization of mirror symmetry.

The key difference between our approach and Hopfish algebra developed in [3] is that a groupoid ALWAYS gives us a spatial algebra, but [3] only gives us a Hopfish algebra if it is a 2-groupoid. Therefore, the spatial algebra can be used in more situations than Hopfish algebras.

In this chapter, all the algebras are assumed to be unital, (if not we replace them by their multiplier algebras). The rotation algebra is denote by A_λ for $\lambda = 2\pi\theta$ for the consistent with [3].

3.2 Background on C^*- functor

3.2.1 C^*-correspondence

Definition 3.2.1. [47, 48] Let A and B be C^*-algebras. By a C^*- **correspondence from A to B**, we mean a pair (E, ϕ) satisfying:

1. E is a left B-C^*-module, *i.e.* an inner product with values on the **target algebra** B.
2. ϕ is a non-degenerate $*-$homomorphism of A into $L_B(E)$.

Notice that [14] Connes also defined the correspondence for Von Neumann algebras. We can compose a C^*-correspondence from A to B with one from B to C to obtain another C^*-correspondence from A to C. Therefore, the set of C^*- algebras associated with C^*-correspondences becomes a category, denoted by ($C^* -$ Algebra, Corress). Because any C^*-morphism $\Phi : A \to B$ corresponds to a $(B, A)-$ bimodule structure on B, with A-module structure given by the multiplication by Φ and B-value inner product is ${}_B\langle b_1, b_2\rangle = b_2^* b_1$, and any C^*-correspondence is a trivial $KK-$functor [2] (*i.e.* the Fredholm operator F vanishes), this category is a subcategory of the $KK-$category, and contains the familiar C^*-category with C^*-morphisms A.2.3.

3.2.2 Groupoid Correspondence

Definition 3.2.2. [48, 37] Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be second countable locally compact groupoids and Z a second countable locally compact Hausdorff space. The space Z is a **Groupoid Correspondence (Hilsum-Skandalis map, also generalized morphism)** from $\mathcal{G}_1$ to $\mathcal{G}_2$ if it satisfies the following conditions:

1. There exists a right proper action of $\mathcal{G}_1$ on Z such that the momentum map ρ_1 is an open,
2. There exists a left proper action of $\mathcal{G}_2$ on Z.
3. The $\mathcal{G}_1$ and $\mathcal{G}_2$ actions commute
4. The map ρ_2 induces a bijection from $Z/\mathcal{G}_2$ onto $\mathcal{G}_1^0$, *i.e.* a homeomorphism on the **source side**.

Definition 3.2.3. If the space Z is associated with two-sided correspondence structure, *i.e.* both of the two side momentum map ρ_1 and ρ_2 are open, and ρ_2 induces a bijection of $Z/\mathcal{G}_1$ on G_2^0, then we call Z a **Morita equivalent bimodule between groupoids** $\mathcal{G}_1$ and $\mathcal{G}_2$. In this case, these groupoids are called groupoid Morita equivalent.

As mentioned in Appendix A.1, the equivalent class of groupoids is equivalent to a stack. Because it is possible to compose a Hilsum-Skandalis map with a Morita bimodule (in fact, an invertible Hilsum-Skandalis map) to obtain another groupoid correspondence, a Hilsum-Skandalis map can be viewed as a presentation of morphism between stacks. As a result, the space of Morita equivalent classes of groupoids together with equivalence classes of Hilsum-Skandalis maps forms a category, called **category of stacks.**

It is important to notice that in Definition 3.2.2, the Hilsum-Skandalis map is also called a generalized morphism. Thus, it is natural to ask, what is a non-generalized morphism of groupoids?

Definition 3.2.4. [59]A **strict homomorphism** $\phi : G \to H$ of topological groupoids is just a continuous functor, *i.e.* it is given by a pair of continuous maps $\phi_0 : G_0 \to H_0$ and $\phi_1 : G_1 \to H_1$ commuting with the structure maps of G and H.

Thus, it is necessary to notice the result in [59], any strict morphism $\phi : G \to H$ between groupoids induces a generalized correspondence $H_1 \times_{t,H_0,\Phi,} G_0$.

$$\begin{tikzcd}
H_1 \arrow[r, bend left] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & H_1 \underset{t,H_0,\phi_0}{\times} G_0 \arrow[dl, "s\circ pr_1"'] \arrow[dr, "pr_2"] & G_1 \arrow[l, bend right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
H_0 & & G_0 \arrow[ll, dashed]
\end{tikzcd} .$$

where the left action of G is given by $(h, s(g)).g = (h.\phi_1(g), t(g))$.

Clearly, the class of generalized morphisms contains strict homomorphisms. The definition of strict homomorphisms at the very first sight is the most natural notion of morphisms between groupoids. However, a substantial weakness of strict homomorphisms is that in many cases, there is no nontrivial morphism between groupoids although it is still necessary to study morphisms between their coarse moduli spaces. Therefore, it is necessary to pass through correspondence category, where the composition of a strict morphism with a Morita bimodule produces again a Hilsum-Skandalis morphism.

The similar story also appears in noncommutative geometry [14], where there is no nontrivial C^*-homomorphism from $M_n(A)$ to A for a C^*- algebra A. Therefore, it is natural to introduce bimodules to compare these objects.

In the thesis, the groupoid morphisms are from the right to the left and C^*-algebra correspondences are from the left to the right. We also write these correspondences in terms of $(target, source)-$bimodules.

Therefore, locally compact groupoids with Hilsum-Skandalis maps forms a category (Groupoid, Corress). If the space Z is equipped with a two sided-correspondence structure, *i.e.* both of the two side momentum map ρ_1 and ρ_2 are open, and ρ_2 induces a bijection from $Z/\mathcal{G}_1$ to $\mathcal{G}_2^0$, then we call Z a Morita bimodule between groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$.

3.2.3 C^*-Functor

Given a groupoid G with a Haar system, there exists a C^*-algebra of functions with convolution over G called groupoid C^*-algebra of G (see [70]) and denoted by $C^*(G)$. It is wellknown that the C^*-functor is a contra-variant functor from the groupoid category to the C^*-category with correspondences or even the $KK-$category.

Theorem 3.2.5. *[48, 47] Let G_1 and G_2 be locally compact groupoids with Haar system λ^i such that G_1^0 and G_2^0 are Hausdorff and (Z, ρ, σ) is a correspondence from G_1 to G_2.*

Then one can construct a C^*-correspondence (E_Z, π) from $C_r^*(G_2) \to C_r^*(G_1)$. Moreover, if (Z, ρ, σ) is proper, then π maps $C_r^*(G_2)$ to $K(E_Z)$, which defines an element of $KK(C_r^*(G_2), C_r^*(G_1))$.

For $\xi, \eta \in C_c(Z)$, $a \in C_c(G_1)$, Macho also constructs a bimodule structure on $C_c(Z)$ with the actions of $C_r^*(G_i)$

$$\xi.a(z) = \int_{G_2} a(\gamma).\xi(\gamma.z)d\lambda^1_{\rho(z)}(\gamma),$$

and the $C_r^*(G_1)-$ inner product structure is given by

$$\langle \xi, \eta\rangle(\gamma) = \int_{G_2} \overline{\xi(z.\gamma'^{-1})}.\eta(\gamma^{-1}z\gamma'^{-1})d\lambda^2_{\sigma(z)}(\gamma'), \tag{3.2.1}$$

for $\gamma \in G_1$, $\gamma' \in G_2$ and $z \in E$. Please notice that the construction does not depend on the choice of z because of condition (4) in Definition 3.2.2.

Then the C^*- correspondence bimodule is obtained by completing $C_c(Z)$ with respect to the inner product in Equation 3.2.1 and $C^*(G_i)$ with *the $L^2(G_i, \lambda_i)-$norms.*

In summary, the groupoid C^*- algebra functor is a contra-variant functor from the category of groupoids to $KK-$category, which factors through the Rieffel Category in 2.2.3. Some other authors have also proved the same result for the full groupoid C^*- algebras as in [45, 61] or [55] for a clearer treatment.

Example 3.2.6. Any continuous map $f : X \to Y$ can be viewed as a Hilsum-Skandalis bimodule between groupoids from $X \rightrightarrows X$ to $Y \rightrightarrows Y$ via its graphs

$$\begin{tikzcd}
Y \arrow[r, bend left] \arrow[d, shift left] \arrow[d, shift right] & \text{Graph(X,f)} \arrow[dl, "f"] \arrow[dr] & X \arrow[l, bend right] \arrow[d, shift left] \arrow[d, shift right] . \\
Y & & X
\end{tikzcd}$$

The result of the C^*- algebra functor 3.2.5 on the correspondence is a diagram of C^*-algebra morphisms from $C^*(Y \rightrightarrows Y) \cong C_0(Y)$ to $C^*(X \rightrightarrows X) \cong C_0(X)$,

$$\begin{tikzcd}
C_0(Y) \arrow[r, bend left] & C_0(X) & C_0(X) \arrow[l, bend right] \arrow[d, leftrightarrow, "\cong"] . \\
C_0(Y) \arrow[u, "\cong"'] \arrow[ur, "f^*"'] \arrow[rr, dashed] & & C_0(X) \arrow[ul]
\end{tikzcd}$$

Therefore, a continuous maps $f : X \to Y$ is equivalent to a $(C_0(X), C_0(Y)-$ bimodules structure on $C_0(X)$ which induces a functor from the $C_0(Y)-$Mod to $C_0(X)-$Mod.

Example 3.2.7. On the other hand, a group homomorphism $f : G \to H$ induces a C^*-morphism $f^* : C^*(G) \to C^*(H)$. Thus, there exists a C^*-correspondence $C_c(H)$ from $C^*(H)$ to $C^*(G)$ (notice the wrong way morphism!) defined as in the following diagram. The $C^*(H)$ acts on $C_c(H)$ by left translation, and $C^*(G)$ act on the right of $C_c(H)$ by composing the right translation with f^*.

As a result, it induces a functor $P \mapsto_{C^*(G)} C_0(H) \otimes_{C^*(H)} P$ from $C^*(H)-$Mod to $C^*(G)-$Mod, which is the wellknown reduction technique in representation theory.

$$\begin{tikzcd}
C^*(H) \arrow[r, bend left] & C(H) & C^*(G) \arrow[l, bend right, "f"'] \\
\mathbb{C} \arrow[u] \arrow[ur] & - - - - - - > & \mathbb{C} \arrow[u] \arrow[ul]
\end{tikzcd}$$

3.3 Spatial Structure

3.3.1 Definition and examples

With the background on C^*- functor in the the previous section, we come up with the following definition.

Definition 3.3.1. A **spatial structure** over a C^*-algebra A is a C^*-correspondence Δ from $A \otimes A$ to A, an unit ϵ which is a correspondence from k to A, and satisfying the following properties:

1. (Associativity Condition) The $C^* - (A, A \otimes A \otimes A)$ bimodules $\Delta \otimes_{A \otimes A} (\Delta \otimes A)$ and $\Delta \otimes_{A \otimes A} (A \otimes \Delta)$ are isomorphic.

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\Delta \otimes A"] \arrow[dr, "\Delta \otimes_{A \otimes A} (\Delta \otimes A)"'] & A \otimes A \arrow[d, "\Delta"] & A \otimes (A \otimes A) \arrow[l, "\Delta \otimes A"'] \arrow[dl, "\Delta \otimes_{A \otimes A} (A \otimes \Delta)"] \\
& A &
\end{tikzcd} .$$

1. (Projection) The following $C^* - (A, A)$ bimodules are isomorphic.

$$({}_A\Delta_{A \otimes A}) \otimes_{A \otimes A} (A \otimes \epsilon)_A \cong_A A_A,$$

$$({}_A\Delta_{A \otimes A}) \otimes_{A \otimes A} (\epsilon \otimes A)_A \cong_A A_A,$$

i.e. the following diagram is commutative.

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "\Delta"'] \\
& A \arrow[ul, "Id"] \arrow[u, "\Pi_k"'] \arrow[ul, phantom, shift left=3, "\circlearrowright"]
\end{tikzcd} .$$

Remark 3.3.2. The first condition is the generalization of the well known identity $x \mapsto (x, x) \mapsto (x, x, x)$ and the second condition is the generalization of the equation $\pi_k \circ \Delta = Id$ for $k = 1, 2$ of the diagonal map.

Definition 3.3.3. An C^*-algebra with a spatial structure is called **Spatial C^*-Algebra**. If we forget the C^*-condition, we call it simply spatial algebra.

Example 3.3.4. (Spatial Structure of the topological space) For any locally compact space, viewed as a trivial compact groupoid $X \rightrightarrows X$, the groupoid C^*-algebra is $C(X)$. The spatial structure can be taken as $\Delta = C(X)$ with the multiplication of $C(X)$ on the left, and $C(X) \otimes C(X) \overset{\cong}{\to} C(X \times X)$ on the right.

Proof. We can verify that

$$({}_A\Delta_{A\otimes A}) \otimes_{A\otimes A} (A \otimes A)_A \cong C(X) \underset{C(X)\otimes C(X)}{\otimes} C(X \times X)_{C(X)}$$

is isomorphic to $C(X \underset{X\times X}{\times} (X \times X))$. However the fiber-product is isomorphic to X, so it is isomorphic to $C(X)$ as $(C(X), C(X))$ bimodule. The action of the $C(X) \otimes C(X)$ on the bimodule is given by the diagonal map. The second condition is trivial to verify.

We can show that the inner product from Theorem 3.2.5 with $C(X)-$valued is

$$\langle \xi, \eta \rangle (x, x) = \bar{\xi}(x).\eta(x). \tag{3.3.1}$$

QED. ∎

For $X = S^1$, the above diagonal map

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, bend left] \arrow[d, shift right] \arrow[d, shift left] & S^1 \arrow[dl, "\Delta"'] \arrow[dr] & S^1 . \arrow[l, bend right] \arrow[d, shift right] \arrow[d, shift left] \\
S^1 \times S^1 & & S^1 \arrow[ll, dashed]
\end{tikzcd}$$

implies a correspondence from $C(S^1 \times S^1)$ to $C(S^1)$. Therefore, it is a $(C(S^1), C(S^1) \otimes C(S^1))$ bimodule structure on $C(S^1)$. However, it is not the unique spatial structure on $C(S^1)$, because the isomorphism $C(S^1) \cong C^*(\mathbb{Z})$ gives us a new one.

Example 3.3.5. (Spatial Structure of the Classifying Stack): Let G be a locally compact group and $\mathcal{G}$ be its corresponding groupoid $G \rightrightarrows \{pt\}$. Then the spatial bimodule can be given by $\Delta = C_0(G \times G) \cong C_0(G) \otimes C_0(G)$. In fact, applying the classifying functor, the diagonal group homomorphism $\delta : G \to G \times G$ yields the stacky diagonal $\Delta : BG \to BG \times BG$.

Proof. We study the C^*-version of the stacky diagonal map $\Delta : BG \to BG \times BG$. The Hilsum-Skandalis bimodule of the diagonal maps can be realized as:

$$\begin{tikzcd}
G \times G \arrow[r, bend left] \arrow[d, shift left] \arrow[d, shift right] & G \times G \arrow[dl, "\Delta"'] \arrow[dr] & G \arrow[l, bend right] \arrow[d, shift left] \arrow[d, shift right] \\
\{pt\} & & \{pt\} \arrow[ll, dashed]
\end{tikzcd}$$

Applying the C^*- functor to the groupoids, we obtain $C_0(G) \otimes C_0(G)$ as a correspondence from $C^*(BG \times BG)$ to $C^*(BG)$ or equivalently a $(C^*(G), C^*(G) \otimes C^*(G))$ bimodule. It is easy to verify that

$${}_A\Delta \underset{A\otimes A}{\otimes} (A \otimes A)_{\pi_1(A)} \cong C_0(G \times G) \underset{C^*(G)\otimes C^*(G)}{\otimes} (C^*(G) \otimes C^*(G))_{C^*(G)})$$

as $(C^*(G), C^*(G))-$bimodules. The $C^*(G)-$inner product is given by the formula:

$$\langle \xi, \eta \rangle(g) = \int_G \bar{\xi}(pk^{-1}, ql^{-1}).\eta(pgk^{-1}, qgl^{-1}).d\lambda(k)d\lambda(l).$$

Also, because the right action of G is on the first coordinate of $C_0(G \times G)$, and the left action is diagonal multiplication, we obtain the above verification (after taking the completion with the canonical C^*-norm). The full module condition is trivial. QED. $\square$

3.3.2 Spatial structure from groupoids

Given a locally compact groupoid, we can construct a canonical spatial algebra.

Theorem 3.3.6. *Let $\mathcal{G}$ be a locally compact groupoid together with a Haar system, such that the diagonal map $\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ is proper. Then the composition of the diagonal map Δ and the groupoid C^*-algebra functor 3.2.5 induces a spatial structure on $C_r^*(\mathcal{G})$. Furthermore, it induces an element in $KK(C_r^*(\mathcal{G}) \otimes C_r^*(\mathcal{G}), C_r^*(\mathcal{G}))$. The same result also hold for the full groupoid C^*-algebra version.*

Proof. Let $\mathcal{G}$ be the groupoid. Then canonically, the diagonal map induces a strict homomorphism of groupoids, $\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$, which maps $g \mapsto (g, g)$. Then $s(\Delta(g))$ =$s((g), s(g))$ and $t(\Delta(g)) = (t(g), t(g))$.

Apply the result in [59], any strict morphism $\Phi : \mathcal{G} \to \mathcal{H}$ between groupoids induces a generalized correspondence $H_1 \times_{t,H_0,\Phi,} G_0$.

$$\begin{tikzcd} H_1 \arrow[r, bend left] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & H_1 \underset{t,H_0,\Phi}{\times} G_0 \arrow[dl, "s\circ pr_1"'] \arrow[dr, "pr_2"] & G_1 \arrow[l, bend right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ H_0 & < - - - - - & G_0 \end{tikzcd} .$$

to the case $\mathcal{H} = \mathcal{G} \times \mathcal{G}$, we obtain

$$\begin{tikzcd} G_1 \times G_1 \arrow[r, bend left] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & G_1 \times G_1 \times_{t,G_0\times G_0,\Delta} G_0 \arrow[dl] \arrow[dr] & G_1 \arrow[l, bend right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_0 \times G_0 & < - - - - - & G_0 \end{tikzcd} .$$

Then Δ can be computed as $H_1 \times_{t,H_0,\Phi,} G_0$, which means

$$G_1 \times G_1 \times_{t,G_0\times G_0,\Delta} G_0 \cong \{(g_1, g_2) \mid t(g_1) = t(g_2)\}.$$

The groupoid $G_1 \times G_1$ acts on the left by composing with the arrows on the left of Δ, and the right action of G_1 is by the regular right multiplication $(g_1, g_2) \mapsto (g_1 g, g_2 g)$. The left moment map is the source of $G_1 \times G_1$ components and the right moment maps is just the regular projection on G_0 component.

Applying the C^*-functor in Theorem 3.2.5 to the correspondence $\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$, we obtain a C^*- correspondence Δ from $C^*(\mathcal{G}) \times C^*(\mathcal{G})$ to $C^*(\mathcal{G})$. The equations $\pi_i \circ \Delta \cong Id$ of the canonical projections $\pi_i : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ make the rest follows. The same fashion hold for the reduced C^*- algebra. QED. □

Example 3.3.7. Let $S^1 \rtimes_\lambda Z \rightrightarrows S^1$ be the groupoid corresponding to the rotation algebra A_λ. There exists a spatial structure on the rotation algebra.

Proof. We fix the notation for the groupoid $\begin{array}{c} S^1 \times \mathbb{Z} \\ \downdownarrows \\ \mathbb{Z} \end{array}$ for the rest of the thesis.

The source and target morphisms s and t map the element (θ, k) of $S^1 \times \mathbb{Z}$ into $s(\theta, k) = \theta$ and $t(\theta, k) = \theta + k\lambda$. The bimodule Z is given as

$$S^1\mathbb{Z}.S^1\mathbb{Z} \times_{t,S^1.S^1,\Delta} S^1 \cong \{(x, y, n_1, n_2, z) \mid x + n_1\lambda = y + n_2\lambda = z\} \cong \mathbb{Z}.\mathbb{Z}.S.$$

For the elements (n_1, n_2, z) with $n_1, n_2 \in \mathbb{Z}, z \in S^1$, the left moment map gives

$$J_{S^1 \times S^1}(n_1, n_2, z) = (x, y) = (z - n_1\lambda, z - n_2\lambda).$$

The right moment map gives

$$J_{S^1}(n_1, n_2, z) = z.$$

The action of the generator of $\mathbb{Z}$ is given as

$$(z, n_1, n_2).j \mapsto (z + j\lambda, n_1 + j, n_2 + j), \tag{3.3.2}$$

and the action of the generators of $\mathbb{Z} \times \mathbb{Z}$ by

$$(j_1, j_2).(z, m, n) \mapsto (z, m + j_1, n + j_2). \tag{3.3.3}$$

where the notation is described as

$$\begin{tikzcd}
S^1 \times S^1 \times \mathbb{Z} \times \mathbb{Z}(x, y, j_1, j_2) \arrow[r, bend left] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & S^1 \times \mathbb{Z} \times \mathbb{Z}(z, n_1, n_2) \arrow[dl] \arrow[dr] & S^1 \times \mathbb{Z}\ (z, j)). \arrow[l, bend right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
S^1 \times S^1(z - n_1\lambda, z - n_2\lambda) & & S^1(z) \arrow[ll, dashed]
\end{tikzcd}$$

We choose the basis of $C(S^1 \times \mathbb{Z})$ as a collection of vectors

$$a_{m,j}(\theta, k) = e^{im\theta}\delta_{jk}.$$

The convolution of the groupoid yields a product on $C_c(S^1 \times \mathbb{Z})$, which can be written as

$$a * b(z, k) = \sum_l a(z + \lambda l, k - l).b(z, l).$$

Namely

$$a_{m_1,j_1} * a_{m_2,j_2} = e^{im_1 j_2 \lambda} a_{m_1+m_2, j_1+j_2}. \tag{3.3.4}$$

We fix the basis

$$d_{l,n_1,n_2}(\theta, k_1, k_2) = e^{il\theta}\delta_{k_1 n_1}\delta_{k_2 n_2},$$

of the bimodule $C(S^1 \times \mathbb{Z} \times \mathbb{Z})$. Then the right multiplication can be computed from Equation 3.3.5

$$d * a(\theta, k_1, k_2) = \sum_r d(\theta - r\lambda, k_1 - r, k_2 - r).a(\theta - r\lambda, r),$$

which gives

$$d_{l,n_1,n_2} * a_{m,j} = e^{-i\lambda j(l+m)}.d_{l+m,n_1+j,n_2+j}. \tag{3.3.5}$$

The left multiplication can also be computed from Equation 3.3.3 as

$$(a_1 \otimes a_2) * d(\theta, k_1, k_2) = \sum_{r_1, r_2} a_1(\theta - (k_1 + r_1)\lambda, r_1) a_2(\theta - (k_2 + r_2)\lambda, r_2) d(\theta, k_1 + r_1, k_2 + r_2)$$

which is induced from the groupoid multiplication

$$(\theta, k_1, k_2) = [(\theta - (k_1 + r_1)\lambda, r_1)(\theta - (k_2 + r_2)\lambda, r_2)].(\theta, k_1 + r_1, k_2 + r_2).$$

Simplifying this formula, we obtain the left multiplication

$$(a_{m_1,j_1} \otimes a_{m_2,j_2}) * d_{l,n_1,n_2} = e^{-i\lambda[m_1 n_1 + m_2 n_2]} d_{l+m_1+m_2,n_1+j_1,n_2+j_2}. \tag{3.3.6}$$

QED. □

3.3.3 Spatial structure and Morita equivalence

Theorem 3.3.8. *Let A be a C^*- algebra with a spatial structure (Δ, ϵ). Let B be an C^*- algebra that is Morita equivalent to A. Then B is also a spatial C^*-algebra.*

Proof. We assume that $({}_A\Delta_{A\otimes A}, \epsilon)$ is a spatial structure for the algebra A. Let P be the Morita $(B, A)-$ bimodule and Q the inverted Morita $(A, B)-$bimodule, then we define

$$\widehat{{}_B\Delta_{B\otimes B}} = {}_B P \otimes_A \Delta \underset{A\otimes A}{\otimes} (Q \otimes_k Q)_{B\otimes B}$$

and ${}_B\hat{\epsilon}_k = {}_B P_A \otimes_A \epsilon_k$. Then $\widehat{\Delta}, \hat{\epsilon}$ is a $(B, B \otimes B)$ bimodule, which satisfies the condition of the spatial structure.

Taking tensor product of the equation

$$\Delta \otimes_{A\otimes A} (\Delta \otimes A) \cong \Delta \otimes_{A\otimes A} (A \otimes \Delta)$$

with $P \otimes_A$ on the left and $\otimes_{A\otimes A}(Q \otimes Q)$ on the right, we obtain the balance condition for $\hat{\Delta}$. The projection condition in the definition 3.3.1 is just a routine check. The condition for $\hat{\Delta}$ to be a C^*- correspondence is also deduced from classical Rieffel theory [72]. QED. □

Definition 3.3.9. Let $(A, \Delta^A, \epsilon^A)$ and $(B, \Delta^B, \epsilon^B)$ be two spatial C^*- algebras. Then $(A, \Delta^A, \epsilon^A)$ is **spatial-Morita equivalent** to $(B, \Delta^B, \epsilon^B)$ if there exist a Rieffel $(B, A)-$bimodule ${}_BP_A$ and a $(A, B)-$bimodule ${}_AQ_B$ satisfying

1. $P \otimes_A Q = B$ and $Q \otimes_B P = A$,
2. $\epsilon^B = P \otimes_A \epsilon^A$ as $(B, k)-$bimodules,
3. $\Delta^B =_B P_A \otimes_A \Delta^A \otimes_{A \otimes A} (Q \otimes Q)_{B \otimes B}$ as $(B, B \otimes B)-$ bimodule,
4. The Rieffel inner product on Δ^B and the one on Δ^A obtained by composing the series of Morita bimodule in the condition (3) coincide.

The main purpose of the spatial structure is to study the structure of the stack that we lost under groupoid C^*-functor. It is important to see that Morita equivalent groupoids yield spatial Morita equivalent C^*-algebras.

Lemma 3.3.10. *Let G and H be Morita equivalent groupoids with Haar systems λ_G and λ_H, and (M, σ_1, σ_2) the corresponding Morita bimodule. Then the spatial algebras defined by G and H are spatial Morita equivalent.*

Proof. The proof is just a routine check.

$$
\begin{tikzcd}
 & G_1 \times G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & G_1 \times G_1 \times_{t, G_0 \times G_0, \Delta} G_{0(*)} \arrow[dl] \arrow[dr] & G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \\
 & G_0 \times G_0 & & G_0 & \\
M \times M_{(***)} \arrow[ur] \arrow[ddr] & \dashleftarrow & \dashleftarrow & \dashleftarrow & M_{(**)} \arrow[ul] \arrow[ddl] \\
 & H_1 \times H_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & H_1 \times H_1 \times_{t, H_0 \times H_0, \Delta} H_0 \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \\
 & H_0 \times H_0 & & H_0 &
\end{tikzcd}
$$

By the definition of Morita equivalence of groupoids, $M \times_{H_0} M = G \times_{G_0} M$, $M/H_1 \cong G_0$, and $M/G_1 = H_0$. Therefore, we would like to compute the product of the groupoid correspondences

$$G_1 \times G_1 \setminus [M \times M] \times_{G_0 \times G_0} \times [G_1 \times G_1] \times_{G_0 \times G_0} [G_0 \times_{G_0} M] / G_1$$

which gives us back $[H_1 \times H_1] \times_{t,H_0 \times H_0 \Delta} H_0$. We apply Theorem 3.2.5 to the diagram above, and obtain that: A and B can be taken as $C^*(G)$ and $C^*(H)$, ${}_BP_A$ can be taken to be the completion of $C_c(M)$ for the $A-$norm 2.2.2. The spatial structures are

$$\Delta^A = C_c(G_1 \times G_1 \times_{t,G_0 \times G_0,\Delta} G_0)$$

$$\Delta^B = C_c(H_1 \times H_1 \times_{t,H_0 \times H_0,\Delta} H_0)$$

and the bimodule P is given by $C_c(M)$. The inner products then can be written down by Theorem 3.2.5. Taking the completion, we obtain the lemma. □

Morita equivalent groupoids determine spatial-Morita C^*-algebras. On the other hand, non-Morita equivalent groupoids which correspond to the same C^*-algebra have different spatial structures.

3.3.4 Monoidal category

Definition 3.3.11. For a spatial C^*- algebra, there exists a monoidal structure on the category of left modules, given by $M *_A N :=_A \Delta \otimes_{A\otimes A} (M \otimes N)$. We call it **monoidal spatial structure**.

Theorem 3.3.12. *Let $(A, \Delta^A, \epsilon^A)$ and $(B, \Delta^B, \epsilon^B)$ be two spatial-Morita equivalent C^*- algebras. Then monoidal categories built on the spatial-Morita structures are equivalent.*

Proof. Fixing the notations as in Definition 3.3.9, we need to prove that the Morita functor $f(M) =_B P_A \otimes_A M$ is an equivalence of categories. Because

$$\begin{aligned}
{}_BP_A \otimes_A (M *_A N) :&= {}_BP_A \otimes_A ({}_A\Delta_{A\otimes A}) \otimes_{A\otimes A} (M \otimes N) \\
&= {}_B\Delta_{B\otimes B} \otimes_{B\otimes B} \otimes (Q \otimes Q) \otimes_{A\otimes A} (M \otimes M) \\
&= \Delta_{B\otimes B} \otimes_{B\otimes B} (Q \otimes_A M) \otimes (Q \otimes_A N).
\end{aligned}$$

We obtain $f(M *_A N) \cong f(M) *_B f(N)$. The compatibility of C^*-norms can be deduced from the Hilbert module structure of the C^*-correspondence. The inverse direction is trivial. QED. □

Theorem 3.3.13. *Let $(A_i, \Delta^{A_i}, \epsilon^{A_i})$ be spatial C^*-algebras. Then their tensor products $(A_1 \otimes A_2, \Delta^{A_1} \otimes \Delta^{A_2}, \epsilon^{A_1} \otimes \epsilon^{A_2})$ and $(A_1 \oplus A_2, \Delta^{A_1} \oplus \Delta^{A_2}, \epsilon^{A_1} \oplus \epsilon^{A_2})$ are also spatial algebras.*

Proof. The proof is just a routine check, with the full C^*-norms defining on the tensor products. $\square$

Theorem 3.3.14. *Let X be a locally compact space, viewed as trivial groupoid $\begin{matrix} X \\ \downdownarrows \\ X \end{matrix}$. Then the monoidal structure on category of $C_0(X)-$modules is isomorphic to the tensor product of sheaves.*

Proof. We proceed as in Example 3.3.4. Let E and F be $C_0(X)-$modules and Δ be the canonical spatial structure $\Delta = C_0(X \times X)$ as in Example 3.3.4.

Then $E * F$ is isomorphic to $C_0(X) \otimes_{C_0(X)\otimes C_0(X)} E \otimes_{\mathbb{C}} F$ as Hermitian modules over $C_0(X)$. Because we can identify X with its diagonal map inside $X \times X$, the monoidal product $E * F$ is isomorphic to $E \otimes F$ as modules over $C_0(\Delta(X)) \cong C_0(X)$. $\square$

Example 3.3.15. Let G be any locally compact group. Then the monoidal structure induced from the groupoid $G \rightrightarrows \{pt\}$ is isomorphic to the tensor product of the category of group representations.

Proof. We proceed like in 3.3.5. The spatial structure is determined by $\Delta = C_0(G \times G)$, and $\epsilon^A = \mathbb{C}$. The action of $C^*(G \times G)$ on Δ given by the left translation and the right action of $C^*(G)$ is induced from the diagonal morphism $g \mapsto (g, g)$ from G to $G \times G$. The conditions of spatial structure is followed from Theorem 3.2.5, so we can apply the procedure of the standard construction. The spatial product is given as

$$P * Q =_{C^*(G)} [C_0(G) \otimes_{\mathbb{C}} C_0(G)] \otimes_{C^*(G)\otimes C^*(G)} P \otimes Q,$$

it is easy to show that $P * Q$ is tensor product of group representations. The representation ϵ^A, *i.e.* the trivial representation, plays the role of identity in the monoidal category. Finally, the representation theory of a locally compact group G and its group C^* - algebra $C^*(G)$ are equivalent [14]. $\square$

Corollary 3.3.16. *From Examples 3.3.15 and 3.3.14, we conclude that there exist many non-isomorphic monoidal structures on the category of representations of a fixed C^*-algebra A, depending on the groupoids that the C^*-algebra A represents.*

Remark 3.3.17. If the group G is Abelian, $\hat{G}$ is also Abelian group and the tensor product of irreducible representations is nothing but the group structure on $\hat{G}$. Then, there exist two different monoidal structures on the category of representation of $C^*(G)$.

Example 3.3.18. Assume that G is a locally compact Abelian group. If $C^*(G)$ is presented as the groupoid C^*-algebra of the space $\hat{G}$, the diagonal map induces a correspondence from $C_0(\hat{G} \times \hat{G})$ to $C_0(\hat{G})$, which is a $(C_0(\hat{G}), C_0(\hat{G}) \otimes C_0(\hat{G}))$ bimodule. It is known that all the irreducible representations of $C_0(\hat{G})$ are one dimensional representations, and are identified with the evaluation maps at one point $\theta \in \hat{G}$. Then the new tensor product we want to compute is isomorphic to

$$H_\theta * H_\phi =_{C_0(\hat{G})} C_0(\hat{G}) \otimes_{C_0(\hat{G}) \otimes C_0(\hat{G})} (H_\theta \otimes H_\phi)$$

which can be reduced to $C_0(\hat{G} \times_{\hat{G} \times \hat{G}} \{pt\}))$ with the embedding $pt \hookrightarrow \{\theta, \phi\}$.

As a $C_0(\hat{G})-$module, $H_\theta * H_\phi$ is isomorphic to H_θ if $\theta = \phi$ and degenerates to the trivial representation if $\theta \neq \phi$.

If $C^*(G) \cong C_0(\hat{G})$ is viewed as a groupoid C^*-algebra of the classifying stack BG, the computation above proves that the monoidal structure is equivalent to the group multiplication of $\hat{G}$, the unitary dual of G. Therefore, at least two spatial structures exist on a single C^*- algebra of a locally compact Abelian group.

Remark 3.3.19. Unlike multiplication of groups, the monoidal spatial structure does not necessary induces homomorphisms between cohomology groups. For example, it does not necessary induce an algebra structure on the $K-$theory.

Corollary 3.3.20. *On the classical torus $C(\mathbb{T}^2) = C(S^1 \times S^1)$, there exist the following spatial structures, which are products of spatial structures on $C(S^1)$.*

We want to compute the product $P_{(x_1,y_1)} * P_{(x_2,y_2)}$ for $P_{(x,y)}$ the line bundle over the point (x, y). Then, by Examples 3.3.18 and 3.3.15,

Case 1. Two copies of spatial algebras $(C(S^1), \Delta = C(S^1), \epsilon = C(S^1))$. The monoidal structure is the tensor product of sheaves over $\mathbb{T}^2$. In this case $P_{(x_1,y_1)} * P_{(x_2,y_2)} = \delta_{x_1,x_2}\delta_{y_1,y_2}P_{(x_1,y_1)}$. [1]

[1] δ_{x_1,x_2} is equal to zero if $x_1 \neq x_2$, and equal to 1 if $x_1 = x_2$.

Especially, the operator induced on $K^0(\mathbb{T}^2)$ is just the tensor product of vector bundles. In terms of periodic cyclic homology groups, $HP_\bullet(C(\mathbb{T}^2)) \cong H^\bullet(\mathbb{T}^2)$, then $H^2(\mathbb{T}^2) = \langle dx \wedge dy\rangle$, $H^1(\mathbb{T}^2) = \langle u^{-1}du, v^{-1}dv\rangle = \langle dx, dy\rangle$, and $H^0(\mathbb{T}^2) = \langle 1\rangle$ as showed in [63]. The monoidal structure induces an algebra structure on $HP_\bullet(C(\mathbb{T}^2))$, which can be computed as $k.u^{-1}du * lv^{-1}dv = k.l.dx * dy = k.l.dx \wedge dy$, $1 * dx \wedge dy = dx \wedge dy$.

Case 2. Two copies of $(C^*(\mathbb{Z}), \Delta = C^*(\mathbb{Z}\times\mathbb{Z}), \epsilon = C(S^1))$.
In this case $P_{(x_1,y_1)} * P_{(x_2,y_2)} = P_{(x_1+x_2,y_1+y_2)}$.
However, the monoidal structure behaves in a very different way, in the level of K_0-groups. The monoidal product of two vector bundles E and F is isomorphic to

$$(\Gamma(E)\otimes\Gamma(F))\otimes_{C^*(\mathbb{Z}^4)} C^*(\mathbb{Z}^4)_{C^*(\mathbb{Z}^2)} \cong \Gamma(E)\otimes\Gamma(F)_{C^*(\mathbb{Z}^2)}.$$

The C^*- morphism $C^*(\mathbb{Z}^2) \to C^*(\mathbb{Z}^4)$ is equivalent to the C^*-homomorphism $C(\mathbb{T}^2) \to C(\mathbb{T}^4)$, which is induced from the group multiplication $m : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$, $m(x, y, z, t) = (x + z, y + t)$. Therefore, the monoidal product of two vector bundles $E \otimes F$ on $\mathbb{T}^4$ is obtained by pushing forward the exterior tensor product $E \times F$ via the multiplication m. But the process results in an infinite dimensional vector bundle, *i.e.* not a projective module over $C^*(\mathbb{Z}\times\mathbb{Z})$. Therefore, the monoidal structure does not lie in $K-$theory in the classical sense.

Case 3. One copy of the spatial algebra $(C(S^1), \Delta = C(S^1), \epsilon = C(S^1))$ and one copy of $(C^*(\mathbb{Z}), \Delta = C^*(\mathbb{Z}\times\mathbb{Z}), \epsilon = C(S^1))$. In this case, the product is $P_{(x_1,y_1)} * P_{(x_2,y_2)} = \delta_{x_1,x_2} P_{(x_1,y_1+y_2)}$.

We want to emphasize that, the group $GL(2,\mathbb{Z})$ acts on the classical torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, so it also acts on the moduli space of spatial structures. The spatial structures in case 1 and 2 are invariant under the $GL(2,\mathbb{Z})$ actions, but the case 3 is not. In fact, the group $GL(2,\mathbb{Z})$ transforms the $x-$direction (the tensor one) and $y-$ directions (the addition one) of the torus into other directions.

Example 3.3.21. To get a feeling about different spatial structures on the same C^*-algebra, let $G \cong \mathbb{Z}\times\mathbb{Z}/n_1\mathbb{Z}\times..\times\mathbb{Z}/n_k\mathbb{Z}$. First, as an Abelian group, the $C^*(G)$ has a structure of spatial algebra given in Example 3.3.5. The second spatial structure $\Delta_{n_1,n_2,..n_k}$ is given by $C^*(G) \cong C(S^1) \otimes C^*(\mathbb{Z}/n_1\mathbb{Z})..\otimes C^*(\mathbb{Z}/n_k\mathbb{Z})$, which is isomorphic to the product of the spatial structures on the space S^1 (which is the tensor product of sheaves) and the ones of groups $\mathbb{Z}/n_1\mathbb{Z} \times .. \times \mathbb{Z}/n_k\mathbb{Z}$, (which is the addition along the finite groups).

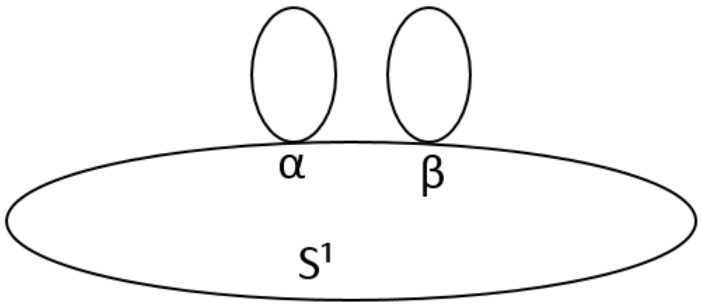


Figure 3.3.1: Spatial product

Given a simple module H over $C^*(G)$, we may see that there exists $\alpha \in S^1$ such that H corresponds to a element of the fiber $\mathbb{Z}/n_1\mathbb{Z} \times .. \times \mathbb{Z}/n_k\mathbb{Z}$ lying over α. Then by example 3.3.18, their tensor product $H_\alpha *_{\Delta_{n_1,n_2,..n_k}} K_\beta \cong 0$ if α and β belong to different fibers of the $n_1.n_2...n_k-$covering map S^1 on S^1. In case they belong to the same fiber, we just add them together to get another representation M_α using the group structure on $\mathbb{Z}/n_1\mathbb{Z} \times .. \times \mathbb{Z}/n_k\mathbb{Z}$.

Example 3.3.22. If $G = K \ltimes H$, and H is an Abelian group, then $C^*(G)$ is Morita equivalent to $C^*(H) \rtimes K \cong C_0(\hat{H}) \rtimes K \cong C^*\left(\begin{array}{c}\hat{H} \rtimes K \\ \downdownarrows \\ \hat{H}\end{array}\right)$ and Theorem 3.3.6 can be applied. Thus, we obtain another spatial structure on the same C^*-algebra $C^*(G)$, besides the ones in 3.3.15.

3.3.5 Crossed Product of Spatial Algebras

Example 3.3.23. We show that the notion of crossed product also exists for the spatial algebras.

Theorem 3.3.24. *Let $(A, \Delta^A, \epsilon^A)$ be a spatial C^*- algebra and G a locally compact group acting on A by α.*

We assume that:

1-There also exists a right representation u of G on Δ, satisfying the condition

$$u_g.\pi(a).u_g^{-1} = \pi(\alpha_g(a)).$$

2-There also exists a right representation v of G on ϵ^A, which is co-variant with respect to the representation of A on ϵ^A viewed as an A-right module.

$$v_g.\pi_\epsilon(a).v_g^{-1} = \pi_\epsilon(\alpha_g(a)).$$

Then there exists a co-variant spatial structure on $A \rtimes G$, denoted by $Ind_A^{A\rtimes G}\Delta$.

Proof. We keep in mind the example of the transformation groupoid $G \ltimes X \rightrightarrows X$, where the spatial bimodule over $C_0(X) \rtimes G$ is $\tilde{\Delta} = Ind_{C_0(X)}^{C_0(X)\rtimes G}\Delta^A = (L^2(G \times G, C_0(X)), \epsilon^A = C_0(X)$. The group G acts on X component of $X \times G \times G$ and acts diagonally on $G \times G$. It is obvious that the left action of $G \times G$ is given as the regular representation on $X \times G \times G$. Our construction is a noncommutative generalization of this example.

Right Module: The bimodule $\tilde{\Delta}$ is defined to be $L^2(G \times G, \Delta)$, *i.e.* the L^2-space of functions on $G \times G$ with the values on Δ, with respect to the Haar measure on G. Denote the $*-$ representation of $A \otimes A$ on Δ by π, we define the right co-variant representation $(\tilde{\pi}, U)$ of $(A \otimes A, G \times G, \Delta)$ on $\tilde{\Delta}$ as:

$$(\tilde{\pi}(a_1 \otimes a_2)f)(x, y) = \pi(\alpha^{-1}_{(g,h)}(a_1 \otimes a_2)).(f(x, y),$$

$$(U_{g_1,h_1}f)(x, y) = f(g_1^{-1}x, h_1^{-1}y),$$

for all $a_1, a_2 \in A$, d in $L^2(G \times G, \Delta)$. We can verify that

$$U_{g_1,h_1}.\tilde{\pi}(a_1 \otimes a_2)U^*_{g_1,h_1} = \tilde{\pi}(\alpha_{g_1}(a_1) \otimes \alpha_{g_2}(a_2)).$$

It is known that in ([17]) any co-variant representation of noncommutative dynamical system (A, G, α) is equivalent to a $*-$ representation of the crossed product $\tilde{A} = A \rtimes_\alpha G$. Thus, there exists a right $(A \rtimes G) \otimes (A \rtimes G)-$ module structure on $\tilde{\Delta}$.

Left Module: Let $d \in L^2(G \times G, \Delta)$ and $a, b \in C_c(G, A)$. We define naturally the right representations of A and G on $\tilde{\Delta}$

$$(\Pi(a)d)(x, y) = \pi(a).d(x, y),$$

$$(U_g d)(x, y) = u_g(d(x, y)).$$

Then naturally, condition 3.3.24 extends to $U_g \Pi(a).U_g^* = \Pi(\alpha_g(a))$, which can be integrated to a right representation of $A \rtimes G$ on $\tilde{\Delta}$. Thus, the construction gives the $(\tilde{A}, \tilde{A} \otimes \tilde{A})-$bimodule structure on $\tilde{\Delta}$.

Identity: Without any difficulty, condition 3.3.24 guarantees that integrating the co-variant representation $(\pi_\epsilon, v_\epsilon)$ of (A, G, ϵ^A) yields a right representation of $A \rtimes G$ on ϵ^A. The verification that $(L^2(G \times G, \Delta), \epsilon^A)$ satisfies the axioms of the spatial structure:

$$\tilde{\Delta} \otimes_{A \rtimes G \otimes A \rtimes G} (\tilde{\Delta} \otimes (A \rtimes G)) \cong \tilde{\Delta} \otimes_{A \rtimes G \otimes A \rtimes G} ((A \rtimes G) \otimes \tilde{\Delta}),$$

as $(A \rtimes G, A \rtimes G \otimes A \rtimes G \otimes A \rtimes G)-$bimodules and

$$\tilde{\Delta} \otimes_{A \rtimes G} (\epsilon^A \otimes A \rtimes G) \cong A \rtimes G,$$

as $(A \rtimes G, A \rtimes G)$ modules is then a routine check. □

Remark 3.3.25. We would like to mention that the example of spatial structure of the deformation quantization of the Poisson and Dirac tori is discussed in 2.9.18.

Example 3.3.26. Let G a locally compact group, with a two-cocycle *i.e.* a function such that $\sigma : G \times G \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$,

$$\sigma(g, h)\sigma(gh, k) = \sigma(g, hk)\sigma(h, k) \text{ for all } g, h, k \in G.$$

Then we can twist the spatial group C^*-algebra in Example 3.3.5 to obtain a spatial algebra structure on $C^*(G, \sigma)$. The bimodule is still $C_c(G \times G)$, but the action is twisted. $\sum a_{(g_1,g_2)} *_\sigma b_g = \sum a_{(g_1 g, g_2 g)}.\sigma(g_1, g).\sigma(g_2, g)$. The left action is given analogously.

A well known example is the construction of the noncommutative two-torus via the a cocycle over $\mathbb{Z}^2$.

Given a group H acting on a groupoid $G_1 \rightrightarrows G_0$, we can build up a bigger groupoid $H \times G_1 \rightrightarrows H \times G_0$ to study the stack quotient $[G_1/G_0]//H$ as well as its monoidal category of representations. The group H also preserves the spatial $C^*(G_1 \rightrightarrows G_0)$ and Theorem 3.3.24 can be applied.

However, there are many known examples that a group cannot act on a groupoid, but still acts on the groupoid C^*-algebra. A typical example brought to us by Echterhoff via private correspondences is the classical torus $\mathbb{T}^2$ can acts ergodically on the noncommutative two-torus A_λ, but $A_\lambda \rtimes \mathbb{T}^2 \cong K(L^2(\mathbb{T}^2))$ is Morita equivalent to $\mathbb{C}$ or groupoid C^*- algebra of a point! The monoidal structure of the category of modules over $A_\lambda \rtimes \mathbb{T}^2$, therefore, is trivial. The the reason is that $\mathbb{T}^2$ does not preserve the spatial structure of A_λ.

A more trivial example is that there is no nontrivial action of S^1 on the stack $B\mathbb{Z}$, but there is one on the category of representations of $\mathbb{Z}$. Up to now, we do not know any example of any group action preserving a spatial algebra, but not preserving the underlying groupoid.

3.4 Monoidal Structure For Rotation Algebras

We compute the monoidal structure for Example 3.3.7 in the category of cyclic modules, *i.e.* the modules generated by one element, analog to the monoidal structure coming from Hopfish algebras in [3].

Let $T^\alpha_{p,q}$ be $A/(u-1)A$, which is a left cyclic A-module with $u = e^{-i\alpha} a_{p,q}$.

Lemma 3.4.1. *[3] $T^\alpha_{p,q}$ is a simple module if and only if p and q are relatively prime. Furthermore, the modules $T^\alpha_{p,q}$ and $T^\beta_{r,s}$ are isomorphic iff $(p,q) = \pm(r,s)$ and $\alpha = \pm\beta + n$ for some $n \in \mathbb{Z}$.*

We try to use the same terminology in reference [3] to see the analog between our spatial structure and the Hopfish structure, which is expected to be a candidate for the quantum mirror symmetry.

Let T be the inner product space spanned by the orthonormal basis $\{\xi_n \mid n \in \mathbb{Z}\}$. Then for $p \neq 0$, $T^\alpha_{p,q}$ is isomorphic to the action given by

$$\begin{aligned} \xi_n a_{10} &= e^{\frac{i}{p}[\alpha+\lambda n+\frac{1}{2}\lambda q(p+1)]}\xi_{n+q} \\ \xi_n a_{01} &= \xi_{n-p}. \end{aligned} \tag{3.4.1}$$

Sometimes, for convenience of computation, we switch the roles of a_{10} and a_{01}. Then $T^{\alpha}_{p,q}$ is isomorphic to

$$\begin{aligned} \eta_n a_{10} &= \eta_{n+q} \\ \eta_n a_{01} &= e^{\frac{i}{q}[\alpha+\lambda n+\frac{1}{2}\lambda p(q+1)]} \quad \eta_{n-p} = DD^{\alpha}_{p,q,n}\eta_{n-p}. \end{aligned} \tag{3.4.2}$$

While for $p=0$

$$\begin{aligned} \xi_n a_{10} &= \xi_{n+q} \\ \xi_n a_{01} &= e^{\frac{i}{q}[\alpha+\lambda n]}\xi_n. \end{aligned} \tag{3.4.3}$$

Notice that we denote $e^{\frac{i}{p}[\alpha+\lambda n+\frac{1}{2}\lambda q(p+1)]}$ by $CC^{\alpha}_{p,q,n}$ for shorter computation.

For $q=0$

$$\begin{aligned} \xi_n a_{10} &= e^{\frac{i}{p}[\alpha+\lambda n]}\xi_n \\ \xi_n a_{01} &= \xi_{n-p}. \end{aligned}$$

The cyclic modules $T^{\alpha}_{p,q} = A/(e^{-i\alpha}a_{pq}-1)A$ admit the action $(U,V)\mapsto(\theta^q U, \theta^{-p}V)$ of the one dimensional sub-torus (θ^q,θ^{-p}) of $\{(\theta_1,\theta_2)\}$, giving the spectral decomposition $\oplus_k T^{\alpha}_{p,q,k}$.

Compute $T = T^{\alpha_1}_{p_1,q_1}\otimes T^{\alpha_2}_{p_2,q_2}\otimes_{A\otimes A}\Delta$.

To compute the tensor product, we compute the action of the noncommutative two-tori on the generators of the cyclic modules. As a vector space, the generators of the spatial tensor product $T = T^{\alpha_1}_{p_1,q_1}\otimes T^{\alpha_2}_{p_2,q_2}\otimes_{A\otimes A}\Delta$ can be fixed as $\eta^1_{k_1}\otimes\eta^2_{k_2}\otimes d_{l,n_1,n_2}$. The main tool for our computation is Example 3.3.7.

From equation 3.3.6, we obtain

$$(a_{m_1,-n_1}\otimes a_{-l-m_1,-n_2}) * d_{l,n_1,n_2} = e^{-i\lambda[m_1 n_1-(m_1+l)n_2]}d_{0,0,0} \tag{3.4.4}$$

which implies that T is generated by elements $t_{k_1,k_2} = \eta^1_{k_1}\otimes\eta^2_{k_2}\otimes d_{0,0,0}$ where $\eta^i_{k_i}$ are basis for the cyclic module $T^{\alpha_i}_{p_i,q_i}$. However, in order to obtain the expected modules from the generators t_{k_1,k_2}, we still need to mod out the bimodule by the action of the generator $a_{1,0}\otimes a_{-1,0}$, which corresponds to the different choices of m_1. The equivalence relations can be read

$$\eta^1_{k_1}\otimes\eta^2_{k_2}.a_{1,0}\otimes d_{0,0,0} = \eta^1_{k_1}.a_{1,0}\otimes\eta^2_{k_2}\otimes d_{0,0,0}. \tag{3.4.5}$$

To clarify how the relationship behaves, we distinguish three cases, corresponding to the behavior of the cyclic modules.

3.4.1 Case $q_1 \neq 0$ and $q_2 \neq 0$:

The right hand side of Equation 3.4.5 is equivalent to

$$t_{k_1,k_2}.(a_{1,0} \otimes Id) = \eta^1_{k_1+q_1} \otimes \eta^2_{k_2} \otimes d_{000} \tag{3.4.6}$$

while the left hand side reads

$$t_{k_1,k_2}.(Id \otimes a_{1,0}) = \eta^1_{k_1} \otimes \eta^2_{k_2+q_2} \otimes d_{000}. \tag{3.4.7}$$

Equivalently,

$$t_{k_1+q_1,k_2} = t_{k_1,k_2+q_2}. \tag{3.4.8}$$

Using the formula 3.3.5 for the right action of $a_{m,j}$

$$d_{l,n_1,n_2} * a_{m,j} = e^{-i\lambda j(l+m)}.d_{l+m,n_1+j,n_2+j},$$

we can rephrase 3.4.6 as

$$\begin{aligned} t_{k_1,k_2} * a_{10} &= \eta^1_{k_1} \otimes \eta^2_{k_2} \otimes d_{000} * a_{10} \\ &= \eta^1_{k_1} \otimes \eta^2_{k_2} \otimes d_{1,0,0} \qquad (3.4.9) \\ &= (\eta^1_{k_1} \otimes \eta^2_{k_2})(a_{1,0} \otimes a_{00}) * d_{000} \\ &= (\eta^1_{k_1} \otimes \eta^2_{k_2})(a_{0,0} \otimes a_{10}) * d_{000} \\ &= t_{k_1+q_1,k_2} = t_{k_1,k_2+q_2} \qquad (3.4.10) \end{aligned}$$

While Equation 3.4.7 is equivalent to

$$\begin{aligned} \eta^1_{k_1} \otimes \eta^2_{k_2} \otimes d_{000} * a_{0,1} &= \eta^1_{k_1} \otimes \eta^2_{k_2} \otimes d_{0,1,1} \qquad (3.4.11) \\ &= \eta^1_{k_1} \otimes \eta^2_{k_2} \otimes a_{0,1} \otimes a_{0,1} * d_{000} \\ &= DD^{\alpha_1}_{p_1,q_1,n_1} DD^{\alpha_2}_{p_2,q_2,n_2} \eta^1_{k_1-p_1} \otimes \eta^2_{k_2-p_2} \otimes d_{0,0,0}. \end{aligned}$$

or

$$t_{k_1,k_2} a_{0,1} = DD^{\alpha_1}_{p_1,q_1,n_1} DD^{\alpha_2}_{p_2,q_2,n_2} t_{k_1-p_1,k_2-p_2}.$$

Equation 3.4.10 can be thought of as the equation for sections of a line bundle on the discrete torus of period (q_1, q_2). Thus, these vectors can be parametrized by $\mathbb{Z}^2/(q_1, -q_2)\mathbb{Z}$ to be compatible with the relation 3.4.8

$$H_{[k_1,k_2]} := t_{(k_1,k_2)}, \qquad [k_1, k_2] = (k_1, k_2) + \mathbb{Z}(p_1, -p_2)$$

$$H_{[k_1,k_2]}.a_{10} = H_{k_1+q_1,k_2} = H_{k_1,k_2+q_2},$$
$$H_{[k_1,k_2]}.a_{01} = DD^{\alpha_1}_{p_1,q_1,n_1} DD^{\alpha_2}_{p_2,q_2,n_2} H_{[k_1-p_1,k_2-p_2]}.$$

Fixing the bijection

$$v : \mathbb{Z}^2/(q_1,-q_2)\mathbb{Z} \xrightarrow{\cong} (\mathbb{Z}/gcd(q_1,q_2)\mathbb{Z}) \times \mathbb{Z}$$

$$[k_1,k_2] \mapsto (p_2k_1 - p_1k_2 \ mod \ gcd(q_1,q_2), \frac{q_2k_1+q_1k_2}{gcd(q_1,q_2)})$$

as in [3] but for different parameters, and also relabeling the basis by setting $H^{(m)}_n = H_{v^{-1}(m,n)}$. Under this new form,

$$H^{(m)}_n.a_{10} = H^{(m)}_{n+q}$$
$$H^{(m)}_n.a_{01} = DD^{\alpha_1}_{p_1,q_1,n_1} DD^{\alpha_2}_{p_2,q_2,n_2} H^{(m)}_{n-p}. \tag{3.4.12}$$

Here $q = lcm(q_1,q_2)$ and $p = \frac{q_1p_2+q_2p_1}{gcd(q_1,q_2)}$. It is easy to see that from two equations,

$$H^{(m)}_n a^q_{01} a^p_{10} = e^{i\alpha}.H^{(m)}_n,$$

for some constant α. Therefore, the tensor product should be isomorphic to $\mathbb{T}^{\alpha}_{p,q}$.

Equivalently, we obtain a new basis for the modules T, for which the action of the torus is very similar to Equation 3.4.1. Naturally, we would like to to compare Equation 3.4.12 to the standard cyclic module

$$\begin{aligned} \eta_n a_{10} &= \eta_{n+q} \\ \eta_n a_{01} &= e^{\frac{i}{q}[\alpha+\lambda n+\frac{1}{2}\lambda p(q+1)]} \quad \eta_{n-p} = DD^{\alpha}_{p,q,n}\eta_{n-p}, \end{aligned} \tag{3.4.13}$$

by finding α, p, q, k satisfying the equation

$$DD^{\alpha_1}_{p_1,q_1,n_1} DD^{\alpha_2}_{p_2,q_2,n_2} = [DD^{\alpha}_{p,q,n}]$$

so that the tensor module is isomorphic to a cyclic one $T^{\alpha}_{p,q}$. But what value may α take? Recall that in [3], all the cyclic modules are isomorphic under the transformation $\alpha \mapsto \pm\alpha + k\lambda$, and it is possible to choose different k among the generator of the cyclic modules, so equivalently

$$\frac{\alpha+\lambda n+\frac{1}{2}\lambda p(q+1)}{q} = \frac{\alpha_1+\lambda n_1+\frac{1}{2}\lambda p_1(q_1+1)}{q_1} + \frac{\alpha+\lambda n_1+\frac{1}{2}\lambda p_1(q_1+1)}{q_1}$$

modulo $2\pi\mathbb{Z}$. Or equivalently

$$\frac{\alpha}{q} = [\frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}] + \frac{1}{2}\lambda\frac{p_1}{q_1}(q_1+1)] + \frac{1}{2}\lambda\frac{p_2}{q_2}(q_2+1)] - \frac{1}{2}\lambda\frac{p}{q}(q+1) \text{ modulo } 2\pi\mathbb{Z}. \tag{3.4.14}$$

or

$$\frac{\alpha}{q} = [\frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}] + \frac{\lambda}{2}(\frac{p_1}{q_1} + \frac{p_2}{q_2} - \frac{p}{q}) + \frac{1}{2}\lambda(p_1 + p_2 - p) \text{ modulo } 2\pi\mathbb{Z}. \tag{3.4.15}$$

Finally, we obtain

$$T^{\alpha_1}_{p_1,q_1} *_\Delta T^{\alpha_2}_{p_2,q_2} = gcd(q_1,q_2)T^{\alpha}_{p,q}.$$

3.4.2 Only one of q_1 or $q_2 = 0$:

We can verify that this case is totally similar to the above and we obtain the same result.

3.4.3 Case $q_1 = q_2 = 0$:

This case turns out to be totally different, mostly because the relationship 3.4.8 degenerates to

$$CC^{\alpha_1}_{p_1,0,k_1}.t_{k_1,k_2} = CC^{\alpha_2}_{p_2,0,k_2}.t_{k_1,k_2}.$$

This condition is not always satisfied and constitutes a condition for the vanishing of the product. From the definition of $CC^{\alpha}_{p,q,k}$, the condition can be rewritten as

$$\frac{1}{p_1}[\alpha_1 + \lambda k_1] - \frac{1}{p_2}[\alpha_2 + \lambda k_2] \in 2\pi\mathbb{Z},$$

or

$$\frac{1}{2\pi}(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}) + \frac{\lambda}{2\pi}(\frac{k_1}{p_1} - \frac{k_2}{p_2}) \in \mathbb{Z}, \tag{3.4.16}$$

or

$$\frac{\alpha_1 p_2 - \alpha_2 p_1 + \lambda(k_1 q_2 - k_2 q_1)}{p_1 p_2} = 0 \ mod \ 2\pi\mathbb{Z}.$$

Assume that $0 \leq \alpha_1, \alpha_2 < 2\pi$, then monoidal product of modules always vanishes unless

$$r := -\frac{\alpha_1 p_2 - \alpha_2 p_1}{\lambda . gcd(p_1, p_2)}$$

is an integral modulo multiple of $lcm(q_1, q_2)\frac{2\pi}{\lambda}$.

How does the action of a_{10} and a_{01} behave? Recall that $t_{k_1,k_2} a_{0,1} = t_{k_1-p_1,k_2-p_2}$, and $t_{k_1,k_2} a_{10}$=C$C^{\alpha_1}_{p_1,q_1,k_1} t_{k_1,k_2}$, so we carry out analogously as the case $q_1, q_2 \neq 0$ and in the same line with [3]. Let s_1, s_2 be integers such that

$$\frac{s_1 p_2 - s_2 p_1}{gcd(p_1, p_2)} = 1.$$

In this case, the basis for the tensor modules is given by

$$B := \{t_{k_1,k_2} \mid k_1, k_2 \in \mathbb{Z}, \frac{k_1 p_2 - k_2 p_1}{gcd(p_1, p_2)} = r\}.$$

We choose a map v to mod out all the elements in the equivalent class $t_{k_1,k_2} a_{0,1} = t_{k_1-p_1,k_2-p_2}$

$$v : \mathbb{Z} \to \{(k_1, k_2) \in \mathbb{Z} \mid \frac{k_1 p_2 - k_2 p_1}{gcd(p_1, p_2)} = r\}$$

$$n \mapsto r(s_1, s_2) + \frac{k}{gcd(p_1, p_2)}(p_1, p_2)$$

and relabel the basis by $t_n := t_{v(n)}$. In this new basis, we choose $p := gcd(p_1, p_2), \alpha := s_1\alpha_2 - s_2\alpha_1$.

$$t_n a_{10} = e^{\frac{i}{p}[\alpha + \lambda n)]} t_n$$

$$t_n a_{0,1} = t_{n-p}.$$

Then, $t_n a_{10}^p = e^{i(\alpha + p\lambda n)}$. Choose $n = 0$, we obtain the module $T^{\alpha}_{p,0}$. In summary of all the computation above, we obtain a Decomposition Theorem of cyclic modules according to the standard tensor product on rotation algebra.

Theorem 3.4.2. (Decomposition Theorem) *Let $T^{\alpha_1}_{p_1 q_1}$ and $T^{\alpha_2}_{p_2 q_2}$ be two cyclic modules over an rotation algebra, defined by the relation $T^{\alpha}_{p,q} = A/(e^{-i\alpha} a_{pq} - 1)A$. Their spatial product determined by the spatial structure on noncommutative two-torus via the groupoid presentation $S^1 \rtimes_\theta \mathbb{Z}$ is:*

- If q_1 or q_2 is nonzero, then 3.4.2

$$T^{\alpha_1}_{p_1,q_1} *_\Delta T^{\alpha_2}_{p_2,q_2} = gcd(q_1,q_2)T^{\alpha}_{p,q}$$

with $q = lcm(q_1,q_2)$ and $p = \frac{q_1p_2+q_2p_1}{gcd(q_1,q_2)}$ and

$$\frac{\alpha}{q} = [\frac{\alpha_1}{q_1}+\frac{\alpha_2}{q_2}]+\frac{\lambda}{2}(\frac{p_1}{q_1}+\frac{p_2}{q_2}-\frac{p}{q})+\frac{1}{2}\lambda(p_1+p_2-p) \text{ modulo } 2\pi\mathbb{Z}. \quad (3.4.17)$$

α can be choose up to a transformation of the forms $\alpha \mapsto \pm\alpha + k\lambda$.

- If $q_1 = q_2 = 0$ then

$$T^{\alpha_1}_{p_1,0} *_\Delta T^{\alpha_2}_{p_2,,0} \quad = \quad \begin{cases} T^{\alpha}_{p,0} \text{ if } \quad \phi = \frac{1}{2\pi}(\frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}) \in \mathbb{Z} \\ 0 \text{ otherwise} \end{cases} \quad (3.4.18)$$

with $p = gcd(p_1,p_2)$,$\alpha := s_1\alpha_2 - s_2\alpha_1$, $\frac{s_1q_2-s_2q_1}{gcd(p_1,p_2)} = 1$.

Remark 3.4.3. Most of the notations in this section are similar to those in [3], since we want to establish mirror symmetry correspondence between Hopfish algebras and Spatial algebras. The unknown correspondence is analog to the correspondence between the normal product and the convolution product of functions over $\mathbb{R}^n$, but the nature of our computation is different. Compare to the decomposition of cyclic modules with respect to Hopfish structure in [3], the formula for p and q in our computation can be rephrased as

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p}{q}$$

but in the Hopfish monoidal structure, the formula is

$$\frac{q_1}{p_1} + \frac{q_2}{p_2} = \frac{q}{p}$$

(the formula in Remark 2 of [3] contains a typo, the fraction should be written inversely).

The difference comes from the fact that we use spatial structure to kill the stackiness in the U direction, and use Hopfish algebra to maintain the group structure on the U direction and forget the stackiness of the V direction. But we can switch the role of p and q, with an automorphism of the rotation algebras $U \mapsto V, V \mapsto U$. Therefore, after twisting with an automorphism, we obtain the categorical equivalent between monoidal categories generated by cyclic modules with respect to Hopfish and Spatial structures.

Corollary 3.4.4. *In formula 3.4.17, if λ jumps $2\pi m$, then α increases $\pi m(\frac{p_1}{q_1} + \frac{p_2}{q_2} - \frac{p}{q})q + \pi m(q_1 + q_2 - q)q$. Thus, the monoidal structure is stable under the isomorphism $A_\lambda \to A_{\lambda+2\pi}$. But clearly, the spatial structure is not stable under Morita equivalent, especially the inversion $\theta \mapsto \frac{1}{\theta}$.*

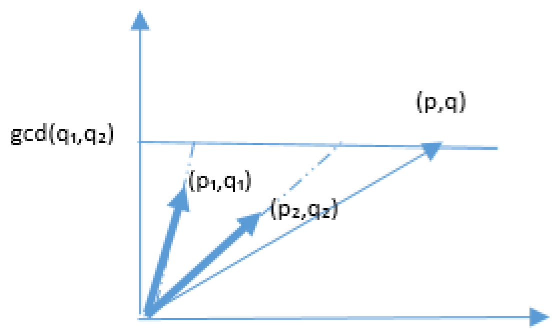


Figure 3.4.1: Spatial monoidal product

The decomposition of the cyclic modules with respect to the Hopfish structure of $A_{\frac{1}{\theta}}$ coincides with the decomposition with respect to the Spatial Structure of A_θ.

However, from the corollary, we do not know any information about the full correspondence between two categories, because the categories of representations are very huge and cannot be classified. The fact is due to the liminal property of the noncommutative two-tori. But the result still suggests that there exists a kind of mirror symmetry for the noncommutative two-tori that relates two monoidal structures. We will return to the same problem from another point of view in future work.

3.5 Spatial structures and continuous fields

We return to the section of the continuous fields of noncommutative two-tori to see how the spatial structure appears in the picture. Given a bundle of tori $A = A_{\theta(t)}$ with $\theta(t)$ a continuous function from a contractible locally compact space X into S^1, there exists a bundle of groupoids with fibers $S^1 \times_{\theta(t)} \mathbb{Z}$ corresponding to the C^*-algebra A. The bundle can be given as $G = (S^1 \times X) \rtimes_{\theta(t)} \mathbb{Z}$, which means that $C^*(G)$ can be associated with a spatial structure Δ. In the form of an $(A \otimes A, A)-$bimodule, the spatial structure Δ can be decomposed into bundles of bimodules over bundles of noncommutative two-tori.

In chapter 2, we have used the Morita bimodules to glues continuous fields to obtain the bundles of stacks. However, as we showed in this chapter, Morita equivalence does not preserve spatial structure. Therefore, the process in the chapter 2 is just the gluing of categories of modules, not of monoidal categories.

Because $S^1 \rtimes_\theta \mathbb{Z}$ and $S^1 \rtimes_{\theta+1} \mathbb{Z}$ are isomorphic, the classification of the continuous fields of noncommutative two-tori up to a spatial Morita equivalence reduces to the classification up to an isomorphisms.

Theorem 3.5.1. *Let $H_3(\mathbb{Z})$ be the Heisenberg group over $\mathbb{Z}$*

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in Z \right\}$$

and $H_3(\mathbb{R})$ be the same group with z in $\mathbb{R}$.

Proof. We fix the generators of $H_3(\mathbb{Z})$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that any element $g \in H_3(\mathbb{Z})$ can be written

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = V^y.U^x.W^z.$$

We denote the element $V^y.U^x.W^z$ by (x, y, z). Then the equations determining the Heisenberg group are

$$Z = UVU^{-1}V^{-1}, UW = WU, WV = VW \tag{3.5.1}$$

and the multiplication is given by $(a, b, c) * (x, y, z) = (a+x, b+y, c+z+ay)$.

The canonical embedding i of $N = \langle W \rangle$ into the center of $H_3(\mathbb{Z})$ induces $i^* : C^*(N) \to C^*(H_3(\mathbb{Z}))$. As known in classical harmonic analysis, the Fourier transform $C^*(N) \cong C(\hat{N}) \cong C(S^1)$ decomposes $C^*(H_3(\mathbb{Z}))$ into a bundle of noncommutative two-tori [38].

A typical section of the bundle is of the form

$$a = \sum a_{n,m,k} V^m U^n W^k = \sum a_{n,m,k} V^m U^n e^{ikt}.$$

Let I_θ be the left ideal generated by $\langle W = \theta \rangle$, then the image of $a \in C^*(H_3)$ under the projection onto $C^*(H_3(\mathbb{Z}))/C^*(H_3(\mathbb{Z}))I_\theta$ has the value $\sum a_{n,m,k} V^m U^n \theta^k$.

The fiber algebra then is isomorphic to

$$< U, V, W | UVU^{-1}V^{-1} = W > / < W - \theta = 0 > \cong \langle UV = \theta VU \rangle.$$

Quantization of multiplication of Poisson Tori

We study the spatial structure coming from the group $H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{Z})$. Let $\delta : H_3(\mathbb{Z}) \to H_3(\mathbb{Z}) \times H_3(\mathbb{Z})$ be the diagonal map of the group, which induces $\Delta : A \to A \otimes A$.

$$\begin{tikzcd}
C^*(G) \arrow[r] & C^*(G) \otimes C^*(G) \\
C^*(\mathbb{Z}) \arrow[r] \arrow[u] & C^*(\mathbb{Z}) \otimes C^*(\mathbb{Z}) \arrow[u] \\
C(S^1) \arrow[r] \arrow[u] & C(S^1) \arrow[u]
\end{tikzcd}$$

It is clear that the map has a restriction to the center,

$$\Delta : C^*(\mathbb{Z}) \to C^*(\mathbb{Z}) \otimes C^*(\mathbb{Z}),$$

$$\delta(\sum a_n \delta_n) = \sum a_n \delta_n \otimes \delta_n.$$

By the Fourier transform, it induces $\Delta : C(S^1) \to C(S^1 \times S^1)$, namely mapping $e^{inx} \mapsto e^{inu}.e^{inv} = e^{in(u+v)}$. As a result, Δ is the group structure of S^1. Therefore, Δ^{-1} maps ideals of $A \otimes A$ to ideals of A.

$$\Delta^{-1}(I_{\theta_1} \otimes I_{\theta_2}) = I_{\theta_1+\theta_2} \trianglelefteq C(S^1).$$

Hence it induces the morphism between fiber algebras

$$\Delta_{\theta_1,\theta_2} : A/(I_{\theta_1+\theta_2} A) \to A \otimes A/(I_{\theta_1} A \otimes I_{\theta_2} A),$$

i.e.

$$\Delta_{\theta_1,\theta_2} : A_{\theta_1+\theta_2} \longrightarrow A_{\theta_1} \otimes A_{\theta_2}.$$

From the viewpoint of Poisson geometry, the family of maps $\Delta_{\theta_1,\theta_2}$ is the quantization of the maps

$$(\mathbb{T}^2, \theta_1\partial_x \wedge \partial_y) \times (\mathbb{T}^2, \theta_2\partial_x \wedge \partial_y) \to (\mathbb{T}^2, (\theta_1 + \theta_2)\partial_x \wedge \partial_y).$$

However, we have never tried to compute the monoidal product for this spatial structure. □

The next example has been proved in another form in 3.3.7

Example 3.5.2. On the $C(S^1 \times S^1) \rtimes \mathbb{Z}$, with the action given as $(\theta_1, \theta_2).n = (\theta_1, \theta_2 + n.\theta_1)$, there exists a spatial structure Δ_Z. In fact, the localization of the spatial structures at the point θ_1 gives us back example 3.3.7. There are many ways to construct this spatial structure, the easiest way is to apply Theorem 3.3.24 to the groupoid $(S^1 \times \mathbb{R}) \rtimes \mathbb{Z}$, the second way is to build the spatial structure directly from Theorem 3.3.6.

On the extreme side of the *Space* $\leftrightarrow$ *Group* correspondence, we can show that quantization of a Poisson manifold has a natural spatial structure. The work can be done in the same way with Theorem 2.9.16.

Example 3.5.3. The final example is the quantum Heisenberg manifold $M = H_3(\mathbb{R}, \mathbb{R}, \mathbb{R})/H_3(\mathbb{Z}, \mathbb{Z}, \mathbb{Z})$. But instead of doing calculation in the lines of theorem 2.9.16, it is possible to establish a general approach, using the idea of strict deformation quantization with the torus action by Rieffel [34, 74]. We will publish the result in a forthcoming paper.

Appendix A

Appendix

A.1 Groupoids and Stacks

In the section, we collect some background on groupoid and stack. None of them is original and can be found in literature [49] or from the thesis of Canez [9]

A Groupoid is a category such that all the arrows are invertible.

A **topological groupoid** is one whose the space of arrows forms a topological space and the structure maps are continuous.

A **Haar system** on a locally compact groupoid is a continuous family of measures on the range fibers of the groupoid, which is invariant under left translation [70]. In the thesis, all the groupoids are assumed to be locally compact groupoids with Haar systems.

A **Lie groupoid** is a groupoid object in the category of smooth manifolds such that the source and target maps are surjective submersions.

A Lie groupoid $G \rightrightarrows M$ is **etale** if l and r are etale maps, is proper if the maps $l \times r : G \to M \times M$ is proper.

A Lie Algebroid over a smooth manifold M is a vector bundle E over M, together with: $[.,.] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ an $\mathbb{R}-$ linear map, a bundle map $\rho : E \to TM$ called the anchor, such that the Leibniz rule is satisfied: $[a, fb] = f[a, b] + \rho(a)f.b$ for any $a, b \in \Gamma(E)$, $f \in C^\infty(M)$.

An **action** of a Lie Groupoid $G \rightrightarrows M$ on a smooth manifold N consists of a moment map $J : N \to M$ and a smooth map $\Phi : G \times_N M \to M$ such that

- $J(\Phi(g)m) = l(g)$ for $g \in G$ and $n \in J^{-1}(r(g))$

- $\Phi(e(p), n) = n$ for any $p \in M$and $n \in J^{-1}(p)$.
- $\Phi(g, \Phi(h)n) = \Phi(gh, n)$ for any composable $g, h \in G$ and $n \in J^{-1}(r(h))$.

Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be two Lie Groupoid. A $(G, H)-$**bibundle** is a smooth manifold B equipped with commuting left and right actions of G and H.

B is called **right principal** if the moment maps $J_B : B \to M$ is surjective submersion and the $H-$action is free and transitive on its fibers. The left principal bibundle is defined similarly. A bibundle is **biprincipal** if it is both left and right principal.

Two Lie groupoids $G \rightrightarrows M$ and $H \rightrightarrows N$ are called **Morita equivalent** if and only if there exists a biprincipal bimodule between G and H.

A differential (topological, algebraic) **Stack** is a Morita Equivalence class of Lie (topological, algebraic) groupoids. A groupoid lying in the equivalent class is called a **presentation** of a stack. We denote by $M//G$ the stack presented by the groupoid $G \rightrightarrows M$.

A **Symplectic Groupoid** is a symplectic manifold M with symplectic form ω such that the graph of the multiplication is a Lagrangian submanifold inside $M \times M \times \bar{M}$.

A **Poisson Groupoid** is a Poisson manifold M with symplectic form ω such that the graph of the multiplication is a coisotropic submanifold inside $M \times M \times \bar{M}$.

A.2 $KK-$category

There are two ways to define a $KK-$category, the first one is using universal property and the second one is the concrete way for some specific purpose.

Abstract Approach

Given a category of C^*-algebras with C^*-morphism, there is a natural question to ask, what is the universal bi-invariant theory satisfying some good enough properties? Among them there should be **C^*-stable property** (invariant under tensor product with algebra of compact operators), and split-exact.

Definition A.2.1. A functor F from the C^*- category to any additive category C is called **split exact**, if for any split exact sequence of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] & A \arrow[r, "\pi"] & A/I \arrow[l, bend right, "s"'] \arrow[r] & 0 \end{tikzcd},$$

the map $i^* \oplus s^* : F(I) \oplus F(A/I) \to F(A)$ is invertible.

The answer was found by Cuntz and Higson.

Theorem A.2.2. *(Joachim Cuntz and Nigel Higson). Bivariant $KK-$theory is the universal C^***-stable, split-exact** bi-functor on the category of separable C^* - algebras. That is, a functor from the category of separable C^* - algebras to some additive category factors through $KK-$theory if and only if it is C^*-stable and split-exact, and this factorization is unique if it exists.*

$$\begin{tikzcd} C^* - Alg \arrow[r, "i"] \arrow[dr, "F'"] & KK - Category \arrow[d, "F"] \\ & C \end{tikzcd},$$

Therefore, the $KK-$category exists naturally as a C^*-stable and split exact theory with the separable C^*-algebras as objects. The space of the morphisms between two C^*-algebras A and B in the category is called $KK_\bullet(A, B)$.

The most important feature of the $KK-$category is that it is triangulated, so that we can do homological algebra.

Using the abstract approach, it is possible to define that the C^*-correspondence is a subcategory of the $KK-$category.

Theorem A.2.3. *[60]The C^*-correspondence category is the universal C^***-stable** functor on the category of separable C^* - algebras. That is, a functor from the category of separable C^* - algebras to some additive category factors through $KK-$theory if and only if it is C^*-stable, and this factorization is unique if it exists.*

$$\begin{tikzcd} C^* - Alg \arrow[r, "i"] \arrow[dr, "F'"] & C^* - Correspondence\ Category \arrow[d, "F"] \\ & C \end{tikzcd}$$

Concrete Approach

Another approach is the standard way to construct the $KK-$groups concretely as in [2].

Definition A.2.4. Given two C^*-algebras A and B. By a $KK-$**cycle**, we mean a triple (E, ρ, F) such that

1. E is a countably graded C^*-graded module over B.
2. ρ is a $*-$representation of A on E as bounded operators, commuting with B.
3. F is a bounded operator on E of degree 1, commuting with B.
4. For any $a \in A$, the operators $[F, \rho(a)]$, $(F^2 - 1)\rho(a)$ and $(F - F^*)\rho(a)$ are all B-compact operators (the norm limit of sums of finite rank operators $x \mapsto \langle x, b_1 \rangle b_2$).

A cycle is said to be a d**egenerate** $KK-$**cycle** if all three expressions vanish.

In some senses, we think of a $KK-$cycle as a bivariant spectral triple, in which the actions of the C^*-algebras on both sides are to count the infinite dimensional indexes.

Two cycles are called to be homotopic if there exist a cycles from A to the C^*-algebra $IB = C([0,1], B)$, such that the 0-end of the cycle is unitary equivalent to the first cycle, and the 1-end one is unitary equivalent to the second cycle.

Example A.2.5. A very important example is the correspondence between C^*-algebras, with trivial operator F.

Definition A.2.6. The $KK-$group from A to B, denoted $KK_0(A, B)$, is the set of the $KK-$cycles modulo homotopy equivalence. The odd KK group is then defined to be $KK_1(A, B) := KK(A \otimes C_0(\mathbb{R}), B)$.

We can view the graded groups $KK_\bullet(A, B)$ between a pair of C^*-algebras A and B as the space of generalized morphisms, and the space of C^*-algebras with $KK-$groups then forms the so-call $KK-$category. The composition of morphism is given by the Kasparov's technical lemma, where Kasparov computed the formula of the product Fredholm operator.

Lemma A.2.7. *(Kasparov Technical lemma). Let A, B and C be C^*- algebras. There exists a Kasparov product*

$$\otimes_B : KK(A,B) \times KK(B,C) \to KK(A,C)$$

satisfying the following properties.

1. Bi-additivity, *i.e.* $(x_1 + x_2) \otimes_B y = x_1 \otimes_B y + x_2 \otimes_B y$.
2. Associativity, *i.e.* $(x \otimes_B y) \otimes_C z = x \otimes_B (y \otimes_C z)$.
3. Unit elements, if we define $1_A := [Id_A]$,$1_B := [Id_B]$ then
$$\forall x \in KK(A,B) : 1_A \otimes x = x \otimes_B 1_B = x$$
4. Functoriality, if $f : A \to B$ and $g : B \to C$ are graded $*-$homomorphism, then $\forall x \in KK(A,B), x \otimes_B [g] = g_*(x)$ and $y \in KK(B,C), [f] \otimes_B y = f_*(y)$.
5. (Triviality) if $(E_1, \phi_1, 0) \in E(A,B)$, and $(E_2, \phi_2, 0) \in E(B,C)$ then $[(E_1, \phi_1, 0) \otimes_B (E_2, \phi_2, 0)] = (E_1 \otimes_B E_2, \phi_1, 0)$.

In fact, a generalized version was found by Kasparov [2]

Lemma A.2.8. *Let A, B, C, D, E be C^*-algebras. Then there exists a product*

$$* : KK(A, B \otimes E) \times KK(B \otimes D, C) \to KK(A \otimes D, C \otimes D).$$

We notice that an invertible Kasparov cycle with trivial Fredholm operator is a Morita bimodule.

A.3 Torus bundle

In the section, we review some basic definition of the torus bundles from Kahn [39].

Given a torus bundle $\xi : \mathbb{T}^n \to Y \to X$ over a connected manifold X. Then there exists a representation of the fundamental group $\pi_1(X)$ on the automorphism group of $\mathbb{T}^n$ as well as automorphism groups of its homology/cohomology. Denote the map by

$$\rho : \pi_1(X) \to Aut(H_1(\mathbb{T}^n)) \cong Aut(\mathbb{Z}^n) \cong GL(n, \mathbb{Z}).$$

Then we have the following description of the torus fibration.

Theorem A.3.1. *Assume that X is a compact connected manifold, and choose any representation $\rho : \pi_1(X) \to GL(n, \mathbb{Z})$. Then there exists a natural bijective correspondence between the equivalence classes of torus bundles over X with the monodromy representation ρ, and element $c(\xi)$ of $H^2(X, \mathbb{Z}^n_\rho) \cong H^2(X, H_1(\mathbb{T}^n))$. The element $c(\xi)$ is called the characteristic class of the torus bundle ξ.*

Remark A.3.2. If $\rho(\pi_1(X))$ is trivial, then ξ is a principal torus bundle, and $c(\xi)$ reduces to the first Chern class. $c(\xi)$ vanishes if and only if ξ admits a nontrivial section.

If the tori are associated with symplectic structures, then

Theorem A.3.3. *Assume that X is a compact connected manifold, and choose any representation $\rho : \pi_1(X) \to SL(n, \mathbb{Z})$. Then there exists a natural bijective correspondence between the equivalence classes of* ***symplectic*** *torus bundles over X with the monodromy representation ρ, and element $c(\xi)$ of $H^2(X, \mathbb{Z}^n_\rho) \cong H^2(X, H_1(\mathbb{T}^n))$. The element $c(\xi)$ is called the characteristic class of the torus bundle ξ.*

Corollary A.3.4. *Every principle torus bundle has a canonical structure as a symplectic torus bundle.*

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Index

$AF-$algebra, 36
$C_0(X) - algebra$, 20
$KK(A, B)$, 130
$KK-$cycle, 130

a bundle of C^*-stack., 88
action of a Lie Groupoid, 127

$B-A$ correspondence, 17
B-compact operators, 130
base space, 21
bibundle, 128
biprincipal, 128
boundary points, 40
bundle of C^*-stacks., 88

C^*- dynamical system, 21
C^*-module, 17
C^*-stable property, 128
C^*correspondence, 98
C^*-Morita equivalent, 18
category of stacks, 99
character map, 67
co-variant representation, 21
continuous field, 20
contraction, 58
cyclic modules, 116

Decomposition Theorem, 121
degenerate $KK-$cycle, 130
dequantization, 62
DGLA, 49

Effros-Shen algebra, 36
etale, 127
extreme points, 40

fibers, 21
Formal Deformation Quantization, 23
Full C^*-module, 17
full crossed product, 22

Gauss-Manin connection, 68
generalized morphism, 98
Gerstenhaber bracket, 56
Goldman-Millson groupoid, 54
Groupoid, 127
groupoid C^* - algebra, 100
Groupoid Correspondence, 98
groupoid Morita bimodule, 99
groupoid Morita equivalent, 99

Heisenberg group, 124
Hermitian module, 16
Hilsum-Skandalis map, 98
Hochschild chain complex, 54
Hochschild cochain complex, 55
Hochschild cohomology, 55
Hochschild homology, 55
homotopically trivial family, 32

internal multiplication, 58

J-curve, 25

Kontsevich morphism, 63

L_∞ - algebra structure, 49
L_∞-morphism, 53
$L_\infty - algebra$, 50
$L_\infty - module$, 51
Lie Algebroid, 127
Lie groupoid, 127

Maurer-Cartan, 53
mirror symmetry, 1
monoidal spatial structure, 109
Morita equivalent, 15, 16, 128

non-degenerate, 16
noncommutative two-torus bundle, 44

periodic cyclic homology, 60
Poisson algebra, 23
Poisson Groupoid, 128
presentation, 128

quasi-isomorphic, 49
quasi-isomorphism, 49

reduced crossed product., 22
Rieffel category., 18
Rieffel tensor product, 17
right Hermitian module, 16

section algebra, 21
smooth element, 24
smooth noncommutative two-torus, 12
Spatial C^*-algebra, 103
Spatial structure, 102
Spatial-Morita equivalent, 108
split exact, 128
Stack, 128
strict deformation quantization, 24
strict homomorphism, 99
structure map, 20
Symplectic Groupoid, 128

target algebra, 98
topological groupoid, 127
trace of the noncommutative two-torus, 14
twisted DGLA, 65
twisted-DGLA, 65