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by

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Abstract

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We define a new class of symplectic spaces called "pumpkin domains", which roughly speaking comprise a Liouville domain and a Liouville hypersurface of its boundary. To such an object we assign an A_∞-category called its partially wrapped Fukaya category. An exact Landau-Ginzburg model gives rise to a pumpkin domain, and the partially wrapped Fukaya category of this pumpkin domain is meant to agree with the Fukaya category one is supposed to assign to the Landau-Ginzburg model. As evidence, we prove a formula that relates the partially wrapped Fukaya category of a pumpkin domain to the wrapped Fukaya category of its underlying Liouville domain. This operation is mirror to removing a divisor.

Contents

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Chapter 1

Introduction

1.1 Background: mirror symmetry and Fukaya categories

Mirror symmetry was first discovered as a relationship between pairs of Calabi-Yau manifolds which exchanges two parameters on an associated field theory. It was conjectured that every Calabi-Yau manifold belongs to such a mirror pair, and this conjecture was later extended to Landau-Ginzburg models. For the purposes of mirror symmetry, we can take a Landau-Ginzburg model to be a Kähler space X with a holomorphic function $W\colon X \to \mathbb{C}$ called the superpotential. The best understood case is where one member is a toric variety X^n and the other is a Landau-Ginzburg model $\check{X} = \big((\mathbb{C}^*)^n, \check{W}\big)$, where the superpotential is given by an explicit formula depending on the fan of X. This formula was extended to complete intersections in toric varieties by Givental [15] and Hori–Vafa [16].

Kontsevich's 1994 homological mirror symmetry (HMS) conjecture [17] predicted that mirror symmetry should interchange two triangulated categories associated to the mirror Calabi-Yaus X and $\check{X}$. Specifically, the derived category of coherent sheaves on X from algebraic geometry should be equivalent to the derived Fukaya category of $\check{X}$ from symplectic geometry, and vice versa. Subsequent work extended the HMS conjecture to Landau-Ginzburg models [18]. Moreover, although homological mirror symmetry was first stated in the setting of derived categories, it has since become well understood that in fact, these derived equivalences should lift to chain-level A_∞ quasi-equivalences.

Another less direct approach to mirror symmetry is to assign to X a singular space Λ which is meant to be the skeleton of some Weinstein manifold M. The pair (M, Λ) then has two flavors of Fukaya category, called partially wrapped and infinitesimally wrapped, which are meant to be equivalent to the dg-categories $Coh(X)$ and $Perf(X)$, respectively. The equivalence proceeds by combining the coherent-constructible correspondence [9] and the Nadler–Zaslow correspondence [22]. This approach is explored by Fang–Liu–Treumann–Zaslow in [10].

The starting point in all definitions of Fukaya categories is the construction, due to Floer

[11], of a cochain complex $CF^*(L_0, L_1)$ associated to a pair of Lagrangian submanifolds (L_0, L_1) in a sufficiently well-behaved symplectic manifold M. His construction considers Morse theory on the space of paths from L_0 to L_1 and interprets the gradient flow equation on this space as a holomorphic curve equation for maps $u\colon \mathbb{R}\times[0,1]\to M$. Fukaya described how to enhance the Floer cochain complex of (L, L) to an A_∞-algebra, whose structure is given by a sequence of multilinear maps

$$\mu^d\colon\ (CF^*(L,L))^{\otimes d}\to CF^*(L,L)$$

for $d\geq 1$ satisfying certain relations. A systematic study of the existence of these Fukaya A_∞-algebras was undertaken by Fukaya–Oh–Ohta–Ono in [13].

The construction of Fukaya categories is analogous to this, where instead of a single Lagrangian we have multiple Lagrangians, and the operations μ^d take the form

$$\mu^d\colon CF^*(L_{d-1},L_d)\otimes\cdots\otimes CF^*(L_0,L_1)\to CF^*(L_0,L_d)$$

and satisfy the same algebraic relations. In this case, the Fukaya category of M, denoted $\mathcal{F}(M)$, is the A_∞-category with objects the properly embedded Lagrangians in M and with $\hom(L_0,L_1) := CF^*(L_0,L_1)$. The operations μ^d provide the A_∞ structure, with μ^1 defined as the differential on the Floer cochain complex and μ^2 acting as composition in the category.

In noncompact manifolds, Lagrangian Floer theory depends on a choice of Hamiltonian perturbations, and as a result there are many types of Fukaya categories of noncompact symplectic manifolds. For our purposes the most immediately relevant definitions are due to Seidel [23] and Abouzaid–Seidel [6]. The first is known as the Fukaya-Seidel category of a Lefschetz fibration, and it is the Fukaya category associated to a Landau-Ginzburg model with finitely many nondegenerate critical points. The second is the wrapped Fukaya category, which is the Fukaya category associated to open Calabi-Yau manifolds. These may be viewed as Landau-Ginzburg models with superpotential $W = 0$. There is an A_∞ functor from the Fukaya-Seidel category to the wrapped Fukaya category called the acceleration functor, and this corresponds to forgetting the superpotential, or more precisely to deleting all of its terms. Abouzaid and Seidel give a universal characterization of this functor as a certain localization in [5]. The goal of this thesis is to generalize this result. To that effect, we define a symplectic object called a *pumpkin domain* and construct its *partially wrapped Fukaya category*. This is meant to capture the Floer theory of an exact Landau-Ginzburg model which is not necessarily a Lefschetz fibration. We then prove a “stop removal” formula, which characterizes the acceleration functor as a quotient and identifies its kernel.

1.2 Pumpkin domains

A Liouville domain is a compact exact symplectic manifold with boundary $(M, \omega = d\lambda)$, such that the Liouville vector field Z defined by $\imath_Z\omega = \lambda$ points out along the boundary. Attaching the positive part of the symplectization of ∂M canonically extends M to a noncompact exact symplectic manifold $\hat{M}$ in such a way that Z becomes a complete vector field. Standard examples of Liouville domains are cotangent bundles with their canonical symplectic forms and smooth affine complex varieties with the restriction of the standard symplectic form $\sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathbb{C}^n$.

For $\rho > 0$, let $\mathbb{H}_\rho$ be the half-plane $\{\Re(z) \geq -\rho\}$ with the Liouville form

$$\lambda_{\mathbb{H}_\rho} = \frac{1}{2}xdy - \frac{1}{2}ydx.$$

A *stop* is a proper exact symplectic embedding $\sigma\colon \hat{F} \times H_\rho \to \hat{M}$, where $\hat{F}$ is the completion of some Liouville domain of dimension 2 less than M. F is called the *fiber* of σ, and $D_\sigma := \sigma|_{\hat{F}\times\{0\}}$ is called its *divisor*. It is useful to think of D_σ as a symplectic submanifold of ∂M, which we can pretend is the intersection of $\sigma(\hat{F} \times \mathbb{R}_+)$ with ∂M. A *pumpkin domain* is a Liouville domain M along with a finite collection $\boldsymbol{\sigma}$ of disjoint stops.

We prove in Proposition 2.2.4 that any Liouville hypersurface of ∂M can be perturbed to be the divisor of a stop. In particular, if $W\colon \hat{M} \to \mathbb{C}$ is a superpotential, then the generic fiber of W can be extended to a stop σ_W, and the triple $(M, \lambda_M, \{\sigma_W\})$ is the pumpkin domain associated to the Landau-Ginzburg model (M, W). Similarly, if M is equipped with a singular Lagrangian skeleton $\Lambda \subset M$ with $\partial\Lambda$ smooth, then we may thicken $\partial\Lambda \subset \partial M$ in the contact directions to obtain a Liouville hypersurface, which in turn gives rise to a stop σ_Λ. The triple $(M, \lambda_M, \{\sigma_\Lambda\})$ is the pumpkin domain associated to the Lagrangian skeleton (M, Λ). In the situations where W and Λ are mirror to smooth projective toric varieties, $\partial\Lambda$ is a skeleton for the generic fiber of W, so they give rise to the same pumpkin domain.

For a Liouville domain M and a field $\mathbb{K}$ of characteristic 2 we can form the wrapped Fukaya category $\mathcal{W}(M)$ with coefficients in $\mathbb{K}$: this is the A_∞ category whose objects are the exact Lagrangian submanifolds of $\hat{M}$ which are Z-invariant near infinity, and whose morphism spaces $\hom(L_0, L_1)$ are defined to be the free $\mathbb{K}$-vector space generated by the time 1 chords of a Hamiltonian vector field X_H. Here, H is a positive Hamiltonian which is quadratic in the symplectization coordinate and chosen to make L_0 and L_1 transverse.

The A_∞ operations are defined using perturbed holomorphic disks as follows. For μ^1, we consider maps $u\colon \mathbb{R} \times [0,1] \to \hat{M}$ such that $u|_{\mathbb{R}\times\{i\}}$ maps to L_i and which satisfy Floer's equation

$$\partial_s u + J(\partial_t u - X_H) = 0,$$

where s and t are the coordinates on $\mathbb{R}$ and $[0,1]$, respectively, and J is an almost complex structure which is chosen to be compatible with ω and well-behaved at infinity. When such a map is isolated (up to $\mathbb{R}$-translation) and asymptotic to X_H-chords $\gamma_\pm$ as $s \to \pm\infty$, it contributes one term of γ_- to $\mu^1(\gamma_+)$. The higher operations μ^d are defined by counting similarly perturbed holomorphic $(d+1)$-gons. Together, they satisfy the A_∞ associativity equations

$$\sum_{k=1}^{d} \sum_{i=1}^{k} \mu^k(\gamma_d, \ldots, \gamma_{i+d-k+1}, \mu^{d-k+1}(\gamma_{i+d-k}, \ldots, \gamma_i), \gamma_{i-1}, \ldots, \gamma_1) = 0.$$

When M is equipped with a pumpkin structure $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_k\}$, one can carefully choose the quadratic Hamiltonians so that all intersections of X_H chords with the spikes $\sigma(\hat{F} \times \mathbb{R}_+)$ occur in the positive (counterclockwise) direction, and similarly one can choose the complex structures so that all holomorphic polygons intersect the divisors positively. Once this is done, each stop $\sigma_i \in \boldsymbol{\sigma}$ induces a filtration by $\mathbb{N}$ on the morphisms in wrapped category. The *partially wrapped Fukaya category* $\mathcal{W}_{\boldsymbol{\sigma}}(M)$ is the subcategory of $\mathcal{W}(M)$ generated by those chords which intersect no stops.

Conjecture 1.2.1. *If $(M, \lambda_M, \boldsymbol{\sigma})$ is the pumpkin domain associated to a Landau-Ginzburg model (M, W), then $\mathcal{W}_{\boldsymbol{\sigma}}(M)$ is the Fukaya category one is meant to associate to (M, W). In particular, if (M, W) is mirror to a Fano variety X, then*

$$dgCoh(X) \cong Tw^{\pi}\mathcal{W}_{\boldsymbol{\sigma}}(M),$$

where $dgCoh(X)$ is a dg-category enhancing $D^bCoh(X)$, and $Tw^{\pi}\mathcal{W}_{\boldsymbol{\sigma}}(M)$ is a triangulated enlargement of $\mathcal{W}_{\boldsymbol{\sigma}}(M)$.

Similarly, if $(M, \lambda_M, \boldsymbol{\sigma})$ is the pumpkin domain associated to a Lagrangian skeleton $\Lambda \subset M$, then $\mathcal{W}_{\boldsymbol{\sigma}}(M)$ coincides with the partially wrapped Fukaya category of (M, Λ).

1.3 Stop removal

The main result of this paper, Theorem 4.2.7, relies on the technical assumption is that a stop is *nondegenerate*. In analogy to Ganatra's terminology [14] which calls a Liouville domain nondegenerate if it admits a collection of Lagrangians satisfying Abouzaid's generation criterion [1], a stop is nondegenerate if it supports a collection of Lagrangians satisfying a similar condition. For a precise definition of nondegeneracy, see Section 4.2. In a future version of this paper [26], we will prove that any stop whose fiber is a Riemann surface or a cotangent bundle is nondegenerate.

An approximate version of the stop removal formula (Theorem 4.2.7) can be stated as follows:

Theorem 1.3.1. *Let M be a pumpkin domain, and let $\sigma \in \boldsymbol{\sigma}$ be a nondegenerate stop. Let $\mathcal{B} \subset \mathcal{W}_{\boldsymbol{\sigma}}(M)$ be the full subcategory of objects supported near the image of σ. Then there is a fully faithful functor*

$$\mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B} \hookrightarrow \mathcal{W}_{\boldsymbol{\sigma}\setminus\{\sigma\}}(M),$$

where the quotient is a quotient of an A_∞ category by a full subcategory in the sense of Lyubashenko-Ovsienko [20]. In particular, if $\boldsymbol{\sigma}$ contains only one stop σ, then this quotient recovers the ordinary wrapped category.

The key ingredient of the proof of Theorem 4.2.7 is an auxiliary filtration on $\mathcal{W}_{\boldsymbol{\sigma}\setminus\{\sigma\}}(M)$, presented as the trivial quotient $\mathcal{A} = \mathcal{W}_{\boldsymbol{\sigma}\setminus\{\sigma\}}(M)/\mathcal{B}$. The benefit of this quotient presentation is that it naturally contains the category $\mathcal{A}_0 = \mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B}(\sigma)$ as the minimally filtered part, which makes it possible to build a homotopy which retracts $\mathcal{A}$ onto $\mathcal{A}_0$. The homotopy itself requires a filtered version of the annulus trick, which was introduced in [1] and extended in [14] and [4]. Specifically, one factors the identity operation as a composition of a product and a coproduct, where the coproduct is required to have one output land in the partially wrapped complex.

As an application, consider the Landau-Ginzburg model $(M, W) = (\mathbb{C}^3, xyz)$, which is mirror to the pair of pants. This has generic fiber $(\mathbb{C}^*)^2$, whose wrapped Fukaya category is generated by the single Lagrangian $L = (\mathbb{R}_+)^2$. It turns out that in Theorem 4.2.7 it suffices to consider those objects of $\mathcal{B}$ which are parallel transports of Lagrangians in the fiber. Thus, in this case, we can replace $\mathcal{B}$ with a single Lagrangian $\widetilde{L}$ which is the parallel transport of L over the arc that curves around σ. This expresses the trivial category $\mathcal{W}(\mathbb{C}^3)$ as a quotient of $\mathcal{W}_{\boldsymbol{\sigma}}(M)$ by $\widetilde{L}$, which means $\widetilde{L}$ generates $\mathcal{W}_{\boldsymbol{\sigma}}(M)$. This is predicted by mirror symmetry in [3], where Abouzaid and Auroux compute the endomorphism algebra of $\widetilde{L}$.

More abstractly, in view of Conjecture 1.2.1, Theorem 1.3.1 can be thought of as a characterization of the acceleration functor

$$A\colon Fuk(M, W) \to Fuk(M, 0).$$

This characterization is dual to that of [5] and can be thought of as extending Abouzaid and Seidel's result to more general Landau-Ginzburg models. In fact, one could dream of a situation in which the theory of pumpkin domains has been extended to intersecting stops. In this case, a theorem analogous to Theorem 1.3.1 would give a strong refinement of the acceleration functor.

Conjecture 1.3.2. *Suppose $W = \sum_{i=1}^{d} W_i$ is a sum of monomials. Then σ_{W_i} and σ_{W_j} are generically expected to intersect. However, if partially wrapped Fukaya categories are developed for intersecting stops, one expects*

$$\mathcal{W}_{\{\sigma_W\}}(M) \cong \mathcal{W}_{\boldsymbol{\sigma}}(M),$$

where $\boldsymbol{\sigma} = \{\sigma_{W_1}, \ldots, \sigma_{W_d}\}$. In this case, deleting a stop corresponds to deleting a monomial from W. For (M, W) mirror to a toric variety X, this in turn corresponds to deleting a toric divisor from X.

1.4 Outline of the thesis

In Chapter 2, we define stops and pumpkin domains. We prove Proposition 2.2.4, which shows that stops exist, and we use it to give basic examples of pumpkin domains. We then describe how to glue pumpkin domains along stops. This will be used in an upcoming paper [25] where we will explain how to recover the partially wrapped Fukaya category of a gluing from the partially wrapped Fukaya categories of the original pumpkin domains and certain functors associated to the stops.

In Chapter 3, we define partially wrapped Fukaya categories. We then construct various Floer theoretic operations on Fukaya categories, with the objective of proving that partially wrapped Fukaya categories are invariant under isotopies of the stops. The reader who is willing to take for granted that one can construct equivalences between Fukaya categories by counting holomorphic polygons may safely skip Sections 3.4 and 3.5.

Finally, in Chapter 4, we state the precise version of Theorem 1.3.1 and give its proof. We begin by constructing the filtration and the coproduct operation. Then we construct a sequence of smaller homotopies which interpolate between the composition of product with coproduct and a projection to the partially wrapped part. The key observation here is that every time a long X_H-chord intersects the spike of a stop σ, it does so by first entering the stop, then intersecting the spike, and then leaving the stop. Thus, by carefully choosing incidence conditions with the boundary of the image of σ, we construct in Section 4.9 an operation which looks like the identity but is homotopic to zero.

Chapter 2

Geometric setup

2.1 Liouville domains

Our basic object of study will be Liouville domains (M, λ_M), which are compact manifolds with boundary such that $\omega_M := d\lambda_M$ is symplectic, and such that the Liouville vector field Z_M defined by $\imath_{Z_M}\omega_M = \lambda_M$ points outward along the boundary. This implies that $\alpha = \lambda_M|_{\partial M}$ is a contact form, and flowing along $-Z_M$ gives a collar

$$(U, \lambda_M) \cong ((0, 1] \times \partial M, r\alpha).$$

Thus, we can attach the rest of the symplectization of ∂M to obtain the **completion** $\hat{M} := M \cup_{\partial M} [1, \infty) \times \partial M$. $\hat{M}$ comes with a natural 1-form $\hat{\lambda}_M$, symplectic form $\hat{\omega}_M$, and Liouville vector field $\hat{Z}_M$.

A good class of mappings between Liouville domains F and M is that of **Liouville maps**, which are proper embeddings $\phi\colon \hat{F} \hookrightarrow \hat{M}$ such that

$$\begin{aligned} &\phi^*\hat{\lambda}_M = \hat{\lambda}_F + df && \text{for some compactly supported } f\text{, and} \\ &\phi_*\hat{Z}_F = \hat{Z}_M && \text{away from a compact set.} \end{aligned} \tag{2.1.1}$$

Note that the second condition is redundant for codimension zero maps. In general, it can be rephrased as saying that the symplectic orthogonal of the image of ϕ lies in the kernel of $\hat{\lambda}_M$. A Liouville isomorphism, then, is just a Liouville map that is a diffeomorphism. Two Liouville maps into the same target are said to be **orthogonal** if, along their intersection, the symplectic orthogonal to each is tangent to the other.

A version of Moser's lemma holds in this setting [7]:

Lemma 2.1.1. *Let (F, λ_F^t) and (M, λ_M^t) be smooth families of Liouville domains for $t \in [0, 1]$. Suppose there exists a Liouville map $\phi\colon (F, \lambda_F^0) \to (M, \lambda_M^0)$. Then ϕ extends to an isotopy of Liouville maps $\phi^t\colon (F, \lambda_F^t) \to (M, \lambda_M^t)$.* □

Occasionally we will use the stronger notion of an **isomorphism of exact symplectic manifolds**, which is a diffeomorphism $\phi\colon M \to M'$ of exact symplectic manifolds (not necessarily Liouville domains or their completions) satisfying $\phi^*\lambda_{M'} = \lambda_M$.

Given two Liouville domains M and M', one can attempt to form their product. The result is an exact symplectic manifold with corners. One can non-canonically round the corners to obtain a Liouville domain. The result completes to $(\hat{M} \times \hat{M}', \hat{\lambda}_M + \hat{\lambda}_{M'})$, so the product is at least defined up to isomorphism. We'll use $M \times M'$ to denote the resulting Liouville domain for any choice of boundary.

If we additionally have a Liouville map $\phi\colon F \to M$, then the product

$$\phi \times \mathrm{id}_{M'}\colon F \times M' \to M \times M'$$

is not quite a Liouville map, but it becomes one if we replace λ_F by $\phi^*\lambda_M$. By definition, this doesn't change the Liouville isomorphism class of F, and in the sequel we'll often make such compactly supported changes implicitly when talking about products. If $\phi' : F' \to M'$ is another Liouville map, then $\phi \times \mathrm{id}_{M'}$ and $\mathrm{id}_M \times \phi'$ are orthogonal.

A collection of pairwise orthogonal Liouville maps $\sigma_i\colon F_i \to M$ induces further Liouville maps

$$\sigma_{ji}\colon F_{ji} = \sigma_i(F_i) \cap \sigma_j(F_j) \to F_i \tag{2.1.2}$$

for $i \neq j$. If i, j, k as above are distinct, then σ_{ji} and σ_{ki} are orthogonal.

2.2 Stops

Symplectic manifolds often come with additional data, such as a global meromorphic function or a distinguished collection of Lagrangians. For Floer theoretic purposes, this data can often be encoded as a set of framed complex hypersurfaces.

Definition 2.2.1. Let (M^{2n}, λ_M) and (F^{2n-2}, λ_D) be Liouville domains. For $\rho > 0$, denote by $\mathbb{H}_\rho$ the set $\{z \in \mathbb{C} \mid \Re(z) \geq -\rho\}$ with the standard exact symplectic structure coming from $\mathbb{C}$. A **stop** of width ρ in M with fiber F is a proper embedding $\sigma\colon \hat{F} \times \mathbb{H}_\rho \to \hat{M}$ satisfying

$$\sigma^*\hat{\lambda}_M = \hat{\lambda}_F + \lambda_{\mathbb{H}_\rho} + df$$

for some compactly supported f. If σ is a stop, then $D_\sigma := \sigma|_{\hat{F}\times\{0\}}$ is a Liouville map, which we'll call its **divisor**. In what follows, we'll often identify D_σ with its image.

The requirement that a stop be a proper map is important. It means that all of the data lives on the boundary, which will be needed to obtain well behaved gluing operations. The notion of width, on the other hand, is just a notational convenience. Specifically, if $\rho' = t\rho$, then $\mathbb{H}_\rho$ and $\mathbb{H}_{\rho'}$ are isomorphic as exact symplectic manifolds via

$$(x, y) \mapsto (tx, t^{-1}y).$$

We will also sometimes wish to narrow a stop, that is to embed $\mathbb{H}_\rho$ into some enlarged angular sector

$$S_{\rho,s} = \bar{D}^2_\rho \cup \left\{re^{i\theta} \in \mathbb{C} \mid r > 0, |\theta| \leq s\right\}$$

with $\rho, s > 0$. While this can't be done in a way that strictly preserves the Liouville form, it can be done in a way that only modifies the Liouville form in some small annulus around zero and fixes the positive real axis. For this, one can take the large time flow of a Hamiltonian which, outside of the annulus, takes the form $r^2 \sin\theta$. Crossing with the fiber, this might cause the Liouville vector field to fail to point outward along $\partial F \times D^2_\rho$. However, since the modification to $\lambda_{\mathbb{C}}$ is bounded, we replace F with a larger piece of $\hat{F}$ so that $\lambda_{\mathbb{C}}$ is small compared to λ_F, end hence outward pointingness will be preserved at this new boundary. This shows

Lemma 2.2.2. *Let (M, λ_M) be a Liouville domain and $\sigma_0\colon \hat{F} \times S_{\rho,s} \to \hat{M}$ be a proper codimension zero embedding with*

$$\sigma_0^* \hat{\lambda}_M = \hat{\lambda}_F + \lambda_{S_{\rho,s}} + df$$

for some compactly supported f. Then there is a new Liouville form $\hat{\lambda}'_M = \hat{\lambda}_M + dg$, where g is supported in a small tube around $\sigma_0(\hat{F} \times \{0\})$, such that as a map into $(\hat{M}, \hat{\lambda}'_M)$, $\sigma_0|_{(\hat{F}\times\{0\})}$ extends to a stop σ with $\sigma(\hat{F} \times \mathbb{R}_+) = \sigma_0(\hat{F} \times \mathbb{R}_+)$. □

Definition 2.2.3. A map satisfying the properties of σ_0 above will be called a **narrow stop**.

A stop also constrains the behavior of the Liouville form near its divisor. This too will be needed for gluing, though it is not hard to modify a given Liouville map to look like the divisor of a stop. In fact, we have the following:

Proposition 2.2.4. *Let (M^{2n}, λ_M) be a Liouville domain, and let $P \subset \partial M$ be a compact hypersurface with boundary such that $(P, \lambda_M|_P)$ is a Liouville domain. Choose $f\colon P \to [\frac{1}{2}, 1]$ to be a continuous function such that*

1. *f is smooth and less than 1 on the interior of P.*
2. *$f|_{\partial P} = 1$.*
3. *$f^{-1}(t)$ is transversely cut out and contact for $t > \frac{1}{2}$.*
4. *$F = \mathrm{graph}(f) \subset M$ is a smooth submanifold that is parallel to Z to infinite order along its boundary.*

Then $(F, \lambda_M|_F)$ is a Liouville domain, and its inclusion into M extends to a Liouville map ϕ. Moreover, one can construct a new Liouville form $\lambda'_M = \lambda_M + dh$ such that, after moving ∂M out, ϕ becomes the divisor of a stop in (M, λ'_M) with fiber F.

Note that h was not required to vanish in a neighborhood of ∂M. Of course, by Lemma 2.1.1, one can arrange that it does, at the expense of a homotopy of F.

Proof. To see that F is a Liouville domain, it suffices to show that $\omega_M|_F$ is symplectic, since then outward-pointingness is clear from condition 4. In fact, this is automatic near the boundary, since there F is close to the symplectization of the contact manifold ∂P. Thus, we can consider only what happens away from the boundary, which allows us to transport the question to P. For this, let $\widetilde{f}\colon P \to M$ be the graph map $p \mapsto (f, p)$, so that we are interested in whether $\widetilde{f}^*\omega_M$ is symplectic on the interior of P. Then $\widetilde{f}^*\lambda_M = f\lambda_M|_P$, so that $\widetilde{f}^*\omega_M = df \wedge \lambda_M|_P + f\omega_M|_P$. We compute

$$(\widetilde{f}^*\omega_M)^{n-1} = (\omega_M^{n-1})|_P + df \wedge \lambda_M|_P \wedge (\omega_M^{n-2})|_P.$$

The first term is positive because $(P, \lambda_M|_P)$ is a Liouville domain, while the second term is nonnegative by condition 3. This implies that F is a Liouville domain, and it follows from the definitions that ϕ is a Liouville map.

Our next step then is to exhibit F locally as the divisor of a stop. To do this, we will use Moser's argument to modify the Liouville form on M in a neighborhood of F. For that to be effective, we will want to frame F so that, when we try to extend ϕ to a stop, it will know which way points out.

Choose a nonvanishing vector field $X \in \Gamma(T\hat{M}|_{\hat{F}})$ that is symplectically orthogonal to $\hat{F}$ and, in the symplectization coordinates (r, p) on $(0, \infty) \times \partial M$, is of the form $X = (g\frac{\partial}{\partial r}, X_\partial)$, where $g \geq 0$ and X_∂ is tangent to P. The choice is unique up to scaling by a positive function. Next, pick a second vector field $Y \in \Gamma(T\hat{M}|_{\hat{F}})$, also orthogonal to $\hat{F}$, such that the radial component of Y vanishes identically and $\omega_M(X, Y) = 1$. By the symplectic neighborhood theorem on a compact part of $\hat{M}$, we can find a number $\rho > 0$ and a symplectic embedding $\psi\colon F_2 \times D^2_\rho \to \hat{M}$, where F_2 is the part of $\hat{F}$ with $r \leq 2$, such that

(i) $\psi|_{F_2 \times \{0\}} = \phi$

(ii) $\psi_* \frac{\partial}{\partial x} = X$ along F_2

(iii) $\psi_* \frac{\partial}{\partial y} = Y$ along F_2.

where $x = \Re(z)$ and $y = \Im(z)$ are the coordinates on D^2_ρ.

It is time to change λ. Let $\theta = \lambda_{F_2 \times D^2_\rho} - \psi^*\hat{\lambda}_M$. Then θ is closed and $\theta|_{F_2} = 0$, so we can find a primitive h_0 of θ on a neighborhood of F_2 with $h_0|_{F_2} = 0$. Shrinking ρ, we can assume that h_0 is defined on all of $F_2 \times D^2_\rho$. Consider a family of cutoff functions $\kappa_t\colon F_2 \times D^2_\rho \to [0, 1]$ indexed by $t \in (0, \rho)$ and satisfying the following conditions:

(iv) κ_t is independent of the F component and is rotationally invariant and radially nonincreasing in the D^2 component when $r \leq 1$

(v) $\kappa_t = 1$ when $|z| \leq \frac{t}{3}$ and $r \leq 1$

(vi) $\kappa_t = 0$ when $|z| \geq \frac{2t}{3}$ or $r \geq \frac{3}{2}$

(vii) $|d\kappa_t| < \frac{4}{t}$ with respect to some fixed t-independent product metric on $F_2 \times D^2_\rho$ which is Euclidean on the D^2_ρ factor

(viii) $\kappa_{t_0}(p, t_0 z) = \kappa_{t_1}(p, t_1 z)$ for all $(p, z) \in F_2 \times D^2$ and $t_i \in (0, \rho)$.

We can rephrase this last condition as saying that shrinking t corresponds to conjugation by a rescaling of the D^2 component.

We will see that the function h in the statement of the lemma can be taken to be $\psi_*(\kappa_t h_0)$ for sufficiently small t. For now, let us denote that function by h_t. The first thing to notice is that for t sufficiently small, the Liouville vector field Z^t_M associated to $\lambda^t_M = \lambda_M + dh_t$ points out along the boundary of M, so that (M, λ^t_M) is a Liouville domain. To see this, note that λ_M vanishes on the symplectic orthogonal to $[1, 2] \times \partial F$, where $[1, 2] \subset (0, \infty)$ is the symplectization component, so θ does as well. Thus, h_0 vanishes quadratically on $[1, 2] \times \partial F$. This, combined with conditions (vi) and (vii), implies that dh_t has magnitude $O(t)$. Since the condition that Z points outward is open, this gives the desired conclusion.

It remains to find some t for which ϕ extends to a stop in (M, λ^t_M). By Lemma 2.2.2, it is enough to extend ϕ to a narrow stop. By (v), we can find the disk part of a narrow stop, so we need only find an angular sector over which we can finish extending ϕ. The naive solution here is to just pick a small angular sector and flow out via Z, which works, but one needs to ensure that this doesn't get snagged on some interesting piece of M. This is where the framing of ψ becomes important.

For convenience, let's now identify $\hat{F} \times D^2_\rho$ with its image under ψ. Let $\delta > 0$ be such that if $r > 1 - \delta$, then for all sufficiently small t the flow of Z^t_M escapes to infinity. Because of (ii), for t sufficiently small and $r \leq 1 - \delta$, the original Liouville vector field $\hat{Z}_M$ has an x-component $(\hat{Z}_M)_x$ which is positive and bounded away from zero along $\hat{F} \times \{\frac{2t}{3}\}$. By (vi), the same is true of $\hat{Z}^t_M$. By openness, we can find some angle $s_0 \in (0, \frac{\pi}{2})$ such that $\hat{Z}^t_M$ points out of $\hat{F} \times D^2_{\frac{2t}{3}}$ when $r \leq 1 - \delta$ and the angular D^2-coordinate θ belongs to $[-s_0, s_0]$. Since $(\hat{Z}^t_M)_y$ vanishes on F, s_0 can be chosen to be independent of t. Indeed, as $t \to 0$, s_0 could be taken to increase to $\frac{\pi}{2}$. We want to show that there exists some smaller s such that $\hat{Z}^t_M$ takes

$$\hat{F} \times \left\{\tfrac{t}{3} e^{i\theta} \mid -s \leq \theta \leq s\right\}$$

through

$$\left(\hat{F} \times \left\{\tfrac{2t}{3} e^{i\theta} \mid -s_0 \leq \theta \leq s_0\right\}\right) \cup \left(\partial M \times (1 - \delta, \infty)\right).$$

If we can, then we're done, since we know that the flow of $\hat{Z}_M^t$ starting anywhere in the latter set escapes to infinity. To accomplish this, it is enough to show that and there exist positive constants C and D, with D small (strictly less than $\frac{c-1}{2^{c-1}-1}\tan s$), such that

$$\left|\frac{(\hat{Z}_M^t)_y}{(\hat{Z}_M^t)_x}\right| < C \cdot \left|\frac{y}{x}\right| + D \quad \text{and} \quad (\hat{Z}_M^t)_x > 0 \tag{2.2.1}$$

when t is small, $\theta \in [-s_0, s_0]$, and $r \leq 1-\delta$. It turns out we can take $C = 1+\varepsilon$ and $D = \varepsilon$ for arbitrary small $\varepsilon > 0$. Specifically, remember that Z_M^t is the vector field dual to

$$\lambda_M^t = \kappa_t \cdot \left(\hat{\lambda}_F + \frac{xdy - ydx}{2}\right) + (1-\kappa_t)\lambda_M + h_0 d\kappa_t.$$

Note we've switched to the interior part of M, since that is where all our current problems live. Note also that $|(Z_M^t)_y|$ is given by $\left|\lambda_M^t(\frac{\partial}{\partial x})\right|$ and similarly with x and y switched. In the above formula, the $\hat{\lambda}_F$ term doesn't affect (2.2.1), so it can be ignored. Furthermore, to lowest order, the $h_0 d\kappa_t$ term is strictly beneficial from the perspective of (2.2.1). To see this, note that there is a positive function $E \in C^\infty(F)$ such that $dh_0 = -Edy + O(|z|)$ near $F \times \{0\}$, so that $h_0 = -Ey + O(|z|^2)$. Thus, using conditions (vii), we see that

$$h_0 d\kappa_t = -\frac{xy}{|z|}\frac{\partial \kappa_t}{\partial |z|}dx - \frac{y^2}{|z|}\frac{\partial \kappa_t}{\partial |z|}dy + O(t).$$

Now $\frac{\partial \kappa_t}{\partial |z|}$ is nonpositive by assumption, and dx is dual to $-\frac{\partial}{\partial y}$, so the first term leads to a negative y-component when y is positive and a positive y-component when y is negative. This means that its contribution to Z_M^t will never enlarge $|\theta|$. Similarly, the second term is dual to a nonnegative function times $\frac{\partial}{\partial x}$, so it too will never enlarge $|\theta|$. Hence, it suffices to show that (2.2.1) can be satisfied with h_0 replaced by $\widetilde{h}_0 = h_0 + Ey$, i.e. after discarding the leading order term.

For this, we will need to separately consider two pieces. Pick $w \in (\frac{1}{3}, \frac{2}{3})$ to be such that $\kappa_t(p, wt) \neq 1$ and, if $|z| < wt$, $\theta \in [-s_0, s_0]$, and $r \leq 1-\delta$, then

$$\|\widetilde{h}_0 d\kappa_t(p,z)\| < \frac{\varepsilon\kappa_t}{4}x \tag{2.2.2}$$

$$\big(1-\kappa_t(p,wt)\big) \cdot \left\|d\left(\lambda_M\left(\tfrac{\partial}{\partial x}\right)\right)(p,0)\right\| < \frac{\varepsilon\kappa_t}{6}. \tag{2.2.3}$$

When $|z| \geq wt$, there is a positive t-independent lower bound on $(1-\kappa_t)\lambda_M(\frac{\partial}{\partial y})$, whereas after replacing h_0 by $\widetilde{h}_0$ every other term of $\lambda_M^t(\frac{\partial}{\partial x})$ and $\lambda_M^t(\frac{\partial}{\partial y})$ tends uniformly to zero as $t \to 0$. Thus, we can satisfy (2.2.1) as long as we can reach $|z| = wt$. But when $|z| < wt$, we can again shrink t so that (2.2.3) implies

$$(1-\kappa_t)|\lambda_M(\tfrac{\partial}{\partial x})| < \frac{\varepsilon\kappa_t}{5}x. \tag{2.2.4}$$

But now, using the positivity of $\lambda_M(\frac{\partial}{\partial y})$, we have

$$\begin{aligned}\left|\frac{(\hat{Z}_M^t)_y}{(\hat{Z}_M^t)_x}\right| &= \left|\frac{\frac{1}{2}\kappa_t y - (1-\kappa_t)\lambda_M(\frac{\partial}{\partial x}) - \widetilde{h}_0 d\kappa_t(\frac{\partial}{\partial x})}{\frac{1}{2}\kappa_t x + (1-\kappa_t)\lambda_M(\frac{\partial}{\partial y}) + \widetilde{h}_0 d\kappa_t(\frac{\partial}{\partial y})}\right| \\ &\leq \frac{|y| + \frac{2}{5}\varepsilon x + \frac{1}{2}\varepsilon x}{x - \frac{1}{2}\varepsilon x} \\ &< (1+\varepsilon)\cdot\left|\frac{y}{x}\right| + \varepsilon\end{aligned}$$

as desired. □

Example 2.2.5. Let $W\colon M \to \mathbb{C}$ be an exact Lefschetz fibration [23], and let $\gamma\colon [0,\infty) \to \mathbb{C}$ be a properly embedded, asymptotically radial ray that avoids the critical values of W. Then we can modify the Liouville structure on M in a neighborhood of $W^{-1}(\gamma)$ to obtain a stop modeled on $W^{-1}(\gamma(0))$.

More generally, we can do the above for any holomorphic fibration. This is the construction that we will use to define the partially wrapped Fukaya category of a Landau-Ginzburg model.

2.3 Pumpkin domains

Definition 2.3.1. From here on, the basic geometric object we will deal with is a **pumpkin domain**. This is a triple $(M, \lambda_M, \boldsymbol{\sigma})$ with (M, λ_M) a Liouville domain and $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_k\}$ a collection of stops in M such that the images of σ_i and σ_j are disjoint for $i \neq j$. For technical simplicity, we make the additional assumption that every stop σ_i strictly preserves the Liouville form on $\hat{M} \setminus M$. Since a pumpkin domain only has finitely many stops, this can always be achieved by moving ∂M out.

An **equivalence** of pumpkin domains $(M, \lambda_M, \boldsymbol{\sigma})$ and $(M', \lambda_{M'}, \boldsymbol{\sigma'})$ is a homotopy of collections of disjoint stops $\boldsymbol{\sigma}^t = \{\sigma_1^t, \ldots, \sigma_k^t\}$, where σ_i^t is a stop in M with fiber $(F_i, \lambda_{F_i}^t)$, together with a Liouville isomorphism $\psi\colon M \to M'$ such that $\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}$ and $\psi \circ \sigma_i^1 = \sigma_i'$. Here, M and M' are required to have the same number of stops.

We'll usually abuse notation and use M to refer to the pumpkin domain $(M, \lambda_M, \boldsymbol{\sigma})$.

Example 2.3.2. Fix a positive integer n, and consider the map $u_n\colon \mathbb{C} \to \mathbb{C}$ given by

$$u_n(z) = z^{n+1} - 1.$$

Then $u^*\lambda_{\mathbb{C}}$ is almost a Liouville form on $\mathbb{C}$, except that its derivative vanishes at the origin. Choose a cutoff function $\kappa\colon \mathbb{R} \to [0,1]$ with $\kappa(x) = 1$ for $x \leq \frac{1}{4}$ and $\kappa(x) = 0$ for $x \geq \frac{1}{2}$. Next, choose $\varepsilon > 0$ such that $u^*\omega_{\mathbb{C}} + \varepsilon d(\kappa(|z|)\lambda_{\mathbb{C}})$ is symplectic. Let $\boldsymbol{\sigma}^n = \{\sigma_0^n, \ldots, \sigma_n^n\}$ be the set of u_n-lifts of the inclusion $\mathbb{H}_{\frac{1}{2}} \hookrightarrow \mathbb{C}$ ordered counterclockwise with σ_0^n specified by $\sigma_0^n(0) = 1$. Then this data describes a pumpkin domain

$$\mathbb{C}_n := (\mathbb{C}, u^*\lambda_{\mathbb{C}} + \varepsilon\kappa(|z|)\lambda_{\mathbb{C}}, \boldsymbol{\sigma}^n).$$

Its underlying Liouville domain is Liouville isomorphic to $\mathbb{C}$, and it has $n+1$ stops, all with fiber the point. Note that $i\mathbb{R} \subset \mathbb{C}_1$ is invariant under the flow of the Liouville vector field.

Definition 2.3.3. Let M be a Liouville domain. Then the *stabilization* of M is the pumpkin domain

$$\Sigma M = M \times \mathbb{C}_1.$$

As a Liouville domain, ΣM is just isomorphic to the product $M \times \mathbb{C}$. As for the stops, there are two of them, both with fiber M, and their divisors sit over 1 and -1.

Remark 2.3.4. Though we will not deal with them, one is sometimes given manifolds with stops that intersect. In this situation, it is reasonable to ask that the stops are **orthogonal**: it should be the case that if σ_i has fiber F_i and width ρ_i, then there is a Liouville splitting

$$\mathrm{image}(\sigma_1) \cap \mathrm{image}(\sigma_2) = (D_{\sigma_1} \cap D_{\sigma_2}) \times \mathbb{H}_{\rho_1} \times \mathbb{H}_{\rho_2}$$

that induces the splittings given by each of the stops individually. This gives rise to a more natural setting of Liouville domains with stops, not necessarily disjoint, and here the fiber of a stop will again be a Liouville domain with stops. Well definedness is achieved by induction on dimension. This approach has the advantage of being closed under products; in particular it admits arbitrary stabilizations.

2.4 Hamiltonians for stops

Let M be a pumpkin domain. To obtain an invariant Floer theory, we will need to find a class of Hamiltonians on M that is well adapted to the pumpkin structure. To state a compatibility condition, we need a convention for Hamiltonian vector fields, which we set as $dH = -\imath_{X_H}\omega$.

Definition 2.4.1. A **compatible Hamiltonian** on $(M, \lambda_M, \boldsymbol{\sigma})$ is a function $H \in C^\infty(\hat{M})$ such that

1. H is strictly positive.

2. $dH(\hat{Z}_M) = 2H$ outside of a compact set.

3. X_H is tangent to D_σ for each stop $\sigma \in \boldsymbol{\sigma}$.

4. For each stop $\sigma \in \boldsymbol{\sigma}$, $d\theta(X_H)$ is nowhere negative on a neighborhood of $\sigma(\hat{F} \times \mathbb{R}_+)$. Here, θ is the angular coordinate on the right half plane.

In particular, this last condition says that any integral curve for X_H has only positive intersections with $\sigma(\hat{F} \times \mathbb{R}_+)$.

It's worth noting that the space of Hamiltonians compatible with a given pumpkin domain forms a convex cone. Additionally, if one thinks of a Liouville domain M as a pumpkin domain with no stops, then a compatible Hamiltonian on M is just a positive quadratic Hamiltonian, as usual.

Lemma 2.4.2. *Every pumpkin domain admits a compatible Hamiltonian.*

Proof. We need to show that conditions (3) and (4) can be achieved. For this, let $(M, \lambda_M, \boldsymbol{\sigma})$ be a pumpkin domain, and assume without loss of generality that $\boldsymbol{\sigma}$ has only one element σ with fiber F and width ρ. Fix a compatible Hamiltonian g on F. We want to extend this compatibly to all of $\hat{F} \times \mathbb{H}_\rho$, since then we can just patch it into M. For that, choose a nondecreasing smooth function $a\colon \mathbb{R}_{\geq 0} \to [0,1]$ with $a|_{[0,1]} = 0$ and $a|_{[2,\infty]} = 1$, and set $f(z) = |z|^4 a(|z|^4)$ as a function on $\mathbb{H}_\rho$. Define $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $h(x) = \frac{1}{2}a(x)\log x$. Our candidate Hamiltonian is given by

$$\begin{aligned} H_{\mathrm{cand}}(p,z) = e^{2h(f(z))} g\big(\phi_F(-h(f(z)),p)\big) \\ + e^{2h(g(p))} f\big(\phi_{\mathbb{C}}(-h(g(p)),z)\big) \end{aligned} \tag{2.4.1}$$

where $\phi_F(t,\cdot)$ and $\phi_{\mathbb{C}}(t,\cdot)$ are the time t flows of the Liouville vector fields of F and $\mathbb{C}$, respectively. One readily checks that H_{cand} satisfies conditions (1)-(3), so we need only to find some condition under which it satisfies (4). Now, since f is rotationally invariant, condition (4) is equivalent to the requirement that $\frac{\partial}{\partial x} H_{\mathrm{cand}} \geq 0$ for $z = x \in \mathbb{R}_+$. This clearly holds for the second term, and for the first we compute

$$\frac{\partial}{\partial x} e^{2h\circ f} g\big(\phi_F(-h(f(x)),p)\big) = e^{2h\circ f}(h\circ f)'(x) \cdot \big(2g - dg(\hat{Z}_F)\big).$$

Since $h \circ f$ is nonnegative and nondecreasing, it is enough to require $dg(\hat{Z}_F) \leq 2g$ globally. This can be achieved by just making g bigger on the interior of F. $\square$

Remark 2.4.3. We will usually have not one, but a family of compatible Hamiltonians parametrized by some space Σ. In this situation, we require that the compact set in condition 2 in Definition (2.4.1) can be chosen Σ-independently.

There is another class of Hamiltonians we will want to consider, namely those whose vector fields generate Liouville automorphisms. These have a characterization similar to that of compatible Hamiltonians:

Definition 2.4.4. A **linear Hamiltonian** on $(M, \lambda_M, \boldsymbol{\sigma})$ is a function $H^\ell \in C^\infty(\hat{M})$ such that

1. $dH^\ell(\hat{Z}_M) = H^\ell$ outside of a compact set.

2. X_{H^ℓ} is tangent to D_σ for each stop $\sigma \in \boldsymbol{\sigma}$.

A linear Hamiltonian is said to be **transverse** if, outside of a compact piece of $\hat{M}$, it is of the form $b\sqrt{H}$ for $b \in \mathbb{R}$ and H a compatible Hamiltonian. Near infinity, this is equivalent to asking that its flow is either identically zero or transverse to the contact distribution, hence the name.

Lemma 2.4.5. *Let ψ_t be the flow of a time-dependent linear Hamiltonian. Then outside of a compact set, ψ_t can be approximated rel endpoints in C^0 by the flow of a time-dependent transverse Hamiltonian.*

Proof. Fix an auxiliary compatible Hamiltonian H, where H is of the form 2.4.1 with f interpolating from $|z|^2$ to $|z|^4$ instead of from 0 to $|z|^4$. Write H^ℓ for the positive transverse Hamiltonian $\sqrt{H}$ and let $\phi^\tau_{H^\ell}$ be its time τ flow. For sufficiently large $A \in \mathbb{R}$, note that $\phi^{At}_{H^\ell} \circ \psi_t$ is the flow of a transverse Hamiltonian, and fix such an A. Then for $f\colon [0,1] \to [0,1]$ a nondecreasing function of slope at most 2, we have that $\psi^f_t = \phi^{2At}_{H^\ell} \circ \psi_{f(t)}$ is also the flow of a transverse Hamiltonian. Pick f to be as above, constant on many small intervals, and C^0-close to $\mathrm{id}_{[0,1]}$ rel endpoints. Then we can compose ψ^f_t with a large flow in the direction of $-X_{H^\ell}$ on the constant intervals to obtain the desired approximation. □

2.5 Geometric gluing

Let $(M, \lambda_M, \boldsymbol{\sigma})$ and $(M', \lambda_{M'}, \boldsymbol{\sigma}')$ be pumpkin domains. Let F_i and F'_j be the fibers of σ_i and σ'_j, respectively. When F_i and F'_j are isomorphic, we would like to form a new pumpkin domain $M\,{}_{\sigma_i}\#_{\sigma'_j} M'$. To do this, let us fix an isomorphism $\phi\colon F_i \to F'_j$. Replacing the 1-form $\lambda_{F'_j}$ by $(\phi^{-1})^*\lambda_{F_i}$, we can assume ϕ is an isomorphism of exact symplectic manifolds. Due to the noncompact $\mathbb{H}_\rho$ factor in the domain of σ'_j, this causes σ'_j to cease being a stop. To fix that, we need to modify $\lambda_{M'}$. Now σ_i and σ'_j are stops with the *same* fiber F, and so we can make one last modification of λ_M and $\lambda_{M'}$, this one compactly supported and exact, to assume that σ_i and σ'_j themselves strictly preserve the Liouville forms.

That done, we can write down the gluing. Pick a positive number a that is smaller than the widths ρ_i and ρ'_j of σ_i and σ'_j. With this data, we can define the underlying Liouville domain of $M\,{}_{\sigma_i}\#_{\sigma'_j} M'$ as

$$\left(\hat{M} \setminus \sigma_i(\hat{F} \times \{\Re(z) \geq a\})\right) \amalg \left(\hat{M}' \setminus \sigma'_j(\hat{F} \times \{\Re(z) \geq a\})\right) / \sim$$

where $\sim$ is the identification

$$\sigma_i(\hat{F} \times \{-a < \Re(z) < a\}) = \sigma'_j(\hat{F} \times \{-a < \Re(z) < a\})$$

via $(p, z) \mapsto (p, -z)$. The stops are just

$$\boldsymbol{\sigma}_{M\,{}_{\sigma_i}\#_{\sigma'_j} M'} := (\boldsymbol{\sigma} \setminus \{\sigma_i\}) \amalg (\boldsymbol{\sigma}' \setminus \{\sigma'_j\}),$$

which makes sense since the stops are disjoint.

Suppose now that we had a 1-parameter family of pumpkin domains $(M, \lambda_M, \boldsymbol{\sigma}^t)$, that is an equivalence between $(M, \lambda_M, \boldsymbol{\sigma}^0)$ and $(M, \lambda_M, \boldsymbol{\sigma}^1)$. Then the diffeomorphism type of $M_{\sigma_i^t}\#_{\sigma_j'} M'$ is independent of t, and by Moser's lemma we get a family of Liouville isomorphisms $\Psi_t \colon M_{\sigma_i^0}\#_{\sigma_j'} M' \to M_{\sigma_i^t}\#_{\sigma_j'} M'$. Pulling back the stops in $M_{\sigma_i^t}\#_{\sigma_j'} M'$ via Ψ_t, we see that our homotopy of stops in M results only in a homotopy of stops in the gluing. Repeating this on the M' side, we obtain

Lemma 2.5.1. *Gluing descends to an operation on equivalence classes of pumpkin domains, and at this level it depends only on the triple $(\sigma_i, \sigma_j', [\phi])$. Here, $[\phi]$ is the connected component that ϕ belongs to in the space of Liouville isomorphisms from F_i to F_j'.* □

We will often find that the image of a stop is too small to contain interesting global geometric objects. To remedy this, we will often make use of the following construction.

Definition 2.5.2. Let $(M, \lambda_M, \boldsymbol{\sigma})$ be a pumpkin domain, and let $\sigma \in \boldsymbol{\sigma}$ be a stop with fiber F. Then the **trivial gluing** at σ, written $M[\sigma]$, is the pumpkin domain $M_\sigma\#_{\sigma_0} \Sigma F$.

Trivial gluing effectively replaces σ with σ_1 and doesn't change the pumpkin equivalence class of M. Indeed, it can be achieved by homotoping λ_M in the class of Liouville forms and moving σ out. The benefit of trivial gluing is that it gives rise to the Z_M-invariant hypersurface $\hat{F} \times i\mathbb{R}$.

Chapter 3

Partially wrapped Fukaya categories

3.1 Lagrangian Floer cohomology

For convenience of notation, we'll assume everything in sight is graded. Specifically, we require that all of our Liouville domains satisfy $2c_1(M) = 0$, and further that they come with a choice of fiberwise universal cover $\widetilde{LGr}(M)$ of their bundle of unoriented Lagrangian Grassmannians. Given two Liouville domains M_1 and M_2, their product is graded in the unique way that extends $\widetilde{LGr}(M_1) \times_{\mathbb{Z}} \widetilde{LGr}(M_2)$. All codimension zero symplectic embeddings will be assumed to preserve these covers.

Definition 3.1.1. Given a pumpkin domain M, a **Lagrangian** $L \subset M$ is an exact, oriented, properly embedded Lagrangian submanifold of $\hat{M}$ which is parallel to $\hat{Z}_M$ outside of a compact set. It is required to be graded in the standard sense, namely that it is equipped with a lift to $\widetilde{LGr}(M)$ of the natural section $L \to LGr(M)$. For compatibility with the pumpkin structure, we require that L does not intersect any $\sigma_i(\hat{F} \times \mathbb{R}_{\geq 0})$.

An **interior Lagrangian** is a Lagrangian which completely avoids the images of the stops and whose image under the projection $\hat{M} \setminus M \to \partial M$ does as well. It is easy to see that any Lagrangian is isotopic via a linear Hamiltonian to an interior Lagrangian.

Given a compatible Hamiltonian H on M, we want to consider a class $\mathcal{J}(M, H)$ of almost complex structures which are adapted to H. An element $J \in \mathcal{J}(M, H)$ is a smooth almost complex structure on $\hat{M}$ which is compatible with $\hat{\omega}_M$ and satisfies the following three conditions. First, there is some $c > 0$ such that

$$dH \circ J = -cH\hat{\lambda}_M \tag{3.1.1}$$

outside of a compact set. Second, the restriction $J|_{\ker dH \cap \ker \hat{\lambda}_M}$, i.e. the contact portion of J, is asymptotically $\hat{Z}_M$-invariant. Third, for each stop $\sigma \in \boldsymbol{\sigma}$, we require that the projection to $\mathbb{H}_\rho$ is holomorphic along D_σ. In other words, the divisor of each stop is required to be an almost complex submanifold, and the restriction of J to its symplectic orthogonal coincides with multiplication by i in the base.

Lemma 3.1.2. *For any pumpkin domain $(M, \lambda_M, \boldsymbol{\sigma})$ and compatible Hamiltonian H, the space $\mathcal{J}(M, H)$ is contractible and non-empty.*

Proof. We prove only the last part, for which it is enough to construct such an almost complex structure near the divisor of a stop σ. Let F be the fiber of σ, and pick an almost complex structure $J_F \in \mathcal{J}(F, H|_{\hat{F}})$. Since the symplectic orthogonal to D_σ lies in the kernel of both dH and λ_M outside of a compact set, there is no obstruction to extending J_F to $T\hat{M}|_{D_\sigma}$ while satisfying Equation (3.1.1). Now just extend to the rest of $\hat{M}$. □

To endow $\mathcal{J}(M, H)$ with the structure of a complete metric space, one needs to fix the compact set for (3.1.1). This prevents the existence of a sequence of almost complex structures which satisfies (3.1.1) only outside of ever larger compact sets, so that the limit satisfies it nowhere. To obtain transversality results, we will require that H is quadratic and all Lagrangians are $\hat{Z}_M$-invariant outside the compact set. We choose the compact sets implicitly as part of the data of H, for example to equal $H^{-1}((-\infty, r+1])$, where $H^{-1}((-\infty, r])$ is the smallest sublevel set of H outside of which it is strictly quadratic and the Lagrangians are strictly conical.

We will in fact need time-dependent, or more generally domain-dependent almost complex structures. For this, suppose Σ is a smooth manifold, possibly with boundary or corners, and that we've chosen a Σ-parametrized family of compatible Hamiltonians H. Denote by $\mathcal{J}^\Sigma(M, H)$ the set of smooth maps $J\colon \Sigma \to \mathcal{J}(\hat{M}, \hat{\omega}_M)$ satisfying

$$J(z) \in \mathcal{J}(M, H(z))$$

for all $z \in \Sigma$, and such that (3.1.1) holds pointwise outside of a Σ-independent compact subset of $\hat{M}$. Here, $\mathcal{J}(\hat{M}, \hat{\omega}_M)$ is the space of all $\hat{\omega}_M$-compatible almost complex structures. Likewise, for families of domain-dependent almost complex structures, we require that the compact set can be chosen uniformly for the family. In practice, we will choose the compact set implicitly to be a sublevel set for the family of Hamiltonians.

Now suppose L_0 and L_1 are Lagrangians in M, and H is a compatible Hamiltonian. We say that H is **nondegenerate** for the pair (L_0, L_1) if the following two conditions hold. First, $\phi(L_0)$ is transverse to L_1, where ϕ is the time 1 flow of X_H. Second, if $\dim(M) \geq 4$, then no two points of $\phi(L_0) \cap L_1$ are Liouville translates of one another. If H is nondegenerate, set $\mathcal{X}(L_0, L_1; H)$ to be the set of time 1 X_H-chords starting on L_0 and ending on L_1. Since everything was graded, chords $\gamma \in \mathcal{X}(L_0, L_1)$ are equipped with a degree $\deg(\gamma)$ given by topological intersection number with the Maslov cycle.

For $J \in \mathcal{J}^{[0,1]}(M, H)$, we consider maps

$$u\colon Z = \mathbb{R} \times [0, 1] \to \hat{M}$$

mapping $\mathbb{R} \times \{0\}$ to L_0 and $\mathbb{R} \times \{1\}$ to L_1. For fixed γ_+ and γ_- in $\mathcal{X}(L_0, L_1, H)$, let $\widetilde{\mathcal{R}}(\gamma_+; \gamma_-)$ be the collection of such maps satisfying Floer's equation

$$\partial_s u + J(t)(\partial_t u - X_H) = 0 \tag{3.1.2}$$

with s and t the coordinates on $\mathbb{R}$ and $[0,1]$, respectively, and such that

$$\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm. \tag{3.1.3}$$

The transversality arguments in [12] show that

Lemma 3.1.3. *Fix a pumpkin domain (M, λ_M, σ), Lagrangians L_0 and L_1, and a nondegenerate compatible Hamiltonian H. Then there is a comeager subset*

$$\mathcal{J}^{[0,1]}_{reg}(M,H) \subset \mathcal{J}^{[0,1]}(M,H)$$

such that, for any $J \in \mathcal{J}^{[0,1]}_{reg}(M,H)$ and $\gamma_\pm \in \mathcal{X}(L_0,L_1)$, $\widetilde{\mathcal{R}}(\gamma_+;\gamma_-)$ is a smooth manifold of dimension $\deg(\gamma_-) - \deg(\gamma_+)$. *In this case, the translation $\mathbb{R}$-action on $\widetilde{\mathcal{R}}(\gamma_+;\gamma_-)$ is free if and only if $\gamma_+ \neq \gamma_-$.* □

In the above situation, if $\deg(\gamma_-) > \deg(\gamma_+)$, define $\mathcal{R}(\gamma_+,\gamma_-) := \widetilde{\mathcal{R}}(\gamma_+;\gamma_-)/\mathbb{R}$. This is again a smooth manifold, and it has a compactification given by broken Floer trajectories. Specifically, consider the set

$$\mathcal{R}_{(k)}(\gamma_+;\gamma_-) = \coprod_{\substack{\gamma_i\in\mathcal{X}(L_0,L_1),\gamma_0=\gamma_+\\ \deg(\gamma_i)<\deg(\gamma_{i+1})<\deg(\gamma_-)}} \mathcal{R}(\gamma_0;\gamma_1)\times\cdots\times\mathcal{R}(\gamma_k;\gamma_-). \tag{3.1.4}$$

This union is finite, since the sequence γ_i needs to have strictly decreasing action, while there are only finitely many chords with action below that of γ_+ (cf. Section A.1). Now Gromov compactness says that the union

$$\overline{\mathcal{R}}(\gamma_+;\gamma_-) = \coprod_k \mathcal{R}_{(k)}(\gamma_+;\gamma_-)$$

has a natural compact metric space structure in which $\mathcal{R}(\gamma_+;\gamma_-)$ is a dense open subset. In particular, if $\deg(\gamma_-) - \deg(\gamma_+) = 1$, then $\mathcal{R}_{(k)}(\gamma_+;\gamma_-)$ is empty for all $k \geq 1$, which means

$$\mathcal{R}(\gamma_+;\gamma_-) = \overline{\mathcal{R}}(\gamma_+;\gamma_-)$$

is a finite set. To define the Floer complex, we need one more standard result, first proven in [11].

Lemma 3.1.4. *Suppose we are in the situation of Lemma 3.1.3, and that $J \in \mathcal{J}^{[0,1]}_{reg}(M,H)$ and* $\deg(\gamma_-) - \deg(\gamma_+) = 2$. *Then*

$$\overline{\mathcal{R}}(\gamma_+;\gamma_-) = \mathcal{R}_{(0)}(\gamma_+;\gamma_-) \amalg \mathcal{R}_{(1)}(\gamma_+;\gamma_-)$$

is a compact 1-manifold with boundary, and its boundary is precisely $\mathcal{R}_{(1)}(\gamma_+;\gamma_-)$. □

Let $\mathbb{K}$ be a field of characteristic 2. We define a graded vector space $CW^*(L_0, L_1)$ by degree as

$$CW^k(L_0, L_1) = \bigoplus_{\substack{\gamma \in \mathcal{X}(L_0, L_1) \\ \deg(\gamma)=k}} \mathbb{K}\gamma.$$

Fixing $J \in \mathcal{J}^{[0,1]}_{\text{reg}}(M, H)$, we define a differential

$$\delta \colon CW^k(L_0, L_1) \to CW^{k+1}(L_0, L_1)$$

by

$$\delta\gamma_+ = \sum_{\deg(\gamma_-)-\deg(\gamma_+)=1} \#\mathcal{R}(\gamma_+; \gamma_-) \cdot \gamma_-,$$

where $\#\mathcal{R}(\gamma_+; \gamma_-)$ is the mod-2 count of elements of $\mathcal{R}(\gamma_+; \gamma_-)$. Now, δ^2 counts broken trajectories connecting chords of index difference 2, which are precisely elements of some $\mathcal{R}_{(1)}$. By Lemma 3.1.4, this makes up the boundary of some one-dimensional moduli space, so it has an even number of elements. This means $\delta^2 = 0$, so $\big(CW^*(L_0, L_1), \delta\big)$ is a cochain complex, called the **wrapped Floer cochain complex** of L_0 with L_1.

Each stop $\sigma \in \boldsymbol{\sigma}$ induces a filtration by $\mathbb{N}$ on $CW^*(L_0, L_1)$ as follows: Condition (4) in Definition 2.4.1 means that for any $\gamma \in \mathcal{X}(L_0, L_1)$, the intersections of γ with $\sigma(\hat{F} \times \mathbb{R}_+)$ are all positive. Denote the number of such intersections $n_\sigma(\gamma)$.

Lemma 3.1.5. *The Floer differential δ never increases n_σ. In other words, n_σ induces a filtration on wrapped Floer cochain complexes.*

Proof. Suppose $u \in \mathcal{R}(\gamma_+; \gamma_-)$. Since our Lagrangians avoid $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$, the winding number of ∂u about D_σ coincides with the difference $n_\sigma(\gamma_+) - n_\sigma(\gamma_-)$. By definition, this winding number also gives the topological intersection number of u with D_σ. Thus, it is enough to show that u has only positive intersections with D_σ.

Recall Gromov's trick, which interprets H-perturbed holomorphic curves as unperturbed holomorphic sections of $Z \times \hat{M}$, for a special choice of almost complex structure. Since J_t fixes D_σ and X_H is tangent to D_σ, Gromov's trick will present $Z \times D_\sigma$ as an almost complex submanifold of $Z \times \hat{M}$. This means its intersections with the section given by u are all positive, and linear algebra shows that the same holds for the original intersections. $\square$

Combining the above for all the stops $\sigma_i \in \boldsymbol{\sigma}$, we get a filtration on $CW^*(L_0, L_1)$ by $\mathbb{N}^{|\boldsymbol{\sigma}|}$.

Definition 3.1.6. The **partially wrapped Floer cochain complex** of L_0 with L_1, denoted $CW^*_{\boldsymbol{\sigma}}(L_0, L_1)$, is the 0-filtered part of $CW^*(L_0, L_1)$. In other words, it is the subcomplex generated by those H-chords which don't traverse any of the stops.

3.2 A_∞ categories

We'll now construct the Fukaya A_∞-categories that enhance the above Floer complexes. To begin, we establish some notation for associahedra and strip-like ends.

Following [23], for $d \geq 2$, let $\mathcal{R}^{d+1}$ denote the space of disks with $d+1$ boundary punctures, labeled ζ_0 to ζ_d and ordered counterclockwise, modulo conformal equivalence. $\mathcal{R}^{d+1}$ lives naturally as interior of the d'th Stasheff associahedron $\overline{\mathcal{R}}^{d+1}$, where the boundary faces are products of lower dimensional associahedra indexed by irreducible rooted trees with d ordered leaves. To be explicit, by irreducible we mean that the root vertex has valency at least two and the internal vertices have valency at least three.

Associated to the associahedra are their dg-operad of top cells, and an A_∞**-category** is a category over this operad. Explicitly, an A_∞-category $\mathcal{A}$ consists of

1. A collection of objects $\mathrm{Ob}\mathcal{A}$.

2. For each pair of objects $a_0, a_1 \in \mathrm{Ob}\mathcal{A}$, a graded $\mathbb{K}$-vector space $\hom(a_0, a_1)$.

3. For $k \geq 1$ and all sequences of k objects $L_0, \dots, L_k$, a map of degree $2-k$

$$\mu^k\colon \hom(L_{k-1}, L_k) \otimes \cdots \otimes \hom(L_0, L_1) \to \hom(L_0, L_k) \tag{3.2.1}$$

   satisfying the A_∞ **associativity relations**

$$\sum_{k=1}^{d} \sum_{i=1}^{k} \mu^k(\gamma_d, \dots, \gamma_{i+d-k+1}, \mu^{d-k+1}(\gamma_{i+d-k}, \dots, \gamma_i), \gamma_{i-1}, \dots, \gamma_1) = 0. \tag{3.2.2}$$

For an detailed treatment of A_∞-categories, we refer the reader to chapter 1 of [23].

For Σ a boundary-punctured Riemann surface and $\zeta \in \overline{\Sigma}$ a boundary puncture, a **positive strip-like end** is a holomorphic embedding

$$\epsilon\colon Z_+ = \mathbb{R}_{\geq 0} \times [0,1] \to \Sigma \tag{3.2.3}$$

sending $\mathbb{R}_{\geq 0} \times \{0\}$ and $\mathbb{R}_{\geq 0} \times \{1\}$ to $\partial\Sigma$, and satisfying

$$\lim_{s\to\infty} \epsilon(s,t) = \zeta.$$

Similarly, a **negative strip-like end** for ζ is a holomorphic embedding

$$\epsilon\colon Z_- = \mathbb{R}_{\leq 0} \times [0,1] \to \Sigma \tag{3.2.4}$$

sending $\mathbb{R}_{\leq 0} \times \{0\}$ and $\mathbb{R}_{\leq 0} \times \{1\}$ to $\partial\Sigma$, and satisfying

$$\lim_{s\to-\infty} \epsilon(s,t) = \zeta.$$

If Σ^+ has a positive strip-like end ϵ^+ and Σ^- has a negative strip-like end ϵ^-, then we can glue Σ^+ and Σ^- with length $\ell > 0$ by removing $\epsilon^+([\ell, \infty) \times [0,1])$ and $\epsilon^-((-\infty, -\ell]) \times [0,1])$ and identifying, for $s \in (0, \ell)$, $\epsilon^+(s,t)$ with $\epsilon^-(s-\ell, t)$. The resulting glued surface inherits any data on $\Sigma^\pm$ supported away from the images of $\epsilon^\pm$.

A **boundary-punctured Riemann surface with strip-like ends** is a boundary-punctured Riemann surface Σ, along with a choice of a positive or negative strip-like end for each boundary puncture, such that the images of the strip-like ends are pairwise disjoint. For a disk $\Sigma^{d+1} \in \mathcal{R}^{d+1}$, we require this to be a choice of strip-like end ϵ_i for each ζ_i, where ϵ_i is positive for $i > 0$ and negative for $i = 0$. Seidel has shown that we can make a **universal and consistent choice** of strip like ends: we can choose, for all $d \geq 2$, a collection of strip-like ends for each Σ^{d+1} varying smoothly over $\mathcal{R}^{d+1}$, and such that near $\partial\overline{\mathcal{R}}^{d+1}$ they agree with the strip-like ends induced by gluing. See [23] for details.

A universal and consistent choice of strip-like ends gives rise to a **thick-thin decomposition** of each $\Sigma^{d+1} \in \mathcal{R}^{d+1}$, which we modify slightly from Seidel's convention. Namely, for a strip-like end ϵ, define its m**-shift** ϵ^m by

$$\epsilon^m(s,t) = \begin{cases} \epsilon(s+m, t) & \text{if } \epsilon \text{ is a positive strip-like end} \\ \epsilon(s-m, t) & \text{if } \epsilon \text{ is a negative strip-like end.} \end{cases} \tag{3.2.5}$$

Similarly, if $S \in \Sigma^{d+1}$ is a finite-length strip obtained as the overlap from gluing ϵ^+ and ϵ^- with length ℓ, then $S^m \subset S$ is the possibly empty finite-length strip obtained as the overlap from gluing $(\epsilon^+)^m$ and $(\epsilon^-)^m$ with length $\ell - 2m$. Now our thick-thin decomposition can be declared to be the 3-shift of Seidel's. In other words, the thin part of Σ^{d+1} is the union of the images of all 3-shifts of strip-like ends and all 3-shifts of gluing regions, and the thick part is its complement.

Remark 3.2.1. To be properly pedantic, one should first define a gluing region S of length ℓ_S to be **good** if the strip-like ends induced by $\Sigma^\pm$ agree with the strip-like ends on Σ, and to be **very good** if the corresponding gluing region is good and disjoint from all other good gluing regions for all lengths $\ell \geq \ell_S$. Then one defines the thin part to include only the 3-shifts of those gluing regions which are very good. This eliminates further choices from the setup and makes it easy to see that the thick part is nonempty.

In everything that follows, we will assume that that we've fixed a universal and consistent choice of strip-like ends.

Next, we recall Abouzaid's rescaling trick from [1]. Departing slightly from our earlier notation, let ϕ^τ be the diffeomorphism of $\hat{M}$ given by the time $\log \tau$ flow of the Liouville vector field. Note that pullback by ϕ^τ sends Lagrangians to Lagrangians, compatible Hamiltonians to compatible Hamiltonians, and preserves equation (3.1.1). Suppose then that we've fixed Lagrangians L_0 and L_1, along with a nondegenerate Hamiltonian H and regular almost complex structure J. Then we get a natural bijection between solutions to (3.1.2) with boundary conditions (L_0, L_1) and solutions to

$$\partial_s u + (\phi^\tau)^* J(t)(\partial_t u - (\phi^\tau)^* X_H) = 0 \tag{3.2.6}$$

with boundary conditions $\big((\phi^\tau)^* L_0, (\phi^\tau)^* L_1\big)$. The identity

$$(\phi^\tau)^* X_H = X_{\frac{1}{\tau}(\phi^\tau)^* H}$$

lets us rewrite (3.2.6) as

$$\partial_s u + J_\tau(t)(\partial_t u - X_{H_\tau}), \tag{3.2.7}$$

where

$$(J_\tau, H_\tau) = \Big((\phi^\tau)^* J, \frac{1}{\tau}(\phi^\tau)^* H\Big)$$

again satisfies equation (3.1.1). Since $H_\tau = \tau H$ near infinity, we can make H_τ bigger than any other given compatible Hamiltonian by taking τ sufficiently large.

Let us fix, for *each* pair of Lagrangians (L_i, L_j) in M, a nondegenerate Hamiltonian $H^{i,j}$, along with an almost complex structure $J^{i,j} \in \mathcal{J}^{[0,1]}_{\text{reg}}(M, H^{i,j})$. The pair $(H^{i,j}, J^{i,j})$ is known as a Floer datum for (L_0, L_1), and it singles out well defined wrapped and partially wrapped Floer complexes. In the sequel, this choice will be usually be implicit, and we will write, e.g., $\mathcal{X}(L_i, L_j)$ instead of $\mathcal{X}(L_i, L_j, H^{i,j})$.

For $d \geq 2$ and a $d+1$-tuple of Lagrangians $(L_0, \dots, L_d)$, we wish to define a family of maps

$$\mu^d \colon CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_d) \tag{3.2.8}$$

of degree $2-d$ which satisfy an analog of Lemma 3.1.5. Let $\Sigma \in \mathcal{R}^{d+1}$. From our consistent and universal choice, Σ is equipped with a collection of strip-like ends. Let $\partial_i \Sigma$ be the edge of Σ between ζ_i and ζ_{i+1}, or in the case $i = d$ between ζ_d and ζ_0, and label $\partial_i\Sigma$ with the Lagrangian L_i.

The following definition is important to the present situation, but we state it in enough generality that we won't need to rewrite it too many times.

Definition 3.2.2. A **Floer datum** on a boundary-punctured Riemann surface Σ with strip-like ends and Lagrangian labels consists of

1. A positive real number w_i for each puncture ζ_i.
2. A sub-closed 1-form β on Σ satisfying $\beta|_{\partial\Sigma} = 0$ and $(\epsilon_i^1)^*\beta = w_i dt$ for all i.
3. A Σ-parametrized compatible Hamiltonian H on M satisfying a bunch of conditions:
   a) $d^\Sigma H \wedge \beta \leq 0$ outside of a compact set. Here we view H as a function on $\Sigma \times \hat{M}$, and $d^\Sigma H$ is the component of dH in the Σ-direction. Moreover, $d^\Sigma H$ vanishes on outward normal vectors at $\partial\Sigma$, and $d\beta$ is strictly negative and bounded away from zero on the support of $d^\Sigma H$.
   b) For each positive strip-like end ϵ_i, let L_0 and L_1 be the Lagrangians assigned to the boundary components of Σ containing $\epsilon_i(\mathbb{R}_{\geq 0} \times \{0\})$ and $\epsilon_i(\mathbb{R}_{\geq 0} \times \{1\})$, respectively. Then there is a scaling constant $\tau_i > 0$ such that

$$w_i H = H^{0,1}_{\tau_i}$$

on the image of ϵ_i.

c) For each negative strip-like end ϵ_i, let L_0 and L_1 be the Lagrangians assigned to the boundary components of Σ containing $\epsilon_i(\mathbb{R}_{\leq 0} \times \{0\})$ and $\epsilon_i(\mathbb{R}_{\leq 0} \times \{1\})$, respectively. Then there is a scaling constant $\tau_i > 0$ such that

$$w_i H = H^{0,1}_{\tau_i}$$

on the image of ϵ_i.

4. A Σ-parametrized almost complex structure $J \in \mathcal{J}^{\Sigma}(M, H)$ such that

a) For each strip-like end as above, J satisfies

$$J = J^{0,1}_{\tau_i}$$

on the image of ϵ_i^2.

b) Let $c\colon \Sigma \to \mathbb{R}_+$ be the constant in the compatibility condition (3.1.1). Then the support of dc is disjoint from the support of $d^{\Sigma}H$.

5. A smooth function $\tau_E\colon \partial\Sigma \to (0, \infty)$ such that $\tau_E(z) = \tau_i$ for all ends ζ_i and all points $z \in \partial\Sigma \cap \operatorname{image}(\epsilon_i)$.

A Floer datum for a boundary-punctured Riemann surface without Lagrangian labels consists of a Floer datum for every Lagrangian labeling of that Riemann surface.

Lemma 3.2.3. *Let Σ be a boundary-punctured Riemann surface with strip-like ends and Lagrangian labels. Then the space of Floer data on Σ is nonempty and contractible.*

Proof. For existence, choose β, which determines w_i. Then choose H to be Σ-independent outside of the strip-like ends. A choice of H determines τ_i, and from there we can fill in choices of J and τ_E.

For contractibility, we choose data in the order w_i, then β, then τ_i, then H, J, and τ_E. Each space of choices forms a contractible set depending on the previous choices. □

Following Abouzaid, we consider conformal rescalings of Floer data. Namely, we say that the Floer data (β, H, J, τ_E) and $(\beta', H', J', \tau'_E)$ are conformally equivalent if there are constants $C, W > 0$ such that

$$\beta = W\beta', \quad H = \frac{1}{W}(H')_C, \quad J = (J')_C, \quad \tau_E = C\tau'_E. \tag{3.2.9}$$

If Σ^+ has a positive strip-like end ϵ_i^+ and Σ^- has a negative strip-like end ϵ_j^-, and the corresponding Lagrangian labels agree, then Floer data on Σ^+ and Σ^- can be glued by rescaling one and patching together the data. Specifically, one chooses C and W so that $\tau_i^+ = C\tau_j^-$ and $w_i^+ = Ww_j^-$ and uses those constants in (3.2.9) to define a new Floer datum on Σ^+. The precise Floer datum obtained by iterated gluing depends on the order of the gluings, but its conformal equivalence class does not.

We now specialize back to the disks with which we will construct the A_∞ structure. For that we will need coordinate charts near $\partial\mathcal{R}^{d+1}$, which we choose as in [23], but with the exponential gluing profile. In other words, if $S \subset \partial\overline{\mathcal{R}}^{d+1}$ is a boundary stratum corresponding to a rooted tree T with labeled leaves, then a chart for $\overline{\mathcal{R}}^{d+1}$ near $\Sigma \in S$ is

$$\prod_{\text{internal vertices } v} U_v \times \prod_{\text{internal edges } e} [0, a_e). \tag{3.2.10}$$

Here, U_v is a subset of the space $\mathcal{R}^{m+1}$ corresponding to the vertex v, and $[0, a_e)$ is an interval of gluing parameters corresponding to the edge e, where gluing parameter ρ corresponds to the length $\ell = e^{\frac{1}{\rho}}$. The identity map from such a chart to one obtained from the logarithmic gluing profile $\ell = \frac{-1}{\pi}\log\rho$ is smooth, and hence any smooth data on the classical associahedra can be pulled back to smooth data in these charts.

Definition 3.2.4. A **universal and conformally consistent** choice of Floer data for $\mathcal{R}^{d+1}$ consists of, for all $d \geq 2$, a Floer datum $\mathbf{K}(\Sigma) = (\beta, H, J, \tau_E)$ for each Σ varying smoothly over $\mathcal{R}^{d+1}$, and such that near $\partial\overline{\mathcal{R}}^{d+1}$ it satisfies the following consistency condition.

1. For Σ sufficiently close to the boundary of $\mathcal{R}^{d+1}$, $\mathbf{K}(\Sigma)$ coincides on the thin part up to a conformal rescaling with the Floer datum induced by gluing.

2. In a chart of the form (3.2.10), we can consider the restriction of $\mathbf{K}(\Sigma)$ to each piece $\Sigma_i \in \mathcal{R}^{m+1}$ from which Σ is glued. This gives a family of Floer data on Σ_i parametrized by
$$U \times \prod_e (0, a_e) \times E,$$
where $U \subset \mathcal{R}^{m+1}$ is a neighborhood of Σ_i, the intervals consist of the gluing parameters for gluing regions adjacent to Σ_i, and E contains all the remaining terms in (3.2.10). We require that this family extends smoothly to
$$U \times \prod_e [0, a_e) \times E,$$
and that on $U \times \prod_e \{0\} \times E$ it agrees up to a family of conformal rescalings with the family of Floer data that was chosen for $\mathcal{R}^{m+1}$.

Though our situation is slightly different from Abouzaid's, Lemma 4.3 from [1] still holds, namely

Lemma 3.2.5. *Universal and conformally consistent choices of Floer data exist. Moreover, if $\mathbf{K}_0$ is such a choice and K_Σ is another Floer datum on some $\Sigma \in \mathcal{R}^{d+1}$, then K_Σ can be extended to a universal and asymptotically consistent choice that agrees with $\mathbf{K}_0$ on $\mathcal{R}^{m+1}$ for all $m < d$.* $\square$

Let $(L_0, \dots, L_d)$ be a $(d+1)$-tuple of Lagrangians, and let

$$\gamma_i \in \begin{cases} \mathcal{X}(L_{i-1}, L_i) & i \neq 0 \\ \mathcal{X}(L_0, L_d) & i = 0. \end{cases} \tag{3.2.11}$$

Given a Floer datum $\mathrm{K} = (\beta, H, J, \tau_E)$ on some $\Sigma \in \mathcal{R}^{d+1}$, we can consider maps $u\colon \Sigma \to \hat{M}$ satisfying the generalized Floer equation

$$J \circ (du - X_H \otimes \beta) = (du - X_H \otimes \beta) \circ j \tag{3.2.12}$$

and such that $u(\partial_i \Sigma) \subset (\phi^{\tau_E})^* L_i$ and $u(\zeta_i) = (\phi^{\tau_i})^* \gamma_i$ in the sense of (3.1.3). More, given a universal and conformally consistent choice $\mathbf{K}$, we can consider $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$, the space of such maps as Σ varies in $\mathcal{R}^{d+1}$ and K varies with Σ. Note that as with solutions to the Floer's equation on strips, a conformal rescaling of $\mathbf{K}$ induces a *canonical* identification of the corresponding versions of $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$. In the sequel, we will usually make this identification implicitly.

Lemma A.2.2 shows that the maps u as above are constrained to take values in some compact part of $\hat{M}$, so that the Gromov compactness theorem applies. This says that $\mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ has a natural compactification $\overline{\mathcal{R}}^{d+1}(\gamma_d, \dots, \gamma_1; \gamma_0)$ obtained by adding in broken configurations similar to those in equation (3.1.4). We enumerate those broken configurations with exactly two nonconstant components:

$$\begin{matrix} \mathcal{R}^{m+1+1}(\gamma_d, \dots, \gamma_{i+d-m+1}, \widetilde{\gamma}, \gamma_i, \dots, \gamma_1; \gamma_0) \\ \times \mathcal{R}^{d-m+1}(\gamma_{i+d-m}, \dots, \gamma_{i+1}; \widetilde{\gamma}) \end{matrix} \qquad \begin{matrix} 1 \le m \le d-2 \\ 0 \le i \le m \\ \widetilde{\gamma} \in \mathcal{X}(L_i, L_{i+d-m}) \end{matrix} \tag{3.2.13a}$$

$$\begin{matrix} \mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \dots, \gamma_1; \gamma_0) \\ \times \mathcal{R}(\gamma_i; \widetilde{\gamma}) \end{matrix} \qquad \begin{matrix} 1 \le i \le d \\ \widetilde{\gamma} \in \mathcal{X}(L_{i-1}, L_i) \end{matrix} \tag{3.2.13b}$$

$$\mathcal{R}(\widetilde{\gamma}; \gamma_0) \times \mathcal{R}^{d+1}(\gamma_d, \dots, \gamma_1; \widetilde{\gamma}) \qquad \widetilde{\gamma} \in \mathcal{X}(L_0, L_d) \tag{3.2.13c}$$

The first kind occur when a sequence of curves has domains approaching $\partial \mathcal{R}^{d+1}$, and the other two occur when energy escapes through one of the strip-like ends. The configurations with more than two components are in general some combination of the above, but since they don't show up in the construction of Fukaya categories, we won't worry about them. As with Floer trajectories in (3.1.4), there are only finitely many intermediate chords $\widetilde{\gamma}$ for which at least one of the above products is nonempty.

The key analytic ingredient is

Lemma 3.2.6. *There is a subset $\mathcal{K}_{reg}(M)$ of the space of universal and conformally consistent choices of Floer data for $\mathcal{R}^{d+1}$ which is dense and such that any $\mathbf{K} \in \mathcal{K}_{reg}(M)$ has the following properties.*

1. *For all* $d \geq 2$, *all* $L_0, \ldots, L_d$, *and all* γ_i *as in* (3.2.11), *the corresponding moduli space* $\mathcal{R}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ *is a smooth manifold of dimension* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) + d - 2$.

2. *If* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) = 2 - d$, *then* $\mathcal{R}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ *is compact.*

3. *If* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) = 3 - d$, *then* $\overline{\mathcal{R}}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ *is a compact topological 1-manifold with boundary, and its boundary is the union of all binary broken curves* (3.2.13).

Proof. The proof is explained in [23], Section 9. □

Fix an element $\mathbf{K} \in \mathcal{K}_{\text{reg}}(M)$, and hence a moduli space $\mathcal{R}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ for all d and all γ_i. We can now define what will be the A_∞ operations μ^d. Namely, one sets μ^1 to be the Floer differential δ, and

$$\mu^d(\gamma_d, \ldots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathfrak{X}(L_0, L_d) \\ \deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) = 2 - d}} \#\mathcal{R}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0) \cdot \gamma_0$$

if $d \geq 2$. One can check that, when they're zero-dimensional, the products in (3.2.13) encode all possible ways of composing two μ^d's with the given inputs and output. Since these make up the boundary of a compact 1-manifold, the total number of elements is even, so the μ^k satisfy the A_∞ relations (3.2.2).

Definition 3.2.7. The **wrapped Fukaya category** of a pumpkin domain $(M, \lambda_M, \boldsymbol{\sigma})$, denoted $\mathcal{W}(M, \lambda_M)$, is the A_∞-category whose objects are Lagrangians in M, in the sense of Definition 3.1.1, and such that $\hom(L_0, L_1) = CW^*(L_0, L_1)$. The A_∞ structure is given by the μ^d described above.

The **interior wrapped Fukaya category** $\mathcal{W}^{int}(M, \lambda_M)$ is the full subcategory of the wrapped category containing only the interior Lagrangians of M.

Remark 3.2.8. Seidel has observed that while our wrapped Fukaya category of a pumpkin domain embeds a full subcategory of the wrapped Fukaya category of the underlying Liouville domain, the latter category can have strictly more objects. These take the form of Lagrangian submanifolds which intersect the stops in an essential way. This is, however, impossible when M is a Weinstein domain.

As with the Floer differential, the disks defining the higher compositions have only isolated positive intersections with the divisors of the stops. The result is that the A_∞ operations preserve the intersection filtrations induced by the stops.

Lemma 3.2.9. *Let* $\sigma \in \boldsymbol{\sigma}$ *be a stop. Then, for any* $d \geq 1$ *and composable sequence of morphisms* $\gamma_1, \ldots, \gamma_d$, *we have*

$$n_\sigma\left(\mu^d(\gamma_d, \ldots, \gamma_1)\right) \leq \sum_{i=1}^{d} n_\sigma(\gamma_i).$$

□

In particular, this says that the A_∞ operations preserve the partially wrapped complexes, so we can define

Definition 3.2.10. For $(M, \lambda_M, \boldsymbol{\sigma})$ a pumpkin domain, its **partially wrapped Fukaya category** $\mathcal{W}_{\boldsymbol{\sigma}}(M, \lambda_M)$ is the subcategory of $\mathcal{W}(M, \lambda_M)$ with all the same objects and such that

$$\hom_{\mathcal{W}_{\boldsymbol{\sigma}}}(L_0, L_1) = CW^*_{\boldsymbol{\sigma}}(L_0, L_1)$$

The **interior partially wrapped Fukaya category** $\mathcal{W}^{int}_{\boldsymbol{\sigma}}(M, \lambda_M)$ is the full subcategory of the partially wrapped category containing only the interior Lagrangians of M.

As we will see, the two versions of the partially wrapped Fukaya category are in fact quasi-equivalent, but certain functors will be much easier to write down when we have access to both.

3.3 Units and isomorphisms

Here we give a mostly standard review of the construction of isomorphisms inside Fukaya categories. We do this because there are a number of places where it is important that these maps come from holomorphic curves. Specifically, we need to see that there are sufficiently strong maximum principles, as well as analogs of Lemmas 3.1.5 or 3.2.9.

The relevant situation is that we have a family of Lagrangians L_t parametrized by $t \in [0, 1]$. Such an isotopy can be generated by a Hamiltonian H_t which, for all t, is linear up to a term which is locally constant near the ends of L_t. This means there exists a linear Hamiltonian H_t^ℓ whose Hamiltonian vector field agrees with that of H_t near the ends of L_t. We say that the family L_t is **transverse** if H_t^ℓ can be taken to be transverse for all t.

To construct an isomorphism from L_0 to L_1, we need to consider Riemann surfaces with moving Lagrangian labels. The starting point is Definition 3.2.2, which applies as long as the labels don't move inside the strip-like ends, but it isn't quite enough to achieve compactness. For that to work out, we need to control where the label moves and be a bit more careful about the sub-closed 1-form.

Additionally, for lack of a better maximum principle, we will construct holomorphic curves only for transverse families of Lagrangians. However, the proof of Lemma 2.4.5 shows that any family of Lagrangians can be C^0-approximated rel endpoints by a transverse family.

Definition 3.3.1. Let Σ be a boundary-punctured Riemann surface with strip-like ends. For us, a **moving Lagrangian label** on a component E of $\partial\Sigma$ consists of a finite union of closed intervals $T_E \subset E$ which avoids the images of the strip-like ends, along with a smooth transverse E-parametrized family of Lagrangians which is constant outside of T_E.

Definition 3.3.2. Let Σ be a boundary-punctured Riemann surface with strip-like ends and moving Lagrangian labels. A **Floer datum** for Σ consists of a 5-tuple $(\beta, H, J, \tau_E, \beta^\ell)$ with the following properties:

1. (β, H, J, τ_E) satisfies all of the conditions in Definition 3.2.2.
2. For each boundary component $E \subset \partial\Sigma$, $d\beta$ is strictly negative in a neighborhood of T_E.
3. β^ℓ is a 1-form on Σ such that
   a) Outside of a fixed compact subset of $\hat{M}$, $d\beta$ is bounded away from zero on the support of β^ℓ.
   b) β^ℓ vanishes in the strip-like ends of Σ.
   c) Let $c\colon \Sigma \to \mathbb{R}_+$ be the constant in the compatibility condition (3.1.1). Then the support of β^ℓ is disjoint from the support of dc.
   d) There is a compact subset of $\hat{M}$ outside of which, for any $z \in \partial\Sigma$ and $\xi \in T_z\partial\Sigma$, the Hamiltonian vector field associated to $\beta^\ell(\xi)\sqrt{H(z)}$ is tangent to the Lagrangian deformation associated to ξ.

For a family $\mathcal{P}$ of Floer data, we require as usual that all compact subsets of $\hat{M}$ appearing in this definition can be taken $\mathcal{P}$-independently.

If T_E is empty so that Σ has non-moving Lagrangian labels, then a Floer datum on Σ in the sense of Definition 3.2.2 induces one in the new sense by taking $\beta^\ell = 0$.

Before we get to specific instances of moving Lagrangians, it is worth considering the general features of holomorphic curves with moving Lagrangian boundary conditions. For such Floer data, the appropriate version of the holomorphic curve equation is

$$J \circ (du - X_H \otimes \beta - X_{\sqrt{H}} \otimes \beta^\ell) = (du - X_H \otimes \beta - X_{\sqrt{H}} \otimes \beta^\ell) \circ j \qquad (3.3.1)$$

for a map $u\colon \Sigma \to \hat{M}$ with boundary conditions given by the moving Lagrangian labels. Because we've restricted to transverse families of Lagrangians, all the expected properties survive. In particular, Lemma A.2.2 still holds, as does positivity of intersections:

Lemma 3.3.3. *Let $\sigma \in \boldsymbol{\sigma}$ be a stop. If Σ is connected, then any solution to* (3.3.1) *which is not contained in D_σ has only isolated positive intersections with D_σ.* □

Let $D(L_t)$ be a disk with one negative boundary puncture ζ and a moving Lagrangian label corresponding to the family L_t, such that in the coordinates of the strip-like end, $\mathbb{R}_- \times \{0\}$ is labeled with L_0 and $\mathbb{R}_- \times \{1\}$ is labeled with L_1. Denote by $\mathcal{K}^{D(L_t)}(M)$ the space of Floer data on $D(L_t)$. Then we can examine the regularity and compactness of the associated moduli spaces of holomorphic curves:

Lemma 3.3.4. *For $\gamma \in \mathcal{X}(L_0, L_1)$, let $\mathcal{D}(L_t, \gamma)$ denote the space of solutions to* (3.3.1) *with boundary conditions given by the moving Lagrangian label, and which are asymptotic to $(\phi^\tau)^*\gamma$ at ζ. Then there is a comeager subset $\mathcal{K}^{D(L_t)}_{reg}(M) \subset \mathcal{K}^{D(L_t)}(M)$ such that for any $K \in \mathcal{K}^{D(L_t)}_{reg}(M)$, the following hold.*

1. *For all γ, $\mathcal{D}(L_t, \gamma)$ is a smooth manifold of dimension* $\deg(\gamma)$.
2. *If* $\deg(\gamma) = 0$, *then $\mathcal{D}(L_t, J, \gamma)$ is compact.*
3. *If* $\deg(\gamma) = 1$, *then $\mathcal{D}(L_t, J, \gamma)$ has a Gromov compactification $\overline{\mathcal{D}}(L_t, \gamma)$ which is a compact topological 1-manifold with boundary, and there is a canonical identification*

$$\partial\overline{\mathcal{D}}(L_t, \gamma) = \coprod_{\widetilde{\gamma} \in \mathcal{X}(L_0, L_1)} \mathcal{R}(\widetilde{\gamma}; \gamma) \times \mathcal{D}(\widetilde{\gamma}). \tag{3.3.2}$$

In this case, $\widetilde{\gamma}$ necessarily has degree 0.

□

Fix a Floer datum $K \in \mathcal{K}^{D(L_t)}_{\mathrm{reg}}(M)$, and hence a moduli space $\mathcal{D}(L_t, \gamma)$ for all γ. Define an element $e_{L_t} \in \hom_{\mathcal{W}}(L_0, L_1)$ by

$$e_{L_t} = \sum_{\gamma \in \mathcal{X}(L_0, L_1)} \#\mathcal{D}(L_t, \gamma) \cdot \gamma. \tag{3.3.3}$$

By Lemma 3.3.3, $\mathcal{D}(L_t, \gamma)$ is empty if $n(\gamma) > 0$, so that e_{L_t} in fact lies in $\hom_{\mathcal{W}_{\boldsymbol{\sigma}}}(L_0.L_1)$. Further, note that the right hand side of (3.3.2) precisely describes the coefficient of γ in ∂e_{L_t}, from which we conclude that e_{L_t} is closed. For $L_t = L_0$ a constant family, we likewise obtain an element $e_{L_0} \in \hom_{\mathcal{W}_{\boldsymbol{\sigma}}}(L_0, L_0)$.

Lemma 3.3.5. *In the wrapped and partially wrapped Fukaya categories, e_{L_0} is a homology unit for L_0.*

Proof. We show only that $\mu^2(|\gamma|, |e_{L_0}|) = |\gamma|$ for $\gamma \in \hom(L_0, L_1)$ closed, since the proof of the transposed identity is identical. To do this, we will use an interpolating family of holomorphic strips to construct a chain homotopy between $\mu^2(e_{L_0}, \cdot)$ and $\mathrm{id}_{\hom(L_0, L_1)}$.

Let $Z^{0,1}$ be the strip $\mathbb{R} \times [0, 1]$ with $\mathbb{R} \times \{0\}$ labeled by L_0 and $\mathbb{R} \times \{1\}$ labeled by L_1. Let $K = (\beta_q, H_q, J_q, (\tau_E)_q, \beta^\ell_q)$ be a family of Floer data on $Z^{0,1}$ parametrized by $q \in (0, 1)$ and satisfying

1. As q approaches 0, K converges in C^∞ to $(dt, H^{0,1}, J^{0,1}, 1, 0)$.
2. For q sufficiently close to 1, K coincides up to conformal equivalence on the thin part, i.e. 3-shift of the gluing region, with the Floer datum induced by gluing $\zeta \in D(L_0)$ to $\zeta_1 \in \Sigma^{2+1}$ with gluing parameter $1 - q$ (length $\frac{-1}{\pi}\log(1 - q)$).

3. Near $q = 1$, the restriction of K to the pieces $D(L_0)$ and Σ^{2+1} extends smoothly to a $(1-\epsilon, 1]$-parametrized family which agrees up to conformal equivalence at $q = 1$ with the previously chosen data.

As always, among the space of such Floer data, there is a comeager subset for which the moduli space $\mathcal{Z}^{0,1}(\gamma; \gamma')$ of solutions to (3.3.1) with appropriate boundary and asymptotic conditions is a smooth manifold of dimension $\deg(\gamma') - \deg(\gamma) + 1$.

Fixing such a K, we consider the Gromov compactification of the above moduli space when it is 1-dimensional. This gives it the structure of a compact topological 1-manifold with boundary, and the boundary consists of all 0-dimensional configurations of the following spaces:

$$\widetilde{\mathcal{R}}(\gamma; \gamma')$$
$$\mathcal{R}^{2+1}(\gamma, \widetilde{\gamma}; \gamma') \times \mathcal{D}(L_0, \widetilde{\gamma})$$
$$\mathcal{R}(\widetilde{\gamma}; \gamma') \times \mathcal{Z}^{0,1}(\gamma; \widetilde{\gamma})$$
$$\mathcal{Z}^{0,1}(\widetilde{\gamma}; \gamma') \times \mathcal{R}(\gamma; \widetilde{\gamma})$$

The first two correspond to degenerations of the domain as q tends to 0 or 1, respectively, while the last two correspond to energy escaping out one of the strip-like ends. In algebraic terms, the first two correspond to $\mathrm{id}_{\hom(L_0,L_1)}$ and $\mu^2(e_{L_0}, \cdot)$, respectively, while the last two describe a chain homotopy generated by $\mathcal{Z}^{0,1}$. □

Corollary 3.3.6. *The inclusion $\mathcal{W}_{\boldsymbol{\sigma}}(M) \to \mathcal{W}(M)$ is a cohomologically unital functor. In particular, any isomorphism in the partially wrapped Fukaya category is an isomorphism in the fully wrapped Fukaya category.* □

A similar argument, this time interpolating between $\mu^2(e_{L_{1-t}}, e_{L_t})$ and e_{L_0}, shows

Lemma 3.3.7. *e_{L_t} is an isomorphism in the wrapped and partially wrapped Fukaya categories.* □

Using the fact that every Lagrangian is isotopic to an interior Lagrangian, and that any linear Hamiltonian isotopy can be approximated in C^0 by a transverse isotopy, one obtains

Corollary 3.3.8. *The inclusion $\mathcal{W}^{int}_{\boldsymbol{\sigma}}(M, \lambda_M) \hookrightarrow \mathcal{W}_{\boldsymbol{\sigma}}(M, \lambda_M)$ is a quasi-equivalence.* □

3.4 Continuation functors

Here, we sketch a construction of continuation maps and their enhancements to A_∞-functors. These functors provide quasi-equivalences that relate the Fukaya categories obtained by making different universal and consistent choices of Floer data. Our construction aims to be efficient rather than elegant, and to this end we will build our moduli spaces starting with the boundary rather than the interior. The heuristic model which underlies all our choices is the space of ideal polygons in the hyperbolic upper half-plane with a corner at infinity, modulo translation. This space, suitably compactified, realizes Stasheff's multiplihedra. For other, nicer descriptions of the multiplihedra in the context of Floer theory, we refer the reader to [21].

To achieve associativity of strip-like end gluing in the boundary charts, we first need an auxiliary definition.

Definition 3.4.1. An **intrinsic width function** consists, for each $d \geq 2$, of d smooth functions $w_i^d \colon \mathcal{R}^{d+1} \to \mathbb{R}_{\geq 0}$, $i = 1, \dots, d$, with the following properties:

1. $w_1^2 = w_2^2 = 0$.
2. Suppose Σ^{d+1} is obtained by gluing $\zeta_n \in \Sigma^{k+1}$ to $\zeta_0 \in \Sigma^{l+1}$ with length ℓ. Then for all ℓ sufficiently large, we have

$$w_i^d(\Sigma^{d+1}) = \begin{cases} w_i^k(\Sigma^{k+1}) & \text{if } i < n \\ w_{i+1-n}^l(\Sigma^{l+1}) + \ell & \text{if } n \leq i < n+l \\ w_{i+1-l}^k(\Sigma^{k+1}) & \text{if } i \geq n+l. \end{cases}$$

   In other words, for ζ_i not separated from ζ_0 by the long gluing region, w_i^d is unchanged, while for those that are, w_i^d increases by the length of the intervening gluing region.

Intrinsic width functions can be built by induction in d, and consistency near the corners of $\overline{\mathcal{R}}^{d+1}$ amounts to the associativity of addition. Let us fix, once and for all, an intrinsic width function. We are now prepared to construct the multiplihedra as a compactified space of domains.

Construction 3.4.2. Let $\mathcal{S}^{1+1} = \{Z\}$, where Z is equipped with a positive strip-like end ϵ_+ and a negative strip-like end ϵ_- at $+\infty$ and $-\infty$, respectively. For $d \geq 2$ let $\mathcal{S}^{d+1} = \mathcal{R}^{d+1} \times \mathbb{R}_+$, where we temporarily forget all choices of strip-like ends. That is, we allow ourselves to have already chosen strip-like ends for a disk $\Sigma \in \mathcal{R}^{d+1}$ but not for Σ thought of as an element $(\Sigma, w) \in \mathcal{S}^{d+1}$. Instead, we will construct a compactification $\overline{\mathcal{S}}^{d+1}$ to be a model of the multiplihedron, and in doing so we will end up making a universal choice of strip-like ends $\boldsymbol{\epsilon}_{\mathcal{S}}$ for $\mathcal{S}^{d+1}$.

Before we begin, note that the multiplihedra are not in general manifolds with corners, but rather are a slightly more general type of smooth space. In addition to interior and boundary charts, it has charts parametrized by spaces of the form $\mathbb{R}^m \times (V \cap [0, a)^n)$, where $m \leq d - 3$ and $V \subset \mathbb{R}^n$ is a weighted homogeneous variety which is smooth on $(0, a)^n$. We will first describe the honest codimension 1 boundary, and then we will discuss how to fill in the generalized corners.

Suppose then by induction that we have constructed $\overline{\mathcal{S}}^{d+1}$ to have boundary and generalized corners parametrized by associahedra and multiplihedra of lower dimension, and that we have consistently chosen strip-like ends on Σ for every $(\Sigma, w) \in \mathcal{S}^{k+1}$ for $k < d$. Suppose further that these strip-like ends agree *up to shifts* with those chosen for Σ as an element of $\mathcal{R}^{k+1}$. We wish to do the same for $\mathcal{S}^{d+1}$.

Start by constructing a boundary chart which corresponds to taking w to 0, where w is the $\mathbb{R}_+$ parameter. For this, take a small open subset $U \subset \mathcal{R}^{d+1}$. Then for a small and $\rho \in (0, a)$, we attach $\mathcal{S}^{1+1} \times U \times [0, a)$ to $\mathcal{S}^{d+1}$ via $(Z, \Sigma, \rho) \mapsto (\Sigma, \rho)$. The notation is intended to suggest gluing the positive end of Z to the negative end of Σ, and indeed doing this with length $\ell = e^{\frac{1}{\rho}}$ induces a collection of strip-like ends on Σ thought of as $(\Sigma, \rho) \in \mathcal{S}^{d+1}$. Denote by $\mathcal{S}_0^{d+1}$ the space obtained by adding to $\mathcal{S}^{d+1}$ the above boundary faces.

Next, consider the boundary strata that appear when w tends to infinity. Here, a boundary chart is of the form $U \times \prod_{i=1}^{m} U_i \times [0, a)$, where $U \subset \mathcal{R}^{m+1}$ and $U_i \subset \mathcal{S}_0^{d_i+1}$. To describe the attaching map for $\rho \in (0, a)$, we need to produce a pair (Σ, w) from the data $\big(\Sigma^{m+1}, \{(\Sigma^{d_i+1}, w_i)\}\big)$. For the width, set $w = e^{\frac{1}{\rho}}$. For the surface, glue simultaneously and for all i the negative strip-like end of Σ^{d_i+1} to the positive strip-like end ϵ_i in Σ^{m+1} with length $\ell_i = w - w_i - w_i^m(\Sigma^{m+1})$. This choice of gluing length ensures well defined corner charts when we compactify the $\mathcal{R}^{m+1}$ and $\mathcal{S}^{d_i+1}$ components. Once again, this gluing induces a choice of strip-like ends on Σ which varies with w.

The remaining boundary charts are obtained by compactifying all of the $\mathcal{R}^{k+1}$ components. These appear either at $w = 0$ or ∞ on their own, or at $w \in (0, \infty]$ as the non-width part of some $\mathcal{S}^{k+1}$. For an $\mathcal{R}^{k+1}$ coming on its own, compactify it as the associahedron that it is. For the others, following the notation of (3.2.10), a boundary chart for $\mathcal{S}^{k+1}$ is of the form

$$U_{root} \times \prod_{\substack{\text{non-root} \\ \text{internal vertices } v}} U_v \quad \times \prod_{\text{internal edges e}} [0, a_e).$$

Here, U_v is as in (3.2.10), while, U_{root} is a small open subset of $\mathcal{S}^{m+1}$, where m is the valency of the root vertex. When all gluing parameters are nonzero, the identification with a subset of $\mathcal{S}^{d+1}$ is as for $\overline{\mathcal{R}}^{d+1}$, with the extra $\mathbb{R}_+$-factor in $\mathcal{S}^{m+1}$ mapping to the corresponding factor in $\mathcal{S}^{d+1}$ via identity. This gluing induces strip-like ends on Σ for (Σ, w) near the remaining boundary components, and it remains only to attach the generalized corners and choose strip-like ends on the interior.

The compactification at $w \in (0, \infty)$ also gives rise to ordinary corners, for which the inductive hypothesis guarantees that the induced strip-like ends don't depend on the order of gluing. Similarly, we compactify the $\mathcal{R}^{d+1}$ component at $w = 0$ and obtain more ordinary corners, and again the inductive hypothesis guarantees consistency. It remains to consider the corners at $w = +\infty$. When we looked at the codimension 1 portion of this limit, we enforced a correlation among gluing lengths which allowed us to associate a number w to the glued disk. For the corners, we instead consider the space $(0, a)^n$ of all possible combinations of gluing parameters, and observe that the correlations give rise to a submanifold of $(0, a)^n$ for which w is well-defined. This submanifold naturally closes to a singular submanifold of $[0, a)^n$ which is topologically a manifold with boundary. Along with terms associated to the interiors of $\mathcal{R}^{m+1}$ and $\mathcal{S}^{k+1}$, this provides the desired chart for a generalized corner.

To complete the inductive construction, simply choose a collection of $\mathcal{S}^{d+1}$-parametrized shifts which, when applied to the strip-like ends on $\mathcal{R}^{d+1}$, interpolate between those we constructed near $\partial\mathcal{S}^{d+1}$.

Remark 3.4.3. While the boundary charts for $\mathcal{S}^{d+1}$ depended on a family of choices, the smooth structure does not. To see this, note first that any two choices differ by a collection of smooth families of shifts. In the boundary charts, these can be corrected by modifying the gluing parameters and shifting the implicit width function.

Suppose we had, for each pair of Lagrangians (L_i, L_j), two choices of Floer data $(H_0^{i,j}, J_0^{i,j})$ and $(H_1^{i,j}, J_1^{i,j})$. Let $\mathbf{K}_0$ and $\mathbf{K}_1$ be universal and conformally consistent choices of Floer data for $\mathcal{R}^{d+1}$ which are built from these data. For the choice $(\mathbf{K}_\nu, \mathbf{J}_\nu)$, we will will denote the resulting Floer-theoretic doodads with a subscript of ν, e.g. $\mathcal{X}_\nu(L_i, L_j)$, $\mathcal{R}_\nu(\gamma_+;\gamma_-)$, $CW^*_\nu(L_i, L_j)$, or $\mathcal{W}_{\boldsymbol{\sigma},\nu}(M, \lambda_M)$.

Fixing a universal choice of strip-like ends $\boldsymbol{\epsilon}_{\mathcal{S}}$, we proceed to choose data on $\mathcal{S}^{d+1}$ which interpolates between $\mathbf{K}_0$ and $\mathbf{K}_1$. Namely, suppose we are given a $(d+1)$-tuple of Lagrangians $(L_0, \ldots, L_d)$, which induces Lagrangian labels on every disk component of every point of $\overline{\mathcal{S}}^{d+1}$. In this situation, a Floer datum on Σ consists of a 4-tuple (β, H, J, τ_E) which satisfies Definition 3.2.2, except that in conditions 3b and 4a for the positive ends, we replace $(H^{0,1}, J^{0,1})$ with $(H_0^{0,1}, J_0^{0,1})$, while in conditions 3c and 4a for the negative ends, we replace $(H^{0,1}, J^{0,1})$ with $(H_1^{0,1}, J_1^{0,1})$. In other words, we use the "0" data for the inputs and the "1" data for the output.

Consider the space $\mathcal{K}^{\mathcal{S}}(M)$ of **universal and conformally consistent** choices of Floer data for $\mathcal{S}^{d+1}$. Elements $\mathbf{K} \in \mathcal{K}^{\mathcal{S}}(M)$ consist of a choice, for each $(\Sigma, w) \in \mathcal{S}^{d+1}$, of a Floer datum $\mathbf{K}(\Sigma, w)$ on Σ such that the family varies smoothly on $\mathcal{S}^{d+1}$ and satisfies the following analog of Definition 3.2.4:

> Near each boundary stratum as parametrized in Construction 3.4.2, $\mathbf{K}(\Sigma, w)$ coincides up to conformal equivalence on the ϵ_S-thin part with the Floer datum determined by gluing, where all Floer data for disks $\Sigma^{m+1} \in \mathcal{R}^{m+1}$ belong to whichever of $\mathbf{K}_0$ or $\mathbf{K}_1$ will make them a priori gluable. On the thick part, the restriction of $\mathbf{K}(\Sigma, w)$ to each piece extends smoothly to the boundary, where it is conformally equivalent to the previously chosen Floer datum. (3.4.1)

For such a choice $\mathbf{K}$, one can consider

$$\gamma_i \in \begin{cases} \mathcal{X}_0(L_{i-1}, L_i) & i \neq 0 \\ \mathcal{X}_1(L_0, L_d) & i = 0. \end{cases} \tag{3.4.2}$$

and the corresponding moduli space $\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ of solutions to (3.2.12) with the obvious boundary and asymptotic conditions. This space has a Gromov compactification $\overline{\mathcal{S}}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ obtained by adding in appropriate broken configurations. We enumerate those broken configurations with only one $\mathcal{R}$ or $\mathcal{R}^{d+1}$ component.

$$\begin{aligned} &\mathcal{R}_1^{k+1}(\widetilde{\gamma}_k, \ldots, , \widetilde{\gamma}_1; \gamma_0) && m_i \geq 1, \sum m_i = d \\ &\times \prod_{i=1}^{k} \mathcal{S}^{m_i+1}\left(\gamma_{\sum_{j=1}^{i} m_j}, \ldots, \gamma_{\sum_{j=1}^{i-1} m_j+1}; \widetilde{\gamma}_i\right) && \widetilde{\gamma}_i \in \mathcal{X}_1\left(L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^{i} m_j}\right) \end{aligned} \tag{3.4.3a}$$

$$\begin{aligned} &\mathcal{S}^{m+1+1}(\gamma_d, \ldots, \gamma_{i+d-m+1}, \widetilde{\gamma}, \gamma_i, \ldots, \gamma_1; \gamma_0) && 0 \leq i \leq m \leq d-2 \\ &\times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m}) \end{aligned} \tag{3.4.3b}$$

$$\begin{aligned} &\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_1; \gamma_0) && 1 \leq i \leq d \\ &\times \mathcal{R}_0(\gamma_i; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i) \end{aligned} \tag{3.4.3c}$$

$$\mathcal{R}_1(\widetilde{\gamma}; \gamma_0) \times \mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \widetilde{\gamma}) \qquad \widetilde{\gamma} \in \mathcal{X}_1(L_0, L_d) \tag{3.4.3d}$$

The first type comes from the boundary component of $\overline{\mathcal{S}}^{d+1}$ where w tends to infinity, while the second comes from w finite or zero. The other two come from energy escaping through the strip-like ends. The same transversality argument as in Lemma 3.2.6 gives

Lemma 3.4.4. *There is a subset $\mathcal{K}^{\mathcal{S}}_{reg}(M) \subset \mathcal{K}^{\mathcal{S}}(M)$ which is dense and such that any $\mathbf{K} \in \mathcal{K}^{\mathcal{S}}_{reg}(M)$ has the following properties.*

1. *For all $d \geq 1$, all $L_0, \ldots, L_d$, and all γ_i as in* (3.4.2)*, the corresponding moduli space $\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is a smooth manifold of dimension* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) + d - 1$.
2. *If* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) = 1 - d$*, then $\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is compact.*
3. *If* $\deg(\gamma_0) - \sum_{i=1}^{d} \deg(\gamma_i) = 2 - d$*, then $\overline{\mathcal{S}}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is a compact topological 1-manifold with boundary, and its boundary is the union of all broken curves which appear in* (3.4.3).

Remark on proof. The proof of the first two parts is standard. For the third part, when the broken configurations involve only one gluing region, the proof is identical to the proof for the A_∞ operations. For configurations involving several gluing regions with correlated lengths, we proceed as follows. To start, note that an index count shows that the configurations in (3.4.3a) are the only ones which can occur. Thus, we consider the full corner $[0, a)^k$ in which the space of allowed gluing parameters is a 1-dimensional subvariety. By the Whitney extension theorem, we may extend our Floer data to this larger space of domains, and by rerunning the transversality argument we may assume the extension is regular. Since have a broken curve for which each component occurs in index 0, the analytic gluing map with domain $(0, a')^k$ for $a' \ll a$ is bijective to the index k portion of the larger moduli space of maps with the appropriate subset of domains (not just those obtained by gluing with the given lengths, but also by perturbing the unglued domains in $\mathcal{R}^{k+1}$ and $\mathcal{S}^{m_i+1}$). Taking the 1-dimensional subset with correlated gluing lengths then gives the corresponding end of $\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$. □

For $d \geq 1$, define $\mathcal{F}^d\colon CW_0^*(L_{d-1}, L_d) \otimes \cdots \otimes CW_0^*(L_0, L_1) \to CW_1^*(L_0, L_d)$ by

$$\mathcal{F}^d(\gamma_d, \ldots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathcal{X}_1(L_0, L_1) \\ \deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 1-d}} \#\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0) \cdot \gamma_0. \tag{3.4.4}$$

As usual, the characterization in Lemma 3.4.4 of the boundary of the 1-dimensional components of $\mathcal{S}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ implies that

$$\sum_{m=0}^{d-1} \sum_{i=0}^{m} \mathcal{F}^{m+1}\left(\gamma_d, \ldots, \gamma_{i+d-m+1}, \mu_0^{d-m}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}), \gamma_i, \ldots, \gamma_1\right)$$
$$+ \sum_{k=1}^{d} \sum_{\substack{m_i \geq 1 \\ \sum_{i=1}^k m_i = d}} \mu_1^k \left(\mathcal{F}^{m_k}(\gamma_{\sum_{j=1}^k m_j}, \ldots, \gamma_{\sum_{j=1}^{k-1} m_j + 1}), \ldots, \mathcal{F}^{m_1}(\gamma_{m_1}, \ldots, \gamma_1)\right) = 0$$

This collection of equations, as d ranges over the positive integers, is precisely the condition that $\mathcal{F}$ is an A_∞-functor from $\mathcal{W}_0(M, \lambda_M)$ to $\mathcal{W}_1(M, \lambda_M)$.

Definition 3.4.5. $\mathcal{F}$ is called **continuation functor** determined by $(\mathbf{K}, \mathbf{J})$.

By positivity of intersections, we see as in Lemma 3.1.5 that $\mathcal{F}^d$ sends $\mathcal{W}_{\boldsymbol{\sigma},0}(M, \lambda_M)$ to $\mathcal{W}_{\boldsymbol{\sigma},1}(M, \lambda_M)$, so that $\mathcal{F}$ restricts to a functor on partially wrapped Fukaya categories. We also call this restricted functor $\mathcal{F}$, and still refer to it as a continuation functor.

3.5 Homotopies between continuation functors

To show that continuation functors are quasi-equivalences, it is enough to construct homotopies on the linear part, since that shows that they induce isomorphisms on homology. However, to prove that partially wrapped Fukaya categories are invariant under isotopies of the stops, it will be convenient to construct a full A_∞ homotopy between a given pair of continuation functors. To this end, we sketch the construction of a sequence of spaces which parametrize homotopies in the same way that the associahedra and multiplihedra parametrize composition and maps. We call these spaces **homotohedra** for lack of a better term, though we won't prove that they're polyhedra.

Construction 3.5.1. The construction of the dth homotohedron $\overline{\mathcal{T}}^{d+1}$ is similar to that for the multiplihedron in Construction 3.4.2. Namely, one takes the space $\mathcal{S}^{d+1}$, adds in an extra parameter, and compactifies with boundary strata that manifestly induce the A_∞-homotopy equations. In this case, things are even easier than before, since as smooth spaces we can take $\mathcal{T}^{d+1} = \mathcal{S}^{d+1} \times (0, d)$ and $\overline{\mathcal{T}}^{d+1} = \overline{\mathcal{S}}^{d+1} \times [0, d]$. The trickiness is in choosing the strip-like ends.

To begin, suppose we are given two universal choices of strip-like ends $\boldsymbol{\epsilon}_{\mathcal{S}}^{\alpha}$ and $\boldsymbol{\epsilon}_{\mathcal{S}}^{\beta}$ for the multiplihedra, which we think of as two different models of the multiplihedron $\mathcal{S}_\alpha^{d+1}$ and

$\mathcal{S}_\beta^{d+1}$. For $d = 1$, pick a family of strip-like ends on $\mathcal{Z}$ parametrized by $(Z, q) \in \mathcal{T}^{1+1}$ which coincides with $\boldsymbol{\epsilon}_{\mathcal{S}}^{\alpha}$ near $q = 0$ and with $\boldsymbol{\epsilon}_{\mathcal{S}}^{\beta}$ near $q = 1$. Thus, we identify $(Z, 0) \in \overline{\mathcal{T}}^{1+1}$ with $Z \in \mathcal{S}_\alpha^{1+1}$ and $(Z, 1) \in \overline{\mathcal{T}}^{1+1}$ with $Z \in \mathcal{S}_\beta^{1+1}$.

For $d > 1$, we likewise identify $\overline{\mathcal{S}}^{d+1} \times \{0\}$ with $\overline{\mathcal{S}}_\alpha^{d+1}$ and $\overline{\mathcal{S}}^{d+1} \times \{d\}$ with $\overline{\mathcal{S}}_\beta^{d+1}$, and we choose strip-like ends in a small neighborhood of these faces to be independent of the $[0, d]$-parameter. We aim to extend this choice to $\partial\overline{\mathcal{S}}^{d+1} \times (0, d)$ as follows. The boundary at $w = 0$ is of the form $\mathcal{S}^{1+1} \times \mathcal{R}^{d+1} \times (0, d)$, which we identify with $\mathcal{T}^{1+1} \times \mathcal{R}^{d+1}$ via $(Z, \Sigma, q) \mapsto ((Z, \frac{q}{d}), \Sigma)$. The gluing chart is as with the associahedron and doesn't change q. The boundary at $w = +\infty$ is more complicated. Here, unlike with the associahedra, the boundary face is identified with a product of associahedra, multiplihedra, and homotohedra which change *discretely* with q. Specifically, suppose we are in a face of $\overline{\mathcal{T}}$ of the form

$$U \times \prod_{i=1}^{m} U_i \times (0, d)$$

where $U \subset \mathcal{R}^{m+1}$ and $U_i \subset \mathcal{S}^{d_i+1}$. Then, for $q \in \left(\sum_{i=j+1}^m d_i, \sum_{i=j}^m d_i\right)$, we make the identification

$$\mathcal{S}^{d_i+1} = \begin{cases} \mathcal{S}_\alpha^{d_i+1} & \text{if } i < j \\ \mathcal{S}^{d_j+1} \times \left\{q - \sum_{i=j+1}^m d_i\right\} \subset \mathcal{T}^{d_j+1} & \text{if } i = j \\ \mathcal{S}_\beta^{d_i+1} & \text{if } i > j. \end{cases} \tag{3.5.1}$$

For q of the form $\sum_{i=j+1}^m d_i$ for some j, we make the simpler identification

$$\mathcal{S}^{d_i+1} = \begin{cases} \mathcal{S}_\alpha^{d_i+1} & \text{if } i \le j \\ \mathcal{S}_\beta^{d_i+1} & \text{if } i > j. \end{cases} \tag{3.5.2}$$

If, for $k < d$, we have chosen strip-like ends for $\mathcal{T}^{k+1}$ which agree near the above boundary faces with those given by gluing, we can do the same for $\mathcal{T}^{d+1}$ by induction. Extending this family of strip-like ends arbitrarily to the interior, we obtain the desired choice.

Let $\mathbf{K}_\alpha$ and $\mathbf{K}_\beta$ be universal and conformally consistent choices of Floer data for $\mathcal{S}_\alpha^{d+1}$ and $\mathcal{S}_\beta^{d+1}$, respectively. Assume that they both interpolate between $\mathbf{K}_0$ and $\mathbf{K}_1$, and that they are both regular in the sense of Lemma 3.4.4. As with continuation functors, consider the space $\mathcal{K}^{\mathcal{T}}(M)$ whose elements $\mathbf{K}_{\mathcal{T}}$ are families of Floer data for $\mathcal{T}^{d+1}$ which are universal and conformally consistent in the obvious way, with the following strengthening:

> In Definition 3.2.4, we ask that the smooth extension to the $q = 0$ or d boundary strata agrees to infinite order to the family of Floer data induced by gluing. (3.5.3)

This ensures that any consistent choice for $\mathcal{T}^{k+1}$ for $k < d$ can be extended to T^{d+1}. Specifically, it avoids the danger of non-smoothness near the interface strata (3.5.2). Letting $\mathbf{K}_{\mathcal{T}}$ denote such a choice, we examine the corresponding spaces of holomorphic curves

$\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$. Here, γ_i satisfy (3.4.2) as for functors. This space has a Gromov compactification $\overline{\mathcal{T}}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ obtained by adding in possibly-broken configurations. As usual, these include terms which can come from either energy escape through the ends or domain degeneration to the boundary of $\mathcal{T}^{d+1}$. In this case, however, some of the new objects include ordinary non-broken disks coming from the $q = 0$ and $q = d$ components. We enumerate those new configurations with at most one $\mathcal{R}$ or $\mathcal{R}^{d+1}$ component.

$$\begin{aligned}
&\mathcal{R}_1^{k+1}(\widetilde{\gamma}_k, \ldots, , \widetilde{\gamma}_1; \gamma_0) \\
&\times \prod_{i=1}^{r-1} \mathcal{S}_\alpha^{m_i+1}\left(\gamma_{\sum_{j=1}^{i} m_j}, \ldots, \gamma_{\sum_{j=1}^{i-1} m_j + 1}; \widetilde{\gamma}_i\right) && 1 \le r \le k \le d \\
&\times \mathcal{T}^{m_r+1}\left(\gamma_{\sum_{j=1}^{r} m_j}, \ldots, \gamma_{\sum_{j=1}^{r-1} m_j + 1}; \widetilde{\gamma}_r\right) && m_i \ge 1, \textstyle\sum m_i = d \\
&\times \prod_{i=r+1}^{k} \mathcal{S}_\beta^{m_i+1}\left(\gamma_{\sum_{j=1}^{i} m_j}, \ldots, \gamma_{\sum_{j=1}^{i-1} m_j + 1}; \widetilde{\gamma}_i\right) && \widetilde{\gamma}_i \in \mathcal{X}_1\left(L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^{i} m_j}\right)
\end{aligned} \tag{3.5.4a}$$

$$\begin{aligned}
&\mathcal{T}^{m+1+1}(\gamma_d, \ldots, \gamma_{i+d-m+1}, \widetilde{\gamma}, \gamma_i, \ldots, \gamma_1; \gamma_0) && 0 \le i \le m \le d-2 \\
&\times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m})
\end{aligned} \tag{3.5.4b}$$

$$\begin{aligned}
&\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_1; \gamma_0) && 1 \le i \le d \\
&\times \mathcal{R}_0(\gamma_i; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i)
\end{aligned} \tag{3.5.4c}$$

$$\mathcal{R}_1(\widetilde{\gamma}; \gamma_0) \times \mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \widetilde{\gamma}) \qquad \widetilde{\gamma} \in \mathcal{X}_1(L_0, L_d) \tag{3.5.4d}$$

$$\mathcal{S}_\alpha^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0) \tag{3.5.4e}$$

$$\mathcal{S}_\beta^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0) \tag{3.5.4f}$$

$$\begin{aligned}
&\mathcal{R}_1^{k+1}(\widetilde{\gamma}_k, \ldots, , \widetilde{\gamma}_1; \gamma_0) && 0 \le r \le k \\
&\times \prod_{i=1}^{r} \mathcal{S}_\alpha^{m_i+1}\left(\gamma_{\sum_{j=1}^{i} m_j}, \ldots, \gamma_{\sum_{j=1}^{i-1} m_j + 1}; \widetilde{\gamma}_i\right) && 1 \le k \le d \\
& && m_i \ge 1, \textstyle\sum m_i = d \\
&\times \prod_{i=r+1}^{k} \mathcal{S}_\beta^{m_i+1}\left(\gamma_{\sum_{j=1}^{i} m_j}, \ldots, \gamma_{\sum_{j=1}^{i-1} m_j + 1}; \widetilde{\gamma}_i\right) && \widetilde{\gamma}_i \in \mathcal{X}_1\left(L_{\sum_{j=1}^{i-1} m_j}, L_{\sum_{j=1}^{i} m_j}\right)
\end{aligned} \tag{3.5.4g}$$

$$\begin{aligned}
&\mathcal{S}_\alpha^{m+1+1}(\gamma_d, \ldots, \gamma_{i+d-m+1}, \widetilde{\gamma}, \gamma_i, \ldots, \gamma_1; \gamma_0) && 0 \le i \le m \le d-2 \\
&\times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m})
\end{aligned} \tag{3.5.4h}$$

$$\begin{aligned}
&\mathcal{S}_\beta^{m+1+1}(\gamma_d, \ldots, \gamma_{i+d-m+1}, \widetilde{\gamma}, \gamma_i, \ldots, \gamma_1; \gamma_0) && 0 \le i \le m \le d-2 \\
&\times \mathcal{R}_0^{d-m+1}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_i, L_{i+d-m})
\end{aligned} \tag{3.5.4i}$$

$$\begin{aligned}
&\mathcal{S}_\alpha^{d+1}(\gamma_d, \ldots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_1; \gamma_0) && 1 \le i \le d \\
&\times \mathcal{R}_0(\gamma_i; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathcal{X}_0(L_{i-1}, L_i)
\end{aligned} \tag{3.5.4j}$$

$$\begin{aligned} &\mathcal{S}_\beta^{d+1}(\gamma_d, \ldots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_1; \gamma_0) && 1 \le i \le d \\ &\qquad \times \mathcal{R}_0(\gamma_i; \widetilde{\gamma}) && \widetilde{\gamma} \in \mathfrak{X}_0(L_{i-1}, L_i) \end{aligned} \tag{3.5.4k}$$

$$\mathcal{R}_1(\widetilde{\gamma}; \gamma_0) \times \mathcal{S}_\alpha^{d+1}(\gamma_d, \ldots, \gamma_1; \widetilde{\gamma}) \qquad \widetilde{\gamma} \in \mathfrak{X}_1(L_0, L_d) \tag{3.5.4l}$$

$$\mathcal{R}_1(\widetilde{\gamma}; \gamma_0) \times \mathcal{S}_\beta^{d+1}(\gamma_d, \ldots, \gamma_1; \widetilde{\gamma}) \qquad \widetilde{\gamma} \in \mathfrak{X}_1(L_0, L_d) \tag{3.5.4m}$$

Here, one should think of (3.5.4a)-(3.5.4d) as completely analogous to (3.4.3). The next two, (3.5.4e) and (3.5.4f), correspond to degenerations $q \to 0$ and $q \to d$. (3.5.4g) comes from taking $w \to \infty$ as in (3.4.3a), except that q takes a nongeneric value. The remaining configurations come from taking q to 0 or d in (3.5.4b)-(3.5.4d). The usual transversality argument gives

Lemma 3.5.2. *There is a subset $\mathcal{K}^{\mathcal{T}}_{reg}(M) \subset \mathcal{K}^{\mathcal{T}}(M)$ which is dense and such that any $\mathbf{K} \in \mathcal{K}^{\mathcal{T}}_{reg}(M)$ has the following properties.*

1. *For all $d \geq 1$, all $L_0, \ldots, L_d$, and all γ_i as in* (3.4.2), *the corresponding moduli space $\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is a smooth manifold of dimension* $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) + d$.
2. *If* $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = -d$, *then $\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is compact.*
3. *If* $\deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = 1 - d$, *then $\overline{\mathcal{T}}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ is a compact topological 1-manifold with boundary, and its boundary is the union of all configurations of the form* (3.5.4a)-(3.5.4f).

Remark on proof. For the most part, the proof is identical to that for the moduli spaces associated to functors in Lemma 3.4.4. In this case, there are also many broken configurations with only one gluing parameter which don't appear. This happens because they correspond to isolated values of q, and varying q gives another deformation parameter. Therefore, they occur in codimension at least two. Alternatively, one could simply note that the terms with a gluing parameter and no $\mathcal{T}^{k+1}$ would have to occur in negative index, and this is prohibited by the regularity of $\mathbf{K}_\alpha$ and $\mathbf{K}_\beta$. □

For $d \geq 1$, define $T^d \colon CW_0^*(L_{d-1}, L_d) \otimes \cdots \otimes CW_0^*(L_0, L_1) \to CW_1^*(L_0, L_d)$ by

$$T^d(\gamma_d, \ldots, \gamma_1) = \sum_{\substack{\gamma_0 \in \mathfrak{X}_1(L_0, L_1) \\ \deg(\gamma_0) - \sum_{i=1}^d \deg(\gamma_i) = -d}} \#\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0) \cdot \gamma_0. \tag{3.5.5}$$

By definition, we may treat T as a pre-natural transformation from $\mathcal{F}_\alpha$ to $\mathcal{F}_\beta$. The characterization in Lemma 3.5.2 of the boundary strata of the 1-dimensional components of

$\mathcal{T}^{d+1}(\gamma_d, \ldots, \gamma_1; \gamma_0)$ implies that

$$\begin{aligned}
&\mathcal{F}_\beta^{d+1}(\gamma_d, \ldots, \gamma_1) - \mathcal{F}_\alpha^{d+1}(\gamma_d, \ldots, \gamma_1) = \\
&\qquad \sum_{m=0}^{d-1}\sum_{i=0}^{m} T^{m+1}\left(\gamma_d, \ldots, \gamma_{i+d-m+1}, \mu_0^{d-m}(\gamma_{i+d-m}, \ldots, \gamma_{i+1}), \gamma_i, \ldots, \gamma_1\right) \\
&+ \sum_{k=1}^{d}\sum_{r=1}^{k}\sum_{\substack{m_i \geq 1 \\ \sum_{i=1}^k m_i = d}} \mu_1^k\Big(\mathcal{F}_\beta^{m_k}(\gamma_{\sum_{j=1}^k m_j}, \ldots, \gamma_{\sum_{j=1}^{k-1} m_j + 1}), \ldots, \mathcal{F}_\beta^{m_{r+1}}(\gamma_{\sum_{j=1}^{r+1} m_j}, \ldots, \gamma_{\sum_{j=1}^{r} m_j + 1}), \\
&\qquad\qquad T^{m_r}(\gamma_{\sum_{j=1}^{r} m_j}, \ldots, \gamma_{\sum_{j=1}^{r-1} m_j + 1}), \\
&\qquad\qquad \mathcal{F}_\alpha^{m_{r-1}}(\gamma_{\sum_{j=1}^{r-1} m_j}, \ldots, \gamma_{\sum_{j=1}^{r-2} m_j + 1}), \ldots, \mathcal{F}_\alpha^{m_1}(\gamma_{m_1}, \ldots, \gamma_1)\Big).
\end{aligned}$$

This collection of equations, as d ranges over the positive integers, is precisely the condition that $\mathcal{F}_\beta - \mathcal{F}_\alpha = dT$, i.e. T generates an A_∞-homotopy between $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

By positivity of intersections, we see as in Lemma 3.1.5 that T^d sends $\mathcal{W}_{\boldsymbol{\sigma},0}(M, \lambda_M)$ to $\mathcal{W}_{\boldsymbol{\sigma},1}(M, \lambda_M)$, so that T induces a homotopy between $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ on the partially wrapped Fukaya categories as well.

3.6 Moving stops

We are now equipped to prove the following statement.

Proposition 3.6.1. *Suppose $(M, \lambda_M, \boldsymbol{\sigma})$ and $(M', \lambda_{M'}, \boldsymbol{\sigma}')$ are equivalent pumpkin domains. Then $\mathcal{W}_{\boldsymbol{\sigma}}(M)$ is quasi-equivalent to $\mathcal{W}_{\boldsymbol{\sigma}}(M')$.*

For this, it is enough to consider the case where $M = M'$ is a fixed Liouville domain and the equivalence of pumpkin domains arises from an isotopy of stops $\boldsymbol{\sigma}_t$ with varying fiber. Note that if the fiber were nonvarying, then one could find a path of Liouville isomorphisms $M \to M$ which take $\boldsymbol{\sigma}_0$ to $\boldsymbol{\sigma}_t$. For time-dependent fiber, this fails. Since we may decompose the isotopy $\boldsymbol{\sigma}_t$ as a sequence of small isotopies, Proposition 3.6.1 follows from the easier

Lemma 3.6.2. *Let (M, λ_M) be a Liouville domain and $\boldsymbol{\sigma}_t$ a $[0,1]$-parametrized family of pumpkin structures on M. Suppose every interior Lagrangian in $(M, \lambda_M, \boldsymbol{\sigma}_0)$ is a Lagrangian in $(M, \lambda_M, \boldsymbol{\sigma}_t)$ for all t. Then there is a quasi-equivalence $\mathcal{W}^{int}_{\boldsymbol{\sigma}_0}(M) \to \mathcal{W}_{\boldsymbol{\sigma}_1}(M)$.*

The rest of this section is devoted to the proof of Lemma 3.6.2. In what follows, we will use M_0 to denote the pumpkin domain $(M, \lambda_M, \boldsymbol{\sigma}_0)$ and M_1 to denote the pumpkin domain $(M, \lambda_M, \boldsymbol{\sigma}_1)$. From the data of the stops, we obtain two presentations of the fully wrapped Fukaya category $\mathcal{W}(M_0)$ and $\mathcal{W}(M_1)$. These categories are quasi-equivalent via any continuation functor, but the quasi-equivalences don't respect the stop filtrations. To obtain a map of partially wrapped Fukaya categories, we will need to be more careful.

The idea is to construct a sequence of continuation functors associated to ever-slower isotopies of the stops for which we will be able to take Gromov limits. Unlike the continuation functors themselves, these Gromov limits will satisfy positivity of intersections with the divisors, which will allow us to define maps of partially wrapped Fukaya categories. To construct our sequence of functors, we will consider Floer data on the multiplihedra compatible with increasingly shifted strip-like ends. To ensure Gromov convergence, we need to carefully choose our isotopies to be compatible as we move along the sequence. For this, we make the following technical definition.

Definition 3.6.3. A **slowing family** consists of the following data.

- For each integer $n \geq 1$, a universal choice of strip-like ends $\boldsymbol{\epsilon}_{\mathcal{S},n}$ for the multiplihedra.
- For each integer $n \geq 1$ and all $d \geq 1$, a diffeomorphism $\Phi_n^d \colon \mathcal{S}^{d+1} \to \mathcal{S}^{d+1}$ and a family of diffeomorphisms $I_{(\Sigma,w),n} \colon \Sigma \to \Phi_n^d(\Sigma)$ parametrized by $(\Sigma, w) \in \mathcal{S}^{d+1}$.
- For all $d \geq 1$ and each $(\Sigma, w) \in \mathcal{S}^{d+1}$, a function $\mathbf{t}_{(\Sigma,w)} \colon \Sigma \to [0,1]$ varying smoothly on $\mathcal{S}^{d+1}$ with the following properties.

These data are required to satisfy the following conditions.

1. $\Phi_1^d = \mathrm{id}_{\mathcal{S}^{d+1}}$, and Φ_n^d is isotopic to $\mathrm{id}_{\mathcal{S}^{d+1}}$.
2. For (Σ, w) near the boundary of $\mathcal{S}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i < d$ and Σ_j be the disks in $\mathcal{R}^{m_j+1}$ for $m_j \leq d$ from which (Σ, w) is glued. Then $\Phi_n^d(\Sigma, w)$ is glued from $\Phi_n^{k_i}(\Sigma_i, w_i)$ and Σ_j.
3. $I_{(\Sigma,w),1} = \mathrm{id}_\Sigma$, and $I_{(\Sigma,w),n}$ is isotopic to id_Σ. Additionally, $I_{(\Sigma,w),n}$ sends $\boldsymbol{\epsilon}_{\mathcal{S},n}$ to $\boldsymbol{\epsilon}_{\mathcal{S},1}$.
4. For (Σ, w) near the boundary of $\mathcal{S}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i < d$ and Σ_j be the disks in $\mathcal{R}^{m_j+1}$ for $m_j \leq d$ from which (Σ, w) is glued. Let $\Sigma_i^0 \subset \Sigma_i$ be the complement of the strip-like ends. Then under the identifications coming from condition (2), the restrictions to Σ_i^0 of $I_{(\Sigma,w),n}$ and $I_{(\Sigma_i,w_i),n}$ coincide, and the restriction of $I_{(\Sigma,w),n}$ to Σ_j is the identity.
5. For any fixed d and k, the family I over $\mathcal{S}^{d+1} \times \mathbb{Z}_{>0}$ is uniformly bounded in C^k.
6. For all $(\Sigma, w) \in \mathcal{S}^{d+1}$, there is a decomposition $\Sigma = U \amalg S$, where U is open and S is biholomorphic to a disjoint union of rectangles R_i, with the following properties.
   a) $I_{(\Phi_n^d)^{-1}(\Sigma,w),n}$ is holomorphic on $(I_{(\Phi_n^d)^{-1}(\Sigma,w),n})^{-1}(U)$.
   b) R_i can be taken to be of the form $[0, a_i] \times [0,1]$, with $\partial\Sigma \cap S$ mapping to $[0, a_i] \times \{0,1\}$. Moreover, $(I_{(\Phi_n^d)^{-1}(\Sigma,w),n})^{-1}(R_i)$ is biholomorphic to a rectangle $[0, b_{i,n}] \times [0,1]$, and in these coordinates $I_{(\Phi_n^d)^{-1}(\Sigma,w),n}(s,t)$ takes the form $(f_{i,n}(s), t)$.
   c) The above functions $f_{i,n}$ satisfy $\frac{\partial f_{i,n}}{\partial s} \leq 1$ everywhere and $\frac{\partial f_{i,n}}{\partial s} < \frac{1}{n}$ on $f_{i,n}^{-1}([\frac{a_i}{3}, \frac{2a_i}{3}])$.

7. $\mathbf{t}_{(\Sigma,w)}$ is 0 on the positive strip-like ends and 1 on the negative strip-like ends.

8. For (Σ, w) near the boundary of $\mathcal{S}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i < d$ from which (Σ, w) is glued. Then $\mathbf{t}_{(\Sigma,w)}$ agrees with the extension by 1 and/or 0 of the functions $\mathbf{t}_{(\Sigma_i,w_i)}$.

9. In the decomposition of condition (6), $d\mathbf{t}_{(\Sigma,w)}$ is supported on the union of the subrectangles $[\frac{a_i}{3}, \frac{2a_i}{3}] \times [0,1]$, and on these subrectangles $\mathbf{t}_{(\Sigma,w)}$ depends only on the first coordinate.

Note that condition (6c) implies that the rectangle lengths $b_{i,n}$ tend to infinity with n.

To construct a slowing family, one chooses first $\boldsymbol{\epsilon}_{\mathcal{S},1}$ arbitrarily and second the family of functions $\mathbf{t}$ such that $d\mathbf{t}_{(\Sigma,w)}$ is supported near the thin part. Then, one arranges that the embeddings and additional choices of strip-like ends increase the gluing lengths and stretch the support of $d\mathbf{t}_{(\Sigma,w)}$.

From the perspective of Floer theory, a slowing family modifies the compatibility conditions between Hamiltonians or almost complex structures and stops. When the stops were fixed, compatibility was roughly the condition that their divisors were almost complex submanifolds. In the presence of a slowing family, we ask that this holds pointwise in the domain. Concretely, let $\mathbf{t}$ be a family over the multiplihedron of functions $\mathbf{t}_{(\Sigma,w)} \colon \Sigma \to [0,1]$. Then a $\mathbf{t}$-compatible Hamiltonian on $(\Sigma, w) \in \mathcal{S}^{d+1}$ is a Σ-parametrized quadratic Hamiltonian H such that, for all $z \in \Sigma$ and all $\sigma \in \boldsymbol{\sigma}_{\mathbf{t}_{(\Sigma,w)}(z)}$, $X_{H(z)}$ is tangent to D_σ. Likewise, an almost complex structure is adapted to $(\mathbf{t}, H)$ if, for all $z \in \Sigma$, it lies in $\mathcal{J}(M, H(z))$ for the pumpkin structure $\boldsymbol{\sigma}_{\mathbf{t}_{(\Sigma,w)}(z)}$.

Let $\mathbf{K}_0$ and $\mathbf{K}_1$ be universal and conformally consistent choices of Floer data for the A_∞ structure for M_0 and M_1, respectively. Fix a slowing family, and let $\mathbf{K}^1$ be a universal and conformally consistent choice of Floer data for $\boldsymbol{\epsilon}_{\mathcal{S},1}$ which is regular in the sense of Lemma 3.4.4. Here, the notion of a universal and conformally consistent choice of Floer data is as for the multiplihedra in Section 3.4, except that the family of Hamiltonians H is required to be $\mathbf{t}$-compatible and the family of almost complex structures J is required to be adapted to $(\mathbf{t}, H)$. For $n \geq 2$, let $\mathbf{K}^n$ be a universal and conformally consistent choice of Floer data for $\boldsymbol{\epsilon}_{\mathcal{S},n}$ which is regular and 2^{-n}-close to $I_n^*\mathbf{K}^1$ in C^n. Define $\mathcal{F}_n \colon \mathcal{W}^{int}(M_0) \to \mathcal{W}(M_1)$ to be the continuation functor determined by $\mathbf{K}^n$.

In addition to functors, we need homotopies of the same kind. For this, we consider the analog of a slowing family for the homotohedron.

Definition 3.6.4. Fix a slowing family. Then an **interpolating family** consists of the following data.

- For each integer $n \geq 1$, a universal choice of strip-like ends $\boldsymbol{\epsilon}_{\mathcal{T},n}$ for the homotohedra.

- For each integer $n \geq 1$ and all $d \geq 1$, a diffeomorphism $\Psi_n^d \colon \mathcal{T}^{d+1} \to \mathcal{T}^{d+1}$ and a family of diffeomorphisms $I_{(\Sigma,w,q),n} \colon \Sigma \to \Psi_n^d(\Sigma)$ parametrized by $(\Sigma, w, q) \in \mathcal{T}^{d+1}$.

- For all $d \geq 1$ and each $(\Sigma, w, q) \in \mathcal{T}^{d+1}$, a function $\mathbf{t}_{(\Sigma,w,q)} \colon \Sigma \to [0,1]$ varying smoothly on $\mathcal{T}^{d+1}$ with the following properties.

These data are required to satisfy the following conditions.

1. $\boldsymbol{\epsilon}_{\mathcal{T},n}$ interpolates between $\boldsymbol{\epsilon}_{\mathcal{S},n}$ and $\boldsymbol{\epsilon}_{\mathcal{S},n+1}$ in the sense of Construction 3.5.1. That is, it agrees with $\boldsymbol{\epsilon}_{\mathcal{S},n}$ for small q and $\boldsymbol{\epsilon}_{\mathcal{S},n+1}$ for large q.

2. $\Psi_1^d = \mathrm{id}_{\mathcal{T}^{d+1}}$ and Ψ_n^d is isotopic to $\mathrm{id}_{\mathcal{T}^{d+1}}$. Additionally, Ψ_n^d and fixes the q-component.

3. For (Σ, w, q) near the boundary of $\mathcal{T}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i \leq d$ and Σ_j be the disks in $\mathcal{R}^{m_j+1}$ for $m_j \leq d$ and, if applicable, (Σ', w', q') the disk in $\mathcal{T}^{j+1}$ from which (Σ, w, q) is glued. Then $\Psi_n^d(\Sigma, w, q)$ is glued from $\Phi_n^{k_i}(\Sigma_i, w_i)$ for (Σ_i, w_i) of type α, $(\Phi_2^{k_i})^{-1}(\Phi_{n+1}^{k_i}(\Sigma_i, w_i))$ for (Σ_i, w_i) of type β, $\Psi_n^j(\Sigma', w', q')$, and Σ_j. Here, the types α and β refer to the assignments in (3.5.1) and (3.5.2).

4. $I_{(\Sigma,w,q),1} = \mathrm{id}_\Sigma$, and $I_{(\Sigma,w,q),n}$ is isotopic to id_Σ. Additionally, $I_{(\Sigma,w,q),n}$ sends $\boldsymbol{\epsilon}_{\mathcal{T},n}$ to $\boldsymbol{\epsilon}_{\mathcal{T},1}$.

5. For (Σ, w, q) near the boundary of $\mathcal{T}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i \leq d$ and Σ_j be the disks in $\mathcal{R}^{m_j+1}$ for $m_j \leq d$ and, if applicable, (Σ', w', q') the disk in $\mathcal{T}^{j+1}$ from which (Σ, w, q) is glued. Let $\Sigma_i^0 \subset \Sigma_i$ and $(\Sigma')^0 \subset \Sigma'$ be the complements of the strip-like ends. Then under the identifications coming from condition (3), the restriction of $I_{(\Sigma,w,q),n}$ to Σ_i^0 coincides with $I_{(\Sigma_i,w_i),n}$ for (Σ_i, w_i) of type α and $(I_{(\Sigma_i,w_i),2})^{-1} \circ I_{(\Sigma_i,w_i),n+1}$ for (Σ_i, w_i) of type β. Similarly, the restrictions to $(\Sigma')^0$ of $I_{(\Sigma,w,q),n}$ and $I_{(\Sigma',w',q'),n}$ coincide. Finally, the restriction of $I_{(\Sigma,w,q),n}$ to Σ_j is the identity.

6. For any fixed d and k, the family I over $\mathcal{T}^{d+1} \times \mathbb{Z}_{>0}$ is uniformly bounded in C^k.

7. For all $(\Sigma, w, q) \in \mathcal{T}^{d+1}$, there is a decomposition $\Sigma = U \amalg S$, where U is open and S is biholomorphic to a disjoint union of rectangles R_i, with the following properties.
   a) $I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n}$ is holomorphic on $(I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n})^{-1}(U)$.
   b) R_i can be taken to be of the form $[0, a_i] \times [0,1]$, with $\partial\Sigma \cap S$ mapping to $[0, a_i] \times \{0,1\}$. Moreover, $(I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n})^{-1}(R_i)$ is biholomorphic to a rectangle $[0, b_{i,n}] \times [0,1]$, and in these coordinates $I_{(\Psi_n^d)^{-1}(\Sigma,w,q),n}(s,t)$ takes the form $(f_{i,n}(s), t)$.
   c) The above functions $f_{i,n}$ satisfy $\frac{\partial f_{i,n}}{\partial s} \leq 1$ everywhere and $\frac{\partial f_{i,n}}{\partial s} < \frac{1}{n}$ on $f_{i,n}^{-1}([\frac{a_i}{3}, \frac{2a_i}{3}])$.

8. $\mathbf{t}_{(\Sigma,w,q)}$ is 0 on the positive strip-like ends and 1 on the negative strip-like ends.

9. For (Σ, w, q) near the boundary of $\mathcal{T}^{d+1}$, let (Σ_i, w_i) be the disks in $\mathcal{S}^{k_i+1}$ for $k_i \leq d$ and, if applicable, (Σ', w', q') the disk in $\mathcal{T}^{j+1}$ from which (Σ, w, q) is glued. Then $\mathbf{t}_{(\Sigma,w,q)}$ agrees with the extension by 1 and/or 0 of the functions $\mathbf{t}_{(\Sigma_i,w_i)}$ for (Σ_i, w_i) of type α, $(I_{(\Sigma_i,w_i),2})^* \mathbf{t}_{\Phi_2^{k_i}(\Sigma_i,w_i)}$ for (Σ_i, w_i) of type β, and $\mathbf{t}_{(\Sigma',w',q')}$.

10. In the decomposition of condition (7), $d\mathbf{t}_{(\Sigma,w,q)}$ is supported on the union of the subrectangles $[\frac{a_i}{3}, \frac{2a_i}{3}] \times [0,1]$, and on these subrectangles $\mathbf{t}_{(\Sigma,w,q)}$ depends only on the first coordinate.

Fix an interpolating family. As with functors, choose a universal and conformally consistent family of Floer data $\mathbf{K}^1_{\mathcal{T}}$ for $\boldsymbol{\epsilon}_{\mathcal{T},1}$ interpolating between $\mathbf{K}^1$ and $\mathbf{K}^2$ which is regular in the sense of 3.5.2. For $n \geq 2$, let $\mathbf{K}^n_{\mathcal{T}}$ be a universal and conformally consistent choice of Floer data for $\boldsymbol{\epsilon}_{\mathcal{T},n}$ interpolating between $\mathbf{K}^n$ and $\mathbf{K}^{n+1}$ which is regular and 2^{1-n}-close to $I_n^*\mathbf{K}^1_{\mathcal{T}}$ in C^n. Define T_n to be the pre-natural transformation generating a homotopy from $\mathcal{F}_n$ to $\mathcal{F}_{n+1}$ determined by $\mathbf{K}^n_{\mathcal{T}}$.

Proof of Lemma 3.6.2. We begin by replacing $\mathcal{W}(M_1)$ with the limit of the diagram

$$\cdots \xrightarrow{\mathrm{id}} \mathcal{W}(M_1) \xrightarrow{\mathrm{id}} \mathcal{W}(M_1) \xrightarrow{\mathrm{id}} \mathcal{W}(M_1),$$

which can be described in the following way. This is a category $\mathcal{W}^{lim}(M_1)$ whose objects are the same as those of $\mathcal{W}(M_1)$ and whose morphism spaces are homotopy limits of the identity chain map. Concretely, this means

$$\hom^k_{\mathcal{W}^{lim}(M_1)}(L_0, L_1) = \prod_{n=1}^{\infty} \hom^k_{\mathcal{W}^{lim}(M_1)}(L_0, L_1)_n,$$

where

$$\hom^k_{\mathcal{W}^{lim}(M_1)}(L_0, L_1)_n = \hom^k_{\mathcal{W}(M_1)}(L_0, L_1) \oplus \hom^{k-1}_{\mathcal{W}(M_1)}(L_0, L_1)$$

for all n. Let

$$g = \big((\gamma^1, \eta^1), (\gamma^2, \eta^2), \dots\big) \in \hom_{\mathcal{W}^{lim}(M_1)}(L_0, L_1).$$

The differential $\mu^1_{\mathcal{W}^{lim}(M_1)}$ is given by the formula

$$\left(\mu^1_{\mathcal{W}^{lim}(M_1)} g\right)_n = (\partial\gamma^n, \gamma^n + \gamma^{n+1} + \partial\eta^n),$$

where $\partial = \mu^1_{\mathcal{W}(M_1)}$. This can be visualized diagrammatically as

$$\begin{tikzcd}
\cdots \arrow[dr, "\mathrm{id}"] & \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop above, "\partial"] \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop above, "\partial"] \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop above, "\partial"] \arrow[d, "\mathrm{id}"] \\
& \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop below, "\partial"] & \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop below, "\partial"] & \hom^*_{\mathcal{W}(M_1)}(L_0, L_1) \arrow[loop below, "\partial"]
\end{tikzcd}$$

The higher A_∞ maps are defined via

$$\left(\mu^d_{\mathcal{W}^{lim}(M_1)}(g_d,\dots,d_1)\right)_n = \left(\mu^d_{\mathcal{W}(M_1)}(\gamma^n_d,\dots,\gamma^n_1), \sum_{i=1}^d \mu^d_{\mathcal{W}(M_1)}(\gamma^{n+1}_d,\dots,\gamma^{n+1}_{i+1},\eta^n_i,\gamma^n_{i-1},\dots,\gamma^n_1)\right).$$

We will also want the subcategory $\mathcal{W}^{lim}_{\boldsymbol{\sigma}_1}(M)$ generated by sequences g of chords satisfying $n_\sigma(\gamma^i) = n_\sigma(\eta^i) = 0$ for all $\sigma \in \boldsymbol{\sigma}_1$ and all sufficiently large i. In other words, it consists of those generators whose total intersection number with the stops is finite.

There is a strict A_∞-functor $\mathcal{W}(M_1) \to \mathcal{W}^{lim}(M_1)$ which is the identity on objects and sends $\gamma \in \hom_{\mathcal{W}(M_1)}(L_0, L_1)$ to $\big((\gamma,0),(\gamma,0),\dots\big)$. This functor is an isomorphism on homology, and so it is a quasi-equivalence. Similarly, the restriction of this functor to $\mathcal{W}_{\boldsymbol{\sigma}_1}(M)$ is a quasi-equivalence onto $\mathcal{W}^{lim}_{\boldsymbol{\sigma}_1}(M)$. Thus, to construct a functor to $\mathcal{W}_{\boldsymbol{\sigma}_1}(M)$, it suffices to give a functor to $\mathcal{W}^{lim}(M_1)$ whose image lies in $\mathcal{W}^{lim}_{\boldsymbol{\sigma}_1}(M)$.

Now, the sequences $\mathcal{F}_n$ and T_n together define an A_∞-functor $\mathcal{F}\colon \mathcal{W}^{int}(M_0) \to \mathcal{W}^{lim}(M_1)$ given by the formula

$$\mathcal{F}^d = \big((\mathcal{F}^d_1, T^d_1), (\mathcal{F}^d_2, T^d_2), \dots\big).$$

We wish to show that $\mathcal{F}|_{\mathcal{W}^{int}_{\boldsymbol{\sigma}_0}(M)}$ maps into $\mathcal{W}^{lim}_{\boldsymbol{\sigma}_1}(M)$. For this, it suffices to consider a fixed stop $\sigma_t \in \boldsymbol{\sigma}_t$ and fixed inputs and outputs. We want to show that a sequence of holomorphic curves $u_n \in \mathcal{S}^{d+1}_{\mathbf{K}^n}(\gamma_d,\dots,\gamma_1;\gamma_0)$ must eventually have nonnegative topological intersection number with $D_{\sigma_{\mathbf{t}}}$, where $\sigma_{\mathbf{t}}$ is the Σ-parametrized version of σ_t. We also need the same statement with $\mathcal{T}^{d+1}_{\mathbf{K}^n_{\mathcal{T}}}$ instead of $\mathcal{S}^{d+1}_{\mathbf{K}^n}$, but the proof of that is identical.

Assume for a contradiction that there is an increasing sequence n_j such that u_{n_j} has strictly negative topological intersection number with $D_{\sigma_{\mathbf{t}}}$. Without loss of generality, assume further that all compatible Hamiltonians are split near $D_{\sigma_{\mathbf{t}}}$. This is achieved, for example, by the Hamiltonians constructed in the proof of Lemma 2.4.2. Choose a small t-dependent tubular neighborhood of $D_{\sigma_{\mathbf{t}}}$, and let S be the boundary of that neighborhood. For a generic choice of S, $u^{-1}_{n_j}(S)$ is a disjoint union of embedded circles for all j. Let Σ_{n_j} be the domain of u_{n_j}, and let $z_{n_j} \in \Sigma_{n_j}$ be such that $u_{n_j}(z_{n_j}) \in D_{\sigma_{\mathbf{t}(z_j)}}$ and u_{n_j} has negative degree on the smallest circle in $u^{-1}_{n_j}(S)$ surrounding z_{n_j}.

Since all Floer data are approximately pulled back from fixed ones on a compact domain, Arzelà-Ascoli applies to give a subsequence $n_{j'}$ for which $\mathbf{K}^{n_{j'}}$ and the complex structures on the domain converge in C^∞ on an increasing sequence of neighborhoods of $z_{n_{j'}}$. The limit of this data is a boundary-punctured Riemann surface Σ together with a Hamiltonian H, a sub-closed 1-form β, and an almost-complex structure $J \in \mathcal{J}(M,H)$ which are compatible with the stop $\sigma_{\mathbf{t}}$, where $\mathbf{t}$ is now constant. The rectangular coordinates from the definition of a slowing family give rise to strip-like ends on Σ, and in these coordinates β is asymptotic to a 1-form of the form $f(t)dt$.

For this subsequence, the discussion in Appendix A.2 gives a uniform maximum principle, and hence we may apply Gromov compactness at $z_{n_{j'}}$ to get a subsequence $u_{n_{j''}}$ which converges in C^∞_{loc} to some nonconstant curve u with domain Σ. There is no issue of bubbling at $z = \lim z_{n_{j''}}$ because the symplectic form is exact. Now $\mathbf{t}$ is constant on Σ, so positivity of intersections applies. In particular, u has only positive intersections with $D_{\sigma_{\mathbf{t}}}$, and $u(z)$

is such an intersection. Since u has finite energy, it is asymptotic at the punctures to X_H-chords, and because H is split such chords cannot intersect S. Thus, $u^{-1}(S)$ separates z from $\partial\Sigma$.

Identifying $z_{n_{j''}}$ and its neighborhoods with z and subsets of Σ, we see now that the smallest circle of $u_{n_{j''}}^{-1}(S)$ surrounding z is contained in Σ for large j''. Since u is C^0-close to $u_{n_{j''}}$, it has negative winding number about $D_{\sigma_{\mathbf{t}}}$ on this circle. This contradicts positivity of intersections for u and completes the proof. $\square$

Chapter 4

Stop removal

4.1 Closed strings

Theorem 1.3.1 relies on the existence of a closed string version of partially wrapped Floer homology. This will take the form of a filtration on the symplectic homology chain complex, which we now explain.

Let $(M, \lambda_M, \boldsymbol{\sigma})$ be a pumpkin domain. Pick a compatible Hamiltonian $\widetilde{H}$ and an S^1-family of perturbing Hamiltonians $P_t \colon \hat{M} \to \mathbb{R}_{\geq 0}$, where $t \in \mathbb{R}/\mathbb{Z} \cong S^1$, which satisfy the following conditions.

P_t is bounded, and $\|X_{P_t}\|$ decays exponentially in the symplectization coordinate $\sqrt{\widetilde{H}}$ for any metric of the form $\hat{\omega}_M(\cdot, J\cdot)$ with $J \in \mathcal{J}(M, H)$. (4.1.1)

$H_t := \widetilde{H} + P_t$ is **nondegenerate** in the sense that for any 1-periodic orbit x of the time-dependent vector field X_{H_t}, the Poincaré return map of x does not have 1 as an eigenvalue. (4.1.2)

For each $\sigma \in \boldsymbol{\sigma}$, $\widetilde{H}$ is of the form (2.4.1) near D_σ, with $f(z) = c|z|^2$ and $c > 0$. Similarly, P_t is independent of the $\mathbb{H}_\rho$-coordinate near D_σ, and X_{H_t} satisfies condition (4) of Definition 2.4.1. (4.1.3)

Note that perturbing Hamiltonians P can be constructed by taking sums of functions of the form $\kappa \circ H$, where H is a compatible Hamiltonian and $\kappa \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nondecreasing function which is eventually constant.

Let $\mathcal{X}(H_t)$ be the space of 1-periodic orbits of H_t. Because H_t is nondegenerate, this is a discrete space. The symplectic cochain complex $SC^*(M)$ is the graded $\mathbb{K}$-vector space generated by $\mathcal{X}(H_t)$ with grading given by the cohomological Conley-Zehnder index, see [2]. For this situation, we switch to a different convention for almost complex structures. Namely, we follow Ganatra [14] and define the space of almost complex structures of **rescaled contact type** $\mathcal{J}^{S^1}_{resc}(M, H_t)$. This consists of S^1-families of almost complex structures J_t which are $\hat{\omega}_M$-compatible and satisfy the following three conditions. First, there is some

t-independent constant $c_{resc} > 0$ such that, for all t,

$$d\widetilde{H} \circ J_t = -c_{resc}\hat{\lambda}_M \tag{4.1.4}$$

outside of a t-independent compact set. Second, the restriction $J_t|_{\ker dH \cap \ker \hat{\lambda}_M}$ is asymptotically $\hat{Z}_M$-invariant. Third, for each stop $\sigma \in \boldsymbol{\sigma}$, the projection to $\mathbb{H}_\rho$ is holomorphic along D_σ. Given $J_t \in \mathcal{J}^{S^1}_{resc}(M, H_t)$, we can consider maps $u \colon \mathbb{R} \times S^1 \to \hat{M}$ satisfying Floer's equation

$$\partial_s u + J_t(\partial_t u - X_{H_t}) = 0$$

and asymptotic as $s \to \pm\infty$ to orbits $x_\pm \in \mathfrak{X}(H_t)$. The moduli space of such maps, denoted $\widetilde{\mathcal{Q}}(x_+, x_-)$, satisfies the obvious analog of Lemmas 3.1.3 and 3.1.4, i.e.

Lemma 4.1.1. *For generic choices of P_t, there is a comeager subset*

$$\mathcal{J}^{S^1}_{reg}(M, H_t) \subset \mathcal{J}^{S^1}_{resc}(M, H_t)$$

such that, for any $J_t \in \mathcal{J}^{S^1}_{reg}(M, H_t)$ and $x_\pm \in \mathfrak{X}(H_t)$, the following hold.

1. $\widetilde{\mathcal{Q}}(x_+; x_-)$ *is a smooth manifold of dimension* $\deg(x_-) - \deg(x_+)$, *and the translation* $\mathbb{R}$-*action on* $\widetilde{\mathcal{Q}}(x_+; x_-)$ *is free if and only if* $x_+ \neq x_-$. *In this case, write* $\mathcal{Q}(x_+; x_-)$ *for the quotient* $\widetilde{\mathcal{Q}}(x_+; x_-)/\mathbb{R}$.

2. *If* $\deg(x_-) - \deg(x_+) = 1$, *then* $\mathcal{Q}(x_+; x_-)$ *is compact.*

3. *If* $\deg(x_-) - \deg(x_+) = 2$, *then* $\mathcal{Q}(x_+; x_-)$ *admits a Gromov compactification as a topological 1-manifold with boundary, and its boundary is in natural bijection with* $\coprod_{y \in \mathfrak{X}(H_t)} \mathcal{Q}(y; x_-) \times \mathcal{Q}(x_+; y)$.

Remark on proof. Because (4.1.4) is so stringent, we allow small perturbations of P_t. These can be made without changing the set $\mathfrak{X}(H_t)$, and in concert with the freedom to perturb c_{resc} they allow us to achieve transversality even when $\dim M = 2$. This was not an issue for chords because, when $\dim M = 2$, all chords outside of a compact set in a given end live in different relative homotopy classes.

With regards to compactness, our maximum principle does not apply in the presence of a time-dependent perturbing Hamiltonian, but by choosing a symplectization coordinate $r = \sqrt{\widetilde{H}}$ we find ourselves in Ganatra's setup and can apply Theorem A.1 of [14]. $\square$

Fix $J_t \in \mathcal{J}^{S^1}_{\mathrm{reg}}(M, H_t)$. The differential ∂ on $SC^*(M)$ is given by

$$\partial x_+ = \sum_{\deg(x_-) - \deg(x_+) = 1} \#\mathcal{Q}(x_+; x_-) \cdot x_-$$

and satisfies $\partial^2 = 0$ by the usual argument which looks at ends of 1-dimensional moduli spaces. The cohomology $SH^*(M) := H^*(SC^*(M), \partial)$ is known as symplectic cohomology.

The pumpkin structure $\boldsymbol{\sigma}$ endows $SC^*(M)$ with a filtration similar to that for open strings but slightly more subtle due to the fact that orbits can live on the divisor of a stop. We describe a part of it, which will suffice for our purposes. Let $\mathcal{X}_{\boldsymbol{\sigma}}(H_t) \subset \mathcal{X}(H_t)$ be the set of orbits which do not intersect $\sigma(\hat{F} \times \mathbb{R}_+)$ and do not live in D_σ for any $\sigma \in \boldsymbol{\sigma}$. Let $SC^*_{\boldsymbol{\sigma}}(M) \subset SC^*(M)$ be the graded linear subspace generated by $\mathcal{X}_{\boldsymbol{\sigma}}(H_t)$.

Lemma 4.1.2. *$SC^*_{\boldsymbol{\sigma}}(M)$ is a subcomplex of $SC^*(M)$.*

Proof. We need to show that if x_- intersects $\sigma(\hat{F} \times \mathbb{R}_+)$ or lives in D_σ for some $\sigma \in \boldsymbol{\sigma}$, then $\widetilde{Q}(x_+; x_-)$ is empty for any $x_+ \in \mathcal{X}(H_t)$. In the first case, the conclusion follows from positivity of intersections as in Lemma 3.1.5. In the second, the asymptotics in [24], combined with assumption (4.1.3), ensure that x_- behaves as if it had strictly positive winding number around D_σ. This puts us back in the regime where we can use positivity of intersections. □

Definition 4.1.3. $SC^*_{\boldsymbol{\sigma}}(M)$ is called the **partially wrapped symplectic cochain complex**.

We will be interested in holomorphic curves which interpolate between the open and the closed string worlds. For this, we make the following definitions

Definition 4.1.4. A **punctured Riemann surface with boundary, ends, and cylinders** is a Riemann surface

$$\Sigma = \overline{\Sigma} \setminus (Z_{\partial\Sigma} \cup Z_\Sigma),$$

where $\overline{\Sigma}$ is a compact Riemann surface with boundary, $Z_{\partial\Sigma}$ is a finite subset of the boundary of $\overline{\Sigma}$, and Z_Σ is a finite subset of the interior of $\overline{\Sigma}$, together with the following additional data.

1. For each $\zeta \in Z_{\partial\Sigma}$, a positive or negative strip-like end at ζ.
2. For each $\zeta \in Z_\Sigma$, a positive or negative cylindrical end at ζ. These are holomorphic embeddings
$$\epsilon_+ \colon \mathbb{R}_{\geq 0} \times S^1 \to \Sigma \qquad \text{or} \qquad \epsilon_- \colon \mathbb{R}_{\leq 0} \times S^1 \to \Sigma, \tag{4.1.5}$$
respectively, such that $\lim_{s \to \pm\infty} \epsilon_\pm(s,t) = \zeta$.
3. A finite number of **finite cylinders** δ_i. These are holomorphic embeddings
$$\delta_i \colon [a_i, b_i] \times S^1 \to \operatorname{int}(\Sigma).$$

Additionally, we require that all ends and finite cylinders have disjoint images. For cylindrical ends and finite cylinders, we define their m**-shifts** as with strips and define the **thin part** of Σ to be the union of the 3-shifts of all ends, finite cylinders, and, if Σ comes with an implicit gluing decomposition, finite strip-like gluing regions.

A **punctured Riemann surface with labeled boundary, ends, and cylinders** is a punctured Riemann surface Σ with boundary, ends, and cylinders, along with an assignment of a Lagrangian $L_i \subset \hat{M}$ to each boundary component $\partial_i\Sigma$ of Σ.

Definition 4.1.5. Let Σ be a punctured Riemann surface with labeled boundary, ends, and cylinders. A **Floer datum** on Σ is a 5-tuple (β, H, P, J, τ_E), where

- β is a 1-form on Σ
- H is a Σ-parametrized compatible Hamiltonian
- P is a function $P \colon \Sigma \times \hat{M} \to \mathbb{R}_+$
- J is a Σ-parametrized $\hat{\omega}_M$-compatible almost complex structure
- τ_E is a function $\tau_E \colon \partial\Sigma \to \mathbb{R}_+$

with the following properties.

1. Outside the images of the cylindrical ends and finite cylinders, (β, H, J, τ_E) satisfy the conditions of Definition 3.2.2.
2. $d\beta$, $d^\Sigma H \wedge \beta$, and $d^\Sigma P \wedge \beta$ are nonpositive everywhere.
3. For each cylindrical end ϵ_i, $(\epsilon_i^1)^*\beta = w_i dt$ for some positive real number w_i. Similarly, for each finite cylinder δ_i, $(\delta_i^1)^*\beta = w_i dt$ for some positive real number w_i.
4. For each cylindrical end or finite cylinder, there is a scaling constant $\tau_i > 0$ such that
$$w_i H = \widetilde{H}_{\tau_i}$$
on the image of that cylindrical end or finite cylinder.
5. There is some strictly positive function $g \colon \Sigma \times \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$dH \circ J = -g(H)\hat{\lambda}_M$$
outside a Σ-independent compact set.
6. The restriction $J|_{\ker dH \cap \ker \hat{\lambda}_M}$ is asymptotically $\hat{Z}_M$-invariant.
7. For each stop $\sigma \in \boldsymbol{\sigma}$, the projection to $\mathbb{H}_\rho$ is holomorphic along D_σ.
8. For each cylindrical end or finite cylinder,
$$J(s,t) = (J_t)_{\tau_i} := (\phi^{\tau_i})^* J_t$$
in the 2-shift of that cylindrical end or finite cylinder.

9. P is globally bounded, and $\|X_P\|$ decays exponentially in the symplectization coordinate. Moreover, P is locally constant outside the 2-shifts of the cylindrical ends and finite cylinders, and in the 3-shifts of the cylindrical ends and finite cylinders it satisfies

$$w_i P(s,t) = (P_t)_{\tau_i} + A_i := \frac{1}{\tau_i}(\phi^{\tau_i})^* P_t + A_i$$

for some constant A_i depending on the cylindrical region.

A Floer datum for a punctured Riemann surface Σ with boundary, ends, cylinders, but no Lagrangian labels, consists of a Floer datum for each Lagrangian labeling of Σ.

For such Floer data, we again have a notion of conformal equivalence. Two Floer data (β, H, P, J, τ_E) and $(\beta', H', P', J', \tau'_E)$ are **conformally equivalent** if there are constants A, C, W with $C, W > 0$ such that

$$\beta = W\beta', \quad H = \frac{1}{W}(H')_C, \quad P = \frac{1}{W}(P')_C + A, \quad J = (J')_C, \quad \tau_E = C\tau'_E.$$

As before, solutions $u\colon \Sigma \to \hat{M}$ to

$$J \circ (du - X_{H+P} \otimes \beta) = (du - X_{H+P} \otimes \beta) \circ j \tag{4.1.6}$$

with boundary conditions $u(\partial_i E) \subset (\phi^{\tau_E})^* L_i$ are related to solutions $u'\colon \Sigma \to \hat{M}$ to

$$J' \circ (du' - X_{H'+P'} \otimes \beta') = (du' - X_{H'+P'} \otimes \beta') \circ j \tag{4.1.7}$$

with boundary conditions $u'(\partial_i E) \subset (\phi^{\tau'_E})^* L_i$ via Liouville pullback.

4.2 Nondegenerate stops

To make the statement of Theorem 1.3.1 precise, we need to introduce a few notions. To begin, we need the following.

Definition 4.2.1. Let $(M, \lambda_M, \boldsymbol{\sigma})$ be a pumpkin domain, and let $\sigma \in \boldsymbol{\sigma}$ be a stop with fiber F. Note that interior Lagrangians of M and ΣF give rise to interior Lagrangians of $M[\sigma]$ via inclusion. An interior Lagrangian L in $M[\sigma]$ is said to be **supported in** σ if it is isomorphic in $\mathcal{W}_{\boldsymbol{\sigma}}(M[\sigma])$ to an interior Lagrangian of ΣF. Let $\mathcal{B}(\sigma) \subset \mathcal{W}(M[\sigma])$ and $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma) \subset \mathcal{W}_{\boldsymbol{\sigma}}(M[\sigma])$ denote the full subcategories of objects supported in σ. More generally, for a subset $\boldsymbol{\sigma}' \subset \boldsymbol{\sigma}$, let $\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma)$ denote the full subcategory of $\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])$ composed of objects supported in σ.

Although we will generally work in $M[\sigma]$ when dealing with a given stop σ, we will want to state results in M alone. In this case, we abuse notation and denote again by $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma) \subset \mathcal{W}_{\boldsymbol{\sigma}}(M)$ the full subcategory of objects whose image under the quasi-equivalence $\mathcal{W}_{\boldsymbol{\sigma}}(M) \to \mathcal{W}_{\boldsymbol{\sigma}}(M[\sigma])$ lie in $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$.

Note that $\mathcal{B}(\sigma)$ is a contractible subcategory, in the sense that for all $L \in \mathcal{B}(\sigma)$, the unit $e_L \in \hom^1_{\mathcal{W}(M)}(L, L)$ is exact. To see this, assume $L \subset M[\sigma]$ is an interior Lagrangian of ΣF. Then that L can be isotoped clockwise through D_σ to a Lagrangian L' such that L has no chords to L' of small action. This means, by an energy argument, that the isomorphism from L to L' given by Lemma 3.3.7 is the zero morphism, which implies that e_L is exact. Hence, the same holds for any object isomorphic to L. More generally, for $\boldsymbol{\sigma}' \subset \boldsymbol{\sigma}$ not containing σ, $\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma)$ is contractible.

Let $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \setminus \{\sigma\}$. By the universal property of a quotient category [8][19], since the image of $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$ is contractible, the inclusion $\mathcal{W}_{\boldsymbol{\sigma}}(M) \to \mathcal{W}_{\boldsymbol{\sigma}'}(M)$ factors up to homotopy through the quotient $\mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$:

$$\begin{tikzcd} & \mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma) \arrow[dr, dashed, "\mathcal{SR}"] & \\ \mathcal{W}_{\boldsymbol{\sigma}}(M) \arrow[ur] \arrow[rr] & & \mathcal{W}_{\boldsymbol{\sigma}'}(M) \end{tikzcd} \tag{4.2.1}$$

The second notion we need is that of a nondegenerate stop. This is a statement about the image of a certain open-closed string map, which we now describe. The Hochschild homology of $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$ can be given as the homology of a chain complex

$$CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)) = \bigoplus_{d=1}^{\infty} \mathbb{K}\gamma_d \otimes \cdots \otimes \gamma_1 \tag{4.2.2}$$

where $\gamma_i \in \hom(L_i, L_{i+1})$ is a cyclically composable sequence of morphisms in $\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$. The grading is cohomological and is given by

$$\deg(\gamma_d \otimes \cdots \otimes \gamma_1) = \sum_{i=1}^{d} \deg(\gamma_i) + 1 - d.$$

The differential $\delta\colon CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)) \to CC_{*+1}(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ comes from the A_∞ structure on $\mathcal{B}(\sigma)$, namely

$$\begin{aligned} \delta(\gamma_d \otimes \cdots \otimes \gamma_1) = & \sum_{\substack{i,j \geq 0 \\ i+j<d}} \mu^{i+j+1}(\gamma_i, \ldots, \gamma_{d-j}) \otimes \gamma_{d-j-1} \otimes \cdots \otimes \gamma_{i+1} \\ & + \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \gamma_d \otimes \ldots \otimes \gamma_{i+j} \otimes \mu^j(\gamma_{i+j-1}, \ldots, \gamma_i) \otimes \gamma_{i-1} \otimes \cdots \otimes \gamma_1 \end{aligned}$$

where $\gamma_0 := \gamma_d$.

The open-closed string map $\mathcal{OC}\colon CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)) \to SC^*(M[\sigma])$ counts holomorphic annuli, as described in [1]. Following Abouzaid, let $\mathcal{R}^1_d$ be the space of disks with one interior puncture and $d \geq 1$ boundary punctures, one of which is distinguished. For $\Sigma \in \mathcal{R}^1_d$, label the interior puncture by ζ_- and the boundary punctures ζ_1 through ζ_d, ordered counterclockwise, with the distinguished puncture labeled ζ_d. $\mathcal{R}^1_d$ has a natural compactification to a manifold with corners $\overline{\mathcal{R}}^1_d$ whose codimension one faces can be canonically identified with

$$\coprod_{\substack{2\leq k\leq d\\ 1\leq i\leq k}} \overline{\mathcal{R}}^1_{d+1-k} \times \overline{\mathcal{R}}^{k+1,i} \quad \amalg \coprod_{\substack{2\leq k\leq d-1\\ 1\leq i\leq d-k}} \overline{\mathcal{R}}^1_{d+1-k} \times \overline{\mathcal{R}}^{k+1}. \tag{4.2.3}$$

Here, $\overline{\mathcal{R}}^{k+1,i}$ is diffeomorphic to $\overline{\mathcal{R}}^{k+1}$, but if $\Sigma^d \in \overline{\mathcal{R}}^{k+1}$, then the corresponding point of $\overline{\mathcal{R}}^{k+1,i}$ is Σ^d with the additional datum that $\zeta_i \in \Sigma^d$ is distinguished. In other words, it is the space of disks with one negative puncture, d positive punctures, and such that the ith puncture is considered special. The first term in (4.2.3) corresponds then to a collection of punctures which includes ζ_d colliding, while the second corresponds to some other collection colliding. In this case, the additional index i keeps track of where the collision occurred.

A collection of ends for $\Sigma \in \mathcal{R}^1_d$, making it into a punctured Riemann surface with boundary, ends, and cylinders, consists of a positive strip-like end ϵ_i for each boundary puncture ζ_i, along with a negative cylindrical end ϵ_- at ζ_-. In this case, we ask that ϵ_- has a very special form. Specifically, in the holomorphic coordinates on Σ where $\mathrm{int}(\Sigma) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and $\zeta_d = 1$, we require that

$$\epsilon_-(s,t) = ae^{2\pi(s+it)} \qquad \text{with } a \in \mathbb{R} \text{ positive.} \tag{4.2.4}$$

for some positive number $a \in \mathbb{R}$. A **universal family of ends** for $\mathcal{R}^1_d$ consists of a collection of ends on each $\Sigma \in \mathcal{R}^1_d$ for every d, such that near the boundary of $\overline{\mathcal{R}}^1_d$ it agrees up to a rotation of ϵ_- with the collection induced by gluing. This rotation correction is unavoidable, since boundary components $\overline{\mathcal{R}}^1_{d+1-k} \times \overline{\mathcal{R}}^{k+1,i}$ have the same ends for all i, so that without rotation at most one could glue to a configuration which satisfies (4.2.4). However, because we are using an exponential gluing profile the magnitude of the rotation vanishes to infinite order at the boundary, and hence the family of strip-like ends extends smoothly to $\overline{\mathcal{R}}^1_d$. One sees as with $\mathcal{R}^{d+1}$ that universal families of ends for $\mathcal{R}^1_d$ exist, and we fix one once and for all.

A **universal and conformally consistent** choice of Floer data for $\mathcal{R}^1_d$ consists of, for all $d \geq 1$, a Floer datum (β, H, P, J, τ_E) for each $\Sigma \in \mathcal{R}^1_d$ varying smoothly over $\mathcal{R}^1_d$, and such that near $\partial\overline{\mathcal{R}}^1_d$ it agrees to infinite order with the conformal class of not-quite Floer datum determined by gluing. We say not-quite due to the rotation corrections for the strip-like ends, which among other things cause the glued datum to not be a Floer datum in the above sense. Denote by $\mathcal{K}^{\mathcal{OC}}(M[\sigma])$ the space of universal and conformally consistent choices of Floer data for $\mathcal{R}^1_d$.

Given $\mathbf{K} \subset \mathcal{K}^{\mathcal{OC}}(M[\sigma])$, we can consider the resulting spaces of holomorphic curves. Given a collection of Lagrangian labels L_i and asymptotic ends

$$\gamma_i \in \mathfrak{X}(L_i, L_{i+1}) \quad \text{ and } \quad x_- \in \mathfrak{X}(H_t),$$

we are interested in the space

$$\mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; x_-).$$

This consists of all maps $u\colon \Sigma \to \widehat{M[\sigma]}$ for $\Sigma \in \mathcal{R}^1_d$ satisfying (4.1.6) with $u(E_i) \subset (\phi^{\tau_E})^* L_i$, $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$, and $u(\zeta_-) = (\phi^{\tau_-})^* x_-$.

Lemma 4.2.2. *There is a dense subset $\mathcal{K}^{\mathcal{OC}}_{reg}(M[\sigma]) \subset \mathcal{K}^{\mathcal{OC}}(M[\sigma])$ such that, for every universal choice $\mathbf{K} \in \mathcal{K}^{\mathcal{OC}}_{reg}(M[\sigma])$, the following hold.*

1. *For any $d \geq 1$, any sequence of Lagrangians $L_1, \dots, L_d$, and any collection of chords $\gamma_i \in \mathcal{X}(L_i, L_{i+1})$ and orbit $\gamma_- \in \mathcal{X}(H_t)$, $\mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; \gamma_-)$ is a smooth manifold of dimension $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) + d - n - 1$. Here, n is half the dimension of M.*

2. *If $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) = n + 1 - d$, then $\mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; x_-)$ is compact.*

3. *If $\deg(x_-) - \sum_{i=1}^d \deg(\gamma_i) = n + 2 - d$, then $\mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; x_-)$ admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with*

$$\begin{aligned}
&\coprod_{\substack{2\le k\le d\\ 1\le i\le k\\ \widetilde{\gamma}\in\mathcal{X}(L_{d+1-i},L_{k+1-i})}} \mathcal{R}^1_{d+1-k}(\widetilde{\gamma}, \gamma_{d-i}, \dots, \gamma_{1+k-i}; x_-) \times \mathcal{R}^{k+1}(\gamma_{k-i}, \dots, \gamma_{d+1-i}; \widetilde{\gamma}) \\
&\amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_d,L_1)} \mathcal{R}^1_d(\widetilde{\gamma}, \gamma_{d-1}, \dots, \gamma_1; x_-) \times \mathcal{R}(\gamma_d; \widetilde{\gamma}) \\
&\amalg \coprod_{\substack{2\le k\le d-1\\ 1\le i\le d-k\\ \widetilde{\gamma}\in\mathcal{X}(L_i,L_{i+k})}} \mathcal{R}^1_{d+1-k}(\gamma_d, \dots, \gamma_{i+k}, \widetilde{\gamma}, \gamma_{i-1} \dots, \gamma_1; x_-) \times \mathcal{R}^{k+1}(\gamma_{i+k-1}, \dots, \gamma_i; \widetilde{\gamma}) \\
&\amalg \coprod_{\substack{1\le i<d\\ \widetilde{\gamma}\in\mathcal{X}(L_i,L_{i+1})}} \mathcal{R}^1_d(\gamma_d, \dots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1} \dots, \gamma_1; x_-) \times \mathcal{R}(\gamma_i; \widetilde{\gamma}) \\
&\amalg \coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{Q}(\widetilde{x}; x_-) \times \mathcal{R}^1_d(\gamma_d, \dots, \gamma_1; \widetilde{x}).
\end{aligned} \tag{4.2.5}$$

Remark on proof. As with the Floer cylinders, the C^0 estimates in [14] give a bound on how far elements $u \in \mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; \gamma_-)$ can protrude into the symplectization in the cylindrical ends. Everywhere else, we can apply Lemma A.2.2. Together, these show that the image of any u is constrained to lie in a compact set depending only on $\mathbf{K}_\Delta$ and the ends γ_i and γ_-. ☐

Define $\mathcal{OC}\colon CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)) \to SC^{*+n}(M[\sigma])$ by

$$\mathcal{OC}(\gamma_d \otimes \cdots \otimes \gamma_1) = \sum_{\substack{x\in\mathcal{X}(H_t)\\ \deg(x)=\sum_{i=1}^d \deg(\gamma_i)+n+1-d}} \#\mathcal{R}^1_d(\gamma_d, \dots, \gamma_i; x) \cdot x.$$

The boundary strata in (4.2.5) tell us that $\mathcal{OC}$ is a chain map. Further, arguing as in Lemma 4.1.2 gives

Lemma 4.2.3. *The image of $\mathcal{OC}$ lies in $SC^*_{\boldsymbol{\sigma}}(M[\sigma])$.* □

Definition 4.2.4. A stop $\sigma \in \boldsymbol{\sigma}$ is **nondegenerate** if there is a closed Hochschild chain $y \in CC_{1-n}(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ such that $\mathcal{OC}(y) = f_\sigma$, where $f_\sigma \in SC^1_{\boldsymbol{\sigma}}(M[\sigma])$ is a **saddle unit** of σ as described below.

Morally speaking, a saddle unit is any cocycle which lives in the central fiber of $\Sigma F \subset M[\sigma]$ and represents the unit of $SH^*(F)$ when thought of as a chain in $SC^0(F)$. The drop in degree from 1 to 0 comes from the fact that the central fiber lives at a saddle point of the Liouville vector field for $\mathbb{C}_1$, which translates to an index 1 Morse critical point for nice choices of compatible Hamiltonian. However, such a cocycle is often exact in $SC^*_{\boldsymbol{\sigma}}(M[\sigma])$, so the careful definition of f_σ instead involves a count of holomorphic caps.

Concretely, let Σ be $\mathbb{C}$ equipped with the negative cylindrical end ϵ^f asymptotic to ∞ given by

$$\epsilon^f(s,t) = e^{-2\pi(s+it)}$$

Let $\mathcal{K}^{\mathbb{C}}(M[\sigma])$ denote the space of Floer data on Σ. Given a Floer datum $K^f \in \mathcal{K}^{\mathbb{C}}(M[\sigma])$ and an orbit $x \in \mathcal{X}(H_t)$, we are interested in the resulting moduli space space $\mathcal{C}(x)$. This is the space of all maps $u\colon \Sigma \to \widehat{M[\sigma]}$ satisfying (4.1.6) and

$$\lim_{s\to-\infty} u(\epsilon^f(s,t)) = (\phi^\tau)^* x(t),$$

where τ is the conformal factor K^f assigns to $\infty \in \overline{\Sigma}$, and for which

$$u(0) \in Y_\sigma,$$

where $Y_\sigma \subset \hat{M}[\sigma]$ is the hypersurface which comes from $\hat{F} \times i\mathbb{R} \subset \Sigma F$. The last condition is the interesting one. Indeed, that is the only place where the stop σ comes into the definition of f_σ, and without it we would just obtain the unit of symplectic cohomology.

Lemma 4.2.5. *There is a comeager subset $\mathcal{K}^{\mathbb{C}}_{reg}(M[\sigma]) \subset \mathcal{K}^{\mathbb{C}}(M[\sigma])$ such that for any $K^f \in \mathcal{K}^{\mathbb{C}}_{reg}(M[\sigma])$, the following hold.*

1. *For all $x \in \mathcal{X}(H_t)$, $\mathcal{C}(x)$ is a smooth manifold of dimension* $\deg(x) - 1$.
2. *If* $\deg(x) = 1$, *then $\mathcal{C}(x)$ is compact.*
3. *If* $\deg(x) = 2$, *then $\mathcal{C}(x)$ has a Gromov compactification $\overline{\mathcal{C}}(x)$ which is a compact topological 1-manifold with boundary, and there is a canonical identification*

$$\partial\overline{\mathcal{C}}(x) = \coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{Q}(\widetilde{x};x) \times \mathcal{C}(\widetilde{x}).$$

In this case, $\widetilde{x}$ necessarily has degree 1.

□

Definition 4.2.6. A **saddle unit** of σ is any chain

$$f_\sigma = \sum_{\substack{x \in \mathcal{X}(H_t) \\ \deg(x)=1}} \#\mathcal{C}(x) \cdot x$$

obtained from a Floer datum $K^f \subset \mathcal{K}^{\mathbb{C}}_{\mathrm{reg}}(M[\sigma])$. It follows from Lemma 4.2.5 and positivity of intersections that such a chain is in fact a closed element of $SC^1_{\boldsymbol{\sigma}}(M[\sigma])$.

We restate Theorem 1.3.1, which is now technically precise.

Theorem 4.2.7. *Let $(M, \lambda_M, \boldsymbol{\sigma})$ be a pumpkin domain, and let $\sigma \in \boldsymbol{\sigma}$ be a nondegenerate stop. Set $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \setminus \{\sigma\}$. Then the map $\mathcal{SR}\colon \mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma) \to \mathcal{W}_{\boldsymbol{\sigma}'}(M)$ from* (4.2.1) *is fully faithful.*

4.3 A filtration on the quotient category

To prove Theorem 4.2.7, we will work in Lyubashenko–Ovsienko's model for the quotient of an A_∞-category by a full subcategory [20], which is the A_∞ version of Drinfeld's construction for dg-categories [8]. For an A_∞-category $\mathcal{A}$ and a full subcategory $\mathcal{B} \subset \mathcal{A}$, the quotient $\mathcal{A}/\mathcal{B}$ is the A_∞-category with the same objects as $\mathcal{A}$ and whose morphism spaces are given by

$$\hom_{\mathcal{A}/\mathcal{B}}(L_0, L_1) = \bigoplus_{k=0}^{\infty} \bigoplus_{B_i \in \mathcal{B}} \hom_{\mathcal{A}}(B_k, L_1) \otimes \hom_{\mathcal{A}}(B_{k-1}, B_k) \otimes \cdots \otimes \hom_{\mathcal{A}}(L_0, B_1),$$

where for $k = 0$ the right-hand side is just $\hom_{\mathcal{A}}(L_0, L_1)$. The grading is given by

$$\deg(\gamma^k \otimes \cdots \otimes \gamma^0) = \sum_{i=0}^{k} \deg(\gamma^i) - k.$$

The differential $\mu^1_{\mathcal{A}/\mathcal{B}}$ is the bar differential, i.e.

$$\mu^1_{\mathcal{A}/\mathcal{B}}(\gamma^k \otimes \cdots \otimes \gamma^0) = \sum_{0 \le i \le j \le k} \gamma^k \otimes \cdots \otimes \gamma^{j+1} \otimes \mu^{1+j-i}(\gamma^j, \dots, \gamma^i) \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0.$$

The higher operations are similar. Specifically, we have

$$\begin{aligned} &\mu^d_{\mathcal{A}/\mathcal{B}}\Big((\gamma_d^{k_d} \otimes \cdots \otimes \gamma_d^0), \dots, (\gamma_1^{k_1} \otimes \cdots \otimes \gamma_1^0)\Big) \\ &\qquad = \sum_{\substack{0 \le i \le k_1 \\ 0 \le j \le k_d}} \gamma_d^{k_d} \otimes \cdots \otimes \mu^{i+j+d+\sum_{s=2}^{d-1} k_s}(\gamma_d^j, \dots, \gamma_1^{k_1 - i}) \otimes \cdots \otimes \gamma_1^0. \end{aligned}$$

In this model, $\mathcal{W}_{\boldsymbol{\sigma}}(M)/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$ is naturally a subcategory of $\mathcal{W}_{\boldsymbol{\sigma}'}(M)/\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma)$, and $\mathcal{SR}$ is just the inclusion.

Further, we can work in $M[\sigma]$, so in fact we will study the inclusion

$$\mathcal{S}R_{inc}\colon \mathcal{W}_{\boldsymbol{\sigma}}(M[\sigma])/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma) \hookrightarrow \mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])/\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma). \tag{4.3.1}$$

For the sake of readability, we will write $\hom_{\boldsymbol{\sigma}}$ in place of $\hom_{\mathcal{W}_{\boldsymbol{\sigma}}(M[\sigma])/\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)}$ and $\hom_{\boldsymbol{\sigma}'}$ in place of $\hom_{\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])/\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma)}$. Similarly, we'll write $\mu^k_{\boldsymbol{\sigma}'}$ and $\mu^k_{\boldsymbol{\sigma}}$ for the A_∞ operations on the quotient categories.

Theorem 4.2.7 is equivalent to the assertion that $\mathcal{SR}_{inc}$ is fully faithful, which follows from the following statement:

Proposition 4.3.1. *Assume σ is nondegenerate, and let L_0 and L_1 be interior Lagrangians in $M[\sigma]$. Then there is a retraction*

$$R\colon\ \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^*_{\boldsymbol{\sigma}}(L_0, L_1)$$

with the following property:

> *For any finite dimensional subcomplex $C \subset \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$, there is a chain map from $\hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$ to itself which is homotopic to the identity and agrees with R on C.* (4.3.2)

As a topological analogy, this says that R is a deformation retraction on compact subsets. In particular, it is a quasi-isomorphism.

To construct R and prove that it satisfies (4.3.2), we will build an increasing filtration on the morphism spaces of $\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])/\mathcal{B}_{\boldsymbol{\sigma}'}(\sigma)$ and a homotopy that moves us down in the filtration.

Definition 4.3.2. Consider the lexicographic order on $\mathbb{N}^2$, namely $(n, m) < (n', m')$ if $n < n'$ or $n = n'$ and $m < m'$. This has order type ω^2, so in particular it is a well-ordering.

Define $A^*_{n,m} \subset \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$ to be the graded vector subspace generated by

$$\left\{ \gamma^k \otimes \cdots \otimes \gamma^0 \;\middle|\; \left(\sum_{i=0}^{k} n_\sigma(\gamma^i), k\right) < (n, m) \right\}.$$

Then $A^*_{n,m}$ is a subcomplex, and $A^*_{n,m} \subset A^*_{n',m'}$ whenever $(n, m) < (n', m')$. Note that $A^*_{1,0} = \hom^*_{\boldsymbol{\sigma}}(L_0, L_1)$, and that the $A^*_{n,m}$ form an exhausting filtration of $\hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$, which we call the **main filtration**.

To prove Proposition 4.3.1, we will construct a map of graded vector spaces

$$\Delta_y\colon\ \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*-1}_{\boldsymbol{\sigma}'}(L_0, L_1)$$

such that

$$R_y := \mathrm{id} + \mu^1_{\boldsymbol{\sigma}'}\Delta_y + \Delta_y\mu^1_{\boldsymbol{\sigma}'} \tag{4.3.3}$$

is the identity on $A^*_{1,0} = \hom^*_{\boldsymbol{\sigma}}(L_0, L_1)$ and strictly decreases the filtration on $A^*_{n,m}$ for $(n, m) > (1, 0)$. Since the filtration is well-ordered, the sequence

$$\gamma, R_y(\gamma), R_y(R_y(\gamma)), \ldots$$

stabilizes, and hence the infinite iterate

$$R^\infty_y \colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$$

is well defined.

Lemma 4.3.3. *For an element $\gamma \in \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$, let $(n, m)(\gamma)$ be the smallest pair for which $\gamma \in A^*_{m,n}$. Then for any R_y such that*

1. *$R_y(\gamma) = \gamma$ for $\gamma \in A^*_{1,0}$, and*
2. *$(n, m)(R_y(\gamma)) < (n, m)(\gamma)$ whenever $(n, m)_\gamma > (1, 0)$,*

R^∞_y satisfies the conditions on R in Proposition 4.3.1.

Proof. Since the iterates of R_y stabilize for any element, they stabilize after finitely many steps on any finite dimensional subspace. Thus, for any finite dimensional subcomplex C, there is some n such that R^n_y agrees with R^∞_y on C. Since R_y is homotopic to the identity, so is the finite iterate R^n_y. □

4.4 Coproduct disks

The map Δ_y giving rise to R_y will be given by a certain coproduct operation, which will come as always from counts of holomorphic disks. We describe these now.

Definition 4.4.1. For nonnegative integers d, k, l,let $\mathcal{R}^{d;k,l}$ be the abstract **moduli space of coproduct disks**. These are disks Σ with $d + k + l + 3$ boundary punctures labeled in counterclockwise order as follows.

1. ζ_i, for i increasing from $-k$ to $+l$. These will eventually be equipped with positive strip-like ends.
2. ζ_a, which will eventually be equipped with a negative strip-like end.
3. ζ^j for j increasing from 1 to d. These will eventually be equipped with positive strip-like ends.
4. ζ_b, which will eventually be equipped with a negative strip-like end.

The punctures ζ_0, ζ_a, and ζ_b are considered **distinguished points**. Observe that any disk with $n+3$ punctures can made into an element of some $\mathcal{R}^{d;k,l}$ with $d+k+l=n$ by specifying and labeling three distinguished points. The compactified moduli space $\overline{\mathcal{R}}^{d;k,l}$ is diffeomorphic to the associahedron $\overline{\mathcal{R}}^{(d+k+l+2)+1}$, where the identification can be taken to match ζ_0 with ζ_0. The codimension r boundary faces of $\overline{\mathcal{R}}^{d;k,l}$ are identified with products of some lower-dimensional $\mathcal{R}^{d';k',l'}$ with r lower-dimensional associahedra inductively as follows.

For a codimension 1 face, a point corresponds to a pair of disks identified at new boundary punctures $\widetilde{\zeta}$, and we may look at the induced labels of boundary punctures on these two disks. One of these disks, which we call Σ_0, contains two or three distinguished points, while the other disk Σ_1, contains one or zero. In each case Σ_0 will be taken to lie in $\mathcal{R}^{d';k',l'}$, but there are several ways that this can happen.

1. The first possibility is that Σ_0 contains all three distinguished points. In this case it is identified with an element of $\mathcal{R}^{d';k',l'}$ by matching up the distinguished points. Σ_1 is identified with a point of $\mathcal{R}^{m+1}$ by taking $\widetilde{\zeta}$ to be the root.

2. The second possibility, similar to the first, is that Σ_0 contains ζ_a and ζ_b, while Σ_1 contains ζ_0. In this case Σ_0 is identified with the element of $\mathcal{R}^{d';k',l'}$ which has ζ_a and ζ_b in the same place and ζ_0 in the position of $\widetilde{\zeta}$. Σ_1 is again identified with a point of $\mathcal{R}^{m+1}$ by taking $\widetilde{\zeta}$ to be the root. By remembering the distinguished point $\zeta_0 \in \Sigma_1$, we may upgrade it to an element of $\mathcal{R}^{m+1,i}$ for some i.

3. The third and fourth possibilities are that Σ_1 contains ζ_a or ζ_b. We assume that $\zeta_a \in \Sigma_1$, as the other situation is strictly similar. In this case, for Σ_0, $\widetilde{\zeta}$ takes the place of ζ_a as the third distinguished point, while Σ_1 is identified with an element of $\mathcal{R}^{m+1}$ by setting ζ_a to be the root.

For a higher codimension face, we obtain a decomposition by following a sequence of faces, each of which has codimension 1 in the previous. To see that the decomposition is unique, note that an element Σ of the boundary of $\overline{\mathcal{R}}^{d;k,l}$ is a disk with boundary nodes described by a tree T. If at least two distinguished points of Σ live on the same component, then that component is the one identified with an element of $\mathcal{R}^{d';k',l'}$. Otherwise, there is a unique vertex v of T such that every path from v to a vertex containing a distinguished point leaves v along a different edge. The component of Σ identified with a point of $\mathcal{R}^{d';k',l'}$ is the one corresponding to v.

As indicated, a collection of strip-like ends for a disk $\Sigma \in \mathcal{R}^{d;k,l}$ consists of a positive strip-like end at each puncture ζ_i and ζ^j, along with a negative strip-like end at each of ζ_a and ζ_b, such that the images of the ends are pairwise disjoint. A **universal choice** of strip-like ends for $\mathcal{R}^{d;k,l}$ consists of, for all $d,k,l \geq 0$, a collection of strip-like ends for each $\Sigma \in \mathcal{R}^{d;k,l}$ which varies smoothly over $\mathcal{R}^{d;k,l}$ and agrees near the boundary with the collection of ends induced by gluing. As with associahedra, a universal choice of strip-like ends for $\mathcal{R}^{d;k,l}$ can be constructed inductively, and we fix one once and for all.

Similarly, a **universal and conformally consistent** choice of Floer data for $\mathcal{R}^{d;k,l}$ consists of, for all $d,k,l \geq 0$, a Floer datum for each $\Sigma \in \mathcal{R}^{d;k,l}$ which varies smoothly over $\mathcal{R}^{d;k,l}$, and which additionally satisfies the asymptotic consistency condition of Definition 3.2.4 with $\mathcal{R}^{d+1}$ replaced by $\mathcal{R}^{d;k,l}$. Let $\mathcal{K}_\Delta(M[\sigma])$ denote the space of all universal and conformally consistent choices Floer data for $\mathcal{R}^{d;k,l}$.

For any $\mathbf{K}_\Delta \in \mathcal{K}_\Delta(M[\sigma])$, we obtain a perturbed Cauchy-Riemann operator. Given Lagrangians $L_0, \ldots, L_d$ and $B_{-k-1}, \ldots, B_l$ and chords

$$\begin{aligned} \gamma_i &\in \mathcal{X}(B_{i-1}, B_i) & \gamma_a &\in \mathcal{X}(L_0, B_l) \\ \gamma^j &\in \mathcal{X}(L_{j-1}, L_j) & \gamma_b &\in \mathcal{X}(B_{-k-1}, L_d), \end{aligned} \tag{4.4.1}$$

we can consider the space $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$, where $\vec{\gamma}^\star$ and $\vec{\gamma}_\star$ are the tuples $(\gamma^d, \ldots, \gamma^1)$ and $(\gamma_l, \ldots, \gamma_{-k})$, respectively. This consists of all maps $u \colon \Sigma \to \widehat{M[\sigma]}$, with Σ ranging over $\mathcal{R}^{d;k,l}$, satisfying (3.2.12) with

$$\begin{aligned} u(\zeta_i) &= (\phi^{\tau_i})^* \gamma_i & u(\zeta_a) &= (\phi^{\tau_a})^* \gamma_a \\ u(\zeta^j) &= (\phi^{\tau^j})^* \gamma^j & u(\zeta_b) &= (\phi^{\tau_b})^* \gamma_b \end{aligned}$$

and with the appropriate boundary conditions, where τ_i is the rescaling factor assigned to ζ_i, and similarly with τ^j, τ_a, and τ_b. As usual, Lemma A.2.2 tells us that the images of such u are all contained in a fixed compact subset of $\hat{M}$, so Gromov compactness gives $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$ a natural compactification $\overline{\mathcal{R}}^{d;k,l}(\vec{\gamma}^\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$ whose new points are broken configurations consisting of one element of $\mathcal{R}^{d';k',l'}((\vec{\gamma}^\star)', (\vec{\gamma}_\star)'; \gamma_b', \gamma_a')$ for some $d' \leq d$, $k' \leq k$, and $l' \leq l$, along with disks contributing to the A_∞ structure. We list those configurations with exactly two nonconstant components.

$$\begin{aligned} &\mathcal{R}^{d;k+1-m,l}(\vec{\gamma}^\star, (\gamma_l, \ldots, \gamma_{i+m}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_{-k}); \gamma_b, \gamma_a) & &2 \leq m \leq -i \leq k \\ &\qquad \times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \ldots, \gamma_i; \widetilde{\gamma}) & &\widetilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned} \tag{4.4.2a}$$

$$\begin{aligned} &\mathcal{R}^{d;k,l+1-m}(\vec{\gamma}^\star, (\gamma_l, \ldots, \gamma_{i+m}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_{-k}); \gamma_b, \gamma_a) & &2 \leq m \leq 1+l-i \leq l \\ &\qquad \times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \ldots, \gamma_i; \widetilde{\gamma}) & &\widetilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned} \tag{4.4.2b}$$

$$\begin{aligned} &\mathcal{R}^{d;k+i,l-i+1-m}(\vec{\gamma}^\star, (\gamma_l, \ldots, \gamma_{i+m}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_{-k}); \gamma_b, \gamma_a) & &-k \leq i \leq 0 \\ &\qquad \times \mathcal{R}^{m+1}(\gamma_{i+m-1}, \ldots, \gamma_i; \widetilde{\gamma}) & &\min\{2, 1-i\} \leq m \leq l-i+1 \\ & & &\widetilde{\gamma} \in \mathcal{X}(B_{i-1}, B_{i+m-1}) \end{aligned} \tag{4.4.2c}$$

$$\begin{aligned} &\mathcal{R}^{d;k,l}(\vec{\gamma}^\star, (\gamma_l, \ldots, \gamma_{i+1}, \widetilde{\gamma}, \gamma_{i-1}, \ldots, \gamma_{-k}); \gamma_b, \gamma_a) & &-k \leq i \leq l \\ &\qquad \times \mathcal{R}(\gamma_i; \widetilde{\gamma}) & &\widetilde{\gamma} \in \mathcal{X}(B_{i-1}, B_i) \end{aligned} \tag{4.4.2d}$$

$$\begin{aligned} &\mathcal{R}^{d+1-m;k,l}((\gamma^d, \ldots, \gamma^{i+m}, \widetilde{\gamma}, \gamma^{i-1}, \ldots, \gamma^1), \vec{\gamma}_\star; \gamma_b, \gamma_a) & &2 \leq m \leq d+1-i \leq d \\ &\qquad \times \mathcal{R}^{m+1}(\gamma^{i+m-1}, \ldots, \gamma^i; \widetilde{\gamma}) & &\widetilde{\gamma} \in \mathcal{X}(L_{i-1}, L_{i+m-1}) \end{aligned} \tag{4.4.2e}$$

$$\begin{gathered}\mathcal{R}^{d;k,l}((\gamma^d,\ldots,\gamma^{i+1},\widetilde{\gamma},\gamma^{i-1},\ldots,\gamma^1),\vec{\gamma}_\star;\gamma_b,\gamma_a)\\ \times\mathcal{R}(\gamma^i;\widetilde{\gamma})\end{gathered} \qquad \begin{gathered}1\le i\le d\\ \widetilde{\gamma}\in\mathcal{X}(L_{i-1},L_i)\end{gathered} \tag{4.4.2f}$$

$$\begin{gathered}\mathcal{R}^{i+j+1+1}(\gamma_l,\ldots,\gamma_{l-j+1},\widetilde{\gamma},\gamma^i,\ldots,\gamma^1;\gamma_a)\\ \times\mathcal{R}^{d-i;k,l-j}((\gamma^d,\ldots,\gamma^{i+1}),(\gamma_{l-j},\ldots,\gamma_{-k});\gamma_b,\widetilde{\gamma})\end{gathered} \qquad \begin{gathered}0\le i\le d\\ \max\{0,1-i\}\le j\le l\\ \widetilde{\gamma}\in\mathcal{X}(L_i,B_{l-j})\end{gathered} \tag{4.4.2g}$$

$$\mathcal{R}(\widetilde{\gamma};\gamma_a)\times\mathcal{R}^{d;k,l}(\vec{\gamma}^\star,\vec{\gamma}_\star;\gamma_b,\widetilde{\gamma}) \qquad \widetilde{\gamma}\in\mathcal{X}(L_0,B_l) \tag{4.4.2h}$$

$$\begin{gathered}\mathcal{R}^{i+j+1+1}(\gamma^d,\ldots,\gamma^{d+1-i},\widetilde{\gamma},\gamma_{j-k-1},\ldots,\gamma_{-k};\gamma_b)\\ \times\mathcal{R}^{d-i;k-j,l}((\gamma^{d-i},\ldots,\gamma^1),(\gamma_l,\ldots,\gamma_{j-k});\widetilde{\gamma},\gamma_a)\end{gathered} \qquad \begin{gathered}0\le i\le d\\ \max\{0,1-i\}\le j\le k\\ \widetilde{\gamma}\in\mathcal{X}(B_{j-k-1},L_{d-i})\end{gathered} \tag{4.4.2i}$$

$$\mathcal{R}(\widetilde{\gamma};\gamma_b)\times\mathcal{R}^{d;k,l}(\vec{\gamma}^\star,\vec{\gamma}_\star;\widetilde{\gamma},\gamma_a) \qquad \widetilde{\gamma}\in\mathcal{X}(B_{-k-1},L_d) \tag{4.4.2j}$$

This looks like a lot, but the first four are just different ways that an A_∞ disk can break off on the "subscript" side, of which the first three differ only in the placement of the marked input. The others, in pairs, describe the possible breakings of an A_∞ disk on the "superscript" side, at ζ_a, and at ζ_b.

Lemma 4.4.2. *There is a dense subset $\mathcal{K}_{\Delta,reg}(M[\sigma])\subset\mathcal{K}_\Delta(M[\sigma])$ such that, for every $\mathbf{K}_\Delta\in\mathcal{K}_{\Delta,reg}(M[\sigma])$, the following hold.*

1. *For any Lagrangians $L_0,\ldots,L_d$ and $B_{-k-1},\ldots,B_l$ and any chords γ_j,γ^j, γ_a, and γ_b as in 4.4.1, $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star,\vec{\gamma}_\star;\gamma_b,\gamma_a)$ is a smooth manifold of dimension*

$$\deg(\gamma_a)+\deg(\gamma_b)-\sum_{r=d}^{d}\deg(\gamma^r)-\sum_{s=-k}^{l}\deg(\gamma_s)+d+k+l-n.$$

2. *Whenever*

$$n_\sigma(\gamma_a)+n_\sigma(\gamma_b)>\sum_{r=1}^{d}n_\sigma(\gamma^r)+\sum_{s=-k}^{l}n_\sigma(\gamma_s),$$

   $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star,\vec{\gamma}_\star;\gamma_b,\gamma_a)$ is empty.

3. *If*

$$\deg(\gamma_a)+\deg(\gamma_b)-\sum_{r=1}^{d}\deg(\gamma^r)-\sum_{s=-k}^{l}\deg(\gamma_s)=n-d-k-l,$$

   then $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star,\vec{\gamma}_\star;\gamma_b,\gamma_a)$ is compact.

4. *If*

$$\deg(\gamma_a)+\deg(\gamma_b)-\sum_{r=1}^{d}\deg(\gamma^r)-\sum_{s=-k}^{l}\deg(\gamma_s)=n-d-k-l+1,$$

then $\mathcal{R}^{d;k,l}(\vec{\gamma}^{\star}, \vec{\gamma}_{\star}; \gamma_b, \gamma_a)$ admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with the configurations (4.4.2).

□

4.5 The main homotopy

We are now prepared to begin constructing the operation Δ_y which is used in the definition of the basic retraction R_y in (4.3.3). Concretely, we will give a formula for Δ_v for $v = \gamma_q \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ and extend to all of $CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ by linearity. For now, we define a coproduct operation Δ_v^0. This is the **main homotopy**. Later on, we will define a second operation h_v and set $\Delta_v = \Delta_v^0 + h_v$.

For a morphism $\gamma = \gamma^m \otimes \cdots \otimes \gamma^0 \in \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$ define

$$\Delta_v^0(\gamma) = \sum_{\substack{0 \le i \le m+1 \\ 0 \le d \le m+1-i \\ k+l<q \\ n_\sigma(\gamma^r)=0\,\forall r<i}} \sum_{\substack{\gamma_a,\gamma_b \\ n_\sigma(\gamma_a)=0 \\ \deg(\gamma_a)+\deg(\gamma_b)=\sum_{r=i}^{i+d-1}\deg(\gamma^r)+\sum_{s=q-k}^{l}\deg(\gamma_s)+n-d-k-l}} \#\mathcal{R}^{d;k,l}((\gamma^{i+d-1}, \ldots, \gamma^i), (\gamma_l, \ldots, \gamma_{q-k}); \gamma_b, \gamma_a) \cdot \widehat{\gamma}, \qquad (4.5.1)$$

where tuples with increasing or nonexistent indices are the empty tuple (), and

$$\widehat{\gamma} := \gamma^m \otimes \cdots \otimes \gamma^{i+d} \otimes \gamma_b \otimes \gamma_{q-k-1} \otimes \cdots \otimes \gamma_{l+1} \otimes \gamma_a \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \qquad (4.5.2)$$

is required to be composable. Again the indices for the Hochschild chain v are cyclically ordered. A straightforward calculation gives that Δ_v^0 is homogeneous of degree $\deg(v)+n-2$. In particular, Δ_v^0 has degree -1 when $\deg(v) = 1-n$, which is precisely the grading required in Definition 4.2.4.

Remark 4.5.1. The key condition here is that, in the output, all chords starting with γ_a must have crossing number zero with σ. This is what allows the resulting chain map to interact with the intersection filtration. In particular, we will see that $\mu^1_{\boldsymbol{\sigma}'}\Delta^0_y + \Delta^0_y\mu^1_{\boldsymbol{\sigma}'}$ is nontrivial precisely at the smallest r such that $n_\sigma(\gamma^r) \neq 0$, where it is homotopic to the identity up to terms lower in the main filtration.

We begin by examining the configurations of holomorphic disks which appear in $\Delta_v^0\mu^1_{\boldsymbol{\sigma}'}(\gamma)$. These come in two types.

The first type occurs when the superscript inputs for Δ_v^0 do not include the output for $\mu^1_{\boldsymbol{\sigma}'}$. In this case, there are two components which are disjoint and do not want to glue together. (4.5.3)

The second type occurs when the superscript inputs for Δ_v^0 do include the output for $\mu^1_{\boldsymbol{\sigma}'}$. In this case, the configuration is a broken disk of the form (4.4.2e) or (4.4.2f). (4.5.4)

Next, we examine the configurations of holomorphic disks which appear in $\mu^1_{\boldsymbol{\sigma}'}\Delta^0_v(\gamma)$. These come in five types.

The first type occurs when the inputs for $\mu^1_{\boldsymbol{\sigma}'}$ do not include any of γ_a, γ_b, or the γ_i coming from the unused components of v. In this case, there are two components which are disjoint and do not want to glue together. These configurations are exactly the same as those in (4.5.3), so their contributions to $\Delta^0_v\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}\Delta^0_v$ cancel. (4.5.5)

The second type occurs when the inputs for $\mu^1_{\boldsymbol{\sigma}'}$ consist of one or more of the γ_i coming from the unused components of v. In this case, the configuration consists of two disjoint disks, each of which uses different portions of the Hochschild chain v. (4.5.6)

The third type occurs when the inputs for $\mu^1_{\boldsymbol{\sigma}'}$ include γ_a but not γ_b. In this case, the configuration is a broken disk of the form (4.4.2g) or (4.4.2h). (4.5.7)

The fourth type occurs when the inputs for $\mu^1_{\boldsymbol{\sigma}'}$ include γ_b but not γ_a. In this case, the configuration is a broken disk of the form (4.4.2i) or (4.4.2j). (4.5.8)

The fifth type occurs when the inputs for $\mu^1_{\boldsymbol{\sigma}'}$ include both γ_a and γ_b. In this case, the configuration is formally an annulus with two nodes. The outside of this annulus is labeled with some substring of γ and an output chord $\widetilde{\gamma}$, while the inside is labeled with the entire Hochschild chain v. Let (4.5.9)

$$A_v\colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*+\deg(v)+n-1}_{\boldsymbol{\sigma}'}(L_0, L_1)$$

be the linear map obtained by counting only such annuli.

We claim that, modulo terms which decrease the main filtration,

$$\Delta^0_v\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}\Delta^0_v = \Delta^0_{\delta v} + A_v. \tag{4.5.10}$$

For this, it suffices to consider those contributions to $\Delta^0_v\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}\Delta^0_v$ which come from configurations which avoid D_σ. Considering only such configurations, we want to see that the terms coming from (4.5.3)-(4.5.8) add up to the corresponding portion of $\Delta^0_{\delta v}$. To do so, note that there are two types of configurations which contribute to $\Delta^0_{\delta v}$. The first type occurs when the output of the A_∞ disk contributing to δv is not an input of $\mathcal{R}^{d;k,l}(\vec{\gamma}^\star, \vec{\gamma}_\star; \gamma_b, \gamma_a)$, so instead it appears in $\widehat{\gamma}$. In this case, the configuration is precisely what is counted in (4.5.6). The remaining type of configuration occurs when the output of the A_∞ disk contributing to δv is a component of $\vec{\gamma}_\star$. In this case, the broken configuration is one of (4.4.2a)-(4.4.2d). Because the spaces in (4.4.2) form the boundary of a compact 1-manifold, we are left with terms coming from configurations of the form (4.4.2e)-(4.4.2j). On the other hand, for spaces of coproduct disks which do not intersect D_σ, the condition that $n_\sigma(\gamma_a) = 0$ is preserved under breaking off an A_∞ disk. Thus, the operation coming from these configurations coincides precisely with the sum of the remaining terms (4.5.4), (4.5.7), and (4.5.8).

Since we will eventually be interested in replacing v with the closed chain y, we can ignore the δv term in (4.5.10). Moreover, since our goal is to show that R_y satisfies the conditions of Lemma 4.3.3, we may ignore all terms of $\Delta_v^0\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}\Delta_v^0$ which strictly decrease the main filtration. Because all operations involved satisfy positivity of intersections and y is made up of chords with $n_\sigma = 0$, the only way in which they can fail to decrease the filtration is by failing to decrease the length of a generator $\gamma^m \otimes \cdots \otimes \gamma^0$. For the annulus term A_y, this only happens when the broken annuli are labeled with zero or one γ^j input. Write $A_y = \sum_{\nu,\mu} A_y^{\nu,\mu}$, where $A_y^{\nu,\mu}$ is the operation coming from those broken holomorphic annuli with μ superscript inputs and intersection number ν with D_σ. This means, for $A_y^{\nu,\mu}(\gamma) \neq 0$, we have

$$(n,m)(A_y^{\nu,\mu}(\gamma)) = (n,m)(\gamma) + (-\nu, 1-\mu).$$

We conclude

Lemma 4.5.2. *Let y be a closed element of $CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$. Then, up to terms which decrease the main filtration, $\Delta_y^0\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}\Delta_y^0 = A_y^{0,0} + A_y^{0,1}$.* □

4.6 Closed-open maps

The eventual objective will be to show that $A_y^{0,0}+A_y^{0,1}$ is homotopic to a closed-open operation $\mathcal{CO}_x^{filt}$ depending on a cochain $x \in SC^*_{\boldsymbol{\sigma}}(M[\sigma])$, where it will turn out that $x = \mathcal{OC}(y)$. We now construct this operation.

Let $\mathcal{R}_1^{0+1}$ be the singleton set containing a disk $D_1^{0,1}$ with one interior puncture ζ_+ and one boundary puncture ζ_0. Up to biholomorphism, there is a unique such disk. Equip ζ_0 with a negative strip-like end ϵ_0 and ζ_+ with a positive cylindrical end ϵ_+. As with the punctured disks giving rise to $\mathcal{OC}$, we ask that ϵ_+ has a very special form. Specifically, in the holomorphic coordinates on $D_1^{0,1}$ where $\operatorname{int}(\Sigma) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and $\zeta_0 = -1$, we require that

$$\epsilon_+(s,t) = ae^{-2\pi(s+it)} \qquad \text{with } a \in \mathbb{R} \text{ positive.} \tag{4.6.1}$$

Going up in dimension, let $\mathcal{R}_1^{1+1}$ be the space of disks with one interior puncture ζ_+ and two boundary punctures ζ_0 and ζ_1. The corresponding compactified moduli space $\overline{\mathcal{R}}_1^{1+1}$ is

$$\overline{\mathcal{R}}_1^{1+1} = \left(\mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}\right) \amalg \mathcal{R}_1^{1+1} \amalg \left(\mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}\right), \tag{4.6.2}$$

where the two broken configurations correspond to the two ways of attaching the negative end of $D_1^{0,1}$ to one of the two positive ends of the unique disk $\Sigma^{2+1} \in \mathcal{R}^{2+1}$. Choose smooth, disjoint $\mathcal{R}_1^{1+1}$-parametrized families of positive strip-like ends ϵ_1 for ζ_1, negative strip-like ends ϵ_0 for ζ_0, and positive cylindrical ends ϵ_+ for ζ_+, which satisfy the compatibility conditions

1. In the gluing charts of the form $[0,a) \times \mathcal{R}^{2+1} \times \mathcal{R}_1^{0+1}$ with gluing length $\ell = e^{\frac{1}{\rho}}$, where $\rho \in [0,a)$, the families of ends agree to infinite order at $\rho = 0$ with those induced by gluing.

2. For all $\Sigma \in \mathcal{R}_1^{1+1}$, in the holomorphic coordinates on Σ where the interior of Σ is the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and $\zeta_0 = -1$, ϵ_+ satisfies (4.6.1).

Unlike with open-closed maps, for careful choices of ends elsewhere the agreement to infinite order could be strengthened to agreement in a neighborhood of the boundary, but there is no benefit to doing so. One could also extend the above and construct a map to Hochschild cohomology as in [14], but in our application the higher terms would reduce the main filtration, so we ignore them.

A **conformally consistent** choice of Floer data for the closed-open maps consists of a Floer datum on $D_1^{0,1}$, along with a Floer datum on Σ for each $\Sigma \in \mathcal{R}_1^{1+1}$ varying smoothly over $\mathcal{R}_1^{1+1}$, and such that near $\partial\overline{\mathcal{R}}_1^{1+1}$ it agrees to infinite order with the conformal class of not-quite Floer data determined by gluing. Denote by $\mathcal{K}^{\mathcal{CO}}(M[\sigma])$ the space of conformally consistent choices of Floer data for the closed-open maps.

Given $\mathbf{K} \subset \mathcal{K}^{\mathcal{CO}}(M[\sigma])$, we can consider the resulting holomorphic curves. Given Lagrangian labels L_i and asymptotic ends γ_i as in (3.2.11) and $x_+ \in \mathcal{X}(H_t)$, we are interested in the spaces

$$\begin{gathered}\mathcal{R}_1^{0+1}(x_+;\gamma_0)\\ \mathcal{R}_1^{1+1}(x_+,\gamma_1;\gamma_0).\end{gathered}$$

These consists of all maps $u\colon \Sigma \to \widehat{M[\sigma]}$ for $\Sigma \in \mathcal{R}_1^{0+1}$ or $\Sigma \in \mathcal{R}_1^{1+1}$, respectively, satisfying (4.1.6) with $u(E_i) \subset (\phi^{\tau_E})^* L_i$, $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$, and $u(\zeta_+) = (\phi^{\tau_+})^* x_+$.

Lemma 4.6.1. *There is a dense subset $\mathcal{K}^{\mathcal{CO}}_{reg}(M[\sigma]) \subset \mathcal{K}^{\mathcal{CO}}(M[\sigma])$ such that, for every $\mathbf{K} \in \mathcal{K}^{\mathcal{CO}}_{reg}(M[\sigma])$, the following hold.*

1. *For any Lagrangian L, chord $\gamma_0 \in \mathcal{X}(L,L)$, and orbit $x_+ \in \mathcal{X}(H_t)$, $\mathcal{R}_1^{0+1}(x_+;\gamma_0)$ is a smooth manifold of dimension* $\deg(\gamma_0) - \deg(x_+)$. *Additionally, it is empty unless $n_\sigma(\gamma_0) \leq n_\sigma(x_+)$.*

2. *If* $\deg(\gamma_0) - \deg(x_+) = 0$, *then $\mathcal{R}_1^{0+1}(x_+;\gamma_0)$ is compact.*

3. *If* $\deg(\gamma_0) - \deg(x_+) = 1$, *then $\mathcal{R}_1^{0+1}(x_+;\gamma_0)$ admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with*

$$\coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \left(\mathcal{R}_1^{0+1}(\widetilde{x};\gamma_0) \times \mathcal{Q}(x_+;\widetilde{x})\right) \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L,L)} \left(\mathcal{R}(\widetilde{\gamma};\gamma_0) \times \mathcal{R}_1^{0+1}(x_+;\widetilde{\gamma})\right). \tag{4.6.3}$$

4. *For any pair of Lagrangians L_0 and L_1, any pair of chords $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$, and any orbit $x_+ \in \mathcal{X}(H_t)$, $\mathcal{R}_1^{1+1}(x_+,\gamma_1;\gamma_0)$ is a smooth manifold of dimension*

$$\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) + 1.$$

It is empty unless $n_\sigma(\gamma_0) \leq n_\sigma(\gamma_1) + n_\sigma(x_+)$.

5. *If* $\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) = -1$, *then* $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$ *is compact.*

6. *If* $\deg(\gamma_0) - \deg(\gamma_1) - \deg(x_+) = 0$, *then* $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$ *admits a Gromov compactification as a compact topological 1-manifold with boundary, and its boundary is in natural bijection with*

$$\coprod_{\widetilde{x} \in \mathcal{X}(H_t)} \left(\mathcal{R}_1^{1+1}(\widetilde{x}, \gamma_1; \gamma_0) \times \mathcal{Q}(x_+; \widetilde{x})\right) \amalg \coprod_{\widetilde{\gamma} \in \mathcal{X}(L_0, L_1)} \left(\mathcal{R}_1^{1+1}(x_+, \widetilde{\gamma}; \gamma_0) \times \mathcal{R}(\gamma_1; \widetilde{\gamma})\right)$$
$$\amalg \coprod_{\widetilde{\gamma} \in \mathcal{X}(L_0, L_1)} \left(\mathcal{R}(\widetilde{\gamma}; \gamma_0) \times \mathcal{R}_1^{1+1}(x_+, \gamma_1; \widetilde{\gamma})\right)$$
$$\amalg \coprod_{\widetilde{\gamma} \in \mathcal{X}(L_1, L_1)} \left(\mathcal{R}^{2+1}(\widetilde{\gamma}, \gamma_1; \gamma_0) \times \mathcal{R}_1^{0+1}(x_+; \widetilde{\gamma})\right) \amalg \coprod_{\widetilde{\gamma} \in \mathcal{X}(L_0, L_0)} \left(\mathcal{R}^{2+1}(\gamma_1, \widetilde{\gamma}; \gamma_0) \times \mathcal{R}_1^{0+1}(x_+; \widetilde{\gamma})\right). \tag{4.6.4}$$

□

Definition 4.6.2. Suppose $u \in \mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$ with $x_+ \in SC^*_{\boldsymbol{\sigma}}(M[\sigma])$ and $n_\sigma(\gamma_0) = n_\sigma(\gamma_1)$. Then, by positivity of intersections, u doesn't pass through D_σ. Let Σ be the domain of u, and let $e\colon [0,1] \to \Sigma$ be a path with $e(0) \in E_0$ and $e(1) = \zeta_+$. Since u avoids D_σ, so does $u \circ e$, and hence the topological intersection number of $u \circ e$ with $\sigma(\hat{F} \times \mathbb{R}_+)$ is well defined and independent of the choice of e. Let $n_\sigma^{\mathcal{CO}}(u)$ be this number. The **filtered closed-open moduli space** $\mathcal{R}_1^{1+1,filt}(x_+, \gamma_1; \gamma_0)$ is the connected component of $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$ consisting of u with $n_\sigma^{\mathcal{CO}}(u) = 0$.

For $n_\sigma(\gamma_0) < n_\sigma(\gamma_1)$, we take $\mathcal{R}_1^{1+1,filt}(x_+, \gamma_1; \gamma_0)$ to be empty, though one could just as well take it to be all of $\mathcal{R}_1^{1+1}(x_+, \gamma_1; \gamma_0)$.

For $x \in SC^*_{\boldsymbol{\sigma}}(M[\sigma])$, define $\mathcal{CO}_x^{filt}\colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*+\deg(x)-1}_{\boldsymbol{\sigma}'}(L_0, L_1)$ to depend linearly on x and, for $x \in \mathcal{X}(H_t)$ a generator, to satisfy

$$\begin{aligned} \mathcal{CO}_x^{filt}(\gamma^m \otimes \cdots \otimes \gamma^0) = &\sum_{\substack{0 \le i \le m+1 \\ n_\sigma(\gamma^r) = 0\, \forall r < i \\ \widetilde{\gamma} \text{ making the result composable} \\ \deg(\widetilde{\gamma}) = \deg(x)}} \#\mathcal{R}_1^{0+1}(x; \widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ &+ \sum_{\substack{0 \le i \le m \\ n_\sigma(\gamma^r) = 0\, \forall r < i \\ \widetilde{\gamma} \text{ making the result composable} \\ \deg(\widetilde{\gamma}) = \deg(x) + \deg(\gamma^i) - 1}} \#\mathcal{R}_1^{1+1,filt}(x, \gamma^i; \widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned} \tag{4.6.5}$$

Note, as with the coproduct, that we have only allowed configurations where all long chords occur after the new one. The filtered moduli space is the object which captures the contributions of Δ_y^0 for which $n_\sigma(\gamma_a) = 0$ while $n_\sigma(\gamma_b) > 0$.

4.7 Annuli, part 1

To relate $A_y^{0,0} + A_y^{0,1}$ with $\mathcal{CO}^{filt}_{\mathcal{OC}(y)}$, we follow Abouzaid's construction in [1]. Specifically, we will coherently extend his first and second homotopies to allow for one outer input and verify that the result is a homotopy on $\hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$. This section constructs the first homotopy. The result is a homotopy h_y^1 such that, for $\delta y = 0$, the operation $A_y^{0,0} + A_y^{0,1} + h_y^1\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}h_y^1$ counts analytically gluable broken annuli.

For that, let

$$\mathcal{P}_d^0 = \coprod_{\substack{k,l\geq 0\\ m\geq 2\\ k+l+m=d+1}} [0,1] \times \overline{\mathcal{R}}^{m+1} \times \overline{\mathcal{R}}^{0;k,l}.$$

Note that this differs slightly from Abouzaid's terminology. First, the superscript of zero means that there are no outer inputs, meaning that we should think of gluing the first input ζ_1 of the A_∞ disk to the first output ζ_a of the coproduct, and likewise we should glue the last input ζ_m of the A_∞ disk to the last output ζ_b of the coproduct. Second, we do not bother identifying paired boundary components. When we consider holomorphic curves with domains in $\mathcal{P}_d^0$, this means that there will be extra boundary terms which cancel in pairs.

Similarly, let

$$\mathcal{P}_d^1 = \coprod_{\substack{k,l\geq 0\\ m\geq 2\\ k+l+m=d+1}} [0,1] \times \overline{\mathcal{R}}^{m+1} \times \overline{\mathcal{R}}^{1;k,l} \amalg \coprod_{\substack{k,l\geq 0\\ m\geq 3\\ k+l+m=d+2}} [0,1] \times \overline{\mathcal{R}}^{m+1,1} \times \overline{\mathcal{R}}^{0;k,l} \amalg \coprod_{\substack{k,l\geq 0\\ m\geq 3\\ k+l+m=d+2}} [0,1] \times \overline{\mathcal{R}}^{m+1,m} \times \overline{\mathcal{R}}^{0;k,l}.$$

Here, the first term is as before except with an outer input in the coproduct disk. For the other two, we have an extra distinguished input on the A_∞ disk. In this case, we attach ζ_a to the first nondistinguished input and ζ_b to the last nondistinguished input.

Definition 4.7.1. Before we can start to choose Floer data, we need some auxiliary definitions. For any disk

$$\Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \amalg \overline{\mathcal{R}}^{m+1,m},$$

let

$$\zeta_{first} = \begin{cases} \zeta_1 & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,m} \\ \zeta_2 & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1,1} \end{cases}$$

$$\zeta_{last} = \begin{cases} \zeta_m & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \\ \zeta_{m-1} & \text{for } \Sigma \in \overline{\mathcal{R}}^{m+1,m}. \end{cases}$$

For a two-component stable disk

$$\Sigma \in \partial\left(\overline{\mathcal{R}}^{m+1} \amalg \overline{\mathcal{R}}^{m+1,1} \amalg \overline{\mathcal{R}}^{m+1,m}\right),$$

the **main component** of Σ is the component which contains at least two of $\{\zeta_0, \zeta_{first}, \zeta_{last}\}$. For a stable disk with more than two components, the main component is the one for which ζ_0, ζ_{first}, and ζ_{last} all lie in different directions.

Definition 4.7.2. For $p = (t, \Sigma^{m+1}, \Sigma^{j;k,l}) \in \mathcal{P}^i_d$, a Lagrangian labeling of p consists of a Lagrangian labeling for each of Σ^{m+1} and $\Sigma^{j;k,l}$ such that the labels at $\zeta_{first} \in \Sigma^{m+1}$ agree with the labels at $\zeta_a \in \Sigma^{j;k,l}$, and similarly with ζ_{last} and ζ_b. A **universal and conformally consistent** choice of Floer data $\mathbf{K}^{\mathcal{P}}$ for the first homotopy consists, for all $d \geq 1$ and $i \in \{0, 1\}$ and each $p = (t, \Sigma^{m+1}, \Sigma^{j;k,l}) \in \mathcal{P}^i_d$ with Lagrangian labels, of a Floer datum $\mathbf{K}^{\mathcal{P}}(p)$ on Σ^{m+1} with the corresponding labels, such that $\mathbf{K}^{\mathcal{P}}$ varies smoothly on $\mathcal{P}^i_d$ and has the following properties.

1. For $t = 0$, $\mathbf{K}^{\mathcal{P}}(p)$ agrees up to conformal rescaling with the Floer datum on Σ^{m+1} chosen for the A_∞ structure.

2. For $t = 1$, the configuration is gluable to an annulus after a conformal rescaling. Concretely, let $r_\Delta \colon \overline{\mathcal{R}}^{j;k,l} \to (0, \infty)$ be the unique smooth function with $r_\Delta(\Sigma) = \frac{\tau_b}{\tau_a}$ for $\Sigma \in \mathcal{R}^{j;k,l}$. Similarly, let
$$r_\mu \colon \coprod \mathcal{P}^i_d \to (0, \infty)$$
be the unique smooth function with
$$r_\mu(p) = \frac{\tau_{last}(p)}{\tau_{first}(p)},$$
where τ_{first} and τ_{last} are the rescaling factors that $\mathbf{K}^{\mathcal{P}}$ assigns to the ends $\zeta_{first} \in \Sigma^{m+1}$ and $\zeta_{last} \in \Sigma^{m+1}$, respectively. We require that
$$r_\mu(1, \Sigma^{m+1}, \Sigma^{j;k,l}) = r_\Delta(\Sigma^{j;k,l}).$$

3. If Σ^{m+1} is a nontrivial stable disk, then on every component of Σ^{m+1} aside from the main component, $\mathbf{K}^{\mathcal{P}}(p)$ is conformally equivalent to the Floer datum chosen for that disk as an element of the associahedron.

4. If Σ^{m+1} is a nontrivial stable disk, let Σ_{main} be its main component. If Σ_{main} doesn't contain ζ_{first}, let Σ_{first} be the possibly-nodal connected piece of $\Sigma^{m+1} \setminus \Sigma_{main}$ containing ζ_{first}. Likewise, if Σ_{main} doesn't contain ζ_{last}, let Σ_{last} be the possibly-nodal connected piece of $\Sigma^{m+1} \setminus \Sigma_{main}$ containing ζ_{last}. Define a probably-nodal disk
$$\Sigma_{big} := \left(\Sigma_{first} \amalg \Sigma^{j;k,l} \amalg \Sigma_{last}\right) / (\zeta_a = \zeta_{first}, \zeta_b = \zeta_{last}) \in \overline{\mathcal{R}}^{j';k',l'}.$$
Then the restriction of $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma^{j;k,l})$ to Σ_{main} is conformally equivalent to the Floer datum $\mathbf{K}^{\mathcal{P}}(t, \Sigma_{main}, \Sigma_{big})$.

5. Suppose $\Sigma^{j;k,l}$ is a nontrivial stable disk, and that $\Sigma_{leaf} \subset \Sigma^{j;k,l}$ is an irreducible A_∞-type component which is only attached to the rest of $\Sigma^{j;k,l}$ at the negative puncture. In other words, Σ_{leaf} has one nodal negative puncture, zero other negative punctures, and its positive punctures are all honest positive punctures of $\Sigma^{j;k,l}$ instead of nodes. Define
$$\Sigma_{small} = \Sigma^{j;k,l} \setminus \Sigma_{leaf}.$$
Then $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma^{j;k,l})$ is conformally equivalent to $\mathbf{K}^{\mathcal{P}}(t, \Sigma^{m+1}, \Sigma_{small})$.

Denote by $\mathcal{K}^{\mathcal{P}}(M[\sigma])$ the space of universal and conformally consistent choices of Floer data for the first homotopy.

Suppose we have picked some universal choice $\mathbf{K}^{\mathcal{P}} \subset \mathcal{K}^{\mathcal{P}}(M[\sigma])$. For a generator
$$v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma)),$$
let $B_i \in \mathcal{B}_{\boldsymbol{\sigma}}(\sigma)$ be such that $\gamma_i \in \mathcal{X}(B_{i-1}, B_i)$. For L an interior Lagrangian of $M[\Sigma]$ and $\gamma_0 \in \mathcal{X}(L, L)$, define
$$\mathcal{P}^0_d(v;\gamma_0) = \coprod_{\substack{k,l\geq 0\\ k+l<d}} \mathcal{P}^{0,0,0;k,l}_d(v;\gamma_0)$$
where $P^{0,0,0;k,l}_d(v;\gamma_0)$ is the union over all
$$p = (t, \Sigma_\mu, \Sigma_\Delta) \in [0,1] \times \mathcal{R}^{(d-k-l+1)+1} \times \mathcal{R}^{0;k,l}$$
of the space of all maps
$$u\colon \Sigma_\Delta \amalg \Sigma_\mu \to \widehat{M[\sigma]}$$
satisfying the following conditions.

1. Write $u_\Delta := u|_{\Sigma_\Delta}$ and $u_\mu := u|_{\Sigma_\mu}$. Then
$$u_\Delta \in \mathcal{R}^{0;k,l}\left((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a\right)$$
for some $\gamma_a \in \mathcal{X}(L, B_l)$ and $\gamma_b \in \mathcal{X}(B_{d-k-1}, L)$.

2. u_μ satisfies (3.2.12) for the Floer datum $\mathbf{K}^{\mathcal{P}}(p)$.

3. Let $\tau_E(p)$ be the boundary rescaling function assigned to Σ_μ by $\mathbf{K}^{\mathcal{P}}(p)$. Then
$$u\left(\partial_i \Sigma_\mu\right) \in \begin{cases} \left(\phi^{\tau_E(p)}\right)^* L & \text{for } i = 0 \text{ or } d-k-l+1 \\ \left(\phi^{\tau_E(p)}\right)^* B_{l+i-1} & \text{otherwise}\end{cases}$$
where $\partial_i \Sigma_\mu$ is the portion of the boundary between ζ_i and ζ_{i+1}, ordered cyclically.

4. Let τ_i be the rescaling factor assigned by $\mathbf{K}^{\mathcal{P}}(p)$ to $\zeta_i \in \Sigma_\mu$. Then

$$\begin{aligned} u(\zeta_0) &= (\phi^{\tau_0})^* \gamma_0 \\ u(\zeta_1) &= (\phi^{\tau_1})^* \gamma_a \\ u(\zeta_i) &= (\phi^{\tau_i})^* \gamma_{l+i-1} \qquad \text{for } 2 \leq i \leq d-k-l \\ u(\zeta_{d-k-l+1}) &= (\phi^{\tau_{d-k-l+1}})^* \gamma_b. \end{aligned}$$

For any fixed $v \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, by the discussion in Appendix A.1, there are only finitely many choices of intermediate chords (γ_a, γ_b) which u_Δ can approach. The maximum principle (Lemma A.2.2) and the usual Gromov compactness argument then imply that the spaces $\mathcal{P}_d^{0,0,0;k,l}(v;\gamma_0)$ have Gromov compactifications $\overline{\mathcal{P}}_d^{0,0,0;k,l}(v;\gamma_0)$ obtained by allowing either or both of u_Δ and u_μ to break.

Lemma 4.7.3. *There is a dense subset $\mathcal{K}^{\mathcal{P}}_{reg}(M[\sigma]) \subset \mathcal{K}^{\mathcal{P}}(M[\sigma])$ such that, for any choice $\mathbf{K}^{\mathcal{P}} \in \mathcal{K}^{\mathcal{P}}_{reg}(M[\sigma])$, the following hold.*

1. *For any $d \geq 1$, any Lagrangian L, any $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, and any chord $\gamma_0 \in \mathcal{X}(L,L)$, $\mathcal{P}_d^{0,0,0;k,l}(v;\gamma_0)$ is a smooth manifold with boundary of dimension $\deg(\gamma_0) - \deg(v) + 1 - n$. Additionally, it is empty unless $n_\sigma(\gamma_0) = 0$. The boundary comes from the fact that the space of domains, even before compactification, has a $[0,1]$ factor.*

2. *If $\deg(\gamma_0) - \deg(v) = n-1$, then $\mathcal{P}_d^{0,0,0;k,l}(v;\gamma_0)$ is compact.*

3. *If $\deg(\gamma_0) - \deg(v) = n$, then $\overline{\mathcal{P}}_d^{0,0,0;k,l}(v;\gamma_0)$ is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations u of the following types.*

   In the first type of configuration, $t \in (0,1)$,

   $$u_\Delta \in \partial \overline{\mathcal{R}^{0;k,l}}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

   while u_μ is a map $u_\mu \colon \Sigma_\mu \to \widehat{M[\sigma]}$ satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for $(t, \Sigma_\mu, \Sigma_\Delta)$ with the corresponding boundary and asymptotic conditions, where Σ_Δ is the nodal disk which is the domain of u_Δ. (4.7.1)

In the second type of configuration, $t \in (0,1)$,

$$u_\Delta \in \mathcal{R}^{0;k,l}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

while u_μ *has broken, such that one component is an honest* A_∞ *disk* u_μ^0 *which has neither* γ_a *nor* γ_b *as an input. Such a disk is either a portion of the Hochschild differential on* v *or a Floer strip which outputs* γ_0*. The other disk,* u_μ^{main}*, is a map* $u_\mu^{main} \colon \Sigma_\mu^{main} \to \widehat{M[\sigma]}$ *satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for* $(t, \Sigma_\mu^{main}, \Sigma_\Delta)$ *with the induced boundary and asymptotic conditions.* (4.7.2)

In the third type of configuration, $t \in (0,1)$,

$$u_\Delta \in \mathcal{R}^{0;k,l}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

while u_μ *has broken, such that one component is an honest* A_∞ *disk* u_μ^{in} *which has* γ_a *or* γ_b *as an input but does not have* γ_0 *as its output. The other,* u_μ^{out}*, is a map* $u_\mu^{out} \colon \Sigma_\mu^{out} \to \widehat{M[\sigma]}$ *satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for* $(t, \Sigma_\mu^{out}, \Sigma_{big})$ *with the induced boundary and asymptotic conditions, where* Σ_{big} *is the broken disk formed by joining the domains of* u_Δ *and* u_μ^{in}*.* (4.7.3)

In the fourth type of configuration, $t = 0$*, in which case* u *is a two-component broken annulus of the type contributing to* $A_v^{0,0}$*.* (4.7.4)

In the fifth type of configuration, $t = 1$*, in which case* u *is a two-component broken annulus which can be glued into an honest perturbed holomorphic annulus.* (4.7.5)

In the previous notation, these cases comprise seventeen types of boundary strata, each of which is described by a formula slightly different from the others.

□

For v as above and $\gamma^1 \in \mathfrak{X}(L_0, L_1)$ with L_0 and L_1 interior Lagrangians in $M[\Sigma]$, define

$$\mathcal{P}_d^1(v;\gamma_0) = \coprod_{\substack{k,l\geq 0\\ k+l<d}} \left(\mathcal{P}_d^{1,0,0;k,l}(v,\gamma^1;\gamma_0) \amalg \mathcal{P}_d^{0,1,0;k,l}(v,\gamma^1;\gamma_0) \amalg \mathcal{P}_d^{0,0,1;k,l}(v,\gamma^1;\gamma_0)\right)$$

for certain spaces $\mathcal{P}_d^{q_a,j,q_b;k,l}(v,\gamma^1;\gamma_0)$. These describe broken annuli with q_a outer inputs between the output and the a-node, j outer inputs on the coproduct, and q_b outer inputs between the b-node and the output. Concretely, for $q_a + j + q_b = 1$, this is the union over all

$$p = (t, \Sigma_\mu, \Sigma_\Delta) \in [0,1] \times \mathcal{R}^{(d-k-l+q_a+q_b+1)+1} \times \mathcal{R}^{j;k,l}$$

of the space of all maps

$$u\colon \Sigma_\Delta \amalg \Sigma_\mu \to \widehat{M[\sigma]}$$

satisfying essentially the same conditions as for $\mathcal{P}^{0,0,0;k,l}(v;\gamma_0)$:

1. Write $u_\Delta := u|_{\Sigma_\Delta}$ and $u_\mu := u|_{\Sigma_\mu}$. Then

$$u_\Delta \in \begin{cases} \mathcal{R}^{0;k,l}\left((),(\gamma_l,\dots,\gamma_{d-k});\gamma_b,\gamma_a\right) & \text{for } j=0 \\ \mathcal{R}^{1;k,l}\left((\gamma^1),(\gamma_l,\dots,\gamma_{d-k});\gamma_b,\gamma_a\right) & \text{for } j=1 \end{cases}$$

for some $\gamma_a \in \mathcal{X}(L_{q_a}, B_l)$ and $\gamma_b \in \mathcal{X}(B_{d-k-1}, L_{1-q_b})$.

2. u_μ satisfies (3.2.12) for the Floer datum $\mathbf{K}^{\mathcal{P}}(p)$.

3. Let $\tau_E(p)$ be the boundary rescaling function assigned to Σ_μ by $\mathbf{K}^{\mathcal{P}}(p)$. Then

$$u\left(\partial_i\Sigma_\mu\right) \in \begin{cases} \left(\phi^{\tau_E(p)}\right)^* L_0 & \text{for } i=0 \text{ and, if } q_b=1, \text{ for } i=d-k-l+1 \\ \left(\phi^{\tau_E(p)}\right)^* L_1 & \text{for } i=d-k-l+q_a+q_b+1 \text{ and, if } q_a=1, \text{ for } i=1 \\ \left(\phi^{\tau_E(p)}\right)^* B_{l+i-q_a-1} & \text{otherwise} \end{cases}$$

where $\partial_i\Sigma_\mu$ is the portion of the boundary between ζ_i and ζ_{i+1}, ordered cyclically.

4. Let τ_i be the rescaling factor assigned by $\mathbf{K}^{\mathcal{P}}(p)$ to $\zeta_i \in \Sigma_\mu$. Then

$$\begin{aligned} u(\zeta_0) &= (\phi^{\tau_0})^*\gamma_0 \\ u(\zeta_{1+q_a}) &= (\phi^{\tau_{1+q_a}})^*\gamma_a \\ u(\zeta_i) &= (\phi^{\tau_i})^*\gamma_{l+i-q_a-1} \qquad \text{for } 2+q_a \le i \le d-k-l+q_a \\ u(\zeta_{d-k-l+q_a+1}) &= (\phi^{\tau_{d-k-l+q_a+1}})^*\gamma_b. \end{aligned}$$

If $q_a=1$, then we further require $u(\zeta_1) = (\phi^{\tau_1})^*\gamma^1$. Similarly, if $q_b=1$, we require $u(\zeta_{d-k-l+2}) = (\phi^{\tau_{d-k-l+2}})^*\gamma^1$.

Once again Gromov compactness holds, and the compactified spaces $\overline{\mathcal{P}}_d^{q_a,j,q_b;k,l}(v,\gamma^1;\gamma_0)$ are obtained by allowing either or both of u_Δ and u_μ to break.

Lemma 4.7.4. *For an appropriate choice of $\mathcal{K}^{\mathcal{P}}_{reg}(M[\sigma])$ in Lemma 4.7.3, the following hold for every $\mathbf{K} \in \mathcal{K}^{\mathcal{P}}_{reg}(M[\sigma])$.*

1. *For any $d \ge 1$ and all $q_a, j, q_b \ge 0$ with $q_a + j + q_b = 1$, any Lagrangians L_0, L_1, any $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, and any chords $\gamma_0, \gamma^1 \in \mathcal{X}(L_0, L_1)$, $\mathcal{P}_d^{q_a,j,q_b;k,l}(v,\gamma^1;\gamma_0)$ is a smooth manifold with boundary of dimension $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) + 2 - n$. Additionally, it is empty unless $n_\sigma(\gamma_0) \le n_\sigma(\gamma^1)$.*

2. *If $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) = n - 2$, then $\mathcal{P}_d^{q_a,j,q_b;k,l}(v,\gamma^1;\gamma_0)$ is compact.*

3. *If* $\deg(\gamma_0) - \deg(\gamma^1) - \deg(v) = n-1$, *then* $\overline{\mathcal{P}}_d^{q_a,j,q_b;k,l}(v, \gamma^1; \gamma_0)$ *is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations* u *of the following types.*

   In the first type of configuration, $t \in (0,1)$,

   $$u_\Delta \in \partial\overline{\mathcal{R}^{j;k,l}}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

   while u_μ *is a map* $u_\mu \colon \Sigma_\mu \to \widehat{M[\sigma]}$ *satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for* $(t, \Sigma_\mu, \Sigma_\Delta)$ *with the corresponding boundary and asymptotic conditions.* (4.7.6)

   In the second type of configuration, $t \in (0,1)$,

   $$u_\Delta \in \mathcal{R}^{j;k,l}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

   while u_μ *has broken, such that one component is an honest* A_∞ *disk* u_μ^0 *which has neither* γ_a *nor* γ_b *as an input. Such a disk is either a portion of the Hochschild differential on* v, *or it is a 1- or 2-input disk involving* γ^1 *and/or* γ_0. *The other disk,* u_μ^{main}, *is a map* $u_\mu^{main} \colon \Sigma_\mu^{main} \to \widehat{M[\sigma]}$ *satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for* $(t, \Sigma_\mu^{main}, \Sigma_\Delta)$ *with the induced boundary and asymptotic conditions. This configuration is still part of* $\mathcal{P}_d^1(v, \gamma^1; \gamma_0)$ *unless* u_μ^0 *involves both* γ^1 *and* γ_0, *in which case the configuration is instead part of* $\mathcal{P}^{0;k,l}(v, \widetilde{\gamma})$ *with* $\widetilde{\gamma}$ *an input of* u_μ^0. (4.7.7)

   In the third type of configuration, $t \in (0,1)$,

   $$u_\Delta \in \mathcal{R}^{j;k,l}((), (\gamma_l, \ldots, \gamma_{d-k}); \gamma_b, \gamma_a),$$

   while u_μ *has broken, such that one component is an honest* A_∞ *disk* u_μ^{in} *which has* γ_a *or* γ_b *as an input but does not have* γ_0 *as its output. The other,* u_μ^{out}, *is a map* $u_\mu^{out} \colon \Sigma_\mu^{out} \to \widehat{M[\sigma]}$ *satisfying the Cauchy-Riemann equation perturbed by the Floer datum chosen for* $(t, \Sigma_\mu^{out}, \Sigma_{big})$ *with the induced boundary and asymptotic conditions, where* Σ_{big} *is the broken disk formed by joining the domains of* u_Δ *and* u_μ^{in}. (4.7.8)

   In the fourth type of configuration, $t = 0$, *in which case* u *is a two-component broken annulus of the type contributing to* $A_v^{0,1}$. *Indeed,* $A_v^{0,1}$ *is a count of precisely such annuli satisfying either* $n_\sigma(\gamma^1) = 0$ *or* $q_a = 0$, $n_\sigma(\gamma_a) = 0$, *and* $n_\sigma(\gamma_0) = n_\sigma(\gamma^1)$. (4.7.9)

> *In the fifth type of configuration, $t = 1$, in which case u is a two-component broken annulus which can be glued into an honest perturbed holomorphic annulus.* (4.7.10)

□

While $\mathcal{P}^0_d(v;\gamma_0)$ already extends the moduli space giving rise to $A^{0,0}_v$, for $A^{0,1}_v$ we need to look at a connected component of $P^1_d(v,\gamma^1;\gamma_0)$ as hinted by (4.7.9). This is the space $\mathcal{P}^1_{d,filt}(v,\gamma^1;\gamma_0)$ consisting of those $u \in P^1_d(v,\gamma^1;\gamma_0)$ for which either (1) $n_\sigma(\gamma^1) = 0$, or (2) $q_a = 0$, $n_\sigma(\gamma_a) = 0$, and $n_\sigma(\gamma_0) = n_\sigma(\gamma^1)$. For an equivalent description closer in spirit to the filtered closed-open moduli space, choose for all Σ_μ a path $e\colon [0,1] \to \Sigma_\mu$ starting on the edge $\partial_0\Sigma_\mu$ and ending on $\partial_{1+q_a}\Sigma_\mu$. $\mathcal{P}^1_{d,filt}(v,\gamma^1;\gamma_0)$ is the space of all u which avoid D_σ and for which the topological intersection number $u \circ e$ with $\sigma(\hat{F} \times \mathbb{R}_+)$ vanishes.

Define a linear map $h^1_y\colon \hom^*_{\boldsymbol{\sigma}'}(L_0,L_1) \to \hom^{*+\deg(v)+n-2}_{\boldsymbol{\sigma}'}(L_0,L_1)$ to depend linearly on y and, for $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, to be given by

$$\begin{aligned} h^1_v(\gamma^m \otimes \cdots \otimes \gamma^0) \;&= \sum_{\substack{0 \le i \le m+1 \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+n-1}} \#\mathcal{P}^0_d(v;\widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ &+ \sum_{\substack{0 \le i \le m \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-2}} \#\mathcal{P}^1_{d,filt}(v,\gamma^i;\widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned} \tag{4.7.11}$$

In the same way, we define the **gluable annulus maps**

$$\widetilde{A}^{0,0}_y \text{ and } \widetilde{A}^{0,1}_y\colon \hom^*_{\boldsymbol{\sigma}'}(L_0,L_1) \to \hom^{*+\deg(v)+n-1}_{\boldsymbol{\sigma}'}(L_0,L_1)$$

via

$$\widetilde{A}^{0,0}_v(\gamma^m \otimes \cdots \otimes \gamma^0) \;= \sum_{\substack{0 \le i \le m+1 \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+n}} \#\left[\mathcal{P}^0_d(v;\widetilde{\gamma})\right]_{t=1} \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \tag{4.7.12a}$$

$$\widetilde{A}^{0,1}_v(\gamma^m \otimes \cdots \otimes \gamma^0) \;= \sum_{\substack{0 \le i \le m \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-1}} \#\left[\mathcal{P}^1_{d,filt}(v,\gamma^i;\widetilde{\gamma})\right]_{t=1} \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0, \tag{4.7.12b}$$

where the notation $[\cdot]_{t=1}$ refers to the portion of the corresponding moduli space which occurs at $t = 1$. This is the portion of the boundary of the 1-dimensional part of the filtered moduli space described in (4.7.5) and (4.7.10).

Lemma 4.7.5. *Up to terms which decrease the main filtration,*

$$h_v^1\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}h_v^1 = h^1_{\delta v} + A_v^{0,0} + A_v^{0,1} + \widetilde{A}_v^{0,0} + \widetilde{A}_v^{0,1}. \tag{4.7.13}$$

Proof. Write $h_v^1 = (h_v^1)^{0,0} + (h_v^1)^{0,1}$, where $(h_v^1)^{0,0}$ is the part of h_v^1 coming from the first sum in (4.7.11), while $(h_v^1)^{0,1}$ is the part coming from the second sum. We begin by analyzing the part of $h_v^1\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}h_v^1$ which increases word length. This consists of all ways of applying $(h_v^1)^{0,0}$ and $\mu^1_{\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])}$ in some order. All such terms cancel except those in which $\mu^1_{\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])}$ is applied to the output of h_v^1, which we see constitute the part of (4.7.2) which do not contribute to the Hochschild differential. We therefore examine the rest of the boundary of the corresponding 1-dimensional moduli space.

The rest of the Hochschild differential appears in (4.7.1), giving rise to a $(h^1_{\delta v})^{0,0}$. The rest of (4.7.1) comes from the breaking of an A_∞ disk outputting ζ_a or ζ_b. Such disks precisely form the contribution of (4.7.3) for a different connected component, and so they cancel in pairs. The remaining terms (4.7.4) and (4.7.5) correspond precisely to $A_v^{0,0}$ and $\widetilde{A}_v^{0,0}$, respectively, which confirms the portion of (4.7.13) which increases word length.

For the portion which preserves word length, in order to avoid leaving the filtered moduli space, we consider only the part of $\mu^1_{\boldsymbol{\sigma}'}$ which does not decrease intersection number. Among such terms, we are interested in all ways of performing both $\mu^2_{\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])}$ and $(h_v^1)^{0,0}$ or both $\mu^1_{\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])}$ and $(h_v^1)^{0,1}$ in some order. These again cancel when the operations take place at different places in $\gamma^m \otimes \cdots \otimes \gamma^0 \in \hom_{\boldsymbol{\sigma}'}(L_0, L_1)$. The remaining terms include not just the non-Hochschild part of (4.7.7), but also the component of (4.7.6) given by a Floer strip escaping at the outer input of the coproduct, as in (4.4.2f). The rest of the argument proceeds as above. $\square$

4.8 Annuli, part 2

We can now construct the homotopy between $\widetilde{A}_y^{0,0} + \widetilde{A}_y^{0,1}$ and $\mathcal{CO}^{filt}_{\mathcal{OC}(y)}$ for a Hochschild cycle $y \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$. This is the induced effect of the Cardy relation for wrapped Floer theory [1][14] on the quotient category, modified to interact with the intersection filtration.

For $d \geq 1$, let $\mathcal{A}_d^{0+1}$ be the space of conformal annuli Σ with the following data

1. d punctures on the inner boundary, labeled ζ_1 through ζ_d in clockwise order. Note that this becomes a standard counterclockwise ordering after exchanging the inner and outer boundary components.

2. One puncture ζ_0 on the outer boundary component, such that in coordinates

$$\mathrm{int}(\Sigma) = \{z \in \mathbb{C} \mid 1 < |z| < R\} \tag{4.8.1}$$

for some $R > 1$ with $\zeta_d = 1$, we have $\zeta_0 = -R$.

$\mathcal{A}_d^{0+1}$ admits a Deligne-Mumford compactification $\overline{\mathcal{A}}_d^{0+1}$ which is a manifold with stratified boundary. In addition to ordinary smooth corners, it has boundary strata of codimension at least two for which neighborhoods are subvarieties of the standard corner $[0,a)^i \times (-a,a)^j$. This is similar to the situation with multiplihedra. Its codimension 1 boundary components come in three types.

1. The first type occurs as some of the inner boundary punctures come together while R remains finite and strictly greater than 1. Such configurations are described by

$$\coprod_{\substack{2\leq k\leq d\\ 1\leq i\leq k}} \mathcal{A}_{d+1-k}^{0+1} \times \mathcal{R}^{k+1,i} \quad \amalg \coprod_{\substack{2\leq k\leq d-1\\ 1\leq i\leq d-k}} \mathcal{A}_{d+1-k}^{0+1} \times \mathcal{R}^{k+1} \tag{4.8.2}$$

   as in (4.2.3). As before, the index i in the second term keeps track of where the punctures collided.

2. The second type occurs as R tends to 1. Because of the anti-alignment condition on ζ_0 and ζ_d, Σ has to break into a nodal configuration in which ζ_0 and ζ_d are on different irreducible components. Thus, Σ must have at least two components, and the codimension 1 condition is that it breaks into exactly two components. Such configurations are described by

$$\coprod_{\substack{k,l\geq 0\\ m\geq 2\\ k+l+m=d+1}} \mathcal{R}^{m+1} \times \mathcal{R}^{0;k,l},$$

   which we identify with $\mathrm{int}[\mathcal{P}_d^0]_{t=1}$, the interior of the portion of $\mathcal{P}_d^0$ lying over $t=1$.

3. The third type occurs as R tends to ∞. In this case, we obtain two disks attached nodally at their centers, and the anti-alignment condition gives rise to a preferred angular gluing parameter. The configuration is thus parametrized by

$$\mathcal{R}_1^{0+1} \times \mathcal{R}_d^1,$$

   where the alignment conditions on the cylindrical ends for $\mathcal{CO}$ and $\mathcal{OC}$ implements the restriction on gluing angles. We then obtain boundary charts by introducing a gluing parameter ρ satisfying $\rho = \frac{1}{\log \ell}$, where ℓ is the gluing length for the cylindrical ends.

The higher codimension strata are either combinations of the above or paired boundary strata of $[\mathcal{P}_d^0]_{t=1}$, or in other words configurations which arise as the boundary of two different components of $[\mathcal{P}_d^0]_{t=1}$.

Next, let $\mathcal{A}_d^{1+1}$ for $d \geq 1$ be the space of conformal annuli Σ with the following data

1. d punctures on the inner boundary, labeled ζ_1 through ζ_d in clockwise order. Note that this becomes a standard counterclockwise ordering after exchanging the inner and outer boundary components.

2. Two punctures ζ_0 and ζ^1 on the outer boundary component, such that in coordinates (4.8.1) with $\zeta_d = 1$, we have $\zeta_0 = -R$.

$\mathcal{A}_d^{1+1}$ admits a Deligne-Mumford compactification $\overline{\mathcal{A}}_d^{1+1}$ which is again a manifold with stratified boundary. Its codimension 1 boundary components come in four types, the first three of which are essentially the same as for $\overline{\mathcal{A}}_d^{0+1}$.

1. The first type occurs as some of the inner boundary punctures come together while R remains finite and strictly greater than 1. Such configurations are described by

$$\coprod_{\substack{2\le k\le d\\ 1\le i\le k}} \mathcal{A}_{d+1-k}^{1+1} \times \mathcal{R}^{k+1,i} \quad \amalg \coprod_{\substack{2\le k\le d-1\\ 1\le i\le d-k}} \mathcal{A}_{d+1-k}^{1+1} \times \mathcal{R}^{k+1}. \tag{4.8.3}$$

2. The second type occurs as R tends to 1. Such configurations can be identified with $\mathrm{int}[\mathcal{P}_d^1]_{t=1}$, the interior of the portion of $\mathcal{P}_d^1$ lying over $t = 1$.

3. The third type occurs as R tends to ∞ and is parametrized by

$$\mathcal{R}_1^{1+1} \times \mathcal{R}_d^1.$$

4. The fourth type occurs when ζ^1 collides with ζ_0 while $R \in (1, \infty)$. This case is formally similar to the first, but here we reduce the number of outer punctures. The configurations are described by

$$\mathcal{R}^{2+1} \times \mathcal{A}_d^{0+1}.$$

As before, the higher codimension strata are either combinations of the above or paired boundary strata of $[\mathcal{P}_d^1]_{t=1}$.

A collection of strip-like ends for an annulus $\Sigma \in \mathcal{A}_d^{j+1}$ consists of positive strip-like ends ϵ_i at ζ_i for $i \in \{1, \dots, d\}$ and, if applicable, ϵ^1 at ζ^1, along with a negative strip-like end ϵ_0 at ζ_0, such that the images of the ends are pairwise disjoint. A cylinder for Σ is a finite cylinder $\delta\colon [a, b] \times S^1 \to \Sigma$ which is disjoint from the strip-like ends and, in the coordinates (4.8.1) with $\zeta_d = 1$ and $\zeta_0 = -R$, takes the form

$$\delta(s, t) = ce^{-2\pi(s+it)} \qquad \text{with } c \in \mathbb{R} \text{ positive}.$$

A **universal choice of ends and cylinders** for $\mathcal{A}_d^{j+1}$ consists, for all $d \geq 1$ and $j \in \{0, 1\}$, of a collection of strip-like ends for each $\Sigma \in \mathcal{A}_d^{j+1}$ which varies smoothly over $\mathcal{A}_d^{j+1}$, along with a cylinder for Σ whenever $R \geq 2$ which also varies smoothly over $\mathcal{A}_d^{j+1}$, which satisfy

1. The strip-like ends agree to infinite order at the boundary with the collection of strip-like ends induced by gluing.

2. Near $R = \infty$, the cylinder agrees with the finite cylinder induced by gluing.

3. When $R = 2$, the width $b - a$ of the cylinder is zero.

Fix once and for all a universal choice of strip-like ends and cylinders for $\mathcal{A}_d^{j+1}$.

Similarly, a **universal and conformally consistent** choice of Floer data for $\mathcal{A}_d^{j+1}$ consists, for all $d \geq 0$ and $j \in \{0, 1\}$, of a Floer datum for each $\Sigma \in \mathcal{A}_d^{j+1}$ varying smoothly over $\mathcal{A}_d^{j+1}$, and such that at the boundary it agrees to infinite order with the conformal class of Floer data induced by gluing. It is easy to see that conformal consistency can be achieved, at least away from the $R = 1$ boundary of $\overline{\mathcal{A}}_d^{j+1}$. At the $R = 1$ boundary, one needs the observation that, for fixed d, the Floer data on paired boundary strata of $\mathcal{P}_d^j$ agree up to a global conformal factor, so consistency can be extended across the corresponding strata of $\partial\overline{\mathcal{A}}_d^{j+1}$. Let $\mathcal{K}^{\mathcal{A}}(M[\sigma])$ denote the space of all universal and conformally consistent choices of Floer data for $\mathcal{A}_d^{j+1}$.

Given $\mathbf{K}^{\mathcal{A}} \in \mathcal{K}^{\mathcal{A}}(M[\sigma])$, we obtain spaces of holomorphic annuli. For $j = 0$, these are specified by a generator $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ and a chord $\gamma_0 \in \mathfrak{X}(L, L)$ with L an interior Lagrangian in $M[\sigma]$. The resulting moduli space

$$\mathcal{A}_d^{0+1}(v; \gamma_0)$$

is the space of all maps $u\colon \Sigma \to \widehat{M[\sigma]}$ for $\Sigma \in \mathcal{A}_d^{0+1}$ satisfying (4.1.6) such that $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$ and with the corresponding boundary conditions. Similarly, if L_0 and L_1 are interior Lagrangians of $M[\sigma]$ and $\gamma_0, \gamma^1 \in \mathfrak{X}(L_0, L_1)$, then we obtain

$$\mathcal{A}_d^{1+1}(v, \gamma^1; \gamma_0),$$

the space of perturbed holomorphic curves u with domain in $\mathcal{A}_d^{1+1}$ such that $u(\zeta_i) = (\phi^{\tau_i})^*\gamma_i$ and $u(\zeta^1) = (\phi^{\tau^1})^*\gamma^1$, and which satisfy the appropriate boundary conditions.

As usual, $\mathcal{A}_d^{0+1}(v; \gamma_0)$ and $\mathcal{A}_d^{1+1}(v; \gamma^1, \gamma_0)$ have Gromov compactifications $\overline{\mathcal{A}}_d^{0+1}(v; \gamma_0)$ and $\overline{\mathcal{A}}_d^{1+1}(v; \gamma^1, \gamma_0)$, respectively, which are obtained by including broken configurations.

Lemma 4.8.1. *There is a dense subset $\mathcal{K}^{\mathcal{A}}_{reg}(M[\sigma]) \subset \mathcal{K}^{\mathcal{A}}(M[\sigma])$ such that, for every universal choice $\mathbf{K}^{\mathcal{A}} \in \mathcal{K}^{\mathcal{A}}_{reg}(M[\sigma])$, the following hold.*

1. *For any $d \geq 1$, any Lagrangian L, any generator $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, and any chord $\gamma_0 \in \mathfrak{X}(L, L)$, $\mathcal{A}_d^{0+1}(v; \gamma_0)$ is a smooth manifold of dimension*

$$\deg(\gamma_0) - \deg(v) + 1 - n.$$

Additionally, it is empty unless $n_\sigma(\gamma_0) = 0$.

2. *If $\deg(\gamma_0) - \deg(v) = n - 1$, then $\mathcal{A}_1^{0+1}(v; \gamma_0)$ is compact.*

3. *If* $\deg(\gamma_0) - \deg(v) = n$, *then* $\overline{\mathcal{A}}_1^{0+1}(v;\gamma_0)$ *is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations of the following types.*

   The first type corresponds to the domain hitting a boundary stratum of $\overline{\mathcal{A}}_d^{0+1}$ *of the form* (4.8.2) *or a Floer strip breaking off at a puncture on the inner boundary. In symbols, these are essentially the same as the first four terms of* (4.2.5)*, though there we separated the chords making up* v. (4.8.4)

   The second type comes from a Floer strip breaking off at γ_0 *and is parametrized by*

$$\coprod_{\widetilde{\gamma}\in\mathcal{X}(L,L)} \mathcal{R}(\widetilde{\gamma};\gamma_0)\times\mathcal{A}_d^{0+1}(v;\widetilde{\gamma}). \tag{4.8.5}$$

   The third type comes from the domain hitting the $R=1$ *boundary and is precisely* $[\mathcal{P}_d^0(v;\gamma_0)]_{t=1}$. (4.8.6)

   The fourth type comes from the domain hitting $R=\infty$ *and is parametrized by*

$$\coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{R}_1^{0+1}(\widetilde{x};\gamma_0)\times\mathcal{R}_d^1(\gamma_d,\dots,\gamma_1;\widetilde{x}) \tag{4.8.7}$$

4. *For any* $d\geq 1$ *any Lagrangians* L_0, L_1, *any generator* $v=\gamma_d\otimes\cdots\otimes\gamma_1\in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, *and any chords* $\gamma_0,\gamma^1\in\mathcal{X}(L_0,L_1)$, $\mathcal{A}_1^{1+1}(v,\gamma^1;\gamma_0)$ *is a smooth manifold of dimension* $\deg(\gamma_0)-\deg(\gamma^1)-\deg(v)+2-n$. *Additionally, it is empty unless* $n_\sigma(\gamma_0)\leq n_\sigma(\gamma^1)$.

5. *If* $\deg(\gamma_0)-\deg(\gamma^1)-\deg(v)=n-2$, *then* $\mathcal{A}_1^{1+1}(v,\gamma^1;\gamma_0)$ *is compact.*

6. *If* $\deg(\gamma_0)-\deg(\gamma^1)-\deg(v)=n-1$, *then* $\overline{\mathcal{A}}_1^{1+1}(v,\gamma^1;\gamma_0)$ *is a compact topological 1-manifold with boundary, and its boundary consists of all broken configurations of the following types.*

   The first type corresponds to the domain hitting a boundary stratum of $\overline{\mathcal{A}}_d^{1+1}$ *of the form* (4.8.3) *or a Floer strip breaking off at a puncture on the inner boundary. In symbols, these are also essentially the same as the first four terms of* (4.2.5). (4.8.8)

The second type comes from a Floer strip breaking off at γ^1 or γ_0 or from a collision of ζ^1 with ζ_0. Such configurations are parametrized by

$$\coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_1)} \mathcal{A}_d^{1+1}(v,\widetilde{\gamma};\gamma_0)\times\mathcal{R}(\gamma^1;\widetilde{\gamma}) \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_1)} \mathcal{R}(\widetilde{\gamma};\gamma_0)\times\mathcal{A}_d^{1+1}(v,\gamma^1;\widetilde{\gamma})$$
$$\amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_0)} \mathcal{R}^{2+1}(\gamma^1,\widetilde{\gamma};\gamma_0)\times\mathcal{A}_d^{0+1}(v;\widetilde{\gamma}) \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_1,L_1)} \mathcal{R}^{2+1}(\widetilde{\gamma},\gamma^1;\gamma_0)\times\mathcal{A}_d^{0+1}(v;\widetilde{\gamma}). \tag{4.8.9}$$

The third type comes from the domain hitting the $R=1$ boundary and is precisely $[\mathcal{P}_d^1(v,\gamma^1;\gamma_0)]_{t=1}$. (4.8.10)

The fourth type comes from the domain hitting $R=\infty$ and is parametrized by

$$\coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{R}_1^{1+1}(\widetilde{x},\gamma^1;\gamma_0)\times\mathcal{R}_d^1(\gamma_d,\ldots,\gamma_1;\widetilde{x}) \tag{4.8.11}$$

□

To extend the filtered versions of the moduli spaces for the closed-open maps and the first homotopy, choose for all $\Sigma\in\mathcal{A}_d^{1+1}$ a path $e\colon[0,1]\to\Sigma$ such that $e(0)$ is on the outer boundary component to the right of ζ_0 and $e(1)$ is on the inner boundary. Then for any γ_0 and γ^1 with $n_\sigma(\gamma_0)=n_\sigma(\gamma_1)$ and any $u\in\mathcal{A}_d^{1+1}(v;\gamma^1;\gamma_0)$, $u\circ e$ is a path between interior Lagrangians which avoids D_σ, so it has a well defined intersection number with $\sigma(\hat{F}\times\mathbb{R}_+)$. Since the chords γ_i for $i>0$ have $n_\sigma(\gamma_i)=0$, we can homotope the end of e through ζ_i without changing the intersection number. This implies that the intersection number is independent of the choice of e, and so we call it $n_\sigma^{\mathcal{A}}(u)$. The space

$$\mathcal{A}_{d,filt}^{1+1}(v,\gamma^+;\gamma_0)$$

consists of all $u\in\mathcal{A}_d^{1+1}(v;\gamma^1;\gamma_0)$ which avoid D_σ and satisfy $n_\sigma^{\mathcal{A}}(u)=0$.

The space $\mathcal{A}_{d,filt}^{1+1}(v,\gamma^+;\gamma_0)$ is a union of connected components of $\mathcal{A}_d^{1+1}(v,\gamma^+;\gamma_0)$, and its boundary inherits the filtered condition. In other words, they are the same except in the following two ways. First, all annuli, broken annuli, and closed-open maps are replaced by their filtered versions. Second, for $n_\sigma(\gamma_0)=n_\sigma(\gamma^1)>0$, the terms

$$\coprod_{\widetilde{\gamma}\in\mathcal{X}(L_1,L_1)} \mathcal{R}^{2+1}(\widetilde{\gamma},\gamma^1;\gamma_0)\times\mathcal{A}_d^{0+1}(v;\widetilde{\gamma})$$

in (4.8.9) no longer contribute.

Define a linear map $h^2_y\colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*+\deg(v)+n-2}_{\boldsymbol{\sigma}'}(L_0, L_1)$ to depend linearly on y and, for $v = \gamma_d \otimes \cdots \otimes \gamma_1 \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$, to be given by

$$\begin{aligned} h^2_v(\gamma^m \otimes \cdots \otimes \gamma^0) \; = & \sum_{\substack{0\le i\le m+1 \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+n-1}} \#\mathcal{A}^{0+1}_d(v;\widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\ & + \sum_{\substack{0\le i\le m \\ n_\sigma(\gamma^r)=0\,\forall r<i \\ \widetilde{\gamma}\text{ making the result composable} \\ \deg(\widetilde{\gamma})=\deg(v)+\deg(\gamma^i)+n-2}} \#\mathcal{A}^{1+1}_{d,filt}(v, \gamma^i; \widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0. \end{aligned} \tag{4.8.12}$$

By essentially the same argument as for Lemma 4.7.5, we conclude

Lemma 4.8.2. *Up to terms which decrease the main filtration,*

$$h^2_v \mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'} h^2_v = h^2_{\delta v} + \widetilde{A}^{0,0}_v + \widetilde{A}^{0,1}_v + \mathcal{CO}^{filt}_{\mathcal{OC}(v)}. \tag{4.8.13}$$

□

4.9 The last homotopy

Our goal now is to construct, for a saddle unit $f_\sigma \in SC^1_{\boldsymbol{\sigma}}(M[\sigma])$, a homotopy $h^3_{f_\sigma}$ between $\mathcal{CO}^{filt}_{f_\sigma}$ and an operation id_σ which, while not the identity, induces the identity on the portion of the associated graded of $\hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$ which does not lie in $\hom^*_{\boldsymbol{\sigma}}(L_0, L_1)$.

Thus, let K^f be a Floer datum on $\mathbb{C}$ giving rise to f_σ as in 4.2.6. Denote by D^Y the closed unit disk with a puncture ζ^Y_0 at -1, which we equip with a negative strip-like end ϵ^Y_0 and a family of finite cylinders δ^Y as follows. Let $D^{0,1}_1$ be as in Section 4.6 with negative strip-like end ϵ_0. Then ϵ_0 induces ϵ^Y_0 via the unique biholomorphism $D^{0,1}_1 \to D^Y \setminus \{0\}$. For the cylinders, we are interested in a $(0, \frac{1}{2}]$-parametrized family

$$\delta^{D,Y}(\rho)\colon [a_\rho, b_\rho] \times S^1 \to \mathrm{int}(D^Y)$$

which satisfies

1. For ρ close to 0, $\delta^{D,Y}(\rho)$ agrees with the finite cylinder obtained by gluing ϵ^f on $\mathbb{C}$ to ϵ_+ on $D^{0,1}_1$ with length $e^{\frac{1}{\rho}}$. Here, we are implicitly using the biholomorphism from the glued surface to D^Y which sends $0 \in \mathbb{C}$ to $0 \in D^Y$ and ζ_0 to ζ^Y_0.

2. $b_{\frac{1}{2}} = a_{\frac{1}{2}}$. In other words, at $\rho = \frac{1}{2}$, the cylinder has width zero.

We will think of this data as a $(0, 1]$-parametrized space of Riemann surfaces with boundary, ends, and cylinders $D^Y(\rho)$, which for $\rho \in (0, \frac{1}{2}]$ is equipped with the strip-like end ϵ^Y_0 and finite cylinder $\delta^{D,Y}(\rho)$ and for $\rho > \frac{1}{2}$ is equipped only with ϵ^Y_0.

We consider smooth families of Floer data $\mathbf{K}^{D,Y}(\rho)$ on $D^Y(\rho)$ such that, in a gluing chart near $\rho = 0$, $\mathbf{K}^{D,Y}$ extends smoothly to 0, where it is conformally equivalent to the Floer data chosen for $\mathbb{C}$ and $D_1^{0,1}$. Let $\mathcal{K}^{D,Y}(M[\sigma])$ denote the space of such families.

In addition to the above data, choose $p_D\colon (0,1] \to [0,1]$ to be a nondecreasing smooth function which is 0 on $(0, \frac{1}{3}]$ and 1 on $[\frac{1}{2}, 1]$. Similarly, choose a smooth isotopy $Y_\sigma(\rho)$ of properly embedded hypersurfaces in $\widehat{M[\sigma]}$ which avoid $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$ for all ρ and satisfy the following conditions. First, $Y_\sigma(\rho) = Y_\sigma$ for $\rho \leq \frac{1}{4}$. Second, $Y_\sigma(\rho) \in \operatorname{image}(\sigma)$ for $\rho \geq \frac{1}{3}$, and moreover $Y_\sigma(1)$ is transverse to all chords between interior Lagrangians which appear in $\mathcal{W}_{\boldsymbol{\sigma}'}(M[\sigma])$.

Let L be an interior Lagrangian of $M[\sigma]$, and let $\gamma \in \mathfrak{X}(L, L)$. Given a family of Floer data $\mathbf{K}^{D,Y} \in \mathcal{K}^{D,Y}(M[\sigma])$, let $\mathcal{D}^Y(\gamma)$ be the union over all $\rho \in (0,1]$ of the spaces $\mathcal{D}^Y_\rho(\gamma)$ of maps

$$u\colon D^Y \to \widehat{M[\sigma]}$$

satisfying the following conditions.

1. u satisfies (4.1.6) for $\mathbf{K}^{D,Y}(\rho)$.
2. $u(z) \in (\phi^{\tau_E(\rho)(z)})^* L$ for $z \in \partial D^Y$, and $u(\zeta_0^Y) = (\phi^{\tau(\rho)})^*\gamma$, where τ is the conformal factor that $\mathbf{K}^{D,Y}$ assigns to ζ_0^Y.
3. $u(p_D(\rho)) \in Y_\sigma(\rho)$.

Gromov compactness applies to $\mathcal{D}^Y(\gamma)$, and in fact the situation is better than expected. Namely, suppose ρ_i is a sequence with $p_D(\rho_i) < 1$ but $\lim p_D(\rho_i) = 1$, and that $u_i \in \mathcal{D}^Y_{\rho_i}(\gamma)$ is a Gromov convergent sequence. Then we expect the Gromov limit to contain a bubble component with the incidence condition. However, in this case everything is exact, so all bubbles are constant. Thus, the incidence condition on the bubble is equivalent to an incidence condition on ∂D^Y, which means the incidence condition is a point of $L \cap Y_\sigma(\lim \rho_i)$. However, $Y_\sigma(\rho)$ lies in the image of σ whenever $p_D(\rho) = 1$, while L is an interior Lagrangian, which means that no such point exists. This shows that, in fact, $\mathcal{D}^Y_\rho(\gamma)$ is empty for $p_D(\rho)$ sufficiently close to 1.

Applying the usual transversality argument, we now obtain

Lemma 4.9.1. *There is a comeager subset $\mathcal{K}^{D,Y}_{reg}(M[\sigma]) \subset \mathcal{K}^{D,Y}(M[\sigma])$ such that for any $\mathbf{K}^{D,Y} \in \mathcal{K}^{D,Y}_{reg}(M[\sigma])$, the following hold.*

1. *For all interior Lagrangians L and all $\gamma \in \mathfrak{X}(L, L)$, $\mathcal{D}^Y(\gamma)$ is a smooth manifold of dimension $\deg(\gamma)$. It is empty if $n_\sigma(\gamma) > 0$.*
2. *If $\deg(\gamma) = 0$, then $\mathcal{D}^Y(\gamma)$ is compact.*

3. *If* $\deg(\gamma) = 1$*, then* $\mathcal{D}^Y(\gamma)$ *has a Gromov compactification* $\overline{\mathcal{D}}^Y(\gamma)$ *which is a compact topological 1-manifold with boundary, and there is a canonical identification*

$$\partial\overline{\mathcal{D}}^Y(\gamma) = \coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{R}_1^{0+1}(\widetilde{x};\gamma)\times\mathcal{C}(\widetilde{x}) \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L,L)} \mathcal{R}(\widetilde{\gamma};\gamma)\times\mathcal{D}^Y(\widetilde{\gamma})$$

□

Fix $\mathbf{K}^{D,Y} \in \mathcal{K}^{D,Y}_{\mathrm{reg}}(M[\sigma])$. We now repeat the above with one input. Choose a diffeomorphism $\Psi\colon [0,1] \to \overline{\mathcal{R}}_1^{1+1}$ such that $\Psi(0)$ is nodal at the first positive puncture of $\Sigma^{2+1} \in \mathcal{R}^{2+1}$ and $\Psi(1)$ is nodal at the second positive puncture of Σ^{2+1}. Let Z^Y be the strip $\mathbb{R}\times[0,1]$, where $\mathbb{R}\times\{i\}$ is $\partial_i Z^Y$, and the ends $-\infty$ and $+\infty$ are labeled ζ_0^Y and ζ_1^Y, respectively. Then Ψ induces a $(0,1)$-parametrized family of strip-like ends $\epsilon_q^{Z,Y}$ on Z^Y by specifying, for $q \in (0,1)$, the embedding $\Psi(q) \hookrightarrow Z^Y$ which sends $\partial\Psi(q)$ to ∂Z^Y, ζ_i to ζ_i^Y, and ζ_+ to a point on $\{0\}\times(0,1)$. Extend this to a $(0,1]\times(0,1)$-parametrized family $\boldsymbol{\epsilon}^{Z,Y}$ with the following properties

1. For $(\rho,q) \in (0,1]\times(0,1)$ with ρ small or $q \le \frac{1}{4}$ or $q \ge \frac{3}{4}$, $\boldsymbol{\epsilon}^{Z,Y}(\rho,q)$ agrees with $\epsilon_q^{Z,Y}$.
2. For ρ close to 1 and $q \in [\frac{1}{4},\frac{3}{4}]$, $\boldsymbol{\epsilon}^{Z,Y}(\rho,q)$ agrees up to shift with the canonical strip-like ends on Z.

Similarly, choose a smooth $(0,\frac{1}{2}]\times(0,1)$-parametrized family of finite cylinders $\delta^{Z,Y}$ on Z^Y as follows.

1. For $(\rho,q) \in (0,1]\times(0,1)$ with ρ small, $\delta^{Z,Y}(\rho,q)$ agrees with the finite cylinder obtained by gluing ϵ^f on $\mathbb{C}$ to ϵ_+ on $\Psi(q)$ with length $e^{\frac{1}{\rho}}$.
2. For q close to 0 or 1, $\delta^{Z,Y}(\rho,q)$ agrees with the finite cylinder induced by gluing $D^Y(\rho)$ to the appropriate input of Σ^{2+1} with length dictated by consistency with Ψ.
3. For $\rho = \frac{1}{2}$ and any q, the cylinder $\delta^{Z,Y}(\rho,q)$ has width zero.

We then think of this data as a $(0,1]\times(0,1)$-parametrized space of Riemann surfaces with boundary, ends, and cylinders $Z^Y(\rho,q)$, which for $\rho \in (0,\frac{1}{2}]$ is equipped with the strip-like ends $\boldsymbol{\epsilon}^{Z,Y}$ and finite cylinder $\delta^{D,Y}(\rho)$ and for $\rho > \frac{1}{2}$ is equipped only with $\boldsymbol{\epsilon}^{Z,Y}$.

Consider the space $\mathcal{K}^{Z,Y}(M[\sigma])$ of smooth, $(0,1]\times(0,1)$-parametrized families of Floer data $\mathbf{K}^{Z,Y}$ on Z^Y with the following properties.

1. For $(\rho,q) \in (0,1]\times(0,1)$ with ρ small, $\mathbf{K}^{Z,Y}(\rho,q)$ extends smoothly to $\rho = 0$, where it agrees up to conformal equivalence with the Floer data chosen for $\mathbb{C}$ and $\Psi(q)$.
2. For q close to 0 or 1, $\mathbf{K}^{Z,Y}(\rho,q)$ is conformally close in the sense of Definition 3.2.4 to the Floer datum induced by gluing $D^Y(\rho)$ to the appropriate input of Σ^{2+1} with length dictated by consistency with Ψ.

3. For $\rho = 1$ and $q \in [\frac{1}{4}, \frac{3}{4}]$, $\mathbf{K}^{Z,Y}(\rho, q)$ is conformally equivalent to the Floer perturbation $(H^{0,1}, dt, 1)$ which gives rise to the Floer differential.

Finally, we choose a function $p_Z\colon (0,1]\times(0,1) \to Z^Y$ extending p_D in the following sense. For q near 0 or 1, $p_D(\rho)$ can be thought of as a point on the thick part of D^Y, and we require $p_Z(\rho, q)$ to be the point of Z^Y with corresponds to $p_D(\rho)$ under gluing. For ρ close to 0, $p_Z(\rho, q)$ agrees with the point of $Z^Y(\rho, q)$ coming from the origin in $\mathbb{C}$ under the gluing of $\mathbb{C}$ to $\Psi(q)$. For ρ close to 1, we require that $p_Z(\rho, q)$ is ρ-independent and depends on q in the following way. For $q \le \frac{1}{3}$ or $q \ge \frac{2}{3}$, $p_Z(\rho, q)$ is on the boundary of Z^Y. For $q \in [\frac{1}{3}, \frac{2}{3}]$, the $[0,1]$-component of p_Z increases monotonically from 0 to 1.

Let L_0 and L_1 be interior Lagrangians of $M[\sigma]$, and let $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$. For a universal choice $\mathbf{K}^{Z,Y} \in \mathcal{K}^{Z,Y}(M[\sigma])$, the corresponding space of holomorphic strips is called $\mathcal{Z}^Y(\gamma_1; \gamma_0)$ and is the union over all $(\rho, q) \in (0,1] \times (0,1)$ of the spaces $\mathcal{Z}^Y_{\rho,q}(\gamma)$ of maps

$$u\colon Z^Y \to \widehat{M[\sigma]}$$

satisfying the conditions

1. u satisfies (4.1.6) for $\mathbf{K}^{Z,Y}(\rho, q)$.
2. $u(z) \in (\phi^{\tau_E(\rho)(z)})^* L_i$ for $z \in \partial_i D^Y$, and $u(\zeta_i^Y) = (\phi^{\tau_i(\rho,q)})^*\gamma_i$, where τ_i is the conformal factor that $\mathbf{K}^{Z,Y}$ assigns to ζ_i^Y.
3. $u(p_Z(\rho, q)) \in Y_\sigma(\rho)$.

The compactness situation is the same as before, and we have

Lemma 4.9.2. *There is a comeager subset $\mathcal{K}^{Z,Y}_{reg}(M[\sigma]) \subset \mathcal{K}^{Z,Y}(M[\sigma])$ such that for any $\mathbf{K}^{Z,Y} \in \mathcal{K}^{Z,Y}_{reg}(M[\sigma])$, the following hold.*

1. *For all interior Lagrangians L_0, L_1 and all $\gamma_0, \gamma_1 \in \mathcal{X}(L_0, L_1)$, $\mathcal{Z}^Y(\gamma_1;\gamma_0)$ is a smooth manifold of dimension* $\deg(\gamma_0) - \deg(\gamma_1) + 1$. *It is empty if $n_\sigma(\gamma_0) > n_\sigma(\gamma_1)$.*
2. *If* $\deg(\gamma_0) - \deg(\gamma_1) = -1$, *then $\mathcal{Z}^Y(\gamma_1;\gamma_0)$ is compact.*
3. *If* $\deg(\gamma_0) - \deg(\gamma_1) = 0$, *then $\mathcal{Z}^Y(\gamma_1;\gamma_0)$ has a Gromov compactification $\overline{\mathcal{Z}}^Y(\gamma_1;\gamma_0)$ which is a compact topological 1-manifold with boundary, and its boundary is in natural bijection with*

$$\begin{gathered}\coprod_{\widetilde{x}\in\mathcal{X}(H_t)} \mathcal{R}^{1+1}_1(\widetilde{x}, \gamma_1; \gamma_0) \times \mathcal{C}(\widetilde{x}) \ \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_1)} \mathcal{Z}^Y(\widetilde{\gamma}; \gamma_0) \times \mathcal{R}(\gamma_1; \widetilde{\gamma}) \\ \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_1,L_1)} \mathcal{R}^{2+1}(\widetilde{\gamma}, \gamma_1; \gamma_0) \times \mathcal{D}^Y(\widetilde{\gamma}) \ \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_0)} \mathcal{R}^{2+1}(\gamma_1, \widetilde{\gamma}; \gamma_0) \times \mathcal{D}^Y(\widetilde{\gamma}) \\ \amalg \coprod_{\widetilde{\gamma}\in\mathcal{X}(L_0,L_1)} \mathcal{R}(\widetilde{\gamma}; \gamma_0) \times \mathcal{Z}^Y(\gamma_1; \widetilde{\gamma}) \ \amalg \coprod_{\substack{t\in(0,1)\\ \gamma_0(t)\in Y_\sigma(1)}} \widetilde{\mathcal{R}}(\gamma_1; \gamma_0).\end{gathered}$$

Of course, by Lemma 3.1.3, the last term only occurs when $\gamma_0 = \gamma_1$. □

As with the other homotopies, there is a filtered version of $\mathcal{Z}^Y(\gamma_1;\gamma_0)$. This is obtained by choosing, for all (ρ, q), a path $e\colon [0,1] \to Z^Y$ such that $e(0) \in \partial_0 Z^Y$ and $e(1) = p_Z(\rho, q)$. The filtered component $\mathcal{Z}^Y_{filt}(\gamma_1;\gamma_0)$ consists of all $u \in \mathcal{Z}^Y(\gamma_1;\gamma_0)$ such that u avoids D_σ and $u \circ e$ has topological intersection number zero with $\sigma(\hat{F} \times \mathbb{R}_+)$. The condition that $Y_\sigma(\rho)$ avoids $\sigma(\hat{F} \times \mathbb{R}_{\geq 0})$ ensures that this is indeed a connected component of $\mathcal{Z}^Y(\gamma_1;\gamma_0)$.

Fixing $\mathbf{K}^{Z,Y} \in \mathcal{K}^{Z,Y}_{\mathrm{reg}}(M[\sigma])$, define a linear map $h^3_{f_\sigma}\colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*-1}_{\boldsymbol{\sigma}'}(L_0, L_1)$ by

$$
\begin{aligned}
h^3_{f_\sigma}(\gamma^m \otimes \cdots \otimes \gamma^0) = &\sum_{\substack{0 \leq i \leq m+1 \\ n_\sigma(\gamma^r) = 0\, \forall r < i \\ \widetilde{\gamma} \text{ making the result composable} \\ \deg(\widetilde{\gamma}) = 0}} \#\mathcal{D}^Y(\widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^i \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0 \\
&+ \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r) = 0\, \forall r < i \\ \widetilde{\gamma} \text{ making the result composable} \\ \deg(\widetilde{\gamma}) = \deg(\gamma^i) - 1}} \#\mathcal{Z}^Y_{filt}(\gamma^i;\widetilde{\gamma}) \cdot \gamma^m \otimes \cdots \otimes \gamma^{i+1} \otimes \widetilde{\gamma} \otimes \gamma^{i-1} \otimes \cdots \otimes \gamma^0.
\end{aligned}
\tag{4.9.1}
$$

We are finally rewarded for the bizarre filtered moduli spaces:

Lemma 4.9.3. *Let f_σ be a saddle unit, and let $\gamma = \gamma^m \otimes \cdots \otimes \gamma^0 \in \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1)$ have $\sum_{i=0}^m n_\sigma(\gamma_i) > 0$. Then, up to terms which decrease the main filtration,*

$$
\left(h^3_{f_\sigma}\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}h^3_{f_\sigma}\right)(\gamma) = \mathcal{CO}^{filt}_{f_\sigma}(\gamma) + \gamma \tag{4.9.2}
$$

Proof. Following the usual argument, we obtain

$$
\left(h^3_{f_\sigma}\mu^1_{\boldsymbol{\sigma}'} + \mu^1_{\boldsymbol{\sigma}'}h^3_{f_\sigma}\right)(\gamma) = \mathcal{CO}^{filt}_{f_\sigma}(\gamma) + \sum_{\substack{0 \leq i \leq m \\ n_\sigma(\gamma^r) = 0\, \forall r < i}} \sum_{\substack{t \in (0,1) \\ \gamma^i(t) \in Y_\sigma(1) \\ \gamma^i(t') \notin \sigma(\hat{F} \times \mathbb{R}_+) \text{ for any } t' < t}} \#\widetilde{\mathcal{R}}(\gamma^i;\gamma^i) \cdot \gamma.
$$

The coefficient $\#\widetilde{\mathcal{R}}(\gamma^i;\gamma^i)$ is of course 1, but we include it for clarity. Examining the conditions on the sums, we see that the only i which contributes is the smallest i such that $n_\sigma(\gamma^i) \neq 0$. For this γ^i, let $t_0 \in (0,1)$ be the first time at which γ^i intersects $\sigma(\hat{F} \times \mathbb{R}_+)$. Since γ^i starts outside the image of σ, it crosses $Y_\sigma(1)$ topologically once before t_0, and hence the sum contributes a total coefficient of 1. □

For $y \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ closed with $\mathcal{OC}(y) = f_\sigma$ a saddle unit, set $h_y := h^1_y + h^2_y + h^3_{f_\sigma}$. This is the last ingredient we need to prove the stop removal formula:

Proof of Theorem 4.2.7. Since σ is nondegenerate, there is some $y \in CC_*(\mathcal{B}_{\boldsymbol{\sigma}}(\sigma))$ such that $\delta y = 0$ and $\mathcal{OC}(y)$ is a saddle unit. Define a linear map $\Delta_y\colon \hom^*_{\boldsymbol{\sigma}'}(L_0, L_1) \to \hom^{*-1}_{\boldsymbol{\sigma}'}(L_0, L_1)$ by

$$
\Delta_y(\gamma) = \begin{cases} \Delta^0_y + h_y & \text{if } (n,m)(\gamma) > (1,0) \\ 0 & \text{otherwise.} \end{cases}
$$

Let $R_y = \mathrm{id} + \mu^1_{\boldsymbol{\sigma}'}\Delta_y + \Delta_y\mu^1_{\boldsymbol{\sigma}'}$ be the basic retraction as in Section 4.3. It suffices to show that R_y satisfies the conditions of Lemma 4.3.3, as this would prove Proposition 4.3.1 and hence Theorem 4.2.7. Condition (1) is trivial. To prove Condition (2), combine Equation 4.5.10 with Lemmas 4.7.5, 4.8.2, and 4.9.3. $\square$

Appendix A

Energy and compactness

To prove the various compactness claims, we need to know that, given a set of input chords or orbits, there are only finitely many possible outputs, and that these constrain the resulting holomorphic curves to some compact subset of our ambient space M. Appendix A.1 addresses the first part of this. For the second, we use Appendix A.2 away from cylindrical ends and interpolate to Ganatra's setup ([14] Appendix A) near those. Our maximum principle coincides with his for the "unperturbed region" on a separating open set, and so they patch together. All of the included proofs are essentially standard.

A.1 Action inequalities

Let Σ be a Riemann surface with boundary components $\partial_i\Sigma$, interior punctures z_j, and boundary punctures ζ^k. Let ϵ_j be cylindrical ends at z_j, and likewise let ϵ^k be strip-like ends at ζ^k, such that the images of the ends are pairwise disjoint.

Suppose further we are given a Liouville domain (M, λ_M), and that $\partial_i\Sigma$ is labeled with a smooth transverse family of Lagrangians $\{L_i(z) \in \hat{M} \mid z \in \partial_i\Sigma\}$ which is constant in the strip-like ends. Additionally, we have a Σ-parametrized compatible Hamiltonian $\widetilde{H}$, a Σ-parametrized perturbing Hamiltonian P, along with 1-forms β and β^ℓ with the following properties.

1. On the cylindrical and strip-like ends, let t be the coordinate of the compact S^1 or $[0,1]$ factor, and let s be the coordinate on the $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$ factor. Then $\widetilde{H}$ is both s- and t-independent, P is s-independent, and $\beta = w dt$ for some positive constant w depending on the end.

2. $d\beta \leq 0$ and $\beta|_{\partial\Sigma} = 0$.

3. Outside of a compact set, $d^\Sigma \widetilde{H} \wedge \beta \leq 0$, and further there is some $c < 0$ such that $\widetilde{H} d\beta + d^\Sigma \widetilde{H} \wedge \beta \leq c$ on the support of β^ℓ.

4. For any $z \in \partial_i\Sigma$ and $\xi \in T_z\partial\Sigma$, the vector field $\beta^\ell(\xi) \cdot X_{\sqrt{\widetilde{H}(z)}}$ is tangent outside of a compact set to the deformation of $L_i(z)$ associated to ξ.
5. P is globally bounded, and $\|X_P\|$ decays exponentially in the symplectization coordinate $\sqrt{\widetilde{H}}$ for any metric coming from some $J \in \mathcal{J}^\Sigma(M, \widetilde{H})$.
6. $d^\Sigma P \wedge \beta \leq 0$, and the supports of β^ℓ and dP are disjoint.

Finally, we suppose we have some Σ-parametrized $\hat{\omega}_M$-compatible almost complex structure J.

Set $H = \widetilde{H} + P$. We consider the space $\mathcal{M}$ of maps $u\colon \Sigma \to \hat{M}$ sending every $z \in \partial_i\Sigma$ to a point $L_i(z)$ and satisfying the inhomogeneous Cauchy-Riemann equation

$$J \circ (du - X_H \otimes \beta - X_{\sqrt{\widetilde{H}}} \otimes \beta^\ell) = (du - X_H \otimes \beta - X_{\sqrt{\widetilde{H}}} \otimes \beta^\ell) \circ j$$

where j is the almost-complex structure on Σ. Given a Kähler metric on Σ, define the **geometric energy** of such u as

$$E^{geom}(u) = \int_\Sigma \|du - X_H \otimes \beta - X_{\sqrt{\widetilde{H}}} \otimes \beta^\ell\|^2 dvol,$$

where the metric on $\hat{M}$ is Σ-dependent and is obtained from J. This is independent of the choice of Kähler metric on Σ and is in fact given by

$$E^{geom}(u) = \int_\Sigma \left(u^*\hat{\omega}_M - u^*(d^{\hat{M}} H) \wedge \beta - u^*(d^{\hat{M}} \sqrt{\widetilde{H}}) \wedge \beta^\ell \right).$$

In fact, this formula would hold and the theory would go through if $\sqrt{\widetilde{H}}$ were replaced with any other linear Hamiltonian H^ℓ satisfying $[H, H^\ell] = 0$, but it is hard to find such H^ℓ.

We also define the **topological energy** of u by

$$\begin{aligned} E^{top}(u) &= \int_\Sigma \left(u^*\hat{\omega}_M - d(u^* H\beta) - d(u^*\sqrt{\widetilde{H}}\beta^\ell) \right) \\ &= E^{geom}(u) - \int_\Sigma \left(u^* H d\beta + u^*(d^\Sigma H) \wedge \beta + u^*\sqrt{\widetilde{H}} d\beta^\ell + u^*(d^\Sigma \sqrt{\widetilde{H}}) \wedge \beta^\ell \right). \end{aligned}$$

Lemma A.1.1. *There is some constant $C \in \mathbb{R}$ depending only on Σ, $\widetilde{H}$, P, β, and β^ℓ for which*

$$E^{top}(u) \geq E^{geom}(u) + C$$

for all $u \in \mathcal{M}$.

Proof. We begin by noting that $Hd\beta$, $d^\Sigma H \wedge \beta$, $\sqrt{\widetilde{H}} d\beta^\ell$, and $d^\Sigma \sqrt{\widetilde{H}} \wedge \beta^\ell$ all belong to $\Omega^2(\Sigma; C^\infty(\hat{M}))$ and are compactly supported in the Σ-direction. Moreover, $Hd\beta$ and $d^\Sigma \widetilde{H} \wedge \beta$

are asymptotically quadratic in the symplectization coordinate r, while $d^\Sigma\sqrt{\widetilde{H}} \wedge \beta^\ell$ and $\sqrt{\widetilde{H}}d\beta^\ell$ are linear in r and $d^\Sigma P \wedge \beta$ is nonpositive by assumption. Thus, condition 3 above shows that

$$Hd\beta + (d^\Sigma H) \wedge \beta + \sqrt{\widetilde{H}}d\beta^\ell + (d^\Sigma\sqrt{\widetilde{H}}) \wedge \beta^\ell$$

is nonpositive outside of a compact subset of $\Sigma \times \hat{M}$. This means that it is bounded above, and because it is compactly supported in the Σ-direction this gives the desired conclusion. □

We now examine the relation between action and topological energy . For any $u \in \mathcal{M}$ with $E^{geom}(u) < \infty$, u converges at ϵ_j to a 1-periodic orbit x_j of $w_j X_H$ and at ϵ^k to a chord γ^k of $w^k X_H$, for appropriate 1-parameter specializations of H. Before we define the action, choose for each boundary component $\partial_i\Sigma$ a smooth family of functions $f_i(z) \in C^\infty(L_i(z))$ which satisfy $d[f_i(z)] = \hat{\lambda}_M|_{L_i(z)}$ and are independent of z in the strip-like ends. For an orbit x, define

$$A(x) = \int_{S^1} \left(x^*\hat{\lambda}_M - wx^*Hdt\right).$$

For a chord γ, let L_0 and L_1 be the Lagrangians containing $\gamma(0)$ and $\gamma(1)$, respectively, and let f_0 and f_1 be the primitives chosen above for L_0 and L_1. Define

$$\begin{aligned} A_0(\gamma) &= \int_{[0,1]} \left(\gamma^*\hat{\lambda}_M - w\gamma^*Hdt\right) \\ A(\gamma) &= \int_{[0,1]} \left(\gamma^*\hat{\lambda}_M - w\gamma^*Hdt\right) + f_0(\gamma(0)) - f_1(\gamma(1)). \end{aligned}$$

With this set up, we have

Lemma A.1.2. *There is some constant $D \in \mathbb{R}$ depending only Σ, $\widetilde{H}$, β^ℓ, $\{L_i\}$, and $\{f_i\}$ for which*

$$E^{top}(u) \leq \sum_{\substack{\text{positive}\\ \text{cylindrical}\\ \text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive}\\ \text{strip-like}\\ \text{ends } \epsilon^k}} A(\gamma^k) - \sum_{\substack{\text{negative}\\ \text{cylindrical}\\ \text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative}\\ \text{strip-like}\\ \text{ends } \epsilon^k}} A(\gamma^k) + D$$

for all $u \in \mathcal{M}$.

Proof. To begin, pick an element $\eta \in \Omega^1(\partial\Sigma; \Gamma(T\hat{M}))$ which agrees with $X_{\sqrt{\widetilde{H}}}\beta^\ell$ outside of a compact subset of $\partial\Sigma \times \hat{M}$ and such that, for all $z \in \partial_i\Sigma$ and $\xi \in T_z\partial\Sigma$, the vector field $\eta(z)(\xi)$ is tangent to the deformation of $L_i(z)$ associated to ξ. For any $u \in \mathcal{M}$ of finite energy, $du|_{\partial\Sigma} - \eta(u(z))$ is then valued at $z \in \partial_i\Sigma$ in vector fields tangent to the Lagrangian submanifold $L_i(z)$.

Next, for each i, let $L_{(i)}$ be the smooth manifold underlying $L_i(z)$, so that we may view $L_i(z)$ as a family of exact Lagrangian embeddings $\Lambda_i(z)\colon L_{(i)} \hookrightarrow \hat{M}$. Moreover, we may arrange that for any vector field $\xi \in \Gamma(T\partial_i\Sigma)$, we have

$$\mathcal{L}_\xi\Lambda_i = \eta(\xi)$$

After pulling back by Λ_i, we may view $f_i(z)$ as functions on $L_{(i)} \times \partial_i\Sigma$.

For any $u \in \mathcal{M}$ of finite energy, set

$$A_0(u) = \sum_{\substack{\text{positive}\\\text{cylindrical}\\\text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive}\\\text{strip-like}\\\text{ends } \epsilon^k}} A_0(\gamma^k) - \sum_{\substack{\text{negative}\\\text{cylindrical}\\\text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative}\\\text{strip-like}\\\text{ends } \epsilon^k}} A_0(\gamma^k)$$

$$A(u) = \sum_{\substack{\text{positive}\\\text{cylindrical}\\\text{ends } \epsilon_j}} A(x_j) + \sum_{\substack{\text{positive}\\\text{strip-like}\\\text{ends } \epsilon^k}} A(\gamma^k) - \sum_{\substack{\text{negative}\\\text{cylindrical}\\\text{ends } \epsilon_j}} A(x_j) - \sum_{\substack{\text{negative}\\\text{strip-like}\\\text{ends } \epsilon^k}} A(\gamma^k).$$

We evaluate

$$\begin{aligned}
E^{top}(u) &= \int_{\partial\overline{\Sigma}} \left(u^*\hat{\lambda}_M - u^*H\beta - u^*\sqrt{\widetilde{H}}\beta^\ell\right) \\
&= A_0(u) + \int_{\partial\Sigma} \left(u^*\hat{\lambda}_M - u^*\sqrt{\widetilde{H}}\beta^\ell\right) \\
&= A_0(u) + \int_{\partial\Sigma} \left(\langle\hat{\lambda}_M, du - \eta(u)\rangle + \langle\hat{\lambda}_M, \eta(u)\rangle - u^*\sqrt{\widetilde{H}}\beta^\ell\right) \\
&\leq A_0(u) + \sum_i \int_{\partial_i\Sigma} \left(\langle d^{L_{(i)}} f_i, du - \eta(u)\rangle + \langle\hat{\lambda}_M, X_{\sqrt{\widetilde{H}}}(u)\beta^\ell\rangle - u^*\sqrt{\widetilde{H}}\beta^\ell\right) + D_1 \\
&\leq A_0(u) + \sum_i \int_{\partial_i\Sigma} \langle d^{L_{(i)}} f_i, du - \eta(u)\rangle + D_2 \\
&= A_0(u) + \sum_i \int_{\partial_i\Sigma} u^* d^{L_{(i)}} f_i + D_2 \\
&= A_0(u) + \sum_i \int_{\partial_i\Sigma} \left(u^* df_i - d^{\partial_i\Sigma} f_i(u)\right) + D_2 \\
&= A(u) - \sum_i \int_{\partial_i\Sigma} d^{\partial_i\Sigma} f_i(u) + D_2
\end{aligned}$$

As a 1-form on $\partial_i\Sigma$, this final integrand is compactly supported and globally bounded independently of u, so it can be bounded by an additive constant. This completes the proof. □

Because $E^{geom}(u) \geq 0$ for any $u \in \mathcal{M}$, Lemmas A.1.1 and A.1.2 give an upper bound on the action of the output chords and orbits in terms of the data on Σ and the action of the input chords and orbits. To obtain finiteness, we note that for any orbit x on the portion of

$\hat{M}$ where H is quadratic, we have

$$\begin{aligned}A(x) &= \int_{S^1} \left(x^*\hat{\lambda}_M - wx^*Hdt\right) \\ &= \int_{S^1} \left(\hat{\lambda}_M(wX_H)dt - wx^*Hdt\right) \\ &= w\int_{S^1} \left(x^*(2\widetilde{H})dt - x^*\widetilde{H}dt\right) + w\int_{S^1} \left(\hat{\lambda}_M(X_P)dt - x^*Pdt\right) \\ &= w\int_{S^1} \widetilde{H}(x)dt + w\int_{S^1} \left(\hat{\lambda}_M(X_P) - x^*P\right) dt.\end{aligned}$$

The first integral is very nearly the same as $w\widetilde{H}(x(t_0))$, while the second is globally bounded. In particular A is proper and bounded below on the space of X_H-orbits. A similar result holds for $A(\gamma)$ or $A_0(\gamma)$. Thus, the space of possible outputs for a given choice of inputs is compact. Since our Hamiltonians are nondegenerate, this shows that it is finite.

A.2 A maximum principle

With data as above, we further assume that $P = 0$, J is asymptotically $\hat{Z}_M$-invariant on the contact planes $\ker dH \cap \ker \hat{\lambda}_M$, and that there is some compact subset $K \subset \hat{M}$ such that, outside $\Sigma \times K$,

$$d^{\hat{M}} H \circ J = -g(H)\hat{\lambda}_M \tag{A.2.1}$$

for some strictly positive function g. Additionally, we require that there is some k such that $d\beta < k < 0$ on the supports of $\partial^\Sigma H$ and β^ℓ.

Note that $H(z)$ is quadratic for all $z \in \Sigma$, which means we can write $d^\Sigma H = H\nu$ for some $\nu \in \Omega^1(\Sigma, C^\infty(\hat{M}))$. Moreover, we may enlarge K to assume ν is $\hat{Z}_M$-invariant on the complement of $\Sigma \times K$.

We are interested in two regimes. In the first, $\partial^\Sigma H = 0$ and $\beta^\ell = 0$, but g is allowed to be Σ-dependent. In the second, $g(x) = cx$ for some $c > 0$, and no assumptions are made on $\partial^\Sigma H$ or on β^ℓ. In either case, we will use the Hopf maximum principle, which states that if some function $F\colon \Sigma \to \mathbb{R}$ satisfies $\Delta F \geq 0$ modulo dF, then the local maxima of F all occur on $\partial\Sigma$, and that at such local maxima the outward normal derivative of F is strictly positive. We take $F = H \circ u$, and compute

$$\begin{aligned}d^c F &= d^\Sigma H(u) \circ j + d^{\hat{M}} H \circ du \circ j \\ &= F\nu(u) \circ j + d^{\hat{M}} H \circ (du - X_H\beta - X_{\sqrt{H}}\beta^\ell) \circ j \\ &= F\nu(u) \circ j - g(F)\hat{\lambda}_M \circ (du - X_H\beta - X_{\sqrt{H}}\beta^\ell) \\ &= F\nu(u) \circ j - g(F) \cdot \left(u^*\hat{\lambda}_M - 2F\beta - \sqrt{F}\beta^\ell\right).\end{aligned}$$

In the first regime, $\nu = 0$ and $\beta^\ell = 0$, so that modulo dF we have

$$\begin{aligned}\Delta F \cdot dvol &= -dd^c F \\ &= d^\Sigma g(F) \wedge \left(u^* \hat{\lambda}_M - 2F\beta\right) + g(F) \cdot (u^* \hat{\omega}_M - 2F d\beta) \\ &= d^\Sigma g(F) \wedge \frac{-1}{g(F)} d^c F + g(F) \cdot \Big([u^* \hat{\omega}_M - u^*(d^{\hat{M}} H) \wedge \beta - u^*(d^{\hat{M}} \sqrt{H}) \wedge \beta^\ell] \\ &\qquad\qquad + u^*(d^{\hat{M}} H) \wedge \beta - 2F d\beta \Big).\end{aligned}$$

The $d^c F$ term is linear in dF and can be ignored, and in the first regime $dH = d^{\hat{M}} H$, so modulo dF we are left with

$$\begin{aligned}\Delta F \cdot dvol &= g(F) \cdot \left(\|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 + u^*(dH) \wedge \beta - 2F d\beta \right) \\ &= g(F) \cdot \left(\|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 + dF \wedge \beta - 2F d\beta \right) \\ &= g(F) \cdot \left(\|du - X_H \beta - X_{\sqrt{H}} \beta^\ell\|^2 - 2F d\beta \right).\end{aligned}$$

The right hand side is globally nonnegative, so the Hopf maximum principle applies.

In the second regime, we instead have

$$d^c F = F \nu \circ j - cF u^* \hat{\lambda}_M + 2cF^2 \beta + cF^{\frac{3}{2}} \beta^\ell.$$

For convenience of notation, we will write in local coordinates

$$-\nu(u) \circ j = c\nu_1(s, t, u(s,t)) ds + c\nu_2(s, t, u(s,t)) dt,$$

where ν_1 and ν_2 are functions on $\Sigma \times \hat{M}$ which are $\hat{Z}_M$-invariant outside $\Sigma \times K$. Modulo

dF, this gives

$$
\begin{aligned}
-dd^cF &= cF\cdot\Big((D_1\nu_2 - D_2\nu_1)ds\wedge dt + \langle D_3\nu_1, du\rangle\wedge ds + \langle D_3\nu_2, du\rangle\wedge dt \\
&\qquad\qquad + u^*\hat{\omega}_M - 2Fd\beta - \sqrt{F}d\beta^\ell\Big) \\
&= cF\cdot\Big(\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 + u^*(d^{\hat{M}}H)\wedge\beta + u^*(d^{\hat{M}}\sqrt{H})\wedge\beta^\ell \\
&\quad + \langle D_3\nu_1, du\rangle\wedge ds + \langle D_3\nu_2, du\rangle\wedge dt - 2Fd\beta - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds\wedge dt\Big) \\
&= cF\cdot\Big(\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 + dF\wedge\beta + d\sqrt{F}\wedge\beta^\ell - d^\Sigma H(u)\wedge\beta - d^\Sigma\sqrt{H}(u)\wedge\beta^\ell \\
&\quad + \langle D_3\nu_1, du - X_H\beta - X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, du - X_H\beta - X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad - 2Fd\beta - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds\wedge dt\Big) \\
&= cF\cdot\Big(\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 \\
&\quad + \langle D_3\nu_1, du - X_H\beta - X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, du - X_H\beta - X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad - 2Fd\beta - F\nu(u)\wedge\beta - \frac{1}{2}\sqrt{F}\nu(u)\wedge\beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds\wedge dt\Big).
\end{aligned}
$$

Now $D_3\nu_1$ and $D_3\nu_2$ vanish on $\hat{Z}_M$ and are $\hat{Z}_M$-invariant. Since the metric $\hat{\omega}_M(\cdot, J\cdot)$ grows with H in the ∂M directions, $\|D_3\nu_1\|$ and $\|D_3\nu_2\|$ tend to zero as H tends to infinity. Thus, we can apply Cauchy-Schwarz and obtain that, modulo dF,

$$
\begin{aligned}
-dd^cF &\geq cF\cdot\Big(\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\|^2 dvol - \frac{1}{c}\|D_3\nu\|\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\| dvol \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad - 2Fd\beta - F\nu(u)\wedge\beta - \frac{1}{2}\sqrt{F}\nu(u)\wedge\beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds\wedge dt\Big) \\
&= cF\cdot\Big(\big(\|du - X_H\beta - X_{\sqrt{H}}\beta^\ell\| - \frac{1}{2c}\|D_3\nu\|\big)^2 dvol - 2Fd\beta - F\nu(u)\wedge\beta \\
&\quad + \langle D_3\nu_1, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge ds + \langle D_3\nu_2, X_H\beta + X_{\sqrt{H}}\beta^\ell\rangle\wedge dt \\
&\quad - \frac{1}{2}\sqrt{F}\nu(u)\wedge\beta^\ell - \sqrt{F}d\beta^\ell + (D_1\nu_2 - D_2\nu_1)ds\wedge dt - \frac{1}{4c^2}\|D_3\nu\|^2 dvol\Big)
\end{aligned}
$$

On the other hand, $-2Fd\beta$ and $-F\nu(u)\wedge\beta$ are assumed to be nonnegative, and moreover $-2Fd\beta$ grows faster than any term in the last two lines and is assumed to be strictly positive on the support of those terms. Thus, the right hand side is nonnegative for sufficiently large F. In particular, we obtain

Lemma A.2.1. *In the above setup, suppose there is some k such that $d\beta < k < 0$ on the supports of $d^{\Sigma}H$ and β^{ℓ}, and that*

$$\mathrm{support}(d^{\Sigma}g) \cap \big(\mathrm{support}(d^{\Sigma}H) \cup \mathrm{support}(\beta^{\ell})\big) = \emptyset.$$

Then there is some R depending on g, H, β, and β^{ℓ}, but not on u, for which u^*H satisfies the Hopf maximum principle outside of $H^{-1}((-\infty, R))$. □

It remains to prevent maxima on $\partial\Sigma$. For this, let ξ be a vector field along $\partial\Sigma$ which points in the negative direction, so that $j\xi$ points outward. We calculate

$$\begin{aligned} dF(j\xi) = d^cF(\xi) &= d^{\Sigma}H(u)(j\xi) - g(F) \cdot \Big(\hat{\lambda}_M(du(\xi)) - 2F\beta(\xi) - \sqrt{F}\beta^{\ell}(\xi)\Big) \\ &= d^{\Sigma}H(u)(j\xi) - g(F)\hat{\lambda}_M\left(du(\xi) - X_{\sqrt{H}}\beta^{\ell}(\xi)\right) \\ &= d^{\Sigma}H(u)(j\xi). \end{aligned}$$

This gives

Lemma A.2.2. *In the situation of Lemma A.2.1, if additionally $d^{\Sigma}H$ vanishes on outward normal vectors at $\partial\Sigma$, then u^*H has no local maxima outside $H^{-1}((-\infty, R))$. In other words, u^*H is bounded by the larger of R and the values of H on the asymptotic X_H chords and orbits.*

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