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On Riemannian Submersions and Diffeomorphism Stability

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Curtis Christopher Pro

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Dissertation Committee:

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Committee Chairperson

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To Wilbur Frank Pro

ABSTRACT OF THE DISSERTATION

On Riemannian Submersions and Diffeomorphism Stability

by

Curtis Christopher Pro

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2012
Professor Frederick Wilhelm, Chairperson

This thesis consists of work that was carried out in three separate papers that were written during my time at UC, Riverside.

Abstract of chapter II: If $\pi : M \to B$ is a Riemannian Submersion and M has non-negative sectional curvature, O'Neill's Horizontal Curvature Equation shows that B must also have non-negative curvature. We find constraints on the extent to which O'Neill's horizontal curvature equation can be used to create positive curvature on the base space of a Riemannian submersion. In particular, we study when K. Tapp's theorem on Riemannian submersions of compact Lie groups with bi-invariant metrics generalizes to arbitrary manifolds of non-negative curvature.

Abstract of Chapter III: Though Riemannian submersions preserve non-negative sectional curvature this does not generalize to Riemannian submersions from manifolds with non-negative Ricci curvature. We give here an example of a Riemannian submersion $\pi : M \to B$ for which $\mathrm{Ricci}_p(M) > 0$ and at some point $p \in B$, $\mathrm{Ricci}_p(B) < 0$.

Abstract of Chapter IV: The smallest r so that a metric r–ball covers a metric space M is called the radius of M. The volume of a metric r-ball in the space form of constant curvature k is an upper bound for the volume of any Riemannian manifold with sectional curvature $\geq k$ and radius $\leq r$. We show that when such a manifold has volume almost equal to this upper bound, it is diffeomorphic to a sphere or a real projective space.

Contents

Chapter 1

Introduction

1.1 Flats and Submersions in Non-Negative Curvature

Until very recently all examples of compact, positively curved manifolds were constructed as the image of a Riemannian submersion of a Lie group with a bi-invariant metric ([9, 27, 48]). Earlier constructions of positive curvature in [1, 3, 4], and [10, 12, 11] combined the fact that Lie groups with bi-invariant metrics are non-negatively curved with the so called Horizontal Curvature Equation,

$$\sec_B(x,y) = \sec_M(\tilde{x},\tilde{y}) + 3\left|A_{\tilde{x}}\tilde{y}\right|^2$$

[16, 42]. Here $\pi : M \to B$ is a Riemannian submersion, $\{x, y\}$ is an orthonormal basis for a plane in a tangent space to B, $\{\tilde{x},\tilde{y}\}$ is a horizontal lift of $\{x, y\}$, and A is the "integrability tensor" for the horizontal distribution—that is,

$$A_{\tilde{x}}\tilde{y} \equiv \frac{1}{2}[\tilde{X},\tilde{Y}]^{\mathrm{vert}}$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary extensions of $\tilde{x}$ and $\tilde{y}$ to horizontal vector fields.

Since the Horizontal Curvature Equation decomposes $\sec_B(x, y)$ into the sum of two non-negative quantities, we see immediately that Riemannian submersions preserve non-negative curvature. In addition, if *either* term on the right is positive, then $\sec_B(x, y) > 0$. Naively, one might expect positively curved examples to be constructed by exploiting the full power of the Horizontal Curvature Equation; however, a survey of the examples shows that this has never been done. In the context in which the examples in [1, 3, 4, 10, 12, 11], and [57] were constructed, it is impossible for a Riemannian submersion to create positive curvature via the A–tensor alone. In fact, in [56] Tapp shows

Theorem[Tapp] *Let $\pi : G \to B$ be a Riemannian submersion of a compact Lie group with a bi-invariant metric. Then*

1 *Every zero-curvature plane of B exponentiates to a flat (meaning a totally geodesic immersion of $\mathbb{R}^2$ with a flat metric), and*

2 *Every horizontal zero-curvature plane of G projects to a zero-curvature plane of B.*

In the case of bi-quotients of Lie groups, this is a consequence of an equation in [19]. This was first observed explicitly in [60].

Recall, if σ is a zero-curvature plane in a Lie group G with bi-invariant metric, then $\exp(\sigma)$ is a (globally) flat submanifold of G. So it is natural to ask about the extent to which Tapp's theorem holds if σ is assumed to be a horizontal zero-curvature plane whose exponential image is a flat submanifold of M. More formally, we pose:

Problem 1 *If $\pi : M \to B$ is a Riemannian submersion of a compact, non-negatively* *curved manifold M and σ is a horizontal zero-curvature plane in M such that* $\exp(\sigma)$ *is a flat submanifold, does it follow that $\pi_*(\sigma)$ is a zero-curvature plane in B?*

We emphasize that the given flat is not assumed to be globally horizontal.

The following easy consequence of Lemma 1.5 in [55] shows that an affirmative answer to our problem implies that both M and B have a lot of additional structure.

Theorem 2 *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds. Let σ be a zero-curvature plane in B and $\tilde{\sigma}$ a horizontal lift of σ so that* $\exp(\tilde{\sigma})$ *is a flat in M. Then*

1 *The plane σ exponentiates to a flat in B, and*

2 *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

If we assume the fibers of the submersion are totally geodesic, then, even in the non-compact case, the conclusion of Tapp's theorem holds.

Theorem 3 *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds with totally geodesic fibers. Let $\tilde{\sigma}$ be a horizontal zero-curvature plane in M such that* $\exp(\tilde{\sigma})$ *is a flat. Then*

1 *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*

2 *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

We also give an affirmative answer to Problem 1 in the special case when the submersion is induced by an isometric group action with only principal orbits.

Theorem 4 *Let a compact Lie group G act by isometries on a compact, non-negatively curved manifold M. Suppose all of the orbits are principal, and let $\pi : M \to M/G$ be the induced Riemannian submersion.*

Suppose $\tilde{\sigma}$ is a horizontal zero-curvature plane in M such that $\exp_p(\tilde{\sigma})$ is a flat. Then

- ***1*** *$\tilde{\sigma}$ projects to a zero-curvature plane σ in M/G that exponentiates to a flat submanifold of M/G, and*
- ***2*** *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

Example 17 shows that this result does not hold if we remove the hypothesis that M is compact. On the other hand, appropriate associated bundles also inherit this property.

Corollary 5 *Let G be a compact Lie group, P be compact, and $\pi_P : P \to B \equiv P/G$ a principal G–bundle with non-negatively curved G–invariant metric. Let F be a non-negatively curved manifold that carries an isometric G–action and $\pi : E := P \times_G F \to B$ the corresponding associated bundle with fiber F. Give E and B the corresponding non-negatively curved metrics so that π and $Q : P \times F \to P \times_G F = E$ become Riemannian submersions.*

If $\tilde{\sigma}$ is a π–horizontal zero-curvature plane in E such that $\exp_p(\tilde{\sigma})$ is a flat, then

- ***1*** *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*
- ***2*** *Every horizontal lift of σ exponentiates to a horizontal flat in E.*

Example 6 *Grove and Ziller have shown how to lift the product metric on* $S^2 \times S^2$ *and Cheeger's metric on* $\mathbb{C}P^2 \# -\mathbb{C}P^2$ *to various principal* $SO(k)$ *bundles and hence to all of the associated bundles [30]. According to Lemma 23 (below) the flat tori in* $S^2 \times S^2$ *lift to flats in all of these non-negatively curved bundles. Similarly, the flat Klein bottles in Cheeger's* $\mathbb{C}P^2 \# -\mathbb{C}P^2$ *must also lift to flats in all of the non-negatively curved bundles of [30]. It follows from the construction of the metric that the principal bundles all have totally geodesic fibers. Therefore the principal bundles give examples of Theorems 2, 3, and 4. The associated bundles give examples of Theorem 2 and Corollary 5.*

1.2 Riemannian Submersions Need Not Preserve Positive Ricci Curvature

One might ask if something similar to O'Neill's horizontal curvature equation exists for Riemannian submersions in the Ricci curvature case. However, given the difference between Ricci and sectional curvature, it is not a surprise that Riemannian submersions need not preserve a lower Ricci cuvature bound. Yet, an example of this appears to be absent from the literature. We give an example that shows this can fail severely, that is,

Theorem 7 *For any* $C > 0$*, there is a Riemannian submersion* $\pi : M \to B$ *for which* M *is compact with positive Ricci curvature and* B *has some Ricci curvatures less than* $-C$*.*

The examples are constructed as a warped product $S^2 \times_\nu F$, where F is any manifold that admits a metric with Ricci curvature ≥ 1, and the metric on S^2 is C^1–close to any predetermined positively curved rotationally symmetric metric on S^2.

1.3 The Diffeomorphism Type Of Manifolds with Almost Maximal Volume

Any closed Riemannian n–manifold M has a lower bound for its sectional curvature, $k \in \mathbb{R}$. This gives an upper bound for the volume of any metric ball $B(x,r) \subset M$,

$$\operatorname{vol} B(x,r) \leq \operatorname{vol} \mathcal{D}_k^n(r),$$

where $\mathcal{D}_k^n(r)$ is an r–ball in the n–dimensional, simply connected space form of constant curvature k. If rad M is the smallest number r such that a metric r–ball covers M, it follows that

$$\operatorname{vol} M \leq \operatorname{vol} \mathcal{D}_k^n(\operatorname{rad} M).$$

The invariant rad M is known as the *radius* of M and can alternatively be defined as

$$\operatorname{rad} M = \min_{p \in M} \max_{x \in M} \operatorname{dist}(p,x).$$

In the event that $\operatorname{vol} M$ is almost equal to $\operatorname{vol} \mathcal{D}_k^n(\operatorname{rad} M)$, we determine the diffeomorphism type of M.

Theorem 8 *Given $n \in \mathbb{N}, k \in \mathbb{R}$, and $r > 0$, there is an $\varepsilon > 0$ so that every closed Riemannian n–manifold M with*

$$\begin{aligned} \sec M &\geq k, \\ \operatorname{rad} M &\leq r, \text{ and} \\ \operatorname{vol} M &\geq \operatorname{vol} \mathcal{D}_k^n(r) - \varepsilon \end{aligned} \tag{1.1}$$

is diffeomorphic to S^n or $\mathbb{R}P^n$.

Grove and Petersen obtained the same result with *diffeomorphism* replaced by *homeomorphism* in [25]. They also showed that for any $\varepsilon > 0$ and $M = S^n$ or $\mathbb{R}P^n$ there are Riemannian metrics that satisfy (1.1) except when $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$.

For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, Grove and Petersen also computed the optimal upper volume bound for the class of manifolds M with

$$\sec M \geq k \quad \text{and} \quad \operatorname{rad} M \leq r. \tag{1.2}$$

It is strictly less than vol $\mathcal{D}_k^n(r)$ [25]. For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, manifolds satisfying (1.2) with almost maximal volume are already known to be diffeomorphic to spheres [28]. The main theorem in [43] gives the same result when $r = \frac{\pi}{\sqrt{k}}$.

For $k > 0$ and $r = \frac{\pi}{\sqrt{k}}$, the maximal volume vol $\mathcal{D}_1^n\left(\frac{\pi}{\sqrt{k}}\right)$ is realized by the n-sphere with constant curvature k. For $k > 0$ and $r = \frac{\pi}{2\sqrt{k}}$, the maximal volume vol $\mathcal{D}_1^n\left(\frac{\pi}{2\sqrt{k}}\right)$ is realized by $\mathbb{R}P^n$ with constant curvature k. Apart from these cases, there are no Riemannian manifolds M satisfying (1.2) and vol M = vol $\mathcal{D}_k^n(r)$. Rather, the maximal volume is realized by one of two types of Alexandrov spaces. [25]

Example 9 ***(Crosscap)*** *The constant curvature k Crosscap, $C_{k,r}^n$, is the quotient of $\mathcal{D}_k^n(r)$ obtained by identifying antipodal points on the boundary. Thus $C_{k,r}^n$ is homeomorphic to $\mathbb{R}P^n$. There is a canonical metric on $C_{k,r}^n$ that makes this quotient map a submetry. The universal cover of $C_{k,r}^n$ is the double of $\mathcal{D}_k^n(r)$. If we write this double as $\mathbb{D}_k^n(r) := \mathcal{D}_k^n(r)^+ \cup_{\partial \mathcal{D}_k^n(r)^\pm} \mathcal{D}_k^n(r)^-$, then the free involution*

$$A : \mathbb{D}_k^n(r) \longrightarrow \mathbb{D}_k^n(r)$$

that gives the covering map $\mathbb{D}_k^n(r) \longrightarrow C_{k,r}^n$ *is*

$$A : (x, +) \longmapsto (-x, -)\,,$$

where the sign in the second entry indicates whether the point is in $\mathcal{D}_k^n(r)^+$ *or* $\mathcal{D}_k^n(r)^-$.

Example 10 ***(Purse)*** *Let* $R : \mathcal{D}_k^n(r) \to \mathcal{D}_k^n(r)$ *be reflection in a totally geodesic hyperplane* H *through the center of* $\mathcal{D}_k^n(r)$*. The Purse,* $P_{k,r}^n$*, is the quotient space*

$$\mathcal{D}_k^n(r) \,/\, \{v \sim R(v)\}\,, \text{ provided } v \in \partial \mathcal{D}_k^n(r)\,.$$

Alternatively we let $\{\mathcal{H}\mathcal{D}_k^n(r)\}^+ \cup \{\mathcal{H}\mathcal{D}_k^n(r)\}^- = D_k^n(r)$ *be the decomposition of* $\mathcal{D}_k^n(r)$ *into the two half disks on either side of* H*. Then* $P_{k,r}^n$ *is isometric to the double of* $\{\mathcal{H}\mathcal{D}_k^n(r)\}^+$.

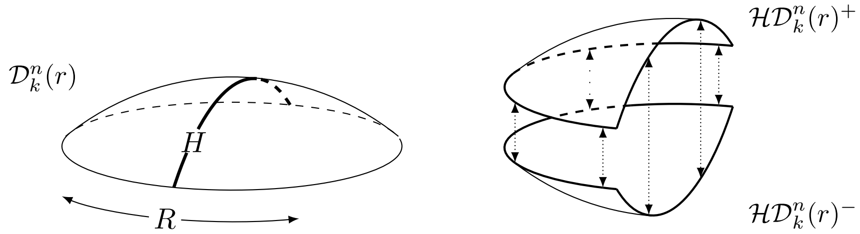

Figure 1.1: Two equivalent constructions of $P_{1,r}^2$

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed n-manifolds satisfying sec $M \geq k$ and rad $M \leq r$ and $\{\text{vol} M_i\}$ converging to vol $\mathcal{D}_k^n(r)$ where $r \leq \frac{\pi}{2\sqrt{k}}$ if $k > 0$. Grove and Petersen showed that $\{M_i\}$ has a subsequence that converges to either $C_{k,r}^n$ or $P_{k,r}^n$ in the Gromov-Hausdorff topology [25]. The main theorem follows by combining this with the following *diffeomorphism stability theorems.*

Theorem 11 *Let* $\{M_i\}_{i=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $\sec M_i \geq k$ *so that*

$$M_i \longrightarrow C_{k,r}^n$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_i*s are diffeomorphic to* $\mathbb{R}P^n$*.*

Theorem 12 *Let* $\{M_i\}_{i=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $\sec M_i \geq k$ *so that*

$$M_i \longrightarrow P_{k,r}^n$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_i*s are diffeomorphic to* S^n*.*

Remark 13 *One can get Theorem 12 for the case* $k = 1$ *and* $r > \operatorname{arccot}\left(\frac{1}{\sqrt{n-3}}\right)$ *as a corollary of Theorem C in [29]. Theorem 11 when* $k = 1$ *and* $r = \frac{\pi}{2}$ *follows from the main theorem in [61] and the fact that* $C_{1,\frac{\pi}{2}}^n$ *is* $\mathbb{R}P^n$ *with constant curvature* 1 *. With minor modifications of our proof, the hypothesis* $\sec M_i \geq k$ *in Theorems 11 and 12 can replaced, except in one case, with an arbitrary uniform lower curvature bound. The exceptional case, is Theorem 11 in dimension 4, specifically in Proposition 56. For ease of notation, we have written all of the proofs for* $\{M_i\}_{i=1}^{\infty}$ *with* $\sec M_i \geq k$ *converging to* $C_{k,r}^n$ *or* $P_{k,r}^n$*.*

Remark 14 *We mention here that the space of directions at every point* $p \in C_{k,r}^n$ *is isometric the round sphere* S^{n-1}*. It has been shown in [35] that diffoemorphism stability holds when the limit space has all of its space of directions being Euclidean and therefore Theorem*

11 follows immediately from this. However, the proof here differs greatly from what is in [35] and we leave it for this reason.

Chapter 2

Flats and Submersions in Non-Negative Curvature

2.1 Background

By the implicit function theorem, the fibers of a smooth submersion $\pi : M \to B$ are smooth submanifolds of M. We call the distribution $\mathcal{V}$ defined by

$$\mathcal{V} := \mathrm{ker}\pi_*$$

the *vertical distribution.* If M is a Riemannian manifold, we denote by $\mathcal{H}$ the distibution defined as the orthogonal compliment to $\mathcal{V}$ and call $\mathcal{H}$ the *horizontal distribution.* We then have a decomposition of the tangent bundle

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

If, in addition, B is Riemannian, π is called a *Riemannian submersion* if $\pi_*|_{\mathcal{H}}$ is an isometry.

For a vector $v \in TM$, we write

$$v = v^v + v^h$$

to denote the vertical and horizontal components of v, respectively. In [42] O'Neill generalizes the classical second fundamental form for immersions by defining two tensors A and T on M. Here, for vector fields E, F on M these are defined as

$$A_E F := (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h$$

$$T_E F := (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v$$

Just like the case for the second fundamental form of an immersion, these tensors measure the geometric complexities of the submersion.

2.2 Examples

Our goal is to generalize Tapp's theorem to Riemannian submersions from more than just compact Lie groups G with a biinvariant metric. So we begin by giving examples of how the conclusions of Tapp's theorem can fail to hold in this new setting. Recall that Tapp shows

Theorem [Tapp] *Let $\pi : G \to B$ be a Riemannian submersion of a compact Lie group with a bi-invariant metric. Then*

1 *Every zero-curvature plane of B exponentiates to a flat (meaning a totally geodesic immersion of $\mathbb{R}^2$ with a flat metric), and*

2 *Every horizontal zero-curvature plane of G projects to a zero-curvature plane of B.*

The following examples show that this theorem fails if the Lie group G is replaced by an arbitrary, compact, non-negatively curved Riemannian manifold M. The inhomogeneous metrics of these examples have zero-planes whose exponentials are locally, but not globally, flat.

Example 15 *(Fish Bowl) Let* $\psi : [0, \pi] \longrightarrow \mathbb{R}$ *be a smooth, concave-down function that satisfies*

$$\psi(t) = \begin{cases} t & \text{for } t \in \left[0, \frac{\pi}{4}\right] \\ \pi - t & \text{for } t \in \left[\frac{3\pi}{4}, \pi\right] \end{cases}$$

Consider the warped product metric

$$g_\psi = dt^2 + \psi^2 d\theta^2$$

on $S^2 = [0, \pi] \times_\psi S^1$. *As before,* S^1 *acts isometrically on* $\left(S^2, g_\psi\right)$, *so we get a Riemannian submersion*

$$\left(S^2, g_\psi\right) \times S^1 \longrightarrow \left(S^2, \bar{g}_\psi\right),$$

where $\bar{g}_\psi$ *is the metric induced by the submersion. Notice that* $\left(S^2, g_\psi\right) \times S^1$ *is flat in a neighborhood of the set* $\{0, \pi\} \times S^1$, *but, as in Example 17,* $\left(S^2, \bar{g}_\psi\right)$ *is positively curved in the image of this neighborhood. If, in addition,*

$$\psi''|_{\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)} < 0,$$

then $\left(S^2, \bar{g}_\psi\right)$ *is positively curved. This shows that even in the compact case, the* $A-$ *tensor can be responsible for creating positive curvature and that conclusion 2 of Tapp's Theorem fails for arbitrary Riemannian submersions of compact, nonnegatively curved manifolds.*

Example 16 *To see how conclusion 1 of Tapp's theorem can fail to hold, choose* ψ *in the previous example to be constant in a neighborhood of* $\pi/2$. *This makes* $\left(S^2, g_\psi\right)$ *isometric to a flat cylinder near a neighborhood of the equator. In the Cheeger deformed metric, the image of this region is a smaller flat cylinder. Since the base,* $\left(S^2, \bar{g}_\psi\right)$, *is not flat, we have zero–curvature planes near the equator that do not exponentiate to flats.*

In Theorem 2, we do not require that M is compact; on the other hand, without compactness, the answer to Problem 1 is “no”, even when M is a Lie group.

Example 17 *Let* $(\mathbb{R}^2, \bar{g})$ *be the Cheeger deformation of* $\mathbb{R}^2$ *obtained from the standard* S^1 *action on* $\mathbb{R}^2$. *Let* s *and* g *be the usual metrics on* S^1 *and* $\mathbb{R}^2$, *respectively. Recall that* $\bar{g}$ *is defined so that the quotient map,*

$$Q : (S^1 \times \mathbb{R}^2, s + g) \to (\mathbb{R}^2, \bar{g})$$

given by $Q(z, q) = \bar{z}q$ *is a Riemannian submersion. This new metric is positively curved and is a paraboloid asymptotic to a cylinder of radius 1. All horizontal planes have zero curvature, but each projects to a positively curved plane. So positive curvature is created via the A-tensor alone.*

2.3 Jacobi Fields Along Geodesics Contained In Flats

To prove Theorems 3 and 4 we establish a main lemma on holonomy fields, whose defintion we recall from [20].

Definition 18 *Given a Riemannian submersion* $\pi : M \to B$ *let* A *and* T *be the corresponding fundamental tensors as defined in [42]. A Jacobi field* J *along a horizontal* *geodesic $c : I \to M$ is said to be a holonomy field if $J(0)$ is vertical and satisfies*

$$J'(0) = A_{\dot{c}(0)} J(0) + T_{J(0)} \dot{c}(0). \tag{2.1}$$

Main Lemma 1 *Let $\pi : M \to B$ be a Riemannian submersion of complete, non-negatively curved manifolds so that each holonomy field is bounded. Let $\tilde{\sigma}$ be a horizontal zero-curvature plane in M such that* $\exp(\tilde{\sigma})$ *is a flat. Then*

1 *$\tilde{\sigma}$ projects to a zero-curvature plane σ in B that exponentiates to a flat submanifold of B, and*

2 *Every horizontal lift of σ exponentiates to a horizontal flat in M.*

The symmetries of the curvature tensor imply that the map $X \longmapsto R(X,W)W$ is self-adjoint. This combined with the spectral theorem yields the following result, which appears implictly in [48].

Proposition 19 *Let* $\operatorname{span}\{X, W\}$ *be a zero curvature plane in a nonnnegatively curved manifold, then*

$$R(X,W)W = R(W,X)X = 0.$$

In a compact Lie group G with bi-invariant metric, solutions to the Jacobi equation along a geodesic $\gamma(t)$ have the form

$$J(t) = E_0 + tF_0 + \sum_{i=0}^{l} \left(\cos(\sqrt{k_i} t) E_i + \sin(\sqrt{k_i} r) F_i \right),$$

where E_i and F_i are parallel along γ (see [41]). We generalize this decomposition in the following way:

Lemma 20 *Suppose γ is a geodesic in a complete, non-negatively curved manifold M, and suppose J_0 is a normal, parallel, Jacobi field along γ, then any normal Jacobi field J along γ can be written as*

$$J(t) = (a + bt)J_0(t) + W(t), \tag{2.2}$$

where $a, b \in \mathbb{R}$ and W and W' are perpendicular to J_0.

Proof. Extend J_0 to an orthonormal basis $\{J_0, E_2, ..., E_{n-1}\}$ of normal, parallel fields along γ. Since $J_0(t)$ and $\gamma'(t)$ span a zero-curvature plane and M is non-negatively curved, $R(J_0, \gamma')\gamma' = 0$, by Proposition 19. Therefore, if we write

$$J(t) = f(t)J_0(t) + \sum_{i=2}^{n-1} f_i(t)E_i(t),$$

we have

$$\begin{aligned} J''(t) &= -R(J(t), \gamma'(t))\gamma'(t) \\ &= -\sum_{i=2}^{n-1} f_i(t)R(E_i, \gamma'(t))\gamma'(t) \end{aligned}$$

and

$$\langle R(E_i, \gamma')\gamma', J_0\rangle = \langle R(J_0, \gamma')\gamma', E_i\rangle = 0$$

by a symmetry of the curvature tensor. Thus $J'' \perp J_0$. Since $\{J_0, E_2, ..., E_{n-1}\}$ is parallel and orthogonal, we also have

$$J''(t) = f''(t)J_0(t) + \sum_{i=2}^{n-1} f_i''(t)E_i(t).$$

Combining this with $J'' \perp J_0$, we see that $f'' = 0$ as claimed.

Since $W' = \sum_{i=2}^{n-1} f_i'(t)E_i(t)$, we also have $W' \perp J_0$. ■

Given a Riemannian submersion $\pi : M \to B$, let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions. As holonomy fields are the variational fields arising from horizontal lifts of geodesics in B, they never vanish, they remain vertical, and they satisfy (2.1) for all time. In fact, we can find a collection $\{J_i(t)\}$ of such fields that span $\mathcal{V}$ along c. This description of $\mathcal{V}$ allows one to determine precisely when a field along a curve in M has values in $\mathcal{H}$. In particular, we have the following, as observed by Tapp when M is a Lie group.

Lemma 21 *Suppose $\pi : M \to B$ is a Riemannian submersion of a complete, non-negatively curved manifold M, γ is a horizontal geodesic in M, and J_0 is a parallel Jacobi field along γ such that $J_0(0)$ is horizontal. If all holonomy fields V along γ have bounded length, then J_0 is everywhere horizontal.*

Proof. Let V be a holonomy field. Since V is always vertical, the decomposition in Lemma 20 simplifies to

$$V(t) = btJ_0(t) + W(t).$$

Since V has bounded length, $b = 0$ and therefore $V(t) = W(t)$, which is perpendicular to J_0. As the collection of all holonomy fields spans the vertical distribution along γ, the result follows. ■

Part 1 of the main lemma is a consequence of the next result.

Lemma 22 *Suppose $\pi : M \to B$ is a Riemannian submersion of a complete, non-negatively curved manifold M, and all holonomy fields of π have bounded length. Suppose $\tilde{\sigma}$ is a horizontal zero curvature plane and $\exp(\tilde{\sigma})$ is a totally geodesic flat.*

Then $\sigma := d\pi(\tilde{\sigma})$ *has a zero curvature and* $\exp(\sigma)$ *is a totally geodesic flat submanifold of* B*.*

Proof. Let $\{X, Y\}$ be any orthonormal pair in $\tilde{\sigma}$. Let γ be the geodesic: $t \longmapsto \exp(tX)$, and let J be the parallel Jacobi field along γ with $J(0) = Y$. Then by the previous Lemma, $J(t)$ is horizontal for all t. Hence $\exp(\tilde{\sigma})$ is everywhere horizontal, and, by assumption, a totally geodesic flat.

It follows from the Horizontal Curvature Equation that $\pi(\exp(\tilde{\sigma}))$ is also flat, and from the formula for covariant derivatives of horizontal fields it follows that $\pi(\exp(\tilde{\sigma}))$ is totally geodesic. Since horizontal geodesics project to geodesics, $\pi(\exp(\tilde{\sigma})) = \exp(d\pi(\tilde{\sigma})) = \exp(\sigma)$. So $\exp(\sigma)$ is a totally geodesic flat submanifold of B. ■

The following lemma is probably a well known application of the Horizontal Curvature Equation. We include it as it establishes part 2 of our main lemma and is also used in the proof of Theorem 2.

Lemma 23 *Let* $\pi : M \to B$ *be a Riemannian submersion of a complete, non-negatively curved manifold* M*. Let* σ *be a tangent plane to* B *so that* $\exp(\sigma)$ *is a totally geodesic flat.*

Then for any horizontal lift $\tilde{\sigma}$ *of* σ, $\exp(\tilde{\sigma})$ *is a totally geodesic flat that is everywhere horizontal.*

Proof. The Horizontal Curvature Equation implies that any horizontal lift $\hat{\tau}$ of a plane τ tangent to $\exp(\sigma)$ satisfies

$$\sec_M(\hat{\tau}) = 0 \text{ and } A(\hat{\tau}) = 0.$$

In particular, the collection of all such $\hat{\tau}$s gives us an integrable 2-dimensional distribution that is horizontal. The vanishing A–tensor combined with our hypothesis that $\exp(\sigma)$ is totally geodesic gives us that all the integral submanifolds of this distribution are also totally geodesic. If $\tilde{\sigma}$ is a horizontal lift of σ, then it follows that $\exp(\tilde{\sigma})$ is tangent to this distribution and hence is a totally geodesic flat that is everywhere horizontal. ■

We now proceed with proofs of theorems 3 and 2.

Proof of Theorem 3. When the fibers of a Riemannian submersion are totally geodesic, the T-tensor for the submersion vanishes. If V is a holonomy field along a horizontal geodesic γ, by (2.1) we have

$$\langle V(t), V(t)\rangle' = 2\langle V(t), V'(t)\rangle = 2\langle V(t), T_{V(t)}\gamma'(t)\rangle = 0,$$

so V has constant norm. An applicaiton of the main lemma completes the proof. ■

In contrast to our other results the proof of Theorem 2 does not use the main lemma. Instead we exploit the infinitesimal geometry of the submersion.

Proof of Theorem 2.

Let σ be a zero-curvature plane in B and $\tilde{\sigma}$ a horizontal lift of σ so that $\exp(\tilde{\sigma})$ is contained in a flat of M. Let γ be a geodesic in $\exp(\tilde{\sigma})$ and J_0 be a parallel Jacobi field along γ such that

$$\tilde{\sigma} = \operatorname{span}\left\{\gamma'(0), J_0(0)\right\}.$$

Now $A_{\gamma'(0)} J_0(0) = 0$ because $\sec_M(\tilde{\sigma}) = \sec_B(\sigma) = 0$; so for any holonomy field V, we have

$$\begin{aligned} \left.\langle J_0(t), V'(t)\rangle\right|_{t=0} &= \left.\langle J_0(t), A_{\gamma'(t)} V(t)\rangle\right|_{t=0}, \text{ since } J_0(0) \text{ is horizontal} \\ &= -\left.\langle A_{\gamma'(t)} J_0(t), V(t)\rangle\right|_{t=0} \\ &= 0. \end{aligned}$$

On the other hand, differentiating the right hand side of $V(t) = btJ_0(t) + W(t)$, we find

$$\begin{aligned} \left.\langle J_0(t), V'(t)\rangle\right|_{t=0} &= \langle J_0(t), bJ_0(t)\rangle|_{t=0} + \left.\langle J_0(t), W'(t)\rangle\right|_{t=0} \\ &= b\,|J_0(0)|^2 . \end{aligned}$$

Therefore $b = 0$ and $V = W$, and it follows that $N := \exp(\tilde{\sigma})$ is everywhere horizontal. Thus its projection, $\exp(\sigma)$, is a totally geodesic flat in B.

By Lemma 23, every horizontal lift of σ exponentiates to a horizontal flat in M.

■

2.4 The Holonomy of π

In this section we prove Theorem 4 by showing that such submersions have bounded holomomy fields and hence satisfy the hypotheses of the main lemma. At the end of the section we prove Corollary 5.

Given a point $b \in B$, we define the *holonomy group* $\mathrm{hol}(b)$ to be the group of all diffeomorphisms of the fiber $\pi^{-1}(b)$ that occur as holonomy diffeomorphisms $h_c : \pi^{-1}(b) \to \pi^{-1}(b)$ obtained by lifting piecewise smooth loops c at b. If M is compact, the T tensor is globally bounded in norm. It follows that each holonomy diffeomorphism h_c is Lipschitz with Lipschitz constant dependent only on the length of c (see [21], Lemma 4.2). Since this Lipschitz constant can actually depend on the length of c, this is generally not enough to conclude that the the holonomy fields are uniformly bounded (see [56], Example 6.1]).

On the other hand, if B is compact and $\mathrm{hol}(b)$ is a compact, finite-dimensional Lie group, then there is a uniform Lipschitz constant for all of $\mathrm{hol}(b)$. Thus the holonomy fields are uniformly bounded ([56], Proposition 6.2). So to prove theorem 4, it suffices to show that $\mathrm{hol}(b)$ is a compact, finite-dimensional Lie group.

Proof of Theorem 4. Set $B = M/G$, and for $p \in M$, let G_p denote the isotropy subgroup of G. Note that the map $f : G/G_p \to M$ given by $f(gG_p) = g(p)$ is an imbedding onto the orbit $G(p)$ of p. Now take any piecewise smooth curve $c : [0, 1] \to B$. The holonomy diffeomorphism

$$h_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$$

is defined by

$$h_c(p) = \bar{c}(1),$$

where $\bar{c}$ is the unique horizontal lift of c starting at p. By assumption, G acts isometrically on M, so $g\bar{c}$ is also horizontal. Since $(g\bar{c})(1) = g(\bar{c}(1))$, we have that

$$h_c(gp) = gh_c(p).$$

In other words, h_c is G-equivariant.

By the above, $\mathrm{hol}(b)$ is a subgroup of the collection $\mathrm{Diff}_G(\pi^{-1}(b))$ of all G-equivariant diffeomorphisms of the fiber $\pi^{-1}(b)$. Take any $p \in \pi^{-1}(b)$. Set $H \equiv G_p$, and identify $\pi^{-1}(b)$

with G/H. Then $\mathrm{Diff}_G(G/H)$ is isomorphic to the Lie group $N(H)/H$, where $N(H)$ is the normalizer of H (see [20], Lemma 2.3.3).

In [59], Wilking associates to a given metric foliation $\mathcal{F}$ the so-called *dual foliation* $\mathcal{F}^{\#}$. The dual leaf through a point $p \in M$ is defined as all points $q \in M$ such that there is a piecewise smooth, horizontal curve from p to q. Let $L_p^{\#}$ be the dual leaf through p.

We shall see that for any $p \in M$, $\mathrm{hol}(b)$ is homeomorphic to $L_p^{\#} \cap \pi^{-1}(b)$.

We have the continuous map

$$\mathrm{ev}_p : \mathrm{hol}(b) \to L_p^{\#} \cap \pi^{-1}(b)$$

defined by

$$\mathrm{ev}_p : h_c \mapsto h_c(p).$$

To construct the inverse, let q be in $L_p^{\#} \cap \pi^{-1}(b)$. There is a piecewise smooth, horizontal curve $\bar{c}$ from p to q. Now $\pi \circ \bar{c}$ is a piecewise smooth loop at b and

$$h_{\pi \circ \bar{c}}(p) = q.$$

We therefore propose to define ev_p^{-1} by

$$\mathrm{ev}_p^{-1} : q \longmapsto h_{\pi \circ \bar{c}}.$$

To see that ev_p^{-1} is well-defined, suppose $\tilde{c}$ is another piecewise smooth, horizontal curve from p to q. By construction, we have $h_{\pi \circ \bar{c}}(p) = h_{\pi \circ \tilde{c}}(p)$. Since all holonomy diffeomorphisms are G-equivariant and G acts transitively on $\pi^{-1}(b)$, it follows that

$$h_{\pi \circ \bar{c}} = h_{\pi \circ \tilde{c}}.$$

Now take a sequence of points $q_i \in L^{\#} \cap \pi^{-1}(b)$ converging to $q_0 \in L^{\#} \cap \pi^{-1}(b)$. There are horizontal curves $\bar{c}_i$ from p to q_i such that $h_{\pi\circ\bar{c}_i}(p) = q_i$. Again by G-equivariance and the transitive action of G, these holonomy diffeomorphisms are completely determined by their behavior at a point. Thus $h_{\pi\circ\bar{c}_i} \to h_{\pi\circ\bar{c}_0}$, and so ev_p^{-1} is continuous. Therefore $\mathrm{hol}(b)$ is homeomorphic to $L^{\#} \cap \pi^{-1}(b)$.

Since $\mathcal{F}$ is given by the orbit decomposition of an isometric group action, the dual foliation has complete leaves ([59], Theorem 3(a)). In particular, this says $L^{\#} \cap \pi^{-1}(b) \cong \mathrm{hol}(b)$ is a closed subset of the compact space $\pi^{-1}(b)$ and hence is also compact. It follows that $\mathrm{hol}(b)$ is closed in the Lie group $\mathrm{Diff}_G(G/H) \cong N(H)/H$, so is a Lie subgroup of $\mathrm{Diff}_G(G/H)$. Thus $\mathrm{hol}(b)$ is a compact, finite-dimensional Lie group. $\blacksquare$

Remark 24 *In general,* $\mathrm{hol}(b)$ *need not even be a Lie group, let alone a compact Lie group [56]. However, it is shown in [22] that when the fibers come from principal G-actions,* $\mathrm{hol}(b)$ *is always a Lie group.*

Recall (see [20], p.92) that if P is the total space of the principal G–bundle $\pi_P : P \to B := P/G$ and F is a manifold that carries a G–action, then G acts freely on the product $P \times F$. In particular, if P and F have G-invariant metrics of non-negative curvature, G acts by isometries on the product $P \times F$. As a result, the total space $E = P \times_G F := (P \times F)/G$ of the associated bundle inherits a metric of non-negative curvature such that the quotient map $Q : P \times F \to P \times_G F$ is a Riemannian submersion [8]. Similarly, B inherits a metric of non-negative curvature such that $\pi_P : P \to B$ is a Riemannian submersion. If $\pi_1 : P \times F \to P$ is projection onto the first factor, the diagram

$$\begin{tikzcd} P \times F \arrow[r, "Q"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ P \arrow[r, "\pi_P"'] & B \end{tikzcd}$$

commutes and so $\pi : E \to B$ is also a Riemannian submersion.

Proof of Corollary 5: . Consider the composition

$$\pi_P \circ \pi_1 : P \times F \longrightarrow B.$$

The holonomy fields for $\pi_P \circ \pi_1$ are the products of holonomy fields for $\pi_P : P \to B$ and π_1. The former are bounded by the proof of Theorem 4, the latter are bounded because the fibers of π_1 are totally geodesic.

Now suppose that $\tilde{\sigma}$ is a horizontal zero-curvature plane for $\pi : E \longrightarrow B$ such that $\exp_p(\tilde{\sigma})$ is a flat. Apply Lemma 23 to $Q : P \times F \to E$ to conclude that any horizontal lift $\tilde{\sigma}_{P \times F}$ of $\tilde{\sigma}$ exponentiates to a (Q–horizontal) flat. Since the holonomy fields of $\pi_P \circ \pi_1 = \pi \circ Q$ are bounded, we can apply Lemma 22 to conclude that $\sigma := d(\pi \circ Q)(\tilde{\sigma}_{P \times F}) = d\pi(\tilde{\sigma})$ is a zero plane that exponentiates to a flat. Applying Lemma 23 to $\pi : E \to B$ we conclude that every horizontal lift of σ is a horizontal flat. ■

Remark 25 *Combining the Main Lemma with the concept of projectable Jacobi fields from [20] one gets a shorter (but more learned) proof of the Corollary.*

Chapter 3

Riemannian Submersions Need Not Preserve Positive Curvature

3.1 Vertical Warping

Given a Riemannian submersion $\pi : M \to B$, the *vertical* and *horizontal* distributions are defined as $\mathcal{V} := \ker \pi_*$ and $\mathcal{H} := (\ker \pi_*)^{\perp}$, respectively. This gives a splitting of the tangent bundle as

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

If g is the metric on M, we denote by g^h and g^v the restrictions of g to $\mathcal{H}$ and $\mathcal{V}$. Define a new metric $g_\nu := e^{2\nu} g^v + g^h$, where ν is any smooth function on B. Note that both $\mathcal{H}$ and g^h are unchanged, so $\pi : (M, g_\nu) \to B$ is also Riemannian.

The calculations that give important geometric quantities associated to g_ν in terms of g and ν are carried out in ([20], p. 45). In particular, the $(0, 2)$ Ricci tensor Ric_ν of g_ν

is given in detail. When $M = B^m \times F^k$ with g a product metric, these quantities reduce to the following (Corollary 2.2.2 [20]):

For horizontal X, Y and vertical U, V, we have

$$\begin{aligned}
\mathrm{Ric}_\nu(X,Y) &= \mathrm{Ric}_B(X,Y) - k(\mathrm{Hess}\,\nu(X,Y) + g(\nabla\nu, X)g(\nabla\nu, Y)), && (3.1)\\
\mathrm{Ric}_\nu(X,U) &= 0, && (3.2)\\
\mathrm{Ric}_\nu(U,V) &= \mathrm{Ric}_F(U,V) - g(U,V)e^{2\nu}(\Delta\nu + k|\nabla\nu|^2). && (3.3)
\end{aligned}$$

Here we denote by the same letter those fields which are π_1-related where $\pi_1 : B \times F \to B$ is projection onto the first factor. We write $B \times_\nu F$ to denote the warped product metric g_ν on $B \times F$.

3.2 $S^2_\varphi \times_\nu F$

Choose $\varphi : [0, \pi] \to [0, \infty)$ so that S^2 equipped with the metric $g_\varphi = dr^2 + \varphi^2 d\theta^2$ is a smooth Riemannian manifold denoted by S^2_φ. Let $\nu : [0, \pi] \to \mathbb{R}$ be a function on S^2_φ that only depends on r. Consider the warped product $S^2_\varphi \times_\nu F$ where (F, g_F) is any k-dimensional manifold ($k \geq 2$) with $\mathrm{Ric}_F \geq 1$. Using the notation $\dot\nu = \partial_r \nu$, since ν only depends on r

$$\nabla\nu = \dot\nu \partial_r.$$

If L denotes Lie derivative we have,

$$\begin{aligned} 2\mathrm{Hess}\,\nu &= L_{\nabla\nu}g_{\varphi} \\ &= L_{\dot{\nu}\partial_r}g_{\varphi} \\ &= \dot{\nu}L_{\partial_r}g_{\varphi} + d\dot{\nu}dr + drd\dot{\nu} \\ &= 2\dot{\nu}\mathrm{Hess}\,r + 2\ddot{\nu}dr^2, \end{aligned}$$

and so the Hessian of ν is given by

$$\mathrm{Hess}\,\nu = \ddot{\nu}dr^2 + \dot{\nu}\varphi\dot{\varphi}d\theta^2.$$

The Ricci tensor of S^2_{φ} is (see p. 69 [47]) given as

$$\mathrm{Ric}_{S^2_{\varphi}} = -\frac{\ddot{\varphi}}{\varphi}g_{\varphi}.$$

Let Ric^h_{ν} and Ric^v_{ν} denote Ric_{ν} restricted to the horizontal and vertical distribution, respectively. Equation (3.1) can be written as

$$-\mathrm{Ric}^h_{\nu} = \left[\frac{\ddot{\varphi}}{\varphi} + k(\ddot{\nu} + \dot{\nu}^2)\right]dr^2 + \varphi\left[\ddot{\varphi} + k\dot{\nu}\dot{\varphi}\right]d\theta^2 \tag{3.4}$$

and equation (3.3) can be written as

$$\mathrm{Ric}^v_{\nu} = \mathrm{Ric}_F - e^{2\nu}(\ddot{\nu} + \frac{\dot{\varphi}\dot{\nu}}{\varphi} - k\dot{\nu}^2)g_F. \tag{3.5}$$

Notice that since $\mathrm{Ric}_F \geq 1$, if Ric^h_{ν} is positive, then these equations together with Equation 3.2 imply that $S^2_{\varphi}\times_{\nu+\ln\lambda}F$ has positive Ricci curvature, provided λ is a sufficiently small positive constant.

By requiring that $\ddot{\varphi}(p) > 0$ for some point $p \in (0, \pi)$, the projection $\pi_1 : S^2_\varphi \times_\nu F \longrightarrow S^2_\varphi$ is a Riemannian submersion for which the base has points of negative Ricci curvature.

To describe a Riemannian submersion that does not preserve non-negative Ricci curvature, it suffices to find functions φ and ν, and a metric g_F on F so that

1. S^2_φ is smooth and has points of negative curvature, i.e.,

$$\varphi^{(\text{even})}(0) = \varphi^{(\text{even})}(\pi) = 0,$$

$$\dot{\varphi}(0) = -\dot{\varphi}(\pi) = 1,$$

$$\ddot{\varphi}(p) = \eta > 0$$

   for some point $p \in (0, \pi)$,

2. $\mathrm{Ric}^h_\varphi \geq 0$, i.e.,

$$\begin{aligned} \ddot{\nu} + \dot{\nu}^2 &\leq -\frac{\ddot{\varphi}}{k\varphi} \\ \dot{\varphi}\dot{\nu} &\leq -\frac{\ddot{\varphi}}{k}. \end{aligned}$$

   and

3. $\mathrm{Ric}^v_\varphi \geq 0$, i.e.,

$$\mathrm{Ric}_F \quad \geq \quad e^{2\nu}(\ddot{\nu} + \varphi\dot{\varphi}\dot{\nu} - k\dot{\nu}^2)g_F$$

Assuming F admits a metric with positive Ricci curvature, once functions satisfying (1) and (2) are found, we can scale down to a metric g_F that strictly satisfies (3). So all that remains is to find functions φ and ν satisfying (1) and (2).

In fact, we show $S^2_\varphi \times_\nu F$ can have positive Ricci curvature by showing the existence of functions satisfying (1) and (2) with strict inequalities. This will follow by finding smooth functions φ and ν and numbers $a, \varepsilon > 0$, and $b > p + \varepsilon$ such that

(A) φ satisfies (1) and

$$\ddot{\varphi} < 0$$

on $(0, \pi) \setminus [p - \varepsilon, p + \varepsilon]$,

$$\dot{\varphi} \geq a$$

on $[0, b]$.

(B) $\dot{\nu} = 0$ on $[0, \pi] \setminus (p/2, b)$ and

$$\begin{aligned} \dot{\nu} &< -\frac{\eta}{ka} \\ \ddot{\nu} + \dot{\nu}^2 &< -\frac{\eta}{k\varphi(p - \varepsilon)} \end{aligned}$$

on $(p - \varepsilon, p + \varepsilon)$.

(C)

$$\ddot{\nu} + \dot{\nu}^2 < -\frac{\ddot{\varphi}}{k\varphi}$$

on $(p + \varepsilon, b]$.

The only constraint for a function φ that satisfies all conditions of (A) is that

$$\int_0^b \ddot{\varphi}\, dr \geq a - 1.$$

On $(0, p - \varepsilon)$, $|\ddot{\varphi}|$ may be chosen arbitrarily small, so this constraint may be written as

$$\int_{p+\varepsilon}^b \ddot{\varphi}\, dr \geq a - 1. \tag{3.6}$$

Take ν so that on $[0, p/2]$, $\dot{\nu} = 0$. On some subinterval of $[p/2, p-\varepsilon]$, require $\ddot{\nu}$ small enough so that

$$\begin{aligned} \ddot{\nu} + \dot{\nu}^2 &\leq 0, \\ \dot{\nu}(p-\varepsilon) &= -\frac{2\eta}{ka}, \end{aligned}$$

and on $(p-\varepsilon, p+\varepsilon)$, we may set

$$\ddot{\nu} = \frac{-2\eta}{k\varphi(p-\varepsilon)} - \left(\frac{2\eta}{ka}\right)^2.$$

Then

$$\dot{\nu}(p+\varepsilon) = -\frac{2\eta}{ka} + O(\varepsilon)$$

and therefore on $(p-\varepsilon, p+\varepsilon)$,

$$\ddot{\nu} + \dot{\nu}^2 \leq -\frac{2\eta}{k\varphi(p-\varepsilon)} + O(\varepsilon).$$

So (B) will be satisfied provided ε is small enough and $\dot{\nu} = 0$ on $[b, \pi]$. This last constraint can be written as

$$\begin{aligned} \int_{p+\varepsilon}^{b} \ddot{\nu}\, dr &= -\dot{\nu}(p+\varepsilon) \\ &= \frac{2\eta}{ka} - O(\varepsilon). \end{aligned} \tag{3.7}$$

On $[p+\varepsilon, b)$, $\varphi \geq (p+\varepsilon)a$. So (3.6) says

$$\int_{p+\varepsilon}^{b} -\frac{\ddot{\varphi}}{k\varphi}\, dr \leq \frac{1-a}{k(p+\varepsilon)a}.$$

On the same interval,

$$\dot{\nu} \geq -\frac{2\eta}{ka} + O(\varepsilon).$$

This, together with (3.7) says

$$\int_{p+\varepsilon}^{b} \ddot{\nu} + \dot{\nu}^2 \, dr \leq \frac{2\eta}{ka} + \left(\frac{2\eta}{ka}\right)^2 (b - (p + \varepsilon)) + O(\varepsilon).$$

Therefore, on $(p + \varepsilon, b]$, if we take $\ddot{\varphi}$ small enough so that

$$\ddot{\varphi} < -ka(p + \varepsilon)(\ddot{\nu} + \dot{\nu}^2),$$

(C) will be satisfied provided

$$\frac{2\eta}{ka} + \left(\frac{2\eta}{ka}\right)^2 (b - (p + \varepsilon)) + O(\varepsilon) < \frac{1 - a}{k(p + \varepsilon)a}.$$

If $\eta < 1/(4(p+\varepsilon))$, this will be satisfied, for example, by taking $a = 1 - 4(p+\varepsilon)$, $b = p + 2\varepsilon$ and ε sufficiently small.

Notice that η can be taken large and $\varphi(p)$ will be small provided p is sufficiently small. Given $C > 0$, the metric S^2_φ may be taken to have points of curvature less than $-C$.

Chapter 4

The Diffeomorphism Type of Manifolds with almost maximal volume

Section 4.1 introduces notations and conventions. Section 4.2 is review of necessary tools from Alexandrov geometry. Section 4.3 develops machinery and proves Theorem 11 in the case when $n \neq 4$. Theorem 11 in dimension 4 is proven in Section 4.4, and Theorem 12 is proven in Section 4.5.

Throughout the remainder of the paper, we assume without loss of generality, by rescaling if necessary, that $k = -1, 0$ or 1.

4.1 Conventions and Notations

We assume a basic familiarity with Alexandrov spaces, including but not limited to [5]. Let X be an n–dimensional Alexandrov space and $x, p, y \in X$.

1. We call minimal geodesics in X *segments*. We denote by px a segment in X with endpoints p and x.

2. We let Σ_p and T_pX denote the space of directions and tangent cone at p, respectively.

3. For $v \in T_pX$ we let γ_v be the segment whose initial direction is v.

4. Following [46], $\Uparrow_x^p \subset \Sigma_x$ will denote the set of directions of segments from x to p, and $\uparrow_x^p \in \Uparrow_x^p$ denotes the direction of a single segment from x to p.

5. We let $\sphericalangle(x, p, y)$ denote the angle of a hinge formed by px and py and $\tilde{\sphericalangle}(x, p, y)$ denote the corresponding comparison angle.

6. Following [43], we let $\tau : \mathbb{R}^k \to \mathbb{R}_+$ be any function that satisfies

$$\lim_{x_1,\ldots,x_k \to 0} \tau\left(x_1, \ldots, x_k\right) = 0,$$

and abusing notation we let $\tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be any function that satisfies

$$\lim_{x_1,\ldots,x_k \to 0} \tau\left(x_1, \ldots, x_k | y_1, \ldots, y_n\right) = 0,$$

provided that $y_1, \ldots, y_n$ remain fixed.

When making an estimate with a function τ we implicitly assert the existence of such a function for which the estimate holds.

7. We denote by $\mathbb{R}^{1,n}$ the Minkowski space $(\mathbb{R}^{n+1}, g)$, where g is the semi-Riemannian metric defined by

$$g = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$$

for coordinates $(x_0, x_1, \cdots, x_n)$ on $\mathbb{R}^{n+1}$.

8. We reserve $\{e_j\}_{j=0}^m$ for the standard orthonormal basis in both euclidean and Minkowski space.

9. We use two isometric models for hyperbolic space,

$$H_+^n := \left\{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \middle| -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1,\ x_0 > 0 \right\}$$

and

$$H_-^n := \left\{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \middle| -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1,\ x_0 < 0 \right\}.$$

10. We obtain explicit double disks, $\mathbb{D}_k^n(r) := \mathcal{D}_k^n(r)^+ \cup_{\partial\mathcal{D}_k^n(r)^\pm} \mathcal{D}_k^n(r)^-$, by viewing $\mathcal{D}_k^n(r)^+$ and $\mathcal{D}_k^n(r)^-$ explicitly as

$$\mathcal{D}_k^n(r)^+ := \left[\begin{array}{cc} \left\{ z \in H_+^n \subset \mathbb{R}^{1,n} \middle| \operatorname{dist}_{H_+^n}(e_0, z) \le r \right\} & \text{if } k = -1 \\ \left\{ z \in \{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(e_0, z) \le r \right\} & \text{if } k = 0 \\ \left\{ z \in S^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(e_0, z) \le r \right\} & \text{if } k = 1, \end{array} \right.$$

and

$$\mathcal{D}_k^n(r)^- := \left[\begin{array}{cc} \left\{ z \in H_-^n \subset \mathbb{R}^{1,n} \middle| \operatorname{dist}_{H_-^n}(-e_0, z) \le r \right\} & \text{if } k = -1 \\ \left\{ z \in \{-e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(-e_0, z) \le r \right\} & \text{if } k = 0 \\ \left\{ z \in S^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(-e_0, z) \le r \right\} & \text{if } k = 1. \end{array} \right.$$

Since $r < \frac{\pi}{2}$ when $k = 1$, $\mathcal{D}_k^n(r)^+$ and $\mathcal{D}_k^n(r)^-$ are disjoint in all three cases.

4.2 Basic Tools From Alexandrov Geometry

The notion of strainers [5] in an Alexandrov space forms the core of the calculus arguments used to prove our main theorem. In this section, we review this notion and its relevant consequences. In some sense the idea can be traced back to [43], and some of the ideas that we review first appeared in other sources such as [58] and [62].

Definition 26 *Let X be an Alexandrov space. A point $x \in X$ is said to be (n, δ, r)–strained by the strainer $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$ provided that for all $i \neq j$ we have*

$$\widetilde{\sphericalangle}(a_i, x, b_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}(a_i, x, b_i) > \pi - \delta,$$

$$\widetilde{\sphericalangle}(a_i, x, a_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}(b_i, x, b_j) > \tfrac{\pi}{2} - \delta, \text{ and}$$

$$\min_{i=1,\dots,n} \{\mathrm{dist}(\{a_i, b_i\}, x)\} > r.$$

We say a metric ball $B \subset X$ is an (n, δ, r)–strained neighborhood with strainer $\{a_i, b_i\}_{i=1}^n$ provided every point $x \in B$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$.

The following is observed in [62].

Proposition 27 *Let X be a compact n-dimensional Alexandrov space. Then the following are equivalent.*

1 *There is a (sufficiently small) $\eta > 0$ so that for every $p \in X$*

$$\mathrm{dist}_{G-H}\left(\Sigma_p, S^{n-1}\right) < \eta.$$

2 *There is a (sufficiently small) $\delta > 0$ and an $r > 0$ such that X is covered by finitely many (n, δ, r)–strained neighborhoods.*

Theorem 28 *([5] Theorem 9.4) Let X be an n–dimensional Alexandrov space with curvature bounded from below. Let $p \in X$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Provided δ is small enough, there is a $\rho > 0$ such that the map $f : B(p, \rho) \to \mathbb{R}^n$ defined by*

$$f(x) = (\operatorname{dist}(a_1, x), \operatorname{dist}(a_2, x), \dots, \operatorname{dist}(a_n, x))$$

is a bi-Lipschitz embedding with Lipschitz constants in $(1 - \tau(\delta, \rho), 1 + \tau(\delta, \rho))$.

If every point in X is (n, δ, r)–strained, we can equip X with a C^1–differentiable structure defined by Otsu and Shioya in [44]. The charts will be smoothings of the map from the theorem above and are defined as follows: Let $x \in X$ and choose $\sigma > 0$ so that $B(x, \sigma)$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$. Define $d_{i,x}^\eta : B(x, \sigma) \to \mathbb{R}$ by

$$d_{i,x}^\eta(y) = \frac{1}{\operatorname{vol}(B(a_i, \eta))} \int_{z \in B(a_i, \eta)} \operatorname{dist}(y, z).$$

Then $\varphi_x^\eta : B(x, \sigma) \to \mathbb{R}^n$ is defined by

$$\varphi_x^\eta(y) = (d_{1,x}^\eta(y), \dots, d_{n,x}^\eta(y)). \tag{4.1}$$

If B is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, any choice of $2n$–directions $\left\{\left(\uparrow_x^{a_i}, \uparrow_x^{b_i}\right)\right\}_{i=1}^n$ where $x \in B$ will be called a set of straining directions for Σ_x. As in, [5, 62], we say an Alexandrov space Σ with $\operatorname{curv} \Sigma \geq 1$ is globally (m, δ)-strained by pairs of subsets $\{A_i, B_i\}_{i=1}^m$ provided

$$|\operatorname{dist}(a_i, b_j) - \tfrac{\pi}{2}| < \delta, \quad \operatorname{dist}(a_i, b_i) > \pi - \delta,$$

$$|\operatorname{dist}(a_i, a_j) - \tfrac{\pi}{2}| < \delta, \quad |\operatorname{dist}(b_i, b_j) - \tfrac{\pi}{2}| < \delta$$

for all $a_i \in A_i$, $b_i \in B_i$ and $i \neq j$.

Theorem 29 *([5] Theorem 9.5, cf also [43] Section 3) Let* Σ *be an* $(n-1)$*–dimensional Alexandrov space with curvature* ≥ 1*. Suppose* Σ *is globally strained by* $\{A_i, B_i\}$*. There is a map* $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ *so that* $\Psi : \Sigma \to S^{n-1}$ *defined by*

$$\Psi(x) = \tilde{\Psi} \circ (\mathrm{dist}\,(A_1, x)\,, \mathrm{dist}\,(A_2, x)\,, \ldots, \mathrm{dist}\,(A_n, x))$$

is a bi-Lipschitz homeomorphisms with Lipshitz constants in $(1-\tau\,(\delta)\,, 1+\tau\,(\delta))$*.*

Remark 30 *The description of* $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ *in [5] is explicit but is geometric rather than via a formula. Combining the proof in [5] with a limiting argument, one can see that the map* Ψ *can be given by*

$$\Psi(x) = \left(\sum \cos^2\left(\mathrm{dist}\,(A_i, x)\right)\right)^{-1/2} \left(\cos\left(\mathrm{dist}\,(A_1, x)\right), \ldots, \cos\left(\mathrm{dist}\,(A_n, x)\right)\right).$$

In particular, the differentials of $\varphi_x^\eta : B(x, \sigma) \subset X \longrightarrow \varphi(B(x, \sigma))$ *are almost isometries.*

Next we state a powerful lemma showing that for an (n, δ, r) strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

Lemma 31 *([5] Lemma* 5.6*) Let* $B \subset X$ *be* $(1, \delta, r)$*–strained by* (y_1, y_2)*. For any* $x, z \in B$

$$\left|\tilde{\sphericalangle}\,(y_1, x, z) + \tilde{\sphericalangle}\,(y_2, x, z) - \pi\right| < \tau\,(\delta, \mathrm{dist}\,(x, z)\,|r)$$

In particular, for $i = 1, 2$*,*

$$\left|\sphericalangle\,(y_i, x, z) - \tilde{\sphericalangle}\,(y_i, x, z)\right| < \tau\,(\delta, \mathrm{dist}\,(x, z)\,|r)\,.$$

Corollary 32 *Let* $B \subset X$ *be* $(1, \delta, r)$*–strained by* (a, b)*. Let* $\{X^\alpha\}_{\alpha=1}^\infty$ *be a sequence of Alexandrov spaces with* $\mathrm{curv} X^\alpha \geq k$ *such that* $X^\alpha \longrightarrow X$*. For* $x, z \in B$*, suppose that* $a^\alpha, b^\alpha, x^\alpha, z^\alpha \in X^\alpha$ *converge to* $a, b, x,$ *and* z *respectively. Then*

$$\left|\sphericalangle\left(a^\alpha, x^\alpha, z^\alpha\right) - \sphericalangle\left(a, x, z\right)\right| < \tau\left(\delta, \mathrm{dist}\left(x, z\right), \tau\left(1/\alpha | \mathrm{dist}\left(x, z\right)\right) \mid r\right).$$

Proof. The convergence $X^\alpha \longrightarrow X$ implies that we have convergence of the corresponding comparison angles. The result follows from the previous lemma. ■

Lemma 33 *Let* $B \subset X$ *be* (n, δ, r)*–strained by* $\{(a_i, b_i)\}_{i=1}^n$*. Let* $\{X^\alpha\}_{\alpha=1}^\infty$ *have* $\mathrm{curv} X^\alpha \geq k$ *and suppose that* $X_\alpha \longrightarrow X$*. Let* $\{(\gamma_{1,\alpha}, \gamma_{2,\alpha})\}_{\alpha=1}^\infty$ *be a sequence of geodesic hinges in the* X^α *that converge to a geodesic hinge* (γ_1, γ_2) *with vertex in* B*. Then*

$$\left|\sphericalangle\left(\gamma_{1,\alpha}'\left(0\right), \gamma_{2,\alpha}'\left(0\right)\right) - \sphericalangle\left(\gamma_1'\left(0\right), \gamma_2'\left(0\right)\right)\right| < \tau\left(\delta, \tau\left(1/\alpha | \mathrm{len}\left(\gamma_1\right), \mathrm{len}\left(\gamma_2\right)\right) \mid r\right).$$

Remark 34 *Note that without the strainer,* $\liminf_{\alpha\to\infty} \sphericalangle\left(\gamma_{1,\alpha}'\left(0\right), \gamma_{2,\alpha}'\left(0\right)\right) \geq \sphericalangle\left(\gamma_1'\left(0\right), \gamma_2'\left(0\right)\right)$ *[24], [5].*

Proof. Apply the previous corollary with $x^\alpha = \gamma_{1,\alpha}\left(0\right),\ z^\alpha = \gamma_{1,\alpha}\left(\varepsilon\right),\ x^\alpha \to x,$ and $z^\alpha \to z$ to conclude

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma_{1,\alpha}'\left(0\right)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma_1'\left(0\right))\right| < \tau\left(\delta, \mathrm{dist}\left(x, z\right), \tau\left(1/\alpha | \mathrm{dist}\left(x, z\right)\right) \mid r\right).$$

Similar reasoning with $x^\alpha = \gamma_{2,\alpha}\left(0\right),\ z^\alpha = \gamma_{2,\alpha}\left(\varepsilon\right),\ x = \lim_{\alpha\to\infty} x^\alpha$, and $z = \lim_{\alpha\to\infty} z^\alpha$ gives

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma_{2,\alpha}'\left(0\right)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma_2'\left(0\right))\right| < \tau\left(\delta, \mathrm{dist}\left(x, z\right), \tau\left(1/\alpha | \mathrm{dist}\left(x, z\right)\right) \mid r\right).$$

Since $\mathrm{dist}\left(x, z\right)$ may be as small as we please, the result then follows from Theorem 29. ■

Lemma 35 *([62] Lemma 1.8.2) Let* $\{(a_i, b_i)\}_{i=1}^n$ *be an* (n, δ, r)*–strainer for* $B \subset X$*. For any* $x \in B$ *and* $\mu > 0$*, let* Σ_x^μ *be the set of directions* $v \in \Sigma_x$ *so that* $\gamma_v|_{[0,\mu]}$ *is a segment. For any sufficiently small* $\mu > 0$*,* Σ_x^μ *is* $\tau(\delta, \mu)$*–dense in* Σ_x*.*

Corollary 36 *Suppose* $X^\alpha \longrightarrow X$*,* $\{(a_i, b_i)\}_{i=1}^n$ *is an* (n, δ, r)*–strainer for* $B \subset X$*, and* (n, δ, r)*–strainers* $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n$ *for* $B^\alpha \subset X^\alpha$ *satisfy*

$$\left(\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n, B^\alpha\right) \longrightarrow \left(\{(a_i, b_i)\}_{i=1}^n, B\right).$$

For any fixed $\mu > 0$ *and any sequence of directions* $\{v^\alpha\}_{a=1}^\infty \subset \Sigma_{x^\alpha}$ *with* $x^\alpha \in B^\alpha$*, there is a sequence* $\{w^\alpha\}_{a=1}^\infty \subset \Sigma_{x^\alpha}^\mu$ *with*

$$\sphericalangle(w^\alpha, v^\alpha) < \tau(\delta, \mu)$$

so that a subsequence of $\{\gamma_{w^\alpha}\}_{\alpha=1}^\infty$ *converges to a geodesic* $\gamma : [0, \mu] \longrightarrow X$*.*

From Arzela-Ascoli and Hopf-Rinow, we conclude

Proposition 37 *Let* X *be an Alexandrov space and* $p, q \in X$*. For any* $\varepsilon > 0$*, there is a* $\delta > 0$ *so that for all* $x \in B(p, \delta)$ *and all* $y \in B(q, \delta)$ *and any segment* xy*, there is a segment* pq *so that*

$$\operatorname{dist}(xy, pq) < \varepsilon.$$

We end this section by showing that convergence to a compact Alexandrov space X without collapse implies the convergence of the corresponding universal covers, provided $|\pi_1(X)| < \infty$. For our purposes, when $X = C_{k,r}^n$, it would be enough to use [52] or [15].

The key tools are Perelman's Stability and Local Structure Theorems and the notion of first systole, which is the length of the shortest closed non-contractible curve.

Perelman's proof of the Local Structure Theorem can be found in [45], this result is also a corollary to his Stability Theorem, whose proof is published in [33].

Theorem 38 *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of* n*–dimensional Alexandrov spaces with a uniform lower curvature bound converging to a compact,* n*–dimensional Alexandrov space* X*. If the fundamental group of* X *is finite, then*

1 *A subsequence of the universal covers,* $\{\tilde{X}_i\}_{i=1}^{\infty}$*, of* $\{X_i\}_{i=1}^{\infty}$ *converges to the universal cover,* $\tilde{X}$*, of* X*.*

2 *A subsequence of the deck action by* $\pi_1(X_i)$ *on* $\{\tilde{X}_i\}_{i=1}^{\infty}$ *converges to the deck action of* $\pi_1(X)$ *on* $\tilde{X}$*.*

Proof. In [45], Perelman shows X is locally contractible. Let $\{U_j\}_{j=1}^{n}$ be an open cover of X by contractible sets and let μ be a Lebesgue number of this cover. By Perelman's Stability Theorem, there are $\tau\left(\frac{1}{i}\right)$–Hausdorff approximations

$$h_i : X \longrightarrow X_i$$

that are also homeomorphisms. Therefore, if i is sufficiently large, $\{h_i(U_j)\}_{j=1}^{n}$ is an open cover for X_i by contractible sets with Lebesgue number $\mu/2$. It follows that the first systoles of the X_is are uniformly bounded from below by μ. Since the minimal displacement of the deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ is equal to the first systole of X_i, this displacement is also uniformly bounded from below by μ. By precompactness, a subsequence of $\{\tilde{X}_i\}$ converges to a length space Y. From Proposition 3.6 of [15], a subsequence of the actions $\left(\tilde{X}_i, \pi_1(X_i)\right)$ converges to an isometric action by some group G on Y. By Theorem

2.1 in [14], $X = Y/G$. Since the displacements of the (nontrivial) deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ are uniformly bounded from below, the action by G on Y is properly discontinuous. Hence $Y \longrightarrow Y/G = X$ is a covering space of X. By the Stability Theorem, Y is simply connected, so Y is the universal cover of X. ■

Remark 39 *When the X_i are Riemannian manifolds, one can get the uniform lower bound for the systoles of the X_is from the generalized Butterfly Lemma in [23]. The same argument also works in the Alexandrov case but requires Perelman's critical point theory, and hence is no simpler than what we presented above.*

Lens spaces show that without the noncollapsing hypothesis this result is false even in constant curvature.

4.3 Cross Cap Stability

The main step to prove Theorem 11 is the following.

Theorem 40 *Let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed Riemannian n–manifolds with* sec $M^\alpha \geq k$ *so that*

$$M^\alpha \longrightarrow C^n_{k,r}$$

in the Gromov-Hausdorff topology. Let $\tilde{M}^\alpha$ be the universal cover of M^α. Then for all but finitely many α, there is a C^1 embedding

$$\tilde{M}^\alpha \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$$

that is equivariant with respect to the deck transformations of $\tilde{M}^\alpha \longrightarrow M^\alpha$ and the Z_2–action on R^{n+1} generated by $-id$.

Two and three manifolds have unique differential structures up to diffeomorphism; so in dimensions two and three Theorems 11 and 40 follow from the main result of [25]. We give the proof in dimension 4 in section 4.4. Until then, we assume that $n \geq 5$.

Proof of Theorem 11 modulo Theorem 40.. By Perelman's Stability Theorem all but finitely many $\{\tilde{M}^{\alpha}\}_{\alpha=1}^{\infty}$ are homeomorphic to S^n (cf [25]). Combining this with Theorem 40 and Brown's Theorem 9.7 in [39] gives an H–cobordism between the embedded image of $\tilde{M}^{\alpha} \subset \mathbb{R}^{n+1}$ and the standard S^n. Modding out by $\mathbb{Z}_2$, we see that M^{α} and $\mathbb{R}P^n$ are H–cobordant. Since the Whitehead group of $\mathbb{Z}_2$ is trivial ([32], [40], p. 373), any H–cobordism between M_{α} and $\mathbb{R}P^n$ is an S–cobordism and hence a product, which completes the proof. [2, 38, 53] ■

The proof of Theorem 11 does not exploit any a priori differential structure on the Crosscap. Instead we exploit a model embedding of the double disk

$$\mathbb{D}_k^n(r) \hookrightarrow \mathbb{R}^{n+1},$$

whose restriction to either half, $\mathcal{D}_k^n(r)^+$ or $\mathcal{D}_k^n(r)^-$, is the identity on the last n–coordinates. By describing the identity $\mathcal{D}_k^n(r) \longrightarrow \mathcal{D}_k^n(r)$ in terms of distance functions, we then argue that this embedding can be lifted to all but finitely many of a sequence $\{M^{\alpha}\}$ converging to $\mathbb{D}_k^n(r)$.

The Model Embedding

Let $A : \mathbb{D}_k^n(r) \to \mathbb{D}_k^n(r)$ be the free involution mentioned in Example 9. For $z \in \mathbb{D}_k^n(r)$, we define $f_z : \mathbb{D}_k^n(r) \to \mathbb{R}$ by

$$f_z(x) = h_k \circ \operatorname{dist}(A(z), x) - h_k \circ \operatorname{dist}(z, x) \tag{4.2}$$

where $h_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$h_k(x) = \begin{cases} \frac{1}{2\sinh r}\cosh(x) & \text{if } k = -1 \\ \frac{x^2}{4r} & \text{if } k = 0 \\ \frac{1}{2\sin r}\cos(x) & \text{if } k = 1. \end{cases}$$

Recall that we view $\mathcal{D}_k^n(r)^{\pm}$ as metric r-balls centered at $p_0 = e_0$ and $A(p_0) = -e_0$ in either $H_{\pm}^n$, $\{\pm e_0\} \times \mathbb{R}^n$, or S^n. For $i = 1, 2, \ldots, n$ we set

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases} \tag{4.3}$$

The functions $\{f_i\}_{i=1}^n := \{f_{p_i}\}_{i=1}^n$ are then restrictions of the last n–coordinate functions of $\mathbb{R}^{n+1}$ to $\mathcal{D}_k^n(r)^{\pm}$. We set $f_0 := f_{p_0}$. In contrast to $f_1, \ldots, f_n$, our f_0 is not a coordinate function. On the other hand its gradient is well defined everywhere on $\mathbb{D}_k^n(r) \setminus \{p_0, A(p_0)\}$, even on $\partial\mathcal{D}_k^n(r)^+ = \partial\mathcal{D}_k^n(r)^-$ where it is normal to $\partial\mathcal{D}_k^n(r)^+ = \partial\mathcal{D}_k^n(r)^-$.

Define $\Phi : \mathbb{D}_k^n(r) \to \mathbb{R}^{n+1}$, by

$$\Phi = (f_0, f_1, f_2, \cdots, f_n),$$

and observe that

Proposition 41 *Φ is a continuous, $\mathbb{Z}_2$–equivariant embedding.*

Proof. Write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ and let $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be projection. Since $f_1, f_2, \cdots, f_n$ are coordinate functions, the restrictions

$$\pi \circ \Phi|_{\mathcal{D}_k^n(r)^{\pm}} : \mathcal{D}_k^n(r)^{\pm} \longrightarrow \mathbb{R}^n$$

are both the identity. From this and the definition of f_0, we conclude that Φ is one–to–one. Since $\mathbb{D}_k^n(r)$ is compact, it follows that Φ is an embedding. The $\mathbb{Z}_2$–equivariance is immediate from definition 4.2. ■

Lifting the Model Embedding

To start the proof of Theorem 40 let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed Riemannian n–manifolds with $\sec M^\alpha \geq k$ so that

$$M^\alpha \longrightarrow C_{k,r}^n,$$

and we let $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ denote the corresponding sequence of universal covers. From Theorem 38, a subsequence of $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ together with the deck transformations $\tilde{M}^\alpha \longrightarrow M^\alpha$ converge to $(\mathbb{D}_k^n(r), A)$. For all but finitely many α, $\pi_1(M^\alpha)$ is isomorphic to $\mathbb{Z}_2$. We abuse notation and call the nontrivial deck transformation of $\tilde{M}^\alpha \longrightarrow M^\alpha$, A.

First we extend definition 4.2 by letting $f_z^\alpha : \tilde{M}^\alpha \to \mathbb{R}$ be defined by

$$f_z^\alpha(x) = h_k \circ \operatorname{dist}(A(z), x) - h_k \circ \operatorname{dist}(z, x). \tag{4.4}$$

Let $p_i^\alpha \in \tilde{M}^\alpha$ converge to $p_i \in \mathbb{D}_k^n(r)$, and for some $d > 0$ define $f_{i,d}^\alpha : \tilde{M}^\alpha \to \mathbb{R}$ by

$$f_{i,d}^\alpha(x) = \frac{1}{\operatorname{vol} B(p_i^\alpha, d)} \int_{q^\alpha \in B(p_i^\alpha, d)} f_{q^\alpha}^\alpha(x). \tag{4.5}$$

Differentiation under the integral gives

Proposition 42 *The $f_{i,d}^\alpha$ are C^1 and $\left|\nabla f_{i,d}^\alpha\right| \leq 2$.*

We now define $\Phi_d^\alpha : \tilde{M}^\alpha \to \mathbb{R}^{n+1}$ by

$$\Phi_d^\alpha = \left(f_{0,d}^\alpha, f_{1,d}^\alpha, f_{2,d}^\alpha, \cdots, f_{n,d}^\alpha\right).$$

As $\alpha \to \infty$ and $d \to 0$, Φ_d^α converges to Φ in the Gromov–Hausdorff sense. Since Φ is an embedding it follows that Φ_d^α is one–to–one in the large. More precisely,

Proposition 43 *For any* $\nu > 0$, *if* α *is sufficiently large and* d *is sufficiently small, then*

$$\Phi_d^\alpha (x) \neq \Phi_d^\alpha (y),$$

provided $\operatorname{dist}(x, y) > \nu$.

Since the $\mathbb{Z}_2$-equivariance of Φ_d^α immediately follows from definition 4.5, all that remains to prove Theorem 40 is the following proposition:

Proposition 44 *There is a* $\rho > 0$ *so that* Φ_d^α *is one to one on all* ρ*–balls, provided that* α *is sufficiently large and* d *is sufficiently small.*

This is a consequence of Key Lemma 46 (stated below), whose statement and proof occupy the remainder of this section.

Uniform Immersion

The proof of the Inverse Function Theorem in [50] gives

Theorem 45 *(Quantitative Immersion Theorem) Let*

$$\mathbb{R}^n_{\hat{i}} := \{(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n+1})\} \subset \mathbb{R}^{n+1}$$

and let

$$P_{\hat{i}} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n_{\hat{i}}$$

be orthogonal projection.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be a C^1 map so that for some $a \in \mathbb{R}^n$, $\lambda > 0$, and $\rho > 0$, there is an $i \in \{1, \ldots, n+1\}$ so that

$$\left| d\left(P_{\hat{\imath}} \circ F\right)_a (v) \right| \geq \lambda \left| v \right|$$

and

$$\left| d\left(P_{\hat{\imath}} \circ F\right)_a (v) - d\left(P_{\hat{\imath}} \circ F\right)_x (v) \right| < \frac{\lambda}{2} \left| v \right|$$

for all $x \in B\left(a, \rho\right)$ and $v \in \mathbb{R}^n$, then $\left(P_{\hat{\imath}} \circ F\right)|_{B(a,\rho)}$ is a one–to–one, open map.

We note that every space of directions to $\mathbb{D}^n_k(r)$ is isometric to S^{n-1}. By proposition 27, there are $r, \delta > 0$ so that every point in the double disk has a neighborhood B that is (n, δ, r)–strained. If $B \subset \mathbb{D}^n_k(r)$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, by continuity of comparison angles, we may assume there are sets $B^\alpha \subset \tilde{M}^\alpha$ (n, δ, r)–strained by $\{a_i^\alpha, b_i^\alpha\}_{i=1}^n$ such that

$$\left(\left\{\left(a_i^\alpha, b_i^\alpha\right)\right\}_{i=1}^n, B^\alpha\right) \longrightarrow \left(\left\{\left(a_i, b_i\right)\right\}_{i=1}^n, B\right).$$

Given $x^\alpha \in B^\alpha$, we let $\varphi^\eta_{x^\alpha}$ be as in 4.1.

To prove Proposition 44 it suffices to prove the following.

Key Lemma 46 *There is a $\lambda > 0$ and $\rho > 0$ so that for all $x^\alpha \in \tilde{M}^\alpha$ there is an $i_{x^\alpha} \in \{0, 1, \ldots, n\}$ such that the function $F := \Phi^\alpha_d \circ \left(\varphi^\eta_{x^\alpha}\right)^{-1}$ satisfies*

1.

$$\left| d(P_{\hat{\imath}_{x^\alpha}} \circ F)_{\varphi^\eta_{x^\alpha}(x^\alpha)} (v) \right| > \lambda \left| v \right|$$

and

2.

$$\left| d\left(P_{\hat{\imath}_{x^\alpha}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(y)}(v) - d\left(P_{\hat{\imath}_{x^\alpha}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(x^\alpha)}(v) \right| < \frac{\lambda}{2}|v|$$

for all $y \in B(x^\alpha, \rho)$ *and* $v \in \mathbb{R}^n$*, provided that* α *is sufficiently large and* d *and* η *are sufficiently small.*

We show in the next subsection that part 1 of Key Lemma 46 holds, and in the following subsection we show that part 2 holds.

Lower bound on the differential

We begin by illustrating that, in a sense, the first part of the key lemma holds for the model embedding.

Lemma 47 *There is a* $\lambda > 0$ *so that for all* $v \in T\mathbb{D}^n_k(r)$ *there is a* $j(v) \in \{0, 1, \dots, n\}$ *so that*

$$\left|D_v f_{j(v)}\right| > \lambda |v|.$$

Proof. Recall that the double disk $\mathbb{D}^n_k(r)$ is the union of two copies of $\mathcal{D}^n_k(r)$ that we call $\mathcal{D}^n_k(r)^+$ and $\mathcal{D}^n_k(r)^-$—glued along their common boundary—that throughout this section we call $\mathcal{S} := \partial\mathcal{D}^n_k(r)^{\pm}$.

If $x \in \mathbb{D}^n_k(r) \setminus \mathcal{S}$, then for $i \neq 0$, ∇f_i is unambiguously defined; moreover,

$$\{\nabla f_i(x)\}_{i=1}^n$$

is an orthonormal basis. Thus the lemma certainly holds on $\mathbb{D}^n_k(r) \setminus \mathcal{S}$.

For $x \in \mathcal{S}$ and $i \in \{1, \ldots, n\}$, we can think of the gradient of f_i as multivalued. More precisely, for $x \in \mathcal{S}$, we view

$$\mathcal{S} \subset \mathcal{D}_k^n(r)^{\pm} \subset \begin{cases} H_{\pm}^n & \text{if } k = -1 \\ \{\pm e_0\} \times \mathbb{R}^n & \text{if } k = 0 \\ S^n & \text{if } k = 1 \end{cases}$$

and define $\nabla f_i^{\pm}$ to be the gradient at x of the coordinate function that extends f_i to either $H_{\pm}^n, \{\pm e_0\} \times \mathbb{R}^n$, or S^n.

From definition 4.2, for any $v \in T_x \mathbb{D}_k^n(r)$

$$D_v f_i = \begin{cases} \left\langle \nabla f_i^+, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^+ \\ \left\langle \nabla f_i^-, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^-. \end{cases}$$

Notice that the projections of ∇f_i^+ and ∇f_i^- onto $T_x\mathcal{S}$ coincide, so for $v \in T_x\mathcal{S}$ we have $D_v f_i = \left\langle \nabla f_i^+, v \right\rangle = \left\langle \nabla f_i^-, v \right\rangle$. As $\left\{ \nabla f_i^+ \right\}_{i=1}^n$ is an orthonormal basis, the lemma holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$. Since ∇f_0 is well defined on $\mathcal{S}$ and normal to $\mathcal{S}$, for any unit $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we have $|D_v f_0| > 0$. The lemma follows from the compactness of the set of unit vectors in $T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$. ■

Notice that at p_k and $A(p_k)$ the gradients of f_k and f_0 are colinear. Using this we conclude

Addendum 4.3.1 *Let p_k be any of $p_1, \ldots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x\mathbb{D}_k^n(r)$, the index $j(v)$ in the previous lemma can be chosen to be different from k.*

Lemma 48 *There is a $\lambda > 0$ so that for all $v \in T_x\mathbb{D}_k^n(r)$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so* *that*

$$|D_v f_z| > \lambda\, |v|$$

for all $z \in B(p_{j(v)}, d)$, provided d is sufficiently small.

Proof. If not then for each $i = 0, 1, \ldots, n$ there is a sequence $\{z_i^j\}_{j=1}^{\infty} \subset \mathbb{D}_k^n(r)$ with $\mathrm{dist}(z_i^j, p_i) < \frac{1}{j}$ and a sequence of unit $v^j \in T_{x^j}\mathbb{D}_k^n(r)$ so that

$$\left| D_{v^j} f_{z_i^j} \right| < \frac{1}{j}.$$

Choose the segments $x^j z_i^j$ and $x^j A\left(z_i^j\right)$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow_{x^j}^{z_i^j},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{z_i^j},\ v^j\right) \text{ and} \\ \sphericalangle\left(\uparrow_{x^j}^{A\left(z_i^j\right)},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{A\left(z_i^j\right)},\ v^j\right). \end{aligned}$$

After passing to subsequences, we have $v^j \to v$, $x^j \to x$ and

$$\begin{aligned} x^j z_i^j &\to x p_i \\ x^j A\left(z_i^j\right) &\to x A\left(p_i\right), \end{aligned}$$

for some choice of segments $x p_i$ and $x A\left(p_i\right)$. Using Lemma 33 and Corollary 36 we conclude

$$\begin{aligned} \left| \sphericalangle\left(\uparrow_{x^j}^{z_i^j}, v^j\right) - \sphericalangle\left(\uparrow_x^{p_i}, v\right) \right| &< \tau\left(\delta, \tau\left(\frac{1}{j} \middle| \mathrm{dist}\left(x, p_i\right)\right)\right), \\ \left| \sphericalangle\left(\uparrow_{x^j}^{A\left(z_i^j\right)}, v^j\right) - \sphericalangle\left(\uparrow_x^{A(p_i)}, v\right) \right| &< \tau\left(\delta, \tau\left(\frac{1}{j} \middle| \mathrm{dist}\left(x, A\left(p_i\right)\right)\right)\right). \end{aligned} \tag{4.6}$$

If $x \notin \mathcal{S}$, then the segments $x p_i$ and $x A\left(p_i\right)$ are unambiguously defined, and so the previous inequality and the hypothesis $\left| D_{v^j} f_{z_i^j} \right| < \frac{1}{j}$, contradict the previous lemma and its addendum.

If $x \in \mathcal{S}$ and $v \in T_x\mathcal{S}$, then

$$\sphericalangle\left(\uparrow_x^{p_i}, v\right) \text{ and } \sphericalangle\left(\uparrow_x^{A(p_i)}, v\right)$$

are independent of the choice of the segments xp_i and $xA\left(p_i\right)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ together with the Inequalities 4.6 contradict the previous lemma and its addendum. Thus our result holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$.

For a unit vector $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we saw in the proof of the previous lemma that for some $\lambda > 0$

$$|D_v f_0| > \lambda. \tag{4.7}$$

For $x \in \mathcal{S}$, we have unique segments xp_0 and $xA\left(p_0\right)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ and inequalities 4.6 contradict Inequality 4.7. ■

Combining the proof of the previous lemma with Addendum 4.3.1, we get

Addendum 4.3.2 *Let p_k be any of $p_1, \ldots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B\left(p_k, \varepsilon\right) \cup B\left(A\left(p_k\right), \varepsilon\right)$ and all $v \in T_x\mathbb{D}_k^n(r)$, the index $j\left(v\right)$ in the previous lemma can be chosen to be different from k.*

Lemma 49 *There is a $\lambda > 0$ so that for all $v \in T\tilde{M}^\alpha$ there is a $j\left(v\right) \in \{0, 1, \ldots, n\}$ so that*

$$D_v f_{j(v),d}^{\alpha} > \lambda \left|v\right|,$$

provided α is sufficiently large and d is sufficiently small.

Proof. If the lemma were false, then there would be a sequence of unit vectors $\{v^\alpha\}_{\alpha=1}^\infty$ with $v^\alpha \in T_{x^\alpha}\tilde{M}^\alpha$ such that for all i,

$$\left|D_{v^\alpha} f_{i,d}^\alpha\right| < \tau\left(\frac{1}{\alpha}, d\right).$$

Let $\lim_{\alpha\to\infty} x^\alpha = x \in \mathbb{D}_k^n(r)$. By Corollary 36, for any $\mu > 0$ there is a sequence $\{w^\alpha\}_{\alpha=1}^\infty$ with $w^\alpha \in \Sigma_{x^\alpha}^\mu$ such that

$$\sphericalangle\left(v^\alpha, w^\alpha\right) < \tau\left(\delta, \mu\right).$$

Since $\left|\nabla f_{i,d}^\alpha\right| \leq 2$,

$$\left|D_{w^\alpha} f_{i,d}^\alpha\right| < \tau\left(\delta, \mu, \frac{1}{\alpha}, d\right) \tag{4.8}$$

for all i. After passing to a subsequence, we conclude that $\left\{\gamma_{w^\alpha}|_{[0,\mu]}\right\}_{\alpha=1}^\infty$ converges to a segment $\gamma_w|_{[0,\mu]}$. By the previous lemma, there is a $\lambda > 0$ and a $j\left(w\right)$ so that for all $z \in B(p_{j(w)}, d)$,

$$\left|D_w f_z\right| > \lambda\left|w\right|, \tag{4.9}$$

provided d is small enough. Moreover, by Addendum 4.3.2 we may assume that

$$\begin{aligned} \operatorname{dist}\left(x, p_{j(w)}\right) &> 100d > \mu \text{ and} \\ \operatorname{dist}\left(x, A\left(p_{j(w)}\right)\right) &> 100d > \mu. \end{aligned} \tag{4.10}$$

By the Mean Value Theorem, there is a $z_{j(w)}^\alpha \in B\left(p_{j(w)}^\alpha, d\right)$ with

$$D_{w^\alpha} f_{z_{j(w)}^\alpha}^\alpha = D_{w^\alpha} f_{j(w),d}^\alpha. \tag{4.11}$$

Choose segments $x^\alpha z_{j(w)}^\alpha$ and $x^\alpha A(z_{j(w)}^\alpha)$ in $\tilde{M}^\alpha$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow_{x^\alpha}^{z_{j(w)}^\alpha}, w^\alpha\right) &= \sphericalangle\left(\Uparrow_{x^\alpha}^{z_{j(w)}^\alpha}, w^\alpha\right) \text{ and} \\ \sphericalangle\left(\uparrow_{x^\alpha}^{A(z_{j(w)}^\alpha)}, w^\alpha\right) &= \sphericalangle\left(\Uparrow_{x^\alpha}^{A(z_{j(w)}^\alpha)}, w^\alpha\right). \end{aligned}$$

After passing to a subsequence, we may assume that for some $z_{j(w)} \in B(p_{j(w)}, d)$, $x^{\alpha} z^{\alpha}_{j(w)}$ and $x^{\alpha} A(z^{\alpha}_{j(w)})$ converge to segments $x z_{j(w)}$ and $x A(z_{j(w)})$, respectively. By Lemma 33,

$$\begin{aligned}
\left| \sphericalangle(\Uparrow_{x^{\alpha}}^{z^{\alpha}_{j(w)}}, \gamma'_{w^{\alpha}}(0)) - \sphericalangle(\Uparrow_{x}^{z_{j(w)}}, \gamma'_{w}(0)) \right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, z_{j(w)}\right)\right)\right) \\
\left| \sphericalangle(\Uparrow_{x^{\alpha}}^{A(z^{\alpha}_{j(w)})}, \gamma'_{w^{\alpha}}(0)) - \sphericalangle(\Uparrow_{x}^{A\left(z_{j(w)}\right)}, \gamma'_{w}(0)) \right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, A\left(z_{j(w)}\right)\right)\right)\right).
\end{aligned}$$

Combining the previous two sets of displays with 4.10

$$\left| D_{w^{\alpha}} f^{\alpha}_{z^{\alpha}_{j(w)}} - D_{w} f_{z_{j(w)}} \right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right). \tag{4.12}$$

So by Equation 4.11,

$$\left| D_{w^{\alpha}} f^{\alpha}_{j(w),d} - D_{w} f_{z_{j(w)}} \right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right),$$

but this contradicts Inequalities 4.8 and 4.9. ■

The first claim of Key Lemma 46 follows by combining the previous lemma with the fact that the differentials of the $\varphi^{\eta}_{x^{\alpha}}$'s are almost isometries.

Remark 50 *Note that when x^{α} is close to p_k or $A\left(p_k\right)$, the desired estimate*

$$\left| d(P_{\hat{\imath}_{x^{\alpha}}} \circ F)_{\varphi^{\eta}_{x^{\alpha}}(x^{\alpha})}(v) \right| > \lambda \left| v \right|$$

holds with $P_{\hat{\imath}_{x^{\alpha}}} = P_{\hat{k}}$. This follows from Addendum 4.3.2 and the proof of the previous lemma.

Equicontinuity of Differentials

In this subsection, we establish the second part of the key lemma. If x^{α} is not close to one of the $p_k s$ or $A\left(p_k\right) s$ we will show the stronger estimate

$$\left| d\left(F\right)_{\varphi^{\eta}_{x^{\alpha}}(y)}(v) - d\left(F\right)_{\varphi^{\eta}_{x^{\alpha}}(x^{\alpha})}(v) \right| < \frac{\lambda}{2} \left| v \right|. \tag{4.13}$$

So at such points, the second part of the key lemma holds with *any* choice of coordinate projection $P_{\hat{i}_{x^\alpha}}$.

For x^α close to p_k or $A\left(p_k\right)$, we will show

$$\left|d\left(P_{\hat{k}} \circ F\right)_{\varphi^\eta_{x^\alpha}(y)}(v) - d\left(P_{\hat{k}} \circ F\right)_{\varphi^\eta_{x^\alpha}(x^\alpha)}(v)\right| < \frac{\lambda}{2}\left|v\right|, \tag{4.14}$$

where λ is the constant whose existence was established in the previous section. Together with remark 50, this will establish the key lemma.

Suppose $B \subset \mathbb{D}^n_k(r)$ is (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Let $x, y \in B$ and let

$$\varphi^\eta : B \longrightarrow \mathbb{R}^n$$

be the map defined in 4.1 and [44]. Set

$$P_{x,y} := \left(d\varphi^\eta\right)_y^{-1} \circ \left(d\varphi^\eta\right)_x : T_x\mathbb{D}^n_k(r) \to T_y\mathbb{D}^n_k(r).$$

It follows that $P_{x,y}$ is a $\tau(\delta, \eta)$–isometry.

Lemma 51 *Let $B \subset \mathbb{D}^n_k(r)$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Given $\varepsilon > 0$ and $x \in B$, there is a $\rho\left(x, \varepsilon\right) > 0$ so that the following holds.*

For all $k \in \{0, 1, \ldots, n\}$, there is a subset $E_{k,x} \subset \{B\left(p_k, d\right) \cup B\left(A\left(p_k\right), d\right)\}$ with measure $\mu\left(E_{k,x}\right) < \varepsilon$ so that for all $z \in B\left(p_k, d\right) \setminus E_{k,x}$, all $y \in B\left(x, \rho\left(x, \varepsilon\right)\right)$, and all $v \in \Sigma_x$,

$$\begin{aligned} \left|\sphericalangle\left(v, \uparrow_x^z\right) - \sphericalangle\left(P_{x,y}\left(v\right), \uparrow_y^z\right)\right| &< \tau\left(\varepsilon, \delta, \eta | \operatorname{dist}\left(x, z\right)\right) \text{ and} \\ \left|\sphericalangle\left(v, \uparrow_x^{A(z)}\right) - \sphericalangle\left(P_{x,y}\left(v\right), \uparrow_y^{A(z)}\right)\right| &< \tau\left(\varepsilon, \delta, \eta | \operatorname{dist}\left(x, A(z)\right)\right). \end{aligned}$$

Proof. Let $C_x = \{z | z \in \text{Cutlocus}\,(x) \ \text{ or } A\,(z) \in \text{Cutlocus}\,(x)\}$ and set

$$E_{k,x} = B\left(C_x, \nu\right) \cap \left\{B\left(p_k, d\right) \cup B\left(A\left(p_k\right), d\right)\right\}.$$

Choose $\nu > 0$ so that $\mu\left(E_{k,x}\right) < \varepsilon$.

By Proposition 37, for each $z \in B\left(p_k, d\right) \setminus E_{k,x}$, there is a $\rho\left(x, z, \varepsilon\right)$ so that for all $y \in B\left(x, \rho\left(x, z, \varepsilon\right)\right)$ and any choice of segment zy,

$$\text{dist}\left(zx, zy\right) < \varepsilon,$$

where zx is the unique segment from z to x.

Making $\rho\left(x, z, \varepsilon\right)$ smaller and using Corollary 32, it follows that for any $\tilde{a}_i, \bar{a}_i \in B\left(a_i, \eta\right)$,

$$\begin{aligned} \left|\sphericalangle\left(\Uparrow_x^{\tilde{a}_i}, \uparrow_x^z\right) - \sphericalangle\left(\Uparrow_y^{\bar{a}_i}, \uparrow_y^z\right)\right| &< \tau\left(\delta, \varepsilon, \eta | \text{dist}\left(x, z\right), \text{dist}\left(y, z\right)\right) \\ &= \tau\left(\delta, \varepsilon, \eta | \text{dist}\left(x, z\right)\right). \end{aligned}$$

It follows that

$$\left|\left(d\varphi^{\eta}\right)_x\left(\uparrow_x^z\right) - \left(d\varphi^{\eta}\right)_y\left(\uparrow_y^z\right)\right| < \tau\left(\delta, \varepsilon, \eta | \text{dist}\left(x, z\right)\right),$$

and hence

$$\sphericalangle\left(P_{x,y}\left(\uparrow_x^z\right), \uparrow_y^z\right) = \sphericalangle\left(\left(d\varphi^{\eta}\right)_y^{-1} \circ \left(d\varphi^{\eta}\right)_x\left(\uparrow_x^z\right), \left(\uparrow_y^z\right)\right) < \tau\left(\delta, \varepsilon, \eta | \text{dist}\left(x, z\right)\right).$$

So for any $v \in \Sigma_x$,

$$\begin{aligned} \left|\sphericalangle\left(v, \uparrow_x^z\right) - \sphericalangle\left(P_{x,y}\left(v\right), \uparrow_y^z\right)\right| &\leq \left|\sphericalangle\left(v, \uparrow_x^z\right) - \sphericalangle\left(P_{x,y}\left(v\right), P_{x,y}\left(\uparrow_x^z\right)\right)\right| + \\ &\quad \left|\sphericalangle\left(P_{x,y}\left(v\right), P_{x,y}\left(\uparrow_x^z\right)\right) - \sphericalangle\left(P_{x,y}\left(v\right), \uparrow_y^z\right)\right| \\ &< \tau\left(\delta, \eta\right) + \tau\left(\varepsilon, \delta, \eta | \text{dist}\left(x, z\right)\right) \\ &= \tau\left(\varepsilon, \delta, \eta | \text{dist}\left(x, z\right)\right). \end{aligned}$$

Using Proposition 37 and the precompactness of $B\left(p_k,d\right)\backslash E_{k,x}$, we can then choose $\rho\left(x,z,\varepsilon\right)$ to be independent of $z\in B\left(p_k,d\right)\setminus E_{k,x}$. A similar argument gives the second inequality. ■

Corollary 52 *Given any $\varepsilon>0$, there is a $\rho(\varepsilon)>0$ so that for any $x\in\mathbb{D}_k^n(r)$, $y\in B(x,\rho(\varepsilon))$, and $z\in B(p_i,d)\setminus E_{i,x}$, we have*

$$\left|D_v f_z - D_{P_{x,y}(v)} f_z\right| < \tau\left(\varepsilon,\delta,\eta|\operatorname{dist}\left(z,x\right),\operatorname{dist}\left(A\left(z\right),x\right)\right)$$

for all unit vectors $v\in\Sigma_x$.

Proof. Since $\mathbb{D}_k^n(r)$ is compact, the $\rho(\varepsilon,x)$ from the previous lemma can be chosen to be independent of x.

Given $x\in\mathbb{D}_k^n(r)$, $y\in B(x,\rho(\varepsilon))$, and $v\in\Sigma_x$, choose segments yz and $yA(z)$ so that

$$\begin{aligned}\sphericalangle\left(\uparrow_y^z,P_{x,y}\left(v\right)\right) &= \sphericalangle\left(\Uparrow_y^z,P_{x,y}\left(v\right)\right) \text{ and}\\ \sphericalangle\left(\uparrow_y^{A(z)},P_{x,y}\left(v\right)\right) &= \sphericalangle\left(\Uparrow_y^{A(z)},P_{x,y}\left(v\right)\right).\end{aligned}$$

Since the segments xz and $xA(z)$ are unique, the result follows from the formula for directional derivatives of distance functions, the previous lemma, and the chain rule. ■

We can lift a strainer from $\mathbb{D}_k^n(r)$ to any $\tilde{M}^\alpha$ if $\operatorname{dist}_{GH}\left(\tilde{M}^\alpha,\mathbb{D}_k^n(r)\right)$ is sufficiently small. So if x^α and y^α are sufficiently close, we define

$$P_{x^\alpha,y^\alpha}:=\left(d\varphi^\eta\right)_{y^\alpha}^{-1}\circ\left(d\varphi^\eta\right)_{x^\alpha}:T_{x^\alpha}\tilde{M}^\alpha\to T_{y^\alpha}\tilde{M}^\alpha.$$

Lemma 53 *Let i be in $\{0,\ldots,n\}$. There is a $\rho>0$ so that for any $x^{\alpha}\in\tilde{M}^{\alpha}$, any $y^{\alpha}\in B(x^{\alpha},\rho)$, and any unit $v^{\alpha}\in T_{x^{\alpha}}\tilde{M}^{\alpha}$ we have*

$$\left|D_{v^{\alpha}}f_{i,d}^{\alpha}-D_{P_{x^{\alpha},y^{a}}(v^{\alpha})}f_{i,d}^{\alpha}\right|<\tau\left(\rho,\frac{1}{\alpha},\delta,\eta|\operatorname{dist}\left(x^{\alpha},p_{i}^{\alpha}\right),\operatorname{dist}\left(x^{\alpha},A\left(p_{i}^{\alpha}\right)\right)\right),$$

provided d is sufficiently small.

Proof. If not, then for any $\rho>0$ and some $i=0,1,\ldots,n$, there would be a sequence of points $x^{\alpha}\to x\in\mathbb{D}_{k}^{n}(r)$, a sequence of unit vectors $\{v^{\alpha}\}_{\alpha=1}^{\infty}$ and a constant $C>0$ that is independent of α, δ, and η so that

$$\begin{aligned}\left|D_{v^{\alpha}}f_{i,d}^{\alpha}-D_{P_{x^{\alpha},y^{a}}(v^{\alpha})}f_{i,d}^{\alpha}\right| &\geq C,\\ \operatorname{dist}\left(x,p_{i}\right) &\geq C,\text{ and}\\ \operatorname{dist}\left(x,A\left(p_{i}\right)\right) &\geq C \qquad (4.15)\end{aligned}$$

for some $y^{\alpha}\in B(x^{\alpha},\rho)$. Choose $\varepsilon>0$ and take $\rho<\rho(\varepsilon)$ where $\rho(\varepsilon)$ is from the previous corollary. We assume $B\left(x,\rho(\varepsilon)\right)$ is (n,δ,r)–strained. Let $y=\lim y^{\alpha}$ and $\mu>0$ be sufficiently small. By corollary 36, there are sequences $\{w^{\alpha}\}_{\alpha=1}^{\infty}\in\Sigma_{x^{\alpha}}^{\mu}$ and $\{\tilde{w}^{\alpha}\}_{\alpha=1}^{\infty}\in\Sigma_{y^{\alpha}}^{\mu}$ so that

$$\begin{aligned}\sphericalangle\left(v^{\alpha},w^{\alpha}\right) &< \tau\left(\delta,\mu\right)\\ \sphericalangle\left(P_{x^{\alpha},y^{a}}\left(w^{\alpha}\right),\tilde{w}^{\alpha}\right) &< \tau\left(\delta,\mu\right) \qquad (4.16)\end{aligned}$$

and subsequences $\{\gamma_{w^{\alpha}}\}_{\alpha=1}^{\infty}$ and $\{\gamma_{\tilde{w}^{\alpha}}\}_{\alpha=1}^{\infty}$ converging to segments γ_{w} and $\gamma_{\tilde{w}}$ that are parameterized on $[0,\mu]$. Since $|\nabla f_{i,d}^{\alpha}|\leq 2$, we may assume for a possibly smaller constant C that

$$\left|D_{w^{\alpha}}f_{i,d}^{\alpha}-D_{\tilde{w}^{\alpha}}f_{i,d}^{\alpha}\right|\geq C.$$

Thus for some $z^{\alpha} \in B(p_i^{\alpha}, d)$ with $\operatorname{dist}_{\text{Haus}}(z^{\alpha}, E_{i,x}) > 2\nu$,

$$|D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| \geq \frac{C}{2}. \tag{4.17}$$

Passing to a subsequence, we have $z^{\alpha} \to z \in B(p_i, d) \setminus E_{i,x}$. As in the proof of Lemma 49 (Inequality 4.12), we have

$$\begin{aligned} |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_w f_z| &< \tau(\delta, \tau(1/\alpha|\mu)) \text{ and} \\ |D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}} f_z| &< \tau(\delta, \tau(1/\alpha|\mu)). \end{aligned}$$

Thus,

$$\begin{aligned} |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| &\leq |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_w f_z| + |D_w f_z - D_{\tilde{w}} f_z| + |D_{\tilde{w}} f_z - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| \\ &< |D_w f_z - D_{\tilde{w}} f_z| + \tau(\delta, \tau(1/\alpha|\mu)) \\ &\leq \left|D_w f_z - D_{P_{x,y}(w)} f_z\right| + \left|D_{P_{x,y}(w)} f_z - D_{\tilde{w}} f_z\right| + \tau(\delta, \tau(1/\alpha|\mu)) \\ &\leq \tau(\varepsilon, \delta, \mu, \eta, \tau(1/\alpha|\mu)) \end{aligned}$$

by the previous corollary and Inequalities 4.15 and 4.16. Choosing $\varepsilon, \delta, \eta$, μ, and $1/\alpha$ small enough, we have a contradiction to 4.17. ■

The previous lemma, together with the definitions of Φ_d^{α}, $(\varphi^{\eta})^{-1}$ and $P_{x^{\alpha}, y^{a}}$ establishes the estimates 4.13 and 4.14 and hence the second part of Key Lemma, completing the proof of Theorem 11, except in dimension 4.

4.4 Recognizing $\mathbb{R}P^4$

To prove Theorem 11 in dimension 4, we exploit the following corollary of the fact that $\operatorname{Diff}_+(S^3)$ is connected [7].

Corollary 54 *Let M be a smooth 4–manifold obtained by smoothly gluing a 4–disk to the boundary of the nontrivial 1–disk bundle over $\mathbb{R}P^3$. Then M is diffeomorphic to $\mathbb{R}P^4$.*

To see that our M^αs have this structure, we use standard triangle comparison and argue as we did in the part of Section 4.3 titled "Lower Bound on Differential" to conclude

Proposition 55 *For any fixed $\rho_0 > 0$, $f^\alpha_{0,d}$ does not have critical points on $M^\alpha \backslash \{B(p^\alpha_0, \rho_0) \cup B(A(p^\alpha_0), \rho_0)\}$, and $\nabla f^\alpha_{0,d}$ is gradient-like for* $\mathrm{dist}(A(p^\alpha_0), \cdot)$ *and* $-\mathrm{dist}(p^\alpha_0, \cdot)$, *provided α is sufficiently large and d is sufficiently small.*

Finally, using Swiss Cheese Volume Comparison (see 1.1 in [25]) we will show

Proposition 56 *There is a $\rho_0 > 0$ so that* $\mathrm{dist}(p^\alpha_0, \cdot)$ *does not have critical points in $B(p^\alpha_0, \rho_0)$, provided α is sufficiently large.*

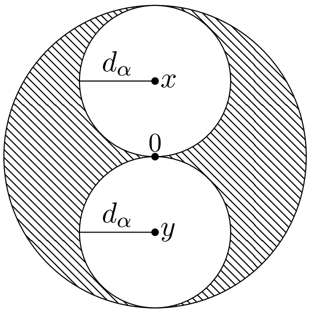

Figure 4.1: The model $\mathcal{D}^n_k(2d_\alpha)$.

Proof. Since $\mathrm{vol}\, M^\alpha \to \mathrm{vol}\, \mathcal{D}^n_k(r)$, $\mathrm{vol}\, B(p^\alpha_0, r) \to \mathrm{vol}\, \mathcal{D}^n_k(r)$. Via Swiss Cheese Volume Comparison (see 1.1 in [25]) we shall see that the presence of a critical point close to p^α_0 contradicts $\mathrm{vol}\, B(p^\alpha_0, r) \to \mathrm{vol}\, \mathcal{D}^n_k(r)$. Suppose q_α is critical for $\mathrm{dist}(p^\alpha_0, \cdot)$, and $\mathrm{dist}(p^\alpha_0, q_\alpha) = d_\alpha \to 0$. Let x, y be points in $\partial \mathcal{D}^n_k(d_\alpha)$ at maximal distance. By Swiss Cheese Comparison and 1.4 in [25],

$$\begin{aligned}\mathrm{vol}\, (B(q_\alpha, 2d_\alpha) \setminus B(p^\alpha_0, d_\alpha)) &\leq \mathrm{vol}\, (\mathcal{D}^n_k(2d_\alpha) \setminus \{B(x, d_\alpha) \cup B(y, d_\alpha)\}) \\ &= \mathrm{vol}\, (\mathcal{D}^n_k(2d_\alpha)) - 2\mathrm{vol}\, (\mathcal{D}^n_k(d_\alpha)).\end{aligned}$$

Since

$$\operatorname{vol} B\left(p_0^\alpha, d_\alpha\right) \leq \operatorname{vol}\, \mathcal{D}_k^n\left(d_\alpha\right),$$

we conclude

$$\begin{aligned}
\operatorname{vol}\left(B\left(q_\alpha, 2d_\alpha\right)\right) &\leq \operatorname{vol}\, \left(\mathcal{D}_k^n\left(2d_\alpha\right)\right) - \operatorname{vol}\, \left(\mathcal{D}_k^n\left(d_\alpha\right)\right) \\
&< \kappa \cdot \operatorname{vol}\, \mathcal{D}_k^n\left(2d_\alpha\right)
\end{aligned}$$

for some $\kappa \in (0,1)$. By relative volume comparison for $\rho \geq 2d_\alpha$,

$$\kappa > \frac{\operatorname{vol} B\left(q_\alpha, 2d_\alpha\right)}{\operatorname{vol}\, \mathcal{D}_k^n\left(2d_\alpha\right)} \geq \frac{\operatorname{vol} B\left(q_\alpha, \rho\right)}{\operatorname{vol}\, \mathcal{D}_k^n\left(\rho\right)}$$

or

$$\kappa \cdot \operatorname{vol}\, \mathcal{D}_k^n\left(\rho\right) > \operatorname{vol} B\left(q_\alpha, \rho\right).$$

Since

$$\begin{aligned}
B\left(p_0^\alpha, r\right) &\subset B\left(q_\alpha, r + d_\alpha\right), \\
\operatorname{vol} B\left(p_0^\alpha, r\right) &< \kappa \cdot \operatorname{vol}\, \mathcal{D}_k^n\left(r + d_\alpha\right).
\end{aligned}$$

Letting $d_\alpha \to 0$, we conclude that

$$\operatorname{vol} B\left(p_0^\alpha, r\right) < \kappa \cdot \operatorname{vol}\, \mathcal{D}_k^n\left(r\right),$$

a contradiction. ■

An identical argument shows

Proposition 57 *There is a* $\rho_0 > 0$ *so that* $\operatorname{dist}\left(A\left(p_0^\alpha\right), \cdot\right)$ *does not have critical points in* $B\left(A\left(p_0^\alpha\right), \rho\right)$, *provided* α *is sufficiently large.*

Combining the previous three propositions, we see that $(f^{\alpha}_{0,d})^{-1}(0)$ is diffeomorphic to S^3. By Geometrization, $(f^{\alpha}_{0,d})^{-1}(0)\,/\,\{\mathrm{id}, A\}$ is diffeomorphic to $\mathbb{R}P^3$. If ρ_0 is as in Proposition 55, it follows that $(f^{\alpha}_{0,d})^{-1}([-\rho_0,\rho_0])/\,\{\mathrm{id}, A\}$ is the nontrivial 1–disk bundle over $\mathbb{R}P^3$. $\tilde{M}^{\alpha}\setminus(f^{\alpha}_{0,d})^{-1}([-\rho_0,\rho_0])$ consists of two smooth 4–disks that get interchanged by A. Thus M^{α} has the structure of Corollary 54 and is hence diffeomorphic to $\mathbb{R}P^4$.

Remark 58 *The proof of Perelman's Parameterized Stability Theorem [33] can substitute for Geometrization to allow us to conclude that* $f^{-1}(0)\,/\,\{\mathrm{id}, A\}$ *is homeomorphic and therefore diffeomorphic to* $\mathbb{R}P^3$*. The need to cite the proof rather than the theorem stems from the fact that the definition of admissible functions in [33] excludes* $f^{\alpha}_{0,d}$ *. It is straightforward (but tedious) to see that the proof goes through for an abstract class that includes* $f^{\alpha}_{0,d}$*.*

The fact that $\mathbb{R}P^4$ *admits exotic differential structures can be seen by combining [31] with either [6] or [13].*

4.5 Purse Stability

We let Γ^n denote the group of twisted n–spheres. Recall that there is a filtration

$$\{e\}\subset\Gamma^n_{n-1}\subset\cdots\subset\Gamma^n_1=\Gamma^n$$

by subgroups, which are called Gromoll groups [18]. Rather than using the definition of the Γ^n_qs from [18], we use the equivalent notion from Theorem D in [29].

Definition 59 *Let*

$$f: S^{q-1}\times S^{n-q}\longrightarrow S^{q-1}\times S^{n-q}$$

be a diffeomorphism that satisfies

$$p_{q-1} \circ f = p_{q-1},$$

where

$$p_{q-1} : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1}$$

is projection to the first factor. Then Γ_q^n *consists of those smooth manifolds that are diffeomorphic to*

$$D^q \times S^{n-q} \cup_f S^{q-1} \times D^{n-q+1}. \tag{4.18}$$

Theorem 60 *Let* $\{M^\alpha\}_{\alpha=1}^\infty$ *be a sequence of closed, Riemannian n–manifolds with*

$$\sec M^\alpha \geq k$$

so that

$$M_\alpha \longrightarrow P_{k,r}^n$$

in the Gromov-Hausdorff topology. Then for α *sufficiently large,* $M_\alpha \in \Gamma_{n-1}^n$.

Notice that a diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ so that $p_{n-2} \circ f = p_{n-2}$ gives rise to an element of $\pi_{n-2}\left(\mathrm{Diff}_+\left(S^1\right)\right)$. If two such diffeomorphisms give the same homotopy class, then the construction 4.18 yields diffeomorphic manifolds (cf [29]). Since the group of orientation preserving diffeomorphisms of the circle deformation retracts to $SO\left(2\right)$, it follows that for $n \geq 4$, $\Gamma_{n-1}^n = \{e\}$. Since $\Gamma^n = \{e\}$ for $n = 1, 2, 3$, we have $\Gamma_{n-1}^n = \{e\}$ for all n. Thus all but finitely many of the M^αs in Theorem 60 are diffeomorphic to S^n, and to prove Theorem 12 it suffices to prove Theorem 60.

The Model Submersion

Recall that we view $\mathcal{D}^n_k(r)$ as a metric r-ball centered at $p_0 = e_0$ in either $H^n_+ \subset \mathbb{R}^{1,n}$, $\{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$, or $S^n \subset \mathbb{R}^{n+1}$, and we defined

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases}$$

We let the totally geodesic hyperplane $H \subset \mathcal{D}^n_k(r)$ that defines $P^n_{k,r}$ be the one containing $p_0, p_1, \ldots, p_{n-1}$. We denote the singular subset of $P^n_{k,r}$ by $\mathcal{S}$, that is, $\mathcal{S}$ is the copy of S^{n-2} which is the boundary of the $(n-1)$–disk $\mathcal{D}^n_k(r) \cap H$. Thus $\{p_i\}_{i=1}^{n-1} \subset \mathcal{S}$.

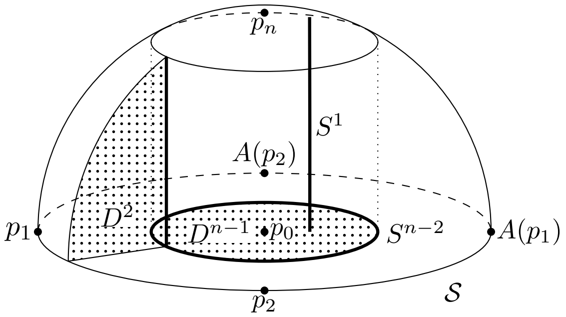

Figure 4.2: One side of $P^n_{k,r}$ for $n = 3$ and $k = 0$.

As the antipodal map $A : \mathcal{D}^n_k(r) \longrightarrow \mathcal{D}^n_k(r)$ commutes with the reflection R in H, it induces a well-defined involution of $P^n_{k,r}$, which we also call A. Note that $A : P^n_{k,r} \longrightarrow P^n_{k,r}$ restricts to the antipodal map of $\mathcal{S}$ and fixes the circle at maximal distance from $\mathcal{S}$.

For $i = 1, \ldots, n-1$, we view $\mathcal{S} \subset \mathcal{D}^n_k(r)$ and define f_i as in 4.2

$$f_i(x) := h_k \circ \operatorname{dist}(A(p_i), x) - h_k \circ \operatorname{dist}(p_i, x).$$

We let $\Psi : P^n_{k,r} \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi = (f_1, f_2, \ldots, f_{n-1}) .$$

Lifting The Model Submersion

Let $\{M^\alpha\}_{\alpha=1}^{\infty}$ be a sequence of closed, Riemannian n–manifolds with

$$\sec M^\alpha \geq k$$

so that

$$M_\alpha \longrightarrow P^n_{k,r}.$$

In contrast to the situation for the Crosscap, the isometry $A : P^n_{k,r} \longrightarrow P^n_{k,r}$ need not lift to an isometry of M^α. We nevertheless let $A : M^\alpha \longrightarrow M^\alpha$ denote any map that is Gromov-Hausdorff close to $A : P^n_{k,r} \longrightarrow P^n_{k,r}$.

As before, we define $f^\alpha_{i,d} : M^\alpha \longrightarrow \mathbb{R}$ by

$$f^\alpha_{i,d}(x) = \int_{z \in B\left(A\left(p^\alpha_i\right),d\right)} h_k \circ \operatorname{dist}(z, x) - \int_{z \in B\left(p^\alpha_i,d\right)} h_k \circ \operatorname{dist}(z, x) . \tag{4.19}$$

We let $\Psi^\alpha_d : M^\alpha \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi^\alpha_d = (f^\alpha_{1,d}, \ldots, f^\alpha_{n-1,d}).$$

The Handles

We identify $\mathbb{R}^{n-1}$ with

$$\mathbb{R}^{n-1} \equiv \operatorname{span}\{e_1, \ldots, e_{n-1}\} \subset \begin{cases} \mathbb{R}^{1,n} & \text{if } k = -1 \\ \mathbb{R}^{n+1} & \text{if } k = 0 \\ \mathbb{R}^{n+1} & \text{if } k = 1. \end{cases}$$

For small $\varepsilon > 0$, we set

$$\begin{aligned} E_0(\varepsilon) &:= (\Psi)^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_0^{\alpha}(\varepsilon) &:= (\Psi_d^{\alpha})^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_1(\varepsilon) &:= (\Psi)^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \text{ and} \\ E_1^{\alpha}(\varepsilon) &:= (\Psi_d^{\alpha})^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \end{aligned}$$

where $\overline{A^{n-1}(0, r-\varepsilon, 2r)}$ is the closed annulus in $\mathbb{R}^{n-1}$ centered at 0 with inner radius $r-\varepsilon$ and outer radius $2r$, and $D^{n-1}(0, r-\varepsilon)$ is the closed ball in $\mathbb{R}^{n-1}$ centered at 0 with radius $r-\varepsilon$.

Theorem 60 is a consequence of the next two lemmas.

Key Lemma 61 *For any sufficiently small* $\varepsilon > 0$,

$$\Psi_d^{\alpha} : E_0^{\alpha}(\varepsilon) \longrightarrow D^{n-1}(0, r-\varepsilon)$$

is a trivial S^1–bundle, provided α is sufficiently large and d is sufficiently small.

Let $\mathrm{pr} : \overline{A^{n-1}(0, r-\varepsilon, 2r)} \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$ be radial projection and set

$$\begin{aligned} g &:= \mathrm{pr} \circ \Psi : E_1(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) \\ g_d^{\alpha} &:= \mathrm{pr} \circ \Psi_d^{\alpha} : E_1^{\alpha}(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right). \end{aligned}$$

Key Lemma 62 *There is an $\varepsilon > 0$ so that*

$$g_d^{\alpha} : E_1^{\alpha}(\varepsilon) \longrightarrow \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$

is a trivial D^2*–bundle over* $\partial\left(D^{n-1}(0,r-\varepsilon)\right)=S^{n-2}$*, provided* α *is sufficiently large and* d *is sufficiently small.*

Since every space of directions of $P^n_{k,r}$ contains an isometrically embedded, totally geodesic copy of S^{n-3}, and every space of directions of $P^n_{k,r}\setminus\mathcal{S}$ contains an isometrically embedded, totally geodesic copy of S^{n-1}, we get the following. (Cf Proposition 27.)

Proposition 63 *There are* $r,\delta>0$ *so that every point in the purse* $P^n_{k,r}$ *has a neighborhood* B *that is* $(n-2,\delta,r)$*–strained.*

For any neighborhood U *of* $\mathcal{S}$*, there are* $r,\delta>0$ *so that every point in* $P^n_{k,r}\setminus U$ *has a neighborhood* B *that is* (n,δ,r)*–strained.*

Remark 64 *For* $x\in\mathcal{S}$*, the strainer* $\{(a_i,b_i)\}_{i=1}^{n-2}$ *can be chosen to lie in* $\mathcal{S}$*.*

Because the $f_i:P^n_{k,r}\longrightarrow\mathbb{R}$ are coordinate functions, $\Psi|_{\mathcal{D}^n_k(r)\cap H}$ differs from the identity by a translation. Using this and ideas from Section 4.3, we will be able to prove

Proposition 65 *There is a neighborhood* U *of* $\mathcal{S}\subset P^n_{k,r}$ *so that for any family of open sets* $U^\alpha\subset M^\alpha$ *with* $U^\alpha\to U$*,* $g^\alpha_d|_{U^\alpha}$ *is a submersion, provided* α *is sufficiently large and* d *is sufficiently small.*

We will show that our key lemmas hold for any $\varepsilon>0$ so that

$$\Psi^{-1}\left(\overline{A^{n-1}(0,r-\varepsilon,r)}\right)\subset U.$$

Since $\{f_i\}_{i=1}^{n-1}$ are the $(n-1)$–coordinate functions for the standard embedding of $\mathcal{S}=S^{n-2}\subset\mathbb{R}^{n-1}$, we have

Lemma 66 *There is a $\lambda > 0$ so that for all $v \in T\mathcal{S}$, there is an j so that the j^{th}–component function of g satisfies*

$$\left|D_v\left(g_j\right)\right| > \lambda\left|v\right|.$$

As in Section 4.3, we have

Addendum 4.5.1 *Let p_k be any of $p_1, \ldots p_{n-1}$. There is an $\varepsilon > 0$ so that for all $x \in B\left(p_k, \varepsilon\right) \cup B\left(A\left(p_k\right), \varepsilon\right)$ and all $v \in T_x\mathcal{S}$, the index j in the previous lemma can be chosen to be different from k.*

To lift Lemma 66 to the M^αs, we need an analog of $T\mathcal{S}$ within each M^α, or better a notion of g_d^α–almost horizontal for each $U^\alpha \subset M^\alpha$. To achieve this, cover $\mathcal{S}$ by a finite number of $(n-2, \delta, r)$–strained neighborhoods $B \subset P_{k,r}^n$ with strainers $\{(a_i, b_i)\}_{i=1}^{n-2} \subset \mathcal{S}$. Let U be the union of this finite collection, and let $U^\alpha \subset M^\alpha$ converge to U.

Given $x^\alpha \in U^\alpha$, we now define a g_d^α–almost horizontal space at x^α as follows. Let B^α be a $(n-2, \delta, r)$–strained neighborhood for x^α with strainers $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{n-2}$ that converge

$$\left(B^\alpha, \{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{n-2}\right) \longrightarrow \left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right),$$

where $\left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right)$ is part of our finite collection of $(n-2, \delta, r)$–strained neighborhoods for points in $\mathcal{S} \subset P_{k,r}^n$. We set

$$H_{x^\alpha}^{g_d^\alpha} := \operatorname{span}_{i \in \{1, \ldots, n-2\}}\left\{\uparrow_{x^\alpha}^{a_i^\alpha}\right\},$$

where $\uparrow_{x^\alpha}^{a_i^\alpha}$ is the direction of *any* segment from x^α back to a_i^α. Regardless of this choice, $H_{x^\alpha}^{g_d^\alpha}$ satisfies the following Lemma, from which Proposition 65 follows.

Lemma 67 *There is a* $\lambda > 0$ *so that for all* $x^\alpha \in U^\alpha$ *and all* $v \in H_{x^\alpha}^{g_d^\alpha}$, *there is an* j *so that the* j^{th}*–component function of* g_d^α *satisfies*

$$\left| D_v \left(\left(g_d^\alpha \right)_j \right) \right| > \lambda \left| v \right|,$$

provided U *and* d *are sufficiently small and* α *is sufficiently large. In particular,* $g_d^\alpha|_{U^\alpha}$ *is a submersion.*

Proof. Let $x_\alpha \to x$, and for all $j = 1, \ldots, n-1$, let $z_j^\alpha \to z_j \in B\left(p_j, d\right)$. If $x_\alpha z_j^\alpha$ converges to $x z_j$, then by Corollary 32,

$$\left| \sphericalangle \left(\uparrow_{x^\alpha}^{a_i^\alpha}, \uparrow_{x^\alpha}^{z_j^\alpha} \right) - \sphericalangle \left(\uparrow_x^{a_i}, \uparrow_x^{z_j} \right) \right| < \tau \left(\delta, 1/\alpha | \mathrm{dist}\left(x, z_j \right) \right).$$

Similarly for a sequence of segments $x_\alpha A\left(z_j^\alpha \right)$ converging to $x A\left(z_j \right)$, we have

$$\left| \sphericalangle \left(\uparrow_{x^\alpha}^{a_i^\alpha}, \uparrow_{x^\alpha}^{A\left(z_j^\alpha \right)} \right) - \sphericalangle \left(\uparrow_x^{a_i}, \uparrow_x^{A\left(z_j \right)} \right) \right| < \tau \left(\delta, 1/\alpha | \mathrm{dist}\left(x, A\left(z_j \right) \right) \right).$$

Arguing as in the proof of Lemma 49, we have for all i and j,

$$\left| D_{\uparrow_{x^\alpha}^{a_i^\alpha}} \left(g_d^\alpha \right)_j - D_{\uparrow_x^{a_i}} \left(g \right)_j \right| < \tau \left(\delta, d, 1/\alpha | \mathrm{dist}\left(x, p_j \right), \mathrm{dist}\left(x, A\left(p_j \right) \right) \right).$$

Since $v \in H_{x^\alpha}^{g_d^\alpha} = \mathrm{span}_{i \in \{1, \ldots, n-2\}} \left\{ \uparrow_{x^\alpha}^{a_i^\alpha} \right\}$, the lemma follows from the previous display together with Lemma 66, Addendum 4.5.1, and the hypothesis that U is sufficiently small.

■

Let $p_n \in \mathcal{D}_k^n\left(r \right)$ be as in 4.3, and let $Q : \mathcal{D}_k^n\left(r \right) \longrightarrow P_{k,r}^n$ be the quotient map. We abuse notation and call $Q\left(p_n \right)$, p_n. We define $f_n : P_{k,r}^n \to \mathbb{R}$ by

$$f_n(x) := h_k \circ \mathrm{dist}\left(\left(p_n \right), x \right) - h_k \circ \mathrm{dist}\left(p_0, x \right).$$

With a slight modification of the proof of Proposition 27, we get

Lemma 68 *There are $\delta, r > 0$ so that for all $x \in E_0(\varepsilon/2)$ there is an (n, δ, r)–strainer $\{(a_i, b_i)\}_{i=1}^n$ with*

$$\{(a_i, b_i)\}_{i=1}^{n-1} \subset f_n^{-1}(l)$$

for some $l \in \mathbb{R}$.

We cover $E_0(\varepsilon/2)$ by a finite number of such (n, δ, r)–strained sets and make

Definition 69 *For $x \in E_0(\varepsilon/2)$, set*

$$H_x^{\Psi} := \operatorname{span}_{i \in \{1,\ldots,n-1\}} \{\uparrow_x^{a_i}\},$$

where $\{(a_i, b_i)\}_{i=1}^{n-1}$ is as in the previous lemma.

Since $\Psi : E_0(\varepsilon/2) \longrightarrow D^{n-1}(r - \varepsilon/2)$ is simply orthogonal projection, we have

Lemma 70 *There is a $\lambda > 0$ so that for all $x \in E_0(\varepsilon/2)$ and all $v \in H_x^{\Psi}$, there is an i so that*

$$|D_v f_i| > \lambda |v|.$$

To lift this lemma to the M^{α}s, we need a notion of Ψ_d^{α}–almost horizontal for each M^{α}. Given $z^{\alpha} \in E_0^{\alpha}(\varepsilon/2)$, we define a Ψ_d^{α}–almost horizontal space at z^{α} as follows. Let B^{α} be a (n, δ, r)–strained neighborhood for z^{α} with strainers $\{(a_i^{\alpha}, b_i^{\alpha})\}_{i=1}^n$ that converge

$$\left(B^{\alpha}, \{(a_i^{\alpha}, b_i^{\alpha})\}_{i=1}^n\right) \longrightarrow \left(B, \{(a_i, b_i)\}_{i=1}^n\right),$$

where $\left(B, \{(a_i, b_i)\}_{i=1}^n\right)$ is part of our finite collection of (n, δ, r)–strained neighborhoods for points in $E_0(\varepsilon/2)$ that comes from Lemma 68. We set

$$H_{z^{\alpha}}^{\Psi_d^{\alpha}} := \operatorname{span}_{i \in \{1,\ldots,n-1\}} \left\{\uparrow_{z^{\alpha}}^{a_i^{\alpha}}\right\},$$

where $\uparrow_{z^\alpha}^{a_i^\alpha}$ is the direction of *any* segment from z^α back to a_i^α. Regardless of this choice, $H_{z^\alpha}^{\Psi_d^\alpha}$ satisfies the following Lemma, whose proof is nearly identical to the proof of Lemma 49.

Lemma 71 *There is a* $\lambda > 0$ *so that for all* $z^\alpha \in E_0^\alpha(\varepsilon/2)$ *and all* $v \in H_{z^\alpha}^{\Psi_d^\alpha}$*, there is an* $i \in \{1, \ldots, n-1\}$ *so that*

$$\left|D_v f_{i,d}^\alpha\right| > \lambda |v|,$$

provided α *is sufficiently large and* d *is sufficiently small. In particular,* $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ *is a submersion.*

Proposition 72 $E_1^\alpha(\varepsilon)$ *is homeomorphic to* $S^{n-2} \times D^2$*, and* $E_0^\alpha(\varepsilon)$ *is homeomorphic to* $D^{n-1} \times S^1$*, provided* α *is sufficiently large and* d *is sufficiently small.*

Proof. First we show that $E_0^\alpha(\varepsilon)$ is connected. By the Stability Theorem [33], we have homeomorphisms $h_\alpha : P_k^n(r) \longrightarrow M^\alpha$ that are also Gromov–Hausdorff approximations (cf [23], [25] and [45]). Thus for α sufficiently large, we have

$$E_0^\alpha(\varepsilon) \subset h_\alpha(E_0(\varepsilon/2)).$$

Let $\rho^\alpha : M^\alpha \longrightarrow \mathbb{R}$ be defined by

$$\rho^\alpha(x) := |\Psi_d^\alpha(x)|.$$

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ is a submersion, it follows that ρ^α does not have critical points on $E_0^\alpha(\varepsilon/2) \setminus E_0^\alpha(2\varepsilon)$. By construction, the flow lines of $\nabla\rho^\alpha$ are transverse to the boundary of $E_0^\alpha(\varepsilon)$ and hence can be used to move $h_\alpha(E_0(\varepsilon/2))$ onto $E_0^\alpha(\varepsilon)$. It follows that $E_0^\alpha(\varepsilon)$ is connected.

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, it is a fiber bundle with contractible base $D^{n-1}(0, r-\varepsilon)$. Since the fiber is 1–dimensional and the total space is connected, we conclude that $E_0^\alpha(\varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$.

We choose a homeomorphism $h_0 : E_0(\varepsilon/2) \longrightarrow E_0^\alpha(\varepsilon/2)$ so that

$$\begin{tikzcd}
E_0(\varepsilon/2) \arrow[rr, "h_0"] \arrow[dr, "\Psi_d"'] & & E_0^\alpha(\varepsilon/2) \arrow[dl, "\Psi_d^\alpha"] \\
& D^{n-1} &
\end{tikzcd}$$

commutes. Using the proof of the Gluing Theorem ([33], Theorem 4.6), we construct a homeomorphism $h : P_k^n(r) \longrightarrow M^\alpha$ so that

$$h = \begin{cases} h_0 & \text{on } E_0(\varepsilon) \\ h_\alpha & \text{on } E_1(\varepsilon/4). \end{cases}$$

It follows that $h(E_1(\varepsilon)) = E_1^\alpha(\varepsilon)$. Since $E_1(\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, the result follows. ■

Proof of Key Lemma 62. By Proposition 65, $g_d^\alpha : E_1^\alpha(\varepsilon) \longrightarrow \partial D^{n-1}(0, r-\varepsilon) = S^{n-2}$ is a submersion. Since g_d^α is proper, g_d^α is a fiber bundle with two-dimensional fiber F. From the long exact homotopy sequence and Proposition 72, we conclude that F is a 2–disk. For $n \neq 4$, every D^2–bundle over S^{n-2} is trivial by Theorem 1 of [36]. When $n = 4$, $E_1^\alpha(\varepsilon)$ is a D^2–bundle over S^2 whose total space is homeomorphic to $S^2 \times D^2$. It follows for example from [54] that $E_1^\alpha(\varepsilon)$ is trivial in all cases, completing the proof of Key Lemma 62. ■

Proof of Key Lemma 61. Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, $(E_0^\alpha(\varepsilon), \Psi_d^\alpha)$ is a fiber bundle over $D^{n-1}(0, r-\varepsilon)$ with one-dimensional fiber F. Since $E_0^\alpha(\varepsilon)$ is also homeomorphic to $D^{n-1} \times S^1$, it follows that the fiber is S^1. The base is contractible, so the bundle is trivial. This completes the proof of Key Lemma 61 and hence the proofs of Theorems 60 and 12, establishing our Main Theorem. $\blacksquare$

Double Disk Stability

The proof of Theorem 11 also yields

Corollary 73 *Let* $\{M_i\}_{i=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $\sec M_i \geq k$ *so that*

$$M_i \longrightarrow \mathbb{D}_k^n(r)$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_i*s are diffeomorphic to* S^n*.*

Proof. In contrast to Theorem 40, we do not necessarily have an isometric involution of the M_is. Instead, we let $A : M_i \longrightarrow M_i$ be any map which is Gromov-Hausdorff close to $A : \mathbb{D}_k^n(r) \longrightarrow \mathbb{D}_k^n(r)$. We then define $f_{i,d}^{\alpha} : M_i \longrightarrow \mathbb{R}$ as in 4.19 and proceed as in the proof of Theorem 11. $\blacksquare$

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