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Instantons and odd Khovanov homology

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Christopher William Scaduto

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Abstract of the Dissertation

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Professor Ciprian Manolescu, Chair

We construct a spectral sequence from the reduced odd Khovanov homology of a link converging to the framed instanton homology of the double cover branched over the link, with orientation reversed. Framed instanton homology counts certain instantons on the cylinder of a 3-manifold connect-summed with a 3-torus. En route, we provide a new proof of Floer's surgery exact triangle for instanton homology using metric stretching maps, and generalize the exact triangle to a link surgeries spectral sequence. Finally, we relate framed instanton homology to Floer's instanton homology for admissible bundles.

The dissertation of Christopher William Scaduto is approved.

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To my parents, Anne and Michael,

and my brother, Mike

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Vita

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CHAPTER 1

Introduction

1.1 Floer's instanton homology

Instanton homology is a gauge-theoretic invariant for 3-manifolds defined by Andreas Floer in the late 1980's [13, 14]. Its construction is motivated by applying the ideas of finite-dimensional Morse homology to the Chern-Simons functional, whose domain is an infinite-dimensional space of connections on a fixed SO(3)-bundle over the base 3-manifold Y. The result (up to small perturbations) is a chain complex whose generators are flat connections on Y, and whose differential counts isolated gradient flow lines of the Chern-Simons functional. These flow lines can be interpreted as connections on the cylinder $\mathbb{R} \times Y$ that are *anti-self-dual* (ASD), otherwise known as *instantons* – that is, connections with anti-self-dual curvature. The study of instantons on closed 4-manifolds had been exploited prior to Floer's work with magnificent success by Donaldson [8].

Due to technical obstacles caused by the presence of reducible connections, Floer's construction was first only carried out for homology 3-spheres Y. A *homology 3-sphere* is a 3-manifold with the same integral homology as the 3-sphere S^3. The resulting invariant $I(Y)$, called the *instanton homology* of Y, is a $\mathbb{Z}/8$-graded abelian group. Floer then extended his construction to include some 3-manifolds with $b_1(Y) > 0$. In this latter case, however, the extra data of an SO(3)-bundle $\mathbb{Y}$ over Y is required, and, to avoid reducible connections, the bundle $\mathbb{Y}$ must be *admissible* in the following sense.

Definition 1.1.1. *An $SO(3)$-bundle $\mathbb{Y}$ over a closed, connected and oriented 3-manifold Y is said to be* admissible *if either (i) Y is a homology 3-sphere (in which case $\mathbb{Y}$ is trivial) or (ii) there exists an orientable surface $\Sigma \subset Y$ with $\mathbb{Y}|_{\Sigma}$ non-trivial. In case (ii), we say $\mathbb{Y}$ is* non-trivial admissible.

The case (ii) is equivalent to the conditions that $w_2(\mathbb{Y})$ is non-zero and lifts to a non-torsion class in $H^2(Y;\mathbb{Z})$. Floer defined an abelian group $I(\mathbb{Y})$ for any admissible bundle over a 3-manifold. In the non-trivial admissible case, the group is only *relatively* $\mathbb{Z}/8$-graded.

The groups $I(\mathbb{Y})$ are difficult to compute in general. Fintushel and Stern [12] computed $I(Y)$ when Y is a homology 3-sphere Brieskorn sphere $\Sigma(p,q,r)$, and closed formulae are now known for any homology 3-sphere Seifert-fibered space [37, Thm 6.28]. Some other computations of $I(\mathbb{Y})$ are made possible by Floer's exact triangle, which will be discussed in Chapter 3.

1.2 Framed instanton homology

From Floer's instanton homology $I(\mathbb{Y})$ for admissible bundles one can construct an invariant $I^{\#}(Y)$ defined for any closed, connected, oriented 3-manifold Y. The trick, due to Kronheimer and Mrowka, is the following: making some inessential choices, we construct a bundle $\mathbb{Y}^{\#}$ over $Y\#T^3$ by gluing together a trivial bundle over Y and a non-trivial bundle over T^3, the 3-torus. The bundle $\mathbb{Y}^{\#}$ is always admissible, and the group $I(\mathbb{Y}^{\#})$, it turns out, is always 4-periodic. The *framed instanton homology of* Y, written $I^{\#}(Y)$, is the $\mathbb{Z}/4$-graded abelian group isomorphic to four consecutive gradings of $I(\mathbb{Y}^{\#})$. The terminology *framed* is from [21], and comes from interpreting $I^{\#}(Y)$ as a Morse-Bott homology theory of the Chern-Simons functional on the space of framed connections.

The framed instanton homology $I^{\#}(Y)$ is isomorphic to the sutured instanton group $\mathrm{SHI}(M,\gamma)$ introduced by Kronheimer and Mrowka in [22], where M is the complement of an open 3-ball in Y and γ is a suture on the 2-sphere boundary. We restate a conjecture of Kronheimer and Mrowka, transferred from the sutured setting, cf. [22, Conj. 7.24]:

Conjecture 1.2.1. *The framed instanton homology $I^{\#}(Y)$ is isomorphic to the Heegaard-Floer hat homology $\widehat{HF}(Y)$.*

Heegaard-Floer homology was introduced by Ozsváth and Szabó [34] in the early 2000's, and has enjoyed much more computational success over $I^{\#}(Y)$. A related gauge-theoretic Floer theory is $\widetilde{\mathrm{HM}}(Y)$, a version of monopole-Floer homology, as defined by Bloom [3] using the machinary of Kronheimer and Mrowka [23]. It has been shown, after much work, that the monopole-Floer group $\widetilde{\mathrm{HM}}(Y)$ is isomorphic to $\widehat{\mathrm{HF}}(Y)$, cf. [25, 6, 40].

1.3 A spectral sequence from odd Khovanov homology

Khovanov homology, defined originally by Khovanov [19] in 2000, is a combinatorially defined link invariant that comes in the form of a bigraded abelian group $\mathrm{Kh}(L)$. From it, one can recover the Jones polynomial. The first instance of a structural relation between a Floer homology and a combinatorial link homology was given by Ozsváth and Szabó:

Theorem 1.3.1 (Ozsváth-Szabó [35]). *Given a link L, there is a spectral sequence with E^2-page $\overline{Kh}(L;\mathbb{F}_2)$, the reduced Khovanov homology of L, that converges to $\widehat{HF}(\overline{\Sigma(L)};\mathbb{F}_2)$, where $\Sigma(L)$ is the double cover of S^3 branched over L.*

We have written $\mathbb{F}_2$ for the field $\mathbb{Z}/2$. An overline $\overline{Y}$ over a manifold indicates orientation reversal. In the sequel, it will be convenient to write this result as

$$E^2 = \overline{\mathrm{Kh}}(L;\mathbb{F}_2) \quad \rightsquigarrow \quad \widehat{\mathrm{HF}}(\overline{\Sigma(L)};\mathbb{F}_2).$$

The notation $A \rightsquigarrow B$ is an abbreviation for the existence of a spectral sequence with some starting page A converging to B.

Ozsváth and Szabó speculated in [35] that their spectral sequence, if lifted to $\mathbb{Z}$-coefficients, would not have reduced Khovanov homology as the E^2-page, but would have some other link homology with altered signs in the differentials. With this is mind, Ozsváth, Rasmussen and Szabó [33] defined an abelian group $\mathrm{Kh}'(L)$ called the *odd Khovanov homology of L*. With $\mathbb{F}_2$-coefficients, it coincides with Khovanov homology – but they are very different with $\mathbb{Z}$-coefficients.

Odd Khovanov homology is bigraded by a homological grading, called t, and a quantum grading, called q. There is a splitting

$$\mathrm{Kh}'(L)_{t,q} \simeq \overline{\mathrm{Kh}'}(L)_{t,q-1} \oplus \overline{\mathrm{Kh}'}(L)_{t,q+1},$$

where $\overline{\mathrm{Kh}'}(L)$ is called the *reduced* odd Khovanov homology. The bigraded group $\overline{\mathrm{Kh}'}(L)_{t,q}$ categorifies the Jones polynomial J_L in the sense that

$$J_L(x) = \sum_{t,q} (-1)^t \mathrm{rk}_{\mathbb{Z}}(\overline{\mathrm{Kh}'}(L)_{t,q}) x^q.$$

Here, $J_{\text{unknot}}(x) = 1$. The authors of odd Khovanov homology put forth

Conjecture 1.3.2 (Ozsváth-Rasmussen-Szabó [33])**.** *Given a link L, there is a spectral sequence with E^2-page $\overline{Kh'}(L)$, the reduced odd Khovanov homology of L, converging to $\widehat{HF}(\overline{\Sigma(L)})$.*

One of the main results of this thesis, as motivated by Conjecture 1.2.1, is the construction of an instanton analogue of Conjecture 1.3.2:

Main Theorem (see Ch. 6)**.** *Given a link L, there is a spectral sequence with E^2-page $\overline{Kh'}(L)$, the reduced odd Khovanov homology of L, that converges to the framed instanton homology $I^\#(\overline{\Sigma(L)})$.*

A novelty of our spectral sequence is its compatibility with $\mathbb{Z}/4$-gradings. Indeed, the $\mathbb{Z}/4$-grading of framed instanton homology has as of yet no Heegaard-Floer or monopole-Floer counterpart, which only come equipped with absolute $\mathbb{Z}/2$-gradings.

We define a mod 4 grading $\delta^\#$ on $\overline{\mathrm{Kh}'}(L)$ by the formula

$$\delta^\# := \frac{3}{2}q - t + \frac{1}{2}(\sigma + \nu) \mod 4,$$

where σ and ν are the signature and nullity of L, respectively. The grading $\delta^\#$ is carried via our spectral sequence to the mod 4 grading on $I^\#(\overline{\Sigma(L)})$.

1.4 The signs: composing homology orientations

The main difficultly in the proof of our spectral sequence is the computation of signs in the relevant differential in framed instanton homology at the E^1-page – that is, the ability to work with $\mathbb{Z}$-coefficients. In fact, the $\mathbb{F}_2$-coefficient version of our spectral sequence is rather straightforward, given the material of Chapter 3: the proof is nearly complete after the short discussion in §6.3. To work with $\mathbb{Z}$-coefficients, however, we require a firm understanding of how signs work in the morphisms of framed instanton homology.

Given a 4-dimensional cobordism $X : Y_1 \to Y_2$ between two 3-manifolds, we obtain an induced group homomorphism $I^\#(X) : I^\#(Y_1) \to I^\#(Y_2)$ on framed instanton homology. This morphism, however, is only well-defined up to an overall sign. To fix a sign, we need the extra data of a *homology orientation* of X – an orientation of the real vector space

$$H_1(Y_1; \mathbb{R}) \oplus H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R}).$$

Thus homology orientations are part of the morphism data in an appropriate category on which framed instanton homology is a functor. So to understand the signs in our differentials, we require an understanding of how to compose homology orientations. This is the undertaking of Chapter 5, in which we introduce an algebro-topological way of composing homology orientations. The composition rule for homology orientations we define is perhaps the main technical novelty in this thesis, and is what allows us to verify the construction of our spectral sequence with $\mathbb{Z}$-coefficients.

1.5 Double branched covers of quasi-alternating links

Our spectral sequence allows us to compute $I^{\#}(Y)$ when Y is the double branched cover of a non-split alternating link, and, more generally, that of a quasi-alternating link L. For these cases, the spectral sequence actually collapses at the E^2-page because the gradings are supported in the even degrees $0, 2 \bmod 4$. We obtain

$$I^{\#}(\Sigma(L)) \simeq \mathbb{Z}_{(0)}^{\frac{1}{2}(\det(L)+2^{\#L-1})} \oplus \mathbb{Z}_{(2)}^{\frac{1}{2}(\det(L)-2^{\#L-1})} \tag{1.1}$$

where $\mathbb{Z}_{(i)}$ stands for a copy of $\mathbb{Z}$ supported in grading $i \bmod 4$, and $\#L$ is the number of components of the link L. Note here that the total rank of $I^{\#}(\Sigma(L))$ is given by the determinant of L. This result verifies Conjecture 1.2.1 for the class of 3-manifolds that are double branched covers over quasi-alternating links.

To better understand the meaning of (1.1), let L be the 2-bridge link of type (p, q), which is a non-split alternating link. Then $\Sigma(L)$ is the lens space $L(p, q)$. We have $\det(L) = |H_1(L(p, q))| = p$ and $\#L = 1$ if p is odd and $\#L = 2$ if p is even. The formula (1.1) yields

$$I^{\#}(L(p, q)) \simeq \mathbb{Z}_{(0)}^{\lceil \frac{p+1}{2} \rceil} \oplus \mathbb{Z}_{(2)}^{\lfloor \frac{p-1}{2} \rfloor}.$$

We then observe that this is none other than the $\mathbb{Z}/4$-graded abelian group

$$H_*(\mathrm{Hom}(\pi_1, \mathrm{SU}(2)))$$

where $\pi_1 = \pi_1(L(p, q)) \simeq \mathbb{Z}/p$. Indeed, $\mathrm{Hom}(\mathbb{Z}/p, \mathrm{SU}(2))$ is a collection of 1 point and $\left(\frac{p-1}{2}\right)$ 2-spheres if p is odd and 2 points and $\left(\frac{p}{2} - 1\right)$ 2-spheres if p is even. This observation aligns with the interpretation of framed instanton homology as a Morse-Bott theory for framed connections, as $\mathrm{Hom}(\pi_1, \mathrm{SU}(2))$ is, via the holonomy correspondence, the space of flat *framed* connections on Y up to equivalence (the usual, unframed correspondence involves modding out by conjugation).

1.6 The relationship between $I^{\#}(Y)$ and $I(Y)$

For examples of when our spectral sequence does not collapse, we look to relate $I^{\#}(Y)$ to Floer's instanton homology $I(Y)$ when Y is a homology 3-sphere, and then use the calculations of $I(Y)$ available in the literature. As $I^{\#}(Y)$ is determined by the instanton homology of an admissible bundle over $Y\#T^3$, we can apply a modified version of Fukaya's instanton connected sum theorem of [17].

The resulting relationship between $I^{\#}(Y)$ and $I(Y)$ is best understood using Frøyshov's reduced instanton groups $\widehat{I}(Y)$ from [16], which are obtained from $I(Y)$ by considering interactions with the trivial connection. They come equipped with an absolute $\mathbb{Z}/8$-grading and a degree 4 endomorphism u. Frøyshov's Theorem 10 says $(u^2-64)^n = 0$ for some $n > 0$, when the coefficient ring contains an inverse for 2. We find, for example, that

$$I^{\#}(Y;\mathbb{Q}) \simeq \ \ker(u^2-64) \otimes \big(\mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(3)}\big) \oplus \mathbb{Q}_{(0)} \tag{1.2}$$

as $\mathbb{Z}/4$-graded vector spaces, where u^2-64 is acting on $\bigoplus_{j=0}^{3} \widehat{I}(Y;\mathbb{Q})_j$. The coefficient ring $\mathbb{Q}$ can be replaced by any field of characteristic not equal to 2. There are cases in which the relationship between $I^{\#}(Y)$ and $I(Y)$ is more exact: for instance, when Y is ±1-surgery on a knot of genus at most 2, and Y happens to be homology cobordant to S^3, we obtain

$$I^{\#}(Y;\mathbb{Q})^{\oplus 2} \simeq I(Y;\mathbb{Q}) \otimes \big(\mathbb{Q}_{(0)} \oplus \mathbb{Q}_{(3)}\big) \oplus \mathbb{Q}_{(0)}.$$

This result, combined with the computations of Fintushel and Stern [12], allows us to compute the framed instanton homology of the Brieskorn spheres $\Sigma(2,3,6k\pm1)$:

$$I^{\#}(\Sigma(2,3,6k+1);\mathbb{Q}) \simeq \mathbb{Q}_{(0)}^{\lfloor k/2\rfloor+1} \oplus \mathbb{Q}_{(1)}^{\lfloor k/2\rfloor} \oplus \mathbb{Q}_{(2)}^{\lceil k/2\rceil} \oplus \mathbb{Q}_{(3)}^{\lceil k/2\rceil},$$

$$I^{\#}(\Sigma(2,3,6k-1);\mathbb{Q}) \simeq \mathbb{Q}_{(0)}^{\lceil k/2\rceil} \oplus \mathbb{Q}_{(1)}^{\lceil k/2\rceil-1} \oplus \mathbb{Q}_{(2)}^{\lfloor k/2\rfloor} \oplus \mathbb{Q}_{(3)}^{\lfloor k/2\rfloor}.$$

The 3-manifolds $\Sigma(2,3,6k\pm1)$ are branched double covers over the $(3,6k\pm1)$ torus knots, and as such these computations provide many examples in which our spectral sequence does not collapse.

The result (1.2), along with a simple argument involving Floer's exact triangle, also allows us to compute the Euler characteristic of $I^\#(Y)$. We find that

$$\chi(I^\#(Y)) = |H_1(Y;\mathbb{Z})|,$$

where $|S|$ stands for the cardinality of S when S is finite, and is 0 otherwise. This confirms that the Euler characteristics of Conjecture 1.2.1 match up, i.e., $\chi(I^\#(Y)) = \chi(\widehat{\mathrm{HF}}(Y))$.

1.7 Another proof of Floer's exact triangle

The proof of the main theorem, our spectral sequence, requires an understanding of how instanton homology behaves with respect to Dehn-surgery on 3-manifolds. The first result in this direction is Floer's, in the form of an exact sequence

$$\cdots I(\mathbb{Y}) \to I(\mathbb{Y}_0) \to I(\mathbb{Y}_1) \to I(\mathbb{Y}) \to \cdots$$

Here $\mathbb{Y}$, $\mathbb{Y}_0$ and $\mathbb{Y}_1$ are certain admissible bundles over 3-manifolds Y, Y_0 and Y_1, where Y_0 and Y_1 are the results of 0- and 1-surgery, respectively, on some fixed framed knot in Y. Because of the 3-periodicity of this sequence, we call his result the *exact triangle* in instanton homology.

An essential ingredient in the construction of Ozsváth and Szabó's spectral sequence of Theorem 1.3.1 is a *link surgeries spectral sequence*. This is a generalization of the exact triangle in which one does surgery on a link $L \subset Y$ instead of just a knot. We provide the instanton analogue of this result, which takes the form of a spectral sequence

$$E^1 = \bigoplus_{v\in\{0,1\}^{\#L}} I(\mathbb{Y}_v) \quad \rightsquigarrow \quad I(\mathbb{Y}) \tag{1.3}$$

in which $\mathbb{Y}_v$ is a bundle over the result of doing 0- and 1-surgery on each component of L according to the vector v. There are conditions as to which bundles are allowed here. We mention that Bloom [3] constructed a link surgeries spectral sequence in the setting of monopole Floer homology, and from this obtained a spectral sequence from $\overline{\mathrm{Kh}}(L;\mathbb{F}_2)$ to $\widetilde{\mathrm{HM}}(\overline{\Sigma(L)};\mathbb{F}_2)$.

Floer's exact triangle was studied by Braam and Donaldson in [5], where a detailed proof following Floer's ideas can be found. In Chapter 3 we provide an alternative proof. The proof relies on the *triangle detection lemma*, first used by Ozsváth and Szabó [35], which requires the input of maps between three chain complexes satisfying certain properties. The maps we choose count instantons on families of metrics that are parameterized by convex polytopes. This approach was used by Kronheimer, Mrowka, Ozsváth and Szabó to prove a surgery exact sequence in the monopole case [24]. Our proof is largely an adaptation of Kronheimer and Mrowka's proof in [20] of an exact triangle in singular instanton homology, although the analysis of instanton counting is different. As in their case, our proof easily generalizes to prove (1.3). Especially relevant here is the work of Bloom [3], who studied the combinatorics of link surgeries spectral sequences.

1.8 Relation to singular instanton homology

Kronheimer and Mrowka [20] introduced the singular instanton homology groups $I^{\#}(Y,L)$ and $I^{\natural}(Y,L)$, where L is a link in Y. The framed group $I^{\#}(Y)$ is obtained from $I^{\#}(Y,L)$ by taking L to be empty. The construction of these more general groups involves counting instantons on $\mathbb{R}\times Y$ singular along $\mathbb{R}\times L$. Writing $I^{\#}(L) = I^{\#}(S^3,L)$ and $I^{\natural}(L) = I^{\natural}(S^3,L)$, Kronheimer and Mrowka produced spectral sequences

$$E^2 = \mathrm{Kh}(L) \;\rightsquigarrow\; I^{\#}(L), \qquad E^2 = \overline{\mathrm{Kh}}(L) \;\rightsquigarrow\; I^{\natural}(L).$$

Here $\mathrm{Kh}(L)$ and $\overline{\mathrm{Kh}}(L)$ are unreduced and reduced Khovanov homology, respectively. Their spectral sequences respect $\mathbb{Z}/4$-gradings. Using the latter spectral sequence, they proved in [20] that Khovanov homology detects the unknot.

1.9 Outline

The outline of this thesis is as follows. We begin in Chapter 2 by reviewing Floer's instanton homology for admissible bundles. Then, in Chapter 3, we state and prove Floer's instanton exact triangle and its generalization, the link surgeries spectral sequence. Next, we focus our attention on framed instanton homology, studying its basic properties and constructions in Chapter 4. In Chapter 5, we define a rule for composing homology orientations, which is needed to understand the signs of morphisms in framed instanton homology. We then use the machinary developed so far to prove the main theorem, our spectral sequence, in Chapter 6. In this chapter the definition of reduced odd Khovanov homology is also given. In Chapter 7, we study the relationship between $I(Y)$ and $I^{\#}(Y)$ for homology 3-spheres via the connected sum theorem of Fukaya. Finally, in Chapter 8, we perform some computations in framed instanton homology, including the determination of $I^{\#}(Y)$ for double branched covers of quasi-alternating links and for the Brieskorn spheres $\Sigma(2, 3, 6k \pm 1)$ – as well as the computation of the Euler characteristic $\chi(I^{\#}(Y))$.

CHAPTER 2

Background on Floer's instanton homology

In this chapter we review the relevant aspects of instanton homology for admissible bundles, introduced by Floer [13, 14]. Our main technical references are [7, 20]. Other useful references include [13, 14, 5, 16, 37]. The purpose of this chapter is mainly to fix conventions and notation.

2.1 Instanton groups

Let $\mathbb{Y}$ be an SO(3)-bundle over a closed, connected, oriented Riemannian 3-manifold Y. The group $I(\mathbb{Y})$ is heuristically a Morse homology group computed using a suitably perturbed Chern-Simons functional $\mathbf{cs}_\pi : \mathscr{C}(\mathbb{Y}) \to \mathbb{R}$ modulo a group of gauge transformations:

$$\mathbf{cs}_\pi(a) = -\frac{1}{8\pi^2}\int_{[0,1]\times Y} \operatorname{tr}(F_A^2) + f_\pi(a).$$

Here A is a connection on $[0,1]\times\mathbb{Y}$ which restricts to a base connection a_0 on $\{0\}\times\mathbb{Y}$ and the connection a on $\{1\}\times\mathbb{Y}$, and f_π is a small perturbation, see [20, §3.4]. We have written $\mathscr{C}(\mathbb{Y})$ for the space of smooth connections on $\mathbb{Y}$, an affine space modelled on $\Omega^1(\mathbb{Y}_{\text{ad}})$, where $\mathbb{Y}_{\text{ad}} = \mathbb{Y}\times_{\text{ad}}\mathfrak{so}(3)$ is the adjoint bundle of $\mathbb{Y}$.

Let $\mathbb{X}$ be an SO(3)-bundle over an n-dimensional manifold X. In our constructions we do not use the full automorphism group $\mathscr{G}(\mathbb{X})$ of $\mathbb{X}$, but rather, following the terminology of [16], we use the subgroup $\mathscr{G}_{\text{ev}} = \mathscr{G}_{\text{ev}}(\mathbb{X})$ of *even* gauge transformations. Elements of $\mathscr{G}_{\text{ev}}$ are called determinant-1 gauge transformations in [20] and restricted gauge transformations in [5]. Viewing gauge transformations as sections of the bundle $\mathbb{X} \times_{\mathrm{Ad}} \mathrm{SO}(3)$, the even transformations are the ones that lift to sections of $\mathbb{X} \times_{\mathrm{Ad}} \mathrm{SU}(2)$. There is an exact sequence

$$1 \longrightarrow \mathscr{G}_{\mathrm{ev}}(\mathbb{X}) \longrightarrow \mathscr{G}(\mathbb{X}) \xrightarrow{\eta} H^1(X; \mathbb{F}_2) \longrightarrow 1, \tag{2.1}$$

where η measures the obstruction to deforming a gauge transformation over the 1-skeleton of X. For a connection A on $\mathbb{X}$ we write $h^0(A)$ for the dimension of its $\mathscr{G}_{\mathrm{ev}}$-stabilizer. The possible values of $h^0(A)$ are $0, 1, 3$. For us, the only stabilizers that will appear will be $1, S^1, \mathrm{SO}(3)$. We call A *irreducible* if $h^0(A) = 0$, and *reducible* otherwise.

We will write $a, b, c, \ldots$ for typical connections on bundles over 3-manifolds and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ for their respective $\mathscr{G}_{\mathrm{ev}}$-classes. A typical connection on a bundle over a 4-manifold X is written as A, and simply $[A]$ for its $\mathscr{G}_{\mathrm{ev}}$-class.

Let $\mathscr{B}(\mathbb{Y})$ denote the quotient $\mathscr{C}(\mathbb{Y})/\mathscr{G}_{\mathrm{ev}}$. The functional $\mathbf{cs}_\pi$ induces a map $\mathbf{cs}'_\pi : \mathscr{B}(\mathbb{Y}) \to \mathbb{R}/\mathbb{Z}$. The set of critical points of $\mathbf{cs}'_\pi$ is denoted $\mathfrak{C}$ or $\mathfrak{C}(\mathbb{Y})$; when the perturbation π is zero this is the set of flat connection classes on $\mathbb{Y}$. We write $h^1(a)$ for the dimension of the Zariski tangent space of $\mathfrak{a}$ in $\mathfrak{C}$. Following [7], when $h^0(a) = h^1(a) = 0$, the connection a is called *acyclic*. Let $\mathfrak{C}^{\mathrm{irr}}$ denote the subset of irreducibles in $\mathfrak{C}$. When $\mathbb{Y}$ is admissible and a suitable perturbation is chosen, $\mathfrak{C}^{\mathrm{irr}}$ is a finite set of acyclic classes, and it is in fact all of $\mathfrak{C}$ or is missing only the trivial class, according to whether $b_1(Y) \neq 0$ or not, respectively. We assume such a perturbation is chosen.

Fix a base connection a_0 on $\mathbb{Y}$. We define the chain group

$$\mathrm{C}(\mathbb{Y}) = \bigoplus_{\mathfrak{a} \in \mathfrak{C}^{\mathrm{irr}}} \mathbb{Z}\Lambda(\mathfrak{a})$$

where $\Lambda(\mathfrak{a})$ is the 2-element set of orientations of the real line $\det(D_A)$, where A is a connection on $\mathbb{R} \times \mathbb{Y}$ with $A|_{\mathbb{Y} \times \{t\}}$ equivalent to a_0 for $t \ll 0$ and in the class $\mathfrak{a}$ for $t \gg 0$, and D_A is the Fredholm operator $-d_A \oplus d_A^+$ defined on suitable Sobolev spaces in §2.3; see also [20, §3.6]. Here $\mathbb{Z}\Lambda(\mathfrak{a})$ means the infinite cyclic group with generators the elements of $\Lambda(\mathfrak{a})$. We often think of $\mathrm{C}(\mathbb{Y})$ as generated by $\mathfrak{C}^{\mathrm{irr}}$; when doing this it is understood that we have chosen distinguished elements from each set $\Lambda(\mathfrak{a})$.

A connection A on an SO(3)-bundle over a Riemannian 4-manifold is an *instanton* or is *anti-self-dual* (ASD) if its curvature F_A satisfies

$$\star F_A = -F_A$$

where $\star$ is the Hodge star. The *energy* of a connection A is given by $\|F_A\|_{L^2}^2 = -\int \mathrm{tr}(F_A \wedge \star F_A)$. Instantons on $\mathbb{X} = \mathbb{R} \times \mathbb{Y}$ may be interpreted as gradient flow-lines for the Chern-Simons functional. In actuality we consider a perturbed instanton equation involving π, and call the solutions instantons as well. Given acyclic $a, b \in \mathscr{C}(\mathbb{Y})$ we let $M(a,b)$ be the space of $\mathscr{G}_{\mathrm{ev}}$-classes of finite-energy instantons on $\mathbb{X}$ asymptotic at $-\infty$ to a and at $+\infty$ to b. When the perturbation is zero, elements $[A] \in M(a,b)$ are distinguished by the property $\frac{1}{8\pi^2}\|F_A\|_{L^2}^2 = \mathbf{cs}(b) - \mathbf{cs}(a)$.

For a small, generic perturbation $M(a,b)$ is a smooth manifold, and we write

$$\mu(a,b) = \dim M(a,b).$$

Passing to $\mathscr{G}_{\mathrm{ev}}$-classes, the number $\mu(\mathfrak{a},\mathfrak{b})$ is well-defined modulo 8, and equips $\mathrm{C}(\mathbb{Y})$ with a relative $\mathbb{Z}/8$-grading given by $\mathrm{gr}(\mathfrak{a}) - \mathrm{gr}(\mathfrak{b}) \equiv \mu(\mathfrak{a},\mathfrak{b})$. The space $M(a,b)$ has an $\mathbb{R}$-action by translation along the $\mathbb{R}$-factor of $\mathbb{R} \times \mathbb{Y}$, and we write

$$\breve{M}(a,b) = M(a,b)/\mathbb{R}.$$

The data of $\mathfrak{a}, \mathfrak{b}$ and the lift of $\mu(\mathfrak{a},\mathfrak{b}) \in \mathbb{Z}/8$ to $d \in \mathbb{Z}$ are sufficient to describe $M(a,b)$; viewing $[A] \in M(a,b)$ as a path in $\mathscr{B}(\mathbb{Y})$, the index d faithfully records the homotopy class of $[A]$ relative to the endpoints $\mathfrak{a}, \mathfrak{b}$. That said, if $d = \mu(a,b)$, we also write $M(\mathfrak{a},\mathfrak{b})_d$ for the space $M(a,b)$, and similarly $\breve{M}(\mathfrak{a},\mathfrak{b})_{d-1}$ for $\breve{M}(a,b)$. Thus $M(\mathfrak{a},\mathfrak{b})_d$ is a d-dimensional component of instanton classes whose limits are in the classes $\mathfrak{a}$ and $\mathfrak{b}$.

Suppose $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}^{\mathrm{irr}}$ with $\mu(\mathfrak{a},\mathfrak{b}) \equiv 1$. With suitable perturbation, $\breve{M}(\mathfrak{a},\mathfrak{b})_0$ is a finite set, and as explained in [20, §3.6], each of its elements determines an isomorphism

$\Lambda(\mathfrak{a}) \to \Lambda(\mathfrak{b})$. Denoting the induced isomorphism $\mathbb{Z}\Lambda(\mathfrak{a}) \to \mathbb{Z}\Lambda(\mathfrak{b})$ corresponding to $[A] \in \check{M}(\mathfrak{a}, \mathfrak{b})_0$ by the symbol $\epsilon[A]$, the differential ∂ for $\mathrm{C}(\mathbb{Y})$ is defined in pieces by

$$\partial|_{\mathbb{Z}\Lambda(\mathfrak{a}) \to \mathbb{Z}\Lambda(\mathfrak{b})} = \sum_{[A] \in \check{M}(\mathfrak{a}, \mathfrak{b})_0} \epsilon[A].$$

If we choose an element from each $\Lambda(\mathfrak{a})$, then we may view ∂ as a map on $\mathfrak{C}^{\text{irr}}$ and write $\langle \partial \mathfrak{a}, \mathfrak{b} \rangle = \# \check{M}(\mathfrak{a}, \mathfrak{b})_0$, where $\#$ indicates a signed count. The differential lowers the relative $\mathbb{Z}/8$-grading by 1. The identity $\partial^2 = 0$ is obtained by interpreting the boundary of a 1-dimensional moduli space $\check{M}(\mathfrak{a}, \mathfrak{b})_1$ as a disjoint union of broken trajectories $\check{M}(\mathfrak{a}, \mathfrak{c})_0 \times \check{M}(\mathfrak{c}, \mathfrak{b})_0$. The relatively $\mathbb{Z}/8$-graded abelian group $I(\mathbb{Y})$ is defined to be $H_*(\mathrm{C}(\mathbb{Y}), \partial)$.

In defining the complex $\mathrm{C}(\mathbb{Y})$ we have chosen a Riemannian 3-manifold Y, an admissible SO(3)-bundle $\mathbb{Y}$ over Y, a perturbation π, and a base connection a_0 on $\mathbb{Y}$. When working with the chain group we always assume such data is chosen. The isomorphism class of the relatively $\mathbb{Z}/8$-graded group $I(\mathbb{Y})$ depends only on the oriented homeomorphism type of Y and $w_2(\mathbb{Y})$.

2.2 Maps from cobordisms

Let $X : Y_1 \to Y_2$ be a cobordism from Y_1 to Y_2. That is, X is a compact, connected, oriented 4-manifold with an orientation preserving diffeomorphism $\partial X \simeq Y_2 \sqcup \overline{Y}_1$. As before, each Y_i is connected. Assume X is equipped with a metric that is product-like near its boundary. Suppose further that $\mathbb{X}$ is an SO(3)-bundle over X with $\mathbb{X}|_{Y_i} = \mathbb{Y}_i$ where each $\mathbb{Y}_i$ is admissible. We abbreviate this setup as $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$. To obtain a chain map

$$m(\mathbb{X}) : \mathrm{C}(\mathbb{Y}_1) \to \mathrm{C}(\mathbb{Y}_2),$$

first form the bundle $(\mathbb{R}_{\leq 0} \times \mathbb{Y}_1) \cup \mathbb{X} \cup (\mathbb{R}_{\geq 0} \times \mathbb{Y}_2)$ over the Riemannian 4-manifold obtained from X by attaching cylindrical ends to the boundary. We define $M(a, \mathbb{X}, b)$ to be the space of $\mathscr{G}_{\text{ev}}$-classes of finite-energy instantons on this bundle. With suitable perturbations chosen, $M(a, \mathbb{X}, b)$ is a smooth manifold, and we write $\mu(a, \mathbb{X}, b) =$

$\dim M(a, \mathbb{X}, b)$. As before, $\mu(\mathfrak{a}, \mathbb{X}, \mathfrak{b})$ is well-defined modulo 8, and we write $M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_d$ for $M(a, \mathbb{X}, b)$ where $d = \mu(a, \mathbb{X}, b)$.

Now suppose $\mathfrak{a} \in \mathfrak{C}^{\text{irr}}(\mathbb{Y}_1)$ and $\mathfrak{b} \in \mathfrak{C}^{\text{irr}}(\mathbb{Y}_2)$ with $\mu(\mathfrak{a}, \mathbb{X}, \mathfrak{b}) \equiv 0$. With suitable perturbations, $M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_0$ is a finite set of points. In defining $\mathrm{C}(\mathbb{Y}_i)$, basepoint connections $a_{i,0}$ are chosen. Let A be a connection on $\mathbb{X}$ (with cylindrical ends attached) with limits at the ends equivalent to the $a_{i,0}$. An orientation of the line $\det(D_A)$ will be called an *I-orientation* of $\mathbb{X}$, following [20, Def. 3.9]. With an I-orientation of $\mathbb{X}$, an element $[A] \in M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_0$ determines an isomorphism $\epsilon[A] : \mathbb{Z}\Lambda(\mathfrak{a}) \to \mathbb{Z}\Lambda(\mathfrak{b})$, and $m(\mathbb{X})$ is defined in pieces by

$$m(\mathbb{X})|_{\mathbb{Z}\Lambda(\mathfrak{a}) \to \mathbb{Z}\Lambda(\mathfrak{b})} = \sum_{[A] \in M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_0} \epsilon[A].$$

In shorthand, $\langle m(\mathbb{X})\mathfrak{a}, \mathfrak{b}\rangle = \# M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_0$. When $\mu(\mathfrak{a}, \mathbb{X}, \mathfrak{b}) \not\equiv 0$ this part of the differential is zero. Different choices of I-orientations only affect the overall sign of the map $m(\mathbb{X})$. The notation we use for composing bundle cobordisms is

$$\mathbb{X}_2 \circ \mathbb{X}_1 = \mathbb{X}_1 \cup_{\mathbb{Y}_2} \mathbb{X}_2 : \mathbb{Y}_1 \to \mathbb{Y}_3$$

where $\mathbb{X}_i : \mathbb{Y}_i \to \mathbb{Y}_{i+1}$ for $i = 1, 2$. We write $I(\mathbb{X}_1) : I(\mathbb{Y}_1) \to I(\mathbb{Y}_2)$ for the map on homology induced by $m(\mathbb{X}_1)$. Having assumed Y_i is connected for $i = 1, 2$, we have the composition law

$$I(\mathbb{X}_2 \circ \mathbb{X}_1) = I(\mathbb{X}_2) \circ I(\mathbb{X}_1).$$

There is a well-defined notion of composing I-orientations using (2.2) below, and this is needed to make sense of this expression. For a general discussion of the composition law involving disconnected 3-manifolds see [20, §5.2]. We mention that the composition law follows from the homotopy formula (2.5) below, using a 1-dimensional family of metrics that stretches along Y_2.

2.3 Index formulae

The numbers $\mu(a,b)$ and $\mu(a,\mathbb{X},b)$ above are more properly described as the indices of certain Fredholm operators. Let $\mathbb{X}:\mathbb{Y}_1\to\mathbb{Y}_2$ as above. The $\mathbb{Y}_i$ are not assumed to be admissible. Let a and b be connections on $\mathbb{Y}_1$ and $\mathbb{Y}_2$, respectively. Attach cylindrical ends to $\mathbb{X}$ as above and call the result $\mathbb{X}$ as well. Choose a connection A on $\mathbb{X}$ with $A|_{\mathbb{Y}_1\times\{t\}}$ equal to a for $t\ll 0$ and $A|_{\mathbb{Y}_2\times\{t\}}$ equal to b for $t\gg 0$, and consider the operator

$$D_A=-d_A^*\oplus d_A^+ : L^p_{s,\phi}(\Lambda^1\otimes\mathbb{X}_{\mathrm{ad}})\to L^p_{s-1,\phi}((\Lambda^0\oplus\Lambda^+)\otimes\mathbb{X}_{\mathrm{ad}})$$

where $L^p_{s,\phi}=\phi L^p_s$ are Sobolev spaces weighted by the real function ϕ, equal to $e^{-\epsilon t}$ for some sufficiently small $\epsilon>0$ on the ends $\mathbb{R}_{\leq 0}\times Y_1$ and $\mathbb{R}_{\geq 0}\times Y_2$, and equal to 1 otherwise. This operator arises from linearizing the instanton equation and using a Coulomb gauge condition. If $\mathbb{X}':\mathbb{Y}_2\to\mathbb{Y}_3$ and A' is a connection on $\mathbb{X}'$ with limit b over $\mathbb{Y}_2$, there is a natural isomorphism

$$\det(D_A)\otimes\det(D_{A'})\simeq\det(D_{A\cup A'}) \tag{2.2}$$

and the index relation $\mathrm{ind}(D_A)+\mathrm{ind}(D_{A'})=\mathrm{ind}(D_{A\cup A'})$ holds, see for example [7, Prop. 5.11]. In the definition of $\mathrm{C}(\mathbb{Y})$ in §2.1 we take $\mathbb{X}=[0,1]\times\mathbb{Y}$ to define the operator D_A.

Note that the two ends $\mathbb{Y}_1$ and $\mathbb{Y}_2$ of the cobordism $\mathbb{X}$ have opposite Sobolev weights in the description of D_A. If we instead view $\mathbb{X}:\emptyset\to\mathbb{Y}_2\sqcup\overline{\mathbb{Y}}_1$ then the construction yields a different operator D_A'. That is, D_A' differs from D_A by using the weight function ϕ' in place of ϕ, where ϕ' is obtained by altering ϕ over $\mathbb{R}_{\leq 0}\times Y_1$ from $e^{-\epsilon t}$ to $e^{+\epsilon t}$. We have the relation

$$\mathrm{ind}(D_A')-\mathrm{ind}(D_A)=h^0(a)+h^1(a),$$

cf. [7, Prop. 3.10]. When there is one cylindrical end, the number $\mathrm{ind}(D_A')$ is the same as $\mathrm{ind}^-(A)$ in the notation of [5] and $\mathrm{ind}^+(A)$ in the notation of [7].

The index $\mathrm{ind}(D'_A)$ is the expected dimension of the moduli space $M(a, \mathbb{X}, b)^{\mathrm{irr}}$ of irreducible instanton classes. It is this number that we refer to in computations, so we define

$$\mu(a, \mathbb{X}, b) = \mu(A) = \mathrm{ind}(D'_A),$$

and this agrees with our earlier usage of $\mu(a, \mathbb{X}, b)$. Note that the order of the symbols $a, \mathbb{X}, b$ does not matter, and is only suggestive of the situation in mind. If $\mathbb{X}_1$ and $\mathbb{X}_2$ are bundles over cobordisms and are composable, we have the gluing formula

$$\mu(a, \mathbb{X}_2 \circ \mathbb{X}_1, c) = \mu(a, \mathbb{X}_1, b) + \mu(b, \mathbb{X}_2, c) + h^0(b) + h^1(b). \tag{2.3}$$

If $\mathbb{X}$ is over a closed 4-manifold X then we also have

$$\mu(\mathbb{X}) = -2p_1(\mathbb{X}) - 3(1 - b_1(X) + b_+(X)). \tag{2.4}$$

Here $b_+(X)$ is the dimension of a maximal positive definite subspace for the intersection form on $H_2(X; \mathbb{R})$. The term $1 - b_1(X) + b_+(X)$ may also be written as $(\chi(X) + \sigma(X))/2$, where χ is the Euler characteristic and σ the signature.

2.4 Maps from families of metrics on cobordisms

This section extracts formulae due to Kronheimer and Mrowka from [20, §3.9]. We first consider families of metrics in a general context. Let X be any smooth manifold and S a hypersurface in the interior of X. We assume S has a neighborhood $N \subset X$ diffeomorphic to $(-1, 1) \times S$. A *metric on X cut along S* is a Riemannian metric g on $X \setminus S$ that on the neighborhood N is of the form

$$dr^2/r^2 + g_0,$$

where g_0 is a metric on S and r is the parameter of $(-1, 1)$. We also call g simply a *cut metric*. We may regard a Riemannian manifold with a cut metric as one with two opposing cylindrical ends that along the cut hypersurface S meet only at infinity.

Given a collection of hypersurfaces $\mathcal{H} = \{S_i\}$ in the interior of X with similar neighborhoods we construct a set of metrics $G = G(\mathcal{H})$ on X that are cut along various subsets of $\mathcal{H}$. The construction is intuitively simple: stretch an initial metric in all possible ways along each hypersurface.

First, suppose that $\mathcal{H}$ has no intersecting hypersurfaces. We will parameterize the family G by $[0,1]^d$ where $d = |\mathcal{H}|$. Let b_t be a family of positive smooth functions on $[-1,1]$ parameterized smoothly by $t \in [0,1)$ such that $b_t(r)$ approaches $1/r^2$ as t goes to 1. For some fixed ε, $0 < \varepsilon < 1$, we require that $b_t(r) = 1$ for $|r| > \varepsilon$. We also require $b_t \neq b_s$ when $t \neq s$. We choose the initial metric $G(0)$ on X so that it is of the form $dr^2 + g_i$ in the neighborhood of $S_i \subset X$ diffeomorphic to $(-1,1) \times S_i$. Here $S_i \in \mathcal{H}$ and g_i is a metric on S_i. For $t \in [0,1]^d$ we define $G(t)$ on X by changing $G(0)$ in the neighborhood of S_i to $b_{t_i}(r)dr^2 + g_i$.

Now consider an arbitrary set of hypersurfaces $\mathcal{H}$. Let $\mathcal{H}_0$ be a subset of $\mathcal{H}$ with no intersecting hypersurfaces. We have constructed a family $G(\mathcal{H}_0)$ for each such $\mathcal{H}_0$. We glue the hypercubes $[0,1]^{d_0}$ where $d_0 = |\mathcal{H}_0|$ together to form a space in the obvious way: when two points correspond to the same metric, identify them. This defines the family $G(\mathcal{H})$.

Now suppose $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ as in §2.2. Let $G = G(\mathcal{H})$ be a family of metrics on X constructed as above. We extend G to a family of metrics on X with cylindrical ends attached, product-like on the ends, which we also call G. Let $M_G(a, \mathbb{X}, b)$ be the moduli space of pairs $([A], g)$ where $g \in G$ and A is a finite-energy instanton with respect to g. The meaning of this is staightforward if g is an uncut, smooth metric. An instanton with a metric cut along $S \subset X$ is an instanton on the complement of S, with its limits on the two cylindrical ends $[0, \infty) \times \mathrm{S}$ agreeing. More details can be found in [20, §3.9].

Let $G = G(\mathcal{H})$ be a family of metrics on X as constructed above. In the cases in which we are interested, G will have the structure of a convex polytope. The metrics parameterized by a face of G consist of cut metrics, cut along a hypersurface in $\mathcal{H}$. The expected dimension of $M_G(a, \mathbb{X}, b)$ is $\mu(a, \mathbb{X}, b) + \dim G$. We then obtain a map

$$m_G(\mathbb{X}) : \mathrm{C}(\mathbb{Y}_1) \to \mathrm{C}(\mathbb{Y}_2),$$

defined just as for cobordisms. To fix the sign of $m_G(\mathbb{X})$, in addition to an I-orientation of $\mathbb{X}$, we must orient the metric family G. The following three formulae are due to Kronheimer and Mrowka, [20, §3.9], and arise from understanding the compactification and gluing of certain moduli spaces. First,

$$(-1)^{\dim G} m_G(\mathbb{X})\partial - \partial m_G(\mathbb{X}) = m_{\partial G}(\mathbb{X}). \tag{2.5}$$

In writing this we have inherited the orientation conventions of [20], with the exception that the quotients $\check{M}(a,b)$ are oriented oppositely, changing the signs of the maps ∂. For the polytopes G that we will consider, ∂G decomposes into a union of faces $G(S)$, one for each hypersurface $S \in \mathcal{H}$. In this case

$$m_{\partial G}(\mathbb{X}) = \sum_{S\in\mathcal{H}} m_{G(S)}(\mathbb{X}). \tag{2.6}$$

Finally, suppose $\mathbb{X}$ is the composite of two bundle cobordisms: $\mathbb{X} = \mathbb{X}_2 \circ \mathbb{X}_1$. Also suppose that $G = G_1 \times G_2$ where G_1 is a family of metrics that only varies on X_1 and G_2 on X_2, and all metrics are cut along $X_1 \cap X_2$. Then

$$m_G(\mathbb{X}) = (-1)^{\dim G_1 \dim G_2} m_{G_2}(\mathbb{X}_2) m_{G_1}(\mathbb{X}_1) \tag{2.7}$$

where we interpret G_1 as a family of metrics on X_1 and G_2 as a family on X_2. Here the metric families are oriented, and $G = G_1 \times G_2$ is an orientation preserving identification.

2.5 Index bounds

The following discussion is based on [5, §3.4] and [7, §4], with the material of [20, §3.9] in mind. So far we have only mentioned moduli spaces for which the limiting connections are acyclic. This guarantees, in particular, that all instantons are irreducible.

For simplicity, suppose $\mathbb{X}$ has one cylindrical end. We consider moduli spaces $M(\mathbb{X}, a)$ where a is any almost flat connection (i.e., an element of $\mathfrak{C}$), where the finite-energy instantons exponentially approach a over the cylindrical end. Then, with suitable perturbation, the subset of irreducibles $M(\mathbb{X}, a)^{\text{irr}}$ is a smooth manifold of dimension $\mu(\mathbb{X}, a)$. In this case, the existence of $[A] \in M(\mathbb{X}, a)^{\text{irr}}$ implies $\mu(\mathbb{X}, a) = \mu(A) \geq 0$. On the other hand, if all the instantons are reducible with common isotropy group Γ, the space $M(\mathbb{X}, a)$ has dimension $\mu(\mathbb{X}, a) + \dim \Gamma$. Recall $h^0(A) = \dim \Gamma$. In this case, after perturbation, the existence of an instanton $[A]$ in the moduli space implies the bound

$$\mu(A) + h^0(A) \geq 0. \tag{2.8}$$

More generally, suppose $([A], g) \in M_G(\mathbb{X}, a)$ for a family of metrics G. Then

$$\mu(A) + h^0(A) + \dim G \geq 0. \tag{2.9}$$

We also consider the case in which some of the limiting connections are allowed to vary. Suppose $[0, \infty) \times \mathbb{Y}$ is the cylindrical end of $\mathbb{X}$, and consider a smooth manifold $\mathfrak{F} \subset \mathfrak{C}(\mathbb{Y})$ of critical points to which the Chern-Simons functional is non-degenerate transverse. We consider $M(\mathbb{X}, \mathfrak{F})$, the instanton classes that exponentially approach the set $\mathfrak{F}$. The irreducibles within typically form a smooth manifold whose components have dimensions mod 8 congruent to $\mu(\mathbb{X}, \mathfrak{a}) + \dim \mathfrak{F}$, where $\mathfrak{a} \in \mathfrak{F}$. We write $M(\mathbb{X}, \mathfrak{F})^{\text{irr}}_d$ for the d-dimensional component.

We can introduce metrics into all of these situations. The most general situation we consider is the following. Suppose $\mathfrak{F}$ is as above, and consider the moduli space $M_G(\mathbb{X}, \mathfrak{F})$. If $([A], g)$ is a member, in the generic case we obtain a bound

$$\mu(A) + h^0(A) + \dim G + \dim \mathfrak{F} \geq 0. \tag{2.10}$$

We write $M_G(\mathbb{X}, \mathfrak{F})^{\circ}_d$ for the d-dimensional moduli space of instantons $([A], g)$ with d equal to the left side of (2.10) and where $\circ = \text{irr}, \text{red}, \text{flat}$ describes the respective stabilizer-types $h^0(A) = 0, 1, 3$. One can drop the assumption that $\mathfrak{F}$ is smooth and obtain moduli spaces that are stratified according to the structure of $\mathfrak{F}$. Such spaces have been studied in [39, 30].

2.6 Gradings

In addition to the relative $\mathbb{Z}/8$-grading on $I(\mathbb{Y})$, we can define an absolute $\mathbb{Z}/2$-grading following [16, §2.1] and [7, §5.6]. It is more generally defined on the critical sets $\mathfrak{C}$. If $\mathfrak{a} \in \mathfrak{C}$, its grading is given by

$$\mathrm{gr}(\mathfrak{a}) = b_1(E) + b_+(E) + \mu(\mathbb{E}, \mathfrak{a}) \mod 2,$$

where $\mathbb{E} : \emptyset \to \mathbb{Y}$ is an SO(3)-bundle over a connected 4-manifold E with $\partial E = Y$ that restricts to $\mathbb{Y}$ over Y. The differential of $\mathrm{C}(\mathbb{Y})$ shifts this grading by 1. A map $m(\mathbb{X}) : \mathrm{C}(\mathbb{Y}_1) \to \mathrm{C}(\mathbb{Y}_2)$ shifts the grading by the parity of

$$\deg(X) = -\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)), \tag{2.11}$$

cf. [20, §4.5]. More generally, a map $m_G(\mathbb{X})$ shifts the grading by $\deg(X) + \dim G$. As an example, suppose $\mathbb{T}^3$ is the bundle over T^3 with $w_2(\mathbb{T}^3)$ Poincaré dual to an S^1-factor. Then $I(\mathbb{T}^3)$ is two copies of $\mathbb{Z}$ supported in the even grading. Note that the trivial connection θ on S^3 has $\mathrm{gr}(\theta) \equiv 1$. We note that $I(\overline{\mathbb{Y}})_i$ is the same as the cohomology group $I(\mathbb{Y})^{b_1(Y)+1+i}$, where $\overline{\mathbb{Y}}$ means the orientation of the base space Y is reversed. For our conventions regarding the absolute $\mathbb{Z}/8$-grading in the case that Y is a homology 3-sphere, see §7.1.

CHAPTER 3

Surgery in instanton homology

The instanton surgery exact triangle was introduced by Floer [14] soon after his inception of instanton homology. Let K be a framed knot in a closed, connected, oriented 3-manifold Y. That is, K has a preferred meridian and longitude. Let ω be a *geometric representative* for $\mathbb{Y}$ – that is, an unoriented, embedded 1-manifold in Y with $[\omega] \in H_1(Y;\mathbb{F}_2)$ Poincaré-dual to $w_2(\mathbb{Y})$. Choose ω disjoint from K. Denote by Y_0 and Y_1 the results of performing 0- and 1-surgery on K, respectively. We can view ω inside Y_0 and Y_1 by keeping it away from the surgery neighborhood. Let $\omega_0 = \omega \cup \mu \subset Y_0$ where μ is a core for the induced framed knot in Y_0, and let $\omega_1 = \omega \subset Y_1$. Finally, for $i = 0, 1$, choose a bundle $\mathbb{Y}_i$ over Y_i geometrically represented by ω_i. If the ordered triple of bundles $\mathbb{Y}, \mathbb{Y}_0, \mathbb{Y}_1$ can be geometrically represented this way, we say they form a *surgery triad.* The exact triangle is

Theorem 3.0.1 (Floer)**.** *There is an exact sequence*

$$\cdots I(\mathbb{Y}) \to I(\mathbb{Y}_0) \to I(\mathbb{Y}_1) \to I(\mathbb{Y}) \cdots$$

provided all three bundles are admissible and form a surgery triad.

The loop μ in Y_0, pushed out of the surgery solid torus, becomes a small meridional loop around the surgered neighborhood of K in Y_0. This is depicted in the top row of Figure 3.1 in a local surgery diagram for Y_0. One can view Y_1 (resp. Y) as obtained from 0-surgery on the induced framed knot in Y_0 (resp. Y_1), see §3.1. Thus we obtain two more local surgery diagram depictions of where μ may be placed, listed in the bottom two rows of Figure 3.1. See also §3.1.7.

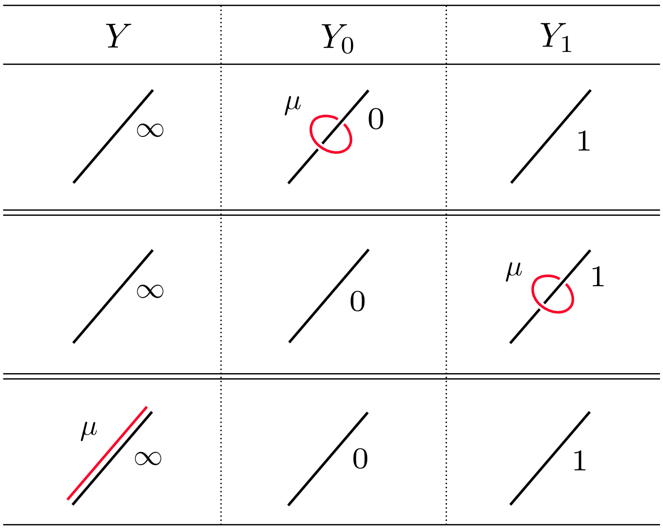

Figure 3.1: Local surgery diagrams. The slanted line in each case is the knot K. Each row represents a possible construction for a surgery triad.

Floer's exact triangle was studied by Braam and Donaldson in [5], where a detailed proof following Floer's ideas can be found. In this chapter we provide an alternative proof. The proof relies on an algebraic lemma which was first used by Ozsváth and Szabó [35]. The lemma requires the input of maps between the three relevant chain complexes satisfying certain properties. The maps we choose count instantons on families of metrics that are parameterized by convex polytopes. This approach was used by Kronheimer, Mrowka, Ozsváth and Szabó to prove a surgery exact sequence in the monopole case [24]. Our proof is largely an adaptation of Kronheimer and Mrowka's proof in [20] of an analogous exact triangle in singular instanton knot homology.

This method of proof leads to a generalization of Floer's theorem to a so-called link surgeries spectral sequence, as was first done by Ozsváth and Szabó in Heegaard Floer homology [35]. Let L be a framed link in Y with components $L_1, \ldots, L_m$. For each $v \in \{0, 1, \infty\}^m$ let Y_v be the result of v_i surgery on L_i for $1 \leq i \leq m$. Briefly, we say Y_v is the result of v-surgery on L. Choose a geometric representative ω for $\mathbb{Y}$ disjoint from L. Let $\omega_v \subset Y_v$ be ω together with a core for the knot in Y_v induced by L_i for each i with $v_i = 0$. Let $\mathbb{Y}_v$ be bundles over Y_v geometrically represented by the ω_v. If the bundles $\mathbb{Y}_v$ can be geometrically represented according to these rules we say that they form a *surgery cube.*

Theorem 3.0.2. *Suppose the bundles $\mathbb{Y}_v$ for $v \in \{0,1,\infty\}^m$ are admissible and that they form a surgery cube. Then there is a spectral sequence*

$$E^1 = \bigoplus_{v\in\{0,1\}^m} I(\mathbb{Y}_v) \quad \rightsquigarrow \quad I(\mathbb{Y}).$$

That is, the left side is the E^1-page and the sequence converges to the right side.

A more detailed statement is provided in Theorem 3.3.1. An analogous result in monopole Floer homology was proved by Bloom [3] with $\mathbb{F}_2$-coefficients, and in singular instanton knot homology by Kronheimer and Mrowka [20].

The goal of this chapter is to prove Theorems 3.0.1 and 3.0.2. In §3.1 we discuss how Dehn surgery extends to SO(3)-bundles over 3-manifolds, from which we construct the bundles and cobordisms that appear in Theorem 3.0.1. In §3.2 we prove Theorem 3.0.1 and in §3.3 we prove Theorem 3.0.2.

3.1 Bundles in the exact triangle

In this section we introduce the manifolds and bundles that feature in the proof of Floer's exact triangle. We take a systematic approach to the bundles $\mathbb{Y}_i$ that appear in Floer's exact triangle by extending Dehn surgery to SO(3)-bundles. This viewpoint was Floer's [14], and is expanded upon in [5]. The construction of surgery cobordism bundles $\mathbb{X}_{ij}$ in §3.1.3 is straightforward in this setting. These bundles induce the maps in the exact triangle. We then introduce some hypersurfaces in X_{ij} that yield useful metric families; these were used in [24, 3, 20]. In §3.1.7 we relate our new setup to that of geometric representatives.

In this section, we write $A \cup_f B$ for the space obtained from the disjoint union of A and B, with points identified using the map f. Our convention is that the gluing map

f is always from a subset of B to a subset of A. We freely use isomorphisms of the form $A \cup_f B \simeq A \cup_{fg} C$, where g is an isomorphism from a subset of C to a subset of B. All constructions that are not smooth have a canonical smoothing, as mentioned in [18, Rmk. 1.3.3]. All (principal) SO(3)-bundles have right actions. Thus our bundle gluing maps, in order to be equivariant, always involve left multiplication on trivialized fibers.

3.1.1 Dehn surgery with bundles

Let $\mathbb{Y}$ be an SO(3)-bundle over a closed, oriented 3-manifold Y. Let $K : S^1 \times D^2 \to Y$ be an embedding. We refer to K as a framed knot in Y. We consider equivariant embeddings $\mathbb{K} : S^1 \times D^2 \times \mathrm{SO}(3) \to \mathbb{Y}$ that lie above K, i.e. $\mathbb{K}/\mathrm{SO}(3) = K$. We refer to $\mathbb{K}$ as a framed knot in $\mathbb{Y}$. The space of bundle automorphisms of $S^1 \times D^2 \times \mathrm{SO}(3)$ fixing the base space has two connected components. An automorphism τ not isotopic to the identity is

$$\tau(w, z, a) = (w, z, c(w)a)$$

where $(w, z) \in S^1 \times D^2$, $a \in \mathrm{SO}(3)$, and c is a standard inclusion $S^1 \to \mathrm{SO}(3)$ of a maximal torus. In particular, c is a homomorphism and generates $\pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}/2$. If $\mathbb{K}$ is one embedding, another embedding lying above K is given by $\mathbb{K}\tau$.

We generalize Dehn surgery to surgery on the framed knots $\mathbb{K}$. For $\Omega = (A, b) \in \mathrm{SL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2)^2$ we define an automorphism ψ_Ω of $S^1 \times \partial D^2 \times \mathrm{SO}(3)$ by

$$\psi_\Omega(w, z, a) = (w^{A_{11}} z^{A_{12}}, w^{A_{21}} z^{A_{22}}, c(w)^{b_1} c(z)^{b_2} a).$$

Let $\mathbb{K}'$ be the interior of the image of $\mathbb{K}$. The result of Ω-surgery on $\mathbb{K}$ is then defined to be the identification space

$$\mathbb{Y}_\Omega(\mathbb{K}) = (\mathbb{Y} \setminus \mathbb{K}') \cup_{\mathbb{K}\psi_\Omega} (S^1 \times D^2 \times \mathrm{SO}(3)).$$

There is an induced framed knot $\Omega(\mathbb{K})$ in $\mathbb{Y}_\Omega(\mathbb{K})$ given by the inclusion of $S^1 \times D^2 \times \mathrm{SO}(3)$ into the above expression for $\mathbb{Y}_\Omega(\mathbb{K})$. The product of elements in $G =$

$\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2)^2$ is given by $(A',b')(A,b) = (A'A, b'A+b)$. The assignment $\Omega \mapsto \psi_\Omega$ induces an isomorphism from G to the group of isotopy classes of orientation preserving equivariant automorphisms of $S^1 \times \partial D^2 \times \mathrm{SO}(3)$. Note that we have an associativity rule

$$\mathbb{Y}_{\Omega'\Omega}(\mathbb{K}) \simeq (\mathbb{Y}_{\Omega'}(\mathbb{K}))_\Omega(\Omega'(\mathbb{K})).$$

The space $\mathbb{Y}_\Omega(\mathbb{K})$ is naturally a bundle over $Y_{p/q}(K)$, the result of p/q Dehn surgery on the framed knot K in Y, where $\Omega = (A,b)$, $p = A_{22}$, $q = A_{12}$ and of course $K = \mathbb{K}/\mathrm{SO}(3)$. Note that the automorphism τ above restricts to ψ_Θ where $\Theta = (1_{2\times 2}, (1,0)) \in G$. We have the transformation rule $\mathbb{Y}_\Omega(\mathbb{K}\tau) \simeq \mathbb{Y}_{\Theta\Omega}(\mathbb{K})$.

3.1.2 The surgery bundle $\mathbb{Y}_i$

There is a particular choice of surgery parameter Ω that Floer used in the setting of his exact triangle:

$$\Lambda = \left(\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, (1,0) \right). \tag{3.1}$$

To understand this, write $\Lambda = \Psi\Lambda'$, where

$$\Psi = (1_{2\times 2}, (0,1)), \quad \Lambda' = \left(\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, (0,0) \right). \tag{3.2}$$

First, Ψ twists the trivialization around ∂D^2. Then, Λ' performs 0-surgery on K, leaving bundles alone. Note that $\Lambda^3 = 1$. With $\mathbb{Y}$ and $\mathbb{K}$ fixed, we define for $i \in \mathbb{Z}$ the surgery bundles $\mathbb{Y}_i = \mathbb{Y}_{\Lambda^{i+1}}(\mathbb{K})$, the surgery base manifolds $Y_i = \mathbb{Y}_i/\mathrm{SO}(3)$, and the induced embeddings $\mathbb{K}_i = \Lambda^{i+1}(\mathbb{K})$. The index offset is here so that Y_0 and Y_1 are simply 0- and 1-surgery on $K \subset Y$, respectively. Because $\Lambda^3 = 1$, there are isomorphisms $\mathbb{Y}_i \simeq \mathbb{Y}_{i+3}$.

3.1.3 The surgery cobordism $\mathbb{X}_{ij}$

Our goal is to construct cobordism bundles $\mathbb{X}_{ij} : \mathbb{Y}_i \to \mathbb{Y}_j$ for $i < j$. Each $\mathbb{X}_{ij}$ will be an SO(3)-bundle over a standard surgery cobordism $X_{ij} : Y_i \to Y_j$. We first construct

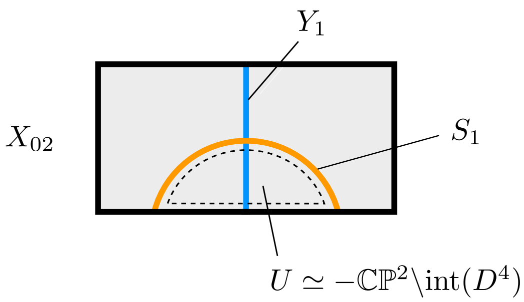

Figure 3.2: The two hypersurfaces Y_1 and S_1 in the interior of X_{02}. The 3-sphere S_1 separates off a copy of $-\mathbb{CP}^2$ minus a 4-ball.

$\mathbb{X}_{ij}$ when $j - i = 1$ and use these as building blocks for the general construction. Write $\mathbb{H} = D^2 \times D^2 \times \mathrm{SO}(3)$. We view $\mathbb{H}$ as a 2-handle thickened by $\mathrm{SO}(3)$. Write

$$\partial\mathbb{H} = \mathbb{H}_1 \cup \mathbb{H}_2,$$
$$\mathbb{H}_1 = \partial D^2 \times D^2 \times \mathrm{SO}(3),$$
$$\mathbb{H}_2 = D^2 \times \partial D^2 \times \mathrm{SO}(3).$$

Viewing $\mathbb{K}_0$ as a map $\mathbb{H}_1 \to \{1\} \times \mathbb{Y}_0$, we define $\mathbb{X}_{01}$ by setting

$$\mathbb{X}_{01} = ([0,1] \times \mathbb{Y}_0) \cup_{\mathbb{K}_0} \mathbb{H}.$$

The definition of $\mathbb{X}_{ij}$ for general $j - i = 1$ is similar. We want to define $\mathbb{X}_{02}$ as $\mathbb{X}_{01} \cup_{\mathbb{Y}_1} \mathbb{X}_{12}$. To make sense of this expression we give an explicit identification of $\partial\mathbb{X}_{01} \setminus \mathbb{Y}_0$ with $\mathbb{Y}_1$. Let the interior of the image of $\mathbb{K}_0$ in $\mathbb{Y}_0$ be denoted $\mathbb{K}_0'$. Note

$$\partial\mathbb{H}_1 = \mathbb{H}_1 \cap \mathbb{H}_2 = \partial\mathbb{H}_2$$

is a trivial bundle over a 2-torus. Now we write

$$\partial\mathbb{X}_{01} \setminus \mathbb{Y}_0 = (\mathbb{Y}_0 \setminus \mathbb{K}_0') \cup_{\mathbb{K}_0|\mathbb{H}_1\cap\mathbb{H}_2} \mathbb{H}_2.$$

Let $\psi : \mathbb{H}_1 \to \mathbb{H}_2$ be an isomorphism. Then

$$\partial\mathbb{X}_{01} \setminus \mathbb{Y}_0 \simeq (\mathbb{Y}_0 \setminus \mathbb{K}_0') \cup_{\mathbb{K}_0\psi|\mathbb{H}_1\cap\mathbb{H}_2} \mathbb{H}_1 = (\mathbb{Y}_0)_{\psi|\mathbb{H}_1\cap\mathbb{H}_2}(\mathbb{K}_0).$$

To identify this bundle with $\mathbb{Y}_1 = (\mathbb{Y}_0)_\Lambda(\mathbb{K}_0)$ we need ψ such that $\psi|_{\mathbb{H}_1 \cap \mathbb{H}_2} = \psi_\Lambda$. For this we choose

$$\psi : \mathbb{H}_1 \to \mathbb{H}_2, \quad \psi(w, z, a) := (\overline{w}z, \overline{w}, c(w)a).$$

Making this choice, we have identified $\partial\mathbb{X}_{01} \setminus \mathbb{Y}_0$ with $\mathbb{Y}_1$. Finally, to construct $\mathbb{X}_{ij}$ for $j - i > 1$, we inductively define $\mathbb{X}_{ij} = \mathbb{X}_{i,j-1} \cup_{\mathbb{Y}_{j-1}} \mathbb{X}_{j-1,j}$, where the gluing is done according to the same identification process.

3.1.4 The bundle $\mathbb{S}_i$

We construct a subset $\mathbb{S}_1 \subset \mathbb{X}_{02}$ which is a bundle over a 3-sphere $S_1 \subset X_{02}$. One gets $\mathbb{S}_i$ inside $\mathbb{X}_{i-1,i+1}$ for each i in a similar fashion. Write

$$\mathbb{X}_{02} = ([0,1] \times \mathbb{Y}_0 \cup_{\mathbb{K}_0} \mathbb{H}) \cup_{\mathbb{Y}_1} ([0,1] \times \mathbb{Y}_1 \cup_{\mathbb{K}_1} \mathbb{H}) \simeq ([0,1] \times \mathbb{Y}_0) \cup_{\mathbb{K}_0} \mathbb{H} \cup_\psi \mathbb{H} \quad (3.3)$$

with notation as in the construction of $\mathbb{X}_{01}$. Introduce the subset

$$\mathbb{H}(r, s) = D^2(r) \times D^2(s) \times \mathrm{SO}(3) \subset \mathbb{H}$$

where $D^2(r)$ is the disk of radius r, $0 < r \leq 1$, and consider the following restriction bundles of $\mathbb{X}_{02}$:

$$\mathbb{U} = \mathbb{H}(1/2, 1) \cup \mathbb{H}(1, 1/2) \subset \mathbb{H} \cup_\psi \mathbb{H}, \quad \mathbb{S}_1 = \partial\mathbb{U}.$$

It is well-known that the base space U of $\mathbb{U}$ is diffeomorphic to $-\mathbb{CP}^2$ minus an embedded 4-ball, cf. [18, Ex. 4.2.4]. It follows that $\mathbb{S}_1$ is a trivial bundle over a 3-sphere S_1. We see that we can decompose X_{02} along S_1 into a connected sum of $-\mathbb{CP}^2$ with a manifold whose boundary is $Y_2 \sqcup \overline{Y}_0$. The intersection $S_1 \cap Y_1$ is 2-torus. This decomposition is depicted in Figure 3.2.

We claim that $\mathbb{U}$ is a non-trivial bundle. We check that the restriction of $\mathbb{U}$ to an essential sphere is non-trivial. Define $\mathbb{D}_1 = D^2 \times \{0\} \times \mathrm{SO}(3)$ and $\mathbb{D}_2 = \{0\} \times D^2 \times \mathrm{SO}(3)$ as subsets of $\mathbb{H}$. Consider

$$\mathbb{D}_2 \cup_{\psi|\partial\mathbb{D}_1} \mathbb{D}_1 \subset \mathbb{U}.$$

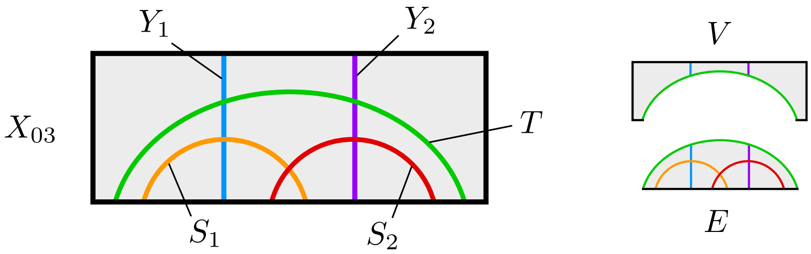

Figure 3.3: The intersections of the five hypersurfaces in the interior of X_{03}. The $S^1 \times S^2$ hypersurface T divides X_{03} into two pieces, V and E. This picture first appeared in [24].

This is isomorphic to $D^2 \times \mathrm{SO}(3) \cup_f D^2 \times \mathrm{SO}(3)$ where f is the automorphism of $\partial D^2 \times \mathrm{SO}(3)$ given by $f(z,a) = (\overline{z}, c(z)a)$. This is a nontrivial bundle over a 2-sphere.

3.1.5 The bundle $\mathbb{T}$

We construct a subset $\mathbb{T} \subset \mathbb{X}_{03}$ which is a trivial bundle over $T \subset X_{03}$ where T is diffeomorphic to $S^1 \times S^2$. By iterating (3.3) and stretching the ends we write

$$\mathbb{X}_{03} \simeq ([0,1] \times \mathbb{Y}_0) \cup_{\mathbb{K}_0} \mathbb{H} \cup_\psi \mathbb{H} \cup_\psi \mathbb{H} \cup_{\mathbb{Y}_0} ([0,1] \times \mathbb{Y}_0).$$

Identifying $\mathbb{X}_{03}$ with the expression on the right, we define the restriction bundles

$$\mathbb{E} = \mathbb{H} \cup_\psi \mathbb{H} \cup_\psi \mathbb{H}, \quad \mathbb{T} = \partial \mathbb{E},$$

and their respective base spaces E and T. We have an isomorphism

$$f : \mathbb{T} = \mathbb{H}_1 \cup_{(\psi|_{\mathbb{H}_1 \cap \mathbb{H}_2})^2} \mathbb{H}_2 \to S^1 \times S^2 \times \mathrm{SO}(3) \tag{3.4}$$

where, viewing $S^2 \subset S^1 \times S^2$ as $\mathbb{C} \cup \infty$, we set

$$f|_{\mathbb{H}_1} = \mathrm{id}, \quad f|_{\mathbb{H}_2}(z,w,a) = (\overline{w}, \overline{w}/\overline{z}, c(w)a).$$

The triviality of the bundle $\mathbb{T}$ is also seen from the observation that it is the restriction of a bundle on a space in which T is contractible. We note that we could have also trivialized $\mathbb{T}$ by using a similar isomorphism in which $f|_{\mathbb{H}_2} = \mathrm{id}$. These two isomorphisms determine trivializations that differ by a non-even gauge transformation.

We remark that the intersections $T \cap Y_1$ and $T \cap Y_2$ are 2-tori. We illustrate the arrangement of intersections in Figure 3.3. We note that T may be described as the boundary of a regular neighborhood of the union of the two essential spheres inside the copies of $-\mathbb{CP}^2$ divided off by S_1 and S_2. The hypersurface T separates X_{03} into two 4-manifolds, E and V, where E is diffeomorphic to $-\mathbb{CP}^2$ minus a neighborhood of an unknotted circle, and V is diffeomorphic to $[0, 1] \times Y_0$ minus a neighborhood of $\{1/2\} \times K$.

3.1.6 An involution of $\mathbb{E}$

We construct an involution $\sigma : \mathbb{E} \to \mathbb{E}$. We write

$$\mathbb{E} = \mathbb{H}^{-1} \cup_\psi \mathbb{H}^0 \cup_\psi \mathbb{H}^{+1}$$

where the superscripts have been added to distinguish the copies of $\mathbb{H}$. We write $[w, z, a, i] \in \mathbb{E}$ for the point represented by $(w, z, a) \in \mathbb{H}^i$. Define σ by

$$\sigma[w, z, a, i] = [\overline{z}, \overline{w}, c(w)^{i(i+1)} c(z)^{i(i-1)} a, -i].$$

Here we have extended $c^2 : S^1 \to \mathrm{SO}(3)$ to a map $c^2 : D^2 \to \mathrm{SO}(3)$ such that $c^2(\overline{w}) = (c^2(w))^{-1}$. Note that σ interchanges the outer copies of $\mathbb{H}$ and fixes the middle copy of $\mathbb{H}$. It is straightforward that σ is well-defined: writing σ as three maps $\sigma_i : \mathbb{H}^{+i} \to \mathbb{H}^{-i}$, one uses the relations

$$\sigma_0^2 = \mathrm{id}, \quad \sigma_{\pm 1} = \psi^{\pm 1} \sigma_0 \psi^{\pm 1}, \quad \psi^3 = \mathrm{id},$$

whenever these compositions are defined. The involution σ is a bundle automorphism that restricts to an orientation-preserving diffeomorphism of E. It fixes $\mathbb{T}$ and swaps $\mathbb{S}_1$ with $\mathbb{S}_2$.

Let us look at how the involution σ affects $\mathbb{T}$. Recall the isomorphism (3.4). We have the relation

$$f \sigma f^{-1}(w, z, a) = (w, w/z, c(\overline{w}) d(z) a) \tag{3.5}$$

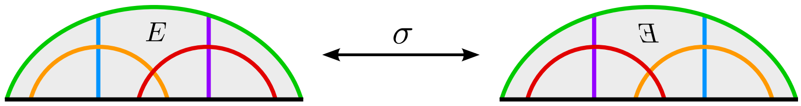

Figure 3.4: The involution σ.

where $w \in S^1, z \in \mathbb{C} \cup \infty$, $a \in \mathrm{SO}(3)$ and $d : S^2 \to \mathrm{SO}(3)$ is the double of $c^2 : D^2 \to \mathrm{SO}(3)$. It is easily seen that $f \sigma f^{-1}$ is isotopic to $\theta \circ \upsilon$, where

$$\theta(w, z, a) = (w, w/z, a), \quad \upsilon(w, z, a) = (w, z, c(w)a).$$

The map θ is a diffeomorphism of $S^1 \times S^2$ that, with respect to our trivialization, is extended in trivial way to the overlying bundle. Note υ is a non-even gauge transformation of the trivial bundle over $S^1 \times S^2$. The involution σ will be useful in the proof of the exact triangle.

3.1.7 Geometric representatives

Let ω be an embedded loop in Y. Extend this to an embedding $\mathbb{K}_\omega : S^1 \times D^2 \times \mathrm{SO}(3) \to Y \times \mathrm{SO}(3)$. Let $\Psi = (1_{2\times 2}, (0, 1))$ as in (3.2). Then the result of Ψ-surgery on $\mathbb{K}_\omega$ as a framed knot in $Y \times \mathrm{SO}(3)$ is a bundle geometrically represented by ω. More generally, ω can be a collection of embedded loops, and Ψ-surgery for each component gives a bundle geometrically represented by ω.

This relates our current framework to the statement of the theorems in the introduction to this chapter. Let ω be a closed, unoriented 1-manifold in Y, and K a framed knot in Y disjoint from ω. We set

$$\mathbb{Y} = (Y \times \mathrm{SO}(3))_\Psi(\mathbb{K}_\omega), \tag{3.6}$$

where it is understood that if ω has multiple components, we do Ψ-surgery for each component. This description of $\mathbb{Y}$ gives a preferred trivialization away from a neighborhood of ω. We let $\mathbb{K}$ be the SO(3)-thickening of K using this preferred data, precomposed with τ. That is,

$$\mathbb{K} = (K \times \mathrm{id}_{\mathrm{SO}(3)})\tau.$$

Recall that τ restricts to ψ_Θ where $\Theta = (1_{2\times 2}, (1,0))$, and that $\mathbb{Y}_0$ is defined as $\mathbb{Y}_\Lambda(\mathbb{K})$. Using $\Theta\Lambda = \Lambda'\Psi$ with notation as in (3.2), we have

$$\mathbb{Y}_0 \simeq \mathbb{Y}_{\Lambda'\Psi}(K \times \mathrm{id}_{\mathrm{SO}(3)}).$$

Because Λ' is 0-surgery without bundle-twisting, we see $\mathbb{Y}_0$ is of the form (3.6), where Y is replaced by Y_0 and ω replaced by $\omega \cup K_0$, where K_0 is the induced knot in Y_0. Thus $\mathbb{Y}_0$ is geometrically represented by $\omega \cup K_0$. Pushing K_0 away from the surgered neighborhood makes it a small meridional loop μ as in Figure 3.1, by the nature of 0-surgery.

We may deduce that $\mathbb{Y}_1$ is geometrically represented by $\omega \subset Y_1$ by either of two ways. First, we may interpret Y_1 as 0-surgery on the induced knot $K_0 \subset Y_0$ and iterate the rule already established, forgetting about bundles altogether. Alternatively, we can repeat the above argument for Λ^2 in place of Λ. The difference in this case is that $\Theta\Lambda^2 = (\Lambda')^2$. This is 1-surgery on K without bundle-twisting.

3.2 Proving the exact triangle

In this section we prove Theorem 3.0.1, Floer's exact triangle.

3.2.1 The triangle detection lemma

The following statement is adapted from [20, §7.1] and first appeared in [35].

Lemma 3.2.1. *Let $(\mathrm{C}_i, \partial_i)$ be a sequence of complexes, $i \in \mathbb{Z}$. Suppose that there are chain maps $f_i : \mathrm{C}_i \to \mathrm{C}_{i+1}$ and maps $h_i : \mathrm{C}_i \to \mathrm{C}_{i+2}$ satisfying*

$$f_{i+1}f_i + \partial_{i+2}h_i + h_i\partial_i = 0.$$

Suppose further that each sum

$$f_{i+2}h_i + h_{i+1}f_i$$

induces an isomorphism $H(\mathrm{C}_i) \to H(\mathrm{C}_{i+3})$. *Then*

$$\cdots \to H(\mathrm{C}_i) \xrightarrow{H(f_i)} H(\mathrm{C}_{i+1}) \xrightarrow{H(f_{i+1})} H(\mathrm{C}_{i+2}) \to \cdots$$

is an exact sequence. Furthermore, the anti-chain map $f_i \oplus h_i : \mathrm{C}_i \to \mathrm{Cone}(f_{i+1})$ *is a quasi-isomorphism for each* $i \in \mathbb{Z}$.

To apply this lemma, we use the notation of §3.1, so that we have a 3-periodic sequence of surgery bundles $\mathbb{Y}_i$, $i \in \mathbb{Z}$, and surgery cobordism bundles $\mathbb{X}_{ij} : \mathbb{Y}_i \to \mathbb{Y}_j$ whenever $j > i$. We let $(\mathrm{C}_i, \partial_i)$ be the instanton chain complex $\mathrm{C}(\mathbb{Y}_i)$ with its differential. We take f_i to be $m(\mathbb{X}_{i,i+1}) : \mathrm{C}(\mathbb{Y}_i) \to \mathrm{C}(\mathbb{Y}_{i+1})$. The map h_i is defined in §3.2.2, and in §3.2.3 we define a chain homotopy k_i from $f_{i+2}h_i + h_{i+1}f_i$ to an intermediate map, and then show that this intermediate map is chain homotopic to the identity map of C_i up to sign. All maps are of the form $m_G(\mathbb{X})$.

3.2.2 The h_i maps

We define $h_0 : \mathrm{C}_0 \to \mathrm{C}_2$ in this section. Recall from §3.1.4 that we can write

$$X_{02} = W \cup_{S_1} U$$

where U is diffeomorphic to $-\mathbb{CP}^2$ minus a 4-ball, and W has boundary $Y_2 \sqcup \overline{Y}_0 \sqcup S_1$. The map h_0 is taken to be $m_G(\mathbb{X}_{02})$ where G is a family of metrics on X_{02} induced by the set of two intersecting hypersurfaces $\mathcal{H} = \{S_1, Y_1\}$. Thus G is parameterized by an interval, with endpoint metrics $G(S_1)$ and $G(Y_1)$, cut along S_1 and Y_1, respectively, as depicted in Figure 3.5. Equations (2.5) and (2.6) yield

$$-h_0\partial_0 - \partial_2 h_0 = m_{G(S_1)}(\mathbb{X}_{02}) + m_{G(Y_1)}(\mathbb{X}_{02}).$$

By equation (2.7), we also have $m_{G(Y_1)}(\mathbb{X}_{02}) = m(\mathbb{X}_{12})m(\mathbb{X}_{01}) = f_1 f_0$. It remains to show that $m_{G(S_1)}(\mathbb{X}_{02}) = 0$.

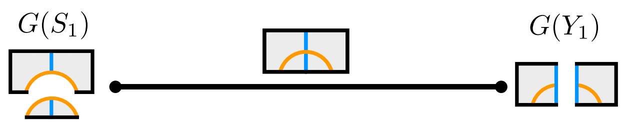

Figure 3.5: The family of metrics on X_{02} used to define the h_0 map.

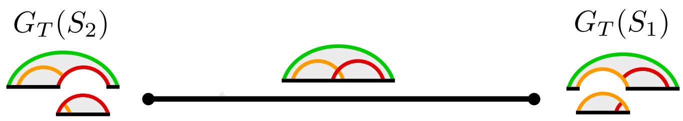

Figure 3.6: The family of metrics G_T on $E \subset X_{03}$.

Let a and b be given with $\mu(a, \mathbb{X}_{02}, b) = 0$. To show that $m_{G(S_1)}(\mathbb{X}_{02}) = 0$, it suffices to show that $M_{G(S_1)}(a, \mathbb{X}_{02}, b)$ is empty for any such a, b. We prove this by contradiction. Suppose $[A] \in M_{G(S_1)}(a, \mathbb{X}_{02}, b)$. Write $\mathbb{U}$ and $\mathbb{W}$ for the restriction of $\mathbb{X}_{02}$ to U and W, respectively. Because $G(S_1)$ is cut along S_1, $[A]$ is a pair $[A_W], [A_U]$ in $M(a, \mathbb{W}, b, c) \times M(c, \mathbb{U})$ for some flat connection c on $\mathbb{S}_1$. We arrange that the perturbation data near S_1 is 0. The gluing formula (2.3) says

$$\mu(A) = \mu(A_W) + \mu(A_U) + h^0(c) + h^1(c).$$

The flat connection c is on a 3-sphere, so $h^1(c) = 0$ and $h^0(c) = 3$. Since a and b are irreducible, so is A_W. It follows that $\mu(A_W) \geq 0$, see inequality (2.9). The connection A_U may be reducible to S^1, but no further, because $\mathbb{U}$ is non-trivial, so $h^0(A_U) \leq 1$. It follows from (2.8) that $\mu(A_U) \geq -1$, implying $\mu(A) = \mu(a, \mathbb{X}_{02}, b) \geq 2$, a contradiction.

3.2.3 The k_i maps

We define $k_0 : \mathrm{C}_0 \to \mathrm{C}_0$ in this section. Recall from §3.1.5 that we have five hypersurfaces Y_1, Y_2, S_1, S_2, T in X_{03} that intersect one another as in Figure 3.3. We define k_0 to be $m_G(\mathbb{X}_{03})$ where G is the family of metrics on X_{03} induced by the set of hypersurfaces $\mathcal{H} = \{Y_1, Y_2, S_1, S_2, T\}$. The family G is parameterized by a pentagon and has faces $G(Y_1), G(Y_2), G(S_1), G(S_2), G(T)$, each of which is an interval

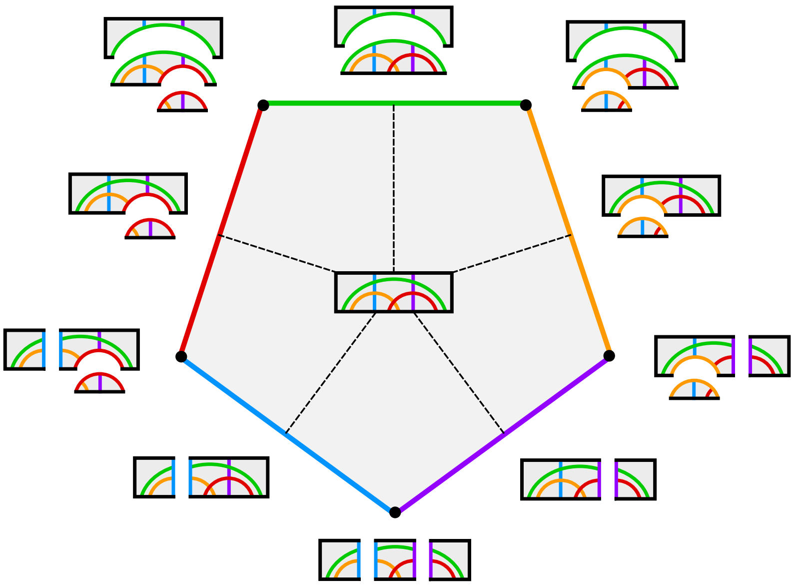

Figure 3.7: The family of metrics on X_{03} used to define the k_0 map. This picture is modelled on Bloom's from [3].

of metrics broken along the indicated hypersurface. See Figure 3.7. Equations (2.5) and (2.6) yield

$$k_0\partial_0 - \partial_0 k_0 = \sum_{S\in\mathcal{H}} m_{G(S)}(\mathbb{X}_{03})$$

and the argument from §3.2.2 shows that $m_{G(S_1)}(\mathbb{X}_{03}) = m_{G(S_2)}(\mathbb{X}_{03}) = 0$. We also have $m_{G(Y_1)}(\mathbb{X}_{03}) = h_1 f_0$ and $m_{G(Y_2)}(\mathbb{X}_{03}) = f_2 h_0$ by (2.7). Thus

$$k_0\partial_0 - \partial_0 k_0 = m_{G(T)}(\mathbb{X}_{03}) + f_2 h_0 + h_1 f_0,$$

or in other words, k_0 is a chain homotopy from $-m_{G(T)}(\mathbb{X}_{03})$ to $f_2 h_0 + h_1 f_0$. The proof is thus complete if we establish

Lemma 3.2.2. *$m_{G(T)}(\mathbb{X}_{03})$ is chain homotopic to $\pm\mathrm{id} : \mathrm{C}_0 \to \mathrm{C}_0$.*

The remainder of this section goes towards proving this lemma. From §3.1.5, we know the hypersurface T induces a decomposition

$$X_{03} = V \cup_T E$$

where E is diffeomorphic to $-\mathbb{CP}^2$ minus a regular neighborhood of an unknotted S^1. Let $\mathbb{V}, \mathbb{E}$ be the restrictions of $\mathbb{X}_{03}$ to V, E, respectively. The restriction of $G(T)$

to V is a single metric. On the other hand, the restriction of $G(T)$ to E is an interval of metrics, and we denote this family by G_T, see Figure 3.6. We arrange that the perturbations used near T are zero, so that the relevant limiting connections are flat.

The map $m_{G(T)}(\mathbb{X}_{03})$ is defined by counting $[A] \in M_{G(T)}(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0$. That is,

$$\langle m_{G(T)}(\mathbb{X}_{03})\mathfrak{a}, \mathfrak{b}\rangle = \# M_{G_T}(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0$$

where $\#$ means a signed count determined by orienting moduli spaces. Note that $\mu(A) = -1$ since $\dim G(T) = 1$. Let a and b be the limiting connections of A on $\mathbb{Y}_0$ and $\mathbb{Y}_3$, so $[a] = \mathfrak{a}$ and $[b] = \mathfrak{b}$. Each such A can be written as a pair

$$A_V, (A_E, g) \tag{3.7}$$

where A_V is an instanton on $\mathbb{V}$ with limit a over $\mathbb{Y}_0$, b over $\mathbb{Y}_3$, and some flat limit c over $\mathbb{T}$; and A_E is a g-instanton on $\mathbb{E}$ where $g \in G_T$, and A_E has the same flat limit c over $\mathbb{T}$.

First, let us understand $\mathfrak{T} = \mathfrak{C}(\mathbb{T})$, the space of $\mathscr{G}_{\text{ev}}$-classes of flat connections on $\mathbb{T}$. Recall that $\mathbb{T}$ is a trivial SO(3)-bundle over an $S^1 \times S^2$. Choose a spin structure for $\mathbb{T}$, i.e. a lift to an SU(2)-bundle. Lifting connections sets up a bijection between flat SO(3)-connections modulo $\mathcal{G}_{\text{ev}}$ on $\mathbb{T}$ with flat SU(2)-connections modulo SU(2) gauge transformations. It is well-known that this latter set is in correspondence with $\text{Hom}(\pi_1(T), \text{SU}(2))$ modulo conjugation, which is essentially the set of conjugacy classes of SU(2). The space of conjugacy classes of SU(2) is $[-1, 1]$, given by the trace map divided by 2.

The isomorphism $\mathfrak{T} \simeq [-1, 1]$ depends on the spin structure of $\mathbb{T}$ chosen. There are two such choices, and they are related by any *non*-even gauge transformation of $\mathbb{T}$; using such a transformation the isomorphisms $\mathfrak{T} \simeq [-1, 1]$ are related by reflecting $[-1, 1]$ about 0. The choice of isomorphism can also be determined by choosing a trivial holonomy flat connection on $\mathbb{T}$; this choice corresponds to $1 \in [-1, 1]$. We record the following.

Lemma 3.2.3. *A choice of spin structure for* $\mathbb{T}$ *determines an isomorphism* $\mathfrak{T} \simeq [-1,1]$*. The action on the space of flat connection classes* $\mathfrak{T}$ *by* $\mathscr{G}/\mathscr{G}_{ev} \simeq \mathbb{Z}/2$ *under this isomorphism is reflection about* 0*.*

We can now understand the structure of the relevant moduli space following basic index computations. Write $\mathfrak{T}^0$ for the interior of $\mathfrak{T}$, and G_T^0 for the interior of G_T.

Lemma 3.2.4. *After a suitable perturbation, the moduli space* $M(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0$ *can be identified with the fiber product*

$$M(\mathfrak{a}, \mathbb{V}, \mathfrak{b}, \mathfrak{T}^0)_0 \times_{\mathfrak{T}^0} M_{G_T^0}(\mathfrak{T}^0, \mathbb{E})_1^{red}.$$

The moduli space on the right is the space of pairs $([A_E], g)$ where $g \in G_T^0$ and A_E is a g-instanton on $\mathbb{E}$ (exponentially decaying over the ends), such that the flat limit class of A_E over T lies in the interior of $\mathfrak{T}$; $h^0(A_E) = 1$, i.e. A_E has gauge-stabilizer S^1; and $\mu(A_E) = 1 - h^0(A_E) - \dim G_T^0 - \dim \mathfrak{T}^0 = -2$. In other words, the lemma says that in the pair (3.7) representing $[A] \in M(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0$, we have the constraints

$$\mathfrak{c} = [c] \in \mathfrak{T}^0, \quad g \in G_T^0, \quad \mu(A_V) = -1, \quad \mu(A_E) = -2. \tag{3.8}$$

The fiber product is taken with respect to limit maps $\lambda : M \to \mathfrak{T}^0$ that send an instanton class to its flat limit class over $\mathbb{T}$, where M is one of the two moduli spaces appearing in the lemma. This fiber product description is an application of the Morse-Bott gluing theory as discussed in [7, §4.5.2] and [31, 30, 39]. Our situation, that of instantons broken along $S^1 \times S^2$ with flat limits in $\mathfrak{T} \simeq [-1,1]$, is similar to that of Fintushel and Stern's in [11], where results of Mrowka's thesis [31] are used, and we will refer the reader to these sources for more details. We mention that for the above fiber product it is important that the stabilizers of c and A_E, each a circle, can be identified. In general, one must record a gluing parameter in $\Gamma_c/\Gamma_{A_V} \times \Gamma_{A_E}$ where Γ_A is the stabilizer of A. For instance, if both $[A_V]$ and $[A_E]$ were irreducible, there would be more than one choice of such a parameter. We proceed to prove that the constraints (3.8) characterize the possible gluing data.

Proof of Lemma 3.2.4. We first show $\mathfrak{c} \in \mathfrak{T}^0$. For convenience we set

$$h(c) = (h^0(c) + h^1(c))/2.$$

We note that $h(c) = 1$ or 3, depending on whether $\mathfrak{c}$ is in the interior or boundary of $\mathfrak{T}$, respectively, cf. [11, §3]. By assumption $\mu(A) = -1$, so (2.3) yields

$$-1 = \mu(A) = \mu(A_V) + \mu(A_E) + 2h(c).$$

Let $A_{S^1\times D^3}$ be a connection on the trivial bundle over $S^1 \times D^3$ with one cylindrical end attached. We identify the bundle over cross-sections of the end with $\mathbb{T}$, with the base having the opposite orientation of T. Suppose $A_{S^1\times D^3}$ has flat limit c. We glue $A_{S^1\times D^3}$ to A_E to obtain a connection $A_{-\mathbb{CP}^2}$ on a non-trivial bundle $\mathbb{E}'$ over $-\mathbb{CP}^2$. The isomorphism class of $\mathbb{E}'$ depends on c, but we know $p_1(\mathbb{E}') = 4k - 1$ for some $k \in \mathbb{Z}$, cf. [8, §4.1.4]. We have

$$\mu(A_E) + \mu(A_{S^1\times D^3}) + 2h(c) = \mu(A_{-\mathbb{CP}^2}).$$

We compute $\mu(A_{S^1\times D^3})$. Two copies of $S^1 \times D^3 \times \mathrm{SO}(3)$, each with a cylindrical end, glue, overlapping the ends, to give $S^1 \times S^3 \times \mathrm{SO}(3)$. Index additivity yields

$$2\mu(A_{S^1\times D^3}) + 2h(c) = \mu(S^1 \times S^3 \times \mathrm{SO}(3)).$$

On the other hand, (2.4) says the right hand side is

$$-3(1 - b_1 + b_2^+)(S^1 \times S^3) = 0.$$

Thus $\mu(A_{S^1\times D^3}) = -h(c)$. This can also be deduced from the Atiyah-Patodi-Singer index theorem, cf. [1, Thm. 3.10]. From (2.4) we obtain $\mu(A_{-\mathbb{CP}^2}) = -8k - 1$, and

$$\mu(A_V) = 8k - h(c), \qquad \mu(A_E) = -8k - 1 - h(c).$$

Suppose for contradiction that $\mathfrak{c}$ is on the boundary of $\mathfrak{T}$, so that $h(c) = 3$. Since A_V is irreducible and the boundary of $\mathfrak{T}$ has dimension 0, we have

$$8k - 3 = \mu(A_V) \geq 0$$

in the generic case, so $k > 0$. Since $\mathbb{E}'$ is nontrivial, $h^0(A_E) \in \{0, 1\}$. Using (2.9),

$$-8k - 4 = \mu(A_E) \geq -\dim G_T - \dim \partial\mathfrak{T} - h^0(A_E) \geq -2.$$

Then $k < 0$, a contradiction. Thus $h(c) = 1$ and $\mathfrak{c} \in \mathfrak{T}^0$. It follows that $\mu(A_V) = 8k - 1$ and $\mu(A_E) = -8k - 2$. Applying (2.9) in this case,

$$\mu(A_E) \geq -\dim \mathfrak{T}^0 - \dim G_T - h^0(A_E) \geq -3,$$

so $k \leq 0$. Similarly, $\mu(A_V) \geq -\dim \mathfrak{T}^0 = -1$ gives $k \geq 0$. Thus $k = 0$, yielding $\mu(A_V) = -1$ and $\mu(A_E) = -2$, as claimed.

Next, we rule out the possibility that $h^0(A_E) = 0$, or in other words, that $[A] \in M_{G(T)}(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0$ can be written as a gluing of $[A_V]$ and $([A_E], g)$ where A_E is *irreducible*, i.e.

$$([A_E], g) \in M_{G_T}(\mathfrak{T}^0, \mathbb{E})_0^{\text{irr}}.$$

Note that if there were such a gluing, we would have to keep track of a gluing parameter, as mentioned earlier. However, this moduli space of irreducibles and $M(\mathfrak{a}, \mathbb{V}, \mathfrak{b}, \mathfrak{T}^0)_0$ are both finite sets after perturbation, by standard compactness results, cf. [11, §5]. Further, the intersection of their flat limits in $\mathfrak{T}^0$ can be made transverse, in which case they have empty intersection. Thus, after a suitable perturbation, $h^0(A_E) = 1$.

Finally, we show $g \in G_T^0$. Suppose for contradiction that $g \in \partial G_T$. Then g is one of two metrics on E, $G_T(S_1)$ or $G_T(S_2)$, cut along S_1 or S_2, respectively. See Figure 3.6. Suppose $g = G_T(S_1)$; the other case is similar. Write

$$E = X \cup_{S_1} U$$

where $U \simeq -\mathbb{CP}^2 \setminus \text{int}(D^4)$ and $X \simeq D^2 \times S^2 \setminus \text{int}(D^4)$. Note that the restriction of $\mathbb{E}$ over X is trivial, while the restriction over U, as in §3.2.2, is non-trivial; write A_X and A_U for the restriction of A_E over these respective bundles. They have a common flat limit d on $\mathbb{S}_1$. In particular, $h^0(d) = 3$ and $h^1(d) = 0$. The connection A_X has the limit c over $\mathbb{T}$ from before.

We compute $\mu(A_X)$ and $\mu(A_U)$. There is only one instanton class on X: the trivial class, cf. [7, §7.4.1]. Thus A_X is trivial, so $h(c) = 3$. Let $A_{S^1 \times D^3}$ be a connection on the trivial bundle over $S^1 \times D^3$ with one cylindrical end attached whose flat limit is c. Then A_X and $A_{S^1\times D^3}$ glue, overlapping ends, to give a connection A_{D^4} over D^4 with one cylindrical end attached. Then (2.3), (2.4) yield

$$\mu(A_X) + \mu(A_{S^1\times D^3}) + 2h(c) = \mu(A_{D^4}) = -3.$$

From above, $\mu(A_{S^1\times D^3}) = -h(c) = -3$. Thus $\mu(A_X) = -6$. With $\mu(A_U) = 8k - 1$ for some $k \in \mathbb{Z}$, we apply (2.3) once more to get

$$\mu(A_E) = \mu(A_X) + \mu(A_U) + 2h(d) = 8k - 4.$$

It follows that $\mu(A_E) \neq -2$, a contradiction. □

Lemma 3.2.5. *The projection $M_{G^0_T}(\mathfrak{T}^0, \mathbb{E})^{red}_1 \to G^0_T$ is a smooth homeomorphism.*

Proof. The moduli space here is topologized as a subset of $\mathscr{B} \times G^0_T$, so the projection map is a continuous, open map. It is also smooth, in the transverse case, by general theory. It suffices to show bijectivity. The argument is a standard account of counting reducible instantons.

Let $([A_E], g)$ be such that $\mu(A_E) = -2, h^0(A_E) = 1$ and $g \in G^0_T$. Because $H^1(E;\mathbb{R}) = 0$, E admits no non-trivial real line bundles. Thus $h^0(A_E) = 1$ implies A_E is compatible with a splitting $\mathbb{L} \oplus \underline{\mathbb{R}}$ of the associated vector bundle of $\mathbb{E}$, where $\mathbb{L}$ is a complex line bundle and $\underline{\mathbb{R}}$ is a trivial real line bundle. Gluing A_E to a connection $A_{S^1\times D^3}$ on a trivial bundle over $S^1 \times D^3$ with one cylindrical end attached gives an instanton $A_{-\mathbb{CP}^2}$ on a bundle $\underline{\mathbb{R}}' \oplus \mathbb{L}'$ over $-\mathbb{CP}^2$ where $\underline{\mathbb{R}}'$ and $\mathbb{L}'$ are extensions of $\underline{\mathbb{R}}$ and $\mathbb{L}$. The gluing formula says

$$\mu(A_E) + \mu(A_{S^1\times D^3}) + h^0(c) + h^1(c) = \mu(A_{-\mathbb{CP}^2}) = -2p_1(\underline{\mathbb{R}}' \oplus \mathbb{L}') - 3. \qquad (3.9)$$

Using that $p_1(\underline{\mathbb{R}}' \oplus \mathbb{L}') = c_1(\mathbb{L}')^2$ we have $\mu(A_E) = -2c_1(\mathbb{L}')^2 - 4$. Since $\mu(A_E) = -2$, we conclude that $c_1(\mathbb{L}')^2 = -1$. Let $P(E)$ denote the image of the map

$H^2(E,\partial E;\mathbb{Z}) \to H^2(E;\mathbb{Z})$. Note that inclusion $E \to -\mathbb{CP}^2$ induces an isomorphism of intersection forms from $H^2(-\mathbb{CP}^2;\mathbb{Z})$ to $P(E)$, both negative definite of rank 1, under which $c_1(\mathbb{L}')$ is sent to $c_1(\mathbb{L})$. Thus $c_1(\mathbb{L})$ is a generator of $H^2(E;\mathbb{Z})$.

There are thus two choices of $\mathbb{L}$ corresponding to the choices of generator for $H^2(E;\mathbb{Z})$. To get one from the other take the conjugate $\mathbb{L}^*$. The choice we make does not matter in the end, as we can relate the two by an even gauge transformation, by combining the conjugation map $\mathbb{L} \to \mathbb{L}^*$ with the involution of $\underline{\mathbb{R}}$ that reflects each fiber. Note that $\mathscr{G} = \mathscr{G}_{\mathrm{ev}}$ for $\mathbb{E}$.

We are left with the problem of finding g-instantons on $\mathbb{L}$. According to [1, Prop. 4.9], the space of L^2 harmonic 2-forms on E is isomorphic to the image of $H^2(E,\partial E;\mathbb{R}) \to H^2(E;\mathbb{R})$, and under this isomorphism a harmonic form x corresponds to its de Rham class $[x]$. In our case this map is an isomorphism $\mathbb{R} \to \mathbb{R}$. Further, any such harmonic x satisfies $\star x = -x$, as follows: $\star x$ is L^2 harmonic, so $\star x = cx$ for some $c \in \mathbb{R}$; then $\star^2 = 1$, $\int x \wedge x < 0$, and $0 \leq \|x\|^2_{L^2} = \int x \wedge \star x = c \int x \wedge x$ imply that $c = -1$. Conversely, a closed L^2 2-form x satisfying $\star x = -x$ is easily seen to be L^2 harmonic.

The arguments from [8, §2.2.1] easily adapt here, since $H^1(E;\mathbb{R}) = 0$, to show that given a closed L^2 2-form x on E, there is a connection A on $\mathbb{L}$ with curvature ix which is unique up to gauge equivalence. In this way, the unique L^2 harmonic 2-form representing $-2\pi c_1(\mathbb{L})$ specifies a unique g-instanton class on $\mathbb{L}$. □

Lemma 3.2.6. *The moduli space $M_{\partial G_T}(\partial\mathfrak{T},\mathbb{E})_0^{red}$ consists of two points, and is the natural boundary of the open interval $M_{G_T^0}(\mathfrak{T}^0,\mathbb{E})_1^{red}$.*

Proof. The previous lemma tells us that the ends of the latter moduli space are essentially the ends of G_T. There are two endpoint metrics of G_T, labelled $G_T(S_1)$ and $G_T(S_2)$, each broken along the indicated 3-sphere. Any instanton A on $\mathbb{E}$ compatible with $G_T(S_1)$ is a gluing of the trivial instanton on the trivial bundle over $X \simeq D^2 \times S^2 \setminus \mathrm{int}(D^4)$ with two cylindrical ends attached and an instanton A_U on $U \simeq -\mathbb{CP}^2 \setminus \mathrm{int}(D^4)$ with one cylindrical end attached. By the removable singularities theorem of Uhlenbeck, cf. [8, Thm. 4.4.12], the instanton A_U uniquely extends to an instanton A on a bundle $\mathbb{W}$ over $-\mathbb{CP}^2$. If A is to be a limit of elements in M_E, then $p_1(\mathbb{W}) = -1$. There is only one such instanton class on $\mathbb{W}$, cf. [20, §2.7]. Thus $[A]$ is uniquely determined. Similarly, there is one instanton class to add for $G_T(S_2)$. That A is trivial over X implies the flat limits over $\mathbb{T}$ of these two instanton classes lie in $\partial\mathfrak{T}$. □

Note that the map in Lemma 3.2.5 extends to a homeomorphism of closed intervals. We write $M_{G_T}(\mathfrak{T}, \mathbb{E})_1^{\text{red}}$ for the completed closed interval moduli space. We call a map between closed intervals *proper* if it sends boundary to boundary. A proper map between oriented, closed intervals has a well-defined degree, which is 0 or ± 1. Indeed, one can define the degree by looking at the induced map $S^1 \to S^1$ obtained by identifying boundary points.

Lemma 3.2.7. *The map $\lambda : M_{G_T}(\mathfrak{T}, \mathbb{E})_1^{red} \to \mathfrak{T}$ defined by sending an instanton class to its flat limit class over $\mathbb{T}$ has degree ± 1.*

Proof. We use the involution $\sigma : \mathbb{E} \to \mathbb{E}$ of §3.1.6. Write M for the moduli space in the lemma. We see that σ induces an action on M, and because $\sigma(\mathbb{T}) = \mathbb{T}$, an action on $\mathfrak{T}$. We can arrange the family of metrics G_T so that σ restricts to an isometry of the base space and reflects G_T, in turn swapping the endpoints of the interval M. If we establish that σ also swaps the endpoints of the interval $\mathfrak{T}$, we are done, because the limit map λ respects the action of σ. From §3.1.6 we know that with respect to a fixed trivialization $\mathbb{T} \simeq S^1 \times S^2 \times \mathrm{SO}(3)$, σ is isotopic to a composition $\theta \circ \upsilon$, where θ is a diffeomorphism of $S^1 \times S^2$ lifted in a trivial way to $S^1 \times S^2 \times \mathrm{SO}(3)$. The diffeomorphism under consideration acts trivially on $\pi_1(T)$, and hence θ acts trivially on $\mathfrak{T}$. The map υ is a non-even gauge transformation, so by Lemma 3.2.3, it reflects the interval $\mathfrak{T}$. It follows that σ reflects $\mathfrak{T}$. □

Proof of Lemma 3.2.2. By Lemma 3.2.4 we can write

$$\# M_{G_T}(\mathfrak{a}, \mathbb{X}_{03}, \mathfrak{b})_0 = \pm \sum \varepsilon(x)\varepsilon(y)$$

where the sum is over pairs

$$(x,y) \in M(\mathfrak{a},\mathbb{V},\mathfrak{b},\mathfrak{T}^0)_0 \times M_{G_T^0}(\mathfrak{T}^0,\mathbb{E})_1^{\mathrm{red}}$$

having equal flat limit class $\lambda(x) = \lambda(y) \in \mathfrak{T}^0$. Each x and y has a sign, $\varepsilon(x)$ and $\varepsilon(y)$ respectively, prescribed by orienting moduli spaces. In the generic case, the sum of the $\varepsilon(y)$ for a fixed value $\lambda(y)$ equals $\pm\deg(\lambda) = \pm 1$. In this way we obtain

$$\# M_{G_T}(\mathfrak{a},\mathbb{X}_{03},\mathfrak{b})_0 = \pm \# M(\mathfrak{a},\mathbb{V},\mathfrak{b},\mathfrak{T}^0)_0$$

where the sign does not depend on the pair $(\mathfrak{a},\mathfrak{b})$. Thinking of cobordisms as morphisms, we abbreviate $[0,1]\times\mathbb{Y}_0$ to $1_{\mathbb{Y}_0}$. Write $1_{\mathbb{Y}_0} = \mathbb{V}\cup_{\mathbb{T}}\mathbb{W}$ where $\mathbb{W}$ is a trivial bundle over $W = S^1 \times D^3$. We choose the perturbation data for $\mathbb{W}$ to be 0. Let Q be the family of metrics on $[0,1]\times Y_0$ induced by $\mathcal{H} = \{T\}$. The boundary of Q consists of an initial product metric on $[0,1]\times Y_0$ and a metric $Q(T)$ cut along T. Thus (2.5) and (2.6) yield

$$-m_Q(1_{\mathbb{Y}_0})\partial_0 - \partial_0 m_Q(1_{\mathbb{Y}_0}) = m_{Q(T)}(1_{\mathbb{Y}_0}) + m(1_{\mathbb{Y}_0}).$$

Of course, $m(1_{\mathbb{Y}_0})$ is the identity. It remains to show that $m_{Q(T)}(1_{\mathbb{Y}_0})$ is equal to $m_{G(T)}(\mathbb{X}_{03})$ up to an overall sign, or equivalently

$$\# M_{Q(T)}(\mathfrak{a},\mathfrak{b})_0 = \pm \# M(\mathfrak{a},\mathbb{V},\mathfrak{b},\mathfrak{T}^0)_0 \tag{3.10}$$

where the sign does not depend on the pair $(\mathfrak{a},\mathfrak{b})$. In the spirit of our previous arguments, we establish this by arguing that $M(\mathfrak{a},\mathbb{V},\mathfrak{b},\mathfrak{T}^0)_0$ can be written as a fiber product

$$M(\mathfrak{a},\mathbb{V},\mathfrak{b},\mathfrak{T}^0)_0 \times_{\mathfrak{T}^0} M(\mathfrak{T}^0,\mathbb{W})_1^{\mathrm{flat}}.$$

Here $M(\mathfrak{T}^0,\mathbb{W})_1^{\mathrm{flat}}$ is the 1-dimensional family of flat connection classes on $\mathbb{W}$ with arbitrary flat limit class in $\mathfrak{T}^0$. Indeed, any flat connection class on $\mathbb{T}$ uniquely extends to a flat connection class on $\mathbb{W}$ over $S^1\times D^3$. We conclude that all instantons on $\mathbb{W}$ are flat, cf. [7, §7.4]. In particular, the limit map $\lambda : M(\mathfrak{T}^0,\mathbb{W})_1^{\mathrm{flat}} \to \mathfrak{T}^0$ is a smooth homeomorphism. Now suppose $[A] \in M_{Q(T)}(\mathfrak{a}, \mathfrak{b})_0$ restricts to a pair $[A_V], [A_{\mathrm{W}}]$ of instantons on $\mathbb{V}$ and $\mathbb{W}$, respectively, with equal limit $\mathfrak{c}$ over $\mathbb{T}$. Then

$$0 = \mu(A) = \mu(A_V) + \mu(A_{\mathrm{W}}) + 2h(c).$$

We saw in Lemma 3.2.4 that $\mu(A_W) = -h(c)$, so $\mu(A_V) = -h(c)$. The space of $[A_V]$ with $\mu(A_V) = -2$ is generically empty, so we conclude that $\mu(A_V) = -1$. It follows that $\mathfrak{c} \in \mathfrak{T}^0$. Because the stabilizer of each A_W is SU(2), the gluing parameter space is trivial, and our fiber product description is verified, cf. [11, §4]. Because the limit map $\lambda : M(\mathfrak{T}^0, \mathbb{W})_1^{\text{flat}} \to \mathfrak{T}^0$ is a homeomorphism, our fiber product yields (3.10). This completes the proof of Lemma 3.2.2, and consequently the proof of Theorem 3.0.1. □

3.3 A link surgeries spectral sequence

In this section we prove Theorem 3.0.2. We follow [20] and [3]. In [20], Kronheimer and Mrowka work over $\mathbb{Z}$, taking care with signs, and we adapt many of the details from their setup. Bloom's paper [3] is especially descriptive of the combinatorics involved here, and provides many illustrations. The idea for this spectral sequence originates from Ozsváth and Szabó's paper [35].

3.3.1 The cobordisms & metric families

Let $\mathbb{Y}$ be an admissible bundle over Y and $L \subset Y$ a framed link with m components $L_1, \ldots, L_m$. Suppose we have admissible bundles $\mathbb{Y}_v$ for $v \in \{\infty, 0, 1\}^m$ that form a surgery cube as in the introduction to this chapter. We conflate the subscript ∞ with -1 and write $\mathbb{Y}_v$ for $v \in \{-1, 0, 1\}^m$. Further, we write $\mathbb{Y}_v$ for $v \in \mathbb{Z}^m$ by taking the modulo 3 reduction of v. Define the norms

$$|v|_1 = \sum_{i=1}^{m} |v_i|, \qquad |v|_\infty = \max_{1 \le i \le m} \{|v_i|\}.$$

We use the partial order on $\mathbb{Z}^m$ that says $v \le w$ whenever $v_i \le w_i$ for $i = 1, \ldots, m$.

Since the $\mathbb{Y}_v$ form a surgery cube, they can be generated by the data of $\mathbb{Y}$ and a framed link $\mathbb{L} = \mathbb{L}_1 \cup \cdots \cup \mathbb{L}_m$ in $\mathbb{Y}$ as in §3.1.1, where each $\mathbb{L}_i$ is an equivariant embedding of $S^1 \times D^2 \times \mathrm{SO}(3)$ into $\mathbb{Y}$. For $v < w$ we have surgery bundle cobordisms $\mathbb{X}_{vw} : \mathbb{Y}_v \to \mathbb{Y}_w$ constructed by iterating the construction for $\mathbb{X}_{ij}$ from §3.1.3 for each $\mathbb{L}_i$. To give a definition, first set $k = |w - v|_1$. We choose a maximal chain $v = v(0) < v(1) < \cdots < v(k) = w$. Each $\mathbb{X}_{v(i)v(i+1)}$ may be viewed as a surgery bundle as defined in §3.1.3, and we may set

$$\mathbb{X}_{vw} = \mathbb{X}_{v(k-1)v(k)} \circ \cdots \circ \mathbb{X}_{v(0)v(1)}.$$

The choice of maximal chain does not affect the isomorphism type of $\mathbb{X}_{vw}$. In fact, the identification of (3.3) lends a more invariant interpretation: we may view $\mathbb{X}_{vw}$ as $\mathbb{Y}_v \times [0, 1]$ with, for each $i = 1, \ldots, m$, a copy of $\mathbb{H} \cup_\psi \cdots \cup_\psi \mathbb{H}$ ($w_i - v_i$ copies of $\mathbb{H}$) attached to $\mathbb{Y}_v \times \{1\}$ via the framed knot $\Lambda^{v_i+1}(\mathbb{L}_i)$. We have the isomorphism

$$\mathbb{X}_{vw} \simeq \mathbb{X}_{uw} \circ \mathbb{X}_{vu}$$

whenever $v < u < w$. We write $\mathbf{0}$ for the element of $\mathbb{Z}^m$ with all zeros, and similarly $\mathbf{n}$ for the element with all elements equal to $n \in \mathbb{Z}$. Note that $\mathbb{X}_{\mathbf{03}}$ is not $\mathbb{X}_{\mathbf{00}} = \mathbb{Y}_{\mathbf{0}} \times [0, 1]$, but for instance $\mathbb{X}_{\mathbf{01}} \simeq \mathbb{X}_{\mathbf{34}}$. The base space of $\mathbb{X}_{vw}$ is written X_{vw}. In the sequel we will only consider $\mathbb{X}_{vw}$ with $|w - v|_\infty \leq 3$.

As in the case when L had one component, we have distinguished hypersurfaces in the interior of X_{vw}. Of course, the 3-manifolds $Y_u \subset X_{vw}$ for $v < u < w$ are the first examples. Note that Y_u and $Y_{u'}$ are disjoint if and only if $u < u'$ or $u' < u$. For each $i \in \{1, \ldots, m\}$ and k with $v_i < k < w_i$ we have a 3-sphere S^i_k in X_{vw} which generalizes $S_1 \subset X_{02}$ from §3.1.4. The spheres S^i_k and S^j_l intersect if and only if $i = j$ and $|k - j| \leq 1$, and S^i_k intersects Y_u if and only if $u_i = k$. For $v, w \in \mathbb{Z}^m$ with $v < w$ and $|w - v|_\infty \leq 2$ we define a set of hypersurfaces in X_{vw}:

$$\mathcal{H}_{vw} = \{Y_u : v < u < w\} \cup \{S^i_k : 1 \leq i \leq m,\ v_i < k < w_i\}.$$

Note that the second set is empty if $|w - v|_\infty < 2$.

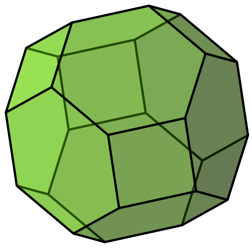

Figure 3.8: The permutahedron P_4.

We obtain a family of metrics $G_{vw} = G(\mathcal{H}_{vw})$ on X_{vw} as constructed in §2.4. The space of metrics G_{vw} is a convex polytope called a graph-associahedron, and

$$\dim G_{vw} = |w - v|_1 - 1,$$

as Bloom explains in [3, Thm. 5.3]. In fact, when $|w-v|_\infty < 2$, G_{vw} is the permutahedron P_N, the convex polytope defined as the convex hull in $\mathbb{R}^N$ of all permutations of $(1, 2, \dots, N) \in \mathbb{R}^N$ where $N = |w - v|_1$. For example, P_3 is a hexagon, and the polytope P_4 is shown (hollowed out) in Figure 3.8. Write $m_{vw} = m_{G_{vw}}(\mathbb{X}_{vw})$ and ∂_v for the differential of $\mathrm{C}(\mathbb{Y}_v)$. From the formulae in §2.4 we obtain

$$(-1)^{|w-v|_1-1} m_{vw}\partial_v - \partial_w m_{vw} = \sum_{v<u<w} m_{G(Y_u)}(\mathbb{X}_{vw}) + \sum_{\substack{1\le i\le m \\ w_i<k<v_i}} m_{G(S^i_k)}(\mathbb{X}_{vw}). \quad (3.11)$$

As in §3.2.2, each $m_{G(S^i_k)}(\mathbb{X}_{vw}) = 0$. Also, the family $G(Y_u)$ can be identified with the product $G_{vu} \times G_{uw}$. Before we apply equation (2.7), we discuss the arrangement of signs.

It is possible to choose I-orientations μ_{vw} for $\mathbb{X}_{vw}$ such that $\mu_{vw} = \mu_{uw} \circ \mu_{vu}$ whenever $v < u < w$, and we do so. For a proof, see [20, Lemma 6.1]. We can orient each G_{vw} such that the identification of $G_{vu} \times G_{uw}$ with $G(Y_u) \subset \partial G_{vw}$ has orientation deficiency $(-1)^{\dim G_{vu}}$. That is, the product orientation for $G_{vu} \times G_{uw}$ using our chosen orientations differs from the boundary orientation as induced from G_{vw} by the sign $(-1)^{\dim G_{vu}}$. This essentially follows from the discussion in [20] following Prop. 6.4. With this understood, equation (2.7) yields

$$m_{G(Y_u)}(\mathbb{X}_{vw}) = (-1)^{(\dim G_{uw}+1)\dim G_{vu}} m_{uw} m_{vu}.$$

Writing $m_{vv} = \partial_v$, equation (3.11) becomes

$$\sum_{v \leq u \leq w} (-1)^{|w-u|_1(|u-v|_1-1)} m_{uw} m_{vu} = 0.$$

We remind the reader that this holds under the assumptions that $v < w$ and $|w - v|_\infty \leq 2$. The case $v = w$ also holds, encoding the relation $\partial_v^2 = 0$.

3.3.2 Constructing the spectral sequence

We now construct the spectral sequence of Theorem 3.0.2. We define a chain complex $(\mathbf{C}, \boldsymbol{\partial})$ with a filtration $F^i\mathbf{C}$. The filtration will induce the spectral sequence we desire. To begin, set

$$\mathbf{C} = \bigoplus_{v \in \{0,1\}^m} \mathrm{C}(\mathbb{Y}_v), \quad \boldsymbol{\partial} = \sum_{v \leq w} \partial_{vw} \tag{3.12}$$

where $\partial_{vw} = (-1)^{s(v,w)} m_{vw}$. The sign here is given by

$$s(v, w) = (|w - v|_1^2 - |w - v|_1)/2 + |v|_1,$$

as lifted from [20, eq. 38]. We compute the $\mathrm{C}(\mathbb{Y}_v) \to \mathrm{C}(\mathbb{Y}_w)$ part of $\boldsymbol{\partial}^2$ to be

$$(-1)^{s(v,w)+|w|_1} \sum_{v \leq u \leq w} (-1)^{|w-u|_1(|u-v|_1-1)} m_{uw} m_{vu} = 0.$$

We call $(\mathbf{C}, \boldsymbol{\partial})$ the *link surgeries complex associated to* $(\mathbb{Y}, \mathbb{L})$, with the understanding that the necessary auxiliary choices we've made have been fixed.

We define the filtration on $(\mathbf{C}, \boldsymbol{\partial})$ by setting

$$F^i\mathbf{C} = \bigoplus_{|v| \geq i} \mathrm{C}(\mathbb{Y}_v) \subseteq \mathbf{C}. \tag{3.13}$$

Since $\boldsymbol{\partial}$ involves only terms with $v \leq w$, it is immediate that $\boldsymbol{\partial} F^i\mathbf{C} \subseteq F^i\mathbf{C}$. This filtered complex induces a spectral sequence whose E^1-page and E^1-differential d^1 are given by

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I(\mathbb{Y}_v), \quad d^1 = \sum_{\substack{v < w \\ |w-v|_1 = 1}} (-1)^{\delta(v,w)} m(\mathbb{X}_{vw}),$$

where $\delta(v,w) \equiv \sum_{1\leq i\leq j} v_i$, in which j is the unique index where v and w differ. This carries over from the discussion following [20, Cor. 6.9]. To prove Theorem 3.0.2 it remains to identify the E^∞-page: we must show that the homology of $(\mathbf{C}, \boldsymbol{\partial})$ is the instanton homology $I(\mathbb{Y})$.

3.3.3 Convergence

Let $(\mathbf{C}, \boldsymbol{\partial})$ be the link surgeries complex associated to $(\mathbb{Y}, \mathbb{L})$. For $i \in \mathbb{Z}$ define the chain complex $(\mathbf{C}_i, \boldsymbol{\partial}_i)$ to be the link surgeries complex associated to $(\mathbb{Y}_{\Lambda^{i+1}}(\mathbb{L}_1), \mathbb{L} \setminus \mathbb{L}_1)$. Recall that the notation $\mathbb{Y}_{\Lambda^{i+1}}(\mathbb{L}_1)$ is from §3.2, and stands for Λ^{i+1}-surgery on $\mathbb{L}_1$ in $\mathbb{Y}$. We conflate ∞ and -1 in the following. Note that for $i = \infty, 0, 1$ and $a, b \in \mathbb{Z}^{m-1}$ we have $(\boldsymbol{\partial}_i)_{ab} = \partial_{vw}$ where $v = (i, a)$ and $w = (i, b)$. Thus we can work exclusively with the maps ∂_{vw} with $v, w \in \mathbb{Z}^m$. Consider $\mathbf{f}_0 : \mathbf{C}_0 \to \mathbf{C}_1$ given by

$$\mathbf{f}_0 = \sum_{\substack{v,w\in\{0,1\}^m \\ v_1=0,w_1=1}} \partial_{vw}.$$

It should be understood that $\partial_{vw} = 0$ if $v \not\leq w$. In words, $\mathbf{f}_0$ is the sum of the components in the differential $\boldsymbol{\partial}$ that correspond to surgery-cobordisms that include surgery on L_1. This is an anti-chain map, and the larger complex $(\mathbf{C}, \boldsymbol{\partial})$ is the cone-complex of $\mathbf{f}_0$. That is,

$$\mathbf{C} = \mathbf{C}_0 \oplus \mathbf{C}_1, \quad \boldsymbol{\partial} = \begin{pmatrix} \boldsymbol{\partial}_0 & 0 \\ \mathbf{f}_0 & \boldsymbol{\partial}_1 \end{pmatrix}.$$

Define a map $\mathbf{F} : \mathbf{C}_\infty \to \mathbf{C}$ by

$$\mathbf{F} = \sum_{\substack{v_1=-1 \\ w_1\in\{0,1\}}} \partial_{vw}.$$

This is an anti-chain map: the relation $\mathbf{F}\boldsymbol{\partial}_\infty + \boldsymbol{\partial}\mathbf{F} = 0$ is an encoding of (3.11):

$$\sum_{\substack{v_1=u_1=-1 \\ w_1\in\{0,1\}}} \partial_{uw}\partial_{vu} + \sum_{\substack{v_1=-1 \\ u_1,w_1\in\{0,1\}}} \partial_{uw}\partial_{vu} = 0.$$

Equip $\mathbf{C}$ and $\mathbf{C}_\infty$ with filtrations as in (3.13) but using the sum $\sum_{i=2}^m v_i$ instead of $|v|_1$. Then $\mathbf{F}$ respects these filtrations, and on the E_p^0-components of the induced spectral sequences, the map induced by $\mathbf{F}$ takes the form

$$\mathbf{F}_p^0 : \bigoplus_{\substack{v_1=-1 \\ \sum_{i\geq 2} v_i = p}} \mathrm{C}(\mathbb{Y}_v) \to \bigoplus_{\substack{v_1\in\{0,1\} \\ \sum_{i\geq 2} v_i = p}} \mathrm{C}(\mathbb{Y}_v)$$

and for v with $v_1 = -1$ is given by

$$\mathbf{F}_p^0|_{\mathrm{C}(\mathbb{Y}_v)} = \partial_{vv'} \oplus \partial_{vv''}$$

where v', v'' have $v_1' = 0$ and $v_1'' = 1$, and otherwise agree with v. But $\partial_{vv'}$ is the map f_{-1} in §3.2.2 for the surgery triangle involving $\mathbb{Y}_v$ and L_1; and likewise $\partial_{vv''}$ is the map h_{-1}. It follows from Lemma 3.2.1 that $\mathbf{F}^0$ is a quasi-isomorphism, and hence so is $\mathbf{F}$. By removing each link component as we have just done for L_1, and composing the m maps $\mathbf{F}$ associated to each removal, we get a quasi-isomorphism $\mathbf{Q}$ from $(\mathrm{C}(\mathbb{Y}), \partial)$ to $(\mathbf{C}, \boldsymbol{\partial})$, completing the proof of Theorem 3.0.2.

3.3.4 Gradings

We follow Bloom's [3] treatment of gradings for the spectral sequence. We refer to the mod 2 grading on the complex $\mathrm{C}(\mathbb{Y})$ defined in §2.6 as $\mathrm{gr}[\mathbb{Y}]$. We define a grading $\mathrm{gr}[\mathbf{C}]$ on the complex $\mathbf{C}$ in (3.12). For $x \in \mathrm{C}(\mathbb{Y}_v) \subset \mathbf{C}$ with homogeneous $\mathrm{gr}[\mathbb{Y}_v]$ grading, we define

$$\mathrm{gr}[\mathbf{C}](x) \equiv \mathrm{gr}[\mathbb{Y}_v](x) + \deg(X_{\boldsymbol{\infty} v}) + |v|_1 \mod 2. \tag{3.14}$$

We conflate $\boldsymbol{\infty}$ with $-\mathbf{1} \in \mathbb{Z}^m$. Let $\pi_w : \mathbf{C} \to \mathrm{C}(\mathbb{Y}_w)$ be the projection. Note that

$$\mathrm{gr}[\mathbf{C}](\pi_w(\boldsymbol{\partial}(x))) \equiv \mathrm{gr}[\mathbb{Y}_w](m_{vw}(x)) + \deg(X_{\boldsymbol{\infty} w}) + |w|_1 \mod 2. \tag{3.15}$$

We have the additivity relation $\deg(X_{\boldsymbol{\infty} w}) \equiv \deg(X_{\boldsymbol{\infty} v}) + \deg(X_{vw})$, and also

$$\mathrm{gr}[\mathbb{Y}_w](m_{vw}(x)) = \mathrm{gr}[\mathbb{Y}_v](x) + \dim(G_{vw}) + \deg(X_{vw}).$$

Knowing $\dim(G_{vw}) = |w-v|_1 - 1$ shows that the expressions (3.14) and (3.15) differ by 1 mod 2, and thus the differential $\boldsymbol{\partial}$ alters $\mathrm{gr}[\mathbf{C}]$ by 1.

The quasi-isomorphism $\mathbf{Q} : \mathrm{C}(\mathbb{Y}) \to \mathbf{C}$ is a composition of m maps $\mathbf{F}$ as in the previous section. Thus it is a sum of maps of the form $m_G(\mathbb{X}_{\boldsymbol{\infty} v})$, where $v \in \{0,1\}^m$ and $G = G_1 \times \cdots \times G_m$. Here $G_i = G_{v(i)v(i+1)}$ only varies on $X_{v(i)v(i+1)} \subset X_{\boldsymbol{\infty} v}$ and $\boldsymbol{\infty} = -\mathbf{1} = v(1) < v(2) < \cdots < v(m+1) = v$. Using $\dim(G_{vw}) = |w-v|_1 - 1$ for $v < w$, we find $\dim(G) = |v|_1$. Since the $\mathrm{gr}[\mathbb{Y}]$ to $\mathrm{gr}[\mathbb{Y}_v]$ degree of $m_G(\mathbb{X}_{\boldsymbol{\infty} v})$ is $\dim(G) + \deg(X_{\boldsymbol{\infty} v})$, it follows that $\mathbf{Q}$ preserves the $\mathbb{Z}/2$-gradings.

There is also a $\mathbb{Z}$-grading on $\mathbf{C}$ given by the vertex weight $|v|_1$ for a homogeneous element in $\mathrm{C}(\mathbb{Y}_v) \subset \mathbf{C}$, and by construction $\boldsymbol{\partial}$ increases this by 1. We summarize a more detailed statement of Theorem 3.0.2; compare [20, Cors. 6.9, 6.10] and [3, Thm. 1.1].

Theorem 3.3.1. *Let L be an oriented, framed link with m components in Y. For each $v \in \{\infty, 0, 1\}^m$ denote by Y_v the result of v-surgery on L and let $\mathbb{Y}_v$ be an admissible bundle over Y_v such that the total collection of $\mathbb{Y}_v$ forms a surgery cube. For $v < w$ there are surgery cobordism bundles $\mathbb{X}_{vw}$ from $\mathbb{Y}_v$ to $\mathbb{Y}_w$ with I-orientations μ_{vw} satisfying $\mu_{uw} \circ \mu_{vu} = \mu_{vw}$ whenever $v < u < w$, such that there is a spectral sequence (E^r, d^r) with*

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I(\mathbb{Y}_v), \qquad d^1 = \sum_{\substack{v<w \\ |w-v|_1 = 1}} (-1)^{\delta(v,w)} I(\mathbb{X}_{vw})$$

where $\delta(v,w) = \sum_{1 \le i \le j} v_j$, in which j is the unique index where v and w differ. The spectral sequence is graded by $\mathbb{Z}/2 \times \mathbb{Z}$, where d^r has bi-degree $(1, r)$. The $\mathbb{Z}/2$-grading is given by (3.14) while the $\mathbb{Z}$-grading is by vertex weight. The spectral sequence converges by the E^{m+1}-page to $I(\mathbb{Y})$, and it induces the usual $\mathbb{Z}/2$-grading on $I(\mathbb{Y})$.

CHAPTER 4

Framed instanton homology: $I^{\#}(Y)$

In this chapter we discuss the basic constructions and properties of the framed instanton groups $I^{\#}(Y)$. These are a special case of the groups $I^{\#}(Y, K)$ introduced by Kronheimer and Mrowka in [20]. Here Y is a 3-manifold and K is a knot or link in Y, and we have $I^{\#}(Y) = I^{\#}(Y, \emptyset)$. The name *framed instanton homology* comes from [21]. The group $I^{\#}(Y)$ is isomorphic to the sutured instanton group $\mathrm{SHI}(M, \gamma)$ from [22], where M is the complement of an open 3-ball in Y and γ is a suture on the 2-sphere boundary.

4.1 Framed instanton groups

Let Y be a connected, oriented, closed 3-manifold. Consider an SO(3)-bundle $\mathbb{Y}^{\#}$ over $Y \# T^3$ with $\mathbb{Y}^{\#}$ trivial over Y and non-trivial over T^3. To make the construction of $\mathbb{Y}^{\#}$ from Y more precise, we can once and for all pick a point $x \in T^3$, a bundle $\mathbb{T}^3$ over T^3 geometrically represented by an S^1-factor, and an isomorphism $\mathbb{T}^3_x \simeq \mathrm{SO}(3)$. Then, up to inessential choices, $\mathbb{Y}^{\#}$ can be constructed from Y and a basepoint $y \in Y$. Indeed, we can perform the connected sum $Y \# T^3$ between 3-balls around y and x, and glue the bundles $Y \times \mathrm{SO}(3)$ and $\mathbb{T}^3$ by expanding the isomorphism $\mathrm{SO}(3) \simeq \mathbb{T}^3_x$ near x.

We describe a useful operation for cobordisms in this context. Let $X : Y_1 \to Y_2$ be a cobordism and let $\gamma : [0, 1] \to X$ be a properly embedded path with $\gamma(0)$ and $\gamma(1)$ being the chosen basepoints in Y_1 and Y_2, respectively. Given another such pair

X', γ' where $X' : Y_1' \to Y_2'$, we form a cobordism

$$X \bowtie X' : Y_1 \# Y_1' \to Y_2 \# Y_2'$$

as follows. Let Γ be a neighborhood of γ diffeomorphic to $\text{int}(D^3) \times [0,1]$. Write

$$\partial(X \setminus \Gamma) \setminus (Y_1 \cup Y_2 \setminus \partial\Gamma) = S^2 \times [0,1];$$

do the same for X'. Then identify the copies of $S^2 \times [0,1]$ by an orientation reversing homeomorphism. See Figure 4.1. We omit the paths from the notation $X \bowtie X'$ because for all of our cobordisms there will be a natural choice of path up isotopy relative to the boundaries. The operation $\bowtie$ extends to glue together cobordisms of bundles $\mathbb{X}$ and $\mathbb{X}'$ if a path of isomorphisms $\mathbb{X}_{\gamma(t)} \simeq \mathbb{X}'_{\gamma'(t)}$ is chosen.

Let g be a gauge transformation of $\mathbb{Y}^{\#}$ with $\eta(g) \in H^1(Y \# T^3; \mathbb{F}_2)$ Poincaré dual to a 2-torus $\Sigma \subset T^3$ over which $\mathbb{Y}^{\#}$ is non-trivial. Here $\eta : \mathscr{G}(\mathbb{X}) \to H^1(X; \mathbb{F}_2)$ is from the exact sequence (2.1). Such a transformation may be constructed explicitly as in [9, Lemma A.2]. Define the framed gauge transformations $\mathscr{G}^{\#}$ to be the subgroup of $\mathscr{G}(\mathbb{Y}^{\#})$ generated by $\mathscr{G}_{\text{ev}}(\mathbb{Y}^{\#})$ and g. We let $\mathfrak{C}^{\#}$ denote the critical set of a perturbed Chern-Simons functional $\mathbf{cs}_\pi$ on $\mathscr{C}/\mathscr{G}^{\#}$. Note that $\mathfrak{C}^{\#}$ is obtained from $\mathfrak{C}(\mathbb{Y}^{\#})$ by modding out by the $\mathbb{Z}/2$-action of degree 4 induced by the gauge transformation g.

We define the chain complex $\mathrm{C}^{\#}(Y)$ for $I^{\#}(Y)$ following ideas from [20, §4.4]. This definition transparently replaces the notion of an I-orientation with that of a homology orientation. Fix once and for all a bundle $\mathbb{W} : S^3 \times \mathrm{SO}(3) \to \mathbb{T}^3$ over $T^2 \times D^2 \setminus \text{int}(D^4) : S^3 \to T^3$ extending $\mathbb{T}^3$. Fix a path γ in W beginning in S^3, ending at $x \in T^3$, and a path of isomorphisms $\mathbb{W}_{\gamma(t)} \simeq \mathrm{SO}(3)$, the isomorphisms at the ends being the natural choices. We define

$$\mathrm{C}^{\#}(Y) = \bigoplus_{\mathfrak{a} \in \mathfrak{C}^{\#}} \mathbb{Z}\Lambda^{\#}(\mathfrak{a})$$

where $\Lambda^{\#}(\mathfrak{a})$ is the 2-element set of orientations of the line $\det(D_A)$; here A is a connection on $[0,1] \times \mathbb{Y} \bowtie \mathbb{W}$ (with cylindrical ends attached) where the limit of A

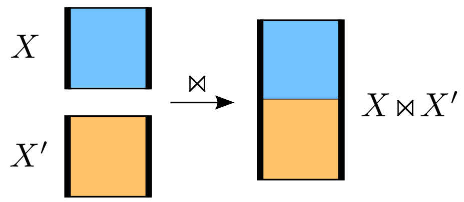

Figure 4.1: A schematic depiction of the $\bowtie$ operation. The thicker lines represent actual boundary components.

over the $\mathbb{R} \times \mathbb{Y}$ cylindrical end is equivalent to the trivial connection, and the limit of A over the $\mathbb{R} \times \mathbb{Y}^\#$ end is in the class $\mathfrak{a}$. The operator D_A is as in §2.3.

The differential for $\mathrm{C}^\#(Y)$ is straight-forward to define, following the construction of the differential for $I(\mathbb{Y})$ in §2.1, which followed [20, §3.6]. Note that a base connection as in the definition for $\mathrm{C}(\mathbb{Y})$ is no longer needed. In summary, given Y with a basepoint, with suitable metric and perturbation, the complex $\mathrm{C}^\#(Y)$ and hence the group $I^\#(Y)$ are determined. The isomorphism class of $I^\#(Y)$ depends only on Y.

4.2 Maps from cobordisms

We describe how a cobordism $X : Y_1 \to Y_2$ with a path γ as above gives rise to a map $I^\#(X) : I^\#(Y_1) \to I^\#(Y_2)$. Again, we omit γ from the notation because there will always be a natural choice for us. We always assume X and Y_1, Y_2 are connected. Take the path in $T^3 \times [0,1]$ given by $t \mapsto (x,t)$. Using this we form

$$X^\# = X \bowtie (T^3 \times [0,1]) : Y_1 \# T^3 \to Y_2 \# T^3.$$

Further, there is a natural choice for bundle $\mathbb{X}^\#$ over $X^\#$ by performing the $\bowtie$ operation between $X \times \mathrm{SO}(3)$ and $\mathbb{T}^3 \times [0,1]$ using the constant path of isomorphisms $\mathrm{SO}(3) \simeq \mathbb{T}^3_x$. We enlarge the even gauge transformation group used for $\mathbb{X}^\#$ to include gauge transformations whose restriction to each T^3 is of the form g from §4.1 above. See [20, §5.1] for a general discussion. Then, in the usual way, we obtain a chain map $m^{\#}(X) : \mathrm{C}^{\#}(Y_1) \to \mathrm{C}^{\#}(Y_2)$ and an induced map $I^{\#}(X)$ on homology.

The data of an I-orientation may be replaced by an orientation of the line $\det(D_A)$ where A is the trivial connection on $X \times \mathrm{SO}(3)$. Following [21, §3.8], but using homology instead of cohomology, this amounts to an orientation of the vector space

$$H_1(Y_1; \mathbb{R}) \oplus H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R}),$$

where $H_2^+(X; \mathbb{R})$ is a maximal positive definite subspace for the intersection form on $H_2(X; \mathbb{R})$. A choice of such an orientation is called a *homology orientation* for the cobordism X, and is typically denoted μ_X. In summary, given $X : Y_1 \to Y_2$, a path γ from the basepoint of Y_1 to the basepoint of Y_2, a suitable perturbation and metric, and a homology orientation μ_X, the chain map $m^{\#}(X)$ is determined. The induced map $I^{\#}(X)$ depends on X, μ_X, and presumably γ.

We define $I^{\#}(\emptyset) = I^{\#}(S^3)$, and when $X : \emptyset \to \partial X$, we define the map $I^{\#}(X)$ by deleting a 4-ball in X. In particular, when X is a compact, connected, oriented 4-manifold with connected boundary, and an orientation of $H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R})$ is chosen, we obtain an element

$$[X]^{\#} \in I^{\#}(\partial X).$$

We also obtain a map $[X]_{\#} : I^{\#}(\overline{\partial X}) \to \mathbb{Z}$ by viewing $X : \overline{\partial X} \to \emptyset$ and orienting $H_1(\partial X; \mathbb{R}) \oplus H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R})$. If X is a closed, connected, oriented 4-manifold and $H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R})$ is oriented, then we have a number $[X]^{\#} \in \mathbb{Z}$.

Finally, we mention another topological operation that arises naturally in this setting. This is the boundary sum $W \natural W'$ of two 4-manifolds with boundary, as used in [18]; one deletes a model half-4-ball along the boundaries of W and W' and glues them together with an orientation-reversing homeomorphism, so that $\partial(W \natural W') = \partial W \# \partial W'$. We have

$$(X \bowtie X') \circ (W \natural W') \simeq (X \circ W) \natural (X' \circ W')$$

where compositions involved are of course assumed to make sense, and the same relation holds with the compositions reversed. See Figure 4.2.

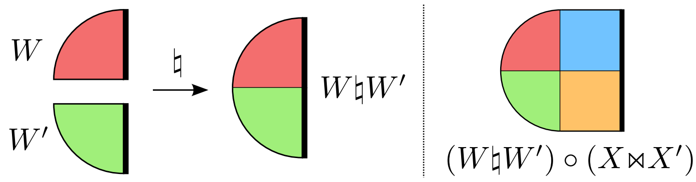

Figure 4.2: On the left, a schematic depiction of the boundary sum ♮ operation. On the right, we compose the ⋈ operation against ♮, and the result may be interpreted as involving only ♮. The thick lines represent actual boundary components.

4.3 Gradings

We now define the absolute $\mathbb{Z}/4$-grading on $I^{\#}(Y)$. Let $\mathbb{W}'$ be a completion of $\mathbb{W}$ from §4.1 with the 4-ball filled in, so that it is a non-trivial bundle over $T^2 \times D^2$, and we may write $\mathbb{W}' : \emptyset \to \mathbb{T}^3$. Fix an integer k. For $\mathfrak{a} \in \mathfrak{C}^{\#}(Y)$ we define

$$\mathrm{gr}(\mathfrak{a}) := -\mu(\mathbb{E} \natural \mathbb{W}', \mathfrak{a}) - b_1(E) + b_+(E) - b_1(Y) + k \mod 4$$

where $E : \emptyset \to Y$ is a 4-manifold with boundary Y and $\mathbb{E} = E \times \mathrm{SO}(3)$. We choose k such that $I^{\#}(S^3)$ is supported in grading 0. The proof that this grading is well-defined is the same as the case of the absolute mod 2 grading for $I(\mathbb{Y})$ as for example in [7]; we get $\mathbb{Z}/4$ instead of $\mathbb{Z}/2$ because the characteristic classes of the bundles are uniformly controlled in this case. We give the argument for completeness, and compute the degrees of cobordism maps. We have chosen our conventions so that the degree formula aligns with that of [20, Prop. 4.4].

Proposition 4.3.1. *The assignment $\mathfrak{a} \mapsto \mathrm{gr}(\mathfrak{a})$ gives a well-defined $\mathbb{Z}/4$-grading on $C^{\#}(Y)$ for which the differential has degree -1 and thus descends to a $\mathbb{Z}/4$-grading on $I^{\#}(Y)$. Given a cobordism $X : Y_1 \to Y_2$ equipped with the data to form $X^{\#}$ as in §4.2, the degree of the induced map $I^{\#}(X) : I^{\#}(Y_1) \to I^{\#}(Y_2)$ is given by the expression for deg(X) in (2.11) taken modulo 4. More generally, if $\mathbb{Y}_i = Y_i \times SO(3)$ and $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ is possibly non-trivial and comes equipped with the data to form $\mathbb{X}^{\#}$, then the degree of the induced map $I^{\#}(\mathbb{X}) : I^{\#}(Y_1) \to I^{\#}(Y_2)$ is given by*

$$-\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)) + 2\mathscr{P}(\mathbb{X}) \mod 4 \tag{4.1}$$

where the invariant $\mathscr{P}(\mathbb{X}) \in \mathbb{Z}/2$ *is defined by*

$$\mathscr{P}(\mathbb{X}) \equiv [S] \cdot [S] \mod 2.$$

Here $S \subset X$ *is a surface in the interior of* X, $[S] \in H_2(X; \mathbb{F}_2)$, *and the image of* $[S]$ *in* $H_2(X, \partial X; \mathbb{F}_2)$ *is Poincaré dual to* $w_2(\mathbb{X})$.

Proof. Let $E' : Y \to \emptyset$ and $\mathbb{E}' = E' \times \mathrm{SO}(3)$, and let $\mathbb{W}''$ be the reverse of $\mathbb{W}'$. In particular, we may write $\mathbb{W}'' : \mathbb{T}^3 \to \emptyset$. Then by (2.3) we have

$$\mu(\mathbb{E} \natural \mathbb{W}', \mathfrak{a}) + \mu(\mathfrak{a}, \mathbb{E}' \natural \mathbb{W}'') = \mu((\mathbb{E}' \circ \mathbb{E}) \# (\mathbb{W}'' \circ \mathbb{W}')). \tag{4.2}$$

By (2.3) we may write the right hand side as

$$\mu(\mathbb{E}' \circ \mathbb{E}) + 3 + \mu(\mathbb{W}'' \circ \mathbb{W}').$$

Note $\mathbb{W}'' \circ \mathbb{W}'$ is a bundle over $T^2 \times S^2$, which necessarily has p_1 congruent to 0 mod 4. Also, $(1 - b_1 + b_+)(T^2 \times S^2) = 0$. By (2.4) we conclude that $\mu(\mathbb{W}'' \circ \mathbb{W}')$ is congruent to 0 mod 4. Since $\mathbb{E}' \circ \mathbb{E}$ is a trivial bundle, (4.2) is mod 4 congruent to

$$\mu(\mathbb{E}' \circ \mathbb{E}) + 3 = 3(b_1 - b_+)(E' \circ E)$$

which by a Mayer-Vietoris argument (see §5.1) is mod 4 congruent to

$$-b_1(E) - b_1(E') + b_+(E) + b_+(E') + b_1(Y).$$

It follows that the expression

$$\mathrm{gr}(\mathfrak{a}) - \mu(\mathfrak{a}, \mathbb{E}' \natural \mathbb{W}'') = b_1(E') - b_+(E') - 2b_1(Y) + k \mod 4$$

is independent of $\mathbb{E}$, and thus so is $\mathrm{gr}(\mathfrak{a})$. In other words, $\mathrm{gr}(\mathfrak{a})$ is a well-defined $\mathbb{Z}/4$-grading on $\mathrm{C}^\#(Y)$. Suppose $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}^\#(Y)$ with $\mu(\mathfrak{a}, \mathbb{R} \times \mathbb{Y}^\#, \mathfrak{b}) = 1$. Then

$$\mu(\mathbb{E} \natural \mathbb{W}', \mathfrak{a}) + \mu(\mathfrak{a}, \mathbb{R} \times \mathbb{Y}^\#, \mathfrak{b}) = \mu(\mathbb{E} \natural \mathbb{W}', \mathfrak{b})$$

yields $\mathrm{gr}(\mathfrak{b}) - \mathrm{gr}(\mathfrak{a}) = -1$. It follows that the differential lowers the grading by 1. Now we compute the degree of a map $I^\#(X)$ induced by a cobordism $X : Y_1 \to Y_2$. Let

$\mathbb{X} = X \times \mathrm{SO}(3)$ and form $\mathbb{V} = \mathbb{X} \bowtie (\mathbb{T}^3 \times [0,1])$. Let $\mathfrak{a} \in \mathfrak{C}^{\#}(Y_1)$ and $\mathfrak{b} \in \mathfrak{C}^{\#}(Y_2)$ with $\mu(\mathfrak{a}, \mathbb{V}, \mathfrak{b}) = 0$. Let $E : \emptyset \to Y_1$ and $\mathbb{E} = E \times \mathrm{SO}(3)$. Then (2.3) and $\mu(\mathfrak{a}, \mathbb{V}, \mathfrak{b}) = 0$ yield $\mu(\mathbb{V} \circ (\mathbb{E} \natural \mathbb{W}'), \mathfrak{b}) = \mu(\mathbb{E} \natural \mathbb{W}', \mathfrak{a})$. Thus $\deg(X) \equiv \mathrm{gr}(\mathfrak{b}) - \mathrm{gr}(\mathfrak{a})$ is given by

$$-b_1(X \circ E) + b_+(X \circ E) - b_1(Y_2) + b_1(E) - b_+(E) + b_1(Y_1).$$

From the discussion in §5.1, $-b_1(X \circ E) + b_+(X \circ E)$ is equal to

$$-b_1(E) - b_1(X) + b_+(E) + b_+(X) + b_1(Y_1).$$

We obtain the simplified expression

$$\deg(X) \equiv -b_1(X) + b_+(X) + 2b_1(Y_1) - b_1(Y_2) \mod 4. \tag{4.3}$$

Using the assumption that X, Y_1 and Y_2 are connected and non-empty, we have $\chi(X) = 1 - b_1(X) + b_2(X) - b_3(X)$. Poincaré-Lefschetz duality tells us $b_3(X) = b_1(X, \partial X)$, and by the long exact sequence for the pair $(X, \partial X)$ with real coefficients we obtain

$$d - b_2(X) + b_1(\partial X) - b_1(X) + b_1(X, \partial X) - b_0(\partial X) + b_0(X) = 0,$$

where d is the dimension of the image of the map $H_2(X) \to H_2(X, \partial X)$. Note $b_0(\partial X) = 2$ and $b_0(X) = 1$. On the other hand, $d = b_+(X) + b_-(X)$ and $\sigma(X) = b_+(X) - b_-(X)$. We obtain

$$\chi(X) = -2b_1(X) + b_1(Y_1) + b_1(Y_2) + d, \quad \sigma(X) = 2b_+(X) - d.$$

Plugging this data into expression (2.11), rewritten here as

$$-\frac{3}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}(b_1(Y_2) - b_1(Y_1)),$$

yields, modulo 4, the expression for $\deg(X)$ in (4.3). Now we approach the more general statement, supposing that $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ is possibly non-trivial. We write

$$\deg(\mathbb{X}) \equiv \deg(X) + 2\mathscr{P}(\mathbb{X}) \mod 4,$$

where $\mathscr{P}(\mathbb{X})$ is to be determined. Let $E_1 : \emptyset \to Y_1$ and $\mathbb{E}_1 = E_1 \times \mathrm{SO}(3)$. Let $\mathfrak{a} \in \mathfrak{C}^{\#}(Y_1)$, $\mathfrak{b} \in \mathfrak{C}^{\#}(Y_2)$ with $\mu(\mathfrak{a}, \mathbb{X}^{\#}, \mathfrak{b}) \equiv 0$. Write $\mathbb{X}_{\mathrm{tr}} = X \times \mathrm{SO}(3)$. Then

$$\deg(\mathbb{X}) - \deg(X) \equiv \mu(\mathbb{X} \circ \mathbb{E}_1 \natural \mathbb{W}', \mathfrak{b}) - \mu(\mathbb{X}_{\mathrm{tr}} \circ \mathbb{E}_1 \natural \mathbb{W}', \mathfrak{b}).$$

After closing up bundles using some $E_2 : Y_2 \to \emptyset$ with $\mathbb{E}_2 = E_2 \times \mathrm{SO}(3)$ and cancelling out the contribution from the bundle over $T^2 \times S^2$ as above, this difference is seen from (2.4) to be

$$-2p_1(\mathbb{E}_2 \circ \mathbb{X} \circ \mathbb{E}_1) = \frac{1}{4\pi^2} \int_{E_2 \circ X \circ E_1} \mathrm{tr}(F_A^2),$$

where A is any connection. We can choose A to be trivial away from the interior of X, thus

$$\mathscr{P}(\mathbb{X}) \equiv \frac{1}{8\pi^2} \int_X \mathrm{tr}(F_A^2) \mod 2$$

where A is any connection on $\mathbb{X}$ that restricts to trivial connections on each $\mathbb{Y}_i$. In other words, $\mathscr{P}(\mathbb{X}) \equiv p_1(\mathbb{X}') \mod 2$, where $\mathbb{X}'$ is any trivial extension of $\mathbb{X}$ over a closed 4-manifold. Thus

$$\mathscr{P}(\mathbb{X}) \equiv \widetilde{w}_2(\mathbb{X})^2 \mod 2,$$

where $\widetilde{w}_2(\mathbb{X})$ is a lift of $w_2(\mathbb{X})$ to $H^2(X, \partial X; \mathbb{F}_2)$. The result follows. □

4.4 Duality

The chain group $\mathrm{C}^{\#}(\overline{Y})$ is the same as $\mathrm{C}^{\#}(Y)$ but with the differential maps transposed. It follows that $I^{\#}(Y)$ and $I^{\#}(\overline{Y})$ are isomorphic over $\mathbb{Q}$. More precisely, given a homology orientation of Y, i.e. an orientation of $H_1(Y; \mathbb{R})$, we get an isomorphism

$$I^{\#}(\overline{Y}; \mathbb{Q})_i \simeq I^{\#}(Y; \mathbb{Q})^*_{b_1(Y)-i}. \tag{4.4}$$

The homology orientation is required to identify the chain groups. The grading shift in (4.4) is explained as follows. Let $E_1 : \emptyset \to Y$ and $E_2 : Y \to \emptyset$, and $\mathfrak{a} \in \mathfrak{C}^{\#}(Y)$. Write $\overline{\mathfrak{a}}$ for the corresponding class in $\mathfrak{C}^{\#}(\overline{Y})$. From (2.3) and (2.4) we obtain

$$\mu(\mathbb{E}_1 \natural \mathbb{W}', \mathfrak{a}) + \mu(\overline{\mathfrak{a}}, \mathbb{E}_2 \natural \mathbb{W}'') = 3(b_1(E) - b_+(E))$$

where $\mathbb{E}_i = E_i \times \mathrm{SO}(3)$, $E = E_2 \circ E_1$, and $\mathbb{W}''$ is the reverse of $\mathbb{W}'$. The bundle $\mathbb{W}'' \circ \mathbb{W}'$ over $T^2 \times S^2$ has been removed from the expression just as in §4.3. Using that $b_1(E) - b_+(E)$ is equal to

$$b_1(E_1) + b_1(E_2) - b_+(E_1) - b_+(E_2) - b_1(Y),$$

see §5.1, we obtain $\mathrm{gr}(\mathfrak{a}) + \mathrm{gr}(\bar{\mathfrak{a}}) \equiv b_1(Y) + 2k$. We claim k is even. Let $\mathfrak{a}$ be the generator of $I^{\#}(S^3)$, represented by a flat connection on $T^3 \simeq S^3 \# T^3$. Recall that k is chosen so that $I^{\#}(S^3)$ is supported in grading 0, so we have $\mathrm{gr}(\mathfrak{a}) = 0$ (also see §4.7). In the definition of $\mathrm{gr}(\mathfrak{a})$, choose $\mathbb{E} : \emptyset \to S^3$ to be a trivial bundle over a 4-ball. Then

$$0 \equiv \mathrm{gr}(\mathfrak{a}) \equiv -\mu(\mathbb{W}', \mathfrak{a}) + k \mod 4.$$

Recall from the proof of Prop. 4.3.1 that $\mu(\mathbb{W}'' \circ \mathbb{W}') \equiv 0 \bmod 4$, where $\mathbb{W}'' : \mathbb{T}^3 \to \emptyset$ is the reverse bundle-cobordism of $\mathbb{W}'$. By the index gluing formula (2.3) we then have $-\mu(\mathbb{W}', \mathfrak{a}) \equiv \mu(\mathfrak{a}, \mathbb{W}'') \bmod 4$. Since W' is diffeomorphic to its orientation reversal, which is W'', we also have $\mu(\mathbb{W}', \mathfrak{a}) \equiv \mu(\mathfrak{a}, \mathbb{W}'') \bmod 4$, as follows from the Atiyah-Patodi-Singer index formula [1, Thm. 3.10]. Thus $k \equiv \mu(\mathbb{W}', \mathfrak{a}) \equiv 0 \bmod 2$. It follows that

$$\mathrm{gr}(\mathfrak{a}) + \mathrm{gr}(\bar{\mathfrak{a}}) \equiv b_1(Y) \mod 4,$$

establishing the grading shift in (4.4).

4.5 Twisted framed groups $I^{\#}(Y; \lambda)$

To state a framed instanton exact triangle, it is necessary to allow non-trivial bundles in our constructions. In the above sections of this chapter, take $\mathbb{Y}^{\#}$ to be geometrically represented by $\lambda \cup \omega$ where $\lambda \subset Y$ is a closed, unoriented 1-manifold and ω is an S^1-factor of T^3. We obtain a group $I^{\#}(Y; \lambda)$ that is now only relatively $\mathbb{Z}/4$-graded. We refer to $I^{\#}(Y; \lambda)$ as the *framed instanton homology of Y twisted by λ*. It is isomorphic to four consecutive gradings of the relatively $\mathbb{Z}/8$-graded group $I(\mathbb{Y}^{\#})$.

The isomorphism class of $I^{\#}(Y;\lambda)$ depends only on the oriented homeomorphism type of Y and the class $[\lambda] \in H_1(Y;\mathbb{F}_2)$.

4.6 Exact triangles

In this section we state a few exact triangles for framed instanton homology. Let Y be a closed, connected, oriented 3-manifold and $\lambda \subset Y$ a closed, unoriented 1-manifold. Let K be a framed knot in Y disjoint from λ. Denote by Y_i the result of i-surgery on K. Let μ be the core of the knot K as viewed in Y_0. Then we have an exact triangle

$$\cdots I^{\#}(Y;\lambda) \to I^{\#}(Y_0;\lambda \cup \mu) \to I^{\#}(Y_1;\lambda) \to I^{\#}(Y;\lambda) \cdots$$

There are two other exact triangles corresponding to the two other rows in Figure 3.1. For example, if we view μ as the core of the knot inside Y_i where $i = \infty$ or $i = 1$, the exact sequence has μ appearing in the twisting for the group of Y_i, and not the other two. Each of these is an application of Floer's original exact triangle, Theorem 3.0.1, obtained by connected summing each 3-manifold with T^3 and performing the surgeries away from T^3, with the appropriate overlying bundles.

By changing the framing of K, we obtain variants of the above triangles that are computationally handy. Let l and m be the longitude and meridian of K, respectively. Suppose the meridian is unchanged but the longitude is changed to $-pm + l$. Then we have

$$\cdots I^{\#}(Y;\lambda) \to I^{\#}(Y_p;\lambda \cup \mu) \to I^{\#}(Y_{p+1};\lambda) \to I^{\#}(Y;\lambda) \cdots$$

where again the core μ can be arranged in two other ways. Alternatively, keep the longitude the same but change the meridian to $m - ql$. Then we have

$$\cdots I^{\#}(Y_0;\lambda) \to I^{\#}(Y_{1/(q+1)};\lambda \cup \mu) \to I^{\#}(Y_{1/q};\lambda) \to I^{\#}(Y_0;\lambda) \cdots$$

where the same freedom with the placement of μ is understood. For other variants, we refer the reader to [23, §42.1].

For an alternative perspective, one can begin with a 3-manifold Z with torus boundary and consider the possible ordered triplets of Dehn fillings of Z that are compatible with a surgery triangle description. This is the viewpoint taken in [23, §42.1] and [35].

We mention that the mod 2 degrees of the the cobordism maps in these exact triangles is the same as the monopole case, and is explained in [23, §42.3]. There are always non-trivial bundles amongst the 3 cobordism maps, even if the three framed groups are untwisted. For in this case the composite of three consecutive cobordism bundles, call it $\mathbb{X}_{03}$ as in §3.1.3, has $\mathscr{P}(\mathbb{X}_{03}) \equiv 1 \mod 2$. This is because $\mathbb{X}_{03}$ is trivial away from a copy of $-\mathbb{CP}^2$ minus a thickened S^1; over this area it restricts to a non-trivial bundle $\mathbb{E}$ which is easily seen to have $\mathscr{P}(\mathbb{E}) \equiv 1$. Then, by the additivity of $\mathscr{P}(\mathbb{X})$, at least one of $\mathbb{X}_{i,i+1}$ has $\mathscr{P}(\mathbb{X}_{i,i+1}) \equiv 1$. Note that, after computing $\deg(X_{03}) = 1$, we see $\deg(\mathbb{X}_{03}) \equiv -1 \mod 4$.

4.7 Examples

In this section we consider the framed instanton homology of S^3 and $S^1 \times S^2$. To compute $I^\#(S^3)$ it suffices to compute $I(\mathbb{T}^3)$. This is well-known and elementary, see [5]. Let N be a regular neighborhood of the geometric representative for $\mathbb{T}^3$. The flat connections modulo even gauge on $\mathbb{T}^3$ are in correspondence with the set

$$\{\rho \in \mathrm{Hom}(\pi_1(T^3 \setminus N), \mathrm{SU}(2)) \mid \rho(\nu) = -1\}/\mathrm{SU}(2),$$

where ν is a small meridian around N, and the SU(2)-action is by conjugation. A computation shows that this set consists of two elements; these two elements are non-degenerate and irreducible. The two classes as generators for $\mathrm{C}(\mathbb{T}^3)$ differ by degee 4. It follows that $\mathrm{C}^\#(S^3)$ has one generator, and we obtain

$$I^\#(S^3) \simeq \mathbb{Z}_0,$$

where, as usual, the subscript indicates the grading. We usually assume a distinguished generator for $I^\#(S^3)$ has been fixed.

Next, we compute $I^\#(S^1 \times S^2)$. For this we adapt [20, Lemma 8.3]. By placing the twisting μ at an S^3, we have an exact sequence

$$\cdots I^\#(S^3) \xrightarrow{\alpha} I^\#(S^1 \times S^2) \xrightarrow{\beta} I^\#(S^3) \xrightarrow{\gamma} I^\#(S^3) \cdots$$

We apply the grading formula (2.11). The map α comes from the cobordism $D^2 \times S^2 \setminus \mathrm{int}(D^4)$ from S^3 to $S^1 \times S^2$. The overlying bundle is necessarily trivial. We compute $\deg(\alpha) \equiv -1$. The map β is the same cobordism, but reversed, and $\deg(\beta) \equiv -2$. By the previous section, we know the sum of the degrees of the three maps is -1 mod 4, so $\deg(\gamma) \equiv 2$. This can be computed directly by observing that γ comes from the cobordism $-\mathbb{CP}^2$ minus two 4-balls, from S^3 to S^3, with a non-trivial bundle. Because $\gamma : \mathbb{Z}_0 \to \mathbb{Z}_0$ has degree 2, it must be 0. By exactness, we conclude

$$I^\#(S^1 \times S^2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_3,$$

where, as usual, the subscripts indicate gradings. As is evident by the above computation, a canonical generator in grading 3 for $I^\#(S^1 \times S^2)$ is given by $[D^2 \times S^2]^\#$. Recall from §4.2 that $[D^2 \times S^2]^\#$ is the notation for the relative invariant induced by the cobordism $D^2 \times S^2 : \emptyset \to S^1 \times S^2$. A canonical homology orientation is used here. The element $[S^1 \times D^3]^\#$ generates the summand in grading 2. This is seen by identifying $S^1 \times S^2$ with its orientation-opposite in a standard way, and viewing $[D^2 \times S^2]_\#$ as a map $I^\#(S^1 \times S^2) \to \mathbb{Z}$. For then

$$[D^2 \times S^2]_\# [S^1 \times D^3]^\# = \pm 1,$$

since $S^1 \times D^3$ and $D^2 \times S^3$ glue along $S^1 \times S^2$ to give S^4. However, the element $[S^1 \times D^3]^\#$ is not canonically homology oriented; it requires an orientation of $H_1(S^1 \times S^2; \mathbb{R})$. Thus a generator for $\mathbb{Z}_2 \subset I^\#(S^1 \times S^2)$ is distinguished by orienting the vector space $H_1(S^1 \times S^2; \mathbb{R})$.

4.8 The Künneth formula

Let Y and Y' be closed, oriented and connected 3-manifolds. If either one of $I^{\#}(Y)$ or $I^{\#}(Y')$ is torsion-free, there is a graded isomorphism

$$I^{\#}(Y\#Y') \simeq I^{\#}(Y) \otimes I^{\#}(Y').$$

This is a special case of [20, Cor. 5.9], and follows from Floer's original excision theorem. Further, this isomorphism is natural for split cobordisms, in the following sense. Let $X : Y_1 \to Y_2$ and $X' : Y_1' \to Y_2'$ be cobordisms with paths chosen so that the composite $X \bowtie X'$ is defined. Suppose the above product isomorphism holds for $Y_1\#Y_1'$ and $Y_2\#Y_2'$; then we have a commutative diagram

$$\begin{array}{ccc} I^{\#}(Y_1\#Y_1') & \xrightarrow[\simeq]{} & I^{\#}(Y_1) \otimes I^{\#}(Y_1') \\ {\scriptstyle I^{\#}(X\bowtie X')}\downarrow & & \downarrow{\scriptstyle I^{\#}(X)\otimes I^{\#}(X')} \\ I^{\#}(Y_2\#Y_2') & \xrightarrow{\simeq} & I^{\#}(Y_2) \otimes I^{\#}(Y_2') \end{array}$$

We do not address the arrangement of homology orientations, as we will not require it in the sequel.

4.9 A connected sum of $S^1 \times S^2$'s

Let Y be a 3-manifold with $Y \simeq \#^k S^1 \times S^2$. From the Künneth formula it is clear that $I^{\#}(Y) \simeq \otimes^k(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$. The subscripts here indicate gradings. Let μ_Y be an orientation of $H_1(Y;\mathbb{R})$. In this section we establish the following.

Proposition 4.9.1. *There is an isomorphism of abelian groups*

$$\phi : \textstyle\bigwedge^*(H_1(Y;\mathbb{Z})) \to I^{\#}(Y)$$

which only depends on Y and μ_Y, not the decomposition $Y \simeq \#^k S^1 \times S^2$. The choice of μ_Y only affects the overall sign of ϕ. For $x \in \bigwedge^i(H_1(Y;\mathbb{Z}))$ with $b_1(Y) = k$, the grading of $\phi(x)$ in $I^{\#}(Y)$ is given by $2k + i$ mod 4.

Proof. Choose oriented, closed, embedded curves $c_1, \dots, c_k$ in Y such that there exists a diffeomorphism $Y \simeq \#_{i=1}^k S^1 \times S^2$ sending c_i to $S^1 \times \mathrm{pt}$ in the i^{th} copy of $S^1 \times S^2$. Given $J = \{i_1, \dots, i_l\} \subset \{1, \dots, k\}$ we define a cobordism $X_J : \emptyset \to Y$ by starting with $Y \times [0,1]$ and attaching a 2-handle to each $c_i \times \{0\}$ if $i \in J$, and on top of this, attaching 3-handles and a 4-handle such that $\partial X_J = Y \times \{1\}$ and

$$X_J \simeq X_1 \natural \cdots \natural X_k, \qquad X_i = \begin{cases} S^1 \times D^3 & \text{if } i \notin J \\ D^2 \times S^2 & \text{if } i \in J \end{cases} \tag{4.5}$$

with ∂X_i the i^{th} copy of $S^1 \times S^2$ in the decomposition $Y \simeq \#_{i=1}^k S^1 \times S^2$. Let $\{1, \dots k\} \setminus J = \{i_{l+1}, \dots, i_k\}$ be such that $\mu_Y = [c_{i_1} \wedge \cdots \wedge c_{i_k}]$. To homology orient X_J we orient $\mathcal{L}(X_J) = H_1(X_J; \mathbb{R})$ by $[c_{i_{l+1}} \wedge \cdots \wedge c_{i_k}]$. Now we define the map $\psi : \bigwedge^*(c_1, \dots, c_k) \to I^\#(Y)$ by

$$\psi(c_{i_1} \wedge \cdots \wedge c_{i_l}) = [X_J]^\#.$$

This map is an isomorphism by the case $k = 1$ and the Künneth formula.

With the help of the orientation μ_Y of $H_1(Y; \mathbb{R})$, we can define a bilinear form

$$\langle \cdot, \cdot \rangle : I^\#(Y) \otimes I^\#(Y) \to \mathbb{Z},$$

see also (4.4). The elements $[X_J]^\#$ as J runs over subsets of $\{1, \dots, k\}$ form a basis for $I^\#(Y)$, so it suffices to define the form on these. Given $J, K \subset \{1, \dots, k\}$, let X_J and X_K be as above with homology orientations μ_J and μ_K, respectively. Then we have elements $[X_J]^\#, [X_K]^\# \in I^\#(Y)$. Consider $\overline{X}_K : Y \to \emptyset$ and homology orient it by $\mu_Y \wedge \mu_K$. This yields $[\overline{X}_K]_\# : I^\#(Y) \to \mathbb{Z}$. Then $\langle \cdot, \cdot \rangle$ is given by

$$\langle [X_K]^\#, [X_J]^\# \rangle = [\overline{X}_K]_\# [X_J]^\# = [\overline{X}_K \circ X_J]_\# = A_{JK} \in \mathbb{Z}.$$

Now observe that

$$X_K \circ X_J \simeq X_1 \# \cdots \# X_k, \qquad X_i \simeq \begin{cases} S^1 \times S^3 & \text{if } i \notin J \cup K \\ S^2 \times S^2 & \text{if } i \in J \cap K \\ S^4 & \text{otherwise} \end{cases} \tag{4.6}$$

Note $[S^2 \times S^2]^{\#} = 0$, because the degree of the cobordism $S^3 \to S^3$ given by $S^2 \times S^2$ minus two 4-balls is odd, and similarly for $[S^1 \times S^3]^{\#}$. Using the naturality with respect to split cobordisms of the Künneth formula, we conclude that $A_{JK} \neq 0$ if and only if J and K are complementary, and in this case $A_{JK} = \pm 1$. This sign may be determined by using Definition 5.1.1, but we will not need it. It is clear that this bilinear form is non-degenerate. Note that $\langle \cdot, \cdot \rangle$ depends on $c_1, \ldots, c_k$ (which determine an identification of Y with $\overline{Y}$).

We argue that ψ is independent of the 2-handle framings chosen to construct the X_J. First construct cobordisms X_J for each subset $J \subset \{1, \ldots, k\}$ as above. Choose some J, and construct a cobordism X'_J by attaching the 2-handles using possibly different framings as was done for X_J, subject to the constraint that X'_J is of the form (4.5). Then

$$\overline{X}_K \circ X'_J \simeq X_1 \# \cdots \# X_k$$

just as in (4.6), except now if $i \in J \cap K$ then X_i is a possibly non-trivial S^2-bundle over S^2, in which case $[X_i]^{\#} = 0$. We homology orient X'_J in the same way as X_J. It is easily seen that $[X'_J]^{\#}$ has all the same values A_{JK} as $[X_J]^{\#}$ under the bilinear pairing, and thus $[X'_J]^{\#} = [X_J]^{\#}$.

Now we see how ψ changes when we change the loops c_i. Consider replacing the oriented loop c_1 by an oriented connected sum $c_1 \# c_2$. There are many ways of forming this connected sum. Let $X_{c_1 \# c_2}$ be the cobordism $\emptyset \to Y$ obtained by attaching to $Y \times [0,1]$ a 2-handle along $c_1 \# c_2 \times \{0\}$ and 3-handles and a 4-handle as above. Supposing $\mu_Y = [c_1 \wedge \cdots \wedge c_k]$, we homology orient $X_{c_1 \# c_2}$ by $[c_2 \wedge \cdots \wedge c_k]$, just as we homology orient $X_{\{1\}}$ and $X_{\{2\}}$. Then

$$[X_{c_1 \# c_2}]^{\#} = [X_{\{1\}}]^{\#} + [X_{\{2\}}]^{\#}.$$

Viewing $X_{c_1 \# c_2} : \emptyset \to Y$ and $\overline{X}_J : Y \to \emptyset$, this follows from computing

$$\overline{X}_J \circ X_{c_1 \# c_2} \simeq X \# X_3 \# \cdots \# X_k, \qquad \begin{cases} X \simeq S^4 & \text{if } |\{1,2\} \cap J| = 1 \\ \deg(X) \text{ is odd} & \text{otherwise} \end{cases}$$

where each $X_i \simeq S^4$ if $i \in J$ and $\deg(X_i)$ is odd otherwise, and then appealing to the non-degeneracy of our bilinear form. A similar argument shows $[X_{c_1\#c_2 \cup J}]^\# = [X_{\{1\}\cup J}]^\# + [X_{\{2\}\cup J}]^\#$ where J is any subset of $\{3, \ldots, k\}$.

As a consequence, ψ induces a well-defined isomorphism

$$\phi : \bigwedge^*(H_1(Y;\mathbb{Z})) \to I^\#(Y).$$

This is because any two sets of loops $c_1, \ldots, c_k$ in Y as above (having the property that there exists a diffeomorphism $Y \simeq \#^k S^1 \times S^2$ sending each c_i to a factor $S^1 \times \mathrm{pt}$) are related by sequences of connected sums (and the reverse operation) as in the previous paragraph. Indeed, these are just handle-slides, and a result of Laudenbach and Poénaru [26], as cited in [18, Rmk. 4.4.1], says that any self-diffeomorphism of $\#^k S^1 \times S^2$ extends to a diffeomorphism of $\natural^k S^1 \times D^3$, a bounding 1-handlebody, which can be written as a composite of 1-handle slides. In fact, this result also says that the way in which the 3-handles and 4-handle are attached to construct X_J above is essentially unique.

In summary, ϕ is defined by choosing an orientation μ_Y of $H_1(Y;\mathbb{R})$, a diffeomorphism $Y \simeq \#^k S^1 \times S^2$, oriented loops $c_1, \ldots, c_k$ corresponding to the $S^1 \times \mathrm{pt}$ factors, and setting

$$\phi([c_{i_1}] \wedge \cdots \wedge [c_{i_l}]) = [X_J]^\#$$

where the element $[X_J]^\#$ is defined as above. The content of the above discussion is that this map is well-defined and is an isomorphism. The statement about gradings is easily verified. □

4.10 A link surgeries spectral sequence

The spectral sequence of Theorem 3.3.1 leads to one for the groups $I^\#(Y)$. The setup is as follows. Again we have an m-component framed link L in Y. We view L as a link in $Y\#T^3$, and we choose a family of bundles over the surgered manifolds $Y_v\#T^3$ which for $v \in \{0,1\}^m$ are of the form $(Y_v \times \mathrm{SO}(3))\#\mathbb{T}^3$, at the expense of having possibly non-trivial bundles (so twisted framed groups) for the indices $v \in \{0,1,\infty\}^m \setminus \{0,1\}^m$. We are using the third row of Figure 3.1 to achieve this setup. This forces the bundle over $Y\#T^3$ to be geometrically represented by the link L together with an S^1-factor of T^3. More general spectral sequences may be obtained by allowing twisting in the E^1-page.

Before stating the resulting theorem, we discuss how to lift the previous $\mathbb{Z}/2$-grading $\mathrm{gr}[\mathbf{C}]$ for the E^1-page of the link surgeries spectral sequence to a $\mathbb{Z}/4$-grading, in the special case where $[L] = 0 \in H_1(Y;\mathbb{F}_2)$. Write $\mathrm{gr}[Y]$ for the $\mathbb{Z}/4$-grading on $I^\#(Y)$ and $\mathbb{Y}_v^\# = \mathbb{Y}_v\#\mathbb{T}^3$, where for $v \in \{0,1\}^m \cup \{\boldsymbol{\infty}\}$ we have $\mathbb{Y}_v = Y_v \times \mathrm{SO}(3)$. Recall that we conflate $\boldsymbol{\infty}$ and $-\mathbf{1}$. Also write $\mathbb{X}_{vw}^\# = \mathbb{X}_{vw} \bowtie (\mathbb{T}^3 \times [0,1])$ for the surgery cobordism bundles. For $v \in \{0,1\}^m \cup \{\boldsymbol{\infty}\}$ we may view each $\mathrm{C}(\mathbb{Y}^\#)$ as two copies of $\mathrm{C}^\#(Y_v)$, $\mathbb{Z}/4$-graded by $\mathrm{gr}[Y_v]$. For $v \in \{0,1\}^m$ and $x \in \mathrm{C}(\mathbb{Y}_v^\#) \subset \mathbf{C}$ of homogeneous $\mathrm{gr}[Y_v]$ grading, we define

$$\mathrm{gr}[\mathbf{C}](x) = \mathrm{gr}[Y_v](x) - \deg(\mathbb{X}_{\boldsymbol{\infty} v}) - |v|_1 \mod 4. \tag{4.7}$$

The verification that $\boldsymbol{\partial}$ lowers this grading by 1, and that the quasi-isomorphism $\mathbf{Q} : \mathrm{C}(\mathbb{Y}^\#) \to \mathbf{C}$ preserves the relevant $\mathbb{Z}/4$-gradings, is the same as in §3.3.4.

Theorem 4.10.1. *Let L be an oriented framed link with m components in Y and for each $v \in \{\infty,0,1\}^m$ denote by Y_v the result of v-surgery on L in Y. There are surgery cobordisms X_{vw} for $v < w$ from Y_v to Y_w with homology orientations μ_{vw} satisfying $\mu_{uw} \circ \mu_{vu} = \mu_{vw}$ whenever $v < u < w$, and an appropriate bundle $\mathbb{X}_{vw}$ over each X_{vw}, such that there is a spectral sequence (E^r, d^r) with*

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I^\#(Y_v), \qquad d^1 = \sum_{\substack{v<w \\ |w-v|_1=1}} (-1)^{\delta(v,w)} I^\#(\mathbb{X}_{vw})$$

where $\delta(v,w)$ is as in Theorem 3.3.1. The spectral sequence is graded by $\mathbb{Z}/2 \times \mathbb{Z}$, where d^r has bi-degree $(1,r)$, and it converges by the E^{m+1}-page to the possibly twisted framed instanton group

$$I^\#(Y;L).$$

The $\mathbb{Z}/2$*-grading induced by the spectral sequence agrees with the* $\mathbb{Z}/2$*-grading of* $I^{\#}(Y;L)$*. If* $[L] = 0 \in H_1(Y;\mathbb{F}_2)$*, then we can lift the* $\mathbb{Z}/2$*-grading of the* E^1*-page to a* $\mathbb{Z}/4$*-grading by (4.7), such that the induced* $\mathbb{Z}/4$*-grading agrees with the one on* $I^{\#}(Y)$*. The differential for the* $\mathbb{Z}/4 \times \mathbb{Z}$*-grading has bi-degree* $(-1, r)$*.*

CHAPTER 5

Composing homology orientations

Given a cobordism $X : Y_1 \to Y_2$ between two 3-manifolds, the induced morphism $I^{\#}(X) : I^{\#}(Y_1) \to I^{\#}(Y_2)$ on framed instanton homology is only well-defined up to an overall sign. To fix this sign, the extra data of a *homology orientation* of X is required. When all manifolds under consideration are closed, oriented and connected, a homology orientation is an orientation of the real vector space

$$\mathcal{L}(X) := H_1(Y_1; \mathbb{R}) \oplus H_1(X; \mathbb{R}) \oplus H_2^+(X; \mathbb{R}),$$

where $H_2^+(X; \mathbb{R})$ is a maximal positive definite subspace for the intersection form on $H_2(X; \mathbb{R})$. In this chapter, we describe an algebro-topological way of composing homology orientations. In the next chapter, we will construct a spectral sequence from reduced odd Khovanov homology to the framed instanton homology of the double branched cover. The composition rule described here allows us to understand and control the signs of that construction. As the signs in the differentials are what makes odd Khovanov homology distinct from ordinary Khovanov homology, this chapter develops the most important technical tool for the construction of that spectral sequence.

We define our composition rule of homology orientations in §5.1. In §5.2 we relate our composition rule to the one described by Kronheimer and Mrowka in the more abstract setting of Fredholm determinant line bundles.

5.1 The composition rule

Suppose we are given $X_1 : Y_1 \to Y_2$ and $X_2 : Y_2 \to Y_3$. Let $X_{12} = X_2 \circ X_1$. In this section we describe the rule we use to orient $\mathcal{L}(X_{12})$ given orientations of $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$. Typically, an orientation of $\mathcal{L}(X_i)$ will be denoted μ_i. Although the composition of homology orientations originates from the determinants of the relevant Fredholm operators, in our applications we prefer to have a concrete, algebro-topological description of such a rule. Perhaps the two most important formal properties of a composition rule compatible with a construction of framed instanton homology are associativity and the existence of units. In other words,

$$(\mu_3 \circ \mu_2) \circ \mu_1 = \mu_3 \circ (\mu_2 \circ \mu_1)$$

whenever μ_i is a homology orientation of $X_i : Y_i \to Y_{i+1}$ for $i = 1, 2, 3$, and for $Y \times [0, 1]$ there exists a distinguished homology orientation μ_Y^{id} such that

$$\mu_Y^{\text{id}} \circ \mu = \mu, \qquad \mu \circ \mu_Y^{\text{id}} = \mu$$

whenever μ is a homology orientation and these compositions make sense. We will first define a composition rule in an algebro-topological fashion and then show it has these two properties. At the end of this section, we will describe how the rule we have defined can be described using Fredholm determinant line bundles, using the setup of Kronheimer and Mrowka [23, §20.2], ensuring that our rule is compatible with a construction of framed instanton homology. In this section all homology groups are assumed to have real coefficients.

We proceed to define the composition rule. We assume that the X_i and Y_i are closed, oriented and connected. For background on the following setup, see [10, Thm. 27.5] and [2, §7]. Let $f_{12} : H_1(Y_2) \to H_1(X_1) \oplus H_1(X_2)$ be the map in the Mayer-Vietoris sequence. Consider the following exact sequences:

$$0 \to \mathrm{im}(f_{12}) \to H_1(X_1) \oplus H_1(X_2) \to H_1(X_{12}) \to 0 \tag{5.1}$$

$$0 \to \ker(f_{12}) \to H_1(Y_2) \to \mathrm{im}(f_{12}) \to 0 \tag{5.2}$$

$$0 \to H_2^+(X_1) \oplus H_2^+(X_2) \to H_2^+(X_{12}) \to \ker(f_{12}) \to 0 \tag{5.3}$$

The first exact sequence is extracted from the Mayer-Vietoris sequence, and the second is naturally associated to the map f_{12}. Our convention is that $f_{12}(x) = (x, -x)$ on the chain level. For the third sequence, we choose the positive definite subspace $H_2^+(X_{12})$ so that it contains the image of $H_2^+(X_1) \oplus H_2^+(X_2)$ under the map $H_2(X_1) \oplus H_2(X_2) \to H_2(X_{12})$. The map $H_2^+(X_{12}) \to \ker(f_{12})$ is a restriction of the Mayer-Vietoris boundary map $H_2(X_{12}) \to H_1(Y_2)$.

There is a concrete interpretation of (5.3). Upon splitting the sequence it says we can write

$$H_2^+(X_{12}) = H_2^+(X_1) \oplus H_2^+(X_2) \oplus \ker(f_{12}).$$

To interpret the summand $\ker(f_{12})$, we write down a section s for the map $H_2^+(X_{12}) \to \ker(f_{12})$. We define $s : \ker(f_{12}) \to H_2^+(X_{12})$ on a basis of 1-cycle classes $[\gamma]$ in $\ker(f_{12}) \subset H_1(Y_2)$ as follows. For each such 1-cycle γ in Y_2, choose a 2-cycle Σ in Y_2 such that $\#(\gamma \cap \Sigma) = 1$, and extend γ to a 2-cycle Γ in X_{12}. Then $s[\gamma] = [\Gamma] + [\Sigma]$.

Choosing splittings of the above three exact sequences, summing, cancelling a copy of $\ker(f_{12})$ on both sides, and then moving summands around yields an identification

$$\mathcal{L}(X_{12}) \oplus \mathrm{im}(f_{12})^{\oplus 2} = \mathcal{L}(X_1) \oplus \mathcal{L}(X_2). \tag{5.4}$$

Thus we can orient $\mathcal{L}(X_{12})$ by using given orientations of $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ and equipping the two copies of $\mathrm{im}(f_{12})$ with the same orientation. We will give an explicit rule for doing this, designed so as to be associative. We choose splittings of the above exact sequences, in their respective order:

$$F_{12} : \mathrm{im}(f_{12}) \oplus H_1(X_{12}) \xrightarrow{\sim} H_1(X_1) \oplus H_1(X_2), \tag{5.5}$$

$$G_{12} : \ker(f_{12}) \oplus \mathrm{im}(f_{12}) \xrightarrow{\sim} H_1(Y_2), \tag{5.6}$$

$$H_{12} : H_2^+(X_1) \oplus H_2^+(X_2) \oplus \ker(f_{12}) \xrightarrow{\sim} H_2^+(X_{12}). \tag{5.7}$$

The space of such splittings is contractible, so these particular choices do not matter for the following definition.

Definition 5.1.1. *For $i = 1, 2$ let $X_i : Y_i \to Y_{i+1}$ be two connected, oriented cobordisms between connected, oriented, closed, non-empty 3-manifolds. Write $X_{12} = X_2 \circ X_1$. Suppose μ_i is a homology orientation of X_i, i.e. an orientation of $\mathcal{L}(X_i)$, for $i = 1, 2$. Write $\mu_i = \beta_i \wedge \alpha_i \wedge \gamma_i$ where α_i is an orientation for $H_1(Y_i)$, β_i for $H_1(X_i)$, and γ_i for $H_2^+(X_i)$. Choose any orientation δ_{12} of im(f_{12}). Choose splittings of the exact sequences (5.1)-(5.3) written as in (5.5)-(5.7). Equip $H_1(X_{12})$ with an orientation β_{12} given by the condition*

$$F_{12}(\delta_{12} \wedge \beta_{12}) = \beta_1 \wedge \beta_2.$$

Similarly, equip ker(f_{12}) with an orientation ζ_{12} which satisfies

$$G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2.$$

Then define the composition of μ_1 with μ_2, which is an orientation of $\mathcal{L}(X_{12})$, by

$$\mu_2 \circ \mu_1 = (-1)^s \beta_{12} \wedge \alpha_1 \wedge H_{12}(\gamma_1 \wedge \gamma_2 \wedge \zeta_{12}),$$

$$s = \frac{1}{2}\left(d_{12}^2 - d_{12}\right) + b_1(X_1)b_1(Y_2) + b_1(X_1)b_2^+(X_2) + b_1(Y_2)b_2^+(X_2).$$

Here $d_{12} = \dim\left[im(f_{12})\right]$.

Proposition 5.1.1. *The composition rule of Definition 5.1.1 is associative.*

Proof. We first rephrase the problem in terms of linear algebra. For $i = 1, 2$ consider quadruples $\mathscr{A}_i = (A_i, B_i, C_i, \mu_i)$ where A_i, B_i, C_i are vector spaces and μ_i is an orientation of $A_i \oplus B_i \oplus C_i$. In our application we have $A_i = H_1(Y_i)$, $B_i = H_1(X_i)$, and $C_i = H_2^+(X_i)$. Given a linear map

$$f_{12} : A_2 \to B_1 \oplus B_2,$$

we can compose $\mathscr{A}_1$ and $\mathscr{A}_2$ along f_{12} to form

$$\mathscr{A}_2 \circ_{f_{12}} \mathscr{A}_1 = (A_1, \operatorname{coker}(f_{12}), C_1 \oplus C_2 \oplus \ker(f_{12}), \mu_2 \circ \mu_1).$$

The orientation $\mu_{12} = \mu_2 \circ \mu_1$ is adapted from Definition 5.1.1 as follows. Write $\mu_i = \beta_i \wedge \alpha_i \wedge \gamma_i$ where $\alpha_i, \beta_i, \gamma_i$ are respective orientations of A_i, B_i, C_i. Choose an orientation δ_{12} of $\operatorname{im}(f_{12})$. Choose isomorphisms

$$F_{12} : \operatorname{im}(f_{12}) \oplus \operatorname{coker}(f_{12}) \xrightarrow{\sim} B_1 \oplus B_2, \tag{5.8}$$

$$G_{12} : \ker(f_{12}) \oplus \operatorname{im}(f_{12}) \xrightarrow{\sim} A_2 \tag{5.9}$$

that are splittings of the naturally associated exact sequences. Orient $\operatorname{coker}(f_{12})$ by β_{12} and $\ker(f_{12})$ by ζ_{12} using the conditions

$$F_{12}(\delta_{12} \wedge \beta_{12}) = \beta_1 \wedge \beta_2, \quad G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2.$$

Then the composition μ_{12} is given by

$$\mu_{12} = (-1)^{s_{12}} \beta_{12} \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \zeta_{12},$$

$$s_{12} = b_1 a_2 + b_1 c_2 + a_2 c_2 + (d_{12}^2 - d_{12})/2,$$

where $a_i = \dim A_i$, $b_i = \dim B_i$, $c_i = \dim C_i$, and $d_{12} = \dim [\operatorname{im}(f_{12})]$. Now suppose we have a third quadruple $\mathscr{A}_3 = (A_3, B_3, C_3, \mu_3)$ and a linear map $f_{23} : A_3 \to B_2 \oplus B_3$. Consider

$$f = f_{12} + f_{23} : A_2 \oplus A_3 \to B_1 \oplus B_2 \oplus B_3.$$

The map f induces further maps

$$f_{1,23} : A_2 \to B_1 \oplus \mathrm{coker}(f_{23}), \quad f_{12,3} : A_3 \to \mathrm{coker}(f_{12}) \oplus B_3.$$

We write F_{23}, G_{23} for the isomorphisms associated to f_{23} as in (5.8), (5.9); $F_{12,3}, G_{12,3}$ associated to $f_{12,3}$; and $F_{12,3}, G_{12,3}$ to $f_{1,23}$. We have identifications

$$\mathrm{coker}(f_{1,23}) = \mathrm{coker}(f) = \mathrm{coker}(f_{12,3}), \tag{5.10}$$

$$\ker(f_{23}) \oplus \ker(f_{1,23}) = \ker(f) = \ker(f_{12}) \oplus \ker(f_{12,3}). \tag{5.11}$$

The cokernel identifications are natural. The kernel identifications depend on some choices. For instance, $\ker(f_{12}) \oplus \ker(f_{12,3}) = \ker(f)$ is established as follows. Clearly $\ker(f_{12}) \subset \ker(f)$. Now suppose $a \in \ker(f_{12,3}) \subset A_3$. Then $\pi_{12}(f(a)) \in \mathrm{im}(f_{12})$ where π_{12} projects onto $B_1 \oplus B_2$. Thus $\pi_{12}(f(a)) = f_{12}(b)$ for some $b \in A_2$. Let $\sigma_{12} : \mathrm{im}(f_{12}) \to A_2$ be such that $f_{12}\sigma_{12} = \mathrm{id}_{\mathrm{im}(f_{12})}$. Then we may take $b = \sigma_{12}(\pi_{12}(f(a)))$, and the assignment $a \mapsto (-b, a)$ injects $\ker(f_{12,3})$ into $\ker(f)$. In this way we obtain a map from $\ker(f_{12}) \oplus \ker(f_{12,3})$ to $\ker(f)$ which is easily seen to be an isomorphism. With these identifications, the associativity of our rule in Definition 5.1.1 is nearly equivalent to

$$\mathscr{A}_3 \circ_{f_{12,3}} (\mathscr{A}_2 \circ_{f_{12}} \mathscr{A}_1) = (\mathscr{A}_3 \circ_{f_{23}} \mathscr{A}_2) \circ_{f_{1,23}} \mathscr{A}_1. \tag{5.12}$$

We have only left out the roles of the H_{12} maps; these are not essential and we remark on their absence at the end of the proof. We proceed to establish (5.12). Let us write out $\mu_{12,3} = \mu_3 \circ \mu_{12}$, the orientation associated to the left side of (5.12). Let $\mu_3 = \beta_3 \wedge \alpha_3 \wedge \gamma_3$ where $\alpha_3, \beta_3, \gamma_3$ are orientations of A_3, B_3, C_3, respectively. Let $\delta_{12,3}$ orient $\mathrm{im}(f_{12,3})$. Orient $\mathrm{coker}(f_{12,3})$ by $\beta_{12,3}$ and $\ker(f_{12,3})$ by $\zeta_{12,3}$, where

$$F_{12,3}(\delta_{12,3} \wedge \beta_{12,3}) = \beta_{12} \wedge \beta_3, \quad G_{12,3}(\zeta_{12,3} \wedge \delta_{12,3}) = \alpha_3.$$

Then we use our composition rule to obtain

$$\mu_{12,3} = (-1)^{s_{12,3}} \beta_{12,3} \wedge \alpha_1 \wedge (\gamma_1 \wedge \gamma_2 \wedge \zeta_{12}) \wedge \gamma_3 \wedge \zeta_{12,3},$$

$$s_{12,3} = s_{12} + b_{12}a_3 + b_{12}c_3 + a_3c_3 + (d_{12,3}^2 - d_{12,3})/2.$$

Here $d_{12,3} = \dim[\dim(f_{12,3})]$ and $b_{12} = \dim[\mathrm{coker}(f_{12})]$, so in particular

$$b_{12} = b_1 + b_2 - d_{12}.$$

Now write out the orientation associated to the right side of (5.12). First,

$$\mu_{23} = \mu_3 \circ \mu_2 = (-1)^{s_{23}} \beta_{23} \wedge \alpha_2 \wedge \gamma_2 \wedge \gamma_3 \wedge \zeta_{23},$$

$$s_{23} = b_2 a_3 + b_2 c_3 + a_3 c_3 + (d_{23}^2 - d_{23})/2,$$

where, given an orientation δ_{23} of $\mathrm{im}(f_{23})$, we have imposed

$$F_{23}(\delta_{23} \wedge \beta_{23}) = \beta_2 \wedge \beta_3, \quad G_{23}(\zeta_{23} \wedge \delta_{23}) = \alpha_3.$$

Now we can also write

$$\mu_{1,23} = (-1)^{s_{1,23}} \beta_{1,23} \wedge \alpha_1 \wedge \gamma_1 \wedge (\gamma_2 \wedge \gamma_3 \wedge \zeta_{23}) \wedge \zeta_{1,23},$$

$$s_{1,23} = s_{23} + b_1 a_2 + b_1 c_{23} + a_2 c_{23} + (d_{1,23}^2 - d_{1,23})/2,$$

where $c_{23} = \dim[C_2 \oplus C_3 \oplus \ker(f_{23})]$, so that

$$c_{23} = c_2 + c_3 + a_3 - d_{23},$$

and, given an orientation $\delta_{1,23}$ of $\mathrm{im}(f_{1,23})$, we have the conditions

$$F_{1,23}(\delta_{1,23} \wedge \beta_{1,23}) = \beta_1 \wedge \beta_{23}, \quad G_{1,23}(\zeta_{1,23} \wedge \delta_{1,23}) = \alpha_2.$$

We will now show that $\mu_{12,3} = \mu_{1,23}$. We choose identifications

$$\mathrm{im}(f_{12}) \oplus \mathrm{im}(f_{12,3}) = \mathrm{im}(f) = \mathrm{im}(f_{23}) \oplus \mathrm{im}(f_{1,23}).$$

These depend on F_{12} and F_{23}. For instance, let $\tau_{12} : \mathrm{coker}(f_{12}) \to B_1 \oplus B_2$ be the map extracted from F_{12} (and conversely it may define F_{12}). Then $\mathrm{im}(f_{12,3})$ maps into $\mathrm{im}(f)$ by $a \mapsto (\tau_{12}(\pi(a)), \pi_3(a))$ where π projects onto $\mathrm{coker}(f_{12})$ and π_3 onto B_3. Since $\mathrm{im}(f_{12})$ is naturally a subset of $\mathrm{im}(f)$, we then obtain a map from

$\mathrm{im}(f_{12}) \oplus \mathrm{im}(f_{12,3})$ into $\mathrm{im}(f)$ which yields the above identification. We can thus orient $\mathrm{im}(f)$ by $\delta_{12} \wedge \delta_{12,3}$ or by $\delta_{23} \wedge \delta_{1,23}$. It suffices to show

$$\delta_{12} \wedge \delta_{12,3} \wedge \mu_{12,3} \wedge \delta_{12} \wedge \delta_{12,3} = \delta_{23} \wedge \delta_{1,23} \wedge \mu_{1,23} \wedge \delta_{23} \wedge \delta_{1,23} \tag{5.13}$$

as orientations of $\mathrm{im}(f) \oplus V \oplus \mathrm{im}(f)$, where V is the total space of either side of (5.12) for which $\mu_{1,23}$ and $\mu_{12,3}$ are orientations. Compute the left side of (5.13):

$$\begin{aligned}
&(-1)^{s_{12,3}+d_{12}d_{12,3}}\delta_{12} \wedge \delta_{12,3} \wedge \beta_{12,3} \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \zeta_{12} \wedge \gamma_3 \wedge \zeta_{12,3} \wedge \delta_{12,3} \wedge \delta_{12} \\
&\quad = (-1)^{s_{12,3}+d_{12}d_{12,3}+d_{12}(a_3+c_3)+a_2c_3} \left[(\mathrm{id}_{\mathrm{im}(f_{12})} \oplus F_{12,3}^{-1})(F_{12}^{-1} \oplus \mathrm{id}_{B_3})\right] (\beta_1 \wedge \beta_2 \wedge \beta_3) \\
&\qquad\quad \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \left[G_{12}^{-1} \oplus G_{12,3}^{-1}\right] (\alpha_2 \wedge \alpha_3).
\end{aligned}$$

Now, choose splitting isomorphisms

$$F : \mathrm{im}(f) \oplus \mathrm{coker}(f) \xrightarrow{\sim} B_1 \oplus B_2 \oplus B_3,$$

$$G : \ker(f) \oplus \mathrm{im}(f) \xrightarrow{\sim} A_2 \oplus A_3$$

for the naturally associated short exact sequences. We claim we have

$$\left[(\mathrm{id}_{\mathrm{im}(f_{12})} \oplus F_{12,3}^{-1})(F_{12}^{-1} \oplus \mathrm{id}_{B_3})\right] (\beta_1 \wedge \beta_2 \wedge \beta_3) = F^{-1}(\beta_1 \wedge \beta_2 \wedge \beta_3), \tag{5.14}$$

$$\left[G_{12}^{-1} \oplus G_{12,3}^{-1}\right] (\alpha_2 \wedge \alpha_3) = G^{-1}(\alpha_2 \wedge \alpha_3). \tag{5.15}$$

We consider (5.14). To abstract the underlying problem, consider a linear map $\phi : V \to W$ and distinguished subspaces $V' \subset V$ and $W' \subset W$ such that $\phi(V') \subset W'$. In other words, we have a relative linear map $\phi : (V, V') \to (W, W')$. Choose an isomorphism

$$\Phi : \mathrm{im}(\phi) \oplus \mathrm{coker}(\phi) \xrightarrow{\sim} W$$

associated to the natural short exact sequence. Similarly, choose

$$\Phi' : \mathrm{im}(\phi') \oplus \mathrm{coker}(\phi') \xrightarrow{\sim} W'$$

where $\phi' : V' \to W'$ is a restriction of ϕ. Also, with $\phi'' : V/V' \to \mathrm{coker}(\phi') \oplus W/W'$ choose

$$\Phi'' : \mathrm{im}(\phi'') \oplus \mathrm{coker}(\phi'') \xrightarrow{\sim} \mathrm{coker}(\phi') \oplus W/W'.$$

We can identify $\mathrm{coker}(\phi'') = \mathrm{coker}(\phi)$ and $\mathrm{im}(\phi) = \mathrm{im}(\phi') \oplus \mathrm{im}(\phi'')$ just as we have done in our setting above. We also choose an identification $W/W' \oplus W' = W$. Then (5.14) is equivalent to

$$\det\Big[\Phi^{-1}(\Phi' \oplus \mathrm{id}_{W/W'})(\Phi'' \oplus \mathrm{id}_{\mathrm{im}(\phi')})\Big] > 0,$$

by setting $\phi = f$, $V = A_2 \oplus A_3$, $V' = A_2$, $W = B_1 \oplus B_2 \oplus B_3$, and $W' = B_1 \oplus B_2$. In fact, we can choose the data so that, under these identifications,

$$\Phi = (\Phi' \oplus \mathrm{id}_{W/W'})(\Phi'' \oplus \mathrm{id}_{\mathrm{im}(\phi')}). \tag{5.16}$$

This can be seen as follows. We may equip V and W with inner products so that we may freely take complements. In the following, we use the notation $V_1^\perp \subset V_2$ to mean that the complement $V_1^\perp$ (with V_2 possibly inside a larger space) was taken inside V_2. We may then identify $\mathrm{coker}(\phi) = \mathrm{im}(\phi)^\perp \subset W$, $\mathrm{coker}(\phi') = \mathrm{im}(\phi')^\perp \subset W'$ and $\mathrm{im}(\phi'') = \mathrm{im}(\phi')^\perp \subset \mathrm{im}(\phi)$. We also identify W/W' with $W'^\perp \subset W$. We use these identifications to define Φ, Φ', Φ'' in the natural way. Then Φ is just the identification $\mathrm{im}(\phi) \oplus \mathrm{im}(\phi)^\perp = W$. On the other hand, we view $\Phi'' \oplus \mathrm{id}_{\mathrm{im}(\phi')}$ as a map

$$\mathrm{im}(\phi) \oplus \mathrm{im}(\phi)^\perp \to \mathrm{im}(\phi') \oplus \mathrm{im}(\phi')^\perp \oplus W'^\perp$$

where $\mathrm{im}(\phi')^\perp \subset W'$. This last expression uses the identification $\mathrm{im}(\phi) = \mathrm{im}(\phi') \oplus \mathrm{im}(\phi')^\perp$ where $\mathrm{im}(\phi')^\perp \subset \mathrm{im}(\phi)$, followed by the identification $\mathrm{im}(\phi')^\perp \oplus \mathrm{im}(\phi)^\perp = \mathrm{im}(\phi')^\perp \oplus W'^\perp$, where on the left $\mathrm{im}(\phi')^\perp \subset \mathrm{im}(\phi)$ but on the right we have the larger complement $\mathrm{im}(\phi')^\perp \subset W'$. These are just two different decompositions of $\mathrm{im}(\phi')^\perp \subset W$. Then, $\Phi' \oplus \mathrm{id}_{W/W'}$, viewed as a map

$$\mathrm{im}(\phi') \oplus \mathrm{im}(\phi')^\perp \oplus W'^\perp \to W,$$

where again $\mathrm{im}(\phi')^\perp \subset W'$, first uses the identification $\mathrm{im}(\phi') \oplus \mathrm{im}(\phi')^\perp = W'$, and then the identification $W' \oplus W'^\perp = W$. From this perspective, from which everything happens inside W and uses its various orthogonal decompositions, (5.16) is clear, and thus (5.14) is established; (5.15) is similar. We return to establishing (5.13). We now know the left hand side is

$$(-1)^{s_{12,3}+d_{12}d_{12,3}+d_{12}(a_3+c_3)+a_2c_3}F^{-1}(\beta_1\wedge\beta_2\wedge\beta_3)$$
$$\wedge\,\alpha_1\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3\wedge G^{-1}(\alpha_2\wedge\alpha_3).$$

We can also compute the right side of (5.13):

$$\begin{aligned}(-1)^{s_{1,23}+d_{23}d_{1,23}}&\delta_{23}\wedge\delta_{1,23}\wedge\beta_{1,23}\wedge\alpha_1\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3\wedge\zeta_{23}\wedge\zeta_{1,23}\wedge\delta_{1,23}\wedge\delta_{23}\\ &=(-1)^{s_{1,23}+d_{23}d_{1,23}+d_{23}(b_1+a_2)+a_2a_3}F^{-1}(\beta_1\wedge\beta_2\wedge\beta_3)\\ &\qquad\wedge\alpha_1\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3\wedge G^{-1}(\alpha_2\wedge\alpha_3).\end{aligned}$$

We have used the necessary analogues of (5.14) and (5.15). Thus (5.13) holds if

$$s_{1,23}+d_{23}(d_{1,23}+b_1+a_2)+a_2(a_3+c_3)+s_{12,3}+d_{12}(d_{12,3}+a_3+c_3)$$

is even. Using $d_{23}+d_{1,23}=d_{12}+d_{12,3}$, this is easily verified. This establishes (5.12). Finally, we remark on the absence of the H_{12} maps in our setup. In our application, we can choose the relevant maps H_{12} and $H_{12,3}$ so that

$$(H_{12,3})(H_{12}\oplus\mathrm{id}_{H_2^+(X_3)\oplus\ker(f_{12,3})})=H$$

where we are using some chosen map

$$H:H_2^+(X_2)\oplus H_2^+(X_2)\oplus H_2^+(X_3)\oplus\ker(f)\xrightarrow{\sim}H_2^+(X_{123})$$

associated to the natural short exact sequence, and the identifications (5.11). This is established just as was (5.14). H_{23} and $H_{1,23}$ can be chosen similarly, and this compatibility allows the above argument to carry through. $\square$

Now we define distinguished identity homology orientations. If $X=Y\times[0,1]$, then $\mathcal{L}(X)=H_1(Y)\oplus H_1(X)$. Let α be any orientation of $H_1(Y)$, and choose an orientation β of $H_1(X)$ such that $\alpha=\beta$ under the natural identification of $H_1(X)$ with $H_1(Y)$. Then define

$$\mu_Y^{\mathrm{id}}:=(-1)^{\frac{1}{2}(b_1(Y)^2+b_1(Y))}\beta\wedge\alpha$$

to be the distinguished identity homology orientation of $Y\times[0,1]$.

Proposition 5.1.2. *Whenever μ is a homology orientation of a cobordism X with incoming boundary Y, we have $\mu \circ \mu_Y^{id} = \mu$. Similarly, if X has outgoing boundary Y, then $\mu_Y^{id} \circ \mu = \mu$.*

Proof. Suppose X has incoming boundary Y, i.e. $X : Y \to Y'$. We let $X_1 = Y \times [0,1]$ and $X_2 = X$ and use the notation of Definition 5.1.1. We have $\mathrm{im}(f_{12}) = H_1(Y)$ and thus $d_{12} = b_1(Y)$. We identify X_{12} with $X_2 = X$. Choose the section of the exact sequence (5.1), which is a map $H_1(X) \to H_1(Y \times [0,1]) \oplus H_1(X)$, to be of the form $y \mapsto (0, y)$. The induced isomorphism $F_{12} : H_1(Y) \oplus H_1(X) \to H_1(Y \times [0,1]) \oplus H_1(X)$ is of the form $(x, y) \mapsto (x, y - \pi(x))$ where $\pi : H_1(Y) \to H_1(X)$ is induced by inclusion. Let $\mu = \mu_2 = \beta_2 \wedge \alpha_2 \wedge \gamma_2$ where $\beta_2, \alpha_2, \gamma_2$ are respective orientations of $H_1(X), H_1(Y), H_2^+(X)$. Write $\mu_1 = \mu_Y^{\mathrm{id}} = (-1)^{\frac{1}{2}(b_1(Y)^2 + b_1(Y))} \beta_1 \wedge \alpha_1$ as above, where $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Choose $\delta_{12} = \alpha_1$. Then

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12}$$

where $\beta_{12} = \beta_2$. We can choose $\alpha_2 = \alpha_1$ so that the condition $\zeta_{12} \wedge \delta_{12} = \alpha_2$ (G_{12} implicit) forces ζ_{12} to be the canonical $+1$ orientation of the 0-vector space. Similarly, γ_1 is taken to be $+1$, and the expression $H_{12}(\gamma_1 \wedge \gamma_2 \wedge \zeta_{12})$ may be regarded as equal to γ_2. The sign s in Definition 5.1.1 is equal to $\frac{1}{2}(b_1(Y)^2 + b_1(Y))$, and so cancels with the sign in μ_Y^{id}. All together, Definition 5.1.1 yields

$$\mu \circ \mu_Y^{\mathrm{id}} = \beta_2 \wedge \alpha_2 \wedge \gamma_2 = \mu.$$

Next, suppose X has outgoing boundary Y, i.e. $X : Y' \to Y$. Now we write $X = X_1 = X_{12}$ and $Y \times [0,1] = X_2$ and, correspondingly, we swap the indices for the above orientations and write $\mu = \mu_1 = \beta_1 \wedge \alpha_1 \wedge \gamma_1$ and $\mu_Y^{\mathrm{id}} = (-1)^{\frac{1}{2}(b_1(Y)^2 + b_1(Y))} \beta_2 \wedge \alpha_2 = \mu_2$. Choose the section of the exact sequence (5.1), which is a map $H_1(X) \to H_1(X) \oplus H_1(Y \times [0,1])$, to be of the form $y \mapsto (y, 0)$. Now the induced map $F_{12} : H_1(Y) \oplus H_1(X) \to H_1(X) \oplus H_1(Y \times [0,1])$ is of the form $(x, y) \mapsto (y + \pi(x), -x)$. Choose $\delta_{12} = \alpha_2 = \beta_2$ and so on, just as above. Then

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12}$$

where $\beta_{12} = (-1)^{b_1(Y)b_1(X)+b_1(Y)}\beta_1 = (-1)^t\beta_1$. The exponent s in Definition 5.1.1 is given by

$$\frac{1}{2}\left(b_1(Y)^2 - b_1(Y)\right) + b_1(Y)b_1(X) \mod 2.$$

We see that $s + t \equiv \frac{1}{2}(b_1(Y)^2 + b_1(Y)) \mod 2$. This cancels with the sign put in front of μ_Y^{id}, and we obtain from Definition 5.1.1 the identity $\mu_Y^{\mathrm{id}} \circ \mu = \mu$, just as before. □

5.2 Relation to the Fredholm description

In this section, we describe how our composition rule can be described in the setting of Fredholm determinant line bundles, as in [23, §20.2], the purpose of which is to show that our rule is compatible with a construction of instanton homology. As such, the following details are not needed to understand the rest of the paper.

In the Fredholm setting, a homology orientation of X is an orientation of $\det(D)$, where D is the operator $-d^* \oplus d^+$ acting on suitably weighted Sobolev spaces over X with cylindrical ends attached. Recall that

$$\det(D) = \bigwedge^{\max}(\ker(D)) \otimes \bigwedge^{\max}(\mathrm{coker}(D)^*).$$

The Sobolev weights are chosen such that we have natural identifications

$$\ker(D) = H^1(X), \quad \mathrm{coker}(D) = H^1(Y) \oplus H^2_+(X),$$

where Y is the incoming end of X, cf. [7, Prop. 3.15]. Note that an orientation of a vector space induces, in a natural way, an orientation of its dual space. Since we are working with real coefficients, homology and cohomology groups are dual to one another, so an orientation of $\det(D)$ is the same as an orientation of $\mathcal{L}(X)$.

Let us now suppose we are in the situation of Definition 5.1.1, so that μ_i is an orientation of $\mathcal{L}(X_i)$, or equivalently $\det(D_i)$, for $i = 1, 2$. We again write $\mu_i = \beta_i \wedge \alpha_i \wedge \gamma_i$ where now we view β_i as orienting $\ker(D)$ and $\alpha_i \wedge \gamma_i$ as orienting $\mathrm{coker}(D)$

(or its dual). We will denote the composition of μ_1 and μ_2 as given in this setting by

$$\mu_2 \mathbin{\overline{\circ}} \mu_1$$

to distinguish it from our previous rule. The composition $\mu_2 \mathbin{\overline{\circ}} \mu_1$ goes in two steps. First, we use the μ_i to orient $\det(D_1 \oplus D_2)$, which is identified with

$$\bigwedge^{\max}(\ker(D_1) \oplus \ker(D_2)) \otimes \bigwedge^{\max}(\operatorname{coker}(D_1) \oplus \operatorname{coker}(D_2))^*.$$

We use the following general rule for doing this: if $K_i \wedge C_i$ is an orientation for $\det(D_i)$ where K_i orients $\ker(D_i)$ and C_i orients $\operatorname{coker}(D_i)$ (or its dual), then we orient $\det(D_1 \oplus D_2)$ by

$$(-1)^{\dim \operatorname{coker}(D_2) \operatorname{index}(D_1)}(K_2 \wedge K_1) \wedge (C_1 \wedge C_2).$$

This is a slight modification of the rule in [23, Lemma 20.2.1] but is easily seen to be associative; the difference between the two rules is the sign $(-1)^s$ where

$$s = \dim \operatorname{coker}(D_1) \dim \ker(D_2) + \operatorname{index}(D_2) \dim \ker(D_1).$$

Applying this procedure to μ_1 and μ_2, we obtain the orientation

$$\mu' := (-1)^{(a_2+c_2)(a_1+b_1+c_1)}(\beta_2 \wedge \beta_1) \wedge (\alpha_1 \wedge \gamma_1 \wedge \alpha_2 \wedge \gamma_2)$$

of $\det(D_1 \oplus D_2)$, where $a_i = \dim H^1(Y_i)$, $b_i = \dim H^1(X_i)$ and $c_i = \dim H^2_+(X_i)$.

The second step in describing the composition rule in this setting involves relating $\det(D_1 \oplus D_2)$ to $\det(D_{12})$ by means of a (Fredholm) homotopy from the operator $D_1 \oplus D_2$ to D_{12}, where D_{12} is the operator associated to X_{12}. We will use the notation of [23, §20.2]. Let P_s for $s \in [0,1]$ be such a homotopy, so that $P_0 = D_1 \oplus D_2$ and $P_1 = D_{12}$. To be precise, we should understand these two aforementioned operators as having the same domain and codomain; this may be achieved using the finite cylinder setup as in [23]. Denoting our codomain by B, choose $J \subset B$ so that $P_s^{-1}J + J = B$ for all s. We have for each s an exact sequence

$$0 \to \ker(P_s) \xrightarrow{j} P_s^{-1}J \xrightarrow{k} J \xrightarrow{l} \operatorname{coker}(P_s) \to 0. \qquad (5.17)$$

We use the following general rule for orienting $\det(P_s)$ given an orientation μ'' of the line $\bigwedge^{\max} P_s^{-1}J \otimes \bigwedge^{\max} J^*$ using the exact sequence (5.17): write

$$\mu'' = (K \wedge D) \wedge (k(D) \wedge C) \tag{5.18}$$

where K is an orientation of $\mathrm{im}(j)$, D of $\mathrm{im}(j)^\perp$, and C of $k(\mathrm{im}(j)^\perp)^\perp$; then orient $\det(P_s)$ by

$$(-1)^{\phi(d)} j^{-1}(K) \wedge l(C) \tag{5.19}$$

where $\phi(x) := (x^2 - x)/2$ and $d := \dim(\mathrm{im}(j)^\perp)$. In our situation, we choose J to be a complement of $\mathrm{im}(P_0) = \mathrm{im}(D_1 \oplus D_2)$, and we make the identification

$$J = H^1(Y_1) \oplus H^2_+(X_1) \oplus H^1(Y_2) \oplus H^2_+(X_2).$$

We choose the homotopy so that $P_s^{-1}J = \ker(P_0) = \ker(D_1 \oplus D_2)$ for all s, so that

$$P_s^{-1}J = H^1(X_1) \oplus H^1(X_2).$$

In particular, we have an identification of $\bigwedge^{\max} P_1^{-1}J \otimes \bigwedge^{\max} J^*$ with $\det(D_1 \oplus D_2)$, which is oriented by μ'. Noting that the maps in (5.17) for $s = 1$ come from the Mayer-Vietoris maps as in Definition 5.1.1, we can write μ'' from μ' as in (5.18):

$$\mu'' = (-1)^t (\beta_{12} \wedge \delta_{12}) \wedge (\delta_{12} \wedge \gamma_{12}).$$

In this expression, and in all to follow, the maps F_{12}, G_{12} and H_{12} from Definition 5.1.1 as well as the maps in (5.17) will be implicitly understood, e.g. $F_{12}(\delta_{12} \wedge \beta_{12})$ is the same as $\delta_{12} \wedge \beta_{12}$. The orientation β_{12} plays the role of K above, γ_{12} that of C, and δ_{12} that of D. The sign $(-1)^t$ is given by

$$t = (a_2 + c_2)(a_1 + b_1 + c_1) + d_{12}(b_1 + b_2 + d_{12}) + b_1 b_2,$$

where d_{12} is as in Definition 5.1.1. The first term in t is from μ' and the rest are added to ensure that β_{12} is defined by the condition $\delta_{12} \wedge \beta_{12} = \beta_1 \wedge \beta_2$, to match Definition 5.1.1. The orientation γ_{12} is defined by the condition $\delta_{12} \wedge \gamma_{12} = \alpha_1 \wedge \gamma_1 \wedge \alpha_2 \wedge \gamma_2$. The general rule that takes μ'' to (5.19), applied to our μ'', tells us the final orientation of $\det(D_{12})$:

$$\mu_2 \,\overline{\circ}\, \mu_1 = (-1)^{\phi(d_{12})+t}\beta_{12} \wedge \gamma_{12}.$$

Now write $\alpha_2 = \zeta_{12} \wedge \delta_{12}$ as in Definition 5.1.1. We compute

$$\mu_2 \,\overline{\circ}\, \mu_1 = (-1)^r \beta_{12} \wedge \alpha_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \zeta_{12},$$

$$r = \phi(d_{12}) + t + d_{12}(a_2 + d_{12} + c_1 + a_1) + c_2(d_{12} + a_2).$$

The sign given by r does not match the sign given by s in Definition 5.1.1, and so this composition rule is not the same as the one previously defined. However, there is an automorphism $\mu \mapsto \overline{\mu}$ on the class of all homology orientations that intertwines the two rules. Given a homology orientation μ of a cobordism X, set

$$\overline{\mu} = (-1)^{\phi(b_1(X))+\phi(b_1(Y)+b_2^+(X))}\mu$$

where Y is the incoming end of X. Then we have

$$\overline{(\overline{\mu_1} \,\overline{\circ}\, \overline{\mu_2})} = \mu_1 \circ \mu_2.$$

The verification is a straightforward computation that we omit. It follows that the composition rule $\mu_1 \circ \mu_2$ of Definition 5.1.1 is compatible with a construction of Floer homology.

CHAPTER 6

From odd Khovanov homology to $I^{\#}(Y)$

In this chapter we prove the main result of this thesis:

Theorem 6.0.1. *Given an oriented link L in S^3, there is a spectral sequence whose second page is $\overline{Kh'}(L)$ that converges to $I^{\#}(\overline{\Sigma(L)})$. Each page of the spectral sequence comes equipped with a $\mathbb{Z}/4$-grading, which on $\overline{Kh'}(L)$ is given by*

$$\delta^{\#} := \frac{3}{2}q - t + \frac{1}{2}(\sigma + \nu) \mod 4, \tag{6.1}$$

where σ and ν are the signature and nullity of L, respectively, and the induced $\mathbb{Z}/4$-grading on $I^{\#}(\overline{\Sigma(L)})$ is the usual one.

Our convention is that the signature of the right-handed trefoil is $+2$. The theorem immediately implies the four rank inequalities

$$\mathrm{rk}_{\mathbb{Z}}\overline{\mathrm{Kh}'}(L)_j \geq \mathrm{rk}_{\mathbb{Z}} I^{\#}(\overline{\Sigma(L)})_j \tag{6.2}$$

where $j \in \mathbb{Z}/4$ and the gradings are as in the Theorem.

In §6.1, following Ozsváth and Szabó [35], we apply the framed instanton link surgeries spectral sequence, Theorem 4.10.1, to the situation of branched double covers. In §6.2 we review the definition of reduced odd Khovanov homology given by Bloom [4]. In §6.3 we complete the proof of Theorem 6.0.1 up to gradings, our main tool being the composition of homology orientations from Chapter 5. Finally, in §6.4, we discuss the mod 4 gradings of Theorem 6.0.1.

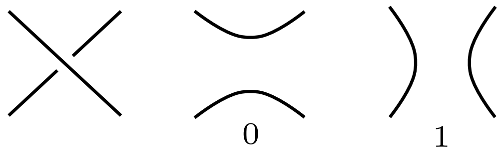

Figure 6.1: From the diagram D to a resolution diagram D_v.

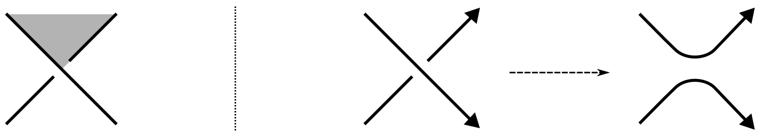

Figure 6.2: On the left, we want to color the regions of the diagram so that at each crossing exactly one of the four regions is colored. On the right, we go from an oriented diagram to a disjoint union of oriented circles.

6.1 Branched double covers

Let L be a link in S^3 and $\Sigma(L)$ the double cover of S^3 branched over L. Let D be a planar diagram for L with m crossings. For each $v \in \{0,1\}^m$ there is a resolution diagram D_v which is a disjoint union of circles, obtained by performing 0- and 1-resolutions according to Figure 6.1. Each branched cover $\Sigma(D_v)$ is diffeomorphic to $\#^k S^1 \times S^2$ where D_v has $k+1$ circles. Further, there is a link $L' \subset \overline{\Sigma(L)}$ and a framing on L' such that $\Sigma(D_v)$ is the result of v-surgery on L'. If we draw a small arc between each crossing in D, the preimages in the branched cover $\Sigma(L)$ are loops, and the link L' is the union of these preimages.

With this setup, from Theorem 4.10.1 we have a spectral sequence

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I^\#(\Sigma(\mathrm{D}_v)) \quad \rightsquigarrow \quad I^\#(\overline{\Sigma(L)}; L'). \tag{6.3}$$

We claim that $[L'] \in H_1(\Sigma(L); \mathbb{F}_2)$ is zero, so that the target of this spectral sequence is in fact $I^\#(\overline{\Sigma(L)})$. The diagram D divides the plane into regions. To show $[L'] = 0$, it suffices to color the regions black and white in a way such that each crossing touches exactly one black region. See Figure 6.2. For then the black regions can be lifted to a surface in $\Sigma(L)$ whose boundary is L', implying $[L'] = 0$.

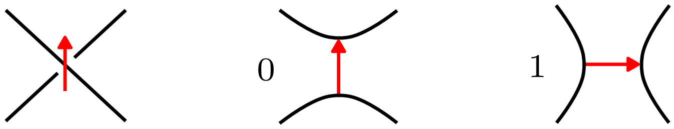

Figure 6.3: Resolution conventions for the arc-decorated diagrams in reduced odd Khovanov homology. There two choices for the placement of an arc at a given crossing; in the left-most picture, the arc can be pointing up (as depicted) or down. In the latter case, the arcs in the resolution pictures are correspondingly reversed.

To color the regions, we follow an argument communicated to the author by Jianfeng Lin. We proceed as if performing the algorithm to construct a Seifert surface, as in [36, §5.4]. First, we orient L. Then we resolve each crossing as in Figure 6.2. We assign to each circle z in the resolved diagram two signs, a_z and b_z. The first sign a_z is $+1$ if z is oriented counter-clockwise in the plane, and -1 otherwise. The second sign b_z is given by $(-1)^N$ where N is the number of circles that surround z. Now color, with black, the regions that are directly interior to each circle z with $a_z b_z = +1$. Transferring the coloring back to the unresolved diagram, each crossing touches exactly one such region.

This reduces the proof of Theorem 6.0.1 to identifying the E^1-page of (6.3) and then understanding the gradings. We can compute the groups $I^{\#}(\Sigma(D_v))$ using Lemma 4.9.1, and we can compute the E^1-differential, with the help of Chapter 5, because the cobordism maps involved are topologically simple. This is carried out in §6.3, where we identify the E^1-page as the chain complex used to compute $\overline{\mathrm{Kh}'}(L)$ from the diagram D. We then check in §6.4 that the relevant gradings are preserved, completing the proof of Theorem 6.0.1. But first, we review the definition of reduced odd Khovanov homology.

6.2 Reduced odd Khovanov homology

Let L be an oriented link and D a planar diagram for L. Suppose D has m crossings. We assume that each crossing has an arrow drawn over it, as in Figure 6.3. Then for each $v \in \{0,1\}^m$ we can define a resolution diagram D_v according to the rules of Figure 6.3. Each D_v is a disjoint union of planar-embedded unoriented circles together with a disjoint union of planar-embedded oriented arcs, each arc beginning and ending at a circle. Suppose D_v has $k+1$ circles. Then we have a rank k abelian group V_v defined by

$$V_v = \mathbb{Z}\{\text{arcs}\}/\ker(\mathbb{Z}\{\text{arcs}\} \to \mathbb{Z}\{\text{circles}\})$$

where the map involved sends an arc to the circle at which it begins minus the circle at which it ends. A basis for V_v is given by any k arcs that touch all $k+1$ circles in D_v. Otherwise said, a basis is given by the edges of any spanning tree of the graph whose vertices are the circles of D_v and edges are the arcs. We define

$$\mathrm{C}_v = \bigwedge\nolimits^*(V_v), \qquad \mathrm{C} = \bigoplus_{v\in\{0,1\}^m} \mathrm{C}_v.$$

For each $v, w \in \{0,1\}^m$ with $v < w$ and $|w-v|_1 = 1$ we introduce a map $\partial'_{vw} : \mathrm{C}_v \to \mathrm{C}_w$. There is a single arc x_{vw} in each of D_v and D_w that changes from a 0-resolution position to a 1-resolution position. There are two cases to consider, corresponding to two circles merging or splitting:

$$\partial'_{vw}(x) := \begin{cases} x_{vw} \wedge x & \text{if } 0 = x_{vw} \in \mathrm{C}_v \text{ (split)} \\ x & \text{if } 0 \neq x_{vw} \in \mathrm{C}_v \text{ (merge)} \end{cases}$$

In these expressions we use the symbol x_{vw} to stand both for an arc and its equivalence class in V_v. We call the collection of ∂'_{vw} the *pre-differential*. The differential for C is defined by

$$\partial = \sum \partial_{vw} = \sum \varepsilon_{vw}\partial'_{vw}$$

where each ε_{vw} is $+1$ or -1, and the sums are over v, w with $v < w$ and $|w-v|_1 = 1$. The signs ε_{vw} are chosen to satisfy two conditions. The first condition is that $\partial^2 = 0$.

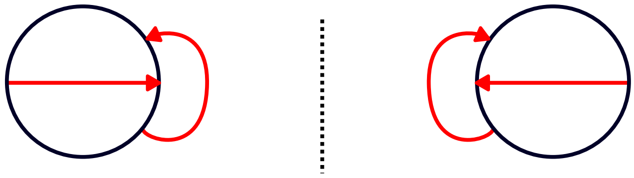

Figure 6.4: The configurations of the two relevant arcs in the initial vertex of a type X and type Y face.

The second condition is as follows. Let $v < t, u$ with $|t - v|_1 = |u - v|_1 = 1$ be three vertices where the arcs x_{vu} and x_{vt} are arranged in D_v as in the left of Figure 6.4. Let w be the vertex with $w > t, u$ and $|w - t|_1 = |w - u|_1 = 1$. Any four such vertices v, u, t, w will be called a *type X* face. A *type Y* face is obtained by reversing one of either x_{vu} or x_{vt}. The second condition is that for a type X face, the sign

$$\varepsilon_{vu}\varepsilon_{vt}\varepsilon_{tw}\varepsilon_{uw}$$

is always $+1$ or always -1; and the same product for a type Y face is also always $+1$ or always -1, and is minus the type X sign. We call the collection of ε_{vw} a *valid edge assignment* if it satisfies these two conditions. The reduced odd Khovanov homology of L is then defined to be $\overline{\mathrm{Kh}'}(L) = H_*(\mathrm{C}, \partial)$. The well-defined-ness and invariance is proved in [33].

The group $\overline{\mathrm{Kh}'}(L)$ is bigraded by a homological grading t and and quantum grading q. For an element $x \in \bigwedge^{|x|}(V_v)$ where $k = \dim(V_v)$, these are defined by

$$t(x) = |v|_1 - n_-,$$

$$q(x) = k - 2|x| + n_+ - 2n_- + |v|_1.$$

Here $n_\pm$ is the number of $\pm$ crossings in D. We are interested in the $\mathbb{Z}/4$-grading

$$\delta^\# := \frac{3}{2}q - t + \frac{1}{2}(\sigma + \nu) \mod 4 \tag{6.4}$$

where σ is the signature of L and ν the nullity.

6.3 Computing the E^1-page

In this section we identify the E^1-page of (6.3) with the chain complex that computes reduced odd Khovanov homology. We fix as before a diagram D for the m-component link L with crossings decorated by arcs as in §6.2. We let $Y_v = \Sigma(D_v)$ for each $v \in \{0,1\}^m$ so that Y_v is homeomorphic to $\#^k S^1 \times S^2$ when D_v has $k+1$ circles. The E^1-page and differential of our spectral sequence are given by

$$E^1 = \bigoplus_{v \in \{0,1\}^m} I^\#(Y_v), \quad d^1 = \sum (-1)^{\delta(v,w)} I^\#(X_{vw}),$$

where the sum runs over $v < w$ with $|w - v|_1 = 1$ and $v, w \in \{0,1\}^m$. In writing d^1, we have chosen homology orientations μ_{vw} of the X_{vw} so that $\mu_{uw} \circ \mu_{vu} = \mu_{tw} \circ \mu_{vt}$ always holds. We are also using that the relevant bundles $\mathbb{X}_{vw}$ are trivial. This is because each such bundle lies over a cobordism which is $D^2 \times S^2 \setminus \text{int}(D^4)$ running along a product cobordism, see (6.5); since we have arranged that the restriction of each such bundle over the boundary is trivial, for topological reasons the bundle must be trivial.

Let $\mathrm{C} = \bigoplus \mathrm{C}_v$ be the reduced odd Khovanov chain group for the diagram D and $\partial' = \sum \partial'_{vw}$ its pre-differential. For each $v \in \{0,1\}^m$ we define an isomorphism

$$\Phi_v : \mathrm{C}_v \to I^\#(Y_v)$$

defined as a composition $\Phi_v = \phi_v \circ \rho_v$ where $\phi_v : \bigwedge^*(H_1(Y_v;\mathbb{Z})) \to I^\#(Y_v)$ is from §4.9 and $\rho_v : \mathrm{C}_v \to \bigwedge^*(H_1(Y_v;\mathbb{Z}))$ is defined by lifting arcs in D_v to loops in Y_v, and is explained in the following paragraph. For the ϕ_v maps, we fix orientations μ_v for each $H_1(Y_v;\mathbb{R})$. We write $\Phi : \mathrm{C} \to E^1$ for the sum of the Φ_v maps.

Recall $\mathrm{C}_v = \bigwedge^*(V_v)$, and that Y_v is branched over $D_v \subset S^3$. Let S be the union of disks in the plane enclosed by the circles in D_v. They can be pushed out so that they are disjoint and form a Seifert surface for the union of circles. Let N be a neighborhood of the circles, a union of solid tori. Then Y_v can be written as

$$Y_v = Y_- \cup N \cup Y_+$$

where $Y_\pm = S^3 \setminus (S \cup N)$. Distinguishing one of the copies of $S^3 \setminus (S \cup N)$, say Y_+, allows us to lift an arc x in D_v to an *oriented* loop $\widetilde{x}$ in Y_v: the orientation is obtained by locally lifting the orientation of x to the part of $\widetilde{x}$ in Y_+. Then $x \mapsto [\widetilde{x}]$ is an isomorphism from V_v to $H_1(Y_v;\mathbb{Z})$, and ρ_v is taken to be the extension of this map to exterior algebras. We can construct the ρ_v in this way so that it is uniform among all v, in the sense that there are natural ways of identifying Y_v with Y_w away from surgery (or resolution) areas, and in these areas we can lift arcs the same way.

In summary, the map Φ_v is described as follows. Let $x = x_1 \wedge \cdots \wedge x_i$ be a wedge of arcs in C_v. Lift the arcs to embedded loops $\widetilde{x}_j$ in the branched double cover Y_v as above. Choose $x_{i+1}, \ldots, x_k$ and their lifts such that $\mu_v = [\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_k]$. Attach 2-handles to $\widetilde{x}_1, \ldots, \widetilde{x}_i$ and 3-handles and a 4-handle as in §4.9 to obtain a cobordism $X : \emptyset \to Y_v$ homology oriented by $[\widetilde{x}_{i+1} \wedge \cdots \wedge \widetilde{x}_k]$. Then $\Phi_v(x) = [X]^\#$. The following completes the proof of Theorem 6.0.1 up to gradings, which are dealt with in the next section.

Lemma 6.3.1. $\Phi^{-1} d^1 \Phi = \sum \varepsilon_{vw} \partial'_{vw}$ *where* ε_{vw} *is a valid edge assignment.*

Proof. Let $v, w \in \{0,1\}^m$ with $v < w$ and $|w - v|_1 = 1$. There are two cases to consider, depending on whether D_{vw} is a split or a merge diagram. We retain the convention from §5.1 that singular homology $H_*(X)$ is taken with real coefficients. For most of the proof, we conflate the symbols x and $\widetilde{x}$, where x is an arc (usually viewed as a class in V_v) and $\widetilde{x}$ is its lift to Y_v (usually viewed as a class in $H_1(Y_v)$). That is, the maps ρ_v from above are implicit. Suppose first we are in the split case. Let $k = b_1(X_{vw})$. Note that $b_2^+(X_{vw}) = 0$, and that the cobordism $X_{vw} : Y_v \to Y_w$ is homeomorphic to

$$(Y_v \times [0,1]) \bowtie \left(D^2 \times S^2 \setminus \mathrm{int}(D^4)\right). \tag{6.5}$$

We note that we may also view X_{vw} as the branched double cover of a pair of pants properly embedded in $S^3 \times [0,1]$. We have $\mathcal{L}(X_{vw}) = H_1(Y_v) \oplus H_1(X_{vw})$. We will follow the notation of Definition 5.1.1, setting $X_1 = X$ and $X_2 = X_{vw}$. Choose orientations α_2 and β_2 of $H_1(Y_v)$ and $H_1(X_{vw})$, respectively. We can identify

$H_1(Y_v) = H_1(X_{vw})$ using the map induced by inclusion, and we choose to impose the condition $\alpha_2 = \beta_2$. Define $\varepsilon'_{vw} = \pm 1$ by

$$\mu_{vw} = \varepsilon'_{vw}\beta_2 \wedge \alpha_2.$$

Let $x = x_1 \wedge \cdots \wedge x_i \in \mathrm{C}_v$. Recall that $\Phi_v(x) = [X]^\#$ where X is obtained by attaching 2-handles to $x_1, \ldots, x_i$ along with some 3-handles and a 4-handle. Choose $x_{i+1}, \ldots, x_k$ so that $\mu_v = [x_1 \wedge \cdots \wedge x_k]$. Then $\mathcal{L}(X) = H_1(X)$ is generated by $x_{i+1}, \ldots, x_k$ and X is homology oriented by $\beta_1 := [x_{i+1} \wedge \cdots \wedge x_k]$. We can identify $\mathcal{L}(X_{vw} \circ X) = H_1(X_{vw} \circ X)$. Note that $\mathrm{im}(f_{12}) = H_1(Y_v)$, so $d_{12} = k$. Choose the section in the exact sequence (5.1), which in this case is a map $H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw})$, to be of the form $y \mapsto (y, 0)$. The induced isomorphism $F_{12} : H_1(Y_v) \oplus H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw})$, written as in (5.5), can be written

$$F_{12} : \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_{i+1}, \ldots, x_k\} \to \mathbb{R}\{x_{i+1}, \ldots, x_k\} \oplus \mathbb{R}\{x_1 \ldots, x_k\},$$

$$F_{12}(x_p, x_q) = (x_q + \pi(x_p), -x_p),$$

where $\pi : H_1(Y_v) \to H_1(X)$ is induced by inclusion. Writing $\beta_2 = \delta_{12}$, we have

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = (-1)^k \beta_1 \wedge \beta_2 = \delta_{12} \wedge \beta_{12}$$

where $\beta_{12} = (-1)^{(k-i)k+k}\beta_1 = (-1)^{ki}\beta_1$. Using Definition 5.1.1, we obtain

$$I^\#(X_{vw})\Phi_v(x) = (-1)^{(k^2+k)/2}\varepsilon'_{vw}[X_{vw} \circ X]^\#$$

where $X_{vw} \circ X$ is homology oriented by β_1. The sign $(-1)^{(k^2+k)/2}$ is obtained by computing

$$ki + \left((k^2 - k)/2 + (k - i)k\right),$$

where the term ki is from β_{12}, and the the expression inside the parentheses is from Definition 5.1.1. We mention that the condition $G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2$ holds by $\alpha_2 = \delta_{12} = \beta_2$ and setting ζ_{12} to be the canonical $+1$ orientation of the 0 vector space. Note that $[X_{vw} \circ X]^\# = \Phi_w(x_{vw} \wedge x)$ if and only if $\mu_w = [x_{vw} \wedge x_1 \wedge \cdots \wedge x_k] = x_{vw} \wedge \mu_v$;

otherwise they differ in sign. We record a sign $\varepsilon''_{vw} = \pm 1$ measuring this possible discrepancy between μ_v and μ_w:

$$\mu_w = \varepsilon''_{vw} x_{vw} \wedge \mu_v.$$

Recalling that $d^1_{vw} = (-1)^{\delta(v,w)} I^{\#}(X_{vw})$ and $\partial'_{vw}(x) = x_{vw} \wedge x$, we conclude

$$\Phi_w(\partial'_{vw}(x)) = \varepsilon_{vw} d^1_{vw}(\Phi_v(x))$$

where $\varepsilon_{vw} = \pm 1$ is given by

$$\varepsilon_{vw} = (-1)^{(k^2+k)/2+\delta(v,w)} \varepsilon'_{vw} \varepsilon''_{vw}.$$

Now suppose we are in the merge case. Again, let $k = b_1(X_{vw})$. As before, $b_2^+(X_{vw}) = 0$ and the cobordism $X_{vw} : Y_v \to Y_w$ is now homeomorphic to

$$(Y_w \times [0,1]) \bowtie \left(D^2 \times S^2 \setminus \operatorname{int}(D^4)\right).$$

We identify $H_1(X_{vw}) = H_1(Y_w)$, and write $\mathcal{L}(X_{vw}) = H_1(Y_v) \oplus H_1(Y_w)$. Note the natural codimension 1 inclusion $H_1(Y_w) \subset H_1(Y_v)$. A complement for $H_1(Y_w)$ is generated by x_{vw}. Let α_2 be an orientation for $H_1(Y_v)$. Define $\varepsilon'_{vw} = \pm 1$ by

$$\mu_{vw} = \varepsilon'_{vw} \beta_2 \wedge \alpha_2, \quad \beta_2 = \alpha_2 \llcorner x_{vw}.$$

The condition $\beta_2 = \alpha_2 \llcorner x_{vw}$ is equivalently expressed (or is defined) by $\beta_2 \wedge x_{vw} = \alpha_2$. Let $x = x_1 \wedge \cdots \wedge x_i \in \bigwedge^i(V_v)$. If x_{vw} is among $x_1, \ldots, x_i$ (or linearly dependent on them), the 4-manifold X constructed by attaching 2-handles to $x_1, \ldots, x_i$ and some 3-handles and a 4-handle, once paired with X_{vw} to form $X_{vw} \circ X$, contains a non-trivial S^2-bundle over S^2 as in §4.9, so $[X_{vw} \circ X]^{\#} = 0$. Choose $x_{i+1}, \ldots, x_{k+1}$ so that $\mu_v = [x_1 \wedge \cdots \wedge x_{k+1}]$; we may assume that $x_{vw} = x_{k+1}$. We may also set $\alpha_2 = \mu_v$, so that $\beta_2 = [x_1 \wedge \cdots \wedge x_k]$. Recall $\Phi_v(x) = [X]^{\#}$ where X is homology oriented by $\beta_1 = [x_{i+1} \wedge \cdots \wedge x_{k+1}]$. There is a codimension 1 inclusion $H_1(X_{vw} \circ X) \subset H_1(X)$. The vector space $H_1(X_{vw} \circ X)$ is generated by $x_{i+1}, \ldots, x_k$ and a complement for $H_1(X_{vw} \circ X)$ in $H_1(X)$ is generated by $x_{vw} = x_{k+1}$. Choose the section in the exact

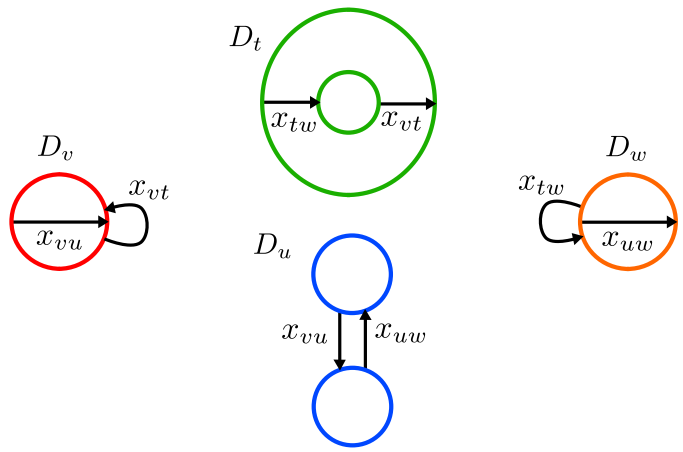

Figure 6.5: Local pictures for four diagrams appearing in a type X face, starting at the diagram D_v and ending at D_w. The circles in each diagram are colored so as to distinguish their roles in Figure 6.6.

sequence (5.1), which is a map $H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw})$, to be of the form $y \mapsto (y, 0)$. As in the split case, $\mathrm{im}(f_{12}) = H_1(Y_v)$. We obtain an isomorphism $F_{12} : H_1(Y_v) \oplus H_1(X_{vw} \circ X) \to H_1(X) \oplus H_1(X_{vw})$ that takes the form

$$F_{12} : \mathbb{R}\{x_1, \ldots, x_{k+1}\} \oplus \mathbb{R}\{x_{i+1}, \ldots, x_k\} \to \mathbb{R}\{x_{i+1}, \ldots, x_{k+1}\} \oplus \mathbb{R}\{x_1, \ldots, x_k\},$$

$$F_{12}(x_p, x_q) = (x_q + \pi_1(x_p), -\pi_2(x_p)),$$

where $\pi_1 : H_1(Y_v) \to H_1(X)$ and $\pi_2 : H_1(Y_v) \to H_1(X_{vw})$ are projections induced by inclusion maps. In particular, $\pi_1(x_p) = x_p$ if $p \geq i+1$ and is otherwise 0, and $\pi_2(x_p) = x_p$ if $p \neq k+1$ and $\pi_2(x_{k+1}) = 0$. Recalling that $\beta_2 = \alpha_2 \llcorner x_{vw}$ and choosing $\delta_{12} = \alpha_2$, we have

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = (\beta_1 \llcorner x_{vw}) \wedge \alpha_2 = \delta_{12} \wedge \beta_{12}$$

where $\beta_{12} = (-1)^{ki+i}(\beta_1 \llcorner x_{vw})$. Note $\beta_1 \llcorner x_{vw} = [x_{i+1} \wedge \cdots \wedge x_k]$. From Definition 5.1.1 we obtain

$$I^{\#}(X_{vw})[X]^{\#} = (-1)^{(k^2-k)/2+1} \varepsilon'_{vw} [X_{vw} \circ X]^{\#}$$

where $X_{vw} \circ X$ is homology oriented by $\beta_1 \llcorner x_{vw}$. We have computed the sign

$(-1)^{(k^2-k)/2+1}$ from

$$ki + i + \big(((k+1)^2 - (k+1))/2 + (k+1-i)(k+1)\big)\,,$$

where $ki + i$ is from β_{12}, and the expression inside the parentheses is from the sign in Definition 5.1.1. On the other hand, $[X_{vw} \circ X]^{\#} = \Phi_w(x)$ exactly when $\mu_w = [x_1 \wedge \cdots \wedge x_k] = \mu_v \llcorner x_{vw}$. Accounting for this, we define $\varepsilon''_{vw} = \pm 1$ by

$$\mu_v = \varepsilon''_{vw} \mu_w \wedge x_{vw}.$$

Recalling that $\partial'_{vw}(x) = x$, we conclude

$$\Phi_w(\partial'_{vw}(x)) = \varepsilon_{vw} d^1_{vw}(\Phi_v(x))$$

where $\varepsilon_{vw} = \pm 1$ is given by

$$\varepsilon_{vw} = (-1)^{(k^2-k)/2+1+\delta(v,w)} \varepsilon'_{vw} \varepsilon''_{vw}.$$

In summary, we have shown that

$$\Phi^{-1} d^1 \Phi = \sum_{\substack{v<w \\ |w-v|_1=1}} \varepsilon_{vw} \partial'_{vw}$$

where we have determined ε_{vw} in the split and merge cases separately. It remains to show that ε_{vw} is a valid edge assignment. The first condition, that the total differential squares to zero, already falls out from the spectral sequence. We now show that the ε_{vw} satisfy the second condition, that is, if v, u, t, w form a type X face, then the product

$$\varepsilon_{vu} \varepsilon_{vt} \varepsilon_{uw} \varepsilon_{tw} \tag{6.6}$$

is always +1 or -1, independently of the particular face chosen; and if they form a type Y face, the same is true, and the sign is opposite the type X case. We fix such a type X face. Note

$$\delta(v,u) + \delta(v,t) + \delta(u,w) + \delta(t,w) \equiv 0 \mod 2.$$

Next we consider the ε''_{vw} terms. We compute

$$x_{vu} \wedge \mu_v = \varepsilon''_{vu}\mu_u = \varepsilon''_{vu}\varepsilon''_{uw}\mu_w \wedge x_{uw}.$$

Since $x_{vu} = -x_{uw}$ in D_u, the above can be abbreviated to $\mu_v = (-1)^{k+1}\varepsilon''_{vu}\varepsilon''_{uw}\mu_w$. Similarly, we obtain $\mu_v = (-1)^{k+1}\varepsilon''_{vt}\varepsilon''_{tw}\mu_w$, implying $\varepsilon''_{vu}\varepsilon''_{vt}\varepsilon''_{uw}\varepsilon''_{tw} = 1$. A similar argument in the type Y case yields $\varepsilon''_{vu}\varepsilon''_{vt}\varepsilon''_{uw}\varepsilon''_{tw} = 1$ as well. To summarize, we may reconsider the problem with (6.6) replaced by the expression $\varepsilon'_{vu}\varepsilon'_{vt}\varepsilon'_{uw}\varepsilon'_{tw}$.

Note $\mathcal{L}(X_{vu}) = H_1(Y_v) \oplus H_1(X_{vu})$, and, as this is a split cobordism, we have a natural identification $H_1(Y_v) = H_1(X_{vu})$. Choose respective orientations α_1 and β_1 of $H_1(Y_v)$ and $H_1(X_{vu})$ that agree under this identification. Recall that ε'_{vu} has been defined by the relation

$$\mu_{vu} = \varepsilon'_{vu}\beta_1 \wedge \alpha_1.$$

On the other hand, $\mathcal{L}(X_{uw}) = H_1(Y_u) \oplus H_1(X_{uw})$, and, as this is a merge cobordism, there is a codimension 1 inclusion $H_1(X_{uw}) \subset H_1(Y_u)$ with a complement generated by x_{uw}. Let α_2 be an orientation of $H_1(Y_u)$ and set $\beta_2 = \alpha_2 \llcorner x_{uw}$. Then ε'_{uw} has been defined by

$$\mu_{uw} = \varepsilon'_{uw}\beta_2 \wedge \alpha_2.$$

In this situation, the map f_{12} of (5.1) has a 1-dimensional kernel spanned by x_{uw}. In this way $\mathrm{im}(f_{12})$ can be identified with $H_1(Y_v)$ and $H_1(X_{vu})$. Let a section for the exact sequence (5.1), here a map $H_1(X_{vw}) \to H_1(X_{vu}) \oplus H_1(X_{uw})$, be given by $y \mapsto (y, 0)$. The map $F_{12} : \mathrm{im}(f_{12}) \oplus H_1(X_{vw}) \to H_1(X_{vu}) \oplus H_1(X_{uw})$ of (5.5) can be written

$$F_{12} : \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_1, \ldots, x_k\} \to \mathbb{R}\{x_1, \ldots, x_k\} \oplus \mathbb{R}\{x_1, \ldots, x_k\},$$

$$F_{12}(x_p, x_q) = (x_q + x_p, -x_p).$$

Proceeding with the conditions of Definition 5.1.1, we find

$$F_{12}^{-1}(\beta_1 \wedge \beta_2) = \delta_{12} \wedge \beta_{12}$$

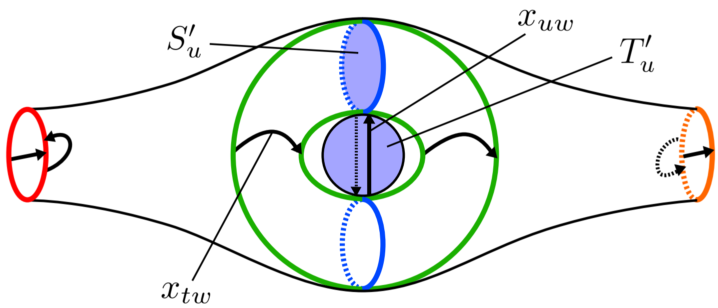

Figure 6.6: This is an illustration (missing a dimension) of S^4 minus two 4-balls, with a properly embedded surface F, a torus with two disks removed, with the local portions of the diagrams of the type X face from Figure 6.5 embedded; the circles of the diagrams lie on F, while only the endpoints of the arcs lie on F. The cobordism X_{vw} is the double cover over S^4 minus two 4-balls branched over F. The disk S'_u lifts to a 2-sphere $S_u \subset X_{vw}$ intersecting $\widetilde{x}_{uw}$ (the lift of x_{uw}) in one point. The disk T'_u lifts to a 2-sphere $T_u \subset X_{vw}$ intersecting Y_u in $\widetilde{x}_{uw}$.

where $\delta_{12} = \alpha_1 = \beta_2$ and $\beta_{12} = \beta_1$. We can arrange that $\beta_2 = \beta_1$ under the appropriate identification. The condition $G_{12}(\zeta_{12} \wedge \delta_{12}) = \alpha_2$, having that $\alpha_2 = \beta_2 \wedge x_{uw}$, yields $\zeta_{12} = (-1)^k x_{uw}$. Using Definition 5.1.1 we obtain

$$\mu_{uw} \circ \mu_{vu} = (-1)^{(k^2-k)/2+k(k+1)+k} \varepsilon'_{vu}\varepsilon'_{uw}\beta_1 \wedge \alpha_1 \wedge H^u_{12}(x_{uw})$$

where we've used $k = b_1(X_{vu}) = d_{12}$. The superscript u in H^u_{12} distinguishes this map from the map H^t_{12} which appears when u is replaced by t. We obtain a similar equation for $\mu_{tw} \circ \mu_{vt}$ with x_{uw} replaced by x_{tv} and $\varepsilon'_{vu}\varepsilon'_{uw}$ replaced by $\varepsilon'_{vt}\varepsilon'_{tw}$. Because our setup includes the conditions $\mu_{tw} \circ \mu_{vt} = \mu_{uw} \circ \mu_{vu}$, we conclude

$$\varepsilon'_{vu}\varepsilon'_{vt}\varepsilon'_{tw}\varepsilon'_{uw} = H^u_{12}(x_{uw})/H^t_{12}(x_{uw}) =: \varepsilon.$$

In summary, we see that ε is the sign determined by comparing the result of orienting $H_2^+(X_{vw})$ by x_{uw} versus the result by using x_{tw}. Using the interpretation of the splitting map (5.3) from §5.1, we obtain the following interpretation of ε. Here is a suitable moment to reintroduce the distinction between each arc x and its lift $\widetilde{x}$. Choose an oriented surface $S_u \subset Y_u$ transverse to $\widetilde{x}_{uw}$ with intersection product

$[S_u] \cdot [\widetilde{x}_{uw}] = 1$. Choose an oriented surface T_u with $T_u \cap Y_u = \widetilde{x}_{uw}$. Then

$$[S_u] + [T_u] = H^u_{12}(\widetilde{x}_{uw}).$$

To illustrate this, we supply Figure 6.6, where we use that X_{vw} is a double cover of S^4 minus two 4-balls branched over a properly embedded torus with two disks removed. Similarly, we can write $[S_t] + [T_t] = H^t_{12}(\widetilde{x}_{tw})$. The sign ε is then the intersection product of these classes:

$$\varepsilon = ([S_u] + [T_u]) \cdot ([S_t] + [T_t]).$$

In fact, $[T_t] = \varepsilon[S_u]$. From this it is clear that ε only depends on the topology of the type X configuration. A type Y face is obtained from a type X face by reversing the direction of either $\widetilde{x}_{uw}$ or $\widetilde{x}_{tw}$, and ε correspondingly changes sign. $\square$

6.4 Gradings

In this section we prove that the spectral sequence preserves the relevant $\mathbb{Z}/4$-gradings, completing the proof of Theorem 6.0.1. As usual, let $k = \dim(V_v)$. For $x \in \bigwedge^i(V_v) \subset \mathrm{C}$, the grading of $\Phi(x)$ in E^1 is given in (4.7) by

$$\mathrm{gr}[E^1](\Phi(x)) \equiv \mathrm{gr}[Y_v](\Phi(x)) - \deg(\mathbb{X}_{\boldsymbol{\infty} v}) - |v|_1 \mod 4. \tag{6.7}$$

We know, by the remark at the end of §4.9, that $\mathrm{gr}[Y_v](\Phi(x)) \equiv 2k + i$. We have $\deg(\mathbb{X}_{\boldsymbol{\infty} v}) = \deg(\mathbb{X}_{\boldsymbol{\infty}\mathbf{1}}) - \deg(X_{v\mathbf{1}})$, since $\mathbb{X}_{v\mathbf{1}}$ is trivial. From (2.11) we compute

$$\deg(X_{v\mathbf{1}}) = -\frac{3}{2}(m - |v|_1) + \frac{1}{2}(b_1(Y_{\mathbf{1}}) - k)$$

using $\chi(X_{vw}) = |w - v|_1$, $\sigma(X_{v\mathbf{1}}) = 0$ and $b_1(Y_v) = k$. We also compute

$$\deg(X_{\boldsymbol{\infty}\mathbf{1}}) = -\frac{3}{2}(2m + \sigma(X_{\boldsymbol{\infty}\mathbf{1}})) + \frac{1}{2}(b_1(Y_{\mathbf{1}}) - b_1(\Sigma(L)))$$

knowing $\Sigma(L) = \overline{Y}_{\boldsymbol{\infty}}$. Recall from (4.1) that $\deg(\mathbb{X}_{\boldsymbol{\infty}\mathbf{1}}) \equiv \deg(X_{\boldsymbol{\infty}\mathbf{1}}) + 2\mathscr{P}(\mathbb{X}_{\boldsymbol{\infty}\mathbf{1}})$.

Lemma 6.4.1. *$\mathscr{P}(\mathbb{X}_{\infty\mathbf{1}}) \equiv \sigma(X_{\infty\mathbf{1}}) \mod 2$.*

Before proving this lemma, we make our conclusion. In [3], Bloom computes $\sigma(X_{\mathbf{0}\infty}) = \sigma - n_+$ and $b_1(\Sigma(L)) = \nu$, where σ and ν are the signature and nullity of L, respectively, and $n_\pm$ is the number of $\pm$ crossings of the diagram D. Note that $X_{\infty\mathbf{1}}$ and $X_{\mathbf{1}\infty}$ compose along $Y_{\mathbf{1}}$ to give a cobordism which, away from a manifold of signature 0, has m copies of $-\mathbb{CP}^2$ connected summed to it (cf. E from §3.1.4). In addition, since $\sigma(X_{\mathbf{01}}) = 0$, we have $\sigma(X_{\infty\mathbf{1}}) = \sigma(X_{\infty\mathbf{0}})$. Additivity of the signature again implies that $\sigma(X_{\infty\mathbf{1}}) = -m - \sigma + n_+$. Note $m = n_+ + n_-$. All together, (6.7) computes to

$$i + 2n_- + \frac{3}{2}(n_+ + k) + \frac{1}{2}(|v|_1 + \nu + \sigma) \mod 4,$$

which is congruent to (6.4). This completes the proof of Theorem 6.0.1.

Proof of Lemma 6.4.1. By additivity and the fact that $\mathscr{P}(\mathbb{X}_{\mathbf{01}}) \equiv \sigma(\mathbb{X}_{\mathbf{01}}) \equiv 0 \mod 2$, it suffices to show that $\mathscr{P}(\mathbb{X}_{\infty\mathbf{0}}) \equiv \sigma(X_{\infty\mathbf{0}})$. Write $\mathbb{X} = \mathbb{X}_{\infty\mathbf{0}}$ and X for its base space. We have

$$\mathbb{X} = ([0,1] \times \mathbb{Y}) \cup_{\mathbb{L}} (\cup_{i=1}^m \mathbb{H})$$

where $\mathbb{L} = \mathbb{L}_1 \cup \cdots \cup \mathbb{L}_m$ is an SO(3)-thickening of $L = L_1 \cup \cdots \cup L_m$, and each $\mathbb{L}_i : \mathbb{H}_1 \to \mathbb{Y} \times \{1\}$ is as in §3.1.3. Here we are viewing

$$\mathbb{Y} = (Y \times \mathrm{SO}(3))_\Psi(\mathbb{L})$$

as a bundle over $Y = Y_\infty = \overline{\Sigma(L)}$ built from the geometric representative L as in §3.1.7. In §6.1 we saw that L is the boundary of a surface $S \subset Y$, so in fact $\mathbb{Y}$ is a trivial bundle. Note $\mathbb{X}$ is reducible to an S^1-bundle by its very construction. Let $\mathscr{L}$ be the associated complex line bundle. The Poincaré dual of a pre-image of $c_1(\mathscr{L}) \in H^2(X;\mathbb{Z})$ in $H^2(X, \partial X; \mathbb{Z})$ is represented by the closed surface $S' \subset \mathrm{int}(X)$ which is the union of the cores of the 2-handles together with $S \subset Y \times \{1\}$. Indeed, it is straightforward to define a section of $\mathscr{L}$ with zero set S'. By the definition of $\mathscr{P}(\mathbb{X})$, it suffices to show that

$$[S'] \cdot [S'] \equiv \sigma(X) \mod 2$$

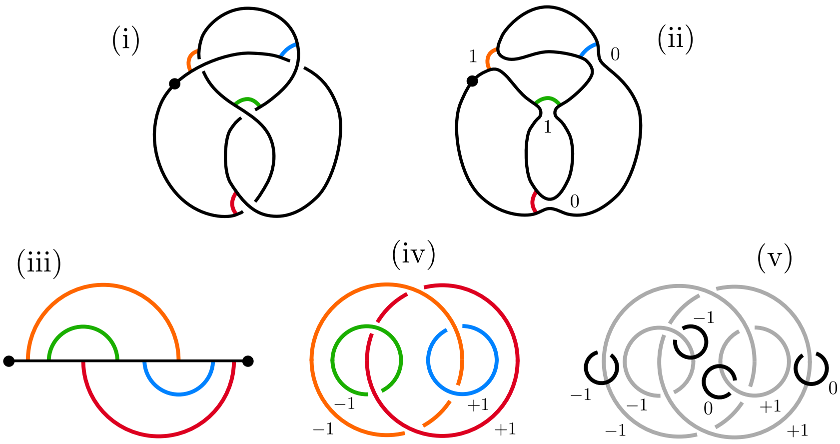

Figure 6.7: To obtain a relative Kirby diagram for $(X_{\infty \mathbf{0}}, Y_\infty)$ where $Y_\infty = \overline{\Sigma(L)}$, we borrow some constructions from Bloom [3]. With a diagram of the figure eight knot in (i) as an example, we first choose a resolution that yields one connected circle as in (ii), drawing small arcs where crossings used to be. We then cut the connected circle at the dot, and straighten it out, as in (iii). Reflecting this picture across the line, we obtain a surgery diagram (iv) for Y_∞ by choosing a $+1$ framing for each circle corresponding to a 0-resolution, and a -1 framing for 1-resolution circles. Finally, the relative Kirby diagram (v) is obtained by placing a small meridional circle on each circle in (iv) framed by 0 or -1, depending on whether the circle corresponds to a 0- or 1-resolution, respectively.

where $[S'] \cdot [S']$ is the intersection product. To do this we write down a relative Kirby diagram for (X, Y). We start by writing a surgery diagram for $Y = \overline{\Sigma(L)}$ using the chosen diagram D. For this we follow Bloom [3]. First, choose $v \in \{0,1\}^m$ for which the resolution D_v has 1 circle. We can always choose D so that there is such a resolution. Then, in D_v, having placed arcs where crossings once were, cut the lone circle at an isolated point p and unravel it, with the arcs attached, into a horizontal segment; then double it as in Figure 6.7 (iv). Place a $+1$ framing on a circle in the resulting picture if that circle came from a 0-resolution, and a -1 framing otherwise. This gives a surgery diagram for $Y = \overline{\Sigma(L)}$.

To turn this into a relative Kirby diagram for (X, Y), we simply add small meridians around each circle, framed with a 0 if the circle is $+1$ framed and a -1 if the circle is -1 framed. The intersection number $[S'] \cdot [S']$ is concentrated at the attaching locations of the 2-handles, represented by the meridional circles in the relative Kirby diagram. Thus there is a -1 contribution to $[S'] \cdot [S']$ from each 1-resolution in D_v. We conclude $[S'] \cdot [S'] = -|v|_1$. According to [3] Prop. 1.7 and Lemma 9.4, the signature $\sigma(X)$ is mod 2 congruent to the vertex weight of a 1-circle resolution. □

CHAPTER 7

The relationship between $I^{\#}(Y)$ and $I(Y)$

A closed, connected, oriented 3-manifold Y is called an integral homology 3-sphere if $H_1(Y;\mathbb{Z})=0$, or equivalently, if Y has the same integral homology as the 3-sphere. In this section we study $I^{\#}(Y)$ when Y is an integral homology 3-sphere. As $I^{\#}(Y)$ is determined by the instanton homology of an admissible bundle over $Y\#T^3$, we can apply a modified version of Fukaya's instanton connected sum theorem of [17]. As a result, we relate the $\mathbb{Z}/4$-graded group $I^{\#}(Y)$ to Floer's original $\mathbb{Z}/8$-graded instanton homology $I(Y)$ through the trivial connection and u-maps studied by Donaldson [7] and Frøyshov [16]. Twisted framed groups $I^{\#}(Y;\lambda)$ are also considered.

To state the result, it is convenient to employ Frøyshov's reduced instanton groups $\widehat{I}(Y)$ from [16]. They come equipped with an absolute $\mathbb{Z}/8$-grading and a degree 4 endomorphism u. The main result is

Theorem 7.0.2. *Let F be a field with char$(F)\neq 2$, and suppose all homology groups are taken with F-coefficients, unless indicated otherwise. If $H_1(Y;\mathbb{Z})=0$,*

$$I^{\#}(Y)\simeq \ker(u^2-64)\otimes H_*(S^3)\oplus H_*(pt.)$$

as $\mathbb{Z}/4$-graded F-modules, where u^2-64 is acting on $\bigoplus_{j=0}^{3}\widehat{I}(Y)_j$. If $\mathbb{Y}$ is non-trivial admissible with geometric representative λ, then

$$I^{\#}(Y;\lambda)\simeq \ker(u^2-64)\otimes H_*(S^3)$$

as relatively $\mathbb{Z}/4$-graded F-modules, where u^2-64 is acting on four consecutive gradings of the relatively $\mathbb{Z}/4$-graded F-module $I(\mathbb{Y})$.

In this chapter we prove this theorem. In §7.1, we fix our convention for the absolute $\mathbb{Z}/8$-grading on $I(Y)$. In §7.2, we describe maps on the instanton chain complex that are used to define Frøyshov's reduced groups $\widehat{I}(Y)$ in §7.3. In §7.4 we state and prove Fukaya's connected sum theorem for $I(Y)$ following Donaldson. Then, in §7.5, we give modified statements and proofs of Fukaya's theorem to handle the presence of non-trivial admissible bundles. In §7.6 we prove Theorem 7.0.2. Finally, in §7.7 we discuss rank inequalities that result from Theorem 7.0.2.

7.1 Gradings

Let Y be an integral homology 3-sphere. Let θ be the distinguished trivial connection on $Y \times \mathrm{SO}(3)$. We can use θ to fix an absolute $\mathbb{Z}/8$-grading on $I(Y)$ as was done in Floer's original construction [13]. For a connection a on $Y \times \mathrm{SO}(3)$ we set

$$\mathrm{gr}(a) = -3 - \mu(\theta, a)$$

and on $\mathscr{G}$-classes $\mathfrak{a}$ this descends to a function with $\mathrm{gr}(\mathfrak{a}) \in \mathbb{Z}/8$. Note that $\mathscr{G}_{\mathrm{ev}} = \mathscr{G}$ in this setting. When a is irreducible, $\mathrm{gr}(a) = \mu(a, \theta)$. The trivial connection has $\mathrm{gr}(\theta) = 0$. The differential shifts this grading by -1 and the grading descends to the $\mathbb{Z}/2$-grading defined in §2.6. We write $\mathfrak{t}$ for the $\mathscr{G}$-class of θ. Our $I(Y)_i$ agrees with Donaldson's $\mathrm{HF}(Y)_i$ in [7]. Note that $I(\overline{Y})_i$ is the same as the cohomology group $I(Y)^{5-i}$. In particular, by the universal coefficients theorem, the vector spaces $I(\overline{Y})_i \otimes \mathbb{Q}$ and $I(Y)_{5-i} \otimes \mathbb{Q}$ are isomorphic. Our $I(Y)_i$ is the same as Frøyshov's $\mathrm{HF}(Y)^{5-i} = \mathrm{HF}(\overline{Y})_i$ from [16].

7.2 Other boundary maps

From here on we fix a field F which has $\mathrm{char}(F) \neq 2$ and take all homology with F-coefficients. With an integral homology 3-sphere Y fixed, we write $\mathrm{C}_i = \mathrm{C}(Y)_i$

and $I_i = I(Y)_i$. Following [7, 16] we have maps

$$\delta : \mathrm{C}_1 \to F, \qquad \delta' : F \to \mathrm{C}_4$$

defined using the trivial connection. For $\mathfrak{a} \in \mathfrak{C}^{\mathrm{irr}}(Y)$ with $\mathrm{gr}(a) \equiv 1$ we define $\delta(\mathfrak{a}) = \#\breve{M}(\mathfrak{a}, \mathfrak{t})_0$, and for $\mathfrak{b}$ with $\mathrm{gr}(\mathfrak{b}) \equiv -4$, we define $\langle \delta'(1), \mathfrak{b} \rangle = \#\breve{M}(\mathfrak{t}, \mathfrak{b})_0$. More precisely, one writes $F = F\Lambda(\mathfrak{t})$ and $\epsilon[A] : \Lambda(\mathfrak{t}) \to \Lambda(\mathfrak{a})$, and $\delta = \sum \epsilon[A]$ for each $[A] \in \breve{M}(\mathfrak{a}, \mathfrak{t})_0$, and so on, as in §2.1. We will often conflate δ' with $\delta'(1) \in \mathrm{C}_4$. These are chain maps, in the sense that $\delta\partial = \partial\delta' = 0$, and we write

$$\boldsymbol{\delta} : I_1 \to F, \qquad \boldsymbol{\delta}' : F \to I_4$$

for the induced maps on homology.

We also have maps that record data from 3-dimensional moduli spaces $\breve{M}(\mathfrak{a}, \mathfrak{b})_3$,

$$v : \mathrm{C}_i \to \mathrm{C}_{i+4}.$$

Our v is $1/2$ times the v of Frøyshov, and 4 times the U of Donaldson. That is, it is defined, roughly, by evaluating the 4-dimensional class $2\mu(\mathrm{pt})$ over 4-dimensional moduli spaces $M(\mathfrak{a}, \mathfrak{b})_4$. We refer to [16, §3.1] and [7, §7.3.1] for precise definitions of v. We have in mind the following interpretation. First suppose $\breve{M}(\mathfrak{a}, \mathfrak{b})_3$ is connected. We obtain a map $h : \breve{M}(\mathfrak{a}, \mathfrak{b})_3 \to \mathrm{SO}(3)$ by evaluating the holonomy of a connection along the path from $(-\infty, y)$ to (∞, y) on the cylinder $\mathbb{R} \times Y$. With some modifications, see [7, §7.3.2], $\langle v(\mathfrak{a}), \mathfrak{b} \rangle = \deg(h)$. If $\breve{M}(\mathfrak{a}, \mathfrak{b})_3$ has more than one component, the evaluation is done on each component, and then added together.

The map v is not quite a chain map. As explained in [7, §7.3.3], when $\mathrm{gr}(\mathfrak{a}) \equiv 1$ and $\mathrm{gr}(\mathfrak{b}) \equiv -4$, there are ends of $\breve{M}(\mathfrak{a}, \mathfrak{b})_4$ modelled on $\mathrm{SO}(3)$, i.e. cylinders $\mathbb{R} \times \mathrm{SO}(3)$, one for each pair of instantons in $\breve{M}(\mathfrak{a}, \mathfrak{t})_0 \times \breve{M}(\mathfrak{t}, \mathfrak{b})_0$. Each copy of $\mathrm{SO}(3)$ records the choices for gluing parameters. The holonomy at a cross-section is captured by the gluing parameter and has degree 1. Accounting for the other usual ends, modelled on $\mathbb{R} \times \breve{M}(\mathfrak{a}, \mathfrak{c})_i \times \breve{M}(\mathfrak{c}, \mathfrak{b})_j$, where $i = 0$ and $j = 3$, or vice versa, one is led to the relation

$$\partial v - v\partial + \delta'\delta = 0, \tag{7.1}$$

see [16, Thm. 4] and [7, Prop. 7.8]. Here $\delta = 0$ in gradings different from $1 \in \mathbb{Z}/8$. In particular we obtain the maps

$$\begin{aligned} \boldsymbol{v} : I_i \to I_{i+4}, &\qquad i \neq 0, 1 \mod 8 \\ \boldsymbol{v} : I_0 \to \operatorname{coker}(\boldsymbol{\delta}'), &\qquad \boldsymbol{v} : \ker(\boldsymbol{\delta}) \to I_5. \end{aligned}$$

7.3 Reduced instanton groups

Frøyshov defined a $\mathbb{Z}/8$-graded group $\widehat{I} = \widehat{I}(Y)$ by cutting down $I(Y)$ using the maps introduced above. Precisely,

$$\widehat{I}_i = I_i, \quad i \not\equiv 0, 1, 4, 5 \mod 8$$

$$\widehat{I}_0 = I_0 / \left(\sum \operatorname{im}(\boldsymbol{v}^{2k+1} \boldsymbol{\delta}') \right), \qquad \widehat{I}_4 = I_4 / \left(\sum \operatorname{im}(\boldsymbol{v}^{2k} \boldsymbol{\delta}') \right),$$

$$\widehat{I}_1 = \bigcap \ker(\boldsymbol{\delta} \boldsymbol{v}^{2k}) \subset I_1, \qquad \widehat{I}_5 = \bigcap \ker(\boldsymbol{\delta} \boldsymbol{v}^{2k+1}) \subset I_5.$$

Using these groups Frøyshov defined his h-invariant by

$$h(Y) = -\frac{1}{2} \left(\chi(I(Y)) - \chi(\widehat{I}(Y)) \right).$$

This has several nice properties, among them

$$h(\overline{Y}) = -h(Y), \qquad h(Y \# Y') = h(Y) + h(Y').$$

It also descends to a homomorphism $h : \Theta^3_H \to \mathbb{Z}$, where Θ^3_H is the integral homology cobordism group. Frøyshov showed that both I and $\widehat{I}$ are 4-periodic (recall that we are working with F-coefficients). By the chain level relation (7.1) either $\boldsymbol{\delta}$ or $\boldsymbol{\delta}'$ is zero. It follows that, over $\mathbb{Q}$, we can go between I and $\widehat{I}$ using only h. For example, if $h(Y) = 0$, then $\widehat{I} = I$, whereas if $h(Y) > 0$ then $\widehat{I}_i = I_i$ for $i \neq 0, 4$ and $\operatorname{rk}(\widehat{I}_i) = \operatorname{rk}(I_i) - h(Y)$ for $i = 0, 4$.

The maps $\boldsymbol{v}$ above induce maps $\widehat{v} : \widehat{I}(Y)_i \to \widehat{I}(Y)_{4+i}$ for each grading $i \in \mathbb{Z}/8$. As mentioned, this is half of Frøyshov's u mentioned in the introduction:

$$\widehat{v} = u/2.$$

We've chosen this normalization to avoid writing in certain factors of 2. Frøyshov showed that each $\widehat{v}$ is an isomorphism, and that $\widehat{v}^2 - 16$ is nilpotent, i.e.

$$(\widehat{v}^2 - 16)^n = 0$$

for some $n > 0$. If $\mathbb{Y}$ is admissible and $b_1(Y) > 0$, there is no trivial connection to work with, and the maps $v : \mathrm{C}_i \to \mathrm{C}_{i+4}$ are indeed chain maps, inducing maps $\widehat{v} : I(\mathbb{Y})_i \to I(\mathbb{Y})_{i+4}$ for each grading i (here we arbitrarily fix an absolute grading). Again, each $\widehat{v}$ is half of Frøyshov's u, is an isomorphism, and $\widehat{v}^2 - 16$ is nilpotent. The hat notation in this case is used only for uniformity.

7.4 Fukaya's connected sum theorem

In this section we recall the connected sum theorem of Fukaya [17], reviewing the proof exposited by Donaldson in [7, §7.4]. This problem was also considered in [27]. In the following sections we will adapt the proof to the settings of interest to us. Let Y_1 and Y_2 be integral homology 3-spheres. For $i = 1, 2$ write $\mathrm{C}_{(i)} = \mathrm{C}(Y_i)$ and $\partial_{(i)}$ for the corresponding differentials, and $\delta_{(i)}, \delta'_{(i)}, v_{(i)}$ for the relevant boundary maps. For a graded F-module A define the shifted module $A[n]$ by $A[n]_i = A_{i-n}$. We define a chain complex

$$\mathrm{C} = \big(\mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)}\big) \oplus \big(\mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)}\big) \oplus \big(\mathrm{C}_{(1)} \otimes F\big) \oplus \big(F \otimes \mathrm{C}_{(2)}\big)$$

$$\partial = \begin{pmatrix} \partial_{(12)} & 0 & 0 & 0 \\ v_{(12)} & -\partial_{(12)} & 1 \otimes \delta'_{(2)} & \delta'_{(1)} \otimes 1 \\ -1 \otimes \delta_{(2)} & 0 & \partial_{(1)} \otimes 1 & 0 \\ \delta_{(1)} \otimes 1 & 0 & 0 & \epsilon \otimes \partial_{(2)} \end{pmatrix}$$

where $\partial_{(12)} = \partial_{(1)} \otimes 1 + \epsilon \otimes \partial_{(2)}$, $v_{(12)} = v_{(1)} \otimes 1 + 1 \otimes v_{(2)}$, and ϵ is equal, in grading k, to $(-1)^k$ times the identity map on $\mathrm{C}_{(1)}$.

Theorem 7.4.1 (Fukaya)**.** *As $\mathbb{Z}/8$-graded F-modules, $I(Y_1 \# Y_2) \simeq H_*(\mathrm{C}, \partial)$.*

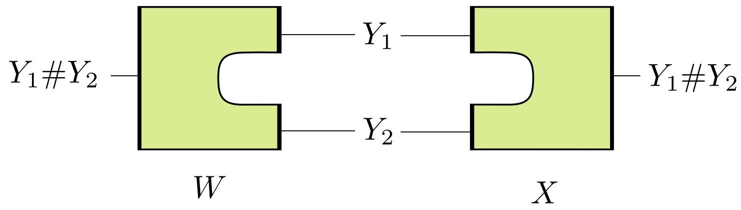

Figure 7.1: The cobordism $W : Y_1\#Y_2 \to Y_1 \sqcup Y_2$ and its reverse, X.

For example, let Y be the Poincaré homology 3-sphere $\Sigma(2,3,5)$. The reader can verify that

$$I(Y\#Y) \simeq F_1^2 \oplus F_5^2, \qquad I(Y\#\overline{Y}) = 0$$

using that $\mathrm{C}(Y) = F_1 \oplus F_5$ and δ, v are isomorphisms. Recall that subscripts indicate gradings. These examples appear in [17]. Note that, generally, the δ, δ', v maps for $\overline{Y}$ are the duals of the maps δ', δ, v for Y, respectively.

We now review the proof that appears in [7, §7.4]. We mention at the outset that to avoid certain factors of 2 that appear in the composition law (since we will glue along a disconnected 3-manifold), we enlarge the gauge transformation group when necessary, as in [20, §5.1]. Let $\mathrm{C}' = \mathrm{C}(Y_1\#Y_2)$ and ∂' be its differential. Let $X : Y_1 \sqcup Y_2 \to Y_1\#Y_2$ be the cobordism which is $([0,1]\times Y_1)\natural([0,1]\times Y_2)$, where the boundary sum is taken near 1, and let $W : Y_1\#Y_2 \to Y_1 \sqcup Y_2$ be the corresponding cobordism when the boundary sum is taken near 0. See Figure 7.1. We define maps

$$m_X : \mathrm{C} \to \mathrm{C}', \quad m_W : \mathrm{C}' \to \mathrm{C}$$

as follows. The map m_X is given by four components:

$$\begin{aligned} v_X &: \mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)} \to \mathrm{C}', \\ i_X &: \mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)} \to \mathrm{C}', \\ \delta'_{X(2)} &: \mathrm{C}_{(1)} \otimes F \to \mathrm{C}', \\ \delta'_{X(1)} &: F \otimes \mathrm{C}_{(2)} \to \mathrm{C}'. \end{aligned}$$

In the following, $\mathfrak{a} \in \mathfrak{C}^{\mathrm{irr}}(Y_1)$, $\mathfrak{b} \in \mathfrak{C}^{\mathrm{irr}}(Y_2)$, and $\mathfrak{c} \in \mathfrak{C}^{\mathrm{irr}}(Y_1\#Y_2)$. The map i_X counts 0-dimensional moduli spaces $M(\mathfrak{a},\mathfrak{b},X,\mathfrak{c})_0$. The map v_X evaluates the holonomy of 3-dimensional moduli spaces $M(\mathfrak{a}, \mathfrak{b}, X, \mathfrak{c})_3$ along a curve γ_X running from Y_1 to Y_2 on the incoming end of X. The map $\delta'_{X(2)}$ counts 0-dimensional moduli spaces $M(\mathfrak{a}, \mathfrak{t}, X, \mathfrak{c})_0$ where $\mathfrak{t}$ is a trivial connection class on Y_2, and $\delta'_{X(1)}$ is defined similarly, with $\mathfrak{t}$ on Y_1. Now, m_X is a chain map because of the following relations. First,

$$i_X \partial_{(12)} = \partial' i_X$$

is the usual relation for the map involving only irreducibles. Second,

$$i_X v_{(12)} + v_X \partial_{(12)} + \delta'_{X(1)}(\delta_{(1)} \otimes 1) - \delta'_{X(2)}(1 \otimes \delta_{(2)}) = \partial' v_X \tag{7.2}$$

records how the holonomy interacts with the ends of a 4-dimensional moduli space $M(\mathfrak{a}, \mathfrak{b}, X, \mathfrak{c})_4$. This is essentially [16, Thm. 6]. See Figure 7.2. Third, the relation

$$i_X(\delta'_{(1)} \otimes 1) + \delta'_{X(1)}(\epsilon \otimes \partial_{(2)}) = \partial' \delta'_{X(1)}$$

and its analogue with indices swapped, records the ends of a 1-dimensional moduli space $M(\mathfrak{t}, \mathfrak{b}, X, \mathfrak{c})_1$ where $\mathfrak{t}$ is the trivial connection class on Y_1. This is a variation of [16, Lemma 1].

The map m_W is defined similarly, this time with components

$$\begin{aligned} i_{\mathrm{W}} &: \mathrm{C}' \to \mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)}, \\ v_{\mathrm{W}} &: \mathrm{C}' \to \mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)}, \\ \delta_{\mathrm{W}(2)} &: \mathrm{C}' \to \mathrm{C}_{(1)} \otimes F, \\ \delta_{\mathrm{W}(1)} &: \mathrm{C}' \to F \otimes \mathrm{C}_{(2)}. \end{aligned}$$

Now, we argue that m_X and m_W are chain homotopy inverse to one another. First consider $m_X m_W$. We have

$$m_X m_W = v_X i_W + i_X m_W + \delta'_{X(2)} \delta_{W(2)} + \delta'_{X(1)} \delta_{W(1)}.$$

We claim that $m_X m_W$ is chain homotopic to the map $m(Z, \gamma) : \mathrm{C}' \to \mathrm{C}'$ obtained by evaluating $2\mu(\gamma)$ on the composite $Z = X \circ W$ where $\gamma = \gamma_X \cup \gamma_W$. This is the the same as the map defined by taking the degrees of modified holonomy maps

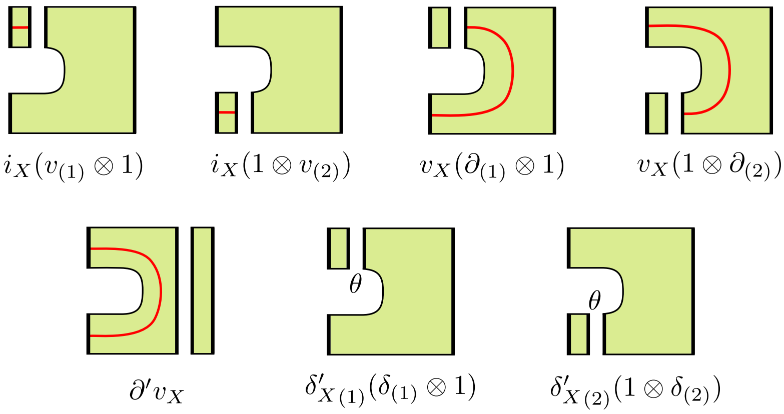

Figure 7.2: A representation of the terms appearing in (7.2). The pieces represent counts of isolated instantons, unless there is a curve present in the interior, indicating a contribution from a v-map. All limiting connections are irreducible, except in the last two diagrams, where trivial limits θ are present. The first two diagrams make up $i_X v_{(12)}$ and the second two make up $v_X \partial_{(12)}$.

$M(\mathfrak{a}, Z, \mathfrak{d})_3 \to \mathrm{SO}(3)$ along γ, see [8, §5.1.2]. The chain homotopy is obtained by stretching the middle copies of Y_1 and Y_2. The 3-dimensional space $M(\mathfrak{a}, Z, \mathfrak{d})_3$ where $\mathfrak{a}, \mathfrak{d}$ are irreducible has four components after stretching:

$$M(\mathfrak{a}, X, \mathfrak{b}, \mathfrak{c})_0 \times M(\mathfrak{b}, \mathfrak{c}, W, \mathfrak{d})_3$$

$$M(\mathfrak{a}, X, \mathfrak{b}, \mathfrak{c})_3 \times M(\mathfrak{b}, \mathfrak{c}, W, \mathfrak{d})_0$$

$$M(\mathfrak{a}, X, \mathfrak{b}, \mathfrak{t})_0 \times \mathrm{SO}(3) \times M(\mathfrak{b}, \mathfrak{t}, W, \mathfrak{d})_0$$

$$M(\mathfrak{a}, X, \mathfrak{t}, \mathfrak{c})_0 \times \mathrm{SO}(3) \times M(\mathfrak{t}, \mathfrak{c}, W, \mathfrak{d})_0$$

As in (7.1), in the last two cases the holonomy is captured by the gluing space SO(3). The four components correspond, in order, to the four components of $m_X m_W$ above. In this way, the chain homotopy from $m_X m_{\mathrm{W}}$ to $m(Z, \gamma)$ may be defined as a map using the 1-dimensional metric family that simultaneously stretches along Y_1, Y_2.

The next step is to use a surgery property, interesting in its own right, due to Donaldson. We state it in a form convenient for our purposes. Let $\mathbb{X} : \mathbb{Y}_1 \to \mathbb{Y}_2$ be an SO(3)-bundle over a cobordism which restricts to admissible bundles over its boundary components. Let γ be a loop in the interior of the base of $\mathbb{X}$. Let $\mathbb{X}_\gamma$ be the bundle obtained by cutting out a neighborhood $S^1 \times D^3 \times \mathrm{SO}(3)$ lying over γ and gluing back in a copy of $D^2 \times S^2 \times \mathrm{SO}(3)$. Denote by $m(\mathbb{X}, \gamma) : \mathrm{C}(\mathbb{Y}_1) \to \mathrm{C}(\mathbb{Y}_2)$ the map obtained by evaluating $\mu(\gamma)$ on 3-dimensional moduli spaces $M(\mathfrak{a}, \mathbb{X}, \mathfrak{b})_3$.

Theorem 7.4.2 (see [7] Thm. 7.16)**.** *$m(\mathbb{X}, \gamma)$ is chain homotopic to $m(\mathbb{X}_\gamma)$.*

In our situation, observe that the surgered manifold Z_γ is the product $[0, 1] \times (Y_1 \# Y_2)$. It follows that $m_X m_W$ is chain homotopic to the identity.

Now consider $m_W m_X$. This has 16 components

$$i_W v_X, \quad v_W i_X, \quad \delta_{W(1)} \delta'_{X(1)}, \quad \ldots$$

It is chain homotopic to a map f that counts similar data on the cobordism $W \circ X$ with metric stretched very long along the internal connected sum portion between $[0, 1] \times Y_1$ and $[0, 1] \times Y_2$. The map f has components corresponding to the components of $m_W m_X$, but most of them vanish. For instance, the 7 components of f corresponding to

$$i_W i_X, \quad \delta_{W(i)} i_X, \quad i_W \delta'_{X(i)}, \quad \delta_{W(i)} \delta'_{X(j)} \quad (i \neq j)$$

all vanish by index arguments. Each counts instantons A with $\mu(A) = 0$ obtained by gluing an instanton A_1 on $\mathbb{R} \times Y_1$ to an instanton A_2 over $\mathbb{R} \times Y_2$ along a S^3. For $i = 1, 2$ at least one of the limits on $\mathbb{R} \times Y_i$ is irreducible. Thus both A_1, A_2 are irreducible. It follows from $0 = \mu(A) = \mu(A_1) + \mu(A_2) + 3$ and $\mu(A_i) \geq 0$ that no such A exist. Similarly, the 4 components of f corresponding to

$$\delta_{W(i)} v_X, \quad v_W \delta'_{X(i)}$$

are zero. These components require 3-dimensional moduli spaces. However, with the neck stretched, the relevant 3-dimensional moduli spaces are $M(\mathfrak{a}, \mathfrak{b})_0 \times \mathrm{SO}(3) \times M(\mathfrak{c}, \mathfrak{d})_0$ where one of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ is the trivial class $\mathfrak{t}$ and $\mathfrak{a}, \mathfrak{b}$ are connection classes on Y_1 and $\mathfrak{c}, \mathfrak{d}$ on Y_2. But $M(\mathfrak{a}, \mathfrak{t})_0$ is empty for any irreducible $\mathfrak{a}$. Next, the 4 components of f corresponding to

$$i_{\mathrm{W}} v_X, \quad v_{\mathrm{W}} i_X, \quad \delta_{\mathrm{W}(i)} \delta'_{X(i)}$$

are identity maps. For instance, the first one uses 3-dimensional spaces modelled on $M(\mathfrak{a},\mathfrak{b})_0 \times \mathrm{SO}(3) \times M(\mathfrak{c},\mathfrak{d})_0$ from gluing; the holonomy map v_X captures the gluing parameter just as in (7.1), leaving us to count $M(\mathfrak{a},\mathfrak{b})_0 \times M(\mathfrak{c},\mathfrak{d})_0$. Of course $M(\mathfrak{a},\mathfrak{b})_0$ forces $\mathfrak{a} = \mathfrak{b}$ and has one translation invariant irreducible flat connection. Finally, we are left with 1 component of f corresponding to

$$v_W v_X$$

which may be nonzero. However, we know that f is the identity plus this off-diagonal term, and thus induces an isomorphism on homology. So $m_W m_X$ also induces an isomorphism on homology. Because $m_X m_W$ induces the identity on $I(Y_1\#Y_2)$, so does $m_{\mathrm{W}} m_X$. This completes the proof.

7.5 Connected sum theorems for non-trivial bundles

In this section we state two variants of the connected sum theorem, when one or both of Y_1 and Y_2 is replaced by a non-trivial admissible bundle. We then explain how the proof above adapts to these cases. These are simpler than the above, having fewer trivial connections to deal with.

We first consider the case where $\mathbb{Y}_1$ is trivial and Y_1 is an integral homology 3-sphere, but $\mathbb{Y}_2$ is non-trivial and admissible. Let $\mathrm{C}_{(1)} = \mathrm{C}(\mathbb{Y}_1)$ with maps $\partial_{(1)}, \delta, \delta', v_{(1)}$. Let $\mathrm{C}_{(2)} = \mathrm{C}(\mathbb{Y}_2)$ with maps $\partial_{(2)}, v_{(2)}$. Define

$$\mathrm{C} = \big(\mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)}\big) \oplus \big(\mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)}\big) \oplus \big(F \otimes \mathrm{C}_{(2)}\big)$$

$$\partial = \begin{pmatrix} \partial_{(12)} & 0 & 0 \\ v_{(12)} & -\partial_{(12)} & \delta' \otimes 1 \\ \delta \otimes 1 & 0 & \epsilon \otimes \partial_{(2)} \end{pmatrix}$$

with notation as before.

Theorem 7.5.1. *Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be admissible bundles, with $\mathbb{Y}_1$ trivial and $\mathbb{Y}_2$ non-trivial. As $\mathbb{Z}/8$-graded F-modules, $I(\mathbb{Y}_1\#\mathbb{Y}_2) \simeq H_*(\mathrm{C},\partial)$.*

Proof. As before, we let $\mathrm{C}' = \mathrm{C}(\mathbb{Y}_1 \# \mathbb{Y}_2)$ and let ∂' be its differential. Let

$$\mathbb{X} : \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \to \mathbb{Y}_1 \# \mathbb{Y}_2$$

be the cobordism bundle obtained from a boundary sum between $[0,1] \times \mathbb{Y}_1$ and $[0,1] \times \mathbb{Y}_2$ near 1, making some inessential choices in gluing the bundles. Let $\mathbb{W}$ be the cobordism in the reverse direction obtained from the boundary sum near 0. We define chain maps

$$m_{\mathbb{X}} : \mathrm{C} \to \mathrm{C}', \quad m_{\mathbb{W}} : \mathrm{C}' \to \mathrm{C}.$$

The map $m_{\mathbb{X}}$ is given by three components:

$$\begin{aligned} v_{\mathbb{X}} &: \mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)} \to \mathrm{C}', \\ i_{\mathbb{X}} &: \mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)} \to \mathrm{C}', \\ \delta'_{\mathbb{X}} &: F \otimes \mathrm{C}_{(2)} \to \mathrm{C}'. \end{aligned}$$

The map $i_{\mathbb{X}}$ counts instantons in 0-dimensional moduli spaces on $\mathbb{X}$ with all limits irreducible; the map $v_{\mathbb{X}}$ evaluates holonomy along a path $\gamma_{\mathbb{X}}$ from Y_1 to Y_2 on 3-dimensional moduli spaces with irreducible limits on $\mathbb{X}$; the map $\delta'_{\mathbb{X}}$ counts 0-dimensional moduli spaces over $\mathbb{X}$ where the limit over Y_1 is trivial. The map $m_{\mathbb{X}}$ is a chain map because of the following. First, we have the usual relation for the map involving only irreducibles, $i_{\mathbb{X}}\partial_{(12)} = \partial' i_{\mathbb{X}}$. Second,

$$i_{\mathbb{X}} v_{(12)} + v_{\mathbb{X}} \partial_{(12)} + \delta'_{\mathbb{X}}(\delta \otimes 1) = \partial' v_{\mathbb{X}}.$$

These relations are the same as before, except that all terms involving a trivial connection on Y_2 do not arise. In particular, all diagrams in Figure 7.2 are relevant except the last. Third, we have the relation

$$i_{\mathbb{X}}(\delta' \otimes 1) = \partial' \delta'_{\mathbb{X}}$$

which again is the same as before but with the term involving a trivial connection on Y_2 absent. The map $m_{\mathbb{W}}$ is defined similarly to m_{W}, with the component involving the trivial connection on Y_2 thrown out.

We proceed as before. The first composite is $m_{\mathbb{X}} m_{\mathbb{W}} = i_{\mathbb{X}} v_{\mathbb{W}} + v_{\mathbb{X}} i_{\mathbb{W}}$, and this is chain homotopic to $m(\mathbb{Z},\gamma)$ where $\mathbb{Z} = \mathbb{X} \circ \mathbb{W}$ and $\gamma = \gamma_{\mathbb{W}} \cup \gamma_{\mathbb{X}}$ by stretching along Y_1, Y_2. The surgery theorem 7.4.2 applies, so $m_{\mathbb{X}} m_{\mathbb{W}}$ is chain homotopic to $m(\mathbb{Z}_\gamma)$, which is the identity. The other composite $m_{\mathbb{W}} m_{\mathbb{X}}$ now has only 9 components. We stretch the neck as before, so that $m_{\mathbb{W}} m_{\mathbb{X}}$ is chain homotopic to a map f; the terms of f corresponding to the 9 components all vanish except the diagonal ones, which are the identity, and possibly $v_{\mathbb{W}} v_{\mathbb{X}}$. As before, $m_{\mathbb{W}}$ and $m_{\mathbb{X}}$ are chain homotopy inverses, and the proof follows through. □

Next, we consider the case where both $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are non-trivial. For $i = 1, 2$ let $\mathrm{C}_{(i)} = \mathrm{C}(\mathbb{Y}_i)$ with maps $\partial_{(i)}, v_{(i)}$. Define

$$\mathrm{C} = \left(\mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)}\right) \oplus \left(\mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)}\right)$$

$$\partial = \begin{pmatrix} \partial_{(12)} & 0 \\ v_{(12)} & -\partial_{(12)} \end{pmatrix}$$

with notation as before.

Theorem 7.5.2. *Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be non-trivial admissible bundles. As $\mathbb{Z}/8$-graded F-modules, $I(\mathbb{Y}_1 \# \mathbb{Y}_2) \simeq H_*(\mathrm{C}, \partial)$.*

Proof. This is the simplest case. Let $\mathrm{C}' = \mathrm{C}(\mathbb{Y}_1 \# \mathbb{Y}_2)$ with differential ∂'. Let

$$\mathbb{X} : \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \to \mathbb{Y}_1 \# \mathbb{Y}_2$$

be the cobordism bundle obtained from a boundary sum between $[0,1] \times \mathbb{Y}_1$ and $[0,1] \times \mathbb{Y}_2$ near 1, making some inessential gluing choices. Let $\mathbb{W}$ be the cobordism in the reverse direction obtained from the boundary sum near 0. As before, we can define chain maps $m_{\mathbb{X}}$ and $m_{\mathbb{W}}$. Here $m_{\mathbb{X}}$ is given by two components, $v_{\mathbb{X}} : \mathrm{C}_{(1)} \otimes \mathrm{C}_{(2)} \to \mathrm{C}'$ and $i_{\mathbb{X}} : \mathrm{C}_{(1)}[3] \otimes \mathrm{C}_{(2)} \to \mathrm{C}'$. As usual, $i_{\mathbb{X}}$ counts 0-dimensional moduli spaces on $\mathbb{X}$ with all limits irreducible, and $v_{\mathbb{X}}$ takes holonomy along a path $\gamma_{\mathbb{X}}$ from Y_1 to Y_2 on 3-dimensional moduli spaces with irreducible limits on $\mathbb{X}$. The relations that make $m_{\mathbb{X}}$ a chain map are just $i_{\mathbb{X}} \partial_{(12)} = \partial' i_{\mathbb{X}}$ and $i_{\mathbb{X}} v_{(12)} + v_{\mathbb{X}} \partial_{(12)} = \partial' v_{\mathbb{X}}$, and follow from the previous cases, with the terms involving trivial connections thrown out. This latter relation is represented by Figure 7.2 with the last two diagrams omitted. The rest of the argument is the same as before. □

7.6 Establishing the relationship

Proof of Theorem 7.0.2. Now we apply the above results to compute $I^{\#}(Y)$ with F-coefficients for an integral homology 3-sphere Y, proving the first part of Theorem 7.0.2. Recall that F is a field with $\mathrm{char}(F) \neq 2$. Let $\mathbb{T}^3$ be a non-trivial bundle over T^3 geometrically represented by an S^1-factor of T^3. Let $V = F_0 \oplus F_4$ be the chain complex that computes $I(\mathbb{T}^3)$. Write $\tau : V \to V$ for the v-map on $\mathbb{T}^3$, with which our normalization may be written as the degree 4 involution that multiplies by 4. Write $\mathrm{C} = \mathrm{C}(Y)$ and $\partial, v, \delta, \delta'$ for its relevant maps. Now let

$$\mathbf{C} = (\mathrm{C} \otimes V) \oplus (\mathrm{C}[3] \otimes V) \oplus (F \otimes V)$$

$$\boldsymbol{\partial} = \begin{pmatrix} \partial \otimes 1 & 0 & 0 \\ v \otimes 1 + 1 \otimes \tau & -\partial \otimes 1 & \delta' \otimes 1 \\ \delta \otimes 1 & 0 & 0 \end{pmatrix}$$

Theorem 7.5.1 tells us that

$$H_*(\mathbf{C}, \boldsymbol{\partial}) \simeq I(\mathbb{Y} \# \mathbb{T}^3) = I^{\#}(Y)[4] \oplus I^{\#}(Y)$$

where $\mathbb{Y}$ is the trivial bundle over Y. Consider the filtration on $(\mathbf{C}, \boldsymbol{\partial})$ given by

$$0 \subset \mathrm{C}[3] \otimes V \subset (\mathrm{C}[3] \otimes V) \oplus (F \otimes V) \subset \mathbf{C}.$$

This induces a spectral sequence with E^2-page

$$(\ker(\boldsymbol{\delta}) \otimes V) \oplus (\ker(\boldsymbol{\delta}')/\mathrm{im}(\boldsymbol{\delta}) \otimes V) \oplus (\mathrm{coker}(\boldsymbol{\delta}')[3] \otimes V)$$

with the only non-zero component of the differential coming from

$$\phi := \boldsymbol{v} \otimes 1 + 1 \otimes \tau : \ker(\boldsymbol{\delta}) \otimes V \to \mathrm{coker}(\boldsymbol{\delta}')[3] \otimes V.$$

We are writing all modules as $\mathbb{Z}/8$-graded modules; for example, $\ker(\boldsymbol{\delta})_i = I(Y)_i$ when $i \neq 1$, and similarly $\operatorname{coker}(\boldsymbol{\delta}')_i = I(Y)_i$ when $i \neq 4$. Also note that the component $\ker(\boldsymbol{\delta}')/\operatorname{im}(\boldsymbol{\delta})$ is supported in grading 0 and is either F or 0. Write

$$\phi_i = (\boldsymbol{v} \oplus \boldsymbol{v} + \sigma)_i : \ker(\boldsymbol{\delta})_i \oplus \ker(\boldsymbol{\delta})_{i+4} \to \operatorname{coker}(\boldsymbol{\delta}')_{i+4} \oplus \operatorname{coker}(\boldsymbol{\delta}')_i$$

where $\sigma(x, y) = (4y, 4x)$ is the degree 4 involution induced by τ. Then

$$I^{\#}(Y)_0 \simeq \ker(\phi_0) \oplus \operatorname{coker}(\phi_1) \oplus \ker(\boldsymbol{\delta}')/\operatorname{im}(\boldsymbol{\delta}),$$
$$I^{\#}(Y)_i \simeq \ker(\phi_i) \oplus \operatorname{coker}(\phi_{i+1}), \quad i = 1, 2, 3.$$

Recall that $\widehat{v}$ is a degree 4 automorphism of $\widehat{I}(Y)$. We claim that

$$\ker(\phi_0) \simeq \ker(\widehat{v}^2 - 16)_0 \oplus \operatorname{im}(\boldsymbol{\delta}')_4,$$
$$\ker(\phi_i) \simeq \ker(\widehat{v}^2 - 16)_i, \quad i = 1, 2, 3.$$

To prove Theorem 7.0.2 it suffices to consider the case in which $h(Y) \leq 0$, so that $\boldsymbol{\delta}' = 0$ and $\widehat{I}_i = I_i$ for $i \neq 1, 5$. For if $h(Y) > 0$, then the theorem applies for $\overline{Y}$, which has $h(\overline{Y}) = -h(Y) < 0$, and the $F[\widehat{v}]$-module $\widehat{I}$ dualizes upon orientation reversal. Thus for $i = 0, 2, 3$ we have

$$\phi_i : I_i \oplus I_{i+4} \to I_{i+4} \oplus I_i$$

and each $\widehat{I}_i = I_i$ with $\boldsymbol{v} = \widehat{v}$ an isomorphism. The isomorphisms $\ker(\phi_i) \simeq \ker(\widehat{v}^2 - 16)_i$ for $i = 0, 2, 3$ are given by $(x, y) \mapsto x$ with inverse $x \mapsto (x, -\widehat{v}x/4)$. Next, consider

$$\phi_1 : \ker(\boldsymbol{\delta})_1 \oplus I_5 \to I_1 \oplus I_5.$$

We have an isomorphism $\ker(\phi_1) \simeq \ker(\boldsymbol{v}^2 - 16)_1 \subset \ker(\boldsymbol{\delta})_1$ given by $(x, y) \mapsto x$ with inverse $x \mapsto (x, -\boldsymbol{v}^{-1}x/4)$, using the isomorphism $\boldsymbol{v} : I_5 \to I_1$. The natural inclusion $\widehat{I}_1 \to I_1$ induces an injection $\ker(\widehat{v}^2 - 16)_1 \to \ker(\boldsymbol{v}^2 - 16)_1$. Note here that $\widehat{v}$ is just the restriction of $\boldsymbol{v}$ to $\widehat{I}$. We show that this is surjective and hence an isomorphism. Given $x \in I_1$ with $\boldsymbol{\delta}x = 0$ and $\boldsymbol{v}^2 x = 16x$ we must show $x \in \ker(\boldsymbol{\delta}\boldsymbol{v}^{2k})$ for all $k > 0$.

But $\boldsymbol{v}^2 x = 16x$ implies $\boldsymbol{\delta v}^{2k} x = 4^k \boldsymbol{\delta} x = 0$. Having computed $\ker(\phi_i)$, dimension counting then yields

$$\operatorname{coker}(\phi_1) \simeq \ker(\widehat{v}^2 - 16)_1 \oplus \operatorname{im}(\boldsymbol{\delta}),$$
$$\operatorname{coker}(\phi_i) \simeq \ker(\widehat{v}^2 - 16)_i, \quad i = 0, 2, 3.$$

Using in our case that $\dim(\operatorname{im}(\boldsymbol{\delta})) + \dim(\ker(\boldsymbol{\delta}')/\operatorname{im}(\boldsymbol{\delta})) = 1$, we deduce that

$$I^{\#}(Y) \simeq \ker(\widehat{v}^2 - 16) \otimes H_*(S^3) \oplus H_*(\text{pt.})$$

where it is understood that $\widehat{v}^2 - 16$ is acting on $\bigoplus_{i=0}^{3} \widehat{I}_i$. This proves the first part of Theorem 7.0.2.

Now we consider the latter part of the theorem. Let $\mathbb{Y}$ be a non-trivial admissible bundle over Y geometricially represented by $\lambda \subset Y$. We now write $I^{\#}(Y;\lambda)$ in terms of $I(\mathbb{Y})$. Let $V = F_0 \oplus F_4$ and $\tau : V \to V$ be as before. Write $\mathrm{C} = \mathrm{C}(\mathbb{Y})$ and ∂, v for its maps, and set

$$\mathbf{C} = (\mathrm{C} \otimes V) \oplus (\mathrm{C}[3] \otimes V)$$
$$\boldsymbol{\partial} = \begin{pmatrix} \partial \otimes 1 & 0 \\ v \otimes 1 + 1 \otimes \tau & -\partial \otimes 1 \end{pmatrix}$$

Theorem 7.5.2 tells us that

$$H_*(\mathbf{C}, \boldsymbol{\partial}) \simeq I(\mathbb{Y} \# \mathbb{T}^3) = I^{\#}(Y;\lambda)[4] \oplus I^{\#}(Y;\lambda).$$

This is a degeneration of the previous case. We want the kernel and cokernel of

$$\widehat{v} \oplus \widehat{v} + \sigma : I[4] \oplus I \to I \oplus I[4].$$

The kernel is isomorphic to $\ker(\widehat{v}^2 - 16)$ by the assignment $(x, y) \mapsto x$, inverse to $x \mapsto (x, -\widehat{v}x/4)$. The cokernel is of course the same, and we obtain

$$I^{\#}(Y;\lambda) \simeq \ker(\widehat{v}^2 - 16) \otimes H_*(S^3) \tag{7.3}$$

as relatively $\mathbb{Z}/4$-graded F-modules, where it is understood that $\widehat{v}^2 - 16$ is acting on four consecutively graded summands of $I(\mathbb{Y})$. This completes the proof of Theorem 7.0.2. □

7.7 Some rank inequalities

We can use Theorem 7.0.2 to deduce relationships between the ranks of $I(Y)$ and $I^{\#}(Y)$. If $f : V \to V$ is a linear map on a vector space V with $f^n = 0$, basic linear algebra tells us that

$$\dim \ker f \geq \frac{1}{n} \dim V. \tag{7.4}$$

Further, if n is the smallest positive integer such that $f^n = 0$, then

$$\dim \ker f \leq \dim V - n + 1. \tag{7.5}$$

Let Y be an integral homology 3-sphere. Define $n(Y)$ to be the smallest positive integer n such that $(u^2 - 64)^n = 0$, where $u^2 - 64$ is acting on $\widehat{I}(Y)$. Write $n = n(Y)$ and $h = h(Y)$ where h is Frøyshov's invariant from §7.3, and also set

$$\begin{aligned} i &= \dim \bigoplus_{k=0}^{3} I(Y)_k = \frac{1}{2} \dim I(Y), \\ \widehat{i} &= \dim \bigoplus_{k=0}^{3} \widehat{I}(Y)_k = \frac{1}{2} \dim \widehat{I}(Y), \\ i^{\#} &= \dim \bigoplus_{k=0}^{3} I^{\#}(Y)_k = \dim I^{\#}(Y). \end{aligned}$$

Note that we have the relation

$$|h| = i - \widehat{i}.$$

From Theorem 7.0.2, combined with (7.4) and (7.5), we easily obtain

$$1 + \frac{2\,(i - |h|)}{n} \leq i^{\#} \leq 2\,(i - |h| - n) + 3.$$

If (Y, λ) is an admissible pair with $b_1(Y) > 0$ – that is, λ geometrically represents an non-trivial admissible bundle $\mathbb{Y}$ over Y – then we define $n(Y, \lambda)$ similary, but with $u^2 - 64$ acting on $I(\mathbb{Y})$. We also define, as above, $i = \frac{1}{2} \dim I(\mathbb{Y})$ and $i^{\#} = \dim I^{\#}(Y; \lambda)$. Then, in this case, we obtain the simpler inequality

$$\frac{2}{n}(i) \leq i^{\#} \leq 2(i-n+1).$$

Further, if $\mathbb{Y}$ restricts non-trivially to a surface of genus g, then $n(Y, \lambda) \leq n_g$, where n_g is given by

$$n_g := n(S^1 \times \Sigma_g^2, S^1 \times \text{pt}),$$

cf. [16, §6]. Using the exact triangle, one can then deduce that if a homology 3-sphere Y is ± 1-surgery on a genus g knot, then $n(Y) \leq n_g$, cf. §8.2. In forthcoming work with Bill Chen, we will show that $n_g = 2\lceil \frac{g}{2} \rceil - 1$.

We remark that more refined inequalities may be obtained by keeping track of gradings. These, along with the inequalities (6.2) from our spectral sequence of Chapter 6, may facilitate more computations of the groups $I(Y)$ and $I^{\#}(Y)$.

In light of this discussion, we list the following corollary of Theorem 7.0.2:

Corollary 7.7.1. *The rank of $I(Y)$ is zero if and only if the rank of $I^{\#}(Y)$ is 1 and $h(Y) = 0$. As a consequence, $I(Y; \mathbb{Q})$ detects S^3 if and only if the pair $(I^{\#}(Y; \mathbb{Q}),\ h(Y))$ detects S^3.*

CHAPTER 8

Some computations for $I^{\#}(Y)$

In this chapter we use our previous work to make some computations in framed instanton homology. In §8.1, we compute $I^{\#}(Y)$ for double branched covers of quasi-alternating links: this computation comes from a collapsing of the spectral sequence in Chapter 6. In §8.2 we apply the results of Chapter 7 to compute $I^{\#}(Y)$ for some Brieskorn spheres. Finally, in §8.3, we compute the Euler characteristic of framed instanton homology.

8.1 Double branched covers of quasi-alternating links

If rational coefficients are assumed, of the 250 prime knots that have at most 10 crossings, only 7 of them have potentially non-trivial differentials after the E^2-page of Theorem 6.0.1. This follows from the computations of odd Khovanov homology in [33]. In fact, when the link L is quasi-alternating, the spectral sequence collapses, resulting in

Theorem 8.1.1. *If L is a quasi-alternating link, then $I^{\#}(\Sigma(L))$ is free abelian of rank $det(L)$ and is supported in even gradings. The rank in grading $j \in \{0, 2\}$ is*

$$\frac{1}{2}\left[det(L) + (-1)^{j/2} 2^{\#L-1}\right]$$

where $\#L$ is the number of components of L.

Proof. By [29, Thm. 1], [33, §5] and the remarks in [28, §9.3], we know that for a quasi-alternating link L, the gradings q and t of $\overline{\mathrm{Kh}'}(L)$ satisfy $q/2 - t - \sigma/2 = 0$. Let us write $\delta = q/2 - t - \sigma/2$. Then we may say that $\overline{\mathrm{Kh}'}(L)$ is supported in δ-grading 0. Note that $\nu = 0$ when L is quasi-alternating. Further, as is described in [29], the rank of $\overline{\mathrm{Kh}'}(L)_{q,t}$ is given by $|a_q|$, where

$$J_L(x) = \sum a_q x^q$$

is the Jones polynomial, with conventions as given in [33]. The grading $\delta^{\#}$ of Theorem 6.0.1 is given by $\delta^{\#} = \delta + q + \sigma$ for quasi-alternating links. Note that δ and $\delta^{\#}$ agree modulo 2, implying that the spectral sequence collapses at the E^2-page. Write $\overline{\mathrm{Kh}'}(L)_j$ with $j \in \{0, 2\} \subset \mathbb{Z}/4\mathbb{Z}$ for the $\delta^{\#}$-grading. Then

$$\mathrm{rk}_{\mathbb{Z}} \overline{\mathrm{Kh}'}(L)_j = \sum_{q+\sigma \equiv j} |a_q|$$

where the congruence is modulo 4. We remark that $\sum |a_q| = \det(L)$, and the sign of a_q is $(-1)^{q/2+\sigma/2}$, cf. [3, §9.1]. It follows that

$$\mathrm{rk}_{\mathbb{Z}} \overline{\mathrm{Kh}'}(L)_j = \frac{1}{2}\left[\sum |a_q| + (-1)^{j/2} \sum a_q\right].$$

Now we obtain the result of Corollary 8.1.1 using the fact that $J_L(1) = \sum a_q = 2^{m-1}$, where m is the number of components of L. □

8.2 The Brieskorn homology 3-spheres $\Sigma(2, 3, 6 \pm 1)$

In this section we compute $I^{\#}(\Sigma(2, 3, 6 \pm 1))$ for coefficient fields not of characteristic 2. In doing this, we first consider a more general situation. Theorem 7.0.2 suggests that knowledge of the smallest positive integer n such that $(u^2 - 64)^n = 0$ is useful in understanding the relationships between the various instanton groups, cf. §7.7. It is known, cf. [16, §6], that if $\mathbb{Y}$ is non-trivial admissible and restricts non-trivially to a surface of genus ≤ 2, then one can take $n = 1$ (in the notation of §7.7, this says $n_1 = n_2 = 1$).

Using this observation, we first prove a corollary of Theorem 7.0.2. To state it, let $h : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ be Frøyshov's homomorphism from [16], also defined in §7.3, where $\Theta^3_{\mathbb{Z}}$ is the integral homology cobordism group.

Corollary 8.2.1. *Let Y be the result of ± 1-surgery on a knot $K \subset S^3$ with genus ≤ 2. Let F be a field with char$(F) \neq 2$. Then, with all homology taken with F-coefficients, we have an isomorphism*

$$I^{\#}(Y) \simeq H_*(pt.) \oplus H_*(S^3) \otimes \bigoplus_{j=0}^{3} \widehat{I}(Y)_j$$

as $\mathbb{Z}/4$-graded F-modules. In particular, if in addition $h(Y) = 0$, then the groups $\widehat{I}(Y)_j$ on the right can be replaced by $I(Y)_j$.

From this we will deduce the following:

Corollary 8.2.2. *With coefficients in a field F with char$(F) \neq 2$, we have*

$$I^{\#}(\Sigma(2,3,6k+1)) \simeq F_0^{\lfloor k/2 \rfloor + 1} \oplus F_1^{\lfloor k/2 \rfloor} \oplus F_2^{\lceil k/2 \rceil} \oplus F_3^{\lceil k/2 \rceil},$$

$$I^{\#}(\Sigma(2,3,6k-1)) \simeq F_0^{\lceil k/2 \rceil} \oplus F_1^{\lceil k/2 \rceil - 1} \oplus F_2^{\lfloor k/2 \rfloor} \oplus F_3^{\lfloor k/2 \rfloor}$$

where k is a positive integer and the subscripts indicate the gradings.

Proof of Corollary 8.2.1. First, suppose $\mathbb{Y}$ is a non-trivial admissible bundle over Y geometrically represented by λ. As mentioned in [16, §6], when there exists a surface $\Sigma \subset Y$ of genus ≤ 2 with $\mathbb{Y}|_\Sigma$ non-trivial, then $u^2 = 64$ on $I(\mathbb{Y})$. In this case (7.3) yields an isomorphism

$$I^{\#}(Y;\lambda) \otimes H_*(S^4) \simeq I(\mathbb{Y}) \otimes H_*(S^3)$$

as relatively $\mathbb{Z}/4$-graded F-modules. The term $H_*(S^4)$ appears because we take the full $\mathbb{Z}/8$-graded group $I(\mathbb{Y})$ on the right, instead of four consecutive summands as before. Now suppose K is a knot in S^3 of genus ≤ 2. Denote the result of r-surgery on K by Y_r. For $r = 1$, the exact triangle, combined with passing to the reduced groups $\widehat{I}$, yields a map

$$I(\mathbb{Y}_0) \to \widehat{I}(Y_1) \tag{8.1}$$

where $\mathbb{Y}_0$ is a non-trivial bundle over Y_0. This map is an *surjection.* This follows from Frøyshov's observation after Thm. 10 in [16] that, in this situation, when passing to the reduced groups $\widehat{I}$, the surgery triangle retains exactness at the homology 3-spheres (but not $I(\mathbb{Y}_0)$). If $r = -1$, we similarly obtain a *injection* $\widehat{I}(Y_{-1}) \to I(\mathbb{Y}_0)$. In either case, we can form a surface Σ in Y_0 by capping off a Seifert surface for K of genus ≤ 2 by a meridional disk for the new framed knot in Y_0. The bundle $\mathbb{Y}_0|_\Sigma$ is non-trivial, and so $u^2 = 64$ on $I(\mathbb{Y}_0)$. Thus $u^2 = 64$ on $\widehat{I}(Y_{\pm 1})$. With Theorem 7.0.2, this implies Corollary 8.2.1. □

Proof of Corollary 8.2.2. By the remarks in the introduction of [16], we have

$$h(\Sigma(2,3,6k+1)) = 0, \quad h(\Sigma(2,3,6k-1)) > 0.$$

Fintushel and Stern [12] compute

$$I(\Sigma(2,3,6k+1)) = F_1^{\lfloor k/2 \rfloor} \oplus F_3^{\lceil k/2 \rceil} \oplus F_5^{\lfloor k/2 \rfloor} \oplus F_7^{\lceil k/2 \rceil},$$

from which the first part of Corollary 8.2.2 follows, as $\Sigma(2,3,6k+1)$ is $+1$-surgery on a twist knot with k full twists, a knot of genus 1. On the other hand, $\Sigma(2,3,6k-1)$ is -1-surgery on a twist knot K with $2k-1$ half twists. Since K is also genus 1, the inequality of Corollary 1 of [15] yields $h(\Sigma(2,3,6k-1)) = 1$. Combined with Fintushel and Stern's computation from [12], we obtain

$$\widehat{I}(\Sigma(2,3,6k-1)) = F_1^{\lceil k/2 \rceil - 1} \oplus F_3^{\lfloor k/2 \rfloor} \oplus F_5^{\lceil k/2 \rceil - 1} \oplus F_7^{\lfloor k/2 \rfloor}.$$

Now the second part of Corollary 8.2.2 follows from Corollary 8.2.1. □

8.3 The Euler characteristic

In this section we compute the Euler characteristic of $I^\#(Y;\lambda)$:

Proposition 8.3.1. *For any closed, oriented 3-manifold and any unoriented closed 1-manifold $\lambda \subset Y$, we have $\chi(I^\#(Y;\lambda)) = |H_1(Y;\mathbb{Z})|$.*

The expression on the right side means the cardinality of $H_1(Y;\mathbb{Z})$ if it is finite, and is zero otherwise. Setting $\lambda = \emptyset$ yields $\chi(I^\#(Y)) = |H_1(Y;\mathbb{Z})|$.

Proof. We make the abbreviations

$$i(Y;\lambda) = \chi(I^{\#}(Y;\lambda)), \quad |Y| = |H_1(Y;\mathbb{Z})|.$$

Note that §4.8 implies the multiplicativity

$$i(Y;\lambda)i(Y';\lambda') = i(Y\#Y';\lambda\cup\lambda'). \tag{8.2}$$

We also note that $i(Y) = 1$ when $|Y| = 1$ by Theorem 7.0.2.

Next, we claim the result is true for rational homology 3-spheres Y that are obtained by integral surgery on an algebraically split link. That is, Y is the result of $(p_1,\ldots,p_k)$-surgery on a framed link $L = L_1\cup\cdots\cup L_k$ in S^3 whose pairwise linking numbers vanish. Thus $|Y| = |p_1\cdots p_k|$. Assume the result true for $|Y| < n$. Since the case $|Y| = 1$ has already been established, we may assume that Y is not an integral homology 3-sphere, and (by reordering) that $|p_1| > 1$. Let Z_p be $(p,p_2,\ldots,p_k)$-surgery on L. We have an exact sequence

$$\cdots I^{\#}(Z_\infty;\lambda)\to I^{\#}(Z_{p_1-1};\lambda\cup\mu)\to I^{\#}(Z_{p_1};\lambda)\to I^{\#}(Z_\infty;\lambda)\cdots$$

The degree of the first map is odd, while the other two are even, cf. [23, §42.3]. Observing that $Z_{p_1} = Y$, we obtain

$$i(Y;\lambda) = i(Z_{p_1-1};\lambda\cup\mu) + i(Z_\infty;\lambda).$$

By the induction hypothesis, the right side is

$$|(p_1-1)p_2\cdots p_k| + |p_2\cdots p_k| = n,$$

establishing the result for all rational homology 3-spheres which are obtained by integral surgeries on algebraically split links.

We now prove the result for all rational homology 3-spheres Y. We use the fact that for any such 3-manifold, there is a framed algebraically split link $L\subset S^3$ such that some integral surgery on L yields $Z = Y\#Y'$, where Y' is a connected sum of lens spaces of type $L(p,1)$, cf. [32, Cor. 2.5]. Since Y' is integral surgery on an algebraically split link, $i(Y') = |Y'|$. Then (8.2) yields

$$i(Y;\lambda) = i(Z;\lambda)/i(Y') = |Z|/|Y'| = |Y|,$$

establishing the result for all rational homology 3-spheres.

Finally, we consider the case in which $b_1(Y) > 0$. We can always find Z and a framed knot $K \subset Z$ such that Y is 0-surgery on K and $b_1(Z) + 1 = b_1(Y)$. We have an exact sequence

$$\cdots I^{\#}(Y;\lambda) \to I^{\#}(Z_1;\lambda \cup \mu) \to I^{\#}(Z;\lambda) \to I^{\#}(Y;\lambda) \cdots$$

where Z_1 is the result of 1-surgery on K. The degree of the first two maps are even, while the third is odd, again cf. [23, §42.3]. Thus

$$i(Y;\lambda) = i(Z_1;\lambda \cup \mu) - i(Z;\lambda).$$

The proof is again by induction. If $b_1(Y) = 1$, then the right side is known, because Z_1 and Z are rational homology spheres; we have $|Z_1| = |Z|$, so the right side vanishes. Now suppose the result has been proven for $0 < b_1 < n$. If $b_1(Y) = n$, both terms on the right side vanish by the induction hypothesis, and the proof is complete. □

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